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Dynamics of Cosmological Fluids in the Critical Regime of Slow Expansion

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ZUSAMMENFASSUNG

In dieser Dissertation untersuchen wir das Verhalten relativistischer Fluide in langsam expandierenden Raumzeiten. Sie werden durch Systeme hyperbolischer Differentialgleichungen modelliert, die anfällig für die Bildung von Schocksingularitäten sind. Expandiert die zugrundeliegende Raumzeit exponentiell, so wird die Entstehung solcher Singularitäten unterdrückt. Wir untersuchen, welche Expansionsgeschwindigkeit ausreicht, um das Fluid zu stabilisieren. Für lineare barotropische Zustandsgleichungen mit Schallgeschwindigkeitsparameter K zeigen wir nichtlineare Stabilität linear expandierender, räumlich kompakter FLRW-Modelle für die Fälle $0 < K < \frac{1}{3}$ und den Fall ohne Druck ($K = 0$). Diese Resultate sind die ersten ihrer Art im Regime linearer Expansion. Für die Fälle $0 < K < \frac{1}{3}$ zeigen wir zudem die nichtlineare Stabilität der homogenen und isotropen Lösungen der relativistischen Euler-Gleichungen auf einem fixierten, linear expandierenden Hintergrund. Weiters untersuchen wir diese Lösungen mithilfe analytischer und numerischer Methoden in $(1 + 1)$-Symmetrie auf räumlich flachen Raumzeiten mit abnehmender Expansionsgeschwindigkeit. Unter der Annahme eines Potenzgesetzes für den Skalenfaktor, $a(t) = t^\alpha$, zeigen wir Stabilität jenseits der kritischen Expansionrate, gegeben durch die Beziehung $K(\alpha) = 1 - \frac{2}{3\alpha}$. Außerdem diskutieren wir numerische Evidenz für eine Schockbildung unterhalb dieser Expansionsrate. Zusätzlich verallgemeinern wir das Stabilitätsresultat jenseits der kritischen Expansionsrate auf $3 + 1$ Dimensionen. Zuletzt zeigen wir, dass sich in den Fällen von verschwindendem Druck ($K = 0$) und Strahlung ($K = \frac{1}{3}$) auf einem fixierten, räumlich flachen Hintergrund Schocksingularitäten bilden. Für Strahlung gilt dies bei $\alpha \leq 1$ und im drucklosen Regime bei $\alpha \leq \frac{1}{2}$.

ABSTRACT

In this thesis, we investigate the mathematical behavior of relativistic fluids in cosmological spacetimes with slow expansion. These are modeled by systems of hyperbolic differential equations, which are prone to shock formation. Such singularities are suppressed if fluids evolve in an exponentially expanding spacetime. We show that shocks still form if the expansion is not sufficiently fast and investigate where this transition occurs. For a linear barotropic equation of state with speed of sound parameter K, we prove the full nonlinear stability of linearly expanding spatially compact FLRW models as solutions to the Einstein-Euler system in the cases of dust ($K = 0$) and massive fluids ($0 < K < \frac{1}{3}$). These results are the first of their kind in the regime of linear expansion. In the case of massive fluids, we also prove the nonlinear stability of the homogeneous and isotropic solutions to the relativistic Euler equations on fixed backgrounds of linear expansion. Furthermore, we investigate such solutions on spatially flat, decelerated spacetimes in $(1 + 1)$-symmetry, using analytical and numerical methods. In the context of a power law inflation, $a(t) = t^\alpha$, we prove a stability result beyond the critical expansion rate, which is given by the relation $K(\alpha) = 1 - \frac{2}{3\alpha}$. In addition, we provide numerical evidence for shock formation below this rate. Furthermore, we relax this symmetry assumption proving stability in $3 + 1$ dimensions. Finally, we provide generic instability results for fluids close to the homogeneous and isotropic state for radiation ($K = \frac{1}{3}$) and dust on fixed spatially flat background, using the method of characteristics. In particular, we show that radiation fluids are unstable for $\alpha \leq 1$, whereas dust is unstable for $\alpha \leq \frac{1}{2}$.

PREFACE

The mathematical analysis of fluids in the context of general relativity has been an active field of research over the last decades. Besides isolated fluids and models of collapse, perfect fluid solutions to the Einstein equations are the predominant model used in cosmology. To understand the global behavior of solutions close to this class, their nonlinear stability is investigated. While fluids in non-expanding spacetimes generically form shocks, such singularity formation is suppressed for rapidly expanding models. However, the conditions regarding the expansion rate and the speed of sound under which a transition from stability to instability takes place, are currently not known. The present thesis seeks to bridge this gap by investigating solutions to the Einstein-Euler system as well as the relativistic Euler equations in the regimes of linear and decelerated expansion.

In chapter 1 we present a panoramic view of classical and relativistic fluid dynamics. We focus on key physical and mathematical aspects, relevant to the contents of the following chapters. This includes the classical and relativistic Euler equations, mathematical cosmology and systems of hyperbolic equations.

In chapter 2 we discuss the current state of the art regarding the stability of cosmological fluid models. In addition, we discuss how the contents of this thesis tie into the research in this field.

Chapter 3 is based on joint work with David Fajman and Zoe Wyatt and is published as [40]. The key novelty is a proof of the stability of the Milne universe as a solution to the Einstein-dust system.

Chapter 4 is based on joint work with David Fajman, Todd Oliynyk and Zoe Wyatt and is published as [38]. It treats the massive fluid case and is split into two parts. The first part is concerned with the stability of the homogeneous solution to the relativistic Euler equations in linearly expanding backgrounds. The second part generalizes this to the stability of the Milne universe as a solution to the Einstein-Euler system.

Chapter 5 is based on joint work with David Fajman, Maciej Maliborski, Todd Oliynyk and Zoe Wyatt. It is currently under review and available as a preprint [36]. This letter-style article is concerned with solutions to the relativistic Euler equations in the decelerated regime in symmetry. We use analytical methods to identify a stable region depending on the expansion rate and the speed of sound. Furthermore, we present numerical evidence for instabilities in the complement of this region.

Chapter 6 is based on joint work with David Fajman, Todd Oliynyk and Zoe Wyatt. It is currently under review and available as a preprint [37]. This article is mainly concerned with generalizing the stability result of chapter 5 to $3+1$ dimensions. In addition, it presents a new straightforward method for identifying the sharp expansion rate that is necessary for shocks to form in the case of dust and radiation matter as solutions to the relativistic Euler equations on a spatially flat expanding spacetime.

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CHAPTER 1

INTRODUCTION

1.1 The fluid model of the universe

In modern-day cosmology, the matter content of our universe is modeled by a fluid. Individual fluid elements contain stars and galaxies and are assumed to be in thermodynamic equilibrium. This is done in analogy to the classical case, where fluids provide a simplified continuum description of a large number of particles with complex individual dynamics. While the fluid description does not give a complete picture of every microscopic interaction, it provides a very good macroscopic approximation with remarkably few quantities. A handful of variables like the density, velocity and pressure of the fluid give almost complete information about its dynamics.

As is the case in most of physics, there are different levels of approximation for fluids. The simplest model in this regard is one of vanishing viscosity and heat conductivity. This is usually referred to as a perfect fluid. It is governed by the Euler equations, which are a special case of the more general Navier-Stokes equations. The models predominantly used in cosmology describe the matter content of the universe as a perfect fluid. The Euler equations are, however, derived from the laws of classical mechanics and are therefore insufficient to model the dynamics of very massive objects.

Indeed, a relativistic analog is needed for a description on cosmological scales. Hence, Einstein's theory of general relativity is used to formulate the laws of cosmology. The conservation of energy and momentum then imply a new set of equations, which resemble the classical Euler equations. These are referred to as the relativistic Euler equations and reduce to their classical analog in the Newtonian limit. They can be coupled to the Einstein equations that govern the structure of spacetime to form the fully dynamic Einstein-Euler system. This system has many interesting solutions. However, cosmological observations suggest that some are particularly accurate in modeling the large-scale structure of the universe. These solutions originate from a point of singularity, the big bang, and expand continuously over time. They can be derived under the assumption that spacetime as well as the fluid contained in it are highly symmetric at each instant in time. The problem then reduces to solving the Friedman equations that relate the density and pressure of the fluid to the expansion rate of spacetime. We will refer to solutions of this type as Friedman solutions. They are homogeneous and isotropic, meaning that they have no preferred spatial point or direction, respectively.

The post-inflation history of the universe can be split up into three parts, each of them corresponding approximately to one type of fluid component. In the first era, the radiation content of the universe determines the expansion rate. Afterwards, it cools down and the matter content becomes dominant. In the third and final epoch, expansion accelerates, driven by so-called dark energy. This dark energy is modeled by the cosmological constant in Einstein's equations and its physical origin is currently not well understood. However, it explains the current rapid expansion of the universe.

In this thesis we will discuss the stability properties of cosmological solutions to the relativistic Euler equations and the Einstein-Euler system, with a focus on linear and decelerated expansion. Before doing so, however, we will provide an overview of the physical and mathematical concepts involved.

1.2 From classical to relativistic fluid dynamics

We start by introducing some notation. We use the Einstein summation convention. This means that if a term contains the same index as sub- and superscript it is tacitly assumed to be summed over. As an example, consider

$$\delta_{ij}u^i v^j \quad \Big(\equiv \sum_{i,j=1}^{n} \delta_{ij}u^i u^j\Big).$$

In this thesis Greek letters $\alpha, \beta, \gamma \ldots$ denote spacetimes indices, while roman letters $a, b, c, \ldots$ denote spatial indices.

The symbol ∇ denotes the Levi-Civita connection.

Let $f \in C_c(M)$, where (M, g) is an oriented Riemannian manifold. We define

$$\int_{(M,g)} f := \int_M f dV_g,$$

where $\varphi^* dV_g = \sqrt{\det(\varphi^* g)} dx^1 \wedge \cdots \wedge dx^n$ for every oriented chart $\varphi : \mathbb{R}^n \supset U \to O \subset M$. If the context is unambiguous, we will simply write

$$\int f := \int_{(M,g)} f.$$

On subsets of $\mathbb{R}^n$ the metric g is always assumed to be Euclidean.

Minkowski space is the $(n+1)$-dimensional Lorentzian manifold

$$(\mathbb{R}^{n,1}, \eta = -dt^2 + \delta_{ij}dx^i dx^j).$$

Hence, we use the 'mostly plus' convention. For definitions and a review of Lebesgue and Sobolev spaces on manifolds see [56].

As is common in the mathematical literature, we set the speed of light c and the Einstein constant $\kappa = \frac{8\pi G}{c^4}$ to 1. This ensures that the Einstein tensor is equal to the energy-momentum tensor.

1.2.1 The hydrodynamic equations of a perfect fluid

In this section, we will give a concise description of a perfect fluid at the expense of being rather informal. All maps are assumed to have a domain of the type $D = I \times U$, where $I \subset \mathbb{R}$ is an open interval and $U \subset \mathbb{R}^n$, $n \geq 2$, is open. In addition, all maps and boundaries are assumed to be as regular as needed. We will use (t, x) as coordinates on D.

We can think of a fluid as a tuple $(\rho, u, \varepsilon, p)$, where $\rho, \varepsilon, p : D \to \mathbb{R}$ are functions on D and $u : D \to \mathbb{R}^n$ is a vector field. The quantities ρ, u, ε and p are referred to as (mass) density, velocity, specific internal energy and pressure of the fluid, respectively. The mass m contained in a region $W \subset U$ at time t is defined as

$$m(W, t) \coloneqq \int_W \rho(t, \cdot).$$

A fluid particle at x_0 at time s_0 is described by a path $\gamma : I \supset J \to U$, satisfying

$$\dot{\gamma}(s) = u(s, \gamma(s)), \quad \gamma(s_0) = x_0. \tag{1.2.1}$$

Hence, fluid particles follow integral curves of the vector field u. This notion is important, as it allows us to describe the local acceleration

$$\ddot{\gamma}(s) = \Big(D_t u\Big)(s, \gamma(s))$$

in terms of the global quantity u. The operator D_t is defined as

$$D_t f(t, x) \coloneqq \frac{d}{dt} f(t, \gamma(t)) = (\partial_t + u^i(t, x)\partial_{x^i}) f(t, x). \tag{1.2.2}$$

It is called the material derivative.

Remark 1.2.1. This highlights a fundamental property of fluid dynamics. Assume that at some point t_0 in time, the configuration of all particles is known. We can then flow forwards and backwards in time from every point $\xi \in U$ using unique solutions of the ODE given in (1.2.1). The coordinates (t, ξ) are known as the Lagrangian or material description of the fluid. They are related to the coordinates (t, x) of the Eulerian description via

$$(t, x) = (t, \gamma(t)), \quad \gamma(t_0) = \xi,$$

where γ is the integral curve of u. Under suitable conditions this is a diffeomorphism. We will encounter situations in which either one or the other picture is beneficial. From (1.2.2) we see that D_t is the pushforward of $\frac{d}{dt}$ along the transformation from the Lagrangian to the Eulerian picture, justifying its name.

To complete our definition of a fluid we impose the following heuristic physical laws.

(i) Mass can neither be destroyed nor created.
(ii) Newton's second law applies.
(iii) Energy is conserved.
(iv) Every fluid element is in thermodynamic equilibrium.

Furthermore, we say that a fluid is a perfect fluid, if it satisfies the following additional conditions.

(v) For any surface the only source of stress acting on it is pressure, which is normal to the surface.
(vi) Thermal conductivity vanishes.

A perfect fluid satisfies the following set of equations.

$$D_t\rho + \rho \operatorname{div} u = 0, \tag{1.2.3a}$$

$$D_t u + \frac{1}{\rho} \operatorname{grad} p - F = 0, \tag{1.2.3b}$$

$$\rho D_t \varepsilon + p \operatorname{div} u = 0. \tag{1.2.3c}$$

The quantity F represents the external body force per unit mass. The system (1.2.3) represents the equations of motion of a perfect fluid. The continuity equation (1.2.3a) corresponds to conservation of mass (i), whereas the balance of momentum (1.2.3b) follows from (ii) and (v). Equations (1.2.3b) are usually referred to as Euler's equations. The last equation (1.2.3c) describes the evolution of the specific internal energy ε and follows from (iii), (iv) and (vi). For a classical derivation see, e.g., [20] or [104], whereas a particularly elegant derivation via kinetic theory can be found in [92].

Remark 1.2.2. The system (1.2.3) is still somewhat heuristic. It can be derived from simple physical laws and provides a starting point for a rigorous formulation including proper definitions of a domain, boundary, initial data and regularity.

Furthermore, note that (1.2.3) is a system of $n+2$ equations for the $n+3$ variables $\rho, u^i, p, \varepsilon$, where $i = 1, \dots, n$. To close the system, we need a thermodynamic equation of state $p = p(\varepsilon, \rho)$.

A common way of simplifying (1.2.3) is to require the flow to be incompressible, meaning that $\operatorname{div} u = 0$. This leads to conservation of specific internal energy along flowlines via (1.2.3c), making thermodynamic considerations superfluous.

1.2.2 Relativistic perfect fluids

Having discussed the dynamics of a classical compressible perfect fluid, we will now translate this idea to the relativistic regime.

The Einstein equations in their most general form are given by

$$\operatorname{Ric}[g] - \frac{1}{2}R[g] + \Lambda g = T, \tag{1.2.4a}$$

where g is a Lorentzian metric, $\operatorname{Ric}[g]$ and $R[g]$ denote the Ricci tensor and Ricci scalar of g, respectively, $\Lambda \in \mathbb{R}$ is the cosmological constant and T is the energy-momentum tensor. Due to the Bianchi identities, the left-hand side of (1.2.4a) is divergence-free, which implies the desired conservation of energy-momentum,

$$\nabla_\mu T^{\mu\nu} = 0, \quad \text{for all} \quad \nu \in \{0, 1, 2, 3\}. \tag{1.2.4b}$$

Solving the Einstein equations (1.2.4a) for a fixed Λ amounts to finding:

- a Lorentzian manifold (M, g);
- a symmetric $(0, 2)$-tensor field T;

that satisfy (1.2.4a) on all of M. For physical considerations and a discussion of (1.2.4) see for example Einstein's original work [29]. For a more modern treatment we refer to the books by Chruściel [23] and Choquet-Bruhat [17].

We will mostly restrict ourselves to the energy-momentum tensor of a perfect fluid, given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{1.2.5}$$

In the relativistic context ρ denotes the total energy density, p is again the pressure, while u denotes the four-velocity, which is a timelike, future-directed vector field satisfying the normalization condition

$$g(u, u) = -1. \tag{1.2.6}$$

In the context of (1.2.5) and (1.2.6), we refer to (1.2.4) as the Einstein-Euler system.

Physically, the form of T in (1.2.5) is derived from the the asssumptions of a perfect fluid, which imply isotropic pressure and vanishing energy flux. For a detailed discussion see, e.g., [92].

Remark 1.2.3. Note that ρ is the total energy density as opposed to the mass density used in (1.2.3). We can therefore further split

$$\rho = \rho_0(1 + \varepsilon), \tag{1.2.7}$$

where ρ_0 is the rest mass density and ε is the specific internal energy. Without setting $c = 1$, we have that $\rho = \rho_0(c^2 + \varepsilon)$ and see that ρ is indeed an energy density by dimensional considerations.

As for now, we only have the equations of relativistic hydrodynamics in their conservation form given by (1.2.4b). We define the projection tensor Π as

$$\Pi^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu. \tag{1.2.8}$$

It projects any tensor onto the orthogonal complement of u. For the projection of (1.2.4b) we get that

$$\begin{aligned}
0 &= \Pi^\mu{}_\nu \nabla_\alpha T^{\alpha\nu} \\
&= (\delta^\mu{}_\nu + u^\mu u_\nu)\Big(u^\nu u^\alpha \nabla_\alpha(\rho + p) + (\rho + p)(u^\alpha \nabla_\alpha u^\nu + u^\nu \nabla_\alpha u^\alpha) + g^{\nu\alpha}\nabla_\alpha p\Big) \\
&= (\rho + p)u^\alpha \nabla_\alpha u^\mu + \Pi^{\mu\alpha}\partial_\alpha p,
\end{aligned}$$

where we used the normalization $g(u, u) = -1$ and consequently $g(u, \nabla u) = 0$. We have thus found the relation

$$(\rho + p)u^\alpha \nabla_\alpha u^\mu + \Pi^{\mu\alpha}\partial_\alpha p = 0. \tag{1.2.9a}$$

Equations (1.2.9a) are the relativistic Euler equations. Their classical counterpart are (1.2.3b). Note that the gravitational body force is now hidden in the connection ∇.

Similarly, we can project (1.2.4b) onto the vector field u, which yields

$$\begin{aligned}0 &= u_\nu \nabla_\alpha T^{\alpha\nu} \\ &= u_\nu \Big(u^\nu u^\alpha \nabla_\alpha(\rho + p) + (\rho + p)(u^\alpha \nabla_\alpha u^\nu + u^\nu \nabla_\alpha u^\alpha) + g^{\nu\alpha} \nabla_\alpha p \Big) \\ &= -u^\alpha \nabla_\alpha \rho - (\rho + p) \nabla_\alpha u^\alpha.\end{aligned}$$

Therefore, we have found

$$u^\alpha \nabla_\alpha \rho + (\rho + p) \nabla_\alpha u^\alpha = 0. \tag{1.2.9b}$$

In addition to conservation of energy and momentum, represented by (1.2.4b), we need to impose the relativistic version of mass conservation (1.2.3a). This is given by

$$\nabla_\mu J^\mu = \nabla_\mu(\rho_0 u^\mu) = 0. \tag{1.2.9c}$$

The system of equations in (1.2.9) are the equations of motion of relativistic hydrodynamics for a perfect fluid. Using (1.2.9c), we can simplify (1.2.9b), and identify it as the relativistic analog of the energy equation (1.2.3c).

Remark 1.2.4. The equation for $u^0 > 0$ is not independent, as u^0 can be determined from (1.2.6), once u^i are known. The system (1.2.9) in $n + 1$ spacetime dimensions therefore consists of $n + 2$ equations for the $n + 3$ variables $(u^i, \rho_0, \varepsilon, p)$. A thermodynamic equation of state, e.g., $p = p(\rho_0, \varepsilon)$, is needed to close the system. This equation describes the thermodynamic character of the fluid contained in spacetime. For a detailed statistical discussion of different equations of state see [92, Section 2.4].

1.3 Mathematical cosmology

Cosmological observations suggest a high degree of symmetry in our universe on large scales. For various aspects and extensive discussion see for example [7]. Mathematically, this is described by the following conditions.

Definition 1.3.1 (Cosmological principle)**.** A Lorentzian manifold (M, g) is said to satisfy the cosmological principle, if there exists a time function $t_c : M \to \mathbb{R}$ called the cosmic time, such that its level sets Σ_{t_c}, equipped with the induced metric $\iota^* g$, are homogeneous and isotropic Riemannian manifolds. Here, $\iota : \Sigma_{t_c} \hookrightarrow M$ denotes the isometric embedding.

A class of spacetimes satisfying the cosmological principle is of the form $(I \times \mathscr{M}, g)$, where the metric g is given by

$$-dt^2 + a(t)^2 h_{ij} dx^i dx^j = -dt^2 + a(t)^2 \left(\frac{1}{1 - kr^2} dr^2 + r^2 d\Omega^2 \right). \tag{1.3.1}$$

Here $d\Omega^2$ denotes the standard round metric on the 2-sphere. The metric in (1.3.1) will be referred to as the Friedman-Lemaître-Robertson-Walker (FLRW) metric. The manifolds $(\Sigma_t, \iota^* g)$ are spaces of constant curvature, meaning that the sectional curvature at each point is independent of the chosen plane and constant on all of Σ_t. The parameter $k \in \{-1, 0, 1\}$ signifies whether the curvature of the Riemannian hypersurface is negative, zero or positive.

Remark 1.3.1. Spacetimes with an FLRW metric are not the only ones satisfying the cosmological principle. For a discussion of this fact and many other comments see [23, Chapter 6].

1.3.1 Modeling the universe as a solution to Einstein's equations

Note here that the FLRW metrics described by (1.3.1) are just Lorentzian metrics with a high degree of symmetry, but a priori have nothing to do with general relativity. However, they can be used to generate cosmological solutions to Einstein's equations.

One of the first attempts was by Einstein himself in [30]. His considerations start out with the assumptions of a closed spatial universe, which implies that the curvature is positive, i.e., $k = 1$. The matter content in Einstein's static universe is given by $T^{\mu\nu} = \rho u^\mu u^\nu$. Furthermore, the scale factor $a(t) = a_0$ is assumed to be constant. Plugging this into (1.2.4a) yields the relation $\Lambda = \frac{2}{a_0^2}$, which necessitates $\Lambda > 0$.

Another early model was that of de Sitter, see [101] or [17, Chapter 5] for a more modern treatment. In this model, Einstein's approach of a spherical universe was extended to $3+1$ spacetime dimensions. The result is a one-sheeted hyperboloid as a Lorentzian submanifold of $(4+1)$-dimensional Minkowski space. On a subset of this hyperboloid, coordinates can be chosen so that the metric assumes the FLRW form

$$-dt^2 + e^{2t\sqrt{\Lambda/3}}\delta_{ij}dx^i dx^j. \tag{1.3.2}$$

This is referred to as the flat slicing of de Sitter space. It solves the vacuum Einstein equations with $\Lambda > 0$. The de Sitter universe represents a first step towards the ΛCDM-model [7], which is widely accepted at present time. It is based on the idea that our universe is not static, but rather expands continuously over time.

1.3.2 The Friedman models

A systematic approach to generate cosmological solutions to the Einstein-Euler system (1.2.4) was found by Friedman [48] and Lemaître [64]. Another improvement was made by Robertson [95], generalizing the approach to arbitrary constant sectional curvature. For a modern treatment, see also [17, Chapter 4] and [23, Chapter 6]. Plugging the metric (1.3.1) into the Einstein equations (1.2.4a) yields the set of ODEs

$$3\frac{\dot{a}^2(t) + k}{a^2(t)} = \rho(t) + \Lambda, \tag{1.3.3}$$

$$-\frac{k + \dot{a}(t)^2}{a(t)^2} - 2\frac{\ddot{a}(t)}{a(t)} = p(t) - \Lambda. \tag{1.3.4}$$

Equation (1.3.3) is the Friedman equation. Together, (1.3.3) and (1.3.4) imply the evolution equation

$$\frac{d}{dt}\left(\rho(t)a(t)^3\right) + p(t)\frac{d}{dt}\left(a(t)^3\right) = 0. \tag{1.3.5}$$

Given a barotropic equation of state, i.e., $p = p(\rho)$, and choosing a suitable domain and initial data (a_0, ρ_0) the system given by (1.3.3) and (1.3.5) has a unique solution. This, in turn, corresponds to a solution of the Einstein-Euler system.

In cosmology, it is common [7; 92] to assume a barotropic equation of state of the form

$$p(\rho) = K\rho, \quad K \in \left[0, \frac{1}{3}\right]. \tag{1.3.6}$$

We will proceed to discuss some of these models.

The Einstein-de Sitter (EdS) universe. Proposed in [28], this model assumes $k = \Lambda = 0$ and dust, meaning that $K = 0$. The fluid is therefore pressureless and only interacts gravitationally. Given positive initial data (ρ_0, a_0) at $t_0 > 0$, we find the solution

$$\left([t_0, \infty) \times \mathbb{R}^3, -dt^2 + a_0^2 \left(\frac{t}{t_0}\right)^{\frac{4}{3}} \delta_{ij} dx^i dx^j\right), \quad \rho(t) = \rho_0 \left(\frac{t}{t_0}\right)^{-2}, \quad u = \partial_t. \tag{1.3.7}$$

The metric in (1.3.7) becomes singular as $t \to 0$ and expands continuously towards the future. Models of this type are referred to as decelerated, as $\frac{d^2}{dt^2}a(t) < 0$ on the whole domain $[t_0, \infty)$.

The spatially flat radiation universe. Statistical considerations lead to an equation of state $p(\rho) = \frac{1}{3}\rho$ for fluids modeling particles moving at, or close to, the speed of light. This is sometimes referred to as the ultrarelativistic regime [92, Chapter 2]. Analogously to (1.3.7), in the context of $k = \Lambda = 0$, we find the solution

$$\left([t_0, \infty) \times \mathbb{R}^3, -dt^2 + a_0^2 \frac{t}{t_0} \delta_{ij} dx^i dx^j\right), \quad \rho(t) = \rho_0 \left(\frac{t}{t_0}\right)^{-2}, \quad u = \partial_t. \tag{1.3.8}$$

Except for the scale factor $a(t) \propto t^{\frac{1}{2}}$, this model is similar to the EdS universe.

Remark 1.3.2. We can interpolate between the EdS and the radiation model to create new Friedman models by choosing $K \in (0, \frac{1}{3})$. These again feature a decaying energy density $\rho(t) \propto t^{-2}$ and have a scale factor $a(t) \propto t^{\frac{2}{3(1+K)}}$. We refer to these as massive fluids.

Remark 1.3.3. If we fix $t_0 > 0$ and prescribe data (a_0, ρ_0) the metric does not always take the form given in (1.3.7) and (1.3.8), but rather $a(t)$ asymptotes to the respective power law scale factor. However, we can always shift t_0 depending on ρ_0 so that the desired form is achieved. As we are only interested in long-term dynamics, this choice of the time coordinate is irrelevant.

The Milne universe. This model was proposed in [76] and arises as a vacuum solution in the context of $\Lambda = 0$ and $k = -1$. The solution is given by

$$\left((0, \infty) \times M, -dt^2 + \frac{t^2}{9}\gamma\right), \tag{1.3.9}$$

where (M, γ) is a Riemannian Einstein manifold with $\mathrm{Ric}[\gamma] = -\frac{2}{9}\gamma$. This model will be important in the discussion below, as it is the only class of FLRW models with linear expansion, i.e., $\frac{d^2}{dt^2}a(t) = 0$.

For the sake of completeness, we mention the de Sitter model (1.3.2) as an example of an accelerated model, meaning that $\frac{d^2}{dt^2}a(t) > 0$ for all t in the domain.

1.4 Heuristics of stability and shock formation

Whether the solutions discussed above are good candidates for describing our physical universe can be tested by fitting data from observations. However, the symmetry assumptions leading to such models are never satisfied exactly in nature. Clearly, the distribution of matter and spatial curvature of our universe is not completely homogeneous, but rather is subject to small fluctuations.

Mathematical analysis allows us to understand the long-time asymptotics of small perturbations of these solutions. Suppose we start by imposing initial data on some spatial surface at time t_0. This data describes the state at said time and encodes the behavior of the fluid and spacetime at t_0. We denote this initial state by $s(t_0)$, which will typically be an element of some suitable function space. For the Einstein-Euler system $s(t_0)$ includes the spatial metric g, the second fundamental form k, the density ρ and the velocity u at a spatial hypersurface Σ_{t_0}. It is chosen close to the data that corresponds to the state of the exact Friedman solution s_{Fr}, i.e.,

$$\|s(t_0) - s_{\mathrm{Fr}}(t_0)\| < \delta. \tag{1.4.1}$$

Here, $\|\cdot\|$ is some appropriate norm and $\delta > 0$ is a small constant. The Einstein-Euler system can then be understood as a system of evolution equations, transporting this data to a later state $s(t)$ at time t.

As t tends to infinity, several behaviors could potentially be observed. The initial perturbation at time t_0 could decay. In this case, the solution converges back to the Friedman solution, i.e., we have that

$$\lim_{t\to\infty} \|s(t) - s_{\mathrm{Fr}}(t)\| = 0.$$

If there exists a $\delta > 0$ sufficiently small, such that all data satisfying (1.4.1) exhibit this behavior, the solution is referred to as asymptotically stable. Another possibility would be that the perturbation oscillates near the background, i.e., there exists some $\varepsilon > 0$, such that for all $t > t_0$

$$\|s(t) - s_{\mathrm{Fr}}(t)\| < \varepsilon.$$

If this behavior is generic for sufficiently small δ, we refer to this state as Lyapunov stable. While these cases are sometimes distinguished, we will generically refer to them as stable. A case in which the difference grows but is still bounded pointwise is not of particular interest to us.

However, a classical solution to the system could cease to exist in finite time. This means that there exists a $t_* > t_0$, such that the solution exists on $[t_0, t_*)$ but cannot be extended to $[t_0, t_*]$ in a classical sense. A continuation principle will then tell us that at t_* some type of singularity forms, i.e.,

$$\lim_{t\to t_*} \|s(t) - s_{\mathrm{Fr}}(t)\| = \infty.$$

If such solutions satisfying (1.4.1) exist for all $\delta > 0$, we refer to the state as unstable. In the case of fluid dynamics the quantity that blows up would typically be the gradient of the fluid velocity. In this context, we refer to this behavior as shock formation. If shocks form, the solution loses regularity and becomes discontinuous. However, it might be possible to extend the solution in a weak sense, considering the integral counterpart of the differential equation.

Shocks form in nature during supersonic flows. If an object travels faster than the speed of sound of its surrounding medium, the disruption cannot be transported fast enough, leading to a pressure discontinuity. This situation can be described by a unique weak solution of the system. A very simple mathematical model displaying this type of behavior is the inviscid Burgers equation,

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0.$$

It is a one-dimensional conservation law, forming shocks for generic non-trivial initial data on a compact domain.

Whether Friedman solutions to the Einstein-Euler system are stable or unstable is of fundamental interest to physics. Using mathematically rigorous methods, we can infer whether the matter content of the universe forms shocks or stabilizes in a given epoch. The current rate of expansion is exponential and it is known for some time that such models are mathematically stable. However, little is known about the stability of previous matter configurations that are characterized by slower expansion. The present thesis intends to answer some of these open problems.

1.5 The initial value problem for relativistic fluids

From now on, we will focus exclusively on barotropic models with an equation of state as in (1.3.6). This is an important restriction, since it decouples the relativistic Euler equations (1.2.9a) and the energy equation (1.2.9b) from the conservation of mass (1.2.9c). We will now introduce two initial value problems that will form the foundation for the remainder of this thesis. Before that, however, we give a brief exposition of the $3+1$-decomposition.

To pose the Einstein equations as an evolution problem, we need to separate a time direction and split the tensorial equations in (1.2.4a). This analysis is similar to the derivation of (1.2.9a) and (1.2.9b) and is known as a $3+1$-decomposition. One begins by choosing a foliation of the spacetime manifold M with unit normal vector $n \in \mathfrak{X}(M)$, usually done along a time function t. This foliation then provides a natural decomposition of vectors into the normal and tangential part, the latter of which is given by the orthogonal projection (1.2.8) (after replacing u by n). On every foliation hypersurface $\Sigma_t \subset M$, we can decompose

$$\partial_t = \alpha n + \beta \tag{1.5.1}$$

where the lapse function α corresponds to the normal and the shift vector field β to the tangential part of ∂_t.

Remark 1.5.1. Inverting (1.5.1) yields

$$n = \alpha^{-1}(\partial_t - \beta).$$

Heuristically, the lapse function α at a point tells us how fast a normal observer moves through the foliation, while the shift β indicates their spatial displacement from one leaf to another. Given a coordinate system on a leaf Σ_t, and fixing these two quantities gives us a unique coordinate system on the neighboring leaves. In general relativity, this is referred to as a gauge choice.

Given the spatial Riemannian metric γ on the leaf Σ_t, the line element for the local spacetime slab is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{1.5.2}$$

Performing the decomposition we obtain evolution equations for the metric γ and the second fundamental form k as well as the Hamiltonian and momentum constraints, given by

$$\begin{aligned} R[\gamma] + k^2 - k_{ij}\, k^{ij} &= 2E, \\ \nabla[\gamma]_j\, k_i^j - \nabla[\gamma]_i\, k &= p_i. \end{aligned} \tag{1.5.3}$$

Here, $k = \gamma^{ij}\, k_{ij}$, $E = T(n,n)$ and $p_i = -\Pi^\mu{}_i T_{\mu\alpha} n^\alpha$. For a detailed discussion of the $3+1$-formalism, see [53].

With these tools at our disposal we can now formulate the initial value problem.

Definition 1.5.1 (Initial value problem for the Einstein-Euler system)**.** We assume the following:

- $(\Sigma, \mathring{\gamma})$ is a Riemannian 3-manifold.
- $\mathring{k}$ is a symmetric covariant 2-tensor field on Σ.
- $\mathring{\rho}$ is a non-negative function on Σ.
- $\mathring{u}$ is vector field on Σ.

Furthermore, we assume that the constraint equations (1.5.3) are satisfied and the gauge is fixed. In that case we refer to the tuple $(\Sigma, \mathring{\gamma}, \mathring{k}, \mathring{\rho}, \mathring{u})$ as initial data set for the Einstein-Euler system. Given such data, to solve the (initial value problem for the) Einstein-Euler system means to find a four-dimensional Lorentzian manifold (M, g), a function ρ on M and a timelike vector field $u \in \mathfrak{X}(M)$ with $g(u,u) = -1$, such that (1.2.4a), (1.2.9a) and (1.2.9b) are satisfied. In addition, given the isometric embedding $\iota : \Sigma \to M$, the quantities (γ, k, ρ, u) of the embedded submanifold $\iota(\Sigma)$ must agree with the initial data set.

Often, we will be interested in the fluid equations alone. We will define this as a separate problem.

Definition 1.5.2 (The initial value problem for the relativistic Euler equations)**.** Consider a four-dimensional Lorentzian manifold (M, g), a spacelike hypersurface $\Sigma \subset M$, a non-negative function $\mathring{\rho}$ and a vector field $\mathring{u} \in \mathfrak{X}(\Sigma)$. Then $(\mathring{\rho}, \mathring{u})$ is referred to as an initial data set for the relativistic Euler equations. Given such data, to solve the (initial value problem for the) relativistic Euler equations is to find a function ρ on M and a timelike vector field $u \in \mathfrak{X}(M)$ with $g(u,u) = -1$, such that (1.2.9a) and (1.2.9b) are satisfied. In addition, the restriction ρ and the projection of u onto Σ must agree with $\mathring{\rho}$ and $\mathring{u}$, respectively.

Remark 1.5.2. In section 1.2.1, we referred to the momentum balance (1.2.3b) as Euler equations, as these are the original equations written down by Euler [32]. The relativistic analog of momentum balance and therefore the *relativistic Euler equations* are (1.2.9a). However, in the previous definition the system comprised of (1.2.9a) and (1.2.9b) are referred to as the relativistic Euler equations. This slight abuse of terminology is common in the literature centered around the Cauchy problem for relativistic fluids. Hence, we will adopt it for the rest of this thesis.

In section 1.3.2 we discussed several important solutions to the Einstein-Euler system. To gain insight into their plausibility as physical models, we would like to inquire about their stability. As outlined in the informal introduction, this amounts to the question whether solutions launched by initial data close to some symmetric state terminate in finite time. Before we analyze this, however, we have to check if our problem is well-posed. Going back to Hadamard [54], this means that, given suitable initial data,

- there exists a solution;
- the solution is unique;
- it depends continuously on the initial data.

We will now give a summary of the results on well-posedness of the initial value problem concerning the Einstein-Euler system as well as the relativistic Euler equations.

1.5.1 Hyperbolic partial differential equation

Hyperbolic differential equations derive their name from the quadratic form that induces their operator. The prime example is the wave equation

$$(-\partial_t^2 + \partial_x^2)\psi(t,x) = 0, \tag{1.5.4}$$

with its quadratic form $q(x,y) = -x^2 + y^2$. This equation has many interesting properties, well-posedness being the most prominent one. Therefore, 'hyperbolic' is promoted to a more general archetype, meaning that a suitable initial value or Cauchy problem is well-posed. In addition, hyperbolic PDEs are expected to have finite speed of propagation, meaning that information travels along characteristic curves, and hence spatially separated regions can only communicate after a certain amount of time has passed. When studying the initial value problem for the Einstein-Euler system, we formulate it as a hyperbolic set of equations.

We consider a first-order system of the type

$$A^\mu \partial_\mu u = Bu + f, \tag{1.5.5}$$

where A^μ and B are $n \times n$-matrices, f is a vector of dimension n and $u = (u^1, \dots, u^n)$ is the solution vector. In the simplest case, A^μ, B and f are constant. If these quantities are functions on spacetime we speak of the linear problem (with variable coefficients). We will, however, mostly be interested in the case were we will allow A^μ, B and f to be functions of the solution u itself. In standard PDE theory, this is referred to as the quasilinear problem.

There are various notions of hyperbolicity, each suited to specific setups. One of the most widely used and most relevant to our considerations is the following.

Definition 1.5.3 (Symmetric hyperbolic systems)**.** A quasilinear or linear system of the type (1.5.5) is said to be symmetric hyperbolic, if A^μ is symmetric for all μ and A^0 is uniformly positive-definite.

Remark 1.5.3. More generally, if a map S^0 exists, such that $S^0\times$(1.5.5) is symmetric hyperbolic, the system is referred to as symmetrizable. The linear case was famously explored by Friedrichs in [51]. Therein, the author proved existence and uniqueness of weak solutions using functional analysis.

Due to its archetypal significance, we will prove an elementary result here.

Lemma 1.5.1 (Friedrichs-type estimate)**.** *Consider the symmetric hyperbolic system*

$$A^\mu(t,x)\partial_\mu u(t,x) = B(t,x)u(t,x) + f(t,x), \quad (t,x) \in [0,t_*] \times \mathbb{R}^d, \tag{1.5.6}$$

where A^μ, B are smooth matrix-valued functions, that are bounded and have bounded derivatives. Then, there exists a constant $\alpha > 0$, such that for all $u \in C([0,t_*];H^1) \cap C^1([0,t_*];L^2)$ that are solutions to (1.5.6)*, the priori estimate*

$$\|u(t_*,\cdot)\|_{L^2}^2 \leq \alpha\Big(\|u(0,\cdot)\|_{L^2} + \int_0^{t_*} \|f(t,\cdot)\|_{L^2}^2\Big)$$

holds.

Proof. First we look at the evolution of the energy $\frac{1}{2}(A^0u,u)_{L^2}$. Using the equation (1.5.6), we find

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int u^T A^0 u &= \frac{1}{2}\int u^T(\partial_t A^0)u + \int u^T(-A^i\partial_i + B)u + \int u^T f \\ &= \frac{1}{2}\int u^T(\partial_t A^0 + \partial_i A^i)u + \int u^T B u + \int u^T f.\end{aligned}$$

Integrating in time and using Hölder's inequality as well as the fact

$$\max_{(t,x)}(|\partial^\alpha A^\mu|, |B|, |A^0 - \mathbb{1}\,|) < C$$

for $C > 0$, we find that

$$(A^0u(t),u(t))_{L^2} \leq (A^0u(0),u(0))_{L^2} + (4C+1)\int_0^t \|u(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2.$$

For a new constant $C' > 0$ we therefore find that

$$\|u(t)\|_{L^2}^2 \leq C'\Big(\|u(0)\|_{L^2}^2 + \int_0^t \|u(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2\Big).$$

An application of Grönwall's inequality yields

$$\|u(t)\|_{L^2}^2 \leq C'e^{C't}\Big(\|u(0)\|_{L^2}^2 + \int_0^t \|f(t)\|_{L^2}^2\Big).$$

□

This Friedrichs-type estimate provides a powerful tool that can be used to establish local well-posedness in the linear case. The proof relies on pseudo-differential calculus and a duality argument. The obtained weak solution can be upgraded to a strong one of the same regularity as the initial data. For a detailed discussion, see [8, Chapter 2].

Once an existence theory for the linear equation is established, we can proceed to the quasilinear regime. A very general local well-posedness result is due to Kato in [61]. We shall state a simplified version of this, taken from [43]. The proof is based on the existence theory of the linear equation and a fixed-point argument.

Theorem 1.5.1. *Let $s > \frac{d}{2} + 2$, $U \subset H^s$ open and $\delta > 0$. Consider the quasilinear, symmetric hyperbolic system*

$$\partial_t u(t,x) + A^i(t,x,u)\partial_i u(t,x) = f(t,x,u), \qquad (t,x) \in (-\delta,\delta) \times \mathbb{R}^d. \tag{1.5.7}$$

We assume that:

- *for all $u \in U$, we have that $A^i(\cdot,\cdot,u), f(\cdot,\cdot,u) \in H^s$,*
- *A^i, f are rational functions in u with non-zero denominator.*

Then, given $u_0 \in H^s$, there exists $0 < \varepsilon < \delta$, such that there is a unique solution

$$u \in C((-\varepsilon,\varepsilon); H^s) \cap C^1((-\varepsilon,\varepsilon); H^{s-1})$$

to (1.5.7) with $u(0,\cdot) = u_0$. Furthermore, in a suitable neighborhood ε can be chosen uniformly and the solution map $u_0 \mapsto u$ is continuous in the H^s-topology.

A particularly useful tool in the global analysis is given by the following theorem taken from [8, Chapter 10].

Theorem 1.5.2 (Continuation theorem)**.** *Let u be the solution in Theorem 1.5.1 on an interval $[0,T)$. Furthermore, assume that there exists $r > 0$ such that*

$$\sup_{t \in [0,T)} \|u(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^d)} \leq r.$$

Then u can be extended to a solution in $C([0,T']; H^s)$ with $T' > T$.

This is very powerful, as it allows us to show that solutions exist globally, simply by analyzing an appropriate norm. In L^2-based Sobolev spaces, these norms are reminiscent of the kinetic energy of a particle, $\frac{mv^2}{2}$. Hence, analysis of the local and global behavior of solutions using decay estimates of norms is often referred to as the energy method.

Remark 1.5.4. The results discussed in this section are also valid for systems where $A^0 = \mathbb{1}$ and $A^j\xi_j$ has real eigenvalues and a complete set of eigenvectors for all $\xi_j \in \mathbb{R}^n$. This setting is often referred to as strictly hyperbolic. For a recent review, see [25].

1.5.2 Well-posedness theory of relativistic fluids

There are various proofs of local well-posedness for the relativistic Euler equations and the Einstein-Euler system. Many of them are based on results in the spirit of Theorem 1.5.1 or take inspiration from the theory of the classical compressible Euler equations. We shall provide an overview for the most useful ones.

The relativistic Euler equations. We begin the discussion by mentioning a two-part series of articles by Makino and Ukai [72; 73]. Both of them treat the Cauchy problem for the relativistic Euler equations on Minkowski space $(\mathbb{R}^{3,1}, \eta)$ with initial data for which ρ is bounded from below. While [72] focuses on a barotropic linear equation of state $p(\rho) = K\rho$ with parameter range $0 < K < \frac{1}{3}$, [73] considers general barotropic ones satisfying $0 < p'(\rho) < 1$. The proof is based on the construction of a symmetrizer and a Friedrichs-type argument.

A detailed and elegant account of the equations of motion of relativistic fluid- and magneto-fluid dynamics is given in Anile's book [5]. This discussion is not restricted to the barotropic setting and uses the entropy formulation, where, instead of considering mass conservation (1.2.9c), the adiabaticity condition

$$u^\mu \nabla_\mu s = 0 \tag{1.5.8}$$

is used. This warrants an equation of state $p = p(\rho, s)$. The argument presented for local well-posedness is based on the theory developed by Lax and Friedrichs [52] among others. It is achieved by the following observation. Assume we are given a conservation law

$$\partial_t u^j + \partial_k F^{j,k}(u) = 0, \tag{1.5.9}$$

and an additional conservation law

$$\partial_t U(u) + \partial_k f^k(u) = 0. \tag{1.5.10}$$

Simply by applying the chain rule, we find that

$$\partial_{u^j} U \partial_{u^l} F^{j,k} = \partial_{u^l} f^k$$

if and only if (1.5.10) follows from (1.5.9). Taking another derivative ∂_{u^h}, we find that

$$U_{hj} F_l^{j,k} + U_j F_{hl}^{j,k} = f_{hl}^k,$$

implying that $U_{hj} F_l^{j,k}$ is symmetric in $h \leftrightarrow l$. From (1.5.9), we find that

$$U_{hj} \partial_t u^j + U_{hj} F_l^{j,k}(u) \partial_k u^l = 0.$$

Hence, provided that U_{hj} is positive definite, it acts as a symmetrizer. In the case of a perfect fluid, we therefore use the conservation laws

$$\nabla_\mu T^{\mu\nu} = 0, \quad \nabla_\mu(\rho_0 u^\mu) = 0,$$

together with the supplemental conservation law

$$\nabla_\mu(\rho_0 s u^\mu) = 0,$$

to establish local existence. Note that this method also requires $\partial_\rho p(\rho, s) > 0$.

The Einstein-Euler system. Due to its simplicity, the first place to start investigating well-posedness is in vacuum. There, the Einstein equations (1.2.4a) take the simple form

$$R_{\mu\nu} = 0. \tag{1.5.11}$$

Choquet-Bruhat studied this system in her seminal work [45], which remains influential to this day. As discussed in section 1.5, we have to make a choice of gauge for our initial value problem to be well-posed. In [45], the coordinate condition used is given by

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = 0. \tag{1.5.12}$$

In this gauge, the coordinates x^α satisfy $\Box_g x^\alpha = 0$, justifying the name wave coordinates. Sometimes, (1.5.12) is also referred to as harmonic coordinates or isothermal coordinates, as they are called in [45]. In this formulation, (1.5.11) takes the form of second-order hyperbolic system for the metric g. Much of [45] is concerned with developing an existence theory for such systems. For a detailed discussion of this topic see, e.g., [17, Chapter 6]. In contrast to this result, a lot of modern literature resorts to a first-order formulation of the Einstein equations. Hence, we mention [44] by Fischer and Marsden, where the authors prove a well-posedness result for quasilinear symmetric hyperbolic systems. They then proceed to find a first-order formulation of the Einstein equations in vacuum (1.5.11) and apply the aforementioned theorem.

In a follow-up work [46], Choquet-Bruhat proved well-posedness results for the Einstein-Euler system. Therein, she treats dust, $p = 0$, as well as general barotropic fluids for which she uses a formulation based on the Helmholtz equations. Her analysis relies on the theory of higher-order hyperbolic systems introduced by Leray [65].

Lichnerowicz' work [66] is also based on Leray theory. There, the author extends [46] to the non-barotropic setting. It provides a local well-posedness result for the complete Einstein-Euler-entropy system. For a modern discussion of this topic see [24].

The results described above require the density to be positive, i.e., $\rho > 0$. This is, however, very restrictive, as it does not allow for the physically relevant case of compactly supported data. This was pointed out and treated by Rendall [89] for polytropic equations of state $p(\rho) = K\rho^{(n+1)/n}$, $K, n \in \mathbb{R}^+$. It is based on the following observation. One can easily transform the relativistic Euler equations into a system of the form (1.5.5). However, to symmetrize this system, we have to multiply one equation by $(p+\rho)^{-1}$, which is singular in a vacuum region. To avoid this, Makino came up with an approach for the compressible Euler equations in [74]. The key insight is to introduce a new variable $w(\rho) = \sqrt{4Kn(n+1)}\rho^{\frac{1}{2n}}$, which is adapted to the specific equation of state. Rendall then uses this variable to put the system in manifestly symmetric hyperbolic form, proving well-posedness.

We also mention an article by Friedrich and Choquet-Bruhat [19] which proves a local well-posedness result for the Einstein-dust system for compactly supported data. It again uses the theory of Leray [65] to establish hyperbolicity.

1.5.3 Shock formation

To start our discussion on shock formation, we present a classical result.

Lemma 1.5.2 (Blowup of Burgers' equation)**.** *For all non-zero data $u_0 \in C_c^\infty(\mathbb{R})$, the* *unique classical solution to the initial value problem*

$$\begin{cases} \partial_t u(t,x) + \partial_x\left(\frac{u(t,x)^2}{2}\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.5.13}$$

terminates in finite time.

Proof. Commuting (1.5.13) with ∂_x, we find that

$$\partial_t \partial_x u(t,x) + (\partial_x u(t,x))^2 + u(t,x)\partial_x^2 u(t,x) = 0.$$

For any $x_0 \in \mathbb{R}$, we define the physical characteristic γ_0 as the unique solution to the ODE

$$\frac{d}{dt}\gamma_0(t) = u(t, \gamma_0(t)), \quad \gamma_0(0) = x_0.$$

Tracing the gradient $(\partial_x u)(t, \gamma_0(t))$ along a characteristic, we find that

$$\begin{aligned} \frac{d}{dt}(\partial_x u)(t,\gamma_0(t)) &= -(\partial_x u)^2(t,\gamma_0(t)) - u\partial_x^2 u(t,\gamma_0(t)) + \partial_x^2 u(t,\gamma_0(t))\frac{d\gamma_0(t)}{dt} \\ &= -(\partial_x u)^2(t,\gamma_0(t)). \end{aligned} \tag{1.5.14}$$

The solution to (1.5.14) blows up in finite time for non-trivial initial data. But since u_0 is compactly supported and non-trivial, there exists an x_0, such that $(\partial_x u_0)(x_0) \neq 0$. ☐

In general, an equation of the type

$$y'(t) = A_2(t)y^2(t) + A_1(t)y(t) + A_0(t) \tag{1.5.15}$$

is referred to as a Riccati-type ODE. It exhibits blowup, provided the coefficient functions are well behaved. For a detailed discussion see [1].

Note that in the proof of Lemma 1.5.2, $\frac{d}{dt}u(t,\gamma_0(t)) = 0$. Hence, the L^∞ norm of the solution is conserved and the blowup only occurs for the gradient. We refer to this behavior as shock formation.

Shock formation in hyperbolic systems. The theory of shock formation has a long history. For a systematic overview, we refer to [1; 8; 58]. A generic shock formation result for 2×2 first-order hyperbolic system in $1+1$ dimensions was derived by Lax. This was generalized by John [60] to arbitrary first-order strictly hyperbolic systems in $1+1$ dimensions. Both of these results are based on the following idea. Suppose that we are given a strictly hyperbolic system (1.5.5). This requirement guarantees that we have n eigenvalues λ_i that depend on the solution u. These define characteristic speeds for the curve defined via

$$\frac{d\gamma_i}{dt} = \lambda_i(u(\gamma_i(t),t)).$$

Since the corresponding eigenvectors v_i form a basis of $\mathbb{R}^n$, we can split the gradient $w = \partial_x u$ into components $w = w^i v_i$. In general, we find that along γ_i the gradient behaves like

$$\frac{d}{dt}w^i = f^i_{jk}(u)w^j w^k,$$

where f is some function that only depends on the solution u. Studying these relations, John is able to construct a Riccati-type blowup, provided the initial data is sufficiently small.

An interesting early result in arbitrary spacetime dimension was demonstrated in [99]. The author considers hyperbolic conservation laws, showing that shocks form in finite time, provided the coefficient matrix exhibits sufficient growth rates. For a more detailed review of shocks in the context of hyperbolic systems of equations, we refer to [1; 71].

Shock formation for the relativistic Euler equations. The task of obtaining a general shock formation result for the relativistic Euler equations in 3 dimensions is quite challenging. This is due to the fact that results in low dimension do not easily translate to higher dimensions, where dispersion is facilitated. Regarding this matter, we want to highlight the seminal work by Christodoulou [22]. Therein, the author proves that shocks form generically for small compactly supported irrotational perturbations of the homogeneous solutions to the relativistic Euler equations on $(3+1)$-dimensional Minkowski space.

The vorticity ω of a relativistic perfect fluid is defined as

$$\omega = d\beta, \quad \beta_\alpha = \zeta u_\alpha, \tag{1.5.16}$$

where ζ is the specific enthalpy. If $\omega = 0$, the fluid is said to be irrotational. This condition propagates and since β is closed, there is a local potential ϕ satisfying $d\phi = \beta$. The derivatives of ϕ fully encode the information about the fluid and the relativistic Euler equations reduce to the quasilinear wave equation

$$a^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0, \quad a^{\mu\nu} = g^{\mu\nu} + \left(1 - \frac{1}{c_s^2}\right) u^\mu u^\nu. \tag{1.5.17}$$

Here c_s denotes the speed of sound. For a simple derivation see [39]. In [22], the problem is treated by studying the geometric properties of the acoustical metric a. A quantity of particular importance is the inverse foliation density, which is a scalar μ^{-1} depending on the derivative of the velocity u of the solution. It measures the density of leaves of the foliation of characteristic surfaces. Hence, the limit $\mu \to 0$ corresponds to shock formation, meaning an intersection of characteristics. For a review of this proof based on the study of the null condition in quasilinear second-order hyperbolic PDEs, see [57].

CHAPTER 2

STABILITY IN COSMOLOGICAL FLUID MODELS

In the previous chapter, we discussed how perfect fluids tend to form shocks on a flat Minkowski background. However, we can also think about this model as a fluid, filling a spatially flat FLRW metric with constant scale factor $a(t) = 1$. As we will discuss in the next section, Friedman solutions with accelerated expansion are stable solutions to the Einstein-Euler system. This phenomenon is referred to as fluid stabilization. In particular, it concerns Friedman models with $\Lambda > 0$. This naturally poses the question whether cosmological fluid solutions are stable in regimes of linear and decelerated expansion. The present thesis answers this question in a series of articles.

2.1 State of the art

2.1.1 The stabilizing effect of expanding spacetimes

The phenomenon of fluid stabilization for expanding spacetimes was first discovered by Brauer, Rendall and Reula [14]. Therein, the authors study fluids in a Newton-Cartan model of cosmology. At the heart of this discussion lies the following observation. Consider the familiar model of a symmetric hyperbolic system

$$A^0(u)\partial_t u + A^i(u)\partial_i u = 0.$$

As we discussed in section 1.5.3, solutions to this system are prone to the formation of shock singularities. However, if we add the solution vector to the left-hand side,

$$A^0(u)\partial_t u + A^i(u)\partial_i u + u = 0, \tag{2.1.1}$$

this is not the case. Adapting the argument in the proof of Lemma 1.5.1, we find that

$$\frac{d}{dt}\frac{1}{2}\int u^T A^0(u)u = \frac{1}{2}\int u^T(\partial_\mu A^\mu(u))u - \|u\|_{L^2}^2.$$

Estimates of this type can be used to establish global existence for small data in a norm H^s of sufficiently high regularity. In [14] such an argument is used to establish global existence for the fluid in their setup. A key feature of this cosmological model is a positive cosmological constant $\Lambda > 0$. In the equations of motion, this guarantees a structure similar to (2.1.1), as it introduces a damping term $\sqrt{\Lambda/3}u$ for late times.

Remark 2.1.1. Heuristically, we can think about the phenomenon of fluid stabilization in the following way. Hyperbolic systems transport information along characteristics. They exhibit finite speed of propagation. If these curves focus and concentrate over time, gradients become steep and a loss of regularity ensues. However, if physical space expands rapidly over time, characteristics may meet at later points or not at all, preventing the formation of shock singularities.

Another hallmark result was demonstrated by Ringström in [93]. Therein, the author studies the Einstein equations (1.2.4a) coupled to a scalar field ϕ with potential $V(\phi)$. Posing an initial condition $V(0) > 0$ produces a system akin to $\Lambda > 0$. The author then proceeds to show that for sufficiently small data, the solution is future causal geodesically complete.

2.1.2 The Einstein-Euler system with a positive cosmological constant

As we mentioned in section 1.3.1 as well as the previous one, a positive cosmological constant facilitates expansion and stabilization. We will refer to coupled system (1.2.4) together with the assumption of $\Lambda > 0$ as the Einstein-Euler-Λ system.

The first result in this regime was demonstrated by Rodnianski and Speck [96] for irrotational perturbations in the parameter range $0 < K < \frac{1}{3}$. They employ a generalized wave gauge similar to [93] and study the resulting system of quasilinear wave equations. The restriction to the irrotational regime was later lifted by Speck in [102].

The radiation case $K = \frac{1}{3}$ was treated by Lübbe and Valiente-Kroon in [70]. It uses the conformal approach, which was developed and used by Friedrich to show the stability of de Sitter space as a solution to the vacuum Einstein equations [49].

Pressureless dust in the context of $\Lambda > 0$ was shown to be nonlinearly stable by Hadžić and Speck in [55]. Similarly to [93; 96], they use a generalized wave gauge condition to turn the evolution equations for the metric g into a quasilinear wave equation

$$\square_g \phi = \mathscr{N}(\phi, \partial\phi, u, \rho). \tag{2.1.2}$$

In (2.1.2), ϕ schematically replaces the components of the metric tensor g. In addition to (2.1.2), the evolution equations of the matter (1.2.9b) and (1.2.9a) take the schematic form

$$\begin{aligned} D_t u^i &= -2\omega u^i + S^i(\phi, \partial\phi, u), \\ D_t \rho &= -3\omega\rho + S(\phi, \partial\phi, u, \partial u, \rho). \end{aligned} \tag{2.1.3}$$

Again, the coefficient $\omega \to \sqrt{\Lambda/3}$ for large times and is responsible for the stabilization process. Inspecting (2.1.2) and (2.1.3), we immediately run into a problem characteristic to dust. Suppose we want to set up an energy scheme based on the top order of regularity $\|\partial\phi\|_{H^s}$. Taking a glance at (2.1.3), we see that an energy estimate on $\|u\|_{H^s}$ is the best we can achieve. Hence, what we can hope for in terms of ρ is control of the H^{s-1} norm. Now the problem is very apparent. To estimate $\|\partial\phi\|_{H^s}$ requires control of $\|\mathscr{N}(\phi, \partial\phi, u, \rho)\|_{H^s}$. This, in turn, requires control of ρ in H^s, which is not available. As a solution to this loss of regularity, the authors define new top-order norms including $\|(D_t u, D_t\rho)\|_{H^{s-1}}$. Using this formalism, a series of intricate energy estimates shows global existence for small data.

In addition to the previous results we mention [81] by Oliynyk. Therein, the author shows future nonlinear stability of FLRW models in the parameter range $0 < K < \frac{1}{3}$. Contrary to the other mentioned works, it treats the Einstein-Euler-Λ system in a first-order formulation.

This is referred to as Fuchsian approach, as the equation features a global singular factor $\frac{1}{t}$ in front of the decay-inducing terms. The system is studied in a conformal setup using Friedrichs-type energy estimates, which are used to prove global existence. This approach was further generalized and applied to various settings in general relativity in [11].

2.1.3 The Einstein-Euler system in non-accelerated spacetimes

As we discussed in section 1.3.2, for $\Lambda = 0$ we have Friedman models with linear (1.3.9) and decelerated expansion rates (1.3.7), (1.3.8). However, this means that the stability results of the previous section do not straightforwardly translate to these regimes. Indeed, the expansion might be 'too slow' to stabilize the fluid.

In this section our discussion will primarily revolve around the relativistic Euler equations on a spacetime of the type

$$\left([t_0, \infty) \times M, g = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j\right), \tag{2.1.4}$$

where $M = \mathbb{T}^3$ or $M = \mathbb{R}^3$. Whether the spatial manifold is compact is irrelevant in this setting.

In [103], Speck considers the stability of the canonical quiet fluid solutions

$$\rho(t, x) = \rho_0 a(t)^{-3(1+K)}, \quad u^\mu = \delta_0^\mu, \tag{2.1.5}$$

which are the spatially homogeneous and isotropic solutions to (1.2.9b) and (1.2.9a). The author finds an interesting relation between the integrability of $a(t)^{-1}$ and the stability properties of the corresponding class of solutions, depending on the parameter K.

- For dust ($K = 0$), $\|a(t)^{-1}\|_{L^2_t([t_0,\infty))} < \infty$ is a sufficient condition to guarantee small data global existence.
- In the massive parameter range $0 < K < \frac{1}{3}$ and for radiation $K = \frac{1}{3}$, the stronger condition $\|a(t)^{-1}\|_{L^1_t([t_0,\infty))} < \infty$ is sufficient for small data global existence.
- For radiation, the condition $\|a(t)^{-1}\|_{L^1_t([t_0,\infty))} < \infty$ is necessary for stability. Otherwise, the fluid will form shocks in finite time. Therefore, the results regarding radiation are sharp.

The stability results in [103] are derived using energy estimates as well as so-called energy currents $J^\mu[u, \rho]$, introduced in [21]. These vector fields are adapted to the relativistic Euler equations, and their time component $J^0[u, \rho]$ is equivalent to some L^2-based Sobolev norm.

A different approach was taken by Fajman, Oliynyk and Wyatt in [41]. The authors studied the relativistic Euler equations on spacetimes of type (2.1.4) with $a(t) = t$ and parameter range $0 < K < \frac{1}{3}$. They focused exclusively on the irrotational case and reduced the resulting wave equation to a Fuchsian first-order system

$$B^0(u)\partial_t u + (C^i + B^i(u))\nabla_i u = \frac{1}{t}\mathbb{P}u + G(u).$$

The matrix $\mathbb{P}$ on the right-hand side is a projection onto the velocity part of the solution vector u. This term is positive because of the reversed time direction and is again responsible for the decay of the solution. The analysis of this system is similar to [11], but requires new methods because of the additional constant matrix C.

We close this section with figure 2.1. It describes the results in [22; 41; 103] in a spacetime

$$\left([t_0,\infty)\times M, g=-dt^2+t^{2\alpha}\delta_{ij}dx^i dx^j\right), \tag{2.1.6}$$

which we refer to as power law expansion.

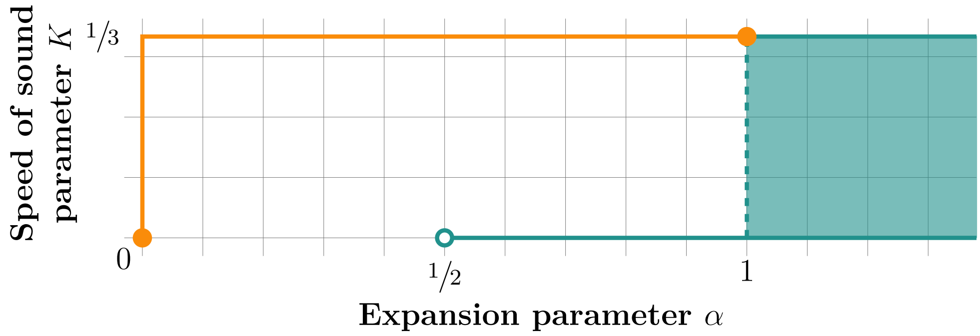

Figure 2.1: This figure summarizes the instability results of [22], [103] and [41] for the relativistic Euler equations. This concerns a power law $a(t)=t^\alpha$ and a linear barotropic equation of state $p(\rho)=K\rho$. The orange line indicates unstable points. The turquoise lines as well as the shaded region are points of stability. In the linear regime with $0<K<\frac{1}{3}$, we have the stability result of [41] for irrotational perturbations, which is indicated by the dashed turquoise line. The aforementioned articles do not treat the white region.

2.2 Summary of the results

While solutions to the relativistic Euler equations may represent a toy problem for the Einstein-Euler system, we shall discuss them separately. There are two reasons for this. In some cases, separately considering the vacuum problem and the matter model is not sufficient. Hence, the methods of analyzing the two models differ significantly, as is the case for dust. The second reason is that the relativistic Euler equations form an interesting model in their own right. Hence, we also consider fluid solutions on expanding backgrounds that are not necessarily part of the Friedman family.

2.2.1 Relativistic Euler equations

Linear expansion. In chapter 4, we consider perturbations of the quiet fluid solutions with $0<K<\frac{1}{3}$ on spacetimes of the form (2.1.6) with $\alpha=1$ and $M=\mathbb{T}^3$. Hence, it essentially lifts the restriction to irrotational data that was treated in [41]. We start by formulating the conformal relativistic Euler equations in a symmetric hyperbolic form

$$B^0\partial_t U+B^k\mathscr{D}_k U=H,$$

see Lemma 4.2.3. A crucial aspect of the proof is the transformation $(\zeta, u) \mapsto (\psi, z)$. By construction, the new coefficient matrices A^0 and A^k have a quadratic dependence on the velocity $|z|$ and a linear dependence on the shift $|\beta|$ in crucial components. For details of this transformation, see Lemma 4.2.4. In similar fashion to [41], the system is extended to a first order evolution equation for $(Z, DZ)^T$, where $Z = (\psi, z)^T$ is the solution vector. In essence, this is a technical feature that ensures that even L^2-order energy estimates close. The key estimates for this system are found in Lemma 4.3.5. It includes various bounds for products of the coefficient matrices B^μ and $\mathbb{P}$, which projects onto the velocity component z of the solution vector Z. These delicate decay estimates are precisely what is needed to apply the global existence theorem from [41].

Decelerated expansion. In chapter 5, we investigate the decelerated regime $\alpha < 1$ and we assume that the initial data is non-trivial only in one spatial direction. This condition propagates. After introducing expansion-normalized variables, we obtain a fairly simple first-order system in (5.2.1). The chapter begins with a numerical analysis. We consider small smooth perturbations of the trivial data and evolve them as long as technically possible. In anticipation of a shock singularity, we track the $\dot{H}^3$-norm, expecting a high sensitivity to steep gradients. Using this method, we scan the (α, K)-parameter space. An extrapolation argument allows us to identify a critical line for a phase transition between stable and unstable behavior at $\alpha = \frac{2}{3(1-K)}$. Below this critical expansion rate, the existence time in relation to the inverse of the initial data size ε^{-1} is a power law, while at the threshold it is exponential. This is in line with the expectation of almost global existence at the transition.

In addition to the numerical evidence, we provide a proof for stability in the region $\alpha > \frac{2}{3(1-K)}$ above the critical expansion rate. This is based on two observations. First, we prove that the mean velocity $\bar{v}$ enjoys a good decay estimate. This can be leveraged, as the problem is posed on a compact domain and thus the Poincaré inequality,

$$\|v - \bar{v}\|_{L^2} \lesssim \|Dv\|_{L^2},$$

can be applied. Naturally, the system provides decay in the velocity component as

$$\partial_t v = -\frac{\alpha(1-3K)}{t} v + r,$$

where $r = r(v\partial_x v, \partial_x L, v^3)$. The density L does not exhibit such a behavior in the standard form (5.2.1) of the equations. We can, however, introduce a new term

$$\frac{1}{2}\alpha(1-3K)(1+K)^{-1} t^{-(1-\alpha)} \int_{\mathbb{S}^1} v \partial_x L dx$$

to the energy, see (5.4.6). Heuristically speaking, this allows us to distribute the decay equally between the velocity v and the density L. This is the second key observation of this argument. As a result, we can prove a global existence and stability result for the region $\alpha > \frac{2}{3(1-K)}$. In chapter 6, we generalize this to data without the symmetry assumptions. The proof is similar, but more technical in nature.

In the final section of chapter 6, we provide generic proofs for shock formation of the fluid for dust, $K = 0$, for $\alpha \leq \frac{1}{2}$ and radiation, $K = \frac{1}{3}$, for $\alpha \leq 1$. Similar results where established in [59] and [103]. We provide a particularly simple proof, based on the method of characteristics. These results complement the stable regions in [103] and are therefore sharp. In figure 2.2 we provide a graphical overview of the additions made by the contents of this thesis.

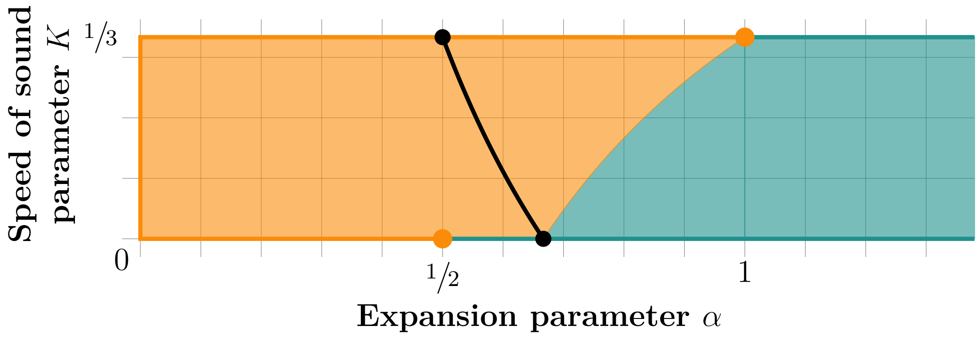

Figure 2.2: This figure shows the stability of the canonical solutions to the relativistic Euler equations with respect to the (α, K) parameter space. Points that lie on the orange boundary lines are unstable, while the turquoise lines and shaded region mark stable points. Furthermore, the orange shaded region is hypothesized to be unstable, supported by numerical evidence. In addition to the results discussed in figure 2.1, those in [36–38] are included. For reference, we have added the line of Friedman solutions discussed in Remark 1.3.3 in black. The upper and lower boundary points correspond to the (1.3.8) and (1.3.7), respectively.

2.2.2 Einstein-Euler system

In the fully dynamic setting, we present proofs of stability of the Milne spacetime (1.3.9) for the parameter ranges $K = 0$ and $0 < K < \frac{1}{3}$. In particular, this is a vacuum solution with linear expansion rate. In both cases we consider the spatial slices to be compact. Regarding the geometry, the gauge choice and estimates are based on work by Andersson and Moncrief [3; 4]. In these two articles the authors use the constant mean curvature spatial harmonic gauge (CMCSH), see Definition 3.2.1. As the name suggests, the manifold is foliated by compact spatial slices of constant mean curvature τ. Since the manifold keeps expanding, τ is a time function. By the choice of this gauge, the Einstein equations become a hyperbolic-elliptic system, with hyperbolic equations for the metric and extrinsic curvature and elliptic equations for the lapse and shift. In [3], the authors prove the stability of the Milne model as a solution to the vacuum Einstein equations.

Einstein-dust. In chapter 3, we prove stability of the Milne model as a solution to the Einstein-Dust system. As discussed in section 2.1.2, dust poses a unique challenge, as a loss of regularity occurs in the standard formulation. To circumvent this issue, we define top-order energies including the material derivative

$$\partial_{\mathbf{u}} = u^0 \partial_T + \tau u^a \nabla_a,$$

where T is an adapted time coordinate. This is inspired by the work of Hadžić and Speck who treat the stability of the Friedman solution to the Einstein-dust system with $\Lambda > 0$ in

[55]. For the definition of the corrected geometric energies based on those found in [4], see section 3.3.6. The challenging aspect of this work lies in adapting the estimates in [3; 4] to include the material derivative $\partial_{\mathbf{u}}$.

Einstein-Euler. A large part of chapter 4 is concerned with the stability of the Milne model as a solution to the Einstein-Euler system in the parameter range $0 < K < \frac{1}{3}$. Here, the most challenging aspect lies in estimating the evolution of the fluid variables. While the L^2-level estimates close, even commuting the equations with the derivatives once already leads to terms like, e.g.,

$$\int_M g^{mn}\nabla_m\psi[\nabla_n, \nabla_a]z^a = \frac{2}{9}\int_M z^m\nabla_m\psi.$$

However, we lack a priori control of the density, which is needed to close these estimates. To deal with this, we set up a correction scheme adding lower-order energies in Definition 4.5.1 and Proposition 4.5.2. This allows us to account for the non-trivial geometry.

2.2.3 Conclusion

The research presented in this thesis significantly advances our knowledge of the mathematical properties of cosmological fluids. It provides a comprehensive discussion of solutions to the relativistic Euler equations in decelerated, spatially flat FLRW-type models. Furthermore it provides proofs of the stability of linearly expanding cosmological models as solutions to the Einstein-Euler system.

CHAPTER 3

SLOWLY EXPANDING STABLE DUST SPACETIMES

Abstract. We establish future nonlinear stability of a large class of FLRW models as solutions to the Einstein-dust system. We consider the case of a vanishing cosmological constant, which in particular implies that the expansion rate of the respective models is linear i.e. has zero acceleration. The resulting spacetimes are future globally regular. These solutions constitute the first generic class of future regular Einstein-dust spacetimes not undergoing accelerated expansion and are thereby the slowest expanding generic family of future complete Einstein-dust spacetimes currently known.

