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Template iterations and maximal cofinitary groups

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The aim of this thesis is to show that the minimal size of a maximal cofinitary group, denoted $\mathfrak{a}_g$ can be of countable cofinality. For this purpose, we present in detail a forcing construction originally due to Shelah, and subsequently developed by Brendle, Fischer and Törnquist. We start with defining a forcing notion $\mathbb{Q}_{A,\rho}$, which is designed to adjoin a maximal cofinitary group of arbitrary uncountable cardinality. This forcing notion has key properties, namely it is product-like and has the strong embedding property, which make it possible to be used within two sided template construction. In the second part of the thesis, we introduce iterations along two-sided templates and show that a special instance of the construction, built with the use of the poset $\mathbb{Q}_{A,\rho}$ and the Localisation forcing notion, leads to the above stated consistency, i.e. the consistency of $\mathfrak{a}_g$ being of countable cofinality.

Das Ziel dieser These ist zu zeigen, dass die minimale Größe einer maximal kofinitären Gruppe, bezeichnet als $\mathfrak{a}_g$, von abzählbarer kofinalität sein kann. Zu diesem Zweck stellen wir detailliert ein Erzwingungskonzept vor, welches ursprünglich von Shelah stammt und später von Brendle, Fischer und Törnquist weiterentwickelt wurde. Wir beginnen mit der Definition eines Erzwingungskonzepts $\mathbb{Q}_{A,\rho}$, welches dazu dient, eine maximal kofinitäre Gruppe beliebiger überabzählbarer Kardinalität hinzuzufügen. Dieses Erzwingungskonzept weist Schlüsseleigenschaften auf, nämlich seine Produktähnlichkeit und die starke Einbettungseigenschaft, die seine Verwendung in der zweiseitigen Template-Konstruktion ermöglichen. Im zweiten Teil der Arbeit führen wir Iterationen entlang zweiseitiger Templates ein und zeigen, dass eine spezielle Instanz der Konstruktion, die unter Verwendung der Halbordnung $\mathbb{Q}_{A,\rho}$ und des Lokalisierungserzwingungskonzepts erstellt wurde, zur oben genannten Konsistenz führt, d.h. zur Konsistenz von $\mathfrak{a}_g$ als abzählbarer Kofinalität.

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Contents

1 Introduction

The goal of this chapter is to introduce the main Theorem and give an outline of the structure of this thesis.

The goal of this thesis is to show that the minimal size of a maximal cofinitary group can be of countable cofinality. We formalize this statement.

Definition 1.1. Let S_∞ be the group of all permutations of ω. A cofinitary group G is a subgroup of S_∞ s.t. each non-identity element has finitely many fixed points.

Definition 1.2. A cofinitary group G is called maximal if it is not contained in a strictly larger cofinitary group. We denote the minimal size of a maximal cofinitary group by $\mathfrak{a}_g$

Cofinitary groups were of interest to algebraists in the 80's who could show that if a cofinitary group is countable, then it is not maximal and that there is always a cofinitary group of cardinality $\mathfrak{c}$. So, they proved $\aleph_1 \leq \mathfrak{a}_g \leq \mathfrak{c}$. The consistency of $\aleph_1 \leq \mathfrak{a}_g = \kappa < \mathfrak{c}$ for κ regular uncountable was obtained via finite support iteration forcing in [BSZ00]. However, the cofinality of $\mathfrak{a}_g$ being of countable cofinality remained open and was solved in [FT13] with the use of the advanced forcing method of template iteration. In this thesis we give a detailed presentation of the following:

Theorem 1.3. *Assume* CH. *Let λ be a singular cardinal of countable cofinality. Then there exists a ccc generic extension in which $\mathfrak{a}_g = \lambda$.*

In chapter 2 we present auxiliary statements and definitions from [Kun11] and [Kun14] such as the forcing notion $\mathbb{L}$.

From chapter 3 onwards we will closely follow [FT13].

In chapter 3 we introduce the forcing notion $\mathbb{Q}_{A,\rho}$, which encode permutations and their finite approximations, and show its key properties, with respect to this thesis, in particular that $\mathbb{Q}_{A,\rho}$ allows us to adjoins a maximal cofinitary group of arbitrary uncountable cardinality.

In chapter 4 we introduce the forcing notion $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$, which allows the iteration along the two-sided template $\mathcal{T}$ of arbitrary representations of the poset classes $\mathbb{Q}$ and $\mathbb{S}$ respectively. We show that the forcing notions $\mathbb{Q}_{A,\rho}$ and $\mathbb{L}$, introduced in the previous chapters, satisfy the definitions of the respective classes $\mathbb{Q}$ and $\mathbb{S}$. Moreover, we establish key properties of $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

In chapter 5 we show two estimates of $\mathfrak{a}_g$ obtained by plugging $\mathbb{Q}_{A,\rho}$ and $\mathbb{L}$ into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$. We show that those estimates depend on the two-sided template $\mathcal{T}$ used to build $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

In chapter 6 we define a specific two-sided template $\mathcal{T}^*$ and show that $\mathbb{P}(\mathcal{T}^*, \mathbb{Q}_{A,\rho}, \mathbb{L})$ forces Theorem 1.3.

2 Preliminaries

The goal of this chapter is to introduce definitions and results, relevant for the proof of Theorem 1.3, which we will not cover in detail. For a detailed discussion, see [Kun11] and [Kun14].

2.1 Set Theory

We will use the notation from [Kun11]. We also assume familiarity with [Kun11] chapter I "Background Material" and the general forcing idea as discussed in [Kun11] chapter IV.2 "Generic Extensions".

2.1.1 Delta System Lemma and Knaster Property

Definition 2.1. A family of sets $\mathcal{A}$ forms a delta system with root R iff $X \cap Y = R$ whenever $X, Y \in \mathcal{A}$ with $X \neq Y$.

Lemma 2.2 ([Kun11] III.6.15 (Delta System Lemma))**.** *Let λ and κ be regular cardinals s.t. $\omega \leq \lambda < \kappa$. Assume that $\forall \theta < \kappa[\theta^{<\lambda} < \kappa]$. Let $\mathcal{A}$ be a family of sets with $|\mathcal{A}| = \kappa$, s.t. $|A| < \lambda$ for all $A \in \mathcal{A}$. Then there is a $\mathcal{B} \in [\mathcal{A}]^\kappa$ s.t. $\mathcal{B}$ forms a delta system.*

Definition 2.3. Let $\mathbb{P}$ be a poset. $L \subseteq \mathbb{P}$ is linked iff for all $p_1, p_2 \in L$, $p_1 \not\perp p_2$.

Definition 2.4. Let $\mathbb{P}$ be a poset. $\mathbb{P}$ has the Knaster property iff for all $E \in [\mathbb{P}]^{\aleph_1}$, there is a linked $L \in [E]^{\aleph_1}$.

2.1.2 Dense open and dense below

Definition 2.5. Let $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{1}_\mathbb{P})$ be a forcing notion with $s \in \mathbb{P}$. Then $s\downarrow$ denotes the forcing notion $(\mathbb{S}, \leq_\mathbb{S}, \mathbb{1}_\mathbb{S})$, where $\mathbb{S} := \{p \in \mathbb{P} : p \leq_\mathbb{P} s\}$, $\mathbb{1}_\mathbb{S} = s$ and $\leq_\mathbb{S}$ is $\leq_\mathbb{P}$ restricted to $\mathbb{S}$.

Definition 2.6. Let $\mathbb{P}$ be a forcing notion and $S \subseteq \mathbb{P}$. Define $S\downarrow := \bigcup\{p\downarrow : p \in S\}$ as the open hull of S.

Reducing $G \cap D \neq \emptyset$ down to $G \cap D\downarrow \neq \emptyset$ for a filter G and $S \subseteq \mathbb{P}$, where $\mathbb{P}$ is a forcing notion, will be useful to show that a filter is generic.

Lemma 2.7. *Let $\mathbb{P}$ be a forcing notion, G a filter on $\mathbb{P}$ and $S \subseteq \mathbb{P}$. Then $S \cap G \neq \emptyset$ iff $S\downarrow \cap G \neq \emptyset$.*

Proof. $\underline{\Rightarrow}$: Follows from $S \subseteq S\downarrow$.
$\underline{\Leftarrow}$: Let $p \in S\downarrow \cap G$. Since $p \in S\downarrow$, there exists a $p' \in S$ s.t. $p' \leq p$. Since G is a filter $p' \in G$. So $p' \in S \cap G$. □

Definition 2.8. Let $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ be a poset $E \subseteq \mathbb{P}$, $p \in \mathbb{P}$. Then E is dense below p iff

$$\forall q \leq p(\exists r \leq q(r \in E)).$$

A subset of a poset meeting a filter, can be reduced to the subset being dense below an element of the filter.

Lemma 2.9. *Let M be a ctm of ZFC, $\mathbb{P} \in M$, $E \subseteq \mathbb{P}$ and $E \in M$. Let G be $\mathbb{P}$-generic over M. Then*

1. *$G \cap E \neq \emptyset$ or $\exists q \in G(\forall r \in E(r \perp q))$.*
2. *If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.*

Proof.

1. Let
$$D = \{p : \exists r \in E(p \leq r)\} \cup \{q : \forall r \in E(r \perp q)\}.$$
Let $q \in \mathbb{P} \setminus D$. Since $q \notin D$, there exists an $r \in E$ s.t. $r \not\perp q$. Let p be the common extension of r and q. Since $r \in E$, $p \in D$. So D is dense in $\mathbb{P}$.
2. Assume $G \cap E = \emptyset$. By Lemma 2.9, there exists a $q \in G$ s.t. $\forall r \in E(r \perp q)$. Since G is a filter, there exists a $q' \in G$ s.t. $q' \leq q$ and $q' \leq p$. Since E is dense below E, there exists an $r \in E$ s.t. $r \leq q'$. Then $r \leq q$. Contradiction to $r \perp q$.

□

2.1.3 Complete Embeddings

The notion of complete embedding captures that certain parts of the structure of a poset $\mathbb{P}$ are contained in the poset $\mathbb{Q}$.

Definition 2.10. Let $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ and $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ be posets and $i : \mathbb{P} \to \mathbb{Q}$. Then i is a complete embedding iff

1. $\forall p_1, p_2 \in \mathbb{P}[p_1 \leq_{\mathbb{P}} p_2 \to i(p_1) \leq_{\mathbb{Q}} i(p_2)]$.
2. $\forall p_1, p_2 \in \mathbb{P}[p_1 \perp_{\mathbb{P}} p_2 \to i(p_1) \perp_{\mathbb{Q}} i(p_2)]$.
3. For all $A \subseteq \mathbb{P}$: If A is a maximal antichain in $\mathbb{P}$, then $i(A)$ is a maximal antichain in $\mathbb{Q}$.

We will write $\mathbb{P} \lessdot \mathbb{Q}$ if a complete embedding $i : \mathbb{P} \to \mathbb{Q}$ exists.

We will abbreviate $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ as $\mathbb{Q}$ whenever it is clear from the context what we are referring to.

We define the notion of a reduction to describe that given certain structural compatibilities for posets $\mathbb{Q}$ and $\mathbb{P}$, compatibility with a $q \in \mathbb{Q}$ might be reduced to extending an $r \in \mathbb{P}$.

Definition 2.11. Let $\mathbb{P}, \mathbb{Q}, i$ satisfy (1),(2) from Definition 2.10, $q \in \mathbb{Q}$. Then $r \in \mathbb{P}$ is a reduction of q to $\mathbb{P}$ if for all $p \in \mathbb{P}$ s.t. $p \leq_{\mathbb{P}} r$, $i(p) \not\perp_{\mathbb{Q}} q$.

Complete embeddings can be characterized by reductions.

Lemma 2.12. *Let $\mathbb{Q}, \mathbb{P}, i$ satisfy (1),(2) from Definition 2.10. Then i is a complete embedding iff every $q \in \mathbb{Q}$ has a reduction to $\mathbb{P}$.*

Proof. $\underline{(\Rightarrow)}$: Assume there exists a $q \in \mathbb{Q}$ s.t. for all $r \in \mathbb{P}$ there is a $p \in \mathbb{P}$ s.t. $p \leq_{\mathbb{P}} r$ and $i(p) \perp_{\mathbb{Q}} q$. Then $A := \{p \in \mathbb{P} : i(p) \perp_{\mathbb{Q}} q\}$ is dense in $\mathbb{P}$. By Zorn's lemma, there exists a maximal antichain $B \subseteq A$ in $\mathbb{P}$. Since $i(p) \perp_{\mathbb{Q}} q$ for all $p \in A$ and $q \notin i(B)$, $i(B)$ is not maximal in $\mathbb{Q}$.

$\underline{(\Leftarrow)}$: Let $A \subseteq \mathbb{P}$ be a maximal antichain in $\mathbb{P}$. Assume $i(A)$ is not maximal in $\mathbb{Q}$. Then there exists a $q \in \mathbb{Q}$ s.t. $i(p) \perp_{\mathbb{Q}} q$ for all $p \in A$. By assumption there exists a reduction r of q to $\mathbb{P}$. So for all $p \in A$, $p \not\leq_{\mathbb{P}} r$. Assume there exists a common extension $t \in \mathbb{P}$ of r and some $p \in A$. Then $t \leq_{\mathbb{P}} r$. So there exists a common extension $q_1 \in \mathbb{Q}$ of $i(t)$ and q. □

Being a reduction is inherited by extension.

Lemma 2.13. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets s.t. $\mathbb{P} \lessdot \mathbb{Q}$, $q \in \mathbb{Q}, p \in \mathbb{P}$ and $q \leq_{\mathbb{Q}} p$.*

1. *Any reduction of q to $\mathbb{P}$ is compatible in $\mathbb{P}$ with p.*
2. *q has a reduction extending p.*

Proof.

1. Let $r \in \mathbb{P}$ be a reduction of q. Assume $r \perp_{\mathbb{P}} p$. Let $x \leq_{\mathbb{P}} r$. Since r is a reduction of q, $x \not\perp_{\mathbb{Q}} q$. Let $x' \in \mathbb{Q}$ be their common extension. Then $x' \leq_{\mathbb{Q}} x \leq_{\mathbb{P}} r$. Since $\mathbb{P} \lessdot \mathbb{Q}$, $x' \leq_{\mathbb{Q}} r$. Since $x' \leq_{\mathbb{Q}} q \leq_{\mathbb{Q}} p$, $x' \leq_{\mathbb{Q}} p$. So $r \not\perp_{\mathbb{Q}} p$. Since $\mathbb{P} \lessdot \mathbb{Q}$, $r \perp_{\mathbb{Q}} p$. Contradiction. So $r \not\perp_{\mathbb{P}} p$.
2. Let r be a reduction of q to $\mathbb{P}$. By (1), $r \not\perp_{\mathbb{P}} p$. Let r_0 be their common extension. Since $r_0 \leq_{\mathbb{P}} r$ and any extension of a reduction is a reduction, r_0 is a reduction of q extending p.

□

We define the notion of a nice name to capture the notion of an increasingly precise approximation for a real. For this purpose we need countably many maximal antichains containing codes for the approximations. Then we order those antichains in a manner s.t. being a more precise approximation can be reduced down to compatibility of codes.

Definition 2.14. Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. For each $n \geq 1$ let $\mathfrak{B}_n$ be a maximal antichain below y. We will say that the set $\{(b, s(b))\}_{b \in \mathfrak{B}_n, n \geq 1}$ is a nice name for a real below y if

1. whenever $n \geq 1, b \in \mathfrak{B}_n$ then $s(b) \in {}^n\omega$,

2. whenever $m > n \geq 1, b \in \mathfrak{B}_n, b' \in \mathfrak{B}_m$ and b, b' are compatible, then $s(b)$ is an initial segment of $s(b')$.

The approximations given by nice names can be encoded in a reduction to a suborder.

Lemma 2.15. *Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a reduction of y to $\mathbb{A}$. Let $\dot{f} = \{(b, s(b))\}_{b\in\mathfrak{B}_n, n\geq 1}$ be a $\mathbb{B}$-nice name for a real below y. Then there is $\dot{g} = \{(a, s(a))\}_{a\in\mathfrak{A}_n, n\geq 1}$, an $\mathbb{A}$-nice name for a real below x, s.t. for all $n \geq 1$, for all $a \in \mathfrak{A}_n$, there is $b \in \mathfrak{B}_n$ s.t. a is a reduction of b and $s(a) = s(b)$.*

Proof. Construct the antichains $\mathfrak{A}_n$ recursively:
Construction of $\mathfrak{A}_1$: Let $t \in \mathbb{A}$ be an extension of x. Since x is a reduction of y, there exists $\hat{t} \in \mathbb{B}$ s.t. $\hat{t} \leq_{\mathbb{B}} t, y$. So there exists $\bar{t}, b \in \mathfrak{B}_1$ s.t. $\bar{t} \leq_{\mathbb{B}} \hat{t}, b$. By Lemma 2.13, there exists an $a \in \mathbb{A}$ s.t. a is a reduction of $\bar{t}$ and $a \leq_{\mathbb{A}} t$. Since $\bar{t} \leq_{\mathbb{B}} b$, a is a reduction of b. Let $s(a) := s(b), a(t) = a$. By Zorn's lemma there exists a maximal antichain $\mathfrak{A}_1$ in the dense below x set $\{a(t) : t \leq x\}$.

Construction of $\mathfrak{A}_{n+1}$ for $n \geq 1$: Assume $\mathfrak{A}_n$ has been defined. Let $a \in \mathfrak{A}_n$ and $t \leq_{\mathbb{A}} a$. By assumption, there exists a $b \in \mathfrak{B}_n$ s.t. a is a reduction of b and $s(a) = s(b)$. So there exists a $\hat{t} \in \mathbb{B}$ s.t. $\hat{t} \leq_{\mathbb{B}} b, t$. So $\hat{t} \leq_{\mathbb{B}} y$. So there exists a $\bar{t} \in \mathfrak{B}_{n+1}$ s.t. $\bar{t} \leq_{\mathbb{B}} \hat{t}, \bar{b}$. Since $\{(b, s(b))\}_{\mathfrak{B}_n, n\geq 1}$ is a nice name, $s(b)$ is an initial segment of $s(\bar{b})$. Since $\bar{t} \leq_{\mathbb{B}} t$, Lemma 2.13 implies that there exists a $\bar{a}$ s.t. $\bar{a}$ is a reduction of $\bar{t}$ and $\bar{a} \leq_{\mathbb{A}} t$. Let $\bar{a}(t) := \bar{a}$, $s(\bar{a}) := s(\bar{b})$. Since $\bar{t} \leq_{\mathbb{B}} \bar{b}$, $\bar{a}$ is a reduction of $\bar{b}$. By Zorn's lemma there exists a maximal antichain $\mathfrak{A}_{n+1,a}$ in the dense below a set $\{\bar{a}(t) : t \leq_{\mathbb{A}} a\}$. Let $\mathfrak{A}_{n+1} := \bigcup_{a\in\mathfrak{A}_n} \mathfrak{A}_{n+1,a}$. □

2.1.4 Two-Step Iteration

Since the construction of $M[G]$ as discussed above yields that $M[G]$ is a ctm, we can iterate this process to produce a ctm $M[G][H]$ satisfying ZFC for a suitable filter H.

Let $\mathbb{P}, \mathbb{Q}$ be forcing notions, s.t. $\mathbb{P} \in M$, G is $\mathbb{P}$-generic over M, $\mathbb{Q} \in M[G]$ and H is $\mathbb{Q}$-generic over $M[G]$. Assuming that we want a model of ZFC satisfying statement φ, which is attained by forcing in $M[G]$, and statement ψ, which is attained by forcing in $M[G][H]$, forcing iteratively in two steps might yield an $M[G][H]$ not satisfying statement φ. To circumvent this we define a method, that allows us to reduce both steps to one.

For this purpose, we define a notion for tangibility of a forcing notion with respect to another forcing notion.

Definition 2.16. If $\mathbb{P}$ is a forcing notion, then a $\mathbb{P}$-name for a forcing notion is a triple of $\mathbb{P}$-names, $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$, s.t. $\dot{\mathbb{1}}_{\mathbb{Q}} \in \mathrm{dom}(\dot{\mathbb{Q}})$ and

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} [\dot{\mathbb{1}}_{\mathbb{Q}} \in \dot{\mathbb{Q}} \wedge \dot{\leq}_{\mathbb{Q}} \text{ is a pre-order of } \dot{\mathbb{Q}} \text{ with largest element } \dot{\mathbb{1}}_{\mathbb{Q}}].$$

We will abbreviate $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ as $\dot{\mathbb{Q}}$, $\dot{\leq}_{\mathbb{Q}}$ as $\dot{\leq}$ and $\dot{\mathbb{1}}_{\mathbb{Q}}$ as $\dot{\mathbb{1}}$ whenever it is clear from the context what we're referring to.

We define the forcing notion used for the two-step iteration in a manner that is compatible with the structure of both underlying forcing notions.

Definition 2.17. Let $\mathbb{P}$ be a forcing notion and $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ a $\mathbb{P}$-name for a forcing notion. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is the triple $(\mathbb{R}, \leq, \mathbb{1})$, where

1. $\mathbb{R} = \{(p, \dot{q}) \in \mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}}) : p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$,
2. $\mathbb{1} = (\mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\mathbb{Q}})$,
3. $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ iff $p_1 \leq_{\mathbb{P}} p_2$ and $p_1 \Vdash [\dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2]$.

We will often write $(p, \dot{q}) \in \mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}})$ for $(p, \dot{q}) \in \mathbb{R}$.
Define $i : \mathbb{P} \to \mathbb{P} \times \dot{\mathbb{Q}}$ by $i(p) = (p, \mathbb{1}_{\mathbb{Q}})$.

This forcing notion is well-defined and well behaved.

Lemma 2.18 ([Kun11] V.3.4). *Using the notation from above with $p_0, p_1 \in \mathbb{P}$ and $\dot{q}_1, \dot{q}_2 \in dom(\dot{\mathbb{Q}})$:*

1. *$\mathbb{P} * \dot{\mathbb{Q}}$ is a forcing notion.*
2. *$p_0 \leq p_1 \leftrightarrow i(p_0) \leq i(p_1)$.*
3. *$i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{P} * \dot{\mathbb{Q}}}$.*
4. *$p_0 \perp p_1 \to (p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$ whenever $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}$.*
5. *$p_0 \perp p_1 \leftrightarrow (p_0, \dot{\mathbb{1}}) \perp (p_1, \dot{q}_1)$ whenever $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}$.*
6. *$p_0 \perp p_1 \leftrightarrow i(p_0) \perp i(p_1)$.*
7. *i is a complete embedding.*

2.2 The forcing notion $\mathbb{L}$

One of the forcing notions we will plug into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is the localization $\mathbb{L}$.

Definition 2.19. The forcing notion $\mathbb{L}$ consists of pairs (σ, ϕ) s.t. $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$, $\phi \in {}^{\omega}({}^{<\omega}[\omega])$ s.t. $\sigma \subseteq \phi, \forall i < |\sigma|(|\sigma(i)| = i)$ and for all $i \in \omega(|\phi(i)| \leq |\sigma|)$. The extension relation is defined as follows: $(\sigma, \phi) \leq (\tau, \psi)$ iff σ end-extends τ and for all $i \in \omega(\psi(i) \subseteq \phi(i))$.

Lemma 2.20. *The poset $\mathbb{L}$ adds a slalom which localizes all ground model reals.*

Proof. Since the ground model V is countable, we can enumerate all reals in V. Let $(f_n)_{n\in\omega}$ be an enumeration of all reals in V. Then $(\sigma, \phi) \in \mathbb{L}$, where

$$\sigma(0) = \emptyset = \phi(0)$$
$$\text{and } \sigma(n+1) = \{f(i)\}_{i \leq n} = \phi(n+1).$$

Let $f_n, n \in \omega$ be a real in V. Then for $m \geq n+1$, $f(m) \in \phi(m)$. □

We define a nice name for an element of $\mathbb{L}$ similar to a nice name for a real.

Definition 2.21. Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. Let $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$ be s.t. $\forall i < |\sigma|(|\sigma(i)| = i)$, and for each $n \geq 1$ let $\mathfrak{B}_n$ be the maximal antichain below y. We will say that the pair $(\check{\sigma}, \dot{\phi})$ is a nice name for an element of $\mathbb{L}$ below y, where $\dot{\phi} = \{\langle b, \sigma(b)\rangle\}_{b\in\mathfrak{B}_n, n\geq 1}$, if the following conditions hold:

1. Whenever $n \geq 1$ and $b \in \mathfrak{B}_n$, then $\sigma(b) \in {}^{n}({}^{<\omega}[\omega])$.
2. Whenever $1 \leq n \leq |\sigma|$ and $b \in \mathfrak{B}_n$, then $\sigma(b) = \sigma \restriction n$.
3. Whenever $n > |\sigma|$, then $\sigma \subsetneq \sigma(b)$ and $\forall i : |\sigma| \leq i < n(|\sigma(b)(i)| \leq |\sigma|)$.
4. Whenever $m > n \geq |\sigma|$, $b \in \mathfrak{B}_n$, $b' \in \mathfrak{B}_m$ and b, b' are compatible, then $\sigma(b)$ is an initial segment of $\sigma(b')$.

If $(\check{\sigma}, \dot{\phi})$ where $\dot{\phi} = \{b, \sigma(b)\}_{b\in\mathfrak{B}_n, n\geq 1}$ is a nice name for an element of $\mathbb{L}$ below y, then $y \Vdash (\check{\sigma}, \dot{\phi}) \in \mathbb{L}$ and for all $n \in \omega$, $b \in \mathfrak{B}_n$ it holds that $b \Vdash \dot{\phi} \restriction n = s(\check{b})$.

Similar to Lemma 2.15:

Lemma 2.22. *Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a projection of y to $\mathbb{A}$. Let $(\check{\sigma}, \dot{\phi})$, where $\dot{\phi} = \{(b, \sigma(b))\}_{b\in\mathfrak{B}_n, n\geq 1}$, be a nice name for an element of $\mathbb{L}$ below y. Then there is an $\mathbb{A}$-nice name $(\check{\sigma}, \dot{\psi})$, where $\dot{\psi} = \{(a, \sigma(a))\}_{a\in\mathfrak{A}_n, n\geq 1}$, for an element in $\mathbb{L}$ below x s.t. for all $n \geq 1$, for all $a \in \mathfrak{A}_n$, there exists a $b \in \mathfrak{B}_n$ s.t. a is a reduction of b and $\sigma(a) = \sigma(b)$.*

Proof. Analogously to Lemma 2.15. □

2.3 Infinitary Combinatorics

We will use some cardinal estimates.

Definition 2.23. Let A be a non-empty set. Then an ideal $\mathcal{I}$ on A is a subset $\mathcal{I} \subseteq \mathcal{P}(A)$ s.t.

1. $\emptyset \in \mathcal{I}$ and $A \notin \mathcal{I}$,
2. $\forall X, Y \in \mathcal{I}[X \cup Y \in \mathcal{I}]$,
3. $\forall X, Y \subseteq A[X \subseteq Y \wedge Y \in \mathcal{I} \rightarrow X \in \mathcal{I}]$.

Definition 2.24. $\mathcal{N}$ is the ideal of Lebesque null subsets of $\mathbb{R}$.
$\mathcal{M}$ is the ideal of first category subsets of $\mathbb{R}$.

Definition 2.25. Let $\mathcal{I}$ be an ideal on A s.t. $[A]^{<\omega} \subseteq \mathcal{I}$. Then

1. $\mathrm{add}(\mathcal{I})$ is the least cardinal κ s.t. $\bigcup \xi \notin \mathcal{I}$ for some $\xi \in [\mathcal{I}]^{\kappa}$,
2. $\mathrm{non}(\mathcal{I})$ is the least cardinal κ s.t. $X \notin \mathcal{I}$ for some $X \in [A]^{\kappa}$,

3. $\mathrm{cof}(\mathcal{I})$ is the least cardinal κ s.t. $(\forall B \in \mathcal{I})(\exists B \in [\mathcal{I}]^{\kappa})(B \subseteq Y)$.

We will assume Cichoń's diagram from [BJ95][Chapter 2]:

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

In this diagram, $x \to y$ should be read as $x \leq y$.

