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Phase transitions in planar statistical mechanics: Ashkin–Teller and related models

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Abstract

This thesis is devoted to the study of phase transitions and the associated behaviour in the Ashkin–Teller (AT) model on the square lattice $\mathbb{Z}^2$ and its related models, including the six- and eight-vertex models, the Potts model and FK percolation. The AT model was introduced in 1943 as a generalisation of the Ising model and may be represented by a pair of Ising spin configurations with coupling constants J, J' for each and U for their pointwise product, the latter describing the interaction within the pair. The thesis can be divided into three parts.

The first part is a study of the isotropic ($J = J'$) ferromagnetic AT model on the square lattice $\mathbb{Z}^2$. We confirm the presence of a *single phase transition* when $J \geq U > 0$ and *two distinct* ones when $U > J > 0$. In the uniqueness case, we identify the transition at the self-dual curve. In the non-uniqueness case, we show that the two distinct transition curves are dual to each other. A significant part is the detailed study of the *self-dual* model via its relations to the six-vertex model and self-dual FK percolation. The main result in this regard is *exponential decay* of correlations of the single spins and *uniform positivity* of correlations of the product, each in finite volume and under the respective least favourable boundary conditions, when $U > J$.

In the second part, we investigate the AT model in the presence of negative coupling constants on the hypercubic lattices $\mathbb{Z}^d$ in any dimension $d \geq 2$. Observing that it suffices to study the case $J, J' \geq 0 > U$, we confirm both *ferromagnetic and antiferromagnetic behaviour* in this regime. We first establish the existence of a partial antiferromagnetic phase in the isotropic model in a perturbative sub-regime. In $d = 2$, we show that the associated staggered six-vertex *height function* is *localised*, although it simultaneously exhibits antiferromagnetic behaviour. We then demonstrate ferromagnetic behaviour by partitioning another sub-regime into a collection of smooth curves along which the model undergoes a subcritically *sharp order-disorder phase transition*, circumventing the difficulty of the lack of general monotonicity properties in the parameters.

The third part builds on the first one and deals with the associated self-dual Potts model with $q > 4$ states on the square lattice $\mathbb{Z}^2$. Under *order-disorder Dobrushin* boundary conditions on a square box of size n, we verify that the *interface* between the ordered and disordered phases is thin, has *fluctuations* of order $\sqrt{n}$ and converges when rescaled to a *Brownian bridge*. This is achieved by a coupling with a graphical representation of the AT model, which allows to relate the interface in the Potts model to a subcritical cluster conditioned to be long, and then developing the celebrated Ornstein–Zernike theory to deduce convergence of the latter. We also show the analogous statements for self-dual FK percolation with $q > 4$. In a subsequent work, which is not part of the thesis, we further build on this project and establish the so-called *wetting phenomenon* in the Potts model. We present the relevant coupling of the Potts model under order-order Dobrushin boundary conditions and a graphical representation of the AT model, conditioned to admit a pair of long clusters, which turn out to be repulsive to each other.

Zusammenfassung

Diese Arbeit widmet sich der Untersuchung von Phasenübergängen und dem damit verbundenen Verhalten im Ashkin–Teller (AT)-Modell auf dem Quadratgitter $\mathbb{Z}^2$ und seinen verwandten Modellen, einschließlich der Sechs- und Acht-Vertex-Modelle, des Potts-Modells und der FK-Perkolation. Das AT-Modell wurde als Verallgemeinerung des Ising-Modells eingeführt und kann durch ein Paar von Ising-Spin-Konfigurationen mit Kopplungskonstanten J, J' für jede von ihnen und U für ihr punktweises Produkt, welches deren Zusammenspiel beschreibt, dargestellt werden. Die Arbeit kann in drei Teile gegliedert werden.

Der erste Teil untersucht das isotrope ($J = J'$) ferromagnetische AT-Modell auf dem Quadratgitter $\mathbb{Z}^2$. Wir bestätigen das Auftreten eines *einzigen Phasenübergangs* wenn $J \geq U > 0$ und von *zwei verschiedenen* wenn $U > J > 0$. Im ersten Fall identifizieren wir den Übergang an der selbst-dualen Kurve. Im zweiten Fall zeigen wir, dass die beiden unterschiedlichen Übergangskurven zueinander dual sind. Ein wesentlicher Teil ist die genaue Untersuchung des *selbst-dualen* Modells durch seine Zusammenhänge mit dem Sechs-Vertex-Modell und der selbst-dualen FK-Perkolation. Das Hauptresultat in diesem Zusammenhang ist die *exponentielle Abnahme* der Korrelationen der einzelnen Spins und die gleichmäßige Positivität der Korrelationen des Produkts, jeweils im endlichen Volumen und unter den ungünstigsten Randbedingungen, wenn $U > J$.

Im zweiten Teil betrachten wir das AT-Modell in der Gegenwart negativer Kopplungskonstanten auf den Ganzzahlgittern $\mathbb{Z}^d$ in beliebigen Dimensionen $d \geq 2$ und wir bestätigen sowohl *ferromagnetisches als auch antiferromagnetisches* Verhalten. Wir weisen zunächst die Existenz einer partiellen antiferromagnetischen Phase im isotropen Modell in einem perturbativen Unterregime nach. In $d = 2$ zeigen wir, dass die zugehörige gestaffelte Sechs-Vertex-*Höhenfunktion lokalisiert* ist, während sie gleichzeitig antiferromagnetisches Verhalten aufweist. Anschließend demonstrieren wir ferromagnetisches Verhalten, indem wir ein weiteres Unterregime in eine Familie von glatten Kurven partitionieren, entlang derer das Modell einen subkritisch *scharfen Ordnungs-Unordnungs-Phasenübergang* durchläuft, wodurch die Schwierigkeit des Fehlens allgemeiner Monotonieeigenschaften in den Parametern umgangen wird.

Der dritte Teil baut auf dem ersten auf und behandelt das zugehörige selbst-duale Potts-Modell mit $q > 4$ Zuständen auf dem Quadratgitter $\mathbb{Z}^2$. Unter *Ordnungs-Unordnungs-Dobrushin*-Randbedingungen auf einer Box der Größe n weisen wir nach, dass die *Grenzfläche* zwischen den geordneten und ungeordneten Phasen dünn ist, *Fluktuationen* der Ordnung $\sqrt{n}$ aufweist und reskaliert zu einer *Brownschen Brücke* konvergiert. Dies wird durch eine Kopplung mit einer graphischen Darstellung des AT-Modells erreicht, die es ermöglicht, die Grenzfläche des Potts-Modells mit einem subkritischen Cluster in Relation zu setzen, der darauf bedingt ist lange zu sein, und dann die renommierte Ornstein–Zernike-Theorie zu entwickeln, um die Konvergenz des Letzteren zu deduzieren. In einer weiteren Arbeit, die nicht Teil dieser Dissertation ist, bauen wir auf diesem Projekt auf und weisen das sogenannte *Benetzungsphänomen* nach. Wir präsentieren die relevante Kopplung des Potts-Modells unter Ordnungs-Ordnungs-Dobrushin-Randbedingungen und einer graphischen Darstellung des AT-Modells unter der Bedingung, dass zwei lange Cluster existieren, welche sich als gegenseitig abstoßend herausstellen.

Organisation and included articles

This thesis is organised into five chapters. The chapters 2-5 each cover the parts outlined in the abstract above.

- Chapter 1 provides a concise introduction to discrete lattice models from statistical mechanics and discusses the specific models that are considered in this thesis. It also outlines the motivation for the studies performed in Chapters 2-5 and presents an informal description of their main results.

- Chapter 2 is devoted to the study of the ferromagnetic isotropic Ashkin–Teller model on the square lattice $\mathbb{Z}^2$ and is based on

  Yacine Aoun, Moritz Dober and Alexander Glazman, *Phase Diagram of the Ashkin–Teller Model.* Commun. Math. Phys. 405, 37 (2024) [ADG24].

- Chapter 3 considers the antiferromagnetic Ashkin–Teller model on the integer lattice $\mathbb{Z}^d$ and, in the planar case $d = 2$, the associated six-vertex height function. It is based on

  Moritz Dober, *On antiferromagnetic regimes in the Ashkin–Teller model.* ArXiv preprint (2024) [Dob24].

- Chapter 4 treats interfaces in the self-dual Potts model with $q > 4$ states on the square lattice $\mathbb{Z}^2$ and under order-disorder Dobrushin boundary conditions, via couplings with the six-vertex and AT models. It is based on

  Moritz Dober, Alexander Glazman and Sébastien Ott, *Discontinuous transition in 2D Potts: I. Order-Disorder Interface convergence.* ArXiv preprint (2025) [DGO25a].

- Chapter 5 presents an analogue of a coupling in Chapter 4 for the Potts model with the same parameters but under order-order Dobrushin boundary conditions, which will be used in a forthcoming work [DGO25b] to verify the wetting phenomenon.

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Contents

Chapter 1

Introduction

Grown out of thermodynamics, statistical mechanics is a branch of theoretical physics that aims to derive the macroscopic behaviour of large physical systems of particles from the microscopic description of the properties of individual particles. Its origins trace back to the works of Maxwell, Boltzmann and Gibbs in the second half of the 19th century.

The complexity of a system, due to the potentially enormous number of variables, complicates this task immensely, presumably rendering it unachievable. This suggests that the focus should be directed towards determining the probable state of the system, as opposed to providing a full deterministic analysis, which constitutes a fundamental principle within this field. The objective is to identify and study probability distributions on the set of all possible states that effectively represent the system's behaviour. This provides a mathematical framework for studying thermodynamic phenomena through the statistical treatment of particle systems. Irrespective of their accuracy in describing the corresponding phenomena in a physically adequate way, the mathematics behind the probabilistic models that have emerged from these considerations has proved to be sophisticated and rich, and ultimately relevant to the physics.

Phase transitions. The likelihood of observing a state in such a system decreases with its total energy. The latter is governed by macroscopic parameters whose variation can lead to abrupt changes in the system's properties, known as a *phase transition*. Classical examples of this phenomenon are the transitions between the solid, liquid and gaseous phases of water. Another, which would provide the motivation for the famous Ising model, was first documented by Pierre Curie in 1895: certain materials undergo a transition from ferromagnetic to paramagnetic behaviour once the temperature exceeds a threshold specific to the substance, now known as the Curie temperature. When a probabilistic model designed to capture this type of behaviour is considered, one of the first fundamental questions that arises is whether or not the model is capable of detecting the phase transition.

1.1 Lattice models

The family of probabilistic models in statistical mechanics can be divided according to different criteria, such as the system's physical nature (classical or quantum) and whether or not it is at equilibrium, that is, whether or not its macroscopic properties are constant over time. Only *classical systems at equilibrium* are considered in this thesis. Further criteria are the topological and geometrical features of the particles' state space and the structure they reside on, as well as the type of interactions between them. When the underlying structure is discrete and regular, the model is called a *lattice model*. Let us describe a generic way to construct such a model:

(i) Fix a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, for example the hypercubic lattice $(\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 1$.

(ii) Take a set S consisting of microscopic *states* placed at the vertices (or alternatively the edges) of the graph, for example $S = \{0, 1\}$ (absent and present) to model percolation phenomena, or the N-sphere $S = \mathbb{S}_N \subset \mathbb{R}^{N+1}$ (spins) to model magnetism. This is referred to as the *(single-)state space*, and the corresponding *configuration spaces* are given by $\Sigma_V = S^V,\ V \subset \mathbb{V}$ (or alternatively $\Omega_E = S^E,\ E \subset \mathbb{E}$).

(iii) Let ν be a finite measure on S, called *reference measure* and taken as the counting measure if not specified. Given a finite subgraph $G = (V, E)$ of $\mathbb{G}$, choose a function $\mathcal{H}_G : \Sigma_V \to \mathbb{R}$, interpreted as *energy* and called *Hamiltonian*. For each parameter $\beta > 0$, interpreted as and called *inverse temperature*, define the associated probability measure $\mu_{G,\beta}$ on Σ_V by

$$d\mu_{G,\beta} := \frac{1}{Z_{G,\beta}} e^{-\beta \mathcal{H}_G} \prod_{v \in V} d\nu, \tag{1.1}$$

where $Z_{G,\beta}$ is the unique normalising constant, called the *partition function*, rendering $\mu_{G,\beta}$ a probability measure. The exponential $e^{-\beta \mathcal{H}_G}$ is called the *Boltzmann weight*, and the measure $\mu_{G,\beta}$ is called the *Gibbs distribution* on G with parameter β.

The measure $\mu_{G,\beta}$ favours configurations of low energy $\mathcal{H}_G$, with this preference intensifying as β is increased. Phase transitions may be detected by considering the averages $\langle f \rangle_{G,\beta}$ of certain observables $f : \Sigma_V \to \mathbb{R}$ under $\mu_{G,\beta}$ when the finite subgraphs G grow and exhaust the whole lattice. Ideally, as G exhausts the lattice, the measure $\mu_{G,\beta}$ converges in a weak sense to a so-called *infinite-volume* measure μ_β on $\Sigma_{\mathbb{V}}$.

Boundary conditions, hardcore constraints and multiple parameters. A fundamental aspect of statistical mechanics is to investigate how assumptions made at the boundary of a given domain affect the behaviour observed in its interior. In order to incorporate into the above setting *boundary conditions* given by configurations $\eta \in \Sigma_{\mathbb{V}}$, one can introduce Hamiltonians $\mathcal{H}_G^\eta$ that induce interactions between $\sigma \in \Sigma_V$ and the restrictions of η to $\mathbb{V} \setminus V$, and then define the measures $\mu_{G,\beta}^\eta$ as in (1.1) but with $\mathcal{H}_G$ replaced by $\mathcal{H}_G^\eta$.

The measures defined by (1.1) assign positive probabilities to products of events that each have positive probability under ν. So-called *hardcore constraints* can be imposed by adding indicators of the respective events as factors of the density in (1.1).

Often the Hamiltonian $\mathcal{H}_G$ is itself subject to parameters, in which case it will be convenient to vary some of them separately while fixing the others. In these cases, the inverse temperature β is taken to be 1 and is replaced by the parameters of the Hamiltonian in the subscript of the respective measures.

Classifications and terminology. The above setting focuses on models whose configurations are assignments of certain values to the vertices of the graph, which will henceforth be called *vertex-models*. When the underlying state space S is finite or a compact subset of $\mathbb{R}^d$, $d \geq 1$, we call S the *(single-)spin space*, its elements the *spins*, and the elements of Σ_V the *spin configurations*. In this case, the model is also called a *lattice spin system*.

On the other hand, if the configurations are assignments of values to the edges, the model is referred to as an *edge-model*. When the state space is given by $S = \{0, 1\}$, the model is termed a *percolation model*, and the elements of Ω_E are called *percolation configurations*.

1.1.1 Examples of lattice spin systems

This section provides some examples of classical lattice spin systems, most of which will be discussed in more detail in the subsequent sections. While the formulations are given in the general setting of an arbitrary fixed graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, one may consider the integer lattices $(\mathbb{Z}^d, \mathbb{E}^d)$ (with $\mathbb{E}^d$ given by edges connecting nearest neighbours) for later purposes. Let $G = (V, E)$ be a finite subgraph of $\mathbb{G}$.

The Ising model was introduced in 1920 by Lenz [Len20] to theoretically study the phase transition from ferromagnetic to paramagnetic behaviour documented by Curie. Ising [Isi25] further developed what would become known as the Ising model and solved it on a linear chain.

In order to construct a (simple) mathematical model of a magnet, one places ± 1-valued *spins* at the vertices of a lattice. The single-spin space is thus defined as $S = \{-1, +1\}$, and the possible (Ising) spin configurations are represented by $\sigma \in \Sigma_V = \{-1, +1\}^V$, where σ_x denotes the spin at vertex $x \in V$. Assuming that only spins at adjacent vertices interact with one another in the absence of an external magnetic field, the associated Ising Hamiltonian on $G = (V, E)$ is defined by

$$\mathcal{H}_G(\sigma) := -\sum_{\{x,y\} \in E} \sigma_x \sigma_y.$$

The associated measure $\mu_{G,\beta}$ is called the (ferromagnetic) Ising measure on G with inverse temperature β and *free boundary conditions* and is denoted by $\mathtt{Ising}^{\mathrm{f}}_{G,\beta}$. The constant $+$ configuration η^+ defined by $\eta^+_x = +1$ for all $x \in V$ constitutes a *ground state*, that is, a configuration of minimal energy. The same holds for the constant $-$ configuration, and in fact there is no other. It is therefore to be expected that, as the inverse temperature β increases, the typical realisations of the measure will increasingly resemble one of these two ground states. This phenomenon will be discussed further in Section 1.2.1.

In order to add an interaction with a boundary condition $\eta \in \Sigma_{\mathbb{V}}$ as described in Section 1.1, we introduce the Hamiltonian

$$\mathcal{H}^\eta_G(\sigma) := -\sum_{\{x,y\} \in E} \sigma_x \sigma_y - \sum_{\substack{x \in V,\, y \in \mathbb{V} \setminus V: \\ \{x,y\} \in \mathbb{E}}} \sigma_x \eta_y.$$

We denote the corresponding measure $\mu^\eta_{G,\beta}$ by $\mathtt{Ising}^\eta_{G,\beta}$ and call it the Ising measure on G with inverse temperature β and boundary condition η. When η is the constant $+$ boundary condition η^+, we simply write $+$ in the superscript of the measure, and similarly for the constant $-$ boundary condition η^-.

The Ashkin–Teller (AT) model was introduced in 1943 by Ashkin and Teller [AT43] as a generalisation of the Ising model to theoretically study phase transitions. Similarly to the latter, one also places spins at the vertices of the graph but with values in a 4-element set $S = \{A, B, C, D\}$. Given $\mathcal{E}_i \in \mathbb{R}$, $i = 0, \ldots, 3$, the Hamiltonian is defined by

$$\mathcal{H}_{G,\mathcal{E}_i}(\sigma) := \sum_{\{x,y\} \in E} \mathcal{E}\big(\sigma_x, \sigma_y\big),$$

where the *interaction energy* $\mathcal{E} : S \times S \to \mathbb{R}$ is the unique symmetric function that satisfies

$$\mathcal{E}(A, A) = \mathcal{E}(B, B) = \mathcal{E}(C, C) = \mathcal{E}(D, D) = \mathcal{E}_0,$$
$$\mathcal{E}(A, B) = \mathcal{E}(C, D) = \mathcal{E}_1, \qquad \mathcal{E}(A, C) = \mathcal{E}(B, D) = \mathcal{E}_2, \qquad \mathcal{E}(A, D) = \mathcal{E}(B, C) = \mathcal{E}_3.$$

Fan [Fan72a] noticed that the model can be represented by pairs of Ising spin configurations. Indeed, representing the state space by $S = \{-1, +1\}^2$ by setting $A = (1, 1)$, $B = (1, -1)$, $C = (-1, 1)$, $D = (-1, -1)$, the interaction energy can be written as

$$\mathcal{E}\big((\tau_x, \tau_x'), (\tau_y, \tau_y')\big) = -\big(J_0 + J\tau_x\tau_y + J'\tau_x'\tau_y' + U\tau_x\tau_x'\tau_y\tau_y'\big),$$

where the so-called *coupling constants* are given by

$$\begin{aligned} J_0 &= -(\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3), \quad & J &= -(\mathcal{E}_0 + \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3), \\ J' &= -(\mathcal{E}_0 - \mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3), \quad & U &= -(\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3). \end{aligned}$$

Identifying $\Sigma_V = S^V$ with $\{-1, +1\}^V \times \{-1, +1\}^V$, the Hamiltonian can be written as

$$\mathcal{H}_{G,\mathcal{E}_i}(\tau, \tau') = -|E|J_0 - \sum_{\{x,y\}\in E} J\,\tau_x\tau_y + J'\,\tau_x'\tau_y' + U\,\tau_x\tau_x'\tau_y\tau_y'.$$

We call the associated measure on $\{\pm 1\}^V \times \{\pm 1\}^V$ with inverse temperature $\beta = 1$ the AT measure on G with coupling constants J, J', U and free boundary conditions, and we denote it by $\mathtt{AT}^{\mathrm{f,f}}_{G,J,J',U}$. It is called *ferromagnetic* if $J, J', U \geq 0$, and *antiferromagnetic* otherwise. Furthermore, it is referred to as *isotropic* when $J = J'$, and *anisotropic* otherwise. The choice $U = 0$ leads to a pair of independent Ising spin configurations with potentially different inverse temperatures. When $J = J' = U > 0$, the interaction depends solely on whether spins at adjacent vertices coincide. This is an instance of the Potts model, which we will introduce next. Given $\eta, \eta' \in \Sigma_{\mathbb{V}}$, we define the corresponding AT measure with boundary conditions η, η' analogously to the Ising model above, and we denote it by $\mathtt{AT}^{\eta,\eta'}_{G,J,J',U}$.

The Potts model, proposed by Domb and introduced by Potts [Pot52] in 1952, represents another generalisation of the Ising model, allowing for an arbitrary integer number of single-spin values. The state space of the q-state Potts model ($q \geq 2$ integer) is given by $S = \{1, \ldots, q\}$. As mentioned before, the interaction only takes into account whether or not spins at adjacent vertices align, whence the Hamiltonian is given by

$$\mathcal{H}_{G,q}(\sigma) := -\sum_{\{x,y\}\in E} \delta_{\sigma_x,\sigma_y},$$

where $\delta_{i,j}$ is the Kronecker delta. The associated measure $\mu_{G,\beta}$ is called the q-state Potts measure on G with inverse temperature β and free boundary conditions and denoted by $\mathtt{Potts}^{\mathrm{f}}_{G,q,\beta}$. As before, we define the measure $\mathtt{Potts}^{\eta}_{G,q,\beta}$ with boundary condition $\eta \in \Sigma_{\mathbb{V}}$ analogously to the Ising case. The boundary conditions that take a constant value $i \in S$ are referred to as monochromatic, and the respective measures are denoted by $\mathtt{Potts}^{i}_{G,q,\beta}$.

The case of each particular q can be considered as its own model, resulting in a collection of *Potts models*. Observe that the choice $q = 2$ leads to the Ising model with inverse temperature $\beta/2$.

Spin $O(N)$ model. The single-spin spaces of the models introduced above share the property of being discrete. Although only such spin systems are studied in this work, we mention the Spin $O(N)$ model as a classical spin system with a continuous single-spin space.

Ising spin configurations are assignments of values in the 0-sphere $\mathbb{S}_0 = \{-1, +1\}$ to the vertices of the lattice. Stanley [Sta68] suggested the generalisation to values in the N-sphere $S =$

$\mathbb{S}_N \subset \mathbb{R}^{N+1}$ with arbitrary $N \geq 0$. The reference measure ν is chosen as the Lebesgue measure on $\mathbb{S}_N$, and the Hamiltonian is defined by

$$\mathcal{H}_{G,N}(\mathbf{S}) = - \sum_{\{x,y\}\in E} \mathbf{S}_x \cdot \mathbf{S}_y,$$

where $\mathbf{S}_x \cdot \mathbf{S}_y$ denotes the scalar product in $\mathbb{R}^N$.

The eight-vertex model was originally introduced [Sut70, FW70] as a model of edge orientations on the square lattice $(\mathbb{Z}^2, \mathbb{E}^2)$, generalising the *six-vertex* (also known as ice-type) model [Pau35, Rys63]. Both admit so-called *staggered* versions, introduced in [WL75, HLW75], which are related to the AT model. The term 'staggered' refers to the underlying weights differing on two sublattices. We will directly introduce the spin representation [Wu71, KW71, Lis22, GP23] of the staggered eight-vertex model, choosing the setting in which Ising spins are assigned to both the vertices of the square lattice and its *dual*, defined below. In this context, the edge orientations would be defined on the *medial graph* of the square lattice.

The vertex-set of the dual square lattice is $(\mathbb{Z}^2)^* = (1/2, 1/2) + \mathbb{Z}^2$ and ist edge-set $\mathbb{E}^*$ is given by edges between nearest neighbours, that is, between vertices at a Eucledian distance 1. Regarded as line-segments between their end-vertices, each edge $e \in \mathbb{E}^2$ intersects a unique edge in $\mathbb{E}^*$, which we denote by e^*. Although we could introduce the measures in the previous setting, we choose to provide a more classical definition. Let $\Delta \subset \mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$ be a finite subset, and let $E \subset \mathbb{E}^2$ be the set of edges $e \in \mathbb{E}^2$ for which $e \cup e^* \subset \Delta$. Moreover, set $\Delta^\bullet = \Delta \cap \mathbb{Z}^2$ and $\Delta^\circ = \Delta \cap (\mathbb{Z}^2)^*$. We consider pairs of spin configurations $(\sigma^\bullet, \sigma^\circ) \in \Sigma_{\Delta^\bullet} \times \Sigma_{\Delta^\circ}$, and define the sets of disagreement edges as

$$E_{\sigma^\bullet} := \big\{\{x,y\} \in E : \sigma^\bullet_x \neq \sigma^\bullet_y\big\} \quad \text{and} \quad E_{\sigma^\circ} := \big\{e \in E : e^* = \{u,v\} \text{ and } \sigma^\circ_u \neq \sigma^\circ_v\big\}.$$

Given weights $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \geq 0$, the staggered eight-vertex spin measure $\mathtt{EightV}_{\Delta,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ (with free boundary conditions) is the probability measure on $\Sigma_{\Delta^\bullet} \times \Sigma_{\Delta^\circ}$ defined by

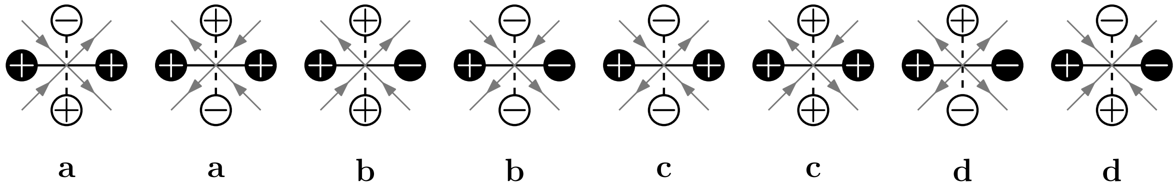
$$\mathtt{EightV}_{\Delta,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}(\sigma^\bullet, \sigma^\circ) = \frac{1}{Z_{\Delta,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}} \mathbf{a}^{|E_{\sigma^\circ} \setminus E_{\sigma^\bullet}|} \mathbf{b}^{|E_{\sigma^\bullet} \setminus E_{\sigma^\circ}|} \mathbf{c}^{|E \setminus (E_{\sigma^\bullet} \cup E_{\sigma^\circ}|)} \mathbf{d}^{|E_{\sigma^\bullet} \cap E_{\sigma^\circ}|}.$$

Boundary conditions are introduced in a similar way to the Ising model, see Section 3.5.1 for details. When $\mathbf{a} = \mathbf{b}$, the model is non-staggered in the sense that the primal and dual lattices play symmetric roles. When $\mathbf{d} = 0$, it holds almost surely that $E_{\sigma^\bullet} \cap E_{\sigma^\circ} = \varnothing$, which is called the *ice rule*. In this case, the model reduces to the (staggered) six-vertex model, which admits a height function representation. For the possible local configurations, see Figure 1.1.

The six-vertex height function is an assignment of integers to the vertices of the square lattice and its dual. Suppose that Δ is connected with respect to the nearest neighbour connectivity of $\mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$. Then a pair $(\sigma^\bullet, \sigma^\circ)$ of spin configurations as above that satisfy the ice rule $E_{\sigma^\bullet} \cap E_{\sigma^\circ} = \varnothing$ uniquely determines the gradient of such a height function by the relation

$$h_u - h_x = \sigma^\bullet_x \sigma^\circ_u \qquad \text{for } x \in \Delta^\bullet,\ u \in \Delta^\circ \text{ with } \|x-u\|_2 = \tfrac{1}{\sqrt{2}}. \tag{1.2}$$

Together with the choice of an integer height $h_x \in \mathbb{Z}$ for some fixed $x \in \Delta$, this relation uniquely determines a function $h \in \mathbb{Z}^\Delta$ that satisfies $|h_x - h_u| = 1$ whenever $\|x-u\|_2 = 1/\sqrt{2}$ (see Figure 1.1). The values of such a function automatically have different parity on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$, and we require them to be even on $\mathbb{Z}^2$ and odd on $(\mathbb{Z}^2)^*$ by convention. The above six-vertex ($\mathbf{d} = 0$) spin measures then induce measures on six-vertex height functions, see Sections 2.3.2 and 3.1.2.



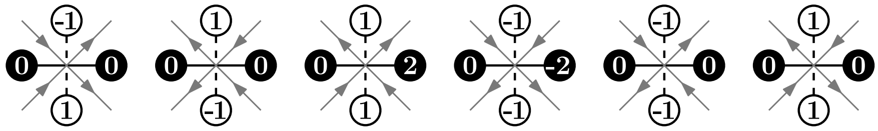


Figure 1.1: *Top:* The eight possible local eight-vertex spin configurations at an horizontal edge $e \in \mathbb{E}^2$ (solid black) and its dual e^* (dashed black) when the spin $\sigma^\bullet$ is fixed to be $+1$ at the left endpoint of e. The corresponding orientations of the edges of the medial graph (solid gray) are obtained by orienting an edge so that the vertex of $\mathbb{Z}^2$ (black) is on its left precisely if the spins separated by the edge coincide. *Centre:* The corresponding local weights. *Bottom:* The height function values corresponding to types 1-6 obtained by (1.2) when the height is fixed to be 0 at the left endpoint of e.

Convention. For ease of notation, the same symbols will be used to represent random variables and their deterministic realisations throughout this thesis.

1.2 Ising, Ashkin–Teller, Potts and six-vertex models

This section provides an exploration of various lattice spin models introduced in Section 1.1.1, including the Ising, Ashkin–Teller, Potts and six-vertex models. It begins with a discussion of phase transitions (Section 1.2.1), focusing on the behaviour of correlation functions. The section then turns to graphical representations of these models (Section 1.2.2), which serve as a tool for studying them and, in the planar case, allow one to obtain duality couplings and with them relations between the models (Section 1.2.3). It also covers correlation inequalities and monotonicity properties (Section 1.2.4), which at the present stage of research are crucial for a rigorous analysis of phase transitions in these systems.

1.2.1 Phase transitions

The study of phase transitions in lattice spin systems can be conducted through the analysis of several characteristics of the model. These include the behaviour of *correlation functions*, the set of infinite-volume *Gibbs measures*, and the analytical properties of quantities such as the *free energy* or *pressure*, all considered in dependence on the underlying parameters. We refer to [FV17] for their connections in the Ising model. The most probabilistic approach is arguably through the study of correlation functions, which is the approach presented here.

We will first consider the Ising model, and then explain the analogues in the AT and Potts models. Subsequently, a different notion of phase transition in the six-vertex height function model will be informally discussed. We fix the underlying lattice to be $(\mathbb{Z}^d, \mathbb{E}^d)$ with $d \geq 1$ to be specified. For simplicity, let us focus on a specific family of subgraphs Λ_n, whose vertex-sets are given by the square boxes $\{-n, \dots, n\}^d$ centred at the origin $0 \in \mathbb{Z}^d$, which we also denote by Λ_n, and whose edge-sets consist of all edges in $\mathbb{E}^d$ that connect vertices within Λ_n. Recall that we conventionally use the same symbols for random variables and their deterministic realisations.

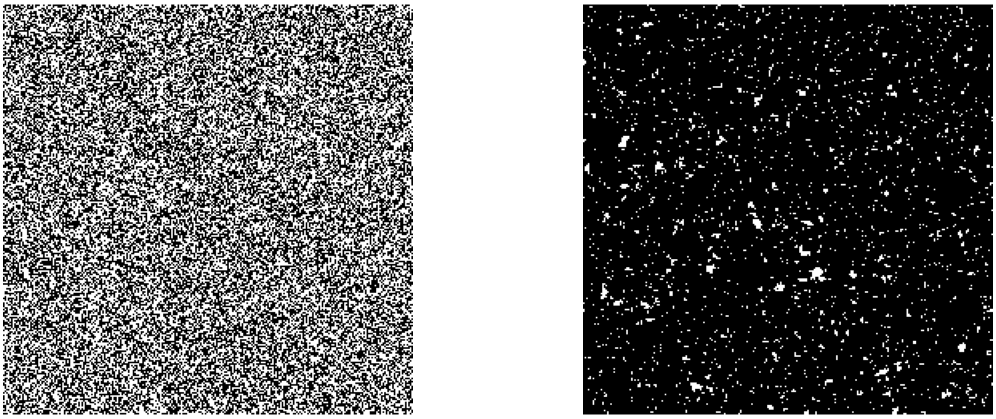


Figure 1.2: Simulations of the Ising model with + boundary condition for $\beta > 0$ very small (*left*) and $\beta > 0$ very large (*right*). Black squares represent the pluses and white squares the minuses.

Ising model

Recall the definition of the Ising measure $\mathtt{Ising}^{\mathrm{f}}_{\Lambda_n,\beta}$ with free boundary conditions, and denote its expectation operator by $\langle\cdot\rangle^{\mathrm{f}}_{\Lambda_n,\beta}$. Clearly, when $\beta = 0$, the measure is uniform and the spins at different vertices are independent. In contrast, when $\beta > 0$, the measure 'favours' configurations in which adjacent spins coincide, with this preference becoming stronger as β is increased. As we mentioned earlier, it is natural to expect that typical realisations increasingly resemble one of the two ground states given by the constant configurations η^+ and η^-.

Let us describe this effect mathematically. It follows directly from the definition that the measure is invariant under the *global spin flip* $\sigma \mapsto -\sigma$. In particular, the average of the spin at a fixed vertex $x \in \Lambda_n$, termed the *magnetisation at* x, vanishes:

$$\langle\sigma_x\rangle^{\mathrm{f}}_{\Lambda_n,\beta} = 0 \qquad \text{for all } \beta > 0.$$

There are essentially two ways to proceed:

- consider instead the 2-point correlations $\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\Lambda_n,\beta}$ of the spins at pairs $x, y \in V$,
- or stick to the magnetisation but under the measure with + boundary condition.

Denote by $\langle\cdot\rangle^{+}_{\Lambda_n,\beta}$ the expectation operator with respect to $\mathtt{Ising}^{+}_{\Lambda_n,\beta}$. While intuitively obvious, it was first proved by Griffiths [Gri67] that both quantities are non-negative:

$$\langle\sigma_x\sigma_y\rangle^{\mathrm{f}}_{\Lambda_n,\beta} \geq 0 \qquad \text{and} \qquad \langle\sigma_x\rangle^{+}_{\Lambda_n,\beta} \geq 0. \tag{1.3}$$

These are specific instances of the so-called *Griffiths'* (or *GKS*) inequalities for correlation functions, named after Griffiths, Kelly and Sherman [KS68]. These inequalities can be used to establish monotonicity of the above quantities in β as well as the existence of infinite-volume limits for the measures with free and + boundary conditions. See Section 1.2.4.

The two proposed strategies turn out to be equivalent in the sense that one will lead to a phase transition precisely when the other does, with the respective transition points coinciding [LL72]. This is a consequence of the convexity of the *free energy* or *pressure* of the Ising model, which will not be further discussed here. We chose to follow the second one, and define the (potentially trivial) *transition point* $\beta_{\mathrm{c}} = \beta_{\mathrm{c}}^{\mathrm{Ising}}(d)$ by

$$\beta_{\mathrm{c}} := \inf\big\{\beta > 0 : \exists\, c_\beta > 0 \text{ such that } \forall\, n \geq 0,\ \langle\sigma_0\rangle^{+}_{\Lambda_n,\beta} > c_\beta\big\}, \tag{1.4}$$

which is also called the *critical inverse temperature*. The model is said to undergo a *order-disorder phase transition* if β_c is non-trivial, that is, if $\beta_c \in (0, \infty)$.

Ising [Isi25] *solved* the model in dimension $d = 1$ in his Ph.D. thesis. He verified the absence of a phase transition and conjectured the same behaviour in any dimension. His result implied that, for any $\beta > 0$, the magnetisation $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ at the origin tends to 0 as $n \to \infty$, that is, $\beta_c = \infty$. In other words, the model exhibits paramagnetic behaviour for all $\beta > 0$. For a decade, the Ising model was considered unsuitable for modelling the ferromagnet. It was not until 1936 that Periels [Pei36] proved, by means of a what is now known as a *Peierls' argument*, that it exhibits a ferromagnetic phase in the dimension $d \geq 2$. To be precise, he showed that the magnetisation at the origin under the measure with $+$ boundary conditions is uniformly positive when β is sufficiently large, that is, $\beta_c < \infty$.

While it was accepted that the model exhibits paramagnetic behaviour in any dimension if $\beta > 0$ is small enough, it was not shown until 1941 by van der Waerden [vdW41], who introduced and utilised the *high-temperature expansion*. In conclusion, the existence of an order-disorder phase transition in dimension $d \geq 2$ has been verified.

Theorem 1 ([Pei36, vdW41]). *For $d \geq 2$, $\beta_c \in (0, \infty)$.*

Furthermore, the phase transition has been proved [ABF87] (seee also [DT16a]) to be (sub-critically) *sharp*, that is, the magnetisation and correlations decay exponentially with the distance when $\beta < \beta_c$. In the above setting, in any dimension $d \geq 2$,

$$\text{for } \beta < \beta_c, \text{ there exists } c_\beta > 0, \text{ such that} \qquad \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \leq e^{-c_\beta n} \qquad \text{for all } n \geq 0. \tag{1.5}$$

In the planar case, the sharpness of the phase transition allows to identify the transition point $\beta_c^{\text{Ising}}(2)$ as the *self-dual point* $\beta_{\text{sd}}^{\text{Ising}} := \ln(1 + \sqrt{2})/2$, predicted by Kramers and Wannier [KW41] and proved by Onsager and Yang [Ons44, Yan52]. See Section 1.2.3 for planar duality and Section 1.4 for the behaviour at the transition points.

AT model

The above considerations can be extended to the AT model in a straightforward manner. One can investigate the presence of a phase transition for each of the spin configurations $\tau, \tau', \tau\tau'$ when varying the parameters J, J', U. The exact analogues in the AT model of the 2-point correlations and the magnetisation (1.3) in the Ising model are

$$\langle \sigma_x \sigma_y \rangle^{\text{f,f}}_{\Lambda_n,J,J',U} \quad \text{and} \quad \langle \sigma_x \rangle^{+,+}_{\Lambda_n,J,J',U} \qquad \text{for } \sigma \in \{\tau, \tau', \tau\tau'\}, \tag{1.6}$$

where $\langle \cdot \rangle^{\text{f,f}}_{\Lambda_n,J,J',U}$ and $\langle \cdot \rangle^{+,+}_{\Lambda_n,J,J',U}$ denote the expectation operators with respect to the AT measures $\mathtt{AT}^{\text{f,f}}_{\Lambda_n,J,J',U}$ and $\mathtt{AT}^{+,+}_{\Lambda_n,J,J',U}$, respectively. In Section 1.2.4, we will see that the quantities in (1.6) are increasing as functions of $(J, J', U) \in [0, \infty)^3$. A convenient way to investigate the presence of phase transitions is to fix the coupling constants J, J', U and multiply the Hamiltonian by an inverse temperature $\beta > 0$, which results in the measures

$$\mathtt{AT}^{\text{f,f}}_{G,\beta} := \mathtt{AT}^{\text{f,f}}_{G,\beta J,\beta J',\beta U} \qquad \text{and} \qquad \mathtt{AT}^{+,+}_{G,\beta} := \mathtt{AT}^{+,+}_{G,\beta J,\beta J',\beta U}. \tag{1.7}$$

One can then define, for $\sigma \in \{\tau, \tau', \tau\tau'\}$, (potentially different) transition points β_c^σ as in (1.4).

We will continue this discussion in Section 1.3. In Section 1.3.1, we discuss the ferromagnetic case and present the phase diagram of the isotropic ($J = J'$) model in dimension $d = 2$. In Section 1.3.2, we discuss the problem in the presence of negative coupling constants.

Potts model

In the case of Potts model, the considerations for the Ising model require minor adjustment. Recall the definition of the Potts measure $\mathtt{Potts}^{\mathrm{f}}_{\Lambda_n,q,\beta}$ and $\mathtt{Potts}^{1}_{\Lambda_n,q,\beta}$ with free and constant 1 boundary conditions, respectively. The analogues of the correlations and the magnetisation in the Ising model are

$$\mathtt{Potts}^{\mathrm{f}}_{\Lambda_n,q,\beta}(\sigma_x=\sigma_y)-\tfrac{1}{q}\qquad\text{and}\qquad\mathtt{Potts}^{1}_{\Lambda_n,q,\beta}(\sigma_0=1)-\tfrac{1}{q}. \tag{1.8}$$

Note that the above quantities vanish for $\beta=0$, since the measures are uniform. As the GKS inequalities are specific for ±1-valued spins, the non-negativity and monotonicity in β of the above quantities must be derived differently. For instance, it can be shown using the positive association property of the *random-cluster representation* of the Potts model, which will be discussed in Sections 1.2.2 and 1.2.4. Again, each of the two types of quantities leads to a phase transition if and only if the other does, and the respective transition points coincide. The arguments [Pei36, vdW41] developed for the Ising model can be adapted to apply for the Potts models or their random-cluster representations to obtain, for $q,d\geq2$, the existence of $\beta_{\mathrm{c}}=\beta_{\mathrm{c}}^{\mathrm{Potts}}(q,d)\in(0,\infty)$ for which

$$\inf_{n\geq0}\ \mathtt{Potts}^{1}_{\Lambda_n,q,\beta}(\sigma_0=1)-\tfrac{1}{q}\ \begin{cases}=0 & \text{if } \beta<\beta_{\mathrm{c}},\\ >0 & \text{if } \beta>\beta_{\mathrm{c}}.\end{cases}$$

We will also see in Section 1.2.4 that the above quantity is non-increasing in n, and thus the infimum can be replaced by a limit. The sharpness of the phase transition for arbitrary q was proved in the planar case in [BD12], where it was used to derive the equality of the transition point and the self-dual point (see Section 1.2.3), and in the general case in [DRT19]. The behaviour of the model at the transition points depends on q and is discussed in Section 1.4.

Six-vertex height function

Let us consider the planar case $d=2$ and describe the phase transition of the six-vertex height function with weights $\mathbf{a}=\mathbf{b}=1$ and $\mathbf{c}\geq1$ in an informal way. Recall the eight-vertex spin measures from Section 1.1.1. When $\mathbf{d}=0$, the spin configurations satisfy the ice rule and the measures reduce to six-vertex measures, which induce measures on (gradients of) height functions via the mapping (1.2). One of the most fundamental questions that can be asked about a model of a random height function is whether it remains flat or exhibits fluctuations. More precisely, the question is whether or not the variance $\mathbf{Var}(h(x)-h(y))$ of the difference of heights at two vertices x,y remains uniformly bounded. If the variance is uniformly bounded, the height function is termed *localised*; otherwise, it is referred to as *delocalised*.

In the setting of the above six-vertex height function, the smaller the weight $\mathbf{c}$, the more the model favours flat height functions. It was shown in several works that the model undergoes a phase transition from delocalisation to localisation at $\mathbf{c}=2$:

- when $\mathbf{c}\in[1,2]$, the height function is delocalised [DCKMO20] (see also [Lis21b, GL23]),
- when $\mathbf{c}>2$, the height function is localised [DGH+21, GP23].

The height function is also expected to be localised when $\mathbf{c}\in(0,1)$, but the invalidity of certain correlation inequalities complicates the rigorous analysis immensely. In Section 1.2.3, we will see that the six-vertex model is in correspondence with the Potts models at their transition points. The behaviour at these points (continuous/ discontinuous transition) is therefore related to the behaviour of these height functions and will be discussed in Section 1.4.

1.2.2 Graphical representations

All the models introduced above are lattice spin systems, in particular vertex-models. One approach to study such a model is to relate it to an edge-model, whose configurations can be regarded as subgraphs given by all the vertices but only those edges of a certain value. This allows to relate properties and relevant quantities of the spin system to counterparts in the edge-model that admit an intuitive graphical interpretation such as the existence of a path connecting two fixed vertices of the lattice. A classical example is the *random-cluster representation* [FK72] of the Potts model, also called Fortuin–Kasteleyn (FK) percolation, which will be presented below, followed by the description of an analogue [PV97, CM97] for the AT model. In Section 1.2.3, we will explain how the latter can be used in the planar case to construct a coupling of the AT and eight-vertex measures. More specifically, we will see that the graphical representation of the AT model also represents the eight-vertex model, in restricted parameter regimes.

Let us first introduce a bit of terminology and notation regarding percolation configurations. As before, we formulate in the setting of an arbitrary graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, but one should have the integer lattices $(\mathbb{Z}^d, \mathbb{E}^d)$ in mind.

Percolation configurations. Given a graph $G = (V, E)$ and $\omega \in \Omega_E = \{0,1\}^E$, an edge is said to be *open* (in ω) precisely if $\omega(e) = 1$, and *closed* otherwise. We identify ω with the set $\{e \in E : \omega(e) = 1\}$ of open edges in ω, as well as with the spanning subgraph (V, ω) of G. For $A, B \subset V$, we write $A \leftrightarrow B$ for the event that there exists a path in ω that connects a vertex in A to a vertex in B. In case of ambiguity, we write $A \overset{\omega}{\leftrightarrow} B$. Singletons are denoted without their braces. The connected components of the graph ω are called *clusters*. We denote by $|\omega| := \sum_{e\in E} \omega_e$ the number of open edges in ω. Furthermore, when G is a subgraph of $(\mathbb{V}, \mathbb{E})$ and $\xi \in \Omega_{\mathbb{E}}$, we write $k_V^\xi(\omega)$ for the number of clusters in $(\mathbb{V}, (\omega \cap E) \cup (\xi \setminus E))$ that intersect V.

FK percolation

Initially invented by Fortuin and Kasteleyn around 1969 with the objective of unifying the (Bernoulli) percolation [BH57], Ising and Potts models, and establishing a connection with electrical networks, FK percolation (also known as the *random-cluster model*) gained prominence in the late '80s and '90s due to its successful application as a tool in the study of the Ising and Potts models [ACCN88], as well as for providing a basis for numerical algorithms [SW87].

The FK percolation measures can be introduced in the setting of Section 1.1, but we choose to provide the classical definition. Let $G = (V, E)$ be a finite subgraph of $(\mathbb{V}, \mathbb{E})$, and let $q > 0$, $p \in [0, 1]$ and $\xi \in \Omega_{\mathbb{E}}$. The FK measure on G with cluster-weight q, edge-weight p and boundary condition ξ is the probability measure $\mathtt{FK}^\xi_{G,q,p}$ on $\Omega_E = \{0,1\}^E$ defined by

$$\mathtt{FK}^\xi_{G,q,p}(\omega) = \frac{1}{Z^\xi_{G,q,p}} p^{|\omega|}(1-p)^{|E\setminus\omega|} q^{k_V^\xi(\omega)}. \tag{1.9}$$

When ξ is constant 0 (respectively, 1), we simply write 0 (respectively, 1) in the superscript and call it the measure with *free* (respectively, *wired*) boundary conditions.

Edwards–Sokal coupling. As FK percolation was introduced to accommodate the Potts model, their relation was already studied in [FK72, For72a, For72b]. It was later extended to a probabilistic coupling [ES88], now known as the Edwards–Sokal coupling, and described as follows. In the interest of simplicity, we will first consider the case of free boundary conditions and then explain the necessary adaptations for the wired case.

Fix $q \geq 2$ integer and a finite subgraph $G = (V, E)$ of $\mathbb{G}$. We say that two configurations $\sigma \in \Sigma_V = \{1, \dots, q\}^V$ and $\omega \in \Omega_E = \{0, 1\}^E$ are *compatible* and write $\sigma \sim \omega$ if and only if,

$$\text{for all } e = \{x, y\} \in E : \quad \omega_e = 1 \quad \text{implies} \quad \sigma_x = \sigma_y.$$

In other words, $\sigma \sim \omega$ precisely if σ takes constant values on clusters of ω. Fix $\beta > 0$, and define a measure $\mathbf{P} = \mathbf{P}_{\mathrm{ES}}^{\mathrm{f}}$ on $\Sigma_V \times \Omega_E$ by setting, for each pair $(\sigma, \omega) \in \Sigma_V \times \Omega_E$,

$$\mathbf{P}(\sigma, \omega) \propto \mathbb{1}_{\{\sigma \sim \omega\}} \big(e^{\beta} - 1\big)^{|\omega|}. \tag{1.10}$$

The measure $\mathbf{P}$ is a coupling of $\mathtt{Potts}_{G,q,\beta}^{\mathrm{f}}$ and $\mathtt{FK}_{G,q,p}^{0}$, where $p = 1 - e^{-\beta}$. Furthermore, the conditional laws can be described as follows.

- For $\sigma \in \Sigma_V$, the conditional law $\mathbf{P}(\,\cdot \mid \sigma)$ on Ω_E is obtained by assigning the values of ω_e independently for each edge $e = \{x, y\} \in E$: if $\sigma_x \neq \sigma_y$, set $\omega_e = 0$; if $\sigma_x = \sigma_y$, set

$$\omega_e = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$

- For $\omega \in \Omega_E$, the conditional law $\mathbf{P}(\,\cdot \mid \omega)$ on Σ_V is obtained by assigning uniformly and independently values in $\{1, \dots, q\}$ to the clusters of ω.

We refer to [Gri06, Chapter 1.4] and [Dum17, Section 1.2.2] for proofs.

Monochromatic boundary conditions for the Potts measure on G induce an interaction of the spins at vertices in V with the fixed values at vertices in the complement $\mathbb{V} \setminus V$ that are adjacent to a vertex in V. Therefore, the FK measure has to be taken with wired boundary conditions on the augmented graph $\bar{G} := (V \cup \partial^{\mathrm{ex}} V, E \cup \partial^{\mathrm{edge}} V)$, where

$$\begin{aligned} \partial^{\mathrm{ex}} V &:= \{y \in \mathbb{V} \setminus V : \exists x \in V \text{ with } \{x, y\} \in \mathbb{E}\}, \\ \partial^{\mathrm{edge}} V &:= \{e \in \mathbb{E} : e \cap V \neq \varnothing \text{ and } e \cap (\mathbb{V} \setminus V) \neq \varnothing\}. \end{aligned}$$

The coupling measure $\mathbf{P}_{\mathrm{ES}}^{1}$ is then defined on $\Sigma_V \times \Omega_{E \cup \partial^{\mathrm{edge}} V}$ as in (1.10) but with an additional indicator function of the event that $\sigma_x = 1$ whenever x is connected to $\partial^{\mathrm{ex}} V$ in ω. It is a coupling of $\mathtt{Potts}_{G,q,\beta}^{1}$ and $\mathtt{FK}_{\bar{G},q,p}^{1}$.

FK relations. Direct consequences of the Edwards–Sokal coupling are the following relations between correlation functions in the Potts model and connection probabilities in the FK model. For any $x, y \in V$,

$$\begin{aligned} \mathtt{Potts}_{G,q,\beta}^{\mathrm{f}}(\sigma_x = \sigma_y) - \tfrac{1}{q} &= \tfrac{q-1}{q}\, \mathtt{FK}_{G,q,p}^{0}(x \leftrightarrow y), \\ \mathtt{Potts}_{G,q,\beta}^{1}(\sigma_x = 1) - \tfrac{1}{q} &= \tfrac{q-1}{q}\, \mathtt{FK}_{\bar{G},q,p}^{1}(x \leftrightarrow \partial^{\mathrm{ex}} V). \end{aligned} \tag{1.11}$$

In the Ising case, these identities can be formulated as

$$\langle \sigma_x \sigma_y \rangle_{G,\beta/2}^{\mathrm{f}} = \mathtt{FK}_{G,2,p}^{0}(x \leftrightarrow y) \qquad \text{and} \qquad \langle \sigma_x \rangle_{G,\beta/2}^{+} = \mathtt{FK}_{\bar{G},2,p}^{1}(x \leftrightarrow \partial^{\mathrm{ex}} V). \tag{1.12}$$

Notice that the above relations imply that all quantities are non-negative.

Percolation thresholds. Let $d \geq 2$, and consider the integer lattices $(\mathbb{Z}^d, \mathbb{E}^d)$. Recall the definition of the transition points $\beta_c = \beta_c^{\text{Potts}}(q,d)$ given in Section 1.2.1, and define $p_c = p_c(q,d) := 1 - e^{-\beta_c}$. The relations (1.11) then imply that

$$\inf_{n\geq 0} \mathtt{FK}^1_{\Lambda_n,q,p}(0 \leftrightarrow \partial^{\text{ex}}\Lambda_n) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases} \tag{1.13}$$

Clearly, the quantity in (1.13) is defined for any $q \in (0,\infty)$. We will see in Section 1.2.4 that it is non-decreasing in p provided that $q \geq 1$. We thus define $p_c = p_c(q,d) \in [0,1]$ for any $q \in [1,\infty)$ by the property (1.13). It is not difficult to show (see, e.g., [Gri99], where it is shown in the case of Bernoulli percolation, $q = 1$, and [Gri06] for the derivation of the statement from the Bernoulli case) that p_c is non-trivial, that is, $p_c \in (0,1)$. The value p_c is called the *percolation threshold*. The FK relations allow the study of order-disorder phase transitions in the Ising and Potts models to be transferred to the study of percolation phase transitions.

Random-cluster representation of the AT model

The random-cluster representation of the AT model was introduced in 1997 independently by Pfister and Velenik [PV97] and by Chayes and Machta [CM97], and we will follow the former here. Since this representation imposes certain restrictions on the parameters of the model, we will formulate it for generic parameters K, K', K'' and spin configurations s, s' in order to apply a change of variables later. For example, when $U > J = J' \geq 0$, we will take $s = \tau$, $s' = \tau\tau'$ and $(K, K', K'') = (J, U, J)$. It should be noted that focusing on only one of the AT spin configurations allows for a wider range of parameters, see Section 3.3.

Fix a finite subgraph $G = (V,E)$ of $\mathbb{G}$, and set $\Sigma_V = \{\pm 1\}^V$ and $\Omega_E = \{0,1\}^E$. Recall the definition of the AT measures in Section 1.1.1. As these are measures on pairs $(s,s') \in \Sigma_V \times \Sigma_V$ of spin configurations, it is natural to consider pairs $(\omega, \omega') \in \Omega_E \times \Omega_E$ of percolation configurations. Let $p_0, p_1, p_1', p_2 \geq 0$ with $p_0 + p_1 + p_1' + p_2 = 1$, and let $\xi, \xi' \in \Omega_{\mathbb{E}}$. The AT random-cluster (ATRC) measure on G with edge-weights $\mathbf{p} = (p_0, p_1, p_1', p_2)$ and boundary conditions ξ, ξ' is the probability measure $\mathtt{ATRC}^{\xi,\xi'}_{G,\mathbf{p}}$ on $\Omega_E \times \Omega_E$ defined by

$$\mathtt{ATRC}^{\xi,\xi'}_{G,\mathbf{p}}(\omega,\omega') = \frac{1}{Z^{\xi,\xi'}_{G,\mathbf{p}}} p_0^{|E\setminus(\omega\cup\omega')|} p_1^{|\omega\setminus\omega'|} p_1'^{|\omega'\setminus\omega|} p_2^{|\omega\cap\omega'|} 2^{k^\xi_V(\omega)+k^{\xi'}_V(\omega')}.$$

We will use the analogous terminology and notation as for FK percolation.

Edwards–Sokal-type coupling. An analogue of the Edwards–Sokal coupling of the AT model and the ATRC model can be constructed as follows. Given $K, K', K'' \in \mathbb{R}$, define

$$\begin{gathered} p_0 := e^{-2(K+K')}, \quad p_1 := e^{-2K'}\big(e^{-2K''} - e^{-2K}\big), \quad p_1' := e^{-2K}\big(e^{-2K''} - e^{-2K'}\big), \\ p_2 := 1 - e^{-2(K+K'')} - e^{-2(K'+K'')} + e^{-2(K+K')}. \end{gathered} \tag{1.14}$$

These quantities are non-negative and sum up to 1 precisely if (see [PV97, Proposition 3.1])

$$\min\{K, K'\} \geq \max\{K'', 0\} \quad \text{and} \quad \tanh K'' \geq -\tanh K \tanh K'. \tag{1.15}$$

Then, the measure $\mathbf{P}^{\text{f,f}}$ on $\Sigma_V \times \Sigma_V \times \Omega_E \times \Omega_E$, defined by

$$\mathbf{P}^{\text{f,f}}(s,s',\omega,\omega') \propto \mathbb{1}_{\{s\sim\omega\}} \mathbb{1}_{\{s'\sim\omega'\}} p_0^{|E\setminus(\omega\cup\omega')|} p_1^{|\omega\setminus\omega'|} p_1'^{|\omega'\setminus\omega|} p_2^{|\omega\cap\omega'|},$$

is a coupling of $\mathtt{AT}^{\text{f,f}}_{G,K,K',K''}$ and $\mathtt{ATRC}^{0,0}_{G,\mathbf{p}}$. Modifications analogous to the FK case allow to construct a coupling of $\mathtt{AT}^{+,+}_{G,K,K',K''}$ and $\mathtt{ATRC}^{1,1}_{G,\mathbf{p}}$.

FK-type relations. The following relations [PV97, Proposition 3.1] (see also Section 3.3) between correlations in the AT and connection probabilities in the ATRC models can be derived from the couplings' properties. Fix K, K', K'' satisfying (1.15) and the associated $\mathbf{p}$, and omit them from the subscripts. Let $\langle\cdot\rangle_G^{\mathrm{f,f}}$ and $\langle\cdot\rangle_G^{+,+}$ be the expectation operators with respect to $\mathtt{AT}_G^{\mathrm{f,f}}$ and $\mathtt{AT}_G^{+,+}$, respectively. Then, for any $x, y \in V$,

$$\begin{gathered}\langle s_x s_y\rangle_G^{\mathrm{f,f}} = \mathtt{ATRC}_G^{0,0}(x \overset{\omega}{\longleftrightarrow} y), \qquad \langle s'_x s'_y\rangle_G^{\mathrm{f,f}} = \mathtt{ATRC}_G^{0,0}(x \overset{\omega'}{\longleftrightarrow} y),\\ \langle s_x s'_x s_y s'_y\rangle_G^{\mathrm{f,f}} = \mathtt{ATRC}_G^{0,0}(x \overset{\omega}{\longleftrightarrow} y \text{ and } x \overset{\omega'}{\longleftrightarrow} y),\end{gathered} \tag{1.16}$$

as well as

$$\begin{gathered}\langle s_x\rangle_G^{+,+} = \mathtt{ATRC}_{\bar G}^{1,1}(x \overset{\omega}{\longleftrightarrow} \partial^{\mathrm{ex}} V), \qquad \langle s'_x\rangle_G^{+,+} = \mathtt{ATRC}_{\bar G}^{1,1}(x \overset{\omega'}{\longleftrightarrow} \partial^{\mathrm{ex}} V),\\ \langle s_x s'_x\rangle_G^{+,+} = \mathtt{ATRC}_{\bar G}^{1,1}(x \overset{\omega}{\longleftrightarrow} \partial^{\mathrm{ex}} V \text{ and } x \overset{\omega'}{\longleftrightarrow} \partial^{\mathrm{ex}} V).\end{gathered} \tag{1.17}$$

Similar to the Potts model and FK percolation, this facilitates the study of the AT model within the framework of a percolation model.