3.1 Introduction

3.1.1 General relativistic hydrodynamics

The Einstein-relativistic Euler system (EES)

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= T_{\mu\nu} \\ \nabla^\mu T_{\mu\nu} &= 0 \\ T_{\mu\nu} &= (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \end{aligned} \tag{3.1.1}$$

describes the dynamical evolution of a four-dimensional spacetime $(\mathscr{M}, g)$ containing a relativistic perfect fluid with pressure p, energy density ρ and 4-velocity vector u^μ. Perfect fluids compatible with relativity were one of the earliest matter models considered in general relativity [15; 84] and have been extensively studied in the context of general relativistic hydrodynamics with numerous applications ranging from astrophysics to cosmological evolution (see e.g. [31; 109]).

The equations (3.1.1) are supplemented by specifying an equation of state which relates the energy density and pressure $p = f(\rho)$. Different choices of the function f encode different behaviour of the fluid. We focus in the following on the class of linear, barotropic equations of state, $p = c_S^2 \rho$, where the constant c_S denotes the speed of sound of the fluid with $0 \le c_S \le 1$. This equation of state contains well-known fluid models: for $c_S = 0$, i.e. $p = 0$, (3.1.1) reduces to the *Einstein-dust* system, the case $c_S = 1/\sqrt{3}$ is the *Einstein-radiation fluid* system and $c_S = 1$ is the *Einstein-stiff fluid* system. The main result of the present paper can be roughly stated as:

Theorem 3.1.1. *All four-dimensional FLRW spacetime models with compact spatial slices and negative spatial Einstein geometry are future stable solutions of the Einstein-dust system.*

To date, all known future stability results establishing the global existence, regularity and completeness of solutions to (3.1.1) concern the regime of accelerated expansion. In such a setting, the fast decay rates of perturbations induced by the expansion have a strong regularization effect on the fluid. For slower expansion rates this effect becomes weaker and global regularity of solutions is less likely to hold.

Theorem 3.1.1 establishes the **first nonlinear future stability result for a coupled Einstein-relativistic Euler system in the absence of accelerated expansion** and thereby initiates the study of the EES in the regime of non-accelerated expansion. Such a regime is also relevant in cosmology. The epoch in the early universe, shortly after a hypothetical inflationary phase, is expected to not initially have exhibited accelerated expansion. It is this epoch, which is not covered by previous results on the EES, which we intend to make accessible by the research initiated in the present paper.

3.1.2 Background and previous results

For the sake of the following presentation we consider four-dimensional FLRW spacetimes of the form

$$\Big((0,\infty)\times M, -\mathrm{d}t_c^2 + a(t_c)^2\cdot\gamma\Big)\,, \tag{3.1.2}$$

where (M,γ) is a complete Riemannian manifold. We remind the reader that in the standard FLRW-models the spatial slices appearing in (3.1.2) have constant sectional curvature $k_M \in \{-1,0,1\}$ and so are taken to be one of $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ or quotients thereof (eg, $\mathbb{T}^3$). We distinguish three classes of scale factors: $\ddot{a}(t_c) > 0$ are referred to as *accelerated expansion*, $\ddot{a}(t_c) < 0$ as *deccelerated expansion* and $\ddot{a}(t_c) = 0$ as *linear expansion*. We introduce the notion of *power law inflation*, where $a(t_c) = (t_c)^p$ for $p > 0$. Finally, we recall that a cosmological constant can be included by adding $+\Lambda g_{\mu\nu}$ to the LHS of (3.1.1). In the following discussion only $\Lambda \geq 0$ is relevant.

Shock formation. Relavistic and non-relativistic fluids are well-known to form shocks in finite time. This was first observed in the general relativistic context by Oppenheimer and Snyder when they investigated the collapse of spherically symmetric clouds of dust [84]. In the terminology of current stability analysis this constitutes the instability of Minkowski spacetime as a particular solution to the Einstein-dust system. Note that a part of Minkowski spacetime corresponds to $M = \mathbb{R}^3, \gamma = \delta, a(t_c) = 1$ in (3.1.2).

More recently, Christodoulou's monograph [22] demonstrated that under a very general equation of state, the constant solutions to the relativistic Euler equations on a fixed background Minkowski space are unstable, i.e. even without gravitational backreaction, fluids form shocks from arbitrarily small initial inhomogeneities in finite time. This suggests that Minkowski spacetime is unstable as a solution to the EES for a large class of equations of state. Note also that Christodoulou's monograph gave a detailed description of the nature of the fluid shock formation, thus providing a major extension beyond work of Sideris [100] on the non-relativistic Euler equations.

Λ-induced accelerated expansion and stabilisation of fluids. As is clear from the previous paragraphs, a powerful dispersive mechanism is required to regularise fluids and to prevent finite-time shock formation. The prime example of such a mechanism comes from cosmological models exhibiting exponential expansion. Heuristically speaking, a cosmological constant $\Lambda > 0$ generates expansion of the form $a(t_c) \sim e^{Ht_c}$ where $H = \sqrt{\Lambda/3}$, for *all* cases of the sectional curvature k_M. The cosmological constant creates damping terms in the equations of motion for the fluid, which dilutes the fluid and causes fluid lines to 'stretch apart', thus preventing shock formation.

This effect was first observed by Brauer, Rendall and Reula [14] who studied Newtonian cosmological models with $\Lambda > 0$ and a perfect fluid (albeit for a slightly different equation of state). They found that the regularising effect from the exponential expansion was strong enough to prevent shock formation for small inhomogeneities of initially uniformly quiet fluid states. See also late-time asymptotics work by Reula [91] and Rendall [87].

Moving to the fully coupled Einstein-relativistic Euler system, there has been much research concerning spacetimes undergoing exponential expansion. The first result is by Rodnianski and Speck [96], who proved future stability to irrotational perturbations of uniformly quiet fluids with $0 < c_S < 1/\sqrt{3}$ on FLRW-spacetimes with underlying spatial manifold $M = \mathbb{T}^3$. The irrotational restriction was later removed by Speck in [102]. An alternative proof of future stability for these FLRW-background solutions was later given by Oliynyk [81], whose Fuchsian techniques were able to uniformly cover the cases $0 < c_S \leq 1/\sqrt{3}$.

Moving to the case of dust $c_S = 0$, stability for FLRW-spacetimes with underlying spatial manifold $M = \mathbb{T}^3$ was given by Hadžić and Speck [55], while more general spatial manifolds were considered by Friedrich [50] using his conformal method. Indeed the work by Lübbe and Valiente-Kroon [70] treated the radiation case with $c_S = 1/3$ using an extension of Friedrich's conformal method. Finally, we note that very recent work of Oliynyk [80] has established stability for ultra-radiation fluids $1/\sqrt{3} < c_S \leq 1/\sqrt{2}$ on a fixed, exponentially expanding spacetime.

Alternative mechanisms for accelerated expansion. A cosmological constant is not the only known mechanism for generating solutions to Einstein's equations with accelerated expansion. In work that predates the references of Subsection 3.1.2, Ringström [93] considered the future global stability of a large class of solutions to the Einstein-nonlinear-scalar field system with a scalar field potential $V(\Phi)$ that satisfied $V(0) > 0, V'(0) = 0, V''(0) > 0$. Roughly speaking $V(0)$ emulates the cosmological constant Λ and so these spacetimes undergo accelerated expansion. Ringström [94] later considered alternative potentials $V(\Phi)$ which relaxed the rate of spacetime expansion to the class of accelerated power law inflation, which in our terminology corresponds to $p > 1$.

Note that in Ringström's papers the global spatial topology becomes irrelevant for the long time behaviour of cosmological spacetimes in the small data regime. This is in sharp contrast to the Einstein *vacuum* equations where the spatial topology does affect the long-time behaviour. In this case, only the Milne geometry with a negative spatial curvature yields future eternally expanding cosmological models with precisely linear expansion rate.

Finally we note that the Chaplygin equation of state, which describes a fluid with negative pressure, can also generate sufficient spatial expansion to ensure future stability results for the coupled EES system [63].

Critical expansion rates. Interpolating between Minkowski space (which can be considered as a cosmological spacetime with non-compact slices and no expansion) and exponentially expanding spacetimes, it is clear that there must be a transition between shock formation and stability. To investigate the expansion rate for which this transition occurs, and how it depends on the equation of state, it is useful to study the stabilisation of fluids on fixed Lorentzian geometries obeying power-law inflation. We consider $M = \mathbb{T}^3$ with $a(t_c) = (t_c)^p$ for $p > 0$. The following table summarises some of the main results concerning linear equation of states $p = c_S^2\rho$ from [41; 103], see also [108]:

Case	Power-law rate	Range of c_S	Behaviour	Reference
No. 1	$p > 1$	$0 < c_S < 1/\sqrt{3}$	Stable	[103]
No. 2	$p = 1$	$c_S = 1/\sqrt{3}$	Shocks	[103]
No. 3	$p = 1$	$0 < c_S < 1/\sqrt{3}$	Stable (irrot.)	[41]
No. 4	$p > \frac{1}{2}$	$c_S = 0$	Stable	[103]

In combination, these results indicate that in spacetimes undergoing power-law inflation whether shocks form from small data depends on the equation of state and, in particular in the linear case, on the speed of sound. Indeed the literature suggests that *slower speeds of sound reduce the tendency of shock formation.* For the particular case of dust ($c_S = 0$), case No. 4 shows that shocks are avoided even in deccelerating spacetimes with scale factors $a(t) = t^{1/2+\delta}$ for $\delta > 0$.

3.1.3 Main results

In the present paper we consider the Einstein-dust system in the regime of linear expansion. For the linearly expanding case, Cases No. 2 and 3 above show that even in the absence of backreaction the speed of sound determines whether shocks form or not. We prove that for the case of dust ($c_S = 0$) shock formation does not occur under the full gravity-fluid dynamics.

Our background geometry is that of the Milne model, which generalises the $k_M = -1$ FLRW vacuum spacetimes. Let (M, γ) be a closed, connected, orientable three-dimensional manifold admitting a Riemannian Einstein metric γ with negative Einstein constant. After rescaling, we suppose that

$$\mathrm{Ric}[\gamma] = -\frac{2}{9}\gamma.$$

The generalised Milne spacetime is the Lorentz cone spacetime $\mathscr{M} = (0, \infty) \times M$ with metric

$$g_M := -\mathrm{d}t_c^2 + \frac{t_c^2}{9}\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b.$$

The spacetime $(\mathscr{M}, g_M)$ is globally hyperbolic and a solution to the four-dimensional vacuum Einstein equations. We formulate the main theorem using terminology introduced in Section 3.3.1. We let $\mathscr{B}_{\varepsilon}^{j,k,l,m}(\frac{t_0^2}{9}\gamma, -\frac{t_0}{9}\gamma, 0, 0)$ denote the ball of radius ε in the space $H^j \times H^k \times H^l \times H^m$ centred at $(\frac{t_0^2}{9}\gamma, -\frac{t_0}{9}\gamma, 0, 0)$. Our main theorem reads:

Theorem 3.1.2. *Let $(\mathscr{M}, g_M)$ be as above. Let $\varepsilon > 0$ and (g_0, k_0, ρ_0, u_0) be initial data for the Einstein-dust system at $t_c = t_0$ such that*

$$(g_0, k_0, \rho_0, u_0) \in \mathscr{B}_{\varepsilon}^{6,5,4,5}\left(\frac{t_0^2}{9}\gamma, -\frac{t_0}{9}\gamma, 0, 0\right).$$

Then, for ε sufficiently small the corresponding future development under the Einstein-dust system is future complete and admits a CMC foliation labelled by $\tau \in [\tau_0, 0)$ such that the induced metric and second fundamental form on constant CMC slices converge as

$$(\tau^2 g, \tau k) \to \left(\gamma, \frac{1}{3}\gamma\right) \quad \text{as } \tau \nearrow 0 \quad \text{i.e. as } t_c \nearrow \infty.$$

If the initial energy density of the dust field is non-negative, $\rho_0 \geq 0$, then it remains so throughout the evolution.

Remark 3.1.1. The Milne model is known to be a stable solution solution to the Einstein vacuum equations [3], the Einstein massive-Vlasov equations [2], the coupled Einstein-Maxwell-scalar field system arising from a Kaluza-Klein reduction [13], and the Einstein Klein-Gordon equations [42; 107].

Remark 3.1.2. Negative spatial curvature is crucial as spherical or toroidal spatial topologies would lead to recollapsing or slowly expanding matter dominated solutions, respectively. The asymptotic behaviour of the solutions in the theorem coincide with the corresponding vacuum solutions.

Structure and key novelties in the proof. The proof of Theorem 3.1.2 consists of three major parts: (i) energy estimates for the perturbation of the spacetime geometry with sources given by the dust variables, (ii) energy estimates for the dust variables in the perturbed spacetime geometry and (iii) a bootstrap argument based on both sets of energy estimates establishing global existence and asymptotic behaviour. This rough approach is standard in the literature on the Milne stability problem (see e.g. [2; 3; 13]), however, for the Einstein dust system there are crucial difficulties caused by a regularity problem inherent to the dust equations, which turns out to affect all parts of the argument. We outline the difficulties and how these are overcome in the following.

To control the perturbed spacetime geometry throughout the evolution we use a CMC time-foliation in combination with a spatial-harmonic gauge [4]. The existence of such a foliation for small perturbations of negative Einstein spaces is non-trivial but standard [35]. The Einstein equations then take the form of an elliptic-hyperbolic system (see (3.2.22)) where the lapse and shift are determined by elliptic PDEs with sources given in terms of metric, second fundamental form and the dust variables. This elliptic system provides Sobolev estimates for the lapse and shift.

The core idea to establish decay for the geometric variables in previous works on Milne stability is a corrected energy (${E^g}_k$ in Definition 3.3.4) based on the modified Einstein-operator ($\mathscr{L}_{g,\gamma}$ in Definition 3.3.2) of the spatial Einstein geometry [2; 3]. For the Einstein-dust system we must deviate from this standard approach due to a regularity issue from the dust model, which in turn affects all parts of the proof.

When expanded, the equations of motion for the dust variables take a form where the source term of the evolution equation for the energy density contains the spatial divergence of the fluid velocity (see (3.2.22d)). Consequently, the fluid energy density can be controlled only in one order of regularity below the order of regularity of the fluid velocity. From the perspective of the Einstein equations this is very problematic as both components of the dust, energy density and fluid velocity, appear at the same order of regularity as source terms of the Einstein equations. As such, they are required to be controlled in suitable Sobolev spaces at the same order as the second fundamental form. Due to the required high regularity of the fluid velocity discussed previously, the velocity then needs to be controlled one order above the second fundamental form. However, the equation of motion for the fluid velocity requires the second fundamental form at the same order of regularity as the velocity itself (see (3.2.22d)). This apparent inconsistency prevents one from establishing a standard and straightforward regularity hierarchy to analyse the fully coupled nonlinear system.

An approach to circumvent this issue has been introduced by Hadžić and Speck in [55] and is modified in the present paper. The central idea is to use a fluid derivative $\partial_{\mathbf{u}} \sim u^\alpha \nabla_\alpha[g_M]$ as a differential operator in the energies for the perturbations of the metric and second fundemental form ($E^g_{\partial_{\mathbf{u}},N-1}$ in Definition 3.3.6). At highest order of regularity, say N, where the loss of derivatives prevents the closure of the system of estimates, the Einstein equations are commuted with $N-1$ spatial derivatives and one fluid derivative. When this derivative acts on the dust source terms in the Einstein equations, in the subsequent calculations for the energy estimates, the equations of motion of the dust variables are used as constraint equations replacing $\partial_{\mathbf{u}}\rho$ and $\partial_{\mathbf{u}}u^j$. In this way, no derivatives are lost and the corresponding auxiliary energies $E^g_{\partial_{\mathbf{u}},N-1}$ for the geometric variables can be estimated in terms of dust variables of one order of regularity below the expected one.

In a follow-up step, we need to show that the auxiliary geometric energies $E^g_{\partial_{\mathbf{u}},N-1}$ in fact control the actual top-order regularity norms of the geometric variables. This is achieved by rewriting the wave-type evolution equation for the metric and second fundamental in terms of an elliptic part and certain mixed spatial and fluid derivative operators (see Proposition 3.6.1). Consequently, the auxiliary energies of the first step provide top-order estimates on the geometric variables (see Corollary 3.6.1) and an overall strategy to close the estimates.

Two final major regularity issues arise when proving energy estimates for the auxiliary energies $E^g_{\partial_{\mathbf{u}},N-1}$ however. We end up needing to estimate one fluid derivative and a critical number of spatial derivatives on certain terms involving the lapse and shift, and we cannot commute the $\partial_{\mathbf{u}}$ operator past the spatial derivatives without exceeding the assumed regularity of the fluid spatial velocity.

To circumvent this problem, we only commute past some of the derivatives and instead derive two auxiliary estimates using the elliptic equations (3.2.22b) for the lapse and shift.

In the estimate on the lapse term (see Proposition 3.7.1) we crucially use the equations of motion of the dust variables to replace a certain matter term $\partial_{\mathbf{u}}\eta$ as a constraint, thus avoiding derivative loss. The estimate for the shift term (see Proposition 3.7.2) proceeds differently, relying on a *remarkable combination* of commutator estimates, the Bianchi identity and the Einstein equations in the CMCSH gauge.

Final remarks. In the regime of non-accelerated expansion, the work [14] indicates that the backreaction between the fluid and the geometry cannot be ignored. The authors consider the case of dust with a Newtonian backreaction, finding that shocks form for arbitrarily small initial data in the regime where the homogeneous background spacetime, which is perturbed, expands like $a(t) = t^{2/3}$. This contrasts *noteably* with case No. 5 above which does not include backreaction. While [14] concerns only Newtonian dynamics, it is nevertheless a fair indication that the fully coupled dynamics under the Einstein-fluid system will likely lead to the formation of shocks. Continuing this line of reasoning, we note that although the work [41] also treated linear expansion, the full coupling between gravity and fluid makes our present work highly nontrivial. Indeed the issues highlighted on the previous Section 3.1.3 are indicative of the substantial technical difficulties that arise in the fully coupled EES.

Finally, it is interesting to recall that Sachs and Wolfe derived a linear *in*stability result for the Einstein-dust equations with $\Lambda = 0$, however their metric had underlying spatial manifold $M = \mathbb{R}^3$ [98]. The fluid plays a major dynamical role in these flat FLRW models. Nevertheless one gleans the importance of the negatively curved spatial slices appearing in our nonlinear stability result.

3.1.4 Outline of the paper

In Section 3.2 we introduce the system of equations and perform a natural rescaling of the variables. In Section 3.3 we introduce function spaces and energy functionals controlling the metric perturbation and shear tensor.

The main theorem is proved using continuous induction. In Section 3.4 we discuss the local existence theory and initiate the bootstrap argument. The remainder of the paper, beginning with Section 3.5, treats the individual estimates necessary to close the bootstrap argument. Section 3.5 gathers various auxiliary estimates, which are used in later sections. Among those are estimates on the source terms of the evolution equations, estimates on the dust-derivative acting on various quantities, commutators of the dust derivative and other operators and estimates on high derivatives combining the dust-derivative and other operators.

Section 3.6 derives the elliptic estimate for the Einstein operator and the evolution equations, which is crucial to turn estimates in terms of the dust derivatives into those in terms of standard energies. These estimates are then given subsequently. In Section 3.7 we provide the estimates on lapse function and shift vector field. A crucial set of lapse and shift estimates on highest order of regularity, involving also the dust derivative, are given here too. In Section 3.8 we derive the central top-order energy estimate for the auxiliary energy controlling the geometric perturbations. In Section 3.9 we derive the estimates for the dust variables and in Section 3.10 we close the bootstrap.

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3.2 Equations of motion

3.2.1 The Einstein–dust system

The Einstein–relativistic Euler system reads

$$\begin{aligned} R_{\mu\nu}[\bar{g}] - \frac{1}{2}R[\bar{g}]\bar{g}_{\mu\nu} &= 2\tilde{T}_{\mu\nu}, \\ \bar{\nabla}_\mu \tilde{T}^{\mu\nu} &= 0, \\ \tilde{T}^{\mu\nu} &= (\tilde{\rho}+\tilde{P})\tilde{u}^\mu\tilde{u}^\nu + \tilde{P}\bar{g}^{\mu\nu}, \end{aligned} \tag{3.2.1}$$

where we set $c = 1$ and $4\pi G = 1$. We use $\bar{\nabla}$ to denote the Levi-Civita connection of the physical metric $\bar{g}$. The four-velocity of the fluid $\tilde{u}^\mu$ is a future-directed timelike vectorfield normalised by

$$\bar{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1\,. \tag{3.2.2}$$

We assume a linear, barytropic fluid equation of state $\tilde{P} = c_S^2\tilde{\rho}$ where $c_S \geq 0$ is a constant, and $\tilde{P} \geq 0$ and $\tilde{\rho} \geq 0$ denote the pressure and energy density respectively. In the present paper, we **restrict ourselves to dust**, which means we set

$$c_S^2 := 0\,.$$

The fluid equations in (3.2.1) can equivalently (for $\tilde{\rho} > 0$) be written as

$$\tilde{u}^\alpha\bar{\nabla}_\alpha \ln\tilde{\rho} + \bar{\nabla}_\alpha\tilde{u}^\alpha = 0\,, \qquad \tilde{u}^\alpha\bar{\nabla}_\alpha\tilde{u}^\mu = 0\,. \tag{3.2.4}$$

The system (3.2.4) is overdetermined in the sense that $\tilde{u}^0$ can be determined from the other fluid velocity components via (3.2.2).

We will study the Einstein-dust equations using the following ADM ansatz for the metric

$$\bar{g} = -\tilde{N}^2\mathrm{d}t^2 + \tilde{g}_{ab}(\mathrm{d}x^a + \tilde{X}^a\mathrm{d}t)(\mathrm{d}x^b + \tilde{X}^b\mathrm{d}t). \tag{3.2.5}$$

Note that $\bar{g}_{ab} = \tilde{g}_{ab}$ but in general $\bar{g}^{ab} \neq \tilde{g}^{ab}$. On t=constant slices, we let τ be the trace of the second fundamental form $\tilde{k}$ with respect to $\tilde{g}$ and define Σ to be the trace-free part of $\tilde{k}$. That is,

$$\tau := \mathrm{tr}_{\tilde{g}}\tilde{k} = \tilde{g}^{ab}\tilde{k}_{ab}, \qquad \tilde{k} := \tilde{\Sigma} + \frac{1}{3}\tau\tilde{g}.$$

We use Roman letters $(a, b, i, j...)$ to denote spatial indices. Let ∇ denote the Levi-Civita connection of the spatial metric $\tilde{g}$. Using (3.2.5) the Christoffel symbols of the 4-metric $\bar{g}$ become (see e.g. [88])

$$\begin{aligned}
{}^{(4)}\tilde{\Gamma}^0_{00} &= \tilde{N}^{-1}(\partial_t\tilde{N} + \tilde{X}^a\nabla_a\tilde{N} - \tilde{k}_{ab}\tilde{X}^a\tilde{X}^b), & {}^{(4)}\tilde{\Gamma}^0_{ab} &= -\tilde{N}^{-1}\tilde{k}_{ab},\\
{}^{(4)}\tilde{\Gamma}^0_{a0} &= \tilde{N}^{-1}(\nabla_a\tilde{N} - \tilde{k}_{ab}\tilde{X}^b), & {}^{(4)}\tilde{\Gamma}^a_{bc} &= \Gamma^a_{bc}[\tilde{g}] + \tilde{N}^{-1}\tilde{k}_{bc}\tilde{X}^a,\\
{}^{(4)}\tilde{\Gamma}^a_{0b} &= -\tilde{N}\tilde{k}^a_b + \nabla_b\tilde{X}^a - \tilde{N}^{-1}\tilde{X}^a\nabla_b\tilde{N} + \tilde{N}^{-1}\tilde{k}_{bc}\tilde{X}^c\tilde{X}^a,\\
{}^{(4)}\tilde{\Gamma}^a_{00} &= \partial_t\tilde{X}^a + \tilde{X}^b\nabla_b\tilde{X}^a - 2\tilde{N}\tilde{k}^a_c\tilde{X}^c + \tilde{N}\nabla^a\tilde{N}\\
&\quad - \tilde{N}^{-1}(\partial_t\tilde{N} + \tilde{X}^b\nabla_b\tilde{N} - \tilde{k}_{bc}\tilde{X}^b\tilde{X}^c)\tilde{X}^a.
\end{aligned}$$

Noting the above, the fluid equations (3.2.4) reduce to

$$\tilde{u}^\alpha\partial_\alpha \ln\tilde{\rho} + \partial_\alpha\tilde{u}^\alpha + {}^{(4)}\tilde{\Gamma}^\alpha_{\alpha\nu}\tilde{u}^\nu = 0, \qquad \tilde{u}^\alpha\partial_\alpha\tilde{u}^\mu + \tilde{u}^{\alpha(4)}\tilde{\Gamma}^\mu_{\alpha\nu}\tilde{u}^\nu = 0.$$

3.2.2 The rescaled Einstein–dust system in CMCSH gauge

Following the work of Andersson and Moncrief [3; 4], we hereon impose the CMCSH gauge which foliates by surfaces of constant mean curvature, taking advantage of the fact that on the Milne background $\tau = \tilde{g}^{ab}\tilde{k}_{ab} = -3/t_c$.

Definition 3.2.1 (CMCSH gauge)**.**

$$t = \tau, \qquad H^a := \tilde{g}^{cb}(\Gamma[\tilde{g}]^a_{cb} - \Gamma[\gamma]^a_{cb}) = 0. \tag{3.2.8}$$

We next rescale our variables with respect to the mean curvature τ.

Definition 3.2.2 (Rescaled variables $(g_{ab}, N, X^a, \Sigma_{ab}, u^a, u^0, \rho, \widehat{N}, \hat{u}^0)$ and logarithmic time T)**.** The rescaled geometric variables are defined as

$$\begin{aligned}
g_{ab} &:= \tau^2\tilde{g}_{ab}, & g^{ab} &:= (\tau^2)^{-1}\tilde{g}^{ab},\\
N &:= \tau^2\tilde{N}, & X^a &:= \tau\tilde{X}^a,\\
\Sigma_{ab} &:= \tau\tilde{\Sigma}_{ab}.
\end{aligned} \tag{3.2.9a}$$

Note $\tilde{X}_a = \tilde{g}_{ab}\tilde{X}^b = \tau^{-3}X_a$. Let $\tilde{u}^0 := \tilde{u}^\tau$. The rescaled matter variables are defined as

$$u^a := \tau^{-2}\tilde{u}^a, \qquad \rho := |\tau|^{-3}\tilde{\rho}, \qquad u^0 := \tau^{-2}\tilde{u}^0. \tag{3.2.9b}$$

Denote $\widehat{N} := \frac{N}{3} - 1$ and $\hat{u}^0 := u^0 - 1/3$. Finally we define the logarithmic time

$$T := -\ln(\tau/(e\tau_0)),$$

which satisfies $\partial_T = -\tau\partial_\tau$.

The above definition means we have the following ranges $\tau_0 \leq \tau \nearrow 0$ and $1 \leq T \nearrow \infty$ where $\tau \nearrow 0$ corresponds to the direction of cosmological expansion (i.e. $t_c \nearrow \infty$).

Lemma 3.2.1. *The normalisation condition* (3.2.2) *implies*

$$u^0 = \frac{1}{(N^2 - X_a X^a)}\Big(\tau X_a u^a + \Big[\tau^2 (X_a u^a)^2 + (N^2 - X_a X^a)(\tau^2 g_{ab} u^a u^b + 1)\Big]^{1/2}\Big).$$

Proof. Using (3.2.2) we have

$$0 = (-\tilde{N}^2 + \tilde{X}_a \tilde{X}^a)(\tilde{u}^0)^2 + 2\tilde{X}_a \tilde{u}^a \tilde{u}^0 + \tilde{g}_{ab}\tilde{u}^a \tilde{u}^b + 1.$$

This is a quadratic equation in $\tilde{u}^0$. The roots are

$$\tilde{u}^0 = \frac{1}{2(-\tilde{N}^2 + \tilde{X}_a \tilde{X}^a)}\Big(- 2\tilde{X}_a \tilde{u}^a \pm \Big[4(\tilde{X}_a \tilde{u}^a)^2 - 4(-\tilde{N}^2 + \tilde{X}_a \tilde{X}^a)(\tilde{g}_{ab}\tilde{u}^a \tilde{u}^b + 1)\Big]^{1/2}\Big).$$

Applying the rescalings from Definition 3.2.2 we find

$$\tilde{u}^0 = \frac{\tau^4}{(N^2 - X_a X^a)}\Big(\tau^{-1} X_a u^a \mp \Big[\tau^{-2}(X_a u^a)^2 + (N^2 - X_a X^a)(\tau^{-2} g_{ab} u^a u^b + \tau^{-4})\Big]^{1/2}\Big).$$

Hence, we introduce the rescaled quantity $u^0 = \tau^{-2}\tilde{u}^0$ for the larger root. □

Let $\nabla, \hat{\nabla}$ denote the Levi-Civita connection of the Riemannian metrics g, γ respectively. The Christoffel symbols of $\bar{g}$ now become (see e.g. [2])

$$\begin{aligned}
{}^{(4)}\tilde{\Gamma}^a_{bc} &= \Gamma^a_{bc}[g] - \Gamma_{bc} X^a, & {}^{(4)}\tilde{\Gamma}^a_{00} &= \tau^{-2}\Gamma^a, \\
{}^{(4)}\tilde{\Gamma}^a_{0b} &= \tau^{-1}\left(-\delta^a_b + \Gamma^a_b\right), & {}^{(4)}\tilde{\Gamma}^0_{00} &= \tau^{-1}(-2 + \Gamma_R), \\
{}^{(4)}\tilde{\Gamma}^0_{ab} &= \tau \Gamma_{ab}, & {}^{(4)}\tilde{\Gamma}^0_{0a} &= \Gamma_a,
\end{aligned}$$

where we have introduced the following rescaled geometric components:

Definition 3.2.3 (Rescaled Christoffel components $\Gamma^a, \Gamma^a_b, \Gamma_R, \Gamma_{ab}, \mathring{\Gamma}^a$)**.**

$$\begin{aligned}
\Gamma^a &:= -\partial_T X^a - X^a - 2\widehat{N} X^a + X^b \mathrm{NewA}_b X^a - 2N\Sigma^a_c X^c + N \mathrm{NewA}^a N \\
&\quad + \Big(N^{-1}\partial_T N - N^{-1} X^b \nabla_b N + N^{-1}\left(\Sigma_{bc} + \frac{1}{3} g_{bc}\right) X^b X^c\Big) X^a, \\
\Gamma^a_b &:= -N\Sigma^a_b - \delta^a_b \widehat{N} + \nabla_b X^a - N^{-1} X^a \nabla_b N + N^{-1}\left(\Sigma_{bc} + \frac{1}{3} g_{bc}\right) X^c X^a, \\
\Gamma_R &:= N^{-1}\big(- \partial_T N + X^a \nabla_a N - (\Sigma_{ab} + \frac{1}{3} g_{ab}) X^a X^b\big), \quad \mathring{\Gamma}^a := \Gamma^a - N\nabla^a N, \\
\Gamma_{ab} &= -N^{-1}(\Sigma_{ab} + \frac{1}{3} g_{ab}), \qquad \Gamma_a := N^{-1}(\nabla_a N - (\Sigma_{ab} + \frac{1}{3} g_{ab}) X^b).
\end{aligned}$$

Remark 3.2.1 (Background solutions). The rescaled background (B) Milne geometry written in CMCSH gauge is

$$(g_{ab}, \Sigma_{ab}, N, X^a)|_B \equiv (\gamma, 0, 3, 0).$$

Furthermore,

$$(\Gamma^a, \Gamma^a_b, \Gamma_R, \Gamma_a)|_B \equiv 0, \quad \Gamma_{ab}|_B \equiv -\frac{1}{9}\gamma_{ab}.$$

Let $\rho_0' > 0$ be a constant. The background, uniformly quiet fluid solution in CMCSH gauge is

$$(u^0, u^i, \rho)|_B = (\frac{1}{3}, 0, \rho_0').$$

See also Appendix 3.11.1.

We next evaluate certain energy momentum and matter source terms arising from the dust.

Definition 3.2.4 (Matter source terms $E, \jmath^a, \eta, S_{ab}, \underline{T}^{ab}$)**.**

$$\begin{aligned} E &:= \rho(u^0)^2 N^2, & \jmath^a &:= \rho N u^0 u^a, \\ \eta &:= E + \rho g_{ab}(u^0 X^a + \tau u^a)(u^0 X^b + \tau u^b), \\ S_{ab} &:= \rho(u^0 X_a + \tau u_a)(u^0 X_b + \tau u_b) + \frac{1}{2}\rho g_{ab}, & \underline{T}^{ab} &:= \rho u^a u^b. \end{aligned}$$

For further details on these definitions see Appendix 3.11.2.

Definition 3.2.5 (Matter source terms F_{u^j}, F_{u^0}, F_ρ)**.**

$$\begin{aligned} F_{u^j} &:= \tau^{-1}(u^0)^2 \overset{\circ}{\Gamma}{}^j + (\Gamma^j_{kl}[g] - \Gamma^j_{kl}[\gamma]) u^k u^l + 2u^0 u^i \Gamma^j_i + \tau u^k u^i \Gamma_{ki} X^j, \\ F_{u^0} &:= (u^0)^2 \Gamma_R + 2\tau u^j u^0 \Gamma_j + \tau^2 u^k u^j \Gamma_{kj}, \\ F_\rho &:= \tau \nabla_i u^i + \Gamma^i_i u^0 - \tau \Gamma_j u^j + \tau \Gamma_{ik} X^i u^k - \tau \frac{u^j}{u^0} \nabla_j u^0 - \tau^2 \frac{u^k u^j}{u^0} \Gamma_{kj}. \end{aligned} \tag{3.2.21}$$

3.2.3 Equations of motion

Bringing together all the previous notation, as well as using the general equations presented in [2], the equations of motion for the Einstein-dust system in CMCSH gauge are the following. We have two constraint equations:

$$\begin{aligned} \mathrm{R}(g) - |\Sigma|_g^2 + \frac{2}{3} &= 4\tau E, \\ \nabla^a \Sigma_{ab} &= 2\tau^2 \jmath_b, \end{aligned} \tag{3.2.22a}$$

and two elliptic equations for the lapse and shift variables

$$\begin{aligned} (\Delta - \frac{1}{3})N &= N\left(|\Sigma|_g^2 - \tau\eta\right) - 1, \\ \Delta X^a + \mathrm{Ric}[g]^a{}_b X^b &= 2\nabla_b N \Sigma^{ba} - \nabla^a \widehat{N} + 2N\tau^2 \jmath^a - (2N\Sigma^{bc} - \nabla^b X^c)(\Gamma[g]^a_{bc} - \Gamma[\gamma]^a_{bc}). \end{aligned} \tag{3.2.22b}$$

We also have evolution equations for the induced metric and trace-free part of the second fundamental form

$$\begin{aligned} \partial_T g_{ab} &= 2N\Sigma_{ab} + 2\widehat{N} g_{ab} - \mathscr{L}_X g_{ab}, \\ \partial_T \Sigma_{ab} &= -2\Sigma_{ab} - N(\mathrm{Ric}[g]_{ab} + \frac{2}{9} g_{ab}) + \nabla_a \nabla_b N + 2N\Sigma_{ac}\Sigma^c_b \\ &\quad - \frac{1}{3}\widehat{N} g_{ab} - \widehat{N}\Sigma_{ab} - \mathscr{L}_X \Sigma_{ab} + N\tau S_{ab}, \end{aligned} \tag{3.2.22c}$$

and, finally, evolution equations for the fluid components:

$$
\begin{aligned}
u^0\partial_T u^j &= \tau u^a\nabla_a u^j + \tau^{-1}(u^0)^2 N\nabla^j N + F_{u^j},\\
u^0\partial_T u^0 &= \tau u^a\nabla_a u^0 + F_{u^0},\\
u^0\partial_T \rho &= \tau u^a\nabla_a \rho + \rho F_\rho.
\end{aligned}
\tag{3.2.22d}
$$

Using notation from [3; 4] we introduce new variables which allow us to rewrite (3.2.22c).

Definition 3.2.6 (Perturbation variables h, v, w and geometric source terms F_h, F_v)**.** Define the variables

$$
h_{ab} := g_{ab} - \gamma_{ab}, \quad v_{ab} := 6\Sigma_{ab}, \quad w := N/3,
$$

and the geometric source terms

$$
\begin{aligned}
(F_h)_{ab} &:= 2\widehat{N}g_{ab} + h_{ac}\hat{\nabla}_b X^c + h_{cb}\hat{\nabla}_a X^c,\\
(F_v)_{ab} &:= \nabla_a\nabla_b N + 2N\Sigma_{ac}\Sigma^c_b - \frac{1}{3}\widehat{N}g_{ab} - \widehat{N}\Sigma_{ab} + N\tau S_{ab} - v_{ac}\hat{\nabla}_b X^c - v_{cb}\hat{\nabla}_a X^c.
\end{aligned}
$$

We start with the following identity from [4]:

$$
\mathscr{L}_X g_{ab} = X^c\hat{\nabla}_c g_{ab} + g_{ac}\hat{\nabla}_b X^c + g_{cb}\hat{\nabla}_a X^c.
$$

Due to rigidity properties of negative Einstein manifolds in three spatial dimensions (see e.g. [3, §1.1]), we have $\partial_T\gamma = 0$. Thus the equations (3.2.22c) reduce to

$$
\begin{aligned}
\partial_T h_{ab} &= wv_{ab} - X^m\hat{\nabla}_m h_{ab} + F_h,\\
\partial_T v_{ab} &= -2v_{ab} - 9w\mathscr{L}_{g,\gamma}h_{ab} - X^c\hat{\nabla}_c v_{ab} + 6F_v.
\end{aligned}
\tag{3.2.26}
$$

In Section 3.8 we will also write the first equation in (3.2.26) as

$$
\partial_T h_{ab} = wv_{ab} + 2\widehat{N}g_{ab} - (\mathscr{L}_X g)_{ab} = wv_{ab} + 2\widehat{N}g_{ab} - g_{am}\nabla_b X^m - g_{bm}\nabla_a X^m. \tag{3.2.27}
$$

Hereon we use the differential equations (3.2.22b), (3.2.22d) and (3.2.26) to analyse the solutions to our Einstein-dust system.

3.3 Preliminary definitions

In this section we present several preliminary definitions concerning Sobolev spaces, norms, elliptic estimates and energy functionals. All of this is standard except for Definitions 3.3.5 and 3.3.6 where we introduce the fluid derivative $\partial_{\mathbf{u}}$ and then the energy functionals for the geometric variables involving this fluid derivative.

3.3.1 Function spaces and norms

Definition 3.3.1 ($L^2_{g,\gamma}$-inner product)**.** Let $\mu_g = \sqrt{\det g}$ denote the volume element on (M,g), similarly for μ_γ. Let V, P be $(0,2)$-tensors on M. Define an inner product by

$$\langle V, P\rangle_\gamma := V_{ij}P_{kl}\gamma^{ik}\gamma^{jl},$$

and define a mixed L^2-scalar product

$$(V,P)_{L^2(g,\gamma)} := \int_M \langle V,P\rangle_\gamma \, \mu_g,$$

with corresponding norm $\|V\|^2_{L^2_{g,\gamma}} := (V,V)_{L^2(g,\gamma)}$.

The following definition follows notation first introduced in [3].

Definition 3.3.2 ($\mathrm{Riem}\,[\gamma]\circ$ and $\mathscr{L}_{g,\gamma}$)**.** Let V be a symmetric $(0,2)$-tensor on M. Define the tensorial contraction

$$(\mathrm{Riem}[\gamma]\circ V)_{ij} := \mathrm{Riem}[\gamma]_{iajb}\gamma^{aa'}\gamma^{bb'}V_{a'b'}\,,$$

where, following the convention of [4], the Riemann tensor is defined by $[\hat\nabla_a, \hat\nabla_i]V_b = (\hat\nabla_a\hat\nabla_i - \hat\nabla_i\hat\nabla_a)V_b := -\mathrm{Riem}[\gamma]^c{}_{bai}V_c$. Define the following differential operators

$$\begin{aligned}
\hat\Delta_{g,\gamma}V_{ij} &:= (\sqrt{\det g})^{-1}\hat\nabla_a(\sqrt{\det g}\cdot g^{ab}\hat\nabla_b V_{ij}),\\
\mathscr{L}_{g,\gamma}V_{ij} &:= -\hat\Delta_{g,\gamma}V_{ij} - 2(\mathrm{Riem}[\gamma]\circ V)_{ij}.
\end{aligned}$$

By using the gauge condition (3.2.1), the operator $\hat\Delta_{g,\gamma}$ can be rewritten as

$$\hat\Delta_{g,\gamma}V_{cd} = g^{ab}\hat\nabla_a\hat\nabla_b V_{cd} - H^a\hat\nabla_a V_{cd}.$$

The operator $\mathscr{L}_{g,\gamma}$ is self-adjoint with respect to the mixed L^2-scalar product (see e.g. [4])

$$(\mathscr{L}_{g,\gamma}V, P)_{L^2(g,\gamma)} = (V, \mathscr{L}_{g,\gamma}P)_{L^2(g,\gamma)}. \tag{3.3.5}$$

A self-adjoint elliptic operator on a compact manifold has a discrete spectrum of eigenvalues. Using eigenvalue estimates from [62], we are led to the following result.

Proposition 3.3.1 (Estimates on λ_0)**.** *Let (M,γ) be a negative Einstein three-manifold with Einstein constant $k = -2/9$. Then the smallest eigenvalue of the operator $\mathscr{L}_{g,\gamma}$ satisfies $\lambda_0 \geq 1/9$ and the operator also has trivial kernel* $\ker(\mathscr{L}_{g,\gamma}) = \{0\}$.

Definition 3.3.3 (Sobolev norms)**.** Let $k \in \mathbb{Z}_{\geq 0}$. For κ a Riemannian metric on M, f a function and V a $(1,1)$-tensor, define

$$\begin{aligned}
|\nabla^k f|^2_\kappa &:= \kappa^{a_1b_1}\cdots\kappa^{a_kb_k}(\nabla_{a_1}\cdots\nabla_{a_k}f)\cdot(\nabla_{b_1}\cdots\nabla_{b_k}f),\\
|\nabla^k V|^2_\kappa &:= \kappa_{ij}\kappa^{kl}\kappa^{a_1b_1}\cdots\kappa^{a_kb_k}(\nabla_{a_1}\cdots\nabla_{a_k}V^i{}_k)\cdot(\nabla_{b_1}\cdots\nabla_{b_k}V^j{}_l).
\end{aligned}$$

The obvious extension to (p,q)-tensors holds. We write

$$\|V\|_{H^k} = \Big(\sum_{0\leq \ell\leq k}\int_M |\nabla^\ell V|^2_g\,\mu_g\Big)^{1/2}.$$

Note that under a global smallness assumption on $g-\gamma$ guaranteed by the bootstrap assumptions, we have the norm equivalence $\|\cdot\|_{L^2} \cong \|\cdot\|_{L^2_{g,\gamma}}$.

Remark 3.3.1. When we write $\|u\|_{H^k}$ we are denoting a sum *only* over the spatial components of the velocity vector-field u^μ.

We frequently, and without comment, use the following product estimate:

Lemma 3.3.1 (Sobolev product estimates)**.** *If $s > n/p = 3/2$ then*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{H^s}.$$

We conclude this subsection with a result concerning elliptic regularity, see e.g. [9, App. H].

Lemma 3.3.2 (Elliptic regularity using $\mathscr{L}_{g,\gamma}$)**.** *Let V be a symmetric $(0,2)$-tensor on M. There exist constants $C_1, C_2 > 0$ such that for all $s \in \mathbb{Z}_{\geq 0}$*

$$C_1\|V\|_{H^{k+2s}} \leq \|\mathscr{L}^s_{g,\gamma}V\|_{H^k} \leq C_2\|V\|_{H^{k+2s}},$$

where $\mathscr{L}^s_{g,\gamma}$ denotes s-copies of $\mathscr{L}_{g,\gamma}$.

3.3.2 Energy for the perturbation of the geometry

As noted in [4], in the spatially harmonic gauge (3.2.1) we have

$$\mathrm{Ric}[g]_{ab} + \frac{2}{9}g_{ab} = \frac{1}{2}\mathscr{L}_{g,\gamma}(g-\gamma)_{ab} + J_{ab},$$

where J_{ab} are higher-order terms (writen as S_{ab} in [4, pg. 22]) satisfying, for $k \geq 1$,

$$\|J\|_{H^{k-1}} \leq C\|g-\gamma\|_{H^k}.$$

Following [2; 3], we define an energy for the geometric perturbation of the first and second fundamental forms by using $\mathscr{L}_{g,\gamma}$. This energy will fulfill a strong decay estimate enabled by the inclusion of certain correction terms $\Gamma_{(m)}$.

Definition 3.3.4 (Geometric energy ${E^g}_k$)**.** Let λ_0 be the lowest eigenvalue of the operator $\mathscr{L}_{g,\gamma}$, with lower bounds given in Proposition 3.3.1. We define the correction parameter $\alpha = \alpha(\lambda_0, \delta_\alpha)$ by

$$\alpha := \begin{cases} 1 & \lambda_0 > 1/9 \\ 1-\delta_\alpha & \lambda_0 = 1/9, \end{cases}$$

where $\delta_\alpha = \sqrt{1 - 9(\lambda_0 - \varepsilon')}$ with $1 \gg \varepsilon' > 0$ remains a variable to be determined in the course of the argument to follow. By fixing ε' once and for all, δ_α can be made suitably small when necessary. The corresponding correction constant, relevant for defining the corrected energies, is defined by

$$c_E := \begin{cases} 1 & \lambda_0 > 1/9 \\ 9(\lambda_0 - \varepsilon') & \lambda_0 = 1/9. \end{cases}$$

We are now ready to define the energy for the geometric perturbation. For $m, k \in \mathbb{Z}_{\geq 1}$ let

$$\mathscr{E}_{(m)} := \frac{1}{2}\left(v, \mathscr{L}^{m-1}_{g,\gamma}(v)\right)_{L^2_{g,\gamma}} + \frac{9}{2}\left(h, \mathscr{L}^{m}_{g,\gamma}(h)\right)_{L^2_{g,\gamma}}, \qquad \Gamma_{(m)} := \left(v, \mathscr{L}^{m-1}_{g,\gamma}(h)\right)_{L^2_{g,\gamma}}.$$

The energy measuring the geometric perturbation is then defined by

$$E^g{}_k := \sum_{1\le m\le k} (\mathscr{E}_{(m)} + c_E \Gamma_{(m)}).$$

We will see later that the corrected geometric energy $E^g{}_k$ is in fact coercive over the standard Sobolev norms of the geometric variables g, Σ.

Definition 3.3.5 (Operators $\hat{\partial}_0, \partial_{\mathbf{u}}$)**.** We define the operators on M

$$\hat{\partial}_0 := \partial_T + \mathscr{L}_X, \qquad \partial_{\mathbf{u}} := u^0 \partial_T - \tau u^a \hat{\nabla}_a.$$

The following identity, taken from [16], holds for some function f on M:

$$\partial_T \int_M f \mu_g = 3 \int_M \widehat{N} f \mu_g + \int_M \hat{\partial}_0(f) \mu_g\,. \tag{3.3.14}$$

Remark 3.3.2 (Regularity parameters ℓ, N)*.* At top-order our bootstrap assumptions will involve Sobolev norms H^N where N is a large integer. It is convenient to require N to be odd so that we can introduce $\ell \in \mathbb{Z}$ satisfying $\ell = \frac{N-1}{2}$.

We end this section with the top-order geometric energy for the geometric variables g, Σ, which crucially involves the fluid operator $\partial_{\mathbf{u}}$. This energy will also fulfill a strong decay estimate enabled by the inclusion of the correction terms.

Definition 3.3.6 ($E^g_{\partial_{\mathbf{u}},2\ell}$ and E_{tot})**.** For $s \in \mathbb{Z}_{\ge 1}$, define

$$\begin{aligned}
\mathscr{E}^g_{\partial_{\mathbf{u}},2s} &:= \frac{9}{2}\Big(\partial_{\mathbf{u}} \mathscr{L}^s_{g,\gamma}(h), \partial_{\mathbf{u}} \mathscr{L}^s_{g,\gamma}(h)\Big)_{L^2_{g,\gamma}} + \frac{1}{2}\Big(\partial_{\mathbf{u}} \mathscr{L}^s_{g,\gamma} v, \partial_{\mathbf{u}} \mathscr{L}^{s-1}_{g,\gamma} v\Big)_{L^2_{g,\gamma}}, \\
\Gamma^g_{\partial_{\mathbf{u}},2s} &:= \Big(\partial_{\mathbf{u}} \mathscr{L}^{s-1}_{g,\gamma} v, \partial_{\mathbf{u}} \mathscr{L}^s_{g,\gamma}(h)\Big)_{L^2_{g,\gamma}}.
\end{aligned}$$

The corrected $\partial_{\mathbf{u}}-$*boosted* geometric energy is then given by

$$E^g_{\partial_{\mathbf{u}},N-1} := \sum_{s=1}^{\ell} \Big(\mathscr{E}^g_{\partial_{\mathbf{u}},2s} + c_E \Gamma^g_{\partial_{\mathbf{u}},2s}\Big).$$

Finally we define

$$E^g_{tot} := E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1}.$$

3.4 The bootstrap argument

In this section we first state the local-existence theory for the Einstein-dust system in CMCSH gauge. Then we introduce the bootstrap assumptions on our solution and give some immediate consequences of these estimates.

3.4.1 Local existence

Theorem 3.4.1. *Let $N \geq 6$. Consider CMC initial data $(g_0, k_0, N_0, X_0, \rho_0, u_0) \in H^N \times H^{N-1} \times H^N \times H^N \times H^{N-2} \times H^{N-1}$ at $T = T_0$ such that the constraints* (3.2.22a) *hold. Then there exists a unique classical solution (g, k, N, X, ρ, u) on $[T_0, T_+)$ for $T_+ > T_0$ to the system* (3.2.22)*, which is consequently also a solution to the Einstein-dust equations. The components have the following regularity features*

$$
\begin{aligned}
g, N, X &\in C^0([T_0, T_+), H^N) \cap C^1([T_0, T_+), H^{N-1}),\\
k &\in C^0([T_0, T_+), H^{N-1}) \cap C^1([T_0, T_+), H^{N-2}),\\
u &\in C^0([T_0, T_+], H^{N-1}),\\
\rho, \partial_{\mathbf{u}}\rho, \partial_{\mathbf{u}}u &\in C^0([T_0, T_+], H^{N-2}).
\end{aligned}
$$

Furthermore, the time of existence and the norms of the solution depend continuously on the initial data. For the maximal time of existence T_∞ we have either $T_\infty = +\infty$ or

$$
\begin{aligned}
\lim_{T \nearrow T_\infty} \sup_{[T_0, T_\infty]} &\|g - \gamma\|_{H^N} + \|\Sigma\|_{H^{N-1}} + \|N - 3\|_{H^N} + \|X\|_{H^N}\\
&+ \|\partial_T N\|_{H^{N-1}} + \|\partial_T X\|_{H^{N-1}} + \|\rho\|_{H^{N-2}} + |\tau| \|u\|_{H^{N-1}} > \delta(\gamma),
\end{aligned}
$$

where $\delta(\gamma)$ is a positive fixed constant depending only on the background metric.

Proof. The proof follows analogous to [55, Theorem 3.5] where the control on the lapse and shift are replaced by the elliptic techniques applied in [34]. The adaption of the regularity scheme to avoid the loss of derivatives from [55] to the present case is executed in detail in the global analysis discussed in the remainder of the paper. The smallness condition in the continuation criterion stems from a smallness requirement in applying the corresponding elliptic equation for the lapse. □

Remark 3.4.1. Since we apply the local-existence theorem only for data close to the background solution the smallness condition required to extend the solution for arbitrarily large times is automatically fulfilled when our smallness conditions hold, which we prove by a bootstrap argument. Since the smallness parameter of the continuation criterion depends only on the background geometry we can choose the smallness of the initial data, which we do accordingly without mentioning it explicitly again.

3.4.2 Bootstrap assumptions

Let μ, λ be fixed positive constants with $\mu \ll 1$ and $\lambda < 1$. We assume that, for all $T_0 \leq T \leq T'$, the following bootstrap assumptions hold (3.2.22):

$$
\begin{aligned}
\|g - \gamma\|_{H^N} + \|\Sigma\|_{H^{N-1}} &\leq C\varepsilon e^{-\lambda T},\\
\|N - 3\|_{H^N} + \|X\|_{H^N} &\leq C\varepsilon e^{-T},\\
\|\partial_T N\|_{H^{N-1}} + \|\partial_T X\|_{H^{N-1}} &\leq C\varepsilon e^{-T},\\
\|\rho\|_{H^{N-2}} &\leq C\varepsilon,\\
\|u\|_{H^{N-1}} &\leq C\varepsilon e^{\mu T},
\end{aligned}
\tag{3.4.3}
$$

where $T' < T_\infty$ is fixed. We hereon assume that (3.4.3) hold and *do not repeat this fact*. Recall also Remark 3.3.1 regarding the norm on u.

Definition 3.4.1 ($\Lambda(T)$)**.** It is convenient to introduce the following notation

$$\Lambda(T) := \|\widehat{N}\|_{H^N}+\|X\|_{H^N}+\|\partial_T N\|_{H^{N-1}}+\|\partial_T X\|_{H^{N-1}}+|\tau|\|\rho\|_{H^{N-2}}+\tau^2\|u\|^2_{H^{N-1}}.$$

Note that, under the bootstrap assumptions (3.4.3), $\Lambda(T) \lesssim \varepsilon e^{-T}$.

We state some immediate consequences of the bootstrap assumptions regarding u^0 which we use without further comment. Using Lemma 3.2.1, we have

$$\begin{aligned} \|\hat{u}^0\|_{H^{N-1}} &\lesssim \varepsilon, \\ \|\nabla\hat{u}^0\|_{H^{N-2}} &\lesssim \|\nabla N\|_{H^{N-2}}+\tau^2\|u^a\nabla u^a\|_{H^{N-2}}+\text{h.o.t} \lesssim \Lambda(T). \end{aligned} \tag{3.4.5}$$

Note the first estimate does not pick up any $\mu-$loss. By the Sobolev embedding $H^2 \hookrightarrow L^\infty$, $\|\hat{u}^0\|_{L^\infty}= \|u^0-1/3\|_{L^\infty}\leq 1/10$ and thus

$$\|u^0\|_{L^\infty}\lesssim 1, \quad \|u^0\|^{-1}_{L^\infty}\lesssim 1.$$

The following Lemma concerning the dust matter components is indicative of the good behaviour that, as we discussed in Section 3.1.2, we roughly expect as the speed of sound is reduced.

Lemma 3.4.1 (Estimates on matter components)**.** *We have*

$$\begin{aligned} |\tau|(\|E\|_{H^{N-2}}+\|\eta\|_{H^{N-2}}+\|S\|_{H^{N-2}}) &\lesssim \Lambda(T), \\ |\tau|^2\|\jmath\|_{H^{N-2}}\lesssim \varepsilon e^{(-1+\mu)T}\Lambda(T), \quad |\tau|^3\|\underline{T}\|_{H^{N-2}} &\lesssim \Lambda(T)^2. \end{aligned}$$

Proof. The estimates are immediate by distributing derivatives across the terms written in Definition 3.2.4 and using Lemma 3.3.1. □

Lemma 3.4.2 (Geometric coercivity estimate)**.** *Let $s \in \mathbb{Z}$ such that $1 \leq s \leq N-1$. There is a $\delta > 0$ and a constant $C > 0$ such that for $(g,\Sigma) \in B_\delta((\gamma,0))$ the following inequality holds*

$$\|g-\gamma\|^2_{H^s}+\|\Sigma\|^2_{H^{s-1}}\leq CE^g_s.$$

Proof. The proof of this lemma follows verbatim from [2, Lemma 19], which itself follows from [3, Lemma 7.2], and the eigenvalue estimates referred to in Proposition 3.3.1. □

The following Lemma is actually only used once in our entire argument. Its significance lies in the fact that it allows us to convert the quadratic derivative term $(\nabla V)^2$ appearing in the H^1 Sobolev norm into just one second-order derivative as $V \cdot \mathscr{L}_{g,\gamma}V$, which will more naturally be controlled by $E^g_{\partial_{\mathbf{u}},N-1}$.

Lemma 3.4.3. *Let V be a symmetric $(0,2)$-tensor on M. Then,*

$$\|V\|_{H^1}\lesssim \|V\|_{L^2}+(V,\mathscr{L}_{g,\gamma}V)^{1/2}_{L^2_{g,\gamma}}.$$

Proof. We integrate by parts, use the closeness between the g and γ metrics and the boundedness of the Riem$[\gamma]$ components:

$$\begin{aligned}\|V\|_{H^1}^2 &= \|V\|_{L^2}^2 + \int_M g^{ab}g^{ij}g^{kl}\nabla_a V_{ik}\nabla_b V_{jl}\,\mu_g \leq \|V\|_{L^2}^2 + \Big| -\int_M g^{ab}g^{ij}g^{kl}V_{ik}\nabla_a\nabla_b V_{jl}\,\mu_g\Big| \\ &\leq \|V\|_{L^2}^2 + \Big|\int_M \langle V, \mathscr{L}_{g,\gamma}V\rangle_\gamma\,\mu_g + 2\int_M \langle V, \mathrm{Riem}[\gamma]\circ V\rangle_\gamma\,\mu_g\Big| \\ &\lesssim \|V\|_{L^2}^2 + (V, \mathscr{L}_{g,\gamma}V)_{L^2_{g,\gamma}}.\end{aligned}$$

□

3.5 Preliminary estimates

This section is concerned with deriving several preliminary estimates that are required for our later energy inequalities. There are estimates on commutator terms (Sections 3.5.1, 3.5.4), estimates on matter terms (Section 3.5.3) and also estimates on geometric variables (Sections 3.5.2, 3.5.5). We also present an integration by parts Lemma 3.5.2, in particular (3.5.12c), which later plays an important role in removing various critical terms that arise during the energy estimates.

3.5.1 First commutator estimates

In this subsection we let V be an arbitrary symmetric $(0,2)$-tensor and ϕ a scalar, unless otherwise specified. We begin with the identity

$$[\partial_T, \nabla_a]V_{ij} = (-\partial_T\Gamma^c_{ai})V_{cj} + (-\partial_T\Gamma^c_{aj})V_{ic}. \tag{3.5.1}$$

The terms $\partial_T\Gamma[g]$ can be estimated (see [2, (10.12)]), for $0 \leq k \leq N-2$, by:

$$\|\partial_T\Gamma(g)\|_{H^k} \lesssim \|\Sigma\|_{H^{k+1}} + \|X\|_{H^{k+2}} + \|\widehat{N}\|_{H^{k+1}}. \tag{3.5.2}$$

We also have the following commutator identities:

$$\begin{aligned}[\partial_T, \nabla_{a_1}\cdots\nabla_{a_k}]\phi &= [\partial_T, \nabla_{a_1}]\nabla_{a_2}\cdots\nabla_{a_k}\phi + \nabla_{a_1}[\partial_T, \nabla_{a_2}]\nabla_{a_3}\cdots\nabla_{a_k}\phi \\ &\quad + \ldots + \nabla_{a_1}\cdots\nabla_{a_{k-2}}[\partial_T, \nabla_{a_{k-1}}]\nabla_{a_k}\phi + \nabla_{a_1}\cdots\nabla_{a_{k-1}}[\partial_T, \nabla_{a_k}]\phi,\end{aligned} \tag{3.5.3}$$

$$\begin{aligned}[\nabla_i, \nabla_{a_1}\cdots\nabla_{a_k}]\phi &= [\nabla_i, \nabla_{a_1}]\nabla_{a_2}\cdots\nabla_{a_k}\phi + \nabla_{a_1}[\nabla_i, \nabla_{a_2}]\nabla_{a_3}\cdots\nabla_{a_k}\phi \\ &\quad + \ldots + \nabla_{a_1}\cdots\nabla_{a_{k-2}}[\nabla_i, \nabla_{a_{k-1}}]\nabla_{a_k}\phi + \nabla_{a_1}\cdots\nabla_{a_{k-1}}[\nabla_i, \nabla_{a_k}]\phi.\end{aligned} \tag{3.5.4}$$

Note the last terms in each of (3.5.3) and (3.5.4) will in fact vanish since the metric g is torsion free.

In the next part of this subsection we state an important estimate, given in (3.5.10), that allows us to turn background $\hat{\nabla}$ derivatives into dynamical ∇ ones.

Definition 3.5.1 (Difference tensor Υ)**.** Recalling that ∇ and $\hat{\nabla}$ are the Levi-Civita symbols of g and γ respectively, we define Υ a (1,2)-tensor by

$$\Upsilon^a_{bc} := \Gamma^a_{bc}[g] - \Gamma^a_{bc}[\gamma].$$

Let V be a vector and P a one-form. Then we have

$$\nabla_a V^i = \hat{\nabla}_a V^i + \Upsilon^i_{ja} V^j, \qquad \nabla_a P_i = \hat{\nabla}_a P_i - \Upsilon^j_{ia} P_j.$$

We will often schematically write tensorial contractions using $*$. For example,

$$\nabla_a V^i = \hat{\nabla}_a V^i + \Upsilon^i_{ja} V^j, \quad \text{becomes} \quad \nabla V = \hat{\nabla} V + \Upsilon * V.$$

In local coordinates the components of the Υ tensor are given by

$$\Upsilon^a_{bc} = -\frac{1}{2}\gamma^{ai}\left(\nabla_b \gamma_{ci} + \nabla_c \gamma_{ai} - \nabla_i \gamma_{bc}\right) = \frac{1}{2}\gamma^{ai}\left(\nabla_b h_{ci} + \nabla_c h_{ai} - \nabla_i h_{bc}\right).$$

Lemma 3.5.1. *For $0 \le k \le N-2$, we have*

$$\|[\hat{\nabla}, \nabla]V\|_{H^k} \lesssim \|V\|_{H^k} + \varepsilon e^{-\lambda T}\|V\|_{H^{k+1}}.$$

Proof. First we see that from Definition 3.5.1, for all $0 \le k \le N-1$,

$$\|\Upsilon\|_{H^k} \lesssim \|g - \gamma\|_{H^{k+1}}. \tag{3.5.10}$$

Thus we compute,

$$[\hat{\nabla}_c, \nabla_a]V_{ij} = [\hat{\nabla}_c, \hat{\nabla}_a]V_{ij} + \Upsilon^b_{ca}\hat{\nabla}_b V_{ij} - ((\hat{\nabla}_c \Upsilon^b_{ai})V_{bj} + (\hat{\nabla}_c \Upsilon^b_{aj})V_{ib}).$$

Using the boundedness of the Riem$[\gamma]$ components, the required estimate then follows. $\square$

We end this subsection with some very useful estimates that come from integration by parts.

Lemma 3.5.2 (Integration by parts)**.** *Let T be an arbitrary vectorfield and V, P arbitrary symmetric $(0,2)$-tensors, on M. Then*

$$(\mathscr{L}_{g,\gamma}V, P)_{L^2(g,\gamma)} = \int_M \langle g^{ab}\hat{\nabla}_a V, \hat{\nabla}_b P\rangle_\gamma \mu_g - 2\int_M \langle Riem[\gamma]_{\bullet a \bullet b} V^{ab}, P\rangle_\gamma \mu_g. \tag{3.5.12a}$$

and

$$|(T^a \hat{\nabla}_a V, P)_{L^2_{g,\gamma}}| \lesssim \|T\|_{H^3}\|V\|_{L^2}\|P\|_{H^1}, \tag{3.5.12b}$$

$$|(T^a \hat{\nabla}_a V, V)_{L^2_{g,\gamma}}| \lesssim \|T\|_{H^3}\|V\|^2_{L^2}. \tag{3.5.12c}$$

Proof. The proof of (3.5.12a) follows by using the gauge condition $H^a = 0$. To show (3.5.12b), recall the Jacobi identity implies

$$\hat{\nabla}_a \sqrt{\det g} = \frac{1}{2}\sqrt{\det g} \cdot g^{ij}(\hat{\nabla}_a g_{ij}).$$

Using this we find,

$$\begin{aligned}
(T^a \hat{\nabla}_a V, P)_{L^2_{g,\gamma}} &= \int_M \gamma^{ij}\gamma^{kl} T^a (\hat{\nabla}_a V_{ik}) P_{jl} \sqrt{\det g} \\
&= -\int_M \gamma^{ij}\gamma^{kl}\hat{\nabla}_a \left(T^a P_{jl}\frac{\sqrt{\det g}}{\sqrt{\det \gamma}}\right) V_{ik}\sqrt{\det \gamma} \\
&= -(V, T^a \hat{\nabla}_a P)_{L^2_{g,\gamma}} - ((\hat{\nabla}_a T^a)V, P)_{L^2_{g,\gamma}} - \frac{1}{2}(V, (T^a g^{bc}\hat{\nabla}_a g_{bc})P)_{L^2_{g,\gamma}}.
\end{aligned}$$

Thus,

$$\begin{aligned}|(T^a\hat{\nabla}_aV,P)_{L^2_{g,\gamma}}| &= |-(V,T^a\hat{\nabla}_aP)_{L^2_{g,\gamma}}-((\hat{\nabla}_aT^a)V,P)_{L^2_{g,\gamma}}-\frac{1}{2}(V,(T^ag^{bc}\hat{\nabla}_ag_{bc})P)_{L^2_{g,\gamma}}| \\ &\lesssim (\|T^a\|_{H^3}+\|T^a\|_{H^2}\|g-\gamma\|_{H^3})\|V\|_{L^2}\|P\|_{H^1}.\end{aligned}$$

Crucially, in the symmetric case, we can bring one term over to the left hand side to show

$$\begin{aligned}|(T^a\hat{\nabla}_aV,V)_{L^2_{g,\gamma}}| &= |-\frac{1}{2}((\hat{\nabla}_aT^a)V,V)_{L^2_{g,\gamma}}-\frac{1}{4}(V,(T^ag^{bc}\hat{\nabla}_ag_{bc})V)_{L^2_{g,\gamma}}| \\ &\lesssim (\|T^a\|_{H^3}+\|T^a\|_{H^2}\|g-\gamma\|_{H^3})\|V\|^2_{L^2}.\end{aligned}$$

□

3.5.2 First estimates on geometric components

We now establish some of our first estimates on the ADM variables N, X and the geometric variables g, Σ.

Lemma 3.5.3 (Dust derivatives of lapse and shift)**.** *For* $2 \le k \le N-1$*,*

$$\|\partial_{\mathbf{u}}N\|_{H^k}+\|\partial_{\mathbf{u}}X\|_{H^k}\lesssim \Lambda(T).$$

Proof. Since the lapse is a scalar, $\partial_{\mathbf{u}}N = u^0\partial_T N - \tau u^c\nabla_c N$. For $\partial_{\mathbf{u}}X^a$ we use (3.5.10). We find

$$\|\partial_{\mathbf{u}}N\|_{H^k}+\|\partial_{\mathbf{u}}X\|_{H^k}\lesssim \|\partial_T N\|_{H^k}+\|\partial_T X\|_{H^k}+\varepsilon e^{(-1+\mu)T}\Lambda(T).$$

□

Lemma 3.5.4 (Estimates on geometric source terms)**.** *We have,*

$$\begin{aligned}\|F_h\|_{H^k} &\lesssim \|\widehat{N}\|_{H^k}+\|X\|^2_{H^{k+1}}+\|g-\gamma\|^2_{H^{k+1}}, && 2\le k\le N-1,\\ \|F_v\|_{H^k} &\lesssim \|\widehat{N}\|_{H^{k+2}}+|\tau|\|\rho\|_{H^k}+\|X\|^2_{H^{k+1}}+\|g-\gamma\|^2_{H^{k+1}}+\|\Sigma\|^2_{H^k}, && 2\le k\le N-2,\end{aligned}$$

and thus

$$\|F_h\|_{H^{N-1}}+\|F_v\|_{H^{N-2}}\lesssim \Lambda(T)+\|g-\gamma\|^2_{H^N}+\|\Sigma\|^2_{H^{N-1}}.$$

Proof. Using the product estimate of Lemma 3.3.1 and (3.5.10) we obtain, for $2\le k\le N-1$,

$$\begin{aligned}\|F_h\|_{H^k} &\lesssim \|\widehat{N}\|_{H^k}+\|h\|_{H^k}\|\nabla X+\Upsilon * X\|_{H^k}\\ &\lesssim \|\widehat{N}\|_{H^k}+\|g-\gamma\|_{H^k}\|X\|_{H^{k+1}}+\|g-\gamma\|^2_{H^{k+1}}\|X\|_{H^k}.\end{aligned}$$

Using in addition Lemma 3.4.1 we see, for $2\le k\le N-2$,

$$\|F_v\|_{H^k}\lesssim \|\widehat{N}\|_{H^{k+2}}+\|\Sigma\|^2_{H^k}+|\tau|\|S_{ij}\|_{H^k}+\|\Sigma\|_{H^k}\|\hat{\nabla}X\|_{H^k}.$$

□

We can now combine the geometric source term estimates from the previous lemma with the equations of motion given in (3.2.26).

Lemma 3.5.5 (Time derivatives of g and Σ)**.** *The following estimates hold*

$$\begin{aligned} \|\partial_T g\|_{H^k} &\lesssim \|\Sigma\|_{H^k}+\|g-\gamma\|_{H^{k+1}}+\|\widehat{N}\|_{H^k}+\|X\|_{H^{k+1}}, && 2 \le k \le N-1, \\ \|\partial_T \Sigma\|_{H^k} &\lesssim \|\Sigma\|_{H^{k+1}}+\|g-\gamma\|_{H^{k+2}}+\|\widehat{N}\|_{H^{k+2}}+\|X\|_{H^{k+1}}+|\tau|\|\rho\|_{H^k}, && 2 \le k \le N-2. \end{aligned}$$

Proof. Using Lemma 3.5.4, (3.2.26) and (3.5.10) we find

$$\begin{aligned} \|\partial_T h\|_{H^k} &\lesssim \|\Sigma\|_{H^k}+\|X(\nabla h + \Upsilon * h)\|_{H^k}+\|F_h\|_{H^k} \\ &\lesssim \|\Sigma\|_{H^k}+\|\widehat{N}\|_{H^k}+\|X\|^2_{H^{k+1}}+\|g-\gamma\|^2_{H^{k+1}}, \end{aligned}$$

and, using additionally Lemma 3.3.2,

$$\begin{aligned} \|\partial_T v\|_{H^k} &\lesssim \|\Sigma\|_{H^k}+\|\mathscr{L}_{g,\gamma} h\|_{H^k}+\|X(\nabla v + \Upsilon * v)\|_{H^k}+\|F_v\|_{H^k} \\ &\lesssim \|\Sigma\|_{H^k}+\|g-\gamma\|_{H^{k+2}}+\|\widehat{N}\|_{H^{k+2}}+\|X\|^2_{H^{k+1}}+\|\Sigma\|^2_{H^{k+1}}+|\tau|\|\rho\|_{H^k}. \end{aligned}$$

□

Corollary 3.5.1 (Dust derivatives of g and Σ)**.** *The following estimates hold*

$$\begin{aligned} \|\partial_{\mathbf{u}} g\|_{H^k} &\lesssim \|\Sigma\|_{H^k}+\|g-\gamma\|_{H^{k+1}}+\Lambda(T), && 2 \le k \le N-1, \\ \|\partial_{\mathbf{u}} \Sigma\|_{H^k} &\lesssim \|\Sigma\|_{H^{k+1}}+\|g-\gamma\|_{H^{k+2}}+\Lambda(T), && 2 \le k \le N-2. \end{aligned}$$

Proof. Writing

$$\partial_{\mathbf{u}} V_{ab} = u^0 \partial_T V_{ab} - \tau u^c \nabla_c V_{ab} - \tau u^c \Upsilon^d_{ca} V_{db} - \tau u^c \Upsilon^d_{cb} V_{ad}$$

and using Lemma 3.5.5 gives the estimates. □

Remark 3.5.1. It is unsurprising that for the geometric variables g, Σ, the fluid derivative estimates in Corollary 3.5.1 do not gain us any improved information compared to the time derivative estimates in Lemma 3.5.5. This will of course change when we consider instead estimates on certain fluid matter variables which are naturally more compatible with the fluid derivative operator $\partial_{\mathbf{u}}$.

3.5.3 First estimates on fluid components

In this subsection we establish further estimates on the various matter variables. Note that we need to estimate both matter components coming from contractions with the stress energy tensor (see Definition 3.2.4) and the fluid source terms terms appearing in the equations of motion (see Definition 3.2.5).

