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A reformulation of Khovanov-Rozansky's braid invariants in terms of fusion functors

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Abstract

In this work, we present an alternative formulation of the Khovanov-Rozansky braid invariants. The construction by Khovanov and Rozansky uses the theory of matrix factorizations. Our formulation is based on fusion functors. We review the underlying theory of R-modules, fusion functors and their cochain complexes from a categorical point of view. In this setting we construct braid invariants, which stand in principle on their own. We will subsequently demonstrate how this formulation is connected to the construction of Khovanov and Rozansky.

Kurzfassung

In dieser Arbeit geben wir eine alternative Formulierung von Khovanov-Rozanskys Zopfinvarianten. Die Konstruktion von Khovanov und Rozansky basiert auf der Theorie der Matrixfaktorisierungen. Die hier dargelegte Formulierung basiert auf Fusionsfunktoren. Wir betrachten die zugrunde liegende Theorie von R-Modulen, Fusionsfunktoren und deren Kettenkomplexe aus kategorieller Sicht. In diesem Rahmen konstruieren wir Zopfinvarianten, die im Prinzip für sich alleine stehen. Wir werden anschließend zeigen, wie diese Formulierung mit der Konstruktion von Khovanov und Rozansky zusammenhängt.

Contents

Introduction

Braids are very physical objects. It is not hard to picture a braid. Most likely one would think of woven ropes or hair, but braids also occur beyond the boundaries of our planet. Pictures from NASA have shown for example that the F-ring of Saturn's rings is braided [SSB$^+$81]. As mathematical objects, braids have first been described by Emil Artin [Art25, Art47]. From the mathematical point of view, material or size of braids are usually neglected and the focus lies on equivalence classes of braids under ambient isotopy, i.e. the topological structure of braids becomes the only relevant aspect. In this setting, one can define braid invariants, for example Khovanov-Rozansky's braid invariants.

From any braid one can obtain a link by connecting the beginning and end of the strands, i.e. by closing the braid. In fact any link can be brought into the form of a closed braid according to Alexander's Theorem. This is one reason why braid invariants can be seen as an intermediate step to obtain link invariants, when proceeding as [GKS23] suggest.

Khovanov and Rozansky came up with a description of link and tangle invariants. As a part of that, braid invariants are described. The original formulation by Khovanov and Rozansky is based on matrix factorizations. In a nutshell they assign to a braid or link a tensor product of cochain complexes of matrix factorizations. They prove that the corresponding braid and link invariants are given by equivalence classes of cochain complexes in a corresponding homotopy category [KR04b].

Matrix factorizations also appear in 2-dimensional topological field theory when considering $N = (2, 2)$ supersymmetric Landau-Ginzburg models [CR16, BR07]. Here, B-type interfaces which separate different regions of a worldsheet are represented by matrix factorizations up to isomorphisms. The fusion of two interfaces results in a new interface that is represented by a tensor product of two matrix factorizations representing the fused interfaces. In this sense computing Khovanov-Rozansky's homology is to some degree fusing defects of Landau-Ginzburg models.

In [Fre24] it is shown that at least some interfaces can be described by module functors instead of matrix factorizations. We will refer to these module functors as fusion functors, since they characterize an interface by its behaviour under fusion. This raises the question if and how Khovanov-Rozansky's braid invariants could be understood in this language.

In the first section 1 we review some basic category theory, including the definition of an additive category and the category of cochain complexes. In section 2 we review the geometric definition of a braid and the definition of Artin's braid group, in order to speak about braid invariants. In section 3 we set up the framework for Khovanov-Rozansky's braid invariants in terms of fusion functors. Starting off with the category of $R-$modules and endofunctors of those, we explain how those endofunctors form an additive strict monoidal category. We build a computational framework in which computations involving those functors and natural transformations are reduced to working with matrices with polynomial entries. We then assign to the generators of the braid group cochain complexes of endofunctors. We will verify the braid relations (3.34), (3.34) and the inverse relation (3.26). This will contain the main computational part of this work, concluding the formulation of braid invariants in terms of fusion functors. In Section 4 we explain the connection to Khovanov and Rozansky's construction. This will in hindsight motivate the form of the endofunctors of section 3, which came a bit out of the blue at first. We give a quick review of matrix factorizations and explain how the endofunctors of section 3 can be applied on matrix factorizations. In Proposition 4.4.3, we give the necessary conditions under which the braid invariants, described in terms of endofunctors, can be understood as a reformulation of Khovanov-Rozansky's braid invariants described in terms of matrix factorizations. We prove Proposition 4.4.3 explicitly in the case of the two stranded braid and explain why it also holds for braids with more than two strands.

1. Preliminary

1.1. Basic category theory

We review some category theory, in particular the structure of an additive category, leading up to cochain complexes, which are fundamental for Khovanov-Rozansky braid invariance.

Definition 1.1.1
A *category* $\mathcal{C}$ consists of

- a set of *objects* denoted $Ob(\mathcal{C})$
- for each pair of objects $a, b \in Ob(\mathcal{C})$, a set $Hom_{\mathcal{C}}(a, b)$, whose elements are referred to as *morphisms* from a to b
- associative *composition maps*:

$$\begin{aligned} Hom_{\mathcal{C}}(a,b) \times Hom_{\mathcal{C}}(b,c) &\to Hom_{\mathcal{C}}(a,c) \\ (\phi, \phi') &\mapsto \phi' \circ \phi \end{aligned}$$

  for all $a, b, c \in Ob(\mathcal{C})$, where associative means that for each $\phi \in Hom_{\mathcal{C}}(a, b)$, each $\phi' \in Hom_{\mathcal{C}}(b, c)$ and each $\phi'' \in Hom_{\mathcal{C}}(c, d)$ we have $(\phi'' \circ \phi') \circ \phi = \phi'' \circ (\phi' \circ \phi) \in Hom_{\mathcal{C}}(a, d)$.

- for each $a \in Ob(\mathcal{C})$ a unit morphism $1_a \in Hom_{\mathcal{C}}(a, a) := End_{\mathcal{C}}(a)$ satisfying

$$1_b \circ \phi = \phi \circ 1_a = \phi$$

  for all $\phi \in Hom_{\mathcal{C}}(a, b)$.

We will often abbreviate the notation for *objects* $Ob(\mathcal{C})$ of the category and simply write $a \in \mathcal{C}$, for an object a of the category $\mathcal{C}$. Instead of writing a morphism $\phi \in Hom_{\mathcal{C}}(a, b)$ we will also use the notation $\phi : a \to b$ or $a \xrightarrow{\phi} b$.

Whenever one defines a mathematical structure such as categories, one is interested in maps between those structures. The following definition arises

Definition 1.1.2
Let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is a map that

- maps objects to objects: $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

- maps morphisms to morphisms: $F : Hom_{\mathcal{C}}(a,b) \to Hom_{\mathcal{D}}(F(a),F(b))$ for all $a,b \in \mathcal{C}$ s.t.

$$F(1_a) = 1_{F(a)}, \qquad F(\phi \circ_{\mathcal{C}} \phi') = F(\phi) \circ_{\mathcal{D}} F(\phi'),$$

for all objects $a \in \mathcal{C}$ and all composable morphisms ϕ, ϕ' in $\mathcal{C}$. Here $\circ_{\mathcal{C}}$ and $\circ_{\mathcal{D}}$ denote the composition maps of $\mathcal{C}$ and $\mathcal{D}$ respectively.

There is also a class of maps between functors.

Definition 1.1.3
Let $\mathcal{C}$ and $\mathcal{D}$ be categories and F and G functors from $\mathcal{C}$ to $\mathcal{D}$, i.e. $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\eta : F \to G$ consists of a family of morphisms:

$$\Big(\eta_a \in Hom_{\mathcal{D}}\big(F(a), G(a)\big)\Big)_{a \in \mathcal{C}},$$

s.t. the diagram

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(\phi)"'] & G(a) \arrow[d, "G(\phi)"] \\ F(b) \arrow[r, "\eta_b"] & G(b) \end{tikzcd},$$

commutes for all $a, b \in \mathcal{C}$ and $\phi \in Hom_{\mathcal{C}}(a,b)$. The maps η_a are referred to as the *components* of η. [Lei16]

Let $\mathcal{C}$ and $\mathcal{D}$ be categories we introduce the category $Fun(C,D)$, with objects being functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms being natural transformations. The composition is given componentwise $(\eta \circ \phi)_c \coloneqq \eta_c \circ \phi_c$ and the identity morphism on a functor F is given by the natural transformation id_F with components defined $id_{F_c} = 1_{F(c)}$.

In the upcoming definitions we add more structure to categories building up to the definition of an additive category, while discussing the interplay between some definitions.

Definition 1.1.4
A category $\mathcal{C}$ is a *category of zero morphisms* if there exists a family of morphisms

$$\{0_{ab} \in Hom_{\mathcal{C}}(a,b) \quad \text{for all} \quad a,b \in \mathcal{C}\}, \quad \text{s.t} \quad \phi \circ 0_{ab} = 0_{ac} \quad \text{and} \quad 0_{bc} \circ \psi = 0_{ac}$$

for all $a,b,c \in \mathcal{C}$ and all $\phi \in Hom_{\mathcal{C}}(b,c)$ and $\psi \in Hom_{\mathcal{C}}(a,b)$.

This family is unique. Let $\{0'_{a,b} \in Hom_{\mathcal{C}}(a,b)$ for all $a,b \in \mathcal{C}\}$ be another family of zero morphisms then

$$0_{ab} = 0_{cb} \circ 0'_{ac} = 0'_{ab}$$

□[Par70]

Definition 1.1.5
Let $\mathcal{C}$ be a *category*. An object T is *terminal* if for every other object $X \in \mathcal{C}$ there exists exactly one morphism $X \to T$. An object I is called *initial* if for every other object $X \in \mathcal{C}$ there exists exactly one morphism $I \to X$. An object that is both *terminal* and *initial* is a *zero* or *null* object.

Remark. A zero object is unique up to unique isomorphism, so it makes sense to talk about the zero object $0 \in \mathcal{C}$. In a category $\mathcal{C}$ with a zero object 0 the unique morphisms $X \to 0 \to Y$ constitute the set of zero morphisms. Conversely this is not true though, a category with zero morphisms does not necessarily have to have zero objects.

Definition 1.1.6
An *Ab−enriched category* or *preaddtive category* is a *catgory* $\mathcal{C}$ s.t. for all objects $a, b \in \mathcal{C}$ the homsets $Hom_{\mathcal{C}}(a, b)$ are *additive abelian groups* and the composition relative to the addition is bilinear, i.e.

$$(\psi + \psi') \circ (\phi + \phi') = \psi \circ \phi + \psi \circ \phi' + \psi' \circ \phi + \psi' \circ \phi'$$

for $\phi, \phi' \in Hom_{\mathcal{C}}(a, b)$ and $\psi, \psi' \in Hom_{\mathcal{C}}(b, c)$.

Remark. In an $Ab-$ *enriched category* the neutral elements of each homset group form the unique zero morphisms of the category, this follows immediately from the bilinearity of the composition, implying that an $Ab-$ *enriched category* in particular forms a *category of zero morphisms* [ML98].

Definition 1.1.7
Let $\mathcal{C}$ be a category with *zero morphisms*, a *finite biproduct* of a finite collection of objects $a_1, a_2, \ldots, a_n$ is an object $\bigoplus_{i=1}^n a_i \in \mathcal{C}$ along with morphisms:

- projections $\pi_i : \bigoplus_{j=1}^n a_j \to a_i$ for each $i = 1, \ldots, n$,
- inclusions $i_i : a_i \to \bigoplus_{j=1}^n a_j$ for each $i = 1, \ldots, n$,

satisfying the following properties:

1. $(\bigoplus_{j=1}^n a_j,\ \pi_1, \ldots, \pi_n)$ forms a product of the collection $a_1, a_2, \ldots, a_n$, i.e. for any object $b \in \mathcal{C}$ and morphisms $\phi_i : b \to a_i$ (for $i = 1, \ldots, n$), there exists a unique morphism $\phi : b \to \bigoplus_{j=1}^n a_j$ such that
$$\pi_i \circ \phi = \phi_i \quad \text{for each } i = 1, \ldots, n.$$
2. $(\bigoplus_{j=1}^n a_j,\ i_1, \ldots, i_n)$ forms a coproduct of the collection $a_1, a_2, \ldots, a_n$, i.e. for any object b and morphisms $\psi_i : a_i \to b$ (for $i = 1, \ldots, n$), there exists a unique morphism $\psi : \bigoplus_{j=1}^n a_j \to b$ such that
$$\psi \circ i_i = \psi_i \quad \text{for each } i = 1, \ldots, n.$$

3. The projections π_i and inclusions i_i satisfy the following relations for all i, j:

$$\pi_i \circ i_j = \begin{cases} 1_{a_i} & \text{if } i = j, \\ 0_{a_i,a_j} & \text{if } i \neq j, \end{cases}$$

where 1_{a_i} is the identity morphism on a_i, and $0_{a_i,a_j}$ denotes the zero morphism from a_i to a_j.

Definition 1.1.8
An *additive category* is a preadditive category which admits all finitary biproducts.

Remark. In a preadditive category a product is also a coproduct and vice versa. So it is sufficient to define an additive category as a preadditive category which admits all finitary products, equivalently all finitary coproducts.

From Definition 1.1.8 it follows that an additive category has a zero object, which is defined as the empty biproduct.

Remark. In a preadditive category the conditions (1) and (2) of Definition 1.1.7 are equivalent to

$$\sum_i^n \iota_i \circ \pi_i = 1_{\bigoplus_i^n a_i} \tag{1.1}$$

for $n > 0$. [ML98]

Thus, we conclude that an additive category is a category which is preadditive, allowing us to build all finitary biproducts, coming with the projection and embedding maps satisfying the conditions (1), (2) in Definition 1.1.8 and condition (1.1).

Given objects $a_1, \dots, a_n$ and $b_1, \dots, b_m$, we can represent morphisms $f : \bigoplus_i^n a_i \to \bigoplus_j^m b_j$ by matrices:

$$\begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \cdots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}, \quad \text{where} \quad f_{ij} = \pi_i^b \circ f \circ \iota_j^a : a_j \to b_i. \tag{1.2}$$

Using that $\sum_i^n \iota_i^a \circ \pi_i^a = 1_{\bigoplus_i^n a_i}$ and $\sum_j^m \iota_i^b \circ \pi_i^b = 1_{\bigoplus_i^m b_i}$, it follows that the addition and composition of morphisms obey the normal rule of matrix addition and multiplication.

Definition 1.1.9
Let $\mathcal{A}$ be an additive category. A *cochain complex* $A^\bullet$

$$\dots \longrightarrow A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \longrightarrow \dots$$

is a sequence of objects $A_i \in \mathcal{A}$ and morphisms $d_i : A_i \to A_{i+1}$ such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. We refer to an object A_i as the degree i component and to the morphism d_i as the degree i differential of the cochain complex.

We say a cochain complex is bounded from above iff for some $n \in \mathbb{Z}$ all components with $i > n$ are the zero object 0 of $\mathcal{A}$, i.e.

$$\dots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{0} 0 \longrightarrow \dots$$

and *bounded from below* iff for some $n \in \mathbb{Z}$ all components with $i < n$ are the zero object 0 of $\mathcal{A}$, i.e.

$$\dots \longrightarrow 0 \xrightarrow{0} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \dots$$

In the following we will omit the unique zero morphisms 0 in the arrows. The zero complex $0^\bullet$ is the chain complex with all components being the zero object $0 \in \mathcal{A}$, i.e.

$$\dots \longrightarrow 0 \xrightarrow{0} 0 \longrightarrow \dots$$

Definition 1.1.10
A *morphism of cochain complexes* $f : A^\bullet \to B^\bullet$ is a family of morphisms $f_i : A_i \to B_i$ such that all the diagrams

$$\begin{tikzcd} A_i \arrow[r, "d_{Ai}"] \arrow[d, "f_i"'] & A_{i+1} \arrow[d, "f_{i+1}"] \\ B_i \arrow[r, "d_{Bi}"'] & B_{i+1} \end{tikzcd}$$

commute, where the differentials of $A^\bullet$ and $B^\bullet$ are denoted by d_{Ai} and d_{Bi} respectively.

Cochain complexes and morphisms of cochain complexes form the *category of cochain complexes* of $\mathcal{A}$, which we denote by $CoCh(\mathcal{A})$. $CoCh(\mathcal{A})$ is an additive category, the abelian group structure of morphisms is carried straight over, i.e. $f + g$ is the family of morphisms $\{f_i + g_i\}_{i\in\mathbb{Z}}$ for two morphisms of chain complexes f, g, the direct sum or biproduct of two chain complexes is formed componentwise, i.e. $A^\bullet \oplus B^\bullet$ is the cochain complex

$$\dots \longrightarrow A_{i-1} \oplus B_{i-1} \xrightarrow{\begin{pmatrix} d_{Ai-1} & 0 \\ 0 & d_{Bi-1} \end{pmatrix}} A_i \oplus B_i \xrightarrow{\begin{pmatrix} d_{Ai} & 0 \\ 0 & d_{Bi} \end{pmatrix}} A_{i+1} \oplus B_{i+1} \longrightarrow \dots$$

using the matrix representation of morphisms (1.2).

Definition 1.1.11
A *homotopy* between morphisms of chain complexes $f, g : A^\bullet \to B^\bullet$ is a family of morphisms $h_i : A_i \to B_{i-1}$, such that

$$f_i - g_i = d_{Bi-1} \circ h_i + h_{i+1} \circ d_{Ai}.$$

From this definition we obtain an equivalence relation on the space of morphisms, i.e. two morphisms are said to be equivalent or homotopic if there exists a homotopy between them. The homotopy category of cochain complexes $HCoCh(\mathcal{A})$ has cochain complexes as objects and homotopy classes of cochain complex morphisms as morphisms, i.e. it has the same objects as $CoCh(\mathcal{A})$ but fewer morphisms.

2. Braids and braid invariants

There have been many different equivalent description of braids. Emil Artin was probably the first to develop a theory of braids in his paper 1925 [Art25] in a more intuitive approach, thinking about braids as physical threads which satisfy certain properties. In his later paper he gave a mathematical refined version of braid theory [Art47]. From a geometric point of view the main property one wants to capture is the topology of a braid, i.e. not caring about the thickness or the material a braid is made of. This leads to the following definition:

Definition 2.0.1
An n-strand geometric braid is a union β of n arcs in $\mathbb{R}^3$, pairwise disjoint, linking n points $(0, i, 0), i = 1, \ldots, n$,to the n points $(1, i, 0), i = 1, \ldots, n$, within the band $[0, 1] \times \mathbb{R}^2$, and whose intersection with every plane $x \times \mathbb{R}^2$ contains exactly n points. The family of n-stranded geometric braids is denoted $\mathcal{GB}_n$. [Deh21]

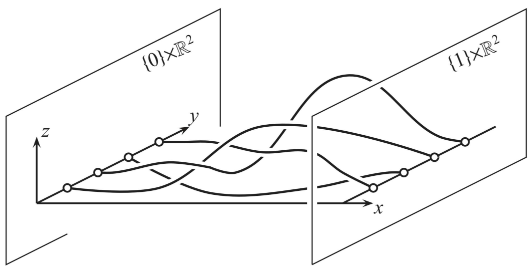


Figure 2.1.: A 4-strand geometric braid, the union of four arcs not intersecting each other going from (0,i,0) to (1,i,0) while strictly increasing in the x-direction, [Deh21]

We may consider the equivalence class of braids under ambient isotopy, although we we will confuse a braid with its equivalence class. It will also be convenient to represent a braid by its braid diagram, i.e. the projection of a braid onto the xy-plane. Such a diagram has then a number of over- and under-crossings indicating which strands goes above the other.

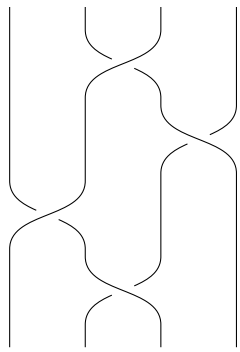


Figure 2.2.: Braid diagram of a 4-stranded braid with 2 over and 2 under crossings

In [Art47, Art25] Artin showed that braids form a group. The identity element may then be represented by the strand whose arcs are all straight lines from $(0, i, 0)$ to $(1, i, 0)$, i.e. they are constant in z and y. The group multiplication of braids is given by the concatenation, so that inverses are constructed by reflecting on the xz-plane.

The n-strand braid group has the representation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \,|\, \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \rangle \tag{2.1}$$

where the generators σ_i and their inverse σ_i^{-1} correspond to the crossings formed between the i and $i+1$ strands as in Figure 2.3. Group elements are formed by words of the generators and their inverse, reflecting the braid one obtains when connecting elements of the types depicted in Figure 2.3. For example the braid in Figure 2.2 is represented by the word $\sigma_2^{-1} \sigma_3 \sigma_1^{-1} \sigma_2$.

Figure 2.3.: The i-th generator σ_i of B_n on the left and its inverse σ_i^{-1} on the right

The relations of the braid group in (2.1) correspond to the relations of braid diagrams (3.34) (3.40) and the inverse relation corresponds to (3.26).

In this language one can construct a braid invariant following the strategy:

- Assign some objects to the generators and their inverse $\sigma_i^{\pm 1}$ and some object to the identity element $1 \in B_n$
- Assign an operation to the group multiplication of B_n
- Verify that the braid relations and the inverse relation hold under the assignments done

Khovanov-Rozansky's braid invariants

In our case we assign to the generators cochain complexes of module functors. The operation assigned to the group multiplication is the tensor product of cochain complexes. We show that this gives a braid invariant considering the homotopy category of cochain complexes, i.e. the cochain complexes obtained for the right and left side of the braid relations are isomorphic in the corresponding category. This construction might come a bit out of the blue. It has its origin in Khovanov-Rozansky's link invariant formulation [KR04b]. Their formulation is done in the language of matrix factorizations. Matrix factorizations also appear in the modelling of 2D Landau Ginzburg models, in these models they represent one dimensional defects. In [BF21] Stefan Fredenhagen and Nicolas Behr came up with a functorial approach to describe such defects, i.e. some defects can be described by module functors with certain properties. Since braid invariants in Khovanov-Rozansky's approach are described by matrix factorizations which can be described by such module functors, it is natural to consider whether one can understand the braid invariants in terms of those module functors.