Since we will only use

$$\mathrm{add}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N}),$$

the other numbers will not be defined here.

Theorem 2.26. *[BJ95]*

1. $\mathrm{add}(\mathcal{N})$ *is the least cardinality of a family* $F \subseteq \omega^\omega$ *s.t. no slalom localizes all members of* F*.*

2. $\mathrm{cof}(\mathcal{N})$ *is the least cardinality of a family* Φ *of slaloms s.t. every member of* ω^ω *is localized by some* $\phi \in \Phi$*.*

Theorem 2.27. *[BSZ00]* $\mathfrak{a}_g \geq \mathrm{non}(\mathcal{M})$*.*

3 The Forcing Notion $\mathbb{Q}_{A,\rho}$

The goal of this chapter is to integrate cofinite groups into a set theoretic framework in a manner that allows us to add a maximal cofinitary group of size λ, where $\lambda < \mathfrak{c}$.

We will often refer to finite approximations of permutations. Since the term finite approximations of permutations is just an auxiliary tool used to understand the underlying mechanisms of the text below and will never be used in a formal manner, it will not be formally defined.
Informally: Let $\sigma : \mathrm{dom}(\sigma) \to \mathrm{ran}(\sigma)$ be a permutation. Then σ' is a finite approximation to σ if $\mathrm{dom}(\sigma') \subseteq \mathrm{dom}(\sigma)$, $|\mathrm{dom}(\sigma')| < \aleph_0$ and $\forall n \in \mathrm{dom}(\sigma')(\sigma'(n) = \sigma(n))$.

The outline for the rest of the chapter:

1. Encode elements of the graph of permutations in S_∞ and finite approximations thereof into sets,

2. process the encoded information by adding the smallest amount of encoded information necessary to construe a permutation in S_∞,

3. decode the information to attain a maximal cofinitary group.

Definition 3.1. 1. Let A be a set. We denote by W_A the set of reduced words in the alphabet $\langle a^i : a \in A, i \in \{-1, 1\}\rangle$. The free group on generator set A is the group $\mathbb{F}_A$ we obtain by giving W_A the concatenate-and-reduce operation. When $A = \emptyset$ then $\mathbb{F}_A$ is by definition the trivial group. A can be naturally identified with a subset of $\mathbb{F}_A$, which generates $\mathbb{F}_A$. Every function $\rho : B \to G$, where G is any group, extends to a group homomorpism $\bar{\rho} : \mathbb{F}_A \to G$.

2. We denote by $\widehat{W_A}$ the set of all $w \in W_A$ s.t. either $w = a^n$ for some $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$, or w starts and ends with a different letter. In the latter case, this means that there is $u \in W_A, a, b \in A, a \neq b$ and $i, j \in \{-1, 1\}$ s.t. $w = a^i u b^j$ without cancelation. Note that any word $w \in W_A$ can be written as $w = u^{-1} w' u$ for some $w' \in \widehat{W_A}$ and $u \in W_A$.

3. For a partial function $f : \omega \to \omega$, let
$$\mathrm{fix}(f) = \{n \in \omega : f(n) = n\}.$$
We denote by $\mathrm{cofin}(S_\infty)$ the set of cofinitary permutations in S_∞, i.e. permutations $\sigma \in S_\infty$ s.t. $\mathrm{fix}(\sigma)$ is finite.

4. For a group G, a cofinitary representation of G is a homomorphism $\varphi : G \to S_\infty$ s.t. $\mathrm{im}(\varphi) \subseteq \{I\} \cup \mathrm{cofin}(S_\infty)$. If B is a set and $\rho : B \to S_\infty$ we say that ρ induces a cofinitary representation of $\mathbb{F}_B$ if the canonical extension of ρ to a homomorphism $\bar{\rho} : \mathbb{F}_B \to S_\infty$ is a cofinitary representation of $\mathbb{F}_B$.

5. To encode a $\sigma \in S_\infty$ via a set A we allocate elements of the corresponding graph to an element of the chosen set. We will refer to an $(n, m) \in \sigma$ as a path

(from n to m). So in the composition $f \circ g$ of $f, g \in S_\infty$ an element $n \in \mathbb{N}$ travels along a path $n \to g(n) \to f(g(n))$ to its destination $f(g((n)) \in \mathbb{N}$. The uncertainty of the term path, since a path might lead to a dead end, is chosen on purpose to reiterate that in the composition $\sigma \in S_\infty$ with a finite approximation f to $\sigma' \in S_\infty$ there will be an $n \in \mathbb{N}$ s.t. there is a path from n to $\sigma(n)$ but no path from $\sigma(n)$ to any $m \in \mathbb{N}$.

Let A be a set and let $s \subseteq A \times \omega \times \omega$. For $a \in A$, let

$$s_a = \{(n, m) \in \omega \times \omega : (a, n, m) \in s\}.$$

The set s_a decodes the paths encoded to $a \in A$.

Let $\sigma, \sigma' \in S_\infty$. If the path from n to $\sigma(n)$ is encoded in $a \in A$, the path from $\sigma(n)$ to $\sigma'(\sigma(n))$ is encoded in $a' \in A$ and the decoding process is capable of decoding finitely many letters in order, then the path from n to $\sigma'(\sigma(n))$ is encoded in $a'a$ (read from right to left). We will say that any set $s \supseteq \{(a, n, \sigma(n)), (a', \sigma(n), \sigma'(\sigma(n)))\}$ is sufficient to describe the path from n to $\sigma'(\sigma(n))$ along the word $w = a'a$. So with respect to any process capable of decoding finitely many words in order, the composition of permutations in S_∞ can be described by words with letters from the set A, which was used to encode the permutations.

Let $\sigma, \sigma' \in S_\infty$. If the path from n to $\sigma(n)$ is encoded in $a \in A$ but the path from $\sigma(n)$ to $\sigma'(\sigma(n))$ is not encoded in $a' \in A$, then the word $w = a'a$ is not sufficient to describe the path from n to $\sigma'(\sigma(n))$. To check if a set $s \subseteq A \times \omega \times \omega$ is sufficient to describe if a particular path is in the graph of a composition of permutations along a particular word w we define the decoding relation $e_w[s]$.

For a word $w \in W_A$, define the relation $e_w[s] \subseteq \omega \times \omega$ recursively by stipulating that for $a \in A$, if $w = a$ than $(n, m) \in e_w[s]$ iff $(n, m) \in s_a$, if $w = a^{-1}$ then $(n, m) \in e_w[s]$ iff $(m, n) \in s_a$, and if $w = a^i u$ for some word $u \in W_A$ and $i \in \{-1, 1\}$ without cancelation, then

$$(n, m) \in e_w[s] \Leftrightarrow (\exists k) e_{a^i}[s](k, m) \wedge e_u[s](n, k).$$

If s_a is a partial injection defined on a subset of ω for all $a \in A$, then $e_w[s]$ is a partial injection defined on some subset of ω, and we call $e_w[s]$ the evaluation of w given s. By definition, let $e_\emptyset[s, \rho]$ be the identity in S_∞.

6. By encoding an additional path into an element a of A, a word w containing a might be sufficient to describe a new path along a word w containing a. So it is useful to define a notation, which can express if a natural number n is in the domain of the permutations along w.

   If $s \subseteq A \times \omega \times \omega$ is s.t. s_a is always a partial injection, and $w \in W_A$, then we will write $e_w[s](n) \downarrow$ when $n \in \mathrm{dom}(e_w[s])$, and $e_w[s](n) \uparrow$ when $n \notin \mathrm{dom}(e_w[s])$.

7. We will work with a set A encoding paths in finite approximations to permutations in S_∞ and a set B encoding paths in permutations in S_∞.

   Let A and B be disjoint sets and let $\rho : B \to S_\infty$ be a function. For a word $w \in W_{A\cup B}$ and $s \subseteq A \times \omega \times \omega$, we define

$$(n, m) \in e_w[s, \rho] \Leftrightarrow (n, m) \in e_w[s \cup \{(b, k, l) : \rho(b)(k) = l\}].$$

   If s_a is a partial injection for all $a \in A$, then $e_w[s, \rho]$ is also a partial injection, and we call it the evaluation of w given s and ρ. The notations $e_w[s, \rho] \downarrow$ and $e_w[s, \rho] \uparrow$ are defined as above.

The evaluation function is capable of decoding the concatenation of words.

Lemma 3.2. *Fix sets A and B s.t. $A \cap B = \emptyset$, and a function $\rho : B \to S_\infty$. Let $w \in W_{A\cup B}$ and $s \subseteq A \times \omega \times \omega$ s.t. s_a is a partial injection for all $a \in A$. Assume $w = uv$ without cancellation for some $u, v \in W_{A\cup B}$. Then*

1. *$n \in \mathrm{dom}(e_w[s, \rho])$ iff $n \in \mathrm{dom}(e_v[s, \rho])$ and $e_v[s, \rho](n) \in \mathrm{dom}(e_u[s, \rho])$.*
2. *If moreover $w \in W_{A\cup B}$ then $n \in \mathrm{fix}(e_w[s, \rho])$ iff $e_v[s, \rho](n) \in \mathrm{fix}(e_{vu}[s, \rho])$. In particular $|\mathrm{fix}(e_w[s, \rho])| = |\mathrm{fix}(e_{vu}[s, \rho])|$.*

Proof. 1. Follows from iterative application of Definition 3.1(5).

2. Let $m := e_v[s, \rho](n)$.
   $\underline{(\Rightarrow)}$: Since $n \in \mathrm{fix}(e_w[s, \rho])$, $e_{vu}[s, \rho](m) = e_v[s, \rho](n) = m$.
   $\underline{(\Leftarrow)}$: Assume $n \notin \mathrm{fix}(e_w[s, \rho])$. Let $n \neq k \in \omega$ be s.t. $e_w[s, \rho](n) = k$. By injectivity of $e_v[s, \rho]$, $e_v[s, \rho](n) \neq e_v[s, \rho](k)$. So

$$e_{vu}[s, \rho](m) = e_v[s, \rho](k) \neq e_v[s, \rho](n).$$

   Contradiction.

□

Our next goal is construe a cofinitary representation of $A \cup B$, where A encodes finite approximations to permutations in S_∞ and B encodes permutations in S_∞. For this, we extend ρ to a function $\rho' : A \cup B \to S_\infty$ inducing a cofinitary representation.
The first subgoal is to encode enough paths into A s.t. we can find a $\sigma \in S_\infty$ for any $a \in A$.

If for two sets of encoded paths $s, t \subseteq A \times \omega \times \omega$, $t \subseteq s$ holds true, then $t_a \subseteq s_a$ for any $a \in A$. So t might be sufficient to describe the path of a fixed point along w, that s is not able to describe along w. Since we want to ensure that adding information will yield an encoded permutation with finitely many fixed points, defining a notion for adding information without adding a new fixed points is useful.

Definition 3.3. Let A and B be sets s.t. $A \cap B = \emptyset$ and a function $\rho : B \to S_\infty$ s.t. ρ induces a cofinitary representation $\bar{\rho} : \mathbb{F}_B \to S_\infty$. We define the forcing notion $\mathbb{Q}_{A,\rho}$ as follows:

1. Conditions of $\mathbb{Q}_{A,\rho}$ are pairs (s, F) where $s \subseteq A \times \omega \times \omega$ is finite and s_a is a finite injection for every $a \in A$, and $F \subseteq \widehat{W}_{A \cup B}$.
2. $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ iff $s \supseteq t$, $F \supseteq E$ and for all $n \in \omega, w \in E$, if $e_w[s, \rho](n) = n$ then already $e_w[t, \rho](n) \downarrow$ and $e_w[t, \rho](n) = n$.

   So extension necessitates that for every fixed point n, that can be described by s and ρ along some word w in E, can be described by t and ρ along the same word.

If $B = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$.

We will abbreviate $\leq_{\mathbb{Q}_{A,\rho}}$ as $\leq$ whenever it is clear from the context what we're referring to.

From here on we assume $A \cap B = \emptyset$, $A \neq \emptyset$ and that $\rho : B \to S_\infty$ induces a cofinitary representation of $\mathbb{F}_B$.

Lemma 3.4. *$\mathbb{Q}_{A,\rho}$ has the Knaster property.*

Proof. For $w \in W_{A \cup B}$, let $\mathrm{oc}(w)$ be the finite set of letters occuring in w. For $F \subseteq W_{A \cup B}$ let $\mathrm{oc}(F) = \bigcup_{w \in F} \mathrm{oc}(w)$. For $C \subseteq A \cup B$ and w, F as above, let $\mathrm{oc}_C(w) = \mathrm{oc}(w) \cap C$ and $\mathrm{oc}_C(F) = \mathrm{oc}(F) \cap C$. For $s \subseteq A \times \omega \times \omega$ let

$$\mathrm{dom}(s) = \{a : \exists n, m \in \omega (a, n, m) \in s\}.$$

Let $\langle (s^\alpha, F^\alpha) \in \mathbb{Q}_{A,\rho} : \alpha < \omega_1 \rangle$ be a sequence of conditions.
<u>1st case</u>: There are $\aleph_1$ many pairwise different conditions in the above sequence. Then we may assume that the sequence consists of pairwise disjoint conditions. By applying the delta system lemma to the sequences $\langle s^\alpha : \alpha < \omega_1 \rangle$, $\langle \mathrm{oc}_A(F^\alpha) : \alpha < \omega_1 \rangle$ and $\langle (\mathrm{dom}(s^\alpha) \cup \mathrm{oc}_A(F^\alpha)) : \alpha < \omega_1 \rangle$ respectively, we may assume that there are finite $A_0, A_1 \subseteq A$ and a finite $t \subseteq A \times \omega \times \omega$ s.t. for all $\alpha \neq \beta$, $s^\alpha \cap s^\beta = t$, $\mathrm{oc}_A(F^\alpha) \cap \mathrm{oc}_A(F^\beta) = A_0$ and

$$(\mathrm{dom}(s^\alpha) \cup \mathrm{oc}_A(F^\alpha)) \cap (\mathrm{dom}(s^\beta) \cup \mathrm{oc}_A(F^\beta)) = A_1.$$

So $\mathrm{dom}(t), A_0 \subseteq A_1$. Let $\alpha, \beta < \omega_1$ s.t. $\alpha \neq \beta$. Assume there is $x \in A_1 \setminus \mathrm{dom}(t)$ s.t. $x \in s_\alpha \cap A_1$ and $x \in s_\beta \cap A_1$. Then $x \in s_\alpha \cap s_\beta$. Contradiction to t being the root of $\langle s^\alpha : \alpha < \omega_1 \rangle$. So each element of $A_1 \setminus \mathrm{dom}(t)$ can only be occupied by one of the pairwise different conditions. Since A_1 is finite and there are $\aleph_1$ many pairwise different conditions, $s^\alpha \cap (A_1 \times \omega \times \omega) = t$ must be true for $\aleph_1$ many conditions. Let $\langle (\bar{s}^\alpha, \bar{F}^\alpha) : \alpha < \omega_1 \rangle$ be the sequence of those conditions. Then $(\bar{s}^\alpha \cup \bar{s}^\beta, \bar{F}^\alpha \cup \bar{F}^\beta) \in \mathbb{Q}_{\mathbb{A},\rho}$ and for $\alpha \neq \beta$, $\bar{s}^\alpha \cap (\mathrm{oc}(\bar{F}^\beta) \times \omega \times \omega) \subseteq t$. Let $w \in \bar{F}^\beta$ and $e_w[\bar{s}^\alpha \cup \bar{s}^\beta, \rho](n) = n$. Since $\bar{s}^\alpha \cap (\mathrm{oc}(\bar{F}^\beta) \times \omega \times \omega) \subseteq t$, $e_w[t \cup \bar{s}^\beta, \rho](n) = n$.

So $e_w[\bar{s}^\beta, \rho(n)] = n$. So $(\bar{s}^\alpha \cup \bar{s}^\beta, \bar{F}^\alpha \cup \bar{F}^\beta) \leq_{\mathbb{Q}_{A,\rho}} (\bar{s}^\alpha, \bar{F}^\alpha)$. $(\bar{s}^\alpha \cup \bar{s}^\beta, \bar{F}^\alpha \cup \bar{F}^\beta) \leq_{\mathbb{Q}_{A,\rho}} (\bar{s}^\beta, \bar{F}^\beta)$ follows analogously.
<u>2nd case</u>: There are not $\aleph_1$ many pairwise different conditions in the above sequence. By the pigeonhole principle there exists a condition (s, F) in the sequence which occurs $\aleph_1$ many times. Then the subsequence $\langle (s, F) : \alpha < \omega_1 \rangle$ is linked. $\square$

For any $\mathbb{Q}_{A,\rho}$-generic G we can encode the paths, that are seen by the people in G, into an extension of ρ. This will yield the function allocating every $a \in A$ to a $\sigma \in S_\infty$.

Definition 3.5. Let G be $\mathbb{Q}_{A,\rho}$-generic over V. Define $\rho_G : A \cup B \to S_\infty$ by

$$\rho_G(x) := \begin{cases} \rho(x) & \text{if } x \in B, \\ \bigcup \{s_x : (\exists F \subseteq \widehat{W}_{A\cup B})(s, F) \in G\} & \text{if } x \in A. \end{cases}$$

The next subgoal is to check that this function is well-defined.

Definition 3.6. Let $a \in A$ and $j \geq 1$. A word $w \in W_{A\cup B}$ is called a-good of rank j if it has the form

$$w = a^{k_j} u_j a^{k_{j-1}} u_{j-1} \cdots a^{k_1} u_1$$

where $u_i \in W_{A\setminus\{a\}\cup B} \setminus \{\emptyset\}$ and $k_i \in \mathbb{Z} \setminus \{0\}$, for $1 \leq i \leq j$.

Lemma 3.7. *Let $s \subseteq A \times \omega \times \omega$ be finite s.t. s_a is a partial injection for all $a \in A$. Let $a \in A$ and $w \in W_{A\cup B}$ be a-good. Then for any $n \in \omega \setminus \operatorname{dom}(s_a)$ and $C \subseteq \omega$ finite there are cofinitely many $w \in \omega$ s.t.*

$$(\forall l \in \omega) e_w[s \cup \{(a, n, m)\}, \rho](l) \in C \Leftrightarrow e_w[s, \rho](l) \downarrow \wedge e_w[s, \rho](l) \in C.$$

Proof. Proof by induction over the rank j:
<u>Induction base</u>: Let $w = a^{k_1} u_1$ be an a-good word of rank 1.
<u>1st case</u>: $k_1 < 0$. Let $C \subseteq \omega$ be finite. Let $n, m \in \omega$ s.t. $n \notin \operatorname{dom}(s_a)$ and $m \notin C$. Assume there exists an $l \in \omega$ s.t. $e_w[s \cup \{(a, n, m)\}, \rho](l) \in C$ and $e_w[s, \rho](l) \uparrow$. Since $e_w[s \cup \{(a, n, m)\}, \rho](l) \downarrow$ and $e_w[s, \rho](l) \uparrow$, there is some $0 \leq i < k_1$ s.t. $e_{a^i u_1}[s, \rho](l) = n$. If $i < k_1 - 1$, then $e_{a^{i+1} u_1}[s \cup \{(a, n, m)\}, \rho](l) \uparrow$. Contradiction to $e_w[s \cup \{(a, n, m)\}, \rho](l) \downarrow$. So $i = k_1 - 1$. Then $e_w[s \cup \{(a, n, m)\}, \rho](l) = m \notin C$. Contradiction.
<u>2nd case</u>: $k_1 < 0$. Let $m \notin \operatorname{ran}(e_a^i u_1[s, \rho])$ for all $k_1 \leq i < 0$. Assume there exists an $l \in \omega$ s.t. $e_w[s \cup \{(a, n, m)\}, \rho](l) \in C$ and $e_w[s, \rho](l) \uparrow$. Since $e_w[s \cup \{(a, n, m)\}, \rho](l) \downarrow$ and $e_w[s, \rho](l) \uparrow$, there is some $k_1 < i \leq 0$ s.t. $e_{a^i u_1}[s, \rho](l) \downarrow$ and $e_{a^{i-1} u_1}[s, \rho](l) \uparrow$. Since $e_{a^i u_1}[s, \rho](l) \neq m$, $e_{a^{i-1} u_1}[s \cup \{(a, n, m)\}, \rho](l) \uparrow$. Contradiction.
<u>Induction assumption</u>: Assume Lemma 3.7 holds for all a-good words of rank j.
<u>Induction step</u>: Let $w = a^{k_j} u_j \bar{w}$ be an a-good word of rank $j + 1$ where $\bar{w}$ is a-good of rank j. Let $C' = e_{u_j^{-1} a^{-k_j}}[s, \rho](C)$. By the induction assumption there is a cofinite $I_0 \subseteq \omega$ s.t. for all $m \in I_0$,

$$(\forall l \in \omega) e_{\bar{w}}[s \cup \{(a, n, m)\}, \rho](l) \in C' \Leftrightarrow e_{\bar{w}}[s, \rho](l) \downarrow \wedge e_{\bar{w}}[s, \rho](l) \in C'.$$

Since $a^{k_i}u_j$ is an a-good word of rank 1, there exists a cofinite $I_1 \subseteq \omega$ s.t. for all $m \in I_1$,

$$(\forall l \in \omega) e_{a^{k_i}u_j}[s \cup \{(a,n,m)\}, \rho](l) \in C \Leftrightarrow e_{a^k u_j}[s,\rho](l) \downarrow \wedge e_{a^{k_i}u_j}[s,\rho](l) \in C.$$

Let $m \in I_0 \cap I_1$ and $l \in \omega$ s.t. $e_w[s \cup \{(a,n,m)\}, \rho](l) \in C$. Since $m \in I_0$, $e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho](l) \in C'$. So $e_{a_{k_i}u_j}[s,\rho](e_{\bar{w}}[s,\rho](l)) \in C$. So

$$e_{a^{k_i}u_j}[s,\rho](e_{\bar{w}}[s,\rho](l)) = e_w[s,\rho](l) \in C.$$

$\square$

Let $a \in A$ and G be $\mathbb{Q}_{A,\rho}$-generic s.t. $\bigcup\{s_a : (\exists F \in \widehat{W}_{A\cup B})(s,F) \in G\}$ does not define a permutation. Since for any $(s,F) \in G$, s is finite, s_a contains only finitely many fixed points for $a \in A$. So adding a path for any $n \in \mathbb{N}$ without adding infinitely many fixed points to s_a is necessary to show that ρ_G is a function.
The next lemma will show an even stronger statement: adding a path for any $n \in \mathbb{N}$ is possible without adding a single new fixed point.

Lemma 3.8. *Let A and B be disjoint sets and $\rho : B \to S_\infty$ a function inducing a cofinitary representation of $\mathbb{F}_B$. Then*

1. *("Domain extension") For any $(s,F) \in \mathbb{Q}_{A,\rho}, a \in A$ and $n \in \omega$ s.t. $n \notin \mathrm{dom}(s_a)$ there are cofinitely many $m \in \omega$ s.t. $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.*
2. *("Range Extension") For any $(s,F) \in \mathbb{Q}_{A,\rho}, a \in A$ and $m \in \omega$ s.t. $m \notin \mathrm{ran}(s_a)$ there are cofinitely many $n \in \omega$ s.t. $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.*

Proof. Both statements hold true for $(\emptyset,\emptyset)$, since any condition extends $(\emptyset,\emptyset)$ by vacuosity. Let $(s,F) \in \mathbb{Q}_{A,\rho}$, s.t. $F = \{w_1,...,w_n\}$. Assume the statements hold true for $(s,w_1),...,(s,w_n)$ and let M_i, N_i, $1 \leq i \leq n$ be the cofinite sets with respect to which the statements hold true respectively. Then $M := \bigcap_{i=1}^n M_i, N := \bigcap_{i=1}^n N_i$ are cofinite. Then the statements also hold true for $(s, \bigcup_{i=1}^n w_i) = (s,F)$ with respect to M,N respectively. So both statements can be reduced to the case $F = \{w\}$. We may also assume that a occurs in w, since else $e_w[s \cup \{(a,n,m)\}, \rho] = e_w[s,\rho]$.

1. <u>1st case</u>: w is a-good. Let $l \in \omega$ be a fixed point of $e_w[s,\rho]$. Then by Lemma 3.7 there are cofinitely many $m \in \omega$ s.t.

   $$e_w[s \cup \{(a,n,m)\}, \rho](l) \in \{l\} \Leftrightarrow e_w[s,\rho](l) \downarrow \wedge e_w[s,\rho](l) \in \{l\}.$$

   Since the other properties are satisfied, $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.
   <u>2nd case</u>: w is not a-good. Then $w = uva^k$, where $u \in W_{A\setminus\{a\}\cup B}$, v is a-good and $k \in \mathbb{Z}$. Let $\bar{w} = va^k u$. Then $\bar{w}$ is a-good. By Lemma 3.7 there exists a cofinite $I \subseteq \omega$ s.t.

   $$(\forall m \in I)(s \cup \{(a,n,m)\}, \{\bar{w}\}) \leq (s, \{\bar{w}\}).$$

   Let $l \in \omega$ be a fixed point of $e_w[s \cup \{(a,n,m)\}, \rho]$. By Lemma 3.2

   $$e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho](e_{va^k}[s \cup \{(a,n,m)\}, \rho](l)) = e_{va^k}[s \cup \{(a,n,m)\}, \rho](l).$$

Since $\bar{w}$ is a-good, Lemma 3.7 implies

$$e_{\bar{w}}[s,\rho](e_{va^k}[s \cup \{(a,n,m)\},\rho](l)) = e_{va^k}[s \cup \{(a,n,m)\},\rho](l).$$

Applying Lemma 3.2 yields $e_w[s,\rho](l) = l$.

2. Let $(s,F) \in \mathbb{Q}_{A,\rho}$, $a \in A$, $m_0 \in \omega \setminus \mathrm{ran}(s_a)$. Define $\bar{s} \subseteq A \times \omega \times \omega$ by

$$(x,n,m) \in \bar{s} \Leftrightarrow (x \neq a \wedge (x,n,m) \in s) \vee (x = a \wedge (x,n,m) \in s).$$

Let $\bar{w}$ be the word in which every occurrence of a is replaced with a^{-1}. Then $e_{\bar{w}}[\bar{s},\rho] = e_w[s,\rho]$ and $m_0 \notin \mathrm{dom}(\bar{s}_a)$. By 1. there exist cofinitely many $n \in \omega$ s.t. $(\bar{s} \cup \{(a,m_0,n)\},\{\bar{w}\}) \leq (\bar{s},\{\bar{w}\})$. So there are cofinitely many $n \in \omega$ s.t. $(s \cup \{(a,n,m_0)\},\{w\}) \leq (s,\{w\})$.

□

So $\rho_G(x)$ is a permutation for all $x \in A \cup B$.

Corollary 3.9. *Let $w \in W_{A \cup B}$ and let $\emptyset \neq A_0 \subseteq A$ be the set of letters from A occurring in w. Then for $(s,F) \in \mathbb{Q}_{A,\rho}$ and finite sets $C_0, C_1 \subseteq \omega$ there exists some $t \subseteq A_0 \times \omega \times \omega$ s.t. $(t \cup s, F) \leq (s,F)$, $\mathrm{dom}(e_w[s \cup t,\rho]) \supseteq C_0$ and $\mathrm{ran}(e_w[s \cup t,\rho]) \supseteq C_1$.*

Proof. Let $C_0 \subseteq \omega$ be finite , $n \in C_0 \setminus \mathrm{dom}(e_w[s,\rho])$. Then there exists an $s_a \in A_0$ s.t. $n \notin \mathrm{dom}(s_a)$. By the domain extension part of Lemma 3.8 there exists an $m \in \omega$ s.t. $(s \cup \{(a,n,m)\},F) \leq (s,F)$.
Since C_0 is finite, iterative application of the above argument yields the statement for C_0.
The statement for C_1 follows analogously by applying the range extension part of Lemma 3.8. □

Someone from $\mathbb{Q}_{A,\rho}$ witnessing the path from n to m along w with the paths encoded in ρ_G determines the path from n along w with the paths encoded in s and the paths from ρ.