1.2.3 Planar duality and relation of the models

In this section, we discuss the concept of *planar duality*. Consider the square lattice $(\mathbb{Z}^2, \mathbb{E}^2)$ and its dual $((\mathbb{Z}^2)^*, \mathbb{E}^*)$ defined by $(\mathbb{Z}^2)^* := (1/2, 1/2) + \mathbb{Z}^2$ with $\mathbb{E}^*$ given by the edges between nearest neighbours, that is, between vertices at Euclidean distance 1. We first consider the Potts model on the square lattice and utilise its graphical representation, FK percolation, to relate it to the same model on the dual square lattice. Subsequently, we address the case of the AT model [MS71, Fan72c]. The duality relation in the Ising model was discovered by Kramers and Wannier [KW41] using so-called *tranfer matrices*. It should be noted that, in the case of the Ising and AT models, these duality relations can alternatively be derived via a combination of the low-temperature [Pei36] and high-temperature [vdW41] expansions. See [FV17, Section 3.10.1] and [PV97, Section 2] for modern approaches in the Ising and AT models, respectively.

Regarded as line-segments between their end-vertices, each edge $e \in \mathbb{E}^2$ intersects a unique edge in $\mathbb{E}^*$ that we denote by e^*. For a subset $E \subseteq \mathbb{E}^2$ and a configuration $\omega \in \Omega_E = \{0,1\}^E$, set $*E := \{e^* : e \in E\}$, and denote by ω^* the configuration in Ω_{*E} given by $\omega^*_{e^*} = 1 - \omega_e$. Given a finite subgraph $G = (V, E)$ of $(\mathbb{Z}^2, \mathbb{E}^2)$, define its dual G^* as the subgraph of $((\mathbb{Z}^2)^*, \mathbb{E}^*)$ with edge-set $*E$ and vertex-set given by the endpoints of its edges (see Figure 1.3 below).

Potts model. It can be shown that a random variable ω with values in Ω_E satisfies

$$\omega \sim \mathtt{FK}^{\xi}_{G,q,p} \quad \text{if and only if} \quad \omega^* \sim \mathtt{FK}^{\xi^*}_{G^*,q,p^*}, \tag{1.18}$$

where $p^* = p^*(q)$ is the unique solution of

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

We refer to [Dum17, Proof of Proposition 2.17] and [Gri06, Section 6.1] for proofs. Together with the Edwards–Sokal coupling from Section 1.2.2, this allows to couple a Potts (in particular an Ising) measure on a graph G with free boundary conditions and a Potts measure on the dual graph G^* with monochromatic boundary conditions. The respective inverse temperatures β and β^* are related to each other via the above formula for $p = 1 - e^{-\beta}$ and $p^* = 1 - e^{-\beta^*}$.

For each $q > 0$, the maps $p \mapsto p^*$ and $\beta \mapsto \beta^*$ have unique positive fixed points $p_{\mathrm{sd}} = p_{\mathrm{sd}}^{\mathrm{FK}}(q)$ and $\beta_{\mathrm{sd}} = \beta_{\mathrm{sd}}^{\mathrm{Potts}}(q)$, respectively, given by

$$p_{\mathrm{sd}} = \frac{\sqrt{q}}{1+\sqrt{q}} \qquad \text{and} \qquad \beta_{\mathrm{sd}} = \ln(1+\sqrt{q}). \tag{1.19}$$

They are called the *self-dual points*, and they were shown [BD12] to coincide with the transition points p_{c} and β_{c}, respectively, whenever $q \geq 1$.

AT model. The above approach can be adapted to the AT and ATRC models in a straightforward fashion. A random tuple (ω, ω') with values in $\Omega_E \times \Omega_E$ satisfies

$$(\omega, \omega') \sim \mathtt{ATRC}_{G,\mathbf{p}}^{\xi,\xi'} \quad \text{if and only if} \quad (\omega'^*, \omega^*) \sim \mathtt{ATRC}_{G^*,\mathbf{p}^*}^{\xi'^*,\xi^*}, \tag{1.20}$$

where $\mathbf{p} = (p_0, p_1, p_1', p_2)$ and $\mathbf{p}^* = (p_0^*, p_1^*, p_1'^*, p_2^*)$ are related by

$$\mathbf{p}^* = C(p_2, 2p_1, 2p_1', 4p_0), \qquad C = (p_2 + 2p_1 + 2p_1' + 4p_0)^{-1}.$$

This was shown in [PV97, Section 3.4] in the case $\eta = \eta' = 0$. The proof for arbitrary boundary conditions is a straight forward modification (compare with [Dum17, Proof of Proposition 2.17]). As in the Potts case, the Edwards–Sokal-type coupling of the AT and ATRC measures in Section 1.2.2 then allows to couple an AT measure on a graph with free boundary conditions and an AT measure on the dual graph with + boundary conditions. The respective coupling constants are related by the above and (1.14), see (2.4) for the isotropic case and [PV97, Section 2.3] for the general case. The invariant manifold [MS71, Fan72c] of this duality map is known as the *self-dual manifold* and is given by

$$e^{2U} \sinh(2(J+J')) = \cosh(2J) + \cosh(2J'). \tag{1.21}$$

Based on the assumption of a unique phase transition in the AT model, Fan [Fan72c] initially conjectured that the transition points are given by the self-dual manifold. However, as Wegner [Weg72] subsequently noted, this should only be expected when the largest two coupling constants coincide, see Section 1.3. The intersection of this manifold with the isotropic plane $J = J'$ is called the (isotropic) *self-dual curve* and is given by

$$\sinh(2J) = e^{-2U}. \tag{1.22}$$

The duality transformation can also be used to establish a connection with the *eight-vertex model*, which in special cases can be further related to FK percolation and the Potts mode. This will be discussed below.

Relation of the AT, six-vertex, FK and Potts models

In the above setting, applying the duality transformation to only one of the configurations ω or ω' while retaining the other leads to a coupling [Fan72b, Weg72] with the *staggered eight-vertex model* introduced in Section 1.1.1. We refer to Section 3.5.1 for details of this coupling in quite a wide range of parameters. The resulting eight-vertex model is non-staggered when the associated AT model is self-dual, and it may reduce to the *six-vertex model* when two of the AT coupling constant coincide. If both are the case, the associated (non-staggered) six-vertex model is further related to FK percolation and the Potts models at their transition points via the Baxter–Kelland–Wu (BKW) correspondence [BKW76]. Let us describe this procedure in more detail in the case of the self-dual isotropic AT, in view of its relevance in Sections 1.3.1 and 1.4 as well as in Chapers 2 and 4-5.

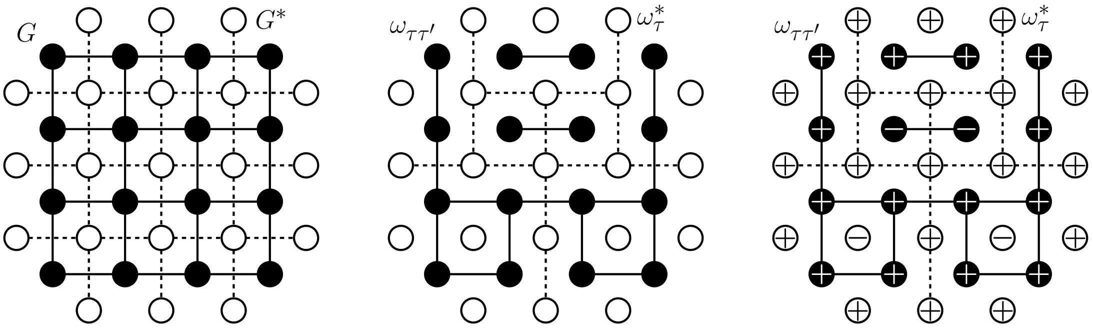


Figure 1.3: *Left:* A subgraph G (black vertices and solid edges) of $(\mathbb{Z}^2, \mathbb{E}^2)$ and its dual G^* (white vertices and dashed edges). Center: ATRC configurations $\omega_\tau, \omega_{\tau\tau'}$ on G satisfying $\omega_\tau \subset \omega_{\tau\tau'}$. The configuration $\omega_{\tau\tau'}$ is drawn on the primal graph G and ω_τ^* on its dual G^*. *Right:* A pair $(\sigma^\bullet, \sigma^\circ)$ of spin configurations obtained by assigning ± 1 to the clusters of $\omega_{\tau\tau'}$ and ω_τ^*.

The self-dual isotropic case. Let $J = J', U > 0$ with $\sinh(2J) = e^{-2U}$, and assume first that $J < U$. Let $(\omega_\tau, \omega_{\tau\tau'})$ be distributed according to an ATRC measure on a finite graph and with weights given by (1.14) with $(K, K', K'') = (J, U, J)$. The choice $K = K''$ results in $p_1 = 0$, which implies $\omega_\tau \subset \omega_{\tau\tau'}$. We do not specify the boundary conditions. Assigning ± 1 spin values uniformly at random to the clusters of $\omega_{\tau\tau'}$ and ω_τ^* as in the Edwards–Sokal coupling in Section 1.2.2 results in a pair $(\sigma^\bullet, \sigma^\circ)$ of spin configurations, with $\sigma^\bullet$ defined on the original primal graph (and distributed according to the associated AT product $\tau\tau'$) and σ° on the dual graph, see Figure 1.3. This pair then follows the law of the spin representation of the six-vertex model (more precisely, the *F model*) with weighs $\mathbf{a} = \mathbf{b} = 1$ and $\mathbf{c} > 2$ introduced in Section 1.1.1. The ice rule is guaranteed since $\omega_\tau \subset \omega_{\tau\tau'}$.

In the case $J \geq U > 0$, the weights of the above ATRC measure become negative, but one can still make sense of the first marginal (ω_τ) in order to apply duality to it (see Section 3.3). This leads to a coupling with the six-vertex model with $\mathbf{c} \in (\sqrt{2}, 2]$. In fact, this approach does not rely on $U > 0$, and the range $U \leq 0$ corresponds in the six-vertex model to $\mathbf{c} \in (1, \sqrt{2}]$.

See Section 3.5.1 for details of this coupling as described above. For the reverse direction, see Section 2.4.2 ('modified' ATRC marginal) and Sections 4.4.2 and 5.2 ('modified' ATRC pairs under Dobrushin conditions).

The BKW correspondence relates the six-vertex model with $\mathbf{c} > \sqrt{2}$ to FK percolation and the Potts models with $q > 0$ at their self-dual points β_{sd}. The regime $\mathbf{c} \geq 2$ corresponds to $q \geq 4$, in which case the correspondence can be formulated as a probabilistic coupling. Let us briefly and informally describe how, in this case, the six-vertex height function can be obtained from the FK percolation. Given ω distributed according to the self-dual wired FK measure with $q \geq 4$ on a box Λ_n, constant integer values are assigned to the clusters of both ω and its dual ω^* by increasing (respectively, decreasing) the value by 1 with probability $e^\lambda/\sqrt{q}$ (respectively, $e^{-\lambda}/\sqrt{q}$) when transitioning from the exterior to the interior of nested clusters (see Figure 1.4), where $\lambda \geq 0$ is the unique solution of $e^\lambda + e^{-\lambda} = \sqrt{q}$. The resulting function has the law of a six-vertex height function with $\mathbf{a} = \mathbf{b} = 1$ and $\mathbf{c} = e^{\lambda/2} + e^{-\lambda/2} \geq 2$ but with a modified boundary weight $\mathbf{c}_{\mathrm{b}} = e^{\lambda/2}$. We refer to Sections 2.3.6 and 4.5.1 for details on the BKW coupling with $\mathbf{c} \geq 2$.

When $q < 4$, that is $\mathbf{c} < 2$, the weights become complex-valued and the correspondence is no longer a probabilistic coupling. However, it can still be used to transfer information between the models, see [Bax89, Chapter 12.4] and [Dub11, Lis21b, GL23]). The relationship between the regimes in each model is summarised in the table below.

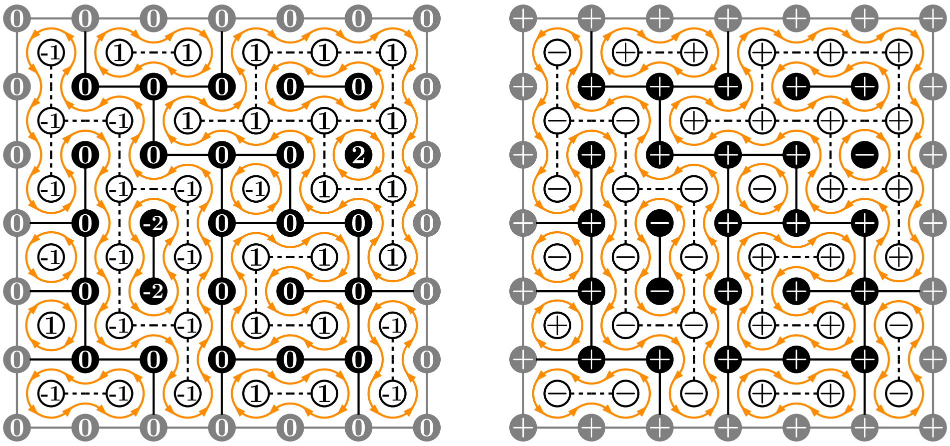


Figure 1.4: Visualisation of the BKW coupling on Λ_3. Orange loops separate primal (black vertices and solid edges) and dual (white vertices and dashed edges) clusters of $\omega \sim \mathtt{FK}^1_{\Lambda_2,q,p_{\mathrm{sd}}}$. *Left:* The height is chosen to be 0 on the boundary. Changing it by ± 1 when traversing a loop can be interpreted as assigning orientations to them. The arrows then induce the edge orientations of the six-vertex model. *Right:* The associated spin configurations chosen to be $+1$ on the boundary.

isotropic self-dual AT	**six-vertex**	**self-dual FK and Potts**
$U > J > 0$	$\mathbf{c} > 2$	$q > 4$
$J \geq U > 0$	$\mathbf{c} \in (\sqrt{2}, 2]$	$q \in (0, 4]$
$J > 0 \geq U$	$\mathbf{c} \in (1, \sqrt{2}]$	$q \leq 0$

The BKW coupling is particularly *sensitive to boundary conditions*. The corresponding percolation measure on the FK side only takes the form (1.9) when the six-vertex and AT measures are considered under specific boundary conditions and with modified weights $\mathbf{c}_{\mathrm{b}}$ and $J_{\mathrm{b}}, U_{\mathrm{b}}$ at the boundary (see Sections 2.3.6, 2.4.2, 4.4.2 and 5.2). Other boundary conditions and unmodified weights at the boundary lead to a different weight for clusters touching the boundary in the associated FK model (see [GP23, Section 2.6]), and this weight may even depend on how often the cluster touches the boundary. As in the AT model, the six-vertex spin (and height function) measures are subject to *pairs of boundary conditions* — one on the primal and one on the dual graph — and we denote them by pairs in $\{0,1\}^2$, where 0 stands for free and 1 for fixed constant boundary conditions. The following table provides an informal description of the effects of the boundary conditions and boundary weights in the ATRC model.

isotr. self-dual ATRC	**six-vertex**	**self-dual FK**
$(0,1)$, modified $J_{\mathrm{b}}, U_{\mathrm{b}}$	$(1,1)$, modified $\mathbf{c}_{\mathrm{b}}$	standard
$(0,1)$, normal J, U	$(1,1)$, normal $\mathbf{c}$	modified constant boundary weight
$(1,1)$, normal J, U	$(0,1)$, normal $\mathbf{c}$	modified non-constant boundary weight

It is worth mentioning that, for example, one can use the exploration techniques in the six-vertex model of Section 2.5.2, the results of [GP23] and the BKW coupling to show that the FK measure with modified non-constant boundary weight, corresponding to the ATRC measure with wired-wired boundary condition, converges to the average of the free and wired infinite-volume FK measures (see Section 1.2.4 below).

1.2.4 Correlation inequalities and monotonicity

This section contains a brief discussion of certain correlation inequalities for the spin systems and their graphical representations introduced in Sections 1.1.1 and 1.2.2, respectively. Such inequalities permit the derivation of *monotonicity* properties of the measures with respect to the underlying parameters and systems. These properties ensure that quantities such as correlations and magnetisations are monotone functions of the parameters, a property vital for studying phase transitions. Furthermore, they enable the deduction of (weak) convergence of the measures to *infinite-volume* or *thermodynamic* limits as the systems exhaust the entire lattice.

Correlation inequalities in the Ising and AT models

An important tool in the study of the ferromagnetic Ising and AT models are the *Griffiths'* or *GKS* inequalities named after Griffiths [Gri67], Kelly and Sherman [KS68]. When exploring a potential phase transition in the Ising model by means of correlations and magnetisations in Section 1.2.1, we already encountered a special case (1.3) of *Griffiths' first inequality*, which confirmed the validity of the intuition that the corresponding quantities are non-negative. This inequality, together with a second one, which we will state shortly, was initially proved by Griffiths for the ferromagnetic Ising model and subsequently extended by Kelly and Sherman to allow for arbitrary *n-body interactions*. Fix a finite set V, and introduce the notation

$$\sigma^A := \prod_{x\in A} \sigma_x, \qquad \sigma \in \Sigma_V = \{\pm 1\}^V,\ A \subseteq V. \tag{1.23}$$

A measure on Σ_V with expectation operator $\langle \cdot \rangle$ is said to satisfy the GKS inequalities if

$$\langle \sigma^A \rangle \geq 0 \qquad \text{for all } A \subseteq V, \tag{GKS1}$$
$$\langle \sigma^A \sigma^B \rangle \geq \langle \sigma^A \rangle \langle \sigma^B \rangle \qquad \text{for all } A, B \subseteq V. \tag{GKS2}$$

This definition will be applied to the setting of the AT measures by identifying $\Sigma_V \times \Sigma_V$ with $\Sigma_{V \sqcup V'}$ where V' is a disjoint copy of V. Take $\mathbf{J} = (J_A)_{A\subseteq V}$ with $J_A \in \mathbb{R}$, and let $\mu_{V,\mathbf{J}}$ be the measure on Σ_V defined by (1.1) with

$$\mathcal{H}_V(\sigma) = -\sum_{A\subseteq V} J_A \sigma^A.$$

Denote the associated expectation operator by $\langle \cdot \rangle_{V,\mathbf{J}}$.

Theorem 2 ([Gri67, KS68])**.** *If $J_A \geq 0$ for all $A \subseteq V$, then $\mu_{V,\mathbf{J}}$ satisfies the GKS inequalities.*

By differentiation, these inequalities can be used to derive that $J_A \mapsto \langle \sigma^B \rangle_{V,\mathbf{J}}$ is non-decreasing on $(0, \infty)$ for all fixed $A, B \subseteq V$, which in turn can be used to deduce the existence of weak limits of the measures $\mu_{V,\mathbf{J}}$ as V exhausts some infinite set. This will be addressed below for the Ising and AT measures.

As the Ising and AT measures with free and + boundary conditions can be represented by measures of the form $\mu_{V,\mathbf{J}}$, we obtain the following corollary.

Corollary 1.2.1. *For any finite graph G, the following measures satisfy the GKS inequalities:*

$$\mathtt{Ising}^{\mathrm{f}}_{G,\beta},\ \mathtt{Ising}^{+}_{G,\beta} \quad \textit{for } \beta \geq 0, \qquad \textit{and} \qquad \mathtt{AT}^{\mathrm{f,f}}_{G,J,J',U},\ \mathtt{AT}^{+,+}_{G,J,J',U} \quad \textit{for } J, J', U \geq 0.$$

The above measures are also *positively associated* or satisfy the *FKG* inequality, a property we will introduce in the subsequent section. We do not use this inequality for spins and do not discuss it further here. For more details on the GKS and FKG inequalities in the Ising model, the reader is referred to [FV17, Section 3.6].

Antiferromagnetic AT model. While negative coupling constants render the GKS inequalities generally invalid, in the AT model, the effect of the negativity of one coupling constant may be counterbalanced by the positivity of others. For example, the relations (1.16) and (1.17), which remain valid when $K, K' \gg 0 > K''$, clearly imply that the respective 2-point correlation functions and the magnetisations are non-negative. More generally, the coupling of the AT and ATRC models from Section 1.2.2 and the positive and negative association properties satisfied by the latter (see below) can be used to deduce that each of the two AT marginals is still positively correlated itself, but negatively correlated with the other.

Proposition 1.2.2 ([PV97]). *Let $J, J' > 0 > U$ be such that (1.15) is satisfied for $(K, K', K'') = (J, J', U)$, and let $G = (V, E)$ be a finite graph. Let $\langle\cdot\rangle$ be the expectation operator with respect to $\mathtt{AT}^{\mathrm{f,f}}_{G,J,J',U}$ or $\mathtt{AT}^{+,+}_{G,J,J',U}$. Then, for all $A \subseteq V$,*

$$\langle\sigma^A\rangle \geq 0 \qquad \textit{for } \sigma \in \{\tau, \tau', \tau\tau'\}.$$

Furthermore, for all $A, B \subseteq V$,

$$\langle\sigma^A\sigma^B\rangle \geq \langle\sigma^A\rangle\langle\sigma^B\rangle \qquad \textit{for } \sigma \in \{\tau, \tau'\},$$
$$\langle\tau^A\tau'^B\rangle \leq \langle\tau^A\rangle\langle\tau'^B\rangle.$$

The first inequality for $\sigma = \tau$ is generalised in Corollary 3.3.5 to hold for $J \geq \min\{|J'|, |U|\}$, see also Section 1.3.2 and Section 3.3.2.

Eight-vertex model. As the eight-vertex measures can also be represented by measures of the form $\mu_{V,\mathbf{J}}$, it can be concluded that they also satisfy the GKS inequalities within a certain range of the weights. Alternatively, this can be derived in part from the analogue in the AT model via the coupling presented in Section 1.2.3. Furthermore, again in restricted parameter regimes, the eight-vertex spin marginals also satisfy the FKG inequality (introduced below), for which we refer to [GP23, Theorem 2.5] and [GL23, Proposition 3.10].

Positive association in graphical representations

The positive association property or (Harris-) FKG inequality [Har60, FKG71] of the FK and ATRC measures is a tremendously useful tool in the study of the respective models, with a significant proportion of results relying on them.

Let I be an index-set, let $S_i \subset \mathbb{R}$, $i \in I$, and define $\Omega = \prod_{i\in I} S_i$. Equip Ω with the natural coordinate-wise order: $\omega \leq \omega'$ if and only if $\omega_i \leq \omega'_i$ for all $i \in I$. A function $f : \Omega \to \mathbb{R}$ is called increasing if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$. A measure μ on Ω is said to be positively associated or to satisfy the (Harris-) FKG inequality if

$$\mathbf{E}_\mu(fg) \geq \mathbf{E}_\mu(f)\mathbf{E}_\mu(g) \qquad \text{for all increasing } f, g : \Omega \to [0, \infty). \tag{FKG}$$

A sufficient condition was given in [FKG71] provided that S and I are finite. In this case, we say that a measure μ on Ω satisfies the *FKG-lattice condition* if

$$\mu(\omega \vee \omega')\mu(\omega \wedge \omega') \geq \mu(\omega)\mu(\omega') \qquad \text{for all } \omega, \omega' \in \Omega, \tag{FKG-L}$$

where $\omega \vee \omega'$ and $\omega \wedge \omega'$ denote the coordinate-wise maxima and minima, respectively. There are several equivalent properties, for which we refer to [Gri06, Chapter 2].

In the context of FK and ATRC measures on a graph with edge-set E, we respectively chose $I = E$ and $I = E \sqcup E'$ with E' a disjoint copy of E, and $S_i = \{0, 1\}$ for all i. It was proved respectively in [FKG71, Section 3] (see also [For72a, Theorem 1]) and [PV97, Lemma 4.1] that the FK and AT measures under any boundary condition satisfy (FKG-L) and therefore (FKG).

Proposition 1.2.3 ([FKG71, PV97]). *Let G be a finite subgraph of $(\mathbb{V}, \mathbb{E})$. The following measures satisfy* (FKG-L)*:*

$$\begin{aligned} &\mathtt{FK}^{\xi}_{G,q,p} && \textit{for all } p \in (0,1),\ q \geq 1,\ \xi \in \Omega_{\mathbb{E}}, \\ &\mathtt{ATRC}^{\xi,\xi'}_{G,\mathbf{p}} && \textit{for all } \mathbf{p} = (p_0, p_1, p_1', p_2) \textit{ with } p_2 p_0 \geq p_1 p_1',\ \textit{ and } \xi, \xi' \in \Omega_{\mathbb{E}}. \end{aligned}$$

The FKG lattice condition for the ATRC marginals remains valid when $p_2 p_0 < p_1 p_1'$, but the two configurations are *negatively correlated* [PV97]. In other words, the law of $(\omega, 1 - \omega')$ satisfies (FKG-L). Via the relations (1.14), the condition $p_2 p_0 \geq p_1 p_1'$ translates to $U \geq 0$. We refer to Sections 1.3.2 and 3.3.3 for details.

Together with the couplings from Section 1.2.2, the FKG inequality for the above measures can be used to derive the GKS inequalities for the Ising and AT measures stated in Corollary 1.2.1 and Proposition 1.2.2. As mentioned before, the ferromagnetic Ising and AT measures under any boundary conditions also satisfy (FKG-L), see [FV17, Section 3.6] for the Ising case.

Monotonicity and infinite-volume limits

The correlation inequalities (GKS and FKG) are related to the monotonicity properties of the models, both in the underlying parameters and the graphs. Let us first introduce the notion of *stochastic domination*.

Stochastic domination. In the setup of the previous section, given two measures μ, ν on Ω, μ is said to be stochastically dominated by ν (or ν stochastically dominates μ), denoted by $\mu \leq_{\text{st}} \nu$ (or $\nu \geq_{\text{st}} \mu$) if and only if $\mathbf{E}_\mu(f) \leq \mathbf{E}_\nu(f)$ for all increasing $f : \Omega \to [0, \infty)$. Holley [Hol74] introduced a sufficient condition for verifying stochastic domination. By [Gri06, Theorem 2.27], the FKG lattice condition that the FK and ATRC measures are stochastically increasing in the boundary conditions. We refer to [Gri06, Chapter 2] and [GHM01, Section 4] for surveys.

Monotonicity in the parameters. As a first application, we explain how to derive monotonicity in the parameters of the relevant quantities when studying phase transitions in the respective models.

Corollary 1.2.4. *For any $n \geq 0$ and all $x, y \in \Lambda_n$, the quantities in*

(i) (1.3) *and* (1.8) *are non-decreasing as functions of $\beta \in [0, \infty)$,*

(ii) (1.6) *are non-decreasing as functions of $(J, J', U) \in [0, \infty)^3$.*

The statements hold more generally for any finite subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$ and $x, y \in V$, for sequences of parameters indexed by the edges, and for arbitrary k-point functions.

In the Ising and AT cases, the statements are derived by differentiating the respective quantities with respect to β and J, J', U respectively, and then applying the inequality (GKS2). In case of the Potts model, one can use the identities (1.11) and differentiate the analogue in the FK model with respect to β and then apply (FKG). This argument also provides monotonicity of the associated FK measures in the parameter β (or the edge-weight p) in the sense of stochastic domination, which can alternatively be derived using the Holley criterion [Hol74]. The analogue holds for the ATRC measures. See Section 1.3.1 and Chapter 2 for the isotropic case.

Monotonicity in the domain and infinite-volume limits. The FKG and *spatial Markov* properties of the FK and ATRC measures directly imply the following monotonicity in the domains, which in turn implies the existence of infinite-volume limits of the respective measures with free and monochromatic boundary conditions.

Corollary 1.2.5. *Let G be a finite subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$, and let H be a subgraph of G. Then,*

$$\begin{array}{llll} \mathtt{FK}^0_{G,q,p} \geq_{\rm st} \mathtt{FK}^0_{H,q,p} & \textit{and} & \mathtt{FK}^1_{G,q,p} \leq_{\rm st} \mathtt{FK}^1_{H,q,p} & \textit{for all } q \geq 1,\ p \in [0,1], \\ \mathtt{ATRC}^{0,0}_{G,\mathbf{p}} \geq_{\rm st} \mathtt{ATRC}^{0,0}_{H,\mathbf{p}} & \textit{and} & \mathtt{ATRC}^{1,1}_{G,\mathbf{p}} \leq_{\rm st} \mathtt{ATRC}^{1,1}_{H,\mathbf{p}} & \textit{for all } \mathbf{p} \textit{ with } p_2 p_0 \geq p_1 p_1'. \end{array}$$

Consequently, as the graph G exhausts $(\mathbb{Z}^d, \mathbb{E}^d)$, each of the above measures converges weakly (with respect to the natural product topology) to a limiting measure, called an *infinite-volume* or *thermodynamic* limit. In the case of the ATRC measures with $p_2 p_0 < p_1 p_1'$, the positive association property of $(\omega, 1-\omega')$, mentioned after Proposition 1.2.3, allows to derive the convergence of the measures $\mathtt{ATRC}^{1,0}_{G,\mathbf{p}}$, see Section 3.6.1 for details.

Formally, one has to consider all measures on the corresponding infinite configuration spaces. To lighten notation, we will for example identify the measure $\mathtt{FK}^0_{G,q,p}$ on $\{0,1\}^E$ with the measure on $\{0,1\}^{\mathbb{E}^d}$ defined by setting, for any $\omega \in \{0,1\}^{\mathbb{E}^d}$,

$$\overline{\mathtt{FK}}^0_{G,q,p}(\omega) = \mathtt{FK}^0_{G,q,p}(\omega) \mathbb{1}_{\{\omega_e = 0 \text{ for all } e \in \mathbb{E}^d \setminus E\}}. \tag{1.24}$$

Via the couplings in Section 1.2.2, the convergence of the FK and ATRC measures can be used to derive the analogue for the associated Potts and AT measures under free and monochromatic boundary conditions. This argument relies on the (almost sure) *existence of at most one infinite cluster* in the infinite-volume FK and ATRC measures (see [Gri06, Section 4.6]), which follows from the robust *Burton–Keane argument* [BK89]. In the cases of the Ising and AT models, alternative approaches are via the monotonicity in the parameters, Corollary 1.2.4, or by applying the FKG inequality for the spins directly.

1.3 Phase transitions in the Ashkin–Teller model

Relating the partition function of the (ferromagnetic) AT model on the square lattice to that of the eight-vertex model [Sut70, FW70], Wegner conjectured that the AT model generally undergoes *two distinct phase transitions* [Weg72]. The occurrence of a single one is expected only if the two largest coupling constants are equal to each other. This issue will be explored in Section 1.3.1 within the framework of the ferromagnetic model, using the correlation inequalities from Section 1.2.4 and the graphical representation from Section 1.2.2.

In contrast to the Ising model (without external field), the AT model on bipartite graphs cannot always be transformed into a purely ferromagnetic one, and the competition between ferromagnetic and antiferromagnetic interactions has to be considered. Depending on which type dominates, one may expect either kind of behaviour. Nevertheless, the bipartite structure still allows the study to be restricted to a certain partial ferromagnetic regime, only in parts of which certain correlation inequalities still hold. This topic will be addressed in Section 1.3.2.

Recall the spin representation of the AT model with coupling constants J, J', U introduced in Section 1.1.1 and the brief discussion in Section 1.2.1. We consider the model on the hypercubic lattice $(\mathbb{Z}^d, \mathbb{E}^d)$ of arbitrary dimension $d \geq 2$, restricting ourselves to the case $d = 2$ only when planar duality is applied.

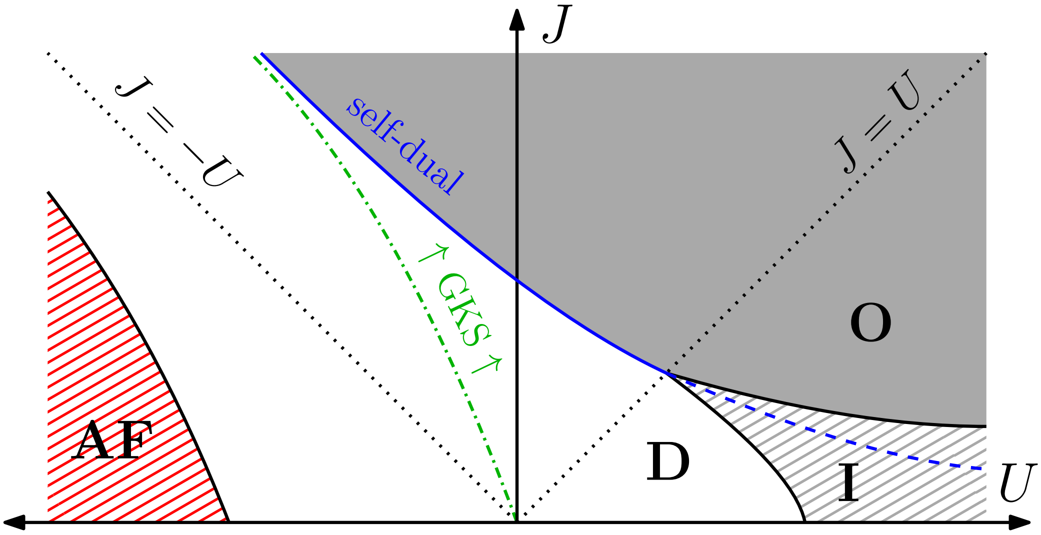


Figure 1.5: Conjectured phase diagram of the isotropic AT model for $U \geq 0$. Ordered phase (**O**): both τ and $\tau\tau'$ are ordered. Disordered phase (**D**): both τ and $\tau\tau'$ are disordered, admitting exponential decay of correlations. Intermediate phase (**I**): τ is disordered, whereas $\tau\tau'$ is ordered. Antiferromagnetic phase (**AF**): τ is disordered, whereas $\tau\tau'$ is antiferromagnetically ordered. The GKS inequalities (see Section 1.2.4) for the marginals are valid to the right of the dashed green curve. The ferromagnetic part is fully established in Chapter 2 (see also Section 1.3.1). The existence of **AF** is verified in Chapter 3 (see also Section 1.3.2), where the presence of a sharp phase transition at the self-dual curve when $U < 0$ is also supported.

1.3.1 The ferromagnetic case: main results of Chapter 2

Let us first discuss the ferromagnetic case $J, J', U \geq 0$ and previous rigorous results concerning the above conjecture. Recall the definition (1.7) of the measures with coupling constants J, J', U multiplied by an inverse temperature $\beta > 0$. In Section 1.2.4, we argued that these measures admit infinite-volume limits, whose expectation operators we denote by $\langle \cdot \rangle_\beta^{\mathrm{f,f}}$ and $\langle \cdot \rangle_\beta^{+,+}$, respectively. Recall the monotonicity in the parameters (Corollary 1.2.4) and observe that it trivially extends to infinite-volume. Then, adaptations of the Peierls' argument [Pei36] and the high-temperature expansion [vdW41] of the Ising model (applied to the AT or ATRC models) allow to derive, in analogy to (1.4), the existence of non-trivial transition points $\beta_c^\sigma = \beta_c^\sigma(J, J', U) \in (0, \infty)$ for $\sigma \in \{\tau, \tau', \tau\tau'\}$ that satisfy

$$\langle \sigma_0 \rangle_\beta^{+,+} \begin{cases} = 0 & \text{if } \beta < \beta_c^\sigma, \\ > 0 & \text{if } \beta > \beta_c^\sigma, \end{cases} \qquad \sigma \in \{\tau, \tau', \tau\tau'\}. \tag{1.25}$$

To gain some intuition, let us first consider the special case where one of the coupling constants vanishes, say $U = 0$, which is known as the *XOR Ising* model. In this case, the model reduces to a pair of *independent* Ising configurations τ and τ', which undergo a phase transitions at the multiples $\beta_c^{\mathrm{Ising}}/J$ and $\beta_c^{\mathrm{Ising}}/J'$ of the Ising transition point, respectively. In particular, if $J \neq J'$, two distinct transitions occur. Moreover, the transition point for the product clearly satisfies $\beta_c^{\tau\tau'} = \max\{\beta_c^\tau, \beta_c^{\tau'}\}$. Before investigating when else the model undergoes more than one phase transition, we first rule out the possibility of more than two.

Proposition 1.3.1. *For $d \geq 2$ and $J, J', U \geq 0$, it holds that*

$$\beta_c^\tau \leq \max\{\beta_c^{\tau'}, \beta_c^{\tau\tau'}\}, \qquad \beta_c^{\tau'} \leq \max\{\beta_c^\tau, \beta_c^{\tau\tau'}\}, \qquad \beta_c^{\tau\tau'} \leq \max\{\beta_c^\tau, \beta_c^{\tau'}\}.$$

In particular,

$$|\{\beta_c^\tau, \beta_c^{\tau'}, \beta_c^{\tau\tau'}\}| \leq 2.$$

Proof. By (GKS2), which holds by Corollary 1.2.1 on finite graphs and extends to the infinite-volume limit, we have for any $\beta > 0$

$$\langle \tau_0 \rangle_\beta^{+,+} = \langle \tau_0 \tau_0' \tau_0' \rangle_\beta^{+,+} \geq \langle \tau_0 \tau_0' \rangle_\beta^{+,+} \langle \tau_0' \rangle_\beta^{+,+}.$$

This implies that $\beta_c^\tau \leq \max\{\beta_c^{\tau'}, \beta_c^{\tau\tau'}\}$. The other inequalities follow by symmetry. □

The first rigorous proof of the occurrence of two distinct phase transitions in the non-independence case was provided by Pfister [Pfi82] in rather high generality, relying only on the GKS inequalities and symmetry.

Theorem 3 ([Pfi82])**.** *Let $d \geq 2$ and $J, J', U > 0$.*

- *If $J + J' < U$, then $\beta_c^\tau = \beta_c^{\tau'} > \beta_c^{\tau\tau'}$.*
- *If $J > J' + U$, then $\beta_c^\tau < \beta_c^{\tau'} = \beta_c^{\tau\tau'}$.*
- *If $J' > J + U$, then $\beta_c^{\tau'} < \beta_c^\tau = \beta_c^{\tau\tau'}$.*

Observe that, by applying the changes of variables $(\tau, \tau') \mapsto (\tau', \tau\tau')$ and $(\tau, \tau') \mapsto (\tau, \tau\tau')$, the first statement of the above theorem implies the other two. The first statement should hold more generally when $\max\{J, J'\} < U$. We will continue this discussion for the isotropic case below, where we present the full phase diagram. Before turning to the antiferromagnetic case, let us demonstrate a simple application of the ATRC representation in the study of the above transition points. It excludes the possibility that the transition for the configuration with the smallest coupling constant 'precedes' any of the others.

Lemma 1.3.2. *For $d \geq 2$ and $J, J' \geq U \geq 0$, it holds that $\beta_c^{\tau\tau'} \geq \max\{\beta_c^\tau, \beta_c^{\tau'}\}$.*

Proof. Denote by $\mathtt{ATRC}_\beta^{1,1}$ the limit of the ATRC measures associated to $\mathtt{AT}_{G,\beta}^{+,+}$ via (1.14) with $(K, K', K'') = \beta \cdot (J, J', U)$, which exists by Corollary 1.2.5. Also by Corollary 1.2.5, the relations (1.17) for $(s, s') = (\tau, \tau')$ extend to infinite-volume by replacing $0 \leftrightarrow \partial^{\mathrm{ex}} V$ with the event that 0 is part of an infinite cluster, which we denote by $0 \leftrightarrow \infty$ (see [Gri06, Proposition 5.11] for a proof of the analogue in FK percolation). Therefore,

$$\begin{aligned}
\langle \tau_0 \tau_0' \rangle_\beta^{+,+} &= \mathtt{ATRC}_\beta^{1,1}\big(0 \overset{\omega}{\longleftrightarrow} \infty, 0 \overset{\omega'}{\longleftrightarrow} \infty\big) \\
&\leq \min\big\{\mathtt{ATRC}_\beta^{1,1}\big(0 \overset{\omega}{\longleftrightarrow} \infty\big), \mathtt{ATRC}_\beta^{1,1}\big(0 \overset{\omega'}{\longleftrightarrow} \infty\big)\big\} \\
&= \min\big\{\langle \tau_0 \rangle_\beta^{+,+}, \langle \tau_0' \rangle_\beta^{+,+}\big\},
\end{aligned}$$

which directly implies the claim. □

It is noteworthy that the above proof can be adapted for the case where $J, J' > 0 > U$ provided that $(K, K', K'') = \beta \cdot (J, J', U)$ satisfy (1.15). In fact, the same proof applies to all finite volume ATRC measures, which in the case $J, J' > 0 > U$ converge when the boundary conditions are + for τ and free for τ', or vice versa (see the paragraph under Corollary 1.2.5, and also Section 3.6.1). Nevertheless, the definition of the transition points is problematic due to the absence of monotonicity in the parameters (see Section 1.3.2 below).

Together with Proposition1.3.1, Lemma 1.3.2 implies the following corollary.

Corollary 1.3.3. *For $d \geq 2$ and $J, J' \geq U \geq 0$, it holds that $\beta_c^{\tau\tau'} = \max\{\beta_c^\tau, \beta_c^{\tau'}\}$.*

The isotropic case: main results of Chapter 2

Let us restrict to the isotropic regime $J = J'$. In this case, the marginals (laws of τ and τ') coincide by symmetry, and $\beta_c^\tau = \beta_c^{\tau'}$ holds trivially. We can therefore disregard $\beta_c^{\tau'}$ in this section. Then, Corollary 1.3.2 implies the *uniqueness* of the phase transition in *any dimension* when $J \geq U$.

Corollary 1.3.4. *For $d \geq 2$ and $J \geq U \geq 0$, it holds that $\beta_c^\tau = \beta_c^{\tau\tau'}$.*

The verification of the strict inequality $\beta_c^\tau > \beta_c^{\tau\tau'}$ when $J < U$ is much more challenging. Observe that the weak inequality is also a consequence of Proposition 1.3.1, and that the strict inequality follows from Theorem 3 if $2J < U$.

The phase diagrams of the isotropic model in dimensions 2 and 3 were predicted in [DBGK80]. We confirm the validity of their predictions in the planar case $d = 2$ in Chapter 2 and informally describe below our main results and the methods used to obtain them. We refer to [DBGK80, DG04] and [Bax89, Section 12.9] for the expected behaviour at the transition points.

Main result. Chapter 2 focuses on the planar case $d = 2$. The main result is the validation of the strict inequality $\beta_c^\tau > \beta_c^{\tau\tau'}$ for all $J < U$, with the self-dual point β_{sd} given by $\sinh(2\beta J) = e^{-2\beta U}$ (see Section 1.2.3) lying strictly in between (Theorem 6). We also verify that the two distinct transition points are dual to each other (Theorem 7). Furthermore, in the case $J \geq U$, we identify the transition point $\beta_c^\tau = \beta_c^{\tau\tau'}$ as the self-dual point β_{sd} (Theorem 8). See Figure 1.5 for the phase diagram. We also show that the transitions for the free and monochromatic measures occur simultaneously (Theorems 7-8), and we establish the sharpness of all transitions (as in (1.5)) in any dimension $d \geq 2$ (Appendix 2.A). Although the locations of the transition points in the non-uniqueness case were considered in [WL74], there are no precise conjectures.

Methods. All statements are proved via their analogues in the ATRC model and the Edwards–Sokal-type coupling in Section 1.2.2. For $J \geq U$ we use the representation with $(s, s') = (\tau, \tau')$, whereas for $J < U$ we use it with $(s, s') = (\tau, \tau\tau')$. The reason for restricting to $d = 2$ is that the proof goes through couplings that rely on planarity, described in Section 1.2.3. The first is a duality coupling [Fan72b, Weg72, GP23] of the self-dual AT model and the six-vertex model [Pau35, Rys63]. The second is a version of the Baxter–Kelland–Wu correspondence [BKW76] of the latter with FK percolation and the Potts model. Relying on the discontinuity of the phase transition in FK percolation when $q > 4$ [DGH+21] (see Section 1.4 below), we improve the recent result [GP23, Theorem 2.8] on the intermediate behaviour on the self-dual curve (1.22) in the AT model when $J < U$. We establish a finite size criterion (Proposition 2.1.1) and extend the result to an open neighbourhood of the curve. The sharpness statements are proved by adapting the OSSS approach [DRT19]. The equality $\beta_c^\tau = \beta_{\mathrm{sd}}$ for $J \geq U$ is derived from the sharpness and the fact that the free and wired ATRC measures converge to the same limit for Lebesgue-almost every $J, U \geq 0$, which is proved by modifying the elegant argument for FK percolation in [Dum17, Proof of Theorem 1.12].

1.3.2 The antiferromagnetic case: main results of Chapter 3

Now consider the case where at least one of the coupling constants is negative. The first observation is that this renders the GKS inequalities generally invalid, which complicates the rigorous analysis of the model enormously. Recall that we have relied on these inequalities in the ferromagnetic case to derive monotonicity in the parameters, the existence of infinite-volume limits, and even to formally rule out rather obscure behaviour such as the presence of three distinct phase transitions.

On a more positive note, like the Ising model, the AT model on bipartite graphs (such as $(\mathbb{Z}^d, \mathbb{E}^d)$) enjoys a certain invariance under *spin flips* on the partite classes of the vertices. A vertex in $\mathbb{Z}^d$ is called *odd* if the sum of its coordinates is odd, otherwise it is called *even*. Let $G = (V, E)$ be a finite subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$, and consider the map $F_{\text{odd}} : \Sigma_V \to \Sigma_V$ that reverses the sign of Ising spin configurations at the *odd* vertices in V, defined by

$$\big(F_{\text{odd}}(\sigma)\big)_x = \begin{cases} -\sigma_x & \text{if } x \text{ is odd,} \\ \sigma_x & \text{if } x \text{ is even,} \end{cases} \qquad \sigma \in \Sigma_V,\ x \in V.$$

Let us first demonstrate the invariance of the Ising measures under F_{odd}.

The Ising antiferromagnet. Let $\eta^\pm$ be the image of the constant $+$ configuration under F_{odd}, and simply write $\pm$ in the superscripts of the measures. A simple computation shows that a random variable σ with values in $\{\pm 1\}^V$ has law $\mathtt{Ising}^{+}_{G,\beta}$ precisely if $F_{\text{odd}}(\sigma)$ has law $\mathtt{Ising}^{\pm}_{G,-\beta}$. The transition of the ferromagnetic model at $\beta_c = \beta_c^{\text{Ising}}(d)$ therefore implies that

$$\langle \sigma_x \rangle^{\pm}_{\Lambda_n,\beta} \begin{cases} \xrightarrow{n\to\infty} 0 & \text{if } \beta \in (-\beta_c, 0), \\ \geq c_\beta > 0 & \text{if } \beta < -\beta_c \text{ and } x \text{ is even,} \\ \leq -c_\beta < 0 & \text{if } \beta < -\beta_c \text{ and } x \text{ is odd,} \end{cases} \tag{1.26}$$

in which case we say that the model undergoes an *antiferromagnetic order-disorder transition*. We call the above behaviour for $\beta < -\beta_c$ (bipartite) *antiferromagnetic order*.

Returning to the AT model, another simple computation verifies the following invariances of the AT measures, which we formulate for free boundary conditions for simplicity. If (τ, τ') is distributed according to $\mathtt{AT}^{\text{f,f}}_{G,J,J',U}$, then

- $\big(F_{\text{odd}}(\tau), \tau'\big)$ has law $\mathtt{AT}^{\text{f,f}}_{G,-J,J',-U}$,
- $\big(\tau, F_{\text{odd}}(\tau')\big)$ has law $\mathtt{AT}^{\text{f,f}}_{G,J,-J',-U}$,
- $\big(F_{\text{odd}}(\tau), F_{\text{odd}}(\tau')\big)$ has law $\mathtt{AT}^{\text{f,f}}_{G,-J,-J',U}$.

Therefore, any AT measure can be transformed into another AT measure whose first two coupling constants are non-negative, whence it suffices to study the regime

$$J, J' \geq 0 > U. \tag{1.27}$$

It is natural to believe that the ferromagnetic interaction imposed by $J, J' > 0$ dominates the opposite one imposed by $U < 0$ if $J, J' > |U|$, in which case one might expect order-disorder phase transitions as observed in the ferromagnetic case. Conversely, for $J, J' < |U|$, the antiferromagnetic effect becomes dominant, presumably causing disorder in τ and τ', and an antiferromagnetic order-disorder transition in the product $\tau\tau'$, as in (1.26).

Recall the positive and negative correlation inequalities [PV97] from Proposition 1.2.2 that hold in the restricted part of the above regime where (1.15) holds for $(K, K', K'') = (J, J', U)$. Investigating monotonicity in β of the measures $\mathtt{AT}^{\text{f,f}}_{G,\beta}$ from (1.7) demonstrates the competition between these positive and negative correlations:

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\langle \tau^A \rangle^{\text{f,f}}_{G,\beta} = \sum_{\{x,y\}\in E} J \underbrace{\mathbf{Cov}^{\text{f,f}}_{G,\beta}(\tau^A, \tau_x\tau_y)}_{\geq 0} + J' \underbrace{\mathbf{Cov}^{\text{f,f}}_{G,\beta}(\tau^A, \tau'_x\tau'_y)}_{\leq 0} + U \underbrace{\mathbf{Cov}^{\text{f,f}}_{G,\beta}(\tau^A, \tau_x\tau_y\tau'_x\tau'_y)}_{?},$$

where $\mathbf{Cov}^{\mathrm{f,f}}_{G,\beta}$ denotes the covariance with respect to $\mathtt{AT}^{\mathrm{f,f}}_{G,\beta}$. By Proposition 1.2.2, the first summand is non-negative, the second is non-positive, and no conclusions can be drawn about the third. In order to circumvent this issue, one may search for curves in the phase diagram along which the model is increasing. This strategy is described further below and implemented in Chapter 3.

Main results of Chapter 3

In Chapter (3), we verify both ferromagnetic and antiferromagnetic behaviour in the AT model on $(\mathbb{Z}^d, \mathbb{E}^d)$ for arbitrary $d \geq 2$ in different parts of the regime (1.27). It contains two main results and a number of by-products that we derive along the way.

(i) The first one is the existence of a partial antiferromagnetic phase in the isotropic model when $J \geq 0$ is sufficiently small and $U < 0$ is sufficiently large in absolute value (Theorem 10). See Figure 1.5 for the phase diagram. Specifically, in this regime the configuration τ is disordered, admitting exponential decay of the magnetisation and correlations, whereas the product $\tau\tau'$ is antiferromagnetically ordered as in (1.26) for $\beta < -\beta_c$. The statement can be proved in exactly the same way for the anisotropic model, provided that additionally $J' \in [-J, J]$ (see the methods below).

In the planar case, we also prove an analogue of this result for the related staggered six-vertex [WL75] height function. We show that it is localised while simultaneously displaying antiferromagnetic order on one partite class of the graph (Theorem 11). To this end, we construct a coupling with the staggered eight-vertex model [HLW75] in fairly high generality.

(ii) The second one is that the anisotropic model accommodates sharp order-disorder phase transitions for τ and τ' in the FKG regime given by (1.15) with $(K, K', K'') = (J, J', U)$, thereby displaying ferromagnetic behaviour (Theorem 12). We partition this regime into a family of smooth curves along which the model is monotone and undergoes the aforementioned transition. The validity of monotonicity properties in the isotropic model would enable us to deduce the presence of such a transition in the antiferromagnetic isotropic model as well, and we also present some indications of such.

Methods. The main tool for proving these results is the ATRC representation. In the setting of Section 1.2.2, we consider this representation for s only and extend it to the regime $K \geq |K''|$ (see Section 3.3). As a bi-product, we derive (GKS1) for τ in the regime $J \geq \min\{|J'|, |U|\}$ (Corollary 3.3.5). For (i), we apply classical energy entropy arguments to different versions of this representation. For (ii), we use the representation in its FKG regime (where it coincides with the one introduced in Section 1.2.2), and adapt the recent approach [DRT19] utilising the OSSS inequality.

1.4 Properties of the Potts models at their transition points

In this section, the behaviour of Potts models (in particular the Ising model) and FK percolation at their transition points is discussed. We first give a brief summary of known results on $(\mathbb{Z}^d, \mathbb{E}^d)$ in any dimension $d \geq 2$ and then restrict ourselves to the planar case, where significant progress has been made in the last decades. Recall the phase transitions of the Ising and Potts models from Section 1.2.1 and those of FK percolation from 1.2.2. We simply write β_c and p_c for the respective transition points and thresholds, as it will be clear from the context which model, parameter and dimension is being considered. In Section 1.2.4, we argued that the Ising and Potts measures with free and monochromatic boundary conditions and the FK percolation measures with free and wired boundary conditions admit infinite-volume limits on $(\mathbb{Z}^d, \mathbb{E}^d)$, which we denote by omitting the graphs from the subscripts of the respective measures. Let us first address the case of the Ising model. Denote by $\langle\cdot\rangle_\beta^+$ the expectation operator with respect to $\mathtt{Ising}_\beta^+$.

In the Ising model, it is relatively easy to see that the function $\beta \mapsto \langle\sigma_0\rangle_\beta^+$ is right-continuous, and continuous everywhere except possibly at β_c. As it is constant 0 for $\beta < \beta_c$ by definition, continuity of the function is equivalent to

$$\langle\sigma_0\rangle_{\beta_c}^+ = 0. \tag{1.28}$$

We refer to the above property as *continuity of the phase transition*. It is known to hold in any dimension $d \geq 2$. This was first proved in the planar case $d = 2$ by Yang [Yan52], who computed the limit for any $\beta > 0$ using the methods of [Ons44, Kau49]. The absence of *long range order*, now known [ADS15] to be equivalent to (1.28), in $d = 2$ was previously proved in [KO50]. A short proof for (1.28) in $d = 2$ was recently provided by Werner [Wer09]. Continuity in high dimensions $d \geq 4$ was established in [AF86] relying on reflection positivity. The remaining case $d = 3$ was treated in [ADS15], where a proof for any $d \geq 3$ was given using the random current expansion [Aiz82] of the Ising model and also reflection positivity.

The Potts model offers more variety in the sense that the type of the transition depends on the dimension d as well as on the number q of states. For $d, q \geq 2$, the transition is said to be

$$\begin{cases} \text{continuous} & \text{if} \quad \mathtt{Potts}^1_{q,\beta_c}(\sigma_0 = 1) - \frac{1}{q} = 0, \\ \text{discontinuous} & \text{if} \quad \mathtt{Potts}^1_{q,\beta_c}(\sigma_0 = 1) - \frac{1}{q} > 0. \end{cases}$$

In any fixed dimension $d \geq 2$, the transition is known [KS82] to be discontinuous for sufficiently many states $q \geq q_c(d)$. On the other hand, it was shown [BCC06] that, for $q \geq 3$ fixed, the transition is discontinuous in sufficiently large dimension $d \geq d_c(q)$. In fact, the transition in dimension $d \geq 3$ is expected to be continuous exactly when $q = 2$. In other words, given the above results on the Ising model, one expects that $q_c(d) = 3$ for $d \geq 3$.

For FK percolation with cluster-weight $q \geq 1$, the transition is referred to as

$$\begin{cases} \text{continuous} & \text{if} \quad \mathtt{FK}^1_{q,p_c}(0 \leftrightarrow \infty) = 0, \\ \text{discontinuous} & \text{if} \quad \mathtt{FK}^1_{q,p_c}(0 \leftrightarrow \infty) > 0, \end{cases}$$

where $0 \leftrightarrow \infty$ denotes the event that the cluster of the origin is infinite. By the FK relations (1.11) and the fact that $\mathtt{FK}^1_{\Lambda_n,q,p}(0 \leftrightarrow \partial^{\text{ex}}\Lambda_n)$ converges to $\mathtt{FK}^1_{q,p}(0 \leftrightarrow \infty)$ as $n \to \infty$ (see [Gri06, Proposition 5.11]), the types of the phase transition for the Potts model and FK percolation with $q \geq 2$ integer are the same.

From now on, until the end of this section, we will consider the planar case.

$$\boxed{\textbf{Assumption: } d = 2}$$

In the planar case, the type of the phase transition has been determined for any $q \geq 1$ [DCST17, DGH+21]. A fundamental step towards understanding the behaviour at the transition point was the following dichotomy statement.

Theorem 4 ([DCST17, Theorem 3]). *Let $d = 2$ and $q \in [1, \infty)$. One of the following holds.*

- *Continuous transition:*

$$\mathtt{FK}^1_{q,p_c}(0 \leftrightarrow \infty) = 0, \qquad \mathtt{FK}^0_{q,p_c} = \mathtt{FK}^1_{q,p_c} \qquad \textit{and} \qquad \lim_{n\to\infty} \tfrac{1}{n} \ln \mathtt{FK}^0_{q,p_c}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda_n) = 0.$$

- *Discontinuous transition:*

$$\mathtt{FK}^1_{q,p_c}(0 \leftrightarrow \infty) > 0, \qquad \mathtt{FK}^0_{q,p_c} \neq \mathtt{FK}^1_{q,p_c} \qquad \textit{and} \qquad \lim_{n\to\infty} \tfrac{1}{n} \ln \mathtt{FK}^0_{q,p_c}(0 \leftrightarrow \partial^{\mathrm{ex}}\Lambda_n) > 0.$$

The main statement of the above theorem is that the presence of an infinite cluster in $\mathtt{FK}^1_{q,p_c}$ is equivalent to exponential decay of connection probabilities in $\mathtt{FK}^0_{q,p_c}$. It should be noted that the original theorem [DCST17, Theorem 3] is stronger in the sense that it contains more properties such as *Russo–Seymour–Welsh* [Rus78, SW78] behaviour in the continuous case, while it also states that the properties in each case are in fact equivalent. In a previous work [Dum12], the first scenario was shown to occur when $1 \leq q \leq 4$ (see also [GL23]). It was later shown [DGH+21] (see also [RS20]) that the second scenario occurs when $q > 4$.

Theorem 5 ([DCST17, DGH+21]). *Let $d = 2$. The phase transition of the Potts model and FK percolation is continuous if $1 \leq q \leq 4$, and discontinuous if $q > 4$.*

Relation to the transition in the six-vertex model. Recall the transition of the six-vertex height function from delocalisation to localisation discussed in Section 1.2.1 and the BKW correspondence with self-dual FK percolation discussed in Section 1.2.3. When $q > 4$, which corresponds to $\mathbf{c} > 2$, the self-dual wired FK measure admits a unique macroscopic cluster on which the six-vertex height function is constant, forcing the latter to be flat. On the other hand, for $q \leq 4$, which corresponds to $\mathbf{c} \leq 2$, there are macroscopic circuits in both the primal and dual configurations, leading to fluctuations in the six-vertex height function.