Remark. Khovanov-Rozansky's formulation is a broader description since it also describes invariants for tangles and links. To obtain a formulation of a link invariant in the formulation of module functors one needs to give a certain operation for the closing of a braid which satisfies the Markov moves. This might be done by considering the Hochschild homology of the cochain complexes of modules which are obtained by applying the cochain complexes of functors onto a module, for example the ground module.

3. Khovanov-Rozansky's braid invariants in terms of fusion functors

Following the strategy to describe braid invariants outlined in Section 2 we assign to the generators σ_i and their inverse σ_i^{-1} the cochain complexes of module functors

$$\sigma_i \mathrel{\widehat{=}} C(\sigma_i) = \; 0 \longrightarrow \underline{U} \xrightarrow{\Phi_{i,i+1}} V_{i,i+1} \longrightarrow 0 \tag{3.1}$$

$$\sigma_i^{-1} \mathrel{\widehat{=}} C(\sigma_i^{-1}) = \; 0 \longrightarrow V_{i,i+1} \xrightarrow{\Psi_{i,i+1}} \underline{U} \longrightarrow 0 \; . \tag{3.2}$$

To the group multiplication, we associate the tensor product of cochain complexes. This framework yields braid invariants as isomorphism classes within the relevant homotopy category $HCoCh(End(R - mod_f))$.

The process begins with an additive category of R-modules, from which we examine endofunctors that themselves constitute objects of an additive strict monoidal category $End(R - mod_f)$, with morphisms being natural transformations. Subsequently, we evaluate tensor products of cochain complexes of module functors and verify the braid relations within $HCoCh(End(R - mod_f))$.

3.1. Category of R-modules

The category of R-modules is an example of an additive category or in other words: An additive category is a generalisation of the category of R-modules.

Definition 3.1.1
Let R be a unitary ring. An R-module is an additive group $(M, +)$ together with a map

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto r \cdot m \end{aligned}$$

called the action of R on M, s.t.

$$\begin{aligned} r \cdot (m_1 + m_2) &= r \cdot m_1 + r \cdot m_2 \\ (r_1 * r_2) \cdot (m) &= r_1 \cdot (r_2 \cdot m) \\ (r_1 + r_2) \cdot m &= r_1 \cdot m + r_2 \cdot m \\ 1 \cdot m &= m, \end{aligned}$$

where we denote the multiplication of the ring by $*$, the unitary element of R with respect to the multiplication by 1 and $+$ the addition in R and M respectively.

Definition 3.1.2
Let M and N be two R-modules. A morphism of modules is a map $f : M \to N$, compatible with the R-action, respecting the addition of the modules, i.e.

$$f(r \cdot m) = r \cdot f(m)$$
$$f(m_1 + m_2) = f(m_1) + f(m_2).$$

Modules and morphisms between modules form the category of R-modules denoted by $R - Mod$. $R - Mod$ is an additive category.

The addition of morphisms $f, g : M \to N$ is defined element wise, i.e.

$$(f + g)(m) \coloneqq f(m) + g(m)$$

making the homsets into an additive abelian group, where unit elements are defined as the zero morphisms

$$\begin{aligned} 0_{MN} : M &\to N \\ m &\mapsto 0_N \end{aligned}$$

where 0_N is the neutral group element of N. It is straightforward to check that this is a morphism of modules. Furthermore the addition of morphisms is bilinear under composition making $R - Mod$ into a preadditive category.

Let M and N be R-modules, the direct sum module $M \oplus N$ is the R-module with addition and R-action being defined

$$(m_1 \oplus n_1) + (m_2 \oplus n_2) \coloneqq (m_1 + m_2 \oplus n_1 + n_2)$$
$$r \cdot (m \oplus n) \coloneqq (r \cdot m \oplus r \cdot n).$$

The projection and embedding maps $\pi_M : M \oplus N \to M$, $\pi_N : M \oplus N \to N$ and $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus N$ defined by

$$\begin{aligned} \pi_M(m \oplus n) &\coloneqq m \\ \pi_N(m \oplus n) &\coloneqq n \\ \iota_M(m) &\coloneqq m \oplus 0_N \\ \iota_N(n) &\coloneqq 0_M \oplus n, \end{aligned}$$

satisfy

$$\pi_M \circ \iota_N = 0_{NM}, \pi_N \circ \iota_M = 0_{MN}$$
$$\pi_M \circ \iota_M = 1_M, \pi_N \circ \iota_N = 1_N$$
$$\iota_M \circ \pi_M + \iota_N \circ \pi_N = 1_{M \oplus N},$$

following from their definition. Since we can build arbitrary direct sums of modules, i.e. the category of $R - Mod$ admits all finitary biproducts, the category $R - Mod$ is an additive category.

We denote the zero module (zero object of $R - Mod$) by 0 which is the module with a single element * and the single morphisms $1_0 : * \mapsto *$. The zero object is uniquely isomorphic to the subgroup 0_M of any M viewed as a module itself with the single element 0_M, the unit element of M.

3.2. An additive monoidal category of functors

Let R be a polynomial ring over $\mathbb{C}$ and let $R-mod_f$ be the category of free finite rank R-modules, i.e. objects are R-modules of finite rank admitting a basis. We consider functors U from $R-mod_f$ to $R-mod_f$ whose action on morphisms is $\mathbb{C}-linear$. Let $End(R-mod_f)$ be the category whose objects are $\mathbb{C}-linear$ endofunctors and whose morphisms are natural transformations. $End(R-mod_f)$ is an additive category as we shall explain in the following.

Let $\eta_1, \eta_2 : U \to V$ be natural transformations, their sum is defined by the addition in $R-mod_f$, i.e $\eta_1 + \eta_2$ has components

$$\left((\eta_1 + \eta_2)_M \coloneqq \eta_{1M} + \eta_{2M}\right)_{M \in R-mod_f}.$$

The neutral element of $Hom_{End(R-mod_f)}(U, V)$ is the natural transformation 0_{UV} with the components

$$\left(0_{UV_M} = 0_{U(M)V(M)}\right)_{M \in R-mod_f},$$

thus satisfying $\eta + 0_{UV} = \eta$. Thus $End(R-mod_f)$ is a preadditve category.

Furthermore $End(R-mod_f)$ admits all finitary biproducts. Let $U, V \in End(R-mod_f)$ we define the functor $U \oplus V$ on modules

$$(U \oplus V)(M) = U(M) \oplus V(M)$$

and on any morphisms $\phi : M \to N$

$$\begin{aligned}(U \oplus V)(\phi) =& \iota_{V(N)} \circ V(\phi) \circ \pi_{V(M)} + \iota_{U(N)} \circ U(\phi) \circ \pi_{U(M)} \\ &\in Hom_{R-mod_f}(U(M) \oplus V(M), U(N) \oplus V(N))\end{aligned}$$

where $\pi_{U(M)}$ and $\pi_{V(M)}$ denote the projection maps from $U(M) \oplus V(M)$ to $U(M)$ and $V(M)$, respectively, and $\iota_{U(N)}$ and $\iota_{V(N)}$ the embedding maps from $U(N)$ and $V(N)$ to $U(N) \oplus V(N)$, respectively. We define the natural transformation $\pi_U : U \oplus V \to U$ consisting of components

$$\left(\pi_{U_M} \coloneqq \pi_{U(M)} \in Hom_{R-mod_f}(U(M) \oplus V(M), U(M))\right)_{M \in R-mod_f}$$

and similarly π_V. The embedding $\iota_U : U \to U \oplus V$ has the components

$$\left(\iota_{U_M} \coloneqq \iota_{U(M)} \in Hom_{R-mod_f}(U(M), U(M) \oplus V(M))\right)_{M \in R-mod_f}$$

and similarly ι_V. They satisfy the relations

$$\begin{aligned}\pi_V \circ \iota_V &= 1_V \\ \pi_U \circ \iota_U &= 1_U \\ \pi_U \circ \pi_V &= 0_{VU} \\ \pi_V \circ \iota_U &= 0_{UV} \\ \iota_U \circ \pi_U + \iota_V \circ \pi_V &= 1_{U \oplus V}.\end{aligned}$$

The zero functor 0 is defined on modules

$$0 : M \to 0$$

and on morphisms

$$0(\phi) = 0_0 = 1_0$$

for any $\phi : M \to N$.

The category $End(R - mod_f)$ is a monoidal category where the monoidal product of objects is given by the composition of functors, its unit object is the identity functor. Let U_1, U_2, V_1, V_2 be functors and $\eta_1 : U_1 \to V_1$ and $\eta_2 : U_2 \to V_2$ be natural transformations. The monoidal product or horizontal composition

$$\eta_2 \otimes \eta_1 : U_2 \circ U_1 \to V_2 \circ V_1$$

is given by their components defined as

$$(\eta_2 \otimes \eta_1)_M := \eta_{2V_1(M)} \circ U_2(\eta_{1M}) : U_2(U_1(M)) \to V_2(V_1(M)). \tag{3.3}$$

In fact $End(R-mod_f)$ is a strict monoidal category since the composition of functors is associative by definition. To see $(\eta_3 \otimes \eta_2) \otimes \eta_1 = \eta_3 \otimes (\eta_2 \otimes \eta_1) : U_3 \circ U_2 \circ U_1 \to V_3 \circ V_2 \circ V_1$ one just writes out their components as defined in (3.3) and uses functoriality of the functors, where $\eta_3 : U_3 \to V_3$. Similarly one finds $\eta \circ id_U = \eta = id_U \circ \eta$ for all functors U and morphisms $\eta : U \to U$.

3.3. A monoidal category of cochain complexes

We have seen in section 3.2 that $End(R - mod_f)$ constitutes an additive strict monoidal category. We therefore can consider the category $HCoCh(End(R - mod_f))$ to be the category of cochain complexes of $End(R - mod_f)$ up to homotopies. $HCoch(End(R - mod_f))$ is a monoidal category. Let $F^\bullet, G^\bullet \in HCoCh(End(R - mod_f))$, their monoidal or tensor product $(F^\bullet \otimes G^\bullet)$ is given by the cochain complex with components

$$\left(F^\bullet \otimes G^\bullet\right)_n := \bigoplus_{i+j=n} F_i \circ G_j \tag{3.4}$$

and differentials

$$d_{F^\bullet \otimes G^\bullet n} := \sum_{i+j=n} \iota_{F_{i+1} \circ G_j} \circ (d_{F_i} \otimes id_{G_j}) \circ \pi_{F_i \circ G_j} + (-1)^i \iota_{F_i \circ G_{j+1}} \circ (id_{F_i} \otimes d_{G_j}) \circ \pi_{F_i \circ G_j} \tag{3.5}$$

where $\pi_{F_i \circ G_j} : \bigoplus_{i+j=n} F_i \circ G_j \to F_i \circ G_j$ for all i, j, and the embedding $\iota_{F_{i+1} \circ G_j} : F_{i+1} \circ G_j \to \bigoplus_{k+l=n+1} F_k \circ G_l$ and similarly $\iota_{F_j \circ G_{j+1}}$ for all $i + j = n$. Using the matrix representation of morphisms in an additive category (1.2) the entries of the matrices are morphisms of the form $d_{F_i} \otimes id_{G_j}$ and $(-1)^i (id_{F_i} \otimes d_{G_j})$.

As stated before the tensor product of cochain complexes is associative up to isomorphism making $HCoCh(End(R-mod_f))$ into a monoidal but not strict monoidal category, i.e.

$$A^\bullet \otimes (B^\bullet \otimes C^\bullet) \simeq (A^\bullet \otimes B^\bullet) \otimes C^\bullet \simeq A^\bullet \otimes B^\bullet \otimes C^\bullet \tag{3.6}$$

where the rightmost tensor product is defined as the cochain complex with the components

$$(A^\bullet \otimes B^\bullet \otimes C^\bullet)_n := \bigoplus_{a+b+c=n} A_a \circ B_b \circ C_c$$

and differentials

$$\begin{aligned} d^n_{A^\bullet \otimes B^\bullet \otimes C^\bullet} := \sum_{a+b+c=n} \big(& \iota_{(a+1)bc} \circ (d_{A_a} \otimes 1_{B_b} \otimes 1_{C_c}) + (-1)^a \iota_{a(b+1)c} \circ (1_{A_a} \otimes d_{B_b} \otimes 1_{C_c}) \\ & + (-1)^{a+b} \iota_{ab(c+1)} \circ (1_{A_a} \otimes 1_{B_b} \otimes d_{C_c}) \big) \circ \pi_{abc}. \end{aligned}$$

This bracket less definition of the tensor product extends to tensor products of more then three cochain complexes in an analogous way and will be useful for computations later on.

Remark. Showing isomorphisms of (3.6) involves basic properties of the canonical embedding and projection morphisms and properties of the horizontal composition of natural transformations.

The following lemma will be useful when investigating isomorphism classes of cochain complexes:

Lemma 3.3.1
Let $A^\bullet \simeq A'^\bullet$ and $B^\bullet \simeq B'^\bullet$ be cochain complexes, then $A^\bullet \otimes B^\bullet \simeq A'^\bullet \otimes B'^\bullet$.

3.3.1. Isomorphism classes in $HCoCh(End(R-mod_f))$

We turn our attention to isomorphism classes in $HCoCh(End(R-mod_f))$, since they form the braid invariants described. Two cochain complexes $F^\bullet$ and $G^\bullet$ are isomorphic in $HCoCh(End(R-mod_f)$ if

$$\begin{tikzcd} \cdots \arrow[r] & F_{i-1} \arrow[r, "d_{F_{i-1}}"] \arrow[d, shift right, "\phi_{i-1}"'] & F_i \arrow[r, "d_{F_i}"] \arrow[d, shift right, "\phi_i"'] & F_{i+1} \arrow[r] \arrow[d, shift right, "\phi_{i+1}"'] & \cdots \\ \cdots \arrow[r] & G_{i-1} \arrow[r, "d_{G_{i-1}}"] \arrow[u, shift right, "\phi'_{i-1}"'] & G_i \arrow[r, "d_{G_i}"] \arrow[u, shift right, "\phi'_i"'] & G_{i+1} \arrow[r] \arrow[u, shift right, "\phi'_{i+1}"'] & \cdots \end{tikzcd} \tag{3.7}$$

commutes and there exist a family of natural transformations $\{h_{G_i}\}$ and $\{h_{F_i}\}$ s.t.

$$\begin{aligned} \phi'_i \circ \phi_i + h_{F_{i+1}} \circ d_{F_i} + d_{F_{i-1}} \circ h_{F_i} &= 1_{F_i} \\ \phi_i \circ \phi'_i + h_{G_{i+1}} \circ d_{G_i} + d_{G_{i-1}} \circ h_{G_i} &= 1_{G_i} \end{aligned} \tag{3.8}$$

where 1_{F_i} and 1_{G_i} denote the identity morphisms of F_i and G_i respectively. Since all morphisms involved are natural transformations, this in principle means that for all

$M \in R-mod_f$ the morphisms admit components s.t. for all components the diagram (3.7) commutes and (3.8) holds. We will demonstrate that it is sufficient for (3.7) to commute and (3.8) to be satisfied for the R-components of the natural transformations involved.

Proposition 3.3.2
Let F, G be any functors in $End(R-mod_f)$. A natural transformation η: $F \to G$ is already specified by its component η_R of the ground module R.

Proof. Consider the following commutative diagrams for any free finite rank module M

$$\begin{tikzcd} F(R) \arrow[r, "\eta_R"] \arrow[d, "F(\iota^i_M)"'] & G(R) \arrow[d, "G(\iota^i_M)"] \\ F(M) \arrow[r, "\eta_M"'] & G(M) \end{tikzcd}, \qquad \begin{tikzcd} F(M) \arrow[r, "\eta_M"] \arrow[d, "F(\pi^i_M)"'] & G(M) \arrow[d, "G(\pi^i_M)"] \\ F(R) \arrow[r, "\eta_R"'] & G(R) \end{tikzcd}, \tag{3.9}$$

with the embedding maps $\iota^i_M : R \to M$ and projection maps $\pi^i_M : M \to R$ where $i = 1, ..., m$ and m is the rank of M. We therefore have

$$\eta_M = 1_{G(M)} \circ \eta_M = \sum_i^m G(\iota^i_M) \circ G(\pi^i_M) \circ \eta_M = \sum_i^m G(\iota^i_M) \circ \eta_R \circ F(\pi^i_M) \tag{3.10}$$

using (3.9) and linearity and functoriality of G and F. $\square$

The addition of natural transformations is specified by the addition of their R-components

$$\begin{aligned} \eta_M + \eta'_M &= \sum_i^m F(\iota^i_M) \circ \eta_R \circ G(\pi^i_M) + \sum_k^m (F(\iota^k_M) \circ \eta'_R \circ G(\pi^k_M) \\ &= \sum_i^m F(\iota^i_M) \circ (\eta_R + \eta'_R) \circ G(\pi^i_M) \end{aligned}$$

where $\eta, \eta' : F \to G$ and we used bilinearity of the addition relative to the composition.

Furthermore the composition of natural transformations is also specified by the composition of their R-components, i.e.

$$\begin{aligned} \eta_M \circ \varphi_M &= \sum_i^m F(\iota^i_M) \circ \eta_R \circ G(\pi^i_M) \circ \sum_k^m G(\iota^k_M) \circ \varphi_R \circ H(\pi^k_M) \\ &= \sum_{i,k}^m F(\iota^i_M) \circ \eta_R \circ G(\pi^i_M \circ \iota^k_M) \circ \varphi_R \circ H(\pi^k_M) \\ &= \sum_{i,k}^m F(\iota^i_M) \circ \eta_R \circ \delta_{ik} \circ \varphi_R \circ H(\pi^k_M) \\ &= \sum_i^m F(\iota^i_M) \circ \eta_R \circ \varphi_R \circ H(\pi^i_M) \end{aligned}$$

for $\eta : F \to G$ and $\varphi : G \to H$, we used bilinearity of the addition relative to the composition and functoriality. The delta morphism δ_{ik} is defined to be 1_R if $i = k$ and 0_R if $i \neq k$. Proposition 3.3.2 together with the compatibility of addition and composition of the components of natural transformations lets us conclude that two cochain complexes in $HCoCh(End(R - mod_f))$ are isomorphic if (3.8) holds for the R-component, i.e.

$$\phi'_{iR} \circ \phi_{iR} + h_{F_{i+1}R} \circ d_{F_iR} + d_{F_{i-1}R} \circ h_{F_iR} = id_{F_iR}$$
$$\phi_{iR} \circ \phi'_{iR} + h_{G_{i+1}R} \circ d_{G_iR} + d_{G_{i-1}R} \circ h_{G_iR} = id_{G_iR}.$$

3.4. Isomorphism classes in $End(R - mod_f)$

As we have seen in Proposition 3.3.2 a natural transformation is completely characterized by its $R-$component. This already indicates that the determining property of a functor lies in its action on the ground module R. We have the following proposition:

Proposition 3.4.1
A functor is specified up to isomorphism by its action on the ground module R and action on maps

$$\begin{aligned} m_p : & R \to R \\ & r \mapsto p \cdot r \end{aligned}$$

Proof. Let F and $\tilde{F}$ be functors with $F(R) = \tilde{F}(R)$ and $F(m_p) = \tilde{F}(m_p)$. There exists a natural isomorphism α with components $\alpha_M = \sum_i^m \tilde{F}(\iota^i_M) \circ F(\pi^i_M)$ such that

$$\begin{tikzcd} F(M) \arrow[r, "\alpha_M"] \arrow[d, "F(f)"] & \tilde{F}(M) \arrow[d, "\tilde{F}(f)"] \\ F(N) \arrow[r, "\alpha_N"] & \tilde{F}(N) \end{tikzcd}$$

commutes. We calculate

$$\begin{aligned} \tilde{F}(f) \circ \alpha_M = & \tilde{F}(\iota^k_N \circ \pi^k_N \circ f \circ \iota^j_M \circ \pi^j_M) \circ \tilde{F}(\iota^i_M) \circ F(\pi^i_M) \\ = & \tilde{F}(\iota^k_N) \circ \tilde{F}(\pi^k_N \circ f \circ \iota^j_M) \circ \delta_{ji} \circ F(\pi^i_M) \\ = & \tilde{F}(\iota^k_N) \circ \tilde{F}(\pi^k_N \circ f \circ \iota^j_M) \circ F(\pi^j_M) \\ = & \tilde{F}(\iota^k_N) \circ F(\pi^k_N \circ f \circ \iota^j_M) \circ F(\pi^j_M) \\ = & \tilde{F}(\iota^k_N) \circ F(\pi^k_N) \circ F(f) \circ F(1_M) \\ = & \alpha_N \circ F(f) \end{aligned}$$

where we used that $\tilde{F}(\pi^k_N \circ f \circ \iota^j_M) = F(\pi^k_N \circ f \circ \iota^j_M)$, i.e. on maps from R to R the functors F and $\tilde{F}$ acting the same way. To complete the proof we give the inverse natural transformation α^{-1} with components $\alpha^{-1}_M = \sum_i^m F(\iota^i_M) \circ \tilde{F}(\pi^i_M)$. $\square$

Remark. In fact if $F(R) \simeq \tilde{F}(R)$ and there is an isomorphism $\varphi : F(R) \to \tilde{F}(R)$ such that $F(m_p) = \varphi^{-1} \circ \tilde{F}(m_p) \circ \varphi$ we have $[F] = [\tilde{F}]$. In this case the natural isomorphism is given by $\alpha_M = \sum_i^m \tilde{F}(\iota_M^i) \circ \varphi \circ F(\pi_M^i)$.