The proof of this statement, and more formally the proof of Lemma 3.10, is done by induction over the number of letters from a in w.
In the induction step we will formulate an argument by contradiction. In particular, assuming that someone from $\mathbb{Q}_{A,\rho}$ witnesses the path from n to m along w with the paths encoded in ρ_G but no path from m along w with the paths encoded in s and ρ ending in n. So there must exists an $a \in A$ s.t. $w = ua^iv$ and the paths encoded in s and ρ along v are sufficient to describe a path from n to some $n' \in \mathbb{N}$ but n' is not in the domain of any paths encoded in a^i.
Then we encode sufficient paths in a^i s.t. the paths from s and ρ along w can describe a path from n to $m' \neq m$ without adding a new fixed point, which will result in a contradiction to the induction assumption.
To encode the new paths in such a manner we will employ the algorithm below.

Algorithm 1 Pseudocode for the algorithm used in the proof of Lemma 3.10

Input: $(s, F), w$
Output: s' ▷ s' will satisfy the given criteria

$n \leftarrow$ number of occurences of a in ua^i
$i \leftarrow 1$
$x \leftarrow (s, F)$
$z' \leftarrow ua^i$ ▷ where $w = ua^i v$
$s' \leftarrow \emptyset$
while $i \leq n$ **do** ▷ Break condition is used to convey that the algorithm only attends to the a's in ua^i, the algorithm will always reach a return case.
  if u is the empty word **then**
    $s' \leftarrow s_1$ ▷ s_1 defined as in the proof
    return s'
  else
    if $w = w_1 w_0$ where w_0 is a-good and a does not occur in w_1 **then**
      $s' \leftarrow s' \cup \{(a, e_v[s, \rho](n), m''')\}$ ▷ m''' defined as in the proof
      $x \leftarrow (s \cup s', F)$
      $l \leftarrow$ rightmost letter after a^i
      if $l = a^i$ **then**
        if $m''' \in \mathrm{dom}(s_l)$ **then**
          $z \leftarrow z'$ ▷ z' defined as in the proof
          $i \leftarrow i + 1$
        else
          $s' \leftarrow s' \cup \{(a, m''', n')\}$ ▷ m''' redefined as in the proof
          $x \leftarrow (s \cup s', F)$
          $z \leftarrow z'$ ▷ z' defined as in the proof
          $i \leftarrow i + 1$
        end if
      else
        if l is a subword of w_1 **then**
          $s' \leftarrow s_1$ ▷ s_1 defined as in the proof
          return s'
        else
          $s' \leftarrow s' \cup \{(a, m''', n')\}$ ▷ m''', n' defined as in the proof
          $x \leftarrow (s \cup s'.F)$

$z \leftarrow u'a^{i_1}$ ▷ $u'a^{i_1}$ defined as in the proof
$i \leftarrow i + 1$
end if
end if
else ▷ w is of the form $w'a^i$
$s' \leftarrow s' \cup \{(a, n, m''')\}$ ▷ m''' defined as in the proof
$x \leftarrow (s \cup s', F)$
$l \leftarrow$ rightmost letter in z' after a^i
if l is a letter in w_1 **then**
$s' \leftarrow s_1$ ▷ s_1 defined as in the proof
return s'
else
$s' \leftarrow s' \cup \{(a, n, m''')\} \cup \{(a, m''', n')\}$ ▷ m''', n' defined as in the proof
$x \leftarrow (s \cup s', F)$
$z' \leftarrow w'a^{k-1}$
end if
end if
end if
end while

Lemma 3.10. *Let $w \in \widehat{W}_{A \cup B}$ and assume $(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[p_{\dot{G}}](n) = m$ for some $n, m \in \omega$. Then $e_w[s, \rho](n) \downarrow$ and $e_w[s, \rho](n) = m$.*

Proof. Proof by induction over the number of letter from A occuring in w:
Induction base: No letter from A occurs in w. Then the statement reduces to $e_w[\rho] = e_w[\rho]$. Which holds true.
Induction assumption: The statement holds true for all words with at most k letters from A.
Induction step: Let w be a words with $k + 1$ letters from A occuring. Assume that $e_w[s, \rho](n) \uparrow$ and $(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_{\dot{G}}](n) = m$. By the induction assumption, there exists an $a \in A$ s.t. $w = ua^iv$ without cancellation, $i \in \{-1, 1\}$ and $u, v \in W_{A \cup B}$ are s.t. $e_v[s, \rho](n) \downarrow$ and $e_v[s, \rho](n) \notin \text{dow}(e_{a^i}[s, \rho])$.
Applying the following algorithm to the letters of ua^i starting from the rightmost to the leftmost will yield that there exists some finite $s_1 \subseteq \{a\} \times \omega \times \omega$ satisfying

1. $(s \cup s_1, F) \leq (s, F)$,
2. $e_{w_0}[s \cup s_1, \rho](n) \downarrow$,
3. $n_1 = e_{w_0}[s \cup s_1, \rho](n) \neq e_{w_1}[s, \rho]^{-1}(m)$ if $e_{w_1}[s, \rho]^{-1}(m)$ is defined.

1st case: u is the empty word. Apply the domain extension part of Lemma 3.8 to attain an $s_1 \subseteq \{a\} \times \omega \times \omega$ satisfying the given criteria.
2nd case: u is not the empty word.

1st subcase: We can write w as $w_1 w_0$, without cancellation, where w_0 is a-good and a does not occur in w_1. Assume that w_0 has rank j. Then

$$w_0 = a^{k_j} u_j a^{k_{j-1}} u_{j-1} \cdots a^{k_1} u_1,$$

where a does not occur in any u_i for $1 \leq i \leq j$. Since a does not occur in w_1, v is a subword of w_0. Let a^i occur in a^{k_i} for some $1 \leq i \leq j$. By Lemma 3.7 there are cofinitely many $m' \in \omega$ s.t.

$$\begin{aligned}(\forall l \in \omega) & e_{a^i v}[s \cup \{(a, e_v[s,\rho](n), m')\}, \rho](l) \in \{e_u[s,\rho]^{-1}(m)\} \\ \Leftrightarrow & e_{a^i v}[s,\rho](l) \downarrow \wedge e_{a^i v}[s,\rho](l) \in \{e_u[s,\rho]^{-1}(m)\}.\end{aligned}$$

Since $e_{a^i v}[s,\rho](n) \uparrow$,

$$e_{a^i v}[s \cup \{(a, e_v[s,\rho](n), m')\}, \rho](n) \notin \{e_u[s,\rho]^{-1}(m)\}$$

if $e_u[s,\rho]^{-1}(m)$ is defined. By the domain extension part of Lemma 3.8 there exist cofinitely many $m'' \in \omega$ s.t.

$$(s \cup \{(a, e_v[s,\rho](n), m'')\}, F) \leq (s, F).$$

So there exist cofinitely many m''' satisfying

$$\begin{aligned} & e_{a^i v}[s \cup \{(a, e_v[s,\rho](n), m''')\}, \rho](n) \notin \{e_u[s,\rho]^{-1}(m)\} \\ \text{and } & (s \cup \{(a, e_v[s,\rho](n), m''')\}, F) \leq (s, F).\end{aligned}$$

Let l be the rightmost letter after a^i in w.
1st subsubcase: $l = a^i$ for $i \in \{1, -1\}$.
1st subsubsubcase: $m''' \in \operatorname{dom}(s_l)$. Apply the algorithm to the subword z' of w and $(s \cup \{(a, e_v[s,\rho](n), m''')\}, F)$, where $z = z' a^i$.
2nd subsubsubcase: $m''' \notin \operatorname{dom}(s_l)$. Apply the range extension part of Lemma 3.8 to l to attain

$$(s \cup \{(a, e_v[s,\rho](n), m''')\} \cup \{(a, m''', n')\}, F) \leq (s, F),$$

where $n' \in \operatorname{dom}(s_l)$, choosing a different m''' if necessary. Such an m''' exists since the range extension part of Lemma 3.8 and the subcase 1 part of the algorithm return cofinitely many m'''. Apply the algorithm to z' and

$$(s \cup \{(a, e_v[s,\rho](n), m''')\} \cup \{(a, m''', n')\}, F),$$

where $z = z' a^i$.
2nd subsubcase: $l \neq a^i$ for $i \in \{1, -1\}$.
1st subsubsubcase: l is a subword of w_1. Apply the domain extension part of Lemma 3.8, to attain an $s_1 \subseteq \{a\} \times \omega \times \omega$ satisfying the given criteria.
2nd subsubsubcase: l is not a subword of w_1. Let w' be s.t. $w = u' a^{i_1} w' a^{i_2} v$ where a not occur as in w'. Since $(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_{\dot{G}}](n) = m$, (s, F) satisfies

$$(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_{u' a^i w'}[\rho_{\dot{G}}](e_{a^i v}[\rho_{\dot{G}}](n)) = m.$$

By the induction hypothesis, $\mathrm{dom}(e_{u'a^iw'}[s,\rho]) \neq \emptyset$. Let $n' \in \mathrm{dom}(e_{u'a^iw'}[s,\rho])$. Since $e_v[s,\rho](n) \notin \mathrm{dom}(e_{a^i}[s,\rho])$, $e_{w'a^iv}[s,\rho](n) \notin \mathrm{dom}(e_{a^i}[s,\rho])$. Apply the domain extension part of Lemma 3.8, to attain

$$(s \cup \{(a, e_v[s,\rho](n), m''')\} \cup \{(a, m''', n')\}, F) \leq (s, F)$$

choosing a different m''' if necessary. Such an m''' exists since the range extension part of Lemma 3.8 and the subcase 1 part of the algorithm return cofinitely many m'''. Apply the algorithm to $u'a^{i_1}$ and

$$(s \cup \{(a, e_v[s,\rho](n), m''')\} \cup \{(a, m''', n')\}, F).$$

<u>2nd subcase</u>: w is of the form $w'a^k$, where $w' \in W_{A\cup B}$ and $k \in \mathbb{Z} \setminus \{0\}$. Analogously to case 1, there exist cofinitely many m''' satisfying

$$\begin{aligned} &e_{a^i}[s \cup \{(a,n,m''')\}, \rho](n) \notin \{e_{wa^{k-1}}[s,\rho]^{-1}(m)\} \\ \text{and } &(s \cup \{(a,n,m''')\}, F) \leq (s,F). \end{aligned}$$

Let l be the rightmost letter in w after the rightmost a^i.
<u>1st subsubcase</u>: l is a letter in w_1. Apply the domain extension part of Lemma 3.8 to attain an $s_1 \subseteq \{a\} \times \omega \times \omega$ satisfying the given criteria.
<u>2nd subsubcase</u>: l is not a letter in w_1. Apply the range extension part of Lemma 3.8 to attain

$$(s \cup \{(a,n,m''') \cup (a,m''',n')\}, F) \leq (s,F).$$

choosing a different m''' if necessary. Such an m''' exists since the range extension part of Lemma 3.8 and the subcase 2 part of the algorithm return cofinitely many m'''. Apply the algorithm to $w'a^{k-1}$ and

$$(s \cup \{(a,n,m''') \cup (a,m''',n')\}, F).$$

This algorithm breaks after finite steps, since w contains finite letters.
Since $(s,F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_{\dot{G}}](n) = m$ and $(s \cup s_1, F) \Vdash e_{w_0}[\rho_{\dot{G}}](n) = n_1$,

$$(s \cup s_1, F) \Vdash e_{w_1}[\rho_{\dot{G}}](n_1) = m.$$

By the induction assumption $e_{w_1}[s \cup s_1, \rho](n_1) = m$. Since a does not occur in w_1, $e_{w_1}[s,\rho](n_1) = m$. Contradiction. □

The first part of the proof of the next proposition will satisfy our current subgoal. The statement will also satisfy our next subgoal: ρ_G induces a cofinitary representation.

Proposition 3.11. *Let G be $\mathbb{Q}_{A,\rho}$-generic. Then ρ_G is a function $A \cup B \to S_\infty$ s.t. $\rho_G \restriction B = \rho$, and ρ_G induces a cofinitary representation $\hat{\rho}_G : \mathbb{F}_{A\cup B} \to S_\infty$ satisfying $\hat{\rho}_G \restriction \mathbb{F}_B = \hat{\rho}$.*

Proof. For $a \in A$, $n \in \omega$, $w \in \widehat{W}_{A\cup B}$ define

$$\begin{aligned} D_{a,n} &:= \{(s,F) \in \mathbb{Q}_{A,\rho} : (\exists m)(a,n,m) \in s\}, \\ R_{a,n} &:= \{(s,F) \in \mathbb{Q}_{A,\rho} : (\exists m)(a,m,n) \in s\}, \\ D_w &:= \{(s,F) \in \mathbb{Q}_{A,\rho} : w \in F\}. \end{aligned}$$

Let $(s, F) \in \mathbb{Q}_{A,\rho}$. Then $(s, F \cup \{w\}) \leq (s, F)$ and $(s, F \cup \{w\}) \in D_w$. So D_w is dense in $\mathbb{Q}_{A,\rho}$.
<u>1st case</u>: $(\exists m)(a, n, m) \in s$. Then $(s, F) \in D_{a,n}$ and $(s, F) \leq (s, F)$.
<u>2nd case</u>: $(\forall m)(a, n, m) \notin s$. Then Lemma 3.8 implies, that there exists an $m \in \omega$ s.t. $(s \cup \{(a, n, m)\}, F) \leq (s, F)$. Also $(s \cup \{(a, n, m)\}, F) \in D_{a,n}$
So $D_{a,n}$ is dense in $\mathbb{Q}_{A,\rho}$. $R_{a,n}$ being dense in $\mathbb{Q}_{A,\rho}$ follows analogously. Since G is $\mathbb{Q}_{A,\rho}$-generic, $\rho_G : A \cup B \to S_\infty$ is well defined.
Let $w \in W_{A\cup B}$. Then there exist $w' \in \widehat{W}_{A\cup B}$ and $u \in W_{A\cup B}$ s.t. $w = u^{-1}wu$. Since $D_{w'}$ is dense in $\mathbb{Q}_{A,\rho}$, there exists a $(s, F) \in G$ s.t. $w' \in F$. Assume that for $n \in \omega$, $V[G] \models e_{w'}[\rho_G](n) = n$. Then there exists a $(t, E) \in G$ s.t. $(t, E) \leq (s, F)$ and $(t, E) \Vdash e_{w'}[\rho_G](n) = n$. By Lemma 3.10, $e_{w'}[t, \rho](n) = n$. Since $(t, E) \leq (s, F)$, $e_{w'}[s, \rho](n) = n$. So

$$|\mathrm{fix}(e_{w'}[\rho_G])| \leq |\mathrm{fix}(e_{w'}[s, \rho])| < \aleph_0.$$

Since $\mathrm{fix}(e_w[\rho_G]) = e_u[\rho_G]^{-1}(\mathrm{fix}(e_{w'}[\rho_G]))$, $|\mathrm{fix}(e_w[\rho_G])| < \aleph_0$. So ρ_G induces a cofinitary representation. □

The last subgoal is the goal of this chapter: $\mathrm{im}(\rho_G)$ is a maximal cofinitary group under certain conditions.

Given a set A used to encode paths and a subset $A_0 \subseteq A$, it will be useful to determine which information is lost and which information is retained using only the paths encoded in A_0.

Definition 3.12. 1. Let $s \subseteq A \times \omega \times \omega, A_0 \subseteq A$. Then we write $s \restriction A_0$ for $s \cap A_0 \times \omega \times \omega$.

2. Let $p = (s, F) \in \mathbb{Q}_{A,\rho}$. Then we write $p \restriction A_0$ for $(s \restriction A_0, F)$ and $p \,||\!\restriction A_0$ for $(s \restriction A_0, F \cap \widehat{W}_{A_0\cup B})$.

Lemma 3.13. *If $\emptyset \neq A_0 \subseteq A$ then $\mathbb{Q}_{A_0,\rho}$ is completely embedded in $\mathbb{Q}_{A,\rho}$.*

Proof. Let $A_1 = A \setminus A_0$. If $A_1 = \emptyset$, then $\mathbb{Q}_{A_0,\rho} = \mathbb{Q}_{A,\rho}$. Then $\mathbb{Q}_{A_0,\rho} \subseteq \mathbb{Q}_{A,\rho}$. If $p \leq_{\mathbb{Q}_{A_0,\rho}} q$, then $p \leq_{\mathbb{Q}_{A,\rho}} q$. Let $p \perp_{\mathbb{Q}_{A_0,\rho}} q$. Assume that $p \not\perp_{\mathbb{Q}_{A,\rho}} q$. Then there exists an $r \in \mathbb{Q}_{A,\rho}$ s.t. $r \leq_{\mathbb{Q}_{A,\rho}} p, q$. Then $r \,||\!\restriction \leq_{\mathbb{Q}_{A_0,\rho}} p, q$. Contradiction. So $p \perp_{\mathbb{Q}_{A,\rho}} q$.
<u>Claim:</u> For every $(s, F) \in \mathbb{Q}_{A,\rho}$ there exists $t_0 \supseteq s \restriction A_0 \times \omega \times \omega$, s.t. whenever $(t, E) \leq_{\mathbb{Q}_{A_0\cup B,\rho}} (t_0, F \cap \widehat{W}_{A_0})$ then $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$. So for any $q \in \mathbb{Q}_{A,\rho}$ there is $p_0 \leq_{\mathbb{Q}_{A_0,\rho}} q \,||\!\restriction A_0$ s.t. whenever $p \leq_{\mathbb{Q}_{A_0,\rho}} p_0$ then $p \not\perp_{\mathbb{Q}_{A,\rho}} q$.
Proof: Let $\{w_1, ..., w_n\} = F \setminus W_{A_0\cup B}$. Then each word w_i can be written as

$$w_i = u_{i,k_i} v_{i,k_i} \cdots u_{i,1} v_{i,1} u_{i,0},$$

where $u_{i,j} \in W_{A_0}$, $v_{i,j} \in W_{A_1}$ are nonempty except possibly u_{i,k_i} and $u_{i,0}$. So $\mathrm{ran}(e_{v_{i,j}}[s, \rho])$ and $\mathrm{dom}(e_{v_{i,j+1}}[s, \rho])$ are finite for $1 \leq j < k_i$. Applying Corollary 3.9

iteratively to w_i yields that there exists a $t_0 \subseteq A_0 \times \omega \times \omega$ s.t. for $1 \leq j < k_i$

$$\begin{aligned}&\mathrm{dom}(e_{u_{i,j}}[s \cup t_0, \rho]) \supseteq \mathrm{ran}(e_{v_{i,j}}[s, \rho]),\\&\mathrm{ran}(e_{u_{i,j}}[s \cup t_0, \rho]) \supseteq \mathrm{dom}(e_{v_{i,j+1}}[s, \rho])\\&\text{and } (s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F).\end{aligned}$$

<u>1st case</u>: $u_{i,0}$ is the empty word. Then $\mathrm{ran}(e_{v_{i,0}}[s, \rho]) = \omega$. So

$$\mathrm{ran}(e_{u_{i,0}}[s \cup t_0, \rho]) \supseteq \mathrm{dom}(e_{v_{i,1}})$$

<u>2nd case</u>: $u_{i,0}$ is not the empty word. Then $\mathrm{ran}(e_{v_{i,0}}[s, \rho])$ is finite. So Corollary 3.9 implies that

$$\mathrm{ran}(e_{u_{i,0}}[s \cup t_0, \rho]) \supseteq \mathrm{dom}(e_{v_{i,1}}).$$

Analogously

$$\mathrm{dom}(e_{u_{i,k_i}}[s \cup t_0, \rho]) \supseteq \mathrm{ran}(e_{v_{i,k_i}}[s, \rho]).$$

We may assume that $t_0 \supseteq s \restriction A_0$. Assume that $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t_0, F \cap \widehat{W}_{A_0 \cup B})$. Let $n \in \omega$ be s.t. $e_{w_i}[s \cup t, \rho](n) \downarrow$. Since $t \subseteq A_0 \times \omega \times \omega$, $e_{w_i}[s \cup t_0, \rho](n) \downarrow$. So for any $n \in \omega$ s.t. $e_{w_i}[s \cup t, \rho](n) = n$, $e_{w_i}[s \cup t_0, \rho](n) = n$ holds true. Since $(s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$, $e_{w_i}[s, \rho](n) = n$. So $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$. □
The reduction property follows from the claim above. □

Lemma 3.14. *Let $A = A_0 \cup A_1$. If $(t, E) \in \mathbb{Q}_{A_0,\rho}$ and*

$$(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} (s_0, F_0) \leq_{\mathbb{Q}_{A_1,\rho_{\dot{G}}}} (s_1, F_1)$$

then $(t \cup s_0, F_0) \leq_{\mathbb{Q}_{A,\rho}} (t \cup s_1, F_1)$.

Proof. Let $w \in F_1$ and suppose $e_w[t \cup s_0, \rho](n) = n$. Let G be $\mathbb{Q}_{A_0,\rho}$-generic over V s.t. $(t, E) \in G$. Since $(t, E) \in G$,

$$V[G] \models e_w[s_0, \rho_G](n) = n.$$

Since $(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} (s_0, F_0) \leq_{\mathbb{Q}_{A_1,\rho_{\dot{G}}}} (s_1, F_1)$,

$$V[G] \models e_w[s_q, \rho_G](n) = n.$$

By Lemma 3.10, $e_w[t \cup s_1, \rho](n) = n$. Since $\subseteq$ is absolute for transitive models,

$$(t \cup s_0, F_0) \leq_{\mathbb{Q}_{A,\rho}} (t \cup s_1, F_1).$$

□

Decomposing the encoding process of ρ_G into a two-step process over a partition of A is compatible with respect to certain filters.

Lemma 3.15. *Let G be $\mathbb{Q}_{A,\rho}$-generic over V, $A = A_0 \cup A_1$ where $A_0 \cap A_1 = \emptyset$ and $A_0, A_1 \neq \emptyset$. Then $H = G \cap \mathbb{Q}_{A_0,\rho}$ is $\mathbb{Q}_{A_0,\rho}$-generic over V and*

$$K = \{p \restriction A_1 : p \in G\} = \{(s \restriction A_1, F) : (s, F) \in G\}$$

is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$. Also $\rho_G = (\rho_H)_K$.

Proof. To show that H is a filter on $\mathbb{Q}_{A_0,\rho}$:

1. Let $p, q \in H$. Then $p, q \in G$. Since G is a filter, $p \not\perp q$. Since $\mathbb{Q}_{A_0,\rho} \lessdot \mathbb{Q}_{A,\rho}$ and $p \not\perp_{\mathbb{Q}_{A,\rho}} q$, $p \not\perp_{\mathbb{Q}_{A_0,\rho}} q$.
2. Let $p, q \in \mathbb{Q}_{A_0,\rho}$ s.t. $p \geq_{\mathbb{Q}_{A_0,\rho}} q$ and $q \in H$. Since $H \subseteq G$ and G is a filter, $q \in G$. So $q \in H$.

So H is a filter on $\mathbb{Q}_{A_0,\rho}$.

Let $D \in V$ be s.t. $D \subseteq \mathbb{Q}_{A_0,\rho}$ is dense in $\mathbb{Q}_{A_0,\rho}$. Assume that $D \cap H = \emptyset$. Then by Lemma 2.7, $D\downarrow \cap H = \emptyset$.
<u>1st case</u>: $D\downarrow$ is dense in $\mathbb{Q}_{A,\rho}$. Then $D\downarrow \cap G \neq \emptyset$. So $D\downarrow \cap H \neq \emptyset$. Contradiction.
<u>2nd case</u>: $D\downarrow$ is not dense in $\mathbb{Q}_{A,\rho}$. Since $D\downarrow$ is dense in $\mathbb{Q}_{A_0,\rho}$, there necessarily exists a $q \in \mathbb{Q}_{A_1,\rho}$ s.t. $\forall p \in D\downarrow (p \not\leq_{\mathbb{Q}_{A,\rho}} q)$. By Lemma 3.13 there exists a reduction $r \in \mathbb{Q}_{A_0,\rho}$ of q. Since $D\downarrow$ is dense in $\mathbb{Q}_{A_0,\rho}$, there exists a $d \in D\downarrow$ s.t. $d \leq_{\mathbb{Q}_{A_0,\rho}} r$. Let d' be the common extension of q and d. Since $D\downarrow$ is dense open, $d' \in D\downarrow$. Contradiction to $\forall p \in D\downarrow (p \not\leq_{\mathbb{Q}_{A,\rho}} q)$.
So H is $\mathbb{Q}_{A_0,\rho}$-generic.

K being a filter follows from G being a filter.
Let $D \in V[H]$ s.t. $D \subseteq \mathbb{Q}_{A_1,\rho_H}$ is dense in $\mathbb{Q}_{A_1,\rho_H}$. Let $D'' := D\downarrow$ Define

$$D' := \{p \in \mathbb{Q}_{A,\rho} : p \mid\restriction A_0 \Vdash_{\mathbb{Q}_{A_0,\rho}} \dot{p} \restriction A_1 \in \dot{D}''\}$$

and let $p_0 \in H$ be a condition s.t. $p_0 \Vdash_{\mathbb{Q}_{A_0,\rho}}$ "D is dense". Let $(s, F) = p \leq_{\mathbb{Q}_{A,\rho}} p_0$. By the claim used in the proof of Lemma 3.13 there exists a $p_1 \leq_{\mathbb{Q}_{A_0,\rho}} p \mid\restriction A_0$ s.t. for any $p_2 \leq_{\mathbb{Q}_{A_0,\rho}} p_1$, $p_2 \not\perp_{\mathbb{Q}_{A,\rho}} p$. Since D is dense the truth lemma implies that there exists $q = (s_0, F_0) \in \mathbb{Q}_{A_1,\rho_H}$ and $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} p_1$ s.t.

$$(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} \dot{q} \in \dot{D} \wedge \dot{q} \leq_{\mathbb{Q}_{A_1,\rho_{\dot{H}}}} \dot{p} \restriction A_1.$$

By Lemma 3.14, $(s_0 \cup t, F_0) \leq_{\mathbb{Q}_{A,\rho}} (s \restriction A_1 \cup t, F)$. Since $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} p \mid\restriction A_0$,

$$(s_0 \cup t, F_0 \cup E) \leq_{\mathbb{Q}_{A,\rho}} (s, F).$$

Since $(s_0 \cup t, F_0 \cup E) \in D'$, D' is dense below p_0. Since $p_0 \in G$, there exists a $q' \in D' \cap G$. So in $V[H]$ it holds true that $q' \restriction A_1 \in D\downarrow$. So $K \cap D\downarrow \neq \emptyset$. By Lemma 2.7 $K \cap D \neq \emptyset$

Let $x \in A \cup B$.
<u>1st case</u>: $x \in B$. Then $(\rho_H)_K(x) = \rho_H(x) = \rho(x) = \rho_G(x)$.
<u>2nd case</u>: $x \in A_0$. Then

$$\begin{aligned}(\rho_H)_K(x) &= \rho_H(x) \\ &= \bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A_0 \cup (A_1 \cup B)})(s, F) \in G \cap \mathbb{Q}_{A_0,\rho}\} \\ &= \bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A_0 \cup (A_1 \cup B)})(s, F) \in G\} \\ &= \rho_G(x).\end{aligned}$$

3rd case: $x \in A_1$. Then

$$\begin{aligned}(\rho_H)_K(x) &= \bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A_1\cup(A_0\cup B)})(s,F) \in K\}\\ &= \bigcup\{s_x : (\exists F \subseteq \widehat{W}_{A_1\cup(A_0\cup B)})(s,F) \in G\}\\ &= \rho_G(x).\end{aligned}$$

So $(\rho_H)_K = \rho_G$. □

The main theorem of this section allows us to add a cofinitary group of size $|\mathrm{im}(\rho_G)|$ in $V[G]$ for certain filter G.

Theorem 3.16. *Suppose $\rho : B \to S_\infty$ induces a cofinitary representation of $\mathbb{F}_B$. If $|A| > \aleph_0$ and G is $\mathbb{Q}_{A,\rho}$-generic over V, then $\mathrm{im}(\rho_G)$ is a maximal cofinitary group in $V[G]$.*

To prove the above theorem, we will make use of the following Lemma.