Lemma 3.5.6 (Fluid source term estimates)**.** *We have, for some $\nu > 0$,*

$$\begin{aligned} \|F_\rho\|_{H^{N-2}} &\lesssim |\tau|\|u\|_{H^{N-1}}+\|\Sigma\|_{H^{N-2}}+\Lambda(T), \\ \|F_{u^0}\|_{H^{N-1}} &\lesssim \|\Sigma\|^2_{H^{N-1}}+\Lambda(T), \\ \|F_{u^j}\|_{H^{N-1}} &\lesssim |\tau|^{-1}\Lambda(T) + \varepsilon^2 e^{-\nu T}. \end{aligned}$$

Proof. For $2 \leq k \leq N-2$ we distribute derivatives across the terms given in Definitions 3.2.3 and 3.2.5. This yields

$$\begin{aligned}\|F_\rho\|_{H^k} &\lesssim |\tau|\|\nabla u^j\|_{H^k}+\|u^0\|_{L^\infty}\|\Gamma^i_i\|_{H^k}+|\tau|\|\Gamma_j\|_{H^k}\|u\|_{H^k}+|\tau|\|\Gamma_{ik}\|_{H^k}\|X\|_{H^k}\|u\|_{H^k}\\&\quad+\tau^2\|u\|^2_{H^k}\|\Gamma_{kj}\|_{H^k}\\&\lesssim |\tau|\|u\|_{H^{k+1}}+\|\Sigma\|_{H^k}+\|X\|_{H^{k+1}}.\end{aligned}$$

Similarly, for $2 \leq k \leq N-1$,

$$\begin{aligned}\|F_{u^0}\|_{H^k} &\lesssim \|\Gamma_R\|_{H^k}+|\tau|\|u\|_{H^k}\|\Gamma_j\|_{H^k}+\tau^2\|u\|^2_{H^k}\|\Gamma_{jk}\|_{H^k}\\&\lesssim \|\partial_T N\|_{H^k}+\|\Sigma\|^2_{H^k}+\|\widehat{N}\|^2_{H^{k+1}}+\|X\|^2_{H^k}+\tau^2\|u\|^2_{H^k}.\end{aligned}$$

Finally, for $2 \leq k \leq N-1$,

$$\begin{aligned}\|F_{u^j}\|_{H^k} &\lesssim |\tau|^{-1}\|\mathring{\Gamma}^j\|_{H^k}+\|u\|^2_{H^k}(\|\Gamma[g]-\Gamma[\gamma]\|_{H^k}+|\tau|\|X\|_{H^k}\|\Gamma_{ki}\|_{H^k})+\|u\|_{H^k}\|\Gamma^i_j\|_{H^k}\\&\lesssim |\tau|^{-1}(\|\partial_T X\|_{H^k}+\|X\|_{H^{k+1}}+\|\widehat{N}\|^2_{H^{k+1}}+\|\Sigma\|^2_{H^k}+\|X\|_{H^k}\|\partial_T N\|_{H^k})\\&\quad+\|u\|^2_{H^k}(\|g-\gamma\|_{H^{k+1}}+|\tau|\|X\|_{H^k})+\|u\|_{H^k}(\|\Sigma\|_{H^k}+\|\widehat{N}\|_{H^k}+\|X\|_{H^{k+1}})\\&\lesssim |\tau|^{-1}\Lambda(T)+\varepsilon^2 e^{(1-2\lambda)T}+\varepsilon^2 e^{(-\lambda+2\mu)T}.\end{aligned}$$

It is convenient also to note that at lower order we can apply Lemma 3.4.2 to find

$$|\tau|\|F_{u^j}\|_{H^{N-2}} \lesssim E^g_{N-2}+\Lambda(T)+|\tau|\|u\|^2_{H^{N-2}}(E^g_{N-1})^{1/2}. \tag{3.5.32}$$

□

In the next lemma we provide estimates for fluid derivatives of certain matter components appearing in Definition 3.2.4. The weights in τ are included for convenience since these expressions appear later on in the energy estimates.

Lemma 3.5.7 (Dust derivatives of matter components)**.** *We have,*

$$\begin{aligned}|\tau|\|\partial_{\mathbf{u}}\eta\|_{H^{N-2}}+|\tau|\|\partial_{\mathbf{u}}S\|_{H^{N-2}} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T)+\Lambda(T)^2,\\|\tau|^2\|\partial_{\mathbf{u}}\jmath\|_{H^{N-2}} &\lesssim \Lambda(T)^2+\varepsilon^2\Lambda(T)e^{(-\lambda+\mu)T}.\end{aligned}$$

Proof. Note that (3.2.22d) can be rewritten as:

$$\partial_{\mathbf{u}}u^j = \tau u^a u^c \Upsilon^j_{ac}+\tau^{-1}(u^0)^2 N\nabla^j N+F_{u^j}, \qquad \partial_{\mathbf{u}}u^0 = F_{u^0}, \qquad \partial_{\mathbf{u}}\rho = \rho F_\rho.$$

Using Definition 3.2.4 we compute:

$$\begin{aligned}\partial_{\mathbf{u}}\eta &= \rho F_\rho\left((u^0)^2N^2+g_{ab}(X^a u^0+\tau u^a)(X^b u^0+\tau u^b)\right)+2\rho(u^0)^2N\partial_{\mathbf{u}}N\\&\quad+\rho(\partial_{\mathbf{u}}g_{ab})(X^a u^0+\tau u^a)(X^b u^0+\tau u^b)+2\rho g_{ab}u^0(X^b u^0+\tau u^b)(\partial_{\mathbf{u}}X^a)\\&\quad+F_{u^0}\left(2\rho u^0 N^2+2\rho g_{ab}X^a(X^b u^0+\tau u^b)\right)+2\rho g_{ab}u^0u^a(X^b u^0+\tau u^b)(\partial_T\tau)\\&\quad+2\rho g_{ab}\tau(X^b u^0+\tau u^b)\left(\tau u^a\Upsilon^j_{ac}u^c+\tau^{-1}(u^0)^2N\nabla^j N+F_{u^j}\right).\end{aligned}$$

So, using Lemma 3.5.3, Corollary 3.5.1 and the matter estimates from Lemma 3.5.6, we obtain

$$\begin{aligned}|\tau|\|\partial_{\mathbf{u}}\eta\|_{H^{N-2}} &\lesssim |\tau|\|\rho\|_{H^{N-2}}\|F_\rho\|_{H^{N-2}}+|\tau|\|\rho\|_{H^{N-2}}\|\partial_{\mathbf{u}}N\|_{H^{N-2}}\\&\quad+|\tau|\|\rho\|_{H^{N-2}}\|F_{u^0}\|_{H^{N-2}}+\text{h.o.t.}\\&\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T)+\Lambda(T)^2.\end{aligned}$$

Again using (3.2.4) we compute

$$\partial_{\mathbf{u}} \jmath^a = (\rho F_\rho) N u^0 u^a + (\partial_{\mathbf{u}} N)\rho u^0 u^a + (F_{u^0})\rho N u^a \\ + (\tau \Upsilon^a_{bc} u^b u^c + \tau^{-1}(u^0)^2 N \nabla^a N + F_{u^a})\rho N u^0.$$

So, by the geometry estimates in Lemma 3.5.3 and 3.5.5, together with the fluid source term estimates in Lemma 3.5.6, we obtain

$$\tau^2 \|\partial_{\mathbf{u}} \jmath^a\|_{H^{N-2}} \lesssim |\tau| \|\rho\|_{H^{N-2}} \|\widehat{N}\|_{H^{N-1}} + |\tau|^2 \|\rho\|_{H^{N-2}} \|F_{u^a}\|_{H^{N-2}} + \text{h.o.t.} \\ \lesssim \varepsilon^2 \Lambda(T) e^{(-\lambda+\mu)T} + \Lambda(T)^2.$$

Finally, from (3.2.4) we find

$$\partial_{\mathbf{u}} S_{ab} = (\partial_{\mathbf{u}} \rho)((u^0 X_a + \tau u_a)(u^0 X_b + \tau u_b) + \frac{1}{2} g_{ab}) + \frac{1}{2}\rho(\partial_{\mathbf{u}} g_{ab}) + \text{h.o.t.}$$

So,

$$|\tau| \|\partial_{\mathbf{u}} S\|_{H^{N-2}} \lesssim |\tau| \|\rho\|_{H^{N-2}} \|F_\rho\|_{H^{N-2}} \lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T} \Lambda(T) + \Lambda(T)^2.$$

□

Remark 3.5.2. The significance of using dust derivatives is made clear by look at the higher regularity appearing in Lemma 3.5.7 compared to the following Lemma 3.5.8 which only concerns time derivatives. In Lemma 3.5.7, we directly computed the $\partial_{\mathbf{u}}$ derivatives using the equations of motion, instead of doing a rough estimate by expanding $\partial_{\mathbf{u}} \sim \partial_T + \tau u^c * \nabla$.

Lemma 3.5.8 (Time derivatives of matter components)**.** *We have,*

$$|\tau| \|\partial_T \eta\|_{H^{N-3}} \lesssim \varepsilon e^{(-1+\mu)T} \Lambda(T) + \Lambda(T)^2, \\ \Lambda(T) \|\partial_T \jmath\|_{H^{N-3}} \lesssim \varepsilon^2 \Lambda(T) + \varepsilon^4 e^{-(1+\nu)T}, \\ |\tau| \|\partial_T S\|_{H^{N-3}} \lesssim \Lambda(T).$$

Proof. We calculate $\partial_T \eta$ explicitly from (3.2.4) as

$$\partial_T \eta = \partial_T E + (X_a X^a (u^0)^2 + 2\tau X_b u^b u^0 + \tau^2 g_{ab} u^a u^b)\partial_T \rho + (2\rho X^a (u^0)^2 + 2\tau \rho u^a u^0)\partial_T X_a \\ + (2u^0 \rho X_a X^a + 2\tau \rho X_b u^b)\partial_T u^0 + (2\tau \rho X_b u^0 + 2\tau^2 \rho u_b)\partial_T u^b \\ + (2\rho X_b u^b u^0 + 2\tau \rho u_a u^a)\partial_T \tau.$$

Using $\partial_T \tau = -\tau$, we obtain

$$|\tau| \|\partial_T \eta\|_{H^{N-3}} \lesssim |\tau| \|\partial_T E\|_{H^{N-3}} + \Lambda(T) |\tau| \|\partial_T \rho\|_{H^{N-3}} + \varepsilon e^{(-1+\mu)T} \Lambda(T) \|\partial_T X\|_{H^{N-3}} \\ + \Lambda(T)^2 \|\partial_T u^0\|_{H^{N-3}} + \varepsilon e^{(-1+\mu)T} \Lambda(T) \|\tau \partial_T u^b\|_{H^{N-3}} + \Lambda(T)^2. \tag{3.5.43}$$

To estimate $\partial_T E$ we use the rescaled continuity equation [2, Eq 10.16]:

$$\partial_T E = (3-N)E - X^a \nabla_a E + \tau N^{-1} \nabla_a (N^2 \jmath^a) - \tau^2 \frac{N}{3} g_{ab} \underline{T}^{ab} - \tau^2 N \Sigma_{ab} \underline{T}^{ab}.$$

So by Lemma 3.4.1, we obtain

$$
\begin{aligned}
|\tau|\|\partial_T E\|_{H^{N-3}} &\lesssim |\tau|\|E\|_{H^{N-2}}(\|\widehat{N}\|_{H^{N-3}}+\|X\|_{H^{N-3}}) + |\tau|^2\|\jmath\|_{H^{N-2}}+|\tau|^3\|\underline{T}\|_{H^{N-3}} \\
&\lesssim \varepsilon e^{(-1+\mu)T}\Lambda(T) + \Lambda(T)^2.
\end{aligned}
$$

To estimate $\partial_T\rho, \partial_T u^0, \partial_T u^a$ we use the equations of motion (3.2.22d) together with Lemma 3.5.6. We find,

$$
\begin{aligned}
|\tau|\|\partial_T\rho\|_{H^{N-3}} &\lesssim |\tau|\|\rho\|_{H^{N-3}}\|F_\rho\|_{H^{N-3}}+|\tau|^2\|u\|_{H^{N-3}}\|\rho\|_{H^{N-2}} \\
&\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T) + \Lambda(T)^2,
\end{aligned}
\tag{3.5.46a}
$$

and

$$
\begin{aligned}
\|\partial_T u^0\|_{H^{N-2}} &\lesssim \|F_{u^0}\|_{H^{N-2}}+|\tau|\|u\|_{H^{N-2}}\|\nabla u^0\|_{H^{N-2}}\lesssim \Lambda(T) + \|\Sigma\|^2_{H^{N-2}}, \\
|\tau|\|\partial_T u^a\|_{H^{N-2}} &\lesssim |\tau|\|F_{u^a}\|_{H^{N-2}}+\|\widehat{N}\|_{H^{N-1}}+\tau^2\|u\|^2_{H^{N-1}}\lesssim \Lambda(T) + \varepsilon^2 e^{-(1+\nu)T}.
\end{aligned}
\tag{3.5.46b}
$$

Putting all these estimates into (3.5.43) gives, for $2 \leq k \leq N-3$:

$$
|\tau|\|\partial_T\eta\|_{H^k} \lesssim \varepsilon e^{(-1+\mu)T}\Lambda(T) + \Lambda(T)^2.
$$

Next, and again using (3.2.4), we compute

$$
\partial_T\jmath^a = (\partial_T\rho)Nu^0u^a + (\partial_T N)\rho u^0 u^a + (\partial_T u^0)\rho N u^a + (\partial_T u^a)\rho N u^0.
$$

So,

$$
\begin{aligned}
\Lambda(T)\|\partial_T\jmath^a\|_{H^{N-3}} &\lesssim \varepsilon|\tau|\|u\|_{H^{N-3}}\|\partial_T\rho\|_{H^{N-3}}+\varepsilon|\tau|\|\rho\|_{H^{N-3}}\Big(\|\partial_T N\|_{H^{N-3}}\|u\|_{H^{N-3}} \\
&\qquad + \|\partial_T u^0\|_{H^{N-3}}\|u\|_{H^{N-3}}+\|\partial_T u^a\|_{H^{N-3}}\Big) \\
&\lesssim \varepsilon^2\Lambda(T) + \varepsilon^4 e^{-(1+\nu)T}.
\end{aligned}
$$

Finally, from (3.2.4) we calculate (written schematically)

$$
\begin{aligned}
\partial_T S = &(\partial_T\rho)((u^0X^a+\tau u^a)^2 + \frac{1}{2}g) + \frac{1}{2}\rho(\partial_T g) + \rho\partial_T g\cdot(u^0X^c+\tau u^c)^2 \\
&+ \rho\partial_T(u^0X^c+\tau u^c)\cdot(u^0X^d+\tau u^d).
\end{aligned}
$$

So using Lemma 3.5.5 and (3.5.46) we have,

$$
\begin{aligned}
|\tau|\|\partial_T S\|_{H^{N-3}} &\lesssim |\tau|(\|\rho\|_{H^{N-3}}\|F_\rho\|_{H^{N-3}}+|\tau|\|u\|_{H^{N-3}}\|\rho\|_{H^{N-2}}) + |\tau|\|\rho\|_{H^{N-3}}\times(\text{h.o.t.}) \\
&\lesssim |\tau|\|\rho\|_{H^{N-2}}.
\end{aligned}
$$

□

3.5.4 Second commutator estimates

We are now in a position to compute various commutator estimates which are required in the later energy estimates. We let V be a symmetric $(0,2)$-tensor on M unless otherwise specified. The first lemma looks at the commutator between the dust derivative $\partial_{\mathbf{u}}$ with other first-order differential operators.

Lemma 3.5.9. *For $0 \leq k \leq N-2$, we have*

$$\begin{aligned}
\|[\partial_{\mathbf{u}}, \partial_T]V\|_{H^k} &\lesssim \varepsilon e^{(-1+\mu)T}\|V\|_{H^{k+1}} + \varepsilon e^{(-1+\mu)T}\|\partial_T V\|_{H^k}, \\
\|[\partial_{\mathbf{u}}, \hat{\nabla}]V\|_{H^k} &\lesssim \varepsilon e^{(-1+\mu)T}\|V\|_{H^{k+1}} + \Lambda(T)\|\partial_T V\|_{H^k}, \\
\|[\partial_{\mathbf{u}}, \nabla]V\|_{H^k} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{k+1}} + \Lambda(T)\|\partial_T V\|_{H^k}.
\end{aligned}$$

Proof. A computation gives

$$\begin{aligned}
[\partial_{\mathbf{u}}, \partial_T]V_{ij} &= -\tau u^a \hat{\nabla}_a V_{ij} + \tau \partial_T u^a \cdot \hat{\nabla}_a V_{ij} - \partial_T \hat{u}^0 \cdot \partial_T V_{ij}, \\
[\partial_{\mathbf{u}}, \hat{\nabla}_b]V_{ij} &= -\tau u^a [\hat{\nabla}_a, \hat{\nabla}_b]V_{ij} - \hat{\nabla}_b \hat{u}^0 \cdot \partial_T V_{ij} + \tau \hat{\nabla}_b u^a \cdot \hat{\nabla}_a V_{ij}, \\
[\partial_{\mathbf{u}}, \nabla_b]V_{ij} &= u^0[\partial_T, \nabla_b]V_{ij} - \tau u^c [\hat{\nabla}_c, \nabla_b]V_{ij} - \nabla_b \hat{u}^0 \cdot \partial_T V_{ij} + \tau \nabla_b u^c \cdot \hat{\nabla}_c V_{ij}.
\end{aligned} \tag{3.5.53}$$

Thus by (3.4.5) and (3.5.46), if $2 \leq k \leq N-2$,

$$\begin{aligned}
\|[\partial_{\mathbf{u}}, \partial_T]V\|_{H^k} &\lesssim |\tau|(\|u\|_{H^k} + \|\partial_T u^a\|_{H^k})\|\nabla V + \Upsilon * V\|_{H^k} + \|\partial_T u^0\|_{H^k}\|\partial_T V\|_{H^k} \\
&\lesssim \varepsilon e^{(-1+\mu)T}(\|V\|_{H^{k+1}} + \|g-\gamma\|_{H^{k+1}}\|V\|_{H^k}) + \varepsilon e^{(-1+\mu)T}\|\partial_T V\|_{H^k}.
\end{aligned}$$

The estimates for $k = 0, 1$ follow in the same way.

The other two estimates follow in a similar way. Note that for $[\partial_{\mathbf{u}}, \nabla]$ we use Lemma 3.5.1, and equations (3.5.1), (3.5.2) and (3.5.46). □

The next lemma in this subsection investigates the commutator between the second-order operator $\mathscr{L}_{g,\gamma}$ and other first-order operators.

Lemma 3.5.10. *The following estimates hold*

$$\begin{aligned}
\|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]V\|_{H^k} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{k+2}} + \Lambda(T)\|\partial_T V\|_{H^{k+1}}, && 0 \leq k \leq N-3, \\
\|[\hat{\nabla}_m, \mathscr{L}_{g,\gamma}]V\|_{H^k} &\lesssim \varepsilon e^{-\lambda T}\|V\|_{H^{k+2}} + \|V\|_{H^k}, && 0 \leq k \leq N-2, \\
\|[\partial_T, \mathscr{L}_{g,\gamma}]V\|_{H^k} &\lesssim \varepsilon e^{-\lambda T}\|V\|_{H^{k+2}}, && 0 \leq k \leq N-2.
\end{aligned}$$

Also for $k, s \in \mathbb{Z}$ such that $0 \leq k \leq N-2$, $s \geq 1$, and $2(s-1)+k \leq N-1$, we have

$$\|[\partial_T, \mathscr{L}^s_{g,\gamma}]V\|_{H^k} \lesssim \varepsilon e^{-\lambda T}\|V\|_{H^{k+2+2(s-1)}}.$$

Proof. A calculation yields

$$\begin{aligned}
[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]V_{ij} = &-(\partial_{\mathbf{u}} g^{ab})\hat{\nabla}_a \hat{\nabla}_b V_{ij} + \hat{\Delta}_{g,\gamma}\hat{u}^0 \cdot \partial_T V_{ij} + 2g^{ab}\hat{\nabla}_a \hat{u}^0 \hat{\nabla}_b \partial_T V_{ij} \\
&+ \tau u^c g^{ab}(\mathrm{Riem}[\gamma]^k{}_{bac}\hat{\nabla}_k V_{ij} + 4\mathrm{Riem}[\gamma]^k{}_{(i|ac}\hat{\nabla}_b V_{k|j)} \\
&+ 2\hat{\nabla}_a \mathrm{Riem}[\gamma]^k{}_{(i|bc} \cdot V_{k|j)}) \\
&- 2\tau g^{ab}\hat{\nabla}_a u^c \hat{\nabla}_b \hat{\nabla}_c V_{ij} - \tau \hat{\Delta}_{g,\gamma} u^c \cdot \hat{\nabla}_c V_{ij} + 2\tau u^c \hat{\nabla}_c \mathrm{Riem}[\gamma]_{iajb} \cdot V^{ab}.
\end{aligned} \tag{3.5.57}$$

It is useful to write this schematically, using that $\mathrm{Riem}[\gamma]$ and its derivatives are bounded by constants:

$$\begin{aligned}
[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]V = &(\partial_{\mathbf{u}} g^{-1}) * \hat{\nabla}^2 V + \hat{\nabla}^2 \hat{u}^0 * \partial_T V + \hat{\nabla} u^0 * \hat{\nabla}\partial_T V \\
&+ \tau(u^c * (V + \hat{\nabla}V) + \hat{\nabla}u^c * \hat{\nabla}^2 V + \hat{\nabla}^2 u^c * \hat{\nabla}V).
\end{aligned}$$

If $2 \leq k \leq N-3$ then by elliptic regularity of Lemma 3.3.2 and the commutator estimates in Lemma 3.5.1,

$$\|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]V\|_{H^k} \lesssim \|\partial_{\mathbf{u}} g\|_{H^k}\|V\|_{H^{k+2}}+\|\hat{u}^0\|_{H^{k+2}}\|\partial_T V\|_{H^{k+1}}+|\tau|\|u\|_{H^{k+2}}\|V\|_{H^{k+2}}. \tag{3.5.59}$$

The conclusion then holds by (3.4.5) and by estimating $\partial_{\mathbf{u}} g$ using Corollary 3.5.1 . The estimates when $k = 0, 1$ follow in a similar way.

Next we compute the identity

$$[\mathscr{L}_{g,\gamma}, \hat{\nabla}_m]V_{ij} = \hat{\nabla}_m g^{ab} \cdot \hat{\nabla}_a \hat{\nabla}_b V_{ij} - g^{ab}[\hat{\nabla}_a\hat{\nabla}_b, \hat{\nabla}_m]V_{ij}, \tag{3.5.60}$$

where we are thinking of the m index as not being free (i.e. contracted with a factor of the shift X^m). Since the commutator involving only $\hat{\nabla}$ will just generate background Riemann curvature components the required estimate follows straightforwardly. We note also that (3.5.60) and the elliptic regularity of Lemma 3.3.2 imply, for $s \in \mathbb{Z}$ such that $1 \leq s \leq N/2$,

$$\begin{aligned}\|[\mathscr{L}^s_{g,\gamma}, \hat{\nabla}_m]V\|_{L^2} &\lesssim \|[\mathscr{L}_{g,\gamma}, \hat{\nabla}]\mathscr{L}^{s-1}_{g,\gamma}V\|_{L^2}+\cdots+\|[\mathscr{L}_{g,\gamma}, \hat{\nabla}]V\|_{H^{2(s-1)}}\\ &\lesssim \|g-\gamma\|_{H^{\max\{3,2s-1\}}}(\|V\|_{H^{2s}}+\|g-\gamma\|_{H^{2s}}\|V\|_{H^{2s-1}}) + \|V\|_{H^{2(s-1)}}\\ &\lesssim \varepsilon e^{-\lambda T}\|V\|_{H^{2s}}+\|V\|_{H^{2(s-1)}}.\end{aligned} \tag{3.5.61}$$

Finally we compute

$$[\partial_T, \mathscr{L}_{g,\gamma}]V_{ij} = -[\partial_T, \hat{\Delta}_{g,\gamma}]V_{ij} = -(\partial_T g^{ab})\hat{\nabla}_a\hat{\nabla}_b V_{ij}.$$

Using Lemma 3.5.5 to estimate $\partial_T g$ this gives, for $2 \leq k \leq N-2$,

$$\|[\partial_T, \mathscr{L}_{g,\gamma}]V\|_{H^k} \lesssim \|\partial_T h\|_{H^k}(\|\nabla(\nabla V + \Upsilon * V)\|_{H^k}+\|\Upsilon(\nabla V + \Upsilon * V)\|_{H^k}) \lesssim \varepsilon e^{-\lambda T}\|V\|_{H^{k+2}}.$$

A similar argument holds for the cases $k = 0, 1$. At higher order, we obtain the identity

$$[\partial_T, \mathscr{L}^s_{g,\gamma}]V_{ij} = -\sum_{1\leq i\leq s} \mathscr{L}^{i-1}_{g,\gamma}(\partial_T g^{a_i b_i}) \cdot \hat{\nabla}_{a_i}\hat{\nabla}_{b_i}(\mathscr{L}^{s-i}_{g,\gamma}(V_{ij})). \tag{3.5.64}$$

This can be estimated, for $2 \leq k \leq N-2$, by

$$\|[\partial_T, \mathscr{L}^s_{g,\gamma}]V\|_{H^k} \lesssim \|\partial_T h\|_{H^{2(s-1)+k}}(\|V\|_{H^{k+2+2(s-1)}}+\|g-\gamma\|_{H^{k+2}}\|V\|_{H^{k+1+2(s-1)}}). \tag{3.5.65}$$

The cases $k = 0, 1$ are treated in a similar way and the conclusion follows from Lemma 3.5.5. □

The next corollary extends the commutator estimates of the previous lemma to higher-orders of $\mathscr{L}^s_{g,\gamma}$, $s \in \mathbb{Z}_{\geq 1}$. Note that the L^2 estimate that appears in the statement will be typically applied with $V = h$, while the lower-order H^1 estimate will be primarily used later on with $V = \Sigma$.

Corollary 3.5.2. *We have,*

$$\begin{aligned}\|[\partial_{\mathbf{u}}, \mathscr{L}^s_{g,\gamma}]V\|_{L^2} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{2s}}+\Lambda(T)\|\partial_T V\|_{H^{2s-1}}, \qquad 1 \leq s \leq \ell,\\ \|[\partial_{\mathbf{u}}, \mathscr{L}^{s-1}_{g,\gamma}]V\|_{H^1} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{2s-1}}+\Lambda(T)\|\partial_T V\|_{H^{2s-2}}, \qquad 2 \leq s \leq \ell.\end{aligned}$$

Proof. By Lemma 3.5.10,

$$\begin{aligned}\|[\partial_{\mathbf{u}},\mathscr{L}^s_{g,\gamma}]V\|_{L^2} &\lesssim \|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}^{s-1}_{g,\gamma}V\|_{L^2}+\|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}^{s-2}_{g,\gamma}V\|_{H^2}+\cdots+\|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]V\|_{H^{2(s-1)}}\\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{2s}}+\Lambda(T)\|\partial_T V\|_{H^{2s-1}}\\ &\quad+\Lambda(T)\sum_{p=0}^{s-2}\|[\partial_T,\mathscr{L}^{s-1-p}_{g,\gamma}]V\|_{H^{1+2p}}.\end{aligned}$$

The conclusion then easily follows and the second estimate follows in the same way. ☐

Corollary 3.5.3.

$$\begin{aligned}\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma)\|_{L^2} &\lesssim (E^g_{N-1})^{1/2}+\Lambda(T),\\ \|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma)\|_{H^1} &\lesssim \|\Sigma\|_{H^{N-1}}+\|g-\gamma\|_{H^N}+\Lambda(T).\end{aligned}$$

Proof. By the $\partial_{\mathbf{u}}\Sigma,\partial_T\Sigma$ estimates of Lemma 3.5.5, Corollary 3.5.1, and the previous commutator estimate of corollary 3.5.2

$$\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma)\|_{L^2}\lesssim \|\partial_{\mathbf{u}}\Sigma\|_{H^{N-3}}+\|[\partial_{\mathbf{u}},\mathscr{L}^{\ell-1}_{g,\gamma}](\Sigma)\|_{L^2}\lesssim \|\Sigma\|_{H^{N-2}}+\|g-\gamma\|_{H^{N-1}}+\Lambda(T).$$

The conclusion then follows by the coercive estimate of Lemma 3.4.2. Similarly,

$$\begin{aligned}\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{H^1} &\lesssim \|\Sigma\|_{H^{N-1}}+\|g-\gamma\|_{H^N}+\Lambda(T)+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\Sigma\|_{H^{N-2}}\\ &\quad+\Lambda(T)\|\partial_T\Sigma\|_{H^{N-3}}\\ &\lesssim \|\Sigma\|_{H^{N-1}}+\|g-\gamma\|_{H^N}+\Lambda(T).\end{aligned}\tag{3.5.70}$$

☐

Remark 3.5.3. Frequently in our energy estimates we will need to study the term $\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma)\|_{H^1}$ appearing in the previous Corollary. However, by looking at the estimate derived in Corollary 3.5.3, we see that we cannot apply Lemma 3.4.2 to the top-order Sobolev norms $\|\Sigma\|_{H^{N-1}}$ and $\|g-\gamma\|_{H^N}$. To estimate these terms by the geometric energy E^g_{tot} we will instead need to use the auxiliary elliptic estimates established in Section 3.6.

We conclude this subsection with a commutator estimate that plays an important role in the proof of the lapse estimate appearing in Proposition 3.7.1.

Lemma 3.5.11. *Let $\Delta := g^{ab}\nabla_a\nabla_b$. Then for $0\le k\le N-3$,*

$$\|[\Delta,\partial_{\mathbf{u}}]V\|_{H^k}\lesssim \Lambda(T)\|\partial_T V\|_{H^{k+1}}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|V\|_{H^{k+2}}.$$

Proof. We compute

$$\begin{aligned}[\Delta,\partial_{\mathbf{u}}]V_{ij} &= \Delta\hat{u}^0\cdot\partial_T V_{ij}+2\nabla^a\hat{u}^0\nabla_a(\partial_T V_{ij})-u^0\partial_T g^{ab}\cdot\nabla_a\nabla_v V_{ij}+u^0g^{ab}[\nabla_a\nabla_b,\partial_T]V_{ij}\\ &\quad+\tau\big(-\Delta u^c\cdot\hat{\nabla}_c V_{ij}-2\nabla_a u^c\nabla^a\hat{\nabla}_c V_{ij}+u^c\hat{\nabla}_c g^{ab}\cdot\nabla_a\nabla_b V_{ij}\big)\\ &\quad+\tau u^c g^{ab}[\hat{\nabla}_c,\nabla_a\nabla_b]V_{ij}.\end{aligned}$$

Let $2\le k\le N-3$. From Lemma 3.5.1 and (3.5.3) we find

$$\begin{aligned}\|[\Delta,\partial_{\mathbf{u}}]V\|_{H^k} &\lesssim \|\nabla\hat{u}^0\|_{H^{k+1}}\|\partial_T V\|_{H^{k+1}}+\|\partial_T h\|_{H^k}\|V\|_{H^{k+2}}+\|\partial_T\Gamma\|_{H^{k+1}}\|V\|_{H^{k+1}}\\ &\quad+|\tau|\|u\|_{H^{k+2}}\|V\|_{H^{k+2}}.\end{aligned}$$

The cases $k=0,1$ follow in the same way. The conclusion follows using Lemma 3.5.5, (3.4.5) and (3.5.2). ☐

3.5.5 Second geometric components estimates

We now reach the final subsection of Section 3.5. The first lemma is an analogue of Lemma 3.4.2 for our top-order $\partial_{\mathbf{u}}-$*boosted* geometric energy. The proof follows those in [2, Lemma 19] and [3, Lemma 7.2].

Lemma 3.5.12. *Let $s \in \mathbb{Z}$ such that $1 \le s \le \ell$. There is a $\delta > 0$ and a constant $C > 0$ such that for $(g, \Sigma) \in B_\delta((\gamma, 0))$ the inequality*

$$(\partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}h, \partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}h)_{L^2_{g,\gamma}} + |(\partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}v, \partial_{\mathbf{u}}\mathscr{L}^{s-1}_{g,\gamma}v)_{L^2_{g,\gamma}}| \le CE^g_{\partial_{\mathbf{u}},2\ell}$$

holds. Furthermore $E^g_{\partial_{\mathbf{u}},2\ell} \ge 0$.

Proof. Recall $E^g_{\partial_{\mathbf{u}},2\ell}$ from Definition 3.3.6. We first note that $E^g_{\partial_{\mathbf{u}},2\ell}|_{(h,v)=(\gamma,0)} = 0$. Next, we see that $(\gamma, 0)$ is a critical point of $E^g_{\partial_{\mathbf{u}},2\ell}$ since the first derivative vanishes. Considering then the second derivative of this energy at $(\gamma, 0)$, we see that the Hessian takes the form

$$\begin{aligned} &D^2(\mathscr{E}^g_{\partial_{\mathbf{u}},2s} + c_E\Gamma^g_{\partial_{\mathbf{u}},2s})((h,k),(h,k)) \\ &\quad = 9(\partial_{\mathbf{u}}\mathscr{L}^s_{\gamma,\gamma}h, \partial_{\mathbf{u}}\mathscr{L}^s_{\gamma,\gamma}h)_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}\mathscr{L}^s_{\gamma,\gamma}k, \partial_{\mathbf{u}}\mathscr{L}^{s-1}_{\gamma,\gamma}k)_{L^2_{g,\gamma}} + c_E(\partial_{\mathbf{u}}\mathscr{L}^{s-1}_{\gamma,\gamma}k, \partial_{\mathbf{u}}\mathscr{L}^s_{\gamma,\gamma}h)_{L^2_{g,\gamma}}. \end{aligned}$$

We claim that the Hessian is non-negative. By expanding in terms of the eigentensors of $\mathscr{L}_{\gamma,\gamma}$, we are left with terms of the type

$$\lambda^{2s-1}\Big(9\lambda(\partial_{\mathbf{u}}P_\lambda h, \partial_{\mathbf{u}}P_\lambda h)_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}P_\lambda k, \partial_{\mathbf{u}}P_\lambda k)_{L^2_{g,\gamma}} + c_E(\partial_{\mathbf{u}}P_\lambda k, \partial_{\mathbf{u}}P_\lambda h)_{L^2_{g,\gamma}}\Big),$$

where P_λ denotes the projection operator onto the λ-eigenspace. The choice of c_E ensures that the bracketed term is non-negative for the smallest eigenvalue λ_0, which in turn implies non-negativity for all eigenvalues. Thus we find

$$D^2(\mathscr{E}^g_{\partial_{\mathbf{u}},2s} + c_E\Gamma^g_{\partial_{\mathbf{u}},2s})((h,k),(h,k)) \ge 0.$$

From this it follows that there is a constant $C = C(\lambda_0, \gamma) > 0$ such that

$$\sum_{s=1}^{\ell}(\partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}h, \partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}h)_{L^2_{g,\gamma}} + |(\partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}k, \partial_{\mathbf{u}}\mathscr{L}^{s-1}_{g,\gamma}k)_{L^2_{g,\gamma}}| \le CE^g_{\partial_{\mathbf{u}},2\ell}.$$

□

The next lemma provides estimates for the Sobolev norms of $\partial_{\mathbf{u}}h, \partial_{\mathbf{u}}\Sigma$ in terms of the geometric energy E^g_{tot}. This is natural given that we have constructed the functional E^g_{tot} to precisely control such Sobolev norms.

Lemma 3.5.13 (Dust derivatives of geometric variables at high-regularity)**.** *We have,*

$$\begin{aligned} &\|\partial_{\mathbf{u}}h\|^2_{H^{N-1}} \lesssim E^g_{\partial_{\mathbf{u}},N-1} + \varepsilon^2 e^{2\max\{-1+\mu,-\lambda\}T}E^g_{N-1} + \Lambda(T)^4, \\ &\|\partial_{\mathbf{u}}\Sigma\|^2_{H^{N-2}} \lesssim E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-1}} + \|g-\gamma\|^2_{H^N}) + \Lambda(T)^2. \end{aligned}$$

Remark 3.5.4. Similar to Remark 3.5.3, we cannot apply Lemma 3.4.2 to the top-order Sobolev norms $\|\Sigma\|^2_{H^{N-1}}$ and $\|g-\gamma\|^2_{H^N}$. To estimate these terms by the geometric energy E^g_{tot} we will instead need to use the auxiliary elliptic estimates of Section 3.6. It is also crucial in later analysis in Section 3.6 that these top-order norms above appear on the right hand side in Lemma 3.5.13 with a smallness factor of ε.

Proof of Lemma 3.5.13. Recall $N := 2\ell + 1$. By the elliptic regularity of Lemma 3.3.2 and Lemma 3.5.12

$$\|\partial_{\mathbf{u}} h\|^2_{H^{2\ell}} \lesssim \|\mathscr{L}^{\ell}_{g,\gamma}\partial_{\mathbf{u}} h\|^2_{L^2} \lesssim E^g_{\partial_{\mathbf{u}},N-1} + \|[\partial_{\mathbf{u}}, \mathscr{L}^{\ell}_{g,\gamma}]h\|^2_{L^2}.$$

We control the commutator term using Corollary 3.5.2, finding

$$\begin{aligned}\|[\partial_{\mathbf{u}}, \mathscr{L}^{\ell}_{g,\gamma}]h\|_{L^2} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|g-\gamma\|_{H^{N-1}} + \Lambda(T)\|\partial_T h\|_{H^{N-2}} \\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(E^g_{N-1})^{1/2} + \Lambda(T)^2,\end{aligned} \tag{3.5.81}$$

where in the final line we used Lemma 3.5.5 and the coercive estimate of Lemma 3.4.2.

Next, by elliptic regularity and Lemma 3.4.3, we have

$$\begin{aligned}\|\partial_{\mathbf{u}}\Sigma\|^2_{H^{2\ell-1}} &\lesssim \|\mathscr{L}^{\ell-1}_{g,\gamma}\partial_{\mathbf{u}}\Sigma\|^2_{H^1} \\ &\lesssim \|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|^2_{L^2} + \|[\partial_{\mathbf{u}}, \mathscr{L}^{\ell-1}_{g,\gamma}]\Sigma\|^2_{L^2} + (\mathscr{L}^{\ell-1}_{g,\gamma}\partial_{\mathbf{u}}\Sigma, \mathscr{L}^{\ell}_{g,\gamma}\partial_{\mathbf{u}}\Sigma)_{L^2_{g,\gamma}}.\end{aligned} \tag{3.5.82}$$

The first term on the RHS of (3.5.82) is treated by Corollary 3.5.3. For the commutator term in (3.5.82), we use Lemma 3.5.5 and Corollary 3.5.2 to find

$$\begin{aligned}\|[\partial_{\mathbf{u}}, \mathscr{L}^{\ell-1}_{g,\gamma}]\Sigma\|_{L^2} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\Sigma\|_{H^{N-3}} + \Lambda(T)\|\partial_T\Sigma\|_{H^{N-4}} \\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(E^g_{N-2})^{1/2} + \Lambda(T)^2.\end{aligned} \tag{3.5.83}$$

Considering next the final term in (3.5.82), we write it as

$$\begin{aligned}(\mathscr{L}^{\ell-1}_{g,\gamma}\partial_{\mathbf{u}}\Sigma, \mathscr{L}^{\ell}_{g,\gamma}\partial_{\mathbf{u}}\Sigma)_{L^2_{g,\gamma}} &= (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma, \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}\Sigma)_{L^2_{g,\gamma}} + ([\mathscr{L}^{\ell-1}_{g,\gamma}, \partial_{\mathbf{u}}]\Sigma, \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}\Sigma)_{L^2_{g,\gamma}} \\ &\quad + (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma, [\partial_{\mathbf{u}}, \mathscr{L}^{\ell}_{g,\gamma}]\Sigma)_{L^2_{g,\gamma}} + ([\mathscr{L}^{\ell-1}_{g,\gamma}, \partial_{\mathbf{u}}]\Sigma, [\partial_{\mathbf{u}}, \mathscr{L}^{\ell}_{g,\gamma}]\Sigma)_{L^2_{g,\gamma}} \\ &=: E_1 + E_2 + E_3 + E_4.\end{aligned}$$

From Lemma 3.5.12, $|E_1| \lesssim E^g_{\partial_{\mathbf{u}},N-1}$. To estimate E_2 we need to integrate by parts one of the derivatives appearing in $\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}\Sigma$. Using (3.5.12a) we find

$$\begin{aligned}E_2 = (g^{ab}\hat{\nabla}_a[\mathscr{L}^{\ell-1}_{g,\gamma}, \partial_{\mathbf{u}}]\Sigma, \hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma)_{L^2_{g,\gamma}} - 2([\mathscr{L}^{\ell-1}_{g,\gamma}, \partial_{\mathbf{u}}]\Sigma, \mathrm{Riem}[\gamma] \circ \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma)_{L^2_{g,\gamma}} \\ + ([\mathscr{L}^{\ell-1}_{g,\gamma}, \partial_{\mathbf{u}}]\Sigma, [\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma)_{L^2_{g,\gamma}}.\end{aligned}$$

By the commutator estimates of Lemma 3.5.10, and Lemma 3.5.5, we have

$$\begin{aligned}\|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{L^2} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\Sigma\|_{H^{N-1}} + \Lambda(T)\|\partial_T\Sigma\|_{H^{N-2}} \\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|_{H^{N-1}} + \|g-\gamma\|_{H^N}) + \Lambda(T)^2.\end{aligned} \tag{3.5.86}$$

Using this, together with (3.5.83) and Corollary 3.5.3, gives

$$\begin{aligned}|E_2| &\lesssim \|[\partial_{\mathbf{u}}, \mathscr{L}^{\ell-1}_{g,\gamma}]\Sigma\|_{H^1}\Big(\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{H^1} + \|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{L^2}\Big) \\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-1}} + \|g-\gamma\|^2_{H^N}) + \Lambda(T)^2.\end{aligned}$$

The terms E_3, E_4 are similarly estimated and inserting all these estimates into (3.5.82) gives the required result. □

We end this subsection with two Lemmas concerning the geometric source terms F_h and F_v.

Lemma 3.5.14 (Time derivative of geometric source terms)**.** *We have,*

$$\|\partial_T F_h\|_{H^{N-2}}+\|\partial_T F_v\|_{H^{N-3}} \lesssim E^g_{N-1} + \Lambda(T).$$

Proof. Let $2 \leq k \leq N-2$. Taking a time derivative of F_h as given in Definition 3.2.6 we see,

$$\begin{aligned}\|\partial_T F_h\|_{H^k} \lesssim \|\partial_T N\|_{H^k}&+(\|\widehat{N}\|_{H^k}+\|X\|_{H^{k+1}}+\|g-\gamma\|_{H^{k+1}}\|X\|_{H^k})\|\partial_T h\|_{H^k}\\ &+ \|g-\gamma\|_{H^k}(\|\partial_T X\|_{H^{k+1}}+\|g-\gamma\|_{H^{k+1}}\|\partial_T X\|_{H^k}).\end{aligned}$$

The first estimate then follows by applying Lemma 3.5.5.

Next, let $2 \leq k \leq N-3$. We take a time derivative of F_v which gives

$$\begin{aligned}\partial_T F_v = \partial_T \nabla_i \nabla_j N &+ \partial_T N \cdot (2\Sigma_{ic}\Sigma^c_j - \frac{1}{3}g_{ij} - \Sigma_{ij} + \tau S_{ij}) + 2N\partial_T(\Sigma_{ic}\Sigma^c_j) - \frac{1}{3}\widehat{N}\partial_T g_{ij}\\ &- \widehat{N}\partial_T \Sigma_{ij} - \tau N S_{ij} + \tau N \partial_T S_{ij} - \partial_T(v_{im}\hat{\nabla}_j X^m + v_{mj}\hat{\nabla}_i X^m).\end{aligned}$$

For the first term we use the commutator identity from (3.5.3) and recall that the lapse is a scalar. We obtain

$$\begin{aligned}\|\partial_T F_v\|_{H^k} \lesssim \|\partial_T N\|_{H^{k+2}}&+\|[\partial_T, \nabla]\nabla N\|_{H^k}+|\tau|(\|S\|_{H^k}+\|\partial_T S\|_{H^k})\\ &+ \|\Sigma\|^2_{H^k}+\|\partial_T\Sigma\|^2_{H^k}+\|\partial_T g\|^2_{H^k}+\Lambda(T)^2.\end{aligned}$$

The conclusion then follows by Lemma 3.5.5 and the matter estimates in Lemma 3.4.1 and Lemma 3.5.8. □

Corollary 3.5.4 (Dust derivative of geometric source terms)**.** *We have,*

$$\begin{aligned}\|\partial_{\mathbf{u}} F_h\|_{H^{N-2}}+\|\partial_{\mathbf{u}} F_v\|_{H^{N-3}} &\lesssim E^g_{N-1} + \varepsilon e^{(-1+\mu)T}(\|g-\gamma\|^2_{H^N}+\|\Sigma\|^2_{H^{N-1}}) + \Lambda(T),\\ \|\partial_{\mathbf{u}} F_h\|_{H^2} &\lesssim E^g_{N-1} + \Lambda(T).\end{aligned}$$

Proof. The estimates follow by expanding out $\partial_{\mathbf{u}} = u^0\partial_T - \tau u^c\hat{\nabla}_c$ and using the F_h, F_v estimates in Lemma 3.5.4 and the $\partial_T F_h, \partial_T F_v$ estimates from Lemma 3.5.14. □

Remark 3.5.5. Since we are dealing with geometric variables in the above corollary, and not matter variables, we roughly estimated the dust derivatives as $\partial_{\mathbf{u}} \sim \partial_T + \tau u^c * \nabla$. Note that doing so introduced the top-order Sobolev norms $\|\Sigma\|^2_{H^{N-1}}$ and $\|g-\gamma\|^2_{H^N}$. Crucially for later analysis in Corollary 3.6.1, however, is that these top-order norms appear with a coefficient of ε.

3.6 Elliptic estimate

In this section we prove an auxiliary elliptic estimate which allows us to control the top-order Sobolev norms of g and Σ in terms of the geometric energy functional E^g_{tot}. Recall only the lower-order Sobolev norms are controlled using Lemma 3.4.2, and so a new idea is indeed needed to cover the top-order of regularity. We also remind the reader that the geometric energy E^g_{tot} will eventually fulfill a strong decay estimate enabled by the inclusion of certain correction terms.

The main result of the section, Corollary 3.6.1, achieves the goal of the previous paragraph. To prove this corollary, we take the first-order equations of motion for $\partial_T h, \partial_T \Sigma$ appearing in (3.2.26) and convert them into second-order equations involving a perturbed wave operator $\mathscr{W}$ and the $\partial_{\mathbf{u}}$ derivatives. We find, very schematically, that

$$\mathscr{L}_{g,\gamma} h \sim \mathscr{W}(h) + \partial_{\mathbf{u}} \partial_T(h) + \tau u^a \hat{\nabla}_a \partial_{\mathbf{u}} h. \tag{3.6.1}$$

By elliptic regularity for $\mathscr{L}_{g,\gamma}$, we can then prove an estimate on $\|g-\gamma\|_{H^N} \sim \|\mathscr{L}_{g,\gamma} h\|_{H^{N-2}}$ by estimating the RHS of (3.6.1), see Proposition 3.6.1. A similar idea holds also for Σ. We note that this idea, albeit for a different gauge, was first introduced by Hadžić and Speck in [55].

Definition 3.6.1 (Operators $\mathscr{W}, \mathscr{H}$)**.** Define the operators

$$\mathscr{W} := \partial_T \partial_T - X^a X^b \hat{\nabla}_a \hat{\nabla}_b + N^2 \mathscr{L}_{g,\gamma}, \qquad \mathscr{H} := N^2 \mathscr{L}_{g,\gamma} - X^a X^b \hat{\nabla}_a \hat{\nabla}_b - \tau^2 \frac{u^a u^b}{(u^0)^2} \hat{\nabla}_b \hat{\nabla}_a,$$

which act on symmetric $(0,2)$-tensors.

Due to the sign convention on $\mathscr{L}_{g,\gamma}$, see Definition 3.3.2, one can think of $\mathscr{W}$ as being a kind of perturbed wave operator.

Lemma 3.6.1 (Wave equations for h, v)**.** *The differential equations* (3.2.26) *for* $h = g - \gamma$ *and* $v = 6\Sigma$ *imply*

$$\mathscr{W}(h) = F_1, \qquad \mathscr{W}(v) = F_2,$$

where,

$$\|F_1\|^2_{H^{N-2}} + \|F_2\|^2_{H^{N-3}} \lesssim E^g_{N-1} + \varepsilon e^{-2\lambda T}(\|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}}) + \Lambda(T)^2.$$

Proof. Rearranging (3.2.26) as $v = w^{-1}(\partial_T h + X^m \hat{\nabla}_m h - F_h)$ and substituting this into (3.2.26) gives

$$-w^{-1}(\partial_T w) v + w^{-1}(\partial_T^2 h + \partial_T(X^m \hat{\nabla}_m h) - \partial_T F_h) = -2v - 9w \mathscr{L}_{g,\gamma} h - X^m \hat{\nabla}_m v + 6F_v.$$

Using again (3.2.26) we note that

$$\partial_T(X^a \hat{\nabla}_a h) = -X^a X^b \hat{\nabla}_a \hat{\nabla}_b h - X^a \hat{\nabla}_a X^b \cdot \hat{\nabla}_b h + \partial_T X^a \cdot \hat{\nabla}_a h + X^m \hat{\nabla}_m (wv + F_h).$$

Rearranging terms (recall $9w^2 = N^2$) we find

$$\begin{aligned} \mathscr{W}(h) = F_1 := X^a \hat{\nabla}_a X^b \cdot \hat{\nabla}_b h - \partial_T X^m \cdot \hat{\nabla}_m h - X^m \hat{\nabla}_m (wv + F_h) + \partial_T F_h + v \partial_T w \\ - 2wv - w X^m \hat{\nabla}_m v + 6w F_v. \end{aligned}$$

Using Lemma 3.5.4 and Lemma 3.5.14 we see

$$\begin{aligned} \|F_1\|_{H^{N-2}} &\lesssim \|\Sigma\|_{H^{N-2}} + \|\partial_T F_h\|_{H^{N-2}} + \|F_v\|_{H^{N-2}} + \Lambda(T) \\ &\lesssim E^g_{N-1} + \|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}} + \Lambda(T). \end{aligned}$$

Although the top-order terms involving $g - \gamma$ and Σ here look worrying, when squaring the estimate we can then apply the bootstraps to gain the crucial factor of ε.

To derive the equation for v we take the ∂_T derivative of (3.2.26):

$$\partial_T^2 v = -2\partial_T v - 9\partial_T w \cdot \mathscr{L}_{g,\gamma} h - 9w\mathscr{L}_{g,\gamma}(\partial_T h) - 9w[\partial_T, \mathscr{L}_{g,\gamma}]h \\ - \partial_T X^m \cdot \hat{\nabla}_m v - X^m \hat{\nabla}_m(\partial_T v) + 6\partial_T F_v.$$

Substituting in the equation of motion (3.2.26) where needed, and expanding as $\mathscr{L}_{g,\gamma}(wv) = w\mathscr{L}_{g,\gamma} v - v\hat{\Delta}_{g,\gamma} w - 2g^{ab}\hat{\nabla}_a w \hat{\nabla}_b v$, we obtain

$$\begin{aligned}\mathscr{W}(v) = F_2 :=\ & 4v + 18w\mathscr{L}_{g,\gamma} h + 2X^m \hat{\nabla}_m v - 12F_v - 9\partial_T w \cdot \mathscr{L}_{g,\gamma} h + 9wv\hat{\Delta}_{g,\gamma} w \\ & + 18wg^{ab}\hat{\nabla}_a w \hat{\nabla}_b v + 9w\mathscr{L}_{g,\gamma}(X^m \hat{\nabla}_m h) - 9w\mathscr{L}_{g,\gamma} F_h - 9w[\partial_T, \mathscr{L}_{g,\gamma}]h \\ & + 6\partial_T F_v - \partial_T X^m \cdot \hat{\nabla}_m v + 2X^m \hat{\nabla}_m v \\ & + 9X^m \hat{\nabla}_m(w\mathscr{L}_{g,\gamma} h) + X^a \hat{\nabla}_a X^m \cdot \hat{\nabla}_m v - 6X^m \hat{\nabla}_m F_v.\end{aligned}$$

Using the F_h, F_v estimates in Lemma 3.5.4 and Lemma 3.5.14, together with the commutator estimate in Lemma 3.5.10, we find

$$\begin{aligned}\|F_2\|_{H^{N-3}} &\lesssim \|\Sigma\|_{H^{N-3}} + \|g-\gamma\|_{H^{N-1}} + \|[\partial_T, \mathscr{L}_{g,\gamma}]h\|_{H^{N-3}} + \|F_v\|_{H^{N-2}} \\ &\quad + \|\partial_T F_v\|_{H^{N-3}} + \|F_h\|_{H^{N-1}} + \Lambda(T) \\ &\lesssim (E^g_{N-1})^{1/2} + \varepsilon e^{-\lambda T}\|g-\gamma\|_{H^{N-1}} + \|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}} + E^g_{N-1} + \Lambda(T).\end{aligned}$$

Note that in the above we also used (3.5.10) to estimate a term of the form:

$$\|\hat{\nabla} h\|_{H^{N-1}} \lesssim \|h\|_{H^N} + \|\Upsilon\|_{H^{N-1}}\|h\|_{H^{N-1}} \lesssim \|g-\gamma\|_{H^N}.$$

□

Proposition 3.6.1 (Elliptic estimate using $\mathscr{W}, \partial_{\mathbf{u}}$ operators)**.** *Let V be an arbitrary $(0,2)$-tensor and $0 \le k \le N-2$. Then,*

$$\|V\|_{H^{k+2}} \lesssim \|\mathscr{W}(V)\|_{H^k} + \|\partial_{\mathbf{u}}\partial_T(V)\|_{H^k} + |\tau|\|u^a \partial_{\mathbf{u}}\hat{\nabla}_a(V)\|_{H^k}.$$

Proof. From (3.3.5) we have,

$$\partial_T V = (u^0)^{-1}\Big(\partial_{\mathbf{u}}(V) + \tau u^a \hat{\nabla}_a V\Big)$$

and thus

$$\partial_T\partial_T V = (u^0)^{-1}\Big(\partial_{\mathbf{u}}(\partial_T V) + \tau u^a \hat{\nabla}_a \partial_T V\Big), \qquad \partial_T \hat{\nabla}_b V = (u^0)^{-1}\Big(\partial_{\mathbf{u}}(\hat{\nabla}_b V) + \tau u^a \hat{\nabla}_a \hat{\nabla}_b V\Big).$$

Recalling $\hat{u}^0 = u^0 - 1/3$, we find

$$\begin{aligned}\mathscr{W}(V) &= \partial_T\partial_T V - X^a X^b \hat{\nabla}_a \hat{\nabla}_b V + N^2 \mathscr{L}_{g,\gamma} V \\ &= (u^0)^{-1}\Big(\partial_{\mathbf{u}}(\partial_T V) + \tau u^a \partial_T \hat{\nabla}_a V\Big) - X^a X^b \hat{\nabla}_a \hat{\nabla}_b V + N^2 \mathscr{L}_{g,\gamma} V \\ &= (u^0)^{-1}\partial_{\mathbf{u}}(\partial_T V) + \frac{\tau u^a}{u^0}\frac{1}{u^0}\Big(\partial_{\mathbf{u}}(\hat{\nabla}_a V) + \tau u^b \hat{\nabla}_b \hat{\nabla}_a V\Big) - X^a X^b \hat{\nabla}_a \hat{\nabla}_b V + N^2 \mathscr{L}_{g,\gamma} V \\ &= (u^0)^{-1}\partial_{\mathbf{u}}(\partial_T V) + \tau u^a (u^0)^{-2}\partial_{\mathbf{u}}(\hat{\nabla}_a V) + \mathscr{H}(V).\end{aligned} \tag{3.6.16}$$

We view $N^{-2}\mathscr{H}$ as a perturbation off the elliptic operator $\mathscr{L}_{g,\gamma}$. For $0 \leq k \leq N-2$,

$$\begin{aligned}
\|(N^{-2}\mathscr{H} - \mathscr{L}_{g,\gamma})V\|_{H^k} &\lesssim \|X^a X^b \hat{\nabla}_a \hat{\nabla}_b V\|_{H^k} + \tau^2 \|\frac{u^a u^b}{(u^0)^2} \hat{\nabla}_b \hat{\nabla}_a V\|_{H^k} \\
&\lesssim \varepsilon^2 e^{(-2+2\mu)T}(\|V\|_{H^{k+2}} + \|\Upsilon\|_{H^{k+1}}\|V\|_{H^{k+1}}) \\
&\lesssim \varepsilon^2 e^{(-2+2\mu)T}\|\mathscr{L}_{g,\gamma}V\|_{H^k}.
\end{aligned}$$

Suppose now $\mathscr{H}(V)=0$. By definition of $\mathscr{H}$, and elliptic regularity of $\mathscr{L}_{g,\gamma}$, this implies

$$\begin{aligned}
\|N^2\mathscr{L}_{g,\gamma}(V)\|_{L^2} &= \|X^a X^b \hat{\nabla}_a \hat{\nabla}_b V + \tau^2 (u^0)^{-2} u^a u^b \hat{\nabla}_b \hat{\nabla}_a V\|_{L^2} \\
&\lesssim \varepsilon^2 e^{2(-1+\mu)T}\|V\|_{H^2} \\
&\leq C(C_1)^{-1}\varepsilon^2 e^{2(-1+\mu)T}\|\mathscr{L}_{g,\gamma}V\|_{L^2},
\end{aligned}$$

for $C>0$ some constant and $C_1>0$ as in Lemma 3.3.2. We also have a lower bound

$$C'\|\mathscr{L}_{g,\gamma}(V)\|_{L^2} \leq \|N^2\mathscr{L}_{g,\gamma}(V)\|_{L^2}.$$

for another constant $C'>0$. Choosing ε sufficiently small so that $C(C_1)^{-1}\varepsilon^2 e^{2(-1+\mu)T} < \varepsilon$, we see these two inequalities imply

$$C'\|\mathscr{L}_{g,\gamma}(V)\|_{L^2} < \varepsilon\|\mathscr{L}_{g,\gamma}(V)\|_{L^2}.$$

For ε sufficiently small this implies $\|\mathscr{L}_{g,\gamma}(V)\|_{L^2} = 0$ and so $V \in \ker \mathscr{L}_{g,\gamma}$. However, $\ker \mathscr{L}_{g,\gamma} = 0$, and so for small data $\mathscr{H}$ also has trivial kernel and thus we obtain

$$\|V\|_{H^{k+2}} \lesssim \|N^{-2}\mathscr{H}(V)\|_{H^k} \lesssim \|V\|_{H^{k+2}}.$$

Putting this together with (3.6.16) we find

$$\|V\|_{H^{k+2}} \lesssim \|N^{-2}\mathscr{W}(V)\|_{H^k} + \|(u^0)^{-1}N^{-2}\partial_{\mathbf{u}}\partial_T(V)\|_{H^k} + |\tau|\|(u^0)^{-2}u^a N^{-2}\partial_{\mathbf{u}}\hat{\nabla}_a(V)\|_{H^k}.$$

□

We can now bring together the previous results and estimate the top-order Sobolev norms of g, Σ in terms of our geometric energy functionals $E^g_{\partial_{\mathbf{u}},N-1}$ and E^g_{N-1}.

Corollary 3.6.1. *We have,*

$$\|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}} \lesssim E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \Lambda(T)^2.$$

Proof. From Proposition 3.6.1,

$$\|g-\gamma\|^2_{H^N} \lesssim \|\mathscr{W}(h)\|^2_{H^{N-2}} + \|\partial_{\mathbf{u}}\partial_T(h)\|^2_{H^{N-2}} + |\tau|^2\|u^a\partial_{\mathbf{u}}\hat{\nabla}_a(h)\|^2_{H^{N-2}}.$$

The first term here is treated using Lemma 3.6.1. For the second term, we begin by using the commutator estimates in Lemma 3.5.9, and the $\partial_T h$ and $\partial_{\mathbf{u}} h$ estimates in Lemma 3.5.5 and Lemma 3.5.13 respectively, to find

$$\begin{aligned}
\|\partial_{\mathbf{u}}\hat{\nabla}(h)\|^2_{H^{N-2}} &\lesssim \|\partial_{\mathbf{u}}h\|^2_{H^{N-1}} + \varepsilon^2 e^{(-2+2\mu)T}\|h\|^2_{H^{N-1}} + \Lambda(T)^2\|\partial_T h\|^2_{H^{N-2}} \\
&\lesssim E^g_{\partial_{\mathbf{u}},N-1} + \varepsilon^2 e^{2\max\{-1+\mu,-\lambda\}T}E^g_{N-1} + \Lambda(T)^3,
\end{aligned} \tag{3.6.25}$$

Next, using the expression for $\partial_T h$ given in (3.2.26), together with Lemma 3.5.3, Corollary 3.6.2, Lemma 3.5.13 and Corollary 3.5.4, we find

$$\begin{aligned}\|\partial_{\mathbf{u}}\partial_T(h)\|^2_{H^{N-2}} \lesssim\ & \|\partial_{\mathbf{u}}\Sigma\|^2_{H^{N-2}}+\|\Sigma\|^2_{H^{N-2}}\|\partial_{\mathbf{u}}N\|^2_{H^{N-2}}+\|\partial_{\mathbf{u}}X\|^2_{H^{N-2}}\|\hat{\nabla}h\|^2_{H^{N-2}}\\ & + \|X\|^2_{H^{N-2}}\|\partial_{\mathbf{u}}\hat{\nabla}h\|^2_{H^{N-2}}+\|\partial_{\mathbf{u}}F_h\|^2_{H^{N-2}}.\\ \lesssim\ & E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^2.\end{aligned}$$

Using again (3.6.25) we find

$$\begin{aligned}|\tau|^2\|u^a\partial_{\mathbf{u}}\hat{\nabla}(h)\|^2_{H^{N-2}} &\lesssim |\tau|^2\|u\|^2_{H^{N-2}}(E^g_{\partial_{\mathbf{u}},N-1}+\varepsilon^2e^{\max\{-2+2\mu,-2\lambda\}T}E^g_{N-1}+\Lambda(T)^3)\\ &\lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\Lambda(T)^4.\end{aligned}$$

Putting this all together

$$\|g-\gamma\|^2_{H^N} \lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\varepsilon(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^2. \tag{3.6.28}$$

We follow the same steps for the Σ estimate. From Proposition 3.6.1,

$$\|\Sigma\|^2_{H^{N-1}} \lesssim \|\mathscr{W}(v)\|^2_{H^{N-3}}+\|\partial_{\mathbf{u}}\partial_T(v)\|^2_{H^{N-3}}+|\tau|^2\|u^a\partial_{\mathbf{u}}\hat{\nabla}_a(\Sigma)\|^2_{H^{N-3}}.$$

The first term here is treated using Lemma 3.6.1. For the second term, we begin by using Lemma 3.5.5, Lemma 3.5.13 and the commutator estimates in Lemma 3.5.9 to show

$$\begin{aligned}\|\partial_{\mathbf{u}}\hat{\nabla}(\Sigma)\|^2_{H^{N-3}} &\lesssim \|\partial_{\mathbf{u}}\Sigma\|^2_{H^{N-2}}+\varepsilon^2e^{(-2+2\mu)T}\|\Sigma\|^2_{H^{N-2}}+\Lambda(T)^2\|\partial_T\Sigma\|^2_{H^{N-3}}\\ &\lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^2.\end{aligned} \tag{3.6.30}$$

Next, by Lemma 3.5.5, Lemma 3.5.13 and the commutator estimate of Lemma 3.5.10, we note

$$\begin{aligned}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}h\|^2_{H^{N-3}} &\lesssim \|\partial_{\mathbf{u}}h\|^2_{H^{N-1}}+\|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]h\|^2_{H^{N-3}}\\ &\lesssim E^g_{\partial_{\mathbf{u}},N-1}+\varepsilon^2e^{\max\{-2+2\mu,-2\lambda\}T}E^g_{N-1}\\ &\quad+\Lambda(T)^2(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^4.\end{aligned}$$

We can now use the expression for $\partial_T v$ given in (3.2.26) and bring together these previous estimates:

$$\begin{aligned}\|\partial_{\mathbf{u}}\partial_T(v)\|^2_{H^{N-3}} \lesssim\ & \|\partial_{\mathbf{u}}\Sigma\|^2_{H^{N-3}}+\|\mathscr{L}_{g,\gamma}h\|^2_{H^{N-3}}\|\partial_{\mathbf{u}}N\|^2_{H^{N-3}}+\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}h\|^2_{H^{N-3}}\\ & + \|\partial_{\mathbf{u}}X\|^2_{H^{N-3}}\|\hat{\nabla}\Sigma\|^2_{H^{N-3}}+\|X\|^2_{H^{N-3}}\|\partial_{\mathbf{u}}\hat{\nabla}\Sigma\|^2_{H^{N-3}}+\|\partial_{\mathbf{u}}F_v\|^2_{H^{N-3}}\\ \lesssim\ & \|\partial_{\mathbf{u}}\Sigma\|^2_{H^{N-3}}+\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}h\|^2_{H^{N-3}}+\|\partial_{\mathbf{u}}F_v\|^2_{H^{N-3}}+\Lambda(T)^2\\ \lesssim\ & E^g_{\partial_{\mathbf{u}},N-1}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^2.\end{aligned}$$

In the above we used Lemma 3.5.3, Corollary 3.6.2, Lemma 3.5.4 and Lemma 3.5.13. Using again (3.6.30) and Lemma 3.5.13, we find

$$\begin{aligned}&|\tau|^2\|u^a\partial_{\mathbf{u}}\hat{\nabla}(\Sigma)\|^2_{H^{N-3}}\\ &\quad\lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T)(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^3.\end{aligned}$$

Finally, putting this all together gives

$$\|\Sigma\|^2_{H^{N-1}} \lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\varepsilon(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})+\Lambda(T)^2. \tag{3.6.34}$$

The conclusion then follows by adding the estimates (3.6.28) and (3.6.34) together and taking ε sufficiently small so that we can absorb the $\varepsilon(\|\Sigma\|^2_{H^{N-1}}+\|g-\gamma\|^2_{H^N})$ term onto the left hand side. □

We now provide new estimates on dust derivatives acting on our variables N, X, g, Σ and, for the latter two variables, apply Corollary 3.6.1.

Corollary 3.6.2. *For I a multi-index,*

$$\sum_{|I|\leq N-1} \|\partial_{\mathbf{u}} \nabla^I X\|_{L^2} + \|\partial_{\mathbf{u}} \nabla^I \widehat{N}\|_{L^2} \lesssim \Lambda(T),$$
$$\sum_{|I|\leq N-1} \|\partial_{\mathbf{u}} \nabla^I (g-\gamma)\|_{L^2} + \sum_{|I|\leq N-2} \|\partial_{\mathbf{u}} \nabla^I \Sigma\|_{L^2} \lesssim (E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T).$$

The same estimates hold with ∇ replaced by $\hat{\nabla}$.

Proof. By Lemma 3.5.13 and Corollary 3.6.1

$$\|\partial_{\mathbf{u}} h\|_{H^{N-1}} + \|\partial_{\mathbf{u}} \Sigma\|_{H^{N-2}} \lesssim (E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T).$$

Next, let $\nabla^I = \nabla_{a_1} \dots \nabla_{a_k}$ where $k := |I| \leq N-1$ and let V be a $(0,2)$-tensor on M. Then, by (3.5.1), (3.5.2) and Lemma 3.5.9,

$$\begin{aligned}\|[\partial_{\mathbf{u}}, \nabla^I]V\|_{L^2} &\lesssim \|[\partial_{\mathbf{u}}, \nabla_{a_1}]\nabla_{a_2} \dots \nabla_{a_k} V\|_{L^2} + \dots + \|[\partial_{\mathbf{u}}, \nabla]V\|_{H^{k-1}} \\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T} \|V\|_{H^k} + \Lambda(T)\|\partial_T V\|_{H^{k-1}} + \Lambda(T)\|\partial_T \Gamma(g)\|_{H^{N-3}} \|V\|_{H^{k-2}}.\end{aligned}$$

This implies

$$\|\partial_{\mathbf{u}} \nabla^I V\|_{L^2} \lesssim \|\partial_{\mathbf{u}} V\|_{H^k} + \varepsilon e^{\max\{-1+\mu,-\lambda\}T} \|V\|_{H^k} + \Lambda(T)\|\partial_T V\|_{H^{k-1}}.$$

A simple check shows that when considering the appropriate commutator expression (see (3.5.53)) on the scalar lapse we will pick up $\|\widehat{N}\|_{H^k}$ terms and not $\|N\|_{H^k}$. The result then follow, using Lemma 3.5.3 and Lemma 3.5.5. □

Lemma 3.6.2. *Let V be a $(0,2)$-tensor on M and let $p \in \{1,2\}$. Then*

$$\|\partial_{\mathbf{u}} V\|_{H^p} \lesssim \sum_{|I|\leq p} \|\partial_{\mathbf{u}} \nabla^I V\|_{L^2} + \varepsilon e^{\max\{-1+\mu,-\lambda\}T} \|V\|_{H^1} + \Lambda(T)\|\partial_T V\|_{H^{p-1}}.$$

Proof. A straight forward application of commutator identities. □

3.7 Lapse and shift estimates

In this section we first establish the basic estimates on the lapse, shift and their time derivatives using elliptic estimates and general formula presented in [2]. The most exciting results lie in the novel top-order lapse and shift estimates given in Section 3.7.1.

Lemma 3.7.1. *For $3 \leq k \leq N$,*

$$\begin{aligned}\|\widehat{N}\|_{H^k} &\lesssim |\tau| \|\rho\|_{H^{k-2}} + \varepsilon^2 e^{-2\lambda T}, \\ \|X\|_{H^k} &\lesssim |\tau| \|\rho\|_{H^{k-3}} + \varepsilon^2 e^{\max\{-2\lambda,-2+\mu\}T}.\end{aligned}$$

Proof. Recall from (3.2.22b) the lapse equation of motion

$$(\Delta - \frac{1}{3})N = N(|\Sigma|_g^2 + \tau\eta) - 1.$$

By elliptic regularity for the standard Laplacian Δ this implies

$$\|\widehat{N}\|_{H^k} \leq C(\|\Sigma\|_{H^{k-2}}^2 + |\tau|\|\eta\|_{H^{k-2}}),$$

and the desired result follows by Lemma 3.4.1.

Similarly from the shift equation of motion (3.2.22b), we find

$$\|X\|_{H^k} \leq C(\|\Sigma\|_{H^{k-2}}^2 + \|g-\gamma\|_{H^{k-1}}^2 + |\tau|\|\eta\|_{H^{k-3}} + \tau^2\|N\jmath\|_{H^{k-2}}),$$

and the desired result follows by Lemma 3.4.1. □

Lemma 3.7.2. *For $4 \leq k \leq N-1$,*

$$\|\partial_T N\|_{H^k} + \|\partial_T X\|_{H^k} \lesssim \|\widehat{N}\|_{H^k} + \|X\|_{H^k} + |\tau|\|\rho\|_{H^{k-2}} + \varepsilon^2 e^{\max\{-2\lambda,-2+\mu\}T}.$$

Proof. Let $4 \leq k \leq N-1$. Following [2], we have

$$\begin{aligned}\|\partial_T N\|_{H^k} &\lesssim \|\widehat{N}\|_{H^k} + \|X\|_{H^k} + \|\Sigma\|_{H^{k-1}}^2 + \|g-\gamma\|_{H^k}^2 + |\tau|(\|S\|_{H^{k-2}} + \|\eta\|_{H^{k-2}} + \|\partial_T\eta\|_{H^{k-2}}) \\ &\lesssim \|\widehat{N}\|_{H^k} + \|X\|_{H^k} + |\tau|\|\rho\|_{H^{k-2}} + \varepsilon^2 e^{\max\{-2\lambda,-2+\mu\}T},\end{aligned}$$

where we used Lemma 3.4.1 and Lemma 3.5.8.

Again, using a general expression given in [2, §7], we have

$$\begin{aligned}\|\partial_T X\|_{H^k} \lesssim\ & \|\widehat{N}\|_{H^k} + \|X\|_{H^k} + \|\Sigma\|_{H^k}^2 + \|g-\gamma\|_{H^k}^2 \\ & + \tau(\|S\|_{H^{k-2}} + \|\eta\|_{H^{k-2}} + \|\partial_T\eta\|_{H^{k-2}}) + \tau^2\|\jmath\|_{H^{k-1}} + \tau^3\|\underline{T}\|_{H^{k-1}} \\ \lesssim\ & \|\widehat{N}\|_{H^k} + \|X\|_{H^k} + |\tau|\|\rho\|_{H^{k-2}} + \varepsilon^2 e^{\max\{-2\lambda,-2+\mu\}T},\end{aligned}$$

where we again used Lemma 3.4.1 and Lemma 3.5.8. □

3.7.1 Additional top-order lapse and shift estimates

In this section we establish important auxiliary estimates in two propositions for the lapse and shift variables. The first proposition is used to estimate $\partial_{\mathbf{u}}\nabla_i\nabla_j N$. This somewhat unusual expression comes from a dust derivative acting on the first term in F_v (see Definition 3.2.6), and appears later on in the energy estimates for E^g_{tot}.