3.4.1. Representatives of the isomorphism classes

When performing calculations it is convenient to work with representatives of functors. In particular we want to fix the action of a functor on the ground module R and on higher rank modules R^m and R^n and R-module maps between them, which are given by $n \times m$-matrices with entries $(f_{ij}) \in R$. One can also fix a functor at least implicitly on all free finite rank modules M and N and morphisms between them, obtaining a representative of an isomorphism class of functors.

Proposition 3.4.2
Let M be a module of rank m and $F \in End(R - mod_f)$. The rank of $F(M)$ is already determined by the rank of $F(R)$, i.e. $rank(F(M)) = mr_F$, where r_F is the rank of $F(R)$.

Proof. There exists a morphism

$$\sum_k^m \iota^k \circ \phi \circ F(\pi_M^k) : F(M) \to R^{mr_F} \tag{3.11}$$

with an isomorphism $\phi : F(R) \to R^{r_F}$ and the projection maps $\pi_M^k : M \to R$ and the embedding map $\iota^k : R^{r_F} \to R^{mr_F}$. The morphism (3.11) has an inverse

$$\sum_i^m F(\iota_M^i) \circ \phi^{-1} \circ \pi^i,$$

with $\pi^i : R^{mr_F} \to R^{r_F}$ and $\iota_M^i : R \to M$, thus $F(M) \simeq R^{mr_F}$. □

Let $\tilde{U} \in End(R - mod_f)$, there is an isomorphic functor U, whose action on the ground module R is $U(R) \coloneqq R^{r_U}$. To be precise, there is an isomorphism $\psi : \tilde{U}(R) \to R^{r_U}$, since $\tilde{U}(R)$ is a finite rank $R-$module. Then $U(m_p) \coloneqq \psi \circ \tilde{U}(m_p) \circ \psi^{-1}$ is an R-module morphism from R^{r_U} to R^{r_U} for all $p \in R$, i.e. given by an $r_U \times r_U$ - matrix with entries in R, satisfying

$$U(m_p \circ m_q) = U(m_p) \circ U(m_q) \tag{3.12}$$

$$U(m_1) = 1_{R^{r_U}}, \tag{3.13}$$

and

$$U(m_{c_1 p} + m_{c_2 q}) = c_1 U(m_p) + c_2 U(m_q), \tag{3.14}$$

for all $p, q \in R$ and $c_1, c_2 \in \mathbb{C}$. Where the first condition is ensuring functoriality of U, the second making U into a $\mathbb{C}$-linear functor. Note that from (3.12) we have that $U(m_p)$ and $U(m_q)$ have to commute for all $p, q \in R$.

In a next step we will fix the action of U further or in other words give the explicit action of U on the higher rank modules R^n for $n \in \mathbb{N}$. We can do this by defining $U(R^n) := R^{nr_U}$. On module morphisms $f : R^m \to R^n$, i.e. matrices with entries $f_{ij} \in R$ we define $U(f)$ to be the matrix

$$\begin{pmatrix} U(f_{11}) & \cdots & U(f_{1m}) \\ \vdots & \ddots & \vdots \\ U(f_{n1}) & \cdots & U(f_{nm}) \end{pmatrix}, \tag{3.15}$$

i.e. every entry of f is "blown up" to be the $r_U \times r_U$-Matrix defined via $U(f_{ij}) := U(m_{f_{ij}})$. Thus $U(f)$ is an $nr_U \times mr_U$-Matrix, hence a map from $R^{mr_U} \to R^{nr_U}$. This componentwise action of the functor U is well defined since functoriality and $\mathbb{C}$-linearity of the functor already follows from (3.12) and (3.14). Fixing the functors this far is already a good enough setting for the computations we will do later on. However, for completeness we still want to discuss a way on how to fix the functors formally completely on the category $R - mod_f$. This is done in the following procedure. We first fix a set of isomorphisms $\{\varphi_M : M \to R^m\}_{M \in R-mod_f}$ and define

$$U(M) := U(R^m), \qquad U(N) := U(R^n) \tag{3.16}$$

where m, n are the ranks of M, N respectively. On morphisms $g : M \to N$ we define

$$U(g) := U(\varphi_N \circ g \circ \varphi_M^{-1}) : U(M) = U(R^m) \to U(N) = U(R^n), \tag{3.17}$$

where $\varphi_N \circ g \circ \varphi_M^{-1}$ is a morphism from R^m to R^n whose action is already defined in (3.15). Thus we constructed a concrete functor $U \in End(R - mod_f)$.

This concludes that given a functor on U and its action on R and action on maps from R to R we can in practice construct a well defined functor which we know how it acts on modules R^n and maps between them explicitly. Fixing isomorphisms $\{\varphi_M\}_{M \in R-mod_f}$ we could define its action on all modules M, N implicitly.

In this setting the components of a natural transformation $\eta : U \to V$ are described by matrices. The $R-$component η_R is given by an $r_V \times r_U$-matrix s.t.

$$V(m_p) \circ \eta_R = \eta_R \circ U(m_p) \tag{3.18}$$

for all $p \in R$. The components of higher rank modules M are

$$\eta_M = \eta_{R^m} := \begin{pmatrix} \eta_R & & \\ & \ddots & \\ & & \eta_R \end{pmatrix} : U(M) \to V(M) \tag{3.19}$$

block diagonal matrices with m blocks η_R. This is consistent with

$$V(f)\eta_M = \eta_M U(f)$$

for $f : M \to N$ due to (3.18). With this configuration all computations are basically calculating with matrices.

3.5. Braid invariants in $HCoCh(End(R-mod_f))$

3.5.1. Explicit functors and natural transformations

For the remainder of this chapter we fix $R = \mathbb{C}[x_1, ..., x_n]$. The number n of variables in the polynomial ring equals the number of strands of the considered braid. For every braid group B_n one can describe braid invariants. In the construction there appear two types of functors, on the one hand the identity functor which we denote by U, i.e. $U \coloneqq Id$, on the other hand functors $V_{i,i+1}$, with $i = 1, ..., n-1$. which are defined

$$V_{i,i+1}(R) \coloneqq R^2, \tag{3.20}$$

$$V_{i,i+1}(m_p) \coloneqq \frac{1}{2}\begin{pmatrix} p_{S_{i,i+1}} & p_{A_{i,i+1}}(x_i - x_{i+1}) \\ p_{A_{i,i+1}}/(x_i - x_{i-1}) & p_{S_{i,i+1}} \end{pmatrix}, \tag{3.21}$$

where we denote m_p as the morphisms from R to R realised by multiplication with the polynomial $p = p(x_1, ..., x_i, x_{i+1}, ...x_n)$. The symmetric and antisymmetric part of the polynomial with respect to variables x_i and x_{i+1} that appear in (3.20) are

$$p_{S_{i,i+1}} = \frac{1}{2}(p(x_1, ..x_i, x_{i+1}, ..x_n) + p(x_1, ..x_{i+1}, x_i, ..x_n))$$
$$p_{A_{i,i+1}} = \frac{1}{2}(p(x_1, ..x_i, x_{i+1}, ..x_n) - p(x_1, ..x_{i+1}, x_i, ..x_n))$$

satisfying

$$p_{S_{i,i+1}} + p_{A_{i,i+1}} = p(x_1, ..x_i, x_{i+1}, ..x_n).$$

The definitions on the ground module R and maps m_p give a well defined isomorphism class of functors. However writing $V_{i,i+1}$ we mean the representative constructed according to the definitions (3.16) and (3.17).

Next we define the natural transformation $\Phi_{i,i+1} : U \to V_{i,i+1}$ by the R-component $\phi_{i,i+1} \coloneqq {\Phi_{i,i+1}}_R$

$$\phi_{i,i+1} \coloneqq \begin{pmatrix} x_i - x_{i+1} \\ 1 \end{pmatrix} : R = U(R) \to V_{i,i+1}(R) = R^2 \tag{3.22}$$

and the natural transformation $\Psi_{i,i+1} : V_{i,i+1} \to U$ by the R-component $\psi_{i,i+1} \coloneqq {\Psi_{i,i+1}}_R$

$$\psi_{i,i+1} \coloneqq \begin{pmatrix} 1 & x_i - x_{i+1} \end{pmatrix} : R^2 = V_{i,i+1}(R) \to U(R) = R. \tag{3.23}$$

3.5.2. Assignment of cochain complexes to elements of B_n

We assign the cochain complexes

$$\sigma_i \mathrel{\widehat{=}} C(\sigma_i) = \; 0 \longrightarrow \underline{U} \xrightarrow{\Phi_{i,i+1}} V_{i,i+1} \longrightarrow 0 \tag{3.24}$$

$$\sigma_i^{-1} \mathrel{\widehat{=}} C(\sigma_i^{-1}) = \; 0 \longrightarrow V_{i,i+1} \xrightarrow{\Psi_{i,i+1}} \underline{U} \longrightarrow 0, \tag{3.25}$$

where we put U in the cohomological degree 0 which is indicated by the underline, to the generators of B_n and their inverse. To the unit element $1 \in B_n$ we assign the cochain complex

$$1 \ \widehat{=}\ C(1) = 0 \longrightarrow \underline{U} \longrightarrow 0.$$

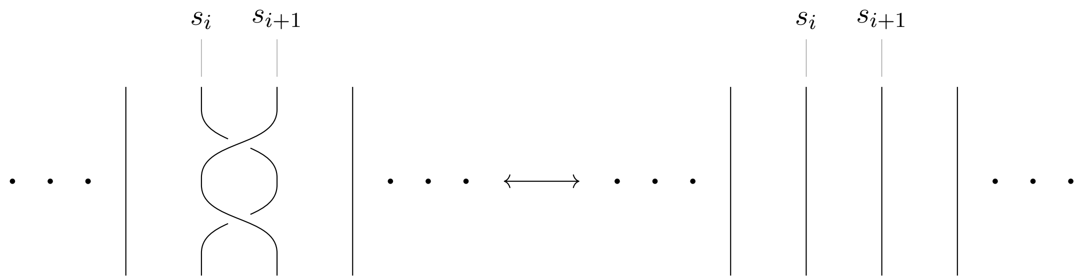
To a braid encoded by the word $\sigma_i\sigma_k\cdots\sigma_l^{-1} \in B_n$ we assign the tensor product of cochain complexes as defined in (3.4) and (3.5), i.e.

$$C(\sigma_i)\otimes C(\sigma_k)\otimes\cdots\otimes C(\sigma_l^{-1}).$$

3.5.3. Verifying relations

We are finally ready to verify the braid relations.

Relation 0: Inverse $\sigma_i\sigma_i^{-1} = 1$



(3.26)

For the right hand side we obtain the cochain complex $R_0^\bullet = C(1)$. We fix $i = 1$ without losing any generality and write for the remainder of this calculation $V \equiv V_{12}$ and $\Phi \equiv \Phi_{12}$, thus $\phi \equiv \phi_{12}$ and $\Psi \equiv \Psi_{12}$, thus $\psi \equiv \psi_{12}$. The cochain complex $C(\sigma_1)\otimes C(\sigma_1^{-1})$ associated to the diagram on the left hand side of (3.26) is isomorphic to $L_0^\bullet$ (3.27)

$$0 \longrightarrow U\circ V \xrightarrow{d_{-1}} U\circ U \ \oplus V\circ V \xrightarrow{d_0} V\circ U \longrightarrow 0\ . \tag{3.27}$$

The differentials of (3.27) are given according to (3.5)

$$\begin{aligned} d_{-1} &= \begin{pmatrix} 1_U\otimes\Psi \\ \Phi\otimes 1_V \end{pmatrix} \\ d_0 &= \begin{pmatrix} \Phi\otimes 1_U & -1_V\otimes\Psi \end{pmatrix}, \end{aligned}$$

and using the matrix representation of morphisms between direct sums (1.2). We picked up the minus sign in the second column of d_0 since V is in odd degree (-1) in $C(\sigma_1)$.

The R-components d_{-1R} and d_{0R} are by (3.3)

$$
\begin{aligned}
d_{-1R} &= \begin{pmatrix} (1_U \otimes \Psi)_R \\ (\Phi \otimes 1_V)_R \end{pmatrix} \\
&= \begin{pmatrix} (1_U)_{U(R)} \circ U(\Psi_R) \\ \Phi_{V(R)} \circ U((1_V)_R) \end{pmatrix} \\
&= \begin{pmatrix} 1_1 \circ \psi \\ \Phi_{R^2} \circ 1_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x_1 - x_2 \\ x_1 - x_2 & 0 \\ 1 & 0 \\ 0 & x_1 - x_2 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
d_{0R} &= \begin{pmatrix} (\Phi \otimes 1_U)_R & (1_V \otimes \Psi)_R \end{pmatrix} \\
&= \begin{pmatrix} \Phi_{U(R)} \circ U((1_U)_R) & -(1_V)_{U(R)} \circ V(\Psi_R) \end{pmatrix} \\
&= \begin{pmatrix} \phi \circ 1_1 & -1_2 \circ V(\psi) \end{pmatrix} \\
&= \begin{pmatrix} x_1 - x_2 & -1 & 0 & 0 & -(x_1 - x_2)^2 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix},
\end{aligned}
$$

using the definitions (3.22) and (3.23) for the R-components of Ψ and Φ and (3.19) for the higher rank component Ψ_{R^2} as well as the componentwise action on morphisms of V given in (3.20). These matrices satisfy the differential property

$$d_{-1R} \circ d_{0R} = 0$$

and the natural transformation properties

$$
\begin{aligned}
(U \circ U \oplus V \circ V)(m_p) \circ d_{-1R} &= d_{-1R} \circ (U \circ V)(m_p) \\
(V \circ U)(m_p) \circ d_{0R} &= d_{0R} \circ (U \circ U \oplus V \circ V)(m_p),
\end{aligned}
$$

with

$$(V \circ U)(m_p) = V(m_p) = \frac{1}{2} \begin{pmatrix} p_S & p_A(x_1 - x_2) \\ p_A/(x_1 - x_2) & p_S \end{pmatrix},$$

and

$$(U \circ U \oplus V \circ V)(m_p) = \begin{pmatrix} p & 0 & 0 & 0 & 0 \\ 0 & p_S & 0 & p_A(x_1 - x_2) & 0 \\ 0 & 0 & p_S & 0 & p_A(x_1 - x_2) \\ 0 & \frac{p_A}{x_1 - x_2} & 0 & p_S & 0 \\ 0 & 0 & \frac{p_A}{x_1 - x_2} & 0 & p_S \end{pmatrix}. \tag{3.28}$$

The degree 0 component of the complex $L_0^\bullet$ in (3.27) has a further direct sum decomposition, i.e.

$$U \oplus V^2 \simeq U \oplus V \oplus V.$$

This isomorphism is found by applying similarity transformations in mathematica to (3.28) to find

$$\begin{aligned} S(U \oplus V^2)(m_p)S^{-1} &= (U \oplus V \oplus V)(m_p) \\ &= \begin{pmatrix} p & 0 & 0 & 0 & 0 \\ 0 & p_S & p_A(x_1-x_2) & 0 & 0 \\ 0 & \frac{p_A}{x_1-x_2} & p_S & 0 & 0 \\ 0 & 0 & 0 & p_S & p_A(x_1-x_2) \\ 0 & 0 & 0 & \frac{p_A}{x_1-x_2} & p_S \end{pmatrix}. \end{aligned}$$

The concrete matrices S and S^{-1} are

$$S = \left(\begin{array}{c:cccc} 1 & 0 & -1 & 0 & x_2-x_1 \\ \hdashline x_2-x_1 & 1 & 0 & 0 & (x_1-x_2)^2 \\ -1 & 0 & 1 & 1 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{3.29}$$

(3.30)

$$S^{-1} = \left(\begin{array}{c:cc:cc} 1 & 0 & 0 & 1 & x_1-x_2 \\ \hdashline x_1-x_2 & 1 & 0 & x_1-x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & x_1-x_2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{3.31}$$

Remark. We want to discuss in a bit more detail how exactly one understands the natural isomorphism $\eta : U\oplus V^2 \to U\oplus V\oplus V$ with $\eta_R = S$ and its inverse $\eta^{-1} : U\oplus V\oplus V \to U\oplus V^2$ with $\eta_R^{-1} = S^{-1}$. In matrix representation we have

$$\eta = \begin{pmatrix} \eta_{UU} & \eta_{V^2U} \\ \eta_{UV} & \eta_{V^2V} \\ \hat{\eta}_{UV} & \hat{\eta}_{V^2V} \end{pmatrix}, \qquad \eta^{-1} = \begin{pmatrix} \eta^{-1}_{UU} & \eta^{-1}_{VU} & \hat{\eta}^{-1}_{VU} \\ \eta^{-1}_{UV^2} & \eta^{-1}_{VV^2} & \hat{\eta}^{-1}_{VV^2} \end{pmatrix}, \tag{3.32}$$

visualising that the blocks of S and S^{-1} indicated by the dashed lines in (3.29) are the R-components of the natural transformations appearing in the matrices (3.32).

Remark. The isomorphism given via S is not unique, for example there is an isomorphism which does not touch the summand U.

The natural isomorphism η induces the cochain complex $L_0'^\bullet$ (3.33)

$$0 \longrightarrow V \xrightarrow{\eta\circ d_{-1}} U \oplus V \oplus V \xrightarrow{d_0\circ\eta^{-1}} V \longrightarrow 0 \tag{3.33}$$

isomorphic to $L_0^\bullet$. The differentials of $L_0'^\bullet$ are of the simple form

$$\eta \circ d_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1_V \end{pmatrix} = \iota_3 : V \to U \oplus V \oplus V \quad \text{since} \quad S \circ d_{-1R} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$d_0 \circ \eta^{-1} = \begin{pmatrix} 0 & -1_V & 0 \end{pmatrix} = -\pi_2 : U \oplus V \oplus V \to V,$$

since

$$d_{0R} \circ S^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We now consider the following morphisms $f : L_0'^\bullet \to R_0^\bullet$ and $\tilde{f} : R_0^\bullet \to L_0'^\bullet$

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\iota_3"] \arrow[d, shift left] & U \oplus V \oplus V \arrow[r, "-\pi_2"] \arrow[d, shift left, "\pi_1"] & V \arrow[r] \arrow[d, shift left] & 0 \\
 & 0 \arrow[r] \arrow[u, shift left] & U \arrow[r] \arrow[u, shift left, "\iota_1"] & 0 \arrow[u, shift left] &
\end{tikzcd}$$

which already satisfy $f \circ \tilde{f} = 1_{R_0^\bullet}$. And with the homotopy

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\iota_3"] \arrow[d, "0"] & U \oplus V \oplus V \arrow[r, "-\pi_2"] \arrow[d, "\iota_1 \circ \pi_1"] \arrow[dl, "\pi_3"'] & V \arrow[r] \arrow[d, "0"] \arrow[dl, "-\iota_2"'] & 0 \\
0 \arrow[r] & V \arrow[r, "\iota_3"] & U \oplus V \oplus V \arrow[r, "-\pi_2"] & V \arrow[r] & 0
\end{tikzcd}$$

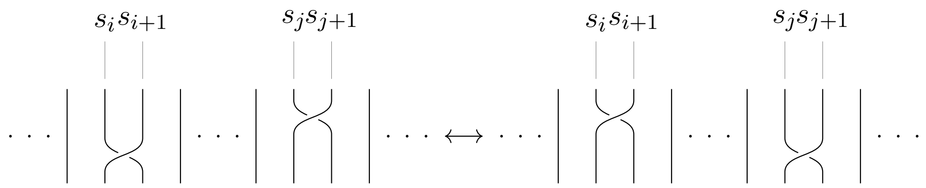
we find that $\tilde{f} \circ f$ is homotopic to $1_{L_0'^\bullet}$ since

$$\begin{aligned}
1_V - 0_V &= \pi_3 \circ \iota_3 \\
1_{U \oplus V \oplus V} - \iota_1 \circ \pi_1 &= \iota_3 \circ \pi_3 + \iota_2 \circ \pi_2 \\
1_V - 0_V &= \pi_2 \circ \iota_2
\end{aligned}$$

Thus $L_0^\bullet \simeq L_0'^\bullet \simeq R_0^\bullet$ in $HCoCh(End(R - mod_f))$. $\square$

Remark. The other direction $\sigma_i^{-1}\sigma_i = 1$ can be shown in a similar way.

Relation I : $\sigma_i^{-1}\sigma_j^{-1} = \sigma_j^{-1}\sigma_i^{-1}$ for all $|i-j| > 1$.