Lemma 3.17. *Suppose $\rho : B \to S_\infty$ induces a cofinitary representation $\hat{\rho} : \mathbb{F}_B \to S_\infty$ and that there is $b_0 \in B$ s.t. $\rho(b_0) \neq I$. Let $(s', F) \in \mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}$ and let $a_0 \in A$. Then there is an $(s,F) \leq_{\mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}} (s',F)$ and an $N \in \omega$ s.t. for all $n \geq N$*

$$(s \cup \{(a_0, n, \rho(b_0)(n))\}, F) \leq_{\mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}} (s,F).$$

Proof. Let $w_1, .., w_l$ be the words in F s.t. a_0 occurs in them. Then each word w_i is of the form

$$w_i = u_{i,j_i} a_0^{k_{i,j_i}} u_{i,j_i-1} a_0^{k_{i,j_i-1}} \cdots u_{i,1} a_0^{k_{i,1}} u_{i,0}$$

where $u_{i,m} \in W_{A\setminus\{a_0\}\cup B\setminus\{b_0\}}$ are nonempty for $m \notin \{j_i, 0\}$. By applying Lemma 3.8 if necessary, we may assume that there is an $(s,F) \leq_{\mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}} (s',F)$ s.t. for all $u_{i,m}$ s.t. $\mathrm{dom}(e_{u_{i,m}}[s,\rho])$ and $\mathrm{ran}(e_{u_{i,m}}[s,\rho])$ are finite

$$\mathrm{dom}(e_{a_0}^{k_{i,m+1}}[s,\rho]) \supseteq \mathrm{ran}(e_{u_{i,m}}[s,\rho])$$

and

$$\mathrm{ran}(e_{a_0}^{k_{i,m}}[s,\rho]) \supseteq \mathrm{dom}(e_{u_{i,m}}[s,\rho]).$$

Let $\bar{w}_i$ be the word s.t. every occurence of a_0 in w_i has been replaced by b_0. Since ρ induces a cofinitary representation, whenever $e_{\bar{w}_i}[s,\rho]$ is totally defined, there are at most finitely many $n \in \omega$ s.t. $e_{\bar{w}_i}[s,\rho](n) = n$. Let C_i be the set of all words s.t. at least one occurrence of a_0 in w_i is substituted by b_0. For every $1 \leq i \leq l$ s.t. $e_{\bar{w}_i}[s,\rho]$ is totally defined, define

$$\bar{w}_{i,m} := u_{i,m} a_0^{k_{i,m}} \cdots u_{i,1} a_0^{k_{i,1}} u_{i,0}$$

and for $\hat{w} \in C_i$

$$\begin{aligned}N_{\hat{w}} := \max\{e_v[s,\rho](k) :& e_{\hat{w}_i}[s,\rho](k) = k \wedge v = a_0^{\mathrm{sign}(k_{i,m+1})p}\bar{w}_{i,m} \wedge \\ & 0 \leq p \leq \mathrm{sign}(k_{i,m+1})k_{i,m+1} - 1 \wedge 0 \leq m \leq j_i\}.\end{aligned}$$

Let $N_i := \bigcup_{\hat{w} \in C_i} N_{\hat{w}}$. Since C_i and $N_{\hat{w}}$ are finite, N_i is finite. Since s_{a_0} and the N_i are finite, there exists an $N \in \omega$ s.t. $N > \max\{N_i; i \leq l\}$, $n \notin \mathrm{dom}(s_{a_0})$ and $\rho(b_0)(n) \notin \mathrm{ran}(s_{a_0})$ for $n \geq N$. Let $n \geq N$. Since $\rho(b_0)^{-1}(\{1, ..., N\})$ is finite, we may assume that $\rho(b_0)(n) \geq N$.
<u>1st case</u>: $e_{\bar{w}_i}[s, \rho]$ is not totally defined. Then there exists an $0 \leq m \leq j_i$ s.t. $u_{i,m}$ contains an $a \in A \setminus \{a_0\}$. Then $\mathrm{dom}(e_{u_{i,m'}}[s, \rho])$ and $\mathrm{ran}(e_{u_{i,m'}}[s, \rho])$ are finite for all $m \leq m' \leq j_i$. Assume there exists a

$$k \in \mathrm{dom}(e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho]) \setminus \mathrm{dom}(e_{w_i}[s, \rho]).$$

<u>1st subcase</u>: $u_{i,j_i} \neq \emptyset$. Let

$$v := a_0^{k_{i,j_i}} u_{i,j_i-1} a_0^{k_{i,j_i-1}} \cdots u_{i,1} a_0^{k_{i,1}} u_{i,0}.$$

Since $\mathrm{dom}(e_{u_{i,j_i}}[s, \rho]) = \mathrm{dom}(e_{u_{i,j_i}}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho])$ and $k \notin \mathrm{dom}(e_{w_i}[s, \rho])$,

$$e_v[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho](k) \notin \mathrm{ran}(e_{a_0}^{k_{i,j_i}}[s, \rho](k)).$$

Since $\mathrm{ran}(e_{a_0}^{k_{i,j_i}}[s, \rho]) \supseteq \mathrm{dom}(e_{u_{i,j_i}}[s, \rho])$,

$$\begin{aligned} e_v[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho](k) \notin & \mathrm{dom}(e_{u_{i,j_i}}[s, \rho]) \\ = & \mathrm{dom}(e_{u_{i,j_i}}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho]) \end{aligned}$$

Contradiction to $k \in \mathrm{dom}(e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho])$.
<u>2nd subcase</u>: $u_{i,j_i} = \emptyset$. Analogously with $\mathrm{dom}(e_{a_0}^{k_{i,j_i}}[s, \rho]) \supseteq \mathrm{ran}(e_{u_{i,j_i-1}}[s, \rho])$ and $\mathrm{ran}(e_{u_{i,j_i-1}}[s, \rho]) = \mathrm{ran}(e_{u_{i,j_i-1}}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho])$.
So

$$\mathrm{dom}(e_{w_i}[s, \rho]) = \mathrm{dom}(e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho]).$$

<u>2nd case</u>: $e_{\bar{w}_i}[s, \rho]$ is totally defined. Assume there exists a $k \in \omega$ s.t.

$$e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho](k) = k \neq e_{w_i}[s, \rho](k).$$

Then there exists a $\hat{w} \in C_i$ s.t. $e_{\hat{w}}[s, \rho](k) = e_{w_i}[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho](k) = k$ and b_0 is only substituted for the leftmost a_0 at the subwords $a_0^i v$ s.t. $w_i = u a_0^i v$ for $i \in \{1, -1\}$ when $e_{a_0^i v}[s, \rho](k) \uparrow$. Then there exists a subword v of $\hat{w}$ s.t. $w_i = u a_0^i v$ for $i \in \{1, -1\}$ and $e_v[s, \rho](k) \in \{n, \rho(b_0)(n)\}$. Assume w.l.o.g. that v is the shortest such. Then

$$v = a_0^{\mathrm{sign}(k_{i,m+1})p} \bar{w}_{i,m}$$

for some $0 \leq p \leq \mathrm{sign}(k_{i,m+1}) k_{i,m+1} - 1 \wedge 0 \leq m \leq j_i$. So $e_v[s, \rho](k) > e_v[s, \rho](k)$. Contradiction.

Since $e_w[s, \rho] = e_w[s \cup \{(a_0, n, \rho(b_0)(n))\}, \rho]$ for $w \in F \setminus \{w_i\}_{i=1}^l$,

$$(s \cup \{(a_0, n, \rho(b_0)(n))\}, F) \leq_{\mathbb{Q}_{A, \rho \restriction B \setminus \{b_0\}}} (s, F).$$

$\square$

We proceed with the proof of Theorem 3.16:

Proof. Let $|A| > \aleph_0$ and G be $\mathbb{Q}_{A,\rho}$-generic. Assume that $\mathrm{im}(\rho_G)$ is not a maximal cofinitary group in $V[G]$. Then there exists a $b_0 \notin B \cup A$ s.t. there is a permutation $\sigma \in (\mathrm{cofin}(S_\infty) \setminus \mathrm{Im}(\rho_G))^{V[G]}$ s.t. $\rho'_G : B \cup \{b_0\} \to S_\infty$

$$\rho'_G(b) := \begin{cases} \rho_G(b) & \text{if } b \neq b_0, \\ \sigma & \text{if } b = b_0 \end{cases}$$

and ρ'_G induces a cofinitary representation of $\mathbb{F}_{B\cup\{b_0\}}$. Let $\dot\sigma$ be a $\mathbb{Q}_{A,\rho}$-name for σ. Since σ contains countably many pairs of natural numbers, at most countably many $p \in \mathbb{Q}_{A,\rho}$ are used to construct $\dot\sigma$. Since each $p \in \mathbb{Q}_{A,\rho}$ is constructed using finitely many $a \in A$, there exists a countable $A_0 \subseteq A$ s.t. $\dot\sigma$ is a $\mathbb{Q}_{A_0,\rho}$-name. So $\dot\sigma_H = \sigma \in V[H]$ where $H := G \cap \mathbb{Q}_{A_0,\rho}$. Let $A_1 = A \setminus A_0$ and K be defined as in Lemma 3.15. Define

$$D_{\sigma,N} = \{(s,F) \in \mathbb{Q}_{A_1,\rho_H} : (\exists n \geq N) s(n) = \sigma(n)\}.$$

Let $\rho'_H : A_1 \cup (A_0 \cup (B \cup \{b_0\})) \to S_\infty$

$$\rho'_H(x) := \begin{cases} \rho_H(x) & \text{if } x \neq b_0, \\ \rho'_G(x) & \text{if } x = b_0 \end{cases}$$

Since $\rho'_H(b_0) = \rho'_G(b_0) = \sigma \neq I$ and ρ'_G induces a cofinitary representation of $\mathbb{F}_{B\cup\{b_0\}}$, Lemma 3.17 implies that $D_{\sigma,N}$ is dense in $\mathbb{Q}_{A_1,\rho_H}$. Since K is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$, for any $a_0 \in A_1$, $(\rho_H)_K(a_0)(n) = \sigma(n)^{V[H][K]}$ holds for infinitely many $n \in \mathbb{N}$. By Lemma 3.15, $(\rho_H)_K = \rho_G$. Since S_∞ is a group, $\sigma^{-1} \circ \rho_G(a_0) \in S_\infty$ for $a_0 \in A_1$. Since $(\rho_H)_K(a_0)(n) = \rho_G(a_0)(n) = \sigma(n)$ for infinitely many $n \in \mathbb{N}$,

$$\sigma^{-1} \circ \rho_G(a_0) \notin \mathrm{cofin}(S_\infty)$$

for any $a_0 \in A_1$. Contradiction to ρ'_G inducing a cofinitary representation. $\square$

4 The Forcing Notion $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$

The goal of this chapter is to define the schema, that will be used to construct the forcing notion $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$, which will be used to force Theorem 1.3.

This construction contains three parts:

1. a good σ-Suslin forcing notion $\mathbb{S}$,
2. a finite function forcing notion with the strong embedding property $\mathbb{Q}$,
3. a two sided template $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$.

At first, we define a good σ-Suslin forcing notion.

For this we define the more general Suslin forcing notion.

Definition 4.1. A Suslin poset is a poset $(\mathbb{S}, \leq_{\mathbb{S}})$ s.t. $\mathbb{S}(\subseteq \omega^\omega), \leq_{\mathbb{S}}, \perp_{\mathbb{S}}$ have $\mathbf{\Sigma}^1_1$ definitions with parameters in the ground model.

Adjoining a Suslin forcing notion with respect to iterated forcing is compatible with complete embeddings.

Lemma 4.2 (Folklore)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and let $\mathbb{S}$ be a c.c.c. Suslin poset. If $\mathbb{P} \lessdot \mathbb{Q}$ then $\mathbb{P} * \dot{\mathbb{S}} \lessdot \mathbb{Q} * \dot{\mathbb{S}}$.*

Definition 4.3. Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion whose conditions can be written in the form (s, f) where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$. We will say that $\mathbb{S}$ is n-suslin if whenever $(s, f) \leq_{\mathbb{S}} (t, g)$ and (t, h) is a condition in $\mathbb{S}$ s.t. $h \restriction (n \cdot |s|) = g \restriction (n \cdot |s|)$ then (s, f) and (t, h) are compatible. A forcing notion is called σ-Suslin if it is n-Suslin for some n.

Note that:

1. If $\mathbb{S}$ is n-Suslin and $m \geq n$, then $\mathbb{S}$ is also m-Suslin.
2. If $\mathbb{S}$ is n-Suslin and $(s, f), (s, g) \in \mathbb{S}$ s.t. $f \restriction (s \cdot |s|) = g \restriction (n \cdot |s|)$, then (s, f) and (s, g) are compatible, so every σ-Suslin forcing notion has the Knaster property.
3. Localization is 2-Suslin.

Definition 4.4. Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose condition can be written in the form (s, f) where $s \in {}^{<\omega}\omega$, $f \in {}^{\omega}\omega$.

1. The pair $(\check{s}, \dot{f})$ is a nice name for a condition in $\mathbb{S}$ below $y \in \mathbb{B}$ if $\dot{f}$ is a nice name for a real below y and $y \Vdash_{\mathbb{B}} (\check{s}, \dot{f}) \in \dot{S}$.

2. Whenever $(\check{s}, \dot{f})$ is a nice name for a condition in $\mathbb{S}$ below $y \in \mathbb{B}$, $x \in \mathbb{A}$ is a reduction of y and $\dot{g}$ is a canonical projection of $\dot{f}$ below x s.t. $x \Vdash_{\mathbb{B}} (\check{s}, \dot{g}) \in \dot{S}$, we will say that $(\check{s}, \dot{g})$ is a canonical projection of the nice name $(\check{s}, \dot{f})$ below x.

3. $\mathbb{S}$ is called good if every nice name for a condition in $\mathbb{S}$ below y has a canonical projection below x, whenever $x \in \mathbb{A}$ is a reduction of $y \in \mathbb{B}$.

By Lemma 2.22, $\mathbb{L}$ is a good σ-Suslin forcing notion.

Next, we define a finite function poset with the strong embedding property.

The next definition contains generalizations of the forcing notion $\mathbb{Q}_{A,\rho}$ from the previous chapter.

Definition 4.5. Let A be a set and $\mathbb{Q}$ a poset of pairs $p = (s^p, F^p)$ s.t. $s^p \subseteq A \times \omega \times \omega$ is finite, for every $a \in A$, $s^p_a = \{(n, m) : (a, n, m,) \in s^p\}$ is a finite partial function and $F^p \in [\widehat{W_A}]^{<\omega}$. For $p \in \mathbb{Q}$ let

$$\begin{aligned} \mathrm{oc}(s^p) &= \{a : \exists n, m \in \omega \text{ s.t. } (a, n, m) \in s^p\} \\ \text{and } \mathrm{oc}(p) &= \mathrm{oc}(s^p) \cup \{a : \ a \text{ is a letter from a word in } F^p\ \}. \end{aligned}$$

For $B \subseteq A$ let

$$\begin{aligned} p \restriction B &= (s^p \cap B \times \omega \times \omega, F^p) \\ p \mid\restriction B &= (s^p \cap B \times \omega \times \omega, F^p \cap \widehat{W_B}) \end{aligned}$$

and let $\mathrm{dom}(\mathbb{Q}) = A$. Then $\mathbb{Q}$ is a finite function poset if:

1. 'Restrictions' whenever $p, q \in \mathbb{Q}, B \subseteq Q$ then
   (a) $p \restriction B, p \mid\restriction B \in \mathbb{Q}$ and $p \restriction B \leq p \mid\restriction B$,
   (b) if $p \leq q$, then $p \mid\restriction B \leq q \mid\restriction B$.
2. 'Extensions' whenever $p = (s, F) \in \mathbb{Q}$
   (a) and $t \subseteq A \times \omega \times \omega$ is finite s.t. $\mathrm{oc}(p) \cap \mathrm{oc}(t) = \emptyset$, then $(s \cup t, F) \leq p$,
   (b) and $E \in [\widehat{W}_A^{<\omega}]$ contains F, then $(s, E) \leq (s, F)$.

If $B \subseteq \mathrm{dom}(\mathbb{Q})$, then we define the suborder $\mathbb{Q}_B := \{p \mid\restriction B : p \in \mathbb{Q}\}$.

Finite function posets are closed with respect to hard restrictions.

Lemma 4.6. *Let $\mathbb{Q}$ be a finite function poset and $B \subseteq \mathrm{dom}(\mathbb{Q})$, then $\mathbb{Q}_B$ is a finite function poset.*

Proof. By definition $\mathbb{Q}_B \subseteq \mathbb{Q}$. Let $p, q \in \mathbb{Q}_B$. Since $\mathbb{Q}_B \subseteq \mathbb{Q}$, $\mathbb{Q}_B$ satisfies all assumption on the elements of $\mathbb{Q}_B$.

1. Let $p, q \in \mathbb{Q}_B, C \subseteq B = \mathrm{dom}(\mathbb{Q}_B)$.

   (a) Since $\mathbb{Q}$ is a finite function poset and $C \subseteq \mathrm{dom}(\mathbb{Q})$, $p \restriction C \in \mathbb{Q}$. So

   $$(p \,\|\!\restriction B) \restriction C = (p \restriction C) \,\|\!\restriction B \in \mathbb{Q}_B.$$

   Analogously $(p \,\|\!\restriction B) \,\|\!\restriction C \in \mathbb{Q}_B$.
   Since $\mathbb{Q}_B \subseteq \mathbb{Q}$ and $C \subseteq \mathrm{dom}(\mathbb{Q})$, $(p \,\|\!\restriction B) \restriction C \leq (p \,\|\!\restriction B) \,\|\!\restriction C$.

   (b) Let $p \leq q$. Since $\mathbb{Q}_B \subseteq \mathbb{Q}$ and $C \subseteq \mathrm{dom}(\mathbb{Q})$, $p \,\|\!\restriction C \leq q \,\|\!\restriction C$.

2. Let $(s, F) \in \mathbb{Q}_B$.

   (a) Follows from $\mathbb{Q}_B \subseteq \mathbb{Q}$ and $B \subseteq \mathrm{dom}(\mathbb{Q})$.

   (b) Follows from $\mathbb{Q}_B \subseteq \mathbb{Q}$ and $B \subseteq \mathrm{dom}(\mathbb{Q})$.

□

The strong embedding property entails a stronger version of reduction.

Definition 4.7. Let $\mathbb{Q}$ be a finite function poset. We say that $\mathbb{Q}$ has the strong embedding property if whenever $A_0 \subseteq \mathrm{dom}(\mathbb{Q})$, and $p = (s, F) \in \mathbb{Q}$, then there is $t_0 \subseteq (\mathrm{oc}(s) \cap A_0) \times \omega \times \omega$ s.t.

1. $s \restriction A_0 \subseteq t_0$,
2. $(t_0, F \cap \widehat{W}_{A_0}) \leq_{\mathbb{Q}_{\mathrm{oc}(p) \cap A_0}} p \,\|\!\restriction A_0$,
3. whenever $(t, E) \leq_{\mathbb{Q}} (t_0, F \cap \widehat{W}_{A_0})$ is s.t. $\mathrm{oc}(t)$ and $\mathrm{oc}(E)$ are disjoint from $\mathrm{oc}(p) \setminus A_0$, then $(t \cup s, F) \leq (s, F)$ and $(t \cup s, E) \leq (t, E)$.

We say that $(t_0, F \cap \widehat{W}_{A_0})$ is a strong reduction of p and $(s \cup t, E \cup F)$ a canonical extension of (s, F) and (t, E).

Note that for any finite function poset with the strong embedding property $\mathbb{Q}$, whenever $A \subseteq B \subseteq \mathrm{dom}(\mathbb{Q}), C \subseteq \mathrm{dom}(\mathbb{Q})$ are s.t. $C \cap B = A$, for every $p \in \mathbb{Q} \restriction B$ there is a $p_0 \leq_{\mathbb{Q} \restriction A} p \restriction A$ s.t. $\mathrm{oc}(p_0) = \mathrm{oc}(p) \cap A$ and if q_0 is a $\mathbb{Q} \restriction C$-extension of p_0, then q_0 is compatible with p.

Definition 4.8. Let $p_0, p, \mathbb{Q}, A$ be as above, then p_0 is called a strong $\mathbb{Q} \restriction A$-reduction of p.

Since we want to plug $\mathbb{Q}_{A,\rho}$ into our schema, $\mathbb{Q}_{A,\rho}$ must be a finite function poset with the strong embedding property.

Lemma 4.9. *$\mathbb{Q}_{A,\rho}$ is a finite function poset with the strong embedding property.*

Proof. $\mathbb{Q}_{A,\rho}$ being a finite function poset follows immediately from the definition. $\mathbb{Q}_{A,\rho}$ having the strong embedding property follows from $t_0 = s \restriction A_0$. □

The construction of $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ will be done recursively along a two-sided template $\mathcal{T}$. In particular, the forcing notion $\mathbb{S}$ will be adjoined in a not necessarily linear manner.

For the purpose of constructing $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$, referring to the elements of L up to a certain $x \in L$ and a notion for a subset $A \subseteq L$ containing all elements of L below A is useful.

Definition 4.10. Let $(L, \leq)$ be a linearly ordered set, $x \in L$. Then

$$\begin{aligned} L_x &:= \{y \in L : y < x\}, \\ L_x^{=} &:= \{y \in L : y \leq x\}. \end{aligned}$$

Let $L_0, A \subseteq L$, then

$$\mathrm{cl}_{L_0}(A) := A \cup \bigcup_{x \in A} L_x \cap L_0$$

is the L_0 closure of A. We will call A L_0 closed, if $\mathrm{cl}_{L_0}(A) = A$

At last, we define two-sided templates as a tool to split a linear order L into two disjoint sets L_0 and L_1, which are entangled in a manner, that is described by a set $\mathcal{I} \subseteq \mathcal{P}(L)$.

Definition 4.11. A two-sided template is a 4-tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ s.t. $(L, \leq)$ is a linearly ordered set, $\mathcal{I} \subseteq \mathcal{P}(L)$, and L_0, L_1 is a disjoint decomposition of L satisfying:

1. $\mathcal{I}$ is closed under finite intersections and unions and $\emptyset, L \in \mathcal{I}$.

2. If $x \in L$, $y \in L_1$ s.t. $x < y$, then there is $A \in \mathcal{I}$ s.t. $A \subseteq L_y$ and $x \in A$.

3. If $A \in \mathcal{I}, x \in L_1 \setminus A$, then $A \cap L_x \in \mathcal{I}$.

4. $\{A \cap L_1 : A \in \mathcal{I}\}$ is well founded with respect to inclusion.

5. All $A \in \mathcal{I}$ are L_0-closed.

For a two-sided template $\mathcal{T}$, $x \in L$, $A \in \mathcal{I}$ define

$$\begin{aligned} \mathcal{I}_A &:= \{B \in \mathcal{I} : B \subseteq A\}, \\ \mathcal{I}_x &:= \{B \in \mathcal{I} : B \subseteq L_x\}, \\ \mathcal{I}_{A,x} &:= \mathcal{I}_A \cap \mathcal{I}_x. \end{aligned}$$

Some proofs in the following text will employ induction over the rank of a two-sided template. The rank of a two-sided template captures how the structural composition of L_1 is tangible to the set $\mathcal{I}$. For this purpose, we define the depth function, which gives us a notion of how the elements of $\mathcal{I}$ are build up with respect to L_1.

$$\mathrm{Dp}: \mathcal{I} \longrightarrow \mathbb{ON} \begin{cases} \mathrm{Dp}(A) = 0 & \text{if } A \subseteq L_0, \\ \mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathcal{I} \wedge B \cap L_1 \subsetneq A \cap L_1\} & \text{else.} \end{cases}$$

If for example, $A \in \mathcal{I}$ contains two elements of L_1 but all subsets of A in $\mathcal{I}$ contain no element of L_1, $\mathrm{Dp}(A) = 1$. This example is highlighted to illustrate that the depth function is not a measure of the amount of elements of L_1 an $A \in \mathcal{I}$ contains, but rather a notion of the construction of A with respect to L_1 tangible to $\mathcal{I}$.

Define the rank of a two-sided template $\mathrm{Rk}(\mathcal{T}) = \mathrm{Dp}(L)$. For $A \subseteq L$, define

$$\mathcal{T}_A := ((A, \leq), \mathcal{I} \restriction A, L_0 \cap A, L_1 \cap A),$$

where $\mathcal{I} \restriction A := \{A \cap B : B \in \mathcal{I}\}$.

Note that in the calculation of $\mathrm{Rk}(\mathcal{T}_A)$, we refer to the function $\mathrm{Dp} : \mathcal{I} \restriction A \to \mathbb{ON}$. This function is not necessarily the same as the function $\mathrm{Dp} : \mathcal{I} \to \mathbb{ON}$. Since it will always be clear from the context which function we are referring to, we will continue to abuse this notation.

The recursive construction of $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ will be over the subtemplates $\mathcal{T}_{\mathcal{A}}$. For this, it is necessary that the subtemplates are well-defined and that the rank function is well behaved.

Lemma 4.12. *Let $\mathcal{T}$ be a two-sided template $A \subseteq L$, then $\mathcal{T}_A$ is a two-sided template. Also $\mathrm{Rk}(\mathcal{T}_A) \leq \mathrm{Rk}(\mathcal{T})$. And if $A \in \mathcal{I}$, $\mathrm{Rk}(\mathcal{T}_A) = \mathrm{Dp}(A)$.*

Proof. Since $A \subseteq L$ and $(L, \leq)$ is linearly ordered, $(A, \leq)$ is linearly ordered. Let $B \cap A \in \mathcal{I} \restriction A$, then $B \cap A \in \mathcal{P}(A)$. Since L_0, L_1 is a disjoint decomposition of L, $L_0 \cap A, L_1 \cap A$ is a disjoint decomposition of A.

1. Let $B_1, B_2 \in \mathcal{I}$. By Definition 4.11, $B_1 \cap B_2, B_1 \cup B_2, \emptyset, L \in \mathcal{I}$. So

$$\begin{aligned} &(A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2) \in \mathcal{I} \restriction A, \\ &(A \cap B_1) \cup (A \cap B_2) = A \cup (B_1 \cap B_2) \in \mathcal{I} \restriction A, \\ &\emptyset \cap A = \emptyset \in \mathcal{I} \restriction A, L \cap A = A \in \mathcal{I} \restriction A. \end{aligned}$$

2. Let $x, y \in A, y \in A \cap L_1, x < y$. By Definition 4.11, there exists $B \in \mathcal{I}$ s.t. $B \subseteq L_y$ and $x \in B$. So

$$A \cap B \in \mathcal{I} \restriction A, A \cap B \subseteq A \cap L_y, x \in A \cap B.$$

3. Let $A \cap B \in \mathcal{I} \restriction A, x \in L_1 \cap A \setminus B \cap A$. Then $x \in L_1 \setminus B$ and $B \in \mathcal{I}$. By Definition 4.11, $B \cap L_x \in \mathcal{I}$. So

$$(A \cap B) \cap A_x = A \cap (B \cap L_x) \in \mathcal{I} \restriction A.$$

4. Let
$$X \subseteq \{B \cap (A \cap L_1) : B \in \mathcal{I} \restriction A\} \neq \emptyset.$$
Then
$$X' := \{B \cap L_1 : B \cap (A \cap L_1) \in X\} \neq \emptyset.$$
Since $X' \subseteq \{B \cap L_1 : B \in \mathcal{I}\}$, it contains a minimal element C with respect to inclusion by Definition 4.11. Then $C \cap A \in X$ and $C \cap A \subseteq C' \cap A$ for all $C' \in X'$. So $C \cap A$ is the minimal element of X with respect to inclusion.