A next step in the study of the properties at the transition points is to examine the behaviour of *interfaces*, such as those imposed by non-constant boundary conditions, known as *Dobrushin* boundary conditions. In the case of a continuous transition and the resulting *critical behaviour*, the fluctuations of the interfaces are of the same order as the size of the system (*Russo–Seymour–Welsh theory*), and their scaling limit is expected to be conformally invariant, proved so far only in the Ising case $d = 2$ [Smi10, CS12, CDCH+14]. The rotational invariance [DCKK+20] at large scales was recently proved for all $q \in [1, 4]$. We focus here on the discontinuous case.

1.4.1 Interfaces in the discontinuous case: main results of Chapter 4

In this section, we consider the Potts models and FK percolation on $(\mathbb{Z}^2, \mathbb{E}^2)$ with $q > 4$ at their transition points, and we omit the parameters from the subscripts of the measures. By Theorems 4 and 5, the free infinite-volume measure $\mathtt{FK}^0$ admits exponential decay of connection probabilities, a property of the model below the transition point (sharpness), whereas the wired infinite-volume measure $\mathtt{FK}^1$ admits an infinite cluster, a property of the model above the transition point. Furthermore they are dual to each other (see Section 1.2.3). The analogue holds for the corresponding Potts measures $\mathtt{Potts}^{\mathrm{f}}$ and $\mathtt{Potts}^1$ with free and constant 1 boundary conditions, respectively.

Below, we consider the Potts and FK measures on square boxes and under Dobrushin boundary conditions (b.c.). We will first discuss the case of *order-disorder* Dobrushin b.c., which in the Potts model is a combination of monochromatic b.c. in the upper half-plane and free b.c. in the lower one. Chapter 4 provides a detailed analysis of these measures and we present the

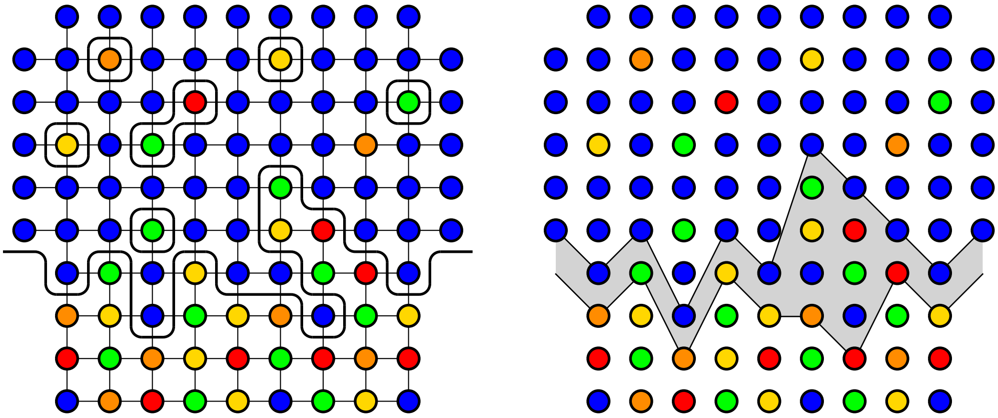


Figure 1.6: *Left:* A configuration of the Potts model on Λ_n, $n = 4$, with order-disorder Dobrushin b.c. (blue represents the constant 1 b.c. in the upper half-plane). The 'Peierls contours' separate the boundary cluster in star connectivity of the blue colour on top from the rest. *Right:* The interpolations of the associated upper and lower envelopes and the area in between (grey). In Chapter 4, we prove that this area is thin (less than $\ln^2 n$), and the envelopes admit fluctuations of order $\sqrt{n}$ and each converges when rescaled to a Brownian bridge.

main results below. Subsequently, we address the case of *order-order* Dobrushin b.c., which in the Potts model is a combination of different monochromatic b.c. in the upper and lower half-planes. The so-called *wetting phenomenon* is expected to occur in this case. We verify this conjecture in a subsequent work [DGO25b], which is not part of this thesis, and present the relevant coupling with the (modified) ATRC model in Chapter 5.

Order-disorder Dobrushin boundary condition

Let us introduce the Potts measure on the square box Λ_n with 1/f Dobrushin b.c., that is, with constant 1 b.c. in the upper half-plane and free b.c. in the lower half-plane. Set $\mathbb{H}_+ = \mathbb{R}\times[0,\infty)$, and define the probability measure $\mathtt{Potts}^{1/\mathrm{f}}_{\Lambda_n}$ on $\Sigma_{\Lambda_n} = \{1,\dots,q\}^{\Lambda_n}$ by (1.1) with $\beta = \beta_\mathrm{c}$ and (see Figure 1.6)

$$\mathcal{H}^{1/\mathrm{f}}_{\Lambda_n}(\sigma) := -\sum_{\substack{x,y\in\Lambda_n\\ \{x,y\}\in\mathbb{E}^2}} \delta_{\sigma_x,\sigma_y} - \sum_{\substack{x\in\Lambda_n,\, y\in\mathbb{H}_+\setminus\Lambda_n\\ \{x,y\}\in\mathbb{E}^2}} \delta_{\sigma_x,1}.$$

Intuitively, one might expect that under this measure, the 'upper part' of the box Λ_n resembles the ordered phase (described by $\mathtt{Potts}^1$) while the 'lower part' resembles the disordered phase (described by $\mathtt{Potts}^0$), and the two are separated by a 'thin interface' (see the left side of Figure 1.7). Before presenting the main result of Chapter 4, which provides a complete resolution of these considerations, let us review the previous results.

In a perturbative regime where q is sufficiently large, the *fluctuations of the interface* between the ordered phase at the top and the disordered phase at the bottom were shown [MMSRS91] to be of *order* $\sqrt{n}$. This was achieved by adapting the theory of Pirogov–Sinai [PS75] to obtain a description of the interface very close to that of an order-order interface in the Ising model above its transition point, which was known [Gal72] to admit $\sqrt{n}$-fluctuations. It was later shown that this related order-order Ising interface under diffusive scaling converges in law to a Brownian bridge — first in a perturbative regime [Hig79] and then down to the transition point [GI05].

By compactness of the configuration space, the above measures admit subsequential limits, each of which is a linear combination of the $q+1$ extremal (translation invariant) *Gibbs measures* [GM23] given by the one free and the q monochromatic measures. This implies that the

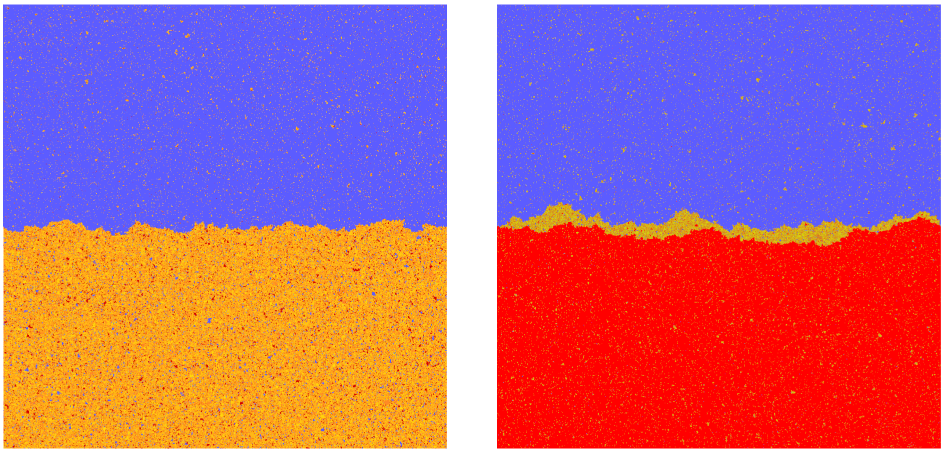


Figure 1.7: Simulations of the Potts model on a 1000x1000 box with 25 colours at $\beta_c(25)$ under different Dobrushin boundary conditions, created by Sébastien Ott using Metropolis-Hastings. Colour 1 is blue, colour 2 is red, and colours 3-25 are interpolations between yellow and orange. *Left:* Blue b.c. in the upper half and free b.c. (no colour favoured) in the lower half, with the order-disorder interface in between. The main result of Chapter 4 is that this interface is thin and converges under diffusive scaling to a Brownian bridge. *Right:* Blue b.c. in the upper half and red b.c. in the lower half, with the order-order interface in between. In a forthcoming work, we show that this interface is thick and its boundaries converge under diffusive scaling to a pair of Brownian bridges conditioned not to intersect.

interface (if defined at all) exhibits *divergent fluctuations* for all $q > 4$.

The main result of Chapter 4 is that for all $q > 4$ this interface, appropriately defined (see Figure 1.6), is *thin* and converges under diffusive scaling to a *Brownian bridge*. More precisely, we consider the boundary clusters of $\sigma = 1$ at the top and of $\sigma \neq 1$ at the bottom (one taken in star connectivity, see Figure 1.6 and Section 4.1), and we prove that the region in between is at most $C \ln^2(n)$ thick with high probability for some $C > 0$, and that their boundary 'envelopes' have fluctuations of order $\sqrt{n}$ and converge when rescaled to a Brownian bridge. This is the content of Theorem 13. We also prove the analogues of these statements for FK percolation with 1/0 Dobrushin b.c. for all $q > 4$ (Theorem 14).

This is achieved by couplings with the six-vertex and AT models (see methods below), and we also derive invariance principles for the corresponding order-order interfaces in the six-vertex and AT models. For this purpose, we establish exponential mixing properties of the corresponding ATRC model (Theorem 16) in order to develop a renewal picture for long clusters. Along the way, we derive sharp Ornstein–Zernike [OZ14] asymptotics for the two-point functions of the AT and ATRC models, which is the content of Theorems 15 and 18.

Methods. The proof proceeds through a modification of the Baxter–Kelland–Wu [BKW76] coupling of the FK percolation and six-vertex measures with Dobrushin b.c., and further through a coupling of the latter with (modifications of) the isotropic AT and ATRC models on the *self-dual* curve (see Section 1.2.3), also under Dobrushin b.c.. We relate the area between the ordered and disordered phases in the Potts model to a *cluster conditioned to be long* in the ATRC model, which admits exponential decay of connection probabilities by the results from Chapter 2. We then apply different versions of these couplings under constant boundary conditions together with the results of Chapter 2 to derive the uniqueness of the infinite-volume ATRC (Gibbs) measure and exponential mixing properties, relying on [Ale98, Ott25]. This is necessary to follow [CIV04, CIV08, AOV24] for developing a kind of *renewal structure*, or more accurately a

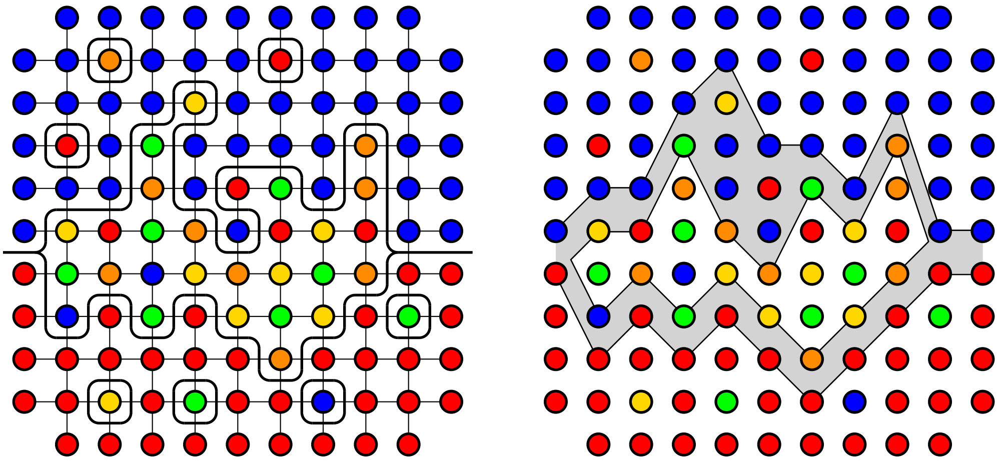


Figure 1.8: *Left:* A configuration of the Potts model on Λ_n, $n = 4$, with order-order Dobrushin b.c. (blue and red represent the constant 1 and 2 b.c. in the upper and lower half-planes, respectively). The 'Peierls contours' separate the upper and lower boundary clusters (one in star connectivity) of the blue and red colours from the rest. *Right:* The interpolations of the associated upper and lower envelopes and the areas in between (grey). In a forthcoming work [DGO25b], we prove that these areas are thin (less than $\ln^c n$ for some $c > 0$), and the, say, uppermost and lowermost envelopes admit fluctuations of order $\sqrt{n}$ and converge when rescaled to a pair of Brownian bridges conditioned not to intersect.

sequence of 'fast mixing kernels', for clusters in the ATRC model à la Ornstein–Zernike [OZ14]. To conclude the proof, we establish the proximity of the interfaces in the FK percolation and (modified) ATRC models by showing uniform exponential decay in the latter, using the results of Chapter 2 and a general argument from [Ale04]. The results for the Potts model are then derived from the results for FK percolation.

Order-order Dobrushin boundary conditions

Consider now the Potts measure on Λ_n under 1/2 Dobrushin b.c., that is, constant 1 b.c. in the upper half-plane and constant 2 b.c. in the lower one. Analogous to the order-disorder case, setting $\mathbb{H}_- = \mathbb{R}^2 \setminus \mathbb{H}_+$, the probability measure $\mathtt{Potts}^{1/2}_{\Lambda_n}$ on Σ_{Λ_n} is defined by (1.1) with $\beta = \beta_c$ and (see Figure 1.8)

$$\mathcal{H}^{1/2}_{\Lambda_n}(\sigma) := -\sum_{\substack{x,y\in\Lambda_n\\ \{x,y\}\in\mathbb{E}^2}} \delta_{\sigma_x,\sigma_y} - \sum_{\substack{x\in\Lambda_n,\, y\in\mathbb{H}_+\setminus\Lambda_n\\ \{x,y\}\in\mathbb{E}^2}} \delta_{\sigma_x,1} - \sum_{\substack{x\in\Lambda_n,\, y\in\mathbb{H}_-\setminus\Lambda_n\\ \{x,y\}\in\mathbb{E}^2}} \delta_{\sigma_x,2}.$$

Let us first observe that this measure is also interesting for any $q \geq 2$ above the transition point. Indeed, it was shown that the two ordered phases (1 at the top and 2 at the bottom) are separated by a *thin interface* (less than $C\ln(n)$ thick with high probability) and fluctuations of order $\sqrt{n}$, which converges when rescaled to a Brownian bridge [CIV03, CIV08].

Wetting. In contrast, at a discontinuous transition point, one would expect a *thick* disordered phase in the middle, separating the two pure phases in the upper and lower parts (see the right side of Figure 1.7 and Figure 1.8). This phenomenon is referred to as *wetting*, whereby the disordered phase in the middle is considered to *wet* the two pure phases above and below. See [DMMR88, MMSRS91] for works on the wetting phenomenon in the Potts model, and [IOSV22] for the *prewetting* in the Ising model.

In a forthcoming work [DGO25b], we verify the occurrence of the wetting phenomenon for all $q > 4$. Via another coupling with the AT model, while relying on the results in Chapters 2 and 4, we show the emergence of a disordered layer of thickness $\sqrt{n}$ between the ordered phases and establish convergence under diffusive scaling of its upper and lower boundaries to a pair of Brownian bridges conditioned not to intersect. The coupling is presented in Chapter 5.

Chapter 2

Phase diagram of the Ashkin–Teller model

The Ashkin–Teller model is a pair of interacting Ising models and has two parameters: J is a coupling constant in the Ising models and U describes the strength of the interaction between them. In the ferromagnetic case $J, U > 0$ on the square lattice, we establish a complete phase diagram conjectured in physics in 1970s (by Kadanoff and Wegner, Wu and Lin, Baxter and others): when $J < U$, the transitions for the Ising spins and their products occur at two distinct curves that are dual to each other; when $J \geq U$, both transitions occur at the self-dual curve. All transitions are shown to be sharp using the OSSS inequality.

We use a finite-size criterion argument and continuity to extend the result of Glazman–Peled [GP23] from a self-dual point to its neighborhood. Our proofs go through the random-cluster representation of the Ashkin–Teller model introduced by Chayes–Machta and Pfister–Velenik and we rely on couplings to FK-percolation.

2.1 Introduction

The Ashkin–Teller (AT) model is named after two physicists who introduced it [AT43] in 1943 and can be viewed as a pair of interacting Ising models [Fan72a]. For a finite subgraph $\Omega = (V, E)$ of $\mathbb{Z}^2$, the AT model is supported on pairs of spin configurations $(\tau, \tau') \in \{\pm 1\}^V \times \{\pm 1\}^V$ and the distribution is defined by

$$\mathtt{AT}_{\Omega, J_\tau, J_{\tau'}, U}(\tau, \tau') = \frac{1}{Z} \cdot \exp\left(\sum_{uv \in E} J_\tau \tau_u \tau_v + J_{\tau'} \tau'_u \tau'_v + U \tau_u \tau'_u \tau_v \tau'_v \right), \tag{2.1}$$

where $J_\tau, J_{\tau'}, U$ are real parameters and $Z = Z(\Omega, J_\tau, J_{\tau'}, U)$ is the unique constant (called *partition function*) that renders the above a probability measure.

In the current chapter, we consider the ferromagnetic symmetric (or isotropic) case

$$J = J_\tau = J_{\tau'} \geq 0, \text{ and } U \geq 0$$

and denote the measure by $\mathtt{AT}_{\Omega, J, U}$.

Important particular cases: $U = 0$ gives two independent Ising models; for $J = 0$, τ reduces to a Bernoulli site percolation with parameter $1/2$, and $\tau\tau'$ to an Ising model, independent of each other; the line $U = J$ corresponds to the 4-state Potts model. These models are very well-studied and their phase diagram is known; see [FV17, Dum17] for excellent surveys. Henceforth in this chapter, we assume that $J, U > 0$. A key observation in the analysis of the AT model on $\mathbb{Z}^2$ is its relation to the six-vertex model [Fan72b, Weg72]. This gives a *non-staggered* six-vertex model (i.e. with shift invariant local weights) only at the *self-dual line* of the AT model: it was found in [MS71] and is described by the equation

$$\sinh(2J) = e^{-2U}. \tag{SD}$$

Outside of this line, the corresponding six-vertex model is staggered and thus the seminal Baxter's solution [Bax71] does not apply. Kadanoff and Wegner [KW71, Weg72], Wu and Lin [WL74], and others conjectured that, when $J < U$, there are two *distinct* transition lines in the AT model: one for correlations of spins τ (or τ') and the other for correlations of products $\tau\tau'$. In the current chapter, we prove this conjecture and establish a complete phase diagram of the AT model in the ferromagnetic regime.

It will be convenient to state the results in infinite volume and to consider also plus boundary conditions. Denote by $\partial\Omega$ the set of boundary vertices of Ω – these are all vertices in Ω that are adjacent to at least one vertex in $\mathbb{Z}^2 \setminus \Omega$. We define the measure with plus boundary conditions by conditioning all boundary vertices to have spin plus in τ and in τ':

$$\mathtt{AT}^{+,+}_{\Omega,J,U} := \mathtt{AT}_{\Omega,J,U}(\cdot \mid \tau_{|\partial\Omega} \equiv \tau'_{|\partial\Omega} \equiv 1).$$

Expectations with respect to the AT measures are denoted by brackets:

$$\langle\cdot\rangle_{\Omega,J,U} := \mathbb{E}_{\Omega,J,U}[\cdot] \qquad \text{and} \qquad \langle\cdot\rangle^{+,+}_{\Omega,J,U} := \mathbb{E}^{+,+}_{\Omega,J,U}[\cdot].$$

The correlations of τ and τ' satisfy the Griffiths–Kelly–Sherman (GKS) inequality [KS68], which states that for any $A, B, C, D \subset V$, one has

$$\langle \tau_A \cdot \tau'_B \cdot \tau_C \cdot \tau'_D \rangle_{\Omega,J,U} \geq \langle \tau_A \cdot \tau'_B \rangle_{\Omega,J,U} \langle \tau_C \cdot \tau'_D \rangle_{\Omega,J,U}, \tag{GKS}$$

where $\tau_A := \prod_{u\in A} \tau_u$ and $\tau'_B := \prod_{v\in B} \tau'_v$, and the same holds under plus boundary conditions. This implies that, for any $A, B \subset V$,

$$\langle \tau_A \cdot \tau'_B \rangle_{\Omega,\beta J,\beta U} \quad \text{and} \quad \langle \tau_A \cdot \tau'_B \rangle^{+,+}_{\Omega,\beta J,\beta U} \quad \text{are increasing in } \beta > 0.$$

Another standard application of the GKS inequality implies that, as $\Omega_n \nearrow \mathbb{Z}^2$, the weak limits under free or plus boundary conditions exist and do not depend on $\{\Omega_n\}$:

$$\mathtt{AT}_{J,U} := \lim_{n\to\infty} \mathtt{AT}_{\Omega_n,J,U} \qquad \text{and} \qquad \mathtt{AT}^{+,+}_{J,U} := \lim_{n\to\infty} \mathtt{AT}^{+,+}_{\Omega_n,J,U}.$$

Similarly to finite-volume measures, we denote by $\langle\cdot\rangle_{J,U}$ and $\langle\cdot\rangle^{+,+}_{J,U}$ the expectations with respect to $\mathtt{AT}_{J,U}$ and $\mathtt{AT}^{+,+}_{J,U}$. It is standard (e.g. can be shown by comparing to the Ising model) that the AT model undergoes a phase transition in terms of correlations of τ and those of $\tau\tau'$. Moreover, a general OSSS inequality [DRT19] can be used to show that both transitions are sharp (see Appendix). That is, for each pair J, U, there exist $\beta_c^\tau, \beta_c^{\tau\tau'} \in (0,\infty)$ and $(c_\beta)_{\beta>0}$ strictly positive, such that

$$\langle \tau_0\tau_x \rangle^{+,+}_{\beta J,\beta U} \begin{cases} \leq e^{-c_\beta \cdot |x|} & \text{if } \beta < \beta_c^\tau \\ \geq c_\beta & \text{if } \beta > \beta_c^\tau \end{cases}, \qquad \langle \tau_0\tau'_0\tau_x\tau'_x \rangle^{+,+}_{\beta J,\beta U} \begin{cases} \leq e^{-c_\beta \cdot |x|} & \text{if } \beta < \beta_c^{\tau\tau'} \\ \geq c_\beta & \text{if } \beta > \beta_c^{\tau\tau'} \end{cases}. \tag{2.2}$$

Symmetry between τ and τ' and the correlation inequalities (GKS) imply directly that

$$\beta_c^\tau \geq \beta_c^{\tau\tau'}. \tag{2.3}$$

There exists a unique β, for which $(\beta J, \beta U)$ is on the line (SD). Denote it by $\beta_{\mathrm{sd}} = \beta_{\mathrm{sd}}(J,U)$. The following theorem states our main result:

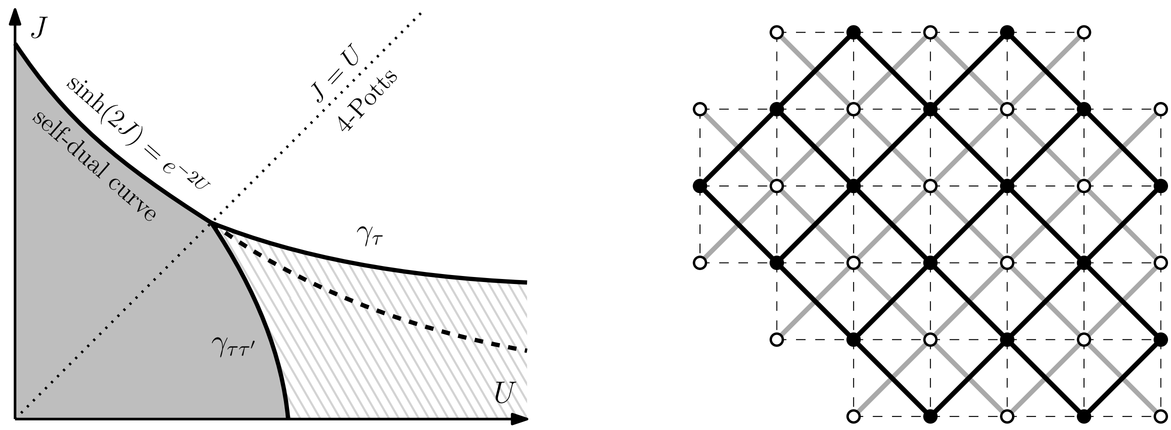


Figure 2.1: *Left:* Phase diagram of the Ashkin–Teller model: when $J \geq U$, transitions for τ and $\tau\tau'$ occur at the self-dual curve (Theorem 8) and when $J < U$, the transition occurs at two distinct curves γ_τ and $\gamma_{\tau\tau'}$ dual to each other (Theorems 6 and 7). There are three regimes: disorder in τ and in $\tau\tau'$ (gray), order in τ and in τ' (white), disorder in τ and order in $\tau\tau'$ (dashed gray). *Right:* Domain Ω (in bold black) in $\mathbb{L}$ and its dual Ω^* (in gray). Notice that Ω^* is not a domain in $\mathbb{L}^*$. The even domain $\mathcal{D}_\Omega$ (dashed) in $\mathbb{Z}^2$.

Theorem 6. *Let* $0 < J < U$. *Then,* $\beta_c^\tau > \beta_{\text{sd}} > \beta_c^{\tau\tau'}$.

This was previously shown when $2J < U$ using a direct comparison to the Ising model [Pfi82]. In addition, in the perturbative regime when J is small enough, the critical exponents associated to the phase transition for the product $\tau\tau'$ have been shown to be the same as for the Ising model [GM05][1]. This is expected to hold for both transitions whenever $J < U$, while the exponents should vary continuously when $J \geq U$. The latter has been established in [Mas04] for U sufficiently small. We refer to [DG04] for a survey on the physics literature on the critical behaviour of the AT model, as well as predictions on critical exponents using the quantum field theory. Recently, Peled and the third author have proven that spins τ (or τ') and the products $\tau\tau'$ exhibit qualitatively different behavior at the self-dual line when $J < U$ [GP23]: products $\tau\tau'$ are ordered, while τ (and τ') exhibits exponential decay of correlations. We derive Theorem 6 by extending this statement to an open neighborhood of the self-dual line when $J < U$. The continuity ideas do not apply directly, since the rate of decay of correlations might, a priori, not be a continuous function in the pair (J, U). To circumvent this problem, we establish exponential decay in finite volume:

Proposition 2.1.1. *Fix* $0 < J < U$ *that satisfy* $\sinh 2J = e^{-2U}$. *Then, there exists* $c := c(J, U) > 0$ *such that*

$$\langle \tau_0 \rangle^{+,+}_{[-n,n]^2,J,U} \leq e^{-cn}.$$

Compared to [GP23], the exponential decay is proven in finite volume and under the largest boundary conditions. This is crucial for applying the so-called "finite-size criterion" (or $\varphi_\beta(S)$) argument [Sim80, Lie80, DT16a, DT16b], since the Simon–Lieb inequality is not available (the Simon inequality is valid for $U \leq 0$ [Lis21a]). This argument, as well as the proof of Proposition 2.1.1, use the random-cluster representation of the AT model (that we call ATRC) introduced by Chayes–Machta [CM97] and Pfister–Velenik [PV97]. As in the seminal Edwards–Sokal coupling for the Potts model [ES88], connectivities in the ATRC describe correlations in the AT model.

[1] Although [GM05] studies the anisotropic case $J_\tau \neq J_{\tau'}$ when U is small, applying the change of variables $(\tau, \tau') \mapsto (\tau\tau', \tau)$ allows to treat the case described above.

Other ingredients in the proof of Proposition 2.1.1 are couplings between the AT and the six-vertex models [Fan72b, Weg72] and between the latter and the FK-percolation [BKW76]. These two couplings were composed for the first time in the work of Peled and the third author [GP23]. We also use $\mathbb{T}$-circuits introduced in [GP23] to apply the non-coexistence theorem [She05, DRT19].

The next result states that the transition lines are dual to each other (see Fig. 2.1) and that the critical points for the measures under the free and plus boundary conditions coincide. We define the critical curves γ_τ and $\gamma_{\tau\tau'}$ as follows:

$$\gamma_\tau := \{(J,U) \in \mathbb{R}^2 : 0 < J < U,\ \beta_c^\tau(J,U) = 1\},$$

and similarly for $\gamma_{\tau\tau'}$. Given a pair of parameters (J,U), we define the dual set of parameters (J^*,U^*) as the unique solutions to the following equations

$$\frac{e^{-2J+2U}-1}{e^{-2J^*+2U^*}-1} = e^{2U}\sinh(2J) = \left[e^{2U^*}\sinh(2J^*)\right]^{-1}. \tag{2.4}$$

This defines an involution. When $J = U$, we replace the first equality by $J^* = U^*$. Finally, we define $\beta_c^{\tau,f}$ and $\beta_c^{\tau\tau',f}$ as in (2.2) but under free boundary conditions.

Theorem 7. *Fix $0 < J < U$. Then, the following holds:*

(i) $\beta_c^\tau = \beta_c^{\tau,f}$ *and* $\beta_c^{\tau\tau'} = \beta_c^{\tau\tau',f}$*;*

(ii) γ_τ *and* $\gamma_{\tau\tau'}$ *are dual in the following sense:* $(J,U) \in \gamma_\tau$ *if and only if* $(J^*,U^*) \in \gamma_{\tau\tau'}$.

In contrast, when $J \geq U$, both transitions in τ and in $\tau\tau'$ occur at the self-dual line.

Theorem 8. *Let $J \geq U > 0$. Then, the following holds:*

(i) $\beta_c^\tau = \beta_c^{\tau,f}$ *and* $\beta_c^{\tau\tau'} = \beta_c^{\tau\tau',f}$*;*

(ii) $\beta_c^\tau = \beta_c^{\tau\tau'} = \beta_{\mathrm{sd}}$.

Indeed, general approach [DRT19] gives sharpness under plus boundary conditions and equality of the transition points $\beta_c^\tau = \beta_c^{\tau\tau'} =: \beta_c$. By standard duality arguments, one deduces $\beta_c \leq \beta_{\mathrm{sd}}$. The bound $\beta_c \geq \beta_{\mathrm{sd}}$ follows from Zhang-type arguments provided the transition points for the free and monochromatic measures coincide. We show the latter by applying to the marginals of the ATRC a neat reformulation of the classical "convexity of the free energy" argument due to Duminil-Copin [Dum17, Theorem 1.12].

Open questions. At the self-dual curve (SD), the Ashkin–Teller model is coupled to the six-vertex model with parameters $a = b = 1$ and $c = \coth 2J$ (see Section 2.3). When $1 \leq c \leq 2$, the height function of the six-vertex model has been recently shown to delocalise [DCKMO20] (see also [GL23] for an alternative argument that does not use Bethe Ansatz). This implies that the transition in the Ashkin–Teller model is continuous for all $J \geq U > 0$: at the critical curve, correlations $\langle\tau_0\tau_x\rangle^{+,+}$ and $\langle\tau_0\tau_0'\tau_x\tau_x'\rangle^{+,+}$ vanish as $|x| \to \infty$. However, the type of the transitions when $J < U$ remains open:

Question 1. *Let $0 < J < U$. Show that the Ashkin–Teller model undergoes two continuous phase transitions: one has $\langle\tau_0\tau_x\rangle^{+,+} \to 0$ at γ_τ and $\langle\tau_0\tau_0'\tau_x\tau_x'\rangle^{+,+} \to 0$ at $\gamma_{\tau\tau'}$, as $|x| \to \infty$.*

As mentioned above [DG04], the correlations should decay like $|x|^{-1/4}$ (as in the Ising model). It would be natural to extend Theorem 8 to the case of a negative U:

Question 2. *Let $J > 0 > U$. Show that the Ashkin–Teller model undergoes a sharp transition at the self-dual curve* (SD) *in terms of correlations of τ and $\tau\tau'$.*

Note that this would imply that the transition is continuous, since the delocalisation results cover this part of the self-dual curve. What is missing to apply the general argument of [DRT19] and prove sharpness is monotonicity of the correlations along some curves in the (J, U) plane when U is negative.

Organisation of the chapter. Sections 2.2–2.6 treat the case $J < U$: in Section 2.2, we introduce the random-cluster representation of the AT model (ATRC) and derive Theorems 6 and 7 from Proposition 2.1.1; Sections 2.3–2.6 are dedicated to proving Proposition 2.1.1. In Section 2.3, we describe the six-vertex and FK-percolation models and give their background, including their relation to the AT model. In Section 2.4, we show that τ exhibits exponential decay of correlations in finite volume under the boundary conditions $\tau = \tau'$. In Section 2.5, we show that τ exhibits no ordering under $\mathtt{AT}^{+,+}_{J,U}$. In Section 2.6, we derive Proposition 2.1.1. Section 2.7 deals with the case $J \geq U$: we introduce the ATRC model and prove Theorem 8. Appendices provide details regarding sharpness for the AT (2.A), exponential relaxation for FK-percolation (2.B), stochastic ordering of the ATRC with respect to its local weights (2.C) and uniqueness of the infinite-volume ATRC measure (2.D).

2.2 From Proposition 2.1.1 to Theorems 6 and 7

From now on, we will consider the AT model on a rotated square lattice that we denote by $\mathbb{L}$: its vertex set is $\{(x, y) \in \mathbb{Z}^2 : x + y \text{ is even}\}$ and edges connect (x, y) to $(x \pm 1, y \pm 1)$, see Figure 2.1. This is more convenient for the coupling with the six-vertex model (Section 2.3).

In this section, we fix $J < U$ and drop them from the notation. In particular, we write $\mathtt{AT}_{\Omega,\beta}$ for the measure $\mathtt{AT}_{\Omega,\beta J,\beta U}$.

We start by defining the random-cluster representation of the AT model (ATRC) introduced by Chayes–Machta [CM97] and Pfister–Velenik [PV97]. Using a $\varphi_\beta(S)$ argument, we prove that $(\beta_c^{\tau\tau'} J, \beta_c^{\tau\tau'} U)$ is strictly above the self-dual line. By duality, this implies that $(\beta_c^{\tau} J, \beta_c^{\tau} U)$ is strictly below the self-dual line which concludes the proof.

2.2.1 ATRC: defintion and basic properties

The ATRC is reminiscent of the Edwards–Sokal [ES88] coupling between FK-percolation and the Potts model. Since the AT model is supported on a pair of spin configurations, the ATRC is supported on a pair of bond percolation configurations.

Percolation configurations. For a finite subgraph $\Omega \subset \mathbb{L}$, the sets of its vertices and edges are denoted by V_Ω and E_Ω, respectively. We view $\omega \in \{0, 1\}^{E_\Omega}$ as a percolation configuration: we say that e is *open* in ω if $\omega(e) = 1$, and otherwise e is *closed*. We identify ω with a spanning subgraph of Ω and edges that are open in ω. Define $|\omega|$ as the number of edges in ω. Boundary conditions for ω are given by a partition η of $\partial\Omega$. We define $k^\eta(\omega)$ as the number of connected components in ω when all vertices belonging to the same element of partition in η are identified. Two important special cases: 1 denotes *wired* b.c. given by a trivial partition consisting of one element $\partial\Omega$; 0 denotes *free* b.c. given by a partition of $\partial\Omega$ into singletons.

Definition of ATRC. A configuration of the ATRC model on Ω is a pair $(\omega_\tau, \omega_{\tau\tau'})$ of percolation configurations on edges of Ω. Formally, the ATRC measure is supported on $(\omega_\tau, \omega_{\tau\tau'}) \in \{0,1\}^{E_\Omega} \times \{0,1\}^{E_\Omega}$. For $\beta > 0$ and partitions $\eta_\tau, \eta_{\tau\tau'}$ of $\partial\Omega$, the ATRC measure is defined by

$$\mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(\omega_\tau,\omega_{\tau\tau'}) = \tfrac{1}{Z} \cdot 2^{k^{\eta_\tau}(\omega_\tau)+k^{\eta_{\tau\tau'}}(\omega_{\tau\tau'})} \prod_{e\in E} a(\omega_\tau(e),\omega_{\tau\tau'}(e)), \tag{2.5}$$

where $Z = Z(\Omega,\beta,J,U,\eta_\tau,\eta_{\tau\tau'})$ is a normalizing constant and

$$a(0,0) := e^{-2\beta(J+U)},\ a(1,0) := 0,\ a(0,1) := e^{-4\beta J} - e^{-2\beta(J+U)},\ a(1,1) := 1 - e^{-4\beta J}. \tag{2.6}$$

Since $J < U$, we have $a(i,j) \geq 0$ for all $i,j \in \{0,1\}$. We will also use the notation $\mathtt{ATRC}_{\Omega,J,U}^{\eta_\tau,\eta_{\tau\tau'}}$ for the measure with $\beta = 1$.

It will be useful to express the measure as

$$\mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(\omega_\tau,\omega_{\tau\tau'}) \propto \mathrm{w}_\tau^{|\omega_\tau|}\, \mathrm{w}_{\tau\tau'}^{|\omega_{\tau\tau'}\setminus\omega_\tau|}\, 2^{k^{\eta_\tau}(\omega_\tau)+k^{\eta_{\tau\tau'}}(\omega_{\tau\tau'})}\, \mathbb{1}_{\{\omega_\tau\subseteq\omega_{\tau\tau'}\}}, \tag{2.7}$$

where

$$\mathrm{w}_\tau = e^{2\beta U}(e^{2\beta J} - e^{-2\beta J}) \quad \text{and} \quad \mathrm{w}_{\tau\tau'} = e^{2\beta(U-J)} - 1. \tag{2.8}$$

In this context, we will refer to the measure as $\mathtt{ATRC}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}^{\eta_\tau,\eta_{\tau\tau'}}$. In Section 2.4.2, we will encounter a version of this measure with non-homogeneous weights.

Remark 2.2.1. The representation can be extended to $J \geq U$ [PV97], see Section 2.7.1.

There are four special types of boundary conditions given by free/wired η_τ and free/wired $\eta_{\tau\tau'}$: $\mathtt{ATRC}_{\Omega,\beta}^{1,1}$ (both wired), $\mathtt{ATRC}_{\Omega,\beta}^{0,0}$ (both free), $\mathtt{ATRC}_{\Omega,\beta}^{1,0}$ (wired for ω_τ, free for $\omega_{\tau\tau'}$), $\mathtt{ATRC}_{\Omega,\beta}^{0,1}$ (free for ω_τ, wired for $\omega_{\tau\tau'}$).

Positive association: general setting. We first introduce the notions of *monotone* measures, *stochastic domination* and *positive association* following [GHM01, Section 4.2]. Let E be a finite set of edges. We introduce a pointwise partial order on percolation configurations on E: we say that $\omega \preceq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for any $e \in E$. A probability measure μ on $\{0,1\}^E$ is called *monotone* if

$$\mu(\omega(e) = 1 \,|\, \omega = \eta \text{ off } e) \leq \mu(\omega(e) = 1 \,|\, \omega = \eta' \text{ off } e) \tag{2.9}$$

for any $e \in E$, and $\eta, \eta' \in \{0,1\}^{E\setminus\{e\}}$ such that $\eta \preceq \eta'$, $\mu(\omega = \eta \text{ off } e) > 0$ and $\mu(\omega = \eta' \text{ off } e) > 0$.

An event $A \subseteq \{0,1\}^E$ is called *increasing* if $\mathbb{1}_A$ is increasing with respect to the partial order. Given two probability measures μ and ν on $\{0,1\}^E$, we say that μ is stochastically dominated by ν (or ν stochastically dominates μ), and write $\mu \leq_{\text{st}} \nu$ (or $\nu \geq_{\text{st}} \mu$), if for every increasing event $A \in \mathcal{A}$, we have $\mu(A) \leq \nu(A)$. Moreover, μ is said to be *positively associated* if, for all increasing bounded functions f and g on $\{0,1\}^E$, we have

$$\mu(f \cdot g) \geq \mu(f)\mu(g). \tag{2.10}$$

This is called the FKG inequality after the work of Fortuin, Kasteleyn and Ginibre [FKG71]. Their fundamental contribution consists in introducing the FKG lattice condition (similar to (2.9)) and showing that it implies positive association (2.10) when μ is strictly positive.

The above setting suffices for the ATRC measure, but for our proofs we need a slightly more general statement. More precisely, we fix $K \in \mathbb{N}^*$ and say that a probability measure μ on $\{-K,\dots,K\}^E$ is *monotone* if and only if

$$\mu(\omega(e) \geq a \,|\, \omega = \eta \text{ off } e) \leq \mu(\omega(e) \geq a \,|\, \omega = \eta' \text{ off } e) \tag{2.11}$$

for any $e \in E$, $a \in \mathbb{R}$ and $\eta, \eta' \in S^{E \setminus \{e\}}$ such that $\eta \preceq \eta'$, $\mu(\omega = \eta \text{ off } e) > 0$ and $\mu(\omega = \eta' \text{ off } e) > 0$. The notions of stochastic domination and positive association extend in a natural way. A probability measure μ on $\{-K, \dots, K\}^E$ is called *irreducible* if, for any $\omega, \omega' \in \{-K, \dots, K\}^E$ such that $\mu(\omega), \mu(\omega') > 0$ there exists a sequence $\omega_0 = \omega$, ω_1, $\dots$, $\omega_N = \omega'$, for some $N > 0$, such that ω_{i-1} and ω_i differ at one coordinate and both have a non-zero probability, for $i = 1, \dots, N$. Assume that μ is monotone, irreducible and the set of $\omega \in \{-K, \dots, K\}^E$ having positive probability contains a unique maximal element. Then, [GHM01, Theorem 4.11] states that μ is positively associated.

Finally, we mention that the positive association property (2.10) naturally extends to probability measures on $\{0,1\}^{\mathbb{N}}$ or $\mathbb{Z}^{\mathbb{N}}$. Moreover, for any increasing sequence of finite sets E_i, if the sequence of positively associated measures μ_i on E_i converges weakly, then the limiting measure is also positively associated. Indeed, (2.10) is preserved under weak limits when f and g are continuous. Then, by Strassen's theorem [Str65], the same holds for any bounded functions (see [GHM01, Theorem 4.6] and the paragraph below [GHM01, Definition 4.10]).

Monotonicity properties of the ATRC measure Consider a natural partial order on pairs of percolation configurations: $(\omega_\tau, \omega_{\tau\tau'}) \preceq (\tilde{\omega}_\tau, \tilde{\omega}_{\tau\tau'})$ if $\omega_\tau(e) \le \tilde{\omega}_\tau(e)$ and $\omega_{\tau\tau'}(e) \le \tilde{\omega}_{\tau\tau'}(e)$ for every edge e of Ω. By [PV97, Proposition 4.1] (and its proof), the measure $\mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}$, for any $\beta > 0$ and any boundary conditions $\eta_\tau, \eta_{\tau\tau'}$, is monotone and hence is positively associated. In particular, for any increasing events A and B,

$$\mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(A \cap B) \ge \mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(A) \cdot \mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(B). \tag{FKG}$$

It is standard that monotone measures are stochastically ordered with respect to their boundary conditions. Indeed, for two partitions η and $\tilde{\eta}$ of $\partial\Omega$, we say that $\eta \ge \tilde{\eta}$ if any two vertices belonging to the same element of $\tilde{\eta}$ also belong to the same element of η. Then, for any $\beta > 0$, and any boundary conditions such that $\eta_\tau \ge \tilde{\eta}_\tau$ and $\eta_{\tau\tau'} \ge \tilde{\eta}_{\tau\tau'}$,

$$\mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau,\eta_{\tau\tau'}} \ge_{\text{st}} \mathtt{ATRC}_{\Omega,\beta}^{\tilde{\eta}_\tau,\tilde{\eta}_{\tau\tau'}}. \tag{CBC}$$

The Holley criterion [Hol74] also allows to show stochastic ordering of the measures in the parameter β: if $\beta_1 \ge \beta_2$, then, for any boundary conditions $\eta_\tau, \eta_{\tau\tau'}$,

$$\mathtt{ATRC}_{\Omega,\beta_1}^{\eta_\tau,\eta_{\tau\tau'}} \ge_{\text{st}} \mathtt{ATRC}_{\Omega,\beta_2}^{\eta_\tau,\eta_{\tau\tau'}}. \tag{MON}$$

See [Gri06, Lemma 11.14] for a proof. In fact, this proof gives a little more. Indeed, it consists of checking inequalities for quantities that are continuous functions of $(\beta_i J, \beta_i U)$ and the inequalities are strict when $\beta_1 > \beta_2$. This implies that the ATRC measure with parameters in a small neighbourhood of $(\beta_1 J, \beta_1 U)$ dominates that with parameters in a small neighbourhood of $(\beta_2 J, \beta_2 U)$. More precisely, for $(x, y) \in \mathbb{R}^2$, define $B_r(x, y)$ as the Euclidian ball of radius r centred at (x, y). If $\beta_1 > \beta_2$, then there exists $\varepsilon > 0$ such that for any $(J_1, U_1) \in B_\varepsilon(\beta_1 J, \beta_1 U)$ and $(J_2, U_2) \in B_\varepsilon(\beta_2 J, \beta_2 U)$,

$$\mathtt{ATRC}_{\Omega,J_1,U_1}^{\eta_\tau,\eta_{\tau\tau'}} \ge_{\text{st}} \mathtt{ATRC}_{\Omega,J_2,U_2}^{\eta_\tau,\eta_{\tau\tau'}}. \tag{MON+}$$

This extension will be useful for our proof of Theorem 7.

Domain Markov property. As in the standard FK-percolation, one can interpret a configuration outside of a subdomain as boundary conditions. Indeed, let $\Omega \subset \Delta$ be two finite subgraphs of $\mathbb{L}$ and $\xi \in \{0,1\}^{E_\Delta \setminus E_\Omega}$ a percolation configuration on $\Delta \setminus \Omega$. Given boundary conditions η on Δ, define a partition $\eta \cup \xi$ of $\partial\Omega$ by first identifying vertices belonging to the same element of η and then identifying vertices belonging to the same cluster of ξ. Then, the following domain Markov property holds:

$$\mathtt{ATRC}_{\Delta,\beta}^{\eta_\tau,\eta_{\tau\tau'}}(\cdot \mid (\omega_\tau,\omega_{\tau\tau'})_{|\Delta\setminus\Omega} = (\xi_\tau,\xi_{\tau\tau'})_{|\Delta\setminus\Omega}) = \mathtt{ATRC}_{\Omega,\beta}^{\eta_\tau\cup\xi_\tau,\eta_{\tau\tau'}\cup\xi_{\tau\tau'}}(\cdot). \tag{DMP}$$

Thus, by (CBC), for any increasing sequence of subgraphs $\Omega_k \nearrow \mathbb{L}$, the measures $\mathtt{ATRC}_{\Omega_k,\beta}^{1,1}$ form a stochastically decreasing sequence. Thus, the weak (or local) limit exists and is unique, by standard arguments. Denote it by $\mathtt{ATRC}_{\beta}^{1,1}$. Define $\mathtt{ATRC}_{\beta}^{0,0}$ analogously. We write $\mathtt{ATRC}_{J,U}^{1,1}$ and $\mathtt{ATRC}_{J,U}^{0,0}$ for the corresponding measures with $\beta = 1$.

Coupling between ATRC and AT. For $X, Y \subset \mathbb{L}$ and a percolation configuration $\omega \in \{0,1\}^{E_\Omega}$, we define $X \xleftrightarrow{\omega} Y$ as an event that X and Y are linked by a path of open edges in ω. If $X = \{x\}$ and $Y = \{y\}$, we simply write $x \xleftrightarrow{\omega} y$. We also use the notation $x \xleftrightarrow{\omega} \infty$ for the event of x belonging to an infinite connected component of ω.

The key property of the ATRC is that connectivities in it describe correlations in the AT model [PV97, Proposition 3.1]: for $\beta > 0$ and any finite subgraph $\Omega \subset \mathbb{L}$ with $x, y \in V_\Omega$,

$$\begin{aligned}
\langle \tau_x\tau_y\rangle_{\Omega,\beta} &= \mathtt{ATRC}_{\Omega,\beta}^{0,0}(x \xleftrightarrow{\omega_\tau} y), & \langle \tau_x\tau'_x\tau_y\tau'_y\rangle_{\Omega,\beta} &= \mathtt{ATRC}_{\Omega,\beta}^{0,0}(x \xleftrightarrow{\omega_{\tau\tau'}} y),\\
\langle \tau_x\rangle_{\Omega,\beta}^{+,+} &= \mathtt{ATRC}_{\Omega,\beta}^{1,1}(x \xleftrightarrow{\omega_\tau} \partial\Omega), & \langle \tau_x\tau'_x\rangle_{\Omega,\beta}^{+,+} &= \mathtt{ATRC}_{\Omega,\beta}^{1,1}(x \xleftrightarrow{\omega_{\tau\tau'}} \partial\Omega).
\end{aligned} \tag{2.12}$$

By the classical Burton–Keane argument [BK89], infinite clusters in ω_τ and $\omega_{\tau\tau'}$ are unique (if they exist). Then, (2.12) and (FKG) imply that $\beta_c^{\tau,f}$ and $\beta_c^{\tau\tau',f}$ are percolation thresholds for ω_τ and $\omega_{\tau\tau'}$ under $\mathtt{ATRC}_{\beta}^{0,0}$. Similarly, the same holds for β_c^{τ} and $\beta_c^{\tau\tau'}$ under $\mathtt{ATRC}_{\beta}^{1,1}$.

Dual ATRC. Define the dual lattice $\mathbb{L}^* := \mathbb{L} + (1,0)$. For each edge e of $\mathbb{L}$, there is a unique edge of $\mathbb{L}^*$ that intersects it: call this edge dual to e and denote it by e^*. Denote by $E_{\mathbb{L}}$ and $E_{\mathbb{L}^*}$ the sets of edges of $\mathbb{L}$ and $\mathbb{L}^*$, respectively. Given a percolation configuration $\omega \in \{0,1\}^{E_{\mathbb{L}}}$, we define its dual configuration $\omega^* \in \{0,1\}^{E_{\mathbb{L}^*}}$ by setting

$$\omega^*(e^*) := 1 - \omega(e).$$

For an ATRC configuration $(\omega_\tau, \omega_{\tau\tau'}) \in \{0,1\}^{E_{\mathbb{L}}} \times \{0,1\}^{E_{\mathbb{L}}}$, we define its dual $(\hat\omega_\tau, \hat\omega_{\tau\tau'}) \in \{0,1\}^{E_{\mathbb{L}^*}} \times \{0,1\}^{E_{\mathbb{L}^*}}$ in the following way,

$$\hat\omega_\tau := \omega_{\tau\tau'}^* \qquad \text{and} \qquad \hat\omega_{\tau\tau'} := \omega_\tau^*. \tag{2.13}$$

We want to emphasize that we are not considering two standard dual percolation configurations but we also swap the order of τ-edges and $\tau\tau'$-edges.

The measures $\mathtt{ATRC}_{J,U}^{0,0} =: \mathtt{ATRC}_{\mathbb{L},J,U}^{0,0}$ and $\mathtt{ATRC}_{J,U}^{1,1} =: \mathtt{ATRC}_{\mathbb{L},J,U}^{1,1}$ on $\mathbb{L}$ can be defined on $\mathbb{L}^*$ in the same manner, and we denote them by $\mathtt{ATRC}_{\mathbb{L}^*,J,U}^{0,0}$ and $\mathtt{ATRC}_{\mathbb{L}^*,J,U}^{1,1}$, respectively. Recall the mapping $(J,U) \mapsto (J^*,U^*)$ defined by (2.4) and note its properties: it is continuous, an involution, identity on the self-dual line (SD), sends every point above (SD) to a point below (SD). The pushforward of the ATRC measure under the duality transformation is also an ATRC measure with the dual parameters:

Lemma 2.2.2 (Prop 3.2 in [PV97]). *Let $0 < J < U$. Let $(\omega_\tau, \omega_{\tau\tau'})$ be distributed according to $\mathtt{ATRC}_{\mathbb{L},J,U}^{1,1}$. Then, the distribution of $(\hat\omega_\tau, \hat\omega_{\tau\tau'})$ is given by $\mathtt{ATRC}_{\mathbb{L}^*,J^*,U^*}^{0,0}$.*

2.2.2 Proof of Theorem 7

We first show that $\mathtt{ATRC}^{0,0}_{J,U}$ and $\mathtt{ATRC}^{1,1}_{J,U}$ coincide for almost every (J,U):

Lemma 2.2.3. *There exists $D \subseteq \{(J,U) \in \mathbb{R}^2 : 0 < J < U\}$ with Lebesgue measure 0 such that, for any $(J,U) \in D^c$, one has*

$$\mathtt{ATRC}^{0,0}_{J,U} = \mathtt{ATRC}^{1,1}_{J,U}. \tag{2.14}$$

Remark 2.2.4. Note that, by (CBC), equation (2.14) implies equality of all Gibbs measures.

The proof goes by applying a version of the classical convexity argument to the marginals of ATRC on ω_τ and $\omega_{\tau\tau'}$, see Appendix 2.D for more details. We are ready to prove part (i) of Theorem 7. Recall that $B_r(x,y)$ is the Euclidean ball of radius r centred at (x,y).

Proof of Theorem 7(i). Fix $J < U$. By (CBC), we have $\beta_c^\tau \leq \beta_c^{\tau,f}$. Assume for contradiction that the inequality is strict, and take $\beta \in (\beta_c^\tau, \beta_c^{\tau,f})$. Then, by (MON+), there exists $\varepsilon > 0$ such that, for any $(J',U') \in B_\varepsilon(\beta J, \beta U)$,

$$\mathtt{ATRC}^{0,0}_{J',U'}(0 \xleftrightarrow{\omega_\tau} \infty) = 0 \quad \text{and} \quad \mathtt{ATRC}^{1,1}_{J',U'}(0 \xleftrightarrow{\omega_\tau} \infty) > 0.$$

This contradicts Lemma 2.2.3. □

Denote by $\mathcal{H}_n^\tau$ (resp. $\mathcal{H}_n^{\tau\tau'}$) the event that the box $[0,2n-1]\times[0,2n-1]$ is crossed horizontally by ω^τ (resp. $\omega^{\tau\tau'}$). Note that the complement of $\mathcal{H}_n^\tau$ is the event that the box $[0,2n-1]\times[0,2n-1]$ is crossed vertically by the dual ω_τ^*. The following lemma states a standard characterisation of non-transition points. It is a consequence of Lemma 2.2.3 and sharpness of the phase transition in the ATRC. The latter can be derived using a robust approach going through the OSSS inequality [DRT19]; see Appendix 2.A for more details.

Lemma 2.2.5. *Let $0 < J < U$. Then, $(J,U) \in \gamma_\tau$ if and only if, for any $\varepsilon > 0$, there exist points (J_0,U_0) and (J_1,U_1) in $B_\varepsilon(J,U)$, such that, as $n \to \infty$,*

$$\mathtt{ATRC}^{0,0}_{J_0,U_0}[\mathcal{H}_n^\tau] \to 1 \quad \textit{and} \quad \mathtt{ATRC}^{1,1}_{J_1,U_1}[\mathcal{H}_n^\tau] \to 0. \tag{2.15}$$

The same holds also when τ is replaced everywhere by $\tau\tau'$.

Proof. Assume $(J,U) \in \gamma_\tau$. By sharpness, $\mathtt{ATRC}^{1,1}_{\beta J,\beta U}[\mathcal{H}_n^\tau] \to 0$, for any $\beta \in (0,1)$. Also, it is standard that part (i) of Theorem 7 and Zhang's argument imply that $\mathtt{ATRC}^{0,0}_{\beta J,\beta U}[\mathcal{H}_n^\tau] \to 1$, for any $\beta > 1$. This gives one direction of the statement.

To show the reverse, assume first that $\beta_c^\tau := \beta_c^\tau(J,U) > 1$ and take $\beta \in (1, \beta_c^\tau)$. By sharpness, $\mathtt{ATRC}^{1,1}_{\beta J,\beta U}[\mathcal{H}_n^\tau] \to 0$ and, by (MON+), the same holds in some neighbourhood of (J,U). The case $\beta_c^\tau < 1$ is analogous. □

Proof of Theorem 7(ii). Let $(J,U) \in \gamma_\tau$ and $\varepsilon > 0$. Since the duality mapping is a continuous involution, we can find $\delta > 0$ such that the image of $B_\delta(J,U)$ is inside $B_\varepsilon(J^*,U^*)$. By Lemma 2.2.5, we get (2.15) for some (J_0,U_0) and (J_1,U_1) in $B_\delta(J,U)$. By duality and symmetry,

$$\mathtt{ATRC}^{1,1}_{J_0^*,U_0^*}[\mathcal{H}_n^{\tau\tau'}] \to 0 \quad \text{and} \quad \mathtt{ATRC}^{0,0}_{J_1^*,U_1^*}[\mathcal{H}_n^{\tau\tau'}] \to 1 \quad \text{as } n \to \infty.$$

Since (J_0^*,U_0^*) and (J_1^*,U_1^*) are in $B_\varepsilon(J^*,U^*)$, Lemma 2.2.5 implies that $(J^*,U^*) \in \gamma_{\tau\tau'}$.

Proving that $(J^*,U^*) \in \gamma_{\tau\tau'}$ implies $(J,U) \in \gamma_\tau$ is analogous. □

2.2.3 $\varphi_\beta(S)$ argument: proof of Theorem 6

Following [Sim80, Lie80] (see also [DT16a]), for a finite subgraph $S \subset \mathbb{L}$ containing 0, define

$$\varphi_\beta(S) = |\partial S| \cdot \mathtt{ATRC}^{1,1}_{S,\beta}(0 \xleftrightarrow{\omega_\tau} \partial S).$$

The following lemma states a key property of $\varphi_\beta(S)$: if it is less than 1 for some S, then ω_τ exhibits exponential decay of connection probabilities. This finite-size criterion allows to use continuity of $\varphi_\beta(S)$ and Proposition 2.1.1 to extend exponential decay of ω_τ beyond (SD). Let Λ_k be the box of size k in $\mathbb{L}$, that is $\Lambda_k = \{u \in \mathbb{L} : \|u\|_1 \leq 2k\}$.

Lemma 2.2.6. *Let $\beta > 0$. Assume that $\varphi_\beta(S) < 1$, for some finite subgraph $S \subset \mathbb{L}$ containing 0. Then, there exists $c := c(\beta, S) > 0$ such that*

$$\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n) \leq e^{-cn}.$$

Remark 2.2.7. 1) Note that the boundary conditions are *free* in [DT16a] and *wired* in our case. The reason is that an analogue of Lemma 2.2.6 is proven in [DT16a] via a modified Simon–Lieb inequality [Lie80, Sim80] for the Ising model. Such inequalities are not available in our case. While Lemma 2.2.6 under wired conditions is elementary, proving exponential decay under wired boundary conditions in finite volume (Proposition 2.1.1) is the subject of Sections 2.3-2.6.