(3.34)

To prove Relation I we make use of Lemma 3.3.1 and instead of taking the tensor product of the generators (3.24) we take the tensor product of cochain complex $C'(\sigma^{-1})$ being defined as

$$C(\sigma^{-1}) \simeq C'(\sigma^{-1}) = 0 \longrightarrow V'_{i,i+1} \xrightarrow{\Psi'_{i,i+1}} \underline{U} \longrightarrow 0.$$

Where this isomorphism is induced by the isomorphism $\eta : V_{i,i+1} \to V'_{i,i+1}$ with the $R-$component

$$\eta_R := \begin{pmatrix} 1 & x_1 - x_2 \\ 0 & 1 \end{pmatrix}$$

thus $V'_{i,i+1}(m_p)$ is given by the similarity transformation

$$V'_{i,i+1}(m_p) = \eta_R V_{i,i+1}(m_p)\eta_R^{-1} = \begin{pmatrix} p_{i,i+1} & 0 \\ \frac{p_{i,i+1}-p_{i+1,i}}{2(x_i - x_{i+1})} & p_{i+1,i} \end{pmatrix} \tag{3.35}$$

where we use the notation

$$p_{i,i+1} := p(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n)$$
$$p_{i+1,i} := p(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n).$$

The induced differential takes the form

$$\Psi'_{i,i+1} = \Psi_{i,i+1} \circ \eta^{-1}, \quad \text{with} \quad \Psi'_{i,i+1R} := \psi'_{i,i+1} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

We specify $i = 1$ and $j = 3$ without losing any generality. The cochain complex $C(\sigma_1^{-1}) \otimes C(\sigma_3^{-1})$ associated to the diagram on the left hand side of (3.34) is thus isomorphic to $L_1^\bullet$ (3.36)

$$0 \longrightarrow V'_{34} \circ V'_{12} \xrightarrow{d_{L_1}^{-2}} U \circ V'_{12} \oplus V'_{34} \circ U \xrightarrow{d_{L_1}^{-1}} U \circ U \longrightarrow 0 \tag{3.36}$$

and the cochain complex $C(\sigma_3) \otimes C(\sigma_1)$ associated to the diagram on the right hand side of (3.34) is isomorphic to $R_1^\bullet$ (3.37)

$$0 \longrightarrow V'_{12} \circ V'_{34} \xrightarrow{\hat{d}_{R_1}^{-2}} U \circ V'_{34} \oplus V'_{12} \circ U \xrightarrow{\hat{d}_{R_1}^{-1}} U \circ U \longrightarrow 0. \tag{3.37}$$

The differentials are

$$d_{L_1}^{-2} = \begin{pmatrix} \Psi'_{34} \otimes id_{V'_{12}} \\ -id_{V'_{34}} \otimes \Psi'_{12} \end{pmatrix}, \qquad d_{L_1}^{-1} = \begin{pmatrix} id_U \otimes \Psi'_{12} & \Psi'_{34} \otimes id_U \end{pmatrix}$$
$$d_{R_1}^{-2} = \begin{pmatrix} \Psi'_{12} \otimes id_{V'_{34}} \\ -id_{V'_{12}} \otimes \Psi'_{34} \end{pmatrix}, \qquad d_{R_1}^{-1} = \begin{pmatrix} id_U \otimes \Psi'_{34} & \Psi'_{12} \otimes id_U \end{pmatrix}.$$

Using (3.3) we calculate the R-component of the differentials and find

$$(d_{L_1}^{-2})_R = (d_{R_1}^{-2})_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$(d_{L_1}^{-1})_R = (d_{R_1}^{-1})_R = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}.$$

The action on $m_p : R \to R$ of the functors is

$$(V'_{34} \circ V'_{12})(m_p) = \begin{pmatrix} p_{1234} & 0 & 0 & 0 \\ \frac{p_{1234}-p_{1243}}{2(x_3-x_4)} & p_{1243} & 0 & 0 \\ \frac{p_{1234}-p_{2134}}{2(x_1-x_2)} & 0 & p_{2134} & 0 \\ \frac{p_{1234}-p_{1243}-p_{2134}+p_{2143}}{4(x_1-x_2)(x_3-x_4)} & \frac{p_{1243}-p_{2143}}{2(x_1-x_2)} & \frac{p_{2134}-p_{2143}}{2(x_3-x_4)} & p_{2143} \end{pmatrix}$$
$$(V'_{12} \circ V'_{34})(m_p) = \begin{pmatrix} p_{1234} & 0 & 0 & 0 \\ \frac{p_{1234}-p_{2134}}{2(x_1-x_2)} & p_{2134} & 0 & 0 \\ \frac{p_{1234}-p_{1243}}{2(x_3-x_4)} & 0 & p_{1243} & 0 \\ \frac{p_{1234}-p_{1243}-p_{2134}+p_{2143}}{4(x_1-x_2)(x_3-x_4)} & \frac{p_{2134}-p_{2143}}{2(x_3-x_4)} & \frac{p_{1243}-p_{2143}}{2(x_1-x_2)} & p_{2143} \end{pmatrix}$$
$$(V'_{12} \oplus V'_{34})(m_p) = \begin{pmatrix} p_{1234} & 0 & 0 & 0 \\ \frac{p_{1234}-p_{2134}}{2(x_1-x_2)} & p_{2134} & 0 & 0 \\ 0 & 0 & p_{1234} & 0 \\ 0 & 0 & \frac{p_{1234}-p_{1243}}{2(x_3-x_4)} & p_{1243} \end{pmatrix}$$
$$(V'_{34} \oplus V'_{12})(m_p) = \begin{pmatrix} p_{1234} & 0 & 0 & 0 \\ \frac{p_{1234}-p_{1243}}{2(x_3-x_4)} & p_{1243} & 0 & 0 \\ 0 & 0 & p_{1234} & 0 \\ 0 & 0 & \frac{p_{1234}-p_{2134}}{2(x_1-x_2)} & p_{2134} \end{pmatrix}.$$

We observe that $(V'_{34} \circ V'_{12})(m_p)$ and $(V'_{12} \circ V'_{34})(m_p)$ can be related by a similarity transformation, i.e.

$$(V'_{12} \circ V'_{34})(m_p) = S \circ (V'_{34} \circ V'_{12})(m_p) \circ S^{-1},$$

with

$$S = S^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

encoding that interchanging row 2 and 3 and column 2 and 3 in $(V'_{12}\circ V'_{34})(m_p)$ one obtains $(V'_{34}\circ V'_{12})(m_p)$ and vice versa (the minus sign is implemented to make it compatible with (3.38) below). We denote the natural isomorphism with R-component S by $s : V'_{34}\circ V'_{12} \to V'_{12}\circ V'_{34}$ and its inverse by $s^{-1} : V'_{12}\circ V'_{34} \to V'_{34}\circ V'_{12}$.

On the other hand $(V'_{12}\oplus V'_{34})(m_p)$ and $(V'_{34}\oplus V'_{12})(m_p)$ can be related by the similarity transformation interchanging the two block matrices, i.e.

$$(V'_{12}\oplus V'_{34})(m_p) = Q\circ (V'_{34}\oplus V'_{12})(m_p)\circ Q^{-1},$$

where

$$Q = Q^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Remark. Q, Q^{-1} are the R-components of the obvious isomorphisms q, q^{-1}

$$\begin{aligned} q &= \begin{pmatrix} 0 & 1_{V'_{12}} \\ 1_{V'_{34}} & 0 \end{pmatrix} : V'_{12}\oplus V'_{34} \to V'_{34}\oplus V'_{12} \\ q^{-1} &= \begin{pmatrix} 0 & 1_{V'_{34}} \\ 1_{V'_{12}} & 0 \end{pmatrix} : V'_{34}\oplus V'_{12} \to V'_{12}\oplus V'_{34} \end{aligned}$$

.

Since S, S^{-1}, Q, Q^{-1} and the R-components of the differentials $d^{-2}_{L_1}, d^{-1}_{L_1}, d^{-2}_{R_1}, d^{-2}_{R_1}$ satisfy the relations

$$(d^{-2}_{R_1})_R \circ S = Q\circ (d^{-2}_{L_1})_R \tag{3.38}$$

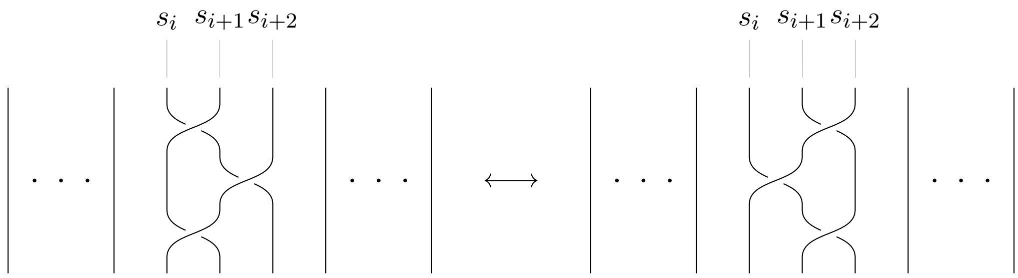
$$(d^{-1}_{L_1})_R = (d^{-1}_{R_1})_R \circ Q \tag{3.39}$$

the natural isomorphisms q and s induce an isomorphism on the cochain complexes $L_1^\bullet$ and $R_1^\bullet$, i.e.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'_{34}\circ V'_{12} & \xrightarrow{d^{-2}_{L_1}} & U\circ V'_{12}\oplus V'_{34}\circ U & \xrightarrow{d^{-1}_{L_1}} & U\circ U & \longrightarrow & 0 \\ & & {\scriptstyle s^{-1}}\uparrow\downarrow{\scriptstyle s} & & {\scriptstyle q^{-1}}\uparrow\downarrow{\scriptstyle q} & & {\scriptstyle 1_U}\uparrow\downarrow{\scriptstyle 1_U} & & \\ 0 & \longrightarrow & V'_{12}\circ V'_{34} & \xrightarrow{\hat{d}^{-2}_{R_1}} & U\circ V'_{34}\oplus V'_{12}\circ U & \xrightarrow{\hat{d}^{-1}_{R_1}} & U\circ U & \longrightarrow & 0 \end{array}$$

This proves Relation I $\square$.

Relation II: $\sigma_i^{-1}\sigma_{i,i+1}^{-1}\sigma_i^{-1} = \sigma_{i,i+1}^{-1}\sigma_i^{-1}\sigma_{i,i+1}^{-1}$ for all $i \in \mathbb{N}$.



(3.40)

We do not lose any information when specifying to $i = 1$. We will use the notation

$$\begin{aligned}
p_{123} &= p(x_1, x_2, x_3, x_4, \dots, x_n)\\
p_{213} &= p(x_2, x_1, x_3, x_4, \dots, x_n)\\
p_{132} &= p(x_1, x_3, x_2, x_4, \dots, x_n)\\
p_{231} &= p(x_2, x_3, x_1, x_4, \dots, x_n)\\
p_{312} &= p(x_3, x_1, x_2, x_4, \dots, x_n)\\
p_{321} &= p(x_3, x_2, x_1, x_4, \dots, x_n)
\end{aligned}$$

Applying our rules we obtain for the left hand side the cochain complex $L_2^\bullet$ (3.41) isomorphic to $C'(\sigma_1) \otimes C'(\sigma_2) \otimes C'(\sigma_1)$

$$0 \longrightarrow V'_{12} \circ V'_{23} \circ V'_{12} \xrightarrow{d_{L_2}^{-3}} \begin{matrix} V'_{12} \circ V'_{23} \circ U \\ \oplus \\ V'_{12} \circ U \circ V'_{12} \\ \oplus \\ U \circ V'_{23} \circ V'_{12} \end{matrix} \xrightarrow{d_{L_2}^{-2}} \begin{matrix} V'_{12} \circ U \circ U \\ \oplus \\ U \circ V'_{23} \circ U \\ \oplus \\ U \circ U \circ V'_{12} \end{matrix} \xrightarrow{d_{L_2}^{-1}} U \circ U \circ U \longrightarrow 0 \tag{3.41}$$

with

$$\begin{aligned}
d_{L_2}^{-3} &= \begin{pmatrix} 1_{V'_{12}} \otimes 1_{V'_{23}} \otimes \Psi'_{12} \\ -1_{V'_{12}} \otimes \Psi'_{23} \otimes 1_{V'_{12}} \\ \Psi'_{12} \otimes 1_{V'_{23}} \otimes 1_{V'_{12}} \end{pmatrix}\\
d_{L_2}^{-2} &= \begin{pmatrix} -1_{V'_{12}} \otimes \Psi'_{23} \otimes 1_U & -1_{V'_{12}} \otimes 1_U \otimes \Psi_{V'_{12}} & 0 \\ \Psi'_{12} \otimes 1_{V'_{23}} \otimes 1_U & 0 & -1_U \otimes 1_{V'_{23}} \otimes \Psi'_{12} \\ 0 & \Psi_{V'_{12}} \otimes 1_U \otimes 1_{V'_{12}} & 1_U \otimes \Psi'_{23} \otimes 1_{V'_{12}} \end{pmatrix}\\
d_{L_2}^{-1} &= \begin{pmatrix} \Psi'_{12} \otimes 1_U \otimes 1_U & 1_U \otimes \Psi'_{23} \otimes 1_U & 1_U \otimes 1_U \otimes \Psi'_{12} \end{pmatrix}
\end{aligned}$$

with the R-components

$$d^{-3}_{L_2 R} = \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{3.42}$$

$$d^{-2}_{L_2 R} = \left(\begin{array}{cccc:cccc:cccc} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{3.43}$$

$$d^{-1}_{L_2 R} = \left(\begin{array}{cc:cc:cc} 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \tag{3.44}$$

The functor $V'_{12} \circ V'_{23} \circ V'_{12}$ is isomorphic to the direct sum functor of V_{12} and some explicit functor Y, i.e.

$$V'_{12} \circ V'_{23} \circ V'_{12} \underset{(f_{-3})^{-1}}{\overset{f_{-3}}{\rightleftarrows}} V_{12} \oplus Y.$$

The R-components of this isomorphism are the matrices f_{-3_R} and $(f_{-3})^{-1}{}_R$ which reflect the similarity transformations performed on $(V'_{12} \circ V'_{23} \circ V'_{12})(m_p)$, i.e.

$$(V_{12} \oplus Y)(m_p) = f_{-3_R} \circ (V'_{12} \circ V'_{23} \circ V'_{12})(m_p) \circ (f_{-3})^{-1}{}_R.$$

Explicitly f_{-3_R} is given by the matrix

$$f_{-3_R} = \left(\begin{array}{cccccccc} 0 & -1 & 0 & 0 & 1 & x_2 - x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & 1 & 1 & 2(x_3 - x_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2(x_3 - x_2) & 1 & 0 & 0 & 0 \\ 1 & 0 & 2(x_3 - x_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

.

The functor Y is defined by its action on modules M of rank m

$$Y(M) = R^{6m}$$

and on morphisms $m_p : R \to R$ with $Y(m_p) =$

$$\begin{pmatrix} p_{231} & 0 & 0 & \frac{p_{132}-p_{231}}{2x_{12}} & 0 & 0 \\ \frac{p_{123}-p_{213}}{2x_{12}}+\frac{p_{213}-p_{231}}{2x_{13}} & p_{123} & \frac{p_{213}-p_{123}}{2x_{12}} & \frac{\frac{p_{123}-p_{132}}{x_{23}}+\frac{p_{231}-p_{213}}{x_{13}}}{4x_{12}} & 0 & 0 \\ \frac{(p_{231}-p_{213})x_{23}}{x_{13}} & 0 & p_{213} & \frac{p_{132}-p_{231}}{2x_{12}}+\frac{p_{231}-p_{213}}{2x_{13}} & 0 & 0 \\ 0 & 0 & 0 & p_{132} & 0 & 0 \\ \frac{p_{231}-p_{213}}{2x_{13}} & 0 & \frac{p_{213}-p_{312}}{2x_{23}} & \frac{(p_{132}-p_{231})x_{23}+p_{213}x_{21}+p_{312}x_{12}}{4x_{12}x_{13}x_{23}} & p_{312} & 0 \\ \frac{\frac{p_{123}-p_{213}}{x_{12}}+\frac{p_{321}-p_{231}}{x_{23}}}{4x_{13}} & \frac{p_{123}-p_{321}}{2x_{13}} & \frac{\frac{p_{321}-p_{123}}{x_{13}}+\frac{p_{213}-p_{312}}{x_{23}}}{4x_{12}} & \frac{p_{123}-p_{132}-p_{213}+p_{231}+p_{312}-p_{321}}{8x_{12}x_{13}x_{23}} & \frac{p_{312}-p_{321}}{2x_{12}} & p_{321} \end{pmatrix}.$$

In a similar way we find an isomorphism f_{-2} with inverse $(f_{-2})^{-1}$

$$\begin{matrix} V'_{12}\circ V'_{23} \\ \oplus \\ V'_{12}\circ V'_{12} \\ \oplus \\ V'_{23}\circ V'_{12} \end{matrix} \underset{\begin{pmatrix} 0 & 1_{V'_{12}\circ V'_{23}} & 0 \\ (f'_{-2})^{-1} & 0 & -(f'_{-2})^{-1}\circ f''_{-2} \\ 0 & 0 & 1_{V'_{23}\circ V'_{12}} \end{pmatrix}}{\overset{\begin{pmatrix} 0 & f'_{-2} & f''_{-2} \\ 1_{V'_{12}\circ V'_{23}} & 0 & 0 \\ 0 & 0 & 1_{V'_{23}\circ V'_{12}} \end{pmatrix}}{\rightleftarrows}} \begin{matrix} \begin{pmatrix} V_{12} \\ \oplus \\ V_{12} \end{pmatrix} \\ \oplus \\ V'_{12}\circ V'_{23} \\ \oplus \\ V'_{23}\circ V'_{12} \end{matrix}$$

with R-components f_{-2R} and $(f_{-2})^{-1}{}_R$ satisfying

$$\begin{aligned} &(V_{12}\oplus V_{12} \oplus V'_{12}\circ V'_{23} \oplus V'_{23}\circ V'_{12})(m_p) \\ &\qquad = f_{-2R}\circ (V'_{12}\circ V'_{23} \oplus V'_{12}\circ V'_{12} \oplus V'_{23}\circ V'_{12})(m_p)\circ (f_{-2})^{-1}{}_R. \end{aligned}$$

The non trivial blocks f'_{-2R} and f''_{-2R} of f_{-2R} are

$$f'_{-2R} = \begin{pmatrix} 0 & 1 & -1 & x_{12} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -x_{12} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad f''_{-2R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -x_{12} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.45}$$

Last but not least using the similarity transformations as in (3.35) we construct the following isomorphisms f_{-1} and $(f_{-1})^{-1}$ depicted in

$$\begin{matrix} V'_{12} \\ \oplus \\ V'_{23} \\ \oplus \\ V'_{12} \end{matrix} \underset{(f_{-1})^{-1}=\begin{pmatrix} -\eta_{12} & \eta_{12} & 0 \\ 0 & 0 & \eta_{23} \\ \eta_{12} & 0 & 0 \end{pmatrix}}{\overset{f_{-1}=\begin{pmatrix} 0 & 0 & \eta_{12}^{-1} \\ \eta_{12}^{-1} & 0 & \eta_{12}^{-1} \\ 0 & \eta_{23}^{-1} & 0 \end{pmatrix}}{\rightleftarrows}} \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix}.$$

The morphisms f_{-3}, f_{-2}, f_{-1} and their respective inverse induce that the cochain complex $L^{\bullet}_{2I}$ (3.46) is isomorphic to $L^{\bullet}_2$ (3.41)

$$0 \longrightarrow \begin{matrix} V_{12} \\ \oplus \\ Y \end{matrix} \xrightarrow{f_{-2} \circ d^{-3}_{L_2} \circ (f_{-3})^{-1}} \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V'_{12} \circ V'_{23} \\ \oplus \\ V'_{23} \circ V'_{12} \end{matrix} \xrightarrow{f_{-1} \circ d^{-2}_{L_2} \circ (f_{-2})^{-1}} \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \xrightarrow{d^{-1}_{L_2} \circ (f_{-1})^{-1}} U \longrightarrow 0 \ . \tag{3.46}$$

Setting $X \coloneqq V'_{12} \circ V'_{23} \oplus V'_{23} \circ V'_{12}$ and writing the differentials as matrices, $L^{\bullet}_{2I}$ takes the form

$$0 \longrightarrow \begin{matrix} V_{12} \\ \oplus \\ Y \end{matrix} \xrightarrow{\begin{pmatrix} 1_{V_{12}} & 0 \\ 0 & 0 \\ \beta & \alpha \end{pmatrix}} \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ X \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 1_{V_{12}} & 0 \\ \gamma & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \end{pmatrix}} \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \xrightarrow{\begin{pmatrix} 0 & \Psi_{12} & \Psi_{23} \end{pmatrix}} \underline{U} \longrightarrow 0 \tag{3.47}$$

with the morphisms satisfying

$$\begin{aligned} \alpha_2 \circ \beta &= 0 \\ \gamma + \alpha_1 \circ \beta &= 0 \\ \alpha_1 \circ \alpha &= 0 \\ \alpha_2 \circ \alpha &= 0 \\ \Psi_{12} \circ \gamma &= 0 \\ \Psi_{23} \circ \alpha_2 + \Psi_{12} \circ \alpha_1 &= 0. \end{aligned} \tag{3.48}$$

For completeness we give the defining R-components of the morphisms involved

$$\beta_R = \begin{pmatrix} 2x_{23} & 2x_{12}x_{23} \\ -1 & x_{13}+x_{23} \\ 1 & x_{12} \\ 0 & 1 \\ 2x_{23} & 2x_{12}x_{23} \\ 1 & x_{12} \\ 0 & 2x_{23} \\ 0 & 1 \end{pmatrix}, \qquad \alpha_R = \begin{pmatrix} 2x_{23} & 4x_{12}x_{23} & -2x_{23} & 1 & 0 & 0 \\ 0 & 2x_{23} & 1 & 0 & 0 & 0 \\ 1 & 2x_{12} & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2x_{23} & 4x_{12}x_{23} & -2x_{23} & 1 & 0 & 0 \\ 1 & 2x_{12} & -1 & 0 & 0 & 0 \\ 0 & 2x_{23} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.49}$$

$$\gamma_R = \begin{pmatrix} x_{12} & -x_{12}^2 \\ -1 & x_{12} \end{pmatrix}, \qquad \alpha_{1R} = \begin{pmatrix} -1 & x_{12} & 0 & 0 & 1 & 0 & -x_{12} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.50}$$

$$\alpha_{2R} = \begin{pmatrix} 1 & 0 & -x_{23} & 0 & -1 & x_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.51}$$

Consider the morphisms of cochain complexes $\tilde{f} : L^\bullet_{2I} \to F^\bullet$ and $\hat{f} : F^\bullet \to L^\bullet_{2I}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & V_{12} \oplus Y \arrow[r, "\left(\begin{smallmatrix} 1_{V_{12}} & 0 \\ 0 & 0 \\ \beta & \alpha \end{smallmatrix}\right)"] \arrow[d, shift left, "(0\ 1_Y)"] & V_{12} \oplus V_{12} \oplus X \arrow[r, "\left(\begin{smallmatrix} 0 & 1_{V_{12}} & 0 \\ \gamma & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \end{smallmatrix}\right)"] \arrow[d, shift left, "(-\beta\ 0\ 1_X)"] & V_{12} \oplus V_{12} \oplus V_{23} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 & 1_{V_{12}} & 0 \\ 0 & 0 & 1_{V_{23}} \end{smallmatrix}\right)"] & \underline{U} \arrow[r] \arrow[d, shift left, "1_U"] & 0 \\
0 \arrow[r] & Y \arrow[r, "\alpha"'] \arrow[u, shift left, "\left(\begin{smallmatrix} 0 \\ 1_Y \end{smallmatrix}\right)"] & X \arrow[r, "\left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)"'] \arrow[u, shift left, "\left(\begin{smallmatrix} 0 \\ 0 \\ 1_X \end{smallmatrix}\right)"] & V_{12} \oplus V_{23} \arrow[r, "(\Psi_{12}\ \Psi_{23})"'] \arrow[u, shift left, "\left(\begin{smallmatrix} 0 & 0 \\ 1_{V_{12}} & 0 \\ 0 & 1_{V_{23}} \end{smallmatrix}\right)"] & \underline{U} \arrow[r] \arrow[u, shift left, "1_U"] & 0,
\end{tikzcd}$$

denoting the lower cochain complex by $F^\bullet$. One quickly makes sure that f and $\hat{f}$ are maps of cochain complexes using (3.48). They satisfy

$$\tilde{f} \circ \hat{f} = 1_{F^\bullet}.$$

and $\hat{f}\circ\tilde{f}$ is homotopic to the identity $1_{L^\bullet_{2I}}$, shown in the following diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & \begin{matrix} V_{12} \\ \oplus \\ Y \end{matrix} \arrow[r, "\begin{pmatrix} 1_{V_{12}} & 0 \\ 0 & 0 \\ \beta & \alpha \end{pmatrix}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 \\ 0 & 1_Y \end{pmatrix}"] & \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ X \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1_{V_{12}} & 0 \\ \gamma & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \end{pmatrix}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_X \end{pmatrix}"] \arrow[dl, "\begin{pmatrix} 1_{V_{12}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"'] & \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{V_{12}} & 0 \\ 0 & 0 & 1_{V_{23}} \end{pmatrix}"] \arrow[dl, "\begin{pmatrix} 0 & 0 & 0 \\ 1_{V_{12}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"'] & \underline{U} \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1_U"] & 0 \\
0 \arrow[r] & \begin{matrix} V_{12} \\ \oplus \\ Y \end{matrix} \arrow[r, "\begin{pmatrix} 1_{V_{12}} & 0 \\ 0 & 0 \\ \beta & \alpha \end{pmatrix}"'] & \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ X \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1_{V_{12}} & 0 \\ \gamma & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \end{pmatrix}"'] & \begin{matrix} V_{12} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"'] & \underline{U} \arrow[r] & 0.
\end{tikzcd}$$

Thus, $L^\bullet_2 \simeq F^\bullet$ in $HCoCh(End(R-mod_f))$.