Proposition 3.7.1 (Top-order auxiliary lapse estimate)**.** *We have*

$$(\Delta - \frac{1}{3})\partial_{\mathbf{u}}\nabla_i\nabla_j N = \mathscr{F}_{N,\mathbf{u}}$$

where,

$$\|\mathscr{F}_{N,\mathbf{u}}\|_{H^{N-4}} \lesssim \Lambda(T) + E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1}.$$

Proof. Start by commuting the lapse equation (3.2.22b) with $\partial_{\mathbf{u}}\nabla_i\nabla_j$:

$$\begin{aligned}(\Delta-\frac{1}{3})\partial_{\mathbf{u}}\nabla_i\nabla_j N &= [\Delta,\partial_{\mathbf{u}}]\nabla_i\nabla_j N+\partial_{\mathbf{u}}[\Delta,\nabla_i\nabla_j]N+\partial_{\mathbf{u}}\nabla_i\nabla_j\Big(N(|\Sigma|_g^2-\tau\eta)\Big)\\ &=: L_1+L_2+L_3=:\mathscr{F}_{N,\mathbf{u}}.\end{aligned}$$

We investigate each of the terms in $\mathscr{F}_{N,\mathbf{u}}$ separately. The first term L_1 we estimate using (3.5.2), (3.5.3) and Lemma 3.5.11

$$\begin{aligned}\|L_1\|_{H^{N-4}} &\lesssim \Lambda(T)\|\partial_T\nabla_i\nabla_j N\|_{H^{N-3}}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\nabla_i\nabla_j N\|_{H^{N-2}}\\ &\lesssim \Lambda(T)\Big(\|\partial_T N\|_{H^{N-1}}+\|[\partial_T,\nabla_i]\nabla_j N\|_{H^{N-3}}\Big)+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\widehat{N}\|_{H^N}\\ &\lesssim \Lambda(T)^2+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T).\end{aligned}$$

For the next term, L_2, we first compute, for ϕ a scalar,

$$\begin{aligned}[\Delta,\nabla_i]\phi &= 2g^{ab}\nabla_{[a}\nabla_{i]}\nabla_b\phi=-\mathrm{Ric}^c{}_i\nabla_c\phi,\\ [\Delta,\nabla_i]\nabla_j\phi &= \mathrm{Ric}^c{}_i\nabla_c\nabla_j\phi-(\nabla^a\mathrm{Riem}^c{}_{jai})\nabla_c\phi-2\mathrm{Riem}^c{}_{jai}\nabla^a\nabla_c\phi.\end{aligned}$$

Thus

$$\begin{aligned}L_2 &= \partial_{\mathbf{u}}([\Delta,\nabla_i]\nabla_j N)+\partial_{\mathbf{u}}\nabla_i([\Delta,\nabla_j N)\\ &= -\Big((\partial_{\mathbf{u}}\nabla^a\mathrm{Riem}^c{}_{jai})+(\partial_{\mathbf{u}}\nabla_i\mathrm{Ric}^c{}_j)\Big)\nabla_c N-\Big(\nabla^a\mathrm{Riem}^c{}_{jai}+\nabla_i\mathrm{Ric}^c{}_j\Big)\partial_{\mathbf{u}}\nabla_c N\\ &\quad+\Big((\partial_{\mathbf{u}}\mathrm{Ric}^c{}_i)\nabla_c\nabla_j N-(\partial_{\mathbf{u}}\mathrm{Ric}^c{}_j)\nabla_i\nabla_c N-2(\partial_{\mathbf{u}}\mathrm{Riem}^c{}_{jai})\nabla^a\nabla_c N\Big)\\ &\quad+\Big(\mathrm{Ric}^c{}_i(\partial_{\mathbf{u}}\nabla_c\nabla_j N)-\mathrm{Ric}^c{}_j(\partial_{\mathbf{u}}\nabla_i\nabla_c N)-2\mathrm{Riem}^c{}_{jai}(\partial_{\mathbf{u}}\nabla^a\nabla_c N)\Big)\\ &=: -(L_{21})-(L_{22})+(L_{23})+(L_{24})\end{aligned}$$

The second and last terms here are easy to estimate using Lemma 3.5.3 and Lemma 3.5.9 (note also that the expressions (3.5.1) and (3.5.53) simplify when calculated for a scalar)

$$\begin{aligned}&\||L_{22}|+|L_{24}|\|_{H^{N-4}}\\ &\lesssim \|\mathrm{Riem}\|_{H^{N-3}}\Big(\|\partial_{\mathbf{u}}N\|_{H^{N-4}}+\|[\partial_{\mathbf{u}},\nabla]N\|_{H^{N-3}}+\|[\partial_{\mathbf{u}},\nabla]\nabla N\|_{H^{N-4}}\Big)\\ &\lesssim \|\mathrm{Riem}\|_{H^{N-3}}\Big(\Lambda(T)+\Lambda(T)\|\partial_T N\|_{H^{N-3}}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|N\|_{H^{N-2}}\Big)\\ &\lesssim \Lambda(T).\end{aligned}$$

For the other two terms, L_{21} and L_{23}, we schematically write

$$\begin{aligned}\partial_{\mathbf{u}}\mathrm{Riem} &= u^0\partial_T(\partial\Gamma+\Gamma\Gamma)-\tau u^c\hat{\nabla}_c\mathrm{Riem}\\ &= u^0(\partial\partial_T\Gamma+\Gamma\partial_T\Gamma)-\tau u^c\nabla_c\mathrm{Riem}+\tau u^c*\Upsilon*\mathrm{Riem},\\ \partial_{\mathbf{u}}\nabla\mathrm{Riem} &= \nabla\partial_{\mathbf{u}}\mathrm{Riem}+[\partial_{\mathbf{u}},\nabla]\mathrm{Riem},\end{aligned}$$

and thus

$$\begin{aligned}\||L_{21}|+|L_{23}|\|_{H^{N-4}} &\lesssim \|\widehat{N}\|_{H^{N-2}}\Big(\|\partial_T\Gamma\|_{H^{N-2}}+\|\partial_T\Gamma\|_{H^{N-3}}\|\Gamma\|_{H^{N-3}}+\|\partial_T\Gamma\|_{H^{N-4}}\|\mathrm{Riem}\|_{H^{N-4}}\\ &\quad+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\mathrm{Riem}\|_{H^{N-2}}+\varepsilon\|\partial_T\mathrm{Riem}\|_{H^{N-4}}\Big)\\ &\lesssim \varepsilon e^{-\lambda T}\|\widehat{N}\|_{H^{N-2}}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\widehat{N}\|_{H^{N-2}}\|\mathrm{Riem}\|_{H^{N-2}}\lesssim\Lambda(T).\end{aligned}$$

Finally we compute

$$\begin{aligned}L_3 &= \nabla_i\nabla_j\partial_{\mathbf{u}}(N(|\Sigma|_g^2{-}\tau\eta)) + \nabla_i[\partial_{\mathbf{u}},\nabla_j](N(|\Sigma|_g^2{-}\tau\eta)) + [\partial_{\mathbf{u}},\nabla_i]\nabla_j(N(|\Sigma|_g^2{-}\tau\eta))\\ &=: L_{31}+L_{32}+L_{33}\end{aligned}$$

By the matter estimates in Lemma 3.4.1 and Lemma 3.5.7, as well as the geometry estimates in Lemma 3.5.3 and Corollary 3.6.2, we find

$$\begin{aligned}\|L_{31}\|_{H^{N-4}} &\lesssim \|\partial_{\mathbf{u}}N\|_{H^{N-2}}(\|\Sigma\|^2_{H^{N-2}}{+}|\tau|\|\eta\|_{H^{N-2}}) + \|\Sigma\|_{H^{N-2}}\|\partial_{\mathbf{u}}\Sigma\|_{H^{N-2}}\\ &\quad + |\tau|\|\eta\|_{H^{N-2}}{+}|\tau|\|\partial_{\mathbf{u}}\eta\|_{H^{N-2}}\\ &\lesssim \Lambda(T) + \|\Sigma\|_{H^{N-2}}((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2}).\end{aligned}$$

Similarly by Lemma 3.4.1, Lemma 3.5.8, Corollary 3.5.1 and the commutator estimate of Lemma 3.5.9,

$$\begin{aligned}\|L_{32}\|_{H^{N-4}} &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|^2_{H^{N-2}}{+}|\tau|\|\eta\|_{H^{N-2}})\\ &\quad + \Lambda(T)\|\partial_T N\|_{H^{N-3}}(\|\Sigma\|^2_{H^{N-3}}{+}|\tau|\|\eta\|_{H^{N-3}})\\ &\quad + \Lambda(T)\|\Sigma\|_{H^{N-3}}\|\partial_T\Sigma\|_{H^{N-3}}{+}\Lambda(T)|\tau|\|\eta\|_{H^{N-3}}{+}\Lambda(T)|\tau|\|\partial_T\eta\|_{H^{N-3}}\\ &\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}E^g_{N-1} + \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T).\end{aligned}$$

The final estimate follows in the same way:

$$\|L_{33}\|_{H^{N-4}} \lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}E^g_{N-1} + \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Lambda(T).$$

Putting this all together we have found

$$\|\mathscr{F}_{N,\mathbf{u}}\|_{H^{N-4}}\lesssim \Lambda(T) + \|\Sigma\|_{H^{N-2}}((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2}),$$

and so the conclusion follows by Lemma 3.4.2. □

Remark 3.7.1. In our energy estimates later on we need to estimate a term of the type $\partial_{\mathbf{u}}\hat{\nabla}^{N-2}F_v$ where $\hat{\nabla}^{N-2}$ indicates $N-2$ covariant derivatives and F_v is given in Definition 3.2.6. We would like to commute the $\partial_{\mathbf{u}}$ operator past these covariant derivatives. However, F_v contains a $\nabla\nabla N$ term, and we cannot commute $\partial_{\mathbf{u}}$ past $N-2$ copies of ∇ as well as the extra two derivatives in $\nabla\nabla N$ since we only control u^a in H^{N-1}. The previous proposition crucially allows us to avoid this issue. Note also that by commuting in the $\partial_{\mathbf{u}}$ operator, instead of doing the rough expansion $\partial_{\mathbf{u}}\sim\partial_T+\tau u^c*\nabla$, we gain an additional derivative in Corollary 3.7.1 compared to Corollary 3.5.4.

Corollary 3.7.1. *For I a multi-index of order $|I|\leq N-2$,*

$$\|\partial_{\mathbf{u}}\hat{\nabla}^I F_v\|_{L^2} \lesssim \Lambda(T) + E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1}.$$

Proof. Write $\hat{\nabla}^I = \hat{\nabla}_{a_1}\dots\hat{\nabla}_{a_k}$ where $k := |I|\leq N-2$. Using the definition of F_v in Definition 3.2.6 and the estimates in Corollary 3.6.2, we see the most subtle terms (from the point of view of regularity) are $\nabla_i\nabla_j N$ and $N\tau S_{ij}$. For these terms we look at what happens when we commute in the $\partial_{\mathbf{u}}$ operator using Lemma 3.5.9:

$$\begin{aligned}\|[\partial_{\mathbf{u}},\hat{\nabla}^I]\nabla_i\nabla_j N\|_{L^2} &\lesssim \|[\partial_{\mathbf{u}},\hat{\nabla}_{a_1}]\hat{\nabla}_{a_2}\dots\hat{\nabla}_{a_k}\nabla_i\nabla_j N\|_{L^2}{+}\dots + \|[\partial_{\mathbf{u}},\hat{\nabla}]\nabla_i\nabla_j N\|_{H^{k-1}}\\ &\lesssim \varepsilon e^{(-1+\mu)T}\|\widehat{N}\|_{H^{k+2}}{+}\Lambda(T)\|\partial_T N\|_{H^{k+1}}{+}\Lambda(T)\|[\partial_T,\nabla_i\nabla_j]N\|_{H^{k-1}}.\end{aligned}$$

So by the commutator estimates (3.5.1) and (3.5.2), as well as Proposition 3.7.1,

$$\begin{aligned}\|\partial_{\mathbf{u}}\hat{\nabla}^I\nabla_i\nabla_j N\|_{L^2} &\lesssim \|\partial_{\mathbf{u}}\nabla_i\nabla_j N\|_{H^k}+\|[\partial_{\mathbf{u}},\hat{\nabla}^I]\nabla_i\nabla_j N\|_{L^2}\\ &\lesssim \|\mathscr{F}_{N,\mathbf{u}}\|_{H^{k-2}}+\Lambda(T)\\ &\lesssim \Lambda(T)+\|\Sigma\|_{H^{N-2}}((E^g_{\partial_{\mathbf{u}},N-1})^{1/2}+(E^g_{N-1})^{1/2}).\end{aligned}$$

Similarly, using the matter estimates in Lemma 3.4.1, Lemma 3.5.7 and Lemma 3.5.8, as well as $\partial_{\mathbf{u}}N$ estimates in Lemma 3.5.3, we find

$$\begin{aligned}\|\partial_{\mathbf{u}}\hat{\nabla}^I(\tau N S_{ij})\|_{L^2_{g,\gamma}} \lesssim{}& |\tau|(\|S\|_{H^k}+\|\partial_{\mathbf{u}}S\|_{H^k}+\|\partial_{\mathbf{u}}N\|_{H^k}\|S\|_{H^k}+\varepsilon e^{(-1+\mu)T}\|S\|_{H^k}\\ &+\Lambda(T)\|\partial_T S\|_{H^{k-1}}+\Lambda(T)\|S\|_{H^{k-1}}+\Lambda(T)\|\partial_T N\|_{H^{k-1}}\|S\|_{H^{k-1}})\\ \lesssim{}& \Lambda(T).\end{aligned}$$

□

In our energy estimates later on we need to estimate a term of the type $\partial_{\mathbf{u}}\hat{\nabla}^{N-1}F_h$. Similar to the issue discussed in Remark 3.7.1, the problematic term here is the $\mathscr{L}_X g$ term appearing in F_h (see Definition 3.2.6). The second proposition of this section estimates this problematic term. Since, however, the shift is not scalar-valued like the lapse, a replication of the ideas used in Proposition 3.7.1 ends up failing. Instead, our proof involves a remarkable combination of commutator estimates, the Bianchi identity and the Einstein equations in the CMCSH gauge.

Proposition 3.7.2 (Top-order estimate for $\mathscr{L}_X g$)**.** *We have*

$$\left|(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Delta\mathscr{L}_X g),\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}\right| \lesssim \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1}+\Lambda(T)E^g_{N-1}+\Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.$$

Proof. Recall $2(\ell-1)=N-3$. We first define $\mathscr{F}^X_a$ by rewriting the shift equation (3.2.22b) as

$$\Delta X_a = -\mathrm{Ric}^c{}_a X_c + \mathscr{F}^X_a.$$

By contracting the Bianchi identity, one finds

$$\nabla^a \mathrm{Riem}_{abcd} = \nabla_c \mathrm{Ric}_{bd} - \nabla_d \mathrm{Ric}_{bc}.$$

Using this, and the fact that ∇ is a torsion-free connection for the metric g, we can show that

$$\begin{aligned}\Delta(\mathscr{L}_X g)_{ab} &= \nabla_a\Delta X_b+\nabla_b\Delta X_a+[\Delta,\nabla_a]X_b+[\Delta,\nabla_b]X_a\\ &= \nabla_a(-\mathrm{Ric}^c{}_b)X_c+\nabla_a\mathscr{F}^X_b+\nabla_b(-\mathrm{Ric}^c{}_a)X_c+\nabla_b\mathscr{F}^X_a\\ &\quad - g^{ij}\nabla_i(\mathrm{Riem}^k{}_{bja})X_k - g^{ij}\nabla_i(\mathrm{Riem}^k{}_{ajb})X_k+2E_{(ab)}\\ &= -2X^k\nabla_k\mathrm{Ric}_{ab}+\nabla_a\mathscr{F}^X_b+\nabla_b\mathscr{F}^X_a+2E_{(ab)},\end{aligned}$$

where we have introduced the error terms

$$E_{ij} := -\mathrm{Ric}^c{}_j\nabla_i X_c - 2\mathrm{Riem}^k{}_{jci}\nabla^c X_k + \mathrm{Ric}_{ki}\nabla^k X_j.$$

As first noted in [4], in the CMCSH gauge, we have

$$\mathrm{Ric}_{ab} = -\frac{2}{9}g_{ab} - \frac{1}{2}\mathscr{L}_{g,\gamma}h_{ab} + J_{ab},$$

and thus

$$\Delta \mathscr{L}_X g_{ab} = X^k \nabla_k \mathscr{L}_{g,\gamma} h_{ab} - 2X^k \nabla_k J_{ab} + 2\nabla_{(a} \mathscr{F}^X_{b)} + 2E_{(ab)}. \tag{3.7.32}$$

We now need to estimate each term in the RHS of (3.7.32). The first term requires the most care. We begin with an identity, valid for V an arbitrary $(0,2)$-tensor and $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\partial_{\mathbf{u}} \mathscr{L}^k_{g,\gamma}(X^m \hat{\nabla}_m V) = X^m \hat{\nabla}_m \partial_{\mathbf{u}} \mathscr{L}^k_{g,\gamma}(V) + X^m \partial_{\mathbf{u}} [\mathscr{L}^k_{g,\gamma}, \hat{\nabla}_m] V + X^m [\partial_{\mathbf{u}}, \hat{\nabla}_m] \mathscr{L}^k_{g,\gamma} V \\ + \partial_{\mathbf{u}} X^m \cdot \mathscr{L}^k_{g,\gamma}(\hat{\nabla}_m V) + \partial_{\mathbf{u}} [\mathscr{L}^k_{g,\gamma}, X^m] \hat{\nabla}_m V.\end{aligned} \tag{3.7.33}$$

Thus

$$\partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma}(X^k \nabla_k \mathscr{L}_{g,\gamma} h_{ij}) = X^m \hat{\nabla}_m \partial_{\mathbf{u}} \mathscr{L}^{\ell}_{g,\gamma}(h_{ij}) + \mathscr{R}^1_{ij} + \mathscr{R}^2_{ij}, \tag{3.7.34}$$

where

$$\begin{aligned}\mathscr{R}^1_{ij} &:= \partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma}(X^m \Upsilon^k_{mi} \mathscr{L}_{g,\gamma} h_{kj}) + \partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma}(X^m \Upsilon^k_{mj} \mathscr{L}_{g,\gamma} h_{ki}) + X^m \partial_{\mathbf{u}} [\mathscr{L}^{\ell-1}_{g,\gamma}, \hat{\nabla}_m] \mathscr{L}_{g,\gamma} h_{ij} \\ &\quad + \partial_{\mathbf{u}}([\mathscr{L}^{\ell-1}_{g,\gamma}, X^m] \hat{\nabla}_m \mathscr{L}_{g,\gamma} h_{ij}), \\ \mathscr{R}^2_{ij} &:= X^m [\partial_{\mathbf{u}}, \hat{\nabla}_m] \mathscr{L}^{\ell}_{g,\gamma} h_{ij} + \partial_{\mathbf{u}} X^m \cdot \mathscr{L}^{\ell-1}_{g,\gamma}(\hat{\nabla}_m \mathscr{L}_{g,\gamma} h_{ij}).\end{aligned}$$

We integrate by parts on the first term in (3.7.34) using (3.5.12b) and Lemma 3.5.12 to find

$$\left| (X^m \hat{\nabla}_m \partial_{\mathbf{u}} \mathscr{L}^{\ell}_{g,\gamma}(h), \partial_{\mathbf{u}} \mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} \right| \lesssim \|X\|_{H^3} \|\partial_{\mathbf{u}} \mathscr{L}^{\ell}_{g,\gamma} h\|^2_{L^2_{g,\gamma}} \lesssim \Lambda(T) E^g_{\partial_{\mathbf{u}}, N-1}.$$

The remaining terms $\mathscr{R}^{1,2}_{ij}$ appearing in (3.7.34) are errors terms. Since we work at high regularity, we can control such error terms using the basic idea of taking low-derivative terms out in L^∞. We briefly present this argument once for the last term in $\mathscr{R}^1_{ij}$:

$$\begin{aligned}\|\partial_{\mathbf{u}}([\mathscr{L}^{\ell-1}_{g,\gamma}, X^m] \hat{\nabla}_m \mathscr{L}_{g,\gamma} h_{ij})\|_{L^2_{g,\gamma}} &\lesssim \sum_{\substack{|I|+|J| \leq 2(\ell-1) \\ |I| \geq 1}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I X * \hat{\nabla}^J \hat{\nabla} \mathscr{L}_{g,\gamma} h)\|_{L^2_{g,\gamma}} \\ &\lesssim \sum_{\substack{|I|+|J| \leq N-3, \\ |I| \geq 1, |J| \geq 3}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I X) \hat{\nabla}^J h\|_{L^2_{g,\gamma}} + \|\hat{\nabla}^I X \partial_{\mathbf{u}}(\hat{\nabla}^J h)\|_{L^2_{g,\gamma}}\end{aligned} \tag{3.7.37}$$

We are now faced with four terms depending on where the derivatives sit. Two of these have high derivatives on the shift X, and so by Sobolev embedding these are controlled by:

$$\sum_{\substack{1 \leq |I| \leq N-3, \\ 3 \leq |J| \leq (N-3)/2}} (\|\partial_{\mathbf{u}} \hat{\nabla}^I X\|_{L^2} \|\hat{\nabla}^J h\|_{H^2} + \|\hat{\nabla}^I X\|_{L^2} \|\partial_{\mathbf{u}} \hat{\nabla}^J h\|_{H^2}).$$

We use Lemma 3.6.2 to exchange the H^2 norm with the $\partial_{\mathbf{u}}$ derivative, and then all terms are then controlled using Corollary 3.6.2 and Lemma 3.5.5 provided $\frac{N-3}{2} + 2 \leq N-1$. The other two terms, where the high derivatives hit the metric, are similarly controlled by:

$$\sum_{\substack{3 \leq |J| \leq N-1, \\ 1 \leq |I| \leq (N-3)/2}} (\|\partial_{\mathbf{u}} \hat{\nabla}^I X\|_{H^2} \|\hat{\nabla}^J h\|_{L^2} + \|\hat{\nabla}^I X\|_{H^2} \|\partial_{\mathbf{u}} \hat{\nabla}^J h\|_{L^2})$$

and once again all these terms are controlled using Lemma 3.5.5, Corollary 3.6.2 and Lemma 3.6.2 provided $\frac{N-3}{2} + 2 \leq N-1$.

Carrying on this way, and using the commutator estimates contained in Lemma 3.5.10, together with Corollary 3.6.2, we find

$$\left|(\mathscr{R}^1, \partial_{\mathbf{u}}\mathscr{L}^\ell_{g,\gamma}(h))_{L^2_{g,\gamma}}\right| \lesssim \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1} + \Lambda(T)E^g_{N-1} + \Lambda(T)^2(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.$$

For the other error term, $\mathscr{R}^2$, we use Lemma 3.5.5, Lemma 3.5.10, Corollary 3.6.1 and Corollary 3.6.2, to find

$$\begin{aligned}\left|(\mathscr{R}^2, \partial_{\mathbf{u}}\mathscr{L}^\ell_{g,\gamma}(h))_{L^2_{g,\gamma}}\right| &\lesssim \|X\|_{H^2}\|\partial_{\mathbf{u}}\mathscr{L}^\ell_{g,\gamma}(h)\|_{L^2_{g,\gamma}}(\varepsilon e^{(-1+\mu)T}\|g-\gamma\|_{H^N}+\Lambda(T)\|\partial_T h\|_{H^{N-1}}) \\ &\quad + \|\partial_{\mathbf{u}}X\|_{H^2}\|g-\gamma\|_{H^N}\|\partial_{\mathbf{u}}\mathscr{L}^\ell_{g,\gamma}h\|_{L^2_{g,\gamma}} \\ &\lesssim \Lambda(T)\Big((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T)\Big)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.\end{aligned}$$

The second term in the RHS of (3.7.32) is estimated by the same expression as for $\mathscr{R}^1$. For the third term in the RHS of (3.7.32), we use the definition of $\mathscr{F}^X$ given in (3.2.22b):

$$\begin{aligned}&\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\nabla_j\mathscr{F}^X_i) \\ &\quad = \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\nabla_j\left(2\hat{\nabla}_c N\Sigma^c_i - \hat{\nabla}_i\widehat{N} + 2N\tau^2 g_{ai}\jmath^a - g_{ai}(2N\Sigma^{bc} - \nabla^b X^c)\Upsilon^a_{bc}\right).\end{aligned}$$

The second term in the large brackets here ($\hat{\nabla}_i\widehat{N}$) decays the slowest, while from a regularity point of view the most subtle term is the matter term ($\jmath^a$). For this latter term, we commute in the $\partial_{\mathbf{u}}$ operator and use the matter estimates of Lemma 3.4.1, Lemma 3.5.7 and Lemma 3.5.8 together with the commutator estimates of Lemma 3.5.9 and Corollary 3.5.2:

$$\begin{aligned}\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\nabla_j(\tau^2 g_{ai}\jmath^a)\|_{L^2} &\lesssim \tau^2(\|\partial_{\mathbf{u}}\jmath^a\|_{H^{N-2}}+\|[\partial_{\mathbf{u}},\mathscr{L}^{\ell-1}_{g,\gamma}]\hat{\nabla}_j\jmath^a\|_{L^2}+\|[\partial_{\mathbf{u}},\nabla_j]\jmath^a\|_{H^{N-3}}) \\ &\lesssim \tau^2(\|\partial_{\mathbf{u}}\jmath^a\|_{H^{N-2}}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|\jmath\|_{H^{N-2}}+\Lambda(T)\|\partial_T\jmath\|_{H^{N-3}}) \\ &\lesssim \varepsilon^2 e^{(-\lambda+\mu)T}\Lambda(T) + \Lambda(T)^2 + \varepsilon^4 e^{-(3+\delta)T}.\end{aligned}$$

All together, and using Lemma 3.5.12, we find

$$\begin{aligned}\left|(2\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\nabla_{(a}\mathscr{F}^X_{b)}, \partial_{\mathbf{u}}\mathscr{L}^\ell_{g,\gamma}(h_{ab}))_{L^2_{g,\gamma}}\right| &\lesssim \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1} + \Lambda(T)E^g_{N-1} \\ &\quad + \Big(\Lambda(T)^2 + \varepsilon^2 e^{(-\lambda+\mu)T}\Lambda(T) + \varepsilon^4 e^{-(3+\delta)T}\Big)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.\end{aligned}$$

Finally, we have

$$\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}E\|_{L^2} \lesssim \sum_{\substack{|I|+|J|\leq N-2 \\ |J|\geq 1}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I \operatorname{Riem} * \hat{\nabla}^J X)\|_{L^2_{g,\gamma}} \lesssim \Lambda(T),$$

where the Riemann terms can be estimated using arguments as in the proof of Proposition 3.7.1. □

3.8 Geometric energy estimates

In this section we establish energy estimates for the geometric energy functionals. We first prove estimates for the time-derivatives of the lower-order energy functional ${E^g}_{N-1}$, and then for the top-order energy functional $\mathscr{E}^g_{\partial_{\mathbf{u}},N-1}$. Recall α, c_E and E^g_{tot} are given in Definitions 3.3.4 and 3.3.2.

Proposition 3.8.1. *There exists a constant $C > 0$ such that*

$$\partial_T {E^g}_{N-1} \leq -2\alpha {E^g}_{N-1} + C\Lambda(T)({E^g}_{N-1})^{1/2} + C({E^g}_{N-1})^{3/2}.$$

Proof. Let $0 \leq k \leq N-1$. Using an estimate from [2, Lemma 20] together with Lemma 3.4.1, we find

$$\begin{aligned}\partial_T {E^g}_k &\leq -2\alpha {E^g}_k + 6({E^g}_k)^{1/2}|\tau| \|NS\|_{H^{k-1}} + C({E^g}_k)^{3/2} \\ &\quad + C({E^g}_k)^{1/2}(|\tau| \|\eta\|_{H^{k-1}} + \tau^2 \|Nj\|_{H^{k-2}}) \\ &\leq -2\alpha {E^g}_k + C({E^g}_k)^{1/2}|\tau| \|\rho\|_{H^{k-1}} + C({E^g}_k)^{3/2} + C\Lambda(T)({E^g}_k)^{1/2}.\end{aligned}$$

□

We now turn to the time evolution of the top-order $\partial_{\mathbf{u}}-$*boosted* geometric energy. The main result is stated in Theorem 3.8.1. For ease of presentation however, the proof relies on the subsequent estimates given in Propositions 3.8.2, 3.8.3, 3.8.4, 3.8.5, and 3.8.6.

Remark 3.8.1. The key auxiliary estimates of the previous section, Corollary 3.7.1 and Proposition 3.7.2, are applied in Proposition 3.8.3 and Proposition 3.8.2 respectively. The important integration by parts identity (3.5.12c) in Lemma 3.5.2, where one term is brought back onto the LHS, is applied in Proposition 3.8.5 (see (3.8.45)).

Theorem 3.8.1. *There exists a constant $C > 0$ such that*

$$\begin{aligned}\partial_T \mathscr{E}^g_{\partial_{\mathbf{u}},N-1} &\leq -2\alpha \mathscr{E}^g_{\partial_{\mathbf{u}},N-1} + C(|\tau| \|u\|_{H^{N-1}} + \Lambda(T)) E^g_{tot} + C\Lambda(T)(E^g_{tot})^{1/2} \\ &\quad + C(E^g_{tot})^{3/2} + C\Lambda(T)^2.\end{aligned}$$

Proof. Recalling Definition 3.3.6, the energy $E^g_{\partial_{\mathbf{u}},N-1}$ consists of a sum over lower-order energies. The top-order is the most subtle, so we focus on this and merely remark that the estimates for the lower-orders follow in the same (or possibly easier) way. The top-order energy consists of two parts, an $\mathscr{E}^g_{\partial_{\mathbf{u}},N-1}$ term and a $c_E \Gamma^g_{\partial_{\mathbf{u}},N-1}$ term. We treat these separately for the moment.

Using (3.3.5)-(3.3.14) and integration by parts, we find

$$\begin{aligned}\partial_T \mathscr{E}^g_{\partial_{\mathbf{u}},2\ell}(T) &\leq (\|\widehat{N}\|_{L^\infty} + \|\nabla X\|_{L^\infty}) \mathscr{E}^g_{\partial_{\mathbf{u}},2\ell}(T) + \Big[9(\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}(\partial_T h), \partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}(h))_{L^2_{g,\gamma}} \\ &\quad + \frac{1}{2}(\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma} \partial_T v, \partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma} v)_{L^2_{g,\gamma}} + \frac{1}{2}(\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma} v, \partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma} \partial_T v)_{L^2_{g,\gamma}}\Big] + G_c,\end{aligned} \tag{3.8.4}$$

where we define

$$\begin{aligned}G_c &:= 9([\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}, \partial_T](h), \partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}(h))_{L^2_{g,\gamma}} \\ &\quad + \frac{1}{2}(\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}(v), [\partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma}, \partial_T](v))_{L^2_{g,\gamma}} + \frac{1}{2}([\partial_{\mathbf{u}} \mathscr{L}^\ell_{g,\gamma}, \partial_T](v), \partial_{\mathbf{u}} \mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}}.\end{aligned}$$

The terms G_c are error terms, and so our main focus is on the terms appearing (3.8.4) in the square bracket. Using the equations of motion (3.2.26) and self-adjointness of $\mathscr{L}_{g,\gamma}$, these become

$$
\begin{aligned}
9(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(\partial_T h), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} &+ \frac{1}{2}(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(\partial_T v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}} \\
&+ \frac{1}{2}(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\partial_T v))_{L^2_{g,\gamma}} \\
&= -2(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}} + G_1 + G_2 + G_3 + G_4,
\end{aligned}
$$

where

$$
\begin{aligned}
G_1 &:= 9(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(2\widehat{N}g - \mathscr{L}_X g), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}, \\
G_2 &:= 3(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(F_v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}} + 3(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(F_v))_{L^2_{g,\gamma}}, \\
G_3 &:= 9\Big[(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(wv), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} \\
&\quad - \frac{9}{2}\Big[(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(w\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(w\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}}\Big], \\
G_4 &:= \frac{1}{2}(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(X^m\hat{\nabla}_m v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v))_{L^2_{g,\gamma}} + \frac{1}{2}(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(X^m\hat{\nabla}_m v))_{L^2_{g,\gamma}}.
\end{aligned}
$$

Similarly using (3.3.5)-(3.3.14) and integration by parts, we find

$$
\begin{aligned}
\partial_T\Gamma^g_{\partial_{\mathbf{u}},2\ell}(T) \leq \varepsilon e^{-T}\Gamma^g_{\partial_{\mathbf{u}},2\ell}(T) &+ \Big[(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\partial_T v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} \\
&+ (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}v, \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}\partial_T h)_{L^2_{g,\gamma}}\Big] + G_{cc}\,,
\end{aligned}
\tag{3.8.7}
$$

where we define

$$
G_{cc} := ([\partial_T, \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}](v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v), [\partial_T, \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}](h))_{L^2_{g,\gamma}}.
$$

Using the equations of motion (3.2.26)

$$
\begin{aligned}
&\Big[(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\partial_T v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}v, \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}\partial_T h)_{L^2_{g,\gamma}}\Big] \\
&\quad = -2(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} - 9(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}} \\
&\qquad + (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(v))_{L^2_{g,\gamma}} + G_5 + G_6 + G_7 + G_8 + G_9,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
G_5 &:= -9(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\widehat{N}\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}, \quad G_6 := (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(\widehat{N}v))_{L^2_{g,\gamma}}, \\
G_7 &:= (\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(2\widehat{N}g - \mathscr{L}_X g))_{L^2_{g,\gamma}}, \quad G_8 := -(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(X^m\hat{\nabla}_m v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}, \\
G_9 &:= 6(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(F_v), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h),)_{L^2_{g,\gamma}}.
\end{aligned}
$$

The errors terms are estimated in the following subsection 3.8.1 and subsection 3.8.2: G_1 and G_7 (Proposition 3.8.2), G_2 and G_9 (Proposition 3.8.3), G_3, G_5 and G_6 (Proposition 3.8.4), G_4 and G_8 (Proposition 3.8.5), G_c and G_{cc} (Proposition 3.8.6). Using these estimates, and adding (3.8.4) and $c_E\times$(3.8.7) together, yields the required inequality. □

3.8.1 Estimates on $\mathbf{G_1}$ to $\mathbf{G_9}$

In this section we prove estimates on the terms G_1 to G_9. We begin with a useful identity. Let $\tilde{V}_{ij}, \tilde{P}_{ij}$ be symmetric $(0,2)$-tensors on M and $k \in \mathbb{Z}_{\geq 1}$. Then, the integration by parts rule (3.5.12a) implies

$$\begin{aligned}
&(\partial_{\mathbf{u}}\mathscr{L}^{k}_{g,\gamma}(\tilde{V}), \partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{P}))_{L^2_{g,\gamma}} \\
&\quad = (g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{V}), \hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{P}))_{L^2_{g,\gamma}} - 2(\mathrm{Riem}[\gamma] \circ \partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{V}), \partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{P}))_{L^2_{g,\gamma}} \\
&\qquad + ([\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}^{k-1}_{g,\gamma}(\tilde{V}), \partial_{\mathbf{u}}\mathscr{L}^{k-1}_{g,\gamma}(\tilde{P}))_{L^2_{g,\gamma}}.
\end{aligned} \tag{3.8.10}$$

Proposition 3.8.2. *We have,*

$$|G_1| + |G_7| \lesssim \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1} + \Lambda(T)E^g_{N-1} + \Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + \Lambda(T)^2.$$

Proof. The first term in G_1 is easily controlled using Lemma 3.5.12 and Corollary 3.6.2:

$$|(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(2\widehat{N}g), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}| \lesssim \sum_{|I| \leq N-1} \|\partial_{\mathbf{u}}\hat{\nabla}^I\widehat{N}\|_{L^2_{g,\gamma}} \cdot \|\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}h\|_{L^2_{g,\gamma}} \lesssim \Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.$$

For the Lie derivative term in G_1, we rewrite it as

$$\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(\mathscr{L}_X g_{ij}) = \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Delta\mathscr{L}_X g_{ij} + \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\mathscr{L}_{g,\gamma} - \Delta)\mathscr{L}_X g_{ij}). \tag{3.8.13}$$

The first term on the RHS of (3.8.13) is precisely what is controlled using Proposition 3.7.2. For the second term, we schematically have

$$\begin{aligned}
(\mathscr{L}_{g,\gamma} - \Delta)\mathscr{L}_X g &= g^{-1}(\hat{\nabla}\hat{\nabla} - \nabla\nabla)\mathscr{L}_X g + \mathrm{Riem}[\gamma] * \mathscr{L}_X g \\
&= \nabla\Upsilon * \nabla X + \Upsilon * \nabla\nabla X + \Upsilon * \Upsilon * \nabla X + \nabla X.
\end{aligned}$$

Using the Υ estimate (3.5.10), we have

$$\begin{aligned}
|(\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}((\mathscr{L}_{g,\gamma} - \Delta)\mathscr{L}_X g), \partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(h))_{L^2_{g,\gamma}}| &\lesssim \sum_{|I|+|J| \leq N-1} \|\partial_{\mathbf{u}}(\hat{\nabla}^I X \hat{\nabla}^J h)\|_{L^2_{g,\gamma}} \cdot \|\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}h\|_{L^2_{g,\gamma}} \\
&\lesssim \Lambda(T)\Big((E^g_{tot})^{1/2} + \Lambda(T)\Big)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + \Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.
\end{aligned}$$

where in the final estimate we used the same ideas as in (3.7.37), namely an application of Corollary 3.6.2, Lemma 3.6.2 and the fact that $2(\ell - 1) = N - 3$. The same arguments clearly apply to G_7 also. □

Proposition 3.8.3. *We have,*

$$|G_2| + |G_9| \lesssim (E^g_{tot})^{3/2} + \Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + \Lambda(T)^2.$$

Proof. The two terms in G_2 are estimated in virtually the same way. We discuss how to estimate the second term, since it is slightly more difficult. We first use the identity (3.8.10) with $\tilde{V} = \Sigma$ and $\tilde{P} = F_v$, in order to transfer derivatives between terms:

$$\begin{aligned}
(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(F_v))_{L^2_{g,\gamma}} X &= (g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma, \hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(F_v))_{L^2_{g,\gamma}} \\
&\quad - 2(\mathrm{Riem}[\gamma] \circ \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma, \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}F_v)_{L^2_{g,\gamma}} + ([\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma, \partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(F_v))_{L^2_{g,\gamma}}.
\end{aligned}$$

We estimate each of these three expressions in turn. For the first expression, we use the various F_v estimates from Lemma 3.5.4, Lemma 3.5.14 and Corollary 3.7.1, together with the commutator estimates from Lemma 3.5.10 and Lemma 3.6.2, to find

$$\begin{aligned}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(F_v)\|_{H^1} &\lesssim \sum_{|I|\leq 1}\|\partial_{\mathbf{u}}\nabla^I\mathscr{L}_{g,\gamma}^{\ell-1}(F_v)\|_{L^2}+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\|F_v\|_{H^{N-2}}\\&\quad+\Lambda(T)\|\partial_T F_v\|_{H^{N-3}}+\Lambda(T)\|[\partial_T,\mathscr{L}_{g,\gamma}^{\ell-1}](F_v)\|_{L^2}\\&\lesssim \Lambda(T)+E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}\\&\quad+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Big(\Lambda(T)+\|g-\gamma\|^2_{H^N}+\|\Sigma\|^2_{H^{N-1}}\Big)\\&\lesssim \Lambda(T)+E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}.\end{aligned}$$

Note in the final line above we used Corollary 3.6.1.

In addition, combining Corollary 3.5.3 with Corollary 3.6.1, we have

$$\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\|_{H^1}\lesssim\|\Sigma\|_{H^{N-1}}+\|g-\gamma\|_{H^N}+\Lambda(T)\lesssim(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}+(E^g_{N-1})^{1/2}+\Lambda(T).$$

So we find

$$|(g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma,\hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(F_v))_{L^2_{g,\gamma}}|\lesssim(E^g_{tot})^{3/2}+\Lambda(T)^2.$$

In a similar way, using Corollary 3.5.3 and the coercive lower-order estimate from Lemma 3.4.2,

$$\begin{aligned}|(\mathrm{Riem}[\gamma]\circ\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}F_v)_{L^2_{g,\gamma}}|&\lesssim\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{L^2}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(F_v)\|_{L^2}\\&\lesssim(E^g_{tot})^{3/2}+\Lambda(T)^2.\end{aligned}$$

Combining (3.5.86) with Corollary 3.6.1 gives

$$\begin{aligned}\|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{L^2}&\lesssim\varepsilon e^{\max\{-1+\mu,-\lambda\}T}(\|\Sigma\|_{H^{N-1}}+\|g-\gamma\|_{H^N})+\Lambda(T)^2\\&\lesssim\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\Big((E^g_{\partial_{\mathbf{u}},N-1})^{1/2}+(E^g_{N-1})^{1/2}+\Lambda(T)\Big).\end{aligned}\tag{3.8.22}$$

Then this gives control over the final term in G_2.

Finally we use the above estimates and Lemma 3.5.12 to estimate G_9 by

$$\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(F_v),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h),)_{L^2_{g,\gamma}}\Big|\lesssim\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(F_v)\|_{L^2}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h)\|_{L^2}\lesssim\Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.$$

□

Remark 3.8.2. In the next proposition, it is crucial that the three terms of G_3 are treated together, since there is an important cancellation that appears.

Proposition 3.8.4. *We have,*

$$\begin{aligned}|G_3|+|G_5|+|G_6|&\lesssim(|\tau|\|u\|_{H^{N-1}}+\Lambda(T))E^g_{tot}+\Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}\\&\quad+(E^g_{tot})^{3/2}+\Lambda(T)^2.\end{aligned}$$

Proof. First note from (3.3.5) that

$$(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(w\mathscr{L}_{g,\gamma}h),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}}=(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(w\mathscr{L}_{g,\gamma}h),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v))_{L^2_{g,\gamma}}+G_3^e,$$

where we have defined

$$
\begin{aligned}
G_3^e := &([\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(w\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}} \\
&+ (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(w\mathscr{L}_{g,\gamma}h), [\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}}.
\end{aligned}
$$

A computation then gives,

$$
\frac{1}{9}G_3 = (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(wv), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}} - (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(w\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v))_{L^2_{g,\gamma}} - \frac{1}{2}G_3^e.
$$

The worst term above occurs when all the derivatives hit the geometric variables v, h instead of the lapse variable w. However, crucially, such terms cancel

$$
(w\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}} - (w\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v))_{L^2_{g,\gamma}} = 0.
$$

So we are left to consider

$$
\begin{aligned}
&(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(N\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}} - (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(\Sigma))_{L^2_{g,\gamma}} \\
&\quad = (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}([\mathscr{L}_{g,\gamma}, N]\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}} - (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}}.
\end{aligned}
\tag{3.8.29}
$$

For the first term on the RHS of (3.8.29),

$$
\begin{aligned}
\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}([\mathscr{L}_{g,\gamma}, N]\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}}\Big| &\lesssim \sum_{\substack{|I|+|J|\le 2\ell \\ |I|\ge 1}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I \widehat{N} \cdot \hat{\nabla}^J \Sigma)\|_{L^2} \|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h)\|_{L^2} \\
&\lesssim \Lambda(T)\Big((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T)\Big)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.
\end{aligned}
$$

For the second term on the RHS of (3.8.29) we need to apply the integration by parts identity (3.5.12a). This gives

$$
\begin{aligned}
&(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}} = \\
&(g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma), \hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}} \\
&- 2(\mathrm{Riem}[\gamma] \circ \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}} \\
&+ ([\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}},
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h))_{L^2_{g,\gamma}}\Big| \\
&\lesssim \Big(\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1} + \|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\|_{L^2}\Big)\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-2}([\mathscr{L}_{g,\gamma}, N]\mathscr{L}_{g,\gamma}h)\|_{H^1} \\
&\lesssim \Big(\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1} + \|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\|_{L^2}\Big) \sum_{\substack{|I|+|J|\le N-1 \\ |I|\ge 1, |J|\ge 2}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I \widehat{N} \hat{\nabla}^J h)\|_{H^1}.
\end{aligned}
$$

All these terms can be controlled by estimates in Corollary 3.5.3, Corollary 3.6.1 and (3.8.22), together with distributing derivatives and applying Lemma 3.6.2 and Corollary 3.5.1 as needed. We refer to the example given in (3.7.37). All together, we find

$$
\begin{aligned}
&\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(N\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}} - (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(\Sigma))_{L^2_{g,\gamma}}\Big| \\
&\lesssim \Lambda(T)\Big(E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \Lambda(T)^2\Big).
\end{aligned}
$$

Finally, we need to estimate the terms in G_3^e. These in fact require the most care. For the first term in G_3^e, when using the commutator identity (3.5.57) on the first factor we see that this has potentially too-many derivatives hitting the metric h

$$[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h)$$

Using the schematic identity given below (3.5.57), we see these problematic terms come when all the derivatives hit the metric h:

$$(\partial_{\mathbf{u}}g^{-1}) * \hat{\nabla}^2\hat{\nabla}^{N-1}h + \hat{\nabla}u^0 * \hat{\nabla}\hat{\nabla}^{N-1}\partial_T h + \tau\hat{\nabla}u^c * \hat{\nabla}^2\hat{\nabla}^{N-1}h.$$

Thus for these terms we integrate by parts using (3.5.12b). For example, we find

$$\begin{aligned}
\Big|\Big((\partial_{\mathbf{u}}g^{ab})\hat{\nabla}_a\hat{\nabla}_b\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\Big)_{L^2_{g,\gamma}}\Big| &\lesssim \|\partial_{\mathbf{u}}g\|_{H^3}\|\mathscr{L}_{g,\gamma}^{\ell}h\|_{H^1}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1},\\
\Big|\Big((\tau(\hat{\nabla}^a u^c)\hat{\nabla}_a\hat{\nabla}_c\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\Big)_{L^2_{g,\gamma}}\Big| &\lesssim |\tau|\|u\|_{H^4}\|\mathscr{L}_{g,\gamma}^{\ell}h\|_{H^1}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1}.
\end{aligned}$$

These terms can be controlled using Corollary 3.5.1, Corollary 3.5.3 and Corollary 3.6.1. We eventually obtain the following estimate for the first term in G_3^e

$$\begin{aligned}
\Big|([\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}}\Big| &\lesssim |\tau|\|u\|_{H^{N-1}}E^g_{\partial_{\mathbf{u}},N-1} + |\tau|\|u\|_{H^{N-1}}E^g_{N-1}\\
&\quad + \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1} + \Lambda(T)E^g_{N-1} + \Lambda(T)^2.
\end{aligned} \tag{3.8.37}$$

For the second term of G_3^e, we use the original commutator estimate (3.5.59) combined with Corollary 3.6.1 to find

$$\begin{aligned}
&\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(w\mathscr{L}_{g,\gamma}h), [\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}}\Big| \lesssim \|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(N\mathscr{L}_{g,\gamma}h)\|_{L^2}\|[\partial_{\mathbf{u}}, \mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\|_{L^2}\\
&\lesssim (E^g_{\partial_{\mathbf{u}},N-1})^{1/2}\Big(\|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}} + \Lambda(T)^2 + |\tau|\|u\|_{H^2}\|\Sigma\|^2_{H^{N-1}}\Big).\\
&\lesssim (E^g_{\partial_{\mathbf{u}},N-1})^{1/2}\Big(E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \Lambda(T)\Big).
\end{aligned}$$

Finally we turn to the estimates for G_5 and G_6. The first of these is straightforward in light of previous estimates:

$$\begin{aligned}
|G_5| = \Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\widehat{N}\mathscr{L}_{g,\gamma}h), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}}\Big| &\lesssim \sum_{\substack{|I|+|J|\leq N-1\\|J|\geq 2}} \|\partial_{\mathbf{u}}(\hat{\nabla}^I\widehat{N}\hat{\nabla}^J h)\|_{L^2}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}h\|_{L^2}\\
&\lesssim \Lambda(T)\Big((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T)\Big)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.
\end{aligned}$$

For G_6 we need to integrate by parts once just as for the first term in G_3^e, however the estimate follows in the same way and so we omit the details. ☐

Proposition 3.8.5. *We have,*

$$|G_4| + |G_8| \lesssim \Lambda(T)E^g_{\partial_{\mathbf{u}},N-1} + \Lambda(T)E^g_{N-1} + \Lambda(T)^2.$$

Proof. The two terms appearing in G_4 are

$$(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(X^m\hat{\nabla}_m v), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}} + (\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(X^m\hat{\nabla}_m v))_{L^2_{g,\gamma}}.$$

We explain the estimate for the first term only since the second one follows in the same way. Using (3.8.10), we obtain

$$
\begin{aligned}
(\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}(X^m\hat{\nabla}_m\Sigma),\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma))_{L^2_{g,\gamma}} &= (g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(X^m\hat{\nabla}_m\Sigma),\hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma))_{L^2_{g,\gamma}} \\
&\quad - 2(\mathrm{Riem}[\gamma]\circ\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(X^m\hat{\nabla}_m\Sigma),\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma))_{L^2_{g,\gamma}} \\
&\quad + ([\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}(X^a\hat{\nabla}_a\Sigma),\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma))_{L^2_{g,\gamma}}.
\end{aligned}
\tag{3.8.42}
$$

We focus on the first term on the RHS of (3.8.42) and use (3.7.33) to write it as

$$
\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(X^m\hat{\nabla}_m\Sigma) = X^m\hat{\nabla}_m\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma) + G^e_4, \tag{3.8.43}
$$

where we have introduced

$$
\begin{aligned}
G^e_4 := {} & \hat{\nabla}_a X^m\cdot\hat{\nabla}_m\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma + X^m[\hat{\nabla}_a,\hat{\nabla}_m]\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma \\
& + \hat{\nabla}_a\Big[\partial_{\mathbf{u}}[\mathscr{L}^{\ell-1}_{g,\gamma},X^m]\hat{\nabla}_m\Sigma + \partial_{\mathbf{u}}(X^m[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}_m]\Sigma) \\
& + \partial_{\mathbf{u}}X^m\cdot\hat{\nabla}_m\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma + X^m[\partial_{\mathbf{u}},\hat{\nabla}_m]\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\Big].
\end{aligned}
$$

By a slight adaption of the integration by parts estimate (3.5.12c), we see

$$
|(g^{ab}X^m\hat{\nabla}_m\hat{\nabla}_aV,\hat{\nabla}_bV)_{L^2_{g,\gamma}}| \lesssim \|X\|_{H^3}\|V\|^2_{H^1}. \tag{3.8.45}
$$

Using this, Corollary 3.5.3 and Corollary 3.6.1, the first term of (3.8.43) is thus estimated as

$$
\begin{aligned}
|(g^{ab}X^m\hat{\nabla}_m\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma),\hat{\nabla}_b\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}(\Sigma))_{L^2_{g,\gamma}}| &\lesssim \|X\|_{H^3}\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|^2_{H^1} \\
&\lesssim \Lambda(T)\Big(E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \Lambda(T)^2\Big).
\end{aligned}
$$

We then turn to the remaining terms in (3.8.43), namely G^e_4. The first, second and fifth terms of G^e_4 are fairly straightfowardly estimated by

$$
\begin{aligned}
\|X\|_{H^3}\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{H^1} + \|\partial_{\mathbf{u}}X\|_{H^3}\|\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{H^2} &\lesssim \Lambda(T)\Big(\|\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma}\Sigma\|_{H^1} + \|\Sigma\|_{H^{N-1}}\Big) \\
&\lesssim \Lambda(T)\Big((E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T)\Big).
\end{aligned}
$$

The third and fourth terms of G^e_4 merely require a careful counting of derivatives. As an example, the fourth term of G^e_4 is estimated by

$$
\|\partial_{\mathbf{u}}(X^m[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}_m](\Sigma))\|_{H^1} \lesssim \|\partial_{\mathbf{u}}X\|_{H^3}\|[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}]\Sigma\|_{H^1} + \|X\|_{H^3}\|\partial_{\mathbf{u}}[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}]\Sigma\|_{H^1}.
$$

The first commutator term here can be studied using Lemma 3.5.10, in particular the same ideas as in (3.5.61) yield

$$
\|[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}_m]\Sigma\|_{H^1} \lesssim \|g-\gamma\|_{H^{N-3}}\|\Sigma\|_{H^{N-2}} + \|\Sigma\|_{H^{N-4}} \lesssim (E^g_{N-1})^{1/2}.
$$

While we use (3.5.60) and the same ideas in the proof of Proposition 3.7.1 to estimate the second commutator term by

$$
\begin{aligned}
\|\partial_{\mathbf{u}}[\mathscr{L}^{\ell-1}_{g,\gamma},\hat{\nabla}]\Sigma\|_{H^1} &\lesssim \sum_{\substack{|I|+|J|\le 2\ell-1\\ |I|\ge 1,|J|\ge 2}}\|\partial_{\mathbf{u}}(\hat{\nabla}^I g\hat{\nabla}^J\Sigma)\|_{L^2} + \sum_{|I|+|J|\le 2(\ell-2)}\|\partial_{\mathbf{u}}(\hat{\nabla}^I\mathrm{Riem}[\gamma]*\hat{\nabla}^J\Sigma)\|_{L^2} \\
&\lesssim (E^g_{\partial_{\mathbf{u}},N-1})^{1/2} + (E^g_{N-1})^{1/2} + \Lambda(T).
\end{aligned}
$$

Finally, for the last term of G_4^e we use the original commutator identity (3.5.53) to estimate it by

$$\begin{aligned}\|X\|_{H^3}\|[\partial_{\mathbf{u}},\hat{\nabla}]\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{L^2} &\lesssim \Lambda(T)\Big(|\tau|\|u\|_{H^3}\|\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1}+\|\hat{\nabla}u^0\|_{H^2}\|\partial_T\Sigma\|_{H^{2(\ell-1)}}\Big)\\ &\lesssim \Lambda(T)\Big(|\tau|\|u\|_{H^{N-1}}(E_{N-1}^g)^{1/2}+\Lambda(T)(E_{N-1}^g)^{1/2}+\Lambda(T)^2\Big).\end{aligned}$$

In summary, we obtain

$$|G_4^3|\lesssim \Lambda(T)E_{\partial_{\mathbf{u}},N-1}^g+\Lambda(T)E_{N-1}^g+\Lambda(T)^2.$$

We now turn to the second term on the RHS of (3.8.42), and estimate it by

$$\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{L^2}\sum_{\substack{|I|+|J|\leq N-2\\ |J|\geq 1}}\|\partial_{\mathbf{u}}(\hat{\nabla}^I X^m\hat{\nabla}^J\Sigma)\|_{L^2}\lesssim \Lambda(T)E_{\partial_{\mathbf{u}},N-1}^g+\Lambda(T)E_{N-1}^g+\Lambda(T)^2.$$

We finally turn to the last term on the RHS of (3.8.42) and use the identity (3.5.57) on the first factor

$$[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(X^a\hat{\nabla}_a\Sigma).$$

Using the schematic identity given below (3.5.57), we see certain problematic terms arise when all the derivatives miss the shift and hit Σ:

$$(\partial_{\mathbf{u}}g^{-1})*X*\hat{\nabla}^2\hat{\nabla}^{N-2}\Sigma+\hat{\nabla}u^0*X*\hat{\nabla}\hat{\nabla}^{N-2}\partial_T\Sigma+\tau\hat{\nabla}u^c*X*\hat{\nabla}^2\hat{\nabla}^{N-2}\Sigma.$$

So, just as for the first term in G_3^e given in the proof of Proposition 3.8.4, we need to integrate by parts using (3.5.12b) on these terms. Given the close similarities, we omit the details and just state the result:

$$\Big|([\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(X^a\hat{\nabla}_a\Sigma),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}}\Big|\lesssim \Lambda(T)E_{\partial_{\mathbf{u}},N-1}^g+\Lambda(T)E_{N-1}^g+\Lambda(T)^2.$$

Note, however, that the above estimate is better than in (3.8.37). This is because of the additional shift term appearing which always gives us additional decay (unlike the lapse which needs at least one derivative acting on it).

Finally we turn to G_8 and remark that

$$\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(X^m\hat{\nabla}_m v),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}}\Big|\lesssim\sum_{\substack{|I|+|J|\leq N-2\\ |J|\geq 1}}\|\partial_{\mathbf{u}}(\hat{\nabla}^I X^m\hat{\nabla}^J\Sigma)\|_{L^2}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}h\|_{L^2}$$

and this is easily estimated using previous ideas. □

3.8.2 Estimates on $\mathbf{G_c}$ and $\mathbf{G_{cc}}$

In this final part of the section we prove estimates on the commutator error terms G_c and G_{cc}. We first prove a preliminary lemma concerning the commutator between the operators ∂_T and $\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}$, and then use this lemma in the subsequent proposition.

Lemma 3.8.1. *We have,*

$$\|[\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma},\partial_T]h\|_{L^2}+\|[\partial_{\mathbf{u}}\mathscr{L}^{\ell-1}_{g,\gamma},\partial_T]\Sigma\|_{H^1}\lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\Lambda(T).$$

Proof. Let V be a $(0,2)$-tensor on M. A computation yields

$$\begin{aligned}[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}]V_{ij}=&-\partial_T u^0\cdot\mathscr{L}_{g,\gamma}\partial_T V_{ij}+\partial_T u^0\partial_T g^{ab}\hat{\nabla}_a\hat{\nabla}_b V_{ij}-\tau u^c\hat{\nabla}_c(\mathscr{L}_{g,\gamma}V_{ij})\\&+\tau\partial_T u^c\hat{\nabla}_c(\mathscr{L}_{g,\gamma}V_{ij})-\partial_{\mathbf{u}}(\partial_T g^{ab})\hat{\nabla}_a\hat{\nabla}_b V_{ij}-u^0\partial_T g^{ab}\cdot\hat{\nabla}_a\hat{\nabla}_b\partial_T V_{ij}\\&+\tau(\partial_T g^{ab})u^c\hat{\nabla}_c\hat{\nabla}_a\hat{\nabla}_b V_{ij}.\end{aligned}\tag{3.8.59}$$

Using Sobolev embedding we find

$$\begin{aligned}\|[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}]V\|_{L^2}\lesssim&\Big(\|\partial_T\hat{u}^0\|_{H^2}\|\partial_T h\|_{H^2}+\|\partial_{\mathbf{u}}\partial_T(g^{-1})\|_{H^2}\Big)\|V\|_{H^2}\\&+|\tau|\Big(\|u\|_{H^2}+\|\partial_T u^c\|_{H^2}\Big)\|V\|_{H^3}+\Big(\|\partial_T\hat{u}^0\|_{H^2}+\|\partial_T g\|_{H^2}\Big)\|\partial_T V\|_{H^2}.\end{aligned}$$

We use (3.2.26) to schematically compute

$$\partial_{\mathbf{u}}\partial_T g^{-1}=\partial_{\mathbf{u}}(g^{-1}g^{-1}\partial_T h)=\partial_{\mathbf{u}}\Sigma+\partial_{\mathbf{u}}\widehat{N}+\partial_{\mathbf{u}}(F_h)+\text{h.o.t.}$$

Using Lemma 3.5.3, Corollary 3.5.1 and Corollary 3.5.4, we obtain

$$\|\partial_{\mathbf{u}}\partial_T(g^{-1})\|_{H^2}\lesssim\|\partial_{\mathbf{u}}\Sigma\|_{H^2}+\|\partial_{\mathbf{u}}\widehat{N}\|_{H^2}+\|\partial_{\mathbf{u}}(F_h)\|_{H^2}+\text{h.o.t.}\ \lesssim(E^g_{N-1})^{1/2}+\Lambda(T).$$

So, by applying the estimates from (3.5.32), (3.5.46) and Lemma 3.5.5,

$$\begin{aligned}\|[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}]V\|_{L^2}\lesssim&\Big(|\tau|\|u\|_{H^2}+|\tau|\|F_{u^a}\|_{H^2}+\Lambda(T)+(E^g_{N-1})^{1/2}\Big)\|V\|_{H^3}\\&+(E^g_{N-2}+\Lambda(T))\|\partial_T V\|_{H^2}\\\lesssim&\Big(|\tau|\|u\|_{H^2}+(E^g_{N-1})^{1/2}+\Lambda(T)\Big)\|V\|_{H^3}+(E^g_{N-2}+\Lambda(T))\|\partial_T V\|_{H^2}.\end{aligned}$$

Thus, from Lemma 3.5.5, Lemma 3.5.10 and Corollary 3.6.1,

$$\begin{aligned}\|[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}]\mathscr{L}^{\ell-1}_{g,\gamma}h\|_{L^2}\lesssim&\Big(|\tau|\|u\|_{H^2}+(E^g_{N-1})^{1/2}+\Lambda(T)\Big)\|g-\gamma\|_{H^N}\\&+(E^g_{N-2}+\Lambda(T))\|\partial_T h\|_{H^{N-1}}\\\lesssim&E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\Lambda(T).\end{aligned}$$

In addition, we note that at higher order $s\in\mathbb{Z}_{\geq1}$, we have the identity

$$\begin{aligned}[\partial_T,\partial_{\mathbf{u}}\mathscr{L}^s_{g,\gamma}]V_{ij}=&[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}](\mathscr{L}^{s-1}_{g,\gamma}V_{ij})+(\partial_{\mathbf{u}}g^{ab})\hat{\nabla}_a\hat{\nabla}_b([\partial_T,\mathscr{L}^{s-1}_{g,\gamma}]V_{ij})\\&+g^{ab}\partial_{\mathbf{u}}\hat{\nabla}_a\hat{\nabla}_b([\partial_T,\mathscr{L}^{s-1}_{g,\gamma}]V_{ij}).\end{aligned}\tag{3.8.65}$$

Using this and the commutator Lemma 3.5.10, we find

$$\begin{aligned}\|[\partial_T,\partial_{\mathbf{u}}\mathscr{L}^{\ell}_{g,\gamma}]h\|_{L^2}\lesssim&\|[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}](\mathscr{L}^{\ell-1}_{g,\gamma}h)\|_{L^2}+\|\partial_{\mathbf{u}}g\|_{H^2}\|[\partial_T,\mathscr{L}^{\ell-1}_{g,\gamma}]h\|_{H^2}\\&+\|g^{ab}\partial_{\mathbf{u}}\hat{\nabla}_a\hat{\nabla}_b([\partial_T,\mathscr{L}^{\ell-1}_{g,\gamma}]h)\|_{L^2}\\\lesssim&E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\Lambda(T)+\|g^{ab}\partial_{\mathbf{u}}\hat{\nabla}_a\hat{\nabla}_b([\partial_T,\mathscr{L}^{\ell-1}_{g,\gamma}]h)\|_{L^2}.\end{aligned}$$

So it remains to control this last term, which we do so using the commutator identity (3.5.64)

$$\begin{aligned}\|g^{ab}\partial_{\mathbf{u}}\hat{\nabla}_a\hat{\nabla}_b([\partial_T,\mathscr{L}_{g,\gamma}^{\ell-1}]h)\|_{L^2} &\lesssim \sum_{i=1}^{\ell-1}\|g^{ab}\partial_{\mathbf{u}}\hat{\nabla}_a\hat{\nabla}_b\Big(\mathscr{L}_{g,\gamma}^{i-1}(\partial_T g^{a_ib_i})\cdot\hat{\nabla}_{a_i}\hat{\nabla}_{b_i}(\mathscr{L}_{g,\gamma}^{\ell-1-i}(h))\Big)\|_{L^2}\\ &\lesssim \sum_{\substack{|I|+|J|\le 2\ell\\ |J|\ge 2}}\|\partial_{\mathbf{u}}(\hat{\nabla}^I\partial_T g)\hat{\nabla}^J h\|_{L^2}+\|\hat{\nabla}^I\partial_T g\cdot\partial_{\mathbf{u}}(\hat{\nabla}^J h)\|_{L^2}\\ &\lesssim E^g_{\partial_{\mathbf{u}},N-1}+E^g_{N-1}+\Lambda(T)^2,\end{aligned}$$

where in the final line we used (3.2.26) to replace $\partial_T g$ and then distributed derivatives and applied Lemma 3.5.5, Corollary 3.6.2 and Lemma 3.6.2 as needed.

The Σ estimate follows in exactly the same way. □

Proposition 3.8.6. *We have,*

$$|G_c|+|G_{cc}|\lesssim (E^g_{tot})^{3/2}+\Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.$$

Proof. There are three terms to estimate in G_c. By Corollary 3.5.1 and Lemma 3.8.1, the first term is easily estimated as

$$\begin{aligned}\Big|([\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell},\partial_T](h),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}}\Big| &\lesssim \|[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell},\partial_T](h)\|_{L^2_{g,\gamma}}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h)\|_{L^2_{g,\gamma}}\\ &\lesssim (E^g_{tot})^{3/2}+\Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2}.\end{aligned}$$

The terms appearing in G_{cc} are similarly easily estimated:

$$\begin{aligned}&\Big|([\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}](v),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h))_{L^2_{g,\gamma}}\Big|+\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v),[\partial_T,\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}](h))_{L^2_{g,\gamma}}\Big|\\ &\lesssim \|[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](\Sigma)\|_{L^2_{g,\gamma}}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(h)\|_{L^2_{g,\gamma}}+\|[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell},\partial_T](h)\|_{L^2_{g,\gamma}}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma)\|_{L^2_{g,\gamma}}\\ &\lesssim (E^g_{tot})^{3/2}+\Lambda(T)(E^g_{tot})^{1/2}+\Lambda(T)^2.\end{aligned}$$

For the second term of G_c, we integrate it by parts using (3.8.10) to obtain

$$\begin{aligned}(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v),[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](v))_{L^2_{g,\gamma}} &= (g^{ab}\hat{\nabla}_a\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v),\hat{\nabla}_b[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](v))_{L^2_{g,\gamma}}\\ &\quad -2(\mathrm{Riem}[\gamma]\circ\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v),[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](v))_{L^2_{g,\gamma}}\\ &\quad +([\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}(v),[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](v))_{L^2_{g,\gamma}}.\end{aligned}$$

So by the estimate (3.5.86), Corollary 3.5.3, Corollary 3.6.1 and Lemma 3.8.1,

$$\begin{aligned}&\Big|(\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}(v),[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1},\partial_T](v))_{L^2_{g,\gamma}}\Big|\\ &\lesssim (\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1}+\|[\partial_{\mathbf{u}},\mathscr{L}_{g,\gamma}]\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{L^2})\|[\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell},\partial_T](\Sigma)\|_{H^1}\\ &\lesssim (E^g_{tot})^{3/2}+\Lambda(T)(E^g_{tot})^{1/2}+\Lambda(T)^2.\end{aligned}$$

For the final term of G_c, namely

$$([\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell},\partial_T](\Sigma),\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}},$$

we need to integrate certain terms by parts since some of the factors contain too many derivatives on Σ. For example, using (3.8.59) and (3.8.65), we see that two such terms are

$$(\partial_T u^0 \mathscr{L}_{g,\gamma} \partial_T(\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}} + (\tau u^c \hat{\nabla}_c(\mathscr{L}_{g,\gamma}^{\ell}\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}}.$$

Using the integration by parts estimate (3.5.12b) and (3.5.46) we find

$$\begin{aligned}\left|(\partial_T u^0 \mathscr{L}_{g,\gamma} \partial_T(\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}}\right| &\lesssim \|\partial_T u^0\|_{H^3}\|\partial_T\Sigma\|_{H^{N-2}}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1} \\ &\lesssim \Lambda E^g_{tot} + (E^g_{tot})^2 + \Lambda(T)^3.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|(\tau u^c \hat{\nabla}_c(\mathscr{L}_{g,\gamma}^{\ell}\Sigma), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(\Sigma))_{L^2_{g,\gamma}}\right| &\lesssim |\tau|\|u\|_{H^3}\|\Sigma\|_{H^{N-1}}\|\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}\Sigma\|_{H^1} \\ &\lesssim |\tau|\|u\|_{H^{N-1}}E^g_{tot} + \Lambda(T)^2.\end{aligned}$$

All the remaining terms can be controlled using the ideas from before. We eventually obtain

$$\begin{aligned}\left|([\partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell}, \partial_T](v), \partial_{\mathbf{u}}\mathscr{L}_{g,\gamma}^{\ell-1}(v))_{L^2_{g,\gamma}}\right| &\lesssim \Big(|\tau|\|u\|_{H^{N-1}}+\Lambda(T)\Big)E^g_{tot} + \Lambda(T)(E^g_{\partial_{\mathbf{u}},N-1})^{1/2} \\ &\quad + (E^g_{tot})^{3/2} + \Lambda(T)^2.\end{aligned}$$

□

3.9 Energy estimates for the dust variables

In this section we establish energy estimates for the dust variables ρ and u^a in two propositions. An integration by parts identity, where due to symmetry a high-derivative term can be brought back onto the LHS, plays a crucial role in both propositions. We write the argument out explicitly for the first proposition, although note in principle the argument is just as in Lemma 3.5.2, (3.5.12c).

Definition 3.9.1 ($\mathscr{E}_k[\rho]$)**.** Define the following functionals

$$E_k[\rho](T) := \frac{1}{2}\int_M |\nabla^k\rho(T,\cdot)|^2\mu_g, \qquad \mathscr{E}_k[\rho](T) \;:=\; \sum_{0\le\ell\le k} E_\ell[\rho](T).$$

Proposition 3.9.1 (Time evolution of $\mathscr{E}_{N-2}[\rho]$ for fluid energy density)**.** *We have,*

$$|\partial_T\mathscr{E}_{N-2}[\rho](T)| \lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\mathscr{E}_{N-2}[\rho](T).$$

Proof. Recall from (3.2.22d) that the equation of motion for ρ is

$$\partial_T\rho = (u^0)^{-1}\rho F_\rho + \tau(u^0)^{-1}u^a\nabla_a\rho.$$

Let $3\le k\le N-2$. Using (3.3.14) and the integration by parts estimate (3.5.12b), we have

$$\begin{aligned}\partial_T E_k[\rho](T) &\lesssim \|\widehat{N}\|_\infty E_k[\rho](T) + \int_M \partial_T(|\nabla^k\rho|^2)\mu_g + \int_M X^c\hat{\nabla}_c(|\nabla^k\rho|^2)\mu_g \\ &\lesssim \Big(\|\widehat{N}\|_{H^2}+\|X\|_{H^3}\Big)E_k[\rho](T) \\ &\quad + \int_M \partial_T\Big(g^{a_1b_1}\cdots g^{a_kb_k}\Big)(\nabla^{a_1}\cdots\nabla^{a_k}\rho)(\nabla_{a_1}\cdots\nabla_{a_k}\rho)\mu_g \\ &\quad + \int_M g^{a_1b_1}\cdots g^{a_kb_k}\Big(\partial_T\nabla_{a_1}\cdots\nabla_{a_k}\rho\Big)(\nabla_{b_1}\cdots\nabla_{b_k}\rho)\mu_g \\ &\lesssim \Big(\|\widehat{N}\|_{H^2}+\|X\|_{H^3}+\|\Sigma\|_{H^2}\Big)E_k[\rho](T) + I_k\end{aligned}$$

where we have defined

$$I_k := \int_M g^{a_1 b_1} \cdots g^{a_k b_k} \Big(\partial_T \nabla_{a_1} \cdots \nabla_{a_k} \rho\Big)(\nabla_{b_1} \cdots \nabla_{b_k} \rho)\mu_g.$$

We focus on the integrand of I_k, commuting the derivatives to find

$$\partial_T \nabla_{a_1} \cdots \nabla_{a_k} \rho = \nabla_{a_1} \cdots \nabla_{a_k} \Big(\frac{\rho}{u^0} F_\rho + \tau \frac{u^i}{u^0} \nabla_i \rho\Big) + [\partial_T, \nabla_{a_1} \cdots \nabla_{a_k}]\rho.$$

We first look at the term $\tau u^i \nabla_i \rho$ which contains the most number of derivatives on ρ. By commuting and integration by parts, this term becomes:

$$\begin{aligned}
&\int_M \tau \frac{u^i}{u^0} (\nabla_{a_1} \cdots \nabla_{a_k} \nabla_i \rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \\
&\quad = \int_M \tau \frac{u^i}{u^0} ([\nabla_{a_1} \cdots \nabla_{a_k}, \nabla_i]\rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \\
&\quad\quad - \int_M \tau \nabla_i (\frac{u^i}{u^0})(\nabla_{a_1} \cdots \nabla_{a_k} \rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \\
&\quad\quad - \int_M \tau \frac{u^i}{u^0} (\nabla^{a_1} \cdots \nabla^{a_k} \rho)([\nabla_i, \nabla_{a_1} \cdots \nabla_{a_k}]\rho)\mu_g \\
&\quad\quad - \int_M \tau \frac{u^i}{u^0} (\nabla^{a_1} \cdots \nabla^{a_k} \rho)(\nabla_{a_1} \cdots \nabla_{a_k} \nabla_i \rho)\mu_g.
\end{aligned}$$

Rearranging, we find

$$\begin{aligned}
&\int_M \tau \frac{u^i}{u^0} (\nabla_{a_1} \cdots \nabla_{a_k} \nabla_i \rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \\
&\quad = \int_M \tau \frac{u^i}{u^0} ([\nabla_{a_1} \cdots \nabla_{a_k}, \nabla_i]\rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g - \frac{1}{2} \int_M \tau \nabla_i (\frac{u^i}{u^0}) |\nabla^k \rho|^2 \mu_g.
\end{aligned}$$

We see that we need to study two commutator error terms. Using (3.5.3), and noting that ρ is a scalar, we see the first error term takes the form

$$|[\partial_T, \nabla_{a_1} \cdots \nabla_{a_k}]\rho| \lesssim \sum_{\substack{|I|+|J|=k-1 \\ |J|\geq 1}} |\nabla^I (\partial_T \Gamma[g])||\nabla^J \rho|.$$

Thus, using (3.5.2),

$$\Big| \int_M ([\partial_T, \nabla_{a_1} \cdots \nabla_{a_k}]\rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \Big| \lesssim \|\rho\|_{H^k}^2 \|\partial_T \Gamma[g]\|_{H^{N-2}} \lesssim \varepsilon e^{-\lambda T} \mathscr{E}_k[\rho](T).$$

For the second commutator error term, we use (3.5.4) to find

$$|[\nabla_i, \nabla_{a_1} \cdots \nabla_{a_k}]\rho| \lesssim \sum_{\substack{|I|+|J|=k-1 \\ |J|\geq 1}} |\nabla^I \mathrm{Riem}||\nabla^J \rho|,$$

and so we have

$$\begin{aligned}
\Big| \int_M \tau u^i ([\nabla_{a_1} \cdots \nabla_{a_k}, \nabla_i]\rho)(\nabla^{a_1} \cdots \nabla^{a_k} \rho)\mu_g \Big| &\lesssim |\tau| \|u\|_{H^2} \|\mathrm{Riem}\|_{H^{k-2}} \|\rho\|_{H^k}^2 \\
&\lesssim \varepsilon e^{(-1+\mu)T} \mathscr{E}_k[\rho](T).
\end{aligned}$$

Putting this all together, and using Lemma 3.5.6, we obtain

$$
\begin{aligned}
|I_k| &\lesssim \|(u^0)^{-1}\rho\|_{H^k}\|\rho\|_{H^k}\|F_\rho\|_{H^k}+|\tau|\|\rho\|_{H^k}\|\nabla(\frac{u^i}{u^0})\cdot\nabla_i\rho\|_{H^{k-1}}+|\tau|\|\frac{u^i}{u^0}\|_{H^3}E_k[\rho](T)\\
&\quad+\varepsilon e^{\max\{-1+\mu,-\lambda\}T}\mathscr{E}_k[\rho](T).\\
&\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\mathscr{E}_k[\rho](T).
\end{aligned}
$$

By summing over k we can conclude

$$
|\partial_T\mathscr{E}_{N-2}[\rho](T)|\lesssim \varepsilon e^{\max\{-1+\mu,-\lambda\}T}\mathscr{E}_{N-2}[\rho](T).
$$

□

Definition 3.9.2 ($\mathscr{E}_k[u]$)**.** Define the functionals

$$
E_k[u](T) := \frac{1}{2}\int_M |\nabla^k u(T,\cdot)|^2\mu_g, \qquad \mathscr{E}_k[u](T) \ := \sum_{0\le\ell\le k} E_\ell[u](T).
$$

Proposition 3.9.2 (Evolution of $\mathscr{E}_{N-1}[u]$ fors spatial fluid velocity components)**.** *The following estimate holds,*

$$
|\partial_T\mathscr{E}_{N-1}[u](T)| \lesssim \varepsilon e^{(-1+\mu)T}\mathscr{E}_{N-1}[u](T)+|\tau|^{-1}\Lambda(T)(\mathscr{E}_{N-1}[u](T))^{1/2}.
$$

Proof. Let $3\le k\le N-1$. The proof is similar to the estimate for $\mathscr{E}_k[\rho](T)$. As in the proof of Proposition 3.9.1, we obtain

$$
\partial_T E_k[u](T) \lesssim \Big(\|\widehat{N}\|_{H^2}+\|X\|_{H^3}+\|\Sigma\|_{H^2}\Big)E_k[u](T)+I'_k,
$$

where we have defined

$$
I'_k := \int_M g_{ij}g^{a_1b_1}\cdots g^{a_kb_k}(\partial_T\nabla_{a_1}\cdots\nabla_{a_k}u^i)(\nabla_{b_1}\cdots\nabla_{b_k}u^j)\mu_g.
$$

The integration by parts analysis on I'_k follows unchanged to I_k. The main difference now arises in the commutator estimates since u^a is a vector while ρ is a scalar. For example, we now have an error term of the form

$$
|[\partial_T,\nabla_{a_1}\cdots\nabla_{a_k}]u^i|\lesssim \sum_{|I|+|J|=k-1}|\nabla^I(\partial_T\Gamma[g])||\nabla^J u^i|.
$$

Thus, using (3.5.2),

$$
\Big|\int_M([\partial_T,\nabla_{a_1}\cdots\nabla_{a_k}]u^i)(\nabla^{a_1}\cdots\nabla^{a_k}u_i)\mu_g\Big| \lesssim \|u^a\|^2_{H^k}\|\partial_T\Gamma(g)\|_{H^{N-2}}\lesssim \varepsilon e^{-\lambda T}\mathscr{E}_k[u^a](T).
$$

Similarly the second commutator error term looks like

$$
|[\nabla_i,\nabla_{a_1}\cdots\nabla_{a_k}]u^i|\lesssim \sum_{|I|+|J|=k-1}|\nabla^I\mathrm{Riem}||\nabla^J u^i|,
$$

and so we have

$$
\begin{aligned}
\Big|\int_M \tau u^a([\nabla_{a_1}\cdots\nabla_{a_k},\nabla_a]u^i)(\nabla^{a_1}\cdots\nabla^{a_k}u_i)\mu_g\Big| &\lesssim |\tau|\|u\|_{H^2}\|\mathrm{Riem}\|_{H^{k-1}}\|u\|^2_{H^k}\\
&\lesssim \varepsilon e^{(-1+\mu)T}\mathscr{E}_k[u](T).
\end{aligned}
$$

Recalling from (3.2.22d) the equation of motion

$$\partial_T u^j = \tau(u^0)^{-1} u^i \nabla_i u^j + \tau^{-1} u^0 N \nabla^j N + (u^0)^{-1} F_{u^j},$$

we obtain

$$\begin{aligned}|I_k'| &\lesssim |\tau| \|(u^0)^{-1} u^a\|_{H^3} E_k[u](T) + |\tau| \|u\|_{H^k} \|\nabla(\frac{u^i}{u^0}) \cdot \nabla_i u^a\|_{H^{k-1}} + |\tau|^{-1} \|\widehat{N}\|_{H^{k+1}} \|u\|_{H^k} \\ &\quad + \|(u^0)^{-1} F_{u^j}\|_{H^k} \|u\|_{H^k} + \varepsilon e^{\max\{-1+\mu, -\lambda\}T} E_k[u](T) \\ &\lesssim \varepsilon e^{(-1+\mu)T} E_k[u](T) + \Big(|\tau|^{-1} \|\widehat{N}\|_{H^{k+1}} + \|F_{u^j}\|_{H^k}\Big) E_k[u](T)^{1/2}.\end{aligned}$$

The conclusion then follows by summing over k and using the source estimates for F_{u^j} given in Lemma 3.5.6. □

3.10 Proof of Theorem 3.1.2

In this final section we bring together our main estimates to conclude the bootstrap argument. One standard, yet technical, aspect of the argument is deferred to Appendix 3.11.3.