2) We point an interested reader to the work [DS87] that introduces a finite-size criterion for the completely analytical interactions.

Proof of Lemma 2.2.6. Let $S \subset \mathbb{L}$ be a finite subgraph containing 0 such that $\varphi_\beta(S) < 1$ and let k be such that $S \subset \Lambda_k$. If $0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}$, then $\partial S \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}$ and $0 \xleftrightarrow{\omega_\tau} \partial S$.

By (CBC) and the union bound,

$$\begin{aligned}
\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}) &\leq \mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial S \,|\, \partial S \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}) \cdot \mathtt{ATRC}^{1,1}_{\beta}(\partial S \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}) \\
&\leq \mathtt{ATRC}^{1,1}_{S,\beta}(0 \xleftrightarrow{\omega_\tau} \partial S) \cdot \sum_{x \in \partial S} \mathtt{ATRC}^{1,1}_{\beta}(x \xleftrightarrow{\omega_\tau} \partial\Lambda_{nk}) \\
&\leq \mathtt{ATRC}^{1,1}_{S,\beta}(0 \xleftrightarrow{\omega_\tau} \partial S) \cdot |\partial S| \mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{(n-1)k}) \\
&= \varphi_\beta(S) \mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{(n-1)k}).
\end{aligned}$$

where we also used translation invariance of $\mathtt{ATRC}^{1,1}_{\beta}$ and that $S \subset \Lambda_k$.

Since $\varphi_\beta(S) < 1$, we get that $\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \Lambda_{nk})$ decays exponentially fast in n by induction. Since for any $m \in \mathbb{N}$, there exists n such that $m \in [nk, (n+1)k]$, we get that $\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_m)$ decays exponentially fast in m. □

We are now ready to derive Theorem 6 from Proposition 2.1.1 and Theorem 7.

Proof of Theorem 6. Fix $J < U$. By Proposition 2.1.1 and (2.12), we can take $n > 1$ such that $\varphi_{\beta_{\mathrm{sd}}}(\Lambda_n) < 1$. Since the function $\beta \mapsto \varphi_\beta(\Lambda_n)$ is increasing and continuous, there exists $\varepsilon = \varepsilon(J, U) > 0$, such that $\varphi_{\beta'}(\Lambda_n) < 1$, for all $\beta' < \beta_{\mathrm{sd}} + \varepsilon$. The latter implies exponential decay by Lemma 2.2.6 and hence $\beta_c^\tau > \beta_{\mathrm{sd}}$. In other words, all points on γ^τ are strictly above the self-dual curve. Hence their images under the duality mapping (2.4) are strictly below the self-dual curve. By Theorem 7, these points are exactly the points of $\gamma^{\tau\tau'}$ and this finishes the proof. □

Remark 2.2.8. Standard arguments similar to the proof of Lemma 2.2.6 show that

$$c_\beta = \lim_{n\to\infty} -\frac{1}{n} \log \mathtt{ATRC}^{1,1}_{\Lambda_n,\beta}[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n]$$

exists and is right-continuous in β, which gives another way to argue that the exponential decay from Proposition 2.1.1 extends to an open neighbourhood of the self-dual line (SD).

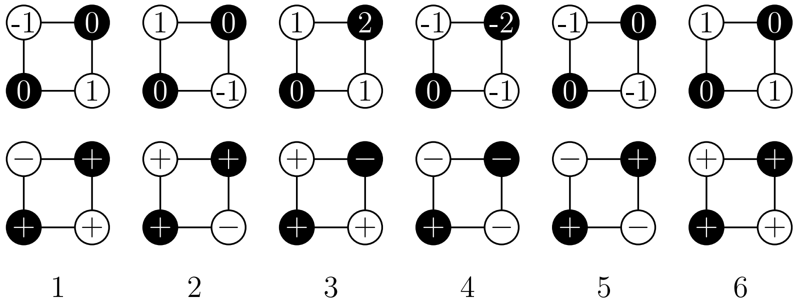


Figure 2.2: *Top:* The height representation of the six-vertex model in the four vertices of a unit square in $\mathbb{Z}^2$, normalized to equal 0 at the lower left vertex. *Bottom:* The spin representation is derived from the heights by setting the spin state at each vertex to $+1$ (resp. -1) if the height modulo 4 equals $0, 1$ (resp. $2, 3$).

2.3 Models, couplings and required input

In this section, we introduce the six-vertex model together with its height and spin representations. We also state couplings of this model with the ATRC model and FK-percolation that will be crucial to our arguments. A combination of these two couplings has been made explicit recently in the work of Peled and the third author [GP23] and we summarize the results of that work that we will rely on.

2.3.1 Graph notation

Dual subgraphs and configurations. For a finite subgraph Ω of $\mathbb{L}$, define its dual graph Ω^* in $\mathbb{L}^*$ formed by edges dual to the edges of Ω. As for primal graphs, we denote the sets of its vertices and edges by V_{Ω^*} and E_{Ω^*}. The boundary $\partial\Omega^*$ is defined in the same way as for subgraphs of $\mathbb{L}$. Given a percolation configuration $\omega \in \{0,1\}^{E_\Omega}$, its dual configuration $\omega^* \in \{0,1\}^{E_{\Omega^*}}$ is defined by $\omega^*(e^*) = 1 - \omega(e)$, $e \in E_\Omega$.

Domains in $\mathbb{L}$. A finite induced subgraph Ω of $\mathbb{L}$ (or $\mathbb{L}^*$) is a *domain* if it is given by vertices and edges within a simple cycle (including the cycle itself). We denote the set of vertices on the surrounding cycle by $\overline{\partial}\Omega$ and call it the *domain-boundary* of Ω. The set of edges on $\overline{\partial}\Omega$ is called *edge-boundary* of Ω and is denoted by $E_{\overline{\partial}\Omega}$.

Domains in $\mathbb{Z}^2$. Given a domain Ω in $\mathbb{L}$, let $\mathcal{D}_\Omega$ be the subgraph of $\mathbb{Z}^2$ induced by vertices in $\Omega \cup \Omega^*$. We call such a domain an *even domain* of $\mathbb{Z}^2$ (see Figure 2.1). Define $\partial^2\mathcal{D}_\Omega := \overline{\partial}\Omega \cup \partial\Omega^*$. Given a domain Ω' on $\mathbb{L}^*$, we define $\mathcal{D}_{\Omega'}$ in the same manner and call it an *odd domain* of $\mathbb{Z}^2$. In particular, odd domains are obtained from even domains by shift by one to the right. We emphasize that we only consider even and odd domains.

Remark 2.3.1. These restrictions stem from the coupling to FK-percolation (Section 2.3.6) that requires two layers of boundary in $\mathcal{D}_\Omega$: inner layer $\overline{\partial}\Omega$ and outer layer $\partial\Omega^*$.

2.3.2 Six-vertex model and its representations

In this section, we define the *six-vertex* model [Pau35] (more precisely, the *F-model*) and its different representations in terms of spins [Rys63] and height functions. For the whole subsection, fix a domain Ω in $\mathbb{L}$ (or $\mathbb{L}^*$) and its corresponding even (odd) domain $\mathcal{D} = \mathcal{D}_\Omega$ in $\mathbb{Z}^2$.

Height functions. A function $h : \mathcal{D} \to \mathbb{Z}$ is called a *height function* (of the six-vertex model) if it satisfies the *ice rule*:

- $|h(u) - h(v)| = 1$ whenever u, v are connected by an edge in $\mathbb{Z}^2$,
- h takes even values on $\mathcal{D} \cap \mathbb{L}$.

This constraint implies that, for each edge e of Ω, the value of h is constant at the endpoints of e or at the endpoints of e^*. Up to an even additive constant, this leaves six local possibilities (*types*), where types 5 and 6 correspond to h taking constant values along both e and e^*, see Figure 2.2.

The *six-vertex height function measure* on $\mathcal{D}$ with parameters $c, c_b > 0$ and boundary conditions $t \in \mathbb{Z}^{\partial^2\mathcal{D}}$ is supported on height functions $h \in \mathbb{Z}^{\mathcal{D}}$ that coincide with t on $\partial^2\mathcal{D}$ and is given by

$$\mathtt{HF}^{t;c_b}_{\mathcal{D},c}[h] = \frac{1}{Z} c^{n^{\mathrm{i}}_{5,6}(h)} c_b^{n^{\mathrm{b}}_{5,6}(h)}, \tag{2.16}$$

where $Z = Z(\mathcal{D}, c, c_b, t)$ is a normalizing constant and $n^{\mathrm{i}}_{5,6}(h)$ (resp. $n^{\mathrm{b}}_{5,6}(h)$) is the number of edges of type 5 or 6 of $E_\Omega \setminus E_{\overline{\partial}\Omega}$ (resp. $E_{\overline{\partial}\Omega}$). When $c = c_b$, we recover the standard six-vertex probability measure that will be denoted by $\mathtt{HF}^{t}_{\mathcal{D},c}$. We write $\mathtt{HF}^{2n,2n+1;c_b}_{\mathcal{D},c}$ for $\mathtt{HF}^{t;c_b}_{\mathcal{D},c}$ with $t \in \{2n, 2n+1\}^{\partial^2\mathcal{D}}$. We define $\mathtt{HF}^{2n,2n-1;c_b}_{\mathcal{D},c}$ analogously.

Finally, we define $\mathtt{HF}^{2n,2n\pm1;c_b}_{\mathcal{D},c}$ as the probability measure given by (2.16) and supported on all height functions in $\mathbb{Z}^{\mathcal{D}}$ that have a fixed value $2n$ on $\partial^2\mathcal{D} \cap \mathbb{L}$. Note that the value on $\partial^2\mathcal{D} \cap \mathbb{L}^*$ is not fixed in this case, so the conditions can be viewed as *semi-free*.

Spin representation. Given a height function $h \in \mathbb{Z}^{\mathcal{D}}$, define $\sigma = \sigma(h) \in \{\pm 1\}^{\mathcal{D}}$ by

$$\sigma(u) = \begin{cases} 1 & \text{if } h(u) \equiv 0, 1 \pmod 4, \\ -1 & \text{otherwise.} \end{cases}$$

The *six-vertex spin measures* $\mathtt{Spin}^{+,+;c_b}_{\mathcal{D},c}$, $\mathtt{Spin}^{+,+}_{\mathcal{D},c}$, $\mathtt{Spin}^{+,\pm}_{\mathcal{D},c}$ are defined as the push-forwards of $\mathtt{HF}^{0,1;c_b}_{\mathcal{D},c}$, $\mathtt{HF}^{0,1}_{\mathcal{D},c}$, $\mathtt{HF}^{0,\pm1}_{\mathcal{D},c}$ under this mapping. The spin measures are supported on all spin configurations $\sigma \in \{\pm1\}^{\mathcal{D}}$ with the following restrictions: $\sigma_{|\partial^2\mathcal{D}} \equiv 1$ under $\mathtt{Spin}^{+,+;c_b}_{\mathcal{D},c}$ and $\mathtt{Spin}^{+,+}_{\mathcal{D},c}$; $\sigma_{|\partial^2\mathcal{D}\cap\mathbb{L}} \equiv 1$ under $\mathtt{Spin}^{+,\pm}_{\mathcal{D},c}$.

2.3.3 From Ashkin–Teller to six-vertex

In this section, we describe the connection between the self-dual Ashkin–Teller model on a domain Ω in $\mathbb{L}$ and the spin representation of the six-vertex model on the corresponding even domain $\mathcal{D}_\Omega$ in $\mathbb{Z}^2$. This relation has first been noticed in [Fan72a] comparing their critical properties, it was made explicit in [Fan72b, Weg72] (see also [HDJS13]), and was upgraded to a coupling in [GP23, Lis22] (we note that [Lis22] treats a more general case of two interacting Potts models). We consider two types of boundary conditions that will play an important role in proving Proposition 2.1.1.

Let Ω be a domain of $\mathbb{L}$, $J < U$ be parameters. We will consider the ATRC measures defined in Section 2.2.1 with boundary conditions on $\overline{\partial}\Omega$ rather than $\partial\Omega$. We write $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,J,U}$ for the corresponding ATRC measure where $\mathbb{1}$ refers to the wired boundary condition on $\overline{\partial}\Omega$.

Let $\eta_\tau, \eta_{\tau\tau'}$ be boundary conditions on $\partial\Omega$ or $\overline{\partial}\Omega$. Consider the marginal of $\mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{\Omega,J,U}$ on ω_τ: this is the probability measure supported on $\{0,1\}^{E_\Omega}$ and defined by

$$\mu^{\eta_\tau,\eta_{\tau\tau'}}_{\Omega,J,U}(\xi) := \mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{\Omega,J,U}(\{\omega_\tau = \xi\}).$$

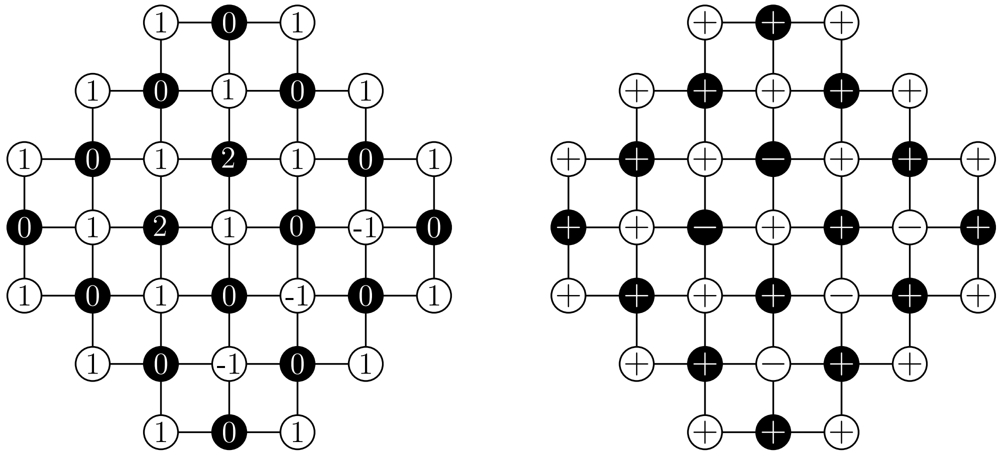


Figure 2.3: *Left:* Height function with $0, 1$ boundary conditions. *Right:* Its spin representation is given by $\sigma^\bullet$ on $\mathbb{L}$ (black circles) and σ° on $\mathbb{L}^*$ (white circles).

Also, given $\sigma \in \{\pm 1\}^{\mathcal{D}_\Omega}$, we write $\sigma^\bullet$ and σ° for the restrictions of σ to Ω and Ω^*, respectively (see Fig 2.3). We define the sets of disagreement edges:

$$E_{\sigma^\bullet} := \{uv \in E_\Omega : \sigma^\bullet(u) \neq \sigma^\bullet(v)\} \quad \text{and} \quad E_{\sigma^\circ} := \{u^*v^* \in E_{\Omega^*} : \sigma^\circ(u^*) \neq \sigma^\circ(v^*)\}.$$

Finally, we define the *compatibility* relation on pairs of $\sigma^\bullet \in \{\pm 1\}^\Omega$ and $\omega \in \{0,1\}^{E_\Omega}$:

$$\sigma \perp \omega \quad \text{if and only if} \quad \sigma(u) = \sigma(v), \text{ for any } uv \in \omega.$$

The compatibility relation on pairs of $\sigma^\circ \in \{\pm 1\}^{\Omega^*}$ and $\omega \in \{0,1\}^{E_{\Omega^*}}$ is defined similarly. The following is a consequence of [GP23, Proposition 8.1] and a remark after it, or may be proved along the same lines:

Proposition 2.3.2. *Let $0 < J < U$ be a point on the self-dual line (SD) and $c = \coth 2J$.*
1) If Ω is a domain in $\mathbb{L}$, then we can couple $\sigma \sim \mathtt{Spin}^{+,+}_{\mathcal{D}_\Omega, c}$ and $\omega_\tau \sim \mu^{0,\mathbb{1}}_{\Omega,J,U}$ by

$$\mathbf{P}[\sigma, \omega_\tau] \propto \left(\tfrac{1}{c-1}\right)^{|\omega_\tau| + |E_{\sigma^\bullet}|} \mathbb{1}_{\{\sigma^\bullet \perp \omega_\tau,\, \sigma^\circ \perp \omega^*_\tau\}} \mathbb{1}_{\{\sigma^\bullet \equiv + \text{ on } \overline{\partial}\Omega,\, \sigma^\circ \equiv + \text{ on } \partial\Omega^*\}}.$$

Thus, σ° is obtained by assigning $+1$ to the clusters of ω^*_τ that intersect $\partial\Omega^*$ and assigning ± 1 uniformly independently to all other clusters.
2) If Ω^* is a domain in $\mathbb{L}^*$, then we can couple $\sigma \sim \mathtt{Spin}^{+,\pm}_{\mathcal{D}_{\Omega^*}, c}$ and $\omega_\tau \sim \mu^{1,1}_{\Omega,J,U}$ by

$$\mathbf{P}[\sigma, \omega_\tau] \propto \left(\tfrac{1}{c-1}\right)^{|\omega_\tau| + |E_{\sigma^\bullet}|} \mathbb{1}_{\{\sigma^\bullet \perp \omega_\tau,\, \sigma^\circ \perp \omega^*_\tau\}} \mathbb{1}_{\{\sigma^\bullet \equiv + \text{ on } \partial\Omega\}}.$$

Remark 2.3.3. Part 1) of Proposition 2.3.2 is a special case of [GP23, Proposition 8.1] while part 2) may be proved in the same way. The proof relies on the following identity:

$$k^1(\omega) - k(\omega^*) = |\omega^*| + \mathrm{const}(\Omega),$$

where $\omega = \omega^*_\tau$ for 1) and $\omega = \omega_\tau$ for 2). This follows from Euler's formula using that either Ω or Ω^* is a domain and our definition of the domain-boundary.

Corollary 2.3.4. *In the setting of part 1) of Proposition 2.3.2, take $(\sigma^\bullet, \sigma^\circ) \sim \mathtt{Spin}^{+,+}_{\mathcal{D}_\Omega,c}$. Sample a percolation configuration ω on E_Ω as follows independently at each edge e: if the endpoints of e^* have opposite values in σ°, then $\omega_e = 1$; if the endpoints of e have opposite values in $\sigma^\bullet$, then $\omega_e = 0$; if σ° agrees on e^* and $\sigma^\bullet$ agrees on e, then*

$$\mathbf{P}(\omega_e = 1) = \tfrac{1}{c}. \tag{2.17}$$

Then, the law of ω is given by $\mu^{0,\mathbb{1}}_{\Omega,J,U}$.

2.3.4 Input from the six-vertex model

In this section, we mention basic properties of six-vertex measures and state some results from [GP23]. The following proposition is a combination of Theorem 2.2, Proposition 6.1 and Lemma 6.2 in [GP23]. We remark that we only consider even and odd domains in $\mathbb{Z}^2$.

For $u \in \mathbb{L}$, $S \subset \mathbb{L}$, define $u \xleftrightarrow{h\neq 0} S$ to be the event that u is connected (in $\mathbb{L}$) to S by a path of heights different from 0. We similarly define $u^* \xleftrightarrow{h\neq 1} S^*$ for $u^* \in \mathbb{L}^*$, $S^* \subset \mathbb{L}^*$.

Proposition 2.3.5 ([GP23])**.** *Fix $c > 2$, and let λ be the unique positive solution of $c = e^{\lambda/2} + e^{-\lambda/2}$. Then, for any sequence of domains $\mathcal{D}_k \nearrow \mathbb{Z}^2$, the measures $\mathtt{HF}^{0,1}_{\mathcal{D}_k,c}$ and $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_k,c}$ converge weakly to the same limit that we denote by $\mathtt{HF}^{0,1}_c$. Moreover, $\mathtt{HF}^{0,1}_c$-a.s. exist unique infinite clusters in $\mathbb{L}$ of height 0 and in $\mathbb{L}^*$ of height 1. Finally, clusters of other heights are exponentially small: for some $\alpha > 0$ uniform in $n \geq 1$, $u \in \mathbb{L}$, $u^* \in \mathbb{L}^*$,*

$$\mathtt{HF}^{0,1}_c\left[u \xleftrightarrow{h\neq 0} u + \partial\Lambda_n\right] < e^{-\alpha n} \quad \text{and} \quad \mathtt{HF}^{0,1}_c\left[u^* \xleftrightarrow{h\neq 1} u^* + \partial\Lambda_n\right] < e^{-\alpha n}.$$

Let us emphasize that, while existence of subsequential limits is a straightforward consequence of discontinuity of the phase transition in FK-percolation, the ordering of *both* even and odd heights is non-trivial. This also implies that the weak limit of $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_k,c}$ remains the same, whether it is taken along even or odd domains. Analogously, for any $n \in \mathbb{Z}$, one obtains limit measures $\mathtt{HF}^{2n,2n+1}_c$ (resp. $\mathtt{HF}^{2n,2n-1}_c$) of $\mathtt{HF}^{2n,2n+1}_{\mathcal{D}_k,c}$ (resp. $\mathtt{HF}^{2n,2n-1}_{\mathcal{D}_k,c}$) satisfying the corresponding properties.

Since the modulo 4 mapping (Section 2.3.2) is local, Propositon 2.3.5 directly implies the following corollary.

Corollary 2.3.6. *Fix $c > 2$, and let λ be the unique positive solution of $c = e^{\lambda/2} + e^{-\lambda/2}$. Then, for any sequence of domains $\mathcal{D}_k$ increasing to $\mathbb{Z}^2$, the measures $\mathtt{Spin}^{+,+}_{\mathcal{D}_k,c}$ and $\mathtt{Spin}^{+,+;e^{\lambda/2}}_{\mathcal{D}_k,c}$ converge weakly to some $\mathtt{Spin}^{+,+}_c$, which is independent of the sequence $\mathcal{D}_k$.*

The height function measures admit useful monotonicity properties and correlation inequalities when $c, c_b \geq 1$, see [GP23, Proposition 5.1].

Proposition 2.3.7. *Let $\mathcal{D}$ be a domain in $\mathbb{Z}^2$, and let $c, c_b \geq 1$. Then, for any boundary condition t, the measure $\mathtt{HF}^{t;c_b}_{\mathcal{D},c}$ is monotone (2.11) and satisfies the FKG inequality (2.10). In particular, if $t \leq t'$, then $\mathtt{HF}^{t;c_b}_{\mathcal{D},c}$ is stochastically dominated by $\mathtt{HF}^{t';c_b}_{\mathcal{D},c}$.*

It has been established in [GP23, Theorem 2.5] that the marginals of $\mathtt{Spin}^{+,+}_{\mathcal{D}_k,c}$ on $\sigma^\bullet$ (resp. σ°) satisfy the FKG inequality with respect to the pointwise order on $\{\pm 1\}^{V_\Omega}$ (resp. $\{\pm 1\}^{V_{\Omega^*}}$). Though [GP23] deals only with boundary conditions specified on the whole boundary, the extension to free or semi-free conditions is straightforward. Indeed, the statement for $\sigma^\bullet$ holds as long as spins σ° on the boundary are not forced to disagree.

Proposition 2.3.8 ([GP23]). *Let Ω be a domain in $\mathbb{L}$, and let $c \geq 1$. The marginals of $\mathtt{Spin}^{+,\pm}_{\mathcal{D}_\Omega,c}$ on $\sigma^\bullet$ and σ° satisfy the FKG inequality* (2.10).

It was also shown in [GP23, Corollary 7.3, Proposition 7.5] that the marginals $\mu^{0,\mathbb{1}}_{\Omega,J,U}$ converge to some infinite-volume state $\mu^{0,1}_{J,U}$ that admits exponential decay of connection probabilities.

Proposition 2.3.9 ([GP23]). *Let $0 < J < U$ be on the self-dual line* (SD) *and Ω_k be a sequence of domains increasing to $\mathbb{L}$. The measures $\mu^{0,\mathbb{1}}_{\Omega_k,J,U}$ converge weakly to some measure $\mu^{0,1}_{J,U}$ on $\{0,1\}^{E_{\mathbb{L}}}$ which is independent of the sequence Ω_k and admits exponential decay of ω_τ-connection probabilities: there exist $M, \alpha > 0$ such that, for any $u, v \in \mathbb{L}$,*

$$\mu^{0,1}_{J,U}[u \xleftrightarrow{\omega_\tau} v] \leq M e^{-\alpha|u-v|}.$$

We sketch the argument given in [GP23].

Sketch of proof of Proposition 2.3.9. The couplings in Proposition 2.3.2 and Corollary 2.4.3 imply convergence of $\mu^{0,\mathbb{1}}_{\Omega,J,U}$, as $\Omega \nearrow \mathbb{L}$, to some $\mu^{0,1}_{J,U}$ that satisfies FKG and is invariant under translations. Thus, it is enough to show that it is exponentially unlikely that ω_τ contains a circuit surrounding Λ_n. Indeed, on this event, the marginal of $\mathtt{Spin}^{+,+}_c$ at vertices in $(\Lambda_n)^*$ is invariant under the spin flip. By Proposition 2.4.2, radii of clusters of minuses have exponential tails and the claim follows. $\square$

We emphasise a difference between Propositions 2.3.9 and 2.1.1: the latter proves exponential decay under the largest boundary conditions and in finite volume. As we saw in Section 2.2.3, this is necessary for the proof of Theorem 6.

2.3.5 FK-percolation

Fortuin–Kasteleyn (FK) percolation [FK72] is an archetypical dependent percolation model. It is well-understood thanks to recent remarkable works; see [Dum17, Gri06] for background. We will transfer some known results from FK-percolation to the six-vertex model via the BKW coupling (Section 2.3.6) and further to the self-dual ATRC via the coupling in Proposition 2.3.2.

Definition 1. Let $\Omega \subset \mathbb{L}$ be a finite subgraph and ξ a partition of $\partial\Omega$. FK-percolation on Ω with parameters $p \in [0,1]$ and $q > 0$ is supported on percolation configurations $\eta \in \{0,1\}^{E_\Omega}$ and is given by

$$\mathtt{FK}^\xi_{\Omega,p,q}(\eta) = \frac{1}{Z} p^{|\eta|}(1-p)^{|E_\Omega|-|\eta|} q^{k^\xi(\eta)},$$

where $Z = Z(\Omega, p, q, \xi)$ is a normalizing constant and $k^\xi(\eta)$ was defined in Section 2.2.1.

The *free* and *wired* FK-percolation measures $\mathtt{FK}^{\mathrm{f}}_{\Omega,p,q}$ and $\mathtt{FK}^{\mathrm{w}}_{\Omega,p,q}$ are defined by free and wired boundary conditions, respectively (as in Section 2.2.1).

We now review several fundamental results about FK-percolation.

Proposition 2.3.10. *Let $p \in [0,1]$, $q > 1$ and $\Omega_k \nearrow \mathbb{L}$ be a sequence of subgraphs. Then, the weak limits of $\mathtt{FK}^{\mathrm{f}}_{\Omega_k,p,q}$ and $\mathtt{FK}^{\mathrm{w}}_{\Omega_k,p,q}$ exist and do not depend on the chosen sequence:*

$$\mathtt{FK}^{\mathrm{f}}_{p,q} := \lim_{k\to\infty} \mathtt{FK}^{\mathrm{f}}_{\Omega_k,p,q} \quad \textit{and} \quad \mathtt{FK}^{\mathrm{w}}_{p,q} := \lim_{k\to\infty} \mathtt{FK}^{\mathrm{w}}_{\Omega_k,p,q}.$$

Moreover, these measures are positively associated, extremal, invariant under translations and satisfy the following ordering, for any finite subgraph $\Omega \subset \mathbb{L}$,

$$\mathtt{FK}^{\mathrm{f}}_{\Omega,p,q} \leq_{\mathrm{st}} \mathtt{FK}^{\mathrm{f}}_{p,q} \leq_{\mathrm{st}} \mathtt{FK}^{\mathrm{w}}_{p,q} \leq_{\mathrm{st}} \mathtt{FK}^{\mathrm{w}}_{\Omega,p,q}.$$

As we will see below, the self-dual AT model with $J < U$ corresponds to FK-percolation with $q > 4$ at $p = p_{\mathrm{sd}}$, where

$$p_{\mathrm{sd}} := \tfrac{\sqrt{q}}{\sqrt{q}+1}.$$

This model is self-dual: if ω has law $\mathtt{FK}^{\mathrm{f}}_{p_{\mathrm{sd}},q}$, then $\omega^*(e^*) := 1 - \omega(e)$ has law $\mathtt{FK}^{\mathrm{w}}_{p_{\mathrm{sd}},q}$.

Theorem 9 ([DGH+21]). *Let $q > 4$. Then, $\mathtt{FK}^{\mathrm{f}}_{p_{\mathrm{sd}},q} \neq \mathtt{FK}^{\mathrm{w}}_{p_{\mathrm{sd}},q}$ and*

(i) $\mathtt{FK}^{\mathrm{w}}_{p_{\mathrm{sd}},q}$(there exists a unique infinite cluster) $= 1$,

(ii) there exists $\alpha > 0$ such that $\mathtt{FK}^{\mathrm{f}}_{p_{\mathrm{sd}},q}(0 \leftrightarrow \partial\Lambda_n) \leq e^{-\alpha n}$, for any $n \geq 1$, .

The second item of this theorem implies *exponential relaxation* at p_{sd}:

Lemma 2.3.11. *Let $q > 4$. Then, there exists $\alpha > 0$ such that, for $n \geq 1$ and any finite subgraph $\Omega \subset \mathbb{L}$ that contains Λ_{2n},*

$$d_{\mathrm{TV}}\left(\mathtt{FK}^{\mathrm{w}}_{\Omega,p_{\mathrm{sd}},q}|_{\Lambda_n}, \mathtt{FK}^{\mathrm{w}}_{p_{\mathrm{sd}},q}|_{\Lambda_n}\right) < e^{-\alpha n},$$

where d_{TV} denotes the total variation distance.

The proof is standard and goes through the monotone coupling; see Appendix 2.B.

2.3.6 Baxter–Kelland–Wu (BKW) coupling

FK-percolation and the six-vertex model were related to each other for the first time by Temperley and Lieb [VL71] on the level of partition functions. BKW [BKW76] turned this relation into a probabilistic coupling when $c > 2$. We follow [GP23] and describe this coupling using a modified boundary coupling constant c_b.

Take $q > 4$, $p = p_{\mathrm{sd}}$. Let $\lambda > 0$ be the unique positive solution to

$$e^{\lambda} + e^{-\lambda} = \sqrt{q}, \text{ and set } c := e^{\lambda/2} + e^{-\lambda/2}.$$

Let Ω be a domain in $\mathbb{L}$ and recall the notations introduced in Section 2.3.1. The measure $\mathtt{FK}^{\overline{\mathrm{w}}}_{\Omega,p_{\mathrm{sd}},q}$ refers to the FK measure with wired boundary conditions on $\overline{\partial}\Omega$. Note that the statements of Proposition 2.3.10 and Lemma 2.3.11 remain valid if we replace w by $\overline{\mathrm{w}}$.

Consider $\eta \sim \mathtt{FK}^{\overline{\mathrm{w}}}_{\Omega,p_{\mathrm{sd}},q}$ and draw loops separating primal and dual clusters within Ω as in Figure 2.4. Given this loop configuration, we define a height function $h \in \mathbb{Z}^{\mathcal{D}_\Omega}$ by:

H1 Set $h = 0$ on $\partial^2\mathcal{D}_\Omega \cap \mathbb{L}$ and $h = 1$ on $\partial^2\mathcal{D}_\Omega \cap \mathbb{L}^*$;

H2 Assign constant heights to primal and dual clusters by going from $\partial^2\mathcal{D}_\Omega$ inside of $\mathcal{D}_\Omega$ and tossing a coin when crossing a loop: the height **increases** by 1 with probability $e^{\lambda}/\sqrt{q}$ and **decreases** by 1 with probability $e^{-\lambda}/\sqrt{q}$, independently of one another.

The following result is classical; see e.g. [GP23, Chapter 3] for a proof in this setup.

Proposition 2.3.12 (BKW coupling). *The resulting height function is distributed according to $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_\Omega,c}$.*

Odd domains. Note that, by symmetry, the whole procedure also works on odd domains with the difference that one needs to replace **H2** by

H2' each time one crosses a loop, the height **decreases** by 1 with probability $e^{\lambda}/\sqrt{q}$ and **increases** by 1 with probability $e^{-\lambda}/\sqrt{q}$, independently of one another.

For a domain Ω' in $\mathbb{L}^*$, this gives a coupling of $\mathtt{FK}^{\overline{\mathrm{w}}}_{\Omega',p_{\mathrm{sd}},q}$ and $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_{\Omega'},c}$.

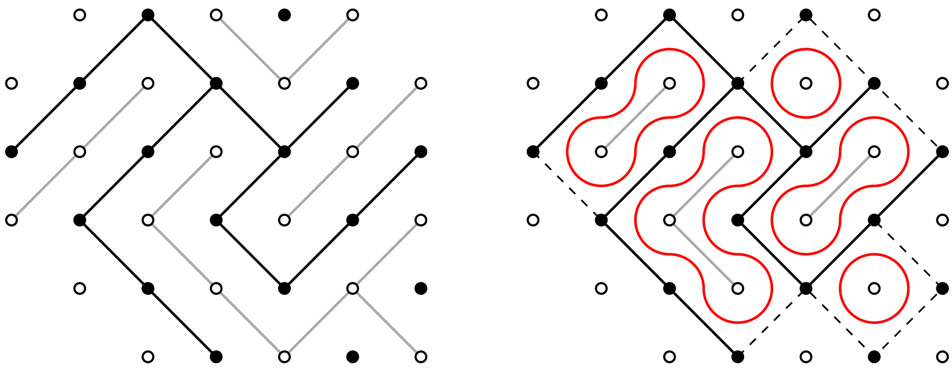


Figure 2.4: *Left:* An edge configuration on the domain $\Omega \subseteq \mathbb{L}$ from Figure 2.1 (in black), and its dual on Ω^* (in gray). *Right:* Loops (in red) separating primal and dual clusters within $\mathcal{D}_\Omega$ after opening all edges in $E_{\overline{\partial}\Omega}$ (dashed).

2.4 Exponential decay for ATRC in finite volume

The goal of this section is to derive exponential decay of connection probabilities for $\mu^{0,1}_{\Omega,J,U}$, which is the marginal of a *finite-volume* ATRC measure on ω_τ, see Section 2.3.3.

Recall that $\Lambda_n = \{u \in \mathbb{L} : \|u\|_1 \leq 2n\}$ is the box of size n in $\mathbb{L}$.

Proposition 2.4.1. *Let $0 < J < U$ be on the self-dual line (SD). There exists $\alpha > 0$ such that, for any $n \geq 1$ and any domain Ω in $\mathbb{L}$ containing Λ_{4n},*

$$\mu^{0,1}_{\Omega,J,U}[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n] \leq e^{-\alpha n}.$$

The proof consists of several steps. We first transfer the exponential relaxation property from FK-percolation (Lemma 2.3.11) to the six-vertex height function (Proposition 2.4.2) and then to the marginal ν_Ω of the ATRC model with modified edge weights on the boundary. Using Proposition 2.3.5, we also show that the limit of ν_Ω is given by $\mu^{0,1}_{J,U}$ (Lemma 2.4.4), and that ν_Ω dominates $\mu^{0,1}_{\Omega,J,U}$ (Lemma 2.4.7). The statement then follows from exponential decay in $\mu^{0,1}_{J,U}$ (Proposition 2.3.9).

2.4.1 Exponential relaxation for height function measures

As in Section 2.3.6, fix $c > 2$ and let $\lambda > 0$ be the unique positive solution of $c = e^{\lambda/2} + e^{-\lambda/2}$. Set $q := (e^\lambda + e^{-\lambda})^2$ and consider $p = p_{\mathrm{sd}}$. For $n \geq 1$, define an even domain $\Delta_{2n} = \mathcal{D}_{\Lambda_n}$.

Proposition 2.4.2. *The convergence of $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D},c}$ towards $\mathtt{HF}^{0,1}_c$ admits exponential relaxation: there exists $\alpha > 0$ such that, for any $n \geq 1$ and any even domain $\mathcal{D} \supset \Delta_{8n}$,*

$$d_{\mathrm{TV}}\left(\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D},c}|_{\Delta_{2n}}, \mathtt{HF}^{0,1}_c|_{\Delta_{2n}}\right) < e^{-\alpha n}. \tag{2.18}$$

Proof. We omit q, p_{sd} from the notation for brevity. We first construct the limiting measure $\mathtt{HF}^{0,1}_c$. Consider $\eta \sim \mathtt{FK}^{\mathrm{w}}$ on $\mathbb{L}$. Using known results about $\eta \sim \mathtt{FK}^{\mathrm{w}}$ (Section 2.3.5) we can sample a height function h as follows. Set $h = 0$ on the unique infinite cluster of η and sample h in its holes according to **H2** in the BKW coupling (Section 2.3.6).

Define $\mathcal{C}_n$ to be the outermost circuit in η surrounding Λ_n and contained in Λ_{2n} (if it does not exist, we set $\mathcal{C}_n := \emptyset$). Stochastic ordering of FK measures, positive association (2.10) and exponential decay in η^* imply existence of $\alpha' > 0$ such that, for any $n \geq 1$ and any domain $\Omega \supset \Lambda_{2n}$,

$$\mathtt{FK}_\Omega^{\overline{\mathrm{w}}}[\mathcal{C}_n \neq \varnothing, \mathcal{C}_n \overset{\eta}{\leftrightarrow} \partial\Omega] \geq \mathtt{FK}^{\mathrm{w}}[\mathcal{C}_n \neq \varnothing, \mathcal{C}_n \overset{\eta}{\leftrightarrow} \infty] > 1 - e^{-\alpha' n}. \tag{2.19}$$

By exponential relaxation of the wired FK measures (Lemma 2.3.11), there exists $\alpha'' > 0$ such that, for any $n \geq 1$ and any domain $\Omega \supset \Lambda_{4n}$,

$$d_{\mathrm{TV}}\left(\mathtt{FK}_\Omega^{\overline{\mathrm{w}}}|_{\Lambda_{2n}}, \mathtt{FK}^{\mathrm{w}}|_{\Lambda_{2n}}\right) < e^{-\alpha'' n}. \tag{2.20}$$

Now, given $\mathcal{C}_n = C$ and $C \overset{\eta}{\leftrightarrow} \infty$, the law of h within C is precisely $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_{\Omega(C)},c}$ where $\Omega(C)$ is the domain in $\mathbb{L}$ induced by the vertices within C (including C). Note that $\Omega(C)$ contains Λ_n, whence $\mathcal{D}_{\Omega(C)}$ contains $\mathcal{D}_{\Lambda_n} = \Delta_{2n}$.

We can also obtain $\mathtt{HF}^{0,1;e^{\lambda/2}}_{\mathcal{D}_{\Omega(C)},c}$ from $\mathtt{FK}_\Omega^{\overline{\mathrm{w}}}$ conditioned on $\mathcal{C}_n = C$ and $C \overset{\eta}{\leftrightarrow} \partial\Omega$ by applying **H1** and **H2**. Together with (2.20) and (2.19), this proves exponential relaxation. □

Recall that the six-vertex spin measures introduced in Sections 2.3.2 and 2.3.4 are the push-forwards of the height function measures under the local modulo 4 mapping.

Corollary 2.4.3. *The convergence of $\mathtt{Spin}^{+,+;e^{\lambda/2}}_{\mathcal{D},c}$ towards $\mathtt{Spin}^{+,+}_c$ admits exponential relaxation: there exists $\alpha > 0$ such that, for any $n \geq 1$ and any even domain $\mathcal{D} \supset \Delta_{8n}$,*

$$d_{\mathrm{TV}}\left(\mathtt{Spin}^{+,+;e^{\lambda/2}}_{\mathcal{D},c}|_{\Delta_{2n}}, \mathtt{Spin}^{+,+}_c|_{\Delta_{2n}}\right) \leq e^{-\alpha n}. \tag{2.21}$$

2.4.2 A modified ATRC marginal

Fix $J < U$ on the self-dual line (SD), take $c = \coth 2J$ and the unique $\lambda > 0$ such that $c = e^{\lambda/2} + e^{-\lambda/2}$. Let Ω be a domain on $\mathbb{L}$ and $\mathcal{D}_\Omega$ be the corresponding even domain on $\mathbb{Z}^2$. Recall the definition of the edge-boundary $E_{\overline{\partial}\Omega}$ in Section 2.3.1. Sample $(\sigma^\bullet, \sigma^\circ)$ from $\mathtt{Spin}^{+,+;e^{\lambda/2}}_{\mathcal{D}_\Omega,c}$. Define ν_Ω as the distribution of $\omega \in \{0,1\}^{E_\Omega}$ sampled independently for each edge e as follows: if the endpoints of e^* have opposite values in σ°, then $\omega_e = 1$; if the endpoints of e have opposite values in $\sigma^\bullet$, then $\omega_e = 0$; if σ° agrees on e^* and $\sigma^\bullet$ agrees on e, then

$$\mathbf{P}(\omega_e = 1) = \begin{cases} e^{-\lambda/2}, & \text{if } e \in E_{\overline{\partial}\Omega}, \\ \frac{1}{c}, & \text{otherwise.} \end{cases} \tag{2.22}$$

We call ν_Ω a *modified ATRC marginal* as it converges to $\mu^{0,1}_{J,U}$ as $\Omega \nearrow \mathbb{L}$. Moreover, this convergence admits exponential relaxation, which is the content of the next lemma.

Lemma 2.4.4. *For any sequence of domains Ω_k increasing to $\mathbb{L}$, the measures ν_{Ω_k} converge to $\mu^{0,1}_{J,U}$. Moreover, this convergence admits exponential relaxation: there exists $\alpha > 0$ such that, for any $n \geq 1$ and any domain $\Omega \supset \Lambda_{4n}$,*

$$d_{\mathrm{TV}}\left(\nu_\Omega|_{\Lambda_n}, \mu^{0,1}_{J,U}|_{\Lambda_n}\right) < e^{-\alpha n}. \tag{2.23}$$

Recall the representation (2.7) of the ATRC measures. The previous lemma becomes more clear once we identify ν_Ω as the marginal of $\mathtt{ATRC}^{0,\mathbb{1}}_{\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$ on ω_τ where the weights are as in (2.8) except that w_τ is modified on the edge-boundary $E_{\overline{\partial}\Omega}$.

Lemma 2.4.5. *For any domain Ω in $\mathbb{L}$, the measure ν_Ω coincides with the marginal of $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$ on ω_τ, where*

$$\mathrm{w}_\tau(e) = \begin{cases} 2\frac{c-1}{e^{\lambda/2}-1}, & \textit{if } e \in E_{\overline{\partial}\Omega}, \\ 2, & \textit{otherwise,} \end{cases} \quad \textit{and} \quad \mathrm{w}_{\tau\tau'} \equiv e^{2(U-J)} - 1 \textit{ on } E_\Omega. \tag{2.24}$$

We now derive Proposition 2.4.1 from Lemmata 2.4.4 and 2.4.5. First of all, the ATRC measure is stochastically increasing in w_τ (see Appendix 2.C for the proof).

Lemma 2.4.6. *Let Ω be a domain in $\mathbb{L}$. The measures $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$ are stochastically increasing in w_τ. More precisely, if $\mathrm{w}_\tau(e) \le \widetilde{\mathrm{w}_\tau}(e)$ for all $e \in E_\Omega$, then the measure $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$ is stochastically dominated by $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\widetilde{\mathrm{w}_\tau},\mathrm{w}_{\tau\tau'}}$.*

This, together with Lemma 2.4.5, implies the following stochastic domination:

Lemma 2.4.7. *For any domain Ω in $\mathbb{L}$, the measure ν_Ω stochastically dominates $\mu^{0,\mathbb{1}}_{\Omega,J,U}$.*

Proof of Proposition 2.4.1. Fix $n \ge 1$ and a domain $\Omega \supset \Lambda_{4n}$ in $\mathbb{L}$. We have

$$\mu^{0,1}_{\Omega,J,U}[0 \leftrightarrow \partial\Lambda_n] \le \mu^{0,\mathbb{1}}_{\Omega,J,U}[0 \leftrightarrow \partial\Lambda_n] \le \nu_\Omega[0 \leftrightarrow \partial\Lambda_n],$$

where we used (CBC) for the first inequality and Lemma 2.4.7 for the second one.

Now, by Lemma 2.4.4 and Proposition 2.3.9, there exist $\alpha, M > 0$ such that

$$\nu_\Omega[0 \leftrightarrow \partial\Lambda_n] \le \mu^{0,1}_{J,U}[0 \leftrightarrow \partial\Lambda_n] + e^{-\alpha n} \le 8nMe^{-\alpha n} + e^{-\alpha n}. \qquad \square$$

It remains to show Lemmata 2.4.4 and 2.4.5.

Proof of Lemma 2.4.4. By construction, ν_Ω can be sampled from $\mathtt{Spin}^{+,+;c_b}_{\mathcal{D}_\Omega,c}$ using (2.22). By Corollary 2.3.4, $\mu^{0,\mathbb{1}}_{\Omega,J,U}$ can be sampled from $\mathtt{Spin}^{+,+}_{\mathcal{D}_\Omega,c}$ using (2.17). The measures $\mathtt{Spin}^{+,+;c_b}_{\mathcal{D}_\Omega,c}$ and $\mathtt{Spin}^{+,+}_{\mathcal{D}_\Omega,c}$ both converge to $\mathtt{Spin}^{+,+}_c$, as $\Omega \nearrow \mathbb{L}$, by Corollary 2.3.6. Also, the rules (2.22) and (2.17) are local and coincide outside of the boundary (which is irrelevant in the limit). Thus, ν_Ω and $\mu^{0,\mathbb{1}}_{\Omega,J,U}$ have the same limit and it can be sampled from $\mathtt{Spin}^{+,+}_c$ using the same rules. Their locality implies that the convergence inherits the exponential relaxation property (2.21) (recall that $\mathcal{D}_{\Lambda_{4n}} = \Delta_{8n}$) and, by Proposition 2.3.9, the limit is $\mu^{0,1}_{J,U}$. $\square$

Proof of Lemma 2.4.5. Fix a domain Ω in $\mathbb{L}$, and take $c_b := e^{\lambda/2}$. Recall that $E_{\sigma^\bullet}$ denotes the set of disagreement edges in $\sigma^\bullet$ (Section 2.3.3).

Step 1: The measure ν_Ω can be written in the following form:

$$\nu_\Omega[\omega] \propto \left(\tfrac{2}{c-1}\right)^{|\omega \setminus E_{\overline{\partial}\Omega}|} \left(\tfrac{2}{c_b-1}\right)^{|\omega \cap E_{\overline{\partial}\Omega}|} 2^{k(\omega)} \sum_{\substack{\sigma^\bullet \in \{\pm 1\}^\Omega \\ \sigma^\bullet \perp \omega,\, \sigma^\bullet|_{\overline{\partial}\Omega} \equiv 1}} \left(\tfrac{1}{c-1}\right)^{|E_{\sigma^\bullet}|}. \tag{2.25}$$

For brevity, we write E for E_Ω and ∂E for $E_{\overline{\partial}\Omega}$. In a slight abuse of notation, we also set $(E_{\sigma^\circ})^* = \{e^* : e \in E_{\sigma^\circ}\}$. The law of (σ,ω) defined by (2.22) satisfies:

$$\begin{aligned}
\mathbf{P}[\sigma,\omega] = \quad & \mathtt{Spin}^{+,+;c_b}_{\mathcal{D}_\Omega,c}[\sigma]\, \mathbb{1}_{\{\sigma^\bullet \perp \omega,\, \sigma^\circ \perp \omega^*\}} \times \left(\tfrac{1}{c}\right)^{|\omega \setminus ((E_{\sigma^\circ})^* \cup \partial E)|} \left(\tfrac{c-1}{c}\right)^{|E \setminus (\omega \cup E_{\sigma^\bullet} \cup \partial E)|} \\
& \qquad \times \left(\tfrac{1}{c_b}\right)^{|\omega \cap \partial E \setminus (E_{\sigma^\circ})^*|} \left(\tfrac{c_b-1}{c_b}\right)^{|\partial E \setminus (\omega \cup E_{\sigma^\bullet})|} \\
\propto \quad & (c-1)^{-|(\omega \cup E_{\sigma^\bullet}) \setminus \partial E|}\, (c_b-1)^{-|(\omega \cup E_{\sigma^\bullet}) \cap \partial E|}\, \mathbb{1}_{\{\sigma|_{\partial^2 \mathcal{D}} \equiv 1\}}\, \mathbb{1}_{\{\sigma^\bullet \perp \omega,\, \sigma^\circ \perp \omega^*\}}.
\end{aligned}$$

Note that $E_{\sigma^\bullet} \cap \partial E = \varnothing$ since $\sigma_{|\partial\Omega} \equiv 1$. Summing over σ then gives

$$\mathbf{P}[\omega] \propto (c-1)^{-|\omega \setminus \partial E|} (c_b-1)^{-|\omega \cap \partial E|}\, 2^{k^1(\omega^*)} \sum_{\substack{\sigma^\bullet \in \{\pm 1\}^\Omega \\ \sigma^\bullet \perp \omega,\, \sigma^\bullet|_{\overline{\partial}\Omega} \equiv 1}} (c-1)^{-|E_{\sigma^\bullet}|}.$$

Finally, by Euler's formula (or induction), $k^1(\omega^*) = k(\omega) + |\omega| + \mathrm{const}(\mathcal{D}_\Omega)$.
Step 2: The marginal of $\mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$ on ω_τ with weights $\mathrm{w}_\tau, \mathrm{w}_{\tau\tau'}$ given by (2.24) coincides with the right-hand side of (2.25).

Given $(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}$, define a spin configuration $\sigma^\bullet \in \{\pm 1\}^\Omega$ by assigning $+1$ to domain-boundary clusters of $\omega_{\tau\tau'}$ and ± 1 to interior clusters of $\omega_{\tau\tau'}$ uniformly independently. Then their joint law can be written as

$$\mathbf{P}[\omega_\tau, \omega_{\tau\tau'}, \sigma^\bullet] \propto \prod_{e\in\omega_\tau} \mathrm{w}_\tau(e) \cdot \mathrm{w}_{\tau\tau'}^{|\omega_{\tau\tau'}\setminus\omega_\tau|}\, 2^{k(\omega_\tau)}\, \mathbb{1}_{\{\omega_\tau\subseteq\omega_{\tau\tau'}\}}\, \mathbb{1}_{\{\sigma^\bullet\perp\omega_{\tau\tau'}\}}\, \mathbb{1}_{\{\sigma^\bullet|_{\overline{\partial}\Omega}\equiv 1\}}.$$

Now, $\sigma^\bullet \perp \omega_{\tau\tau'}$ precisely if $\sigma^\bullet \perp \omega_\tau$ and $(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E_{\sigma^\bullet} = \varnothing$. Sum over $\omega := \omega_{\tau\tau'} \setminus \omega_\tau$:

$$\mathbf{P}[\omega_\tau, \sigma^\bullet] \propto \prod_{e\in\omega_\tau} \mathrm{w}_\tau(e) \cdot 2^{k(\omega_\tau)}\, \mathbb{1}_{\{\sigma^\bullet\perp\omega_\tau\}}\, \mathbb{1}_{\{\sigma^\bullet|_{\overline{\partial}\Omega}\equiv 1\}} \sum_{\omega\subseteq E\setminus(\omega_\tau\cup E_{\sigma^\bullet})} \mathrm{w}_{\tau\tau'}^{|\omega|}.$$

The last term equals $(\mathrm{w}_{\tau\tau'}+1)^{|E|-|\omega_\tau|-|E_{\sigma^\bullet}|}$. Finally, summing over $\sigma^\bullet$, we arrive at

$$\mathbf{P}[\omega_\tau] \propto \prod_{e\in\omega_\tau} \tfrac{\mathrm{w}_\tau(e)}{\mathrm{w}_{\tau\tau'}+1} \cdot 2^{k(\omega_\tau)} \sum_{\substack{\sigma^\bullet\in\{\pm1\}^\Omega \\ \sigma^\bullet\perp\omega_\tau,\, \sigma^\bullet|_{\overline{\partial}\Omega}\equiv 1}} \left(\tfrac{1}{\mathrm{w}_{\tau\tau'}+1}\right)^{|E_{\sigma^\bullet}|}. \tag{2.26}$$

Plugging in the weights (2.24) while using that $\sinh 2J = e^{-2U}$ and $c = \coth(2J)$ gives that (2.26) agrees with (2.25), which finishes the proof. □

2.5 No infinite cluster in the wired self-dual ATRC

Proposition 2.5.1. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. Then, $\mathtt{ATRC}^{1,1}_{J,U}[0 \xleftrightarrow{\omega_\tau} \infty] = 0$.*

The proof of Proposition 2.5.1 again relies on the coupling with the six-vertex model, Proposition 2.3.2. First of all, by the non-coextistence theorem [She05, DRT19], it is sufficient to show that $\mathtt{ATRC}^{1,1}_{J,U}$ admits an infinite ω_τ^*-cluster. If the latter is not the case, the infinite-volume limit of the marginals of $\mathtt{Spin}^{+,\pm}_{\mathcal{D},c}$ on $\{\pm1\}^{\mathbb{L}^*}$ can be shown to be tail-trivial. Exploring clusters of 1 and -1 (in $\mathbb{T}$-connectivity) and using the non-coexistence theorem, we obtain that the limit of $\mathtt{HF}^{0,\pm1}_{\mathcal{D},c}$ is either $\mathtt{HF}^{0,1}_{c}$ or $\mathtt{HF}^{0,-1}_{c}$, thereby contradicting the invariance of $\mathtt{HF}^{0,\pm1}_{\mathcal{D},c}$ under $h \mapsto -h$.

In the following remark, we summarise some basic properties of the ATRC marginals $\mu^{1,1}_{\Omega,J,U}$ (defined in Section 2.3.3) and their infinite-volume limit that we will use in Sections 2.5.1 and 2.5.2.

Remark 2.5.2. Recall that, for domains $\Omega_k \nearrow \mathbb{L}$, the measures $\mathtt{ATRC}^{1,1}_{\Omega_k,J,U}$ form a decreasing sequence and converge to $\mathtt{ATRC}^{1,1}_{J,U}$. In particular, the same holds for the marginals on ω_τ: $\mu^{1,1}_{\Omega_k,J,U}$ converges to $\mu^{1,1}_{J,U}$. It is then standard ([Gri06, Chapter 4.3]) that $\mu^{1,1}_{J,U}$ is invariant under translations and tail-trivial (and hence ergodic). Moreover, $\mathtt{ATRC}^{1,1}_{\Omega_k,J,U}$ (and thus their limit and its marginals) satisfies the finite-energy property. Therefore, the Burton–Keane argument [BK89] and the non-coexistence theorem [She05, DRT19] apply.

2.5.1 Semi-free measures in infinite volume

In this section, we will show weak convergence for some finite-volume spin and height function measures defined in Section 2.3.2.

Lemma 2.5.3. *Let $0 < J < U$ be on the self-dual line* (SD) *and take $c := \coth 2J$. Let $\omega_\tau \sim \mu^{1,1}_{J,U}$. Define $\chi_c^{+,\pm}$ as the distribution on $\{\pm 1\}^{\mathbb{L}^*}$ obtained by assigning ± 1 to every cluster of ω_τ^* uniformly and independently. Then, for any sequence of odd domains $\mathcal{D}_k \nearrow \mathbb{Z}^2$, the marginals of $\mathtt{Spin}^{+,\pm}_{\mathcal{D}_k,c}$ on σ° converge weakly to $\chi_c^{+,\pm}$. Moreover, $\chi_c^{+,\pm}$ is translation-invariant, positively associated and satisfies the finite-energy property.*

Proof. Fix $J < U$. Let $\mathcal{D}_k$ be a sequence of odd domains on $\mathbb{Z}^2$ and Ω_k the corresponding subgraphs of $\mathbb{L}$ such that $\mathcal{D}_k = \mathcal{D}_{(\Omega_k)^*}$. Let ω_τ^k be sampled from $\mu^{1,1}_{\Omega_k,J,U}$. By Proposition 2.3.2, assigning ± 1 to clusters of $(\omega_\tau^k)^*$ uniformly independently gives the marginal of $\mathtt{Spin}^{+,\pm}_{\mathcal{D}_k,c}$ on σ°. Since $\mu^{1,1}_{\Omega_k,J,U}$ converges to $\mu^{1,1}_{J,U}$ that exhibits at most one infinite cluster in ω_τ^*, the marginal of $\mathtt{Spin}^{+,\pm}_{\mathcal{D}_k,c}$ on σ° converges to $\chi_c^{+,\pm}$.

Clearly, $\chi_c^{+,\pm}$ inherits translation-invariance and the finite-energy property from $\mu^{1,1}_{J,U}$. By Proposition 2.3.8, the marginal of $\mathtt{Spin}^{+,\pm}_{\mathcal{D}_k,c}$ on σ° satisfies the FKG inequality. Hence, the same holds for its limit $\chi_c^{+,\pm}$. □

Working with measures on height functions (rather than spins) is more convenient as they satisfy stochastic ordering in boundary conditions. In the proof of Proposition 2.5.1, we use an infinite-volume version of $\mathtt{HF}^{0,\pm 1}_{\mathcal{D},c}$. We show existence of such subsequential limit in the next lemma by sandwiching $\mathtt{HF}^{0,\pm 1}_{\mathcal{D},c}$ between $\mathtt{HF}^{0,-1}_{\mathcal{D},c}$ and $\mathtt{HF}^{0,1}_{\mathcal{D},c}$.

Lemma 2.5.4. *Let $c > 2$. For any sequence of domains $\mathcal{D}_k$ increasing to $\mathbb{Z}^2$, there exists a subsequence (k_ℓ) such that the measures $\mathtt{HF}^{0,\pm 1}_{\mathcal{D}_{k_\ell},c}$ converge weakly to some $\mathtt{HF}^{0,\pm 1}_c$ as ℓ tends to infinity.*

Remark 2.5.5. Proposition 2.5.6 and its proof allow to show that the limiting measure is $\frac{1}{2}(\mathtt{HF}^{0,1}_c + \mathtt{HF}^{0,-1}_c)$ for any (sub)sequence. We do not use this statement and omit the details.