For the right hand side of (3.40) we obtain the cochain complex $R^\bullet_2$ (3.52) isomorphic to $C'(\sigma_2^{-1})\otimes C'(\sigma_1^{-1})\otimes C'(\sigma_2^{-1})$

$$0 \longrightarrow V'_{23}\circ V'_{12}\circ V'_{23} \xrightarrow{d^{-3}_{R_2}} \begin{matrix} V'_{23}\circ V'_{12}\circ U \\ \oplus \\ V'_{23}\circ U\circ V'_{23} \\ \oplus \\ U\circ V'_{12}\circ V'_{23} \end{matrix} \xrightarrow{d^{-2}_{R_2}} \begin{matrix} V'_{23}\circ U\circ U \\ \oplus \\ U\circ V'_{12}\circ U \\ \oplus \\ U\circ U\circ V'_{23} \end{matrix} \xrightarrow{d^{-1}_{R_2}} U\circ U\circ U \longrightarrow 0 \tag{3.52}$$

with

$$d^{-3}_{R_2} = \begin{pmatrix} 1_{V'_{23}}\otimes 1_{V'_{12}}\otimes \Psi'_{23} \\ -1_{V'_{23}}\otimes \Psi'_{12}\otimes 1_{V'_{23}} \\ \Psi'_{23}\otimes 1_{V'_{12}}\otimes 1_{V'_{23}} \end{pmatrix}$$

$$d^{-2}_{R_2} = \begin{pmatrix} -1_{V'_{23}}\otimes \Psi'_{12}\otimes 1_U & -1_{V'_{23}}\otimes 1_U\otimes \Psi_{V'_{23}} & 0 \\ \Psi'_{23}\otimes 1_{V'_{12}}\otimes 1_U & 0 & -1_U\otimes 1_{V'_{12}}\otimes \Psi'_{23} \\ 0 & \Psi_{V'_{23}}\otimes 1_U\otimes 1_{V'_{23}} & 1_U\otimes \Psi'_{12}\otimes 1_{V'_{23}} \end{pmatrix}$$

$$d^{-1}_{R_2} = \begin{pmatrix} \Psi'_{23}\otimes 1_U\otimes 1_U & 1_U\otimes \Psi'_{12}\otimes 1_U & 1_U\otimes 1_U\otimes \Psi'_{23} \end{pmatrix}$$

since

$$\Psi'_{12\,R} = \Psi'_{23\,R} \quad \text{and} \quad 1_{V'_{12}\,R} = 1_{V'_{23}\,R}$$

we have that the R-components of the differentials in (3.41) have the same R-components as the differentials of (3.52) which are given in (3.42).

The first component of (3.52) $V'_{23} \circ V'_{12} \circ V'_{23}$ is isomorphic to the direct sum functor $V_{23} \oplus Y$, i.e. arrows

$$V'_{23} \circ V'_{12} \circ V'_{23} \underset{(g_{-3})^{-1}}{\overset{g_{-3}}{\rightleftarrows}} V_{23} \oplus Y$$

exist, the R-components of g_{-3R} and its inverse $(g_{-3})^{-1}{}_R$ reflecting the similarity transformations performed on $(V'_{23} \circ V'_{12} \circ V'_{23})(m_p)$, i.e.

$$(V'_{23} \circ V'_{12} \circ V'_{23})(m_p) = g_{-3R} \circ (V_{23} \oplus Y)(m_p) \circ (g_{-3})^{-1}{}_R,$$

with the explicit matrix given

$$g_{-3R} = \left(\begin{array}{cccccccc} 0 & -1 & 0 & 0 & 1 & -x_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & 1 & 1 & 0 & 0 & 0 & -2x_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & -2x_{23} & 0 \\ 1 & 2x_{12} & 0 & 0 & -2x_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

We have an isomorphism g_{-2} with inverse $(g_{-2})^{-1}$

$$\begin{matrix} V'_{23} \circ V'_{12} \\ \oplus \\ V'_{23} \circ V'_{23} \\ \oplus \\ V'_{12} \circ V'_{23} \end{matrix} \underset{\begin{pmatrix} 0 & 0 & 1_{V'_{23} \circ V'_{12}} \\ (g'_{-2})^{-1} & -(g'_{-2})^{-1} \circ g''_{-2} & 0 \\ 0 & 1_{V'_{12} \circ V'_{23}} & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 & g'_{-2} & g''_{-2} \\ 0 & 0 & 1_{V'_{12} \circ V'_{23}} \\ 1_{V'_{23} \circ V'_{12}} & 0 & 0 \end{pmatrix}}{\rightleftarrows}} \begin{matrix} \begin{pmatrix} V_{23} \\ \oplus \\ V_{23} \end{pmatrix} \\ \oplus \\ V'_{12} \circ V'_{23} \\ \oplus \\ V'_{23} \circ V'_{12} \end{matrix}$$

with R-components of the non trivial blocks g'_{-2R} and g''_{-2R} given

$$g'_{-2R} = \begin{pmatrix} 0 & 1 & -1 & x_{23} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -x_{23} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad g''_{-2R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -x_{23} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

of similar form as (3.45).

Last but not least we consider the isomorphism g_{-1} with inverse $(g_{-1})^{-1}$

$$\begin{matrix} V'_{23} \\ \oplus \\ V'_{12} \\ \oplus \\ V'_{23} \end{matrix} \underset{\begin{pmatrix} -\eta_{23} & 0 & \eta_{23} \\ 0 & \eta_{12} & 0 \\ \eta_{23} & 0 & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 & 0 & \eta_{23}^{-1} \\ 0 & \eta_{12}^{-1} & 0 \\ \eta_{23}^{-1} & 0 & \eta_{23}^{-1} \end{pmatrix}}{\rightleftarrows}} \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix}.$$

Similar to (3.46) the isomorphisms g_{-3}, g_{-2}, g_{-1} and their respective inverse induce the cochain complex $R_{2I}^{\bullet}$(3.53) isomorphic to $R_2^{\bullet}$ (3.52)

$$0 \longrightarrow \begin{matrix} V_{23} \\ \oplus \\ Y \end{matrix} \xrightarrow{g_{-2}\circ d_{R_2}^{-3}\circ(g_{-3})^{-1}} \begin{matrix} V_{23} \\ \oplus \\ V_{23} \\ \oplus \\ V'_{12}\circ V'_{23} \\ \oplus \\ V'_{23}\circ V'_{12} \end{matrix} \xrightarrow{g_{-1}\circ d_{R_2}^{-2}\circ(g_{-2})^{-1}} \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \xrightarrow{d_{R_2}^{-1}\circ(g_{-1})^{-1}} U \longrightarrow 0\ , \tag{3.53}$$

with $X := V'_{12}\circ V'_{23} \oplus V'_{23}\circ V'_{12}$ (3.53) takes the form

$$0 \longrightarrow \begin{matrix} V_{23} \\ \oplus \\ Y \end{matrix} \xrightarrow{\begin{pmatrix} 1_{V_{23}} & 0 \\ 0 & 0 \\ \beta' & \alpha \end{pmatrix}} \begin{matrix} V_{23} \\ \oplus \\ V_{23} \\ \oplus \\ X \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 1_{V_{12}} & 0 \\ 0 & 0 & \alpha_1 \\ \gamma' & 0 & \alpha_2 \end{pmatrix}} \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \xrightarrow{(\,0\ \Psi_{12}\ \Psi_{23}\,)} \underline{U} \longrightarrow 0\ .$$

Since the morphisms are differentials we have the following relations

$$\begin{aligned} \alpha_1\circ\beta' &= 0 \\ \gamma' + \alpha_2\circ\beta' &= 0 \\ \alpha_1\circ\alpha &= 0 \\ \alpha_2\circ\alpha &= 0 \\ \Psi_{23}\circ\gamma' &= 0 \\ \Psi_{23}\circ\alpha_2 + \Psi_{12}\circ\alpha_1 &= 0. \end{aligned}$$

The R-components of $\alpha, \alpha_1, \alpha_2$ also appeared in $L_{2I}^{\bullet}$ (3.47) and where given in (3.49). The components of γ' and β' are

$$\gamma'_R = \begin{pmatrix} x_{23} & -x_{23}^2 \\ -1 & x_{23} \end{pmatrix}, \qquad \beta'_R = \begin{pmatrix} 2x_{12} & 2x_{12}x_{23} \\ 1 & x_{23} \\ 0 & 2x_{12} \\ 0 & 1 \\ 2x_{12} & 2x_{12}x_{23} \\ -1 & x_{12}+x_{13} \\ 1 & x_{23} \\ 0 & 1 \end{pmatrix}.$$

Consider the following morphisms $\tilde{g} : R^{\bullet}_{2I} \to F^{\bullet}$ and $\hat{g} : F^{\bullet} \to R^{\bullet}_{2I}$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
0 \arrow[r] & \begin{matrix} V_{23} \\ \oplus \\ Y \end{matrix} \arrow[r, "\begin{pmatrix} 1_{V_{23}} & 0 \\ 0 & 0 \\ \beta' & \alpha \end{pmatrix}"] \arrow[d, shift left, "(0\ 1_Y)"] & \begin{matrix} V_{23} \\ \oplus \\ V_{23} \\ \oplus \\ X \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1_{V_{23}} & 0 \\ 0 & 0 & \alpha_1 \\ \gamma' & 0 & \alpha_2 \end{pmatrix}"] \arrow[d, shift left, "(-\beta'\ 0\ 1_X)"] & \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"] \arrow[d, shift left, "\begin{pmatrix} 0 & 1_{V_{12}} & 0 \\ 0 & 0 & 1_{V_{23}} \end{pmatrix}"] & \underline{U} \arrow[r] \arrow[d, shift left, "1_U"] & 0 \\
0 \arrow[r] & Y \arrow[r, "\alpha"'] \arrow[u, shift left, "\begin{pmatrix} 0 \\ 1_Y \end{pmatrix}"] & X \arrow[r, "\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}"'] \arrow[u, shift left, "\begin{pmatrix} 0 \\ 0 \\ 1_X \end{pmatrix}"] & \begin{matrix} V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(\Psi_{12}\ \Psi_{23})"'] \arrow[u, shift left, "\begin{pmatrix} 0 & 0 \\ 1_{V_{12}} & 0 \\ 0 & 1_{V_{23}} \end{pmatrix}"] & \underline{U} \arrow[r] \arrow[u, shift left, "1_U"] & 0
\end{tikzcd}
$$

satisfying

$$\tilde{g} \circ \hat{g} = 1_{F^{\bullet}}.$$

For the other direction we find that $\hat{g} \circ \tilde{g}$ is homotopic to the identity $1_{R^{\bullet}_{2I}}$, shown in following diagram

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & \begin{matrix} V_{23} \\ \oplus \\ Y \end{matrix} \arrow[r, "\begin{pmatrix} 1_{V_{23}} & 0 \\ 0 & 0 \\ \beta' & \alpha \end{pmatrix}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 \\ 0 & 1_Y \end{pmatrix}"] & \begin{matrix} V_{23} \\ \oplus \\ V_{23} \\ \oplus \\ X \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1_{V_{23}} & 0 \\ 0 & 0 & \alpha_1 \\ \gamma' & 0 & \alpha_2 \end{pmatrix}"] \arrow[dl, "\begin{pmatrix} 1_{V_{23}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_X \end{pmatrix}"] & \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"] \arrow[dl, "\begin{pmatrix} 0 & 0 & 0 \\ 1_{V_{23}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{V_{12}} & 0 \\ 0 & 0 & 1_{V_{23}} \end{pmatrix}"] & \underline{U} \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & 0 \\
0 \arrow[r] & \begin{matrix} V_{23} \\ \oplus \\ Y \end{matrix} \arrow[r, "\begin{pmatrix} 1_{V_{23}} & 0 \\ 0 & 0 \\ \beta & \alpha \end{pmatrix}"'] & \begin{matrix} V_{23} \\ \oplus \\ V_{23} \\ \oplus \\ X \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1_{V_{23}} & 0 \\ 0 & 0 & \alpha_1 \\ \gamma' & 0 & \alpha_2 \end{pmatrix}"'] & \begin{matrix} V_{23} \\ \oplus \\ V_{12} \\ \oplus \\ V_{23} \end{matrix} \arrow[r, "(0\ \Psi_{12}\ \Psi_{23})"'] & \underline{U} \arrow[r] & 0
\end{tikzcd}
.
$$

Thus, we conclude $R^{\bullet}_2 \simeq F^{\bullet} \simeq L^{\bullet}_2$. □

We verified the braid relations (Relation I and I) as well as the inverse relation at least in one direction (Relation 0) under the assignments (3.24) explicitly, i.e. we constructed a braid invariant in terms of cochain complexes of module functors.

4. Connection to Khovanov-Rozansky's formalism

Khovanov and Rozansky's description of braid invariants is part of their description of link invariants. In this chapter we will draw the connection from our formalism to theirs. The building blocks in their description are cochain complexes of matrix factorizations of minimal models.

4.1. Background on matrix factorizations

Definition 4.1.1
A $\mathbb{Z}_2$-graded R-module is a pair (M, σ) of an R-module M that decomposes into a direct sum $M^0 \oplus M^1$ together with an R-linear involutive automorphism σ, i.e. $\sigma^2 = 1$ such that for $m \in M^i$, $\sigma(m) = (-1)^i m$. The involution σ characterizes the $\mathbb{Z}_2$-grading.

Let R, R' be polynomial rings and $W \in R, W' \in R'$ be two potentials, i.e. they have an isolated singularity at 0, i.e. $\dim_{\mathbb{C}}(Jac_W) = \dim_{\mathbb{C}}\big(W/(\partial_{x_1} W, \ldots, \partial_{x_n} W)\big) < \infty$, the Jacobian algebra is finite dimensional. We have the following definition:

Definition 4.1.2
A (W, W')-matrix factorization is a tuple (M, σ, Q) consisting of a free $\mathbb{Z}_2$ graded $R \otimes_{\mathbb{C}} R'$-module (M, σ) together with an odd (i.e. $Q\sigma + \sigma Q = 0$) endomorphism Q $: M \to M$, satisfying

$$Q^2 = (W - W') \cdot 1_M,$$

where the action of W and W' on the module M is defined by acting with $W \otimes 1$ and $1 \otimes W'$ respectively. If the module has finite rank we call (M, σ, Q) a finite rank matrix factorization.

Definition 4.1.3
A morphism between matrix factorizations (M_1, σ_1, Q_1) and (M_2, σ_2, Q_2) is given by a module morphism $\phi : M_1 \to M_2$. We distinguish between even $(n = 0)$ and odd $(n = 1)$ morphisms $\phi^{(n)}$ satisfying $\sigma_2 \phi^{(n)} = (-1)^n \phi^{(n)} \sigma_1$. This leads to a category $MF_{W,W'}$ whose subcategory of finite rank matrix factorizations is denoted $mf_{(W,W')}$.

We will sometimes just write Q for a matrix factorization (M, σ, Q). The category $MF_{W,W'}$ is differential $\mathbb{Z}_2$-graded with the differential

$$\delta : Hom_{MF_{W,W'}}(Q_1, Q_2) \to Hom_{MF_{W,W'}}(Q_1, Q_2) \tag{4.1}$$

$$\phi^{(n)} \mapsto Q_2 \phi^{(n)} + (-1)^n \phi^{(n)} Q_1. \tag{4.2}$$

It is straightforward to check that $\delta^2 = 0$. The cohomology space of the morphisms defines the cohomology category $H^\bullet MF_{W,W'}$, which has the same objects as $MF_{W,W'}$ but fewer morphisms. One may further restrict to the homotopy category $HMF_{W,W'}$ which has the same objects as $MF_{W,W'}$ but whose morphisms are the zeroth-δ cohomologies of the morphism spaces of $MF_{W,W'}$, in other words morphisms from Q_1 to Q_2 are even morphisms satisfying $Q_2\phi = \phi Q_1$ modulo morphisms of the form $Q_2\psi + \psi Q_1$ where ψ is odd. Most of the time we will work in the category $HMF_{W,W'}$.

Let (M_Q, σ_Q, Q) be a (W, W')- matrix factorization and (M_P, σ_P, P) be a (W', W'')-matrix factorization both of finite rank. One can build the tensor product matrix factorization

$$Q \boxtimes P = Q \otimes 1_{M_P} + \sigma_Q \otimes P \tag{4.3}$$

realized on the $(R \otimes R'')$-module $M_Q \otimes_R M_P$, where the $R \otimes R''$ action is

$$(r \otimes r'') \cdot (m_P \otimes m_Q) := (r \otimes 1) \cdot m_P \otimes (1 \otimes r'') \cdot m_Q \tag{4.4}$$

and the tensor product over R is understood by the relation

$$((1 \otimes r') \cdot m_P) \otimes m_Q = m_P \otimes ((r' \otimes 1) \otimes m_Q). \tag{4.5}$$

Its straightforwardly checked that the tensor product $Q \boxtimes P$ is a (W, W'')-matrix factorization, which will be of infinite rank even though the factorizations Q and P may have been of finite rank. This is due to forgetting the action of the ring R' when viewing $M_P \otimes_R' M_Q$ as an $R \otimes R''$-module. It has been shown that if Q and P are finite rank matrix factorizations, $Q \boxtimes P$ is isomorphic to a finite rank matrix factorization [KR04b].

Consider the case $R = R'$ and $W = W'$, since it will be of special relevance for the description of braid invariants. In that case the tensor product defined above turns $HMF_{W,W}$ into a monoidal category which has been shown in [CR10]. The unit object has been investigated in [CR10, KR04a, BR07] and been precisely realised up to isomorphisms in [CR10]. We will denote the unit object by I_W. The associators and the left and right unitors $\lambda_Q : Q \mapsto I_W \boxtimes Q$ and $\rho_Q : Q \mapsto Q \boxtimes I_W$ have also been given in [CR10]. In the one variable case, i.e. $R = \mathbb{C}[x]$ we can represent the identity factorization by the matrix

$$I_W = \begin{pmatrix} 0 & x - x' \\ \frac{W(x) - W(x')}{x - x'} & 0 \end{pmatrix} \tag{4.6}$$

acting on the free $R \otimes R$-module

$$M_{I_W} = (R \otimes_{\mathbb{C}} R) \oplus (R \otimes_{\mathbb{C}} R), \qquad \sigma_{I_W} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.7}$$

where we represent the variable of the second tensor factor R by x'. The module M_{I_W} might then be represented by $\mathbb{C}[x, x'] \oplus \mathbb{C}[x, x']$.