5. Let $B \in \mathcal{I}$. Then by Definition 4.11
$$B = B \cup (\bigcup_{x \in B} (L_x \cap L_0)).$$
Let $x \in B \cap A$. Since $L_x \cap L_0 \subseteq B$, $(L_x \cap L_0) \cap A \subseteq B \cap A$. So
$$A \cap B = (A \cap B) \cup (\bigcup_{x \in B \cap A} A_x \cap (L_0 \cap A))).$$

Proof that for all $B \in \mathcal{I}$, $\mathrm{Dp}(B \cap A) \leq \mathrm{Dp}(B)$, by induction over $\lambda = \mathrm{Rk}(\mathcal{T}_A)$:
Induction base: $\lambda = 0$. Then $\mathrm{Dp}(A) = 0$. So $\mathrm{Dp}(B') = 0$ for all $B' \in \mathcal{I} \restriction A$. So $\mathrm{Dp}(B \cap A) = 0 \leq \mathrm{Dp}(B)$ for all $B \in \mathcal{I} \restriction A$.
Induction assumption: For any $B \in \mathcal{I}$ be s.t. $\mathrm{Dp}(B) = \mu < \lambda$, $\mathrm{Dp}(B \cap A) \leq \mathrm{Dp}(B)$.
Successor step: Let $B \in \mathcal{I}$ be s.t. $\mathrm{Dp}(B) = \mu + 1$. Then by induction assumption $\mathrm{Dp}(B' \cap A) \leq \mathrm{Dp}(B')$ for any
$$B' \in \{C : C \in \mathcal{I} \wedge C \cap L_1 \subsetneq B \cap L_1\}.$$
Since any
$$A' \in \{D : D \in \mathcal{I} \restriction A \wedge D \cap (A \cap L_1) \subsetneq B \cap (A \cap L_1)\}$$
is of the form $B' \cap A$ for some $B' \in \{C : C \in \mathcal{I} \wedge C \cap L_1 \subsetneq B \cap L_1\}$, it holds true that $\mathrm{Dp}(B \cap A) \leq \mathrm{Dp}(B)$.
Limit step: Let $B \in \mathcal{I}$ be s.t. $\mathrm{Dp(B)} = \gamma < \lambda$ for some limit ordinal γ. Then $\mathrm{Dp}(A \cap B) \leq \mathrm{Dp}(B)$ follows analogously to the successor step.
So $\mathrm{Rk}(\mathcal{T}_A) \leq \mathrm{Rk}(\mathcal{T})$. $\mathrm{Rk}(\mathcal{T}_A) = \mathrm{Dp}(A)$ follows from the definition of Dp. □

Now we are ready to define $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

Definition 4.13. Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ be a two-sided template, $\mathbb{Q}$ a finite function forcing with the strong embedding property s.t. $L_0 = \mathrm{dom}(\mathbb{Q})$ and $\mathbb{S}$ a good σ-Suslin forcing notion. The poset $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is defined recursively along $\mathrm{Rk}(\mathcal{T})$:

1. If $\mathrm{Rk}(\mathcal{T}) = 0$, then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q}_{L_0}$.

2. Assume $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ has been defined for all $\mathcal{T}$ with $\mathrm{Rk}(\mathcal{T}) < \kappa$. Let $\mathcal{T}$ be a two-sided template of rank κ and $\mathbb{P}_B = \mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$ for $B \in \mathcal{I}$ s.t. $\mathrm{Db}(B) < \kappa$. Then $\mathbb{P} = (\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is defined by:

(a) $\mathbb{P}$ consists of all pairs $P = (p, F^p)$ where p is a finite partial function with $\mathrm{dom}(p) \subseteq L$, $P \restriction L_0 := (p \restriction L_0, F^p) \in \mathbb{Q}$ and if $x_p := \max\{\mathrm{dom}(p) \cap L_1\}$ is defined, then there is $B \in \mathcal{I}_{x_p}$ (called a witness that $P \in \mathbb{P}$) s.t.

$$P \mid\restriction L_{x_p} := (p \restriction L_{x_p}, F^p \cap \widehat{W}_B) \in \mathbb{P}_B,$$

$p(x_p) = (\check{s}^p_x, \dot{f}^p_x)$, where $s^p_x \in {}^{<\omega}\omega$, $\dot{f}^p_x$ is a $\mathbb{P}_B$ name for a real and

$$(P \mid\restriction L_{x_p}, p(x_p)) \in \mathbb{P}_B * \dot{\mathbb{S}}.$$

(b) For $P, Q \in \mathbb{P}$, let $Q \leq_{\mathbb{P}} P$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $Q \restriction L_0 \leq_{\mathbb{Q}} P \restriction L_0$, and if x_p is defined then either

i. $x_p < x_q$ and there exists a $B \in \mathcal{I}_{x_q}$ s.t. $P \mid\restriction L_{x_q}, Q \mid\restriction L_{x_q} \in \mathbb{P}_B$ and $Q \mid\restriction L_{x_q} \leq_{\mathbb{P}_B} P \mid\restriction L_{x_q}$, or

ii. $x_q = x_p$ and $\exists B \in \mathcal{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$, and s.t.

$$(Q \mid\restriction L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B * \dot{\mathbb{S}}} (P \mid\restriction L_{x_p}, p(x_p)).$$

We will call B as in i. or ii. a witness to $Q \leq_{\mathbb{P}} P$.

In the following text we will often show inclusion of some P in $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ or some Q extending some P in $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$. The simplified process:
Check

1. compatibility with respect to the domain,
2. that the L_0 part(s) satisfying the respective property,
3. that if the L_1 part is not empty, then the part below the maximal point in the L_1 part satisfies the respective property for some subtemplate build up by some $A \in \mathcal{I}$.

This construction is well behaved with respect to inclusion of the underlying sets and inclusion of the underlying orders.

Lemma 4.14. *Let $A \in \mathcal{I}$ and $\mathbb{P}_A, \mathbb{P}, \leq_{\mathbb{P}_A}, \leq_{\mathbb{P}}$ be as above. Then $\mathbb{P}_A \subseteq \mathbb{P}$ and $\leq_{\mathbb{P}_A} \subseteq \leq_{\mathbb{P}}$.*

Proof. Follows from $A \subseteq L$ and $L_1 \cap A \subseteq L_1$. $\square$

The main theorem of this chapter shows that the construction is compatible with complete embeddings. It also shows that $\mathbb{P}$ is well-defined and satisfies a stronger version of reduction.

Lemma 4.15. *Let $\mathcal{T} = ((L, \leq).\mathcal{I}, L_0, L_1)$ be a template, $\mathbb{Q}$ a finite function forcing notion with $L_0 = \mathrm{dom}(\mathbb{Q})$ which satisfies the strong embedding property and $\mathbb{S}$ a good σ-Suslin poset. Let $B \in \mathcal{I}, A \subsetneq B$ be closed. Then $\mathbb{P}_B$ is a partial order, $\mathbb{P}_A \subsetneq \mathbb{P}_B$ and even $\mathbb{P}_A \lessdot \mathbb{P}_B$. Also any $P = (p, F^p) \in \mathbb{P}_B$ has a canonical reduction $P_0 = (p_0, F^{p_0}) = p_0(P, A, B) \in \mathbb{P}_A$ s.t.*

1. $\mathrm{dom}(p_0) = \mathrm{dom}(p) \cap A, F^{p_0} = F^p \cap \widehat{W}_A$,

2. $s_x^{p_0} = s_x^p$ *for all* $x \in \mathrm{dom}(p_0) \cap L_1$,

3. $P_0 \restriction L_0 = (p_0 \restriction L_0, F^{p_0})$ *is a strong* $\mathbb{Q}_A$*-reduction of* $P \restriction L_0 = (p \restriction L_0, F^p)$

and whenever $D \in \mathcal{I}, B, C \subseteq D, C$ *is closed,* $C \cap B = A$ *and* $Q_0 \in \mathbb{P}_C$ *extends* P_0*, then there is* $Q \in \mathbb{P}_D$ *extending both* Q_0 *and* P*.*

The proof of Lemma 4.15 requires four lemmas. For the proofs of those lemmas we will assume that $\mathcal{T}, \mathbb{Q}, \mathbb{S}$ are as above and that Lemma 4.15 holds true for all templates $\mathcal{T}'$ s.t. $\mathrm{Rk}(\mathcal{T}') < \mathrm{Rk}(\mathcal{T})$. We work with $\mathbb{P} := \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

One condition extending another can always be traced down to the smallest $x \in L_1$ above the occurrences of both conditions.

Lemma 4.16. *If* $P = (p, F^p)$ *and* $Q = (q, F^q)$ *are conditions in* $\mathbb{P}$ *s.t.* $\mathrm{oc}(P)$ *and* $\mathrm{oc}(Q)$ *are contained in* L_x *for some* $x \in L_1$ *and* $Q \leq_{\mathbb{P}} P$*, then there is* $B \in \mathcal{I}_x$ *s.t.* $Q \leq_{\mathbb{P}_B} P$*.*

Proof. 1st case: x_p is defined and $x_p = x_q$ ($x_p < x_q$). Since $\mathrm{dom}(q) \subseteq \mathrm{oc}(Q) \subseteq L_x$, $x_q < x$. By Definition 4.13, there exists a $B' \in \mathcal{I}_{x_p}$ ($B' \in \mathcal{I}_{x_q}$) s.t. B' is a witness to $Q \leq_{\mathbb{P}} P$. Since $x_q \in L, x \in L_1, x_q < x$, there exists $A \in \mathcal{I}$ s.t. $A \subseteq L_x$ and $x_q \in A$ by Definition 4.11. Let $y \in \mathrm{oc}(Q) \cup \mathrm{oc}(P)$. Since $\mathrm{oc}(Q) \cup \mathrm{oc}(P) \in L_x$ and $x \in L_1$ Definition 4.11 implies that there exists $C_y \in \mathcal{I}$ such that $C_y \subseteq L_x$ and $y \in C_y$. Let

$$C := \bigcup_{y \in \mathrm{oc}(Q) \cup \mathrm{oc}(P)} L_y.$$

Since $\mathrm{oc}(Q) \cup \mathrm{oc}(P)$ is a finite set, $C \in \mathcal{I}$. Let

$$B := B' \cup A \cup C.$$

Since $A \subseteq L_x, B' \in \mathcal{I}_{x_q} \subseteq \mathcal{I}_x, C \subseteq L_x$ and $\mathcal{I}$ is closed under finite unions, $B \in \mathcal{I}_x$. Since $x_q \in A$ and $B' \subseteq L_{x_q}, B' \subseteq B$. So $B' \in \mathcal{I}_{B, x_q}$.
Now we want to show that B' is a witness for $Q \leq_{\mathbb{P}_B} P$. Note that

$$\mathrm{dom}(Q) \subseteq \mathrm{oc}(Q) \subseteq B.$$

Since $Q \in \mathbb{P}, Q \restriction L_0 \in \mathbb{Q}$. Since $B \subseteq L$ and $Q \restriction L_0 \in \mathbb{P}$,

$$Q \restriction (L_0 \cap B) = (Q \restriction L_0) \restriction B \in \mathbb{Q}.$$

Since $\mathrm{oc}(Q) \subseteq B$,

$$\mathrm{dom}(Q) \cap L_1 \cap B = \mathrm{dom}(Q) \cap L_1.$$

So the x_q in $\mathcal{T}$ is the same as the x_q in $\mathcal{T}_B$. Since $\mathrm{dom}(Q) \subseteq B, q \restriction L_{x_p} = q \restriction B_{x_p}$. So

$$Q \mid\restriction L_{x_p} = Q \mid\restriction B_{x_p}.$$

Analogously

$$Q \mid\restriction L_0 = Q \mid\restriction (B \cap L_0).$$

$P \mid\restriction L_{x_p} = P \mid\restriction B_{x_p}$ and $P \mid\restriction L_0 = P \mid\restriction (B \cap L_0)$ follow analogously. So B' satisfying all the conditions to witness $Q \leq_{\mathbb{P}_B} P$ follows from B' satisfying all the conditions to witness $Q \leq_{\mathbb{P}} P$.

2nd case: x_p is not defined in $\mathcal{T}$. Then x_p is also not defined in $\mathcal{T}_B$. Since is holds true that $\mathrm{oc}(Q) \cup \mathrm{oc}(P) \in B$,

$$q \restriction (L_0 \cap B) = q \restriction L_0 \text{ and } p \restriction (L_0 \cap B) = p \restriction L_0.$$

Since $Q \leq_{\mathbb{P}} P$, $P, Q \in \mathbb{P}_B, \mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $(q \restriction L_0, F^q) \leq_{\mathbb{P}_B} (p \restriction L_0, F^p)$. So $Q \leq_{\mathbb{P}} P$. $\square$

Inclusion and extension in $\mathbb{P}$ are compatible with hard restricting below and below-equal any $x_0 \in L$.

Lemma 4.17. *Let $P = (p, F^p) \in \mathbb{P}$, $Q = (q, F^q) \in \mathbb{P}$ and let $x_0 \in L$. Then*

$$Q \mid\restriction L_{x_0} \in \mathbb{P}, Q \mid\restriction L_{x_0}^{=} \in \mathbb{P}$$

and if $Q \leq_{\mathbb{P}} P$ then

$$Q \mid\restriction L_{x_0} \leq_{\mathbb{P}} P \mid\restriction L_{x_0} \text{ and } Q \mid\restriction L_{x_0}^{=} \leq_{\mathbb{P}} P \mid\restriction L_{x_0}^{=}.$$

Proof. The statements $Q \mid\restriction L_{x_0} \in \mathbb{P}$ and $Q \mid\restriction L_{x_0} \leq_{\mathbb{P}} P \mid\restriction L_{x_0}$ follow from induction over $n_q := |\mathrm{dom}(q) \cap L_1|$.

Induction base: $n_q = 0$. Then $\mathrm{dom}(q) \subseteq L_0 = \mathrm{dom}(\mathbb{Q})$ and x_q is not defined. Since $Q \in \mathbb{P}$, $Q \restriction L_0 = Q \in \mathbb{Q}$. Since $L_{x_0} \cap L_0 \subseteq L_0 = \mathrm{dom}(\mathbb{Q})$, Definition 4.5 implies

$$(Q \mid\restriction L_{x_0}) \restriction L_0 = (Q \mid\restriction L_{x_0}) \mid\restriction L_0 = Q \mid\restriction (L_{x_0} \cap L_0) \in \mathbb{Q}.$$

So $Q \mid\restriction L_{x_0} \in \mathbb{P}$. Analogously $P \mid\restriction L_{x_0} \in \mathbb{P}$.
Since $Q \leq_{\mathbb{P}} P, \mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and

$$Q = (q \restriction L_0, F^q) \leq_{\mathbb{Q}} (p \restriction L_0, F^p) = P.$$

Since $L_{x_0} \cap L_0 \subseteq L_0 = \mathrm{dom}(\mathbb{Q})$, Definition 4.5 implies

$$\begin{aligned}(Q \mid\restriction L_{x_0}) \restriction L_0 &= (Q \mid\restriction L_{x_0}) \mid\restriction L_0 \\ &= Q \mid\restriction (L_{x_0} \cap L_0) \\ &\leq_{\mathbb{Q}} P \mid\restriction (L_{x_0} \cap L_0) \\ &= (P \mid\restriction L_{x_0}) \restriction L_0.\end{aligned}$$

So $Q \mid\restriction L_{x_0} \leq_{\mathbb{P}} P \mid\restriction L_{x_0}$.

Induction assumption: $Q \mid\restriction L_{x_0} \in \mathbb{P}$ and $Q \mid\restriction L_{x_0} \leq_{\mathbb{P}} P \mid\restriction L_{x_0}$ hold true for $n_q < n$.

Induction step: Let $n_q = n$. Since $n_q > 0$, x_q is defined.

1st case: $x_q < x_0$. Then $Q \mid\restriction L_{x_0} \in \mathbb{Q}$ follows analogously to the induction base.

Since $Q \in \mathbb{P}$, there exists a witness $B \in \mathcal{I}_{x_p}$ for $Q \in \mathbb{P}$. Let $Q' := Q \mid\restriction L_{x_0}$. Since $x_q < x_0$,

$$\begin{aligned} x_{q'} &= \max(\mathrm{dom}(q') \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L_{x_0} \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L_1) \\ &= x_q. \end{aligned}$$

So $B \in \mathcal{I}_{x_{q'}}$. So B satisfying the conditions to witness $Q' \in \mathbb{P}$ follows from B satisfying the conditions to witness $Q \in \mathbb{P}$.
2nd case: $x_0 \leq x_q$. Since $x_q \notin L_{x_0}$,

$$\begin{aligned} n_{q\restriction L_{x_0}} &= |\mathrm{dom}(q \restriction L_{x_0}) \cap L_1| \\ &< |\mathrm{dom}(q) \cap L_1| \\ &= n_q \\ &= n. \end{aligned}$$

So by the inductive assumption, $Q \mid\restriction L_{x_0} \in \mathbb{P}$.
So $Q \mid\restriction L_{x_0} \in \mathbb{P}$. $P \mid\restriction L_{x_0} \in \mathbb{P}$ follows analogously.
1st case: $\mathrm{dom}(p \restriction L_{x_0}) \subseteq L_0 = \mathrm{dom}(\mathbb{Q})$. Then $x_{p\restriction L_{x_0}}$ is not defined. Since it holds true that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $\mathrm{dom}(p \restriction L_{x_0}) \subseteq \mathrm{dom}(q \restriction L_{x_0})$. Since $Q \leq_{\mathbb{P}} P$, $Q \restriction L_0 \leq_{\mathbb{Q}} P \restriction L_0$. Since $Q \restriction L_0, P \restriction L_0 \in \mathbb{Q}$, $L_{x_0} \cap L_0 \subseteq L_0$ and $Q \restriction L_0 \leq_{\mathbb{Q}} P \restriction L_0$ Definition 4.5 implies

$$\begin{aligned} (Q \mid\restriction L_{x_0}) \restriction L_0 &= (Q \restriction L_0) \mid\restriction L_{x_0} \\ &= (Q \restriction L_0) \mid\restriction (L_{x_0} \cap L_0) \\ &\leq_{\mathbb{Q}} (P \restriction L_0) \mid\restriction L_{x_0} \\ &= (P \mid\restriction L_{x_0}) \restriction L_0. \end{aligned}$$

So $Q \mid\restriction L_{x_0} \leq_{\mathbb{Q}} P \mid\restriction L_{x_0}$.
2nd case: $\mathrm{dom}(p \restriction L_{x_0}) \subsetneq L_0 = \mathrm{dom}(\mathbb{Q})$. Then $x_{p\restriction L_{x_0}}$ is defined. Let $B \in \mathcal{I}_{x_q}$ be a witness to $Q \leq_{\mathbb{P}} P$.
1st subcase: $x_q < x_0$. Then $x_q = x_{q\restriction L_{x_0}}$. So

$$\mathrm{dom}(p \restriction L_{x_0}) = \mathrm{dom}(p) \subseteq \mathrm{dom}(q) = \mathrm{dom}(q \restriction L_{x_0}).$$

$(Q \mid\restriction L_{x_0}) \restriction L_0 \leq_{\mathbb{Q}} (P \mid\restriction L_{x_0}) \restriction L_0$ follows analogously to above. B satisfying the other properties of a witness to $Q \mid\restriction L_{x_0} \leq_{\mathbb{Q}} P \mid\restriction L_{x_0}$ follows from $x_q = x_{q\restriction L_{x_0}}$, $x_p = x_{p\restriction L_{x_0}}$ and B satisfying the properties of a witness to $Q \leq_{\mathbb{Q}} P$.
2nd subcase: $x_0 \leq x_q$. By Definition 4.13 $Q \mid\restriction L_{x_q} \leq_{\mathbb{P}_B} P \mid\restriction L_{x_q}$. And Lemma 4.14 implies that $Q \mid\restriction L_{x_q} \leq_{\mathbb{P}} P \mid\restriction L_{x_q}$.
1st subsubcase: $x_0 = x_q$. Then $Q \mid\restriction L_{x_0} \leq_{\mathbb{P}} P \mid\restriction L_{x_0}$.
2nd subsubcase: $x_0 < x_q$. Since $x_q \notin L_{x_q}$, $n_{q\restriction L_{x_q}} < n_q = n$. So by the induction assumption,

$$\begin{aligned} Q \mid\restriction L_{x_0} &= (Q \restriction L_{x_q}) \mid\restriction L_{x_0} \\ &\leq_{\mathbb{P}} (P \restriction L_{x_q}) \mid\restriction L_{x_0} \\ &= P \mid\restriction L_{x_0}. \end{aligned}$$

So $Q \mid\mid\restriction L_{x_0} \leq_{\mathbb{Q}} P \mid\mid\restriction L_{x_0}$
$Q \mid\mid\restriction L^{=}_{x_0} \leq_{\mathbb{P}} P \mid\mid\restriction L^{=}_{x_0}$ follows analogously by changing the case distinctions in the induction step from $x_q < x_0$ and $x_0 \leq x_q$ to $x_q \leq x_0$ and $x_0 < x_q$. $\square$

For any two conditions $P, Q \in \mathbb{P}$ s.t. w.l.o.g. $\mathrm{dom}(P) \subseteq \mathrm{dom}(Q)$, Q extending P can be reduced to the restrictions below-equal the maximal point in $L_1 \cap \mathrm{dom}(P)$.

Lemma 4.18. *Let $P = (p, F^p) \in \mathbb{P}$ and $Q = (q, F^q) \in \mathbb{P}$. If $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $Q \restriction L_0 \leq_{\mathbb{Q}} P \restriction L_0$ and $Q \mid\mid\restriction L^{=}_{x_p} \leq_{\mathbb{P}} P \mid\mid\restriction L^{=}_{x_p}$, then $Q \leq_{\mathbb{P}} P$.*

Proof. Since $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $x_p \leq x_q$.
<u>1st case</u>: $x_q = x_p$. Let B be a witness to $Q \mid\mid\restriction L^{=}_{x_p} \leq_{\mathbb{P}} P \mid\mid\restriction L^{=}_{x_p}$. By assumption $Q, P \in \mathbb{P}$, $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $Q \restriction L_0 \leq_{\mathbb{Q}} P \restriction L_0$. Let $Q' := Q \mid\mid\restriction L^{=}_{x_q}$. Then

$$\begin{aligned} x_{q'} &= \max(\mathrm{dom}(q') \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L^{=}_{x_q} \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L_1) \\ &= x_q. \end{aligned}$$

Analogously $x_{p'} = x_p$ for $P' = P \mid\mid\restriction L^{=}_{x_q}$. So $x_{q'} = x_{p'}$. B satisfying all the properties of a witness for $Q \leq_{\mathbb{P}} P$ follows from B satisfying all the properties of a witness for $Q' \leq_{\mathbb{P}} P'$.
<u>2nd case</u>: $x_p < x_q$. Let $x := x_p$, $(\mathrm{dom}(q) \cap L_1) \setminus L^{=}_{x_p} = \{x_j\}_{j=1}^n$ be increasing and let $H \in \mathcal{I}_{x_p}$ be a witness to $Q \mid\mid\restriction L^{=}_{x_p} \leq_{\mathbb{P}} P \mid\mid\restriction L^{=}_{x_p}$. Since $y < x_j \in L_1$ for $y \in \mathrm{oc}(Q \mid\mid\restriction L_{x_j}) \cup \mathrm{oc}(P \mid\mid\restriction L_{x_j})$, Definition 4.11 implies that there exists $A_y \in \mathcal{I}$ s.t. $A \subseteq L_{x_j}$ and $y \in A_y$. Let

$$H_0 := H \text{ and } A_j := \bigcup_{y \in \mathrm{oc}(Q \mid\mid\restriction L_{x_j}) \cup \mathrm{oc}(P \mid\mid\restriction L_{x_j})} A_y.$$

Define $H_j := H_{j-1} \cup A_j$ for $1 \leq j \leq n$. Then $\{H_j\}_{j=1}^n$ is an increasing sequence of elements of $\mathcal{I}$ s.t. $H_j \subseteq L_{x_j}$ and $\mathrm{oc}(Q \mid\mid\restriction L_{x_j}), \mathrm{oc}(P \mid\mid\restriction L_{x_j}) \subseteq H_j$. By Lemma 4.17, $Q \mid\mid\restriction L_{x_j}, Q \mid\mid\restriction L_{x_j} \in \mathbb{P}$ for all $1 \leq j \leq n$. Since $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,

$$\mathrm{dom}(p \restriction L_{x_j}) \subseteq \mathrm{dom}(q \restriction L_{x_j})$$

for all $1 \leq j \leq n$. Since $Q \mid\mid\restriction L_0 \leq_{\mathbb{Q}} P \mid\mid\restriction L_0$ and $L_0 \cap L_{x_j} \subseteq L_0$,

$$\begin{aligned} (Q \mid\mid\restriction L_{x_j}) \restriction L_0 &= (Q \mid\mid\restriction L_0) \mid\mid\restriction L_{x_j} \\ &= (Q \restriction L_0) \\ &\leq_{\mathbb{Q}} (P \mid\mid\restriction L_{x_j}) \restriction L_0 \end{aligned}$$

for all $1 \leq j \leq n$. Let $Q'_j := Q \mid\mid\restriction L_{x_j}$. Since $x_j \in L_1$,

$$\begin{aligned} x_{q'_j} &= \max(\mathrm{dom}(q') \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L_{x_j} \cap L_1) \\ &= \max(\mathrm{dom}(q) \cap L_1) \\ &= x_q. \end{aligned}$$

for all $1 \leq j \leq n$. Analogously $x_{p'_j} = x_p$ for all $1 \leq j \leq n$. Since $H_j \in \mathcal{I}_{x_q}$ for $1 \leq j \leq n$, Lemma 4.14 implies that H_{j-1} is a witness for $Q \mathbin{|\!\upharpoonright} L_{x_j} \leq_{\mathbb{P}_{H_j}} P \mathbin{|\!\upharpoonright} L_{x_j}$ for all $1 \leq j \leq n$. In particular H_n is a witness for $Q \leq_{\mathbb{P}} P$. $\square$

Let $Q \in \mathbb{Q}$ be s.t. it has an element $x \in L_1$ as the maximal point of its domain. Then the tail above $L_x^=$ of any $P \in \mathbb{P}$, s.t. Q extends the part of P below-equal x and extending P in $\mathbb{Q}$ can be reduced to extending P in $\mathbb{Q}$ below x, can be adjoined to Q in a manner that yields a common extension of Q and P.