Proof of Lemma 2.5.4. By [Geo11, Proposition 4.9], it suffices to show that $(\mathtt{HF}^{0,\pm 1}_{\mathcal{D}_k,c})_{k\geq 1}$ is locally equicontinuous: for any finite $V \subset \mathbb{Z}^2$ and any decreasing sequence of local events $(A_m)_{m\geq 1}$ supported on V and with $\cap_{m\geq 1} A_m = \varnothing$, it holds that

$$\limsup_{k\to\infty} \mathtt{HF}^{0,\pm 1}_{\mathcal{D}_k,c}[A_m] \to 0 \qquad \text{as } m \to \infty.$$

By Proposition 2.3.7, finite-volume six-vertex height function measures are stochastically ordered with respect to the boundary conditions, whence

$$\mathtt{HF}^{0,-1}_{\mathcal{D}_k,c} \leq_{\text{st}} \mathtt{HF}^{0,\pm 1}_{\mathcal{D}_k,c} \leq_{\text{st}} \mathtt{HF}^{0,1}_{\mathcal{D}_k,c}.$$

Moreover, by Proposition 2.3.5, $\mathtt{HF}^{0,-1}_{\mathcal{D}_k,c}$ converges to $\mathtt{HF}^{0,-1}_c$ and $\mathtt{HF}^{0,1}_{\mathcal{D}_k,c}$ to $\mathtt{HF}^{0,1}_c$ as k tends to infinity. These statements together easily imply the required local equicontinuity. □

2.5.2 Proof of Proposition 2.5.1

As we argued in Remark 2.5.2, it suffices to find a dual infinite cluster:

Proposition 2.5.6. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. Then, $\mathtt{ATRC}^{1,1}_{J,U}[0 \xleftrightarrow{\omega_\tau^*} \infty] > 0$.*

Remark 2.5.7. By the duality relation described in Section 2.2.1, this is equivalent to saying that $\mathtt{ATRC}^{0,0}_{J,U}[0 \xleftrightarrow{\omega_{\tau\tau'}} \infty] > 0$, for any $J < U$ on the self-dual line (SD).

The proof of Proposition 2.5.6 also relies on the non-coexistence theorem – but in the context of site percolation. Following the notation of [GP23], we let $\mathbb{T}^\circ$ be the graph with vertex set $\mathbb{L}^*$ where a vertex $(x, y) \in \mathbb{L}^*$ is adjacent to

$$(x, y) \pm (1, 1),\ (x, y) \pm (1, -1) \text{ and } (x \pm 2, y).$$

Note that $\mathbb{T}^\circ$ is isomorphic to the triangular lattice.

Proof of Proposition 2.5.6. Fix $J < U$. Recall that $\mu^{1,1}_{J,U}$ is the marginal of $\mathtt{ATRC}^{1,1}_{J,U}$ on ω_τ. Assume for contradiction that $\mu^{1,1}_{J,U}$ does not admit an infinite dual cluster.

Set $c = \coth 2J$. Recall that $\chi^{+,\pm}_c$ is obtained from $\omega \sim \mu^{1,1}_{J,U}$ by assigning uniformly independently ± 1 to its dual clusters. Since all of them are finite by our assumption, $\chi^{+,\pm}_c$ inherits ergodicity from $\mu^{1,1}_{J,U}$. Also, by Lemma 2.5.3, $\chi^{+,\pm}_c$ is translation-invariant and satisfies the FKG inequality. Thus, by the non-coexistence theorem, in $\mathbb{T}^\circ$-connectivity, either $\chi^{+,\pm}_c$ admits no infinite cluster of minuses, whence

$$\chi^{+,\pm}_c(\exists \text{ infinitely many disjoint } \mathbb{T}^\circ\text{-circuits of } + \text{ around the origin}) = 1, \tag{2.27}$$

or the same holds for $\mathbb{T}^\circ$-circuits of $-$. By symmetry, we can assume (2.27).

By Lemma 2.5.4, there exists a sequence of odd domains $\mathcal{D}_k$ such that $\mathtt{HF}^{0,\pm 1}_{\mathcal{D}_k,c}$ converge to some infinite-volume height function measure $\mathtt{HF}^{0,\pm 1}_c$ weakly. Recall that $\mathtt{Spin}^{+,\pm}_{\mathcal{D},c}$ is the push-forward of $\mathtt{HF}^{0,\pm 1}_{\mathcal{D},c}$ under the modulo 4 mapping and, by Lemma 2.5.3, the marginals of $\mathtt{Spin}^{+,\pm}_{\mathcal{D},c}$ on σ° converge to $\chi^{+,\pm}_c$ weakly. Then, (2.27) implies that $\mathtt{HF}^{0,\pm 1}_c$ admits infinitely many disjoint $\mathbb{T}^\circ$-circuits around the origin of constant height that is congruent to 1 modulo 4. By Proposition 2.3.7, the measure $\mathtt{HF}^{0,\pm 1}_c$ is between $\mathtt{HF}^{0,-1}_c$ and $\mathtt{HF}^{0,1}_c$ in the sense of stochastic domination. By Proposition 2.3.5, $\mathtt{HF}^{0,-1}_c$ and $\mathtt{HF}^{0,1}_c$ admit infinite clusters of -1 and $+1$, respectively. Hence, the above implies

$$\mathtt{HF}^{0,\pm 1}_c(\exists \text{ infinitely many disjoint } \mathbb{T}^\circ\text{-circuits of } +1 \text{ around the origin}) = 1. \tag{2.28}$$

By a standard exploration argument and FKG inequality (details below), (2.28) implies that $\mathtt{HF}^{0,\pm 1}_c$ stochastically dominates and hence equals $\mathtt{HF}^{0,1}_c$. This leads to a contradiction since $\mathtt{HF}^{0,\pm 1}_c$ is invariant under $h \mapsto -h$ while $\mathtt{HF}^{0,1}_c$ is not.

It remains to show that (2.28) implies $\mathtt{HF}^{0,\pm 1}_c = \mathtt{HF}^{0,1}_c$. It is sufficient to prove $\mathtt{HF}^{0,\pm 1}_c[A] = \mathtt{HF}^{0,1}_c[A]$, for any increasing local event A. Take any $\varepsilon > 0$. Since $\mathtt{HF}^{0,1}_{\mathcal{D},c}$ converges to $\mathtt{HF}^{0,1}_c$ weakly as $\mathcal{D} \nearrow \mathbb{Z}^2$, we can find $n \geq 1$ such that, for any domain $\mathcal{D}$ containing Δ_n,

$$|\mathtt{HF}^{0,1}_{\mathcal{D},c}[A] - \mathtt{HF}^{0,1}_c[A]| < \varepsilon. \tag{2.29}$$

We can find $\mathcal{D} \supseteq \Delta_n$ large enough such that

$$\left|\mathtt{HF}^{0,\pm 1}_c[A] - \mathtt{HF}^{0,\pm 1}_c[A \,|\, \exists\ \mathbb{T}^\circ\text{-circuit of } +1 \text{ in } \mathcal{D} \text{ surrounding } \Delta_n]\right| < \varepsilon. \tag{2.30}$$

Let $\mathcal{C}$ be the outermost $\mathbb{T}^\circ$-circuit of height $+1$ in $\mathcal{D}$ surrounding Δ_n (if such circuit does not exist, we set $\mathcal{C} := \emptyset$). By the domain Markov property and stochastic ordering in boundary conditions, for any $\mathbb{T}^\circ$-circuit C surrounding Δ_n and contained in $\mathcal{D}$,

$$\mathtt{HF}^{0,\pm 1}_c[A \,|\, \mathcal{C} = C] \geq \mathtt{HF}^{0,1}_{\mathcal{D}_C,c}[A], \tag{2.31}$$

where $\mathcal{D}_C$ is the connected component of the origin in the graph obtained from $\mathbb{Z}^2$ after removing all vertices on C or adjacent to it in $\mathbb{Z}^2$. Since $\mathcal{D}_C \supset \Delta_n$, by (2.29), the right-hand side in (2.31) is ε-close to $\mathtt{HF}^{0,1}_c[A]$. Putting this together with (2.30), we get $\mathtt{HF}^{0,\pm 1}_c[A] \geq \mathtt{HF}^{0,1}_c[A] - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $\mathtt{HF}^{0,\pm 1}_c[A] \geq \mathtt{HF}^{0,1}_c[A]$. The opposite inequality follows by the comparison of boundary conditions. $\square$

2.6 Proof of Proposition 2.1.1

Our goal is to use exponential decay under $0,1$ conditions, Proposition 2.4.1, to improve the non-percolation statement, Proposition 2.5.1, and get exponential decay in finite volume, Proposition 2.1.1. We use the approach of [CM10] and [CIV08, Appendix]. Additional difficulties in our case come from a weaker domain Markov property of the ATRC measure.

Fix $J < U$ and $n \geq 1$. For any vertex $x \in \Lambda_n$, define

$$\eta(x) = \mathbb{1}_{\{x \xleftrightarrow{\omega_\tau} \partial\Lambda_n\}}.$$

The next lemma provides a lower bound on the size of the boundary cluster of ω_τ. Recall that we denote by μ the marginal of the ATRC measure on ω_τ.

Lemma 2.6.1. *For any $\delta > 0$, there exists $\alpha := \alpha(\delta, \beta_{\mathrm{sd}}) > 0$ such that*

$$\mu^{1,1}_{\Lambda_n,\beta_{sd}}\left(\sum_{x\in\Lambda_n} \eta_x \geq \delta n^2\right) \leq e^{-\alpha n^2}.$$

Proof. Fix $\delta > 0$. It follows from Proposition 2.5.1 that

$$\lim_{n\to\infty} \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n) = 0.$$

This implies that one can find $M := M(\delta) > 0$ such that

$$\mathbb{E}^{1,1}_{\Lambda_M,\beta_{\mathrm{sd}}}\left[\tfrac{1}{|\Lambda_M|}\sum_{x\in\Lambda_M} \eta_x\right] < \frac{\delta}{2}.$$

Fix $n \gg M$. Without loss of generality, assume that $n = (2k+1)M$. One has

$$\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} \eta_x \leq \frac{1}{|\Lambda_k|}\sum_{x\in\Lambda_k}\left(\frac{1}{|\Lambda_M|}\sum_{y\in 2Mx+\Lambda_M} \mathbb{1}_{y\leftrightarrow\partial(2Mx+\Lambda_M)}\right).$$

Denote the expression in the brackets by $Y_{x,M}$. Then,

$$\begin{aligned}\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\eta_x \geq \delta\right) &\leq \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}\left(\frac{1}{|\Lambda_k|}\sum_{x\in\Lambda_k} Y_{x,M} \geq \delta\right)\\ &\leq \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}\left(\frac{1}{|\Lambda_k|}\sum_{x\in\Lambda_k} Y_{x,M} \geq \delta \,\middle|\, B_M\right),\end{aligned}$$

where $B_M = \bigcap_{x\in\Lambda_k}\{\omega_\tau|_{\partial(2Mx+\Lambda_M)} \equiv 1\}$ and the last inequality uses (FKG) and that B_M is increasing. Note that under $\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(\cdot\,|\,B_M)$, the random variables $Y_{x,M}$ are i.i.d. The statement then follows from Hoeffding's inequality. □

Proof of Proposition 2.1.1. Recall (2.12). We aim to show that, up to an arbitrary small exponential error, there exists a blocking surface of closed edges around $\Lambda_{\frac{4n}{5}}$ in Λ_n.

For each $\ell \in [1, n]$ and $x \in \partial\Lambda_\ell$, define

$$f(x) := \mathbb{1}_{\{x \xleftrightarrow{\omega_\tau,\Lambda_n\setminus\Lambda_\ell} \partial\Lambda_n\}} \quad \text{and} \quad N(\ell) := \sum_{x\in\partial\Lambda_\ell} f(x).$$

Define A_ℓ as the event that $N(\ell) \le \delta n$. Since $f(x) \le \eta(x)$, Lemma 2.6.1 implies that, up to an error $e^{-\alpha n^2}$, event A_ℓ occurs for some $\ell \in [4n/5, n]$, whence

$$\begin{aligned}\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5}) &\le e^{-\alpha n^2} + \sum_{4n/5\le\ell\le n} \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5} \,|\, A_\ell)\cdot\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(A_\ell)\\ &\le e^{-\alpha n^2} + e^{\alpha'\delta n}\sum_{4n/5\le\ell\le n} \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5}, \Lambda_{4n/5} \overset{\omega_\tau}{\not\longleftrightarrow} \partial\Lambda_n \,|\, A_\ell)\cdot\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(A_\ell)\\ &= e^{-\alpha n^2} + e^{\alpha'\delta n}\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5}, \Lambda_{4n/5} \overset{\omega_\tau}{\not\longleftrightarrow} \partial\Lambda_n),\end{aligned}$$

where we used that A_ℓ is measurable with respect to edges of ω_τ in $\Lambda_n \setminus \Lambda_\ell$ and that, conditioned on A_ℓ, there are maximum $4\delta n$ edges that are incident to vertices on $\partial\Lambda_\ell$ that are connected to Λ_n and we can disconnect $\Lambda_{4n/5}$ from $\partial\Lambda_n$ by closing all these edges.

On the event $\{\Lambda_{4n/5} \overset{\omega_\tau}{\not\longleftrightarrow} \partial\Lambda_n\}$, there exists a circuit of closed edges in ω_τ that surrounds $\Lambda_{4n/5}$. Denote the exterior-most such circuit by ζ an explore it from the outside:

$$\begin{aligned}\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5}, \partial\Lambda_{4n/5} \overset{\omega_\tau}{\not\longleftrightarrow} \partial\Lambda_n) &= \sum_C \mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5} \,|\, \zeta = C)\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(\zeta = C)\\ &\le \sum_C \mu^{0,1}_{\Omega_C,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5})\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(\zeta = C),\end{aligned}$$

where the sum is over all possible values of ζ and we define Ω_C as the subgraph of $\mathbb{L}$ bounded by C; the inequality relies on (CBC) and on the $0,1$ boundary conditions being domain Markov for μ. Note that Ω_C can be turned into a domain by consecutively removing vertices of degree 1 – denote it by Ω'_C. Such operations can only increase the measure, whence $\mu^{0,1}_{\Omega_C,\beta_{\mathrm{sd}}} \le_{\mathrm{st}} \mu^{0,1}_{\Omega'_C,\beta_{\mathrm{sd}}}$, and $\Lambda_{4n/5} \subset \Omega'_C$. Thus, the right-hand side in the last equation is exponentially small by Proposition 2.4.1. Combining the bounds, we get

$$\mu^{1,1}_{\Lambda_n,\beta_{\mathrm{sd}}}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_{n/5}) \le e^{-\alpha n^2} + e^{\alpha'\delta n}\cdot e^{-\alpha'' n}.$$

Taking δ small enough finishes the proof. □

2.7 The case $J \ge U$: Proof of Theorem 8

2.7.1 ATRC for $J \ge U$

We fix $J \ge U$ and a finite subgraph Ω of $\mathbb{L}$. The ATRC model is defined via an Edwards–Sokal-type expansion. Since $J \ge U$, the leading terms will correspond to interactions in τ and in τ'. Thus, the ATRC measure on Ω with boundary conditions $\eta_\tau, \eta_{\tau'}$ is supported on pairs of percolation configurations $(\omega_\tau, \omega_{\tau'}) \in \{0,1\}^{E_\Omega} \times \{0,1\}^{E_\Omega}$, and is defined by

$$\mathtt{ATRC}^{\eta_\tau,\eta_{\tau'}}_{\Omega,J,U}(\omega_\tau,\omega_{\tau'}) = \tfrac{1}{Z}\cdot 2^{k^{\eta_\tau}(\omega_\tau)+k^{\eta_{\tau'}}(\omega_{\tau'})}\prod_{e\in E} a(\omega_\tau(e),\omega_{\tau'}(e)), \tag{2.32}$$

where $Z = Z(\Omega, J, U, \eta_\tau, \eta_{\tau'})$ is a normalizing constant and

$$a(0,0) := e^{-4J},\ a(1,0) = a(0,1) := e^{-2(J+U)} - e^{-4J},\ a(1,1) := 1 - 2e^{-2(J+U)} + e^{-4J}. \tag{2.33}$$

Similarly to (2.7), if $J > U$, we can write the measure as

$$\mathtt{ATRC}^{\eta_\tau,\eta_{\tau'}}_{\Omega,J,U}(\omega_\tau,\omega_{\tau'}) \propto \left(\tfrac{a(1,0)}{a(0,0)}\right)^{|\omega_\tau|+|\omega_{\tau'}|}\left(\tfrac{a(0,0)a(1,1)}{a(1,0)^2}\right)^{|\omega_\tau\cap\omega_{\tau'}|} 2^{k^{\eta_\tau}(\omega_\tau)+k^{\eta_{\tau'}}(\omega_{\tau'})}. \tag{2.34}$$

Basic properties. The analogues of the properties (FKG), (CBC), (MON), (MON+) and (DMP) hold in this context as well. In particular, the measures $\mathtt{ATRC}^{0,0}_{\Omega,J,U}$ and $\mathtt{ATRC}^{1,1}_{\Omega,J,U}$ converge weakly to some $\mathtt{ATRC}^{0,0}_{J,U}$ and $\mathtt{ATRC}^{1,1}_{J,U}$, respectively, as $\Omega \nearrow \mathbb{L}$. As before, if the parameters J, U are fixed, we write $\mathtt{ATRC}^{\eta_\tau,\eta_{\tau'}}_{\Omega,\beta}$ for $\mathtt{ATRC}^{\eta_\tau,\eta_{\tau'}}_{\Omega,\beta J,\beta U}$ and analogously for the infinite-volume measures.

Coupling of ATRC and AT. As mentioned above, edges in ω_τ and in $\omega_{\tau'}$ describe interactions in τ and in τ'. In contrast to (2.12), the correlations of the product $\tau\tau'$ are described by simultaneous connections in both ω_τ and $\omega_{\tau'}$: for any $x, y \in V_\Omega$,

$$\langle \tau_x \tau_y \rangle_{\Omega,J,U} = \mathtt{ATRC}^{0,0}_{\Omega,J,U}(x \xleftrightarrow{\omega_\tau} y), \ \langle \tau_x \tau'_x \tau_y \tau'_y \rangle_{\Omega,J,U} = \mathtt{ATRC}^{0,0}_{\Omega,J,U}(x \xleftrightarrow{\omega_\tau} y, x \xleftrightarrow{\omega_{\tau'}} y). \tag{2.35}$$

As in Section 2.2.1, the statement extends to infinite volume in a standard way, and $\beta_c^{\tau,f}$ and $\beta_c^{\tau\tau',f}$ coincide with the corresponding percolation thresholds under $\mathtt{ATRC}^{0,0}_{J,U}$. Similarly, the same holds for $\beta_{\mathrm{c}}^{\tau}$ and $\beta_{\mathrm{c}}^{\tau\tau'}$ under $\mathtt{ATRC}^{1,1}_{J,U}$.

Duality. Given an ATRC configuration $(\omega_\tau, \omega_{\tau'})$, we define its dual $(\hat\omega_\tau, \hat\omega_{\tau'}) := (\omega^*_\tau, \omega^*_{\tau'})$. This extends Lemma 2.2.2 to all $J, U > 0$.

2.7.2 Proof of Theorem 8

Fix $J \geq U$. Positive association (FKG), symmetry and inclusion give

$$\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \infty)^2 \leq \mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \infty, 0 \xleftrightarrow{\omega_{\tau'}} \infty) \leq \mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \infty).$$

Since $\beta_{\mathrm{c}}^{\tau}$ and $\beta_{\mathrm{c}}^{\tau\tau'}$ coincide with the corresponding percolation thresholds, this implies $\beta_{\mathrm{c}}^{\tau} = \beta_{\mathrm{c}}^{\tau\tau'} =: \beta_{\mathrm{c}}$. Analogously, $\beta_{\mathrm{c}}^{\tau,f} = \beta_{\mathrm{c}}^{\tau\tau',f} =: \beta_{\mathrm{c}}^{f}$.

To derive Theorem 8, it remains to show that $\beta_{\mathrm{c}} = \beta_{\mathrm{c}}^{f} = \beta_{\mathrm{sd}}$. The inequality $\beta_{\mathrm{c}} \leq \beta_{\mathrm{sd}}$ follows from a standard argument once the transition is shown to be *sharp*: for $\beta < \beta_{\mathrm{c}}$, there exists $c = c(\beta) > 0$ such that, for every $n \geq 1$,

$$\mathtt{ATRC}^{1,1}_{\beta}(0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n) \leq e^{-cn}. \tag{2.36}$$

This can be derived via a general approach [DRT19], see Appendix 2.A.

The reverse inequality is a consequence of Zhang's argument provided that $\beta_{\mathrm{c}} = \beta_{\mathrm{c}}^{f}$, i.e. the transitions for the free and wired measures occur at the same point. This follows from an analogue of Lemma 2.2.3 (see Appendix 2.D for the proof of both lemmata):

Lemma 2.7.1. *There exists $D \subseteq \{(J,U) \in \mathbb{R}^2 : J \geq U > 0\}$ with Lebesgue measure 0 such that, for any $(J,U) \in D^c$, one has*

$$\mathtt{ATRC}^{0,0}_{J,U} = \mathtt{ATRC}^{1,1}_{J,U}.$$

Proof of Theorem 8. Fix $J \geq U$. Part (i) follows from Lemma 2.7.1 and (MON+) in the same way as for $J < U$, see Section 2.2.2.

Recall the definition of the event $\mathcal{H}^{\tau}_n$ in Section 2.2.2. By duality, symmetry and (CBC),

$$\mathtt{ATRC}^{0,0}_{\beta_{\mathrm{sd}}}(\mathcal{H}^{\tau}_n) \leq \frac{1}{2} \leq \mathtt{ATRC}^{1,1}_{\beta_{\mathrm{sd}}}(\mathcal{H}^{\tau}_n). \tag{2.37}$$

If $\beta_{\mathrm{c}} > \beta_{\mathrm{sd}}$, then, by (2.36), $\mathtt{ATRC}^{1,1}_{\beta_{\mathrm{sd}}}(\mathcal{H}^{\tau}_n)$ converges to 0 as n tends to infinity, If $\beta_{\mathrm{c}} < \beta_{\mathrm{sd}}$, then (since $\beta_{\mathrm{c}} = \beta_{\mathrm{c}}^{f}$) Zhang's argument implies that $\mathtt{ATRC}^{0,0}_{\beta_{\mathrm{sd}}}(\mathcal{H}^{\tau}_n)$ converges to 1 as n tends to infinity. Both statements contradict (2.37). $\square$

2.A Sharpness

The proof of sharpness for FK-percolation via the OSSS inequality [DRT19, OSSS05] adapts to the ATRC. For completeness, we present a sketch of this argument and give details for the steps that are specific for the ATRC.

2.A.1 Sharpness for $J < U$

We start by bounding the derivative in β by a covariance:

Lemma 2.A.1. *Let $0 < J < U$ and $\varepsilon > 0$. Then, there exists $c = c(\varepsilon, J, U) > 0$ such that, for any finite subgraph $\Omega \subseteq \mathbb{L}$, any increasing event A, and any $\beta_0 \in [\varepsilon, \varepsilon^{-1}]$,*

$$\left(\frac{d}{d\beta}\mathtt{ATRC}^{1,1}_{\Omega,\beta}[A]\right)\Bigg|_{\beta=\beta_0} \geq c \sum_{e\in E_\Omega} \mathbf{Cov}[\mathbb{1}_A, \omega_\tau(e)] + \mathbf{Cov}[\mathbb{1}_A, \omega_{\tau\tau'}(e)].$$

Proof. Fix $J < U$. Recall that we can write the measure $\mathtt{ATRC}^{1,1}_{\Omega,\beta}$ as in (2.7) with weights given by (2.8). Then, as in FK-percolation, we get a covariance formula:

$$\frac{d}{d\beta}\mathtt{ATRC}^{1,1}_{\Omega,\beta}[A] = \mathbf{Cov}\left[\mathbb{1}_A, \frac{\mathrm{w}'_\tau(\beta)}{\mathrm{w}_\tau(\beta)}\cdot|\omega_\tau| + \frac{\mathrm{w}'_{\tau\tau'}(\beta)}{\mathrm{w}_{\tau\tau'}(\beta)}\cdot|\omega_{\tau\tau'}\setminus\omega_\tau|\right].$$

Since $J < U$, we have $\mathrm{w}'_{\tau\tau'}(\beta), \mathrm{w}'_\tau(\beta) > 0$ and the statement follows. □

Fix $J < U$. We prove sharpness only for ω_τ, since the proof for $\omega_{\tau\tau'}$ is the same. Recall that $\mu^{1,1}_{\Omega,\beta}$ is the marginal of $\mathtt{ATRC}^{1,1}_{\Omega,\beta}$ on ω_τ. The key step in the proof of sharpness in [DRT19] is the extension of the OSSS inequality [OSSS05] to dependent measures. The inequality holds for any monotone (2.9) measure on $\{0,1\}^E$, for a finite set of edges E. In particular, it applies also to $\mu^{1,1}_{\Lambda_{2n},\beta}$ on $\{0,1\}^{E_{2n}}$ with $E_{2n} = E_{\Lambda_{2n}}$, for $n \geq 1$. Instead of stating the OSSS inequality, we state its consequence that can be derived in the same way as in [DRT19]:

Lemma 2.A.2 ([DRT19], Lemma 3.2)**.** *For any $n \geq 1$, one has*

$$\sum_{e\in E_{2n}} \mathbf{Cov}[\mathbb{1}_{\{0\leftrightarrow\partial\Lambda_n\}}, \omega_e] \geq \frac{n}{16\sum_{k=0}^{n-1}\mu^{1,1}_{\Lambda_{2k},\beta}[0\leftrightarrow\partial\Lambda_k]}\mu^{1,1}_{\Lambda_{2n},\beta}[0\leftrightarrow\partial\Lambda_n]\left(1-\mu^{1,1}_{\Lambda_{2n},\beta}[0\leftrightarrow\partial\Lambda_n]\right),$$

where the covariance is taken with respect to the measure $\mu^{1,1}_{\Lambda_{2n},\beta}$.

We proceed as in [DRT19]. Fix $\beta_0 > 0$. For $n, k \geq 1, \varepsilon < 1$ and $\beta \in [\varepsilon, \varepsilon^{-1}]$, define

$$\theta_k(\beta) := \mu^{1,1}_{\Lambda_{2k},\beta}[0\leftrightarrow\partial\Lambda_k], \qquad S_n := \sum_{k=0}^{n-1}\theta_k.$$

Lemma 2.A.1 applied to $A = \{0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n\}$ and $\Omega = \Lambda_{2n}$ implies

$$\theta_n'(\beta) \geq c\sum_{e\in E_{2n}} \mathbf{Cov}[\mathbb{1}_A, \omega_\tau(e)] + \mathbf{Cov}[\mathbb{1}_A, \omega_{\tau\tau'}(e)] \geq c\sum_{e\in E_{2n}} \mathbf{Cov}[\mathbb{1}_A, \omega_\tau(e)], \tag{2.38}$$

where we used FKG inequality in the last line. By Lemma 2.A.2,

$$\theta_n' \geq c\frac{n}{16S_n}\theta_n(1-\theta_n).$$

By (CBC) and monotonicty in $\beta \leq \varepsilon^{-1}$, we have $\theta_n(\beta) \leq \theta_1(\varepsilon^{-1})$. Then, for some $c_1 > 0$,

$$\theta_n' \geq c_1\frac{n}{S_n}\theta_n. \tag{2.39}$$

By [DRT19, Lemma 3.1], this inequality implies sharpness of the phase transition.

2.A.2 Sharpness for $J \geq U$

The proof is the same as for $J < U$ and we only show the analogue of Lemma 2.A.1.

Lemma 2.A.3. *Let $J \geq U > 0$ and $\varepsilon > 0$. Then, there exists $c = c(\varepsilon, J, U) > 0$ such that, for any finite subgraph $\Omega \subseteq \mathbb{L}$, any increasing event A, and any $\beta_0 \in [\varepsilon, \varepsilon^{-1}]$,*

$$\left(\frac{d}{d\beta}\mathtt{ATRC}^{1,1}_{\Omega,\beta}[A]\right)\Bigg|_{\beta=\beta_0} \geq c\sum_{e\in E_\Omega}\left(\mathbf{Cov}[\mathbb{1}_A, \omega_\tau(e)] + \mathbf{Cov}[\mathbb{1}_A, \omega_{\tau'}(e)]\right).$$

Proof. For $J = U$, the model reduces to FK-percolation with cluster-weight $q = 4$, and the statement follows from [Gri06, Theorem 3.12]. Fix $J > U$. Define $r(\beta), s(\beta) > 0$ by

$$r(\beta) = \tfrac{a(1,0)}{a(0,0)} = \tfrac{a(0,1)}{a(0,0)} \quad \text{and} \quad s(\beta) = \tfrac{a(0,0)a(1,1)}{a(1,0)^2},$$

where the $a(i,j)$ are given by (2.33) evaluated at $(\beta J, \beta U)$. Recall that we can write the ATRC measure as in (2.34). Then, for any $c > 0$ and any increasing event A,

$$\frac{d}{d\beta}\mathtt{ATRC}^{1,1}_{\Omega,\beta}[A] - c\sum_{e\in E_\Omega}\mathbf{Cov}[\mathbb{1}_A, \omega_\tau(e) + \omega_{\tau'}(e)] = \mathbf{Cov}[\mathbb{1}_A, X_c],$$

where

$$X_c = \sum_{e\in E_\Omega}\left(\tfrac{r'}{r} - c\right)(\omega_\tau(e) + \omega_{\tau'}(e)) + \tfrac{s'}{s}\,\omega_\tau(e)\,\omega_{\tau'}(e).$$

It is easy to see that X_c is increasing in ω when $\beta \in [\varepsilon, \varepsilon^{-1}]$ and c is small enough. Then, by (FKG), $\mathbf{Cov}[\mathbb{1}_A, X_c] \geq 0$ and this finishes the proof. □

2.B Proof of Lemma 2.3.11

Proof of Lemma 2.3.11. Fix $q > 4$ and $p = p_{\mathrm{sd}}(q)$ and omit them in the notation below. Let $n \geq 1$ and $\Omega \supset \Lambda_{2n}$ be a finite subgraph of $\mathbb{L}$. By Proposition 2.3.10 and Strassen's theorem [Str65], there exists a coupling $\mathbf{P}$ of $\eta_- \sim \mathtt{FK}^{\mathrm{w}}$ and $\eta_+ \sim \mathtt{FK}^{\mathrm{w}}_\Omega$ such that $\mathbf{P}(\eta_- \leq \eta_+) = 1$.

Define $\mathcal{C}$ to be the outermost circuit of edges in η_- surrounding Λ_n and contained in Λ_{2n} (if there is no such circuit, set $\mathcal{C} := \emptyset$). By exponential decay of connections for the dual η_-^* (Theorem 9), there exists $\alpha > 0$ such that, for any $n \geq 1$,

$$\mathbf{P}[\mathcal{C} = \varnothing] \leq \mathtt{FK}^{\mathrm{w}}[\Lambda_n^* \overset{\eta_-^*}{\longleftrightarrow} \Lambda_{2n}^*] < 8ne^{-\alpha n}.$$

Take any event A depending only on edges in Λ_n. We have

$$\begin{aligned}|\mathtt{FK}^{\mathrm{w}}_\Omega[A] - \mathtt{FK}^{\mathrm{w}}[A]| &\leq |\mathbf{P}[\eta_+ \in A, \mathcal{C} \neq \varnothing] - \mathbf{P}[\eta_- \in A, \mathcal{C} \neq \varnothing]| + 16ne^{-\alpha n}\\ &= \sum_{C\neq\emptyset}|\mathbf{P}[\eta_+ \in A \mid \mathcal{C} = C] - \mathbf{P}[\eta_- \in A \mid \mathcal{C} = C]|\,\mathbf{P}[\mathcal{C} = C] + 16ne^{-\alpha n},\end{aligned}$$

where the sum runs over all realisations $C \neq \emptyset$ of $\mathcal{C}$. Now, the event $\{\mathcal{C} = C\}$ is measurable with respect to the exterior of C. Moreover, on $\{\mathcal{C} = C\}$, the circuit C is open both in η_+ and η_-, whence the distributions of η_+ and η_- in the interior of C are equal. □

2.C Proof of Lemma 2.4.6

Proof of Lemma 2.4.6. By a generalization of the Holley criterion, see [GHM01, Section 4], it suffices to check that, for all $e \in E_\Omega$ and $(\zeta_\tau, \zeta_{\tau\tau'}) \in \{(0,0),(0,1),(1,1)\}^{E_\Omega \setminus \{e\}}$, both

$$f_e(\mathrm{w}_\tau, \zeta_\tau, \zeta_{\tau\tau'}) := \mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}\left[\omega_\tau(e) = 1 \,\middle|\, (\omega_\tau, \omega_{\tau\tau'})|_{E_\Omega\setminus\{e\}} = (\zeta_\tau, \zeta_{\tau\tau'})\right],$$

$$g_e(\mathrm{w}_\tau, \zeta_\tau, \zeta_{\tau\tau'}) := \mathtt{ATRC}^{0,\mathbb{1}}_{\Omega,\mathrm{w}_\tau,\mathrm{w}_{\tau\tau'}}\left[\omega_{\tau\tau'}(e) = 1 \,\middle|\, (\omega_\tau, \omega_{\tau\tau'})|_{E_\Omega\setminus\{e\}} = (\zeta_\tau, \zeta_{\tau\tau'})\right]$$

are increasing in $(\zeta_\tau, \zeta_{\tau\tau'})$ and w_τ.

Fix $e, \zeta_\tau, \zeta_{\tau\tau'}$ as above. Write $\underline{\zeta_\tau}$ and $\overline{\zeta_\tau}$ for the configurations that agree with ζ_τ on $E_\Omega \setminus \{e\}$ while $\underline{\zeta_\tau}(e) = 0$ and $\overline{\zeta_\tau}(e) = 1$, and analogously for $\zeta_{\tau\tau'}$. Then,

$$\begin{aligned} f_e(\mathrm{w}_\tau, \psi, \zeta) &= \frac{\mathrm{w}_\tau(e) 2^{k(\overline{\zeta_\tau}) + k^1(\overline{\zeta_{\tau\tau'}})}}{2^{k(\underline{\zeta_\tau}) + k^1(\underline{\zeta_{\tau\tau'}})} + \mathrm{w}_{\tau\tau'}(e) 2^{k(\underline{\zeta_\tau}) + k^1(\overline{\zeta_{\tau\tau'}})} + \mathrm{w}_\tau(e) 2^{k(\overline{\zeta_\tau}) + k^1(\overline{\zeta_{\tau\tau'}})}} \\ &= \left(\frac{2^{k(\underline{\zeta_\tau}) - k(\overline{\zeta_\tau})} (2^{k^1(\underline{\zeta_{\tau\tau'}}) - k^1(\overline{\zeta_{\tau\tau'}})} + \mathrm{w}_{\tau\tau'}(e))}{\mathrm{w}_\tau(e)} + 1 \right)^{-1}, \end{aligned}$$

which is clearly increasing in w_τ. Moreover $k(\underline{\zeta_\tau}) - k(\overline{\zeta_\tau})$ and $k^1(\underline{\zeta_{\tau\tau'}}) - k^1(\overline{\zeta_{\tau\tau'}})$ are decreasing in $(\zeta_\tau, \zeta_{\tau\tau'})$, respectively.

We now check that $1 - g_e$ is decreasing in w_τ and $(\zeta_\tau, \zeta_{\tau\tau'})$. We have

$$\begin{aligned} 1 - g_e(\mathrm{w}_\tau, \zeta_\tau, \zeta_{\tau\tau'}) &= \frac{2^{k(\underline{\zeta_\tau}) + k^1(\underline{\zeta_{\tau\tau'}})}}{2^{k(\underline{\zeta_\tau}) + k^1(\underline{\zeta_{\tau\tau'}})} + \mathrm{w}_{\tau\tau'}(e) 2^{k(\underline{\zeta_\tau}) + k^1(\overline{\zeta_{\tau\tau'}})} + \mathrm{w}_\tau(e) 2^{k(\overline{\zeta_\tau}) + k^1(\overline{\zeta_{\tau\tau'}})}} \\ &= \left(1 + 2^{k^1(\overline{\zeta_{\tau\tau'}}) - k^1(\underline{\zeta_{\tau\tau'}})} (\mathrm{w}_{\tau\tau'}(e) + \mathrm{w}_\tau(e) 2^{k(\overline{\zeta_\tau}) - k(\underline{\zeta_\tau})})\right)^{-1}, \end{aligned}$$

which is clearly decreasing in w_τ and $(\zeta_\tau, \zeta_{\tau\tau'})$. □

2.D Equality of infinite-volume ATRC measures

In this section, we prove Lemmata 2.2.3 and 2.7.1.

The case $J < U$. The following statement will imply Lemma 2.2.3.

Lemma 2.D.1. *There exist two families of smooth curves $(\gamma_r^\tau)_{r>0}$ and $(\gamma_s^{\tau\tau'})_{s>0}$ such that (i): for any $0 < J < U$, there exist $s, r > 0$ such that $(J, U) \in \gamma_r^\tau \cap \gamma_s^{\tau\tau'}$, and (ii): for any $r, s > 0$, the set of points (J, U) on γ_r^τ (resp. $\gamma_s^{\tau\tau'}$) such that the marginals of $\mathtt{ATRC}^{0,0}_{J,U}$ and $\mathtt{ATRC}^{1,1}_{J,U}$ on ω_τ (resp. $\omega_{\tau\tau'}$) differ is at most countable.*

This implies that the set of pairs $J < U$, for which $\mathtt{ATRC}^{0,0}_{J,U}$ and $\mathtt{ATRC}^{1,1}_{J,U}$ have different marginals on ω_τ or $\omega_{\tau\tau'}$, has Lebesgue measure 0. By a monotone coupling argument, equality of both marginals implies that $\mathtt{ATRC}^{0,0}_{J,U} = \mathtt{ATRC}^{1,1}_{J,U}$, and Lemma 2.2.3 follows. We mention that $(\gamma_r^\tau)_{r>0}$ and $(\gamma_s^{\tau\tau'})_{s>0}$ are dual to each other.

Proof of Lemma 2.D.1. We follow a strategy presented in [Dum17, Theorem 1.12] which is a rephrased version of an argument in [LL72]. For any $J < U$, any finite subgraph $\Omega \subseteq \mathbb{L}$ and any boundary conditions η_τ and $\eta_{\tau\tau'}$, we can write

$$\mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{\Omega,J,U}[\omega_\tau, \omega_{\tau\tau'}] \propto \left(\frac{a(0,0)}{a(0,1)}\right)^{|E_\Omega| - |\omega_{\tau\tau'}|} \left(\frac{a(1,1)}{a(0,1)}\right)^{|\omega_\tau|} 2^{k^{\eta_\tau}(\omega_\tau) + k^{\eta_{\tau\tau'}}(\omega_{\tau\tau'})} \mathbb{1}_{\{\omega_\tau \subseteq \omega_{\tau\tau'}\}}.$$

Consider the curves where $a(0,0)/a(0,1)$ is constant (precisely if $U-J$ is constant). Fix an edge e of $\mathbb{L}$. The Holley criterion [Hol74] easily gives that the function $(J,U)\mapsto \mathtt{ATRC}^{1,1}_{J,U}[\omega_\tau(e)]$ is increasing along these curves, see e.g. the proof of [Gri06, Lemma 11.14]. In particular, the set of discontinuity points is countable along each of them. Fix $C>0$ and $(J,U),(J',U')$ on the curve $a(0,0)/a(0,1)\equiv C$ with $J'<J$ (then also $U'<U$), and assume that (J,U) is a continuity point. Define $a=\mathtt{ATRC}^{0,0}_{J,U}[\omega_\tau(e)]$ and $b=\mathtt{ATRC}^{1,1}_{J',U'}[\omega_\tau(e)]$.

Note that, along the curve $a(0,0)/a(0,1)\equiv C$, the quantity $a(1,1)/a(0,1)$ is strictly increasing. Using this fact, analogous reasoning as in [Dum17] gives that $a\geq b$. Letting (J',U') tend to (J,U) along $a(0,0)/a(0,1)\equiv C$ and using that (J,U) is a continuity point of $(J',U')\mapsto b$, we deduce

$$\mathtt{ATRC}^{0,0}_{J,U}[\omega_\tau(e)]\geq \mathtt{ATRC}^{1,1}_{J,U}[\omega_\tau(e)].$$

Since the reversed inequality follows from (CBC), we obtain equality. By considering a monotone coupling of the corresponding marginals on ω_τ, it is easy to see that this implies that the marginals on ω_τ coincide. The statement for $\omega_{\tau\tau'}$ is derived along the same lines when considering the curves where $a(1,1)/a(0,1)$ is constant.

□

The case $J\geq U$. Below is the analogue of Lemma 2.D.1 that implies Lemma 2.7.1.

Lemma 2.D.2. *There exists a family of smooth curves $(\gamma_r^\tau)_{r>0}$ such that (i): for any $J\geq U>0$, there exists $r>0$ for which $(J,U)\in\gamma_r^\tau$, and (ii): for any $r>0$, there exist only countably many points (J,U) on γ_r^τ for which $\mathtt{ATRC}^{0,0}_{J,U}\neq\mathtt{ATRC}^{1,1}_{J,U}$.*

Proof. For $J=U$, the model reduces to FK-percolation with cluster-weight $q=4$, and the statement follows from [Dum17, Theorem 1.12]. Recall that, for any $J>U$ and any finite subgraph $\Omega\subseteq\mathbb{L}$, we can write the measure as in (2.34). Consider the curves where the weight $\mathrm{w}:=(a(0,0)a(1,1))/(a(1,0)^2)$ is constant. The Holley criterion again allows to show that the ATRC measures are stochastically ordered along these lines. Moreover, the weight $a(1,0)/a(0,0)$ is increasing along each of them.

Fix an edge e of $\mathbb{L}$ and $C\geq 1$, and let (J,U) be a continuity point of $(J',U')\mapsto \mathtt{ATRC}^{1,1}_{J',U'}[\omega_\tau(e)]$ along the curve $\mathrm{w}\equiv C$. Take (J',U') on the same curve with $J'<J$. Analogous reasoning as in the proof of [Dum17, Theorem 1.12] gives

$$\mathtt{ATRC}^{1,1}_{J',U'}[\omega_\tau(e)]\leq \mathtt{ATRC}^{0,0}_{J,U}[\omega_\tau(e)],$$

with the only difference that one has to apply (FKG) to control $|\omega_\tau|$ and $|\omega_{\tau'}|$ simultaneously. Letting (J',U') tend to (J,U) along $\mathrm{w}\equiv C$ from below and using (CBC) gives $\mathtt{ATRC}^{0,0}_{J,U}[\omega_\tau(e)]=\mathtt{ATRC}^{1,1}_{J,U}[\omega_\tau(e)]$. A monotone coupling argument and symmetry between ω_τ and $\omega_{\tau'}$ finish the proof. □

Chapter 3

On antiferromagnetic regimes in the Ashkin–Teller model

The Ashkin–Teller model can be represented by a pair (τ, τ') of Ising spin configurations with coupling constants J and J' for each, and U for their product. We study this representation on the integer lattice $\mathbb{Z}^d$ for $d \geq 2$. We confirm the presence of a partial antiferromagnetic phase in the isotropic case $(J = J')$ when $-U > 0$ is sufficiently large and $J = J' > 0$ is sufficiently small, by means of a graphical representation. In this phase, τ is disordered, admitting exponential decay of correlations, while the product $\tau\tau'$ is antiferromagnetically ordered, which is to say that correlations are bounded away from zero but alternate in sign. No correlation inequalities are available in this part of the phase diagram. In the planar case $d = 2$, we construct a coupling with the six-vertex model and show, in analogy to the first result, that the corresponding height function is localised, although with antiferromagnetically ordered heights on one class of vertices of the graph.

We then return to $d \geq 2$ and consider a part of the phase diagram where $U < 0$ but where correlation inequalities still apply. Using the OSSS inequality, we proceed to establish a subcritical sharpness statement along suitable curves covering this part, circumventing the difficulty of the lack of general monotonicity properties in the parameters. We then address the isotropic case and provide indications of monotonicity.

3.1 Introduction

3.1.1 The Ashkin–Teller and related models

The *Ashkin–Teller* (AT) model [AT43] may be represented [Fan72a] by a pair (τ, τ') of interacting Ising spin configurations, which are random assignments of ± 1 *spins* to the vertices of some graph $\mathbb{G} = (\mathbb{V}_\mathbb{G}, \mathbb{E}_\mathbb{G})$. In its general form, the *spin representation* involves three *coupling constants* $J, J', U \in \mathbb{R}$, where J and J' describe the strength of interaction within each of the Ising configurations τ and τ', while U describes the strength of interaction for their product $\tau\tau'$ and hence the relation between τ and τ'. Its formal *Hamiltonian* is given by

$$-\sum_{\{x,y\} \in \mathbb{E}_\mathbb{G}} J\tau_x\tau_y + J'\tau'_x\tau'_y + U\tau_x\tau'_x\tau_y\tau'_y.$$

When J and J' coincide, the model is referred to as *isotropic*, and *anisotropic* otherwise. In the case $U = 0$, it reduces to two independent Ising models. For $J = J' = U$, the 4-state Potts model is recovered. In the *ferromagnetic* case, where all coupling constants are non-negative, it was conjectured in [Weg72] and later on in [WL74] that the AT model on the square lattice $\mathbb{Z}^2$ generally undergoes two *phase transitions*. This was verified when $U > J + J'$ [Pfi82] and when $U > J = J'$ [ADG24]. The phase diagram of the isotropic model on the square lattice was predicted in [Kno75, DBGK80], see also [Bax89, Chapter 12.9] and [HDJS13, GP23]. Its ferromagnetic part was established in [ADG24]. The presence of negative coupling constants results in preferential disagreement along the edges, which is referred to as *antiferromagnetic* interaction. This further leads to a loss of correlation inequalities and monotonicity properties in the parameters, thereby significantly complicating the analysis of the model.

In case the underlying graph is *bipartite*, such as $\mathbb{Z}^d$, performing spin flips for τ, τ' or both on one class of the vertices leads to a transformation of the AT measures by reversing the signs of any two of the coupling constants, while leaving the third one unchanged. Consequently, the main features of the model when J or J' are negative can be derived from those when both J and J' are positive. The antiferromagnetic regime to be studied is therefore $J, J' > 0 > U$, which is the focus of this chapter. In this regime, the product $\tau\tau'$ is favoured to disagree along the edges, while both τ and τ' are favoured to agree separately. When $J, J' < |U|$, the ferromagnetic effect is dominated by the antiferromagnetic one, and one may therefore expect to observe antiferromagnetic behaviour. In the current chapter, the model is shown to exhibit an antiferromagnetic phase for $\tau\tau'$ when $J = J'$ is small and $|U|$ is large, while also displaying ferromagnetic behaviour through a *sharp order-disorder* phase transition for τ when $J, J' > |U|$.

Planarity and relation to the eight-vertex model. If the underlying graph is *planar*, duality transformations [MS71, Fan72c] applied to two of the AT spin configurations τ, τ' and the product $\tau\tau'$ relate the AT models on the graph and on its *dual* to one another. Furthermore, on the square lattice $\mathbb{Z}^2$, by keeping one of the AT spin configurations fixed and applying a duality transformation to one of the other, a connection [Fan72a, Fan72b, Weg72, GP23] is established with the spin representation [Wu71, KW71, GM21, Lis22, GP23] of the *eight-vertex* model [Sut70, FW70] on the *medial graph*. Unless the initial AT model is self-dual, the weights of the corresponding eight-vertex model differ on two sublattices, in which case it is called *staggered* [HLW75]. In special cases when two of the AT coupling constants coincide, a certain weight vanishes in the related eight-vertex model and it reduces to the (staggered) *six-vertex* model [Pau35, Rys63, WL75]. The latter admits a *height function* representation, which has been studied in the self-dual isotropic case [DCKMO20, Lis21b, GP23, GL23]. In this particular case, the Baxter–Kelland–Wu coupling [BKW76] further relates the corresponding non-staggered six-vertex model to FK-percolation [FK72], see [GP23, ADG24].

3.1.2 Definition of the models

In this section, we introduce the Ashkin–Teller and the six-vertex height function models in a way that allows us to present our main results in the following section. See Section 3.2 for precise definitions regarding graphs and configurations and Section 3.5.1 regarding duality.

The Ashkin–Teller model. We consider the spin representation [Fan72a] of the Ashkin–Teller model [AT43] on the integer lattice $\mathbb{Z}^d$ whose edge-set we denote by $\mathbb{E}_{\mathbb{Z}^d}$. Define the spin configuration space $\Sigma_{\mathbb{Z}^d} := \{-1, 1\}^{\mathbb{Z}^d}$. Fix coupling constants $J, J', U \in \mathbb{R}$. Let $\Lambda \subset \mathbb{Z}^d$ be finite, and let $\eta, \eta' \in \Sigma_{\mathbb{Z}^d}$. Define Σ_Λ^η as the set of all $\tau \in \Sigma_{\mathbb{Z}^d}$ that coincide with η on $\mathbb{Z}^d \setminus \Lambda$. The Ashkin–Teller (AT) measure on Λ with coupling constants J, J', U and *boundary condition* (η, η') is the probability measure on $\Sigma_{\mathbb{Z}^d} \times \Sigma_{\mathbb{Z}^d}$ given by

$$\mathtt{AT}_{\Lambda,J,J',U}^{\eta,\eta'}[\tau, \tau'] := \frac{1}{Z} \exp\left(\sum_{\substack{\{x,y\} \in \mathbb{E}_{\mathbb{Z}^d}: \\ \{x,y\} \cap \Lambda \neq \varnothing}} J\tau_x\tau_y + J'\tau'_x\tau'_y + U\tau_x\tau'_x\tau_y\tau'_y \right) \mathbb{1}_{\{(\tau,\tau') \in \Sigma_\Lambda^\eta \times \Sigma_\Lambda^{\eta'}\}}, \tag{3.1}$$

where the normalisation $Z = Z(\Lambda, J, J', U, \eta, \eta')$ is called the *partition function*. We write $\langle \cdot \rangle^{\eta,\eta'}_{\Lambda,J,J',U}$ for the expectation operator with respect to the above measure. When η or η' are constant $+1$ or -1, we simply write $+$ or $-$ in the superscript, respectively. In the isotropic case, when $J = J'$, we omit J' from the subscripts and simply write $\mathtt{AT}^{\eta,\eta'}_{\Lambda,J,U}$ and $\langle \cdot \rangle^{\eta,\eta'}_{\Lambda,J,U}$.

We will also consider *free* boundary conditions and define $\Sigma^{\mathrm{f}}_\Lambda$ as the set of all $\tau \in \Sigma_{\mathbb{Z}^d}$ that are constant 1 on $\mathbb{Z}^d \setminus \overline{\Lambda}$, where $\overline{\Lambda}$ is the set of vertices in $\mathbb{Z}^d$ that belong to Λ or that are adjacent to vertices in Λ. Notice that the values of τ, τ' on $\mathbb{Z}^d \setminus \overline{\Lambda}$ have no influence on the exponential in (3.1). We then also allow the boundary conditions $\eta = \mathrm{f}$ and $\eta' = \mathrm{f}$ in (3.1). It should be noted that this definition of free boundary conditions is slightly different from the standard one in the Ising model, which we comment on in Remark 3.2.1.

The (staggered) six-vertex height function. Consider the dual square lattice with vertex-set $(\mathbb{Z}^2)^* := (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and edges between nearest neighbours. Regarded as line-segments between their endvertices, each edge e of $\mathbb{Z}^2$ intersects a unique edge e^* of $(\mathbb{Z}^2)^*$, and we call the set $e \cup e^*$ a *quad*. A (six-vertex) height function is a function $h : \mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to \mathbb{Z}$ satisfying

(i) for $x \in \mathbb{Z}^2$ and $x' \in (\mathbb{Z}^2)^*$ belonging to the same quad, $|h_x - h_{x'}| = 1$,

(ii) h takes even values on $\mathbb{Z}^2$: for all $x \in \mathbb{Z}^2$, $h_x \in 2\mathbb{Z}$.

We write Ω_{hf} for the set of all height functions. Denote by $h^\bullet$ and h° the restrictions of h to $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$, respectively. For a height function h and a set E of edges of $\mathbb{Z}^2$, the sets of disagreement edges of $h^\bullet$ and h° in E are defined respectively as

$$\begin{aligned} E_{h^\bullet} &:= \{e \in E : e = \{x, y\} \text{ and } h^\bullet_x \neq h^\bullet_y\}, \\ E_{h^\circ} &:= \{e \in E : e^* = \{x', y'\} \text{ and } h^\circ_{x'} \neq h^\circ_{y'}\}. \end{aligned} \tag{3.2}$$

Observe that condition (i) implies that $E_{h^\bullet}$ and E_{h° are disjoint. Fix $n \geq 1$, and define the boxes $\mathsf{B}_n := \{-n, \dots, n\}^2 \subset \mathbb{Z}^2$ and $\mathsf{B}^*_n := \{-n + \frac{1}{2}, \dots, n - \frac{1}{2}\}^2 \subset (\mathbb{Z}^2)^*$. Given a height function t and weights $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (0, \infty)$, define the corresponding (staggered) six-vertex height function probability measure on Ω_{hf} by

$$\mathtt{HF}^t_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}[h] := \frac{1}{Z}\, \mathbf{a}^{|E_{h^\circ}|}\, \mathbf{b}^{|E_{h^\bullet}|}\, \mathbf{c}^{|E \setminus (E_{h^\bullet} \cup E_{h^\circ})|}\, \mathbb{1}_{\{\forall x \in (\mathsf{B}_n \cup \mathsf{B}^*_n)^c :\, h_x = t_x\}}\, \mathbb{1}_{\{h \in \Omega_{\mathrm{hf}}\}}, \tag{3.3}$$

where $Z = Z(n, \mathbf{a}, \mathbf{b}, \mathbf{c}, t)$ is a normalising constant, and E is the set of edges of $\mathbb{Z}^2$ that intersect B_n. Here, 'staggered' refers to the fact that the measure assigns different weights to disagreements in $h^\bullet$ and h° when $\mathbf{a} \neq \mathbf{b}$, see Section 3.5 for details.

3.1.3 Statement of main results

We now present our main results. The proofs of each are based on a *graphical representation* of the AT model originally introduced in [PV97, CM97] and extended in the current chapter, see Sections 3.1.4 and 3.3. We emphasise that only the results based on duality are specific to the planar case. In particular, Theorems 10 and 12 and Proposition 3.1.1 hold on $\mathbb{Z}^d$ for any $d \geq 2$. Define the boxes $\mathsf{B}_n := \{-n, \dots, n\}^d$, $n \geq 1$.

Partial antiferromagnetic phase in the AT model. Let $d \geq 2$. We provide the first confirmation of the presence of a phase in the AT model in which τ is *disordered*, admitting exponential decay of correlations, whereas the product $\tau\tau'$ is *antiferromagnetically ordered*, which is to say that correlations are bounded away from zero but have alternating signs, see Figure 3.1.

This was predicted in [DBGK80], see also [Bax89, Chapter 12.9] and [HDJS13]. In the corresponding part of the phase diagram, no correlation inequalities such as Griffiths' second [Gri67] or the FKG inequality [FKG71] are available, and we rely on perturbative techniques. To lighten notation and avoid case distinctions, we restrict to the isotropic case $J = J'$.

We say that a subset $\Lambda \subseteq \mathbb{Z}^d$ is connected if its induced subgraph is connected. Define its even and odd parts by

$$\Lambda_{\text{even}} = \Big\{(x_i)_{1\le i\le d} \in \Lambda : \sum_{i=1}^{d} x_i \text{ even}\Big\} \quad \text{and} \quad \Lambda_{\text{odd}} = \Lambda \setminus \Lambda_{\text{even}}.$$

Define an alternating boundary condition $\eta^{\pm}$ by setting, for $x \in \mathbb{Z}^d$,

$$\eta^{\pm}_x := \begin{cases} +1 & \text{if } x \in \mathbb{Z}^d_{\text{even}}, \\ -1 & \text{if } x \in \mathbb{Z}^d_{\text{odd}}. \end{cases}$$

We simply write $\pm$ in the superscripts of the measures.

Theorem 10. *Let $d \ge 2$. There exists $\varepsilon = \varepsilon(d) > 0$ such that the following holds. For $J \in (0, \varepsilon)$ and $U \in (-\infty, -\frac{1}{\varepsilon})$, there exists $c = c(d, J, U) > 0$ such that, for any $n \ge 1$ and any $x \in \mathbb{Z}^d$,*

$$0 \le \langle \tau_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} \le e^{-c(n-\|x\|)} \qquad \textit{and} \qquad \langle \tau_x \tau'_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} \begin{cases} \ge c & \textit{if } x \in \mathbb{Z}^d_{\text{even}}, \\ \le -c & \textit{if } x \in \mathbb{Z}^d_{\text{odd}}, \end{cases} \tag{3.4}$$

where $\|x\|$ is the maximum norm of x. Furthermore, there exists a random connected set $\mathcal{C} \subseteq \mathbb{Z}^d$ satisfying $\tau_x \tau'_x = +1$ for all $x \in \mathcal{C}_{\text{even}}$ and $\tau_y \tau'_y = -1$ for all $y \in \mathcal{C}_{\text{odd}}$, and

$$\mathtt{AT}^{+,\pm}_{\mathsf{B}_n,J,U}[\exists\, \Lambda \subseteq \mathbb{Z}^d \setminus \mathcal{C} \textit{ connected with } \operatorname{diam}(\Lambda) > \log n] \le n^{-c}, \tag{3.5}$$

where $\operatorname{diam}(\Lambda)$ is diameter of Λ with respect to the maximum distance.

Ferroelectric phase in the staggered six-vertex model. Let $d = 2$. We present an analogue of Theorem 10 for the corresponding six-vertex height function. It states that, within the corresponding regime, the heights of even parity are antiferromagnetically ordered, whereas the heights of odd parity exhibit ferromagnetic order, see Figure 3.1. The latter is sufficient to keep the height function flat, meaning its variance is uniformly bounded in the size of the graph, which is termed *localisation* of the height function.

For $n \ge 1$, denote by $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$ the measure $\mathtt{HF}^{t}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$ when $t = 2\mathbb{1}_{\mathbb{Z}^2_{\text{odd}}} + \mathbb{1}_{(\mathbb{Z}^2)^*}$, see Figure 3.1.