Let (M_1, σ_1, Q_1) be a (W_1, W_2)-matrix factorization and (M_2, σ_2, Q_2) be a (W_3, W_4)-matrix factorization. Then we can define the external tensor product like in (4.3) but acting on the $((R_1 \otimes R_2) \otimes (R_3 \otimes R_4))$-module $M_1 \otimes_{\mathbb{C}} M_2$. This factorization will then be a $(W_1 + W_3, W_2 + W_4)$-matrix factorization. We denote the external tensor product of matrix factorizations by $\boxtimes'$. This operation has been investigated by [Yos98, KR04b].

4.2. Fusion functors and matrix factorizations

Definition 4.2.1
A (W', W)-fusion functor U is a $\mathbb{C}$-linear functor from $R-mod_f^{\mathbb{Z}^2}$ to $R'-mod_f^{\mathbb{Z}^2}$ satisfying

$$U(W \cdot f) = W' \cdot U(f) \tag{4.8}$$

for every module homomorphism f.

We can extend $\mathbb{C}$-linear functors on $R - mod_f$ to functors on $R - mod_f^{\mathbb{Z}_2}$ where the grading on $U(M)$ is induced by $U(\sigma)$.[Fre24]

Remark. Under that extension the functors $V_{i,i+1}$ and the identity functor, we denoted by U, become (W, W)-fusion functors for the potential $W = x_1^{k+1} + x_2^{k+1} + \cdots + x_n^{k+1}$, for the n variables and $k \in \mathbb{N}$. This is true due to the fact that W is symmetric under interchanging any variables. In fact this is how the connection to Khovanov-Rozansky's formulation is drawn and why these functors were chosen as the components of the cochain complexes (3.24).

The property (4.8) guarantees that a W-matrix factorization is mapped to a W'-matrix factorization, which is captured in the following proposition.

Proposition 4.2.2
Let (M, σ, Q) be W-matrix factorization, i.e. (M, σ) is a free finite rank $\mathbb{Z}_2$-graded R-module with $Q^2 = W \cdot 1_M$, then for a (W', W)-fusion functor U we have that $(U(M), U(\sigma), U(Q))$ is a W'-matrix factorization.

Proof. $U(Q)$ is odd and $U(\sigma)$ is an involution

$$U(Q)U(\sigma) = U(Q\sigma) = -U(\sigma Q) = -U(\sigma)U(Q), \qquad U(\sigma^2) = U(1_M) = 1_{U(M)}$$

and

$$U(Q)^2 = U(Q^2) = U(W \cdot 1_M) = W' \cdot 1_{U(M)}.$$

□[Fre24]

So far the fusion functors have been defined on finite rank modules only. We would like to define the functors acting on (W, W'')-matrix factorizations (M, σ, Q), viewing the underlying $R \otimes R''$-Module (M, σ) as an $R-$module of infinite rank by forgetting the R'' action, i.e.

$$r \cdot m \coloneqq (r \otimes 1) \cdot m. \tag{4.9}$$

We thus need to extend the functors on modules of infinite rank. In this process let us consider any (W', W)-fusion functor U. One can define a functor $\hat{U}$ by

$$\hat{U}(M) \coloneqq U(R) \otimes_R M \tag{4.10}$$

$$\hat{U}(f) \coloneqq 1_{U(R)} \otimes f, \tag{4.11}$$

where we view $U(R)$ not only as an R'-module which it is by definition but also as an R-module with the action defined as

$$r \cdot u := U(r \cdot 1_R)(u), \tag{4.12}$$

for $u \in U(R)$. Thus the relative tensor product of $U(R) \otimes_R M$ is understood as

$$u \otimes (r \cdot m) = (U(r \cdot 1_R)(u)) \otimes m. \tag{4.13}$$

$\hat{U}$ carries all the necessary structure of a fusion functor, as $\mathbb{C}$-linearity and functoriality, as well as the fusion property follow from the structure of U.

Proposition 4.2.3
Let U be any (W', W)-fusion functor. Then the corresponding functor $\hat{U}$ defined as in (4.10) is isomorphic to U on free finite rank modules, i.e. there exists a natural isomorphism η with inverse η^{-1} such that the diagram

$$\begin{tikzcd} U(M) \arrow[r, "U(\phi)"] \arrow[d, shift left, "\eta_M"] & U(N) \arrow[d, shift left, "\eta_N"] \\ U(R) \otimes_R M \arrow[r, "1 \otimes \phi"] \arrow[u, shift left, "\eta_M^{-1}"] & U(R) \otimes_R N \arrow[u, shift left, "\eta_N^{-1}"] \end{tikzcd} \tag{4.14}$$

commutes for all free finite rank R-modules M, N and morphisms $\phi : M \to N$.

Proof. We choose a basis $\{e_i\}$ in M and $\{b_j\}$ in N. Let p_i^M and ι_i^M be the respective projection and embedding maps defined via the basis $\{e_i\}$, i.e.

$$p_i : M \to R \qquad \iota_i : R \to M \tag{4.15}$$

$$\sum_k m_k e_k \mapsto m_i \qquad r \mapsto r e_i \tag{4.16}$$

and similarly for p_j^N and ι_j^N. The functor U induces maps $U(p_i^M) : U(M) \to U(R)$ and $U(\iota_i^M) : U(R) \to U(M)$. Consider the isomorphisms

$$\eta_M : U(M) \to U(R) \otimes M \qquad \eta_M^{-1} : U(R) \otimes_R M \to U(M) \tag{4.17}$$

$$u \mapsto \sum_i U(p_i^M)(u) \otimes e_i \qquad v \otimes e_i \mapsto U(\iota_i^M)(v). \tag{4.18}$$

We compute

$$\begin{aligned}
\eta_N^{-1} \circ 1 \otimes \phi \circ \eta_M(u) &= \eta_N^{-1} \circ 1 \otimes \phi\Big(\sum_i U(p_i^M)(u) \otimes e_i\Big) \\
&= \eta_N^{-1}\Big(\sum_i U(p_i^M)(u) \otimes \phi(e_i)\Big) \\
&= \eta_N^{-1}\Big(\sum_i U(p_i^M)(u) \otimes \sum_k \phi_{ik} b_k\Big) \\
&= \eta_N^{-1}\Big(\sum_{i,k} U(\phi_{ik}) U(p_i^M)(u) \otimes b_k\Big) \\
&= \sum_{i,k} U(\iota_k^N) U(\phi_{ik}) U(p_i^M)(u) \\
&= \sum_{i,k} U(\iota_k^N \phi_{ik} p_i^M)(u) = U(\phi)(u)
\end{aligned}$$

hence the diagram (4.14) commutes. In the last line we used $\phi_{ik} = p_i^N \circ \phi \circ \iota_k^M$. □

Remark. This also reflects that the module or the fusion functors are well defined up to isomorphism by their action on the ground module R and maps from R to R.

Proposition 4.2.3 provides a canonical way to extend fusion functors from free finite-rank modules to free infinite-rank modules. Starting off with a finite-rank module functor U we define its action on infinite-rank modules M, N and morphisms $f : M \to N$

$$U(M) \coloneqq U(R) \otimes_R M \tag{4.19}$$
$$U(f) \coloneqq 1 \otimes f, \tag{4.20}$$

and for $g : M_f \to N$ or $h : M \to N_f$ where M_f, N_f are of finite rank we define

$$U(g) \coloneqq (1 \otimes g) \circ \eta_{M_f} \tag{4.21}$$
$$U(h) \coloneqq \eta_{N_f}^{-1} \circ (1 \otimes h), \tag{4.22}$$

for some fixed set of isomorphisms $\{\eta_{M_f}\}_{M_f \in R-mod_f}$ depending on a choice of basis as in Proposition4.2.3.

A natural transformation of finite-rank functors extends in a canonical way to the extension of functors of infinite rank. Let $\eta : U \to V$ be a natural transformation between two finite-rank module functors. For an infinite-rank modules M, η extends $\eta_M \coloneqq \eta_R \otimes 1_M$.

Remark. The braid invariants we defined in the category $HCoCh(End(R-mod_f))$ can also be defined in the category $HCoCh(End(R-mod))$ when functors are extended in this way.

This extension of functors on infinite-rank R-modules allows us to act with an (W, W')-fusion functor U also on a (W, W'')-matrix factorization (M, σ, Q) obtaining a (W', W'')-matrix factorization $(U(M), U(\sigma), U(Q))$, realized on an $R' \otimes R''$-Module $(U(M), U(\sigma))$

where the $R' \otimes R''$-action is

$$(r' \otimes r'') \cdot (u \otimes m) \coloneqq r' \cdot u \otimes (1 \otimes r'') \cdot m \tag{4.23}$$

for $u \otimes m \in U(R) \otimes_R M$. We check that $U(Q)$ is indeed an (W', W'')-matrix factorization

$$\begin{aligned}
U(Q)^2 &= U(Q^2) \\
&= U((W - W'') \cdot 1_M) \\
&= U(W \cdot 1_M - W'' \cdot 1_M) \\
&= 1_{U(R)} \otimes W \cdot 1_M - 1_{U(R)} \otimes W'' \cdot 1_M \\
&= U(W \cdot 1_R) \circ 1_{U(R)} \otimes 1_M - 1_{U(R)} \otimes W'' \cdot 1_M \\
&= W' \cdot 1_{U(R)} \otimes 1_M - 1_{U(R)} \otimes (1 \otimes W'') \cdot 1_M \\
&= (W' - W'') \cdot 1_{U(R)} \otimes 1_M \\
&= (W' - W'') \cdot 1_{U(R) \otimes M}.
\end{aligned}$$

The category $FF_{W,W'}$ has fusion functors which are extended on free infinite-rank modules as objects and natural transformations as morphisms.

4.3. Khovanov-Rozansky's formulation

We turn our attention to Khovanov-Rozansky's formulation of braid invariants[KR04b]. In a nutshell they assign to the generators of the braid group the cochain complexes

$$\sigma_i \ \widehat{=}\ KR(\sigma_i) = \ 0 \longrightarrow \underline{I_W} \xrightarrow{\chi_0^i} Q_i \longrightarrow 0 \tag{4.24}$$

$$\sigma_i^{-1} \ \widehat{=}\ KR(\sigma_i^{-1}) = \ 0 \longrightarrow Q_i \xrightarrow{\chi_1^i} \underline{I_W} \longrightarrow 0 \ , \tag{4.25}$$

which are objects in $HCoChHMF_{W,W}$. The components of (4.24) are given by

$$I_W = I_1 \boxtimes' I_2 \boxtimes' \cdots \boxtimes' I_n \tag{4.26}$$

$$Q_i = I \boxtimes' \cdots \boxtimes' I_{i-1} \boxtimes' Q_{i,i+1} \boxtimes' I_{i+2} \boxtimes' \cdots \boxtimes' I_n \tag{4.27}$$

and the corresponding morphisms are given by

$$\chi_0^i = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \chi_0^{i,i+1} \otimes 1 \otimes \cdots \otimes 1 \tag{4.28}$$

$$\chi_0^i = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \chi_1^{i,i+1} \otimes 1 \otimes \cdots \otimes 1, \tag{4.29}$$

where $\chi_0^{i,i+1} : I_i \boxtimes' I_{i+1} \to Q_{i,i+1}$ and $\chi_1^{i,i+1} : Q_{i,i+1} \to I_i \boxtimes' I_{i+1}$ are the i-th factor of the tensor products (4.28).

We will explore later how to represent the factorizations (4.26) and the morphisms (4.28) as we delve into the computational section. At this point, it is only important to mention that the factorizations in (4.26) are (W, W)-factorizations with $W = x_1^{k+1} + x_2^{k+1} +$

$\cdots + x_n^{k+1}$. The identity factorization I_W is considered as the external tensor product of the identity factorizations I_i for the single-variable polynomial x_i^{k+1}, as previously defined in (4.6). $Q_{i,i+1}$ represents a (W, W)-factorization of $W = x_i^{k+1} + x_{i+1}^{k+1}$. It is worth mentioning that the cochain complexes (4.24) are only defined up to isomorphism, since the bracketing of the tensor products in (4.26) is not specified. We will specify the bracketing and determine specific representations for the tensor product $\boxtimes'$ when performing actual calculations.

Khovanov and Rozansky associate to a braid the tensor product of the corresponding elementary cochain complexes (4.24). The tensor product in $HCoChHMF_{W,W}$ is done in the canonical way, i.e. for two cochain complexes

$$\ldots \xrightarrow{d_{-1}^P} P_0 \xrightarrow{d_0^P} P_1 \xrightarrow{d_1^P} \ldots \tag{4.30}$$

$$\ldots \xrightarrow{d_{-1}^Q} Q_0 \xrightarrow{d_0^Q} Q_1 \xrightarrow{d_1^Q} \ldots \tag{4.31}$$

the $i-$th component of the tensor product cochain complex is given by

$$(P \otimes Q)_i \coloneqq \bigoplus_{j+k=i} P_j \boxtimes Q_k \tag{4.32}$$

and the differentials are build accordingly. This concludes in a nutshell how Khovanov and Rozansky formulate their braid invariants. The verification of the braid relations in this formalism was done in [KR04b].

4.4. Khovanov-Rozansky's formulation from fusion functors

We are now ready to finally discuss how one obtains the Khovanov-Rozansky description of braid invariants back. We consider the following functor which has been investigated in [Fre24].

Definition 4.4.1
Let (M, σ, Q) be a finite rank (W, W)-matrix factorization. We define a functor $\hat{\Pi}^Q$ from the category $FF_{W,W}$ to the category $HMF_{W,W}$ by

$$\begin{aligned} \hat{\Pi}^Q : FF_{W,W} &\to HMF_{W,W} \\ U &\mapsto (U(M), U(\sigma), U(Q)) \\ \eta_{UV} &\mapsto \eta_{UV\,M} \end{aligned}$$

for fusion functors U and V and natural transformation $\eta_{UV} : U \to V$.

Since η_{UV} is a natural transformation we have

$$V(\sigma)\eta_{UV\,M} = \eta_{UV\,M} U(\sigma) \tag{4.33}$$

which guarantees that $\hat{\Pi}^{Q}$ is well defined. For two isomorphic factorizations Q_1 and Q_2 the functors $\hat{\Pi}^{Q_1}$ and $\hat{\Pi}^{Q_2}$ are isomorphic, in particular factorizations $U(Q_1)$ and $U(Q_2)$ are isomorphic [Fre24]. A functor $\hat{\Pi}^{Q}$ induces a functor Π^Q from the category of cochain complexes of fusion functors to the category of cochain complexes of matrix factorizations.

Definition 4.4.2
For (M, σ, Q) a (W, W)-matrix factorization there is a functor

$$\begin{aligned} \Pi^Q : HCoCh(FF_{W,W}) &\to HCoCh(HMF_{W,W}) \\ \cdots \to U_0 \xrightarrow{d_0} U_1 \to \ldots &\mapsto \cdots \to U_0(Q) \xrightarrow{d_{0M}} U_1(Q) \to \ldots \\ \{\phi_n\} &\mapsto \{\phi_{nM}\} \\ \{h_n\} &\mapsto \{h_{nM}\}, \end{aligned}$$

where we denoted morphisms by the family $\{\phi_n\}$ and homotopies by $\{h_n\}$.

The functors Π^Q are well defined since all morphisms in $FF_{W,W}$ are natural transformations.

Proposition 4.4.3
Applying the functor Π^{I_W} to the formulation of braid invariants in terms of fusion functors, one obtains back the formulation of Khovanov-Rozansky in terms of matrix factorizations, i.e.

1. Π^{I_W} maps the cochain complexes of fusion functors assigned to the generators of a braid in (3.24) to cochain complexes isomorphic to the cochain complexes Khovanov-Rozansky give in (4.24), i.e. $\Pi^{I_W}(C(\sigma_i^{(-1)})) \simeq KR(\sigma_i^{(-1)})$ in $HMF_{W,W}$, for all $i \in n$.
2. and the functor Π^{I_W} is compatible with taking tensor products, i.e. $\Pi^{I_W}(U^\bullet \otimes V^\bullet) \simeq \Pi^{I_W}(U^\bullet) \otimes \Pi^{I_W}(V^\bullet)$, where $U^\bullet, V^\bullet \in HCoCh(FF_{W,W})$.

Remark. It is expected that Π^{I_W} is in fact a monoidal functor. One might show this by proving compatibility with the associators and left and right unitors as well. For our purpose this is not necessary and the second point of Proposition 4.4.3 is sufficient.

The first part of Proposition 4.4.3 involves a concrete representation of the matrix factorizations and morphisms appearing in the elementary cochain complexes (4.24). For the braid group with two strands we will carry out the concrete computation in a corresponding representation. We will explain rather briefly how one can apply the two strand case also for the case of more than two strands. The second part of Proposition 4.4.3 is more of a general property which does not depend on what kind of factorizations are involved and can be discussed for any (W, W)-fusion functors and (W, W)-factorizations. We postpone the first point and continue with the second one.

The second point of Proposition 4.4.3 is the question of whether the cochain complexes one obtains when first taking the tensor product in $HCoCh(FF_{W,W})$ of two cochain complexes and then applying Π^{I_W} is isomorphic to the cochain complex one obtains when first applying Π^{I_W} to the two cochain complexes and then taking the tensor product of those. This will hold if for any $U, U', V, V' \in FF_{W,W}$ and $\phi : U \to V$ and $\phi' : U' \to V'$ the diagram

$$\begin{tikzcd} U \circ U'(I_W) \arrow[r, "(\phi\otimes\phi')_{M_{I_W}}"] \arrow[d, "\simeq"'] & V \circ V'(I_W) \arrow[d, "\simeq"] \\ U(I_W) \boxtimes U'(I_W) \arrow[r, "\phi_{M_{I_W}} \otimes \phi'_{M_{I_W}}"'] & V(I_W) \boxtimes V'(I_W) \end{tikzcd} \tag{4.34}$$

commutes.

To show this we first consider the diagram

$$\begin{tikzcd} U \circ U'(I_W) \arrow[r, "(\phi\otimes\phi')_{M_{I_W}}"] \arrow[d, "U(\lambda^{-1}_{U'(I_W)})"'] & V \circ V'(I_W) \arrow[d, "V(\lambda^{-1}_{V'(I_W)})"] \\ U(I_W \boxtimes U'(I_W)) \arrow[r, "\phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(1_{M_{I_W}} \otimes \phi'_{M_{I_W}})"'] & V(I_W \boxtimes V'(I_W)) \end{tikzcd} \tag{4.35}$$

which commutes since

$$\begin{aligned} V(\lambda^{-1}_{V'(I_W)}) \circ (\phi \otimes \phi')_{M_{I_W}} &= V(\lambda^{-1}_{V'(I_W)}) \circ \phi_{V'(M_{I_W})} \circ U(\phi'_{M_{I_W}}) \\ &= \phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(\lambda^{-1}_{V'(M_{I_W})}) \circ U(\phi'_{M_{I_W}}) \\ &= \phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(\lambda^{-1}_{V'(M_{I_W})} \circ \phi'_{M_{I_W}}) \\ &= \phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(1 \otimes \phi'_{M_{I_W}} \circ \lambda^{-1}_{U'(M_{I_W})}) \\ &= \phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(1 \otimes \phi'_{M_{I_W}}) \circ U(\lambda^{-1}_{U'(M_{I_W})}) \end{aligned}$$

using naturality of the isomorphisms λ^{-1} as well as functoriality of the functors. We next consider the following diagram

$$\begin{tikzcd} U(I_W) \boxtimes U'(I_W) \arrow[r, "\phi_{M_{I_W}} \otimes \phi'_{M_{I_W}}"] \arrow[d, "\alpha^{(U)}"] & V(I_W) \boxtimes V'(I_W) \arrow[d, "\alpha^{(V)}"] \\ U(I_W \boxtimes U'(I_W)) \arrow[r, "\phi_{M_{I_W} \otimes V'(M_{I_W})} \circ U(1_{M_{I_W}} \otimes \phi'_{M_{I_W}})"'] & V(I_W \boxtimes V'(I_W)) \end{tikzcd}$$

which on the module side we can write using the definition of functors and natural transformations

$$\begin{tikzcd}
(U(R) \otimes_R M) \otimes_R (U'(R) \otimes_R M) \arrow[r, "(\phi_R \otimes 1_M)\otimes(\phi'_R \otimes 1_M)"] \arrow[d, "\alpha^{(U)}"'] & (V(R) \otimes_R M) \otimes_R (V'(R) \otimes_R M) \arrow[d, "\alpha^{(V)}"] \\
U(R) \otimes_R (M \otimes_R (U'(R) \otimes_R M)) \arrow[r, "\phi_R \otimes (1_M \otimes (\phi'_R \otimes 1_M))"'] & (V(R) \otimes_R M) \otimes_R (V'(R) \otimes_R M)
\end{tikzcd} \tag{4.36}$$

where we write $M_{I_W} \equiv M$ for the underlying module of I_W. The bottom morphism from left to right is given since

$$\begin{aligned}
\phi_{M \otimes_R V'(M)} \circ U(1_M \otimes \phi'_M) &= \phi_R \otimes (1_{M \otimes_R V'(M)}) \circ 1_{U(R)} \otimes (1_M \otimes (\phi'_R \otimes 1_M)) \\
&= \phi_R \otimes (1_{M \otimes_R (V'(R) \otimes_R M)}) \circ 1_{U(R)} \otimes (1_M \otimes (\phi'_R \otimes 1_M)) \\
&= \phi_R \otimes (1_M \otimes (1_{V'(R)} \otimes 1_M)) \circ 1_{U(R)} \otimes (1_M \otimes (\phi'_R \otimes 1_M)) \\
&= \phi_R \otimes (1_M \otimes (\phi'_R \otimes 1_M)).
\end{aligned}$$

The diagram (4.36) commutes for the associators $\alpha^{(U)} = \alpha_{U(R),M,U'(R)\otimes_R M}$ and $\alpha^{(V)} = \alpha_{V(R),M,V'(R)\otimes_R M}$ which are isomorphisms by definition. Thus both diagrams (4.35) and (4.36) commute making the diagram (4.34) commute. This concludes the discussion on the second point of Proposition 4.4.3.