Lemma 4.19. *Let $Q = (q, F^q) = Q \mathbin{|\!\upharpoonright} L_x^=$ be s.t. $x = \max\{\mathrm{dom}(q) \cap L_1\}$ is a condition in $\mathbb{P}$ with witness $\bar{D}$. Let $P = (p, F^p)$ be a condition s.t. $(P \mathbin{|\!\upharpoonright} L_x) \upharpoonright L_0$ is a strong $\mathbb{Q}_{L_0 \cap L_x}$-reduction of $P \upharpoonright L_0$ and s.t. $Q \leq_{\mathbb{P}} P \mathbin{|\!\upharpoonright} L_x^=$ with witness $\bar{D}$. Then*

$$Q \bar{\ltimes}_x P = (q \bar{\ltimes}_x p, F^q \cup F^p)$$

is a common extension of Q and P where

$$q \bar{\ltimes}_x p = q \cup p \upharpoonright L \setminus L_x^=.$$

Proof. Since $q \bar{\ltimes}_x p \upharpoonright L_0 = q \upharpoonright L_0 \cup p \upharpoonright L_0 \setminus L_x^=$ and $(P \mathbin{|\!\upharpoonright} L_x) \upharpoonright L_0$ is a strong $\mathbb{Q}_{L_0 \cap L_x}$-reduction of $P \upharpoonright L_0$,

$$(Q \bar{\ltimes}_x P) \upharpoonright L_0 \leq_{\mathbb{Q}} P \upharpoonright L_0.$$

Since $Q \in \mathbb{P}$, $Q \upharpoonright L_0 \in \mathbb{Q}$. Since $Q \mathbin{|\!\upharpoonright} L_x^= = Q$,

$$\mathrm{dom}(p \upharpoonright L_0 \setminus L_x^=) \cap \mathrm{oc}(Q \upharpoonright L_0) = \emptyset.$$

By Definition 4.7, $P \upharpoonright L_0 \in \mathbb{Q} \mathbin{|\!\upharpoonright} (L_x \cap L_0)$. So $F^p \in [\widehat{W}_{L_0}]^{<\omega}$. So $F^q \cup F^p \in [\widehat{W}_{L_0}]^{<\omega}$. By Definition 4.5,

$$\begin{aligned}(Q \bar{\ltimes}_x P) \upharpoonright L_0 &= (q \upharpoonright L_0 \cup p \upharpoonright L_0 \setminus L_x^=, F^q \cup F^p)\\ &\leq_{\mathbb{Q}} (q \upharpoonright L_0 \cup p \upharpoonright L_0 \setminus L_x^=, F^q)\\ &\leq_{\mathbb{Q}} (q \upharpoonright L_0, F^q)\\ &= Q \upharpoonright L_0.\end{aligned}$$

Let $n_p := |\mathrm{dom}(p) \cap L_1 \setminus L_x^=|$. Show that $Q \bar{\ltimes}_x P$ is a common extension of P and Q by induction over n_p:
Induction base: $n_p = 0$. Then

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q \cup p \upharpoonright L \setminus L_x^=) = \mathrm{dom}(q \bar{\ltimes}_x p)$$

and

$$\mathrm{dom}(q) \subseteq \mathrm{dom}(q \cup p \upharpoonright L \setminus L_x^=) = \mathrm{dom}(q \bar{\ltimes}_x p).$$

$(Q \bar{\ltimes}_x P) \upharpoonright L_0 \leq_{\mathbb{Q}} Q \upharpoonright L_0$ and $(Q \bar{\ltimes}_x P) \upharpoonright L_0 \leq_{\mathbb{Q}} P \upharpoonright L_0$ were shown above. Since $\mathrm{dom}(p) \cap L_1 \setminus L_x^= = \emptyset$,

$$\begin{aligned}\max\{\mathrm{dom}(q) \cap L_1\} &= \max\{(\mathrm{dom}(q) \cap L_1) \cup (\mathrm{dom}(p) \cap L_1 \setminus L_x^=)\}\\ &= \max\{\mathrm{dom}(q \bar{\ltimes}_x p) \cap L_1\}.\end{aligned}$$

and

$$\max\{\mathrm{dom(p)} \cap L_1\} \leq \max\{(\mathrm{dom}(q) \cap L_1) \cup (\mathrm{dom}(p) \cap L_1 \setminus L_x^=)\}$$
$$= \max\{\mathrm{dom}(q \bar{\ltimes}_x p) \cap L_1\}$$

if defined.

1st case: $x_{q\bar{\ltimes}_x p}$ is not defined. Then x_q and x_p are not defined. So $Q\bar{\ltimes}_x P \leq_{\mathbb{P}} Q, P$.

2nd case: $x_{q\bar{\ltimes}_x p}$ is defined. Since $\bar{D}$ witnesses $Q \in \mathbb{P}$ and $Q \leq_{\mathbb{P}} P \mid\upharpoonright L_x^=$, $\bar{D}$ witnesses $Q\bar{\ltimes}_x P \leq_{\mathbb{P}} P, Q$.

Induction assumption: $Q\bar{\ltimes}_x P \leq_{\mathbb{P}} P$ and $Q\bar{\ltimes}_x P \leq_{\mathbb{P}} Q$ is true for every $P \in \mathbb{P}$ with $0 \leq n_p < n$.

Induction step: Let $P \in \mathbb{P}$ with $n_p = n$. Since $\mathrm{dom}(q) \subseteq \mathrm{dom}(p \upharpoonright L_x^=) \subseteq \mathrm{dom}(p)$ and $\mathrm{dom}(q) \subseteq L_x^=$,

$$\begin{aligned} x_p &= \max\{\mathrm{dom}(p) \cap L_1\} \\ &> \max\{\mathrm{dom}(p) \cap L_1 \cap L_x^=\} \\ &\geq \max\{\mathrm{dom}(q) \cap L_1\} \\ &= x. \end{aligned}$$

Since $Q \mid\upharpoonright L_x^= = Q \leq_{\mathbb{P}} P \mid\upharpoonright L_x^=$, Lemma 4.17 implies

$$Q = (Q \mid\upharpoonright L_{x_p}) \mid\upharpoonright L_x^= \leq_{\mathbb{P}} (P \mid\upharpoonright L_{x_p}) \mid\upharpoonright L_x^=.$$

Since $x_p \notin \mathrm{dom}(q)$ and $x_p \notin \mathrm{dom}(p \upharpoonright L_x^=)$, the induction hypothesis yields

$$Q\bar{\ltimes}_x P \mid\upharpoonright L_x^= \leq_{\mathbb{P}} Q$$
$$\text{and } Q\bar{\ltimes}_x P \mid\upharpoonright L_x^= \leq_{\mathbb{P}} P \mid\upharpoonright L_x^=.$$

Since $Q \mid\upharpoonright L_x^= = Q$, $(P \mid\upharpoonright L_x^=) \mid\upharpoonright L_x^= = P \mid\upharpoonright L_x^=$ and $x_p > x$,

$$\mathrm{oc}(Q) \cup \mathrm{oc}(P \mid\upharpoonright L_x^=) \in L_{x_p}.$$

By Lemma 4.16, there exists a $B \in \mathcal{I}_{x_p}$ s.t. $Q\bar{\ltimes}_x(P \mid\upharpoonright L_{x_p}) \leq_{\mathbb{P}_B} Q$ and a $B' \in \mathcal{I}_{x_p}$ s.t.

$$Q\bar{\ltimes}_x(P \mid\upharpoonright L_{x_p}) \leq_{\mathbb{P}_{B'}} P \mid\upharpoonright L_{x_p}.$$

Since $B_0 := B \cup B' \in \mathcal{I}_{x_p}$ and $B, B' \in \mathcal{I}_{x_p} \upharpoonright B_0$,

$$Q\bar{\ltimes}_x(P \mid\upharpoonright L_{x_p}) \leq_{\mathbb{P}_{B_0}} Q$$

and

$$Q\bar{\ltimes}_x(P \mid\upharpoonright L_{x_p}) \leq_{\mathbb{P}_{B_0}} P \mid\upharpoonright L_{x_p}.$$

Let $B_1 \in \mathcal{I}_{x_p}$ be a witness to $P \in \mathbb{P}$. Since $((P \mid\upharpoonright L_{x_p}), p(x_p)) \in \mathbb{P}_{B_1} * \dot{\mathbb{S}}$,

$$P \mid\upharpoonright L_{x_p} \in \mathbb{P}_{B_1} \text{ and } P \mid\upharpoonright L_{x_p} \Vdash_{\mathbb{P}_{B_1}} p(x_p) \in \dot{\mathbb{S}}.$$

Let $C := B_0 \cup B_1$. Since $B_0, B_1 \in \mathcal{I}_{x_p}$, $C \in \mathcal{I}_{x_p}$. Since $B_0, B_1 \in \mathcal{I} \upharpoonright C$, Lemma 4.14 implies that $\leq_{\mathbb{P}_{B_1}} \subseteq \leq_{\mathbb{P}_C}$. So

$$Q\bar{\ltimes}_x(P \mid\upharpoonright L_{x_p}) \leq_{\mathbb{P}_C} Q, P \mid\upharpoonright L_{x_p}.$$

So

$$(Q\bar{\ltimes}_x P) \,|\!\restriction L_{x_p} = Q\bar{\ltimes}_x(P \,|\!\restriction L_{x_p}) \Vdash_{\mathbb{P}_C} p(x_p) \in \dot{\mathbb{S}}.$$

So C is a witness for $Q\bar{\ltimes}_x P \in \mathbb{P}_C$. Since $\mathrm{dom}(q) \in L_x^{=}$ and $x_p > x$,

$$\begin{aligned} x_{q\bar{\ltimes}_x p} &= \max\{\mathrm{dom}(q\bar{\ltimes}_x p) \cap L_1\} \\ &= \max\{(\mathrm{dom}(q) \cup \mathrm{dom}(p \restriction L \setminus L_x^{=})) \cap L_1\} \\ &= \max\{\mathrm{dom}(p) \cap L_1\} \\ &= x_p. \end{aligned}$$

Since $B_0 \subseteq C$ and $Q\bar{\ltimes}_x(P \,|\!\restriction L_{x_p}) \leq_{\mathbb{P}_{B_0}} P \,|\!\restriction L_{x_p}$,

$$Q\bar{\ltimes}_x(P \,|\!\restriction L_{x_p}) \leq_{\mathbb{P}_C} P \,|\!\restriction L_{x_p}.$$

So C is a witness for $Q\bar{\ltimes}_x P \leq_{\mathbb{P}} P$. Since $x_q < x_{q\bar{\ltimes}_x p}$, B_0 is a witness for

$$Q\bar{\ltimes}_x P \leq_{\mathbb{P}} Q.$$

□

Explanation to the figure appearing on page 46:

In the proof of Lemma 4.15 we will find a condition T s.t.

$$\begin{aligned} T \Vdash_{\mathbb{P}_{\bar{D}}} {}^{\prime\prime}&(\check{s}_x^{q_0}, \dot{f}_x^{q_0}) \leq_{\dot{\mathbb{S}}} (\check{s}_x^{p_0}, \dot{f}_x^{p_0}) \\ &\wedge \check{s}_x^{p} = \check{s}_x^{p_0} \\ &\wedge \dot{f}_x^{p} \restriction n \cdot m = \dot{f}_x^{p_0} \restriction n \cdot m^{\prime\prime} \end{aligned}$$

for some poset $\bar{D}$. The diagram below shows where those statements are inherited.

We will denote extensions with arrows, reductions with dashed arrows and canonical reductions with arrows with double lines. $p \xrightarrow{\mathrm{A}} q$, $q \overset{A}{\dashrightarrow} p$ and $q \overset{A}{\Longrightarrow} p$ should be understood as "condition p extends condition q in A" or "condition p is a reduction of condition q to A" respectively.

$$
\begin{tikzcd}
U \Vdash \text{"}\dot f_x^{p_0} \restriction n \cdot m = s'\text{"} \arrow[d, "\bar A"] & P_0 \mid\restriction L_x \arrow[dl, "\bar A"] \arrow[rr, "\bar C"] & & \bar Q_0 \Vdash \text{"}(\check s_x^{q_0}, \dot f_x^{q_0}) \leq_{\dot{\mathbb S}} (\check s_x^{p_0}, \dot f_x^{p_0})\text{"} \arrow[dll, "\bar C"] \\
E' \arrow[d, "\bar A"] \arrow[dr, "\bar C"] & \hat Q_0 \arrow[d, "\bar C"] & & \\
\tilde Q_0 \arrow[dr, "\bar A"] & \hat{\hat Q}_0 \arrow[l, dashed, "\bar A"] \arrow[d, dashed, "\bar A"] \arrow[dddd, bend right=60, "\bar C"'] & \check a \Vdash \text{"}\dot f_x^{p_0} \restriction n \cdot m = s(\check a)\text{"} \arrow[dl, "\bar A"] & \check b \Vdash \text{"}\dot f_x^{p} \restriction n \cdot m = s'\text{"} \arrow[l, dashed, "\bar A"] \arrow[ddl, "\bar B"] \\
 & P_0^* \arrow[dr, "\bar B"] \arrow[dd, bend right=50, "\bar A"'] & & \\
 & \widehat P^+ \arrow[d, "\bar A"] & \bar P^+ \arrow[l, Rightarrow, "\bar A"] \arrow[dddl, "\bar D"] & \\
 & \bar P_0^+ \arrow[d, "\bar C"] & & \\
 & \bar Q_0^+ \arrow[d, "\bar D"] & & \\
 & T & &
\end{tikzcd}
$$

Proof of Lemma 4.15:

Proof. Proof by induction over $\mathrm{Rk}(\mathcal{T}_B)$.
<u>Induction base</u>: $\mathrm{Rk}(\mathcal{T}_B) = 0$. Then $B \subseteq L_0$ and $\mathbb{P}_B = \mathbb{Q}_B$. Let $A \subseteq B$ be closed. Then $\mathbb{P}_B = \mathbb{Q}_B$ is by definition a partial order. Since $A \subseteq B$, $\mathrm{Rk}(\mathcal{T}_A) \leq \mathrm{Rk}(\mathcal{T}_B) = 0$ by Lemma 4.12. So $\mathbb{P}_A = \mathbb{Q}_A$. So $\mathbb{P}_A \lessdot \mathbb{P}_B$. Let $P = (p, F^p) \in \mathbb{P}_B = \mathbb{Q}_B$, $P_0 = P \mid\restriction A$. Since $P \in \mathbb{Q}_B$, $P_0 \in \mathbb{Q}_A = \mathbb{P}_A$. Then

1. $\mathrm{dom}(p_0) = \mathrm{dom}(p) \cap A$ and $F^{p_0} = F^p \cap A$,
2. $s_x^{p_0} = s_x^p$ for all $x \in \mathrm{dom}(p_0) \cap L_1$,
3. $P_0 \restriction L_0 = P_0$ is a strong $\mathbb{Q}_A$-reduction of $P \restriction L_0 = P$.

Let $D \in \mathcal{I}$, $B, C \subseteq D$, C closed, $C \cap B = A$ and $Q_0 \in \mathbb{P}_C$ be s.t. it extends P_0. Since $Q_0 \mid\restriction \mathrm{Oc}(P_0) \leq_{\mathbb{Q}_A} Q$,

$$Q_0 \mid\restriction \mathrm{Oc}(P_0) \leq_{\mathbb{Q}_A} P_0 \mid\restriction \mathrm{Oc}(P_0) = P_0$$

and P_0 is a strong $\mathbb{Q}_A$-reduction of $\mathbb{P}$, $Q_0 \mid\restriction \mathrm{Oc}(P_0)$ extends both Q_0 and P.
<u>Induction assumption</u>: Lemma 4.15 holds true for all templates of rank $< \alpha$.
<u>Induction step</u>: Let $\mathrm{Rk}(\mathcal{T}_B) = \alpha$ and $\mathbb{P} = \mathbb{P}_B$.

<u>Transitivity of $\leq_{\mathbb{P}}$</u>: Let $P_0, P_1, P_2 \in \mathbb{P}$ s.t. $P_2 \leq_{\mathbb{P}} P_1$ and $P_1 \leq_{\mathbb{P}} P_0$.
<u>1st case</u>: x_{P_0} is not defined. Since

$$\mathrm{dom}(P_0) \subseteq \mathrm{dom}(P_1) \subseteq \mathrm{dom}(P_2)$$
$$\text{and } P_2 \restriction L_0 \leq_{\mathbb{Q}} P_1 \restriction L_0 \leq_{\mathbb{Q}} P_0 \restriction L_0,$$

$P_2 \leq_{\mathbb{P}} P_0$.
<u>2nd case</u>: x_{P_0} is defined. Then x_{P_1} and x_{P_2} are defined. Let $B_1 \in \mathcal{I}_{x_{P_1}}$ be a witness for $P_1 \leq_{\mathbb{P}} P_0$ and $B_2 \in \mathcal{I}_{x_{P_2}}$ be a witness for $P_2 \leq_{\mathbb{P}} P_1$.
Since $\mathrm{Dp}(B_1 \cup B_2) < \alpha$, the induction assumtion implies that $\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \lessdot \mathbb{P}_{B_1 \cup B_2}$. So $P_i \restriction L_{x_{p_2}} = P_i \restriction B_1 \cup B_2 \in \mathbb{P}_{B_1 \cup B_2}$ for $0 \leq i \leq 2$ and

$$P_2 \mid\restriction L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_1 \mid\restriction L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_0 \mid\restriction L_{x_{p_2}}.$$

Since $\leq_{\mathbb{P}_{B_1 \cup B_2}}$ is transitive by assumption, $P_2 \mid\restriction L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_0 \mid\restriction L_{x_{p_2}}$.
<u>1st case</u>: $x_{p_0} = x_{p_2}$. Then $B_1 \cup B_2$ witnesses $P_2 \leq_{\mathbb{P}} P_0$.
<u>2nd case</u>: Then $p_i(x_{p_2})$ is a $\mathbb{P}_{B_1 \cup B_2}$-name for $0 \leq i \leq 2$. Since $\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \subseteq \mathbb{P}_{B_1 \cup B_2}$,

$$P_1 \mid\restriction L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_1(x_{p_2}) \leq_{\dot{S}} p_0(x_{p_2})$$
$$\text{and } P_2 \mid\restriction L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_2(x_{p_2}) \leq_{\dot{S}} p_1(x_{p_2}).$$

Since $P_2 \mid\restriction L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_1 \mid\restriction L_{x_{p_2}}$,

$$P_2 \mid\restriction L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_1(x_{p_2}) \leq_{\dot{S}} p_0(x_{p_2}).$$

So

$$P_2 \mid\restriction L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_2(x_{p_2}) \leq_{\dot{S}} p_0(x_{p_2}).$$

So

$$(P_2 \,||\!\restriction L_{x_{p_2}}, p_2(x_{p_2})) \leq_{\mathbb{P}_{B_1 \cup B_2} * \dot{S}} (P_0 \,||\!\restriction L_{x_{p_2}}, p_0(x_{p_2})).$$

So $P_2 \leq_{\mathbb{P}} P_0$.

<u>Suborder</u>: Let $B \in \mathcal{I}$ and $A \subseteq B$ be closed. Let $R = (r, F^r) \in \mathbb{P}_A$ and $x = x_r$. By Definition 4.13, there exists $\bar{A} \in \mathcal{I}_{A,x}$ s.t. $R \,||\!\restriction (A \cap L_x) \in \mathbb{P}_A$ and $\dot{f}^r_x$ is a $\mathbb{P}_A$-name. Since $\bar{A} \in \mathcal{I} \restriction A$, there exists $B_0 \in \mathcal{I}$ s.t. $\bar{A} = B_0 \cap A$. Since $A \subseteq B$, $\bar{A} \subseteq B$. So $B_0 \cap A = B_0 \cap B \cap A$. Since $\mathcal{I}$ is closed under finite intersections, $\bar{B} := B_0 \cap B \in \mathcal{I}_B$. So there exists a $\bar{B} \in \mathcal{I}_B$ s.t. $\bar{A} = A \cap \bar{B}$. Since $\bar{A} \in \mathcal{I}_{A,x} \subseteq \mathcal{I}_x$, $\bar{A} \subseteq L_x$. So $x \notin \bar{B}$. Since $x \in L_1 \setminus \bar{B}$, Definition 4.11 implies $\bar{B} \cap L_x \in \mathcal{I}_B$. So we may assume that $\bar{B} \subseteq L_x$. Since $x \in L_1$, $\bar{B} \cap L_1 \subsetneq B \cap L_1$. So $\mathrm{Dp}(\bar{B}) < \mathrm{Dp}(B) = \alpha$. By the induction assumption, $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$. So $\dot{f}^r_x$ is a $\mathbb{P}_{\bar{B}}$-name and $R \,||\!\restriction L_x \in \mathbb{P}_{\bar{B}}$. So $R \in \mathbb{P}_B$.

<u>Complete embedding</u>: Let $P = (p, F^p) \in \mathbb{P}_B, x = x_p$. By definition, there is a $\bar{B} \in \mathcal{I}_{B,x}$ s.t. $P \,||\!\restriction L_x = \bar{P} \in \mathbb{P}_{\bar{B}}$ and $\dot{f}^p_x$ is a $\mathbb{P}_{\bar{B}}$-name. Let $\bar{A} = A \cap \bar{B}$. Then

$$\bar{A} \in \mathcal{I} \restriction A, \bar{A} \subseteq \bar{B}, \bar{A} \in \mathcal{P}(L_x).$$

Since $\bar{A} \in \mathcal{I}$, $\bar{A}$ is L_0-closed. Since $\bar{B} \in \mathcal{I}$ and $\mathrm{Dp}(\bar{B}) < \alpha$, the induction hypothesis implies that $\bar{P}$ has a canonical reduction $\bar{P}_0 = \bar{p}_0(\bar{P}, \bar{A}, \bar{B})$.
Construction of the canonical reduction $P_0 = p_0(P, A, B)$ of P:
Let $F^{p_0} = F^p \cap \widehat{W}_A$. Since $(p \restriction L_0, F^p) \in \mathbb{Q}$ and $A \subseteq L_0 \subseteq \mathrm{dom}(\mathbb{Q})$, Definition 4.8 implies that there exists a strong $\mathbb{Q}_A$-reduction (p', F^{p_0}) of $(p \restriction L_0, F^p)$. Since $\bar{P}_0$ is a canonical extension of $\bar{P}$, $\mathrm{oc}(\bar{p}_0) = \mathrm{oc}(\bar{P}) \cap \bar{A}$. Since $\bar{A} \subseteq A$, for $p_0 \restriction L_0 := p' \cup \bar{p}_0 \restriction L_0$ we have that

$$(p_0 \restriction L_0, F^{p_0}) \text{ is a strong } \mathbb{Q}_A\text{-reduction of } (p \restriction L_0, F^p) \text{ and } p_0 \restriction L_0 \cap \bar{A} \supseteq \bar{p}_0 \restriction L_0.$$

We may assume that $p_0 \restriction L_1 \cap L_x = \bar{p}_0 \restriction L_1 \cap L_x$. Then define $P_0 := (p_0, F^{p_0})$. So $P_0 \,||\!\restriction L_x \leq_{\mathbb{P}_{\bar{A}}} \bar{P}_0$. So $P_0 \,||\!\restriction L_x$ is a canonical reduction of $\bar{P}$. We may assume that $p(x)$ is a nice name for a condition in $\mathbb{S}$ below $\bar{P}$.
<u>1st case</u>: $x \notin A$. Then define $p_0(x) = p(x)$. Then $s^{p_0}_x = s^p_x$ for all $x \in \mathrm{dom}(p_0) \cap L_1$.
<u>2nd case</u>: $x \in A$. Then let $p_0(x)$ be a canonical projection of $p(x)$ below $P_0 \,||\!\restriction L_x$. Then $p_0(x) = p(x)$. So $s^{p_0}_x = s^p_x$ for all $x \in \mathrm{dom}(p_0) \cap L_1$.
Since we may assume that $p_0 \restriction L_1 \setminus L_x = p \restriction (L_1 \setminus L_x) \cap A$, P_0 is a canonical reduction of P.

Let $D \in \mathcal{I}, C \subseteq D$ closed s.t. $B \cup C \subseteq D, A = B \cap C$ and $\mathrm{Dp}(D) = \alpha$. Let $Q_0 = (q_0, F^{q_0}) \leq_{\mathbb{P}_C} P_0$.
Construct a common extension of Q_0 and P:
<u>1st case</u>: $x \notin A$. Since $x \in B$, $x \notin C$. Let $y = \max\{\mathrm{dom}(q_0) \cap L_x \cap L_1\}$. Then

$$y = \max\{\mathrm{dom}(q_0) \cap L_x \cap L_1\} < x.$$

By Lemma 4.17, $Q_0 \restriction L^{=}_y \leq_{\mathbb{P}_C} P_0 \restriction L^{=}_y$. So there exists $\bar{E} \in (\mathcal{I} \restriction C)_y$ witnessing

$$Q_0 \restriction L^{=}_y \leq_{\mathbb{P}_C} P_0 \restriction L^{=}_y.$$

By Definition 4.11 there exist an $\bar{F} \in \mathcal{I}_{D,y}$ s.t. $\bar{E} = \bar{F} \cap C$ and there exists a $\bar{G} \in \mathcal{I}_{D,x}$ s.t. $\mathrm{dom}(q_0) \cap L_x \setminus L_y \subseteq \bar{G}$. Let

$$\bar{D} = \bar{B} \cup \bar{F} \cup \bar{G} \text{ and } \bar{C} = (\bar{G} \cap C) \cup \bar{E} \cup \bar{A}.$$

Since $\bar{B}, \bar{F}, \bar{G} \in \mathcal{I}_{D,x}$, $\bar{D} \in \mathcal{I}_{D,x}$. Since $\bar{E}, \bar{A} \in (\mathcal{I} \restriction C)_x$ and $\bar{G} \in \mathcal{I}_x$, $\bar{C} \in (\mathcal{I} \restriction C)_x$. Since $\bar{E} \subseteq \bar{F}$ and $\bar{A} \subseteq \bar{B}$, $\bar{C} \subseteq \bar{D}$. Since $\bar{A} \subseteq \bar{B}$ and $\bar{A} \subseteq \bar{C}$, $\bar{A} \subseteq \bar{B} \cap \bar{C}$. Since $\bar{B} \subseteq B, \bar{E}, (\bar{G} \cap C) \subseteq C$ and $A = B \cap C$, $\bar{C} \cap \bar{B} \subseteq \bar{A}$. So $\bar{C} \cap \bar{B} = \bar{A}$.
Since $\bar{E}$ witnesses $Q_0 \leq_{\mathbb{P}_C} P_0$, $\bar{E}$ witnesses

$$\bar{Q}_0 := Q_0 \mid\restriction L_x \leq_{\mathbb{P}_{\bar{C}}} P_0 \mid\restriction L_x.$$

We may assume that $\bar{Q}_0 \restriction L_0$ is a strong $\mathbb{Q}_{\bar{C}}$-reduction of $Q_0 \restriction L_0$. Since

$$\mathrm{Dp}_{\mathcal{I}\restriction C}(\bar{C}) \leq \mathrm{Dp}_{\mathcal{I}}(\bar{D}) < \mathrm{Dp}_{\mathcal{I}}(D) = \alpha$$

and $\bar{A}, \bar{B}, \bar{C} \subseteq \bar{D}$, we may apply the inductive hypothesis to $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. So there exists a common extension $\bar{Q} = (\bar{q}, F^{\bar{q}}) \leq_{\mathbb{P}_{\bar{D}}} \bar{Q}_0, P \mid\restriction L_x$. Define $Q = (q, F^q)$ the common extension of Q_0 and P as follows:
Let $q' = \bar{q} \cup \{(x, p(x))\}$, $F^{q'} = F^{\bar{q}}$ and $Q' = (q', F^{q'})$. Then $x_{q'} = x$. So $\bar{D}$ witnesses

$$Q' \in \mathbb{P}_D \text{ and } Q' \leq_{\mathbb{P}_D} \bar{Q}_0 = Q_0 \mid\restriction L_x.$$

By Lemma 4.19 $Q'' = (q'', F^{q'} \cup F^{q_0}) := Q' \bar{\ltimes}_x Q_0$ is a common extension of Q' and Q_0 in $\mathbb{P}_D$. Let $\hat{p} = p \restriction L_0 \setminus \mathrm{Oc}(Q'')$ and $Q = (q, F^q)$ where $q = q'' \cup \hat{q}$, $F^q = F^{q'} \cup F^{q_0}$. Since $\mathrm{Oc}(Q'') \cap \mathrm{dom}(\hat{p}) = \emptyset$, Definition 4.5 implies that for

$$Q := (q'' \cup \hat{p}, F^{q_0} \cup F^{q'})$$

it holds true that $Q \restriction L_0 \leq_{\mathbb{Q}} Q'' \restriction L_0$. Since $\hat{p} \cap L_1 = \emptyset$, $x_q = x_{q''}$. So $Q \leq_{\mathbb{P}} Q''$. So $Q \leq_{\mathbb{Q}} Q_0$. Since $Q'' \restriction L_0 \leq_{\mathbb{Q}} Q_0 \restriction L_0 \leq_{\mathbb{Q}} P_0$ and $P_0 \restriction L_0$ is a strong $\mathbb{Q}_A$-reduction of $P \restriction L_0$,

$$Q \restriction L_0 = (q'' \restriction L_0 \cup \hat{p} \restriction L_0, F^p) \leq_{\mathbb{Q}} P \restriction L_0.$$

So by Lemma 4.18, $Q \leq_{\mathbb{P}} P$.
<u>2nd case</u>: $x \in A$. Then

$$\begin{aligned} x_{p_0} &= \max\{\mathrm{dom}(p_0) \cap L_1\} \\ &= \max\{\mathrm{dom}(p) \cap A \cap L_1\} \\ &= x \\ &= x_p. \end{aligned}$$

Since $x \in A$, $x \in C$. So there is a $\bar{C} \in (\mathcal{I} \restriction C)_x$ witnessing $Q_0 \mid\restriction L_x^= \leq_{\mathbb{P}_C} P_0 \mid\restriction L_x^=$. So

$$\begin{gathered} \bar{Q}_0 = Q_0 \mid\restriction L_x \leq_{\mathbb{P}_{\bar{C}}} P_0 \mid\restriction L_x \leq_{\mathbb{P}_{\bar{C}}} \bar{P}_0 \\ \text{and } Q_0 \mid\restriction L_x \Vdash_{\mathbb{P}_{\bar{C}}} \text{"}(\check{s}_x^{q_0}, \dot{s}_x^{q_0}) \leq_{\dot{\mathbb{S}}} (\check{s}_x^{p_0}, \dot{s}_x^{p_0})\text{"}. \end{gathered}$$

Since $\bar{C} \in (\mathcal{I} \upharpoonright C)_x$ there exists $C_0' \in \mathcal{I}$ s.t. $\bar{C} = C_0' \cap C$. Then $x \notin C_0'$. So by Definition 4.11, $C_0 = C_0' \cap L_x \in \mathcal{I}_x$ and $\bar{C} = C_0 \cap C$. We may assume that $\bar{Q}_0$ is a strong $\mathbb{Q}_{\bar{C}}$-reduction of $Q_0 \upharpoonright L_0$. Since

$$\bar{A} \cup \bar{C} \in (\mathcal{I} \upharpoonright C)_x \text{ and } \mathrm{Rk}(\mathcal{T}_{\bar{A}\cap\bar{C}}) < \mathrm{Rk}(\mathcal{T}),$$

$\mathbb{P}_{\bar{C}} \lessdot \mathbb{P}_{\bar{A}\cap\bar{C}}$. So $\dot{f}_x^{q_0}$ is a $\mathbb{P}_{\bar{A}\cap\bar{C}}$-name. So we may assume that $\bar{A} \subseteq \bar{C}$. Then $\bar{A} = \bar{C} \cap \bar{B}$. For $\bar{D} := D \cap C_0 \in \mathcal{I}_{D,x}$, $\bar{C} = \bar{D} \cap C$. We may assume that $\bar{B} \subseteq \bar{D}$. Since $\mathrm{Dp}_{\mathcal{I}}(\bar{D}) y \mathrm{Dp}_{\mathcal{I}}(D) = \alpha$ and $\bar{A}, \bar{B}, \bar{C} \subseteq \bar{D}$ we may apply the induction hypothesis to $\bar{A}, \bar{B}, \bar{C}, \bar{D}$.