Theorem 11. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} > 0$. There exists $\varepsilon > 0$ such that the following holds for $\frac{\mathbf{a}}{\mathbf{c}} < \varepsilon$ and $\frac{\mathbf{b}}{\mathbf{c}} > \frac{1}{\varepsilon}$. The variance of $h \sim \mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$ at the origin $0 \in \mathbb{Z}^2$ is uniformly bounded: there exists $C = C\left(\frac{\mathbf{a}}{\mathbf{c}}, \frac{\mathbf{b}}{\mathbf{c}}\right) < \infty$ such that, for every $n \ge 1$,*

$$\mathbf{Var}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}(h_0) < C,$$

where the variance is taken with respect to $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$. Furthermore, there exists $\alpha = \alpha\left(\frac{\mathbf{a}}{\mathbf{c}}, \frac{\mathbf{b}}{\mathbf{c}}\right) > 0$ and random connected sets $\mathcal{C} \subseteq \mathbb{Z}^2$ and $\mathcal{C}' \subseteq (\mathbb{Z}^2)^*$ such that:

(i) for any $x, y \in \mathcal{C}$ with $\{x, y\} \in \mathbb{E}_{\mathbb{Z}^2}$, $h_x \ne h_y$,

(ii) for any $x' \in \mathcal{C}'$, $h_{x'} = 1$,

(iii) $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}[\exists\, \Lambda \subseteq \mathbb{Z}^2 \setminus \mathcal{C}$ or $\Lambda \subseteq (\mathbb{Z}^2)^* \setminus \mathcal{C}'$ connected with $\operatorname{diam}(\Lambda) > \log n] \le n^{-\alpha}$.

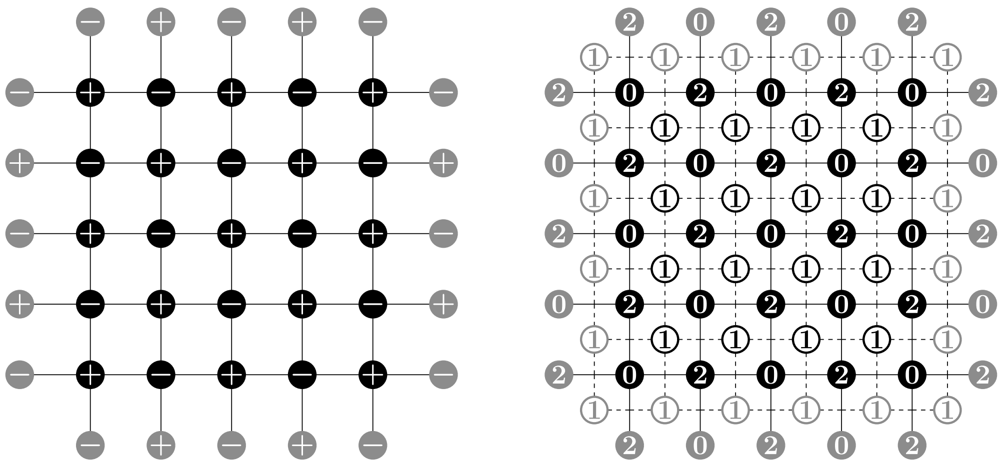


Figure 3.1: *Left:* The unique ground state $\eta^{\pm}$ of $\tau\tau'$ with respect to $\mathtt{AT}^{+,\pm}_{\mathsf{B}_2,J,U}$ on the box $\mathsf{B}_2 \subset \mathbb{Z}^2$ (black vertices) with external boundary (grey vertices) and edges of $\mathbb{Z}^2$ intersecting B_2. By Theorem 10, a typical configuration of $\tau\tau'$ is a small perturbation of $\eta^{\pm}$, whereas τ is disordered. *Right:* The unique ground state of $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$ on B_2 and its dual $\mathsf{B}_2^* \subset (\mathbb{Z}^2)^*$ (black hollow vertices), dual edges (dashed edges) and their endpoints outside B_2^* (grey hollow vertices). By Theorem 11, a typical height function is a small perturbation of this ground state. In relation to Theorem 10, the order in the odd heights corresponds to the disorder in τ, and the antiferromagnetic order in the even heights corresponds to the same property of $\tau\tau'$.

Subcritical sharpness along curves in the antiferromagnetic AT model. Let $d \geq 2$. We consider the part of the AT phase diagram where $U < 0$ but where certain correlation inequalities are still valid [PV97], which corresponds to

$$\min\{J, J'\} > 0 > U \quad \text{and} \quad \tanh U \geq -\tanh J \tanh J'. \tag{AF-FKG}$$

The equation above will be referred to as both the condition and the set of $(J, J', U) \in \mathbb{R}^3$ that satisfy it. One major difficulty in the analysis of this regime is the lack of general monotonicity properties in the coupling constants. To circumvent this issue, we consider the model along smooth curves covering (AF-FKG), and we show that the AT spin configuration τ undergoes a (subcritically) *sharp order-disorder phase transition* along each of them. The curves are chosen in such a way that a graphical representation of the model satisfies monotonicity properties and a suitable Russo-type inequality along them, allowing to run the general argument of [DRT19].

Given a parametrised curve $\gamma : (0,1) \to \mathbb{R}^3$ and $\beta \in (0,1)$, we write $\langle \cdot \rangle^{\eta,\eta'}_{\Lambda,\gamma(\beta)}$ for the expectation operator with respect to the measure in (3.1) with coupling constants given by $\gamma(\beta)$.

Theorem 12. *There exists a family of disjoint smooth curves $\gamma_{\kappa,\kappa'} : (0,1) \to \mathbb{R}^3$, $0 < \kappa < \kappa' \leq 1$, that covers (AF-FKG) and that satisfies the following. For $d \geq 2$ and any $\kappa, \kappa' \in (0,1]$ with $\kappa < \kappa'$, there exists $\beta_c = \beta_c(d, \kappa, \kappa') \in (0,1)$ such that*

- *for $\beta < \beta_c$, there exists $c_\beta = c_\beta(d, \kappa, \kappa') > 0$ such that $0 \leq \langle \tau_0 \rangle^{+,\mathrm{f}}_{\mathsf{B}_n,\gamma_{\kappa,\kappa'}(\beta)} \leq e^{-c_\beta n}$,*
- *there exists $c = c(d, \kappa, \kappa') > 0$ such that, for $\beta > \beta_c$, $\langle \tau_0 \rangle^{+,\mathrm{f}}_{\mathsf{B}_n,\gamma_{\kappa,\kappa'}(\beta)} \geq c(\beta - \beta_c)$.*

The above theorem provides a foundation for validating an analogous statement in the isotropic model when $d = 2$. In fact, once monotonicity properties that we conjecture are confirmed, a sharp phase transition at the self-dual curve $\sinh 2J = e^{-2U}$ [MS71] in the isotropic

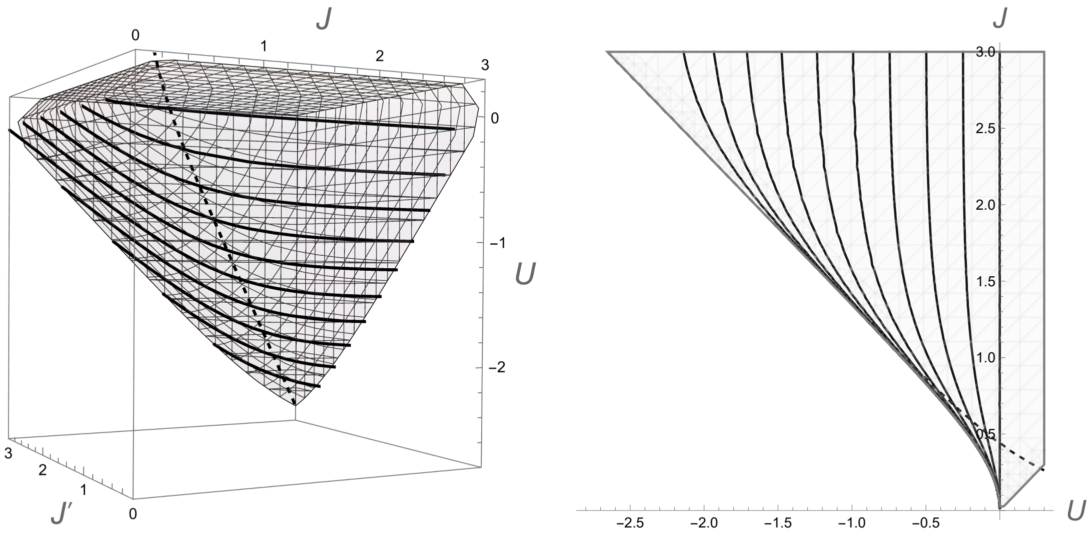


Figure 3.2: *Left:* Part of the regime (AF-FKG) (light grey area) and curves $\gamma_{\kappa,\kappa'}$ (solid black) that intersect the self-dual isotropic curve $\sinh 2J = e^{-2U}$ (dashed black), for ten values of κ, κ'. *Right:* Part of the regime (AF-FKG) (light grey area) in the isotropic phase diagram and curves $\widehat{\gamma}_\kappa$ (solid black) that intersect the self-dual curve (dashed black), for ten values of κ.

model is a consequence of Theorem 12. See Section 3.6.2, where we also show indications of monotonicity in the isotropic regime, which imply, for example, the following.

Proposition 3.1.1. *There exists a family of disjoint smooth curves $\widehat{\gamma}_\kappa : (0,1) \to \mathbb{R}^3$, $\kappa \in (0,1)$, covering the isotropic part of (AF-FKG), and smooth, strictly increasing, bijective functions $f_\kappa : (0,1) \to (0,\infty)$, $\kappa \in (0,1)$, such that the following holds for $d \geq 2$. For any $\kappa \in (0,1)$, any $\beta_1, \beta_2 \in (0,1)$ with $f_\kappa(\beta_1) \leq \kappa\, f_\kappa(\beta_2)$, and any $\Lambda \subset \mathbb{Z}^d$,*

$$\langle \tau_0 \rangle^{+,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta_1)} \leq \langle \tau_0 \rangle^{+,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta_2)}. \tag{3.6}$$

We also prove that, along the same curves $\widehat{\gamma}_\kappa$, a sum of infinite-volume edge densities of the graphical representation is non-decreasing, see Proposition 3.6.4. See Figure 3.2 for an illustration of the regime (AF-FKG) and the curves $\gamma_{\kappa,\kappa'}$ and $\widehat{\gamma}_\kappa$.

3.1.4 Overview and organisation of the chapter

The chapter is organised as follows.

- Section 3.2 is devoted to the introduction and clarification of some notation and conventions.
- In Section 3.3, we revisit a *graphical representation* (GAT) which, in certain parts of the phase diagram, coincides with those introduced in [PV97, CM97]. This is a *percolation* model obtained by means of an FK-Ising-type expansion for one of the AT spin configurations. In contrast to [PV97], where a pair of percolation configurations is considered, we restrict ourselves to one of them to enable greater generality. An immediate consequence is the validation of *Griffiths' first inequality* [Gri67] for the AT spin configuration τ when $J \geq \min\{|J'|, |U|\}$, see Corollary 3.3.5. Subsequently, we discuss the property of *positive association* of the graphical representation and its relation to [PV97] by constructing a coupling of a pair of them.

- In Section 3.4, we utilise the graphical representation to prove Theorem 10.
- In Section 3.5, we make use of the graphical representation again to formulate the connection between the AT and the eight-vertex models by means of coupling. We then discuss the special case of the six-vertex model and the relation of Theorems 10 and 11, and we present the proof of Theorem 11.
- In Section 3.6, we discuss the concept of subcritically sharp order-disorder phase transitions in the AT model. We then use the graphical representation once more to construct the curves from Theorem 12 satisfying the desired properties, and we prove the theorem. Following a brief discussion of the isotropic case, we propose a conjecture concerning monotonicity. The confirmation of this conjecture would enable us to deduce the presence of a sharp order-disorder phase transition in the isotropic model based on our findings for the anisotropic case. Subsequently, we substantiate this conjecture by presenting indications of monotonicity within the isotropic model, see Propositions 3.6.4 and 3.6.5.
- Appendices 3.A–3.D contain the proof of the FKG-lattice condition for the graphical representation (Lemma 3.3.7), the proof of maximality of certain boundary conditions (Lemma 3.6.1), the construction of the curves for Theorem 12 (Lemma 3.6.2), and the proof of a 'jump monotonicity' statement in the antiferromagnetic isotropic regime (Proposition 3.6.5).

3.2 Notation and conventions

Graph notation. Let $\mathbb{G} = (\mathbb{V}_\mathbb{G}, \mathbb{E}_\mathbb{G})$ be a graph. For an edge $\{x, y\} \in \mathbb{E}_\mathbb{G}$, we simply write $xy = \{x, y\}$. Given $V \subset \mathbb{V}_\mathbb{G}$, define

$$\begin{aligned}\overline{V} &:= V \cup \{y \in \mathbb{V}_\mathbb{G} : \exists x \in V,\, xy \in \mathbb{E}_\mathbb{G}\},\\ \overline{\mathbb{E}}_V &:= \{e \in \mathbb{E}_\mathbb{G} : e \cap V \neq \varnothing\}.\end{aligned}$$

A *path* in $\mathbb{G}$ is a sequence $\Gamma = (x_0, \dots, x_\ell)$ of pairwise distinct vertices $x_i \in \mathbb{V}_\mathbb{G}$ such that $x_i x_{i+1} \in \mathbb{E}_\mathbb{G}$ for $i = 0, \dots, \ell - 1$. We call $x_0x_1, \dots, x_{\ell-1}x_\ell$ the edges of Γ, and $\ell =: |\Gamma|$ the length of Γ. When $x_0 = x_\ell$, we call Γ a *circuit*.

The integer lattice has vertex-set $\mathbb{Z}^d = \{(x_i)_{1 \le i \le d} \in \mathbb{R}^d : x_i \in \mathbb{Z} \text{ for all } i\}$ and edges between nearest neighbours, that is, between vertices of Euclidean distance 1. In a slight abuse of notation, we will also write $\mathbb{Z}^d$ for the graph itself. For $n \ge 0$, define $\mathsf{B}_n := \{-n, \dots, n\}^d \subset \mathbb{Z}^d$ and $\mathsf{B}_{-1} = \varnothing$, and, for $x \in \mathbb{Z}^d$, let $\mathsf{B}_n(x)$ be the translate of B_n by x.

Spin and percolation configurations. Let $\mathbb{G} = (\mathbb{V}_\mathbb{G}, \mathbb{E}_\mathbb{G})$ be a graph. Given $V \subseteq \mathbb{V}_\mathbb{G}$, we call $s = (s_x)_{x \in V} \in \{-1, 1\}^V$ a *spin configuration* on V, and we write Σ_V for the set of all such configurations. When $V = \mathbb{V}_\mathbb{G}$, we simply write $\Sigma_\mathbb{G}$ or Σ. Moreover, given $\eta \in \Sigma_\mathbb{G}$, define

$$\begin{aligned}\Sigma_V^\eta &:= \{s \in \Sigma_\mathbb{G} : s_x = \eta_x \text{ for all } x \in \mathbb{V}_\mathbb{G} \setminus V\},\\ \Sigma_V^\mathrm{f} &:= \{s \in \Sigma_\mathbb{G} : s_x = 1 \text{ for all } x \in \mathbb{V}_\mathbb{G} \setminus \overline{V}\}.\end{aligned}$$

Remark 3.2.1. Observe that this definition of free boundary conditions is slightly different from the standard one in the Ising model, and in fact coincides with the latter on the graph $(\overline{V}, \overline{\mathbb{E}}_V)$. The reason for this choice is that the interactions for free boundary conditions and for those given by a configuration η will then occur along the same edges, thus simplifying the presentation and avoiding case distinctions.

The restriction $(s_x)_{x\in V}$ of s to V is denoted by s_V. For a pair $s, s' \in \Sigma_V$, we write $ss' := (s_x s'_x)_{x\in V}$ for their coordinatewise product. Given $E \subseteq \mathbb{E}_\mathbb{G}$, define E_s as the set of *disagreement edges* of s in E, that is,

$$E_s := \{xy \in E : s_x \neq s_y\}. \tag{3.7}$$

We call $\omega = (\omega_e)_{e\in E} \in \{0,1\}^E$ a *percolation configuration* on E, and we write Ω_E for the set of all such configurations. An edge $e \in E$ is said to be *open* (respectively, *closed*) in ω if $\omega_e = 1$ (respectively, $\omega_e = 0$), and a path or a set of edges is open (respectively, closed) if all its edges are open (respectively, closed). We identify ω with the set of open edges $\{e \in E : \omega_e = 1\}$ as well as with the subgraph $(\mathbb{V}_\mathbb{G}, \omega)$. The connected components of ω are called *clusters*. When $E = \mathbb{E}_\mathbb{G}$, we simply write $\Omega_\mathbb{G}$ or Ω. Moreover, given $\omega \in \Omega_\mathbb{G}$, we write Ω_E^ω for the set of percolation configurations on $\mathbb{E}_\mathbb{G}$ that coincide with ω on $\mathbb{E}_\mathbb{G} \setminus E$. The restriction $(\omega_e)_{e\in E}$ of ω to E is denoted by ω_E. Given $V \subset \mathbb{V}_\mathbb{G}$, we denote by $k_V(\omega)$ the number of clusters of ω that intersect $\overline{V}$. For $V, W \subset \mathbb{V}_\mathbb{G}$, we write $V \overset{\omega}{\leftrightarrow} W$ (or simply $V \leftrightarrow W$ when no confusion is possible) if there exists an open path in ω that connects a vertex in V to a vertex in W, and we write $V \overset{\omega}{\leftrightarrow} \infty$ (or simply $V \leftrightarrow \infty$) if some vertex in V belongs to an infinite cluster in ω. In this context, singleton sets are denoted without their braces.

The sets of configurations Σ_V and Ω_E as well as products of them are equipped with the natural product σ-algebra.

Conventions. We use bold letters $\mathbf{P}$, $\mathbf{E}$, $\mathbf{Var}$, $\mathbf{Cov}$ to denote probability measures, expectations, variances and covariances, respectively. To lighten notation, we will use the symbols τ, s, σ, ω for both deterministic configurations and random variables. More precisely, when we consider a random variable X, we may express its law by denoting a deterministic element in its target space by the same symbol X. For a probability measure μ on a measurable space $(M, \mathcal{M})$ and $m \in M$, we write $\mu[m]$ instead of $\mu[\{m\}]$. Given in addition a function $f : M \to [0,\infty)$, we write $\mu[m] \propto f(m)$, $m \in M$, if μ is proportional to f, that is, if there exists a constant $C > 0$ such that $\mu[m] = Cf(m)$ for all $m \in M$.

3.3 Graphical representation of the AT model

3.3.1 The representation

As in the classical Edwards–Sokal coupling [ES88], we are going to define a percolation model whose connection probabilities describe correlations of one of the AT spin configurations. It is obtained by performing an FK-Ising-type expansion for one of the AT spin configurations while keeping the other fixed. For certain ranges of the coupling constants, this representation was already considered in [PV97, CM97, Lis22, RS22, GP23, ADG24], see Remark 3.3.3. As the spin configuration may be either τ, τ' or the product $\tau\tau'$ and we formulate the coupling only once, we shall apply a change of variables later on, which reorders the coupling constants J, J', U. For instance, if (τ, τ') is distributed according to $\mathtt{AT}^{+,+}_{\Lambda,J,J',U}$, then $(\tau, \tau\tau')$ is distributed according to $\mathtt{AT}^{+,+}_{\Lambda,J,U,J'}$. For this purpose, we will use different symbols s, s' for spin configurations and different letters $K, K', K'' \in \mathbb{R}$ for the coupling constants. Define

$$p_1 = 1 - e^{-2(K-K'')} \qquad \text{and} \qquad p_2 = 1 - e^{-2(K+K'')}, \tag{3.8}$$

and, in order for them to be densities, we will make the assumption that

$$K \geq |K''|.$$

Let $\Lambda \subset \mathbb{Z}^d$ be finite, and set $E = \overline{\mathbb{E}}_\Lambda$. Fix boundary conditions $\eta \in \{+, \mathrm{f}\}$ and $\eta' \in \Sigma_{\mathbb{Z}^d} \cup \{\mathrm{f}\}$ arbitrary, and set $\# = \mathbb{1}_{\{\eta=+\}}$. Take (s, s') distributed according to $\mathtt{AT}^{\eta,\eta'}_{\Lambda,K,K',K''}$, and sample a percolation configuration $\omega \in \Omega_{\mathbb{Z}^d}$ as follows. Recall the definition (3.7) of E_s. Define ω_e independently for each edge $e \in \mathbb{E}_{\mathbb{Z}^d}$: if $e \notin E$, set $\omega_e = \#$. Otherwise, set $\omega_e = 0$ for $e \in E_s$, $\omega_e = 1$ with probability p_1 for $e \in E_{s'} \setminus E_s$, and $\omega_e = 1$ with probability p_2 for $e \in E \setminus (E_s \cup E_{s'})$. Formally, let $(U_e)_{e\in E}$ be i.i.d. uniform on $[0, 1]$, independent of (s, s'), and define

$$\omega_e = \begin{cases} \# & \text{if } e \notin E, \\ 0 & \text{if } e \in E_s, \\ \mathbb{1}_{[0,p_1]}(U_e) & \text{if } e \in E_{s'} \setminus E_s, \\ \mathbb{1}_{[0,p_2]}(U_e) & \text{if } e \in E \setminus (E_s \cup E_{s'}). \end{cases} \tag{3.9}$$

We denote the law of ω on $\Omega_{\mathbb{Z}^d}$ by $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$. Recall that, by convention, we use the same symbols for both random variables and their deterministic realisations.

Proposition 3.3.1. *(i) The joint law of (s, s', ω) on $(\Sigma_{\mathbb{Z}^d})^2 \times \Omega_{\mathbb{Z}^d}$ is given by*

$$\begin{aligned} \mathbf{P}\left[s, s', \omega\right] \propto e^{2(K''-K')|E_{s'}|} \left(e^{2(K-K'')} - 1\right)^{|\omega_E \cap E_{s'}|} \left(e^{2(K+K'')} - 1\right)^{|\omega_E \setminus E_{s'}|} \\ \cdot \mathbb{1}_{\{(s,s')\in\Sigma^\eta_\Lambda\times\Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\omega\in\Omega^\#_E\}} \mathbb{1}_{\{\omega\cap E_s=\varnothing\}}. \end{aligned} \tag{3.10}$$

In particular, conditionally on ω, the law of s_Λ is obtained as follows. If $\eta = \mathrm{f}$, assign ±1 uniformly and independently to clusters of ω that intersect Λ. If $\eta = +$, assign $+1$ to clusters of ω that intersect $\mathbb{Z}^d \setminus \Lambda$ and ±1 uniformly and independently to the other ones.

(ii) When $K + K'' > 0$, the law of ω on $\Omega_{\mathbb{Z}^d}$ is given by

$$\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[\omega] \propto \mathrm{w}_1^{|\omega_E|}\, 2^{k_\Lambda(\omega)}\, \mathbb{1}_{\{\omega\in\Omega^\#_E\}} \sum_{s'\in\Sigma^{\eta'}_\Lambda} \mathrm{w}_2^{|E_{s'}\setminus\omega|}\, \mathrm{w}_3^{|E_{s'}\cap\omega|}, \tag{3.11}$$

where the weights are defined as

$$\mathrm{w}_1 := e^{2(K+K'')} - 1, \quad \mathrm{w}_2 := e^{2(K''-K')} \quad \text{and} \quad \mathrm{w}_3 := \frac{e^{-2K'}\left(e^{2K} - e^{2K''}\right)}{e^{2(K+K'')} - 1}. \tag{3.12}$$

(iii) When $K + K'' = 0$, the law of ω on $\Omega_{\mathbb{Z}^d}$ is given by

$$\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[\omega] \propto \left(e^{4K} - 1\right)^{|\omega_E|}\, 2^{k_\Lambda(\omega)}\, \mathbb{1}_{\{\omega\in\Omega^\#_E\}} \sum_{\substack{s'\in\Sigma^{\eta'}_\Lambda:\\ E_{s'}\supseteq\omega_E}} e^{-2(K+K')|E_{s'}|}.$$

As a consequence of part (i) of the above proposition (see, e.g., [Gri06, Theorem 1.16]), correlations of s are described by connection probabilities of ω.

Corollary 3.3.2. *For any $x, y \in \Lambda$,*

$$\langle s_x s_y\rangle^{\eta,\eta'}_{\Lambda,K,K',K''} = \mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[x \leftrightarrow y] \quad \textit{and} \quad \langle s_x\rangle^{+,\eta'}_{\Lambda,K,K',K''} = \mathtt{GAT}^{1,\eta'}_{\Lambda,K,K',K''}[x \leftrightarrow \mathbb{Z}^d \setminus \Lambda].$$

The above proposition also provides an example of non-uniqueness of graphical representations. For instance, consider the AT measure in (3.1) with $J = J' > U > 0$. Then one can choose both $(s, s') = (\tau, \tau')$ and $(s, s') = (\tau, \tau\tau')$ to obtain two different percolation models for which the connection probabilities describe correlations of the AT spin configuration τ. Only the former enjoys the property of positive association, see Section 3.3.3.

Proof of Proposition 3.3.1. (i) Write $s_x s_y = 1 - 2\mathbb{1}_{\{s_x \neq s_y\}}$ and analogously for $s'_x s'_y$ to obtain that

$$\mathtt{AT}^{\eta,\eta'}_{\Lambda,K,K',K''}[s,s'] \propto e^{-2(K+K'')|E_s|}\, e^{-2(K'+K'')|E_{s'}|}\, e^{4K''|E_s \cap E_{s'}|}\, \mathbb{1}_{\{(s,s') \in \Sigma^\eta_\Lambda \times \Sigma^{\eta'}_\Lambda\}}. \tag{3.13}$$

Moreover, by (3.9), we have

$$\begin{aligned}
\mathbf{P}\left[s,s',\omega\right] &= \mathtt{AT}^{\eta,\eta'}_{\Lambda,K,K',K''}[s,s'] \cdot p_1^{|\omega_E \cap E_{s'}|}\,(1-p_1)^{|E_{s'} \setminus (\omega_E \cup E_s)|} \\
&\qquad \cdot p_2^{|\omega_E \setminus E_{s'}|}\,(1-p_2)^{|E \setminus (\omega_E \cup E_s \cup E_{s'})|}\, \mathbb{1}_{\{\omega \in \Omega^\#_E\}}\, \mathbb{1}_{\{\omega \cap E_s = \varnothing\}} \\
&= \mathtt{AT}^{\eta,\eta'}_{\Lambda,K,K',K''}[s,s'] \cdot p_1^{|\omega_E \cap E_{s'}|}\,(1-p_1)^{|E_{s'} \setminus E_s| - |\omega_E \cap E_{s'}|} \\
&\qquad \cdot p_2^{|\omega_E \setminus E_{s'}|}\,(1-p_2)^{|E| - |E_s| - |E_{s'} \setminus E_s| - |\omega_E \setminus E_{s'}|} \\
&\qquad \cdot \mathbb{1}_{\{\omega \in \Omega^\#_E\}}\, \mathbb{1}_{\{\omega \cap E_s = \varnothing\}},
\end{aligned}$$

where we have repeatedly used the fact that ω_E and E_s are disjoint, which is guaranteed by the corresponding indicator. Plugging in (3.8) and (3.13) and collecting like terms, we arrive at (3.10).

(ii) Writing $|\omega_E \setminus E_{s'}| = |\omega_E| - |\omega_E \cap E_{s'}|$ and collecting like terms, equation (3.10) turns into

$$\mathbf{P}\left[s,s',\omega\right] \propto \mathrm{w}_1^{|\omega_E|}\, \mathrm{w}_2^{|E_{s'} \setminus \omega|}\, \mathrm{w}_3^{|E_{s'} \cap \omega|}\, \mathbb{1}_{\{(s,s') \in \Sigma^\eta_\Lambda \times \Sigma^{\eta'}_\Lambda\}}\, \mathbb{1}_{\{\omega \in \Omega^\#_E\}}\, \mathbb{1}_{\{\omega \cap E_s = \varnothing\}}.$$

Summing over (s,s') while noting that there exist $2^{k_\Lambda(\omega) + \mathrm{const}(\Lambda,\#)}$ spin configurations $s \in \Sigma^\eta_\Lambda$ with $E_s \cap \omega = \varnothing$, we obtain (3.11).

(iii) When $K + K'' = 0$, equation (3.10) turns into

$$e^{-2(K+K')|E_{s'}|}\left(e^{4K} - 1\right)^{|\omega_E|}\, \mathbb{1}_{\{\omega_E \subseteq E_{s'}\}}\, \mathbb{1}_{\{(s,s') \in \Sigma^\eta_\Lambda \times \Sigma^{\eta'}_\Lambda\}}\, \mathbb{1}_{\{\omega \in \Omega^\#_E\}}\, \mathbb{1}_{\{\omega \cap E_s = \varnothing\}},$$

and summing over (s,s') as in (ii) finishes the proof. $\square$

Remark 3.3.3. Let ω be distributed according to $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$. The following equalities of laws hold up to boundary conditions and for Λ chosen accordingly.

(i) When $\min\{K,K'\} \geq \max\{K'',0\}$ and $\tanh K'' \geq -\tanh K \tanh K'$ (which implies $K \geq |K''|$), the law of ω coincides with the law of $\underline{n}_\sigma$ under $\nu^+_\Lambda(\,\cdot\,|2,2)$ with $J_\sigma = K$, $J_\tau = K'$ and $J_{\sigma\tau} = K''$ in [PV97, Proposition 3.1].

(ii) When $K > K' = K''$ and $e^{-2K} = \sinh 2K'$, the law of ω_E coincides with [GP23, Equation (7.2)] with $c = \coth 2K'$. Moreover, when $K = K''$ and $\sinh 2K = e^{-2K'}$, it coincides with the law of ξ^* in [GP23, Lemmata 7.1 and 8.1] with $c = \coth 2K$.

(iii) When $K > K' = K'' > 0$, the law of ω_E coincides with [Lis22, Equation (3.6)] for $a = \tanh 2K'$ and $b = (e^{2K} \cosh 2K')^{-1}$, where it appears in the context of the staggered six-vertex model, see Section 3.5.

(iv) When $K > K' = K''$ and $e^{-2K} = \sinh 2K'$, the law of ω coincides with the marginal of [RS22, Equation (5.10)] on η^0 with $\alpha = \coth(2K') - 2$ and $q = 2$.

(v) When $K = K'' \geq K'$ and $\sinh 2K = e^{-2K'}$, the law of ω_E coincides with the law of $\xi^\bullet$ for $a = b = 1$ and $c = \coth 2K \leq 2$ in [GL23, Section 4], where it is decomposed into a sum to recover positive association, which ω itself does not satisfy in that regime, see Section 3.3.3.

The graphical representation satisfies a strong *finite-energy* property which will be crucial for the perturbative analyses in Sections 3.4 and 3.5.2.

Lemma 3.3.4. *Let $K + K'' > 0$. For any $e \in E$ and any $\xi \in \Omega_E^\#$ with $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[\xi] > 0$,*

$$\left(1 + \mathrm{w}_1^{-1} 2\,\frac{\max\{1,\mathrm{w}_2\}}{\min\{1,\mathrm{w}_3\}}\right)^{-1} \le \mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}\left[\omega_e = 1 \,\middle|\, \omega_{\mathbb{E}_{\mathbb{Z}^d}\setminus\{e\}} = \xi_{\mathbb{E}_{\mathbb{Z}^d}\setminus\{e\}}\right] \le \left(1 + \mathrm{w}_1^{-1}\,\frac{\min\{1,\mathrm{w}_2\}}{\max\{1,\mathrm{w}_3\}}\right)^{-1},$$

where the weights w_i are given by (3.12), and where the quantity on the left side is defined to be 0 when $\mathrm{w}_3 = 0$, that is, when $K = K''$.

It should be mentioned that, for the case $\mathrm{w}_3 = 0$, one can also obtain a non-trivial lower bound, which is omitted as it is not used in this chapter.

Proof. Let $F = E \setminus \{e\}$. Equation (3.11) implies

$$\begin{aligned}
&\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}\left[\omega_e = 1 \,\middle|\, \omega_{\mathbb{E}_{\mathbb{Z}^d}\setminus\{e\}} = \xi_{\mathbb{E}_{\mathbb{Z}^d}\setminus\{e\}}\right] \\
&\quad = \left(1 + \mathrm{w}_1^{-1}\, 2^{k_\Lambda(\xi\setminus\{e\}) - k_\Lambda(\xi\cup\{e\})}\,\frac{\sum_{s'} \mathrm{w}_2^{|E_{s'}\setminus(\xi\setminus\{e\})|}\mathrm{w}_3^{|E_{s'}\cap(\xi\setminus\{e\})|}}{\sum_{s'} \mathrm{w}_2^{|E_{s'}\setminus(\xi\cup\{e\})|}\mathrm{w}_3^{|E_{s'}\cap(\xi\cup\{e\})|}}\right)^{-1} \\
&\quad = \left(1 + \mathrm{w}_1^{-1}\, 2^{k_\Lambda(\xi\setminus\{e\}) - k_\Lambda(\xi\cup\{e\})}\,\frac{\sum_{s'} \mathrm{w}_2^{|F_{s'}\setminus\xi|}\mathrm{w}_3^{|F_{s'}\cap\xi|}\mathrm{w}_2^{\mathbb{1}_{\{e\in E_{s'}\}}}}{\sum_{s'} \mathrm{w}_2^{|F_{s'}\setminus\xi|}\mathrm{w}_3^{|F_{s'}\cap\xi|}\mathrm{w}_3^{\mathbb{1}_{\{e\in E_{s'}\}}}}\right)^{-1},
\end{aligned}$$

where the summations are over all $s' \in \Sigma_\Lambda^{\eta'}$. The exponent $k_\Lambda(\xi \setminus \{e\}) - k_\Lambda(\xi \cup \{e\})$ is either 0 or 1, and the claim of the lemma follows. $\square$

3.3.2 Griffiths' first inequality

For this subsection, we return to the terminology of (3.1). As a direct consequence of Proposition 3.3.1, we obtain Griffiths' first inequality [Gri67] for τ when $J \ge \min\{|J'|, |U|\}$, which was previously established when $J, J', U \ge 0$ [KS68] and more generally when $\min\{J, J'\} \ge \max\{U, 0\}$ and $\tanh U \ge -\tanh J \tanh J'$ or the same holds with J' and U interchanged [PV97].

For $\tau \in \Sigma_{\mathbb{Z}^d}$ and $A \subset \mathbb{Z}^d$ finite, define

$$\tau^A := \prod_{x\in A} \tau_x.$$

Corollary 3.3.5. *Let $J, J', U \in \mathbb{R}$ with $J \ge \min\{|J'|, |U|\}$. Then, for $A \subseteq \Lambda \subset \mathbb{Z}^d$ finite,*

$$\langle \tau^A \rangle^{+,+}_{\Lambda,J,J',U} \ge 0. \tag{3.14}$$

Proof. It is classical (see, e.g., [PV97, Proposition 3.1] and its proof) that Proposition 3.3.1(i) implies that, for K, K', K'' with $K \ge |K''|$,

$$\langle s^A \rangle^{+,+}_{\Lambda,K,K',K''} = \mathtt{GAT}^{1,+}_{\Lambda,K,K',K''}[\kappa_A] \ge 0,$$

where κ_A is the event that every finite cluster of ω contains an even number of vertices in A. Applying this for $s = \tau$, $s' = \tau'$ and $(K, K', K'') = (J, J', U)$ gives (3.14) for $J \ge |U|$, while the choice $s = \tau$, $s' = \tau\tau'$ and $(K, K', K'') = (J, U, J')$ gives (3.14) for $J \ge |J'|$. $\square$

Remark 3.3.6. (i) For the isotropic AT model when $J = J'$, the condition in Corollary 3.3.5 reduces to $J \ge 0$, and is therefore optimal.

(ii) Another way to prove Corollary 3.3.5 is to consider τ as an Ising model with random coupling constants and then apply [KS68] to the conditional measures.

3.3.3 Positive association

In some part of the phase diagram, the graphical representation is *positively associated*. We point out that we require this property only in Section 3.6. Fix a graph $\mathbb{G} = (\mathbb{V}_{\mathbb{G}}, \mathbb{E}_{\mathbb{G}})$ and $E \subseteq \mathbb{E}_{\mathbb{G}}$ finite. We equip the set of percolation configurations Ω_E with a partial order $\leq$ which is defined coordinatewise: for $\omega, \omega' \in \Omega_E$, $\omega \leq \omega'$ if and only if $\omega_e \leq \omega'_e$ for all $e \in E$. A random variable $X : \Omega_E \to \mathbb{R}$ is called *increasing* if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. An event $A \subseteq \Omega_E$ is called increasing if its indicator $\mathbb{1}_A$ is increasing. Given two measures μ, ν on Ω_E, we say that μ is (stochastically) *dominated* by ν (or ν dominates μ), and we write $\mu \leq_{\mathrm{st}} \nu$ (or $\nu \geq_{\mathrm{st}} \mu$), if $\mathbf{E}_\mu[X] \leq \mathbf{E}_\nu[X]$ for all increasing $X : \Omega_E \to [0, \infty)$. Furthermore, we say that a measure μ on Ω_E is *positively associated* if it satisfies the *FKG inequality*:

$$\mathbf{E}_\mu[XY] \geq \mathbf{E}_\mu[X]\mathbf{E}_\mu[Y] \qquad \text{for all increasing } X, Y : \Omega_E \to [0, \infty). \tag{FKG}$$

Moreover, μ is said to satisfy the *FKG lattice condition* if

$$\mu[\omega \vee \omega']\mu[\omega \wedge \omega'] \geq \mu[\omega]\mu[\omega'] \qquad \text{for all } \omega, \omega' \in \Omega_E, \tag{FKG-L}$$

where $\omega \vee \omega'$ and $\omega \wedge \omega'$ refer to the coordinatewise maximum and minimum of ω and ω', respectively. If $\mu(\omega) > 0$ for all $\omega \in \Omega_E$, (FKG-L) implies (FKG) [FKG71, Hol74].

In the setting of Section 3.3.1, we can identify $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$ with $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[\omega_E \in \cdot]$ and regard it as a measure on Ω_E. With this identification, within a specific range of the coupling constants, the graphical representation satisfies the FKG lattice condition and, in particular, is positively associated.

Lemma 3.3.7. *Let $K, K', K'' \in \mathbb{R}$ with $K \geq K''$ and $K > -K''$, and let $\Lambda \subset \mathbb{Z}^d$ finite. For $\# \in \{0, 1\}$ and $\eta' \in \{+, \mathrm{f}\}$, the measure $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$ satisfies (FKG-L) if the weights (3.12) satisfy* $\max\{\mathrm{w}_2, \mathrm{w}_3\} \leq 1$, *which is equivalent to*

$$K' \geq K'' \quad \textit{and} \quad \tanh K'' \geq -\tanh K \tanh K'. \tag{3.15}$$

These conditions imply $K' \geq 0$. It is useful to note that, once we additionally impose $K' \neq 0$, the assumptions of the above lemma are symmetric in K and K'. Indeed, it is straightforward to check that they are then equivalent to

$$\min\{K, K'\} > 0, \quad \min\{K, K'\} \geq K'' \quad \text{and} \quad \tanh K'' \geq -\tanh K \tanh K'. \tag{3.16}$$

As we will see below in Section 3.3.4, in the FKG regime $\mathrm{w}_2, \mathrm{w}_3 \leq 1$, one may perform another FK-Ising-type expansion for the partition function in (3.11), which leads to a coupling of graphical representations for s and s'. The resulting joint law is precisely that in [PV97]. In this context, the statement of Lemma 3.3.7 is covered by [PV97, Proposition 4.1], and we provide a proof in Appendix 3.A for completeness.

3.3.4 Coupling of pair of graphical representations in FKG regime

In the setting of Section 3.3.1, fix $K, K', K'' \in \mathbb{R}$ satisfying (3.16), that is, the corresponding weights in (3.12) satisfy $\mathrm{w}_1 > 0$, $\mathrm{w}_2 \in (0, 1]$ and $\mathrm{w}_3 \in [0, 1]$ with $(\mathrm{w}_2, \mathrm{w}_3) \neq (1, 1)$. Let $\Lambda \subset \mathbb{Z}^d$ be finite, and set $E = \overline{\mathbb{E}}_\Lambda$. Fix boundary conditions $\eta, \eta' \in \{+, \mathrm{f}\}$, and set $\# = \mathbb{1}_{\{\eta = +\}}$ and $\#' = \mathbb{1}_{\{\eta' = +\}}$. Then there exist graphical representations for both configurations s and s'. Performing an FK-Ising-type expansion for the partition function in (3.11), we construct a coupling of them.

Define a measure on $\Omega_{\mathbb{Z}^d} \times \Omega_{\mathbb{Z}^d}$ by

$$\mathtt{ATRC}^{\#,\#'}_{\Lambda,K,K',K''}[\omega,\omega'] \propto \mathrm{u}_1^{|\omega_E \setminus \omega'_E|}\, \mathrm{u}_2^{|\omega'_E \setminus \omega_E|}\, \mathrm{u}_3^{|\omega_E \cap \omega'_E|}\, 2^{k_\Lambda(\omega)+k_\Lambda(\omega')}\, \mathbb{1}_{\{\omega \in \Omega_E^{\#}\}}\, \mathbb{1}_{\{\omega' \in \Omega_E^{\#'}\}}, \tag{3.17}$$

where the weights u_i are given by

$$\begin{aligned} \mathrm{u}_1 := \tfrac{\mathrm{w}_1\mathrm{w}_3}{\mathrm{w}_2} = e^{2(K-K'')} - 1, \qquad \mathrm{u}_2 := \tfrac{1}{\mathrm{w}_2} - 1 = e^{2(K'-K'')} - 1, \\ \mathrm{u}_3 := \tfrac{\mathrm{w}_1(1-\mathrm{w}_3)}{\mathrm{w}_2} = e^{2(K+K')} - e^{2(K-K'')} - e^{2(K'-K'')} + 1. \end{aligned} \tag{3.18}$$

Proposition 3.3.8. *The measure* $\mathtt{ATRC}^{\#,\#'}_{\Lambda,K,K',K''}$ *is a coupling of* $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$ *and* $\mathtt{GAT}^{\#',\eta}_{\Lambda,K',K,K''}$.

In Section 3.6.2, we will see how this representation indicates certain monotonicity properties in the parameters of the isotropic model even when $K'' = U$ is negative. Observe that, in the case $K = K''$, one has $\mathrm{u}_1 = 0$, whence $\omega_E \subseteq \omega'_E$ almost surely in the coupling (3.17), and analogously in the case $K' = K''$. As announced before, the measure in (3.17) is exactly that in [PV97], where it is also proved that the pair (ω, ω') is jointly positively associated if $K'' \geq 0$, while the same holds for $(\omega, -\omega')$ if $K'' < 0$.

Proof of Proposition 3.3.8. Consider the law in (3.11) and the weights in (3.12). Assume first that $\mathrm{w}_3 > 0$, that is, $K > K''$. Writing $|E_{s'} \setminus \omega| = |E| - |\omega_E| - |E \setminus (\omega_E \cup E_{s'})|$ and $|E_{s'} \cap \omega| = |\omega_E| - |\omega_E \setminus E_{s'}|$, we obtain

$$\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}[\omega] \propto \big(\tfrac{\mathrm{w}_1\mathrm{w}_3}{\mathrm{w}_2}\big)^{|\omega_E|}\, 2^{k_\Lambda(\omega)}\, \mathbb{1}_{\{\omega \in \Omega_E^{\#}\}} \sum_{s' \in \Sigma_\Lambda^{\eta'}} \big(\tfrac{1}{\mathrm{w}_2}\big)^{|E \setminus (\omega_E \cup E_{s'})|} \big(\tfrac{1}{\mathrm{w}_3}\big)^{|\omega_E \setminus E_{s'}|}.$$

Now, writing $\mathrm{w}_i^{-1} = 1 + (\mathrm{w}_i^{-1} - 1)$ for $i = 2, 3$, expanding and proceeding analogously to the proof of Proposition 3.3.1(ii), we deduce that

$$\begin{aligned} \sum_{s' \in \Sigma_\Lambda^{\eta'}} \big(\tfrac{1}{\mathrm{w}_2}\big)^{|E \setminus (\omega_E \cup E_{s'})|} \big(\tfrac{1}{\mathrm{w}_3}\big)^{|\omega_E \setminus E_{s'}|} &= \sum_{\substack{\omega'_1 \subseteq E \setminus \omega_E \\ \omega'_2 \subseteq \omega_E}} \big(\tfrac{1}{\mathrm{w}_2} - 1\big)^{|\omega'_1|} \big(\tfrac{1}{\mathrm{w}_3} - 1\big)^{|\omega'_2|} \sum_{s' \in \Sigma_\Lambda^{\eta'}} \mathbb{1}_{\{E_{s'} \cap (\omega'_1 \cup \omega'_2) = \varnothing\}} \\ &\propto \sum_{\omega' \in \Omega_E^{\#'}} \big(\tfrac{1}{\mathrm{w}_2} - 1\big)^{|\omega'_E \setminus \omega_E|} \big(\tfrac{1}{\mathrm{w}_3} - 1\big)^{|\omega'_E \cap \omega_E|}\, 2^{k_\Lambda(\omega')}. \end{aligned}$$

Therefore, $\mathtt{GAT}^{\#,\eta'}_{\Lambda,K,K',K''}$ is the first marginal of $\mathtt{ATRC}^{\#,\#'}_{\Lambda,K,K',K''}$. This remains valid when $\mathrm{w}_3 = 0$ (that is, $K = K''$), which is shown by a similar calculation or by taking the limit $\mathrm{w}_3 \to 0$. Then, by symmetry, the second marginal is given by $\mathtt{GAT}^{\#',\eta}_{\Lambda,K',K,K''}$. □

3.4 Proof of Theorem 10

The proof of Theorem 10 is a combination of classical energy-entropy arguments [Pei36] applied to the graphical representation (introduced in Section 3.3.1) of well chosen transformed variables. We mention that, while the antiferromagnetic order in the product $\tau\tau'$ may be verified via a low temperature expansion and a Peierls argument without making use of the graphical representation, a high temperature expansion involves negative weights, see [PV97, Section 2.2]. Before proceeding to the proof, we introduce the *spin-flip* maps and so-called *Peierls contours*. In regard to the latter, we follow [FV17, Chapter 5.7.4] and make appropriate adjustments.

Spin-flip map. For $A \subseteq \mathbb{Z}^d$, define $F_A : \Sigma_{\mathbb{Z}^d} \to \Sigma_{\mathbb{Z}^d}$ by setting

$$(F_A(s))_x = \begin{cases} -s_x & \text{if } x \in A, \\ s_x & \text{if } x \notin A. \end{cases} \tag{3.19}$$

For $A = \mathbb{Z}^d_{\text{odd}}$, we simply write F_{odd}.

Peierls contours. Associate to each $x \in \mathbb{Z}^d$ the closed unit hypercube $\mathcal{S}_x := x + [-\frac{1}{2}, \frac{1}{2}]^d$. Observe that the faces of $\mathcal{S}_x$, $x \in \mathbb{Z}^d$, which are $(d-1)$-dimensional hypercubes, are in one–one correspondence with the edges of $\mathbb{Z}^d$. We call these faces *plaquettes*. For $C \subset \mathbb{Z}^d$ finite and connected, the set $\mathcal{M}(C) := \cup_{x \in C} \mathcal{S}_x \subset \mathbb{R}^d$ is bounded and connected. Therefore, there exists a unique connected component $\gamma(C)$ of the boundary $\partial \mathcal{M}(C)$ (in the Euclidean sense) such that C is contained in the bounded connected component of $\mathbb{R}^d \setminus \gamma(C)$. Let $\mathcal{E}(C)$ be the set of edges of $\mathbb{Z}^d$ that, when regarded as line-segments between their endvertices, intersect $\gamma(C)$. Observe that, for a finite cluster C of some $\omega \in \Omega_{\mathbb{Z}^d}$, the set $\mathcal{E}(C)$ must be closed in ω. We say that a subset $F \subset \mathbb{E}_{\mathbb{Z}^d}$ *blocks* a vertex $x \in \mathbb{Z}^d$ if there exists a connected set $C \subset \mathbb{Z}^d$ with $x \in C$ and $\mathcal{E}(C) = F$. We call $F \subset \mathbb{E}_{\mathbb{Z}^d}$ *blocking* if it blocks some vertex $x \in \mathbb{Z}^d$.

In the proof of Theorem 10, we will need an upper bound on the number of blocking $F \subset E$ with $|F| = k$ for some fixed $E \subset \mathbb{E}_{\mathbb{Z}^d}$. We endow E with a graph structure by declaring two edges in E to be connected if the corresponding plaquettes share a $(d-2)$-dimensional hypercube. It is easy to see that the resulting graph has maximum degree $6(d-1)$. By construction, the set of blocking $F \subset E$ of size k is embedded in the set of connected subgraphs with k vertices of this graph. Using for example [FV17, Lemma 3.38], it is easy to derive the following crude bound.

Lemma 3.4.1. *Let $E \subseteq \mathbb{E}_{\mathbb{Z}^d}$ finite and $k \geq 1$. The number of blocking $F \subseteq E$ with $|F| = k$ is bounded by*

$$|E|(6(d-1))^{6(d-1)k}.$$

We are now ready to prove the theorem.

Proof of Theorem 10. Fix $d \geq 2$ and $n \geq 1$, and set $E = \overline{\mathbb{E}}_{\mathsf{B}_n}$. Assume that $-U > J > 0$ and let (τ, τ') be distributed according to $\mathtt{AT}^{+,\pm}_{\mathsf{B}_n,J,U}$. Observe that the statement of the theorem is trivial for $x \in \mathbb{Z}^d \setminus \mathsf{B}_n$, whence we fix $x \in \mathsf{B}_n$. The proof is divided into two steps.
Step 1. For $J \geq 0$ small enough, there exists $c = c(d, J) > 0$ such that $\langle \tau_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} \leq e^{-c(n - \|x\|)}$.

First, we apply the change of variables $s = \tau$, $s' = \tau\tau'$, and observe that (s, s') is distributed according to $\mathtt{AT}^{+,\pm}_{\mathsf{B}_n,J,U,J}$. The consequence, Corollary 3.3.2, of Proposition 3.3.1(i), applied to (s, s') and with $K = K'' = J$, $K' = U$, implies

$$\langle \tau_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} = \mathtt{GAT}^{1,\pm}_{\mathsf{B}_n,J,U,J}[x \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_n] \leq \sum_{\Gamma : x \to \mathbb{Z}^d \setminus \mathsf{B}_n} \mathtt{GAT}^{1,\pm}_{\mathsf{B}_n,J,U,J}[\Gamma \text{ open}],$$

where we used a union bound, and where the summation index $\Gamma : x \to \mathbb{Z}^d \setminus \mathsf{B}_n$ refers to all paths connecting x to $\mathbb{Z}^d \setminus \mathsf{B}_n$ in $\overline{\mathsf{B}_n}$. Recall Lemma 3.3.4, and observe that, by our assumption on J, U, the weights in (3.12) satisfy $\mathrm{w}_2 > 1$ and $\mathrm{w}_3 = 0$. Hence, for a path $\Gamma = (x_0, \ldots, x_\ell)$ with $e_i = x_i x_{i+1}$, Lemma 3.3.4 gives

$$\mathtt{GAT}^{1,\pm}_{\mathsf{B}_n,J,U,J}[\Gamma \text{ open}] = \prod_{i=0}^{\ell-1} \mathtt{GAT}^{1,\pm}_{\mathsf{B}_n,J,U,J}[\omega_{e_i} = 1 \,|\, \omega_{e_j} = 1 \,\forall\, j < i] \leq \left(1 + \mathrm{w}_1^{-1}\right)^{-\ell}. \tag{3.20}$$

Bounding the number of $\Gamma : x \to \mathbb{Z}^d \setminus \mathsf{B}_n$ of length ℓ crudely by $(2d)^\ell$, we obtain

$$\langle \tau_0 \rangle^{+,\pm}_{\mathsf{B}_n,J,U} \leq \sum_{\ell > n - \|x\|} \sum_{\substack{\Gamma : x \to \mathbb{Z}^d \setminus \mathsf{B}_n, \\ |\Gamma| = \ell}} \mathtt{GAT}^{1,\pm}_{\mathsf{B}_n,J,U,J}[\Gamma \text{ open}] \leq \sum_{\ell > n - \|x\|} \left(\tfrac{2d}{1 + \mathrm{w}_1^{-1}} \right)^\ell,$$

and since $\mathrm{w}_1 = e^{4J} - 1$ tends to zero as $J \to 0$, the claim follows.

Step 2. For $-J - U$ large enough, there exists $c = c(d, J+U) > 0$ such that

$$\langle \tau_x \tau'_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} \begin{cases} \geq c & \text{if } x \in \mathbb{Z}^d_{\text{even}}, \\ \leq -c & \text{if } x \in \mathbb{Z}^d_{\text{odd}}, \end{cases}$$

and $\mathcal{C}$ with the desired properties exists.

Recall the definition of F_{odd} in (3.19), and apply the change of variables $s = F_{\text{odd}}(\tau\tau')$, $s' = \tau$. Observe that the law of (s, s') is given by $\mathtt{AT}^{+,+}_{\mathsf{B}_n,-U,J,-J}$ and that

$$\langle \tau_x \tau'_x \rangle^{+,\pm}_{\mathsf{B}_n,J,U} = \big(\mathbb{1}_{\{x \in \mathbb{Z}^d_{\text{even}}\}} - \mathbb{1}_{\{x \in \mathbb{Z}^d_{\text{odd}}\}} \big) \langle s_x \rangle^{+,+}_{\mathsf{B}_n,-U,J,-J}.$$

Let $\widetilde{\omega}$ be distributed according to $\mathtt{GAT}^{1,+}_{\mathsf{B}_n,-U,J,-J}$, and denote by $\widetilde{\mathrm{w}}_i$ the corresponding weights in (3.12) with $K = -U$, $K' = -K'' = J$. Define $\mathcal{C}_x = \mathcal{C}_x(\widetilde{\omega}) \subseteq \mathbb{Z}^d$ to be the cluster of x in $\widetilde{\omega}$. Then, by Corollary 3.3.2,

$$\langle s_x \rangle^{+,+}_{\mathsf{B}_n,-U,J,-J} = 1 - \mathtt{GAT}^{1,+}_{\mathsf{B}_n,-U,J,-J}[\mathcal{E}(\mathcal{C}_x) \subseteq E] \geq 1 - \sum_{F \subseteq E} \mathtt{GAT}^{1,+}_{\mathsf{B}_n,-U,J,-J}[F \text{ closed}],$$

where we used that $\mathcal{C}_x \subseteq \mathsf{B}_n$ if and only if $\mathcal{E}(\mathcal{C}_x) \subseteq E$, and where the summation is over all non-empty $F \subseteq E$ that block x. The assumption $-U > J > 0$ implies that $\widetilde{\mathrm{w}}_2 < 1 < \widetilde{\mathrm{w}}_3$. Analogously to Step 1, for any $F \subseteq E$, Lemma 3.3.4 gives

$$\mathtt{GAT}^{1,+}_{\mathsf{B}_n,-U,J,-J}[F \text{ closed}] \leq \big(1 + \tfrac{\widetilde{\mathrm{w}}_1}{2}\big)^{-|F|}. \tag{3.21}$$

Furthermore, any $F \subseteq E$ with $|F| = k$ that blocks x necessarily satisfies $F \subseteq \mathbb{E}_{\mathsf{B}_k(x)} =: E^x_k$. Altogether, we obtain

$$\langle s_x \rangle^{+,+}_{\mathsf{B}_n,-U,J,-J} \geq 1 - \sum_{k \geq 2d} \sum_{\substack{F \subseteq E: \\ |F| = k}} \big(1 + \tfrac{\widetilde{\mathrm{w}}_1}{2}\big)^{-k} \geq 1 - \sum_{k \geq 2d} |E^x_k| (6(d-1))^{6(d-1)k} \big(1 + \tfrac{\widetilde{\mathrm{w}}_1}{2}\big)^{-k},$$

where we applied Lemma 3.4.1 to bound the number of blocking $F \subseteq E^x_k$ with $|F| = k$. Since $|E^x_k| = \mathcal{O}(k^d)$ and $\widetilde{\mathrm{w}}_1 = e^{-2(J+U)} - 1 \to \infty$ as $J + U \to -\infty$, the first claim of Step 2 is proved.

Finally, define $\mathcal{C} \subset \mathbb{Z}^d$ as the connected component of $\mathbb{Z}^d \setminus \mathsf{B}_n$ in $\widetilde{\omega} \sim \mathtt{GAT}^{1,+}_{\mathsf{B}_n,-U,J,-J}$, that is,

$$\mathcal{C} := \{ y \in \mathbb{Z}^d : y \overset{\widetilde{\omega}}{\leftrightarrow} \mathbb{Z}^d \setminus \mathsf{B}_n \}.$$

Then, by Proposition 3.3.1(i), $s = F_{\text{odd}}(\tau\tau')$ is constant $+1$ on $\mathcal{C}$, and hence $\tau\tau'$ is constant $+1$ on $\mathcal{C}_{\text{even}}$ and -1 on $\mathcal{C}_{\text{odd}}$. It remains to show (3.5) for $c = c(J+U)$ when $-J-U$ is large enough. If $\Lambda \subseteq \mathsf{B}_n \setminus \mathcal{C}$ is connected, we can assume without loss of generality that $\mathcal{E}(\Lambda) \cap \widetilde{\omega} = \varnothing$ (otherwise, Λ may be enlarged). Noticing that $|\mathcal{E}(\Lambda)|$ is greater than $\operatorname{diam}(\Lambda)$, bounding the number of blocking $F \subseteq E$ of a given size by applying Lemma 3.4.1, and using the bound (3.21), we deduce

$$\mathtt{AT}^{+,\pm}_{\mathsf{B}_n,J,U}[\exists \Lambda \subseteq \mathsf{B}_n \setminus \mathcal{C} \text{ connected with } \operatorname{diam}(\Lambda) > \log n] \leq \sum_{k > \log n} |E| (6(d-1))^{6(d-1)k} \big(1 + \tfrac{\widetilde{\mathrm{w}}_1}{2}\big)^{-k}.$$

Since $|E| = \mathcal{O}(n^d)$ and $\widetilde{\mathrm{w}}_1 \to \infty$ as $J + U \to -\infty$, the proof is complete. $\square$

3.5 Coupling with the eight-vertex model

Duality transformations applied to pairs of AT spin configurations relate the AT model on the square lattice $\mathbb{Z}^2$ and on its dual $(\mathbb{Z}^2)^*$ [MS71, Fan72c, PV97]. On the other hand, keeping one of the AT configurations and applying such a transformation to one of the other, results in a pair of spin configurations, one of which is defined on $\mathbb{Z}^2$ and the other on $(\mathbb{Z}^2)^*$. These configurations then follow the law of the spin representation [Wu71, KW71, GM21, Lis22, GP23] of a staggered (i.e. with non-translation-invariant weights [WL75, HLW75]) eight-vertex model [Sut70, FW70] on the medial graph. In special cases, the latter may be non-staggered or reduces to the six-vertex model [Pau35, Rys63].

The relation between the AT and the eight-vertex models on the square lattice has first been noticed in [Fan72a] comparing their critical properties, and it was made explicit in [Fan72b, Weg72] on a level of partition functions (see also [Wu77, HDJS13] and [Bax89, Chapter 12.9]). In [GP23], a coupling of the self-dual isotropic AT model and the six-vertex model (more precisely, the F-model [Rys63]) was constructed.