Remark. In diagram (4.36) we omitted the additional structure (σ, I_W) the matrix factorization I_W comes with, since they do not effect the argument.

Representation of matrix factorizations

We turn our attention to the first point of Proposition 4.4.3. In order to show that it holds true we need to perform concrete calculations in a certain representation of matrix factorizations. We therefore have to explain how matrix factorizations and morphisms between those can be represented and in particular how we can represent the action of a fusion functor on matrix factorizations, i.e. representing the resulting matrix factorization when a fusion functor applied.

Let (M_Q, σ_Q, Q) and (M'_Q, σ'_Q, Q') be free finite rank (W, W')-matrix factorizations with $W \in \mathbb{C}[x] \equiv \mathbb{C}[x_1, \dots, x_n]$ and $W' \in \mathbb{C}[x'] \equiv \mathbb{C}[x'_1, \dots, x'_{n'}]$. Let M_Q be of rank $2q$ and M'_Q be of rank $2q'$, then we can represent Q by an odd $2q \times 2q$-matrix and an Q' by an odd $2q' \times 2q'$-matrix with polynomial entries in $\mathbb{C}[x, x']$ respectively, i.e.

$$Q = \begin{pmatrix} 0 & Q_1 \\ Q_0 & 0 \end{pmatrix}, \qquad Q' = \begin{pmatrix} 0 & Q'_1 \\ Q'_0 & 0 \end{pmatrix}. \tag{4.37}$$

In this representation a morphism $\phi : Q \to Q'$ is represented by an even $2q' \times 2q$ matrix

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix} \tag{4.38}$$

with polynomial entries in $\mathbb{C}[x, x']$, s.t. $Q'\phi = \phi Q$. Similarly (W', W'')-matrix factorizations (M_P, σ_P, P) and (M'_P, σ'_P, P'), with $W'' \in \mathbb{C}[x'']$ can be represented by an odd $2p \times 2p$-matrix and an odd $2p' \times 2p'$-matrix with polynomial entries in $\mathbb{C}[x', x'']$. The tensor product factorization $Q \boxtimes P$ can be represented by an odd $4qp \times 4qp$-matrix with polynomial entries in $\mathbb{C}[x, x', x'']$ given by

$$Q \boxtimes P = \begin{pmatrix} 0 & 0 & Q_1 \otimes 1_{P_0} & 1_{Q_0} \otimes P_1 \\ 0 & 0 & 1_{Q_1} \otimes P_0 & -Q_0 \otimes 1_{P_1} \\ Q_0 \otimes 1_{P_0} & 1_{Q_1} \otimes P_1 & 0 & 0 \\ 1_{Q_0} \otimes P_0 & -Q_1 \otimes 1_{P_1} & 0 & 0 \end{pmatrix} \tag{4.39}$$

with the tensor product $\otimes$ being realised as the Kronecker product. In this representation a morphism $\phi \otimes \psi : Q \boxtimes P \to Q' \boxtimes P'$ is realised as the matrix

$$\begin{pmatrix} \phi_0 \otimes \psi_0 & 0 & 0 & 0 \\ 0 & \phi_1 \otimes \psi_1 & 0 & 0 \\ 0 & 0 & \phi_1 \otimes \phi_0 & 0 \\ 0 & 0 & 0 & \phi_0 \otimes \psi_1 \end{pmatrix} \tag{4.40}$$

where $\psi : P \to P'$ and the tensor product is realised as the Kronecker tensor product.

Next we want to understand how we can represent the action of a fusion functor on a matrix factorization, in particular how we can represent the resulting factorization that $\hat{\Pi}^{I_W}$ assigns to a fusion functor U. By Definition (4.4.1) and (4.19) we have

$$\hat{\Pi}^{I_W}(U) = U(M_{I_W}, \sigma_{I_W}, I_W) = (U(R) \otimes_R M_{I_W}, 1_{U(R)} \otimes \sigma_{I_W}, 1_{U(R)} \otimes I_W). \tag{4.41}$$

We choose an $(R \otimes R)$-basis of M_{I_W} which we denote by $\{b_i\}$. The canonical R-basis of $U(R) = R^{r_U}$ we denote by $\{e_j\}$. We have

$$1 \otimes I_W : R^{r_U} \otimes_R M_{I_W} \to R^{r_U} \otimes M_{I_W} \tag{4.42}$$

$$e_j \otimes b_i \mapsto e_j \otimes (I_W)_{ik} b_k \tag{4.43}$$

and we calculate

$$e_j \otimes (I_W)_{ik} b_k = e_j \otimes \sum_s (I_{ik}^{(s)}(r_s \otimes r'_s)) b_k \tag{4.44}$$

$$= U(r_s \cdot 1_R) e_j \otimes I_{ik}^{(s)}(1 \otimes r'_s) b_k \tag{4.45}$$

$$= \Big(U(r_s \cdot 1_R) \otimes I_{ik}^{(s)}(1 \otimes r'_s)\Big)(e_j \otimes b_k), \tag{4.46}$$

where we sum over equal indices. In the first line we write out I_W in the basis $\{b_i\}$ as a matrix with entries in $R \otimes R$ where we denote elements of the first tensor factor by r and elements of the second tensor factor by r'. In the second line we used the definition of the balanced tensor product. The last line is just bringing everything back into form so we can read off the action of $1 \otimes I_W$. Representing the tensor product as the Kronecker tensor product, I_W is represented by a matrix with polynomial entries in $\mathbb{C}[x, x']$, i.e. I_W

is represented by a matrix with entries $(I_W)_{ik}= I_{ik}^{(s)}(r_s r_s')$, where $I_{ik}^{(s)} \in \mathbb{C}$. Then from the last line using the Kronecker tensor product realisation we can see that the functor U effectively acts on I_W in this representation by "blowing up" the entries similar as in the finite rank R-module case, acting on the variables x, but leaving the variables x' untouched. In other words starting with a representation of any factorization Q, we obtain a representation of the factorization $U(Q)$ by simply acting on the polynomial entries.

4.4.1. Representation of the matrix factorizations involved in Khovanov-Rozansky's formalism

In this section we review the representation of matrix factorizations in Khovanov-Rozansky's formalism. This will give meaning to the factorizations and morphisms involved in (4.24). In particular we want to understand the objects I_i, $I_i \boxtimes' I_{i+1}$ and $Q_{i,i+1}$ and the morphisms $\chi_0^{i,i+1}, \chi_1^{i,i+1}$. We fix $i = 1$ for simplicity, not loosing any generality. For example I_1 is the factorization of the single variable polynomial x_1^{n+1} which we represent as

$$M_{I_1} = \mathbb{C}[x_1, x_1'] \oplus \mathbb{C}[x_1, x_1'] \tag{4.47}$$

and

$$I_1 = \begin{pmatrix} 0 & x_1 - x_1' \\ \frac{x_1^{n+1} - x_1'^{n+1}}{x_1 - x_1'} & 0 \end{pmatrix}, \qquad \sigma_{I_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.48}$$

The identity factorization $I_1 \boxtimes' I_2$ of the two variable polynomial $x_1^{n+1} + x_2^{n+1}$ can be represented by the matrix

$$P = \begin{pmatrix} 0 & P_1 \\ P_0 & 0 \end{pmatrix} : \mathbb{C}[x]^2 \oplus \mathbb{C}[x]^2 \to \mathbb{C}[x]^2 \oplus \mathbb{C}[x]^2 \tag{4.49}$$

with

$$P_0 = \begin{pmatrix} \pi_{14} & x_{23} \\ \pi_{23} & -x_{14} \end{pmatrix}, \qquad P_1 = \begin{pmatrix} x_{14} & x_{23} \\ \pi_{23} & -\pi_{14} \end{pmatrix} \tag{4.50}$$

using (4.39) and with

$$\pi_{ij} = \frac{{x_i}^{n+1} - {x_j}^{n+1}}{x_i - x_j} \tag{4.51}$$

$$x_{ij} = x_i - x_j, \tag{4.52}$$

where we write $x_1' \equiv x_4$ and $x_2' \equiv x_3$ and thus $\mathbb{C}[x] \equiv \mathbb{C}[x_1, x_2, x_3, x_4]$.

In this scheme the (W, W)-factorization $Q_{1,2}$ with $W = x_1^{n+1} + x_2^{n+1}$ is represented by the matrix

$$Q = \begin{pmatrix} 0 & Q_1 \\ Q_0 & 0 \end{pmatrix} : \mathbb{C}[x]^2 \oplus \mathbb{C}[x]^2 \to \mathbb{C}[x]^2 \oplus \mathbb{C}[x]^2 \tag{4.53}$$

where

$$Q_0 = \begin{pmatrix} u_1 & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1. \end{pmatrix} \tag{4.54}$$

Here, u_1 and u_2 are not uniquely determined. In the original definition of Khovanov and Rozansky they are given by

$$u_1 = \frac{g(x_1 + x_2, x_1x_2) - g(x_3 + x_4, x_1x_2)}{x_1 + x_2 - x_3 - x_4} \tag{4.55}$$

$$u_2 = \frac{g(x_3 + x_4, x_1x_2) - g(x_3 + x_4, x_3x_4)}{x_1x_2 - x_3x_4} \tag{4.56}$$

where the the explicit definiton of $g(s_1, s_2)$ is given

$$g(s_1, s_2) = s_1^{n+1} + (n+1) \sum_{1 \leq i \leq \frac{n+1}{2}} \frac{(-1)^i}{i} \binom{n-i}{i-1} s_2^i s_1^{n+1-2i}. \tag{4.57}$$

The morphism $\chi_0^{1,2} \equiv \chi_0$ is represented by the matrix

$$\begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \tag{4.58}$$

with

$$U_0 = \begin{pmatrix} x_4 - x_2 + \mu(x_1 + x_2 - x_3 - x_4 & 0 \\ a_1 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} x_4 + \mu(x_1 - x_4) & \mu(x_2 - x_3) - x_2 \\ -1 & 1 \end{pmatrix}, \tag{4.59}$$

where

$$a_1 = (\mu - 1)u_2 + \frac{u_1 + x_1u_2 - \pi_{23}}{x_1 - x_4} \tag{4.60}$$

and $\mu \in \mathbb{Z}$. Different μ give homotopic maps. The morphism $\chi_1^{1,2} \equiv \chi_1$ is given by the matrix

$$\begin{pmatrix} V_0 & 0 \\ 0 & V_1 \end{pmatrix} \tag{4.61}$$

with

$$V_0 = \begin{pmatrix} 1 & 0 \\ a_2 & a_3 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & x_3 + \lambda(x_2 - x_3) \\ 1 & x_1 + \lambda(x_4 - x_1) \end{pmatrix}, \tag{4.62}$$

where

$$a_2 = \lambda u_2 + \frac{u_1 + x_1u_2 - \pi_{23}}{x_4 - x_1}, \quad a_3 = \lambda(x_3 + x_4 - x_1 - x_2) + x_1 - x_3. \tag{4.63}$$

with $\lambda \in \mathbb{Z}$. Different choices of λ give homotopic maps. With this we can represent all the factorizations and morphisms in (4.24) as matrices with polynomial entries choosing a corresponding representation for the tensor product $\boxtimes'$ (4.39).

4.4.2. Two strand case

The braid group for two strands has only one generator σ_1. Khovanov-Rozansky assign

$$\sigma \ \widehat{=} \ KR(\sigma) = \ 0 \longrightarrow P \xrightarrow{\chi_0} Q \longrightarrow 0 \tag{4.64}$$

$$\sigma^{-1} \ \widehat{=} \ KR(\sigma^{-1}) = \ 0 \longrightarrow Q \xrightarrow{\chi_1} P \longrightarrow 0 \ . \tag{4.65}$$

We will show that $\Pi^P(C(\sigma^{(\pm 1)})) \simeq KR(\sigma^{(\pm 1)})$. We have

$$0 \longrightarrow \underline{U(P)} \xrightarrow{\Phi_{12M_P}} V_{12}(P) \longrightarrow 0 \tag{4.66}$$

and

$$0 \longrightarrow V_{12}(P) \xrightarrow{\Psi_{12M_P}} \underline{U(P)} \longrightarrow 0 \tag{4.67}$$

where P is the the representation of the identity factorization I_W of the polynomial $W = x_1^{n+1} + x_2^{n+1}$, given in (4.49). The morphisms Φ_{12M_P} are given by

$$\Phi_{12R} \otimes 1_{M_P} = \begin{pmatrix} \phi_{12} & 0 & 0 & 0 \\ 0 & \phi_{12} & 0 & 0 \\ 0 & 0 & \phi_{12} & 0 \\ 0 & 0 & 0 & \phi_{12} \end{pmatrix} \tag{4.68}$$

due to the Kronecker product realisation of the tensor product and $\Phi_{12R} \coloneqq \phi_{12}$ as in chapter 3. Similarly for Ψ_{12M_P}. The factorization $V_{12}(P)$ is then

$$V_{12}(P) = \begin{pmatrix} 0 & V_{12}(P_1) \\ V_{12}(P_0) & 0 \end{pmatrix} \tag{4.69}$$

where we act on the morphisms by acting on the polynomial entries, i.e.

$$V_{12}(P_0) = \begin{pmatrix} V_{12}(\pi_{14}) & V_{12}(x_{23}) \\ V_{12}(\pi_{23}) & V_{12}(-x_{14}) \end{pmatrix} = \begin{pmatrix} \pi_{14S} & x_{12}\pi_{14A} & x_{23S} & x_{12}x_{23A} \\ \frac{\pi_{14A}}{x_{12}} & \pi_{14S} & \frac{x_{23A}}{x_{12}} & x_{23S} \\ \pi_{23S} & x_{12}\pi_{23S} & -x_{14S} & -x_{12}x_{14A} \\ \frac{\pi_{23A}}{x_{12}} & \pi_{23S} & -\frac{x_{14A}}{x_{12}} & -x_{14S} \end{pmatrix} \tag{4.70}$$

and

$$V_{12}(P_1) = \begin{pmatrix} V_{12}(x_{14}) & V_{12}(x_{23}) \\ V_{12}(\pi_{23}) & V_{12}(-\pi_{14}) \end{pmatrix} = \begin{pmatrix} x_{14S} & x_{12}x_{14A} & x_{23S} & x_{12}x_{23A} \\ \frac{x_{14A}}{x_{12}} & x_{14S} & \frac{x_{23A}}{x_{12}} & x_{23S} \\ \pi_{23S} & x_{12}\pi_{23A} & -\pi_{14S} & -x_{12}\pi_{14A} \\ \frac{\pi_{23A}}{x_{12}} & \pi_{23S} & -\frac{\pi_{14A}}{x_{12}} & -\pi_{14S} \end{pmatrix} \tag{4.71}$$

with

$$
\begin{aligned}
x_{14A} &= \frac{1}{2}(x_{14} - x_{24}) = \frac{1}{2}x_{12} \\
x_{14S} &= \frac{1}{2}(x_{14} + x_{24}) = \frac{1}{2}(x_1 + x_2 - 2x_4) \\
x_{23A} &= \frac{1}{2}(x_{23} - x_{13}) = -\frac{1}{2}x_{12} \\
x_{23S} &= \frac{1}{2}(x_{23} + x_{13}) = \frac{1}{2}(x_1 + x_2 - 2x_3) \\
\pi_{14A} &= \frac{1}{2}(\pi_{14} - \pi_{24}) = \frac{1}{2}\Big(\frac{x_1{}^{n+1} - x_4{}^{n+1}}{x_1 - x_4} - \frac{x_2{}^{n+1} - x_4{}^{n+1}}{x_2 - x_4}\Big) \\
\pi_{14S} &= \frac{1}{2}(\pi_{14} + \pi_{24}) = \frac{1}{2}\Big(\frac{x_1{}^{n+1} - x_4{}^{n+1}}{x_1 - x_4} + \frac{x_2{}^{n+1} - x_4{}^{n+1}}{x_2 - x_4}\Big) \\
\pi_{23A} &= \frac{1}{2}(\pi_{23} - \pi_{13}) = \frac{1}{2}\Big(\frac{x_2{}^{n+1} - x_3{}^{n+1}}{x_2 - x_3} - \frac{x_1{}^{n+1} - x_3{}^{n+1}}{x_1 - x_3}\Big) \\
\pi_{23S} &= \frac{1}{2}(\pi_{23} + \pi_{13}) = \frac{1}{2}\Big(\frac{x_2{}^{n+1} - x_3{}^{n+1}}{x_2 - x_3} + \frac{x_1{}^{n+1} - x_3{}^{n+1}}{x_1 - x_3}\Big).
\end{aligned}
\tag{4.72}
$$

We will show that there is an isomorphism $V(P) \simeq Q$ in $HMF_{W,W}$, which induces an isomorphism on the cochain complexes in $CoChHMF_{W,W}$. In other words we find the matrix factorizations and morphisms of the original definition of Khovanov-Rozansky again.

Lemma 4.4.4
Two matrix factorizations P and Q are isomorphic if the matrices (representations) can be related by similarity transformations. *Proof:* $Q = SPS^{-1} \Leftrightarrow QS - SP = 0 \Leftrightarrow PS^{-1} - S^{-1}Q = 0$ □

In our usual block matrix form this reads

$$
\begin{pmatrix} 0 & Q_1 \\ Q_0 & 0 \end{pmatrix} = \begin{pmatrix} S_0 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} 0 & P_1 \\ P_0 & 0 \end{pmatrix} \begin{pmatrix} S_0^{-1} & 0 \\ 0 & S_1^{-1} \end{pmatrix} = \begin{pmatrix} 0 & S_0 P_1 S_1^{-1} \\ S_1 P_0 S_0^{-1} & 0 \end{pmatrix}, \tag{4.73}
$$

i.e. Q_1 is related to P_1 by a number of row and column transformations encoded by the matrices S_0 and S_1^{-1} respectively. Q_0 and P_0 are related by row and column transformations encoded by S_1 and S_0^{-1}. For the calculation this means that we perform transformations on one block matrix, in our case $V_{12}(P_1)$, which already fixes the transformations on the second block matrix $V_{12}(P_0)$. One can find the detailed calculation leading to the following results in the appendix

$$
S_0 V_{12}(P_1) S_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & W_{12} - W_{34} & 0 & 0 \\ 0 & 0 & x_1 + x_2 - x_3 - x_4 & x_1 x_2 - x_3 x_4 \\ 0 & 0 & u_2 & -u_1 \end{pmatrix}, \tag{4.74}
$$

$$S_1 V_{12}(P_0) S_0^{-1} = \begin{pmatrix} W_{12} - W_{34} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_1 & x_1x_2 - x_3x_3 \\ 0 & 0 & u_2 & x_3 + x_4 - x_1 - x_2 \end{pmatrix}. \tag{4.75}$$

Thus $V_{12}(P) \simeq T \bigoplus Q$, where T is the direct sum of two trivial factorizations of $W_{12} - W_{34}$, represented by the module pair $M_T^0 = \mathbb{C}[x]^2 = M_T^1$ and the module maps

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & W_{12} - W_{34} \end{pmatrix}, \qquad T_0 = \begin{pmatrix} W_{12} - W_{34} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.76}$$

The isomorphism S is represented by the matrices

$$S_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -\frac{2\pi_{23A}}{x_{12}} & -2\pi_{23S} & 1 & 2x_{14S} \\ 1 & -2x_{14S} & 0 & 0 \\ \frac{2\pi_{14A} - 2\pi_{23A} + u_2 x_{12}}{x_{12}x_{14S} + x_{12}x_{23S}} & -\frac{2(2\pi_{14A}x_{14S} + 2\pi_{23A}x_{23S} + u_2 x_{12} x_{14S})}{x_{12}(x_{14S} + x_{23S})} & 0 & 2 \end{pmatrix} \tag{4.77}$$

and

$$S_1 = \begin{pmatrix} 1 & 2x_{14S} & -1 & 2x_{23S} \\ 0 & 0 & 0 & -2 \\ 0 & 2x_4 & 1 & -2(x_{14S} + x_4) \\ 0 & -2 & 0 & 2 \end{pmatrix}. \tag{4.78}$$

In the homotopy category $HMF_{W,W}$ a matrix factorization which decomposes into a sum of trivial matrix factorizations and a non trivial factorization is isomorphic to the non trivial part, i.e. $Q \oplus T \simeq Q$. In other words the trivial factorization is isomorphic to the 0-factorization [KR04b].

The isomorphism $S : V_{12}(P) \to T \oplus Q$ induces the cochain complex (4.79) isomorphic to (4.66)

$$0 \longrightarrow P \xrightarrow{S \circ \Phi_{12 M_P}} T \oplus Q \longrightarrow 0 \tag{4.79}$$

which takes the form

$$\begin{tikzcd} & & T \arrow[dr] & \\ 0 \arrow[r] & P \arrow[ur, "f"] \arrow[dr, "\chi_0|_{\mu=0}"'] & \oplus & 0 \\ & & Q \arrow[ur] & \end{tikzcd} \tag{4.80}$$

where f is given by the pair

$$f_0 = \begin{pmatrix} 1 & 0 \\ -\pi_{23} & x_{14} \end{pmatrix}, \qquad f_1 = \begin{pmatrix} x_{14} & x_{23} \\ 0 & -1 \end{pmatrix}. \tag{4.81}$$

As a consequence of T being isomorphic to the zero factorization, f is homotopic to the zero morphism and therefore (4.81) is isomorphic to the first Khovanov-Rozansky cochain complex (4.64), explicitly this reads

$$\begin{tikzcd}
 & & T \arrow[dr] & \\
0 \arrow[r] & P \arrow[ur, "0"] \arrow[dr, "\chi_0|_{\mu=0}"'] \arrow[dd, shift left, "1"] & \oplus & 0 \\
 & & Q \arrow[ur] \arrow[d, shift left, "\pi_Q"] & \\
0 \arrow[r] & P \arrow[r, "\chi_0"] \arrow[uu, shift left, "1"] & Q \arrow[r] \arrow[u, shift left, "\iota_Q"] & 0
\end{tikzcd} \tag{4.82}$$

where $\pi_Q : T \oplus Q \to Q$ is the projection morphism and $\iota_Q : Q \to T \oplus Q$ the injection morphism.