Proof. Smallness of the initial data guarantees the existence of a constant $C_0 > 0$ such that at $T = T_0$

$$\begin{aligned}&\|\widehat{N}\|_{H^N} + \|X\|_{H^N} + \|\partial_T N\|_{H^{N-1}} + \|\partial_T X\|_{H^{N-1}} \\ &\quad + ({E^g}_{N-1}(T_0))^{1/2} + (\mathscr{E}^g_{\partial_{\mathbf{u}}, N-1}(T_0))^{1/2} + \mathscr{E}_{N-2}[\rho](T_0)^{1/2} + \mathscr{E}_{N-1}[u^a](T_0)^{1/2} \le C_0 \varepsilon.\end{aligned}$$

Proposition 3.9.1, together with Grönwall's inequality, implies

$$\|\rho\|_{H^{N-2}} \le \mathscr{E}_{N-2}[\rho](T_0)^{1/2} \exp\Big(\int_{T_0}^T C\varepsilon e^{\max\{-1+\mu, -\lambda\}s} \mathrm{d}s\Big)^{1/2} \le C_0 \varepsilon \exp(C'\varepsilon).$$

Thus by Lemma 3.7.1, and shrinking ε as needed,

$$\|\widehat{N}\|_{H^N} + \|X\|_{H^N} \le |\tau| C \|\rho\|_{H^{N-2}} + \varepsilon^2 C e^{\max\{-2\lambda, -2+\mu\}T} \le CC_0 \varepsilon e^{-T}.$$

Turning next to Lemma 3.7.2, we find

$$\begin{aligned}\|\partial_T N\|_{H^{N-1}} + \|\partial_T X\|_{H^{N-1}} &\le C \|\widehat{N}\|_{H^N} + C \|X\|_{H^N} + |\tau| C \|\rho\|_{H^{N-2}} + \varepsilon^2 C e^{\max\{-2\lambda, -2+\mu\}T} \\ &\le C_0 \varepsilon e^{-T}.\end{aligned}$$

Next, we divide the estimate in Proposition 3.9.2 by the square root of the energy, to obtain

$$|\partial_T \mathscr{E}_{N-1}[u](T)^{1/2}| \le C\varepsilon e^{(-1+\mu)T} \mathscr{E}_{N-1}[u](T)^{1/2} + CC_0 \varepsilon.$$

An application of Grönwall's inequality then produces

$$\begin{aligned}\|u\|_{H^{N-1}} &\le \Big(\mathscr{E}_{N-1}[u](T_0)^{1/2} + \int_{T_0}^T CC_0 \varepsilon \mathrm{d}s\Big) \exp\Big(\int_{T_0}^T C\varepsilon e^{(-1+\mu)s} \mathrm{d}s\Big) \\ &\le C(C_0 \varepsilon + C_0 \varepsilon (T - T_0)).\end{aligned}$$

Up to possibly redefining T_0, see the discussion around (3.11.12) in Appendix 3.11.3, these inequalities now imply

$$\Lambda(T) \leq CC_0\varepsilon e^{-T}.$$

Finally adding together the results of Proposition 3.8.1 and Theorem 3.8.1 and using the above improved estimates, we find

$$\partial_T E^g_{tot} \leq -2\alpha E^g_{tot} + CC_0\varepsilon e^{-1+\mu} E^g_{tot} + CC_0\varepsilon e^{-T}(E^g_{tot})^{1/2} + C(E^g_{tot})^{3/2} + C\Lambda(T)^2.$$

Thus, by the bootstrap argument presented in Appendix 3.11.3, we obtain

$$E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} \leq \frac{1}{2}C_1^2\varepsilon^2 e^{-2\lambda T},$$

where $C_1 \gg C_0$. Finally, as a consequence of Corollary 3.6.1,

$$\|g-\gamma\|^2_{H^N} + \|\Sigma\|^2_{H^{N-1}} \lesssim E^g_{\partial_{\mathbf{u}},N-1} + E^g_{N-1} + \Lambda(T)^2 \leq \frac{3}{4}C_1^2\varepsilon^2 e^{-2\lambda T}.$$

It remains to show that $\tilde{\rho} \geq 0$ given that initially $\tilde{\rho}_0 \geq 0$. We recall an argument given in [50]. We can rewrite the evolution equation for the energy density, to obtain

$$\tilde{u}^\alpha \bar{\nabla}_\alpha \tilde{\rho} + \tilde{\rho}\bar{\nabla}_\alpha \tilde{u}^\alpha = 0.$$

Moreover, the spacetime is ruled by the geodesics tangent to $\tilde{u}^\mu$. Along the geodesics positivity of $\tilde{\rho}$ or $\tilde{\rho} = 0$ is conserved due to the equation above and the regularity of $\tilde{u}^\mu$. Hence $\tilde{\rho} \geq 0$ on the future development holds. □

3.11 Appendix

3.11.1 Background geometry

In this appendix we derive the background solutions given in Remark 3.2.1. On the Milne background the ADM variables induced on a $t_c = const$ slice are

$$(\tilde{g}_{ab}, \tilde{k}_{ab}, \tilde{N}, \tilde{X}^a)|_B = (\frac{t_c^2}{9}\gamma_{ab}, -\frac{1}{t_c}\gamma_{ab}, 1, 0), \quad \tau = \tilde{g}^{ab}\tilde{k}_{ab} = -\frac{3}{t_c}.$$

Condition (3.2.2) implies $(\tilde{u}^{t_c})^2 = 1 + \frac{t_c^2}{9}\gamma(\tilde{u},\tilde{u})$. The fluid equations of motion in cosmological coordinates (t_c, x^i) read

$$\begin{aligned}
\tilde{u}^{t_c}\partial_{t_c}\ln\tilde{\rho} + \tilde{u}^i\partial_i\ln\tilde{\rho} + 3\partial_{t_c}\tilde{u}^0 + \partial_i\tilde{u}^i + 3t_c^{-1}\tilde{u}^{t_c} + \Gamma^i_{ij}[\gamma]\tilde{u}^j &= 0,\\
\tilde{u}^0\partial_{t_c}\tilde{u}^0 + \tilde{u}^i\partial_i\tilde{u}^0 - \frac{1}{t_c}\gamma(\tilde{u},\tilde{u}) &= 0,\\
\tilde{u}^{t_c}\partial_{t_c}\tilde{u}^i + \tilde{u}^i\partial_i\tilde{u}^i - 2\frac{t_c}{9}\tilde{u}^{t_c}\tilde{u}^i + \Gamma^i_{jk}[\gamma]\tilde{u}^j\tilde{u}^k &= 0.
\end{aligned}$$

Picking $\tilde{u}^{t_c} = 1, \tilde{u}^i = 0$ we have $\partial_{t_c}\ln\tilde{\rho} + 3t_c^{-1} = 0$. Thus on the background the fluid solution is

$$(\tilde{u}^{t_c}, \tilde{u}^i, \tilde{\rho})|_B = (1, 0, \tilde{\rho}_0 t_c^{-3})$$

where $\tilde{\rho}_0 > 0$ is a constant. By rescaling the variables as in Definition 3.2.2 we arrive at Remark 3.2.1.

3.11.2 Matter variables

In this appendix we discuss the rescaled components of the energy momentum tensor given in Definition 3.2.4. To see the origin of the various terms, we follow [2] and introduce the following notation for the matter variables

$$\tilde{E} := \tilde{T}^{\mu\nu} n_\mu n_\nu = \tilde{T}^{00}\tilde{N}^2, \qquad \tilde{\jmath}^a := -\tilde{T}^{\mu\nu} \perp^a{}_\mu n_\nu,$$
$$\tilde{\eta} := \tilde{E} + \tilde{g}^{ab}\tilde{T}_{ab}, \qquad \tilde{S}_{ab} := \tilde{T}_{ab} - \frac{1}{2}\mathrm{Tr}_{\bar{g}}\tilde{T}\cdot\tilde{g}_{ab}, \qquad \tilde{T}^{ab} := \tilde{\rho}\tilde{u}^a\tilde{u}^b\bar{g}^{ab}.$$

Using (3.2.1) we find

$$\begin{aligned}
\mathrm{Tr}_{\bar{g}}\tilde{T} &= \bar{g}^{\mu\nu}\tilde{T}_{\mu\nu} = -\tilde{\rho},\\
\tilde{T}_{ab} &= \bar{g}_{a\mu}\bar{g}_{b\nu}\tilde{T}^{\mu\nu} = \tilde{\rho}(\tilde{X}_a\tilde{u}^0 + \tilde{g}_{ai}\tilde{u}^i)(\tilde{X}_b\tilde{u}^0 + \tilde{g}_{bj}\tilde{u}^j),\\
\tilde{g}^{ab}\tilde{T}_{ab} &= \tilde{\rho}(\tilde{X}^a\tilde{X}_a(\tilde{u}^0)^2 + 2\tilde{X}_b\tilde{u}^b\tilde{u}^0 + \tilde{g}_{ab}\tilde{u}^a\tilde{u}^b).
\end{aligned}$$

A calculation then yields

$$\begin{aligned}
\tilde{E} &= |\tau|^3\rho(u^0)^2N^2, \qquad \tilde{\jmath}^a = |\tau|^5\rho u^0u^aN,\\
\tilde{\eta} &= |\tau|^3\rho(u^0)^2N^2 + |\tau|^3\rho\Big(X_aX^a(u^0)^2 + 2\tau X_bu^bu^0 + \tau^2 g_{ab}u^au^b\Big),\\
\tilde{S}_{ab} &= |\tau|\rho\Big(X_aX_b(u^0)^2 + \tau u^0u^cX_bg_{ac} + \tau u^0u^cX_ag_{bc} + \tau^2 g_{ac}g_{bd}u^cu^d\Big) + |\tau|\frac{\rho}{2}g_{ab},\\
\tilde{T}^{ab} &= |\tau|^7\rho u^au^b.
\end{aligned}$$

Finally, we introduce the following rescaled variables

$$\begin{aligned}
E &:= |\tau|^{-3}\tilde{E}, \qquad \jmath^a := |\tau|^{-5}\tilde{\jmath}^a,\\
\eta &:= |\tau|^{-3}\tilde{\eta}, \qquad S_{ab} := |\tau|^{-1}\tilde{S}_{ab}, \quad \underline{T}^{ab} := |\tau|^{-7}\tilde{T}^{ab},
\end{aligned}$$

which yield the expressions given in Definition 3.2.4.

3.11.3 Bootstrap

In this appendix we detail how the bootstrap assumption for $E^g_{tot}(= f)$ is closed. Let $\lambda < 1$ and $\mu \ll 1$ be fixed constants. Let $f : [T_0, T_*) \to [0,\infty)$ be a continuous function, where $T_0 \le T_* \le \infty$. Suppose $f(T_0) \le C_0^2\varepsilon^2$. Suppose also, that for each $T_0 \le T < T_*$ the assumption $f(T) \le C_1^2\varepsilon^2 e^{-2\lambda T}$ allows us to show

$$\partial_T f \le -2\alpha f + CC_0\varepsilon e^{(-1+\mu)T} f + CC_0\varepsilon e^{-T} f^{1/2} + Cf^{3/2} + CC_0^2\varepsilon^2 e^{-2T}.$$

Since $\alpha \in [1-\delta_\alpha, 1]$ we pick a ζ such that $\lambda < \zeta < 1$ and $\lambda < \alpha\zeta < \frac{1}{2}(1+\lambda)$.[1] Define also $\beta > 0$ to be the difference $\beta := \alpha\zeta - \lambda$. We then have,

$$\begin{aligned}
\partial_T(e^{2\alpha\zeta T}f) \le{}& -2\alpha(1-\zeta)e^{2\alpha\zeta T}f + CC_0\varepsilon e^{(-1+\mu+2\alpha\zeta)T}f + CC_0\varepsilon e^{(-1+2\alpha\zeta)T}f^{1/2}\\
&+ Cf^{3/2}e^{2\alpha\zeta T} + CC_0^2\varepsilon^2 e^{(-2+2\alpha\zeta)T}\\
\le{}& -\Big(2\alpha(1-\zeta) - CC_0\varepsilon e^{(-1+\mu)T} - Cf^{1/2}\Big)e^{2\alpha\zeta T}f\\
&+ CC_0\varepsilon e^{(-1+2\alpha\zeta)T}f^{1/2} + CC_0^2\varepsilon^2.
\end{aligned}$$

[1] For example if $\lambda = 0.75, \mu = 0.001$ we can choose ε sufficiently small that $\alpha = 0.98$ and then $\zeta = 0.8$ works.

Substituting in the bootstrap assumption, and choosing ε sufficiently small:

$$\begin{aligned}\partial_T(e^{2\alpha\zeta T}f) &\leq -\Big(2\alpha(1-\zeta) - CC_0\varepsilon - CC_1\varepsilon\Big)e^{2\alpha\zeta T}f + CC_0\varepsilon e^{(-1+2\alpha\zeta)T}f^{1/2} + CC_0^2\varepsilon^2 \\ &\leq CC_1C_0\varepsilon^2 e^{(-1+2\alpha\zeta-\lambda)T} + CC_0^2\varepsilon^2 \\ &\leq CC_1C_0\varepsilon^2.\end{aligned}$$

Thus, by Grönwall's inequality

$$e^{2\alpha\zeta T}f \leq C_0^2\varepsilon^2 + \int_{T_0}^{T} CC_1C_0\varepsilon^2 \mathrm{d}s \quad \Rightarrow f \leq (CC_0^2\varepsilon^2 + CC_0C_1\varepsilon^2(T-T_0))e^{-2\beta T}e^{-2\lambda T}.$$

Let T_0^* be such that for all $T \geq T_0^*$,

$$(CC_0^2 + CC_0C_1(T-T_0))e^{-2\beta T} < \frac{1}{2}C_1^2. \tag{3.11.12}$$

We then *a fortiori* choose $T_0 \geq T_0^*$. So we have shown that for all $T' \in [T_0, T_*)$, if $f(T) \leq C_1^2\varepsilon^2 e^{-2\lambda T}$ for each $T \in [T_0, T']$ then in fact $f(T) \leq \frac{1}{2}C_1^2\varepsilon^2 e^{-2\lambda T}$. Thus $f(T) \leq \frac{1}{2}C_1^2\varepsilon^2 e^{-2\lambda T}$ for all $T \in [T_0, T_*)$.

CHAPTER 4

THE STABILITY OF RELATIVISTIC FLUIDS IN LINEARLY EXPANDING COSMOLOGIES

Abstract. In this paper we study cosmological solutions to the Einstein–Euler equations. We first establish the future stability of nonlinear perturbations of a class of homogeneous solutions to the relativistic Euler equations on fixed linearly expanding cosmological spacetimes with a linear equation of state $p = K\rho$ for the parameter values $K \in (0, 1/3)$. This removes the restriction to irrotational perturbations in earlier work [41], and relies on a novel transformation of the fluid variables that is well-adapted to Fuchsian methods. We then apply this new transformation to show the global regularity and stability of the Milne spacetime under the coupled Einstein–Euler equations, again with a linear equation of state $p = K\rho$, $K \in (0, 1/3)$. Our proof requires a correction mechanism to account for the spatially curved geometry. In total, this is indicative that structure formation in cosmological fluid-filled spacetimes requires an epoch of decelerated expansion.

4.1 Introduction

4.1.1 The Einstein-Euler system

In this paper, we consider the Einstein-relativistic Euler equations

$$\mathrm{Ric}[\bar{g}]_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\mathrm{R}[\bar{g}] = \bar{T}_{\mu\nu}, \tag{4.1.1a}$$

$$\bar{\nabla}_\mu \bar{T}^{\mu\nu} = 0, \tag{4.1.1b}$$

$$\bar{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\bar{v}^\mu \bar{v}^\nu + \bar{p}\bar{g}^{\mu\nu}. \tag{4.1.1c}$$

Here $\bar{\rho}$ is the proper energy density of the fluid, $\bar{p}$ the fluid pressure and $\bar{v}^\mu$ is the fluid 4-velocity, which we assume is normalized by

$$\bar{g}_{\mu\nu}\bar{v}^\mu \bar{v}^\nu = -1. \tag{4.1.2}$$

The system (4.1.1) models a four-dimensional fluid-filled spacetime $(\mathscr{M}, \bar{g})$, whose time-evolution is determined by the coupled interaction between spacetime evolution under the Einstein equations (4.1.1a) and fluid propagation under the relativistic Euler equations (4.1.1b). The Einstein–relativistic Euler equations are particularly relevant in cosmology, where they are used to model the Universe on astrophysical scales (see e.g. [15; 84]). For simplicity, we henceforth drop "relativistic" when referring to the relativistic Euler equations.

From an analytical point of view, the PDE system (4.1.1) presents challenging problems since both the Einstein equations and the Euler equations may form singularities. Singularities arising in the spacetime can occur in the context of gravitational collapse to black holes or a Big Bang, while singularities in solutions of the Euler equations correspond to fluid shock formation [22].

In the cosmological context, singularity formation for the coupled Einstein-Euler equations can be interpreted as the onset of structure formation in the large scale evolution of spacetime [12]. Structure formation in cosmology describes the process by which regions with high matter densities emerge from an initially homogeneous matter distribution. Hence, it describes the formation of structures such as stars, galaxies and galaxy clusters from an initially homogeneous fluid.

4.1.2 Previous work

To close the system (4.1.1), we consider the linear, barotropic equation of state

$$\bar{p} = K\bar{\rho}. \tag{4.1.3}$$

The equation of state parameter K is a constant and gives the fluid speed of sound via $c_s = \sqrt{K}$. On physical grounds, $0 \leq K \leq 1$, however the most common applications in cosmology use $K \in [0, 1/3]$. The case $K = 0$ corresponds to a dust fluid, while $K = 1/3$ corresponds to a radiation fluid.

In the present article we consider cosmological spacetimes with Lorentzian metrics of the form

$$-d\bar{t}^2 + a(\bar{t})^2 g_0, \tag{4.1.4}$$

where (M, g_0) is a closed Riemannian 3-manifold without boundary. The increasing function $a(\bar{t})$ is the *scale factor*. We say the geometry exhibits *accelerated expansion* if $\ddot{a} > 0$, *linear expansion* if $a(\bar{t}) = \bar{t}$ and *decelerated expansion* if $\ddot{a} < 0$. Note that the direction of cosmological expansion corresponds to $a(\bar{t}) \to \infty$ as $\bar{t} \nearrow \infty$. If (4.1.4) solves the Einstein equations, then it is a solution to the Friedman equations describing an isotropic and (locally) homogeneous cosmology. A particularly important example for the present paper is the *Milne model* where $a(\bar{t}) = \bar{t}$ and (M, g_0) is a negative Einstein space satisfying $\mathrm{Ric}[g_0] = -\frac{2}{9} g_0$.

Fluid stabilisation from cosmological expansion. On a fixed Minkowski spacetime, small perturbations of a class of homogeneous solutions to the Euler equations (4.1.1b) are known to form shocks in finite time [22]. This result was shown for a large class of equations of state including (4.1.3). By contrast, on cosmological backgrounds such as (4.1.4), accelerated spacetime expansion is known to suppress shock formation in fluids. This was first discovered in the Newtonian cosmological setting for a class of exponentially expanding spacetimes in [14]. In the context of Einstein-Euler, stability of homogeneous solutions on exponentially expanding cosmological spacetimes was first studied in [96], with several later works [50; 55; 67; 68; 70; 75; 80; 81; 102]. For various other results on fluid stabilization in the regime of accelerated expansion, we refer to [63; 79; 108]. We mention also earlier work [93] concerning the stability of solutions to the Einstein equations undergoing accelerated expansion.

To go below accelerated expansion, i.e. when $\dot{a} > 0$ but $\ddot{a} \leq 0$, it is illuminating to first study the fluid behaviour on fixed metrics of the form (4.1.4). Fluid stabilization depends on the spacetime expansion rate, the fluid parameter K, and the geometry and topology of the expansion-normalized spatial geometry (M, g_0). Roughly speaking, larger speeds of sound and slower expansion rates tend to facilitate singularity formation, while slower speeds of sound and fast expansion rates suppress it.

To be concrete[1], consider homogeneous solutions to the Euler equations (4.1.1b) on fixed cosmological spacetimes undergoing power law inflation, i.e. $a(\bar{t}) = \bar{t}^{\alpha}$ for $\alpha > 0$ with (M, g_0) flat. Note that $\alpha > 1$ corresponds to accelerated expansion. In the case of dust $K = 0$, [103] showed that small perturbations of the homogeneous fluid solutions are globally regular for all $\alpha > 1/2$. If $K \in (0, 1/3)$, work by some of the present authors showed that homogeneous fluid solutions are globally regular under irrotational perturbations for $\alpha = 1$ [40]. The case $\alpha > 1$ without the irrotational restriction was shown in [103]. Finally, if $K = 1/3$, then [103] remarkably showed that radiation fluids do not stabilize for linear expansion i.e. the fluid stabilises if $\alpha > 1$, while shocks develop in finite time if $\alpha = 1$.

Moving next to the coupled Einstein-Euler equations (4.1.1), the only stability result below accelerated expansion is in the case of dust $K = 0$ and linear expansion. More precisely, some of the present authors showed the stability of the Milne model as a solution to the Einstein-dust equations [40]. We mention for context that the Milne model is known to be a stable solution solution to the Einstein vacuum equations [3] as well as a solution to several Einstein-matter models [2; 6; 13; 42; 107].

Note that the case of negative spatial curvature is the only known class of solutions where we can study the fluid dynamics with gravitational backreaction in the regime of linear expansion. For the other FLRW models the long-time dynamics are either recollapsing (in the case of positive spatial curvature) towards a big-crunch singularity or a matter-dominated decelerated expansion in the case of toroidal spatial topology with vanishing curvature [88]. Finally, we emphasise that *without* the backreaction of the fluid on the spacetime, [103] showed dust stability with *de*celerated expansion.

Notation: Our indexing convention is as follows: lower case Greek letters, e.g. μ, ν, γ, will label spacetime coordinate indices that run from 0 to 3 while lower case Latin letters, e.g. i, j, k, will label spatial coordinate indices that run from 1 to 3.

4.1.3 Results and technical advances in the present paper

This paper is roughly divided into three parts. In the first part, outlined in Section 4.1.3, we introduce a *novel transformation* that allows us to treat fluids with non-vanishing rotation. In the second part, see Section 4.1.3 below, we apply our transformation to show fluid

[1]We emphasise though that [103] considers more general conditions on the scale factor than we outline in this paragraph.

stability with $K \in (0, 1/3)$ on *linearly expanding spacetimes*:

Theorem 4.1.1. *The canonical homogeneous fluid solutions to* (4.1.1b) *with a linear equation of state* (4.1.3) *and* $K \in (0, 1/3)$ *are nonlinearly stable in the expanding direction of fixed linearly-expanding FLRW spacetimes of the form*

$$(\mathbb{R} \times \mathbb{T}^3, -d\bar{t}^2 + \bar{t}^2 \delta_{ij} dx^i dx^j).$$

The significance of Theorem 4.1.1 is that it removes the restriction to irrotational fluid perturbations in [41]. Finally in the third part, see Section 4.1.3 below, we consider a fully nonlinear problem including backreaction on spacetimes with negative spatial Einstein geometries. We prove the following theorem:

Theorem 4.1.2. *All four-dimensional FLRW spacetime models with compact spatial slices and negative spatial Einstein geometry are future stable solutions of the Einstein-Euler equations* (4.1.1) *with linear equation of state* (4.1.3) *and* $K \in (0, 1/3)$.

Transformation of the fluid variables. The key advance in this paper is a new version of the Fuchsian method, originally introduced in [41], which is designed to deal with general fluids without the irrotationality restriction. As first identified in [81], the Euler equations (4.1.1b) can be written as a symmetric hyperbolic Fuchsian PDE system as

$$B^0 \partial_t U + B^k \partial_k U = H.$$

Note that the time function t appearing here is related to the cosmological time function $\bar{t}$ by $t = 1/\bar{t}$, and hence timelike infinity is located at $t \to 0$. Precise definitions are given in Section 4.2, however the main idea is that the unknown $U = (\zeta, u^i)^{\mathrm{tr}}$ encodes the fluid energy density and velocity. We then transform to new unknowns $Z = (\psi, z^j)^{\mathrm{tr}}$ by

$$(\zeta, u^i) = (a(\psi, |z|_g^2), b(\psi) z^i).$$

The new variables obey the following PDE:

$$A^0 \partial_t Z + \frac{1}{t} A^k \mathscr{D}_k Z = H', \tag{4.1.5}$$

for H' is the source term as given in Lemma 4.2.4. By design, the functions a, b are chosen so that the matrices A^0, A^k satisfy the relations

$$|\mathbb{P} A^0 \mathbb{P}^\perp|_{\mathrm{op}} = |\mathbb{P}^\perp A^0 \mathbb{P}|_{\mathrm{op}} = \mathrm{O}(|\beta|_g + |z|_g^2), \qquad |\mathbb{P}^\perp A^k \mathbb{P}^\perp|_{\mathrm{op}} = \mathrm{O}(|\beta|_g + |z|_g^2). \tag{4.1.6}$$

In these expression β represents the shift vector of the dynamical metric and $\mathbb{P}$ and $\mathbb{P}^\perp$ are projections that single out specific matrix elements. For a fixed expanding FLRW background metric, or small perturbations thereof, the geometric quantities in H' are negligible and thus the source term can be written as

$$H' = c(K)\mathbb{P}Z + \mathrm{error}.$$

Here $c(K)$ is a strictly positive constant, thanks to our choice of equation of state and $K \in (0, 1/3)$, and the error is of order $|z|_g^2$. Extracting this structure on both sides of the equation (4.1.5) is crucial for the analysis of the system. For further details we refer to Remark 4.3.2.

Application to fluids on linearly expanding spacetimes. In Section 4.3 we apply our novel transformation to establish small data stability for the homogeneous solutions to the Euler equation on the fixed Milne-like (i.e. linearly expanding spatially flat) FLRW background. After the application of the transformation described above, we show, due to (4.1.6) as well as the decomposition of H', that the conditions of the Fuchsian global existence theorem from [41, Thm. 4.5] are satisfied. Note that Theorem 4.1.1 is also true for linearly expanding FLRW-type spacetimes $(\mathbb{R} \times M, -d\bar{t}^2 + \bar{t}^2 g_0)$, where (M, g_0) is an arbitrary closed Riemannian manifold. This extension is discussed in Remark 4.7.1. Finally, we expect that our method is robust enough to be applied to other spacetime geometries and expansion rates than considered here.

Application to Einstein-fluid spacetimes with backreaction. In Sections 4.4 - 4.6 we prove Theorem 4.1.2. The main task is to apply the aforementioned transformation to the setting of a dynamical spacetime geometry. This requires controlling various error terms based on suitable decay properties of the perturbations of the spacetime geometry.

A key challenge arises from the non-trivial spatial curvature of the expansion-normalized spatial metric. In particular, unlike the flat scenario considered in [41] and Theorem 4.1.1, the analysis leads to certain non-vanishing 'problematic' terms that arise from the commutation of curved covariant derivatives. For example, the lowest order problematic term is

$$\int_M \left(g^{mn}\nabla_m z^a[\nabla_n, \nabla_a]\psi + g^{mn}\nabla_m\psi[\nabla_n, \nabla_a]z^a\right).$$

Up to a negligible error term, this reduces to

$$\frac{2}{9}\int_M z^a\nabla_a\psi. \tag{4.1.7}$$

Such a nonlinearity cannot be absorbed into a generic error term due to the structure of the energy estimates. Schematically, the energy estimate looks like

$$\partial_T\left(\|\psi\|_{H^1(M)}+\|z\|_{H^1(M)}\right) \lesssim -C(K)\|z\|_{H^1(M)}+(4.1.7) + \text{error},$$

for some fixed $C(K) > 0$ involving the equation of state parameter K. The expression (4.1.7) is, however, only linear in z and hence it cannot be absorbed into the negative definite term.

To overcome this obstacle, we set up in Section 4.5.1 a correction mechanism for the original Fuchsian energy functionals for the fluid as given in Definition 4.4.9. This technique is usually applied to geometric wave equations and consists of adding an indefinite term to a standard geometric L^2-energy, which leads to a cancellation of the problematic terms in the energy equality while preserving coercivity of the energy (see e.g. [3]). For the first order, our method boils down to adding a **correction** $-\frac{1}{9}\int_M |z|^2$. Note that this correction term is of order zero, i.e. one order lower than the energy. We then need to take the time evolution of the correction term using the equations of motion (4.1.5). Properties of the matrix A^k and projection term $\mathbb{P}$ (given precisely in (4.4.16)) yield a term of the form

$$-\frac{2}{9}\int_M \langle \mathbb{P}Z, \mathbb{P}C\nabla Z\rangle = -\frac{2}{9}\int_M z^a\nabla_a\psi$$

in the energy estimates, which exactly cancels the problematic term (4.1.7). Note that in this case the coefficient in front of the correction is of small modulus and hence does not compromise the equivalence of the corrected energies to the original ones. For details on how to treat more complicated higher-order terms, see Proposition 4.5.2.

4.1.4 Conclusions

Given a spacetime of the form (4.1.4) with $a(\bar{t}) \approx \bar{t}^{\alpha}$, we define, in this paper, the threshold rate $\alpha_c(K)$ to be the critical rate above which fluid stabilization occurs and below which fluid singularities form from arbitrary small perturbations of homogeneous solutions. *A priori* the threshold rate also depends on the presence of gravitational backreaction and whether general fluids are considered or only the irrotational data. From the perspective of cosmology the threshold rate sets limits on the epoch of cosmological evolution where structure formation could have occurred.

The present paper shows that fluid stabilization occurs in the presence of gravitational backreaction in the regime of linear expansion for the full range $K \in (0, 1/3)$. Thus, together with [40], we obtain $\alpha_c(K) < 1$ for $K \in [0, 1/3)$. Since it is known in addition that fluids with $0 \leq K < \frac{1}{3}$ stabilize when the geometry undergoes accelerated expansion, we conclude that structure formation in a cosmological spacetime filled with non-radiation relativistic fluids requires an epoch of *de*celerated expansion.

Finally, we mention that for fluids with $0 \leq K < \frac{1}{3}$ the present paper completes the analysis for linear expansion rates in the neighborhood of the canonical Friedman models.

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4.2 The fluid transformation

4.2.1 The conformal Euler equations

Rather than working with the physical variables $\bar{g}_{\mu\nu}$ and $\bar{v}^{\mu}$, we begin by defining conformal variables.

Definition 4.2.1 (Conformal variables $g, v^{\mu}, \Gamma, \nabla$)**.** For Ψ an arbitrary scalar, define the *conformal metric* $g_{\mu\nu}$ and the *conformal four-velocity* v^{μ} by

$$\bar{g}_{\mu\nu} = e^{-2\Psi} g_{\mu\nu}, \quad v^{\mu} = e^{-\Psi} \bar{v}^{\mu}. \tag{4.2.1}$$

Let $\bar{\Gamma}^{\gamma}_{\mu\nu}$ and $\Gamma^{\gamma}_{\mu\nu}$ denote the Christoffel symbols of $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively, and let ∇_μ denote the Levi-Civita connection of $g_{\mu\nu}$.

Lemma 4.2.1 (The conformal Euler equations)**.** *The Euler equations* (4.1.1b) *can be rewritten in terms of conformal variables as*

$$v^\gamma \partial_\gamma \bar{\rho} + (\bar{\rho} + \bar{p}) L^\gamma_j \delta^j_\nu \nabla_\gamma v^\nu = 3(\bar{\rho} + \bar{p}) v^\gamma \partial_\gamma \Psi, \tag{4.2.2}$$

$$(\bar{\rho} + \bar{p}) M_{ij} \delta^j_\nu v^\gamma \nabla_\gamma v^\nu + L^\gamma_i \partial_\gamma \bar{p} = (\bar{\rho} + \bar{p}) L^\gamma_i \partial_\gamma \Psi, \tag{4.2.3}$$

where

$$L^\gamma_i := \delta^\gamma_i - \frac{v_i}{v_0}\delta^\gamma_0, \quad \textit{and} \quad M_{ij} := g_{ij} - \frac{v_i}{v_0} g_{0j} - \frac{v_j}{v_0} g_{0i} + \frac{v_i v_j}{v_0^2} g_{00}. \tag{4.2.4}$$

Proof. By (4.2.1) it is straightforward to verify that the Christoffel symbols are related via

$$\bar{\Gamma}^{\gamma}_{\mu\nu} - \Gamma^{\gamma}_{\mu\nu} = -g^{\gamma\lambda}(g_{\mu\lambda}\partial_\nu \Psi + g_{\nu\lambda}\partial_\mu \Psi - g_{\mu\nu}\partial_\lambda \Psi).$$

With the help of this relation, we can express the Euler equations (4.1.1b) as

$$\nabla_\mu \bar{T}^{\mu\nu} = 6\bar{T}^{\mu\nu}\partial_\mu \Psi - g_{\lambda\gamma}\bar{T}^{\lambda\gamma} g^{\mu\nu}\partial_\mu \Psi. \tag{4.2.5}$$

We refer to these equations as the *conformal Euler equations.* In [83], see, in particular, equations (2.36)-(2.37) and (2.40)-(2.43) from that article, it is shown that, in an arbitrary coordinate system (x^μ), the conformal Euler equations (4.2.5) reduce to (4.2.2)-(4.2.3). □

Definition 4.2.2 (The ADM decomposition of the conformal metric)**.** Let $(I \times \Sigma, g)$ be a Lorentzian spacetime where I is an interval in $\mathbb{R}$ and Σ is a three manifold. We assume that $t = x^0$ is a time function, that is it takes values in I, and that the level sets $\Sigma_\tau = t^{-1}(\tau)$, $\tau \in I$, are diffeomorphic to Σ. If n denotes the unit conormal to the spatial slices Σ_t, then we can express it as

$$n = -\alpha dt \quad \Longleftrightarrow \quad n_\mu = -\alpha \delta^0_\mu, \tag{4.2.6}$$

where the positive function α is known as the *lapse* and in the following we view it as a time-dependent scalar field on Σ. The *shift vector* $\beta = \beta^i \partial_i$, which we can view as a time-dependent vector field on Σ, is then determined via the expression

$$n^\sharp = \frac{1}{\alpha}(\partial_t - \beta) \quad \Longleftrightarrow \quad n^\mu = \frac{1}{\alpha}(\delta^\mu_0 - \beta^i \delta^\mu_i),$$

that characterises the difference between the covariant form $n^\sharp$ of the normal vector and ∂_t.

The $3+1$ decomposition of the conformal spacetime metric g is then given by

$$g = -\alpha^2 dt \otimes dt + \mathfrak{g}_{ij}(dx^i + \beta^i dt) \otimes (dx^j + \beta^j dt), \tag{4.2.7}$$

where $\mathfrak{g} = \mathfrak{g}_{ij} dx^i \otimes dx^j$ is the induced spatial metric on the spatial slices Σ_t, which we view as a time-dependent Riemmanian metric on Σ, while the shift and lapse are defined by $\beta_j = g_{j0}$ and $\alpha = (-g^{00})^{-\frac{1}{2}}$, respectively. We denote the Levi-Civita connection of $\mathfrak{g}_{ij} = g_{ij}$ by $\mathscr{D}_k$ and the Christoffel symbols γ^k_{ij}.

Spacetime co-vector fields can be projected to three dimensional co-vector fields using the operator

$$P^\mu_i = \delta^\mu_i, \tag{4.2.8}$$

which we note by (4.2.6) satisfies $P^\mu_i n_\mu = 0$. By our conventions, where the spacetime and spatial metrics are respectively used to raise and lower spacetime and spatial indices, we have from (4.2.8) that

$$P^i_\mu = g_{\mu\nu} g^{ij} P^\nu_j. \tag{4.2.9}$$

An explicit formula for this operator is then easily computed from (4.2.7) and (4.2.8) and is given by

$$P^i_\mu = \delta^i_\mu + \beta^i \delta^0_\mu. \tag{4.2.10}$$

It is also worth noting that the spacetime metric can be expressed using the operator (4.2.9) as $g_{\mu\nu} = g_{ij} P^i_\mu P^j_\nu - n_\mu n_\nu$ and that the identities $P^\mu_i P^j_\mu = \delta^j_i$ and $P^i_\mu n^\mu = 0$ hold.

Definition 4.2.3 (Christoffel components Υ^i and Ξ^i_j)**.** A calculation shows that the Christoffel symbols $\Gamma^\gamma_{\mu\nu}$ of the four dimensional conformal metric $g_{\mu\nu}$ are given by the following:

$$\Gamma^0_{00} = \frac{1}{\alpha}(\partial_t \alpha + \beta^j \mathscr{D}_j \alpha - \beta^i \beta^j \mathscr{K}_{ij}), \qquad \Gamma^0_{i0} = \frac{1}{\alpha}(\mathscr{D}_i \alpha - \beta^j \mathscr{K}_{ij}), \qquad \Gamma^0_{ij} = -\frac{1}{\alpha}\mathscr{K}_{ij}, \tag{4.2.11}$$

and

$$\begin{aligned} \Gamma^i_{00} = \Upsilon^i &:= \alpha \mathscr{D}^i \alpha - 2\alpha \beta^j \mathscr{K}^i_j - \frac{1}{\alpha}(\partial_t \alpha + \beta^j \mathscr{D}_j \alpha - \beta^j \beta^k \mathscr{K}_{jk})\beta^i + \partial_t \beta^i + \beta^j \mathscr{D}_j \beta^i, \\ \Gamma^i_{j0} = \Xi^i_j &:= -\frac{1}{\alpha}\beta^i(\mathscr{D}_j \alpha - \beta^k \mathscr{K}_{kj}) - \alpha \mathscr{K}^i_j + \mathscr{D}_j \beta^i, \end{aligned} \tag{4.2.12}$$

and

$$\Gamma^k_{ij} = \gamma^k_{ij} + \frac{1}{\alpha}\beta^k \mathscr{K}_{ij}, \tag{4.2.13}$$

where $\mathscr{K}_{ij} = -\frac{1}{2\alpha}(\partial_t g_{ij} - \mathscr{D}_i \beta_j - \mathscr{D}_j \beta_i)$ and $\mathscr{K}^j_i = g^{ik}\mathscr{K}_{ik}$.

Remark 4.2.1. It is worth noting from (4.2.11)-(4.2.13) that each of the groups of Christoffel symbols define a time-dependent tensor field on Σ except for the last one (4.2.13), which is not a tensor due to the appearance of the Christoffel symbols γ^k_{ij}.

4.2.2 The ADM decomposition of the conformal Euler equations

We first use the normal vector n_μ and the operator P^μ_i to decompose the four-velocity.

Definition 4.2.4 (Decomposition of conformal four-velocity ν, w_j, μ, u^j)**.** We define

$$\nu := -\frac{1}{\alpha} n_\nu v^\nu, \quad w_j := P^\mu_j v_\mu, \quad \mu := (\alpha n^\mu + \beta^i P^\mu_i) v_\mu \quad \text{and} \quad u^j := P^j_\mu v^\mu - \nu \beta^j. \tag{4.2.14}$$

On account of (4.2.6)-(4.2.8) and (4.2.9)-(4.2.10), we have

$$\nu = v^0, \quad w_j = v_j, \quad \mu = v_0 \quad \text{and} \quad u^j = v^j.$$

Furthermore, using this notation, we observe from (4.2.7) and (4.2.14) that M_{ij} can be expressed as

$$M_{ij} = g_{ij} - \frac{1}{\mu}(w_i\beta_j + w_j\beta_i) + \frac{-\alpha^2 + |\beta|_g^2}{\mu^2} w_i w_j. \tag{4.2.15}$$

We further observe that the fields (4.2.14) are not independent due to the relation $v_\mu = g_{\mu\nu}v^\nu$ and, on account of (4.1.2) and (4.2.1), the following normalization condition holds

$$g_{\mu\nu}v^\mu v^\nu = -1. \tag{4.2.16}$$

In fact, ν, w_j and μ can be expressed in terms of u^j. To see why this is the case, we observe from (4.2.14) and (4.2.9) that

$$u_j = P^\mu_j v_\mu - \nu\beta_j = w_j - \nu\beta_j$$

or equivalently

$$w_j = u_j + \nu\beta_j. \tag{4.2.17}$$

Using this, we then have by (4.2.14) that

$$\mu = \beta^i u_i + (|\beta|_g^2 - \alpha^2)\nu. \tag{4.2.18}$$

Additionally, by (4.2.7), (4.2.14) and (4.2.16), we have that

$$(-\alpha^2 + |\beta|_g^2)\nu^2 + 2\beta_j u^j \nu + 1 + |u|_g^2 = 0.$$

Solving this quadratic equation for ν, we find the two roots given by

$$\nu = \frac{1}{-\alpha^2 + |\beta|_g^2}\Big(-\beta_j u^j \pm \sqrt{(\beta_j u^j)^2 + (1 + |u|_g^2)(\alpha^2 - |\beta|_g^2)}\Big). \tag{4.2.19}$$

Remark 4.2.2. The choice of root, i.e. the choice of the $\pm$ sign, in (4.2.19) determines the orientation of the conformal four velocity v^μ. If we want v^μ to point in the direction of *increasing* t then we take the "$-$" sign. On the other hand, if we want v^μ to point in the direction of *decreasing* t, then we take the "+" sign. In either case, we always have that $\mu\nu < 0$, as can be verified from (4.2.18) and (4.2.19).

Lemma 4.2.2 (The conformal Euler equations in ADM variables)**.** *The conformal Euler equations* (4.2.2)-(4.2.3) *can be written as*

$$\begin{aligned} &\partial_t\bar{\rho} - \frac{(\bar{\rho}+\bar{p})}{\nu\mu} w_j \partial_t u^j + \frac{1}{\nu} u^k \mathscr{D}_k \bar{\rho} + \frac{(\bar{\rho}+\bar{p})}{\nu}\mathscr{D}_j u^j \\ &\quad = 3(\bar{\rho}+p)\Big(\partial_t\Psi + \frac{1}{\nu}u^k\mathscr{D}_k\Psi\Big) + (\bar{\rho}+\bar{p})\ell, \end{aligned} \tag{4.2.20}$$

$$\begin{aligned} &(\bar{\rho}+p)M_{ij}\partial_t u^j - \frac{K}{\nu\mu} w_i \partial_t\bar{\rho} + \frac{(\bar{\rho}+\bar{p})}{\nu} M_{ij} u^k \mathscr{D}_k u^j + \frac{K}{\nu}\mathscr{D}_i\bar{\rho} \\ &\quad = (\bar{\rho}+p)\Big(-\frac{1}{\nu\mu} w_i \partial_t\Psi + \frac{1}{\nu}\mathscr{D}_i\Psi\Big) + (\bar{\rho}+\bar{p})m_i, \end{aligned} \tag{4.2.21}$$

where, from the barotropic equation of state $\bar{p} = f(\bar{\rho})$, the square of the sound speed is

$$K = \frac{dp}{d\bar{\rho}} = f'(\bar{\rho}) \tag{4.2.22}$$

and we define

$$\ell := -\frac{1}{\nu\alpha}\mathscr{K}_{jk}\beta^j u^k - \Xi^j_j + \frac{1}{\mu}\Upsilon^j w_j + \frac{1}{\nu\mu}\Xi^j_k w_j u^k \tag{4.2.23}$$

and

$$m_i := -M_{ij}\Big(\frac{1}{\nu\alpha}\mathscr{K}_{lk}\beta^j u^l u^k + \nu\Upsilon^j + 2\Xi^j_k u^k\Big). \tag{4.2.24}$$

Proof. For an arbitrary scalar field h, we observe from (4.2.4) and (4.2.14) that

$$L^\gamma_i \partial_\gamma h = -\frac{1}{\mu}w_i\partial_t h + \mathscr{D}_i u. \tag{4.2.25}$$

We also observe using (4.2.4) and (4.2.14) that

$$\begin{aligned} L^\gamma_j \delta^j_\nu \nabla_\gamma v^\nu &= L^\gamma_j \delta^j_\nu(\partial_\gamma v^\nu + \Gamma^\nu_{\gamma\sigma}v^\sigma) \\ &= -\frac{1}{\mu}w_j\partial_t u^j + \partial_j u^j + \Gamma^j_{j0}\nu + \Gamma^j_{jk}u^k - \frac{1}{\mu}w_j\Gamma^j_{00}\nu - \frac{1}{\mu}w_j\Gamma^j_{0k}u^k. \end{aligned}$$

Then, with the help of (4.2.11), (4.2.12) and (4.2.13), it follows that we can write the above expression as

$$L^\gamma_j \delta^j_\nu \nabla_\gamma v^\nu = -\frac{1}{\mu}w_j\partial_t u^j + \mathscr{D}_j u^j + \frac{1}{\alpha}\mathscr{K}_{jk}\beta^j u^k + \Xi^j_j\nu - \frac{\nu}{\mu}\Upsilon^j w_j - \frac{1}{\mu}\Xi^j_k w_j u^k. \tag{4.2.26}$$

Using a similar calculation, it is also not difficult to verify that

$$\delta^j_\nu v^\gamma \nabla_\gamma v^\nu = \nu\partial_t u^j + u^k\mathscr{D}_k u^j + \frac{1}{\alpha}\mathscr{K}_{ik}\beta^j u^i u^k + \nu^2\Upsilon^j + 2\nu\Xi^j_i u^i. \tag{4.2.27}$$

The identities (4.2.25), (4.2.26) and (4.2.27) then allows us to obtain the required form of the conformal Euler equations (4.2.2)-(4.2.3). □

To proceed, we introduce a modified density variable ζ defined by subtracting 3Ψ from the fluid enthalpy.

Definition 4.2.5 (Modified fluid density variable ζ)**.** For a given barotropic equation of state $\bar{p} = f(\bar{\rho})$, the *modified fluid density* is defined by

$$\zeta := \int_{\bar{\rho}_0}^{\bar{\rho}} \frac{d\xi}{\xi + f(\xi)} - 3\Psi, \tag{4.2.28}$$

where $\bar{\rho}_0$ is any positive constant.

Remark 4.2.3. For the linear equation of state (4.1.3), we note that square of the sound speed K is constant and lies in $[0, 1]$, by assumption, while the the modified fluid density is given by

$$\zeta = \int_{\bar{\rho}_0}^{\bar{\rho}} \frac{d\xi}{(1+K)\xi} - 3\Psi = \frac{1}{1+K}\ln\Big(\frac{\bar{\rho}}{\bar{\rho}_0}\Big) - 3\Psi.$$

Lemma 4.2.3. *The conformal Euler equations* (4.2.20)-(4.2.21) *can be written as*

$$B^0 \partial_t U + B^k \mathscr{D}_k U = H, \tag{4.2.29}$$

where $U = (\zeta, u^j)^{\mathrm{tr}}$*, and the matrices* B^0, B^k, H *are given by*

$$\begin{aligned} B^0 &= \begin{pmatrix} K & -\frac{K}{\nu\mu} w_j \\ -\frac{K}{\nu\mu} w_i & M_{ij} \end{pmatrix}, \quad B^k = \begin{pmatrix} \frac{K}{\nu} u^k & \frac{K}{\nu}\delta^k_j \\ \frac{K}{\nu}\delta^k_i & \frac{1}{\nu} M_{ij} u^k \end{pmatrix}, \\ \text{and} \quad H &= \begin{pmatrix} K\ell \\ \left(\frac{3K-1}{\nu\mu} w_i \partial_t \Psi + \frac{1-3K}{\nu} \mathscr{D}_i \Psi\right) + m_i \end{pmatrix}. \end{aligned} \tag{4.2.30}$$

Proof. Differentiating the expression (4.2.28), we find that

$$\partial_t \zeta = \frac{\partial_t \bar{\rho}}{\bar{\rho} + \bar{p}} - 3\partial_t \Psi \quad \text{and} \quad \mathscr{D}_k \zeta = \frac{\mathscr{D}_k \bar{\rho}}{\bar{\rho} + \bar{p}} - 3\mathscr{D}_k \Psi.$$

With the help of these expressions, a short calculation shows that the conformal Euler equations (4.2.20)-(4.2.21) when expressed in terms of the modified density ζ become the system given in (4.2.29). □

Remark 4.2.4. It is worth noting that (4.2.29) is manifestly symmetric hyperbolic, and consequently represents a symmetric hyperbolic formulation of the conformal Euler equations. The symmetric hyperbolic nature of this formulation guarantees the existence of local-in-time solutions by standard existence results for symmetric hyperbolic systems of equations, e.g. see Propositions 1.4, 1.5 and 2.1 from [106, Chapter 16]. While this at least yields the existence of local solutions, there is still the question of whether or not global solutions exist.

In order to address the long-time existence of solutions, we need to bring the system (4.2.29) into a more favourable form. This will be accomplished by the following change of variables.

Definition 4.2.6 (Fluid transformation)**.** Define $Z = (\psi, z^j)^{\mathrm{tr}}$ and a transformation given by

$$U = (\zeta, u^i)^{\mathrm{tr}} =: (a(\psi, |z|^2_g), b(\psi) z^i)^{\mathrm{tr}}, \tag{4.2.31}$$

where $a(\cdot,\cdot)$ and $b(\cdot)$ are, for the moment, arbitrary functions.

Lemma 4.2.4. *If we choose the functions* $a(\psi, |z|^2_g)$ *and* $b(\psi)$ *as*

$$a(\psi, |z|^2_g) = c_1 - \ln(4) + \ln((\psi + c_2)^2) + \frac{\kappa |z|^2_g}{(\psi + c_2)^2}, \qquad \text{and} \qquad b(\psi) = \frac{2}{\psi + c_2},$$

where $c_{1,2}$ *are arbitrary constants and* $\kappa := K^{-1} - 2$*, then the equations given in* (4.2.29) *take form*

$$A^0 \partial_t Z + \frac{1}{\nu} A^k \mathscr{D}_k Z = Q^{\mathrm{tr}}(H - B^0 Y), \tag{4.2.32}$$

and the following conditions hold:

$$A^0_j = \mathrm{O}(|\beta|_g + |z|^2_g), \qquad A^k_0 = \mathrm{O}(|\beta|_g + |z|^2_g), \tag{4.2.33}$$

and

$$\frac{D_1 a}{b} = c_0 + \mathrm{O}(|\beta|_{g} + |z|_{g}^2), \tag{4.2.34}$$

for c_0 a non-zero constant, where

$$A^0 = \begin{pmatrix} A^0_0 & A^0_j \\ A^0_i & A^0_{ij} \end{pmatrix} \quad \text{and} \quad A^k = \begin{pmatrix} A^k_0 & A^k_j \\ A^k_i & A^k_{ij} \end{pmatrix}.$$

Remark 4.2.5. Note that the explicit form of the functions a and b given in Lemma 4.2.4 are smooth in a neighborhood of $(\psi, |z|_{g}^2) = (0,0)$ for the choice of $c_2 \neq 0$.

Proof. Differentiating U, we find after a short calculation that

$$\partial_t U = Q\partial_t Z + Y \quad \text{and} \quad \mathscr{D}_k U = Q\mathscr{D}_k U, \tag{4.2.35}$$

where

$$Q = \begin{pmatrix} D_1 a & 2D_2 a z_j \\ b' z^i & b\delta^i_j \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} D_2 a \partial_t g_{ij} z^i z^j \\ 0 \end{pmatrix}. \tag{4.2.36}$$

Here $D_1 a$ and $D_2 a$ denote the partial derivatives with respect to the first and second variables of $a = a(\psi, |z|_{g}^2)$. Using (4.2.35) and the fact that $\mathscr{D}_k g_{ij} = 0$, it is then clear that we can write (4.2.29) as

$$A^0 \partial_t Z + \frac{1}{\nu} A^k \mathscr{D}_k Z = Q^{\mathrm{tr}}(H - B^0 Y),$$

where

$$A^0 = Q^{\mathrm{tr}} B^0 Q, \quad A^k = Q^{\mathrm{tr}} A^k Q, \quad \text{and} \quad Q^{\mathrm{tr}} = \begin{pmatrix} D_1 a & b' z^j \\ 2D_2 a z_i & b\delta^i_j \end{pmatrix}.$$

We then have by (4.2.30), (4.2.36) and (4.2.2), that

$$A^0_0 = K(D_1 a)^2 - \frac{2Kb' D_1 a}{\nu\mu} z^k w_k + (b')^2 M_{kl} z^k z^l, \tag{4.2.37}$$

$$A^0_j = 2K D_1 a D_2 a z_j - 2\frac{Kb'}{\nu\mu} D_2 a z_j z^k w_k - \frac{Kb D_1 a}{\nu\mu} w_j + bb' z^k M_{kj},$$

$$A^0_{ij} = b^2 M_{ij} - \frac{2Kb D_2 a}{\nu\mu}(z_i w_j + z_j w_i) + 4K(D_2 a)^2 z_i z_j,$$

$$A^k_0 = K(D_1 a)^2 u^k + 2Kb' D_1 a z^k + (b')^2 M_{lm} z^l z^m u^k, \tag{4.2.38}$$

$$A^k_j = Kb D_1 a \delta^k_j + 2K D_1 a D_2 a u^k z_j + 2Kb' D_2 a z^k z_j + bb' M_{lj} z^l u^k,$$

and

$$A^k_{ij} = b^2 M_{ij} u^k + 2Kb D_2 a(\delta^k_j z_i + \delta^k_i z_j) + 4K(D_2 a)^2 u^k z_i z_j. \tag{4.2.39}$$

Now, our goal is to try and choose a and b so that (4.2.33)-(4.2.34) hold. By (4.2.19) and (4.2.31), we observe that

$$\nu = -\frac{1}{\alpha} + \mathrm{O}(|\beta|_{g} + |z|_{g}^2),$$

which, in turn, implies by (4.2.18) that

$$\mu = \alpha + \mathrm{O}(|\beta|_g + |z|_g^2).$$

Using these, we then see from (4.2.15), (4.2.17) and (4.2.31) that

$$A_j^0 = (KD_1a(2D_2a + b^2) + bb')z_j + \mathrm{O}(|\beta|_g + |z|_g^2). \tag{4.2.40}$$

We see also from (4.2.31) and (4.2.38) that

$$A_0^k = KD_1a(bD_1a + 2b')z^k + \mathrm{O}(|z|_g^2). \tag{4.2.41}$$

By (4.2.40) and (4.2.41), we then see that if a and b are chosen so that

$$bD_1a + 2b' = \mathrm{O}(|z|_g^2), \tag{4.2.42}$$

$$KD_1a(2D_2a + b^2) + bb' = \mathrm{O}(|z|_g^2), \tag{4.2.43}$$

then the conditions (4.2.33)-(4.2.34) are satisfied. Now, writing (4.2.42) as

$$\partial_\psi(a(\psi, |z|_g^2) + \ln(b(\psi)^2)) = \mathrm{O}(|z|_g^2),$$

it is clear that we can ensure that this condition holds by choosing a so that

$$a(\psi, |z|_g^2) = c_1 - \ln(b(\psi)^2) + c(\psi)|z|_g^2, \tag{4.2.44}$$

where $c(\psi)$ is an arbitrary function and c_1 is an arbitrary constant. Inserting this into (4.2.43) and multiplying through by $\frac{b}{b'}$ yields

$$-2K(2c + b^2) + b^2 = \mathrm{O}(|z|_g^2),$$

where here we are viewing K as a function of ψ via[2] (4.2.22), (4.2.28) and (4.2.31). It is then clear that we can guarantee that this condition holds by setting

$$c = \frac{1}{4}\kappa b^2, \quad \kappa = K^{-1} - 2.$$

Now, (4.2.44) implies that

$$a = c_1 - \ln(b^2) + \frac{1}{4}\kappa b^2 |z|_g^2. \tag{4.2.45}$$

Substituting this into (4.2.34), it follows that (4.2.34) will hold provided

$$-2\frac{b'}{b^2} = c_0.$$

Solving this yields

$$b = \frac{2}{c_0\psi + c_2},$$

where c_2 is an arbitrary constant. We then fix the constant c_0 by setting $c_0 = 1$. By (4.2.45), this gives

$$a(\psi, |z|_g^2) = c_1 - \ln(4) + \ln((\psi + c_2)^2) + \frac{\kappa |z|_g^2}{(\psi + c_2)^2}, \quad \text{and} \quad b(\psi) = \frac{2}{\psi + c_2}.$$

By construction, the conditions (4.2.33)-(4.2.34) with $c_0 = 1$ hold. Note that this choice gives

$$D_1a = \frac{2}{\psi + c_2}\Big(1 - \kappa\frac{|z|_g^2}{(\psi + c_2)^2}\Big), \quad D_2a = \frac{\kappa}{(\psi + c_2)^2}, \quad b'(\psi) = \frac{-2}{(\psi + c_2)^2}. \tag{4.2.46}$$

□

[2] It will also, of course be a function of Ψ, but we can for the purpose of this argument treat it as a "constant".

We now apply the transformation of Section 4.2 to the FLRW-type geometries considered in [41] with a linear equation of state (4.1.3).

Definition 4.3.1 (Milne-like FLRW spacetimes)**.** Consider the manifold $\mathbb{R}_{>0} \times \mathbb{T}^3$ with the spacetime metric

$$\bar{g} = \frac{1}{t^2} g, \quad \text{where} \quad g = -\frac{1}{t^2} dt \otimes dt + \delta_{ij} dx^i \otimes dx^j. \tag{4.3.1}$$

We refer to the pair $(\mathbb{R}_{>0} \times \mathbb{T}^3, \bar{g})$ as a *Milne-like FLRW spacetime.*

Note that, with respect to the coordinates used in (4.3.1), the future is located in the direction of decreasing t and future timelike infinity is located at $t = 0$. If we choose $\Psi = \ln(t)$, then the Milne-like FLRW metric $\bar{g}$ is of the form (4.2.1) where the confomal metric is given by g as written in (4.3.1). Moreover, the $3+1$ decomposition of the conformal metric g in (4.3.1) yields

$$\alpha = t^{-1}, \quad \beta_j = 0, \quad g_{ij} = \delta_{ij}. \tag{4.3.2}$$

The spatial manifold on which these fields are defined is $\mathbb{T}^3$.

Lemma 4.3.1. *On the Milne-like FLRW spacetime we have the simplifications*

$$\begin{aligned} &\nu = v^0, \ w_j = v_j, \ \mu = -t^{-2}\nu, \ u^j = v^j, \\ &M_{ij} = \delta_{ij} - \Big(\frac{t}{\nu}\Big)^2 v_i v_j = \delta_{ij} - \Big(\frac{t}{\nu}\Big)^2 b^2 z_i z_j, \quad \text{and} \quad H = \begin{pmatrix} 0 \\ \frac{3K-1}{t\nu\mu} b z_j \end{pmatrix}, \end{aligned} \tag{4.3.3}$$

where we defined $z_i = \delta_{ij} z^j$*. Moreover,* $\mathscr{K}_{ij}, \Upsilon^i, \Xi^i_j, m_i, \ell, Y$ *all identically vanish.*

Proof. This is a straightforward computation using (4.3.2) in the appropriate definitions, and lowering and raising indices with (4.3.2). □

Lemma 4.3.2 (Homogeneous solutions)**.** *The homogeneous background solutions of the Euler equations* (4.1.1b) *on a fixed Milne-like FLRW spacetime* $(\mathbb{R}_{>0} \times \mathbb{T}^3, \bar{g})$ *are given by* $U \equiv 0$*. Furthermore, this reduces to* $Z \equiv 0$ *and on these background solutions we have the simplifications*

$$D_1 a = (2/c_2), \quad D_2 a = \kappa/c_2^2, \quad b' = -2/c_2^2, \quad \nu\mu = -1, \quad \text{and} \quad t/\nu = -1\,. \tag{4.3.4}$$

Proof. The homogeneous solutions considered in [41] are given by

$$(\bar{v}^\mu_H, \bar{\rho}_H) = (-t^2 \delta^\mu_0, ((1-3K)c_H)^{\frac{1+K}{K}} t^{3(1+K)}),$$

where $c_H > 0$ is a constant. On such a homogeneous solution we have

$$v^\mu_H = -t\delta^\mu_0, \quad \zeta_H = 0\,,$$

provided we pick $\rho_0 = ((1-3K)c_H)^{\frac{1+K}{K}}$. This implies that $U \equiv 0$ and $Z = (C_H, 0)$ for some constant C_H. We can in fact set $C_H = 0$ by picking $c_2 = -2\exp(-c_1/2)$.

Inspecting the statements given in (4.2.46) and simply substituting in $Z \equiv 0$ yields the desired result for $D_1 a$, $D_2 a$ and b' in (4.3.4). By Lemma 4.3.1 and using $v^0 = -t$ on the homogeneous solutions, we have that $t/\nu = t/v^0 = -1$ as well as

$$\nu\mu = -t^{-2}\nu^2 = -t^{-2}(v^0)^2 = -1.$$

□

Lemma 4.3.3. *The Euler equations on a fixed Milne-like FLRW spacetime take the form*

$$M_0\partial_t Z + \frac{1}{t}(\mathscr{C}^k + M^k(Z))\partial_k Z = \frac{1}{t}\mathscr{B}(Z)\Pi Z + \frac{1}{t}F(Z)\,, \tag{4.3.5}$$

where

$$\begin{aligned}
M_0(Z) &= \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}\delta_{ij} \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{O}(|z|_g^2) \\ \mathrm{O}(|z|_g^2) & \mathrm{O}(|z|_g^2) \end{pmatrix}, \\
M^k(Z) &= \frac{t}{\nu}\begin{pmatrix} 0 & 0 \\ 0 & K^{-1}\big(\frac{2}{\psi+c_2}\big)\delta_{ij}z^k + \frac{\kappa}{(\psi+c_2)}(\delta_j^k z_i + \delta_i^k z_j) \end{pmatrix} + \mathrm{O}(|z|_g^2)\,, \\
\mathscr{C}^k &= \begin{pmatrix} 0 & \delta_j^k \\ \delta_i^k & 0 \end{pmatrix}, \quad F(Z) = \mathrm{O}(|z|_g^2)\,,
\end{aligned} \tag{4.3.6}$$

and

$$\mathscr{B} := (K^{-1} - 3)\mathbb{1}\,, \qquad \Pi := \begin{pmatrix} 0 & 0 \\ 0 & \delta_j^i \end{pmatrix}, \qquad \Pi^\perp := \mathbb{1} - \Pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 4.3.1. Note that the non-leading order term of the upper left component of M^0 is zero and hence $D\mathbb{P}M^0\mathbb{P}$ for any derivative D.

Proof. From the PDE system (4.2.32), we define

$$\begin{aligned}
M_0(Z) &:= (A_0^0)^{-1}\begin{pmatrix} A_0^0 & A_j^0 \\ A_i^0 & A_{ij}^0 \end{pmatrix}, \quad \mathscr{C}^k := (A_0^0)^{-1}\begin{pmatrix} A_0^k & A_j^k \\ A_i^k & A_{ij}^k \end{pmatrix}\Big|_{z^i\equiv 0}, \\
M^k(Z) &:= \frac{t}{\nu}(A_0^0)^{-1}\begin{pmatrix} A_0^k & A_j^k \\ A_i^k & A_{ij}^k \end{pmatrix} - \mathscr{C}^k.
\end{aligned}$$

Using (4.2.37)-(4.2.39), we compute

$$\begin{aligned}
A_0^0 &= \Big(\frac{2\sqrt{K}}{\psi+c_2}\Big)^2 + \mathrm{O}(|z|_g^2), \quad A_j^0 = \mathrm{O}(|z|_g^2), \quad A_{ij}^0 = \Big(\frac{2}{\psi+c_2}\Big)^2\delta_{ij} + \mathrm{O}(|z|_g^2), \\
A_0^k &= \mathrm{O}(|z|_g^2), \quad A_j^k = \Big(\frac{2\sqrt{K}}{\psi+c_2}\Big)^2\delta_j^k + \mathrm{O}(|z|_g^2), \quad \text{and} \\
A_{ij}^k &= \Big(\frac{2}{\psi+c_2}\Big)^3\delta_{ij}z^k + \Big(\frac{2\sqrt{K}}{\psi+c_2}\Big)^2\frac{\kappa}{(\psi+c_2)}(\delta_j^k z_i + \delta_i^k z_j) + \mathrm{O}(|z|_g^2).
\end{aligned}$$

Multiplying the system (4.2.32) from the left by $(A_0^0)^{-1}$ yields

$$(A_0^0)^{-1}A^0\partial_t Z + \frac{1}{t}\frac{t}{\nu}(A_0^0)^{-1}A^k\partial_k Z = (A_0^0)^{-1}Q^{\mathrm{tr}}(H - B^0 Y).$$

Using that $(A_0^0)^{-1} = (\frac{2\sqrt{K}}{\psi+c_2})^{-2} + \mathrm{O}(|z|_g^2)$, we obtain

$$\begin{aligned} M_0(Z) &= \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}\delta_{ij} \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{O}(|z|_g^2) \\ \mathrm{O}(|z|_g^2) & \mathrm{O}(|z|_g^2) \end{pmatrix}, \quad \mathscr{C}^k = \begin{pmatrix} 0 & \delta_j^k \\ \delta_i^k & 0 \end{pmatrix}, \\ M^k(Z) &= \frac{t}{\nu}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{K}\Big(\frac{2}{\psi+c_2}\Big)\delta_{ij}z^k + \frac{\kappa}{(\psi+c_2)}(\delta_j^k z_i + \delta_i^k z_j) \end{pmatrix} + \mathrm{O}(|z|_g^2). \end{aligned} \tag{4.3.7}$$

Finally, for the right hand side of the equation we use (4.2.2), (4.3.3) and (4.3.4) to compute

$$(A_0^0)^{-1}Q^{\mathrm{tr}}(H-B^0Y) = -\frac{1}{t}(1-3K)(\nu\mu)^{-1}b(A_0^0)^{-1}\begin{pmatrix} b'z^j z_j \\ bz_i \end{pmatrix} = \frac{1}{t}(1-3K)K^{-1}\begin{pmatrix} 0 \\ z_i \end{pmatrix} + \frac{1}{t}F(Z),$$

where

$$F(Z) := t(A_0^0)^{-1}Q^{\mathrm{tr}}H - \mathscr{B}(Z)\cdot\Pi Z = -\frac{(K^{-1}-3)}{(\psi+c_2)}\begin{pmatrix} z^j z_j \\ 0 \end{pmatrix} + \mathrm{O}(|z|_g^2) = \mathrm{O}(|z|_g^2). \tag{4.3.8}$$

□

Definition 4.3.2 (Matrix norm)**.** For any $M \in \mathbb{R}^{d\times d}$ we write $|M|_{\mathrm{op}} = \sup\{|Mv| \colon v \in \mathbb{R}^d\}$. We will use this to estimate inner products by operator norms.

Remark 4.3.2. The projection matrix $\mathbb{P}$ is used to extract particular components of an arbitrary matrix

$$A = \begin{pmatrix} A_0 & A_i \\ A_j & A_{ij} \end{pmatrix}.$$

For example,

$$\mathbb{P}M\mathbb{P} = \begin{pmatrix} 0 & 0 \\ 0 & A_{ij} \end{pmatrix}$$

and so an estimate on $\mathbb{P}A\mathbb{P}$ is precisely an estimate on the lower-diagonal piece A_{ij} of the matrix A. Similarly, $\mathbb{P}A\mathbb{P}^\perp$ extracts the A_i component, $\mathbb{P}^\perp A\mathbb{P}$ the A_j component and $\mathbb{P}^\perp A\mathbb{P}^\perp$ the A_0 component. With this perspective, we can also reinterpret condition (4.2.33) as ensuring that the fluid PDE (4.2.32) enjoys the estimates

$$|\mathbb{P}A^0\mathbb{P}^\perp|_{\mathrm{op}} = |\mathbb{P}^\perp A^0\mathbb{P}|_{\mathrm{op}} = \mathrm{O}(|\beta|_g + |z|_g^2) \quad \text{and} \quad |\mathbb{P}^\perp A^k\mathbb{P}^\perp|_{\mathrm{op}} = \mathrm{O}(|\beta|_g + |z|_g^2).$$

Furthermore, the right hand side of (4.2.32) explicitly reads

$$\begin{aligned} &\begin{pmatrix} D_1 a & b'z^i \\ 2D_2 a z_j & b\delta_j^i \end{pmatrix}\Bigg(\begin{pmatrix} K\ell \\ \Big(\frac{3K-1}{\nu\mu}w_i\partial_t\Psi + \frac{1-3K}{\nu}\mathscr{D}_i\Psi\Big) + m_i \end{pmatrix} \\ &\qquad - \begin{pmatrix} K & -\frac{K}{\nu\mu}w_j \\ -\frac{K}{\nu\mu}w_i & M_{ij} \end{pmatrix}\begin{pmatrix} D_2 a\partial_t g_{ij} z^i z^j \\ 0 \end{pmatrix}\Bigg). \end{aligned}$$

If we assume perturbations of the metric to be negligible, we may set $\ell = m_i = \partial_t g_{ij} = 0$. Furthermore, we may restrict ourselves to the case in which $\Psi = \Psi(t)$ so that $\mathscr{D}\Psi = 0$. In this setting we have

$$Q^{\mathrm{tr}}(H - B^0 Y) \simeq \frac{3K-1}{\nu\mu}\begin{pmatrix} b'z^i w_i\partial_t\Psi \\ bw_j\partial_t\Psi \end{pmatrix}.$$

As can be seen in the next section, an expanding FLRW–type model forces $\nu\mu < 0$ and $w \simeq bz$. Hence, we have that

$$Q^{\mathrm{tr}}(H - B^0 Y) \simeq c(K) \begin{pmatrix} 0 \\ z^j \end{pmatrix} + \mathrm{O}(|z|^2) = c(K)\mathbb{P}Z + \mathrm{O}(|z|^2),$$

for $\partial_t \Psi > 0$ and some fixed constant $c(K) > 0$. Note that this is in the regime of *compactified* time. A switch to physical time translates $c(K) \mapsto -c(K)$ and thus yields a *negative* term.