In Section 3.5.1, we use the graphical representation introduced in Section 3.3.1 to unify the above approaches and construct couplings of the AT and eight-vertex measures. In Section 3.5.2, we explain the relation between Theorems 10 and 11 and we prove the latter.

3.5.1 Duality coupling

Before constructing the coupling, we introduce some notation concerning the dual square lattice $(\mathbb{Z}^2)^*$ and discuss the duality map for percolation configurations.

Dual square lattice. The dual graph of the square lattice has vertex-set $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and edges between nearest neighbours, and we again write $(\mathbb{Z}^2)^*$ for the graph itself. Regarded as line-segments between their endvertices, each edge $e \in \mathbb{E}_{\mathbb{Z}^2}$ intersects a unique edge $e^* \in \mathbb{E}_{(\mathbb{Z}^2)^*}$, and we call e^* the edge dual to e and vice versa. For $E \subseteq \mathbb{E}_{\mathbb{Z}^2}$ and $s \in \Sigma_{V'}$ with $V' \subseteq (\mathbb{Z}^2)^*$, write E_s for the set of edges in E dual to disagreement edges of s, that is,

$$E_s := \{e \in E : e^* = x'y' \text{ and } s_{x'} \neq s_{y'}\}.$$

With each percolation configuration $\omega \in \Omega_{\mathbb{Z}^2}$ is associated a dual percolation configuration $\omega^* \in \Omega_{(\mathbb{Z}^2)^*}$ given by $\omega^*_{e^*} = 1 - \omega_e$, $e \in \mathbb{E}_{\mathbb{Z}^2}$. We call a finite subset $\Lambda \subset \mathbb{Z}^2$ a *domain* if all bounded faces of the graph $(\overline{\Lambda}, \overline{\mathbb{E}}_\Lambda)$ are unit squares, as illustrated in Figure 3.3. In this case, we define its dual $\Lambda^* \subset (\mathbb{Z}^2)^*$ as the set of vertices in $(\mathbb{Z}^2)^*$ at which the bounded faces of $(\overline{\Lambda}, \overline{\mathbb{E}}_\Lambda)$ are centred, as depicted in the figure. Then, for $E = \overline{\mathbb{E}}_\Lambda$ and $\omega \in \Omega^0_E$, the set of faces of $(\overline{\Lambda}, \omega)$ is in one–one correspondence with the set of clusters of ω^* that intersect $\overline{\Lambda^*}$ (see Figure 3.3), and Euler's formula for planar graphs implies

$$k_\Lambda(\omega) = |\overline{\Lambda}| - |\omega_E| + k_{\Lambda^*}(\omega^*) - 1, \tag{3.22}$$

see, e.g., [Gri06, Chapter 6.1].

Duality coupling. We formulate the coupling in the setting of Section 3.3.1. Let $K, K', K'' \in \mathbb{R}$ with $K \geq |K''|$, and define the corresponding eight-vertex weights

$$\begin{aligned} \mathbf{a} &= e^{2(K+K'')} - 1, & \mathbf{b} &= e^{-2K'}\big(e^{2K} + e^{2K''}\big), \\ \mathbf{c} &= e^{2(K+K'')} + 1, & \mathbf{d} &= e^{-2K'}\big(e^{2K} - e^{2K''}\big). \end{aligned} \tag{3.23}$$

Let $\Lambda \subset \mathbb{Z}^2$ be a domain, and set $E = \overline{\mathbb{E}}_\Lambda$. For convenience regarding Theorem 11, we take free boundary conditions $\eta = \mathrm{f}$ and arbitrary $\eta' \in \Sigma_{\mathbb{Z}^2}$. Take (s, s', ω) distributed as in Proposition 3.3.1(i). Sample a pair of spin configurations $\sigma = (\sigma^\bullet, \sigma^\circ) \in \Sigma^{\eta'}_\Lambda \times \Sigma^+_{\Lambda^*}$ as follows:

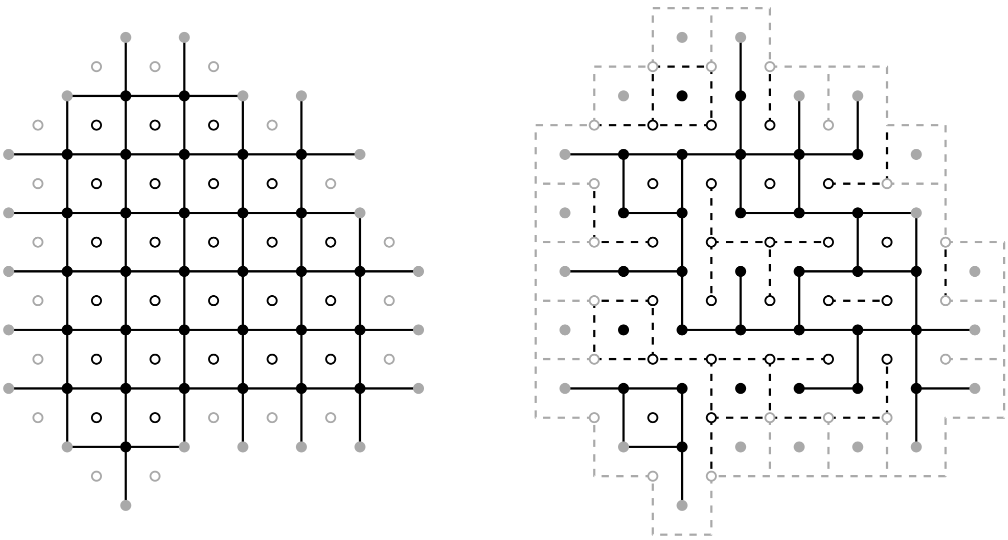


Figure 3.3: *Left:* A domain $\Lambda \subset \mathbb{Z}^2$ (black solid vertices) with external boundary $\overline{\Lambda} \setminus \Lambda$ (grey solid vertices), the edge set $E := \overline{\mathbb{E}}_\Lambda$ (black solid edges), and the dual $\Lambda^* \subset (\mathbb{Z}^2)^*$ (black hollow vertices) with external boundary $\overline{\Lambda^*} \setminus \Lambda^*$ (grey hollow vertices). *Right:* An edge configuration $\omega \in \Omega_E^0$ (black solid edges) and the relevant part of its dual ω^* (black and grey dashed edges, the grey ones are dual to edges outside E).

- set $\sigma^\bullet = s'$,
- sample σ° by assigning $+1$ to the cluster of ω^* that intersects $(\mathbb{Z}^2)^* \setminus \Lambda^*$ and ± 1 uniformly and independently to the clusters of ω^* that are contained in Λ^*.

We denote the law of σ on $\Sigma_{\mathbb{Z}^2} \times \Sigma_{(\mathbb{Z}^2)^*}$ by $\texttt{EightV}^{\eta',+}_{\Lambda,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$, and call it the staggered eight-vertex (spin) measure on $\Lambda \cup \Lambda^*$ with parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and boundary condition $(\eta', +)$. Recall that we conventionally use the same symbols for random variables and their deterministic realisations.

Proposition 3.5.1. *(i) The joint law of (s, s', ω, σ) on $(\Sigma_{\mathbb{Z}^2})^2 \times \Omega_{\mathbb{Z}^2} \times (\Sigma_{\mathbb{Z}^2} \times \Sigma_{(\mathbb{Z}^2)^*})$ is given by*

$$\begin{aligned}\mathbf{P}\left[s, s', \omega, \sigma\right] \propto\; & e^{2(K''-K')|E_{s'}|} \left(e^{2(K-K'')} - 1\right)^{|\omega_E \cap E_{s'}|} \left(e^{2(K+K'')} - 1\right)^{|\omega_E \setminus E_{s'}|} 2^{-k_{\Lambda^*}(\omega^*)} \\ & \cdot \mathbb{1}_{\{(s,s') \in \Sigma^{\mathrm{f}}_\Lambda \times \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\omega \in \Omega^0_E\}} \mathbb{1}_{\{\omega \cap E_s = \varnothing\}} \mathbb{1}_{\{\sigma^\bullet = s'\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}} \mathbb{1}_{\{E_{\sigma^\circ} \subseteq \omega\}}.\end{aligned} \tag{3.24}$$

It is a coupling of

$$(s, s') \sim \texttt{AT}^{\mathrm{f},\eta'}_{\Lambda,K,K',K''}, \quad \omega \sim \texttt{GAT}^{0,\eta'}_{\Lambda,K,K',K''} \quad \text{and} \quad \sigma \sim \texttt{EightV}^{\eta',+}_{\Lambda,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}},$$

where the relation between K, K', K'' and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is given by (3.23).
(ii) The law of σ on $\Sigma_{\mathbb{Z}^2} \times \Sigma_{(\mathbb{Z}^2)^*}$ is given by

$$\texttt{EightV}^{\eta',+}_{\Lambda,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}[\sigma] \propto \mathbf{a}^{|E_{\sigma^\circ} \setminus E_{\sigma^\bullet}|} \mathbf{b}^{|E_{\sigma^\bullet} \setminus E_{\sigma^\circ}|} \mathbf{c}^{|E \setminus (E_{\sigma^\bullet} \cup E_{\sigma^\circ})|} \mathbf{d}^{|E_{\sigma^\bullet} \cap E_{\sigma^\circ}|} \mathbb{1}_{\{\sigma^\bullet \in \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}}. \tag{3.25}$$

To obtain a relation with the classical representation of the eight-vertex model in terms of edge orientations, one may apply the maps in [Wu71, KW71] or [Lis22, GP23]. In the above setting, these are orientations of the edges of the medial graph of $\mathbb{Z}^2$, which is a rotated square lattice. We will only comment on this representation in Figure 3.4, and we focus on those in terms of spin configurations and height functions. We remark that the model is non-staggered,

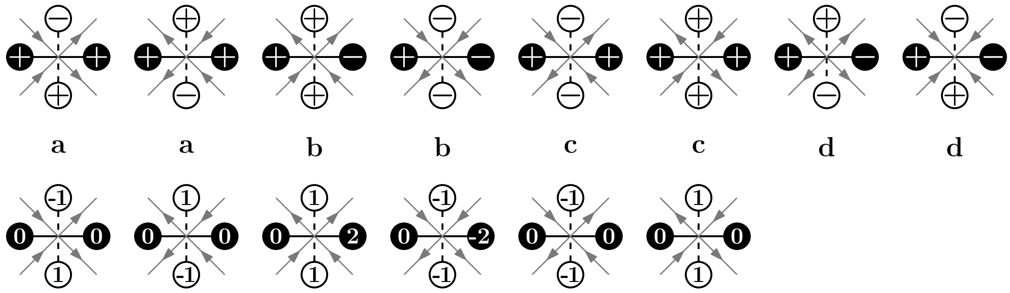


Figure 3.4: *Top:* the eight possible local eight-vertex spin configurations at an horizontal edge e of $\mathbb{Z}^2$ (solid black) and its dual e^* (dashed black) when the spin $\sigma^\bullet$ is fixed to be $+1$ at the left endpoint of e. The corresponding orientations of the edges of the medial graph (solid gray) are obtained by orienting an edge so that the vertex of $\mathbb{Z}^2$ (black) is on its left precisely if the spins separated by the edge coincide. *Centre:* the corresponding local weights. For the same sequence of edge orientations at a vertical edge of $\mathbb{Z}^2$, the weights $\mathbf{a}$ and $\mathbf{b}$ are interchanged. *Bottom:* the height function values corresponding to types 1-6 obtained by (3.27) when the height is fixed to be 0 at the left endpoint of e. In these cases, the edge orientations are visualisations of the discrete gradient of the height function.

meaning in our context that the same weights are assigned to disagreements in $\sigma^\bullet$ and σ°, precisely if $\mathbf{a} = \mathbf{b}$, which corresponds to the self-dual manifold of the AT model [MS71, Fan72c].

We emphasise that our choice of weights differs from the classical one in [Bax89]. In fact, the weights (a, b, c, d) in [Bax89, Equation (10.2.1)] coincide with our weights $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ on one sublattice of the medial graph and with $(\mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{d})$ on the other, see Figure 3.4.

Proof of Proposition 3.5.1. (i) While $\sigma^\bullet$ is uniquely determined by s', there are $2^{k_{\Lambda^*}(\omega^*)-1}$ possibilities for $\sigma^\circ \in \Sigma^+_{\Lambda^*}$ with $E_{\sigma^\circ} \subseteq \omega$ each of which is equally likely. Hence, the joint law of (s, s', ω, σ) is given by (3.24).
(ii) We first sum (3.24) over $s \in \Sigma^{\mathrm{f}}_\Lambda$ with $\omega \cap E_s = \varnothing$, of which there are $2^{k_\Lambda(\omega)}$, and then over $s' \in \Sigma^{\eta'}_\Lambda$, which is uniquely determined by $s' = \sigma^\bullet$, to obtain

$$
\begin{aligned}
\mathbf{P}\left[\omega, \sigma\right] &\propto e^{2(K''-K')|E_{\sigma^\bullet}|} \left(e^{2(K-K'')} - 1\right)^{|\omega_E \cap E_{\sigma^\bullet}|} \left(e^{2(K+K'')} - 1\right)^{|\omega_E \setminus E_{\sigma^\bullet}|} 2^{k_\Lambda(\omega) - k_{\Lambda^*}(\omega^*)} \\
&\qquad \cdot \mathbb{1}_{\{\omega \in \Omega^0_E\}} \mathbb{1}_{\{\sigma^\bullet \in \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}} \mathbb{1}_{\{E_{\sigma^\circ} \subseteq \omega\}} \\
&\propto e^{2(K''-K')|E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K-K'')}-1}{2}\right)^{|\omega_E \cap E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K+K'')}-1}{2}\right)^{|\omega_E \setminus E_{\sigma^\bullet}|} \\
&\qquad \cdot \mathbb{1}_{\{\omega \in \Omega^0_E\}} \mathbb{1}_{\{\sigma^\bullet \in \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}} \mathbb{1}_{\{E_{\sigma^\circ} \subseteq \omega\}},
\end{aligned}
$$

where we also used that $k_\Lambda(\omega) - k_{\Lambda^*}(\omega^*) = -|\omega_E| + \mathrm{const}(\Lambda)$, see (3.22) and above. Summing over $\omega = E_{\sigma^\circ} \cup \omega_1 \cup \omega_2$ with $\omega_1 \subseteq E_{\sigma^\bullet} \setminus E_{\sigma^\circ}$ and $\omega_2 \cap (E_{\sigma^\bullet} \cup E_{\sigma^\circ}) = \varnothing$,

$$
\begin{aligned}
\mathbf{P}\left[\sigma\right] &\propto e^{2(K''-K')|E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K-K'')}-1}{2}\right)^{|E_{\sigma^\circ} \cap E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K+K'')}-1}{2}\right)^{|E_{\sigma^\circ} \setminus E_{\sigma^\bullet}|} \mathbb{1}_{\{\sigma^\bullet \in \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}} \\
&\qquad \cdot \sum_{\omega_1 \subseteq E_{\sigma^\bullet} \setminus E_{\sigma^\circ}} \left(\tfrac{e^{2(K-K'')}-1}{2}\right)^{|\omega_1|} \sum_{\omega_2 \subseteq E \setminus (E_{\sigma^\bullet} \cup E_{\sigma^\circ})} \left(\tfrac{e^{2(K+K'')}-1}{2}\right)^{|\omega_2|} \\
&= e^{2(K''-K')|E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K-K'')}-1}{2}\right)^{|E_{\sigma^\circ} \cap E_{\sigma^\bullet}|} \left(\tfrac{e^{2(K+K'')}-1}{2}\right)^{|E_{\sigma^\circ} \setminus E_{\sigma^\bullet}|} \mathbb{1}_{\{\sigma^\bullet \in \Sigma^{\eta'}_\Lambda\}} \mathbb{1}_{\{\sigma^\circ \in \Sigma^+_{\Lambda^*}\}} \\
&\qquad \cdot \left(\tfrac{e^{2(K-K'')}+1}{2}\right)^{|E_{\sigma^\bullet} \setminus E_{\sigma^\circ}|} \left(\tfrac{e^{2(K+K'')}+1}{2}\right)^{|E \setminus (E_{\sigma^\bullet} \cup E_{\sigma^\circ})|}.
\end{aligned}
$$

Collecting like terms and multiplying by $2^{|E|}$, we arrive at (3.25), and the proof is complete. $\square$

The six-vertex model can be considered a special case of the eight-vertex model. Its characterisation is based on the *ice-rule*, which guarantees that the *height function* observable is well-defined. Via the map [Lis22, GP23], the ice-rule states that $E_{\sigma^\bullet}$ and E_{σ° must be disjoint, which is to say that

$$(\sigma^\bullet_x - \sigma^\bullet_y)(\sigma^\circ_{x'} - \sigma^\circ_{y'}) = 0 \qquad \text{for all } e = xy \in \mathbb{E}_{\mathbb{Z}^2} \text{ with } e^* = x'y'. \tag{3.26}$$

In the above notation, this is equivalent to $\mathbf{d} = 0$, which in turn is equivalent to $K = K''$. We then write $\mathtt{SixV}^{\eta',+}_{\Lambda,\mathbf{a},\mathbf{b},\mathbf{c}}$ for the measure $\mathtt{EightV}^{\eta',+}_{\Lambda,\mathbf{a},\mathbf{b},\mathbf{c},0}$, and we call it the staggered six-vertex (spin) measure on $\Lambda \cup \Lambda^*$ with parameters $\mathbf{a},\mathbf{b},\mathbf{c}$ and boundary condition $(\eta',+)$.

Height function representation of the six-vertex model. Recall the definitions of a quad and a (six-vertex) height function given in Section 3.1.2. A configuration $\sigma = (\sigma^\bullet, \sigma^\circ) \in \Sigma_{\mathbb{Z}^2} \times \Sigma_{(\mathbb{Z}^2)^*}$ that satisfies the ice-rule (3.26) defines a height function h up to an additive constant: for $x \in \mathbb{Z}^2$ and $x' \in (\mathbb{Z}^2)^*$ belonging to the same quad, define

$$h_{x'} - h_x = \sigma^\bullet_x \sigma^\circ_{x'}. \tag{3.27}$$

The ice-rule (3.26) guarantees that the sum of the four gradients (3.27) around each quad is zero. In particular, there is a one–one correspondence between such configurations σ and height functions that take a fixed value at some fixed vertex of $\mathbb{Z}^2$ or $(\mathbb{Z}^2)^*$, see Figure 3.4. Given h and σ related by (3.27), observe that h is constant on a fixed edge of $\mathbb{Z}^2$ or $(\mathbb{Z}^2)^*$ if and only if σ is:

$$\begin{aligned} &\text{for any } xy \in \mathbb{E}_{\mathbb{Z}^2}, \quad h_x = h_y \quad \text{if and only if} \quad \sigma^\bullet_x = \sigma^\bullet_y, \\ &\text{for any } x'y' \in \mathbb{E}_{(\mathbb{Z}^2)^*}, \quad h_{x'} = h_{y'} \quad \text{if and only if} \quad \sigma^\circ_{x'} = \sigma^\circ_{y'}. \end{aligned} \tag{3.28}$$

3.5.2 Relation of Theorems 10 and 11 and proof of Theorem 11

Recall the definition of the $\pm$-boundary condition given in Section 3.1.3. In order to understand the relationship between Theorems 10 and 11, consider the coupling (3.24) with $s = \tau$, $s' = \tau\tau'$, where τ and $\tau\tau'$ are as in Theorem 10 but (for convenience) with boundary conditions taken free for τ and $\pm$ for $\tau\tau'$. The law of the corresponding σ is then given by $\mathtt{SixV}^{\pm,+}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$. Since $\sigma^\bullet = \tau\tau'$ is antiferromagnetically ordered, the corresponding even heights on $\mathbb{Z}^2$ vary along edges of $\mathbb{Z}^2$. On the other hand, τ is disordered, admitting exponential decay of correlations, whence its dual σ° and therefore the corresponding odd heights on $(\mathbb{Z}^2)^*$ are ordered. The latter is sufficient to keep the height function flat, meaning its variance is uniformly bounded. The first step is to identify the law of the corresponding height function pinned to 1 on $(\mathbb{Z}^2)^* \setminus \mathsf{B}^*_n$ as the measure $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$ from Theorem 11.

Lemma 3.5.2. *Let $\mathbf{a},\mathbf{b},\mathbf{c} > 0$ and $n \geq 1$. Let $\sigma = (\sigma^\bullet, \sigma^\circ)$ be distributed according to $\mathtt{SixV}^{\pm,+}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$. There exists a unique random height function h related to σ by (3.27) and satisfying $h = 1$ on $(\mathbb{Z}^2)^* \setminus \mathsf{B}^*_n$. Its law is given by $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$.*

Proof. Since $\sigma^\circ \in \Sigma^+_{\mathsf{B}^*_n}$, there is a unique height function h satisfying (3.27) and $h = 1$ on $(\mathbb{Z}^2)^* \setminus \mathsf{B}^*_n$. Clearly, equation (3.27) and $\sigma^\bullet \in \Sigma^\pm_{\mathsf{B}_n}$ then imply that $h = 0$ on $\mathbb{Z}^2_{\text{even}} \setminus \mathsf{B}_n$ and $h = 2$ on $\mathbb{Z}^2_{\text{odd}} \setminus \mathsf{B}_n$.

Let $E = \overline{\mathbb{E}}_{\mathsf{B}_n}$. By (3.28), we have $E_{\sigma^\bullet} = E_{h^\bullet}$ and $E_{\sigma^\circ} = E_{h^\circ}$. Altogether, we conclude that the law of h is given by (3.3) with $t = 2\mathbb{1}_{\mathbb{Z}^2_{\text{odd}}} + \mathbb{1}_{(\mathbb{Z}^2)^*}$, and the proof is complete. □

Although some parts of the proof of Theorem 11 are now analogous to that of Theorem 10, it is not a direct consequence, whence we provide it for completeness.

Proof of Theorem 11. Fix $\mathbf{a},\mathbf{b},\mathbf{c}>0$ and $n\geq 1$, and set $E=\overline{\mathbb{E}}_{\mathsf{B}_n}$. Consider the coupling of

$$(\tau,\tau\tau')\sim \mathtt{AT}^{\mathrm{f},\pm}_{\mathsf{B}_n,J,U,J},\quad \omega\sim\mathtt{GAT}^{0,\pm}_{\mathsf{B}_n,J,U,J}\quad\text{and}\quad \sigma=(\sigma^\bullet,\sigma^\circ)\sim\mathtt{SixV}^{\pm,+}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$$

in Proposition 3.5.1 for $s=\tau$, $s'=\tau\tau'$, $K=K''=J, K'=U$, $\Lambda=\mathsf{B}_n$ and $\eta'=\eta^\pm$, and let w_i be the corresponding weights in (3.12). Observe that $\frac{\mathbf{a}}{\mathbf{c}}=\tanh 2J$ is small enough and $\frac{\mathbf{b}}{\mathbf{c}}=\frac{e^{-2U}}{\cosh 2J}$ is large enough if and only if $J>0$ small enough and $-U>0$ large enough. We assume henceforth that $-U>J>0$.

By Lemma 3.5.2, there exists a unique height function h related to σ by (3.27) and satisfying $h=1$ on $(\mathbb{Z}^2)^*\setminus\mathsf{B}_n^*$, and its law is given by $\mathtt{HF}^{0/2,1}_{\mathsf{B}_n,\mathbf{a},\mathbf{b},\mathbf{c}}$. The proof is divided into three steps.

Step 1. For $\frac{\mathbf{a}}{\mathbf{c}}$ sufficiently small, the variance of h is uniformly bounded.

Fix $N>0$. We are going to show that h has uniformly bounded Nth moment:

$$\mathbf{E}\left[|h_0|^N\right]=\sum_{k\geq 1}(2k)^N\,\mathbf{P}[|h_0|=2k].$$

Assume $|h_0|=2k$. By the definition of height functions, this implies $|h_{0'}|\geq 2k-1$, where $0'=(\frac12,\frac12)$. Since, by (3.28), h is constant on $e\in\mathbb{E}_{(\mathbb{Z}^2)^*}$ precisely if σ° is, we must have that σ° changes its sign at least $k-1$ times along any path in $(\mathbb{Z}^2)^*$ connecting $0'$ to $(\mathbb{Z}^2)^*\setminus\mathsf{B}_n^*$. However, by (3.24), σ° is constant on clusters of the dual ω^*, whence there exist at least $k-1$ disjoint circuits in ω within B_n that surround $0'$. One of these circuits must be of length $\ell\geq k$, of which there exist at most $\ell 4^\ell$. Thus, by Lemma 3.3.4 (as in (3.20)),

$$\mathbf{P}[|h_0|=2k]\leq\sum_{\ell\geq k}\ell 4^\ell\big(1+\mathrm{w}_1^{-1}\big)^{-\ell},$$

and $\mathrm{w}_1=e^{4J}-1\to 0$ as $J\to 0$. Since $\frac{\mathbf{a}}{\mathbf{c}}=\tanh 2J$, the claim is proved.

Step 2. Existence of c and $\mathcal{C}'$ when $\frac{\mathbf{a}}{\mathbf{c}}$ is sufficiently small.

Define $\mathcal{C}'$ as the set of all $x'\in(\mathbb{Z}^2)^*$ that are connected to $(\mathbb{Z}^2)^*\setminus\mathsf{B}_n^*$ in ω^*. Then σ° and thus (by (3.28)) h are constant on $\mathcal{C}'$. Since $h=1$ on $(\mathbb{Z}^2)^*\setminus\mathsf{B}_n^*\subseteq\mathcal{C}'$, property (ii) follows. If $\Lambda\subset(\mathbb{Z}^2)^*\setminus\mathcal{C}'$ is connected, there must exist a circuit of length $\ell\geq\operatorname{diam}(\Lambda)$ in ω within B_n that surrounds Λ. Similarly to Step 1 in the proof of Theorem 10, using Lemma 3.3.4, we deduce

$$\mathbf{P}[\exists\,\Lambda\subseteq(\mathbb{Z}^2)^*\setminus\mathcal{C}'\text{ connected with }\operatorname{diam}(\Lambda)>\log n]\leq\sum_{\ell\geq\log n}|\mathsf{B}_n|4^\ell\big(1+\mathrm{w}_1^{-1}\big)^{-\ell}.$$

Step 3. Existence of $\mathcal{C}$ when $\frac{\mathbf{b}}{\mathbf{c}}$ is sufficiently large.

Recall the definition of the map F_{odd} given in (3.19). Consider $(s,s')=(F_{\mathrm{odd}}(\tau\tau'),\tau)$ whose law is given by $\mathtt{AT}^{+,\mathrm{f}}_{\mathsf{B}_n,-U,J,-J}$, and let $\widetilde{\omega}$ be distributed according to $\mathtt{GAT}^{1,\mathrm{f}}_{\mathsf{B}_n,-U,J,-J}$. Denote the corresponding weights in (3.12) with $K=-U$, $K'=-K''=J$ by $\widetilde{\mathrm{w}}_i$. Define $\mathcal{C}$ as the set of all $x\in\mathbb{Z}^2$ that are connected to $\mathbb{Z}^2\setminus\mathsf{B}_n$ in $\widetilde{\omega}$. Then $F_{\mathrm{odd}}(\tau\tau')$ is constant on $\mathcal{C}$, and hence $\mathbb{E}_{\mathcal{C}}\subseteq E_{\tau\tau'}=E_{\sigma^\bullet}$. Since h is constant on $e\in\mathbb{E}_{\mathbb{Z}^2}$ if and only if $\sigma^\bullet$ is, property (i) follows. If $\Lambda\subset\mathbb{Z}^2\setminus\mathcal{C}$ is connected, there must exist a circuit of length $\ell\geq\operatorname{diam}(\Lambda)$ in $\widetilde{\omega}^*$ within B_n^* that surrounds Λ. As in Step 2 in the proof of Theorem 10, using Lemma 3.3.4, we deduce

$$\mathbf{P}[\exists\,\Lambda\subseteq\mathbb{Z}^2\setminus\mathcal{C}\text{ connected with }\operatorname{diam}(\Lambda)>\log n]\leq\sum_{\ell\geq\log n}|\mathsf{B}_n|4^\ell\big(1+\tfrac{\widetilde{\mathrm{w}}_1}{2}\big)^{-\ell},$$

and $\widetilde{\mathrm{w}}_1=e^{-2(J+U)}-1\to\infty$ as $J+U\to-\infty$. Since $\frac{\mathbf{b}}{\mathbf{c}}=\frac{e^{-2U}}{\cosh 2J}$, the claim follows. $\square$

3.6 Subcritical sharpness in the AT model

Let $d \geq 2$. In the ferromagnetic regime $J, J', U \geq 0$, the AT model enjoys monotonicity properties in the parameters. For instance, the *magnetisation* at the origin $\langle \tau_0 \rangle^{+,+}_{\mathsf{B}_n,J,J',U}$ is non-decreasing in J, J', U, which follows from the second Griffiths' inequality [Gri67] proved in this generality in [KS68] (see also [PV97]). For fixed $J, J', U > 0$, this implies existence of $\beta_{\mathrm{c}} = \beta_{\mathrm{c}}(d, J, J', U) \in [0, \infty]$ (a priori possibly trivial) such that

$$\langle \tau_0 \rangle^{+,+}_{\mathsf{B}_n,\beta} \begin{cases} \xrightarrow{n\to\infty} 0 & \text{if } \beta < \beta_{\mathrm{c}}, \\ \geq c_\beta > 0 & \text{if } \beta > \beta_{\mathrm{c}}, \end{cases} \tag{3.29}$$

where $\langle \cdot \rangle^{+,+}_{\mathsf{B}_n,\beta}$ is the expectation operator with respect to the AT measure with parameters $\beta(J, J', U)$. Then, the high-temperature expansion and a Peierls' argument [Pei36] (or comparison to the Ising model) show that $\beta_{\mathrm{c}} \in (0, \infty)$, meaning that the model undergoes a non-trivial order-disorder phase transition. The value of β_{c} is known explicitly only when $d = 2$ and $J = J' \geq U > 0$, where it coincides with the self-dual point [ADG24]. However, the strategy in [Gri95] (see also [HK23]) allows to show that the function $(J, J', U) \mapsto \beta_{\mathrm{c}}$ is continuous. The transition is said to be (subcritically) *sharp* if,

$$\text{for } \beta < \beta_{\mathrm{c}}, \text{ there exists } c_\beta = c_\beta(d, J, J', U) > 0 \text{ such that } \langle \tau_0 \rangle^{+,+}_{\mathsf{B}_n,\beta} \leq e^{-c_\beta n}.$$

The general approach [DRT19] also applies in this setting provided a graphical representation admits the following: *FKG lattice condition, maximal boundary condition, Russo-type inequality*, see Section 3.6.1 for details. When $J \geq \min\{J', U\} \geq 0$, these conditions are satisfied, see [ADG24, Appendix A] for the isotropic case. When $0 \leq J < \min\{J', U\}$, sharpness for τ can be derived from the same statement for τ' and $\tau\tau'$. Here we focus on the regime $U < 0$.

3.6.1 The case when U is negative: proof of Theorem 12

By Lemma 3.3.7, even when $U < 0$, there is still a regime in which `GAT` satisfies (FKG-L). However, one difficulty in proving sharpness is attributed to the fact that τ and τ' are negatively correlated [PV97], e.g., for $n \geq 1$ and $x \in \mathsf{B}_n$,

$$\langle \tau_x \tau'_x \rangle^{+,+}_{\mathsf{B}_n,J,J',U} \leq \langle \tau_x \rangle^{+,+}_{\mathsf{B}_n,J,J',U} \langle \tau'_x \rangle^{+,+}_{\mathsf{B}_n,J,J',U}, \tag{3.30}$$

which is strongly related to monotonicity in the parameters in the sense of stochastic domination, and which plays a role when deriving the Russo-type inequality. Below, we present a way to circumvent this issue by applying the strategy of [DRT19] along suitable curves in the AT phase diagram when $U < 0$. From now on, we restrict to the regime (AF-FKG) where $U < 0$ and where the graphical representation satisfies (FKG-L), which corresponds to

$$\big(\min\{J, J'\} > 0 > U \quad \text{and} \quad \tanh U \geq -\tanh J \tanh J'\big) \quad \Leftrightarrow \quad \mathrm{w}_2 < \mathrm{w}_3 \leq 1, \tag{3.31}$$

where the weights w_i are given by (3.12) with $(K, K', K'') = (J, J', U)$. Indeed, we have $U < 0$ precisely if $\mathrm{w}_2 < \mathrm{w}_3$, and the other conditions are those in (3.16). See Figure 3.2 for an illustration of the corresponding part of the phase diagram.

Before proceeding with the proof of Theorem 12 via the argument in [DRT19], it is necessary to identify the boundary condition $(1, \mathrm{f})$ as maximal and to validate a certain Russo-type inequality along suitable curves that cover (AF-FKG).

Maximal boundary condition and infinite-volume limit. In the regime (AF-FKG), the boundary condition $(1, \mathrm{f})$ is maximal for the graphical representation in the following sense.

Lemma 3.6.1. *Let J, J', U satisfy (AF-FKG). Let $\Lambda \subset \Delta \subset \mathbb{Z}^d$ be finite, and set $E_\Lambda = \overline{\mathbb{E}}_\Lambda$ and $E_\Delta = \overline{\mathbb{E}}_\Delta$. For any $\xi \in \Omega^1_{E_\Delta}$,*

$$\mathtt{GAT}^{1,\mathrm{f}}_{\Delta,J,J',U}[\omega_{E_\Lambda} \in \cdot \mid \omega_{E_\Delta \setminus E_\Lambda} = \xi_{E_\Delta \setminus E_\Lambda}] \leq_{\mathrm{st}} \mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,J,J',U}, \tag{3.32}$$

and in particular

$$\mathtt{GAT}^{1,\mathrm{f}}_{\Delta,J,J',U}[\omega_{E_\Lambda} \in \cdot] \leq_{\mathrm{st}} \mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,J,J',U}. \tag{3.33}$$

The proof is via the Holley criterion [Hol74] and is given in Appendix 3.B. Monotonicity in the domain (3.33) implies existence of the weak limit

$$\mathtt{GAT}^{1,\mathrm{f}}_{J,J',U} := \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,J,J',U}, \tag{3.34}$$

and that it is translation-invariant and tail-trivial, and hence mixing and ergodic (see, e.g., [Gri06, Chapter 4.3]).

Russo-type inequality along suitable curves. We introduce a family of curves covering the regime (AF-FKG) along which the weights $\mathrm{w}_2, \mathrm{w}_3$ in (3.12) remain constant, whereas w_1 covers $(0, \infty)$. A collection of these curves is depicted in Figure 3.2.

Given a curve $\gamma : (0, 1) \to \mathbb{R}^3$ with values in (AF-FKG), let $\mathrm{w}_i(\gamma(\beta))$ denote the weights in (3.12) evaluated at $(K, K', K'') = \gamma(\beta)$, and write $\mathrm{w}_i(\gamma)$ for the corresponding function. Moreover, we write $\mathtt{GAT}^{\#,\eta'}_{\Lambda,\gamma(\beta)}$ for the measure in (3.11) with coupling constants given by $\gamma(\beta)$. We use the same notational convention for their infinite volume limit.

Lemma 3.6.2. *There exists a family of disjoint smooth curves $\gamma_{\kappa,\kappa'} : (0, 1) \to \mathbb{R}^3$, $0 < \kappa < \kappa' \leq 1$, that satisfies the following properties:*

(i) J, J', U satisfy (AF-FKG) precisely if there exist κ, κ' and β such that $\gamma_{\kappa,\kappa'}(\beta) = (J, J', U)$,

(ii) for any κ, κ', the weights $\mathrm{w}_2(\gamma_{\kappa,\kappa'})$ and $\mathrm{w}_3(\gamma_{\kappa,\kappa'})$ are constant κ and κ', respectively,

(iii) for any κ, κ', $\mathrm{w}_1(\gamma_{\kappa,\kappa'}) : (0, 1) \to (0, \infty)$ is a bijection, and there exists $\varepsilon = \varepsilon(\kappa, \kappa') > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \log \mathrm{w}_1(\gamma_{\kappa,\kappa'}(\beta)) \geq \varepsilon.$$

The proof is straightforward calculus and is given in Appendix 3.C. By construction, the following Russo-type inequality holds along each curve $\gamma_{\kappa,\kappa'}$.

Lemma 3.6.3. *Let $\kappa, \kappa' \in (0, 1]$ with $\kappa < \kappa'$. There exists $\varepsilon = \varepsilon(\kappa, \kappa') > 0$ such that the following holds. For $\Lambda \subset \mathbb{Z}^d$ finite, $E = \overline{\mathbb{E}}_\Lambda$ and any increasing random variable $X : \Omega^1_E \to [0, \infty)$,*

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \mathbf{E}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}[X] \geq \varepsilon \, \mathbf{Cov}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}(X, |\omega_E|), \tag{3.35}$$

where the expectation and covariance are taken with respect to $\mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}$. In particular, $\mathbf{E}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}[X]$ is non-decreasing in β.

Proof. Fix $\kappa, \kappa', \Lambda, E$ as in the statement. For $X : \Omega^1_E \to [0, \infty)$, define

$$Z_\beta(X) := \sum_{\omega \in \Omega^1_E} \sum_{s' \in \Sigma^{\mathrm{f}}_\Lambda} X(\omega)\, \mathrm{w}_1^{|\omega_E|}\, \mathrm{w}_2^{|E_{s'} \setminus \omega|}\, \mathrm{w}_3^{|E_{s'} \cap \omega|}\, 2^{k_\Lambda(\omega)},$$

where $\mathrm{w}_i = \mathrm{w}_i(\gamma_{\kappa,\kappa'}(\beta))$. Then, since w_2 and w_3 are constant in β by Lemma 3.6.2(ii),

$$\frac{\mathrm{d}}{\mathrm{d}\beta} Z_\beta(X) = Z_\beta(X f_\beta) \quad \text{with} \quad f_\beta(\omega) = \left(\tfrac{\mathrm{d}}{\mathrm{d}\beta} \log \mathrm{w}_1\right) |\omega_E|.$$

Differentiating $\mathbf{E}^{1,\mathrm{f}}_{\Lambda, \gamma_{\kappa,\kappa'}(\beta)}[X] = \frac{Z_\beta(X)}{Z_\beta(1)}$ and applying the quotient rule gives that the left side of (3.35) coincides with $\mathbf{Cov}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}(X, f_\beta)$. By Lemma 3.6.2(iii), there exists $\varepsilon = \varepsilon(\kappa, \kappa') > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}\beta} \log \mathrm{w}_1 \geq \varepsilon$. Moreover, the covariance of X and $|\omega_E|$ is non-negative since both are increasing (in ω) and $\mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\gamma_{\kappa,\kappa'}(\beta)}$ satisfies (FKG) by Lemma 3.3.7. □

We are now ready to run the argument of [DRT19].

Proof of Theorem 12. Consider the curves $\gamma_{\kappa,\kappa'}$ in Lemma 3.6.2. Fix $d \geq 2$, $\kappa, \kappa' \in (0, 1]$ with $\kappa < \kappa'$ and $\beta_0 \in (0, 1)$. For $k, n \geq 0$ and $\beta \in (0, \beta_0)$, define $E_k := \mathbb{E}_{\mathsf{B}_k}$ and

$$\mu_{k,\beta} := \mathtt{GAT}^{1,\mathrm{f}}_{\mathsf{B}_{2k}, \gamma_{\kappa,\kappa'}(\beta)}, \qquad \theta_k(\beta) := \mu_{k,\beta}[0 \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{k-1}], \qquad S_n(\beta) := \sum_{k=0}^{n-1} \theta_k(\beta).$$

Then, by [DRT19, Lemma 3.2] applied to $\mu_{n,\beta}$,

$$\mathbf{Cov}_{\mu_{n,\beta}}(\mathbb{1}_{\{0 \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{n-1}\}}, |\omega_{E_{2n}}|) \geq \frac{n}{4 \max\limits_{x \in \mathsf{B}_n} \sum\limits_{k=0}^{n-1} \mu_{n,\beta}[x \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{k-1}(x)]} \theta_n(\beta)(1 - \theta_n(\beta)). \tag{3.36}$$

For $x \in \mathsf{B}_n$ and $2k \leq n$, Lemma 3.6.1 implies $\mu_{n,\beta}[x \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{k-1}(x)] \leq \theta_k(\beta)$, and thus

$$\sum_{k=0}^{n-1} \mu_{n,\beta}[x \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{k-1}(x)] \leq 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \mu_{n,\beta}[x \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{k-1}(x)] \leq 2 S_n(\beta).$$

Plugging this in (3.36) and combining the result with Lemma 3.6.3, we obtain

$$\theta_n' \geq \varepsilon \frac{n}{8 S_n} \theta_n \, (1 - \theta_n),$$

where $\varepsilon = \varepsilon(\kappa, \kappa') > 0$. By inclusion of events and Lemmata 3.6.1 and 3.6.3, $\theta_n(\beta) \leq \theta_0(\beta) \leq \theta_0(\beta_0) < 1$. We deduce that there exists $c = c(\kappa, \kappa', d, \beta_0) > 0$ such that

$$\theta_n' \geq c \frac{n}{S_n} \theta_n.$$

By [DRT19, Lemma 3.1] applied to $f_n = \theta_n / c$, there exists $\beta_1 = \beta_1(d, \kappa, \kappa') \in [0, \beta_0]$ such that,

- for $\beta < \beta_1$, there exists $c_\beta = c_\beta(d, \kappa, \kappa') > 0$ such that $\theta_n(\beta) \leq e^{-c_\beta n}$,
- for $\beta > \beta_1$, $\theta(\beta) = \lim_{n \to \infty} \theta_n(\beta)$ satisfies $\theta(\beta) \geq c(\beta - \beta_1)$,

where $\lim_{n\to\infty}\theta_n(\beta) = \mathtt{GAT}^{1,\mathrm{f}}_{\gamma_{\kappa,\kappa'}(\beta)}[0 \leftrightarrow \infty]$ as a consequence of (3.33) (see, e.g., [Gri06, Proposition 5.11]). By Lemma 3.6.2(iii), the weight $\mathrm{w}_1(\gamma_{\kappa,\kappa'}(\beta))$ in (3.12) tends to 0 as $\beta \searrow 0$, and it tends to infinity as $\beta \nearrow 1$. Thus, taking β_0 close enough to 1, Lemma 3.3.4 guarantees that $\beta_1 \in (0, \beta_0)$. Finally, Lemma 3.6.1 implies

$$\theta(\beta) \leq \mathtt{GAT}^{1,\mathrm{f}}_{\mathsf{B}_{2n},\gamma_{\kappa,\kappa'}(\beta)}[0 \leftrightarrow \mathbb{Z}^d \setminus \mathsf{B}_{2n}] \leq \theta_n(\beta).$$

Using the consequence, Corollary 3.3.2, of the coupling in Proposition 3.3.1(i), we derive the statement of the theorem for n even, and the case n odd is treated analogously. □

3.6.2 The isotropic case when U is negative

One of the main difficulties in proving sharpness in the isotropic case when $U < 0$ is that there is no known monotonicity in the parameters in the sense of stochastic domination, which is closely related to (3.30). We conjecture the following property which, in conjunction with Theorem 12, implies sharpness (as in Theorem 12) along all strictly increasing curves in the isotropic phase diagram when $d = 2$. Furthermore, the transition must occur at the self-dual curve defined by $\sinh 2J = e^{-2U}$ [MS71], as explained below.

Conjecture 1. *Let $\gamma : (0,1) \to \mathbb{R}^3$ be a curve in (AF-FKG) with all components strictly increasing. Then, for $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 < \beta_2$, there exists $\varepsilon > 0$ such that*

$$\langle \tau_0 \rangle^{+,\mathrm{f}}_{\mathsf{B}_n,J_1,J_1',U_1} \leq \langle \tau_0 \rangle^{+,\mathrm{f}}_{\mathsf{B}_n,J_2,J_2',U_2} \quad \textit{whenever} \quad (J_i, J_i', U_i) \in \mathcal{B}_\varepsilon(\gamma(\beta_i)),\ i = 1,2,$$

where $\mathcal{B}_r(x)$ denotes the three-dimensional Euclidean ball of radius $r > 0$ and centre $x \in \mathbb{R}^3$.

We will only sketch how to use this property to derive sharpness in the isotropic case from Theorem 12 when $d = 2$. The main task is to show that, if the curve $\gamma_{\kappa,\kappa'}$ from Lemma 3.6.2 passes through the self-dual curve in the isotropic phase diagram, then the transition point $\gamma_{\kappa,\kappa'}(\beta_c(\kappa,\kappa'))$ from Theorem 12 coincides with the intersection point with the self-dual curve. Indeed, the argument in the proof of [Dum17, Theorem 1.12] shows that, on the curve $\gamma_{\kappa,\kappa'}$, there are only countably many points at which there is more than one infinite-volume *Gibbs measure*. It is classical (see, e.g., the proof of [ADG24, Theorem 3]) that this property, together with sharpness, Theorem 12, and monotonicity along $\gamma_{\kappa,\kappa'}$, Lemma 3.6.3, implies that the self-dual point coincides with the transition point. Finally, continuity of the curves, Conjecture 1 and Theorem 12 easily imply the desired sharpness in the isotropic model.

We also mention that Conjecture 1 is valid in the ferromagnetic case, that is, when γ takes values in $(0,\infty)^3$. Indeed, differentiating with respect to J, J' and U and using the second Griffiths' inequality [KS68], one sees that the magnetisation at the origin is non-decreasing in each of the coupling constants.

Indications of monotonicity. We conclude this section by presenting indications of monotonicity of the graphical representation along suitable curves that cover the isotropic part of (AF-FKG), by means of the representation in Proposition 3.3.8. Assume henceforth that $J = J'$, U satisfy (AF-FKG), which is equivalent to

$$J = J' > 0 > U \quad \text{and} \quad \cosh 2J \geq e^{-2U}. \tag{iso-AF-FKG}$$

When $K = K' = J$, $K'' = U$ and $(\eta, \eta') = (+, \mathrm{f})$, equation (3.17) may be written as

$$\mathtt{ATRC}^{1,0}_{\Lambda,J,J,U}[\omega,\omega'] \propto \mathrm{u}_1^{|\omega_E|+|\omega'_E|} \left(\tfrac{\mathrm{u}_3}{\mathrm{u}_1^2}\right)^{|\omega_E\cap\omega'_E|} 2^{k_\Lambda(\omega)+k_\Lambda(\omega')}\, \mathbb{1}_{\{\omega\in\Omega^1_E\}}\, \mathbb{1}_{\{\omega'\in\Omega^0_E\}}, \tag{3.37}$$

where the weights are given by (3.12) and (3.18), and they satisfy

$$\mathrm{u}_1 = e^{2(J-U)} - 1 \quad \text{and} \quad \hat{\mathrm{u}}_3 := \tfrac{\mathrm{u}_3}{\mathrm{u}_1^2} = \tfrac{e^{4J}-2e^{2(J-U)}+1}{(e^{2(J-U)}-1)^2}. \tag{3.38}$$

Observe that $\hat{\mathrm{u}}_3 < 1$ precisely if $U < 0$. Now, with respect to this measure, consider the normalised expectation

$$\frac{1}{|E_n|}\mathbf{E}^{1,0}_{\mathsf{B}_n,J,J,U}\left[|\omega_{E_n}| + |\omega'_{E_n}|\right] = \frac{1}{|E_n|}\left(\mathbf{E}^{1,\mathrm{f}}_{\mathsf{B}_n,J,J,U}\left[|\omega_{E_n}|\right] + \mathbf{E}^{0,+}_{\mathsf{B}_n,J,J,U}\left[|\omega_{E_n}|\right]\right), \tag{3.39}$$

where we chose $\Lambda = \mathsf{B}_n$ and $E_n = \overline{\mathbb{E}}_{\mathsf{B}_n}$, and where the expectations on the right side are taken with respect to $\mathtt{GAT}^{1,\mathrm{f}}_{\mathsf{B}_n,J,J,U}$ and $\mathtt{GAT}^{0,+}_{\mathsf{B}_n,J,J,U}$, respectively, see Proposition 3.3.8. It is easy to see that the existence of the limit (3.34) and the fact that it is translation-invariant imply

$$\frac{1}{|E_n|}\mathbf{E}^{1,\mathrm{f}}_{\mathsf{B}_n,J,J,U}\left[|\omega_{E_n}|\right] \quad \xrightarrow{n\to\infty} \quad \mathtt{GAT}^{1,\mathrm{f}}_{J,J,U}[\omega_e = 1],$$

where e is any fixed edge of $\mathbb{Z}^d$. On the other hand, a proof exactly analogous to that of Lemma 3.6.1 shows that the boundary condition $(0,+)$ is minimal in the analogous sense, and hence the limit

$$\mathtt{GAT}^{0,+}_{J,J,U} := \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathtt{GAT}^{0,+}_{\Lambda,J,J,U} \tag{3.40}$$

exists, and it is translation-invariant and tail-trivial. Altogether, we deduce that

$$\frac{1}{|E_n|}\mathbf{E}^{1,0}_{\mathsf{B}_n,J,J,U}\left[|\omega_{E_n}| + |\omega'_{E_n}|\right] \quad \xrightarrow{n\to\infty} \quad \mathtt{GAT}^{1,\mathrm{f}}_{J,J,U}[\omega_e = 1] + \mathtt{GAT}^{0,+}_{J,J,U}[\omega_e = 1]. \tag{3.41}$$

We will consider the measure in (3.37) and the weights in (3.38) with parameters given by curves taking values in (iso-AF-FKG), using the same notational conventions as in the previous section. Similarly to Lemma 3.6.2, there exists a family of curves $\widehat{\gamma}_\kappa : (0,1) \to \mathbb{R}^3$, $\kappa \in (0,1)$, covering (iso-AF-FKG), along which the weight $\hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa)$ in (3.38) is constant κ whereas the weight $\mathrm{u}_1(\widehat{\gamma}_\kappa)$ is increasing and covers $(0,\infty)$. Moreover, along these curves, the coupling constants J and U are strictly increasing and decreasing, respectively. See Figure 3.2 for an illustration. Differentiating the expectation on the left side of (3.39) along these curves, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbf{E}^{1,0}_{\mathsf{B}_n,\widehat{\gamma}_\kappa(\beta)}\left[|\omega_{E_n}| + |\omega'_{E_n}|\right] = \left(\tfrac{\mathrm{d}}{\mathrm{d}\beta}\log \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta))\right)\mathbf{Var}^{1,0}_{\mathsf{B}_n,\widehat{\gamma}_\kappa(\beta)}\left(|\omega_{E_n}| + |\omega'_{E_n}|\right) \geq 0.$$

Let us summarise our findings in the following proposition.

Proposition 3.6.4. *There exists a family of disjoint smooth curves $\widehat{\gamma}_\kappa : (0,1) \to \mathbb{R}^3$, $\kappa \in (0,1)$, covering* (iso-AF-FKG) *along which the sum of the infinite-volume edge-densities*

$$\mathtt{GAT}^{1,\mathrm{f}}_{\widehat{\gamma}_\kappa(\beta)}[\omega_e = 1] + \mathtt{GAT}^{0,+}_{\widehat{\gamma}_\kappa(\beta)}[\omega_e = 1]$$

is non-decreasing in β. For each $\kappa \in (0,1)$, $\mathrm{u}_1(\widehat{\gamma}_\kappa) : (0,1) \to (0,\infty)$ is strictly increasing and bijective, whereas $\hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa)$ is constant κ. Furthermore, along $\widehat{\gamma}_\kappa$, the coupling constants J and U are strictly increasing and decreasing, respectively.

Moreover, the Holley criterion [Hol74] allows to show that, for the same curves $\widehat{\gamma}_\kappa$, the measures $\mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta)}$ satisfy a kind of 'jump monotonicity' in the stochastic sense, and it provides insight into why the regime $U < 0$ is problematic.

Proposition 3.6.5. *Let $\kappa \in (0,1)$, and consider the curve $\widehat{\gamma}_\kappa$ from Proposition 3.6.4. Let $\beta_1, \beta_2 \in (0,1)$ with $\mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1)) \le \kappa\, \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))$. Then, for any finite $\Lambda \subset \mathbb{Z}^d$,*

$$\mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta_1)} \le_{\mathrm{st}} \mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta_2)}. \tag{3.42}$$

There exist analogues of the curves $\widehat{\gamma}_\kappa$ when $U \ge 0$, in which case the weight $\hat{\mathrm{u}}_3$ in (3.38) satisfies $\hat{\mathrm{u}}_3 \ge 1$. The conditions for (3.42) imposed by the Holley criterion are then weaker and reduce to $\mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1)) \le \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))$, that is, $\beta_1 \le \beta_2$. In conclusion, the GAT measures are stochastically ordered along the respective curves when $U \ge 0$. However, when $U < 0$, the Holley criterion guarantees stochastic ordering only if one 'jumps a sufficient distance'.

The proof of Proposition 3.6.5 is given in Appendix 3.D. Notice that, together with Corollary 3.3.2, the above propositions imply Proposition 3.1.1.

3.A FKG lattice condition for GAT

Proof of Lemma 3.3.7. Let K, K', K'' with $K \ge K''$ and $K > -K''$, and omit them from the subscripts. Fix $\Lambda \subset \mathbb{Z}^d$ finite, and set $E = \overline{\mathbb{E}}_\Lambda$. Let $\# \in \{0,1\}$ and $\eta' \in \{+,\mathrm{f}\}$.