Let $n \in \omega$ be s.t. $\mathbb{S}$ is n-Suslin, $m = |s_x^{q_0}|$, $\hat{Q}_0 \leq_{\mathbb{P}_{\bar{C}}} \bar{Q}_0$ and $s' \in {}^{n\cdot m}\omega$ s.t.

$$\hat{Q}_0 \Vdash \text{"} \dot{f}_x^{p_0} \upharpoonright n \cdot m = \check{s}' \text{"}.$$

Let G be $\mathbb{P}_{\bar{C}}$-generic s.t. $\hat{Q}_0 \in G$. Since $\dot{f}_x^{p_0}$ is a $\mathbb{P}_{\bar{A}}$-name and $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{C}}$ the inductive hypothesis implies that $G \cap \bar{A}$ is a $\mathbb{P}_{\bar{A}}$-generic filter and there is $U \in G \cap \bar{A}$ s.t. $U \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \upharpoonright (n \cdot m) = \hat{s}'$. Since $U, P_0 \,||\!\upharpoonright L_x \in G \cap \bar{A}$ they have a common extension $E' \in G \cap \bar{A}$ s.t.

$$E' \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \upharpoonright (n \cdot m) = \hat{s}'.$$

Since $E', \hat{Q}_0 \in G$, they have a common extension $\hat{\hat{Q}}_0 \in G$. Since $\hat{\hat{Q}}_0 \leq_{\mathbb{P}_{\bar{C}}} E'$, $\hat{\hat{Q}}_0$ has a reduction $\tilde{Q}_0 \in \mathbb{P}_{\bar{A}}$ s.t. $\tilde{Q}_0 \leq_{\mathbb{P}_{\bar{A}}} E'$. So

$$\tilde{Q}_0 \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \upharpoonright n \cdot m = \check{s}'$$

and $\tilde{Q}_0 \leq_{\mathbb{P}_{\bar{A}}} P_0 \,||\!\upharpoonright L_x$. So $\tilde{Q}_0$ is compatible with some $a \in \mathcal{A}_{n\cdot m}$. We may assume that $\dot{f}_x^p = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ and $\dot{f}_x^{p_0} = \{(a, s(a))\}_{a \in \mathcal{A}_n, n \geq 1}$. Let P_0^* be a common $\mathbb{P}_{\bar{A}}$-extension of a and $\tilde{Q}_0$. Since $a \Vdash_{\mathbb{P}_{\bar{A}}} \dot{f}_x^{p_0} \upharpoonright n \cdot m = s(\check{a})$,

$$P_0^* \Vdash s' = s(\check{a}).$$

So $s' = s(a)$ in the ground model. Then $P_0^* \leq_{\mathbb{P}_{\bar{A}}} a$ and P_0^* is a reduction of $\hat{\hat{Q}}_0$. Also a is a reduction of some $b \in \mathcal{B}_{n\cdot m}$ s.t. $s(a) = s(b)$. So there exists a common $\mathbb{P}_{\bar{B}}$-extension $\bar{P}^+$ of P_0^* and b. Since by the induction assumption $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$, $\bar{P}^+$ has a canonical reduction $\hat{P}^+ \in \mathbb{P}_{\bar{A}}$. By Lemma 2.13 there exists a common $\mathbb{P}_{\bar{A}}$-extension $\bar{P}_0^+$ of $\hat{P}^+$ and P_0^*. Since $\bar{P}_0^+ \leq_{\mathbb{P}_{\bar{A}}} P_0^*$ and P_0^* is a reduction of $\hat{\hat{Q}}_0$, there exists a common $\mathbb{P}_{\bar{C}}$-extension $\bar{Q}_0^+$ of $\bar{P}_0^+$ and $\hat{\hat{Q}}_0$. Since $\bar{Q}_0^+ \leq_{\mathbb{P}_{\bar{C}}} \bar{P}_0^+$ and $\bar{P}_0^+$ is a canonical reduction of $\bar{P}^+$, there exists a common $\mathbb{P}_{\bar{D}}$-extension T of $\bar{P}^+$ and $\bar{Q}_0^+$. Then

$$\begin{aligned} T \Vdash_{\mathbb{P}_{\bar{D}}} \text{"}&(\check{s}_x^{q_0}, \dot{f}_x^{q_0}) \leq_{\dot{\mathbb{S}}} (\check{s}_x^{p_0}, \dot{f}_x^{p_0}) \\ &\wedge \check{s}_x^p = \check{s}_x^{p_0} \\ &\wedge \dot{f}_x^p \upharpoonright n \cdot m = \dot{f}_x^{p_0} \upharpoonright n \cdot m. \end{aligned}$$

Since $\mathbb{S}$ is n-Suslin, $T \Vdash_{\mathbb{P}_{\bar{D}}} \exists t(x) \in \dot{\mathbb{S}}(t(x) \leq_{\dot{\mathbb{S}}} q_0(x), p(x))$. Let $(\bar{q}^+, F^{\bar{q}^+}) = \bar{Q}^+ \leq_{\mathbb{P}_{\bar{D}}} T$ and $(\check{s}_x^q, \dot{f}_x^q)$ be a nice name for a condition below $\bar{Q}^+$ s.t.

$$\bar{Q}^+ \Vdash_{\mathbb{P}_{\bar{D}}} \text{"}(\check{s}_x^q, \dot{f}_x^q) \leq_{\dot{\mathbb{S}}} (\check{s}_x^{q_0}, \dot{f}_x^{q_0}), (\check{s}_x^p, \dot{f}_x^p)\text{"}.$$

Let $q' = \bar{q}^+ \cup \{x, q(x)\}, F^{q'} = F^{\bar{q}}$ and $Q' = (q', F^{q'})$. Define $Q'', \hat{p}, Q$ as in case 1. Then analogously to case 1, $Q \leq_{\mathbb{P}_D} Q_0, P$. □

Lemma 4.20. *Let $\mathbb{Q}$ be Knaster. Then $\mathbb{P}(\mathcal{T},\mathbb{Q},\mathbb{S})$ is Knaster.*

Proof. Let $\langle Q_\alpha : \alpha < \omega_1\rangle$ be a sequence of conditions in $\mathbb{P}$. Since $\mathbb{Q}$ is Knaster, we may assume that $\langle Q_\alpha \restriction L_0 : \alpha < \omega_1\rangle$ is linked. Since all conditions are finite by the Δ-system lemma we may assume that for all $\alpha,\beta < \omega_1$, $\mathrm{dom}(q_\alpha)\cap\mathrm{dom}(q_\beta) = F$ for some finite set $F \subseteq L$ and $Q_\alpha \restriction L_0 \not\perp Q_\beta \restriction L_0$. We may also assume that for all $x \in F \cap L_1$ there are $s_x \in {}^{<\omega}\omega$ and $t_x \in {}^{n\cdot|s_x|}\omega$ s.t. for a witness B to $Q_\alpha \mid\restriction L_x^= \in \mathbb{P}$,

$$Q_\alpha \mid\restriction L_x \Vdash_{\mathbb{P}_B} \pi_0(q_\alpha(x)) = \check{s}_x \wedge \pi_1(q_\alpha(x)) \restriction (n\cdot|s_x|) = \check{t}_x.$$

Let $\alpha,\beta \in \omega_1$. Since $\mathrm{dom}(q_\alpha),\mathrm{dom}(q_\beta)$ are finite, we depict $(\mathrm{dom}(q_\alpha)\cup\mathrm{dom}(q_\beta))\cap L_1$ as $\{x_i\}_{i=0}^m$ in increasing $<_L$ order. Let $x_{m+1} \in L$ be s.t. $\mathrm{Oc}(Q_\alpha)\cup\mathrm{Oc}(Q_\beta) \subseteq L^=_{x_{m+1}}$.
Proof of $Q_\alpha \not\perp Q_\beta$ by induction over $m+1$:
<u>Induction base</u>: Let $\bar{R} = (\bar{r},\bar{F})$ be a common extension of $Q_\alpha \restriction L_0$ and $Q_\beta \restriction L_0$. Then $(r,F) = R := \bar{R} \restriction L_0 \in \mathbb{Q}$ is a common extension of $Q_\alpha \restriction L_0$ and $Q_\beta \restriction L_0$. Since $Q_\alpha, Q_\beta \in \mathbb{P}$, Lemma 4.17 implies $Q_\alpha \mid\restriction L^=_{x_0}, Q_\beta \mid\restriction L^=_{x_0} \in \mathbb{P}$. So

$$Q_\alpha \mid\restriction L_{x_0} \Vdash_{\mathbb{P}_{L_{x_0}}} q_\alpha(x_0) = (\check{s}_\alpha, \dot{f}_\alpha) \in \dot{\mathbb{S}}$$
$$\text{and } Q_\beta \mid\restriction L_{x_0} \Vdash_{\mathbb{P}} q_\beta(x_0) = (\check{s}_\beta, \dot{f}_\beta) \in \dot{\mathbb{S}}$$

Since $R \mid\restriction L_{x_0}$ is a common extension of $Q_\alpha \mid\restriction L_{x_0}$ and $Q_\beta \mid\restriction L_{x_0}$,

$$\begin{aligned} R \mid\restriction L_{x_0} \Vdash_{\mathbb{P}_{L_{x_0}}} q_\alpha(x_0) &\in \dot{\mathbb{S}} \\ &\wedge q_\beta(x_0) \in \dot{\mathbb{S}} \\ &\wedge \check{s}_\alpha = \check{s}_\beta = \check{s}_{x_0} \\ &\wedge \dot{f}_\alpha \restriction (n\cdot|s_{x_0}|) = \dot{f}_\beta \restriction (n\cdot|s_{x_0}|) = \check{t}_{x_0}. \end{aligned}$$

Since $\mathbb{S}$ is n-Suslin there exist $R_0^* = (r_0^*, F_0^*) \leq_{\mathbb{P}} R \mid\restriction L_{x_0}$ and $t(x_0) \in \dot{\mathbb{S}}$ s.t.

$$R_0^* \Vdash_{\mathbb{P}_{L_{x_0}}} t(x_0) \leq_{\dot{\mathbb{S}}} q_\alpha(x_0), q_\beta(x_0).$$

Let

$$R_0 = (r_0, F_0) = (r_0^* \cup \{(x_0, t(x_0))\} \cup r \restriction L \setminus L_{x_0}, F_0^* \cup F).$$

Since $R \in \mathbb{Q}$, $R \mid\restriction L_{x_0}$ is a strong $\mathbb{Q}_{L_0\cap L_{x_0}}$-reduction of R. So $R_0 \leq_{\mathbb{P}_{L_{x_0}}} R$. By Lemma 4.17, $R_0 \mid\restriction L^=_{x_0} \leq_{\mathbb{P}_{L_{x_1}}} R \mid\restriction L^=_{x_0}$. Since $R \mid\restriction L^=_{x_0} \leq_{\mathbb{P}_{L_{x_1}}} Q_\alpha \mid\restriction L^=_{x_0}, Q_\beta \mid\restriction L^=_{x_0}$, Lemma 4.18 implies

$$R_0 \mid\restriction L_{x_1} \leq_{\mathbb{P}} Q_\alpha \mid\restriction L_{x_1} \text{ and } R_0 \mid\restriction L_{x_1} \leq_{\mathbb{P}} Q_\beta \mid\restriction L_{x_1}.$$

<u>Induction assumption</u>: For $i < j$ there is $R_i \leq_{\mathbb{P}} R$ s.t.

1. $r_i \restriction L \setminus L^=_{x_i} = r \restriction L \setminus L^=_{x_i}$,
2. $(R_i \mid\restriction L_{x_i}) \restriction L_0$ is a strong $\mathbb{Q}_{L_{x_i}}$-extension of R_i,
3. $R_i \mid\restriction L_{x_i} \leq_{\mathbb{P}_{L_{x_i}}} Q_\alpha \mid\restriction L_{x_i}, Q_\beta \mid\restriction L_{x_i}$.

Induction step: Let $i = j$. Analogously to the induction base there exist $t(x_{i+1}) \in \dot{\mathbb{S}}$ and $R_i^* = (r_i^*, F_i^*) \in \mathbb{P}_{L_{x_i}}$ s.t. $R_i^* \leq_{\mathbb{P}_{L_{x_i}}} R_i \mid\restriction L_{x_i}$ and

$$R_i^* \Vdash_{\mathbb{P}_{L_{x_i}}} t(x_{i+1}) \leq_{\dot{\mathbb{S}}} q_\alpha(x_{i+1}), q_\beta(x_{i+1}).$$

Since $(R_i \mid\restriction L_{x_i}) \restriction L_0$ is a strong $\mathbb{Q}_{L_{x_i}}$-reduction of R_i,

$$R_{i+1} = (r_{i+1}, F_{i+1}) := (r_i * \cup\{(x_{i+1}, t(x_{i+1}))\} \cup r \restriction L \setminus L_{x_{i+1}}, F_i^* \cup F_i) \leq_{\mathbb{P}_{L_{x_i}}} R_i.$$

Since $\mathbb{P}_{L_{x_i}}$ witnesses $R_{i+1} \mid\restriction L_{x_i}^{=} \leq_{\mathbb{P}} Q_\alpha \mid\restriction L_{x_i}^{=}, Q_\beta \mid\restriction L_{x_i}^{=}$, Lemma 4.18 implies

$$R_{i+1} \mid\restriction L_{x_{i+1}} \leq_{\mathbb{P}} Q_\alpha \mid\restriction L_{x_{i+1}}, Q_\beta \mid\restriction L_{x_{i+1}}.$$

$R_m \leq_{\mathbb{P}} Q_\alpha, Q_\beta$ follows from $i = m$. $\square$

We end this chapter by showing three properties of iterations along a two-sided template. Those will be used in the next chapters once we plug in particular forcing notions into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

Whenever an $A \in \mathcal{I}$ of a two-sided template is below an element $y \in L_1$, we can adjoin $\mathbb{S}$ to $\mathbb{P}_A$ s.t. it can be completely embedded into $\mathbb{P}$. In particular, we will choose x as the coordinate in $\mathbb{P}$, i.e. for any $P \in A$ and any $(s, f) \in \mathbb{S}$ we will add $(x, (\check{s}, \dot{f}))$ to p.

Lemma 4.21. *Let $x \in L_1, A \in \mathcal{I}_x$. Then the two-step iteration $\mathbb{P}_A * \dot{\mathbb{S}}$ completely embeds into $\mathbb{P}$.*

Proof. Let $B := \mathrm{cl}(A \cup \{x\})$. By Lemma 4.15, $\mathbb{P}_B$ completely embeds into $\mathbb{P}$. We will show that $\mathbb{P}_A * \dot{\mathbb{S}} \lessdot \mathbb{P}_B$:
Define

$$i : \mathbb{P}_A * \dot{\mathbb{S}} \to \mathbb{P}_B, i((P, (\check{s}, \dot{f}))) = (p \cup (x, (\check{s}, \dot{f})), F^p).$$

Since $x \in L_1$, $P \mid\restriction L_x = P$ for all $P \in \mathbb{P}_A$ and $A \in \mathcal{I}_{L_x}$, i is well defined.

Let $(P, (\check{s}, \dot{f})) \leq_{\mathbb{P}_A * \dot{\mathbb{S}}} (Q, (\check{r}, \dot{g}))$. Then

$$x_{i((P,(\check{s},\dot{f})))} = x = x_{i((Q,(\check{r},\dot{g})))}.$$

So $i(P, (\check{s}, \dot{f})) = P \leq_{\mathbb{P}_A} Q = i(Q, (\check{r}, \dot{g}))$ and $i(P, (\check{s}, \dot{f})) = P \Vdash_{\mathbb{P}_A} (\check{s}, \dot{f}) \dot{\leq}_{\dot{S}} (\check{r}, \dot{g})$. So A witnesses $i(P, (\check{s}, \dot{f})) \leq_{\mathbb{P}_B} i(Q, (\check{r}, \dot{g}))$.

Let $(P, (\check{s}, \dot{f})), (Q, (\check{r}, \dot{g})) \in \mathbb{P}_A * \dot{\mathbb{S}}$ s.t. $i((P, (\check{s}, \dot{f}))) \not\perp_{\mathbb{P}_B} i((Q, (\check{r}, \dot{g})))$ with common extension R. Since $R \leq_{\mathbb{P}_B} i((P, (\check{s}, \dot{f})))$ and $x_r = x = x_p$, there exists an $A' \in \mathcal{I}_x$ s.t.

$$\begin{aligned}(R \mid\restriction L_x, r(x)) &\leq_{\mathbb{P}_{A'} * \dot{\mathbb{S}}} (i(P, (\check{s}, \dot{f})) \mid\restriction L_x, (\check{s}, \dot{f})) \\ &= (P, (\check{s}, \dot{f})).\end{aligned}$$

We may assume w.l.o.g. that $A' = A$. So

$$(R \restriction\restriction L_x, r(x)) \leq_{\mathbb{P}_A * \dot{\mathbb{S}}} (P, (\check{s}, \dot{f})).$$

Analogously

$$(R \restriction\restriction L_x, r(x)) \leq_{\mathbb{P}_A * \dot{\mathbb{S}}} (Q, (\check{r}, \dot{g})).$$

So $(P, (\check{s}, \dot{f})) \not\perp_{\mathbb{P}_A * \dot{\mathbb{S}}} (Q, (\check{r}, \dot{g}))$.

Let $P = (p, F^p) \in \mathbb{P}_B$.
<u>1st case</u>: x_p is defined. Then there exists $\bar{B}' \in \mathcal{I}_{x_p}$ s.t. $\bar{P} := P \restriction\restriction L_{x_p} \in \mathbb{P}_{\bar{B}'}$ and $\dot{f}_x^p$ is a $\mathbb{P}_{\bar{B}'}$-name for a real. Then the same holds true for $\bar{B} := \bar{B}' \cup A$. Since $A \subseteq \bar{B}$, Lemma 4.15 implies $\mathbb{P}_A \lessdot \mathbb{P}_{\bar{B}}$. So $\bar{P}$ has a canonical reduction $\bar{P}_0 \in \mathbb{P}_A$. Also there exists a canonical projection $\dot{f}_x^{p_0}$ of $\dot{f}_x^p$ to $\mathbb{P}_A$. Define $P_0 \in \mathbb{P}_A * \dot{\mathbb{S}}$ by $P_0 = (\bar{P}_0, (\check{s}_x^p, \dot{f}_x^{p_0}))$. Analogously to the 2nd case in the proof of Lemma 4.15, any $Q_0 \in \mathbb{P}_A * \dot{\mathbb{S}}$ extending P_0 is compatible with P. So P_0 is a reduction of P to $\mathbb{P}_A * \dot{\mathbb{S}}$.
<u>2nd case</u>: x_p is not defined. Then $P \in \mathbb{Q}_B$. So $P_0 := (P, \mathbb{1}_{\mathbb{S}}) \in \mathbb{P}_A * \dot{\mathbb{S}}$ is a reduction of P to $\mathbb{P}_A * \dot{\mathbb{S}}$. □

$A \cup \{x\}$ is not necessarily closed, so taking the closure is necessary for the proof.

We will work with two-sided templates, where the underlying set L is uncountable. So any countable $A \subseteq L$ can not be cofinal in L. So for any $J \subseteq L$ cofinal in L, there exists an $x \in J$ s.t. $A \in L_x$. Since $L_x \in \mathcal{I}$, L_x is L_0 closed. So $\mathrm{cl}(A) \subseteq L_x$. So $\mathbb{P}_{\mathrm{cl}(A)} \lessdot \mathbb{P}_{L_x}$. So reducing certain properties, such as inclusion or being a name, down to a countable subset of L shows that $\mathbb{P}_{L_x}$ is sufficient to apply those properties.

Lemma 4.22. *For any $P \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ there is a countable set $A \subseteq L$ s.t. $P \in \mathbb{P}_{\mathrm{cl}(A)}$. Similarly, if τ is a $\mathbb{P}$-name for a real then there is a countable $A \subseteq L$ s.t. τ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name.*

Proof. Proof by simultaneous induction on $\mathrm{Dp}(L)$.
<u>Induction base</u>: $\mathrm{Dp}(L) = 0$. Let $A := \{P\}$. Then $A \subseteq L$ is countable. Since $L = L_0$, $\mathrm{cl}(A) = \{P\} \cup L_P = L_P^{=}$. So $P \in \mathbb{P}_{\mathrm{cl}(A)}$.
<u>Induction assumption</u>: Lemma 4.22 holds true for all $L \restriction A$ s.t. $\mathrm{Dp}(L \restriction A) < \alpha$.
<u>Induction step</u>: Let $\mathrm{Dp}(L) = \alpha$.
<u>1st case</u>: x_p is not defined. Then $P \in L_0$ and the statement follows from the induction assumption.
<u>2nd case</u>: x_p is defined. Then there exists $C \in \mathcal{I}_{x_p}$ s.t. $P \restriction\restriction L_{x_p} \in \mathbb{P}_C$ and $\dot{f}_x^p$ is a $\mathbb{P}_C$-name for a real. Since $x_p \notin C$, $\mathrm{Dp}(C) < \alpha$. So by the induction assumption there exists $A_0 \subseteq C$ countable s.t. $P \restriction\restriction L_{x_p} \in \mathbb{P}_{\mathrm{cl}(A_0)}$ and $\dot{f}_x^p$ is a $\mathbb{P}_{\mathrm{cl}(A_0)}$-name. Let $A := A_0 \cup \mathrm{dom}(p)$, $B := \mathrm{cl}(A)$. Since $\mathrm{cl}(A_0) \subseteq \mathrm{cl}(A)$, $P \in \mathbb{P}_B$.
Let $\dot{f}$ be a $\mathbb{P}$-name. By Lemma 4.20, there exists $\{p_{n,i} : i, n \in \omega\} \subseteq \mathbb{P}$ and there exists $\{k_{n,i} \in \omega : i, n \in \omega\}$ s.t.

1. $p_{n,i} \Vdash_{\mathbb{P}} \dot{f}(n) = k_{n,i}$,

2. $\{p_{n,i} : i \in \omega\}$ is a maximal antichain in $\mathbb{P}$ for all $n \in \omega$.

So there exist countable sets $A_{n,i}$ s.t. $p_{n,i} \in \mathbb{P}_{\mathrm{cl}(A_{n,i})}$. Let $A := \bigcup_{n,i} A_{n,i}$, $B = \mathrm{cl}(A)$. By Lemma 2.15, $\dot{f}$ can be contructed as a $\mathbb{P}_B$-name. $\square$

Definition 4.23. Let $((L, \leq), \mathcal{I}, L_0, L_1), ((L, \leq), \mathcal{J}, L_0, L_1)$ be two-sided templates s.t. $\mathcal{J} \subseteq \mathcal{I}$. We say that $\mathcal{I}$ is cofinal in $\mathcal{J}$ if for all $x \in \mathbb{S}$ and all $A \in \mathcal{I} \cap \mathcal{P}(L_x)$, there exists a $B \in \mathcal{J} \cap \mathcal{P}(L_x)$ containing A.

The next lemma will be used in a variation of the isomorphism of names argument. The proof of this isomorphism of names argument is beyond the scope of this thesis.

Lemma 4.24. *Let $\mathcal{J} \subseteq \mathcal{I}$ be s.t. $\mathcal{T}_\mathcal{J} = ((L, \leq), \mathcal{J}, L_0, L_1)$ is a template. Suppose $\mathcal{J}$ is cofinal in $\mathcal{I}$. Then $\mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$ is forcing equivalent to $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.*

Proof. Proof by induction over $\mathrm{Dp}(L)$.
<u>Induction base</u>: Let $\mathrm{Dp}(L) = 0$. Then $L \subseteq L_0$. So $\mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q} = \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$. So $\mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$ is forcing equivalent to $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.
<u>Induction assumption</u>: Lemma 4.24 holds true for all L s.t. $\mathrm{Dp}(L) < \alpha$.
<u>Induction step</u>: Let $\mathrm{Dp}(L) = \alpha$. Let $P \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.
<u>1st case</u>: x_p is defined. Then there exists $A \in \mathcal{I}_x$ s.t. $\bar{P} = P \mid\restriction L_x \in \mathbb{P}_A$ and $\dot{f}^p_x$ is a $\mathbb{P}_A$-name. Since $\mathcal{J}$ is cofinal in $\mathcal{I}$, there exists a $B \in \mathcal{J}_x$ s.t. $A \subseteq B$. By Lemma 4.15, $\mathbb{P}(\mathcal{T}_A, \mathbb{Q}, \mathbb{S}) \lessdot \mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$. Since $x \notin B$, $\mathrm{Dp}(A) \leq \mathrm{Dp}(B) < \mathrm{Dp}(L) = \alpha$. By the induction assumption, $\mathbb{P}(\mathcal{T}_{\mathcal{J},B}, \mathbb{Q}, \mathbb{S})$ is forcing equivalent to $\mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$. So we can construct $\bar{P}$ as a condition in $\mathbb{P}(\mathcal{T}_{\mathcal{J},B}, \mathbb{Q}, \mathbb{S})$ and $\dot{f}^p_x$ as a $\mathbb{P}(\mathcal{T}_{\mathcal{J},B}, \mathbb{Q}, \mathbb{S})$-name. So $P \in \mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$.
<u>2nd case</u>: x_p is not defined. Then $P \in \mathbb{Q}$. So $P \in \mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$.
So both $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ and $\mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$ have a dense embedding into $\mathbb{P}(\mathcal{T}_\mathcal{J}, \mathbb{Q}, \mathbb{S})$. $\square$

5 Estimates for $\mathfrak{a}_g$

For the main result of this thesis, Theorem 1.3, we will plug $\mathbb{Q}_{L_0}$ into $\mathbb{Q}$ and $\mathbb{L}$ into $\mathbb{S}$. For this, we show two auxiliary lemmas in this chapter. Each will plug exactly one of $\mathbb{Q}_{L_0}$ and $\mathbb{L}$ into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$.

$\mathbb{S} = \mathbb{L}$ gives a lower bound for $\mathfrak{a}_\mathfrak{g}$ depending on the two-sided template used.