We conclude this section with some estimates on the quantities appearing in (4.3.5).

Lemma 4.3.4. *Let $\delta > 0$ and assume that $|Z| \leq \delta$. Then the following estimates hold:*

$$|M_0(Z)|_{\mathrm{op}} + |\mathscr{C}^k|_{\mathrm{op}} + |(M_0)^{-1}(Z)|_{\mathrm{op}} \lesssim 1, \qquad |M^k(Z)|_{\mathrm{op}} \lesssim |\Pi Z|, \tag{4.3.9}$$

and

$$\begin{aligned} |\mathbb{P}\partial_a M^k(Z)\mathbb{P}|_{\mathrm{op}} &\lesssim |\Pi Z| + |\Pi D Z|, \\ |\mathbb{P}^\perp \partial_a M^k(Z)\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}^\perp \partial_a M^k(Z)\mathbb{P}^\perp|_{\mathrm{op}} + & \\ |\mathbb{P}\partial_a M^k(Z)\mathbb{P}^\perp|_{\mathrm{op}} + |\partial_a (M_0)(Z)|_{\mathrm{op}} &\lesssim |\Pi Z|^2 + |\Pi D Z|^2, \end{aligned} \tag{4.3.10}$$

and

$$|F(Z)| \lesssim |\Pi Z|^2. \tag{4.3.11}$$

Remark 4.3.3. In this paper, see also the earlier work [81], we write $|f| \lesssim |g|$, or $f = \mathrm{O}(g)$, to denote a universal constant $C > 0$ such that $|f| \leq C|g|$ with f, g some functions. Note that this is not simply an estimate of absolute values but also tacitly implies that $f = h(g)$ for some function $h \in \mathscr{C}^\infty$ with $a < |h| < b$, $a, b > 0$. Thus, for $s \in \mathbb{N}$,

$$\sum_{|l| \leq s} |\nabla^l f| \lesssim \sum_{|l| \leq s} |\nabla^l g|.$$

Proof. It is straightforward to check (4.3.9) using the explicit form of these matrices given in (4.3.6) and (4.3.7). Turning to (4.3.10), we note that the leading order piece of M^k is $\mathbb{P}M^k\mathbb{P}$. Thus, we start with $|\mathbb{P}\partial_a M^k(Z)\mathbb{P}|_{\mathrm{op}}$. Using the expression for ν given in (4.2.19), we have

$$|\partial_a (t/\nu)| = \Big|\frac{t}{\nu}\frac{\partial_a \nu}{\nu}\Big| = \Big|\Big(\frac{t}{\nu}\Big)^2 \frac{v_i \partial_a (v^i)}{\sqrt{1 + |v|^2}}\Big|.$$

Using the transformation $v^j = b(\psi) z^i$, we get

$$|\partial_a (t/\nu) \cdot (A_0^0)^{-1} A^k(Z)|_{\mathrm{op}} \lesssim |z|^2 + |Dz|^2.$$

The other part of $\mathbb{P}\partial_a M^k(Z)\mathbb{P}$ is simple to estimate and in total we see that $|\mathbb{P}\partial_a M^k(Z)\mathbb{P}|_{\mathrm{op}} \lesssim |z| + |Dz|$. The remaining $\mathbb{P}$ and $\mathbb{P}^\perp$ conjugations of $\partial_a M^k$ are easier to estimate. The estimates on $\partial_a M_0$ and (4.3.11) use (4.3.7) and (4.3.8) respectively. $\square$

4.3.1 Extended system

In order to apply the global existence result from [41] we wish to consider the Fuchsian system for $\bar{Z} := (Z, DZ)^T$. Thus, we apply the differential operator $M^0\partial_a(M^0)^{-1}$ to (4.2.32) to get

$$M_0(Z)\partial_t\partial_a Z + \frac{1}{t}(\mathscr{C}^k + M^k(Z))\partial_k\partial_a Z = \frac{1}{t}\mathscr{B}\Pi\partial_a Z + \frac{1}{t}G_a(\bar{Z}),$$

where we introduce

$$\begin{aligned} G_a(\bar{Z}) := M_0(Z)\Big[- \partial_a\left[M_0(Z)^{-1}\right](\mathscr{C}^k + M^k(Z))\partial_k Z - M_0(Z)^{-1}\left(\partial_a M^k(Z)\right)\partial_k Z \\ + \partial_a\left[M_0(Z)^{-1}\right]\mathscr{B}\Pi Z + \partial_a\left[M_0(Z)^{-1}F(Z)\right]\Big]. \end{aligned}$$

Lemma 4.3.5. *The extended system $\bar{Z} := (Z, DZ)^T$ is governed by the PDEs:*

$$B^0(\bar{Z})\partial_t\bar{Z} + \frac{1}{t}(C^k + B^k(\bar{Z}))\partial_k\bar{Z} = \frac{1}{t}\mathscr{B}(\bar{Z})\mathbb{P}\bar{Z} + \frac{1}{t}H(\bar{Z}), \tag{4.3.12}$$

where

$$B^0(\bar{Z}) := \begin{pmatrix} M_0(Z) & 0 \\ 0 & M_0(Z)\end{pmatrix}, \quad C^k := \begin{pmatrix}\mathscr{C}^k & 0 \\ 0 & \mathscr{C}^k\end{pmatrix}, \quad B^k(\bar{Z}) := \begin{pmatrix} M^k(Z) & 0 \\ 0 & M^k(Z)\end{pmatrix},$$

and

$$\mathbb{P} := \begin{pmatrix}\Pi & 0 \\ 0 & \Pi\end{pmatrix}, \quad \mathscr{B}(\bar{Z}) := \begin{pmatrix}\mathscr{B} & 0 \\ 0 & \mathscr{B}\end{pmatrix}, \quad H(\bar{Z}) := \begin{pmatrix} F(Z) \\ G(Z, DZ)\end{pmatrix}.$$

Moreover, let $\delta > 0$ and assume that $|\bar{Z}| \leq \delta$. Then, the following estimates hold:

$$\begin{aligned} |\mathbb{P}(B^0(\bar{Z}) - B^0(0))\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}^\perp(B^0(\bar{Z}) - B^0(0))\mathbb{P}^\perp|_{\mathrm{op}} & \\ + |\mathbb{P}^\perp B^0(\bar{Z})\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}B^0(\bar{Z})\mathbb{P}^\perp|_{\mathrm{op}} &\lesssim |\mathbb{P}\bar{Z}|^2, && (4.3.13)\\ |\mathbb{P}H(\bar{Z})| &\lesssim |\mathbb{P}\bar{Z}|, && (4.3.14)\\ |\mathbb{P}^\perp H(\bar{Z})| &\lesssim |\mathbb{P}\bar{Z}|^2, && (4.3.15)\\ |\mathbb{P}^\perp B^k(\bar{Z})\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}^\perp B^k(\bar{Z})\mathbb{P}^\perp|_{\mathrm{op}} + |\mathbb{P}B^k(\bar{Z})\mathbb{P}^\perp|_{\mathrm{op}} &\lesssim |\mathbb{P}\bar{Z}|^2, && (4.3.16)\\ |\mathbb{P}B^k(\bar{Z})\mathbb{P}|_{\mathrm{op}} &\lesssim |\mathbb{P}\bar{Z}|, && (4.3.17)\\ |\mathbb{P}\mathrm{div}B(t, v, w)\mathbb{P}|_{\mathrm{op}} &\lesssim |t|^{-1}, && (4.3.18)\\ |\mathbb{P}\mathrm{div}B(t, v, w)\mathbb{P}^\perp|_{\mathrm{op}} + |\mathbb{P}^\perp\mathrm{div}B(t, v, w)\mathbb{P}|_{\mathrm{op}} &\lesssim |t|^{-1}|\mathbb{P}v|, && (4.3.19)\\ |\mathbb{P}^\perp\mathrm{div}B(t, v, w)\mathbb{P}^\perp|_{\mathrm{op}} &\lesssim |t|^{-1}|\mathbb{P}v|^2, && (4.3.20) \end{aligned}$$

where

$$\begin{aligned} \mathrm{div}B(t, v, w) := D_v B^0(v)\cdot(B^0(v))^{-1}\left(-\frac{1}{t}(C^k + B^k(v))w_k + \frac{1}{t}\mathscr{B}(v)\mathbb{P}v + \frac{1}{t}H(v)\right) \\ + \frac{1}{t}D_v B^k(v)w_k. \end{aligned}$$

Following the conventions in [11], we use v and w which denote $\bar{Z}$ and $D\bar{Z}$ respectively.

Proof. (4.3.13) immediately follows from the definition of B^0 and M^0 given in (4.3.7). For (4.3.14) we use (4.3.11) for the estimate on F. To estimate G, we define

$$\begin{aligned} C_1 &:= [\Pi, M_0] = \begin{pmatrix} 0 & -A_j^0 \\ A_i^0 & 0 \end{pmatrix}, \\ C_2 &:= [\Pi, M_0^{-1}] = -M_0^{-1}([\Pi, M_0])M_0^{-1}, \\ \text{and} \quad C_3^k &:= [\Pi^\perp, M^k]\,. \end{aligned} \tag{4.3.21}$$

A computation shows that

$$|\partial_a C_2|_{\mathrm{op}} \lesssim |z|^2 + |Dz|^2 \lesssim |\mathbb{P}\bar{Z}|^2\,, \qquad |C_1|_{\mathrm{op}} + |C_2|_{\mathrm{op}} + |C_3|_{\mathrm{op}} \lesssim |z|^2\,.$$

Using (4.3.21) and (4.3.10), we find

$$\begin{aligned} \Pi^\perp G_a = C_1 G_a &+ M_0\left[-\partial_a C_2 \cdot (\mathscr{C}^k + M^k)\partial_k Z - C_2(\partial_a M^k)\partial_k Z + \partial_a C_2 \cdot \mathscr{B}\Pi Z\right] \\ &+ M_0\Big[-\partial_a\left(M_0^{-1}\right)(\Pi^\perp\mathscr{C}^k + \Pi^\perp M^k)\partial_k Z - M_0^{-1}\partial_a\left(\Pi^\perp M^k\Pi^\perp\right)\partial_k Z \\ &\qquad - M_0^{-1}\partial_a\left(\Pi M^k \Pi^\perp\right)\partial_k \Pi Z + \Pi^\perp\partial_a\left(M_0^{-1}F\right)\Big]. \end{aligned}$$

Note that it is crucial that the term $M_0^{-1}\partial_a\left(\Pi M^k\Pi^\perp\right)\partial_k\Pi Z$ is of higher order in $\mathbb{P}Z$. The estimate (4.3.16) and (4.3.17) follow immediately from the estimates in (4.3.9). Using (4.3.9), (4.3.11) and (4.3.10) we may conclude (4.3.14) as well as (4.3.15).

Finally, we study the map

$$\begin{aligned} \mathrm{div}B(t,v,w) = D_v B^0(v)\cdot(B^0(v))^{-1}\left(-\frac{1}{t}(C^k + B^k(v))w_k + \frac{1}{t}\mathscr{B}(v)\mathbb{P}v + \frac{1}{t}H(v)\right) \\ + \frac{1}{t}D_v B^k(v)w_k. \end{aligned}$$

We compute

$$|D_v B^0(v)| \lesssim |D_Z(f(\psi)\cdot|z|_g^2)| \lesssim |\mathbb{P}v|,$$

where f is some smooth bounded function that comes from the choice of transformation. Also, $B^0(v)^{-1} = \mathbb{1} + \mathrm{O}(|\mathbb{P}v|^2)$. Together with (4.3.6), (4.3.9), (4.3.14) as well as Remark 4.3.1, this yields the desired results as given in (4.3.18)–(4.3.20). $\square$

Proposition 4.3.1. *Suppose that $k \geq 4$ and $\bar{Z}_0 \in H^k(\mathbb{T}^3)$ is initial data to the initial value problem posed by* (4.3.12)*. Then, there exists an $\epsilon > 0$ such that if $\|\bar{Z}_0\|_{H_k} < \epsilon$, there exists a unique global solution $\bar{Z}$ to* (4.3.12) *with*

$$\bar{Z} \in C^0((0,T_0], H^k(\mathbb{T}^3)) \cap C^1((0,T_0], H^{k-1}(\mathbb{T}^3)).$$

Proof. After a transformation $t \to -t$, our only task is to check the assumptions from [41, §4.1]. It is straightforward to check that $\mathbb{P}^2 = \mathbb{P}$, $\mathbb{P}^{\mathrm{tr}} = \mathbb{P}$, $\partial_t\mathbb{P} = 0$, $\partial_j\mathbb{P} = 0$ as well as $(C^k)^{\mathrm{tr}} = C^k$, $\partial_t C^k = 0$, $\partial_i C^k = 0$. Using that $0 < K < \frac{1}{3}$, there exist constants $\gamma_1, \gamma_2, \gamma_3 > 0$ such that $\frac{1}{\gamma_1}\mathbb{1} \leq M_0(Z) \leq \frac{1}{\gamma_3}\mathscr{B}(Z) \leq \gamma_2\mathbb{1}$. Since A^0 is symmetric, we have $(B^0(\bar{Z}))^{\mathrm{tr}} = B^0(\bar{Z})$. Also

$$[\mathbb{P}, \mathscr{B}(\bar{Z})] = 0, \qquad \partial_k(\mathbb{P}\mathscr{B}(\bar{Z})) = 0, \quad \text{and} \quad |\mathbb{P}\mathscr{B}(\bar{Z}) - \mathbb{P}\mathscr{B}(0)|_{\mathrm{op}} = 0\,.$$

Since A^k and C^k are symmetric, B^k is also. Together with Lemma 4.3.5 we conclude that all the conditions in [41, §4.1] are met. $\square$

4.4 The Einstein Euler equations near a Milne background

We now consider solutions to the Einstein Euler equations (4.1.1) allowing for the dynamical metric $\bar{g}$ to be a perturbation away from the following linearly expanding Einstein spacetime:

Definition 4.4.1 (The Milne spacetime)**.** Let (M, γ) negative closed Einstein space of dimension 3 and $\mathrm{Ric}[\gamma] = -\frac{2}{9}\gamma$. Then, the Lorentz cone $\mathscr{M} = \mathbb{R} \times M$ with metric

$$g_M = -d\bar{t}^2 + \frac{\bar{t}^2}{9}\gamma \tag{4.4.1}$$

is a Lorentzian solution to the vacuum Einstein equations. We term $(\mathscr{M}, g_M)$ the (compactified) *Milne* spacetime and refer to $\bar{t}$ as cosmological (or physical) time.

In this section we prove the following theorem

Theorem 4.4.1. *Let $(\mathscr{M}, g_M)$ be the Milne spacetime and consider the Einstein-Euler equations* (4.1.1) *with linear equation of state* (4.1.3) *where $K \in (0, 1/3)$. Let (g_0, k_0, ρ_0, u_0) be initial data satisfying the constraint equations* (4.4.5a) *for at physical time t_0 such that $\rho_0 > 0$ and*

$$(g_0, k_0, \rho_0, u_0) \in \mathscr{B}^{6,5,5,5}_{\epsilon}(\frac{t_0^2}{9}\gamma, -\frac{t_0}{9}\gamma, 0, 0). \tag{4.4.2}$$

Then, there exists an $\epsilon > 0$ sufficiently small, such that the future development of (g_0, k_0, ρ_0, u_0) under the Einstein-Euler equations is complete and admits a constant mean curvature foliation labelled by $\tau \in [\tau_0, 0)$, such that the induced metric and second fundamental form on constant mean curvature slices converge as

$$(\tau^2 g, \tau k) \xrightarrow{\tau \to 0} (\gamma, \frac{1}{3}\gamma).$$

Remark 4.4.1. In theorem 4.4.1 above $\mathscr{B}^{6,5,5,5}_{\epsilon}(\cdot, \cdot, \cdot, \cdot)$ denotes the ball of radius ϵ centered at the argument in the set $H^6(M) \times H^5(M) \times H^5(M) \times H^5(M)$, with the canonical Sobolev norms given in Definition 4.4.8 below.

Definition 4.4.2 (ADM decomposition and CMCSH gauge)**.** We reparametrise the physical dynamical metric $\bar{g}$ in terms of the ADM variables via

$$\bar{g} = -\tilde{N}^2 dt^2 + \tilde{g}_{ab}(dx^a + \tilde{X}^a dt)(dx^b + \tilde{X}^b dt). \tag{4.4.3}$$

That is, $\tilde{N}$ is the lapse, $\tilde{X}$ is the shift and $\tilde{g}$ is the induced Riemannian metric on M. Furthermore, we denote the mean curvature and trace-free part of the second fundamental form[3] by

$$\tau := \mathrm{tr}_{\tilde{g}}\tilde{k} = \tilde{g}^{ab}\tilde{k}_{ab}, \qquad \tilde{k}_{ab} := \tilde{\Sigma}_{ab} + \frac{1}{3}\tau\tilde{g}_{ab}.$$

We follow the standard approach for the Milne spacetime [3; 4] and impose the *constant mean curvature* and *spatial harmonic gauge* conditions

$$t = \tau, \qquad H^a := \tilde{g}^{cb}(\Gamma[\tilde{g}]^a_{cb} - \Gamma[\gamma]^a_{cb}) = 0. \tag{4.4.4}$$

[3]We assume that all spatial indices are raised and lowered using the metric g.

Definition 4.4.3 (Rescaled geometric variables)**.** We define

$$g_{ab} := \tau^2 \tilde{g}_{ab}, \quad g^{ab} := (\tau^2)^{-1}\tilde{g}^{ab}, \quad N := \tau^2 \tilde{N}, \quad X^a := \tau \tilde{X}^a, \quad \Sigma_{ab} := \tau \tilde{\Sigma}_{ab},$$

as well as $\hat{N} := N - 3$. We also define the *logarithmic time* T as $T := -\ln(\frac{\tau}{e\tau_0})$ so that T satisfies the relation $\partial_T = -\tau\partial_\tau$.

Remark 4.4.2. The expression for the Milne geometry given in (4.4.1) is with respect to cosmological time $\bar{t}$. Moving to CMC time, given by $\tau = -3/\bar{t}$, we see that the metric takes the form

$$g_M = \frac{1}{\tau^2}\left(-\frac{9}{\tau^2}d\tau^2 + \gamma\right).$$

Thus, the Milne spacetime, when written in CMCSH-gauge and rescaled variables, is given by

$$(g_{ij}, \Sigma_{ij}, N, X^i) = (\gamma_{ij}, 0, 3, 0).$$

We also define various components of the energy momentum tensor that contribute to the dynamics under the Einstein flow, see [2].

Definition 4.4.4 (Matter variables $E, \jmath, \eta, S$)**.** We define the following physical matter quantities

$$\tilde{E} := \bar{T}^{\mu\nu} n_\mu n_\nu, \quad \tilde{\jmath}^a := \tilde{g}^{ab}\tilde{N}\bar{T}^{0\mu}\bar{g}_{b\mu}, \quad \tilde{\eta} := \tilde{E} + \tilde{g}^{ab}\bar{T}_{ab}, \quad \tilde{S}_{ab} := \bar{T}_{ab} - \frac{1}{2}\operatorname{tr}_{\bar{g}}\bar{T}\cdot\tilde{g}_{ab},$$

and the rescaled matter quantities by

$$E := (-\tau)^{-3}\tilde{E}, \quad \jmath^a := (-\tau)^{-5}\tilde{\jmath}^a, \quad \eta := (-\tau)^{-3}\tilde{\eta}, \quad S_{ab} := (-\tau)^{-1}\tilde{S}_{ab}.$$

Definition 4.4.5 (Elliptic operators $\Delta_{g,\gamma}, \mathscr{L}_{g,\gamma}$)**.** Let V be a symmetric $(0,2)$-tensor on M. We define the following self-adjoint, elliptic differential operators

$$\begin{aligned}
\Delta_{g,\gamma}V_{ij} &:= (\sqrt{g})^{-1}\nabla[\gamma]_a(\sqrt{g}g^{ab}\nabla[\gamma]_b V_{ij}),\\
\mathscr{L}_{g,\gamma}V_{ij} &:= -\Delta_{g,\gamma}V_{ij} - 2\mathrm{Riem}[\gamma]_{iajb}\gamma^{ac}\gamma^{bd}V_{cd}.
\end{aligned}$$

The equations of motion for an Einstein-matter system in CMCSH gauge are derived in [2]. Denoting the Levi-Civita connection of g by ∇, the PDEs are as follows.

Lemma 4.4.1 (Equations of motion for the geometric variables)**.** *The Einstein equations* (4.1.1a) *reduce to two constraint equations*

$$\begin{aligned}
R(g) - |\Sigma|_g^2 + \frac{2}{3} &= -4\tau E,\\
\nabla^a\Sigma_{ab} &= 2\tau^2 \jmath_b,
\end{aligned} \tag{4.4.5a}$$

two elliptic equations for the lapse and shift variables

$$\begin{aligned}
(\Delta - \frac{1}{3})N &= N\left(|\Sigma|_g^2 - \tau\eta\right) - 1,\\
\Delta X^a + Ric[g]^a{}_b X^b &= 2\nabla_b N\Sigma^{ba} - \nabla^a\hat{N} + 2N\tau^2\jmath^a - (2N\Sigma^{bc} - \nabla^b X^c)(\Gamma[g]^a_{bc} - \Gamma[\gamma]^a_{bc}),
\end{aligned}$$

and two evolution equations for the induced metric and trace-free part of the second fundamental form

$$\begin{aligned}\partial_T g_{ab} &= 2N\Sigma_{ab} + 2\hat{N} g_{ab} - \mathscr{L}_X g_{ab},\\ \partial_T \Sigma_{ab} &= -2\Sigma_{ab} - N(Ric[g]_{ab} + \frac{2}{9}g_{ab}) + \nabla_a\nabla_b N + 2N\Sigma_{ac}\Sigma^c_b \\ &\quad - \frac{1}{3}\hat{N} g_{ab} - \hat{N}\Sigma_{ab} - \mathscr{L}_X \Sigma_{ab} + N\tau S_{ab}.\end{aligned} \tag{4.4.5b}$$

Remark 4.4.3. In the above equations (4.4.5), $\mathscr{L}_X$ denotes the Lie derivative with respect to X. We also recall from [4] the following decomposition of the curvature term in the spatially harmonic gauge:

$$\mathrm{Ric}[g]_{ab} + \frac{2}{9}g_{ab} = \frac{1}{2}\mathscr{L}_{g,\gamma} g_{ab} + J_{ab}, \tag{4.4.6}$$

where there is a constant $C > 0$ such that $\|J\|_{H^{s-1}} \leq C\|g-\gamma\|_{H^s}$.

4.4.1 Equations of motion for the fluid variables

In CMCSH-coordinates and rescaled variables the dynamical metric (4.4.3) is given by

$$\bar{g} = \frac{1}{\tau^2}\left[-\frac{N^2}{\tau^2}d\tau^2 + g_{ab}(dx^a + \frac{X^a}{\tau}d\tau)(dx^b + \frac{X^b}{\tau}d\tau)\right].$$

This is in the form considered in (4.2.7), provided we make the identifications

$$x^0 \equiv \tau, \quad \alpha \equiv \frac{N}{\tau}, \quad \beta^a \equiv \frac{X^a}{\tau}, \quad g_{ab} \equiv \mathcal{g}_{ab}, \quad \Psi(\tau) = \ln(-\tau).$$

Lemma 4.4.2. *The Euler equations* (4.1.1b) *with respect to the dynamical metric $\bar{g}$ can be written as*

$$B^0\partial_\tau U + B^k\nabla_k U = H, \tag{4.4.7}$$

where $U = (\zeta, u^j)^{\mathrm{tr}}$ and

$$B^0 = \begin{pmatrix} K & -\frac{K}{\nu\mu}w_j \\ -\frac{K}{\nu\mu}w_i & M_{ij}\end{pmatrix}, \quad B^k = \begin{pmatrix} \frac{K}{\nu}u^k & \frac{s^2}{\nu}\delta^k_j \\ \frac{K}{\nu}\delta^k_i & \frac{1}{\nu}M_{ij}u^k\end{pmatrix}, \quad H = \begin{pmatrix} K\ell \\ \left(\frac{1-3K}{\nu\mu}w_i\right) + m_i\end{pmatrix}.$$

Proof. Recall from (4.2.14) that $\nu = v^\tau$, $w_j = v_j$, $\mu = v_\tau$ and $u^j = v^j$. Evaluating the expressions given in (4.2.12), (4.2.15), (4.2.23) and (4.2.24) in CMCSH-coordinates, we compute the geometric quantities to be

$$\begin{aligned}\mathscr{K}_{ij} &= \frac{1}{2N}(\partial_T g_{ij} + \nabla_i X_j + \nabla_j X_i),\\ \Xi^i_j &= \frac{1}{\tau}\Big[-\frac{1}{N}X^i(\nabla_j N - X^k\mathscr{K}_{kj}) - N\mathscr{K}^i_j + \nabla_j X^i\Big],\\ \Upsilon^i &= \frac{1}{\tau^2}\Big[N\nabla^i N - 2NX^j\mathscr{K}^i_j - \frac{X^i}{N}(-\partial_T N - N + X^j\nabla_j N - X^jX^k\mathscr{K}_{jk})\\ &\quad - \partial_T X^i - X^i + X^j\nabla_j X^i\Big],\end{aligned} \tag{4.4.8}$$

while the matter quantities are

$$\begin{aligned} M_{ij} &= g_{ij} - \frac{1}{\tau\mu}(w_i X_j + w_j X_i) + \frac{-N^2 + |X|_g^2}{\tau^2\mu^2} w_i w_j, \\ \ell &= -\frac{1}{\nu N}\mathscr{K}_{jk} X^j u^k - \Xi_j^j + \frac{1}{\mu}\Upsilon^j w_j + \frac{1}{\nu\mu}\Xi_k^j w_j u^k, \\ m_i &= -M_{ij}\Big(\frac{1}{\nu N}\mathscr{K}_{lk} X^j u^l u^k + \nu\Upsilon^j + 2\Xi_k^j u^k\Big). \end{aligned} \tag{4.4.9}$$

The result then follows from Lemma 4.2.3. □

Remark 4.4.4. On the background Milne geometry the tensors $\mathscr{K}_{ij}, \tau\Xi_j^i$ and $\tau^2\Upsilon^i$ identically vanish.

Lemma 4.4.3. *The homogeneous fluid solutions to* (4.1.1b) *are given by* $U = (\zeta, v^a) = 0$.

Proof. The modified density variable as defined in Section 4.2 takes the form

$$\zeta = \frac{1}{1+K}\ln(\tilde{\rho}/\tilde{\rho}_0) - 3\ln(-\tau). \tag{4.4.10}$$

Considering the Euler equations (4.1.1b) with $\bar{g} = g_M$ and $\bar{v}^i = 0$ (i.e. a homogeneous regime) we arrive at the ODEs

$$((\bar{v}^{\bar{t}})^2 - K/(1+K))\partial_{\bar{t}}\bar{\rho} + 2\bar{\rho}\bar{v}^{\bar{t}}\partial_{\bar{t}}\bar{v}^{\bar{t}} + \frac{3}{t}(\bar{v}^{\bar{t}})^2\bar{\rho} = 0, \qquad \partial_i\bar{\rho} = 0.$$

These are satisfied by

$$\bar{v}^{\bar{t}} = 1 \quad \text{and} \quad \tilde{\rho} = \tilde{\rho}_0 \bar{t}^{-3(1+K)}$$

for some constant $\tilde{\rho}_0 > 0$. This implies $\tilde{\rho} = c_h(-\tau)^{3(1+K)}$ for some constant $c_h > 0$ and thus on the background

$$\zeta = \frac{1}{1+K}\ln(\tilde{\rho}/\tilde{\rho}_0) - 3\Psi = 3\ln(-\tau) - 3\Psi = 0,$$

provided we pick $\tilde{\rho}_0 = c_h$. Hence, our homogeneous solution is $U = (\zeta, v^a) = 0$. □

Remark 4.4.5. On the above homogeneous fluid solutions, we have the simplifications

$$\nu = v^\tau = -\frac{\tau}{3}, \quad \mu = v_\tau = \frac{3}{\tau}, \qquad \nu\mu = -1, \quad M_{ij} = \gamma_{ij}. \tag{4.4.11}$$

Additionally, $w_j, u^j, w^j, \tau\ell$ and τm_i identically vanish.

Due to the factor of τ appearing in the expression for ν in (4.4.11), we introduce an additional rescaling.

Definition 4.4.6 ($\hat{v}^\tau$)**.** We rescale the time-component of the conformal fluid four-velocity by $\hat{v}^\tau := (-\tau)^{-1}v^\tau$.

4.4.2 Preliminary notation and computations

In this subsection we assume that (M, g) is a closed 3-dimensional Riemannian manifold with Levi-Civita connection ∇.

Definition 4.4.7 (Inner products)**.** Let v and w be vectors fields on M. We define $\langle \cdot, \cdot \rangle_g$ as

$$\langle v, w \rangle_g := g_{ij} v^i w^j.$$

If $\ell \geq 1$, then we define

$$\langle \nabla^\ell v, \nabla^\ell w \rangle_g := g^{a_1 b_1} \cdots g^{a_\ell b_\ell} \langle \nabla_{a_1} \cdots \nabla_{a_l} v, \nabla_{b_1} \cdots \nabla_{b_\ell} w \rangle_g.$$

This definition is extended in the usual way to arbitrary tensor fields. We also define the modulus as $|v|_g^2 := \langle v, v \rangle_g$. If the subsript is omitted, we assume the bracket to be the Euclidean inner product, i.e. $\langle v, w \rangle := v^T(w)$.

Definition 4.4.8 (Sobolev norms and measures)**.** We denote by μ_g the Riemannian measure associated to g, which is given locally by $\sqrt{\det g} dx^1 \wedge dx^2 \wedge dx^3$. When the context is unambiguous, we will omit the measure μ_g. For a tensor field V and $s \in \mathbb{N}$, we define the *Sobolev norm of order s* as

$$\|V\|_{H^s}^2 := \sum_{0 \leq \ell \leq s} \int_M |\nabla^\ell V|_g.$$

We frequently consider the norms of an abstract vector quantity $\mathscr{V} = (f, v)^T$ which consists of a function f and a spatial vector field v. We simply write

$$\|\mathscr{V}\|_{H^k} := \|f\|_{H^k} + \|v\|_{H^k}.$$

We also require the following result that allows us to commute the time-derivative with integration.

Lemma 4.4.4. *For an arbitrary scalar function f the following identity holds:*

$$\frac{d}{dT} \int_M f \ \mu_g \lesssim \|N - 3\|_{H^2} \|f\|_{L^1} + \|X\|_{H^3} \|f\|_{L^1} + \int_M \partial_T f \ \mu_g.$$

Proof. The following identity from [16], together with integration by parts on the shift term, yields

$$\frac{d}{dt} (\int_M f \ \mu_g) = \int_M (3(N-3)f + \partial_T f - f \nabla_i X^i)) \mu_g.$$

The result then follows by the Hölder inequality and the Sobolev embedding $L^\infty(M) \hookrightarrow H^2(M)$. $\square$

We also state an additional lemma which will be needed later on, when discussing the behavior of the energy functionals.

Lemma 4.4.5. *Suppose that γ is another Riemannian metric on M, such that $Ric[\gamma] = -\frac{2}{9}\gamma$, the harmonic condition* (4.4.4) *holds and also the bounds*

$$\|g-\gamma\|_{H^s} \leq C\epsilon$$

hold for $s \in \mathbb{N}$. Then

$$\|\mathrm{Riem}[g] - \mathrm{Riem}[\gamma]\|_{H^s} \lesssim \|g-\gamma\|_{H^{s+2}}.$$

Proof. By the Ricci decomposition of the Riemann tensor in 3 dimensions, we have that

$$\begin{aligned}\mathrm{Riem}[\gamma]_{ijkl} &= -\frac{\mathrm{R}[\gamma]}{6}(\gamma_{il}\gamma_{jk} - \gamma_{ik}\gamma_{jl}) = \frac{1}{9}(\gamma_{il}\gamma_{jk} - \gamma_{ik}\gamma_{jl}),\\ \mathrm{Riem}[g]_{ijkl} &= -\frac{\mathrm{R}[g]}{6}(g_{il}g_{jk} - g_{ik}g_{jl}) + (V_{il}g_{jk} - V_{jl}g_{ik} - V_{ik}g_{jl} + V_{jk}g_{il}),\end{aligned} \tag{4.4.12}$$

where $V_{jk} := -\mathrm{Ric}[g]_{jk} + \frac{1}{3}\mathrm{R}[g]g_{jk}$ and we used that $\mathrm{Ric}[\gamma] = -\frac{2}{9}\gamma$. From (4.4.6), we have that

$$\mathrm{Ric}[g]_{ab} - \mathrm{Ric}[\gamma]_{ab} = -\frac{2}{9}g_{ab} + \frac{1}{2}\mathscr{L}_{g,\gamma}(g_{ab} - \gamma_{ab}) + J_{ab} + \frac{2}{9}\gamma_{ab}.$$

Thus, by elliptic regularity (see e.g. [9])

$$\|\mathrm{Ric}[g]_{ab} - \mathrm{Ric}[\gamma]_{ab}\|_{H^s} \lesssim \|g-\gamma\|_{H^s} + \|\mathscr{L}_{g,\gamma}(g-\gamma)\|_{H^s} + \|J\|_{H^s} \lesssim \|g-\gamma\|_{H^{s+2}}.$$

To compute the difference of the Riemann tensors, one then expand all of the terms in (4.4.12) including g by $g-\gamma+\gamma$ and $\mathrm{Ric}[g]$ by $\mathrm{Ric}[g]-\mathrm{Ric}[\gamma]+\mathrm{Ric}[\gamma]$. Applying the triangle inequality then shows that $\|V\|_{H^s} \lesssim \|g-\gamma\|_{H^{s+2}}$ and furthermore the desired estimate. □

4.4.3 Local existence and bootstrap assumptions

Theorem 4.4.2. *Let $k \geq 6$ be a fixed integer. At $T = T_0$, suppose that we have CMC initial data satisfying the constraints* (4.4.5a) *with regularity*

$$(g_0, k_0, N_0, X_0, \rho_0, u_0) \in H^k \times H^{k-1} \times H^k \times H^k \times H^{k-1} \times H^{k-1}.$$

Then, there exists a unique classical solution (g, k, N, X, ρ, u) to (4.4.5) *on $[T_0, T_+)$ with $T_+ > T_0$. This local solution satisfies*

$$\begin{aligned} g, N, X &\in C^0([T_0,T_+), H^k) \cap C^1([T_0,T_+), H^{k-1}),\\ k &\in C^0([T_0,T_+), H^{k-1}) \cap C^1([T_0,T_+), H^{k-2}),\\ u, \rho &\in C^0([T_0,T_+), H^{k-1}),\\ \partial_T\rho, \partial_T u &\in C^0([T_0,T_+), H^{k-2}).\end{aligned}$$

In addition, the norms as well as the time of existence T_+ depend continuously on the initial data. By the continuation principle the maximal time of existence T_{max} is either $T_{max} = \infty$, i.e. global existence, or

$$\begin{aligned}\lim_{T\to T_{max}} \sup_{[T_0,T]} &\|g-\gamma\|_{H^k} + \|\Sigma\|_{H^{k-1}} + \|N-3\|_{H^k} + \|X\|_{H^k}\\ &+ \|\partial_T N\|_{H^{k-1}} + \|\partial_T X\|_{H^{k-1}} + \|\rho\|_{H^{k-1}} + \|u\|_{H^{k-1}} > \delta,\end{aligned}$$

where $\delta > 0$ is a fixed constant.

In light of the above local existence theorem, we can now make certain bootstrap assumptions that hold in a non-empty time interval. Let $\lambda < 1$ be a fixed positive constant and, again, $k \geq 6$ a fixed integer. We assume that, for all $T_0 \leq T \leq T'$, where $T' > T_0$, there is a constant $C > 0$ such that the following bootstrap assumptions hold

$$\begin{aligned} \|U\|_{H^{k-1}} &\leq C\epsilon, \\ \|g-\gamma\|_{H^k} + \|\Sigma\|_{H^{k-1}} &\leq C\epsilon e^{-\lambda T}, \\ \|N-3\|_{H^k} + \|X\|_{H^k} + \|\partial_T N\|_{H^{k-1}} + \|\partial_T X\|_{H^{k-1}} &\leq C\epsilon e^{-T}. \end{aligned} \tag{4.4.13}$$

The results in the rest of this paper will be derived under these bootstrap assumptions.

Lemma 4.4.6. *Under* (4.4.13) *we have that*

$$\|\hat{v}^\tau - \frac{1}{3}\|_{H^k} \lesssim \epsilon.$$

Proof. The normalization condition (4.1.2) implies

$$\begin{aligned} \bar{v}^\tau &= \frac{1}{2(-\tilde{N}^2 + \tilde{X}_a\tilde{X}^a)}(-2\tilde{X}_a\tilde{v}^a + [4(\tilde{X}_a\tilde{X}^a)^2 - 4(-\tilde{N}^2 + \tilde{X}_a + \tilde{x}^a)(\tilde{g}_{ab}\tilde{v}^a\tilde{v}^b + 1)]^{\frac{1}{2}}) \\ &= \frac{\tau^2}{X_aX^a - N^2}(-X_av^a + [(X_av^a)^2 + (N^2 - X_aX^a)(g_{ab}v^av^b + 1)]^{\frac{1}{2}}), \end{aligned}$$

and so the final estimate is obtained by applying (4.4.13). □

Remark 4.4.6. From the geometric equations of motion (4.4.5b) we may immediately conclude that

$$\|\partial_T g\|_{H^{k-1}} \lesssim \|N\Sigma\|_{H^{k-1}} + \|\hat{N}g\|_{H^{k-1}} + \|\mathscr{L}_X g\|_{H^{k-1}} \lesssim \epsilon e^{-T}. \tag{4.4.14}$$

We now explain why the bootstrap estimate on U yields a similar estimate on Z.

Lemma 4.4.7. *Under the bootstrap assumptions* (4.4.13) *we have that*

$$\|Z\|_{H^{k-1}} \lesssim \epsilon.$$

Proof. Consider the transformation $Z = (\psi, z^i) \mapsto U = (\xi, u^i)$ as given in (4.2.31) which we will denote by $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$. By the bootstrap (4.4.13) as well as Sobolev embedding we have that

$$\|bz\|_{L^\infty} + \|a\|_{L^\infty} \lesssim \|U\|_{H^{k-1}} \lesssim \epsilon.$$

Using the explicit form of $|b|$, which is bounded from below, we can conclude that $\|z\|_{L^\infty} \lesssim \epsilon$. Choosing $c_{1,2}$ the same as in the proof of Lemma 4.3.2 we see that smallness of a must imply that $\|\psi\|_{L^\infty} \lesssim \epsilon$. Simply calculating the Jacobian of φ, we see that due to smallness of Z, φ is a diffeomorphism. Also note that $z^i = f(\psi)u^i$ is a product of a scalar and a vector and is hence a vector and furthermore that due to the structure of a, ψ then has to be a scalar as well. Let us now consider the $\dot{H}^1$-norm of ψ:

$$\|\psi\|^2_{\dot{H}^1} = \int_M (g^{mn}\nabla_m(\psi(\zeta, |u|^2_g))\nabla_n(\psi(\zeta, |u|^2_g))$$

$$
\begin{aligned}
&= \int_M (g^{mn}[(\partial_1\psi)(\zeta,|u|_g^2)\partial_m\zeta + (\partial_2\psi)(\zeta,|u|_g^2)\partial_m|u|_g^2] \\
&\quad \times [(\partial_1\psi)(\zeta,|u|_g^2)\partial_n\zeta + (\partial_2\psi)(\zeta,|u|_g^2)\partial_n|u|_g^2]) \\
&\lesssim \int_M (g^{mn}\nabla_m\zeta\nabla_n\zeta + g^{mn}u_i\nabla_m u^i u_k\nabla_n u^k + g^{mn}\nabla_m\zeta u_i\nabla_n u^i) \lesssim \|U\|_{H^1},
\end{aligned}
$$

where we used Cauchy-Schwarz and the fact that $\partial_1\psi, \partial_2\psi$ are scalar functions on a compact manifold and hence bounded. Since $z^i = f(\psi)u^i$ we have, by the Leibniz rule of the covariant derivative,

$$
\|z\|_{\dot{H}^1} \lesssim \|U\|_{H^1}.
$$

For higher derivatives the same calculation follows with more applications of the Leibniz rule. □

4.4.4 Transformed Fuchsian system for the Euler equations

In this section we derive the expression for the equations of motion of the fluid.

Lemma 4.4.8. *Under* (4.4.13) *the following estimates hold:*

$$
\begin{aligned}
|\tau\ell| + |\tau m^i|_g + |\mathscr{K}_{ij}|_g + |\tau\Xi^i_j|_g + |\tau^2\Upsilon^i|_g &\lesssim |X|_g + |\nabla X|_g + |\hat{N}| + |\nabla N|_g + |\Sigma|_g + |\partial_T X|_g, \\
|M_{ij} - \gamma_{ij}|_g &\lesssim |g-\gamma|_g + |X|_g + |z|_g^2.
\end{aligned}
$$

Proof. Using the expressions given in (4.4.8) as well as the equation of motion of the geometry, (4.4.5b), we obtain

$$
\begin{aligned}
|\mathscr{K}_{ij}|_g &\lesssim |\partial_T g|_g + |\nabla X|_g \lesssim |\nabla X|_g + |X|_g + |\hat{N}| + |\Sigma|_g, \\
|\tau\Xi^i_j|_g &\lesssim |\partial_T g|_g + |\nabla X|_g + |X\nabla N|_g \lesssim |\nabla X|_g + |\nabla N|_g + |X| + |\hat{N}| + |\Sigma|_g, \\
|\tau^2\Upsilon^i|_g &\lesssim |\nabla N|_g + |\partial_T X|_g + |X|_g.
\end{aligned}
$$

Similarly, using the expressions given in (4.4.9) we find

$$
\begin{aligned}
|M_{ij} - \gamma_{ij}|_g &\lesssim |g-\gamma|_g + |X|_g^2|\nu| + |z|_g|X| + |z|_g^2 \lesssim |g-\gamma|_g + |X|_g + |z|_g^2, \\
|\tau\ell| &\lesssim \left(|\tau/\nu||\mathscr{K}|_g|X|_g + |\tau^2\Upsilon|_g|\tau\mu|^{-1} + |\tau\Xi|_g\right)|z^j|_g + \left(|\tau\Xi|_g + |\tau^2\Upsilon|_g\right)|X|_g|\nu| + |\tau\Xi|_g, \\
|\tau m^i|_g &\lesssim |M||z^j|_g\left(|\tau/\nu||\mathscr{K}|_g|X|_g|z^j|_g + |\tau\Xi|_g\right) + |M||\nu/\tau||\tau^2\Upsilon|_g.
\end{aligned}
$$

□

Lemma 4.4.9 (Fluid equations of motion)**.** *The Euler equations* (4.4.7) *can be rewritten as*

$$
M^0\partial_T Z - (C^k + M^k)\nabla_k Z = -\mathscr{B}\mathbb{P}Z - F, \tag{4.4.15}
$$

where

$$
\begin{aligned}
M^0(Z) &= \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}g_{ij} \end{pmatrix} + e^{-2T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathrm{O}(|X|_g^2) + \begin{pmatrix} 0 & |z|_g^2 \\ |z|_g^2 & |z|_g^2 \end{pmatrix}, \\
M^k(Z) &= \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{O}(|z|_g) \end{pmatrix} + \mathrm{O}(|X|_g^2 + |z|_g^2), \quad \textit{and}
\end{aligned}
$$

$$\mathscr{B} := (K^{-1}-3)\mathbb{1}\,, \quad \mathbb{P} := \begin{pmatrix} 0 & 0 \\ 0 & \delta^i_j \end{pmatrix}, \quad C^k = \begin{pmatrix} 0 & \delta^k_j \\ \delta^k_i & 0 \end{pmatrix}, \tag{4.4.16}$$

and

$$|\mathbb{P}F| + |\mathbb{P}^\perp F| \lesssim \mathrm{O}(|z|^2_g + |X|_g + |\nabla X|_g + |\hat{N}| + |\nabla N|_g + |\Sigma|_g + |\partial_T X|_g). \tag{4.4.17}$$

Proof. We transform to new variables $Z = (\psi, z^a)$ and define $z_i = g_{ij}z^j$. We first compute

$$w_a = v_a = g_{a\mu}v^\mu = X_a v^\tau + g_{ac}z^c b(\psi) = X_a\nu + bz_a\,.$$

Using (4.2.32) and (4.2.46), we compute (using things like $\mu^{-1}\sim\tau\lesssim 1$)

$$\begin{aligned}
A^0_0 &= b^2K\left[1 + \frac{2}{\mu(\psi+c_2)}X_k z^k\right] + \mathrm{O}(|z|^2_g)\,,\\
A^0_j &= b^2K\frac{X_j}{\mu}\left[-1 - \frac{K^{-1}}{(\psi+c_2)}z^k X_k\frac{\nu}{\tau}\left(-2 + \frac{\nu}{\tau\mu}(-N^2+|X|^2_g)\right)\right] + \mathrm{O}(|z|^2_g),\\
A^0_{ij} &= b^2\left[g_{ij} + X_iX_j\frac{\nu}{\tau\mu}\Big(-2 + (-N^2+|X|^2_g)\frac{\nu}{\tau\mu}\Big)\right]\\
&\quad + 2b^2 z_{(i}X_{j)}\left[-\frac{b}{\tau\mu} + (-N^2+|X|^2_g)\frac{b\nu}{\tau\mu} - 2\frac{K}{\mu}\right] + \mathrm{O}(|z|^2_g),\\
A^k_0 &= \mathrm{O}(|z|^2_g), \qquad A^k_j = b^2K\delta^k_j + \mathrm{O}(|z|^2_g),\\
A^k_{ij} &= b^2\left[\Big(\frac{2}{\psi+c_2}\Big)g_{ij}z^k + \frac{\kappa K}{(\psi+c_2)}(\delta^k_j z_i + \delta^k_i z_j)\right]\\
&\quad + b^3X_iX_jz^k\frac{\nu}{\tau\mu}\left[-2 + (-N^2+|X|^2_g)\frac{\nu}{\tau\mu}\right] + \mathrm{O}(|z|^2_g)\,.
\end{aligned}$$

Premultiply the PDE system (4.2.32) by $(A^0_0)^{-1}$ and rewrite it as

$$(A^0_0)^{-1}A^0\partial_\tau Z + \frac{1}{\tau}\frac{\tau}{\nu}(A^0_0)^{-1}A^k\nabla_k Z = (A^0_0)^{-1}Q^{\mathrm{tr}}(H - B^0Y)\,.$$

We define

$$\begin{aligned}
M^0(Z) &:= (A^0_0)^{-1}A^0(Z), \quad C^k := \Big((A^0_0)^{-1}A^k\Big)\,|_{z^i=0},\\
M^k(Z) &:= \frac{\tau}{\nu}(A^0_0)^{-1}A^k(Z) - C^k \quad \text{and} \quad (\mathscr{F}_0\,,\mathscr{F}_j)^{\mathrm{tr}} := \tau(A^0_0)^{-1}Q^{\mathrm{tr}}(H-B^0Y)\,.
\end{aligned}$$

Then, thanks to Lemma 4.2.4, the following PDE holds

$$M^0\partial_\tau Z + \frac{1}{\tau}(C^k + M^k)\nabla_k Z = \frac{1}{\tau}(\mathscr{F}_0, \mathscr{F}_j)^{\mathrm{tr}}.$$

Noting that $(A^0_0)^{-1} = (b^2K)^{-1} + \mathrm{O}(|X|^2_g + |z|^2_g)$ we compute

$$\begin{aligned}
M^0(Z) &= \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}g_{ij} \end{pmatrix} + \begin{pmatrix} 0 & X_j\mu^{-1} \\ X_j\mu^{-1} & \mathrm{O}(|X|^2|z|_g\mu^{-1}) \end{pmatrix} + \mathrm{O}(|X|^2_g + |z|^2_g),\\
&= \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}g_{ij} \end{pmatrix} + |\tau|e^{-T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathrm{O}(|X|^2_g + |z|^2_g),
\end{aligned} \tag{4.4.18}$$

and

$$
\begin{aligned}
M^k(Z) &= \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{K}\left(\frac{2}{\psi+c_2}\right) g_{ij} z^k + \frac{\kappa}{(\psi+c_2)}(\delta^k_j z_i + \delta^k_i z_j) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{O}(|X|^2 |z|_g \mu^{-1}) \end{pmatrix} + \mathrm{O}(|X|^2_g + |z|^2_g) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{O}(|z|_g) \end{pmatrix} + \mathrm{O}(|X|^2_g + |z|^2_g).
\end{aligned} \tag{4.4.19}
$$

Next, we introduce the following notation

$$
R_i := \frac{1-3K}{\mu\nu} w_i \partial_\tau \Psi + m_i + \frac{K}{\mu\nu} w_i D_2 a \partial_\tau g_{lk} \cdot z^l z^k.
$$

Using Lemma (4.4.8) we compute

$$
\begin{aligned}
|\mathscr{F}_0| &\lesssim |\tau (bs)^{-2} \left[K D_1 a (\ell - D_2 a \partial_\tau g_{ij} \cdot z^i z^j) + b' z^j R_j \right] | + \mathrm{O}(|X|^2_g + |z|^2_g) \\
&\lesssim |\psi + c_2| \Big(|\tau \ell| + |\partial_T g| |z|^2_g \Big) + |z|_g |X| |\nu| + |z|_g |\tau m_j|_g + \mathrm{O}(|X|^2_g + |z|^2_g) \\
&\lesssim \mathrm{O}(|X|^2_g + |z|^2_g + |X|_g + |\nabla X|_g + |\hat{N}| + |\nabla N|_g + |\Sigma|_g + |\partial_T X|_g).
\end{aligned}
$$

Similarly, for each j we have

$$
\begin{aligned}
\mathscr{F}_j &= \tau (bs)^{-2} \left[2 K D_2 a z_j (\ell - D_2 a \partial_\tau g_{ij} \cdot z^i z^j) + b R_j \right] + \mathrm{O}(|X|^2_g + |z|^2_g) \\
&\lesssim (3 - K^{-1}) z_j + (3 - K^{-1}) |\nu| |X|_g + |\psi + c_2| |\tau m_i|_g + |z|_g |\tau \ell| + \mathrm{O}(|X|^2_g + |z|^2_g) \\
&\lesssim (3 - K^{-1}) z_j + \mathrm{O}(|X|^2_g + |z|^2_g + |X|_g + |\nabla X|_g + |\hat{N}| + |\nabla N|_g + |\Sigma|_g + |\partial_T X|_g).
\end{aligned}
$$

Defining $F = \mathscr{F} - \mathscr{B}\mathbb{P}Z$ we see that the above bounds on the components of $\mathscr{F}$ imply the estimate on F in (4.4.17). $\square$

4.4.5 Definition and coercivity properties of the fluid energy

Now we are equipped to introduce the higher order energy functionals, which are coercive with respect to the Sobolev-energies of their respective order.

Definition 4.4.9. Let $0 \leq s \leq k-1$. We define the following energy functionals

$$
\begin{aligned}
&E_s(Z) = \frac{1}{2} \sum_{l \leq s} \int_M \langle \nabla^l Z, M^0 \nabla^l Z \rangle_g \, \mu_g, \quad E^p_s(Z) = \frac{1}{2} \sum_{l \leq s} \int_M \langle \mathbb{P} \nabla^l Z, M^0 \mathbb{P} \nabla^l Z \rangle_g \, \mu_g, \\
&\dot{E}_s(Z) = \int_M \langle \nabla^s Z, M^0 \nabla^s Z \rangle_g \, \mu_g.
\end{aligned}
$$

We refer to E^p_s as the *parallel energy* and $\dot{E}_s$ as the *homogeneous energy*, of order s respectively.

Lemma 4.4.10 (Equivalence of the energy norms)**.** *Under the bootstrap assumptions* (4.4.13), *for* $0 \leq s \leq k-1$, *we have that*

$$
E_s(Z) \cong \|Z\|^2_{H^s}, \quad E^p_s(Z) \cong \|\mathbb{P} Z\|^2_{H^s}, \quad \dot{E}_s(Z) \cong \|Z\|_{\dot{H}_s}.
$$

Proof. From (4.4.18) we see that M^0 is approximately the diagonal matrix $\operatorname{diag}(1, g)$. Together with Sobolev embedding, this yields

$$|\dot{E}_s(Z) - \|Z\|^2_{\dot{H}^s}| \lesssim \|Z\|^2_{\dot{H}^s} + (e^{-(1+\lambda)T} + \||X|^2_g + |z|^2_g\|_{L^\infty})\|Z\|^2_{\dot{H}^s} \lesssim \epsilon.$$

The other expressions involve similar estimates, and so we find the energy norms are equivalent to their Sobolev counterparts. $\square$

4.4.6 Fluid energy estimates of lower order

Before we consider the time evolution of the lowest order energy $E(Z)$ we first derive some preliminary estimates and identities for the coefficient matrices.

Lemma 4.4.11 (Properties of the coefficient matrices)**.** *We have that*

$$\begin{aligned}|\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp|_{\mathrm{op}} \\ + |\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}|_{\mathrm{op}} &\lesssim |\mathbb{P}Z| + |\mathbb{P}\nabla Z| + \epsilon e^{-\lambda T}, \\ |\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp|_{\mathrm{op}} &\lesssim |\mathbb{P}Z|^2 + |\mathbb{P}\nabla Z|^2 + \epsilon e^{-\lambda T}.\end{aligned}$$

Furthermore, for any square matrix A, we have the following identity

$$\langle Z, AZ\rangle = \langle \mathbb{P}Z, (\mathbb{P}A\mathbb{P})\mathbb{P}Z\rangle + \langle \mathbb{P}^\perp Z, (\mathbb{P}^\perp A\mathbb{P})\mathbb{P}Z\rangle + \langle \mathbb{P}Z, (\mathbb{P}A\mathbb{P}^\perp)\mathbb{P}^\perp Z\rangle + \langle \mathbb{P}^\perp Z, (\mathbb{P}^\perp A\mathbb{P}^\perp)\mathbb{P}^\perp Z\rangle.$$

Proof. First note that, using the equations of motion (4.4.15), $\partial_T M^0$ can be schematically rewritten using the chain rule as

$$\partial_T M^0(Z, g, X, N) = D_Z M^0 \cdot \partial_T Z + D_g M^0 \cdot \partial_T g + D_X M^0 \cdot \partial_T X + D_N M^0 \partial_T N.$$

The last three terms involving the geometry can be estimated by

$$\|D_g M^0 \partial_T g + D_X M^0 \partial_T X + D_N M^0 \partial_T N\|_{L^\infty} \lesssim \|\partial_T g\|_{H^2} + \|\partial_T X\|_{H^2} + \|\partial_T N\|_{H^2} \lesssim \epsilon e^{-\lambda T},$$

using Sobolev-embedding as well as the bootstrap assumptions and (4.4.14).

Next, we inspect

$$D_Z M^0 \cdot \partial_T Z = D_Z M^0 \cdot (M^0)^{-1}\left((C^k + M^k)\nabla_k Z - \mathscr{B}\mathbb{P}Z - F\right).$$

Every term in this expression is a sum of matrices of the form $f(Z, \nabla Z) D_Z M^0$, where the function f schematically indicates everything coming from

$$(M^0)^{-1}\left((C^k + M^k)\nabla_k Z - \mathscr{B}\mathbb{P}Z - F\right).$$

Now from (4.4.18) we see that to leading order M^0 is given by the diagonal matrix $\operatorname{diag}(1, K^{-1} g_{ij})$. Thus, these leading order terms vanish under the derivative D_Z. The remaining terms in $D_Z M^0$ involve $X, \nabla X$ or $|z|_g$ or $|z|^2_g$, where we used the fact that $D_{z^i}(|z|^2_g) = \mathrm{O}(|z|_g)$. Hence, the first inequality follows for $\partial_T M^0$. As explained in Remark 4.3.1, we again find that $\mathbb{P}^\perp D_Z M^0 \mathbb{P}^\perp = \mathrm{O}(|X|_g)$ and hence the second inequality of the statement is trivially satisfied for $\partial_T M^0$.

Inspecting M^k using (4.4.19), we see the leading order term is the diagonal matrix $\mathrm{diag}(0, \mathrm{O}(|z|_g))$. We immediately infer that each piece of $\nabla_k M^k$, except for $\mathbb{P}\nabla_k M^k \mathbb{P}$, may be estimated in the same way as the error terms of $\partial_T M^0$. For the $\mathbb{P}\nabla_k M^k \mathbb{P}$ part, the leading order term consists of terms of order of $|z|_g$ as well as $|\nabla z|_g$. These are estimated by $|\mathbb{P}Z|$ and $|\mathbb{P}\nabla Z|$.

Finally, using $\mathbb{P}^2 = \mathbb{P}$, $\mathbb{P}^T = \mathbb{P}$ and $\mathbb{1} = \mathbb{P} + \mathbb{P}^\perp$, the identity for $\langle Z, AZ\rangle$ follows. $\square$

Equipped with the preliminary results above we can now establish an estimate for the time evolution of the energy of order zero.

Proposition 4.4.1. *Under* (4.4.13) *there exists a constant* $C > 0$ *such that*

$$\partial_T E_0(Z) \leq (3 - K^{-1} + C\epsilon) E_0^p(Z) + C\epsilon e^{-\lambda T} + C\epsilon E_1^p(Z).$$

Remark 4.4.7. The parameter $K < \frac{1}{3}$ is fixed, and so $3 - K^{-1} + C\epsilon$ is negative provided ϵ is sufficiently small. This overall negative sign of the parallel part of the energy is essential to closing the final energy estimates.

Proof. A computation using the equations of motion (4.4.15) and Lemma 4.4.4 yields

$$\begin{aligned}
\partial_T E_0(Z) &\leq \int_M \langle Z, M^0 \partial_T Z\rangle + \frac{1}{2}\int_M \langle Z, (\partial_T M^0) Z\rangle + C\epsilon e^{-\lambda T} E_0(Z) \\
&= \int_M \langle Z, (C^a + M^a)\nabla_a Z\rangle - \int_M \langle \mathbb{P}Z, \mathscr{B}\mathbb{P}Z\rangle \\
&\quad - \int_M \langle Z, F\rangle + \frac{1}{2}\int_M \langle Z, (\partial_T M^0) Z\rangle + C\epsilon e^{-\lambda T} E_0(Z).
\end{aligned}$$

Using integration by parts and the fact that the matrices C^a and M^a are symmetric we find

$$\int_M \langle Z, (C^a + M^a)\nabla_a Z\rangle = -\frac{1}{2}\int_M \langle Z, \nabla_a (C^a + M^a) Z\rangle = -\frac{1}{2}\int_M \langle Z, (\nabla_a M^a) Z\rangle.$$

Note that a is indeed an index that stems from a spatial vector field, in particular z, which allows for integration by parts. Hence we have

$$\partial_T E_0(Z) \leq (3 - K^{-1}) E_0^p(Z) + \int_M \langle Z, (\partial_T M^0 - \nabla_a M^a) Z\rangle - \int_M \langle Z, F\rangle + C\epsilon e^{-\lambda T} E_0(Z).$$

We start with the term involving F. Using equation (4.4.17), we obtain

$$|\int_M \langle Z, F\rangle| \lesssim \|Z\|_{L^2} \|F\|_{L^2} \lesssim \|Z\|_{L^2} \|F\|_{L^\infty} \lesssim \epsilon^2 e^{-\lambda T}.$$

We next analyse the term involving $\partial_T M^0 - \nabla_k M^k$. Using the matrix identity in Lemma 4.4.11 as well as Hölder's inequality we find

$$\begin{aligned}
\int_M \langle Z, (\partial_T M^0 - \nabla_a M^a) Z\rangle &= \int_M \langle \mathbb{P}Z, \mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}\mathbb{P}Z\rangle \\
&\quad + \int_M \langle \mathbb{P}^\perp Z, \mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}\mathbb{P}Z\rangle + \int_M \langle \mathbb{P}Z, \mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp\mathbb{P}^\perp Z\rangle \\
&\quad + \langle \mathbb{P}^\perp Z, \mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp\mathbb{P}^\perp Z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\lesssim E_0^p(Z)|\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}|_{\mathrm{op}} + \|Z\|_{L^\infty}^2 \int_M |\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp|_{\mathrm{op}} \\
&\qquad + \|Z\|_{L^\infty} \int_M (|\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp|_{\mathrm{op}})|\mathbb{P}Z| \\
&\lesssim E_0^p(Z)(|\mathbb{P}Z| + |\mathbb{P}\nabla Z| + \epsilon e^{-\lambda T}) + \|Z\|_{L^\infty}^2 (E_0^p(Z) + E_1^p(Z) + \epsilon e^{-\lambda T}) \\
&\qquad + \|Z\|_{L^\infty}\|\mathbb{P}Z\|_{L^\infty}(E_0^p(Z) + E_1^p(Z) + \epsilon e^{-\lambda T}).
\end{aligned}
$$

In conclusion, using Sobolev embedding and the smallness assumption, we find that

$$
\int_M \langle Z, (\partial_T M^0 - \nabla_a M^a)Z\rangle \lesssim (\sqrt{\epsilon} + \epsilon e^{-\lambda T})E_0^p(Z) + \epsilon(E_0^p(Z) + E_1^p(Z)) + \epsilon e^{-\lambda T},
$$

where we used Young's inequality and Cauchy-Schwarz in the last step. All together we get

$$
\partial_T E_0(Z) \leq (3 - K^{-1} + C\epsilon)E_0^p(Z) + C\epsilon e^{-\lambda T} + C\epsilon E_1^p(Z).
$$

□

Remark 4.4.8. Note that, due to the inclusion of $E_1^p(Z)$, the energy estimate in Proposition 4.4.1 does not close. However, this is not an issue, since this problem is unique to the first order and this term can be absorbed into a negative definite term of the type $-cE_1^p(Z)$, $c > 0$, appearing in the final estimate.

4.5 Estimates of the higher order fluid energy

In this section we derive an estimate for the higher-order fluid energies. These estimates will be weaker compared to the zero order case. However, in the end this will be remedied by exploiting the lower order estimate. We start be deriving an expression for the time-evolution of the homogeneous part of the energy of order $\ell \geq 1$.

Lemma 4.5.1. *Let $1 \leq \ell \leq k-1$. Under* (4.4.13) *there is a constant $C > 0$ such that*

$$
\begin{aligned}
\frac{1}{2}\partial_T\Big(\int_M \langle \nabla^\ell Z, M^0 \nabla^\ell Z\rangle\Big) &\leq (3 - K^{-1}) \int_M \langle \nabla^\ell Z, \mathbb{P}\nabla^\ell Z\rangle \\
&\quad + \frac{1}{2}\int_M \langle \nabla^\ell Z, (\partial_T M^0 - \nabla_a M^a)\nabla^\ell Z\rangle + \int_M \langle \nabla^\ell Z, G^\ell\rangle + C\epsilon e^{-\lambda T} E_\ell(Z),
\end{aligned}
\tag{4.5.1}
$$

where

$$
\begin{aligned}
G^\ell := &-\mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}((C^a + M^a)\nabla_a Z - \mathscr{B}\mathbb{P}Z - F) \\
&+ \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}(C^a + M^a)]\nabla_a Z - \mathscr{B}\nabla^\ell(\mathscr{B}^{-1}F) \\
&- M^0[\nabla^\ell, \partial_T]Z - (C^a + M^a)[\nabla^\ell, \nabla_a]Z.
\end{aligned}
$$

Proof. Let $\ell \geq 1$. Applying the operator $\mathscr{B}\nabla^\ell\mathscr{B}^{-1}$ to the fluid equations of motion of (4.4.15) yields

$$
M^0\partial_T\nabla^\ell Z - (C^a + M^a)\nabla_a\nabla^\ell Z = -\mathscr{B}\mathbb{P}\nabla^\ell Z + G^\ell.
\tag{4.5.2}
$$

The error term G^ℓ is computed as

$$
G^\ell = -\mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0]\partial_T Z + \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}(C^a + M^a)]\nabla_a Z - \mathscr{B}\nabla^\ell(\mathscr{B}^{-1}F)
$$

$$
\begin{aligned}
&\quad - M^0[\nabla^\ell, \partial_T]Z - (C^a + M^a)[\nabla^\ell, \nabla_a]Z \\
&= -\mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}((C^a + M^a)\nabla_a Z - \mathscr{B}\mathbb{P}Z - F) \\
&\quad + \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}(C^a + M^a)]\nabla_a Z - \mathscr{B}\nabla^\ell(\mathscr{B}^{-1}F) \\
&\quad - M^0[\nabla^\ell, \partial_T]Z - (C^a + M^a)[\nabla^\ell, \nabla_a]Z,
\end{aligned}
$$

where we simply inserted the equations of motion (4.4.15) in the last step.

Using Lemma 4.4.4 and (4.5.2), we have

$$
\begin{aligned}
\frac{1}{2}\partial_T \int_M \langle \nabla^\ell Z, M^0 \nabla^\ell Z\rangle &\le \int_M \langle \nabla^\ell Z, M^0 \partial_T \nabla^\ell Z\rangle + \frac{1}{2}\int_M \langle \nabla^\ell Z, (\partial_T M^0)\nabla^\ell Z\rangle + C\epsilon e^{-\lambda T}E(Z) \\
&= \frac{1}{2}\int_M \langle \nabla^\ell Z, (\partial_T M^0 - \nabla_a M^a)\nabla^\ell Z\rangle - \int_M \langle \nabla^\ell Z, \mathscr{B}\mathbb{P}\nabla^\ell Z\rangle \\
&\quad + \int_M \langle \nabla^\ell Z, G^\ell\rangle + C\epsilon e^{-\lambda T}E(Z).
\end{aligned}
$$

□

We now proceed by introducing some preliminary estimates.

Lemma 4.5.2. *The fluid coefficient matrices and the inhomogeneity F obey the following estimates*

$$
\begin{aligned}
|\nabla \mathbb{P} M^0 \mathbb{P}^\perp|_{\mathrm{op}} + |\nabla \mathbb{P}^\perp M^0 \mathbb{P}|_{\mathrm{op}} &\lesssim e^{-T}\,\mathrm{O}(|X|_g) + \mathrm{O}(|X|_g^2 + |z|_g^2 + |\nabla z|_g^2), \qquad (4.5.3)\\
|\mathbb{P}(M^0)^{-1}\mathbb{P}^\perp|_{\mathrm{op}} + |\mathbb{P}^\perp (M^0)^{-1}\mathbb{P}|_{\mathrm{op}} &\lesssim e^{-T}\,\mathrm{O}(|X|_g) + \mathrm{O}(|X|_g^2 + |z|_g^2), \\
|\mathbb{P}^\perp(M^a\nabla_a Z - \mathscr{B}\mathbb{P}Z - F)\mathbb{P}| + & \\
|\mathbb{P}^\perp(M^a\nabla_a Z - \mathscr{B}\mathbb{P}Z - F)\mathbb{P}^\perp| &\lesssim \epsilon e^{-\lambda T} + \mathrm{O}(|X|_g^2 + |z|_g^2 + |\nabla z|_g^2), \\
|M^a\nabla_a Z - \mathscr{B}\mathbb{P}Z - F| &\lesssim \epsilon e^{-\lambda T} + \mathrm{O}(|z|_g + |X|_g^2), \\
|\mathbb{P}^\perp C^a \nabla_a Z| = |\mathbb{P}^\perp C^a \mathbb{P}\nabla_a Z| &\lesssim \mathrm{O}(|\nabla z|_g).
\end{aligned}
$$

Proof. Recall from Remark 4.3.2 that conjugating matrices with $\mathbb{P}$ and $\mathbb{P}^\perp$ picks out specific matrix elements. Considering the explicit form of M^0 in (4.4.18) we realize that the leading order term is given by $\mathrm{diag}(1, K^{-1}g)$ which is annihilated by the covariant derivative. Hence follows (4.5.3).

Inspecting (4.4.19), we see that the leading order term in M^a is of order $\mathrm{O}(|z|_g)$ and is picked out by the conjugation $\mathbb{P}M^a\mathbb{P}$. In conjunction with (4.4.17) we conclude that the statements about $M^a\nabla_a Z - \mathscr{B}Z - F$ above are true. Note that we treated the ∇Z terms as negligible using Sobolev embedding. The rest of the statements follows in a similar fashion. □

Lemma 4.5.3. *Let $1 \le \ell \le k-1$. Under* (4.4.13)*, there is a constant $C > 0$ such that the following estimates hold*

$$
\begin{aligned}
\int_M \langle \nabla^\ell Z, (\partial_T M^0 - \nabla_a M^a)\nabla^\ell Z\rangle &\le C\epsilon \dot{E}^p{}_\ell(Z) + C\epsilon E^p_{k-1}(Z), \\
\int_M \langle \nabla^\ell Z, G^\ell\rangle &\le C\epsilon e^{\lambda T} E_{k-1}(Z)^{\frac{1}{2}} + C\epsilon E^p_{k-1}(Z) + B^\ell,
\end{aligned}
\qquad (4.5.4)
$$

where the problematic error term B^ℓ is defined as

$$B^\ell := \int_M \langle \mathbb{P}\nabla^\ell Z, \mathbb{P}C^a\mathbb{P}^\perp[\nabla^\ell, \nabla_a]Z\rangle + \langle \mathbb{P}^\perp\nabla^\ell Z, \mathbb{P}^\perp C^a\mathbb{P}[\nabla^\ell, \nabla_a]Z\rangle.$$

Remark 4.5.1. The error terms B^ℓ involve the matrix C^a, defined in (4.4.16). The off-diagonal structure of C^a causes a 'mixing' effect in B^ℓ by leading to terms of the type $\mathrm{O}(\psi \cdot |z|_g)$. Such terms cannot be controlled in terms of the energy E^p since ψ is not controlled by the parallel energy. Furthermore, note that these terms arise due to the curved geometry and are not present in a setting in which derivatives commute. Hence, these terms require additional treatment given below in the form of modified energies.