Similarly S^{-1} induces the cochain complex (4.83) isomorphic to (4.67)

$$0 \longrightarrow Q \xrightarrow{\Psi_{12M_P} \circ S^{-1}} T \oplus P \longrightarrow 0 \tag{4.83}$$

which takes the form

$$\begin{tikzcd}
 & T \arrow[dr, "g"] & & \\
0 \arrow[ur] \arrow[dr] & \oplus & P \arrow[r] & 0 \\
 & Q \arrow[ur, "2\chi_1|_{\lambda=1}"'] & &
\end{tikzcd} \tag{4.84}$$

where g is a null homotopic map given by the pair

$$g_0 = \begin{pmatrix} 2x_{14} & 0 \\ 2\pi_{23} & 2 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} 2 & 2x_{23} \\ 0 & -2x_{14} \end{pmatrix}. \tag{4.85}$$

Explicitly we have the isomorphism to the Khovanov-Rozansky cochain complex (4.64)

$$\begin{tikzcd}
 & T \arrow[dr, "0"] & & \\
0 \arrow[ur] \arrow[dr] & \oplus & P \arrow[r] \arrow[dd, shift left, "\frac{1}{2}"] & 0 \\
 & Q \arrow[ur, "2\chi_1|_{\lambda=1}"'] \arrow[d, shift left, "\pi_Q"] & & \\
0 \arrow[r] & Q \arrow[r, "\chi_1"] \arrow[u, shift left, "\iota_Q"] & P \arrow[r] \arrow[uu, shift left, "2"] & 0.
\end{tikzcd} \tag{4.86}$$

In conclusion we have explicitly shown that in the 2-strand case the first point of the Proposition 4.4.3 holds.

4.4.3. For more than two strands

For more than two strands the computation to show $\Pi^{I_W}(C(\sigma_i^{\pm 1})) \simeq KR(\sigma_i^{\pm 1})$ is not so easy to handle explicitly. However we will give an argument how the computation for two strands can be applied in case for more than two strands, i.e. how to reduce the problem for $n > 2$ to the case $n = 2$, by choosing the representation of the factorizations wisely. First we observe that for $I_W \simeq I'_W$ we also have $\Pi^{I_W}(C(\sigma_i^{\pm 1})) \simeq \Pi^{I'_W}(C(\sigma_i^{\pm 1}))$. For example for σ_i^{-1}

$$\begin{tikzcd} 0 \arrow[r] & V_{i,i+1}(I_W) \arrow[r, "\Phi_{i,i+1}M_{I_W}"] \arrow[d, "V_{i,i+1}(\varphi)"'] & U(I_W) \arrow[r] \arrow[d, "U(\varphi)"] & 0 \\ 0 \arrow[r] & V_{i,i+1}(I'_W) \arrow[r, "\Phi_{i,i+1}M_{I'_W}"] & U(I'_W) \arrow[r] & 0, \end{tikzcd} \tag{4.87}$$

where φ denotes the isomorphism from I_W to I'_W. The diagram commutes due to $\Phi_{i,i+1}$ being a natural transformation. Let $I_W := (...((I_1 \boxtimes' I_2) \boxtimes' I_3) \boxtimes' \cdots \boxtimes' I_n)$ be bracketed from the left as in [CR10]. In this case the cochain complex we obtain for the generator σ_i^{-1} is

$$0 \longrightarrow V_{i,i+1}((...(I_1 \boxtimes' I_2) \boxtimes' ... \boxtimes' I_n)) \xrightarrow{\Phi_{i,i+1}M_{I_W}} U((...(I_1 \boxtimes' I_2) \boxtimes' ... \boxtimes' I_n)) \longrightarrow 0. \tag{4.88}$$

Let $I_W^{(i)} := (I_i \boxtimes' I_{i+1}) \boxtimes' I^{R(i)}$ where $I^{R(i)} = I_1 \boxtimes' \cdots \boxtimes' I_{i-1} \boxtimes' I_{i+2} \boxtimes' \cdots \boxtimes' I_n$ bracketed from the left again. We have $I_W^{(i)} \simeq I_W$. And with (4.87) we have that (4.88) is isomorphic to

$$0 \longrightarrow V_{i,i+1}((I_i \boxtimes' I_{i,i+1}) \boxtimes' I^{R(i)})) \xrightarrow{\Phi_{i,i+1}M_{I_W^{(i)}}} U((I_i \boxtimes' I_{i+1}) \boxtimes' I^{R(i)})) \longrightarrow 0. \tag{4.89}$$

Under the realisation of the tensor product $\boxtimes'$ according to (4.39) and further the Kronecker tensor product one finds that (4.89) is equal to

$$0 \longrightarrow V_{i,i+1}^{(2)}(I_i \boxtimes' I_{i,i+1}) \boxtimes' I^{R(i)} \xrightarrow{\Phi_{i,i+1}^{(2)}M_{I_i \boxtimes' I_{i+1}} \otimes 1_{M_{I^{R(i)}}}} U^{(2)}(I_i \boxtimes' I_{i+1}) \boxtimes' I^{R(i)} \longrightarrow 0, \tag{4.90}$$

where we denoted the functors which occur in the two strand case as $V_{i,i+1}^{(2)}$ and $U^{(2)}$. From (4.90) it is now clear that the isomorphisms we found in the two strand case which where denoted and realised by the matrices S and S^{-1} are realising the isomorphisms $\Pi^{I_W}(C(\sigma_i^{\pm 1})) \simeq KR(\sigma_i^{\pm 1})$. Everything else one does is using the obvious isomorphisms rearranging the factors in the $\boxtimes'$ tensor products or taking different bracketing. In the case (4.90) the isomorphism to $KR(\sigma_i^{-1})$ is given by $S \otimes 1_I^{R(i)}$, where S will be represented by the matrix we had for the two strand case just in the corresponding variables x_i and x_{i+1}. A similar argument can be given for the cochain complexes involving the generators σ_i. This lets us conclude that for $n > 2$ we also have $\Pi^{I_W}(C(\sigma_i^{\pm 1})) \simeq KR(\sigma_i^{\pm 1})$ for $i = 1, \ldots, n$.

5. Discussion

We have shown that we can describe braid invariants in the language of fusion functors in section 3. In section 4 we have shown that there exists a functor Π^{I_W}, which allows us to switch back to Khovanov-Rozansky's formulation in terms of matrix factorizations. Khovanov and Rozansky's formalism describes also link and even tangle invariants. Especially link invariants have been of great interest. The question arises to what extent the formalism in terms of fusion functors can be applied to compute link invariants. Alexander's Theorem tells us that every link can be brought into the from of a closed braid. This leads to the fact that one can obtain link invariants from the description of braid invariants, by finding an assignment to the closing of a braid such that the Markov moves hold [GKS23]. In $HCoCh(End(R - mod_f))$ such an assignment for a closing does not seem to exist, but this is something that could still be investigated or even proven further. From what we know there seems to be at least two options to extend the formalism on links.

The obvious one is to go back to the formalism of matrix factorizations by Khovanov and Rozansky using the functor Π^{I_W}. In this case the computer algorithm of [CM14] could be applied. In a nutshell one would proceed as follows. Firstly one considers the representation of a link as a closed braid. For the corresponding non closed braid one then obtains a cochain complex of fusion functors like in section 3. To this cochain complex Π^{I_W} associates a cochain complex of matrix factorizations. The matrix factorizations can then be represented by matrices with variables x_i, x'_i with $i = 1, \ldots, n$, where n is the number of strands of the braid considered. By identifying $x_i \equiv x'_i$ one obtains a cochain complex associated to the closing of the braid, i.e. a cochain complex of the underlying link. To compute the homology of Khovanov Rozansky one would have to find isomorphism to the finite rank matrix factorizations. This operation has been implemented in a computer program developed by Nils Carqueville and Daniel Murfet [CM14]. Solutions thereof are limited by the computational power, so it would be interesting to analyse whether the computational power needed to compute the link invariants can be reduced when using matrix factorizations and morphisms obtained from Π^{I_W}.

The second option to extend the formalism might be to consider the cochain complexes in $R - Mod$ one obtains by "applying" a cochain complex of fusion functors on the ground module R. The closing might then be given by a Hochschild homology. This is motivated by [GKS23]. Their description of Khovanov Rozansky's link homology is based on Soergel bimodules. One may investigate how exactly the formulation in terms of fusion functors is connected to the formulation in terms of Soergel bimodules.

Another aspect or benefit of the formulation in terms of fusion functors is that in contrast to Khovanov-Rozansky's formulation in terms of matrix factorizations the potential W does not enter at all. In Khovanov-Rozansky's formulation one could already observe that the power k of the polynomials $W = x_1^k + \cdots + x_n^k$ for an n-stranded braid does not enter the computation of verifying the braid relations. The definition of braid invariants in terms of fusion functors allows to consider a different polynomial $W'(x_1, \ldots, x_n)$, which only condition is to be fully symmetric in all its variables, since this is the only condition for the functors given in section 3 to be W', W'-fusion functors. One may investigate what kind of invariants one obtains by applying the functor $\Pi^{I_{W'}}$ on the cochain complexes of fusion functors. By this one would still obtain some kind of braid invariants that might differ from Khovanov-Rozansky's invariants.

As we have remarked in section 4 the functor Π^{I_W} is expected to be a monoidal functor. We did not investigate this any further since it was not necessary for our purpose, however it would still be interesting to investigate if Π^{I_W} is indeed monoidal. We suggest that the way to prove so would be to first investigate if $\hat{\Pi}^{I_W}$ is a monoidal functor. If that is the case, then the step to show that Π^{I_W} is also monoidal should be straightforward.

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A. Appendix

In the following we want to explain how the isomorphism S was found, by giving the concrete row and column transformations performed on $V_{12}(P_1)$, and making sure all the transformations involved on the way are legitimate.

$$V_{12}(P_1) = \begin{pmatrix} x_{14S} & \frac{1}{2}x_{12}^2 & x_{23S} & -\frac{1}{2}x_{12}^2 \\ \frac{1}{2} & x_{14S} & -\frac{1}{2} & x_{23S} \\ \pi_{23S} & x_{12}\pi_{23A} & -\pi_{14S} & -x_{12}\pi_{14A} \\ \frac{\pi_{23A}}{x_{12}} & \pi_{23S} & -\frac{\pi_{14A}}{x_{12}} & -\pi_{14S} \end{pmatrix}$$

$$\Big\downarrow \begin{array}{l} 1R \longleftrightarrow 2R \\ 2C \longleftrightarrow 3C \\ 3R \longleftrightarrow 4R \end{array}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & x_{14S} & x_{23S} \\ x_{14S} & x_{23S} & \frac{1}{2}x_{12}^2 & -\frac{1}{2}x_{12}^2 \\ \frac{\pi_{23A}}{x_{12}} & -\frac{\pi_{14A}}{x_{12}} & \pi_{23S} & -\pi_{14S} \\ \pi_{23S} & -\pi_{14S} & \pi_{23A}x_{12} & -\pi_{14A}x_{12} \end{pmatrix}$$

$$\Big\downarrow \begin{array}{l} 2C \mapsto 2C+1C \\ 3C \mapsto 3C-(2x_{14S})1C \\ 4C \mapsto 4C-(2x_{23S})1C \end{array}$$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ x_{14S} & x_{14S}+x_{23S} & \frac{1}{2}x_{12}^2 - 2x_{14S}^2 & -\frac{x_{12}^2}{2} - 2x_{14S}x_{23S} \\ \frac{\pi_{23A}}{x_{12}} & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{2\pi_{23A}x_{14S}}{x_{12}} + \pi_{23S} & -\frac{2\pi_{23A}x_{23S}}{x_{12}} - \pi_{14S} \\ \pi_{23S} & \pi_{23S}-\pi_{14S} & \pi_{23A}x_{12} - 2\pi_{23S}x_{14S} & -\pi_{14A}x_{12} - 2\pi_{23S}x_{23S} \end{pmatrix}$$

$$\Big\downarrow \begin{array}{l} 2R \mapsto 2R-(2x_{14S})1R \\ 3R \mapsto 3R-(2\frac{\pi_{23A}}{x_{12}})1R \\ 4R \mapsto 4R-(2\pi_{23S})1R \end{array}$$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & x_{14S}+x_{23S} & \frac{1}{2}x_{12}^2 - 2x_{14S}^2 & -\frac{x_{12}^2}{2} - 2x_{14S}x_{23S} \\ 0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{2\pi_{23A}x_{14S}}{x_{12}} + \pi_{23S} & -\frac{2\pi_{23A}x_{23S}}{x_{12}} - \pi_{14S} \\ 0 & \pi_{23S}-\pi_{14S} & \pi_{23A}x_{12} - 2\pi_{23S}x_{14S} & -\pi_{14A}x_{12} - 2\pi_{23S}x_{23S} \end{pmatrix}$$

$$\Big\downarrow{\scriptstyle 4C\mapsto 4C+(2x_{14S})2C+3C}$$

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & x_{14S}+x_{23S} & \frac{1}{2}x_{12}^2-2x_{14S}^2 & 0 \\ 0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{2\pi_{23A}x_{14S}}{x_{12}}+\pi_{23S} & \pi_{23S}-\pi_{14S}-\frac{2\pi_{14A}x_{14S}}{x_{12}}-\frac{2\pi_{23A}x_{23S}}{x_{12}} \\ 0 & \pi_{23S}-\pi_{14S} & \pi_{23A}x_{12}-2\pi_{23S}x_{14S} & \pi_{23A}x_{12}-\pi_{14A}x_{12}-2(\pi_{14S}x_{14S}+\pi_{23S}x_{23S}) \end{pmatrix} \tag{A.1}$$

Using the definitions (4.72)we simplify the $(3,4)$ entry

$$\begin{aligned}
(3,4) =&\pi_{23S}-\pi_{14S}-\frac{2\pi_{14A}x_{14S}}{x_{12}}-\frac{2\pi_{23A}x_{23S}}{x_{12}} \\
=&\frac{1}{x_{12}}((x_1-x_2)(\pi_{23S}-\pi_{14S})-2\pi_{14A}x_{14S}-2\pi_{23A}x_{23S}) \\
=&\frac{1}{2x_{12}}((x_1-x_2)(\pi_{23}+\pi_{13}-\pi_{14}-\pi_{24})-(x_1+x_2-2x_4)(\pi_{14}-\pi_{24}) \\
&-(x_1+x_2-2x_3)(\pi_{23}-\pi_{13})) \\
=&\frac{1}{2x_{12}}(\pi_{13}(2x_1-2x_3)-\pi_{14}(2x_1-2x_4)-\pi_{23}(2x_2-2x_3)+\pi_{24}(2x_2-2x_4)) \\
=&\frac{1}{x_{12}}(x_1^{n+1}-x_3^{n+1}-x_1^{n+1}+x_4^{n+1}-x_2^{n+1}+x_3^{n+1}+x_2^{n+1}-x_4^{n+1}) \\
=&0
\end{aligned}$$

and the $(4,4)$ entry further

$$\begin{aligned}
(4,4) =&\pi_{23A}x_{12}-\pi_{14A}x_{12}-2(\pi_{14S}x_{14S}+\pi_{23S}x_{23S}) \\
=&\frac{1}{2}((x_1-x_2)(\pi_{23}-\pi_{13}-\pi_{14}+\pi_{24})-(x_1+x_2-2x_4)(\pi_{14}+\pi_{24}) \\
&-(x_1+x_2-2x_3)(\pi_{23}+\pi_{13})) \\
=&\frac{1}{2}(-\pi_{23}(2x_2-2x_3)-\pi_{13}(2x_1-2x_3)-\pi_{24}(2x_2-2x_4)-\pi_{14}(2x_1-2x_4)) \\
=&-x_2^n+x_3^n-x_1^n+x_3^n-x_2^n+x_4^n-x_1^n+x_4^n \\
=&-2(W_{12}-W_{34}).
\end{aligned}$$

Thus we continue with A.1 writing $W(x_1,x_2)\equiv W_{12}$, $W(x_3,x_4)\equiv W_{34}$

$$\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0\\
0 & x_{14S}+x_{23S} & \frac{1}{2}x_{12}^2-2x_{14S}^2 & 0\\
0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{2\pi_{23A}x_{14S}}{x_{12}}+\pi_{23S} & 0\\
0 & \pi_{23S}-\pi_{14S} & \pi_{23A}x_{12}-2\pi_{23S}x_{14S} & -2(W_{12}-W_{34})
\end{pmatrix}$$

$$\Big\downarrow\ 4R\mapsto 4R+(2x_{14S})3R-\left(\tfrac{2\pi_{23A}}{x_{12}}\right)2R$$

$$\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0\\
0 & x_{14S}+x_{23S} & \frac{1}{2}x_{12}^2-2x_{14S}^2 & 0\\
0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{2\pi_{23A}x_{14S}}{x_{12}}+\pi_{23S} & 0\\
0 & \pi_{23S}-\pi_{14S}-\frac{2\pi_{14A}x_{14S}}{x_{12}}-\frac{2\pi_{23A}x_{23S}}{x_{12}} & 0 & -2(W_{12}-W_{34})
\end{pmatrix}$$

$$\Big\downarrow\ 3S\mapsto 3S\times\tfrac{1}{2}$$

$$\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0\\
0 & x_1+x_2-x_3-x_4 & x_1x_4-x_1x_2+x_2x_4-x_4^2 & 0\\
0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{\pi_{23A}x_{14S}}{x_{12}}+\frac{1}{2}\pi_{23S} & 0\\
0 & 0 & 0 & -2(W_{12}-W_{34})
\end{pmatrix}$$

$$\Big\downarrow\ \begin{matrix}3S\mapsto 3S-(x_4)2S\\ 1R\mapsto 1R\times 2\\ 4R\mapsto 4R\times(-\frac{1}{2})\end{matrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & x_1+x_2-x_3-x_4 & -x_1x_2+x_3x_4 & 0\\
0 & \frac{\pi_{23A}-\pi_{14A}}{x_{12}} & -\frac{\pi_{23A}(x_{14S}+x_4)-\pi_{14A}x_4}{x_{12}}+\frac{1}{2}\pi_{23S} & 0\\
0 & 0 & 0 & W_{12}-W_{34}
\end{pmatrix}$$

$$\Big\downarrow\ \begin{matrix}3C\mapsto 3C\times(-1)\\ 3R\mapsto 3R\times 2\end{matrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & x_1+x_2-x_3-x_4 & x_1x_2-x_3x_4 & 0\\
0 & 2\frac{\pi_{23A}-\pi_{14A}}{x_{12}} & 2\frac{\pi_{23A}(x_{14S}+x_4)-\pi_{14A}x_4}{x_{12}}-\pi_{23S} & 0\\
0 & 0 & 0 & W_{12}-W_{34}
\end{pmatrix}$$

$$\Big\downarrow\ 3R\mapsto 3R+\tfrac{1}{x_1+x_2-x_3-x_4}\left(u_2-2\tfrac{\pi_{23A}-\pi_{14A}}{x_{12}}\right)2R$$

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & x_1+x_2-x_3-x_4 & x_1x_2-x_3x_4 & 0\\
0 & u_2 & 2\frac{\pi_{23A}(x_{14S}+x_4)-\pi_{14A}x_4}{x_{12}}-\pi_{23S}+\frac{x_1x_2-x_3x_4}{x_1+x_2-x_3-x_4}\left(u_2+2\frac{\pi_{23A}-\pi_{14A}}{x_{12}}\right) & 0\\
0 & 0 & 0 & W_{12}-W_{34}
\end{pmatrix} \tag{A.2}$$

A long calculation gives for the (3,3) entry

$$(3,3)=-\frac{W_{12}-W_{34}+u_2(x_1x_2-x_3x_4)}{x_1+x_2-x_3-x_4}=-u_1 \tag{A.3}$$

where the last equality in (A.3) can be shown by combining the definitions of u_1 and u_2 (4.55). To make sure that the last transformation of (A.2) is legitimate we need to show that $A = u_2 + 2\frac{\pi_{23A}-\pi_{14A}}{x_{12}}$ factorizes in $x_1 + x_2 - x_3 - x_4$, i.e. A=0 for $x_1 + x_2 = x_3 + x_4$.

$$
\begin{aligned}
A =& u_2 + 2\frac{\pi_{23A} - \pi_{14A}}{x_{12}} \\
=& \frac{g(x_3 + x_4, x_1 x_2) - g(x_3 + x_4, x_3 x_4)}{x_1 x_2 - x_3 x_4} + 2\frac{\pi_{23A} - \pi_{14A}}{x_{12}}
\end{aligned}
\tag{A.4}
$$

and using $x_1 + x_2 = x_3 + x_4$ in u_2 we get that

$$
A_{\text{modified}} = \frac{x_1^n + x_2^n - x_3^n - x_4^n}{x_1 + x_2 - x_3 - x_4} + 2\frac{\pi_{23A} - \pi_{14A}}{x_{12}}
\tag{A.5}
$$

this is now in a form mathematica can handle. Converting fractions of A_{modified} to a common denominator one finds that A_{modified} is proportional to $x_1 + x_2 - x_3 - x_4$, i.e.

$$
A_{\text{modified}} = (x_1 + x_2 - x_3 - x_4)\frac{B}{x_{12}x_{14}x_{24}x_{13}x_{23}(x_1 x_2 - x_3 x_4)}.
\tag{A.6}
$$

Thus $A_{|x_1+x_2=x_3+x_4} = 0$ and the last transformation applied in (A.2) is legitimate.

Thus we have shown that the isomorphism S exist. The concrete matrices for S and S^{-1} which encode the transformations are given in (3.29)