Lemma 5.1. *Let $\mathcal{T}$ be a template, $\mathbb{Q}$ a finite function poset with the complete embedding property and $L_0 = \mathrm{dom}(\mathbb{Q})$, $\mathbb{S} = \mathbb{L}$ localization forcing and let μ be a regular uncountable cardinal. Assume $\mu \subseteq L_1$ (as an order), μ is cofinal in L and $L_\alpha \in \mathcal{I}$ for all $\alpha < \mu$. Then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ forces $\mathrm{non}(\mathcal{M}) = \mu$ and $\mathfrak{a}_g \geq \mu$.*

Proof. Let G be $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$-generic over V and work in $V[G]$. Let $\alpha \in \mu$. Since $\mu \subseteq L_1$ is cofinal in L, there exists a $\beta \in \mu \subseteq L_1$ s.t. $L_\alpha \in \mathcal{I}_\beta$. Then Lemma 4.21 implies that $\mathbb{P}_{L_\alpha} * \dot{\mathbb{L}}$ completely embeds into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$. Lemma 2.20 implies that for each $\alpha < \mu$ there exists a slalom $\phi_\alpha \in \mathbb{L}$ which localizes all reals in V. Let $f \in {}^\omega\omega$ be a real in $V[G]$ and $\dot{f}$ its $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$-name. By Lemma 4.22, there exists a countable $A \subseteq L$ s.t. $\dot{f}$ is a $\mathbb{P}(\mathcal{T}_{\mathrm{cl}(A)}, \mathbb{Q}, \mathbb{S})$-name. Since μ is regular and cofinal in L, $\mathrm{cf}(L) = \mathrm{cf}(\mu) = \mu$. Since μ is uncountable $\sup(A) \neq L$. So there exists an $\alpha < \mu$ s.t. $A \subseteq L_\alpha$. So $\mathrm{cl}(A) \subseteq L_\alpha$. So $f \in V[G \cap \mathbb{P}_{L_\alpha}]$. By Theorem 2.26, $\mathrm{cof}(\mathcal{N}) \leq \mu$.

Let $F \subseteq \omega^\omega$ be a family of size $< \mu$ in $V[G]$. Since for any real f in $V[G]$ there exists an $\alpha < \mu$ s.t. $f \in V[G \cap \mathbb{P}_{L_\alpha}]$, the pigeonhole-principle implies that there exists an $\alpha < \mu$ s.t. any real in F is in $V[G \cap \mathbb{P}_{L_\alpha}]$. So ϕ_α localizes all real in F. By Theorem 2.26, $\mathrm{add}(\mathcal{N}) \geq \mu$.

By Cichoń's diagram from chapter 2.3, $\mathrm{non}(\mathcal{M}) = \mu$. By Theorem 2.27, $\mathfrak{a}_g \geq \mu$. $\square$

$\mathbb{Q} = \mathbb{Q}_{L_0}$ gives an upper bound for $\mathfrak{a}_g$ depending on the two-sided template used.

Lemma 5.2. *Let $\mathcal{T}$ be a template, and let $\mathbb{Q} = \mathbb{Q}_{L_0}$ be the poset for adding a cofinitary group with L_0 generators. Assume that L has uncountable cofinality and that L_0 is cofinal in L. Then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ adds a maximal cofinitary group of size $|L_0|$.*

Proof. Let G be $\mathbb{P}$-generic over V. Define

$$\rho_G : L_0 \to S_\infty, \rho_G(x) = \bigcup\{s^p_x : P \in G \wedge P \restriction L_0 = (s^p, F^p)\}.$$

Then

$$\rho_G(x) = \bigcup\{s^p_x : P \in G \cap \mathbb{P}_{L_0}\}.$$

By Proposition 3.11, ρ_G induces a cofinitary representation $\hat{\rho}_G$ of $\mathbb{F}_{L_0}$.
Assume that $\mathrm{im}(\rho_G)$ is not maximal. Then there exist $\sigma \in \mathrm{cofin}(S_\infty)$ and $b_0 \notin L_0$ s.t.
$\rho'_G : L_0 \cup \{b_0\} \to S_\infty$,

$$\rho'_G(x) := \begin{cases} \rho(x) & \text{if } x \in L_0, \\ \sigma & \text{if } x = b_0. \end{cases}$$

induces a cofinitary representation. Let $\dot{\sigma}$ be a $\mathbb{P}$-name for σ in V. Since every permutation is a real, Lemma 4.22 implies that there exists a countable $A \subseteq L$ s.t. $\dot{\sigma}$ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name. Since L_0 is cofinal in L and $\mathrm{cf}(L_0) \geq \aleph_1$, $\mathrm{cf}(L) = \mathrm{cf}(L_0) \geq \aleph_1$. So there exists an $x \in L_0$ s.t. $\mathrm{cl}(A) \subseteq L_x$. By Lemma 4.15, $\mathbb{P}_{\mathrm{cl}(A)} \lessdot \mathbb{P}_{L_x}$. Let $H = G \cap \mathbb{P}_{L_x}$.

Claim:

$$V[H] \models \text{"} D_{\sigma,N,x} = \{P \in (\mathbb{P}/H) : \exists n \geq N(s^p_x(n) = \sigma(n)), \\ \text{where } P \restriction L_0 = (s^p, F^p)\} \text{ is dense in } \mathbb{P}\text{"}$$

Proof: Let $P_0 = (s^{p_0}, F^{p_0}) \in (\mathbb{P}/H)$. Then $P \restriction L_0 \cap L_x \in H_0 := H \cap \mathbb{P}_{L_0 \cap L_x}$. Analogously to the proof of Theorem 3.16,

$$V[H_0] \models \text{"} D^0_{\sigma,N,x} = \{p \in (\mathbb{Q}_{L_0}/\mathbb{Q}_{L_0 \cap L_x}) : (\exists n \geq N) s^p_x(n) = \sigma(n)\} \text{ is dense in } \mathbb{Q}_{L_0}\text{"}.$$

So w.l.o.g. there exists a $(t, E) \leq (s^{p_0} \restriction L_0 \setminus L_x, F^{p_0})$ s.t. $(t, E) \in D^0_{\sigma,N,x}$ i.e. $t_x(n) = \sigma(n)$ for some $n \geq N$. Define P_1 by

$$P_1 \restriction L_x = P_0 \restriction L_x, \\ P_1 \restriction (L_0 \setminus L_x) = (t, E), \\ P_1 \restriction (L_1 \setminus L_x) = P_0 \restriction (L_1 \setminus L_x).$$

Since $(t, E) \in (\mathbb{Q}_{L_0}/\mathbb{Q}_{L_0 \cap L_x})$ and $P_0 \in (\mathbb{P}/H)$, $P_1 \in \mathbb{P}/H$. So in $V[H]$ $P_1 \leq P_0$ and $P_1 \in D_{\sigma,N,x}$. □

By the truth lemma,

$$V[G] \models \text{"} D_{\sigma,N,x} = \{P \in (\mathbb{P}/H) : \exists n \geq N(s^p_x(n) = \sigma(n)), \\ \text{where } P \restriction L_0 = (s^p, F^p)\} \text{ is dense in } \mathbb{P}\text{"}.$$

Let $P \in \mathbb{P}$. So there exists an $x \in L_0$ s.t. $\rho_G(x)(n) = \sigma(n)$ for infinitely many $n \in \mathbb{N}$. But then $\rho'_G(x) \circ \sigma \in \mathrm{cofin}(S_\infty)$. Contradiction. □

Note that both estimates depend on the underlying two-sided template. In the next chapter, we will define the two-sided template used to show Theorem 1.3.

6 Isomorphism of Names Argument

The goal of this chapter is to proof Theorem 1.3. We will do so by defining a specific two-sided template, plugging it into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ and using an isomorphism of names argument.

The isomorphism of names argument used in the proof of Theorem 1.3 is beyond the scope of this thesis. We will present a simpler version of an isomorphism of names argument from [Fis25].

Proposition 6.1. *Assume* CH, *and let* $\lambda = \lambda^\omega$ *be a cardinal. In the forcing extension gotten by adding* λ *Cohen reals, every mad family has either size* $\aleph_1$ *or size* $\mathfrak{c} = \lambda$.

Proof. Let $\omega_2 \cdot 2 \leq \kappa < \lambda$ and assume that $\dot{A}^\alpha, \alpha < \kappa$ are names for countable subsets of ω. Fix $\alpha < \kappa$. Let $\mathbb{C}_\lambda$ denote the forcing notion adding λ Cohen reals. Let $n \in \omega, p_n \in \mathbb{C}_\lambda$.
1st case: $p_n \Vdash_{\mathbb{C}_\lambda} n \in \dot{A}^\alpha$. Then let $p^\alpha_{n,0} := p_n$.
2nd case: $p_n \Vdash_{\mathbb{C}_\lambda} n \in \dot{A}^\alpha$ does not hold.
1st subcase: There exists a $\bar{p}_n \leq_{\mathbb{C}_\lambda} p_n$ s.t. $\bar{p}_n \Vdash_{\mathbb{C}_\lambda} n \in \dot{A}^\alpha$. Then let $p^\alpha_{n,0} := \bar{p}_n$.
2nd subcase: There does not exists a $\bar{p}_n \leq_{\mathbb{C}_\lambda} p_n$ s.t. $\bar{p}_n \Vdash_{\mathbb{C}_\lambda} n \in \dot{A}^\alpha$. Then let $p^\alpha_{n,0} := p_n$.
Since 2 is countable, $\mathbb{C}_\lambda = F_n(\lambda \times \omega, 2)$ is ccc. By Zorn's lemma there exists a countable maximal antichain countaining $p^\alpha_{n,0}$. Repeat the process above for $p^\alpha_{n,1}$ and find a countable maximal antichain containing $\{p^\alpha_{n,0}, p^\alpha_{n,1}\}$. By iterative application of the above process we can find countable maximal antichains $\{p^\alpha_{n,i} : i \in \omega\} \subseteq \mathbb{C}_\lambda$, $n \in \omega$ and $\{k^\alpha_{n,i} \in 2 : i, n \in \omega\}$ s.t.

$$p^\alpha_{n,i} \Vdash_{\mathbb{C}_\lambda} \check{n} \in \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 1$$
$$\text{and } p^\alpha_{n,i} \Vdash_{\mathbb{C}_\lambda} \check{n} \notin \dot{A}^\alpha \text{ iff } k^\alpha_{n,i} = 0.$$

Let $B^\alpha = \bigcup\{\mathrm{dom}(p^\alpha_{n,i}) : i, n \in \omega\}$. Since the $p^\alpha_{n,i}$ are finite for $n, i \in \omega$, B^α is at most countable. So $|\bigcup_\alpha B^\alpha| \leq \kappa < \lambda$. By CH, $\aleph_1^{\aleph_0} = \aleph_1$. So the Δ-system lemma implies that $\{B^\alpha : \alpha < \omega_2\}$ forms a Δ-system with root R, where $|B^\alpha| = |B^\beta|$ for $\alpha, \beta < \omega_2$. We may assume that the bijection $\phi_{\alpha,\beta}$ from B^α to B^β fixes R. Then $\phi_{\alpha,\beta}$ induces a function $\psi_{\alpha,\beta} : \mathbb{C}_{B^\alpha} \to \mathbb{C}_{B^\beta}$ where $\psi_{\alpha,\beta}(f) = g$ iff

$$\mathrm{dom}(g) = \phi_{\alpha,\beta}(\mathrm{dom}(f)) \text{ and}$$
$$g(x) = f(\phi^{-1}_{\alpha,\beta}(x)) \text{ for any } x \in \mathrm{dom}(g).$$

Then $\psi_{\alpha,\beta}$ is an isomorphism between the restrictions of Cohen forcing $\mathbb{C}_{B^\alpha}$ and $\mathbb{C}_{B^\beta}$ as well as between $\mathbb{C}_{B^\alpha}$-names and $\mathbb{C}_{B^\beta}$-names. Since there are only $2^{\aleph_0} = \aleph_1$ many isomorphism types of names, by the pigeonhole principle we may assume that ψ identifies $\dot{A}^\alpha$ with $\dot{A}^\beta$. So $k_{n,i} := k^\alpha_{n,i} = k^\beta_{n,i}$ and $\psi(p^\alpha_{n,i}) = \psi(p^\beta_{n,i})$.

Define a new name $\dot{A}^\kappa$ for a countable $\kappa \subseteq \omega$ by setting

$$B^\kappa \cap \bigcup_{\alpha < \kappa} B^\alpha = R.$$

Any $B^\alpha, \alpha < \omega_2$, is mapped to B^κ by a bijection $\phi_{\alpha,\kappa}$ fixing R. Let $p^\kappa_{n,i}$ be the image of $p^\alpha_{n,i}$ under the induced isomorphism $\psi_{\alpha,\kappa}$ and $k^\kappa_{n,i} = k_{n,i}$.

Let $\beta < \kappa$. Since B^β is countable and CH holds, there exist $\alpha_1 \neq \alpha_2$ s.t.

$$B^{\alpha_1} \cap B^\beta = B^{\alpha_2} \cap B^\beta =: \Delta.$$

Since $\Delta \subseteq B^{\alpha_1} \cap B^{\alpha_2} = R$, there exists an $\alpha < \omega_2$ s.t. $B^\alpha \cap B^\beta \subseteq R$.

<u>Claim:</u> $B^\kappa \cap B^\beta = B^\alpha \cap B^\beta$

Proof: Since $R \subseteq B^\alpha$,

$$R \cap B^\beta \subseteq B^\alpha \cap B^\beta \subseteq R.$$

So $R \cap B^\beta = B^\alpha \cap B^\beta$. Since $R \subseteq B^\kappa$ and $(B^\kappa \setminus R) \cap B^\beta = \emptyset$,

$$\begin{aligned} B^\kappa \cap B^\beta &= (R \cup B^\kappa \setminus R) \cap B^\beta \\ &= R \cap B^\beta \cup (B^\kappa \setminus R \cap B^\beta) \\ &= R \cap B^\beta \\ &= B^\alpha \cap B^\beta. \end{aligned}$$

□

Since $B^\beta = B^\beta \cap R \sqcup B^\beta \setminus R$ and $B^\beta = B^\beta \cap B^\alpha \sqcup B^\beta \setminus B^\alpha$, $B^\beta \setminus R = B^\beta \setminus B^\alpha$. Since $B^\alpha = \Delta \sqcup R \setminus \Delta \sqcup B^\alpha \setminus R$

$$\begin{aligned} B^\alpha \setminus B^\beta &= \Delta \setminus B^\beta \sqcup (R \setminus \Delta) \setminus B^\beta \sqcup (B^\alpha \setminus R) \setminus B^\beta \\ &= \emptyset \sqcup R \setminus \Delta \sqcup B^\alpha \setminus R \\ &= R \setminus \Delta \sqcup B^\alpha \setminus R. \end{aligned}$$

So

$$\begin{aligned} B^\beta \cup B^\alpha &= \Delta \sqcup B^\beta \setminus R \sqcup B^\alpha \setminus R \cup R \setminus \Delta \\ &= \Delta \sqcup B^\beta \setminus B^\alpha \sqcup B^\alpha \setminus B^\beta \\ &= \Delta \sqcup R \setminus \Delta \sqcup B^\beta \setminus R \sqcup B^\alpha \setminus R. \end{aligned}$$

Also

$$\begin{aligned} B^\beta \cup B^\kappa &= B^\beta \sqcup B^\kappa \setminus B^\beta \\ &= (B^\beta \cup R \sqcup B^\beta \setminus R) \sqcup B^\kappa \setminus B^\beta \\ &= \Delta \sqcup B^\beta \setminus R \sqcup B^\kappa \setminus R \sqcup R \setminus \Delta \\ &= B^\beta \sqcup B^\kappa \setminus B^\beta \\ &= \Delta \sqcup R \setminus \Delta \sqcup B^\beta \setminus R \sqcup B^\kappa \setminus R. \end{aligned}$$

Define $\varphi : B^\alpha \cup B^\beta \to B^\kappa \cup B^\beta$ by

$$\varphi(x) := \begin{cases} x & \text{if } x \in B^\beta \cup R \\ \varphi_{\alpha,\kappa}(x) & \text{else.} \end{cases}$$

Let $\psi : \mathbb{C}_{B^\alpha \cup B^\beta} \to \mathbb{C}_{B^\kappa \cup B^\beta}$ be the isomorphism induced by φ. Since $\mathbb{1}_{\mathbb{C}_{B^\alpha\cup B^\beta}} \Vdash |\dot{A}_\alpha \cap \dot{A}_\beta| < \aleph_0$,

$$\psi(\mathbb{1}_{\mathbb{C}_{B^\alpha\cup B^\beta}}) \Vdash |\psi(\dot{A}_\alpha) \cap \psi(\dot{A}_\beta)| < \aleph_0.$$

By definition of ψ,

$$\psi(\mathbb{1}_{\mathbb{C}_{B^\alpha\cup B^\beta}}) = \mathbb{1}_{\mathbb{C}_{B^\kappa\cup B^\beta}},$$
$$\psi(\dot{A}_\beta) = \dot{A}_\beta,$$
$$\psi(\dot{A}_\alpha) = \psi_{\alpha,\kappa}(\dot{A}_\alpha) = \dot{A}_\kappa.$$

So $\mathbb{1}_{\mathbb{C}_{B^\kappa\cup B^\beta}} \Vdash |\dot{A}_\kappa \cap \dot{A}_\beta| < \aleph_0$. So $\mathbb{1}_{\mathbb{C}_\lambda} \Vdash |\dot{A}_\kappa \cap \dot{A}_\beta| < \aleph_0$. So $\{\dot{A}_\alpha\}_{\alpha<\kappa}$ is not maximal. $\square$

Assume CH for the rest of this chapter. Also let λ be an uncountable cardinal s.t. $\lambda = \bigcup_{n\in\omega} \lambda_n$, where $\{\lambda_n\}_{n\in\omega}$ is a strictly increasing sequence of regular cardinals, $\lambda_0 \geq \aleph_2$, $\lambda_n^{\aleph_0} = \lambda_n$ for all $n \in \omega$ and $\kappa^{\aleph_0} < \lambda_n$ for $\kappa < \lambda_n$ for the rest of this chapter.

We begin to work on the definition of the two-sided template used to proof Theorem 1.3. For this purpose, we need a notion of negative ordinals λ^*.

Definition 6.2. Let μ be an ordinal. We denote by μ^* a disjoint copy of μ with the reverse ordering. Let $\leq_\mu$ denote the ordering of μ. We will refer to the elements of μ as positive and the elements of μ^* as negative.

If $\alpha \neq \beta \in \lambda^* \cup \lambda$, we will say that $\alpha <_{\lambda\cup\lambda^*} \beta$ if:

1. $\alpha \in \lambda^*$ and $\beta \in \lambda$,
2. $\alpha, \beta \in \lambda$ and $\alpha <_\lambda \beta$,
3. $\alpha, \beta \in \lambda^*$ and $\alpha <_{\lambda^*} \beta$.

For each $n \in \omega$ fix a partition $\lambda_n^* = \bigcup_{\alpha<\omega_1} S_n^\alpha$, where the S_n^α's are co-initial in λ_n^* and for $m < n$, $S_n^\alpha \cap \lambda_n^* = S_m^\alpha$.

We define the underlying linear order, the sets L_0, L_1 partitioning L and the structural description $\mathcal{I}$ of the two-sided template.

Definition 6.3. Let $L = L(\lambda)$ consist of all finite, nonempty sequences x s.t.

1. $x(0) \in \lambda_0$,
2. $x(n) \in \lambda_n^* \cup \lambda_n$ for $0 < n < |x| - 1$,
3. for $|x| \geq 2$, if $x(|x|-2)$ is positive, then $x(|x|-1) \in \lambda_{|x|-1}^* \cup \lambda$ and if $x(|x|-2)$ is negative, then $x(|x|-1) \in \lambda^* \cup \lambda_{|x|-1}$.

Let $x, y \in L$. Define $x < y$ iff

1. $x \subsetneq y$ and $y(|x|)$ is positive or
2. $y \subsetneq x$ and $x(|y|)$ is negative or
3. $n = \min\{k : x(k) \neq y(k)\}$ is defined and $x(n) <_{\lambda^* \cup \lambda} y(n)$.

Note that $(L, \leq)$ is a linear order.

Definition 6.4. Let $L_1 := \{x \in L : |x| = 1 \text{ or } x(|x|-1) \in \lambda^*_{|x|-1} \cup \lambda_{|x|-1}\}$ and $L_0 := L \setminus L_1$.

So $x \in L_0$ iff $|x| \geq 2$ and if $x(|x|-2)$ is positive, $x(|x|-1) \in [\lambda_{|x|-1}, \lambda)$ and if $x(|x|-2)$ is negative, $x(|x|-1) \in (\lambda^*, \lambda^*_{|x|-1}]$. Also L_0 and L_1 are cofinal in L and uncountable.

Definition 6.5. Let L_{rel} be the subset of L_1 of all x s.t. $|x| \geq 3$ is odd, $x(n) \in \lambda^*_n$ for odd n, $x(n) \in \lambda_n$ for even n, $x(|x|-1) \in \omega_1$ and whenever $n < m$ are even s.t. $x(n), x(m) \in \omega_1$, then there are $\beta < \alpha$ s.t. $x(n-1) \in S^{\alpha}_{n-1}$ and $x(m-1) \in S^{\beta}_{m-1}$. We refer to the elements of L_{rel} as relevant.

For $x \in L_{\text{rel}}$ let $J_x := \{z \in L : x \restriction (|x|-1) \leq z < x\}$. If $x < y$ are relevant, then $J_x \cap J_y = \emptyset$ or $J_x \subseteq J_y$. If $J_x \subseteq J_y$, then $|y| \leq |x|$, $x \restriction (|y|-1) = y \restriction (|y|-1)$ and $x(|y|-1) \leq y(|y|-1)$.

Definition 6.6. Let $\mathcal{I} = \mathcal{I}(\lambda)$ be the collection of all sets of the form:

$$(\bigcup_{\alpha \in I_0} L_\alpha) \cup (\bigcup_{x \in I_1} \text{cl}(J_x)) \cup (\bigcup_{x \in I_2} \text{cl}(\{x\})) \cup (\bigcup_{x \in I_3} L_x \cap L_0),$$

where $I_0 \in [\lambda_0 \cup \{\lambda_0\}]^{<\omega}$, $I_1 \in [L_{\text{rel}}]^{<\omega}$ and $I_2, I_3 \in [L_1]^{<\omega}$.

To plug $\mathcal{T}$ into $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$, we need to check that it is well-defined.

Lemma 6.7. *[FT13][Lemma 5.7.]* $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ *is a two-sided template.*

Proof. Similar to [Bre03][Lemma 2.1.] □

For the rest of this chapter, let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$.

In Chapter 5, we have shown two estimates for $\mathfrak{a}_g$. Those estimates depend on the underlying two-sided template in $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$. For the two-sided template $\mathcal{T}$, defined in this Chapter, we obtain the following estimate.

Lemma 6.8. *In* $V^{\mathbb{P}}$ *there is a maximal cofinitary group of size* λ *and* $\lambda_0 \leq \mathfrak{a}_g$.

Proof. Since L_0 is uncountable and cofinal in L, Lemma 5.2 implies that $\mathbb{P}$ adds a maximal cofinitary group of size $|L_0|$. Since we can allocate every $\alpha \in \lambda$ in an injective manner to an $x \in L_0$, $\lambda \leq |L_0|$. Since

$$|L_0| \leq \sum_{n \in \omega} (\prod_{i=0}^{n} 2^i |\lambda_i|) = \lambda,$$

$\lambda = |L_0|$.

Let $x \in L$. Then $y := x(0) + 1 \in \lambda_0$. So $x \leq y$. So λ_0 is cofinal in L. Since $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$, Lemma 5.1 implies that $\lambda_0 \leq \mathfrak{a}_g$. $\square$

Note that the construction of $\mathcal{T}$ depends on λ, where λ is an uncountable cardinal satisfying a certain construction.

At last, we proof Theorem 1.3 via an isomorphism of names argument.

Definition 6.9. Let g be a real. Then $\dot{g}$ is a good name for g, if there are predense sets $\{p_{n,i}\}_{i\in\omega}$ where $n \in \omega$ and sets of integers $\{k_{n,i}\}_{i\in\omega}$ where $n \in \omega$ s.t. $p_{n,i} \Vdash \dot{g}(n) = k_{n,i}$ for all $n, i \in \omega$.

Whenever $\dot{g}$ is a good name for a real, we will refer to $\bigcup_{n,i\in\omega} \mathrm{dom}(p_{n,i})$ as the L-domain of $\dot{g}$ and denote it $\mathrm{dom}_L(\dot{g})$.

We can assume that all $\mathbb{P}$-names for reals are good.

Lemma 6.10. *[FT13][Lemma 5.8.] Let $\lambda_0 \leq \kappa < \lambda$ and for every $\beta \in \kappa$ let $B^\beta = dom(\dot{g}^\beta)$ be a countable subtree of L, where $\dot{g}^\beta$ is a good name for a cofinitary permutation. Then there is a countable subtree B^κ and a good name for a cofnitary permutation $\dot{g}^\kappa$ s.t.*

1. $\Vdash_{\mathbb{P}} \dot{g}^\kappa \neq \dot{g}^\beta$ *for all* $\beta < \kappa$,
2. $dom_L(\dot{g}^\kappa) = B^\kappa$,
3. *for every* $F \in [\kappa]^{<\omega}$ *there is* $\alpha < \kappa$ *and a partial order isomorphism*

$$\chi_{F,\alpha} : \mathbb{P}_{cl(\bigcup_{\beta\in F} B^\beta \cup B^\alpha)} \to \mathbb{P}_{cl(\bigcup_{\beta\in F} B^\beta \cup B^\kappa)}$$

which maps $\dot{g}^\alpha$ to $\dot{g}^\kappa$ and fixes $\dot{g}^\beta$ for $\beta \in F$.

Proof. Similar to [Bre03][Main Theorem](pp.2646-2648) $\square$

7 $\mathfrak{a}_g$ can be of countable cofinality

We are finally ready to obtain the main result discussed in this thesis and complete the proof of 1.3:

Proof. Let $\mathcal{G}$ be a $\mathbb{P}$-name for a cofinitary group of size κ, where $\lambda_0 \leq \kappa < \lambda$ and let $\{\dot{g}^\beta\}_{\beta\in\kappa}$ be an enumeration of $\mathcal{G}$. Let $B^\beta = \mathrm{dom}_L(\dot{g}^\beta)$ for $\beta < \kappa$. Then B^β is at most a countable subset of L and w.l.o.g. a tree. Let B^β and $\dot{g}^\kappa$ be as in Lemma 6.10 after application to $\{B^\beta\}_{\beta\in\kappa}$ and $\{\dot{g}^\beta\}$.
Show that $\mathcal{H} := \langle \mathcal{G} \cup \{\dot{g}^\kappa\}\rangle$ is a cofinitary group :
Let $\dot{h} \in \mathcal{H}/\mathcal{G}$ and $F_0 \cup \{\kappa\}$ be the indexes of the permutations involved in $\dot{h}$, where $F_0 \in [\kappa]^{<\omega}$. By Lemma 6.10, there exist $\alpha < \kappa$ s.t. the partial order isomorphism

$$\chi = \chi_{F_0,\alpha} : \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F_0} B^\beta \cup B^\alpha)} \to \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F_0} B^\beta \cup B^\kappa)},$$

maps $\dot{g}^\alpha$ to $\dot{g}^\kappa$ and fixes $\dot{g}^\beta$ for any $\beta \in F_0$. Then $\chi^{-1}(h)$ is a name for an element of $\mathcal{G}$. So $|\mathrm{fix}(\chi^{-1}(h))| < \aleph_0$. Since $\mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F_0} B^\beta \cup B^\alpha)}, \mathbb{P}_{\mathrm{cl}(\bigcup_{\beta\in F_0} B^\beta \cup B^\kappa)} \lessdot \mathbb{P}$,

$$V^{\mathbb{P}} \models |\mathrm{fix}(h)| < \aleph_0.$$

$\square$

So Theorem 1.3 is proven.

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