Proof. The proof of the divergence estimate is essentially the same as in lowest order. However, we repeat the calculation and add some details. Using the identity from Lemma 4.4.11

$$\begin{aligned}
&\int_M \langle \nabla^\ell Z, (\partial_T M^0 - \nabla_a M^a)\nabla^\ell Z\rangle \\
&\quad = \int_M \langle \mathbb{P}\nabla^\ell Z, \mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}\mathbb{P}\nabla^\ell Z\rangle + \int_M \langle \mathbb{P}^\perp\nabla^\ell Z, \mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}\mathbb{P}\nabla^\ell Z\rangle \\
&\qquad + \int_M \langle \mathbb{P}\nabla^\ell Z, \mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp\mathbb{P}^\perp\nabla^\ell Z\rangle \\
&\qquad + \int_M \langle \mathbb{P}^\perp\nabla^\ell Z, \mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp\mathbb{P}^\perp\nabla^\ell Z\rangle \\
&\quad \lesssim \dot{E^p}_\ell(Z)\|\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}\|_{L^\infty} + \dot{E}_\ell(Z)\|\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp\|_{L^\infty} \\
&\qquad + \int_M (|\mathbb{P}^\perp(\partial_T M^0 - \nabla_a M^a)\mathbb{P}|_{\mathrm{op}} + |\mathbb{P}(\partial_T M^0 - \nabla_a M^a)\mathbb{P}^\perp|_{\mathrm{op}})|\mathbb{P}\nabla^\ell Z||\nabla^\ell Z|.
\end{aligned}$$

Performing a similar estimate as in the proof of Proposition 4.4.1 we find

$$\int_M \langle \nabla^\ell Z, (\partial_T M^0 - \nabla_a M^a)\nabla^\ell Z\rangle \lesssim \epsilon \dot{E^p}_\ell(Z) + \epsilon E^p_{k-1}(Z) + \epsilon e^{-\lambda T}.$$

Deriving (4.5.4) is slightly more involved. We start be establishing a statement for a general tensor-valued vector V (i.e. the components of V are tensors fields on M):

$$\begin{aligned}
&\int_M \langle \nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}V\rangle \\
&\quad = \int_M \langle \mathbb{P}\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}\mathbb{P}V\rangle + \int_M \langle \mathbb{P}^\perp\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}\mathbb{P}^\perp V\rangle \\
&\qquad + \int_M \langle \mathbb{P}\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}M^0](\mathbb{P} + \mathbb{P}^\perp)(M^0)^{-1}\mathbb{P}^\perp V\rangle \qquad (4.5.5)\\
&\qquad + \int_M \langle \mathbb{P}^\perp\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}^\perp M^0](\mathbb{P} + \mathbb{P}^\perp)(M^0)^{-1}\mathbb{P}V\rangle.
\end{aligned}$$

The first line on the right hand side of (4.5.5) can be estimated by

$$\|\mathbb{P}\nabla^\ell Z\|_{L^2}\|\nabla M^0\|_{H^{k-2}}\|\mathbb{P}V\|_{H^{k-2}} + \|\nabla^\ell Z\|_{L^2}\|\nabla M^0\|_{H^{k-2}}\|\mathbb{P}^\perp V\|_{H^{k-2}},$$

where we used standard estimates for the commutator (see e.g. [11, Theorem A.3]). The second and third terms on the right hand side of (4.5.5) can be expanded as

$$\int_M \langle \mathbb{P}\nabla^\ell Z, \mathscr{B}([\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}M^0\mathbb{P}^\perp](M^0)^{-1}\mathbb{P}^\perp V + [\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}M^0]\mathbb{P}(M^0)^{-1}\mathbb{P}^\perp V)\rangle$$

$$
\begin{aligned}
&+\int_M \langle \mathbb{P}^\perp \nabla^\ell Z, \mathscr{B}([\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}^\perp M^0 \mathbb{P}](M^0)^{-1}\mathbb{P}V + [\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}^\perp M^0]\mathbb{P}^\perp (M^0)^{-1}\mathbb{P}V)\rangle \\
\lesssim\ & \|\mathbb{P}\nabla^\ell Z\|_{L^2}\Big(\|\nabla \mathbb{P} M^0 \mathbb{P}^\perp\|_{H^{k-2}}\|(M^0)^{-1}\mathbb{P}^\perp V\|_{H^{k-2}} \\
&+ \|\nabla \mathbb{P} M^0\|_{H^{k-2}}\|\mathbb{P}(M^0)^{-1}\mathbb{P}^\perp\|_{H^{k-2}}\|V\|_{H^{k-2}}\Big) \\
&+ \|\nabla^\ell Z\|_{L^2}\Big(\|\nabla \mathbb{P}^\perp M^0 \mathbb{P}\|_{H^{k-2}}\|(M^0)^{-1}\mathbb{P}V\|_{H^{k-2}} \\
&+ \|\nabla \mathbb{P}^\perp M^0\|_{H^{k-2}}\|\mathbb{P}^\perp(M^0)^{-1}\mathbb{P}\|_{H^{k-2}}\|V\|_{H^{k-2}}\Big).
\end{aligned}
$$

Utilizing these estimates for $V = (M^0)^{-1}((M^a + C^a)\nabla_a Z - \mathscr{B}\mathbb{P}Z - F)$, in conjunction with Lemma 4.5.2, we conclude that

$$
\int_M \langle \nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^0](M^0)^{-1}((M^a + C^a)\nabla_a Z - \mathscr{B}\mathbb{P}Z - F)\rangle \lesssim \epsilon E^p_{k-1} + \epsilon e^{-\lambda T} E_{k-1}(Z)^{\frac{1}{2}}.
$$

Now we tackle the rest of the inhomogeneity described in G^ℓ.

$$
\begin{aligned}
&\int_M \langle \nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}(C^a + M^a)]\nabla_a Z\rangle \\
&\quad = \int_M \langle \nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}C^a]\nabla_a Z\rangle + \int_M \langle \nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}M^a]\nabla_a Z\rangle \\
&\quad = \int_M \langle \mathbb{P}\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}M^a(\mathbb{P}+\mathbb{P}^\perp)]\nabla_a Z\rangle \\
&\qquad + \int_M \langle \mathbb{P}^\perp\nabla^\ell Z, \mathscr{B}[\nabla^\ell, \mathscr{B}^{-1}\mathbb{P}^\perp M^a(\mathbb{P}+\mathbb{P}^\perp)]\nabla_a Z\rangle \\
&\quad \lesssim \|\mathbb{P}\nabla^\ell Z\|_{L^2}\|\nabla \mathbb{P}Z\|_{H^{k-2}}\|\nabla Z\|_{H^{k-2}} + \|\nabla(\mathbb{P}Z)\|^2_{H^{k-2}}\|\nabla Z\|_{H^{k-2}} + \|X\|^2_{H^{k-2}}\|Z\|^2_{H^{k-2}},
\end{aligned}
$$

where we used that M^a only has leading order terms in the $\mathbb{P}M^a\mathbb{P}$ component, as seen in (4.4.19), and the last term stems from the error terms in M^a. Furthermore, using (4.4.17), we have that

$$
\int_M \langle \nabla^\ell Z, \mathscr{B}\nabla^\ell(\mathscr{B}^{-1}F)\rangle \lesssim \|Z\|_{H^{k-1}}\Big(\|\mathbb{P}Z\|^2_{H^{k-1}} + \|X\|^2_{H^{k-1}} + \epsilon e^{-T}\Big).
$$

Lastly, using $\mathbb{P}C^a\mathbb{P} = \mathbb{P}^\perp C^a \mathbb{P}^\perp = 0$, we find that

$$
\begin{aligned}
\int_M \langle \nabla^\ell Z, (C^a + M^a)[\nabla^\ell, \nabla_a]Z\rangle &= \int_M \langle (\mathbb{P}+\mathbb{P}^\perp)\nabla^\ell Z, (C^a + M^a)(\mathbb{P}+\mathbb{P}^\perp)[\nabla^\ell, \nabla_a]Z\rangle \\
&= \int_M \langle \mathbb{P}\nabla^\ell Z, \mathbb{P}M^a\mathbb{P}[\nabla^\ell, \nabla_a]Z\rangle + \|X\|^2_{H^{k-1}}\|Z\|^2_{H^{k-1}} + B^\ell \\
&\le C\varepsilon\|\mathbb{P}Z\|^2_{H^{k-1}} + C\|X\|^2_{H^{k-1}}\|Z\|^2_{H^{k-1}} + B^\ell.
\end{aligned}
$$

Note that we estimated the non-leading order terms of M^a by $C\|X\|^2_{H^{k-1}}\|Z\|^2_{H^{k-1}}$. Applying the bootstrap assumptions, we see that these terms are exponentially decaying. The last term to control in G^ℓ is the due to the geometry:

$$
\int_M \langle \nabla^\ell Z, M^0[\nabla^\ell, \partial_T]Z\rangle \lesssim \|\partial_T\Gamma\|_{H^{k-1}} E_{k-1}(Z).
$$

The norm of the time derivative of the Christoffel symbols can simply by estimated by $\epsilon e^{-\lambda T}$ by applying the equations of motion of the geometry in (4.4.5b). □

4.5.1 Corrected fluid energies

We now introduce a corrected energy which allows us to compensate for the problematic terms B^ℓ. We start by giving explicit expressions for the first and second order corrected energies and show how these remedy the problem. Then, we present a process to extend these concepts to higher orders.

Definition 4.5.1 (First and second order corrected energies)**.** Define corrected first order energies by

$$\tilde{E}_1(Z) := E_1(Z) - \frac{1}{9}\int_M \langle \mathbb{P}Z, M^0\mathbb{P}Z\rangle, \qquad \dot{\tilde{E}}_1(Z) := \dot{E}_1(Z) - \frac{1}{9}\int_M \langle \mathbb{P}Z, M^0\mathbb{P}Z\rangle.$$

We define the corrected second order energies $\tilde{E}_2$ and $\dot{\tilde{E}}_2$ by

$$\begin{aligned}\tilde{E}_2(Z) &:= E_2 - \frac{4}{9}\int_M \langle \mathbb{P}\nabla Z, M^0\mathbb{P}\nabla Z\rangle - \frac{4}{81}\int_M \langle \mathbb{P}Z, M^0\mathbb{P}Z\rangle,\\ \dot{\tilde{E}}_2(Z) &:= \dot{E}_2 - \frac{4}{9}\int_M \langle \mathbb{P}\nabla Z, M^0\mathbb{P}\nabla Z\rangle - \frac{4}{81}\int_M \langle \mathbb{P}Z, M^0\mathbb{P}Z\rangle.\end{aligned}$$

Remark 4.5.2. Note that while the coefficients of the correction terms are negative, their modulus is less then $1/2$. Hence, the functionals $\tilde{E}$ are equivalent to the functions E and thus also to the Sobolev-norms of their respective order.

With these new energies at our disposal we are now able to formulate improved estimates.

Proposition 4.5.1. *For $\ell = 1, 2$ we have*

$$\partial_T \tilde{E}_\ell(Z) \leq (3 - K^{-1} + C\epsilon)E^p_\ell(Z) + C\epsilon e^{-\lambda T} + C\epsilon E^p_{k-1}(Z).$$

Proof. First, note that by using the explicit form of C^a, the critical term B^ℓ can be written as

$$B^\ell = \int_M \nabla^\ell z^a[\nabla^\ell, \nabla_a]\psi + \nabla^\ell\psi[\nabla^\ell, \nabla_a]z^a.$$

We start by considering the first order case $\ell = 1$. Using Lemma 4.5.3 as well as (4.5.1), we find

$$\partial_T \dot{\tilde{E}}_1 \leq (3 - K^{-1} + C\epsilon)\dot{E}^p{}_1(Z) + C\epsilon e^{-\lambda T} + C\epsilon E^p_{k-1}(Z) + B^1 - \frac{2}{9}\int_M \langle \mathbb{P}Z, M^0\partial_T\mathbb{P}Z\rangle.$$

Note that we used that $|\partial_T M^0| \lesssim \epsilon e^{-\lambda T} + E^p_{k-1}$. When $\ell = 1$, we obtain

$$B^1 = \int_M g^{mn}\nabla_m\psi[\nabla_n, \nabla_a]z^a = \int_M g^{mn}\nabla_m\psi \mathrm{Riem}[g]^a{}_{bna}z^b = -\int_M g^{mn}\nabla_m\psi\mathrm{Ric}[g]_{bn}z^b. \tag{4.5.6}$$

We first consider (4.5.6), when all the geometric quantities are with respect to γ instead of g:

$$-\int_M \gamma^{mn}\partial_m\psi\mathrm{Ric}[\gamma]_{nb}z^b = \frac{2}{9}\int_M \gamma^{mn}\ \gamma_{bn}z^b\nabla_m\psi = \frac{2}{9}\int_M z^m\nabla_m\psi.$$

So we find that the error term is

$$\begin{aligned}\int_M g^{mn}(\mathrm{Ric}[g]_{bn} - \mathrm{Ric}[\gamma]_{bn}) z^b \partial_m \psi \mu_g + \int_M (g-\gamma)^{mn} \mathrm{Ric}[\gamma]_{bn} z^b \partial_m \psi \mu_g \\ \lesssim \|g-\gamma\|_{H^2} \int_M |z \nabla \psi| \lesssim \epsilon e^{-T},\end{aligned}$$

where we used Lemma 4.4.5. Employing (4.4.15), we further calculate

$$\begin{aligned}-\frac{2}{9}\int_M \langle \mathbb{P}Z, \partial_T \mathbb{P}Z\rangle &= -\frac{2}{9}\int_M \langle \mathbb{P}Z, \mathbb{P}((C^a + M^a)\nabla_a Z - \mathscr{B}\mathbb{P}Z - F)\rangle \\ &\le \underbrace{(K^{-1}-3)\frac{2}{9}\int_M \langle \mathbb{P}Z, \mathbb{P}Z\rangle}_{=:(*)} - \frac{2}{9}\int_M z^m \nabla_m \psi + \epsilon\, \mathrm{O}(|z|_g^2 + e^{-\lambda T}).\end{aligned}$$

Note that the positive term $(*)$ from $\mathscr{B}$ is not a problem, since this Lemma gives us the negative term $(3 - K^{-1} + \epsilon)E^p(Z)$ in the estimate for $E_0(Z)$ and it is easy to see that

$$(3 - K^{-1} + C\epsilon)E^p(Z) + (K^{-1}-3)\frac{2}{9}\int_M \langle \mathbb{P}Z, \mathbb{P}Z\rangle \le C(3 - K^{-1} + C\epsilon)E^p(Z),$$

where $C > 0$. Next, we compute the problematic term in the case $\ell = 2$:

$$\begin{aligned}B^2 = \int_M & g^{mn} g^{rs} \nabla_m \nabla_r \psi [\nabla_n \nabla_s, \nabla_a] z^a + g^{mn} g^{rs} \nabla_m \nabla_r z^a [\nabla_n \nabla_s, \nabla_a]\psi \\ &= \int_M g^{mn} g^{rs} \nabla_m \nabla_r \psi (\nabla_n [\nabla_s, \nabla_a] z^a + [\nabla_n, \nabla_a] \nabla_s z^a) \\ &\quad + g^{mn} g^{rs} \nabla_m \nabla_r z^a [\nabla_n, \nabla_a] \nabla_s \psi \\ &= \int_M g^{mn} g^{rs} \nabla_m \nabla_r \psi (\nabla_n (\mathrm{Riem}^a{}_{bsa} z^b) + \mathrm{Riem}^a{}_{bna} \nabla_s z^b - \mathrm{Riem}^b{}_{sna} \nabla_b z^a) \\ &\quad - \int_M g^{mn} g^{rs} \nabla_m \nabla_r z^a \mathrm{Riem}^b{}_{sna} \nabla_b \psi.\end{aligned}$$

We proceed by replacing the Riemann tensor with the one on the background and estimating the result by $\|g-\gamma\|_{H^{k-1}}$ as before. Furthermore, by changing the measure to μ_γ and the connection to $\hat\nabla$, we pick up exponentially decaying error terms as in the case of $l = 1$. We continue the calculation on the background:

$$\begin{aligned}\int_M & \gamma^{mn} \gamma^{rs} \hat\nabla_m \hat\nabla_r \psi (\frac{2}{9}\gamma_{bs} \hat\nabla_n z^b + \frac{2}{9}\gamma_{bn} \hat\nabla_s z^b - \frac{1}{9}(\gamma^b{}_a \gamma_{sn} - \gamma^b{}_n \gamma_{sa}) \hat\nabla_b z^a) \\ &\quad - \frac{1}{9}\int_M \gamma^{mn} \gamma^{rs} \hat\nabla_m \hat\nabla_s z^a (\gamma^b{}_a \gamma_{sn} - \gamma^b{}_n \gamma_{sa}) \hat\nabla_b \psi \\ &= \frac{5}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_b \psi \hat\nabla_n z^b - \frac{1}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_n \psi \hat\nabla_b z^b - \frac{1}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_n z^b \hat\nabla_b \psi \\ &\quad + \frac{1}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_a z^a \hat\nabla_n \psi = \frac{6}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_b \psi \hat\nabla_n z^b + \frac{2}{9}\int_M \gamma^{mn} z^b \hat\nabla_b \hat\nabla_m \hat\nabla_n \psi \\ &= \frac{6}{9}\int_M \gamma^{mn} z^b \hat\nabla_m \hat\nabla_b \psi \hat\nabla_n + \frac{2}{9}\int_M \gamma^{mn} z^b (\hat\nabla_m \hat\nabla_b + [\hat\nabla_b, \hat\nabla_m]) \hat\nabla_n \psi \\ &= \frac{4}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_b \psi \hat\nabla_n z^b - \frac{2}{9}\int_M \gamma^{mn} z^b \mathrm{Riem}[\gamma]^a{}_{nbm} \hat\nabla_a \psi \\ &= \frac{4}{9}\int_M \gamma^{mn} \hat\nabla_m \hat\nabla_b \psi \hat\nabla_n z^b + \frac{4}{81}\int_M z^b \hat\nabla_b \psi.\end{aligned}$$

We also see that, utilizing Lemma 4.5.3,

$$
\begin{aligned}
&-\frac{4}{9}\int_M g^{mn}\langle \mathbb{P}\nabla_m Z, M^0\partial_T \mathbb{P}\nabla_n Z\rangle \\
&\quad = -\frac{4}{9}\int_M g^{mn}\langle \mathbb{P}\nabla_m Z, \mathbb{P}\left((C^a+M^a)\nabla_a\nabla_n Z - \mathscr{B}\mathbb{P}\nabla_n Z + G_m\right)\rangle \\
&\quad \lesssim (K^{-1}-3)\frac{4}{9}\int_M g^{mn}\langle \mathbb{P}\nabla_m Z, \mathbb{P}\nabla_n Z\rangle - \frac{4}{9}\int_M g^{mn}\nabla_m z^b \nabla_n\nabla_b\psi \\
&\qquad - \frac{4}{9}\int_M g^{mn}\underbrace{\langle \mathbb{P}\nabla_m Z, \mathbb{P}C^a[\nabla_a,\nabla_n]Z\rangle}_{=0} + C\epsilon e^{-\lambda T},
\end{aligned}
$$

where, in the last line, we used that $\mathbb{P}C^a[\nabla_a,\nabla_n]Z = (0,[\nabla_i,\nabla_n]\psi)^{\mathrm{tr}} = 0$. In the same spirit as in the case of $l=1$ we can conclude that the term introduced by $\mathscr{B}$ does not affect the overall estimate as it can be absorbed in the negative definite term $(3-K^{-1}+C\epsilon)E_0^p$. Summing our results for $\dot{\tilde{E}}_1$ and $\dot{\tilde{E}}_2$ over $l \leq 1$ and $l \leq 2$ respectively, together with Lemma 4.4.1, yields the desired result. □

Proposition 4.5.2 (Higher order corrections). *Corrected energies $\tilde{E}_\ell$ for all orders $0 \leq \ell \leq k-1$ that are equivalent to the energies in Definition 4.4.9 can be constructed and satisfy the estimate in Proposition 4.5.1, i.e.*

$$
\partial_T \tilde{E}_\ell(Z) \lesssim (3-K^{-1}+\epsilon)E_\ell^p(Z) + \epsilon e^{-\lambda T} + \epsilon E_{k-1}^p(Z).
$$

Remark 4.5.3. Constructing these energies, while in principle straightforward, is a tedious process and does not add in a meaningful way to the arguments. Hence, we omit an explicit construction and instead give a recipe how to construct them and prove why they satisfy the above estimate.

Proof. The ideas presented in the proof of Proposition 4.5.1 can be generalized to higher orders in the following fashion. Schematically, at order ℓ, the error B takes the form

$$
\left|\int_M \nabla^\ell z^a[\nabla^\ell,\nabla_a]\psi + \nabla^\ell\psi[\nabla^\ell,\nabla_a]z^a\right| \lesssim \left|\int_M \nabla^\ell z * \nabla^{\ell-1}\psi + \nabla^\ell\psi * \nabla^{\ell-1}z\right| \lesssim \left|\int_M \nabla^\ell\psi * \nabla^{\ell-1}z\right|,
$$

where we integrated by parts in the last step. By $*$ we denote an arbitrary contraction $T_1 * T_2$ with the metric g. Furthermore, note that z here is always contracted with one of the covariant derivatives, so these tensors are indeed of the same valence. In the expression above there are terms of the form $\nabla\mathrm{Riem}$. We tacitly assumed these to be negligible, since we can always replace any geometric quantities like Riem or ∇ by their background counterparts and $\hat{\nabla}\mathrm{Riem}[\gamma] = 0$ by the Ricci decomposition as done in Lemma 4.4.5.

Now the terms in this $*$-product may involve contractions over multiple derivatives of z or ψ, i.e. $\gamma^{mn}\nabla_m\nabla_n z$ or divergence-like terms, i.e. $\nabla_m\nabla_n z^m$. All the terms are of order ℓ and $\ell-1$. In these cases we can always interchange derivatives, in which case the curvature produces terms of order e.g. $\nabla^{\ell-2}\psi\nabla^{\ell-1}z$, which can then be rearranged using integration by parts. After all these manipulations, one ends up with an error of the type

$$
\int_M c_{\ell\ell}\nabla^{\ell-1}\nabla_a\psi\nabla^{\ell-1}z^a + c_{\ell\ell-1}\nabla^{\ell-2}\nabla_a\psi\nabla^{\ell-2}z + \cdots + c_{\ell 1}z^a\nabla_a\psi.
$$

To illustrate this idea in practice, we note the explicit constants $c_{22} = \frac{4}{9}$ and $c_{21} = \frac{4}{81}$ in the case of $\ell = 2$ are derived explicitly in the proof of Proposition 4.5.1.

In the worst case one of these constants, w.l.o.g. lets say $c_{\ell\ell}$, is large and positive. This forces us to add a negative definite term to our energy. However, we may put a small coefficient in front of the top order term in our energy, i.e.

$$\delta_\ell \int_M \langle \nabla^\ell Z, M^0 \nabla^\ell Z \rangle,$$

so that the coefficient appears as $c_{\ell\ell}\delta_\ell \ll 1$. Thus, the correction term needed is

$$-c_{\ell\ell}\delta_\ell \int_M \langle \mathbb{P}\nabla^{\ell-1} Z, \mathbb{P} M^0 \nabla^{\ell-1} Z \rangle,$$

which conserves the positive definiteness and coercivity of the energy. After deriving this term and employing the equations of motion, another potential problem is the positive definite term

$$c_{\ell\ell}\delta_\ell \int_M \langle \mathbb{P}\nabla^{\ell-1} Z, \mathscr{B} \mathbb{P} \nabla^{\ell-1} Z \rangle.$$

However, this does not pose a threat for closing the energy estimates, since δ_ℓ is fixed and as small as necessary, and hence in the end the term can be absorbed by the negative definite term $(3 - K^{-1}) E^p_{\ell-1}$, as $3 - K^{-1} < 0$ is fixed. Another potential problem is posed by the term generating the error term B initially. We end up with a term of the form

$$-c_{\ell\ell}\delta_\ell \int_M \langle \mathbb{P}\nabla^{\ell-1} Z, \mathbb{P} C^a [\nabla^{\ell-1}, \nabla_a] Z \rangle = -c_{\ell\ell}\delta_\ell \int_M \nabla^{\ell-1} z^i [\nabla^{\ell-1}, \nabla_i] \psi.$$

However, we see that this again straightforwardly falls into our sum

$$\int_M \tilde{c}_{\ell\ell-1} \nabla^{\ell-1}\psi \nabla^{\ell-2} z + \cdots + \tilde{c}_{\ell 1} \nabla\psi z.$$

with new constants $\tilde{c}_{\ell i}$. The same process may now be repeated until the lowest order is reached. In this case the term generated by the matrix C simply vanishes. □

4.6 Geometric energy

Lemma 4.6.1 (Matter source terms)**.** *In the case of a relativistic fluid stress energy tensor* (4.1.1c)*, the rescaled matter quantities take the form*

$$\begin{aligned}
E &= \rho_0 e^{a(1+K)} e^{-3KT} \left((1+K)(\hat{v}^\tau)^2 N^2 + K \right), \qquad j^a = \rho_0 e^{a(1+K)} e^{(1-3K)T} N (1+K) \hat{v}^\tau v^a, \\
\eta &= \tilde{E} + \rho_0 e^{a(1+K)} e^{-3KT} \left((1+K)(|X|_g^2 (\hat{v}^\tau)^2 - 2 g_{ab} X^a v^b v^\tau + |v|_g^2) + 3K \right), \\
S_{ab} &= \rho_0 e^{a(1+K)} e^{(-1-3K)T} ((1+K)(X_a X_b (\hat{v}^\tau)^2 - X_a \hat{v}^\tau v_b - X_b \hat{v}^\tau v_a + v_a v_b) + K(2+K)).
\end{aligned}$$

Proof. Using Definition 4.4.4, a straightforward computation shows that

$$\begin{aligned}
\tilde{E} &= \tilde{T}^{\mu\nu} n_\mu n_\nu = \tilde{\rho} \left((1+K)(\bar{v}^\tau)^2 \tilde{N}^2 + K \bar{g}^{00} \tilde{N}^2 \right), \\
\tilde{j}_a &= \tilde{N} \tilde{T}^{0\mu} \bar{g}_{\mu a} = \tilde{N} \left(\tilde{\rho}(1+K) \bar{v}^\tau \bar{v}^\mu + K \tilde{\rho} \bar{g}^{0\mu} \right) \bar{g}_{a\mu}, \\
\tilde{\eta} &= \tilde{E} + \tilde{g}^{ab} \tilde{T}_{ab} = \tilde{g}^{ab} \bar{g}_{a\mu} \bar{g}_{b\nu} ((1+K) \tilde{\rho} \bar{v}^\mu \bar{v}^\nu + K \tilde{\rho} \bar{g}^{\mu\nu})
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{E} + (1+K)\tilde{\rho}\tau^{-2}\tilde{g}^{ab}\left(X_a X_b(\hat{v}^\tau)^2 - X_a\hat{v}v_b - X_b\hat{v}^\tau v_a + v_a v_b\right) + 3K\tilde{\rho},\\
\tilde{S}_{ab} &= \tilde{T}_{ab} - \frac{1}{2}\operatorname{tr}_{\bar{g}}\tilde{T}\cdot\tilde{g}_{ab} = \tilde{T}_{ab} - \frac{1}{2}\bar{g}_{\mu\nu}\tilde{T}^{\mu\nu}\cdot\tilde{g}_{ab}\\
&= K\tilde{\rho}\tau^{-2}g_{ab} + (1+K)\tilde{\rho}\tau^{-2}\left(X_a X_b(\hat{v}^\tau)^2 - X_a\hat{v}^\tau v_b - X_b\hat{v}^\tau v_a + v_a v_b\right)\\
&\quad - 2\tau^{-2}K\tilde{\rho}g_{ab} + 2\tilde{\rho}(1+K)\tau^{-2}g_{ab}.
\end{aligned}
$$

Furthermore, inverting ζ using (4.4.10), we find

$$
\tilde{\rho} = \rho_0 e^{\zeta(1+K)}(-\tau)^{3(1+K)}.
$$

Using this relation, we end up with the desired results. Note that we used $\tilde{j}^a = \tilde{g}^{ab}\tilde{j}_a = (-\tau)^2 g^{ab}\tilde{j}_{ab}$. □

Lemma 4.6.2 (Estimates on the matter variables)**.** *The matter components obey the following estimates*

$$
\begin{aligned}
|\tau|\|\eta\|_{H^{k-1}} + |\tau|^2\|j\|_{H^{k-1}} + |\tau|\|\partial_T\eta\|_{H^{k-2}} + |\tau|\|\partial_T j\|_{H^{k-2}} &\lesssim \epsilon e^{-(1+3K)T},\\
|\tau|\|S\|_{H^{k-1}} &\lesssim \epsilon e^{-(2+3K)T}.
\end{aligned}
$$

Proof. Using the expressions for the matter components given in Lemma 4.6.1, a straightforward application of Sobolev estimates leads to the following:

$$
\begin{aligned}
|\tau|\|\eta\|_{H^{k-1}} &\lesssim |\tau|^{1+3K}\|e^{a(\psi,|z|_g)}(\hat{v})^2 N^2\|_{H^{k-1}} \lesssim |\tau|^{1+3K}\|Z\|_{H^{k-1}}\|(\hat{v})^2\|_{H^{k-1}}\|N^2\|_{H^{k-1}}\\
&\lesssim \epsilon e^{-(1+3K)T}.
\end{aligned}
$$

The calculation for j and S is analogous. However, the estimate of the time derivatives require a more careful analysis. We have the following bounds on the individual bounds

$$
\begin{aligned}
\|\partial_T(e^{a(\psi,|z|_g)})\|_{H^{k-2}} &\lesssim \|D_1 a\partial_T\psi\|_{H^{k-2}} + \|D_2 a\partial_T|z|^2_g\|_{H^{k-2}} \lesssim \epsilon + \epsilon e^{-\lambda T},\\
\|\partial_T\hat{v}\|_{H^{k-2}} &\lesssim \|\partial_T N\|_{H^{k-2}} + \|\partial_T X\|_{H^{k-2}} + \|\partial_T\mathbb{P}Z\|_{H^{k-2}} \lesssim \epsilon + \epsilon e^{-\lambda T},\\
\|\partial_T g\|_{H^{k-2}} &\lesssim \epsilon e^{-\lambda T}.
\end{aligned}
$$

These rough estimates straightforwardly imply the estimates of the time derivatives. □

4.6.1 Geometric energy

Since $\mathscr{L}_{g,\gamma}$ is an elliptic operator on the compact space M, it has a discrete spectrum of eigenvalues. On account of a result from [62], the smallest eigenvalue λ_0 of the operator $\mathscr{L}_{g,\gamma}$ satisfies $\lambda_0 \geq \frac{1}{9}$ and the operator has trivial kernel.

Definition 4.6.1 (Geometric energy)**.** We define the constant $\alpha(\lambda_0, \delta_\alpha)$ and c_E as

$$
\alpha := \begin{cases} 1, & \lambda_0 < \frac{1}{9},\\ 1-\delta_\alpha, & \lambda_0 = \frac{1}{9}, \end{cases} \quad \text{and} \quad c_E := \begin{cases} 1, & \lambda_0 < \frac{1}{9},\\ 9(\lambda_0 - \xi), & \lambda_0 = \frac{1}{9}, \end{cases}
$$

where $\delta_\alpha = \sqrt{1-9(\lambda_0-\xi)}$ with $0<\xi<1$, which remains to be fixed. Given these constants, we recall from [2; 3] the geometric energy $\mathscr{E}_m$ and the correction term Γ_m of order $m \geq 1$ as

$$\begin{aligned}
E_m^{(g)} &= \frac{1}{2}\int_M \langle 6\Sigma, \mathscr{L}_{g,\gamma}^{m-1} 6\Sigma\rangle + \frac{9}{2}\int_M \langle (g-\gamma), \mathscr{L}_{g,\gamma}^{m}(g-\gamma)\rangle,\\
\Gamma_m^{(g)} &= \int_M \langle 6\Sigma, \mathscr{L}_{g,\gamma}^{m-1}(g-\gamma)\rangle.
\end{aligned}$$

Finally we define the corrected geometric energy E_s of order $s \geq 1$ as

$$\mathscr{E}_s := \sum_{1\leq m\leq s} E_m^{(g)} + c_E \Gamma_m^{(g)}.$$

4.7 Proof of global existence and stability

Proof of Theorem 4.4.1. The smallness condition of the initial data in (4.4.2) implies the existence of a constant C_0 such that

$$\|N-3\|_{H^k} + \|X\|_{H^k} + \|\partial_T N\|_{H^{k-1}} + \|\partial_T X\|_{H^{k-1}} + \tilde{E}_{k-1}(Z) + \mathscr{E}_k \leq C_0\epsilon.$$

Employing the inequality of Proposition 4.5.2 yields

$$\partial_T \tilde{E}_{k-1}(Z) \leq (3 - K^{-1} + C\epsilon) E^p{}_{k-1}(Z) + C\epsilon e^{-\lambda T}.$$

Hence, if ϵ is sufficiently small, and using that $3-K^{-1}+C\epsilon<0$, we have that

$$\|\rho\|_{H^{k-1}} + \|u\|_{H^{k-1}} \lesssim \tilde{E}_{k-1}(Z)(T) \leq C\epsilon \int_{T_0}^T e^{-\lambda s} ds \leq C_0\epsilon,$$

where we potentially redefined T_0. Next, we improve the bootstrap on the lapse and shift using an estimate given in [2, Prop. 17] together with our matter estimates from Lemma 4.6.2:

$$\begin{aligned}
\|N-3\|_{H^k} + \|X\|_{H^k} &\lesssim \|\Sigma\|_{H^{k-2}}^2 + |\tau|\|\eta\|_{H^{k-2}} + |\tau|^2\|Nj\|_{H^{k-2}} + \|g-\gamma\|_{H^{k-1}}^2\\
&\lesssim \epsilon^2 e^{-2\lambda T} + \epsilon e^{-(1+3K)T},
\end{aligned}$$

which closes the bootstrap assumptions on the lapse and shift made in (4.4.13). Similarly, to improve the bootstrap on $\partial_T N, \partial_T X$ we use [2, Lem. 18] (see also the correction in [6, Prop. 7.2]) and the matter estimates of Lemma 4.6.2 to get

$$\begin{aligned}
\|\partial_T N\|_{H^{k-1}} &\lesssim \|\hat{N}\|_{H^{k-1}} + \|X\|_{H^{k-1}} + \|\Sigma\|_{H^{\ell-1}}^2 + \|g-\gamma\|_{H^{k-1}}^2 + |\tau|\|S\|_{H^{k-3}} + |\tau|\|\eta\|_{H^{k-3}}\\
&\quad + |\tau|\|\partial_T\eta\|_{H^{k-3}}\\
&\lesssim \epsilon^2 e^{-2\lambda T} + \epsilon e^{-(1+3K)T},
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_T X\|_{H^{k-1}} &\lesssim \|\hat{N}\|_{H^{k-3}} + \|\partial_T\hat{N}\|_{H^{k-2}} + \|X\|_{H^{k-1}} + \|\Sigma\|_{H^{k-2}}^2 + \|g-\gamma\|_{H^{k-2}}^2 + |\tau|^2\|j\|_{H^{k-3}}\\
&\quad + |\tau|^2\|\partial_T j\|_{H^{k-3}} + |\tau|\|\partial_T\eta\|_{H^{k-3}}
\end{aligned}$$

$$\lesssim \epsilon^2 e^{-2\lambda T} + \epsilon e^{-(1+3K)T}.$$

The evolution of the geometric energy is given by

$$\begin{aligned}\partial_T \mathscr{E}_k &\leq -2\alpha \mathscr{E}_k + 6\mathscr{E}_k^{\frac{1}{2}} \|NS\|_{H^{k-1}} + C\mathscr{E}_k^{\frac{3}{2}} + C\mathscr{E}_k^{\frac{1}{2}} (|\tau| \|\eta\|_{H^{k-1}} + |\tau|^2 \|Nj\|_{H^{k-2}}) \\ &\leq -2\alpha \mathscr{E}_k + 6C\mathscr{E}_k^{\frac{1}{2}} \epsilon^2 e^{-(1+3K)T} + C\mathscr{E}_s^{\frac{3}{2}}.\end{aligned}$$

The first line above is from [2, Lem. 20] and the second uses our matter estimates. This energy estimate can be closed in a similar way as in [2]. This amounts to rescaling the energy by an exponential factor depending on δ_α as introduced in Definition 4.6.1. For further details of this construction see [2, §9]. Also note that the decay in this estimate is stronger due to the additional factor of e^{-KT}. Thanks to [2, Lem. 19], we have the coercivity $\|g-\gamma\|_{H^k} + \|\Sigma\|_{H^{k-1}} \lesssim \mathscr{E}_k$. Finally, future completeness relies on the rate of decay of the perturbation of the unrescaled geometry and matter fields. The exact details are very similar to that given in [2] and the conclusion uses the completeness criterion given in [18]. ☐

Remark 4.7.1. Note that all the arguments in Section 4.4 and following are not specific to a closed manifold close to the *Milne-geometry* but, except for a few details, carry over in a straightforward manner to a *fixed spatial background geometry.* The main motivation to consider constant negative Einstein-curvature is to guarantee stability of the geometry. However, when considering an arbitrary non-dynamic spatial background, we have to adapt our strategy when considering the energy corrections. As noted before, these stem from the non-commutativity of the spatial derivatives. Since we lack the luxury of an explicit background we cannot employ the same idea presented in Lemma 4.5.1, i.e. replace $\mathrm{Ric}[g]$ by $\mathrm{Ric}[\gamma]$ and exploit the fact that their difference is small. However, since our spatial manifold (M, g) is fixed and closed, we know that

$$0 \leq |\mathrm{Ric}[g]|_{\mathrm{op}} \leq \alpha$$

for some $\alpha > 0$. Remember that the first order error term is proportional to

$$\delta_1 \int_M \mathrm{Ric}[g]_{mn} z^n \nabla^m \psi,$$

where $\delta_1 > 0$ is a possibly small constant that we set before the top order part of the norm to scale this error term. We may then correct our energy as

$$\tilde{E}_1(Z) := E_1(Z) - \frac{\delta_1}{2} \int_M \langle \mathbb{P}Z, \mathrm{Ric}[g] M^0 \mathbb{P}Z \rangle,$$

which is still equivalent, given δ_1 is sufficiently small. The time derivative of M^0 and $\mathrm{Ric}[g]$ as well as additional terms from Lemma 4.4.4 are negligible. Employing the equations of motion similar to the proof of Lemma 4.5.1, in addition to the compensating term we also receive the additional

$$\delta_1 \int_M \langle \mathbb{P}Z, \mathrm{Ric}[g] \mathscr{B} \mathbb{P}Z \rangle \leq \alpha \delta_1 E_0^p(Z).$$

Given that δ_1 is sufficiently small, this term gets absorbed into the negative definite term $-cE_0^p(Z)$.

Note that we can always decompose the curvature tensor into $\mathrm{Ric}[g]$ and g. Hence, we can, in a similar fashion as in Proposition 4.5.2, put a small constant δ_l in front of the top order

n derivatives in the energy and compensate for the terms, by adding and appropriate term of order $n-1$ depending only on the fluid velocity. The result then follows as a corollary of Theorem 4.4.1.

CHAPTER 5

PHASE TRANSITION BETWEEN SHOCK FORMATION AND STABILITY IN COSMOLOGICAL FLUIDS

Abstract. We demonstrate a novel phase transition from stable to unstable fluid behaviour for fluid-filled cosmological spacetimes undergoing decelerated expansion. This transition occurs when the fluid speed of sound c_S exceeds a critical value relative to the expansion rate $a(t) = t^\alpha$ of spacetime. We present an explicit relationship between α and c_S, which subdivides the (α, c_S)-parameter space into two regions. Using rigorous techniques, we establish stability of quiet fluid solutions in the stable region. Numerical experiments reveal that the complement of the stable region consists of unstable solutions, implying sharpness of our stability result. We provide a definitive analytical bound and high-precision numerical evidence for the exact location of the critical line separating the stable from the unstable region.

5.1 Introduction

The standard model of cosmology features three key components: radiation, matter (where the pressure is much smaller than the energy density) and the cosmological constant Λ or dark energy (cf. [7; 26; 27]). Radiation and matter components are typically modelled as perfect fluids obeying the relativistic Euler equations

$$\begin{aligned} u^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu u^\mu &= 0 \\ (\rho + p) u^\mu \nabla_\mu u^\nu + (g^{\mu\nu} + u^\mu u^\nu) \partial_\mu p &= 0. \end{aligned} \tag{5.1.1}$$

Radiation and matter are the dominant contributions to the expansion of the Universe in the early and recent stages respectively, where fluid density components lead to expansion rates $a(t) = t^{\frac{1}{2}}$ (radiation) during the early Universe followed by a period with $a(t) = t^{\frac{2}{3}}$ (dust) in the matter phase. Key events for the present large-scale structure of the Universe occurred during such phases and are closely connected to fluid shock formation (cf. [12; 77; 85; 86; 97] for physical implications of fluid shock formation). The cosmological constant or dark energy is considered the main mechanism driving accelerated expansion in later stages of cosmological evolution.

This letter concerns the radiation and matter epochs, and their transition period, for a spatially flat universe, as consistent with the current interpretation of observations, i.e. the post-CMB epochs of evolution according to the standard model. While the development of shocks (discontinuities in the fluid variables) from significant large inhomogeneities seems natural, and is known to occur in particular cases [90], it is unknown under what circumstances shocks form from small inhomogeneities in the fluid. We use the term *fluid instability* to refer to shocks that develop from arbitrarily small inhomogeneities of a quiet fluid solution (i.e. spatially homogeneous solution with $u^\mu \equiv (\partial_t)^\mu$). These shocks are in general caused by self-interaction of the fluid cf. [22]. Our results imply that for *all* decelerated epochs there are unstable fluids.

Fluid stabilization. In the class of barotropic fluids with speed of sound $c_S = \sqrt{K}$ and equation of state

$$p = K\rho, \tag{5.1.2}$$

the regularity of the fluid (in the expanding time direction) is determined by the expansion rate of the Universe, i.e. the scale factor $a(t)$, when the metric is of the form

$$-dt^2 + a(t)^2 g \tag{5.1.3}$$

for a fixed Riemannian 3-metric g. Sufficiently fast expansion induces a damping effect in the fluid, which reduces the tendency for shocks to form. This stabilizing effect was first studied in Newtonian cosmology undergoing accelerated expansion [14]. It competes with the speed of sound, which, if sufficiently large, increases the tendency of shocks forming.

This behaviour is well understood in three regimes. In the absence of expansion ($\dot{a}(t) = 0$) quiet fluids are unstable [22]. For accelerated expansion ($\ddot{a}(t) > 0$) such fluids are stable. Stability means that small inhomogeneities remain uniformly bounded and the fluid remains regular for all times. Stability has been rigorously established during the last two decades in an extensive series of works [50; 55; 63; 70; 79; 81; 96; 102; 108], which rely on the strong expansion-induced dissipation effect. For superradiative fluids cf. [10; 47; 75; 80; 82]. For linear expansion ($a(t) = t$), stability depends non-trivially on the speed of sound: radiation fluids are unstable [103] while all barotropic subradiative fluids ($K < 1/3$) are stable [41; 103]. Subradiative fluids are stable for linear expansion even in the presence of backreaction [38; 40].

The critical regime, where the PDEs describing the fluid dynamics get a completely different structure is that of decelerated expansion. It has so far been poorly understood and is the subject of this letter. It includes the physically important radiation and matter-dominated epochs. We show in particular that the phase transition between stability and shock formation occurs in this regime. Hence, this letter conclusively interpolates between all stable regimes [41; 47; 50; 55; 63; 70; 75; 79–81; 96; 102; 103; 108] for cosmological fluids and Christodoulou's shock formation result [22].

We consider the full range of decelerated power-law expansion rates

$$a(t) = t^\alpha, \qquad 0 < \alpha < 1, \tag{5.1.4}$$

and fluids with speeds of sound exhausting all speeds between dust and radiation

$$0 \leq K \leq 1/3. \tag{5.1.5}$$

We provide strong evidence that the parameter space $\{(\alpha, K) \mid \alpha \in (0,1),\ K \in [0, 1/3]\}$ decomposes into two regions: a region of stable pairs and a region of unstable pairs (cf. Fig. 5.1). A pair (α, K) is *unstable* if, in the spacetime (5.1.3) with α-expansion rate (5.1.4), there exists a sequence of initial data approximating a quiet fluid solution, that

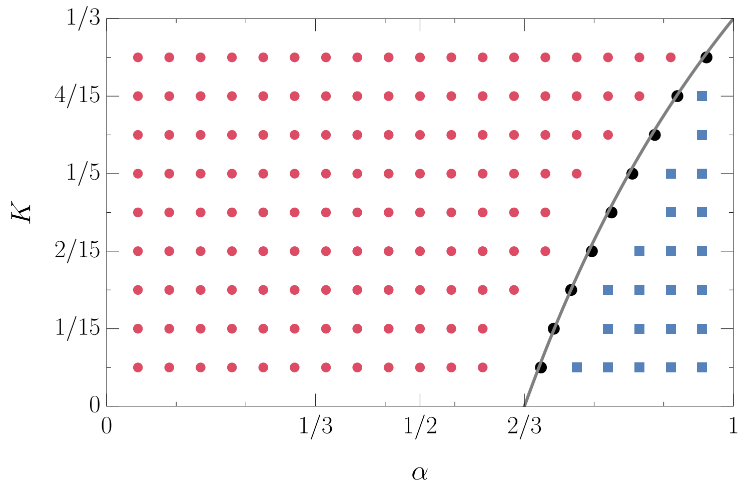

Figure 5.1: Numerical results for the parameter space: Stable (blue squares) and unstable (red circles) regions in parameter space (α, K). The black points are determined from the asymptotic analysis of the unstable points (see section *Numerics on small data shock formation* below) and their location coincide with the critical curve (5.1.6) (grey line). This curve is independently predicted by a rigorous stability analysis as a lower bound for values of $K(\alpha)$, such that (α, K) is unstable (see section *Decay of perturbations in the stable regime*).

launches a sequence of solutions for which shocks form in finite time. Our analytical and numerical studies illustrate that the transition between both regions is located on the *critical line*,

$$K_{\mathrm{crit}}(\alpha) = 1 - \frac{2}{3\alpha}. \tag{5.1.6}$$

For an epoch with expansion rate $\alpha \in (2/3, 1)$ the phase transition of a quiet fluid occurs when the speed of sound passes the critical value $K_{\mathrm{crit}}(\alpha)$. This is the first observation of a generic phase transition for cosmological fluids. This yields the following primary conclusions:

- In the decelerated regime, stability of the fluid depends on the position in parameter space relative to the critical curve (5.1.6).
- All barotropic fluids are unstable when $\alpha < 2/3$.
- For expansion rates $\alpha > 2/3$ the barotropic fluids are stable if $K < K_{\mathrm{crit}}(\alpha)$ and unstable otherwise.
- The behaviour in the decelerated regime is in contrast to the accelerated regime, where fluid stabilization is known to be universal. It follows that relativistic fluids present in the Universe during an epoch of decelerated expansion must have formed shocks, provided their speed of sound was sufficiently small relative to the expansion rate.
- For expansion rates in the regime $1/2 < \alpha \leq 2/3$, dust fluids are stable. On the other hand, fluids with arbitrarily small but positive speed of sound are unstable. Thus in this regime, which interpolates between the standard radiative and dust models, the behaviour of dust does not approximate the behaviour of fluids with non-vanishing

speed of sound, and so using a dust model as an approximation for a fluid with small but non-vanishing speed of sound is misleading.

Methodology. We use high precision numerics and asymptotic analysis to provide strong evidence for shock formation in the region to the left of (5.1.6). To the right the numerics imply stability. Using energy methods for hyperbolic PDEs, we then establish a lower bound for the location of the critical line, which coincides with (5.1.6). We prove that all (α, K) right to that line are stable. This confirms the numerics in the stable regime. Both approaches independently identify the critical line (5.1.6) as the location for the phase transition between stability and shock formation.

5.2 Equations

We study the relativistic Euler equations (5.1.1) on spacetimes of the form (5.1.3) with toroidal topology, $g_{ij} = \delta_{ij}$ and impose $\mathbb{T}^2$-symmetry on the fluid variables, energy density $\rho = \rho(t,x)$ and fluid velocity $u^\mu = (u^0(t,x), u^1(t,x), 0, 0)$, where $u_\mu u^\mu = -1$ and $\rho(t,x) = \rho(t, x+2\pi)$ etc. We introduce *expansion-normalized variables*

$$v := t^\alpha \frac{u^1}{u^0}, \quad L := \log\left(t^{3\alpha(1+K)}\rho\right),$$

to obtain the system

$$\begin{aligned} \partial_t v = &-\frac{\alpha(1-3K)}{t} v - t^{-\alpha}\frac{1-K}{1-Kv^2} v\partial_x v \\ &- t^{-\alpha}\frac{K}{1+K}\frac{(1-v^2)^2}{1-Kv^2}\partial_x L \\ &+ \alpha(1-K)(1-3K)\frac{t^{-1}}{1-Kv^2}\, v^3, \\ \partial_t L = &-\frac{1+K}{1-Kv^2} t^{-\alpha}\partial_x v - \frac{1-K}{1-Kv^2} t^{-\alpha} v \partial_x L \\ &+ \alpha(1+K)(1-3K)\frac{t^{-1}}{1-Kv^2}\, v^2. \end{aligned} \tag{5.2.1}$$

5.3 Numerics on small data shock formation

We first solve (5.2.1) numerically with the method of lines: we use a Fourier pseudospectral approach to discretise the equations in the x-direction on a grid of N collocation points $x_j = 2\pi j/N$, $j = 0, \ldots, N-1$, combined with an adaptive Runge-Kutta algorithm of variable order and step-size to evolve the resulting equations forward in time. For convenience,

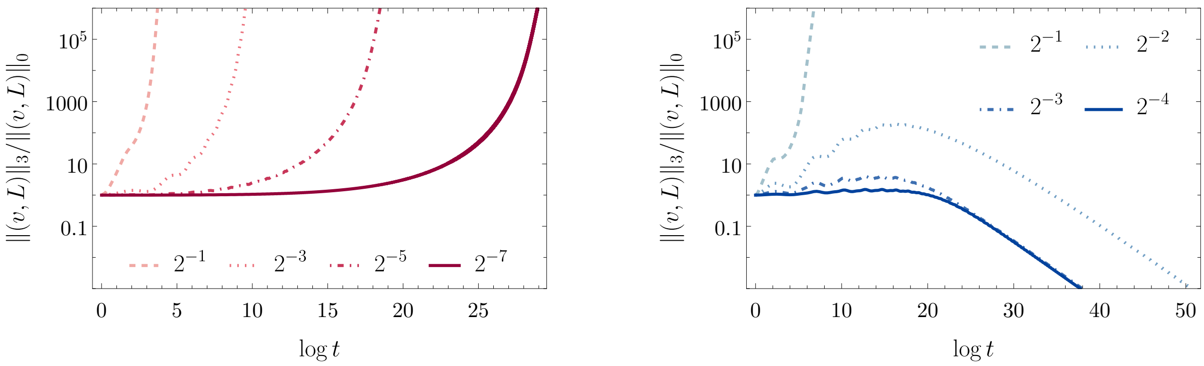

Figure 5.2: Time evolution of $\|(v, L)\|_3/\|(v, L)\|_0$, see Eq. (5.3.1), for initial data $(v, L)|_{t=1}=(\varepsilon \sin x, 0)$ with different amplitudes ε (using various line types). We show both the unstable $(\alpha, K) = (0.7, 1/6)$ (left) and stable $(\alpha, K) = (0.9, 1/6)$ (right) cases.

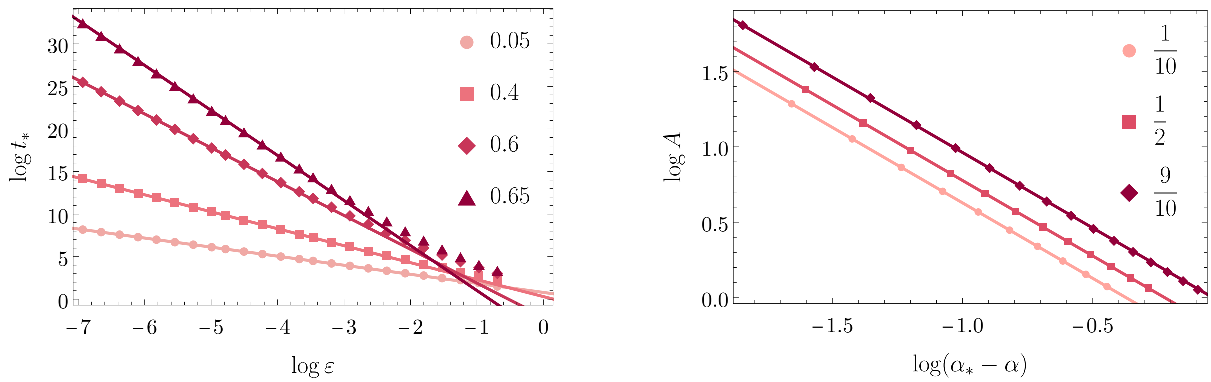

Figure 5.3: Left: The logarithm of the breaking time t_* as a function of initial data amplitude ε for $K = 1/6$ and sample values of α (shown as different point types) left to the transition curve. Points are the numerical data and the lines are the fits of (5.3.2) in the asymptotic region $\varepsilon \to 0$. Right: The behaviour of the proportionality coefficient A in (5.3.2) as a function of the expansion exponent α. On the log-log plot the data points, shown are the results for a few values of K (represented by different markers), lay on the straight line, implying the relation (5.3.3), with α_* values which agree with Eq. (5.1.6).

we work with logarithmic time $\tau = \log t$. In the time-integration we set a strict error tolerance bound to 10^{-14}. For spatial resolution, we typically take $N = 128$, which is as tradeoff between accuracy and the runtime. We stress that for a fixed number of grid points we can follow the shock formation (in the unstable regime) for a finite amount of time. As a consequence, we are unable to get an exact breaking time t_*, as that would require using an adaptive grid. However, by carefully choosing N and the thresholds in the shock formation/stability detection algorithm (see below), we obtain robust and consistent results.

We experimented with several families of initial data. Our classification of the parameter space (α, K) is independent of these choices. For concreteness, we discuss the results with $(v, L)|_{t=1}= (\varepsilon \sin x, 0)$, $\varepsilon > 0$.

During the time-evolution we monitor the quantity $\|(v, L)\|_3/\|(v, L)\|_0$ where

$$\|(v, L)\|_k := (1 + K)^2 \|\partial_x^k v\|^2_{L^2} + K \|\partial_x^k L\|^2_{L^2}\,, \tag{5.3.1}$$

and we classify a solution as smooth or shock forming if this ratio is smaller than 10^{-3} or larger than 10^{6} respectively. See Fig. 5.2, which illustrates a behaviour of solutions in the limit of $\varepsilon \to 0$ for different points in the parameter space (α, K).

We consider individual points in the parameter space (α, K) from within the regime of decelerated expansion rates and speeds of sound below radiation. A direct determination of the boundary between the stable and unstable regions appears particularly difficult as the time evolution in the transition region is especially demanding. Therefore, we rely on the following asymptotic analysis.

We start the analysis from the points with very slow expansion and move towards larger α along the $K = const$ lines. In the unstable regime we observe that the breaking time exhibits the scaling

$$\log t_* \approx -A \log \varepsilon + B\,, \quad \varepsilon \to 0\,, \tag{5.3.2}$$

with (α, K)-dependent constants A and B, which can be determined by fitting the formula to the numerical data, see Fig. 5.3. Next, inspecting the relation between A and α, with fixed K, we find

$$A \approx \frac{D}{\alpha_* - \alpha}\,, \tag{5.3.3}$$

i.e. A diverges as α approaches α_* from below. As before the constants D and α_* can be found by fitting the formula, cf. Fig. 5.3. Such values of α_* all lie on the critical line, see Fig. 5.3. The relative error between α_* and the value from Eq. (5.1.6) is less than 0.8%, and it systematically decreases when we increase both the resolution N and the threshold for shock formation and simultaneously get closer to the critical line. This agrees with the analytical results of following sections, from the side of the unstable region. For D in (5.3.3), we have the following guess $D \approx K + 2/3$. Together (5.3.2) and (5.3.3) imply the following for the breaking time

$$t_* \sim \varepsilon^{(K+2/3)/\left(\frac{2}{3(1-K)} - \alpha\right)}\,, \quad \alpha < \frac{2}{3(1-K)}\,. \tag{5.3.4}$$

For evolution on the critical line $(\alpha, K_{\mathrm{crit}})$, we find evidence that

$$t_* \sim \exp\left(e^B/\varepsilon\right)\,, \quad B = const\,, \quad \alpha = \frac{2}{3(1-K)}\,, \tag{5.3.5}$$

which implies instability as $\varepsilon \to 0$. The points to the right of (or below) the critical line are classified as stable, as in these cases the breaking times $t_* \to \infty$ for $\varepsilon \to \varepsilon_0 > 0$, and the solution exists globally for $\varepsilon < \varepsilon_0$.

The unstable region matches the rigorously known boundary values on the radiation line $K = 1/3$. The vertical rightmost boundary of the stable region (α=1) consists of stable points if $K < 1/3$, and one unstable point for $K = 1/3$, which connects with the unstable region and hence consists of a transition point (both stable and unstable points are in its neighborhood). The lower horizontal boundary of the stable region coincides only partially with the dust line for $\alpha > 2/3$. This suggests that the transition from the unstable region to the dust line for $1/2 < \alpha < 2/3$ is not continuous.

5.4 Decay of perturbations in the stable regime

In the stable regime of parameter space, $(1-K)\alpha > 2/3$, the following theorem establishes stability confirming the numerical results in the stable region. We use Sobolev norms, see e.g. [33], measuring the perturbation from the quiet fluid:

$$\|(v,L)\|_{(2)} := \sum_{k=1}^{2} \|\partial_x^k v\|_{L^2} + \|\partial_x^k L\|_{L^2}. \tag{5.4.1}$$

Theorem 5.4.1. *Let $(1-K)\alpha > 2/3$, $0 < K < 1/3$. Then, for given $\delta > 0$ there exists an $\varepsilon > 0$ such that for initial data (v_0, L_0) with $|\overline{v}_0| + \|(v_0, L_0)\|_{(2)} < \varepsilon$ the emanating solution $(v(t), L(t))$ decays according to*

$$\begin{aligned} \|(v(t), L(t))\|_{(2)} &\lesssim t^{-\alpha(1-3K)/2+\delta} \\ |\overline{v}(t)| &\lesssim t^{-\alpha(1-3K)+\delta}. \end{aligned} \tag{5.4.2}$$

Similar estimates hold for higher order norms.

The mechanism driving decay, which yields Theorem 5.4.1, is presented below; for a complete proof without symmetry restrictions see [37]. Decay is established by deriving an estimate on $\overline{v}$ and on the L^2-energy of derivatives of v and L and combining them using a continuity (or bootstrap) argument. For that purpose, choose a sufficiently small constant $0 < \delta < 2/5K$ obeying $10\delta < \alpha(1-3K) - 2(1-\alpha)$ and consider a sufficiently small initial data size $\varepsilon > 0$ so that the following bounds hold on a sufficiently long time interval.

$$\begin{aligned} &\|(v,L)\|_{(2)} \le C_1 t^{-\alpha(1-3K)/2+\delta} \\ &|\overline{v}| \le C_2 t^{-\alpha(1-3K)+\delta} \end{aligned} \tag{5.4.3}$$

We show that improved bounds hold for all times.

As a direct consequence of (5.2.1), using integration by parts, the mean velocity $\overline{v} = \fint_{\mathbb{S}^1} v dx$ fulfils

$$\begin{aligned} \partial_t \overline{v} = &-\frac{\alpha(1-3K)}{t}\overline{v} - t^{-\alpha} K \int_{\mathbb{S}^1} \frac{v^2(1-K)}{1-Kv^2}\Big[\frac{v\partial_x v}{1-Kv^2} \\ &+ \frac{v^2+K-2}{1-K^2}\partial_x L - \frac{\alpha(1-3K)}{K} t^{\alpha-1} v\Big] dx. \end{aligned} \tag{5.4.4}$$

For the rescaled mean value $\widehat{v} := t^{\alpha(1-3K)-\delta/2}\overline{v}$ Cauchy-Schwarz-, Sobolev- and Poincaré inequality applied to the integral in (5.4.4) imply, replacing $v = \overline{v} + v - \overline{v}$,

$$\frac{d}{dt}\widehat{v} \le -\frac{\delta}{2t}\widehat{v} + \frac{t^{\alpha(1-3K)-\delta/2+1-\alpha}}{t} f(\overline{v}, \|(v,L)\|_{(2)}), \tag{5.4.5}$$

where f is a polynomial, of at least third order, without lower order terms, in the variables $\overline{v}$ and $\|(v,L)\|_{(2)}$.

In consequence, a factor of $t^{3\alpha(1-3K)/2-3\delta}$ can be absorbed into f yielding a term uniformly bounded by (5.4.3). This implies

$$\frac{d}{dt}\widehat{v} \le -\frac{\delta}{2t}\widehat{v} + \frac{C_3}{t^{1+(\alpha(1-3K)-2(1-\alpha))/4}}.$$

As the exponent of t in the denominator is strictly larger than 1 by the condition on α and K, this implies the desired bound for $\overline{v}$.

We then define the following energy functional measuring the deviation from the quiet fluid state:

$$\begin{aligned}E_c[v, L](t) = &\frac{1}{2}\int_{\mathbb{S}^1}(\partial_x v)^2 dx\\ &+ \frac{K}{(1+K)^2}\frac{1}{2}\int_{\mathbb{S}^1}(1-v^2)^2(\partial_x L)^2 dx\\ &+ ct^{-(1-\alpha)}\int_{\mathbb{S}^1} v\cdot\partial_x L dx\end{aligned} \tag{5.4.6}$$

with parameter $c > 0$ chosen as $c = \frac{1}{2}\alpha(1-3K)(1+K)^{-1}$.

For sufficiently large t and sufficiently small $|v|$ this is equivalent to the L^2-norm of $\partial_x v$ and $\partial_x L$ and in particular positive definite. To see this, we have $\int v\partial_x L dx = \int (v - \bar{v})\partial_x L dx$, using integration by parts, and the latter can be estimated up to a constant by the product $\|\partial v\|_{L^2}\cdot\|\partial L\|_{L^2}$ using Hölder's estimate and the Poincaré inequality. For the corrected energy of second order $E_c^{(2)}(t) = E_c[v(t), L(t)] + E_c[\partial_x v(t), \partial_x L(t)]$, we obtain an estimate of the form

$$\begin{aligned}\frac{d}{dt}E_c^{(2)}(t) \le &-\frac{\alpha(1-3K) + C_2 t^{-(1-\alpha)}}{t}E_0^{(2)}(t)\\ &+ C_1\frac{t^{1-\alpha}}{t}\left[\left(E_0^{(2)}\right)^{3/2} + p\left(\sqrt{E_0^{(2)}}, \overline{v}\right)\right],\end{aligned} \tag{5.4.7}$$

where p is a polynomial, of at least third order, without lower order terms. The condition $\alpha(1-3K) > 2(1-\alpha)$ allows a factor $t^{\alpha(1-3K)-\delta/2}$ to be absorbed into the energy to obtain

$$\frac{d}{dt}\left(t^{\alpha(1-3K)-\delta/2}E_c^{(2)}[v, L](t)\right) \le C\frac{t^{1-\alpha}}{t}t^{\alpha(1-3K)-\delta/2}\overline{v}^3 \tag{5.4.8}$$

for sufficiently small initial data. By the condition on α and K, the right-hand side is integrable and this implies, for sufficiently small data, that the energy decays as

$$E_c^{(2)}[v, L](t) \le \frac{C_4}{t^{\alpha(1-3K)-\delta/2}}. \tag{5.4.9}$$

This concludes the sketch of proof of the theorem.

5.5 Discussion

We provide strong evidence for a phase transition of barotropic fluids in decelerated spacetimes and classify all points in the decelerated region of the parameter space with respect to the (in)stability nature of the corresponding barotropic fluid. In combination with previous results on the regime of accelerated and linear expansion, this resolves the fluid stabilization problem for barotropic fluids and provides a first generic example of a phase transition from stable to unstable cosmological fluids. The analytic and numerical approaches are complementary and predict the critical line independently of each other in complete agreement. While this class of fluids is the most relevant in cosmology, there are numerous alternative equations of state used in modelling other regimes of relativistic matter dynamics. One could also look at non-compact spatial topologies where the effect of dispersion would also occur, see for example [105] in the context of Vlasov matter. Given the robust nature of the methods employed in the study of our barotropic case, we expect that similar effects exist for alternative equations of state and for other topologies. There is no a priori reason why the critical behaviour should be limited to the barotropic case.

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CHAPTER 6

STABILITY OF FLUIDS IN SPACETIMES WITH DECELERATED EXPANSION

Abstract. We prove the nonlinear stability of homogeneous barotropic perfect fluid solutions in fixed cosmological spacetimes undergoing decelerated expansion. The results hold provided a specific inequality between the speed of sound of the fluid and the expansion rate of spacetime is valid. Numerical studies in our earlier complementary paper provide strong evidence that the aforementioned condition is sharp, i.e. that instabilities occur when the inequality is violated. In this regard, our present result covers the regime of slowest possible expansion which allows for fluids to stabilize, depending on their speed of sound. Our proof relies on an energy functional which is universal in the sense that it also applies to the case of linear expansion and enables a significantly simplified proof of bounds for fluids on linearly expanding spacetimes. Finally, we consider the special cases of dust and radiation fluids in the decelerated regime and prove shock formation for arbitrarily small perturbations of homogeneous solutions.

6.1 Introduction

6.1.1 The relativistic Euler equations

We consider the relativistic Euler equations

$$\nabla^\mu T_{\mu\nu} = 0\,, \quad T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{6.1.1}$$

with a linear, barotropic equation of state $p = K\rho$ where $c_s = \sqrt{K}$ is the speed of sound of the fluid. Equations (6.1.1) consistute one of the main matter models used in cosmology since the origin of its rigorous study over a century ago [48]. Depending on the value of K, the equations describes radiation ($K = 1/3$), dust ($K = 0$) or a fluid with speed of sound interpolating between both values. We consider the initial value problem for (6.1.1) on the fixed cosmological spacetimes (M, g)

$$M = [t_0, \infty) \times \mathbb{T}^3, g = -dt \otimes dt + a(t)^2 \delta_{ij} dx^i \otimes dx^j, \tag{6.1.2}$$

where $(\mathbb{T}^3, \delta)$ is the standard 3-dimensional torus. Here $a : [t_0, \infty) \to \mathbb{R}_+$ is the scale factor, which is a strictly increasing function of time. In the present paper, we restrict to a polynomial scale factor $a(t) = t^\alpha$ but we expect more general results could be derived in future work. We study the future global behaviour of fluids with sufficiently regular initial data given at $t_0 > 0$, i.e. as $t \nearrow \infty$, which corresponds to the expanding direction of spacetime.

6.1.2 Fluid stabilization

The expansion of spacetime, $\dot{a}(t) > 0$, generates a friction-type term in the equation for the fluid velocity (see (6.3.4a) below). In consequence, for sufficiently fast expansion and sufficiently small initial data, shock formation is suppressed during the evolution and the fluid has future global solutions. We refer to this phenomena as *fluid stabilization*. This effect was first discovered in the work of Brauer, Rendall and Reula [14]. It is known due to a series of rigorous studies over the last 30 years, pioneered by the work of Rodnianski and Speck [96], that in the regime of accelerated expansion $\ddot{a} > 0$, where the stabilizing effect is very strong, fluids stabilize for all values of $K \in [0, 1/3]$[50; 55; 70; 78; 81; 102]. This holds even in the superradiative regime [80] and other types of fluids [63; 69]. On the other hand, the expansion-driven damping effect may be counteracted by the speed of sound, which, if sufficiently large, can cause shock formation. For $K > 1/3$ numerical evidence suggests that instabilities occur [10], while particular solutions are stable despite the high speeds of sound [47; 75].

Looking at slower expansion rates, in the regime of linear expansion where $\ddot{a} = 0$, it is known that radiation fluids ($K = 1/3$) develop shocks in finite time [103]. In contrast, subradiative fluids ($K < 1/3$) remain stable in the linear regime [41], even in the presence of backreaction under the fully nonlinear Einstein–Euler equations [38; 40]. In conclusion, in the linear regime, the future global regularity of the fluid depends in a non-trivial manner on both the expansion rate and the speed of sound. The present paper aims to continue such an investigation into the more physically relevant decelerated regime.

6.1.3 The decelerated regime

The decelerated regime is characterized by $\ddot{a}(t) < 0$ which corresponds in the polynomial class $a(t) = t^\alpha$ to $\alpha < 1$. In the spatially compact case, as considered here, dispersion does not occur. In consequence, other mechanisms of decay are necessary to stabilize the fluid, such as dilution caused by the expansion. In the decelerated regime this dilution effect is weak. This manifests in the expansion-normalized relativistic Euler equations (see (6.3.4) below) by the appearance of weakly-decaying error terms. For example,

$$\partial_t L = \frac{1}{t^\alpha} \partial_j v^j + \cdots .$$

Such terms can only be bounded by terms of the form $t^{-\alpha}\sqrt{E}$, where E is an appropriate Sobolev-type energy of the solution. Here a small polynomial decay of the energy is insufficient to make these terms integrable in time as $1 - \alpha$ is a priori not small. An energy decay of at least $t^{-1+\alpha-\varepsilon}$ for some $\varepsilon > 0$ is necessary to obtain integrability. This is in contrast to the simpler case of linear expansion, where $\alpha = 1$ and hence any small polynomial decay for the energy would make these terms integrable. This slow decay is the first obstruction to prove the decay of the perturbations.

The second difficulty results from the speed of sound of the fluid. The origin of decay of perturbations is the friction-type term in the equation for the expansion normalized fluid velocity:

$$\partial_t v^i = -\frac{\alpha(1-3K)}{t} v^i + \dots$$

The coefficient of this term, which roughly corresponds to the rate of decay of perturbations, vanishes for radiation and is maximal for dust. This corresponds to the interpretation that fluids become more stable for smaller speeds of sound. To achieve a necessary decay rate the speed of sound needs to be sufficiently small.

The balance between these two mechanisms is the key to understand the critical relation between the expansion rate and the speed of sound that allows for fluid stabilization.

6.1.4 Main Result

The main result of the present paper states that under the condition

$$K < 1 - \frac{2}{3\alpha}\,, \tag{6.1.3}$$

the quiet fluid solution

$$(L_q, \vec{v}_q) = (L_0, \vec{0})\,, \tag{6.1.4}$$

where L_0 is a constant, is asymptotically stable as $t \nearrow \infty$. Here, (L, v) respectively denote the expansion normalized energy density and fluid velocity (see (6.3.2) below). Our result is the first stability result in the decelerated regime in the spatially compact setting for fluids with non-vanishing speed of sound. Moreover, numerical studies in our complementary work suggest that if condition (6.1.3) is violated, then the corresponding quiet fluid solutions are unstable [36]. In this sense, our result presented here is presumably sharp and the curve $K(\alpha) = 1 - 2/(3\alpha)$ marks the transition between the stable and the unstable regime. The precise formulation of the theorem is given in Theorem 6.5.1 below.

Our proof uses a formulation of the Euler equations in expansion-normalized variables (L, v) and L^2-based energy functionals. The key idea is the construction of a corrected L^2-energy, where the correction term generates a decay-inducing term for the spatial gradient of the expansion normalized energy density. This term complements the corresponding one for the expansion normalized fluid velocity and in total yields a decay-inducing term for the energy. However, at the same time, the time derivative of the corrected L^2-energy generates a positive term proportional to the velocity part, which reduces the corresponding negative term. The optimal choice for the coefficient of the correction term in the energy balances these two contributions to maximize the resulting energy decay. This process is precisely where the curve (6.1.3) arises as a sufficient criterion for stability.

In addition, our energy functional also requires key cancellations to occur so that we can obtain a closed energy estimate in regards to the regularity. These additional terms, however, do not interfere with the decay mechanism.

6.1.5 Fluids on linearly expansion spacetimes

Uniform bounds on the fluid variables for the range $K \in (0, 1/3)$ were first obtained in the irrotational case in [41]. This was later generalized to the general case in [38] in the presence of non-trivial curvature and gravitational backreaction. In both cases a Fuchsian approach in combination with a transformation of variables was used to obtain sufficient bounds on the fluid variables.

In Section 6.6 we consider a fluid on a linearly expanding background spacetime. We demonstrate that the energy functional, which we discover in the decelerated regime, fulfils a sufficient energy estimate, after a small adaption to the case of linear expansion. This provides a significantly shorter proof of the key steps in the result [38]. This observation suggests that this energy is indeed canonical for relativistic fluids in a cosmological scenario.

6.1.6 Some results on shock formation

In Section 6.7 we address the problem of shock formation for the special cases of dust ($K = 0$) and radiation ($K = 1/3$). In each case we show that there exists initial data arbitrarily close to the respective quiet fluid state, which leads to solutions that develop shocks in finite time. For radiation, this concerns all decelerated expansion rates $\alpha \leq 1$, which was first established in [103]. The proof given in the present article is constructive and makes use of characteristic coordinates. By contrast, the proof in [103] relies on the conformal invariance of the relativistic Euler equations when $K = 1/3$ in order to apply the shock formation result of [22]. For dust, our shock formation result concerns expansion rates $\alpha \leq 1/2$, which complements the range $\alpha > 1/2$ where stability is known [103]. Thus, we show that the phase transition for dust occurs at $\alpha = 1/2$. Remarkably, in comparison with the location of the phase transition for fluids with $K > 0$ at $\alpha = 2(1-K)/3$, this suggests that there is no continuous limit in the fluid behaviour as $K \searrow 0$ for expansion rates $1/2 < \alpha < 2/3$. In that range dust is stable, while the results of [36] suggest that for any $K > 0$ the corresponding fluid is unstable.

6.1.7 Structure of the paper

In Section 6.3 we present the relativistic Euler equations in their standard form and introduce the expansion-normalized rescaling and the relevant equations of motion. In Section 6.4 we present the energy functional and a correction mechanism, which obeys the key energy estimate for solutions of the expansion normalized system. This estimate is used in Section 6.5 to prove the main theorem of this paper, Theorem 6.5.1. This section also contains a corollary stating the decay rates in the original physical variables. In Section 6.6 we adapt the energy functional to the case of linearly expanding spacetimes and give a new short proof of fluid stabilization for the case $K \in (0, 1/3)$. Finally, in Section 6.7 we prove shock formation for dust and radiation fluids in the ranges $\alpha \leq 1/2$ and $\alpha < 1$, respectively.

6.2 Notation, coordinates and definitions

We denote spacetime indices by Greek letters $\mu, \nu, \ldots$, while Roman letters $i, j, \ldots$ denote spacial indices. For spatial submanifolds $\{t = \text{constant}\} =: \Sigma_t \cong \mathbb{T}^3$ slices we choose standard Euclidean coordinates (x^1, x^2, x^3). For a vector field v on Σ_t with components v^i we simply write

$$v_i := \delta_{ij} v^j \quad \text{and} \quad \partial^i := \delta^{ij} \partial_j.$$

In general, for a spacetime function $f : M \to \mathbb{R}$ we write

$$\int f := \int_{\mathbb{T}^3} f(\cdot, x) d^3x. \tag{6.2.1}$$

To define higher-order norms, we adapt the following notation.

Definition 6.2.1 (Spatial derivatives)**.** Let f, g be functions, v, w be vector fields on Σ_t and $\ell \in \mathbb{N}$. Then we write

$$D^\ell_{k_1 \cdots k_\ell} := \partial_{k_1} \cdots \partial_{k_\ell}.$$

In addition, we define the product $(\cdot, \cdot)$ via

$$(D^\ell f, D^\ell g) := \partial_{k_1} \cdots \partial_{k_\ell} f \partial^{k_1} \cdots \partial^{k_\ell} g$$

as well as

$$(D^\ell v, D^\ell w) := \partial_{k_1} \cdots \partial_{k_\ell} v^i \partial^{k_1} \cdots \partial^{k_\ell} w_i.$$

Furthermore, we allow for constructions like

$$(D^\ell v, D^{\ell+1} f) := \partial_{k_1} \cdots \partial_{k_\ell} v^i \partial^{k_1} \cdots \partial^{k_\ell} \partial_i f.$$

In addition we denote $|D^\ell \cdot|^2 = (D^\ell \cdot, D^\ell \cdot)$, and D^1 will simply be shortened to D.

Definition 6.2.2 (Norms)**.** Let f be a function or tensor field on M. We denote the inhomogeneous Sobolev-norm of order r by

$$\|f\|^2_{H^r} := \sum_{n=0}^{r} \int |D^r f|^2.$$

In addition, the Lebesgue-norm L^2 is defined as

$$\|f\|^2_{L^2} := \int |f|^2.$$

In addition to standard Hölder- and Sobolev-type estimates we will frequently use the following.

Lemma 6.2.1 (Poincaré inequality)**.** *Let $(\mathscr{M}, \gamma)$ be a smooth, compact Riemannian manifold. Then*

$$\Big(\int_{\mathscr{M}} |f - \bar{f}|^2 d\mu(\gamma) \Big)^{\frac{1}{2}} \lesssim \Big(\int_{\mathscr{M}} |Df|^2 d\mu(\gamma) \Big)^{\frac{1}{2}},$$

for all $f \in H^1(\mathscr{M})$, where $\bar{f} = \frac{1}{vol(\mathscr{M},\gamma)} \int f d\mu(\gamma)$.

For a proof of Lemma 6.2.1, see e.g. [56, Theorem 2.10].

6.3 Preliminaries

We consider the relativistic Euler equations (6.1.1) on fixed cosmological FLRW spacetimes of the form (6.1.2). The equations take the explicit form

$$u^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu u^\mu = 0, \tag{6.3.1a}$$

$$(\rho + P)u^\mu \nabla_\mu u^\nu + (g^{\mu\nu} + u^\mu u^\nu)\partial_\mu P = 0, \quad \nu \in \{0,1,2,3\}, \tag{6.3.1b}$$

where the metric g is given by (6.1.2) and ∇ denotes its Levi-Civita covariant derivative. The equation of state is used to replace the pressure via $P = K\rho$. We then introduce expansion-normalized variables (L, v) by

$$v^i = \frac{t^\alpha u^i}{\sqrt{1+t^{2\alpha}u^2}} \qquad \text{and} \qquad L = \log\left(t^{3\alpha(1+K)}\rho\right). \tag{6.3.2}$$

For the time-component of the fluid 4-velocity we use the normalization condition $u^\mu u_\mu = -1$ to derive

$$u^0 = \sqrt{1 + t^{2\alpha}|u|^2} = \frac{1}{\sqrt{1-|v|^2}}. \tag{6.3.3}$$

In the expansion-normalized variables the system (6.1.1) takes the following form.

$$\begin{aligned}
\partial_t v^i = &-\frac{\alpha(1-3K)}{t}v^i - \frac{K}{1+K}\frac{t^{1-\alpha}}{t}(1-v^2)\partial_i L \\
&- \frac{t^{1-\alpha}}{t}v^j\partial_j v^i + \frac{t^{1-\alpha}}{t}\left(1 - \frac{1-K}{1-Kv^2}\right)v^i\partial_j v^j \\
&+ \frac{t^{1-\alpha}}{t}\frac{1-K}{1+K}\left(1-\frac{1-K}{1-Kv^2}\right)v^i v^j \partial_j L + \frac{\alpha(1-3K)}{t}\frac{1-K}{1-Kv^2}v^2 v^i,
\end{aligned} \tag{6.3.4a}$$

$$\partial_t L = -\frac{t^{1-\alpha}}{t}\frac{(1+K)}{1-Kv^2}\partial_j v^j - \frac{t^{1-\alpha}}{t}\frac{(1-K)}{1-Kv^2}v^j\partial_j L + \frac{\alpha(1+K)}{t}\frac{1-3K}{1-Kv^2}v^2. \tag{6.3.4b}$$

The following identity is used later on:

$$1 - \frac{1-K}{1-Kv^2} = K\frac{1-v^2}{1-Kv^2}. \tag{6.3.5}$$

We make the following choices and bootstrap assumptions for convenience:

$$\begin{aligned} &t > 1 \\ &|v| < 1/10 \end{aligned}$$

In the following sections we consider a solution (L, v) emanating from initial data $(L_0, v_0) \in H^{N+1}(\mathbb{T}^3) \times H^{N+1}(\mathbb{T}^3)$ with $\|L_0 - L_q\|_{H^{N+1}} + \|v_0\|_{H^{N+1}} < \varepsilon$ for $N \geq 6$ and $\varepsilon > 0$ sufficiently small. Standard local-existence theory implies existence and uniqueness of a local-in-time solution to (6.3.4)

$$\begin{aligned} L &: (t_0, t_1) \times \mathbb{T}^3 \to \mathbb{R} \\ v &: (t_0, t_1) \times \mathbb{T}^3 \to \mathbb{R}^3, \end{aligned} \tag{6.3.7}$$

which can be extended beyond any $t_1 > t_0$ if its Sobolev norm of order $N+1$ remains bounded as $t \nearrow t_1$ [103].

In this section we derive energy estimates which are the key to establish decay estimates for perturbations of $(L_0, \vec{0})$ for the lowest order of regularity. Subsequent higher order energy estimates follow the same structure. The mechanism to establish decay is apparent already at lowest order.

We begin with an estimate on the mean value of the velocity. Using the Poincaré inequality

$$\|f - \bar{f}\|_{L^2} \leq C\|Df\|_{L^2}$$

where $f \in H^1(\mathbb{T}^3)$, we are able to control the L^2-norm of the velocity. We then define and analyze the H^1-energy of the variables L and ρ, which obeys an energy estimate that is closed in regularity as no higher order derivatives appear on the right-hand side. Ensuring that the estimate is closed in regularity is delicate. To upgrade this energy to a decay capturing energy functional we add a suitable correction term and establish the energy estimate for the corrected H^1-energy. We then commute the equations (6.3.4a), (6.3.4b) with spatial derivatives to observe that the resulting equations have a similar structure to the uncommuted equations. This implies that, beside error terms, higher order energies fulfil identical estimates. Finally, we show that the higher order corrected energies are in fact coercive and hence control the solutions in the required Sobolev regularity.

6.4.1 Evolution of the mean velocity

We define the mean velocity field $\overline{v}^i$ by

$$\overline{v}^i = \int v^i \,. \tag{6.4.2}$$

The mean velocity field obeys the following evolution equation.

Lemma 6.4.1. *Let $i \in \{1,2,3\}$ and $v : (t_0, t_1) \to \mathbb{R}^3$ be a solution to* (6.3.4a). *Then*

$$\partial_t \overline{v}^i = -\frac{\alpha(1-3K)}{t}\overline{v}^i + g^i(t), \tag{6.4.3}$$

where $g^i : (t_0, t_1) \times \mathbb{R}^3 \to \mathbb{R}^3$ for $i \in \{1,2,3\}$ and

$$|g(t,.)| \leq C\frac{1+t^{1-\alpha}}{t} p(\overline{v}^j, \|Dv\|_{H^1}, \|Dv\|_{L^\infty}, \|DL\|_{L^2})\,, \tag{6.4.4}$$

where $p(\cdot)$ is a polynomial with terms of at least second order (i.e. vanishing zeroth and first order).

Proof. The time derivative of the mean velocity field is derived using (6.3.4a). We find

$$\begin{aligned}\partial_t \overline{v}^i = &-\frac{\alpha(1-3K)}{t}\overline{v}^i - \frac{K}{1+K}t^{-\alpha}\Big(\int \partial_i L - \int v^2 \partial_i L\Big)\\ &- t^{-\alpha}\int v^j \partial_j v^i + t^{-\alpha}\int (1 - \frac{1-K}{1-Kv^2}) v^i \partial_j v^j\\ &+ t^{-\alpha}\frac{1-K}{1+K}\int (1 - \frac{1-K}{1-Kv^2}) v^i v^j \partial_j L + \frac{\alpha(1-3K)}{t}\int \frac{1-K}{1-Kv^2} v^2 v^i \,.\end{aligned}$$

The first term in the bracket in the first line vanishes due to integration by parts. For terms involving v^i-factors we decompose by $v^i = (v^i - \overline{v}^i) + \overline{v}^i$. For instance, for the second term in the bracket, we have

$$\int v^2 \partial_i L = \int (v_j - \overline{v}_j)(v^j - \overline{v}^j)\partial_i L + 2\overline{v}^j \int (v_j - \overline{v}_j)\partial_i L + \overline{v}^j \overline{v}_j \int \partial_i L \,.$$

The last term of this decomposition vanishes using integration by parts. The other terms in the decomposition can be estimated using the Poincaré inequality. We apply this argument to all terms except to the first one, which is listed explicitly in (6.4.3). □

6.4.2 L^2-energy

In this section we define the canonical L^2-energy for solutions (L, v) of (6.3.4). The specific structure of the energy is chosen so that it obeys an energy estimate which is closed in regularity.

Definition 6.4.1 (L^2-energy)**.** The L^2-energy of a solution (v, L) is defined as

$$\begin{aligned} E_0[v, L] = \frac{1}{2}\int \partial_i v^j \partial^i v_j + \frac{K}{1+K}\int v_m \partial_i v^m \partial^i L \\ + \frac{1}{2}\int \frac{1}{1-v^2} v_m \partial_i v^m v_n \partial^i v^n + \frac{1}{2}\frac{K}{(1+K)^2}\int (1-v^2)\partial^i L \partial_i L \,. \end{aligned} \tag{6.4.7}$$

In the derivation of the energy estimates for (6.4.7) the error terms are of the following type.

Definition 6.4.2 (Perturbative term)**.** We call a term $f(t)$ *perturbative*, if it can be bounded by

$$|f(t)| \leq C\frac{1+t^{1-\alpha}}{t} p(\|DL\|_{L^2}, \|Dv\|_{L^2}, \|v\|_{L^\infty}, \|Dv\|_{L^\infty}), \tag{6.4.8}$$

where $p(\cdot)$ is a polynomial with lowest order terms of at least degree 3.

We compute the time-derivatives of the individual terms of the energy in the following, keeping track of all important terms, while absorbing the remaining expressions into a perturbative term.

Lemma 6.4.2. *Let v be a solution to* (6.3.4a)*. Then, the following identity holds:*

$$\begin{aligned} \partial_t\Big(\frac{1}{2}\int \partial_i v^j \partial^i v_j\Big) = -\frac{\alpha(1-3K)}{t}\int \partial_i v^j \partial^i v_j - \frac{K}{1+K} t^{-\alpha}\int \partial_i v^j ((1-v^2)\partial^i \partial_j L) \\ + t^{-\alpha}\int \partial^i v_j \Big((1 - \frac{1-K}{1-Kv^2}) v^j \partial_i \partial_k v^k\Big) \\ + t^{-\alpha}\frac{1-K}{1+K}\int \partial^i v_j \Big((1 - \frac{1-K}{1-Kv^2}) v^j v^k \partial_i \partial_k L\Big) + f(t)\,, \end{aligned}$$

where $f(t)$ is a perturbative term.

Proof. A straightforward computation using (6.3.4a) yields the following identity.

$$\begin{aligned}\partial_t \frac{1}{2}\int \partial_i v^j \partial^i v_j =& -\frac{\alpha(1-3K)}{t}\int \partial_i v^j \partial^i v_j - t^{-\alpha}\int \partial^i v_j \partial_i (v^k \partial_k v^j)\\ &- \frac{K}{1+K} t^{-\alpha}\int \partial_i v^j \partial^i ((1-v^2)\partial_j L)\\ &+ t^{-\alpha}\int \partial^i v_j \partial_i \left((1-\frac{1-K}{1-Kv^2})v^j \partial_k v^k\right)\\ &+ t^{-\alpha}\frac{1-K}{1+K}\int \partial^i v_j \partial_i \left((1-\frac{1-K}{1-Kv^2})v^j v^k \partial_k L\right)\\ &+ \frac{\alpha(1-3K)}{t}\int \partial_i v^j \partial_i \left(\frac{1-K}{1-Kv^2}v^2 v^j\right).\end{aligned}$$

The second term in the first line on the right-hand side is perturbative after commuting to obtain $\partial^i v_j v^k \partial_k \partial_i v^j$ and applying integration by parts. The term in the last line is immediately perturbative. In all remaining terms we evaluate the derivatives ∂_i and keep only the terms with second order derivatives, while absorbing the perturbative terms into $f(t)$. The resulting explicit terms give the identity in the lemma. □

Lemma 6.4.3. *Let (v, L) be a solution to (6.3.4). Then, the following identity holds:*

$$\partial_t \Big(\frac{1}{2}\frac{K}{(1+K)^2}\int (1-v^2)\partial^i L \partial_i L\Big) = -t^{-\alpha}\frac{K}{(1+K)}\int \frac{1-v^2}{1-Kv^2}\partial^i L \partial_i \partial_j v^j + f(t),$$

where $f(t)$ is a perturbative term.

Proof. We use (6.3.4a) for the first factor yielding only perturbative terms, which are not explicitly listed below, but absorbed into $f(t)$. The other terms are given explicitly using (6.3.4b).

$$\begin{aligned}&\partial_t \Big(\frac{1}{2}\frac{K}{(1+K)^2}\int (1-v^2)\partial^i L \partial_i L\Big)\\ &= \frac{K}{(1+K)^2}\Big[- t^{-\alpha}\int (1-v^2)\partial^i L \partial_i \left(\frac{1+K}{1-Kv^2}\partial_j v^j\right)\\ &- t^{-\alpha}\int (1-v^2)\partial^i L \partial_i \left(\frac{1-K}{1-Kv^2}v^j \partial_j L\right)\\ &+ \frac{\alpha(1+K)}{t}\int (1-v^2)\partial^i L \partial_i \left(\frac{1-3K}{1-Kv^2}v^2\right)\Big] + f(t)\end{aligned}$$

The derivative ∂_i in the first term is evaluated and only the terms with second derivatives are kept explicitly providing the explicit term in the Lemma. The second term is perturbative by an integration by parts argument. The third term is perturbative immediately. □

Lemma 6.4.4. *Let (v, L) be a solution to (6.3.4). Then, the following identity holds:*

$$\begin{aligned}&\partial_t \Big(\frac{K}{1+K}\int v_m \partial_i v^m \partial^i L\Big)\\ &= \frac{K}{1+K}\Big[- \frac{\alpha(1-3K)}{t}\int v^m \partial^i L \partial_i v_m + t^{-\alpha}\int v^m \partial_i v_m \partial^i (v^j \partial_j L)\left(1-\frac{1-K}{1-Kv^2}\right)\\ &\quad + t^{-\alpha}\int v^2 \partial^i L \partial_i \partial_j v^j \left(1-\frac{1-K}{1-Kv^2}\right) - t^{-\alpha}\int v_m \partial^i v^m \partial_i \partial_j v^j \left(\frac{1+K}{1-Kv^2}\right)\Big] + f(t),\end{aligned}$$

where $f(t)$ is a perturbative term.

Proof. We use (6.3.4a) and (6.3.4b) to compute the time-derivatives of the second and first factor. By (6.3.4a) it is immediate that the terms resulting from the first factor are all perturbative.

$$
\begin{aligned}
&\partial_t\Big(\frac{K}{1+K}\int v_m\partial_i v^m\partial^i L\Big)\\
&=\frac{K}{1+K}\Big[-\frac{\alpha(1-3K)}{t}\int v^m\partial^i L\partial_i v_m-\underbrace{\frac{K}{1+K}t^{-\alpha}\int v^m\partial^i L\partial_i\left((1-v^2)\partial_m L\right)}_{(i)}\\
&-\underbrace{t^{-\alpha}\int v^m\partial^i L\partial_i(v^j\partial_j v_m)}_{(iii)}+\underbrace{t^{-\alpha}\int v^m\partial^i L\partial_i\left((1-\frac{1-K}{1-Kv^2})v_m\partial_j v^j\right)}_{(iv)}\\
&+\underbrace{t^{-\alpha}\int v^m\partial^i L\frac{1-K}{1+K}\partial_i\left((1-\frac{1-K}{1-Kv^2})v_m v^j\partial_j L\right)}_{(i)}\\
&+\underbrace{\frac{\alpha(1-3K)}{t}\int v^m\partial^i L\partial_i\left(\frac{1-K}{1-Kv^2}v^2 v_m\right)}_{(ii)}\\
&-\underbrace{t^{-\alpha}\int v_m\partial^i v^m\partial_i\left(\frac{1+K}{1-Kv^2}\partial_j v^j\right)}_{(v)}-\underbrace{t^{-\alpha}\int v_m\partial^i v^m\partial_i\left(\frac{1-K}{1-Kv^2}v^j\partial_j L\right)}_{(iii)}\\
&+\underbrace{\frac{\alpha(1+K)}{t}\int v_m\partial^i v^m\partial_i\left(\frac{1-3K}{1-Kv^2}v^2\right)}_{(ii)}\Big]+f(t)
\end{aligned}
$$

The first term on the right-hand side appears as the first term on the right-hand side in the lemma. The terms marked (i) are perturbative by an integration by parts argument. The terms marked (ii) are directly perturbative. We combine the terms in (iii) to obtain the second term on the right-hand side in the lemma up to perturbative terms. The term (iv) yields the third term on the right-hand side of the lemma up to perturbative terms. Finally the term marked (v) yields the last explicit term in the lemma, again up to perturbative terms. ☐

Lemma 6.4.5. *Let* (v, L) *be a solution to* (6.3.4a)-(6.3.4b). *Then the following identity holds.*

$$
\begin{aligned}
\partial_t\Big(\frac{1}{2}\int\frac{1}{1-v^2}v_m\partial_i v^m v_n\partial^i v^n\Big)&=-\frac{\alpha(1-3K)}{t}\int\frac{v_m\partial^i v^m}{1-v^2}v_n\partial_i v^n\\
-\frac{K}{1+K}t^{-\alpha}&\int v_m\partial^i v^m\partial_i\left(v^n\partial_n L\right)\frac{1-v^2}{1-Kv^2}\\
+t^{-\alpha}&\int\frac{v_m\partial^i v^m}{1-v^2}\partial_i(\partial_j v^j)v^2\left(1-\frac{1-K}{1-Kv^2}\right)+f(t)
\end{aligned}
$$

Proof. We use (6.3.4a) to evaluate the resulting terms when the time-derivative hits the derivative terms. The lower order terms yield only perturbative terms. This implies

$$
\begin{aligned}
\partial_t \frac{1}{2}\Big(\int \frac{1}{1-v^2} v_m \partial_i v^m v_n \partial^i v^n\Big) &= -\frac{\alpha(1-3K)}{t} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i v^n \\
& - \underbrace{\frac{K}{1+K} t^{-\alpha} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i \Big((1-v^2)\partial^n L\Big)}_{(iv)} \\
& + \underbrace{\frac{1-K}{1+K} t^{-\alpha} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i \Big((1-\frac{1-K}{1-Kv^2}) v^n v^j \partial_j L\Big)}_{(iv)} \\
& - \underbrace{t^{-\alpha} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i (v^j \partial_j v^n)}_{(i)} + \underbrace{t^{-\alpha} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i \Big((1-\frac{1-K}{1-Kv^2}) v^n \partial_j v^j\Big)}_{(iii)} \\
& + \underbrace{\frac{\alpha(1-3K)}{t} \int \frac{v_m \partial^i v^m}{1-v^2} v_n \partial_i \Big(\frac{1-K}{1-Kv^2} v^n v^2\Big)}_{(ii)} + f(t).
\end{aligned}
$$

The first term on the right-hand side appears in the lemma. The term (i) is perturbative by an integration by parts argument. The last explicit term (ii) is perturbative. The term (iii) equals the last explicit term in the lemma up to perturbative terms. The terms marked (iv) equal the second term in the lemma up to perturbative terms, where we used the identity (6.3.5). □

We have now evaluated the time-derivatives of all parts of the energy functional defined in (6.4.7). We now combine these in the following proposition, which provides the identity for the L^2-energy.

Proposition 6.4.1. *Let* (v, L) *be a solution to* (6.3.4a), (6.3.4b). *Then the following identity holds.*

$$
\partial_t E_0[v, L] = -\frac{\alpha(1-3K)}{t} \int \partial_i v^j \partial^i v_j + f(t)\,,
$$

where $f(t)$ *is a perturbative term.*

Proof. Combining Lemmas 6.4.2 to 6.4.5 yields the following identity:

$$
\begin{aligned}
\partial_t E_0[v, L] = -\frac{\alpha(1-3K)}{t} \int \partial_i v^j \partial^i v_j & \underbrace{- \frac{K}{1+K} t^{-\alpha} \int \partial_i v^j ((1-v^2)\partial^i \partial_j L)}_{(i)} \\
& + \underbrace{t^{-\alpha} \int \partial^i v_j \Big((1-\frac{1-K}{1-Kv^2}) v^j \partial_i \partial_k v^k\Big)}_{(ii)} \\
& + \underbrace{t^{-\alpha} \frac{1-K}{1+K} \int \partial^i v_j \Big((1-\frac{1-K}{1-Kv^2}) v^j v^k \partial_i \partial_k L\Big)}_{(iii)}
\end{aligned}
$$

$$
\underbrace{-t^{-\alpha}\frac{K}{(1+K)}\int\frac{1-v^2}{1-Kv^2}\partial^i L\partial_i\partial_j v^j}_{(i)}
$$

$$
+\frac{K}{1+K}\Big[-\frac{\alpha(1-3K)}{t}\int v^m\partial^i L\partial_i v_m+\underbrace{t^{-\alpha}\int v^m\partial_i v_m\partial^i(v^j\partial_j L)\left(1-\frac{1-K}{1-Kv^2}\right)}_{(iii)}
$$

$$
+\underbrace{t^{-\alpha}\int v^2\partial^i L\partial_i\partial_j v^j\left(1-\frac{1-K}{1-Kv^2}\right)}_{(i)}+\underbrace{-t^{-\alpha}\int v_m\partial^i v^m\partial_i\partial_j v^j\left(\frac{1+K}{1-Kv^2}\right)}_{(ii)}\Big]
$$

$$
-\frac{\alpha(1-3K)}{t}\int\frac{v_m\partial^i v^m}{1-v^2}v_n\partial_i v^n+\underbrace{-\frac{K}{1+K}t^{-\alpha}\int v_m\partial^i v^m\partial_i\left(v^n\partial_n L\right)\frac{1-v^2}{1-Kv^2}}_{(iii)}
$$

$$
+\underbrace{t^{-\alpha}\int\frac{v_m\partial^i v^m}{1-v^2}\partial_i(\partial_j v^j)v^2\left(1-\frac{1-K}{1-Kv^2}\right)}_{(ii)}+f(t)\,.
$$

The first term on the right-hand side appears in the lemma explicitly. The terms marked by (i), after an integration by parts combine to zero (up to perturbative terms) using the following identity:

$$
\frac{K}{1+K}\Big((1-v^2)-\frac{1-v^2}{1-Kv^2}+v^2\Big(1-\frac{1-K}{1-Kv^2}\Big)\Big)=0\,.
$$

The terms marked by (ii) combine to zero (up to perturbative terms), again after an integration by parts and using the following identity:

$$
1-\frac{1-K}{1-Kv^2}-\frac{K}{1+K}\frac{1+K}{1-Kv^2}+\frac{v^2}{1-v^2}\Big(1-\frac{1-K}{1-Kv^2}\Big)=0\,.
$$

Finally, we collect the terms marked by (iii), which combine to zero up to perturbative terms using the following identity:

$$
\frac{1-K}{1+K}\Big(1-\frac{1-K}{1-Kv^2}\Big)+\frac{K}{1+K}\Big(1-\frac{1-K}{1-Kv^2}\Big)-\frac{K}{1+K}\frac{1-v^2}{1-Kv^2}=0\,.
$$

All other unmarked terms are perturbative. □

6.4.3 Corrected L^2-energy

Proposition 6.4.1 does not include a decay-inducing term on the right-hand side. Therefore, we next construct a correction term which we add to the L^2-energy in order to obtain a decay estimate. Despite the decay-inducing term we need to check that the resulting corrected energy remains coercive and hence equivalent to the standard first Sobolev norm of the fluid variables. We define the correction term by:

$$
t^{\alpha-1}\int v^i\partial_i L. \tag{6.4.21}
$$

The following lemma provides its energy identity.

Lemma 6.4.6. *Let (v, L) be a solution to* (6.3.4)*. Then, the following identity holds:*

$$\partial_t\Big(t^{\alpha-1}\int v^i\partial_i L\Big) = -t^{\alpha-2}(1-3K\alpha)\int v^i\partial_i L - \frac{1}{t}\frac{K}{1+K}\int \partial^i L\partial_i L + \frac{(1+K)}{t}\int(\partial_j v^j)^2 + f(t).$$

Proof. We use the equations (6.3.4a) and (6.3.4b) and observe that most terms that appear are perturbative due to the low-regularity of the correction term. We keep explicitly only the terms proportional to the correction terms and the quadratic terms in the energy. □

Definition 6.4.3 (Corrected L^2-energy)**.** For $c \in \mathbb{R}$, we define

$$E_{0,c}[v, L] := E_0[v, L] + ct^{\alpha-1}\int v^i\partial_i L. \tag{6.4.23}$$

Lemma 6.4.7. *Let (v, L) be a solution to* (6.3.4)*. With $c = 1/2(1+K)^{-1}\alpha(1-3K)$ the following identity holds:*

$$\partial_t E_{0,c}[v, L] = -\frac{\alpha(1-3K)}{t}\Big[\frac{1}{2}\int \partial_i v^j\partial^i v_j + \frac{1}{2}\frac{K}{(1+K)^2}\int \partial^i L\partial_i L\Big] - \frac{1}{2}\frac{\alpha(1-3K)}{t}\int(\mathrm{rot}v)^2 - ct^{\alpha-2}(1-3K\alpha)\int v^i\partial_i L + f(t)\,,$$

where $f(t)$ denotes a perturbative term.

For the proof we need the following standard lemma.

Lemma 6.4.8. *Let v be a C^1-vector field and define* $(\mathrm{rot}v)^2 := [\mathrm{rot}v]^j[\mathrm{rot}v]_j$*. Then*

$$\int(\mathrm{rot}v)^2 = \int|Dv|^2 - \int(\mathrm{div}v)^2\,. \tag{6.4.25}$$

Proof of Lemma 6.4.7. From the previous lemmas we obtain the following identity for arbitrary $c \in \mathbb{R}$.

$$\partial_t E_{0,c}[v, L] = -\Big[\frac{2\alpha(1-3K) - 2c(1+K)}{t}\Big]\frac{1}{2}\int \partial_i v^j\partial^i v_j - 2c(1+K)\frac{1}{t}\frac{1}{2}\frac{K}{(1+K)^2}\int \partial^i L\partial_i L - \frac{1}{2}\frac{\alpha(1-3K)}{t}\int(\mathrm{rot}v)^2 + ct^{\alpha-2}(\alpha-1)\int v^i\partial_i L + f(t)\,.$$

We obtain the desired result with the choice

$$c = \frac{1}{2}\frac{1}{1+K}\alpha(1-3K)\,.$$

□

6.4.4 Higher order energies

In this section, we now consider the evolution equations after commutation with a suitable number of spatial derivatives. We derive higher order energy estimates based on the structure for the L^2-energy (6.4.23). This set up is based on the commuted equations (6.4.28) and (6.4.29), which takes a similar form as the original equations with lower order terms, which can be treated uniformly.

Commuted equations.

Lemma 6.4.9. *Let $t > 1$ and (v, L) be a solution to (6.3.4). Furthermore, let $I \in \mathbb{N}^3$ denote a multi-index $I = (\ell_1, \ell_2, \ell_3)$ and $D^I = \partial_x^{\ell_1}\partial_y^{\ell_2}\partial_z^{\ell_3}$. Then, the following identity holds:*

$$\begin{aligned}\partial_t(D^I v^i) = &-\frac{\alpha(1-3K)}{t}D^I v^i - \frac{K}{1+K}\frac{t^{1-\alpha}}{t}(1-v^2)\partial^i D^I L \\ &- \frac{t^{1-\alpha}}{t}v^j\partial_j D^I v^i + \frac{t^{1-\alpha}}{t}\left(1-\frac{1-K}{1-Kv^2}\right)v^i\partial_j D^I v^j \\ &+ \frac{t^{1-\alpha}}{t}\frac{1-K}{1+K}\left(1-\frac{1-K}{1-Kv^2}\right)v^i v^j \partial_j D^I L \\ &+ \frac{\alpha(1-3K)}{t}\frac{1-K}{1-Kv^2}v^2 D^I v^i + t^{-\alpha}R_v(v, D^{\leq I}v, D^{\leq I}L)\,,\end{aligned} \tag{6.4.28}$$

$$\begin{aligned}\partial_t(D^I L) = &-\frac{t^{1-\alpha}}{t}\frac{(1+K)}{1-Kv^2}\partial_j D^I v^j - \frac{t^{1-\alpha}}{t}\frac{(1-K)}{1-Kv^2}v^j\partial_j D^I L \\ &+ \frac{\alpha(1+K)}{t}\frac{1-3K}{1-Kv^2}D^I v^2 + t^{-\alpha}R_L(v, D^{\leq I}v, D^{\leq I}L)\,,\end{aligned} \tag{6.4.29}$$

where $R_v(.,.,.)$ and $R_L(.,.,.)$ are polynomials with at least second order terms, where the coefficients may contain the factor $1 - \frac{1-K}{1-Kv^2}$. The maximum total number of derivatives appearing in each term is bounded by $|I|+1$.

Proof. The lemma follows from commuting equations (6.3.4) with D^I and applying the Leibniz rule. □

Higher-order energies. We now define the higher-order energies with suitable correction term and derive their energy estimate.

Definition 6.4.4 (Higher-order energy)**.** Let $\ell \in \mathbb{N}$, we define

$$\begin{aligned}E_\ell[v, L] := &\frac{1}{2}\int |D^{\ell+1}v|^2 + \frac{K}{1+K}\int v_m(D^{\ell+1}v^m, D^{\ell+1}L) \\ &+ \frac{1}{2}\int\frac{1}{1-v^2}v_m v_n(D^{\ell+1}v^m, D^{\ell+1}v^n) + \frac{1}{2}\frac{K}{(1+K)^2}\int(1-v^2)|D^{\ell+1}L|^2\,.\end{aligned} \tag{6.4.30}$$

Higher-order energy identities. We define the corrected higher-order energy analogously to what we did at lowest L^2-order.

Definition 6.4.5 (Higher-order corrected energy)**.** The higher-order correction term is given by

$$C_\ell[v, L] = t^{\alpha-1}\int(D^\ell v, D^{\ell+1}L)\,. \tag{6.4.31}$$

The higher-order corrected energy is defined by

$$E_{\ell,c}[v, L] := E_\ell[v, L] + \frac{1}{2}\frac{1}{1+K}\alpha(1-3K)C_\ell[v, L]\,. \tag{6.4.32}$$

Definition 6.4.6 (Higher-order perturbative term)**.** We call $R(t)$ a *higher-order perturbative term*, if it can be bounded by

$$|R(t)| \leq P(\|Dv\|_{H^N}, \|DL\|_{H^N}, \overline{v}), \tag{6.4.33}$$

where $P(\cdot)$ denotes a polynomial where each term is at least of order three and $N \geq \max(6, \ell)$.

Lemma 6.4.10. *Let (v, L) be a solution to (6.3.4). For $c = \frac{\alpha(1-3K)}{2(1+K)}$ the following identity holds:*

$$\begin{aligned}\partial_t E_{\ell,c}[v, L] = &-\frac{\alpha(1-3K)}{t}\Big[\frac{1}{2}\int|D^{\ell+1}v|^2 + \frac{1}{2}\frac{K}{(1+K)^2}\int(1-v^2)|D^{\ell+1}L|^2\Big] \\ &- c(1-3K\alpha)t^{\alpha-2}\int(D^\ell v, D^{\ell+1}L) - \frac{1}{2}\frac{\alpha(1-3K)}{t}\int|D^\ell \mathrm{rot} v|^2 + R(t)\,,\end{aligned}$$

where $R(t)$ is a higher-order perturbative term.

Proof. Using Lemma 6.4.9 to replace the time derivatives of the derivatives of v and L, the identity follows in direct analogy to the proof of Lemma 6.4.7 and the lemmas leading up to it, since all non-perturbative terms have the same structure as in the aforementioned case. The perturbative terms arise from the terms R_v and R_L of Lemma 6.4.9. The perturbative terms involve expressions which contain at least two factors in either v or derivatives of v and L. The critical type of term that one needs to estimate involves precisely two such factors. In that case, we give as an example the method to estimate a term of the following form:

$$\int|D^{\ell+1}v||D^I v||D^J L|, \tag{6.4.35}$$

where I, J are multiindeces. Note that any coefficients, for example of the form $1-\dfrac{1-K}{1-Kv^2}$, can be estimated by K from above. Due to the condition that $|I|+|J| \leq \ell+2$, at least one of the terms (say $|J|$) fulfils the bound

$$|J| \leq \frac{1}{2}(\ell+2)\,.$$

This implies

$$\|D^J L\|_{L^\infty} \leq C\|DL\|_{H^N}\,,$$

by Sobolov-embedding since $|J|+2 \leq \frac{1}{2}(\ell+2)+2 \leq \frac{1}{2}N+3 \leq N$. Consequently, the integral (6.4.35) can be estimated by the above and Cauchy-Schwarz to find

$$C\|DL\|_{H^N}\|Dv\|^2_{H^N}\,.$$

For the remaining higher order terms the derivatives are distributed over more than two terms and all but two terms can be estimated pointwise. Factors in v are expanded via $v = (v-\overline{v}) + \overline{v}$. For example:

$$\int v|D^{\ell+1}v||D^I v||D^J L| = \int(v-\overline{v})|D^{\ell+1}v||D^I v||D^J L| + \overline{v}\int|D^{\ell+1}v||D^I v||D^J L|.$$

The second term is contained by the above discussion in the set of higher-order perturbative terms. In the first term $v-\overline{v}$ is estimated pointwise and the resulting factor is estimated using the Poincaré inequality. ☐

6.4.5 Equivalence of energies and total energy

Before addressing the stability theorem it remains to show that the corrected energies in fact control the perturbation from the homogeneous background solution in standard Sobolev regularity. In order to obtain a uniform energy estimate we also define a total energy, which controls the solution at all orders as well as a corrected version thereof.

Definition 6.4.7 (Total corrected energy)**.** We define the total L^2-based energy of order N by

$$E_N(t) = \sum_{0\leq \ell\leq N} \frac{1}{2}\int |D^{\ell+1}v|^2 + \frac{1}{2}\frac{K}{(1+K)^2}\int (1-v^2)|D^{\ell+1}L|^2 \,. \tag{6.4.40}$$

We define the total energy by

$$\mathbf{E}_N(t) := \overline{v}(t)^2 + E_N(t)\,, \tag{6.4.41}$$

and we define the total corrected energy by

$$\mathbf{E}_{N,c}(t) := \overline{v}(t)^2 + \sum_{0\leq \ell\leq N} E_{\ell,c}[v,L]\,. \tag{6.4.42}$$

The following lemma shows that the corrected energy controls the standard energy.

Lemma 6.4.11. *For t sufficiently large and $\|v\|_{L^\infty}$ sufficiently small there exists a positive constant C such that*

$$\frac{1}{C}\mathbf{E}_{N,c}(t) \leq \mathbf{E}_N(t) \leq C\mathbf{E}_{N,c}(t)\,,$$
$$|\mathbf{E}_{N,c}(t) - \mathbf{E}_N(t)| \lesssim t^{\alpha-1}\mathbf{E}_{N,c}(t) + \mathbf{E}_{N,c}(t)^{3/2}\,.$$

Proof. The lemma follows from applying first Cauchy-Schwarz on the correction terms to obtain

$$|C_\ell[v,L]| \leq t^{\alpha-1}\|v\|_{H^\ell}\|L\|_{H^{\ell+1}} \leq \frac{1}{2}t^{\alpha-1}\Big(\|v\|^2_{H^\ell} + \|L\|^2_{H^{\ell+1}}\Big).$$

Then t is chosen sufficiently large to assure that the right-hand side is smaller than the corresponding terms in the standard energy. □

6.5 Main result

The estimates derived above imply the following theorem, which is the main result of the present paper.

Theorem 6.5.1. *Let $N \geq 6$, $0 < K < 1/3$ and $(1-K)\alpha > 2/3$ and let $\mu > 0$ be chosen such that $\alpha(1-3K) - \mu > 2(1-\alpha)$. Let $(v_0, L_0) \in H^{N+1}(\mathbb{T}^3) \times H^{N+1}(\mathbb{T}^3)$ be a vector field and a function, respectively. Then, there exists an $\varepsilon > 0$ such that if the initial data (v_0, L_0) satisfies*

$$\overline{v}_0 + \|Dv_0\|_{H^N} + \|DL_0\|_{H^N} < \varepsilon\,, \tag{6.5.1}$$

then the solution $(v(t), L(t))$ *to the system* (6.3.4) *exists to the future globally in time and the following decay rates hold:*

$$\begin{aligned} |\overline{v}(t)| &\le C\varepsilon t^{(-\alpha(1-3K)+\mu)/2}\,, \\ \|L(t)-\overline{L}_0\|_{L^\infty} &\le C\varepsilon\,, \\ \|Dv(t)\|_{H^N}+\|DL(t)\|_{H^N} &\le C\varepsilon t^{(-\alpha(1-3K)+\mu)/2}\,. \end{aligned} \tag{6.5.2}$$

Proof. We consider now initial data with the conditions of Theorem 6.5.1, where ε is the size of the initial data. We choose $t_0 > 1$ and ε sufficiently small to assure that

$$|v(t)| < 1$$

on an interval $[t_0, T)$. Using Lemma 6.4.1, Lemma 6.4.7 and Lemma 6.4.10 we obtain an inequality of the form

$$\begin{aligned} \partial_t \mathbf{E}_{N,c}(t) &\le -\frac{2\alpha(1-3K)}{t}\overline{v}(t)^2 - \frac{\alpha(1-3K)}{t}E_N(t) + C\frac{1+t^{1-\alpha}}{t}\mathbf{E}_N(t)^{3/2} + C\frac{t^{\alpha-1}}{t}\mathbf{E}_N(t), \\ &\le -\frac{\alpha(1-3K)}{t}\mathbf{E}_{N,c}(t) + C\frac{1+t^{1-\alpha}}{t}\mathbf{E}_{N,c}(t)^{3/2} + C\frac{t^{\alpha-1}}{t}\mathbf{E}_{N,c}(t) \end{aligned}$$

where we have absorbed any fourth-order or higher order terms into the third order term by assuming sufficient smallness and using the equivalence of energies.

Then for μ with $\alpha(1-3K) - \mu > 2(1-\alpha)$, define

$$\widetilde{\mathbf{E}}_{N,c}(t) := t^{\alpha(1-3K)-\mu}\mathbf{E}_{N,c}(t)\,. \tag{6.5.5}$$

Hence,

$$\begin{aligned} \partial_t \widetilde{\mathbf{E}}_{N,c}(t) &\le -\frac{\mu}{t}\widetilde{\mathbf{E}}_{N,c}(t) + Ct^{\alpha(1-3K)-\mu}\left(\frac{1+t^{1-\alpha}}{t}\mathbf{E}_{N,c}(t)^{3/2} + \frac{t^{\alpha-1}}{t}\mathbf{E}_{N,c}(t)\right) \\ &= -\frac{\mu}{t}\widetilde{\mathbf{E}}_{N,c}(t) + \frac{C}{t}\left(t^{\frac{-\alpha(1-3K)+\mu}{2}}(1+t^{1-\alpha})\widetilde{\mathbf{E}}_{N,c}(t)^{3/2} + t^{\alpha-1}\widetilde{\mathbf{E}}_{N,c}(t)\right). \end{aligned}$$

For sufficiently small initial data this implies

$$\widetilde{\mathbf{E}}_{N,c}(t) \le \widetilde{\mathbf{E}}_{N,c}(t_0)\,,$$

and in turn

$$\mathbf{E}_{N,c}(t) \le (t/t_0)^{-\alpha(1-3K)+\mu}\mathbf{E}_{N,c}(t_0)\,.$$

The theorem follows by the equivalence of energies. The bound for the rescaled energy density L follows by using the decay estimates above in combination with an evolution equation for the mean value of L. □

We next translate the decay rates back into the original “physical" variables (ρ, u). Inverting the transformation (6.3.2) yields

$$u^j = t^{-\alpha}\frac{v^j}{\sqrt{1-v^2}}, \quad \rho = \frac{\exp(L)}{t^{3\alpha(1+K).}} \tag{6.5.9}$$

We define the rescaled energy density by $\boldsymbol{\rho}(t) = \exp(L(t))$ and the rescaled velocity by $\boldsymbol{u}(t) = t^\alpha u(t)$.

Corollary 6.5.1. *The decay rates for the physical variables in Theorem 6.5.1 are the following:*

$$\begin{aligned} |\overline{\boldsymbol{u}}(t)| &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2}\,, \\ |\boldsymbol{\rho}(t)-\boldsymbol{\rho}(t_0)| &\leq C\varepsilon\,, \\ \|\nabla \boldsymbol{u}(t)\|_{H^N}+\|\nabla\boldsymbol{\rho}(t)\|_{H^N} &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2}\,. \end{aligned} \tag{6.5.10}$$

6.6 Implications for fluids on linearly expanding cosmologies

We show in this section that an adapted version of the corrected L^2-energy (6.4.32) can be applied to analyze the behaviour of fluids with $K \in (0,1/3)$ on linearly expanding cosmological spacetimes. In consequence, we obtain an alternative proof of the main stability theorems in [41] and [38]. In comparison with the decelerated regime, where $\alpha < 1$, in the case of linear expansion the power law rate is $\alpha = 1$. This implies that the error terms, which are of higher order in the energy estimates, now have a coefficient of t^{-1}. To compensate these terms in the corresponding energy estimates, it is sufficient to show *any* non-trivial decay of the energy. This is crucial, since the correction term in the energy might otherwise violate coercivity on account of the fact that the decaying time-factor in the correction term (see (6.4.21)) is not present for $\alpha = 1$. We give a new proof of the following theorem, which was obtained in [38] using Fuchsian methods.

Theorem 6.6.1 (Fluid stability under linear expansion)**.** *Let* $0 < K < 1/3$ *and* $\alpha = 1$. *Let* $(v_0, L_0) \in H^{N+1}(\mathbb{T}^3) \times H^{N+1}(\mathbb{T}^3)$ *be a vector field and a function, respectively. Then there exists an* $\varepsilon > 0$ *and a* $\delta > 0$ *such that for initial data* (v_0, L_0) *satisfying*

$$\overline{v}_0 + \|Dv_0\|_{H^N}+\|DL_0\|_{H^N}< \varepsilon, \tag{6.6.1}$$

the solution $(v(t), L(t))$ *to the system* (6.3.4) *exists to the future globally in time and the following decay rates hold:*

$$\begin{aligned} |\overline{v}(t)| &\leq C\varepsilon t^{-\delta/2}\,, \\ \|L(t)-\overline{L}_0\|_{L^\infty} &\leq C\varepsilon\,, \\ \|Dv(t)\|_{H^N}+\|DL(t)\|_{H^N} &\leq C\varepsilon t^{-\delta/2}\,. \end{aligned} \tag{6.6.2}$$

Proof. We sketch the proof in terms of the L^2-energy of lowest order. The higher order case works accordingly. We define the first corrected energy, where $\delta > 0$ is to be determined later, by

$$E_{0,\delta}^{\alpha=1}[v,L] := E_0[v,L] + \delta \int_{\mathbb{T}^3} v^i \partial_i L, \tag{6.6.3}$$

where E_0 is defined in the same way as in (6.4.7).

We choose $\delta > 0$ sufficiently small to assure that the eventual total energy corresponding to (6.4.42) is coercive, despite the fact that no decaying time-factor is present here. We are left to show a suitable decay estimate. We combine Proposition 6.4.1, to obtain the following energy identity.

$$\partial_t E^{\alpha=1}_{0,\delta}[v,L] = -\frac{1-3K}{t}\int \partial_i v^j \partial^i v_j - \delta t^{-1}(1-3K)\int v^i \partial_i L \\ - \frac{\delta}{t}\frac{K}{1+K}\int \partial^i L \partial_i L + \delta\frac{(1+K)}{t}\int (\partial_j v^j)^2 + \tilde{f}(t),$$

where $\tilde{f}(t)$ obeys an estimate of the form (6.4.8) with $\alpha = 1$. We deduce from the above identity

$$\partial_t E^{\alpha=1}_{0,\delta}[v,L] \leq -\frac{\delta/2}{t}E^{\alpha=1}_{0,\delta}[v,L] - \Big(\frac{1-3K}{t} - \frac{\delta(1+K)}{t} - \frac{\delta/4}{t}\Big)\int \partial_i v^j \partial^i v_j \\ + \Big(\frac{\delta^2/2}{t} - \delta t^{-1}(1-3K)\Big)\int v^i \partial_i L \\ - \frac{\delta}{t}\frac{K}{1+K}\Big(1 - \frac{1}{4}\frac{1}{(1+K)}\Big)\int \partial^i L \partial_i L + \tilde{f}(t).$$

Here, we have ignored the additional decay inducing term for the rotation of v. In addition, higher order terms like the second and third in (6.4.7) have been absorbed into $\tilde{f}(t)$. For sufficiently small δ the indefinite term can be absorbed by the manifestly negative terms by using Cauchy-Schwarz, Poincaré's inequality and (possibly) Young's inequality. In combination with the structure of the term $\tilde{f}(t)$ this implies an estimate of the form

$$\partial_t E^{\alpha=1}_{0,\delta}[v,L] \leq -\frac{\delta/2}{t}E^{\alpha=1}_{0,\delta}[v,L] + \frac{C}{t}E^{\alpha=1}_{0,\delta}[v,L]^{3/2}.$$

This implies the desired decay rate. □

6.7 Instability for dust and radiation in the decelerated regime

In this section we show that quiet dust and radiation solutions develop instabilities, in the form of shocks, on fixed FLRW spacetimes (6.1.2) when $\alpha \leq 1/2$ and $\alpha \leq 1$, respectively.

6.7.1 Instability for radiation fluids

In this section we show shock formation for radiation fluids for specific arbitrarily small perturbations of homogeneous data. We use the formulation of the Euler equations as balance laws, as presented by Rendall and Ståhl [90]. Note that our data is small, in contrast to [90] where initial data is *a priori* large.

We consider again spacetimes of the form (6.1.2), where we rename the t-variable by $\bar{t}$ and choose $a(\bar{t}) = \bar{t}^\alpha$ for $\alpha \in (0,1]$. Hence, in the coordinates $(\bar{t}, x^i)$, we have that

$$\bar{g} = -d\bar{t}^2 + a(\bar{t})\delta_{ij}dx^i dx^j. \tag{6.7.1}$$

In these coordinates we compute the second fundamental form $\bar{k}$ in the standard coordinate frame dx^i to be

$$\bar{k}_{ij} = -\frac{1}{2N}\partial_{\bar{t}}\,\bar{g}_{ij} = -\frac{1}{2}\partial_{\bar{t}}\,(\bar{t}^{\,2\alpha}\delta_{ij}) = -\alpha\bar{t}^{\,2\alpha-1}\delta_{ij}\,,$$

where $N = 1$ is the lapse function. The mean curvature is $\mathrm{tr}(\bar{k}) = \bar{t}^{\,-2\alpha}\delta^{ij}\bar{k}_{ij} = -3\alpha/\bar{t}\,$. Thus we introduce the new constant mean curvature time t via

$$t := -\frac{3\alpha}{\bar{t}}, \quad t \in [-\frac{1}{3\alpha}, 0)$$

so that $t \nearrow 0$ is the expanding direction. This gives $dt = \frac{3\alpha}{\bar{t}^{\,2}}d\bar{t}$ and so the metric (6.7.1), expressed in coordinates (t, x^i), takes the form

$$g = -\frac{9\alpha^2}{t^4}dt^2 + \frac{(3\alpha)^{2\alpha}}{|t|^{2\alpha}}\delta_{ij}dx^i dx^j.$$

Hence, again in the coordinate frame,

$$k_{ij} = -\frac{1}{2N}\partial_t g_{ij} = -|t|^{-2\alpha+1}\frac{(3\alpha)^{2\alpha}}{3}\delta_{ij}\,,$$

and we can check $\mathrm{tr}(k) = -|t| = t$. Matching with [90, eq. (1)], where the metric takes the form

$$g = -N^2dt^2 + A^2[(dx + \beta dt)^2 + a^2(dy^2 + dz^2)]$$

we find by comparison

$$N = \frac{3\alpha}{t^2}, \quad A = \frac{(3\alpha)^\alpha}{|t|^\alpha}, \quad \beta = 0, \quad a = 1.$$

Additionally, the orthonormal frame introduced in [90, eq. (2)], becomes in our setting:

$$e_0 = \frac{t^2}{3\alpha}\partial_t, \quad e_1 = \frac{|t|^\alpha}{(3\alpha)^\alpha}\partial_x, \quad e_2 = \frac{t^\alpha}{(3\alpha)^\alpha}\partial_y, \quad e_3 = \frac{t^\alpha}{(3\alpha)^\alpha}\partial_z\,.$$

We compute the second fundamental form $\tilde{k}$ in the co-frame (e^i) to be:

$$k_{ij} = \tilde{k}_{kl}e^k{}_i e^l{}_j = \tilde{k}_{ij}\frac{(3\alpha)^{2\alpha}}{|t|^{2\alpha}} \Rightarrow \tilde{k}_{ij} = \frac{t}{3}\delta_{ij}.$$

Following [90, eq. (3)] we write the second fundamental form in (e^i) as

$$\tilde{k}_{ij} = -\frac{1}{2}(\varkappa - t)\delta_{ij} + \frac{1}{2}(3\varkappa - t)\delta_{i1}\delta_{1j} \tag{6.7.10}$$

Comparing with (6.7.10), we read off that $\varkappa = \frac{t}{3}$.

We assume a linear equation of state $p = K\rho$, and define the *fluid enthalpy* φ by

$$\varphi = \int_{\rho_c}^{\rho}\frac{\sqrt{K}}{1+K}\frac{dm}{m} = \frac{\sqrt{K}}{1+K}\ln(\rho/\rho_c)\,,$$

where $\rho_c > 0$ is a constant. Also following [90, eq (7)], we define the velocity parameter u via

$$u^0 = (1-u^2)^{-1/2}, \quad u^1 = u(1-u^2)^{-1/2}, \quad u^2 = u^3 = 0\,.$$

Observing that $e_1(A) = 0 = e_1(N)$ and following [90, eq (14)], we define the derivative operators

$$D_+ = e_0 + \frac{u+\sqrt{K}}{1+u\sqrt{K}} e_1 = \frac{t^2}{3\alpha}\partial_t + \frac{u+\sqrt{K}}{1+u\sqrt{K}}\Big(\frac{|t|}{3\alpha}\Big)^{\alpha}\partial_x\,,$$
$$D_- = e_0 + \frac{u-\sqrt{K}}{1-u\sqrt{K}} e_1 = \frac{t^2}{3\alpha}\partial_t + \frac{u-\sqrt{K}}{1-u\sqrt{K}}\Big(\frac{|t|}{3\alpha}\Big)^{\alpha}\partial_x\,,$$

and from [90, eq (15)]

$$r = \varphi + \frac{1}{2}\ln\frac{1+u}{1-u} = \ln\left(\left(\frac{\rho}{\rho_c}\right)^{\frac{\sqrt{K}}{1+K}}\Big(\frac{1+u}{1-u}\Big)^{1/2}\right),$$
$$s = \varphi - \frac{1}{2}\ln\frac{1+u}{1-u} = \ln\left(\left(\frac{\rho}{\rho_c}\right)^{\frac{\sqrt{K}}{1+K}}\Big(\frac{1-u}{1+u}\Big)^{1/2}\right).$$

Using $\varkappa = \frac{t}{3}$, the balance laws from [90, eq. (16)] then reduce to

$$D_+ r = t\frac{\sqrt{K}+u/3}{1+\sqrt{K}u}\,, \quad D_- s = t\frac{\sqrt{K}-u/3}{1-\sqrt{K}u}\,. \tag{6.7.13}$$

Transformation to R, S. First, we note that

$$\rho = \rho_c \exp\left(\frac{1+K}{2\sqrt{K}}(r+s)\right)\,, \quad u = \frac{e^r - e^s}{e^r + e^s}\,. \tag{6.7.14}$$

Motivated by this, we introduce the variables $R = e^r$ and $S = e^s$. Multiplying (6.7.13) by e^r and e^s respectively (as well as $(3\alpha)/t^2$) we use (6.7.13) to arrive at the equations

$$\begin{aligned}\partial_t R + \kappa(R,S)\frac{(3\alpha)^{1-\alpha}}{|t|^{2-\alpha}}\partial_x R &= \frac{3\alpha}{t} f_1(R,S)R\,,\\ \partial_t S + \lambda(R,S)\frac{(3\alpha)^{1-\alpha}}{|t|^{2-\alpha}}\partial_x S &= \frac{3\alpha}{t} f_2(R,S)S\,,\end{aligned} \tag{6.7.15}$$

where

$$f_1(R,S) := \frac{1}{3}\frac{(1+3\sqrt{K})R + (3\sqrt{K}-1)S}{(1+\sqrt{K})R + (1-\sqrt{K})S}\,, \quad f_2(R,S) := \frac{1}{3}\frac{(3\sqrt{K}-1)R + (3\sqrt{K}+1)S}{(1-\sqrt{K})R + (1+\sqrt{K})S}\,,$$

and

$$\begin{aligned}\kappa(R,S) &:= \frac{u+\sqrt{K}}{1+\sqrt{K}u} = \frac{(1+\sqrt{K})R - (1-\sqrt{K})S}{(1+\sqrt{K})R + (1-\sqrt{K})S}\,,\\ \lambda(R,S) &:= \frac{u-\sqrt{K}}{1-\sqrt{K}u} = \frac{(1-\sqrt{K})R - (1+\sqrt{K})S}{(1-\sqrt{K})R + (1+\sqrt{K})S}\,.\end{aligned} \tag{6.7.17}$$

For later use, we compute the following

$$\begin{aligned}\partial_1\kappa(x,y) &= \frac{2(1-K)y}{((1+\sqrt{K})x + (1-\sqrt{K})y)^2}\,, \quad \partial_2\kappa(x,y) = \frac{-2(1-K)x}{((1+\sqrt{K})x + (1-\sqrt{K})y)^2}\,,\\ \partial_1\lambda(x,y) &= \frac{2(1-K)y}{((1-\sqrt{K})x + (1+\sqrt{K})y)^2}\,, \quad \partial_2\lambda(x,y) = \frac{-2(1-K)y}{((1-\sqrt{K})x + (1+\sqrt{K})y)^2}\,.\end{aligned} \tag{6.7.18}$$

Coordinate change to physical time. Now that we have the equations of motion (6.7.15) for the fluid, we transform back to physical time $\bar{t}$. Denoting $\bar{f}(\bar{t}) = (f \circ t)(\bar{t})$, we find

$$\partial_{\bar{t}} \bar{R} + \kappa(\bar{R}, \bar{S})\bar{t}^{-\alpha}\partial_x \bar{R} = -3\alpha\bar{t}^{-1} f_1(\bar{R}, \bar{S}), \quad \partial_{\bar{t}} \bar{S} + \lambda(\bar{R}, \bar{S})\bar{t}^{-\alpha}\partial_x \bar{S} = -3\alpha\bar{t}^{-1} f_2(\bar{R}, \bar{S}). \tag{6.7.19}$$

In the case of radiation, a straightforward calculation shows that

$$f_1(x,y)|_{K=\frac{1}{3}} = f_2(x,y)|_{K=\frac{1}{3}} = \frac{1}{\sqrt{3}}.$$

Therefore, (6.7.19) simplifies to

$$\partial_{\bar{t}} \bar{R} + \kappa(\bar{R}, \bar{S})\bar{t}^{-\alpha}\partial_x \bar{R} = -\sqrt{3}\alpha\bar{t}^{-1}, \qquad \partial_{\bar{t}} \bar{S} + \lambda(\bar{R}, \bar{S})\bar{t}^{-\alpha}\partial_x \bar{s} = -\sqrt{3}\alpha\bar{t}^{-1}. \tag{6.7.21}$$

Transformation to $\mathscr{R}, \mathscr{S}$. We introduce

$$R := t^{\sqrt{3}\alpha}\mathscr{R}, \quad S := t^{\sqrt{3}\alpha}\mathscr{S}. \tag{6.7.22}$$

Using (6.7.21) we find

$$\partial_{\bar{t}}\mathscr{R} + \kappa(\mathscr{R}, \mathscr{S})\bar{t}^{-\alpha}\partial_x\mathscr{R} = 0, \quad \partial_{\bar{t}}\mathscr{S} + \lambda(\mathscr{R}, \mathscr{S})\bar{t}^{-\alpha}\partial_x\mathscr{S} = 0. \tag{6.7.23}$$

Note that we used that $\kappa(cR, cS) = \kappa(R, S)$ as well as $\lambda(cR, cS) = \lambda(x, y)$ for any $c \neq 0$.

Characteristic Coordinates. We now introduce characteristic coordinates (τ, ξ)

$$(\bar{t}, x) = (\tau, \phi(\tau, \xi)) \tag{6.7.24}$$

chosen so that

$$\phi(\tau_0, \xi) = \xi, \quad \partial_\tau\phi(\tau, \xi) = \kappa(\tilde{\mathscr{R}}(\tau, \xi), \tilde{\mathscr{S}}(\tau, \xi))\tau^{-\alpha}$$

where for any function $f = f(t, x)$ we write

$$\tilde{f}(\tau, \xi) := f(\tau, \phi(\tau, \xi)).$$

Deriving a Riccati-type equation for radiation. Commuting (6.7.23) with ∂_x and denoting $\partial_x\mathscr{R} = \mathscr{W}$ we find

$$\partial_{\bar{t}}\mathscr{W} = -\bar{t}^{-\alpha}\kappa(\mathscr{R}, \mathscr{S})\partial_x\mathscr{W} - \bar{t}^{-\alpha}(\partial_1\kappa)(\mathscr{R}, \mathscr{S})\mathscr{W}^2 - \bar{t}^{-\alpha}(\partial_2\kappa)(\mathscr{R}, \mathscr{S})\mathscr{W}\partial_x\mathscr{S}.$$

Furthermore, we remember that due to (6.7.23), we have that

$$\partial_{\bar{t}}\mathscr{S} + \bar{t}^{-\alpha}\lambda(R, S)\partial_x\mathscr{S} = 0.$$

Note that this equation is solved by the constant solution $\mathscr{S}(t, x) = c$, where $c \in \mathbb{R}$. Hence, if we assume that solutions $(\mathscr{R}, \mathscr{S})$ exist on an interval $[\bar{t}_0, \bar{t}_1)$, and $\mathscr{S}(\bar{t}_0) = c$, then $\mathscr{S}(\bar{t}) = c$ for all $\bar{t} \in [\bar{t}_0, \bar{t}_1)$ and x. Thus, if $\mathscr{S}$ is chosen to be homogeneous in x initially, then we have that $\partial_x\mathscr{S} \equiv 0$. Therefore, we are left with

$$\partial_{\bar{t}}\mathscr{W} = -\bar{t}^{-\alpha}\kappa(\mathscr{R}, \mathscr{S})\partial_x\mathscr{W} - \bar{t}^{-\alpha}(\partial_1\kappa)(\mathscr{R}, \mathscr{S})\mathscr{W}^2.$$

In coordinates (τ, ξ) as introduced in (6.7.24), we then find that

$$\begin{aligned}\partial_\tau \tilde{\mathscr{W}}(\tau,\xi) &= \partial_\tau \mathscr{W}(\tau,\phi(\tau,\xi)) = (\partial_1 \mathscr{W})(\tau,\phi(\tau,\xi)) + (\partial_2 \mathscr{W})(\tau,\phi(\tau,\xi))\partial_\tau\phi(\tau,\xi) \\ &= -\tau^{-\alpha}\kappa(\tilde{\mathscr{R}},\tilde{\mathscr{S}})(\tau,\xi)(\partial_2\mathscr{W})(\tau,\phi(\tau,\xi)) - \tau^{-\alpha}(\partial_1\kappa)(\tilde{\mathscr{R}},\tilde{\mathscr{S}})\tilde{\mathscr{W}}^2(\tau,\xi) \\ &\quad + \tau^{-\alpha}\kappa(\tilde{\mathscr{R}},\tilde{\mathscr{S}})(\tau,\xi)(\partial_2\mathscr{W})(\tau,\phi(\tau,\xi)) \\ &= -\tau^{-\alpha}(\partial_1\kappa)(\tilde{\mathscr{R}},\tilde{\mathscr{S}})\tilde{\mathscr{W}}^2(\tau,\xi)\,.\end{aligned} \tag{6.7.29}$$

Constructing initial data for blowup. For some constant $\mathring{\mathscr{R}} > 0$, consider initial data

$$\mathscr{R}(1,\cdot) = \mathscr{S}(1,\cdot) = \mathring{\mathscr{R}} > 0\,. \tag{6.7.30}$$

Invoking the translation formula (6.7.14), we find that this data coincides with

$$u(1,\cdot) = 0\,, \quad \rho(1,\cdot) = \rho_c \mathring{\mathscr{R}}^{\frac{1+K}{\sqrt{K}}}\,.$$

Hence, we see that for any $\mathring{\mathscr{R}}$ we recover a quiet fluid solution, launched by this initial data.

Now consider a non-trivial perturbation $\mathscr{R}(1,\cdot) = \mathring{\mathscr{R}} + h$, where $h \in C^\infty(S^1)$ with $h \neq 0$. Then, there exists an $x_0 \in S^1$, such that $\mathscr{W}(1, x_0) < 0$. From (6.7.29) as well as (6.7.18), we have the following Riccati-type ODE for $\tilde{\mathscr{W}}(\,\cdot\,, x_0)$:

$$\frac{d}{d\tau}\tilde{\mathscr{W}}(\tau,x_0) = \frac{12\tilde{\mathscr{S}}}{((3+\sqrt{3})\tilde{\mathscr{R}} - (-3+\sqrt{3})\tilde{\mathscr{S}})^2}\tau^{-\alpha}\mathscr{W}^2(\tau,x_0)\,.$$

Note that $\tilde{\mathscr{R}}(\cdot, x_0)$ is constant which is clear from (6.7.23) and the definition of (τ, ξ) coordinates. In addition, $\mathscr{S}$ is constant everywhere for all time. This ODE exhibits finite-time blowup for all $\alpha \leq 1$ and, by Grönwall's lemma, solution exists for all time given sufficiently small initial data for $\alpha > 1$.

6.7.2 Dust

Finally, we consider the case of dust. We again use the method of characteristics to establish shock formation for arbitrarily small perturbations of homogeneous solutions. For an alternative approach cf. [59]. We consider spacetimes of the form (6.1.2) subject to $\alpha \leq 1/2$. We remark that for $\alpha > 1/2$ it has been shown in [103] that small perturbations of homogeneous dust solutions stabilize. Thus, our result shows that this result from [103] is sharp in the parameter α.

Setup. For the metric (6.1.2) with $a(t) = t^\alpha$ the equation (6.3.1b) with $K = 0$ reduces to

$$u^0\partial_0 u^j + u^i\partial_i u^j + 2\alpha t^{-1}u^0 u^j = 0\,.$$

This is equivalent to

$$\partial_0 u^j + \frac{1}{u^0}u^i\partial_i u^j = -2\alpha t^{-1}u^j\,. \tag{6.7.34}$$

We can easily see from (6.7.34) that if we set $u^2 = u^3 = 0$ initially, this condition propagates. Thus from now on we write simply $u^1 = u$ and the equations simplify to

$$\partial_0 u + \frac{1}{\sqrt{t^{2\alpha}u^2+1}}u\partial_x u = -2\alpha t^{-1}u. \tag{6.7.35}$$

Characteristics of the equations of motion. Assume that we have classical solution $u(t,x)$ on some patch $(1,T)\times\mathbb{T}^3$ and a curve x on said patch. The solution on this curve is given by

$$z(s) = u(s, x(s))\,.$$

The system of (6.7.35) for $t(1) = 1$, $x(1) = x_1$ and $z(1, x_1) = g(x_1)$ is then equivalent to

$$\begin{aligned} &\frac{dt}{ds}(s) = 1 \Rightarrow t = s\,, \\ &\frac{dx}{dt}(t) = \frac{z(t)}{\sqrt{t^{2\alpha}z(t)^2+1}}\,, \qquad \frac{dz}{dt}(t) = -2\alpha t^{-1}z(t)\,. \end{aligned} \tag{6.7.36}$$

Integrating the last equation, we have

$$z(t) = t^{-2\alpha}g(x_1).$$

Furthermore, we have that

$$\dot{x}(t) = \frac{g(x_1)t^{-2\alpha}}{\sqrt{t^{-2\alpha}g(x_1)^2+1}}\,. \tag{6.7.37}$$

Upper bounds for the characteristics. Now consider the solution to the initial value problem

$$\begin{cases} \dot{\zeta}(t) = g(x_1)t^{-2\alpha}, \\ \zeta(1) = x_1. \end{cases} \tag{6.7.38}$$

Assuming that $g(x_1) > 0$, from the fact that

$$|\dot{\zeta}(t)| = |g(x_1)|t^{-2\alpha} > \frac{|g(x_1)|t^{-2\alpha}}{\sqrt{t^{-2\alpha}g(x_1)^2+1}} = |\dot{x}(t)| \tag{6.7.39}$$

we can infer the bound $\zeta(t) > x(t)$ for all $t > 1$.

6.7.3 Case $\alpha = 1/2$

Consider the case $\alpha = \frac{1}{2}$. Assume now without loss of generality that $x_1 < x_2$ and $0 < g(x_2) < g(x_1)$ (i.e. an interval where the initial data is strictly decreasing). Then, the solutions ζ^1, ζ^2 of (6.7.38), with initial data x_1 and x_2 respectively, are given by

$$\begin{aligned} \zeta^1(t) &= g(x_1)\log t + x_1\,, \\ \zeta^2(t) &= g(x_2)\log t + x_2. \end{aligned} \tag{6.7.40}$$

Equating the right-hand side, we find that

$$\log t = -\frac{x_2 - x_1}{g(x_2) - g(x_1)} > 0.$$

Hence, the two curves will meet at time

$$T = \exp\left(-\frac{x_2 - x_1}{g(x_2) - g(x_1)}\right)\,. \tag{6.7.42}$$

By the mean value theorem, we may conclude that there exists a breaking time T_b, such that for any time $T_0 > T_b$ there exist $x_{1,2}$, such that $\zeta_{1,2}$ will have an intersection time $T < T_0$. This breaking time is given by

$$T_b = \exp\left(-\frac{1}{\inf_x g'(x)}\right). \tag{6.7.43}$$

Note, however, that the curves $\zeta^{1,2}$ are only upper bounds for $x^{1,2}$ (the solutions to (6.7.37) with the same initial data), but do not envelope them. Hence, nothing about the intersections of the characteristics of (6.7.35) can be said at this point.

Construction of the enveloping curves. Integrating (6.7.37) in the case of $\alpha = \frac{1}{2}$ with initial data x_1 yields

$$x^1(t) = -2g(x_1)\log(-\sqrt{t} + \sqrt{g(x_1)^2 + t}) + c\,.$$

Prescribing initial data yields

$$c = x_1 + 2g(x_1)\log(-1 + \sqrt{g(x_1)^2 + 1})\,,$$

and hence

$$x^1(t) = x_1 + 2g(x_1)\log\Big(\frac{-1 + \sqrt{g(x_1)^2 + 1}}{-\sqrt{t} + \sqrt{g(x_1)^2 + t}}\Big)\,.$$

Using (6.7.40) we calculate the difference

$$\begin{aligned}|x^1(t) - \zeta^1(t)| &= \Big|2g(x_1)\log\Big(\frac{-1 + \sqrt{g(x_1)^2 + 1}}{-\sqrt{t} + \sqrt{g(x_1)^2 + t}}\Big) - 2g(x_1)\log(\sqrt{t})\Big| \\ &= \Big|2g(x_1)\log\Big(\frac{-1 + \sqrt{g(x_1)^2 + 1}}{-t + \sqrt{tg(x_1)^2 + t^2}}\Big)\Big|\,.\end{aligned}$$

Now let us observe what happens to the denominator as t goes to infinity:

$$-t + \sqrt{tg(x_1)^2 + t^2} = \frac{tg(x_1)^2 + t^2 - t^2}{t + \sqrt{tg(x_1)^2 + t^2}} = \frac{g(x_1)^2}{1 + \sqrt{t^{-1}g(x_1)^2 + 1}} \xrightarrow{t\to\infty} \frac{g(x_1)^2}{2}.$$

The difference $|x^1(t) - \zeta^1(t)|$ is easily seen to be increasing and, by the calculation above, converges to

$$|x^1(t) - \zeta^1(t)| \xrightarrow{t\to\infty} \Big|2g(x_1)\log\Big(\frac{2(-1 + \sqrt{g(x_1)^2 + 1})}{g(x_1)^2}\Big)\Big|. \tag{6.7.48}$$

Note that by l'Hopital, we have that

$$\lim_{g(x_1)\to 0} \frac{2(-1 + \sqrt{g(x_1)^2 + 1})}{g(x_1)^2} = \lim_{g(x_1)\to 0} \frac{2\frac{1}{2}\frac{1}{\sqrt{g(x_1)^2+1}}2g(x_1)}{2g(x_1)} = 1, \tag{6.7.49}$$

and hence, for fixed t,

$$|x^1(t) - \zeta^1(t)| \xrightarrow{t\to\infty} 0.$$

Thus, we have found that, at least pointwise, by decreasing the initial data size we cause ζ^1 and x^1 to converge.

Now since the the two curves ζ and x cannot ever be further apart than the expression given in (6.7.48), we can construct a new curve

$$\xi^1 = \zeta^1 - \left|2g(x_1)\log\Big(\frac{2(-1+\sqrt{g(x_1)^2+1})}{g(x_1)^2}\Big)\right| =: \zeta^1 - m(g(x_1))$$

where m denotes the *margin* between x^1 and ζ^1. Note that, by construction, ξ^1 is a lower bound for the characteristic x^1.

Estimating the time of intersection. Since ξ^1 is a lower bound of x^1 and ζ^2 is an upper bound of x^2, a crossing of these bounds at a point in time indicates that the characteristics $x^{1,2}$ must have already intersected. Equating $\xi^1(t)$ and $\zeta^2(t)$ yields

$$g(x_1)\log t + x_1 - m(g(x_1)) = g(x_2)\log t + x_2,$$

which is equivalent to the statement that

$$\log t = -\frac{x_2 - x_1}{g(x_2) - g(x_1)} + \frac{m(g(x_1))}{g(x_1) - g(x_2)}. \tag{6.7.53}$$

Therefore, the enveloping curves ζ^2 and ξ^1 intersect precisely at

$$T = \exp\Big(-\frac{x_2 - x_1}{g(x_2) - g(x_1)}\Big)\exp\Big(\frac{m(g(x_1))}{g(x_1) - g(x_2)}\Big).$$

Taking the limit $x_1 \to x_2$ is not possible, as it increases the error significantly. However, for fixed $x_{1,2}$, we see that by rescaling the initial data profile to be smaller ($g \to \lambda g$), by (6.7.49) we have that

$$\exp\Big(\frac{m(g(x_1))}{g(x_1) - g(x_2)}\Big) \xrightarrow{\lambda\to 0} 1.$$

Hence, small initial data ensures that shock happens later in time, but increases the accuracy of the prediction of the time of the forming of the shock.

6.7.4 Case $\alpha < \frac{1}{2}$

We consider now $\alpha < \frac{1}{2}$. As before, we know that ζ, the solution to the initial value problem (6.7.38), is an upper bound to the solution (given that $g(x_1) > 0$).
Consider the solution to the initial value problem

$$\begin{cases} \dot\gamma(t) = \frac{g(x_1)}{\sqrt{g(x_1)^2+1}} t^{-2\alpha}, \\ \gamma(1) = x_1. \end{cases} \tag{6.7.56}$$

By an analogous estimate to (6.7.39), we see that

$$\gamma(t) < x(t) \tag{6.7.57}$$

for all $t > 1$, given that $g(x_1) > 0$.

Consider again two solutions launched from $x_{1,2}$ with $0 < g(x_2) < g(x_1)$. By the previous analysis, we see that γ_1 is a lower bound to x^1 and ζ^2 is a upper bound for x^2. Equating these bounds we find the relation

$$x_2 + g(x_2)\frac{t^{-2\alpha+1} - 1}{-2\alpha - 1} = x_1 + \frac{g(x_1)}{\sqrt{g(x_1)^2 + 1}}\frac{t^{-2\alpha+1} - 1}{-2\alpha - 1}.$$

Solving for t yields a breaking time

$$T = \left((-2\alpha + 1)(x_2 - x_1)\frac{1}{\frac{g(x_1)}{\sqrt{g(x_1)^2+1}} - g(x_2)} + 1\right)^{\frac{1}{1-2\alpha}}.$$

Note that all of the terms in this expression are strictly positive with the exception of

$$\frac{g(x_1)}{\sqrt{g(x_1)^2 + 1}} - g(x_2). \tag{6.7.60}$$

The expression above may become negative, if, for example, g does not vary much in x but is large enough so that the denominator is dominant. This can be remedied by the following observation: given any initial data profile g, we define a new smaller data profile by $\tilde{g} = \lambda g$, for some $\lambda > 0$ to be determined later. Then, we calculate

$$\frac{\tilde{g}(x_1)}{\sqrt{\tilde{g}(x_1)^2 + 1}} - \tilde{g}(x_2) = \frac{\lambda g(x_1)}{\sqrt{\lambda^2 g(x_1)^2 + 1}} - \lambda g(x_2) = \frac{\lambda(g(x_1) - g(x_2)\sqrt{\lambda^2 g(x_1)^2 + 1})}{\sqrt{\lambda^2 g(x_1)^2 + 1}}. \tag{6.7.61}$$

Now, choose λ such that

$$\lambda < \frac{1}{g(x_1)}\sqrt{\frac{g(x_1)^2}{g(x_2)^2} - 1}\,.$$

Note that this is always possible as $\frac{g(x_1)}{g(x_2)} > 1$. Then the bracket in the numerator in (6.7.61), and therefore the whole expression, is larger than zero. We have therefore guaranteed that the expression for T exists and is larger than 1. Hence, we have shown that there exist arbitrarily small, smooth initial data that develop shock singularities in finite time.

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