By [Gri06, Theorem 2.22], it suffices to show (FKG-L) for $\omega, \omega' \in \Omega^\#_E$ that agree everywhere except at two edges $e, f \in E$. Regard $\omega \in \Omega^\#_E$ as a set, and assume that $e, f \notin \omega$. We have to show that

$$\mathtt{GAT}^{\#,\eta'}_\Lambda[\omega \cup \{e,f\}]\, \mathtt{GAT}^{\#,\eta'}_\Lambda[\omega] \ge \mathtt{GAT}^{\#,\eta'}_\Lambda[\omega \cup \{e\}]\, \mathtt{GAT}^{\#,\eta'}_\Lambda[\omega \cup \{f\}].$$

The factor $\mathrm{w}_1^{|\omega_E|}\, 2^{k_\Lambda(\omega)}$ in (3.11) satisfies the corresponding inequality (see, e.g., [Gri06, Theorem 3.8]). Setting $Z(\zeta) := \sum_{s' \in \Sigma^{\eta'}_\Lambda} \mathrm{w}_2^{|E_{s'} \setminus \zeta|}\, \mathrm{w}_3^{|E_{s'} \cap \zeta|}$, it remains to show that

$$Z(\omega \cup \{e,f\})\, Z(\omega) \ge Z(\omega \cup \{e\})\, Z(\omega \cup \{f\}). \tag{3.43}$$

Let $F = E \setminus \{e,f\}$, and observe that

$$Z(\zeta) = \sum_{s' \in \Sigma^{\eta'}_\Lambda} \mathrm{w}_2^{|F_{s'} \setminus \zeta|}\, \mathrm{w}_3^{|F_{s'} \cap \zeta|} \mathrm{w}_2^{\mathbb{1}\{e \in E_{s'} \setminus \zeta\} + \mathbb{1}\{f \in E_{s'} \setminus \zeta\}}\, \mathrm{w}_3^{\mathbb{1}\{e \in E_{s'} \cap \zeta\} + \mathbb{1}\{f \in E_{s'} \cap \zeta\}}. \tag{3.44}$$

Define μ as the Ising measure on $\Sigma^{\eta'}_\Lambda$ given by

$$\mu[s'] = \frac{1}{Z_\mu}\, \mathrm{w}_2^{|F_{s'} \setminus \omega|} \mathrm{w}_3^{|F_{s'} \cap \omega|} \qquad \text{with} \qquad Z_\mu := \sum_{s' \in \Sigma^{\eta'}_\Lambda} \mathrm{w}_2^{|F_{s'} \setminus \omega|}\, \mathrm{w}_3^{|F_{s'} \cap \omega|}.$$

Then, dividing both sides by Z_μ^2 and inserting (3.44), the inequality (3.43) turns into

$$\begin{aligned} &\mathbf{E}_\mu\Big[\mathrm{w}_3^{\mathbb{1}\{e \in E_{s'}\} + \mathbb{1}\{f \in E_{s'}\}}\Big]\, \mathbf{E}_\mu\Big[\mathrm{w}_2^{\mathbb{1}\{e \in E_{s'}\} + \mathbb{1}\{f \in E_{s'}\}}\Big] \\ &\qquad\qquad \ge \mathbf{E}_\mu\Big[\mathrm{w}_2^{\mathbb{1}\{f \in E_{s'}\}}\, \mathrm{w}_3^{\mathbb{1}\{e \in E_{s'}\}}\Big]\, \mathbf{E}_\mu\Big[\mathrm{w}_2^{\mathbb{1}\{e \in E_{s'}\}}\, \mathrm{w}_3^{\mathbb{1}\{f \in E_{s'}\}}\Big]. \end{aligned}$$

Applying elementary manipulations, this inequality is equivalent to

$$\big(\mu[e,f \notin E_{s'}]\, \mu[e,f \in E_{s'}] - \mu[e \in E_{s'}, f \notin E_{s'}]\, \mu[e \notin E_{s'}, f \in E_{s'}]\big)\, (\mathrm{w}_2 - \mathrm{w}_3)^2 \ge 0,$$

which is a consequence of Griffiths' second inequality [Gri67], provided that $\mathrm{w}_2, \mathrm{w}_3 \le 1$. The latter is equivalent to (3.15). Indeed, $\mathrm{w}_2 \le 1$ precisely if $K' \ge K''$, and $\mathrm{w}_3 \le 1$ precisely if

$$e^{-2K''} \le \frac{1 + e^{-2(K+K')}}{e^{-2K} + e^{-2K'}}.$$

Applying $x \mapsto \frac{x-1}{x+1}$, the above inequality turns into $-\tanh K'' \le \tanh K \tanh K'$. $\square$

3.B Maximal boundary condition for GAT

Proof of Lemma 3.6.1. Fix $J, J', U, \Lambda, \Delta, E_\Lambda, E_\Delta, \xi$ as in the statement, and define

$$\mu_1 := \texttt{GAT}^{1,\mathrm{f}}_{\Delta,J,J',U}[\omega_{E_\Lambda} \in \cdot \mid \omega_{E_\Delta \setminus E_\Lambda} = \xi_{E_\Delta \setminus E_\Lambda}] \quad \text{and} \quad \mu_2 := \texttt{GAT}^{1,\mathrm{f}}_{\Lambda,J,J',U}.$$

By [Gri06, Theorem 2.6], since μ_2 satisfies (FKG-L) by Lemma 3.3.7, it suffices to show that, for any $e \in E_\Lambda$ and any $\omega \in \Omega_{E_\Lambda}$ (regarded as a set) with $e \notin \omega$,

$$\frac{\mu_1[\omega \cup \{e\}]}{\mu_1[\omega]} \leq \frac{\mu_2[\omega \cup \{e\}]}{\mu_2[\omega]}. \tag{3.45}$$

For $\zeta \in \Omega_{\mathbb{Z}^d}$, define

$$Z^{\mathrm{f}}_\Delta(\zeta) = \sum_{s' \in \Sigma^{\mathrm{f}}_\Delta} \mathrm{w}_2^{|(E_\Delta)_{s'} \setminus \zeta|} \mathrm{w}_3^{|(E_\Delta)_{s'} \cap \zeta|} \quad \text{and} \quad Z^{\mathrm{f}}_\Lambda(\zeta) = \sum_{s' \in \Sigma^{\mathrm{f}}_\Lambda} \mathrm{w}_2^{|(E_\Lambda)_{s'} \setminus \zeta|} \mathrm{w}_3^{|(E_\Lambda)_{s'} \cap \zeta|}.$$

After cancellations, the left side of (3.45) reduces to

$$\mathrm{w}_1 2^{k_\Delta(\omega \cup \{e\} \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda}) - k_\Delta(\omega \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda})} \frac{Z^{\mathrm{f}}_\Delta(\omega \cup \{e\} \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda})}{Z^{\mathrm{f}}_\Delta(\omega \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda})}. \tag{3.46}$$

Similarly, the right side of (3.45) reduces to

$$\mathrm{w}_1 2^{k_\Lambda(\omega \cup \{e\} \cup \mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda) - k_\Lambda(\omega \cup \mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda)} \frac{Z^{\mathrm{f}}_\Lambda(\omega \cup \{e\})}{Z^{\mathrm{f}}_\Lambda(\omega)}. \tag{3.47}$$

It can easily be checked that the exponent of 2 in (3.46) is at most that in (3.47). It remains to treat the ratios of partition functions. Let $F_\Delta = E_\Delta \setminus \{e\}$ and $F_\Lambda = E_\Lambda \setminus \{e\}$, and let ν_1 and ν_2 be the Ising measures on $\Sigma^{\mathrm{f}}_\Delta$ and $\Sigma^{\mathrm{f}}_\Lambda$, respectively, given by

$$\nu_1[s'] \propto \mathrm{w}_2^{|(F_\Delta)_{s'} \setminus (\omega \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda})|} \mathrm{w}_3^{|(F_\Delta)_{s'} \cap (\omega \cup \xi_{\mathbb{E}_{\mathbb{Z}^d} \setminus E_\Lambda})|} \quad \text{and} \quad \nu_2[s'] \propto \mathrm{w}_2^{|(F_\Lambda)_{s'} \setminus \omega|} \mathrm{w}_3^{|(F_\Lambda)_{s'} \cap \omega|}.$$

Then, applying elementary manipulations analogous to those in the proof of Lemma 3.3.7 above, the ratio in (3.46) is at most that in (3.47) if and only if

$$\big(\nu_1[e \notin (E_\Delta)_{s'}] - \nu_2[e \notin (E_\Delta)_{s'}]\big)(\mathrm{w}_3 - \mathrm{w}_2) \geq 0.$$

The first factor is non-negative as a consequence of Griffiths' second inequality [Gri67], and the second factor is non-negative precisely if $U \leq 0$. □

3.C Construction of the curves

Proof of Lemma 3.6.2. For $\kappa, \kappa' \in (0, 1]$ with $\kappa < \kappa'$, define $\widetilde{\gamma}_{\kappa,\kappa'} : (\beta^-_{\kappa,\kappa'}, \beta^+_{\kappa,\kappa'}) \to \mathbb{R}^3$ by

$$\beta^-_{\kappa,\kappa'} = -\tfrac{1}{4}\log(\kappa\kappa'), \quad \beta^+_{\kappa,\kappa'} = -\tfrac{1}{2}\log\kappa,$$
$$\widetilde{\gamma}_{\kappa,\kappa'}(\beta) = \left(\tfrac{1}{2}\log\left(\tfrac{\kappa'-\kappa}{\kappa\kappa' e^{2\beta} - e^{-2\beta}}\right), \beta, \beta + \tfrac{1}{2}\log\kappa\right).$$

It is straightforward to check that these curves satisfy (i) and (ii) in the statement. Moreover, the image of $(\beta^-_{\kappa,\kappa'}, \beta^+_{\kappa,\kappa'})$ under $\mathrm{w}_1(\widetilde{\gamma}_{\kappa,\kappa'})$ is $(0,\infty)$, and

$$-\frac{\mathrm{d}}{\mathrm{d}\beta}\log \mathrm{w}_1(\widetilde{\gamma}_{\kappa,\kappa'}(\beta)) = \frac{4\kappa(\kappa'-\kappa)e^{-4\beta}}{(\kappa\kappa' - e^{-4\beta})(e^{-4\beta} - \kappa^2)} > \frac{4\kappa}{\kappa'-\kappa}.$$

Define $\gamma_{\kappa,\kappa'} : (0,1) \to \mathbb{R}^3$ by $\gamma_{\kappa,\kappa'}(\beta) = \widetilde{\gamma}_{\kappa,\kappa'}(\beta^+_{\kappa,\kappa'} - \beta(\beta^+_{\kappa,\kappa'} - \beta^-_{\kappa,\kappa'}))$. □

3.D Jump monotonicity for GAT

Proof of Proposition 3.6.5. Fix $\kappa \in (0,1)$ and $\Lambda \subset \mathbb{Z}^d$ finite, and set $E = \overline{\mathbb{E}}_\Lambda$. As $\mathtt{GAT}^{1,\mathrm{f}}_{\Lambda,\widehat{\gamma}_\kappa(\beta_i)}$ is the marginal of $\mathtt{ATRC}^{1,0}_{\Lambda,\widehat{\gamma}_\kappa(\beta_i)}$ on the first component by Proposition 3.3.8, the Holley criterion [Hol74, Theorem 6] states that it suffices to show that, for all $\xi, \xi', \zeta, \zeta' \in \Omega_E$,

$$\mathtt{ATRC}^{1,0}_{\Lambda,\widehat{\gamma}_\kappa(\beta_2)}[\xi \vee \zeta, \xi' \vee \zeta']\, \mathtt{ATRC}^{1,0}_{\Lambda,\widehat{\gamma}_\kappa(\beta_1)}[\xi \wedge \zeta, \xi' \wedge \zeta'] \geq \mathtt{ATRC}^{1,0}_{\Lambda,\widehat{\gamma}_\kappa(\beta_2)}[\xi, \xi']\, \mathtt{ATRC}^{1,0}_{\Lambda,\widehat{\gamma}_\kappa(\beta_1)}[\zeta, \zeta'],$$

where $\vee$ and $\wedge$ denote the coordinatewise maximum and minimum, respectively. Recall that the measures can be written as in (3.37) and (3.38). The factor $2^{k(\omega)+k(\omega')}$ satisfies the corresponding inequality (see, e.g., [Gri06, Theorem 3.8]), whence it suffices to check the above inequality for the edge-weights $\mathrm{u}_1^{|\omega_E|+|\omega'_E|}\, \hat{\mathrm{u}}_3^{|\omega_E \cap \omega'_E|}$. Comparing respective factors for each edge $e \in E$, we obtain the following conditions:

$$\begin{aligned} \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2)) &\geq \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1)), & \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))^2\, \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_2)) &\geq \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1))^2\, \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_1)), \\ \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))\, \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_2)) &\geq \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1)), & \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))\, \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_2)) &\geq \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1))\, \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_1)). \end{aligned}$$

Since $\hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_1)) = \hat{\mathrm{u}}_3(\widehat{\gamma}_\kappa(\beta_2)) = \kappa \in (0,1)$, these conditions reduce to $\mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_2))\, \kappa \geq \mathrm{u}_1(\widehat{\gamma}_\kappa(\beta_1))$, and the proof is complete. □

Chapter 4

Discontinuous transition in 2D Potts: I. Order-disorder interface convergence

It is known that the planar q-state Potts model undergoes a discontinuous phase transition when $q > 4$ and there are exactly $q+1$ extremal Gibbs measures at the transition point: q ordered (monochromatic) measures and one disordered (free). We focus on the Potts model under the Dobrushin order–disorder boundary conditions on a finite $N \times N$ part of the square grid. Our main result is that this interface is a well defined object, has $\sqrt{N}$ fluctuations, and converges to a Brownian bridge under diffusive scaling. The same holds also for the corresponding FK-percolation model for all $q > 4$.

Our proofs rely on a coupling between FK-percolation, the six-vertex model, and the random-cluster representation of an Ashkin–Teller model (ATRC), and on a detailed study of the latter. The coupling relates the interface in FK-percolation to a long subcritical cluster in the ATRC model. For this cluster we develop a "renewal picture" *à la* Ornstein-Zernike. This is based on fine mixing properties of the ATRC model that we establish using the link to the six-vertex model and its height function. Along the way, we derive various properties of the Ashkin-Teller model, such as Ornstein-Zernike asymptotics for its two-point function.

In a companion work, we provide a detailed study of the Potts model under order-order Dobrushin conditions. We show emergence of a free layer of width $\sqrt{N}$ between the two ordered phases (*wetting*) and establish convergence of its boundaries to two Brownian bridges conditioned not to intersect.

4.1 Introduction and results

The Potts model is a classical model of statistical mechanics introduced in 1952 [Pot52]. Each vertex of a graph is assigned one of q states (colours), with states of adjacent vertices interacting with a strength depending on the temperature $T > 0$ of the system. At $q = 2$, this corresponds to the seminal Ising model. The Potts model becomes increasingly ordered as the temperature decreases, and a phase transition occurs on lattices $\mathbb{Z}^d$ with $d \geq 2$ at some transition temperature $T_c(q, d) > 0$. Depending on q and d, the transition is either discontinuous (first-order) or continuous (higher-order). Such a rich behaviour has brought a lot of attention to the Potts model.

Interface in the planar Potts model. Our work is restricted to dimension $d = 2$. In this case, planar duality and a correlation inequality (available when $q \geq 1$) have allowed to a watershed of progress in the phase diagram of the Potts model in the last two decades:

- the transition occurs at the self-dual point $T_c(q) := [\ln(1 + \sqrt{q})]^{-1}$ [BD12];

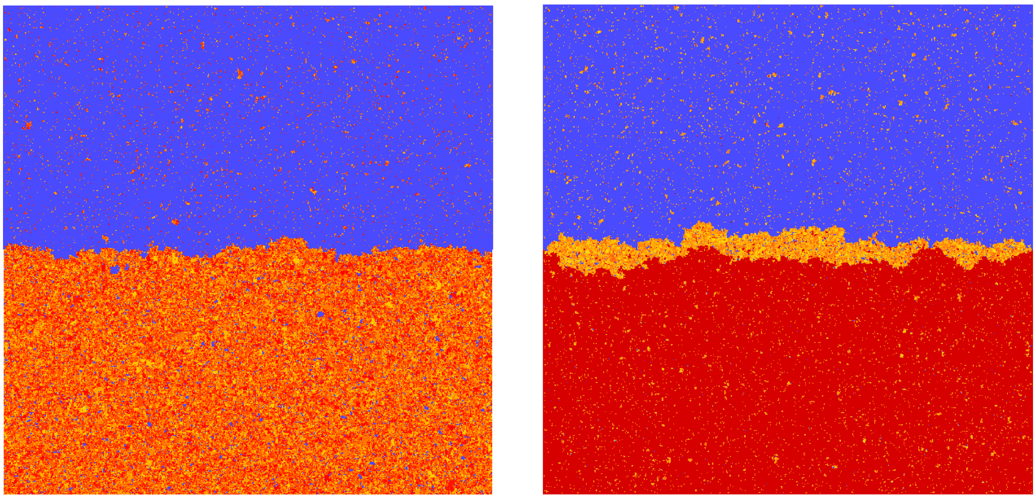


Figure 4.1: Sample of a 600x600 Potts model with 25 colours at $T_c(25)$ with Dobrushin boundary conditions. Left: order-disorder interface; upper part has blue b.c., bottom has white b.c. (no colour favoured), colours are: blue for the first and interpolate between yellow and red for colours 2 to 25. Right: colours are: blue for the first, red for the second, and interpolate between yellow and orange for colours 3 to 25.

- the transition is continuous when $q = 2, 3, 4$, in a sense that, at $T_c(q)$, there is a unique Gibbs measure [DCST17] (see also [GL23] for another argument);
- the transition is discontinuous when $q > 4$ [DGH$^+$21] (see also [RS20] for a short proof), and any Gibbs measure can be written as a linear combination of $q + 1$ extremal Gibbs measures (q monochromatic and one free) [GM23].

We focus on discontinuous transitions ($q > 4$) and study interfaces at $T_c(q)$ separating different states. The structure of extremal Gibbs measures (described above) leads to two natural definitions of Dobrushin boundary conditions:

- *order-disorder*: one half of the boundary is of a fixed colour and the other one is free (no colour assigned);
- *order-order*: both halves of the boundary are assigned different fixed colours.

The phenomenology is quite different in the two cases, see Fig. 4.1.

The main result of the current chapter is the convergence of the interface under order-disorder conditions to the Brownian bridge in the diffusive limit. Results of such precision were previously proven for $T < T_c(q)$ and the order-order interface. Indeed, planar duality transfers this problem to the study of the typical geometry of long clusters at $T > T_c$, for which a quasi-renewal structure was developed in [CIV08]. We do use similar ideas, but the situation is more involved: duality does not help directly as we work at the transition point with order-disorder boundary conditions, where duality only maps the problem to itself. Instead, we build on [BKW76] and [GP23] to construct a sequence of combinatorial mappings relating the interface in the FK-percolation to a suitable long subcritical cluster in the Ashkin-Teller model. We extend the study of [ADG24] to establish mixing properties in the latter model. This opens a way to obtain a renewal picture and invariance principle for long subcritical clusters in the ATRC model using the methods developed in [CIV03, CIV08, AOV24]. Finally, we transport back the convergence result to the FK and Potts models.

Let us also mention the work [MMSRS91] that showed $\sqrt{N}$ fluctuations when q is taken to be large enough. Our results imply this for all $q > 4$. In this generality, the only progress was a recent complete characterisation of Gibbs measures in the planar Potts model [GM23]. This work proves that all Gibbs measures are invariant to translations, and this implies that the interface (if defined at all) exhibits diverging fluctuations.

The case of the order-order conditions is addressed in a subsequent work [DGO25b] (see Section 1.4.1 and Chapter 5): we show emergence of a free layer of width $\sqrt{N}$ between the two ordered phases (*wetting*) and establish convergence of its boundaries to two Brownian bridges conditioned not to intersect.

In particular, we prove that, when $q > 4$, in both cases of Dobrushin conditions, as the mesh of the lattice tends to zero, the scaling limit of the interface in the Potts model is a straight line. To compare with what happens in the case of continuous phase transition, let us describe the expected behaviour for $q = 2, 3, 4$. The mesmerizing physics conjecture from 1980s postulates conformal invariance. Schramm's [Sch00] geometric interpretation of this conjecture asserts that as one takes the scaling limit of the Potts model, the interface converges to a random fractal curve called the Schramm–Loewner Evolution. This has been proven rigorously only at $q = 2$ (Ising model) by Smirnov et al [Smi10, CS11, CDCH$^+$14].

The "Ornstein-Zernike" (OZ) theory is a (non-rigorous) picture introduced in [OZ14, Zer16] of how correlation functions in various models behave. Their main idea was to postulate a suitable *renewal structure* satisfied by correlation functions, which leads to very precise expressions for the asymptotics of the said correlations. The first rigorous implementation of this renewal picture was done by Abraham and Kunz [AK77] using perturbative expansions. The modern approach, which started with the work of Chayes, and Chayes [CC86] on the self-avoiding walk (SAW) and of Campanino, Chayes, and Chayes [CCC91] on Bernoulli percolation, is based on creating a renewal structure *à la* OZ for elongated subcritical objects (SAW or percolation clusters containing a distant point). Both these works rely on heavy combinatorial study, and the renewal steps are "irreducible crossings of slabs".

This idea was further developed by Ioffe [Iof98] for the ballistic phase of the self-avoiding walk, introducing key measure-tilting ideas coming from large deviation theory. His strategy was then extended to Bernoulli percolation in [CI02], to the high-temperature Ising model [CIV03] and to the FK-percolation [CIV08]. Compared to the cases of SAW and Bernoulli percolation where the measure factorizes nicely, the structure obtained in the case of Ising and FK, while still geometrically being a concatenation of "irreducible blocs", is not a real renewal structure. Indeed, it is only a sequence of "fast mixing kernels", which study is heavier and performed in [CIV03]. The last step to finally obtain a true renewal structure from the fast mixing kernel picture was done in [OV18].

These works on subcritical clusters of the FK-percolation all take place in any dimensions. On $\mathbb{Z}^2$, planar duality allows to rewrite the interface of the Potts model at $T < T_c(q)$ as a subcritical FK-percolation cluster conditioned to contain $(0, 0)$ and $(N, 0)$.

Our contribution to this line of works is to derive a "renewal picture" for the long clusters of the planar random-cluster representation of the Ashkin–Teller model, for values of the parameters lying on the self-dual line. This model is then related to the Potts interface using a suitable adaptation of a coupling introduced in [GP23].

We now formally state our main results for the Potts model, the FK percolation and the Ashkin–Teller model.

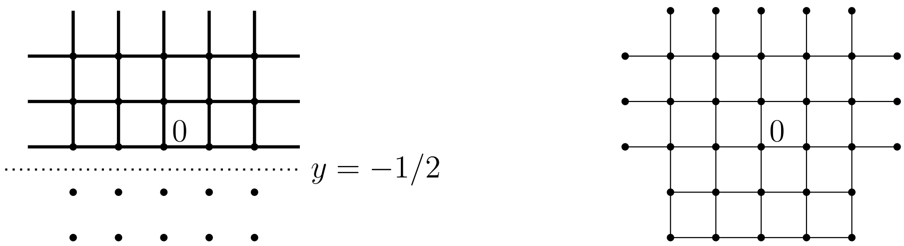


Figure 4.2: Left: wired-free Dobrushin boundary condition. Right: the graph G_2.

4.1.1 Potts model

We view $\mathbb{Z}^2$ both as a set of points on the plane having integer coordinates and as a graph (square grid) with edges linking points at distance one. Denote by $\mathbb{E}$ the set of edges in $\mathbb{Z}^2$ and write $i \sim j$ if $\{i,j\} \in \mathbb{E}$. Let $G = (V, E)$ be a subgraph of $\mathbb{Z}^2$. Take parameters $T > 0$, $q \in \{2, 3, 4, \dots\}$, and boundary conditions $\eta \in \{0, 1, \dots, q\}^{\mathbb{V}}$. The Potts model on G with boundary conditions η is the probability measure on $\{1, \dots, q\}^V$ given by

$$\mathtt{Potts}^{\eta}_{G;T,q}(\sigma) := \tfrac{1}{Z_{\mathtt{Potts}}} \cdot \exp\left[\tfrac{1}{T} \cdot \left(\sum_{\{i,j\} \in E} \delta(\sigma_i, \sigma_j) + \sum_{i \in V, j \in V^c :\, i \sim j} \delta(\sigma_i, \eta_j) \right)\right],$$

where $Z_{\mathtt{Potts}} = Z_{\mathtt{Potts}}(G, T, q, \eta)$ is the unique normalising constant (called *partition function*) that renders the above a probability measure, and $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise. Value 0 for η corresponds to free boundary conditions favoring none of the q possible states of the spins.

We say that η defines the order-disorder (1-free) Dobrushin boundary conditions (and denote it by 1/f) if $\eta((x, y)) = \mathbb{1}_{\geq 0}(y)$. Identify $\Lambda_n := \{-n, \dots, n\}^2$ with the induced subgraph of $\mathbb{Z}^2$ on this set of vertices. The Dobrushin boundary conditions on Λ_n impose existence of an interface between color one and the rest. This can be made explicit, but for brevity we choose to define directly the upper and lower discrete envelopes of this interface: $\Gamma^+_{\mathtt{Potts}}$ and $\Gamma^-_{\mathtt{Potts}}$ respectively. Given $\sigma \in \{1, \dots, q\}^{\Lambda_n}$, define $\bar{\sigma} \in \{0, 1, \dots, q\}^{\mathbb{Z}^2}$ to be its extension outside of Λ_n by the Dobrushin boundary conditions: one on $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and zero on $\mathbb{Z} \times \mathbb{Z}_{<0}$. For $k = -n, \dots, n$, define

$$\Gamma^{+,n}_{\mathtt{Potts}}(k) := \max\{y \in \mathbb{Z} :\ (k, y-1) \xleftrightarrow{\bar{\sigma} \neq 1} (\mathbb{Z} \times \mathbb{Z}_{<0}) \setminus \Lambda_n\},$$
$$\Gamma^{-,n}_{\mathtt{Potts}}(k) := \min\{y \in \mathbb{Z} :\ (k, y+1) \xleftrightarrow{\bar{\sigma} = 1,\text{ diag}} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n\},$$

where $(k, y-1) \xleftrightarrow{\bar{\sigma} \neq 1} (\mathbb{Z} \times \mathbb{Z}_{<0}) \setminus \Lambda_n$ states the existence of a path in $\mathbb{Z}^2$ in going from $(k, y-1)$ to $(\mathbb{Z} \times \mathbb{Z}_{<0}) \setminus \Lambda_n$ and consisting of vertices where $\bar{\sigma} \neq 1$ and $(k, y) \xleftrightarrow{\bar{\sigma} = 1,\text{ diag}} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n$ states the existence of a path in $\mathbb{Z}^2$ *with diagonal connectivity* going from (k, y) to $(\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n$ and consisting of vertices where $\bar{\sigma} = 1$. By $\mathbb{Z}^2$ with diagonal connectivity we mean a graph with the vertex-set $\mathbb{Z}^2$ with edges linking vertices at distance at most $\sqrt{2}$. Define the rescaled linear interpolation of $\Gamma^{+,n}_{\mathtt{Potts}}$ and $\Gamma^{-,n}_{\mathtt{Potts}}$ by

$$\tilde{\Gamma}^{\pm,n}_{\mathtt{Potts}}(t) := \tfrac{1}{\sqrt{n}}\big((1 - \{2tn - n\})\Gamma^{\pm}_{\mathtt{Potts}}(\lfloor 2tn - n \rfloor) + \{2tn - n\}\Gamma^{\pm}_{\mathtt{Potts}}(\lceil 2tn - n \rceil)\big),$$

where $\lfloor\,\rfloor$, $\lceil\,\rceil$, $\{\,\}$ denote respectively lower and upper roundings and fractional part.

Theorem 13. *Let $q > 4$ be integer and take $T = T_c(q)$. For $n \in \mathbb{N}$, sample $\Gamma^{\pm,n}_{\mathtt{Potts}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathtt{Potts}}$ from $\mathtt{Potts}^{1/\mathrm{f}}_{\Lambda_n;T_c(q),q}$ as described above. Then, as n tends to infinity,*

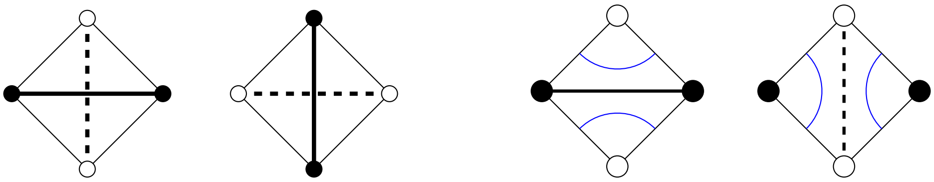


Figure 4.3: Left: Tile associated to a mid-edge. Right: Tile centred at the middle of a horizontal primal edge (solid black) or its associated vertical dual edge (dashed black), with its two possible local loop configurations.

1. *both* $\left(\tilde{\Gamma}^{+,n}_{\texttt{Potts}}(t)\right)_{t\in[0,1]}$ *and* $\left(\tilde{\Gamma}^{-,n}_{\texttt{Potts}}(t)\right)_{t\in[0,1]}$ *converge in law to* $(c_q \mathrm{b}_t)_{t\in[0,1]}$*, where* b_t *is a standard Brownian bridge and* $c_q > 0$ *is some constant;*

2. *the probability that* $\max_k \left|\Gamma^{+,n}_{\texttt{Potts}}(k) - \Gamma^{-,n}_{\texttt{Potts}}(k)\right| \geq C\ln(n)^2$ *tends to zero.*

The constant c_q has an explicit characterization, see Remark 4.8.12.

4.1.2 FK percolation

The main tool in analyzing the Potts model is the Fortuin–Kasteleyn (FK) percolation (or random-cluster) model [FK72] that allows to express the spin-spin correlation function via connection probabilities. We first define it and then state the relation between the two models. Take a finite subgraph $G = (V, E)$ of $\mathbb{Z}^2$, parameters $p \in (0, 1)$, $q > 0$ and boundary conditions $\xi \in \{0, 1\}^{\mathbb{E}}$. We identify any $\omega \in \{0, 1\}^{\mathbb{E}}$ with the set of edges $e \in \mathbb{E}$ for which $\omega_e = 1$ (*open edges*) and with the spanning subgraph of $\mathbb{Z}^2$ defined by the open edges. The FK-percolation model on G with boundary conditions ξ is the probability measure on $\{0, 1\}^{\mathbb{E}}$ given by

$$\mathsf{FK}^{\xi}_{G;p,q}(\eta) := \tfrac{1}{Z_{\mathsf{FK}}} \cdot p^{|\eta\cap E|}(1-p)^{|E\setminus\eta|}\, q^{k_V(\eta)}\, \mathbb{1}_{\eta=\xi \text{ on } \mathbb{E}\setminus E},$$

where $Z_{\mathsf{FK}} = Z_{\mathsf{FK}}(G, p, q, \xi)$ is the partition function and $k_V(\eta)$ is the number of connected components (*clusters*) of $(\mathbb{Z}^2, \eta)$ that intersect V.

The *free* and *wired* measures correspond to the choices $\xi \equiv 0$ and $\xi \equiv 1$, respectively, and we simply write f and w instead of ξ, respectively. We will be interested in the Dobrushin wired/free boundary conditions: $\xi_e = 1$ if and only if $e \subset \mathbb{Z}\times\mathbb{Z}_{\geq 0}$ (see Fig. 4.2). We denote these boundary conditions by 1/0.

For $n \geq 1$, define the graph $G_n = (V_n, E_n)$ by

$$E_n := \{e \in \mathbb{E} : e \subset \Lambda_n\} \cup \{e \in \mathbb{E} : e \subset \mathbb{Z}\times\mathbb{Z}_{\geq 0} \text{ and } e \cap \Lambda_n \neq \varnothing\}, \quad V_n := \bigcup_{e\in E_n} e.$$

The seminal Edwards–Sokal coupling [ES88] states that, when $q \geq 2$ is integer and $p = 1 - \exp[-\frac{1}{T}]$, coloring clusters of $\omega \sim \mathsf{FK}^{1/0}_{G_n;p,q}$ independently in colors $1, 2, \ldots, q$ gives a spin configuration $\sigma \sim \texttt{Potts}^{1/\mathrm{f}}_{\Lambda_n;T,q}$. We will denote this coupling between the Potts model and the FK percolation by $\mathsf{ES}^{1/0}_{\Lambda_n,T,q}$.

We define an interface in the FK percolation forced by the Dobrushin boundary conditions using planar duality. Note that the lattice dual to $\mathbb{Z}^2$ is again a square lattice and we denote it by $(\mathbb{Z}^2)^*$. For each edge e of $\mathbb{Z}^2$, denote by e^* the edge of $(\mathbb{Z}^2)^*$ that is dual to e, i.e. the unique edge of $(\mathbb{Z}^2)^*$ that intersects e; see Fig. 4.3. Given $\omega \in \{0, 1\}^{\mathbb{E}}$, define its dual by

$$\omega^*_{e^*} := 1 - \omega_e.$$

The FK percolation at $p_c(q)$, given by

$$p_c(q) := 1 - \exp[-\tfrac{1}{T_c(q)}] = \tfrac{\sqrt{q}}{\sqrt{q}+1},$$

is known to enjoy the self-duality: if $\omega \sim \mathtt{FK}^{1/0}_{G_n;T_c(q),q}$, then its dual ω^* is also distributed as an FK-percolation with parameters q and $p_c(q)$, but on a dual graph and under dual Dobrushin boundary conditions. This self-dual nature is also revealed when looking at the loop representation of the FK percolation model. Specifically, we draw two arcs next to every primal or dual edge of ω, as shown on Fig. 4.3. These arcs link together into loops separating primal and dual clusters and one interface tracing the boundary of the union of primal clusters attached to the upper boundary of Λ_n. Denote this interface by $\Gamma_{\mathtt{FK}}$. Remarkably, a standard application of the Euler's formula (see eg. [DGH+21, Lemma 3.9]), allows to rewrite the distribution of ω via loops which are symmetric with respect primal and dual configurations:

$$\mathtt{FK}^{1/0}_{G_n;p_c(q),q}(\omega) = \tfrac{1}{Z_{\mathsf{loop}}} \cdot \sqrt{q}^{\#\,\mathrm{loops}}, \tag{4.1}$$

where # loops stands for the number of loops in the loop representation of ω. This is a part of the classical Baxter–Kelland–Wu [BKW76] coupling between the FK percolation and the six-vertex model; see Sections 4.4 and 4.5.1.

As for the Potts model, we define the upper and the lower discrete envelops of $\Gamma_{\mathtt{FK}}$:

$$\Gamma^{+,n}_{\mathtt{FK}}(k) := \max\{y \in \mathbb{Z} : \ (k \pm \tfrac{1}{2}, y - \tfrac{1}{2}) \xleftrightarrow{\omega^*} (\{\mathbb{Z} + \tfrac{1}{2}\} \times \{\mathbb{Z}_{<0} + \tfrac{1}{2}\}) \setminus \Lambda^*_n\},$$
$$\Gamma^{-,n}_{\mathtt{FK}}(k) := \min\{y \in \mathbb{Z} : \ (k, y+1) \xleftrightarrow{\omega} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n\},$$

where $\Lambda^*_n := [-n,n]^2 \cap (\mathbb{Z}^2)^*$. The rescaled linear interpolations $\tilde{\Gamma}^{\pm,n}_{\mathtt{FK}}$ of $\Gamma^{\pm,n}_{\mathtt{FK}}$ are defined in the same way as in the Potts model. Our main result for the FK percolation is an invariance principle for $\tilde{\Gamma}^{\pm,n}_{\mathtt{FK}}$:

Theorem 14. *Let $q > 4$ be a real number and take $p = p_c(q)$. For $n \in \mathbb{N}$, sample $\Gamma^{\pm,n}_{\mathtt{FK}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathtt{FK}}$ from $\mathtt{FK}^{1/0}_{G_n;p_c(q),q}$ as described above. Then, as n tends to infinity, the convergence results from Theorem 13 hold for $\Gamma^{\pm,n}_{\mathtt{FK}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathtt{FK}}$ in place of $\Gamma^{\pm,n}_{\mathtt{Potts}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathtt{Potts}}$.*

We draw attention to the precision of our control of the interface. Previously, at $p_c(q)$ when $q > 4$ (discontinuous transition), it remained open that $\Gamma^n_{\mathtt{FK}}$ does not exhibit linear fluctuations. Our results imply that this is indeed the case and, moreover, the probability that $\Gamma^n_{\mathtt{FK}}$ exhibits linear fluctuations is in fact exponentially small. This should be compared to the behavior of $\Gamma^n_{\mathtt{FK}}$ at $p_c(q)$ when $q \in [1,4]$ (continuous transition): there it does exhibit linear fluctuations [DCST17].

Our proof goes via developing a random walk representation for long clusters in the Ashkin–Teller model. We proceed by introducing the latter model and stating our results for it.

4.1.3 Ashkin–Teller model

Introduced in 1943 [AT43] as a generalization of the Ising model to a four-component system, the Ashkin–Teller (AT) model can be viewed as a pair of interacting Ising models. Take a finite subgraph $G = (V, E)$ of $\mathbb{Z}^2$, parameters $J, U > 0$ and boundary conditions $\sigma, \sigma' \in \{0, \pm 1\}^{\mathbb{Z}^2}$. The AT model on G with parameters $J, U \in \mathbb{R}$ and boundary conditions (σ, σ') is a probability measures on pairs $\tau,\tau'\in\{\pm1\}^V$ given by

$$\mathtt{AT}^{\sigma,\sigma'}_{G,J,U}(\tau,\tau')=\tfrac{1}{Z_{\mathtt{AT}}}\cdot\exp\Big[\sum_{\{i,j\}\in E}J(\tau_i\tau_j+\tau'_i\tau'_j)+U\tau_i\tau_j\tau'_i\tau'_j$$
$$+\sum_{i\in V,j\in V^c:\,i\sim j}J(\tau_i\sigma_j+\tau'_i\sigma'_j)+U\tau_i\sigma_j\tau'_i\sigma'_j\Big]$$

where $Z_{\mathtt{AT}}=Z_{\mathtt{AT}}(G,J,U,\sigma,\sigma')$ is the partition function. Taking $\sigma=\sigma'\equiv1$ we obtain plus-plus boundary conditions and taking $\sigma=\sigma'\equiv0$ we obtain free-free boundary conditions; we denote the corresponding measures by $\mathtt{AT}^{+,+}_{G,J,U}$ and $\mathtt{AT}^{\mathrm{f,f}}_{G,J,U}$ respectively.

We consider only $J>0$, since flipping the sign of J corresponds to flipping the sign of τ' and τ at one of the two partite classes of $\mathbb{Z}^2$. Using the Ising-duality for τ' and then for τ, the *self-dual curve* of the parameters was identified [MS71]:

$$\sinh 2J=e^{-2U}. \tag{4.2}$$

A classical GKS inequality [KS68] implies that the correlations are monotone along the lines of a constant ratio J/U. The case $U=0$ gives two independent Ising models and the line $J=U$ is in direct correspondence with the four-state Potts model and the FK percolation with the cluster-weight $q=4$. In addition, the three models are related on the self-dual line (4.2), with $q>4$ corresponding to $U>J$. A precise coupling was constructed in [GP23] via the six-vertex (square ice) model based on the works of Fan [Fan72a] and Wegner [Weg72] and on the seminal Baxter–Kelland–Wu (BKW) correspondence [BKW76]; see also [HDJS13]. This coupling is crucial to the current work and is described in detail in Sections 4.4.2 and 4.5.1.

When $U>J>0$, correlation inequalities [KS68] guarantee the existence of the infinite-volume limits $\mathtt{AT}^{\mathrm{f,f}}_{J,U}$ and $\mathtt{AT}^{+,+}_{J,U}$ of the free and monochromatic AT measures $\mathtt{AT}^{\mathrm{f,f}}_{G_n,J,U}$ and $\mathtt{AT}^{+,+}_{G_n,J,U}$, respectively, as $G_n\nearrow\mathbb{Z}^2$. Our first result states that their marginals on the single spin τ coincide.

Proposition 4.1.1. *Let $0<J<U$ satisfy $\sinh 2J=e^{-2U}$. Then, the measures $\mathtt{AT}^{\mathrm{f,f}}_{J,U}$ and $\mathtt{AT}^{+,+}_{J,U}$ have the same marginal distribution on τ.*

Denote the expectation operators with respect to $\mathtt{AT}^{\mathrm{f,f}}_{J,U}$ and $\mathtt{AT}^{+,+}_{J,U}$ by $\langle\cdot\rangle^{\mathrm{f,f}}_{J,U}$ and $\langle\cdot\rangle^{+,+}_{J,U}$, respectively, and define the inverse correlation length $\nu_{J,U}$ by setting, for $x\in\mathbb{R}^2$,

$$\nu_{J,U}(x):=-\lim_{n\to\infty}\tfrac{1}{n}\ln\langle\tau_0\tau_{\lfloor nx\rfloor}\rangle^{\mathrm{f,f}}_{J,U}=-\lim_{n\to\infty}\tfrac{1}{n}\ln\langle\tau_0\tau_{\lfloor nx\rfloor}\rangle^{+,+}_{J,U},$$

where $\lfloor\cdot\rfloor$ is the componentwise integer part. The existence of the limit is derived in a standard manner from correlation inequalities and a subadditive argument (see Section 4.7 for more details and references). In [ADG24], it was shown that $\mathtt{AT}^{+,+}_{\Lambda_n}$ admits exponential decay of correlations in τ, which implies that $\nu>0$. The following theorem establishes sharp Ornstein–Zernike-type asymptotics for the 2-point function.

Theorem 15. *Let $0<J<U$ satisfy $\sinh 2J=e^{-2U}$. The inverse correlation length $\nu_{J,U}$ is a norm on $\mathbb{R}^2$. Furthermore, uniformly in $|x|\to\infty$,*

$$\langle\tau_0\tau_x\rangle^{\mathrm{f,f}}_{J,U}=\langle\tau_0\tau_x\rangle^{+,+}_{J,U}=\tfrac{g(x/|x|)}{\sqrt{|x|}}\,e^{-\nu(x)}\,(1+o(1)),$$

where $\nu=\nu_{J,U}$, and $g=g_{J,U}$ is a strictly positive analytic function on $\mathbb{S}^1$.

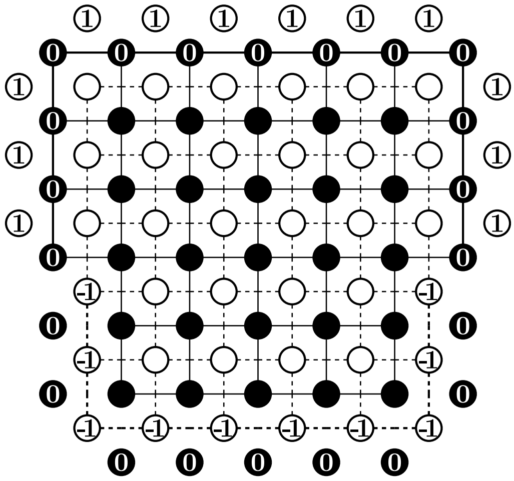


Figure 4.4: Dobrushin boundary condition for the six-vertex height function.

4.1.4 Six-vertex model

The six-vertex model [Pau35, Rys63] is a model of *edge orientations* on a square lattice, which in the setting of this chapter is taken to be the medial graph of $\mathbb{Z}^2$. While it serves merely as an intermediate step and a tool in the derivation of our main results, Theorems 13-15, we also obtain some results on the six-vertex model along the way, which we will *informally* describe in this subsection. We refer to Sections 4.4.1 and 4.5.1 for precise definitions.

The six-vertex model admits a representation in terms of *height functions*. In the above setting, these are assignments of integers to the vertices of both $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$, satisfying the following property. A six-vertex height function $h : \mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to \mathbb{Z}$ takes even values on $\mathbb{Z}^2$ and, for all $i \in \mathbb{Z}^2$ and $u \in (\mathbb{Z}^2)^*$ at a Euclidean distance of $1/\sqrt{2}$, it holds that $|h(i) - h(u)| = 1$. An edge $e = \{i, j\} \in \mathbb{E}$ (or its associated tile) with dual $e^* = \{u, v\}$ is said to be of type 5-6 if both $h(i) = h(j)$ and $h(u) = h(v)$, and otherwise of type 1-4. Given a finite set $E \subset \mathbb{E}$ and a parameter $\mathbf{c} > 0$, each height function is assigned a weight given by

$$\mathrm{w}_{E;\mathbf{c}}(h) := \mathbf{c}^{|\{e \in E \text{ of type 5-6}\}|}.$$

Through the introduction of boundary conditions, measures on the set of height functions are defined by a proportionality relation to the above weight. When $1 \leq \mathbf{c} \leq 2$, the height function under constant boundary condition is known to *delocalise* [DCKMO20] (see also [Lis21a, GL23]) as the system exhausts the whole lattice, whereas it was shown to *localise* [DGH$^+$21, GP23] when $\mathbf{c} > 2$.

In relation to the Potts and FK measures under Dobrushin boundary conditions, we consider six-vertex height function measures with parameter $\mathbf{c} > 2$ on square domains and with the following Dobrushin boundary condition (see Fig. 4.4): $+1$ on $(\mathbb{Z}^2)^*$ in the upper half-plane, -1 on $(\mathbb{Z}^2)^*$ in the lower half-plane, and constant 0 on $\mathbb{Z}^2$. In analogy to Theorems 13-14, we derive the convergence under these height function measures of the Peierls contour between $+1$ and -1 in the odd heights under diffusive scaling. We prove this statement for measures with a modified parameter $\mathbf{c}_\mathrm{b}$ on the boundary, as they can be coupled with the FK and Potts measures, which allows us to transfer the result to the latter and derive Theorems 13-14. However, it should be noted that this boundary weight in fact complicates the analysis in the related AT model. The derivation of the statement for the measures without the modified boundary weight is simpler and can be undertaken using the same method.

It is also noteworthy that we establish exponential relaxation results for the height function measures with constant boundary conditions and potentially inhomogeneous weights $\mathbf{c}$ and $\mathbf{c}_{\mathrm{b}}$ at the boundary (see Section 4.5.2).

4.1.5 Summary of the chapter: what is new?

The main novelty of our work is to study the FK percolation and the Potts models via the graphical (or random-cluster) representation of the Ashkin–Teller model (ATRC model) introduced in [PV97]. At a first glance, the FK percolation model has been by now much better understood, see the classical book [Gri06] and more recent lecture notes [Dum17]. Moreover, the FK model has been used to establish some basic properties for the ATRC model [GP23, ADG24]. However, compared to the FK percolation at its transition point when $q > 4$, the ATRC model at its self-dual line when $U > J$ remarkably exhibits a unique Gibbs measure. This brings more symmetries that play a key role in the study of interfaces.

This comes at some cost: the ATRC model is supported on *pairs* of edge configurations. Thus, the domain Markov property is significantly weaker than in the FK percolation defined on a single edge configuration. In particular, it is highly non-trivial to prove mixing properties of the ATRC model. For this we use a coupling of the ATRC model to the six-vertex model, whose height function enjoys additional monotonicity properties. Using the classical work of Alexander [Ale98] and a new general mixing result [Ott25], we prove, for the regime of parameters considered in the rest of the work, that the ATRC model satisfies

- the exponential ratio weak mixing property (Theorem 16),
- a restricted version of the exponential strong mixing property (Theorem 20).

These mixing properties allow us to extend the arguments of [CIV03, CIV08, OV18, AOV24] to the ATRC model and provide a renewal picture for the long subcritical ATRC clusters in infinite volume (Theorem 21).

Regarding the link with interfaces in FK/Potts, we point out a technical issue that the FK percolation model under standard Dobrushin boundary conditions is directly coupled only to the ATRC with rather involved "boundary conditions" (modified cluster weight on the boundary, we call it a modified ATRC model). Fortunately, this model still satisfies the FKG lattice condition which allows for comparison with normal ATRC measures. Using this, we are able to transfer the results derived for the infinite volume ATRC to the modified, finite volume, measures. This is the content of Section 4.8.

Organisation of the chapter. We now provide the structure of the chapter.

Section 4.2: notation and conventions that will be used throughout the chapter.

Section 4.3: definition of the ATRC model, its basic properties and statement of the results that we establish for this model, including the uniqueness of the ATRC Gibbs measure and mixing properties.

Section 4.4: coupling between the FK percolation and modified ATRC models. The coupling is very sensitive to boundary conditions, eg. note appearance of a different boundary-cluster weight in [GP23]. We extend this coupling to standard Dobrushin boundary conditions in the FK percolation at a price of rather inconvenient conditions in the ATRC model. In particular, we obtain a coupling between the FK interface and a crossing cluster under a modified version of a finite volume ATRC measure.

Section 4.5: proof of new weak mixing properties (Theorem 16) for the ATRC model and, in particular, uniqueness of the ATRC Gibbs measure. We adapt the classical works of Alexander [Ale92, Ale98, Ale04] and use the recent works on the ATRC model [GP23, ADG24].

Section 4.6: derivation of a restricted version of strong mixing (Theorem 20) from the weak mixing of Section 4.5, using a general argument from [Ott25].

Section 4.7: development of the "OZ theory" (renewal picture) for long subcritical cluster in the ATRC model, in infinite volume, following [CIV08, AOV24], and using the mixing established in Sections 4.5 and 4.6. More precisely, we show that the typical geometry of clusters conditioned on containing two distant points is the same as the one of a directed random walk bridge between these points (Theorem 21).

Section 4.8: using stochastic comparison and mixing properties, we couple the "crossing cluster" under the finite volume measures of Section 4.4.2 with an infinite volume long cluster as studied in Section 4.7 (Theorem 23). From there, an invariance principle follows by importing the results of [Kov04].

Section 4.9: wrap up of the proofs of Theorems 13, 14, 15, and 18.

4.2 Notations and conventions

General graphs. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a graph. We simply write $xy = \{x, y\}$ for an edge $\{x, y\} \in \mathbb{E}$. Given finite subsets $\Lambda \subset \mathbb{V}$ and $E \subset \mathbb{E}$, define

$$\mathbb{E}_\Lambda := \{e \in \mathbb{E} : e \subset \Lambda\}, \quad \mathbb{V}_E := \bigcup_{e \in E} e,$$

$$\partial^{\mathrm{in}}\Lambda := \{x \in \Lambda : \exists y \in \mathbb{V} \setminus \Lambda,\, xy \in \mathbb{E}\}, \quad \partial^{\mathrm{ex}}\Lambda := \{y \in \mathbb{V} \setminus \Lambda : \exists x \in \Lambda,\, xy \in \mathbb{E}\},$$

$$\partial^{\mathrm{in}}E := \{e \in E : \exists f \in \mathbb{E} \setminus E, e \cap f \neq \varnothing\}, \quad \partial^{\mathrm{ex}}E := \{e \in \mathbb{E} \setminus E : \exists f \in E, e \cap f \neq \varnothing\},$$

$$\partial^{\mathrm{edge}}\Lambda := \{xy \in \mathbb{E} : x \in \Lambda, y \in \mathbb{V} \setminus \Lambda\}.$$

In case of ambiguity, we add $\mathbb{G}$ as a subscript (and write, for example, $\partial^{\mathrm{in}}_{\mathbb{G}}\Lambda$) to emphasise that the boundary is taken in $\mathbb{G}$. The *interior* of Λ (in $\mathbb{G}$) is given by $\Lambda \setminus \partial^{\mathrm{in}}\Lambda$. The subgraph *induced* by $\Lambda \subset \mathbb{V}$ is given by $(\Lambda, \mathbb{E}_\Lambda)$, and the subgraph induced by $E \subset \mathbb{E}$ is given by $(\mathbb{V}_E, E)$.

Lattices. We will mainly work on $\mathbb{Z}^2$ with nearest-neighbour edges, and on its dual. We will denote the *primal lattice* by $\mathbb{L}_\bullet = \mathbb{Z}^2$ and its *dual* by $\mathbb{L}_\circ = (1/2, 1/2) + \mathbb{Z}^2$. Denote by $\mathbb{E}^\bullet$ the nearest-neighbour edges between sites in $\mathbb{L}_\bullet$ (primal edges), by $\mathbb{E}^\circ$ the nearest-neighbour edges between sites in $\mathbb{L}_\circ$ (dual edges).

Duality. Each edge $e \in \mathbb{E}^\bullet$ intersects a unique edge of $\mathbb{E}^\circ$, we denote it by e^*. For a set of primal (or dual) edges E, define $*E = \{e^* :\ e \in E\}$. We say that $E \subset \mathbb{E}^\bullet$ is *simply lattice-connected* if both the subgraphs induced by E and by $*(\mathbb{E}^\bullet \setminus E)$ are connected. As a convention, sets of edges or of dual edges will be identified with the corresponding sets of mid-points whenever the meaning is clear from the context.

Tiles. To each primal-dual pair of edges e, e^*, associate a *tile* t given by the convex hull of their endpoints and define $e_t := e$; see Fig. 4.3. Define $\mathbb{L}_\diamond$ as the set of all tiles, and let $\mathbb{E}^\diamond$ be the set of all pairs of adjacent tiles. Note that $(\mathbb{L}_\diamond, \mathbb{E}^\diamond)$ is the medial graph of $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$, since tiles can be identified with midpoints of edges in $\mathbb{E}^\bullet$.

Standard rectangular domains. Define the upper and lower half planes by

$$\mathbb{H}^+ := \mathbb{R} \times \mathbb{R}_{\geq 0}, \quad \mathbb{H}^- := \mathbb{R} \times \mathbb{R}_{<0},$$

and, for $n, m \geq 0$, set (see Fig. 4.5)

$$\Lambda_{n,m} := \{-n, \dots, n\} \times \{-m, \dots, m\},$$
$$\Lambda'_{n,m} := \Big(([-n-1, n+1] \times [0, m+1]) \cup ([-n, n] \times [-m, 0]) \Big) \cap \mathbb{L}_\circ.$$

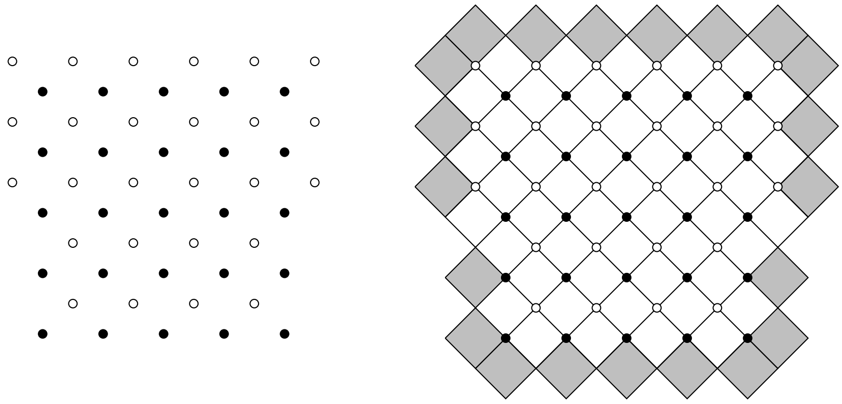


Figure 4.5: Left: the sets $\Lambda_{2,2}$ (solid) and $\Lambda'_{2,2}$ (hollow). Right: the inner tiles $A^{\mathrm{i}}_{2,2}$ (white) and the boundary tiles $\partial A_{2,2}$ (grey).

Define $\mathcal{D}_{n,m} := \Lambda_{n,m} \cup \Lambda'_{n,m}$ and the graph $G_{n,m} = (V_{n,m}, E_{n,m})$ by (see Fig. 4.2)

$$V_{n,m} = \Lambda_{n,m} \cup \left(\partial^{\mathrm{ex}}\Lambda_{n,m} \cap \mathbb{H}^+\right), \quad E_{n,m} = \mathbb{E}_{V_{n,m}} \setminus \mathbb{E}_{\mathbb{L}_\bullet \setminus \Lambda_{n,m}}.$$

Let $A_{n,m} \subset \mathbb{L}_\diamond$ be the set of tiles with at least one corner in $\mathcal{D}_{n,m}$ and $\partial A_{n,m} \subset A_{n,m}$ (boundary tiles) be the set of tiles with precisely one corner in $\mathcal{D}$; see Fig. 4.5. Define also $\partial^+ A = \{t \in \partial A : t \subseteq \mathbb{H}^+\}$, $\partial^- A = \partial A \setminus \partial^+ A$ and $A^{\mathrm{i}}_{n,m} := A_{n,m} \setminus \partial A_{n,m}$.

Define also (see Fig. 4.6 and 4.9)

$$v_L(n) := (-n-1, 0), \quad v_R(n) := (n+1, 0).$$

Augmented rectangular domains. We also introduce $E_{n,m}$ augmented by some *upper* boundary edges. First, define the set of boundary edges and its upper and lower parts by

$$E_{\mathrm{b},n,m} := \{e_t : t \in \partial A_{n,m}\} \qquad \text{and} \qquad E^{\pm}_{\mathrm{b},n,m} = \{e_t : t \in \partial^{\pm} A_{n,m}\}.$$

Then define the augmented sets of edges and vertices by

$$\bar{E}_{n,m} = E_{n,m} \cup E^{+}_{\mathrm{b},n,m} \qquad \text{and} \qquad \bar{V}_{n,m} = \mathbb{V}_{\bar{E}_{n,m}}.$$

We now define the augmented domain and its dual (see Fig. 4.6):

- Set $K_{n,m} := (\bar{V}_{n,m}, \bar{E}_{n,m})$,
- and let $K^*_{n,m}$ be the graph obtained from $(\mathbb{V}_{*\bar{E}_{n,m}}, *\bar{E}_{n,m})$ by identifying the vertices in $\partial^{\mathrm{in}}_{\mathbb{L}_\circ} \mathbb{V}_{*\bar{E}_{n,m}}$.

Connectivity events. Given $\Lambda, \Delta \subset \mathbb{L}_\bullet$ and a percolation configuration $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$, we write $\Lambda \overset{\omega}{\longleftrightarrow} \Delta$ for the event that Λ and Δ are connected by a path in the graph $(\mathbb{L}_\bullet, \omega)$. If $\Lambda = \{i\}$ and $\Delta = \{j\}$, we simply write $i \overset{\omega}{\longleftrightarrow} j$. We omit ω from the notation when it cannot lead to any confusion.

Agreements of spins along edges. For an edge $e = ij \in \mathbb{E}^\bullet$ and a spin configuration $\sigma_\bullet \in \{+1,-1\}^{\mathbb{L}_\bullet}$, we write $\sigma_\bullet \sim e$ if $\sigma_\bullet(i) = \sigma_\bullet(j)$; for $\xi \subset \mathbb{E}^\bullet$, we write $\sigma_\bullet \sim \xi$ if $\sigma_\bullet \sim e$ for every $e \in \xi$ (in other words, $\sigma_\bullet$ is constant on clusters of ξ). We use similar notation for edges in $\mathbb{E}^\circ$ and $\sigma_\circ \in \{+1,-1\}^{\mathbb{L}_\circ}$.

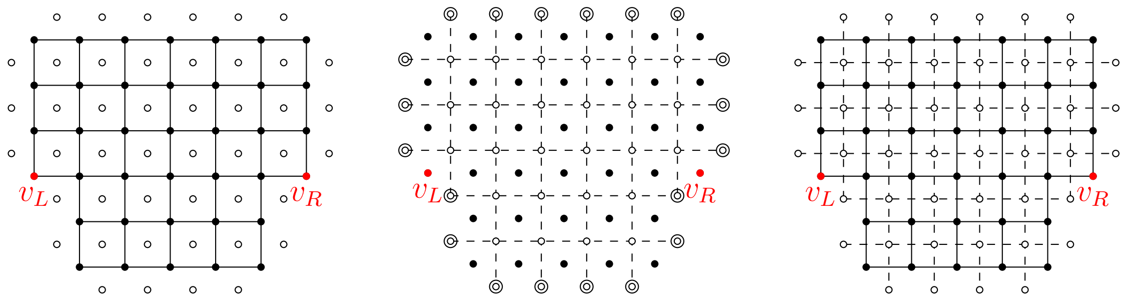


Figure 4.6: Left: the graph $K_{2,2}$. Center: the graph $K^*_{2,2}$ (vertices surrounded by a circle are identified). Right: planar duality relation between their edges.

Parameters. The couplings go through standard "expansion–resummation" of Boltzmann weights combined with extensive use of planarity. As for each step one will write the weight associated with a given model as a sum of weights for a "more expanded" model, several parameters will come into play. Below we list the parameters, as well as the algebraic relations linking them:

$$\begin{gathered} q > 4, \quad \beta = \beta_c(q) = \ln(1+\sqrt{q}), \quad \lambda > 0, \quad \mathbf{c} > 2, \quad U > J > 0, \\ \sqrt{q} = e^{\lambda} + e^{-\lambda}, \quad \mathbf{c} = e^{\lambda/2} + e^{-\lambda/2} = \coth(2J), \\ \sinh(2J) = e^{-2U}. \end{gathered} \tag{4.3}$$

The parameter $\mathbf{c}$ also has a "boundary version":

$$\mathbf{c}_{\mathrm{b}} = e^{\lambda/2} > 1. \tag{4.4}$$

For the remainder of the chapter, we fix $q > 4$, along with the corresponding parameters above (which are uniquely determined by q).

Constants. Constants like $c, c_1, C, C_1, C', \ldots$ are constants which can change from line to line and which can depend on the parameters unless explicitly stated. They are independent of the system size, n, which will be our main "variable" quantity.

4.3 Ashkin–Teller random-cluster model

Like the Potts models, the AT model has a random-cluster (RC) representation, called the *ATRC model*, and introduced in [CM97, PV97]. We will first introduce the model and state some of its basic properties. This will be followed by the statement of our results.

4.3.1 Definition and basic properties

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a graph, and let $G = (V, E)$ be a finite subgraph of $\mathbb{G}$. Let $\eta_\tau, \eta_{\tau\tau'} \in \{0,1\}^{\mathbb{E}}$ with $\eta_\tau \subseteq \eta_{\tau\tau'}$. The ATRC model on G with parameters $U > J > 0$ under boundary conditions $(\eta_\tau, \eta_{\tau\tau'})$ is the probability measure on $\{0,1\}^{\mathbb{E}} \times \{0,1\}^{\mathbb{E}}$ given by

$$\begin{aligned} \mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{G;J,U}(\omega_\tau, \omega_{\tau\tau'}) = \tfrac{1}{Z} \cdot \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \cdot \mathrm{w}_\tau^{|\omega_\tau \cap E|}\, \mathrm{w}_{\tau\tau'}^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|}\, 2^{k_V(\omega_\tau) + k_V(\omega_{\tau\tau'})} \\ \cdot \prod_{e \in \mathbb{E}\setminus E} \mathbb{1}_{\omega_\tau(e) = \eta_\tau(e)} \mathbb{1}_{\omega_{\tau\tau'}(e) = \eta_{\tau\tau'}(e)}, \end{aligned} \tag{4.5}$$

where $Z = Z_{\mathtt{ATRC}}^{\eta_\tau,\eta_{\tau\tau'}}(G,J,U)$ is the partition function, $k_V(\cdot)$ is the number of clusters that intersect V, and the weights are given by

$$\mathrm{w}_\tau = e^{2U}(e^{2J} - e^{-2J}) \quad \text{and} \quad \mathrm{w}_{\tau\tau'} = e^{2(U-J)} - 1. \tag{4.6}$$

When $\eta_\tau \equiv 0$ (resp. $\eta_\tau \equiv 1$), we write 0 (resp. 1) instead of η_τ in the superscript, and analogously for $\eta_{\tau\tau'}$. Given finite subsets $\Lambda \subset \mathbb{V}$ and $E \subset \mathbb{E}$, we write $\mathtt{ATRC}_{\Lambda;J,U}^{\eta_\tau,\eta_{\tau\tau'}}$ and $\mathtt{ATRC}_{E;J,U}^{\eta_\tau,\eta_{\tau\tau'}}$ for the measures on the graphs $(\Lambda, \mathbb{E}_\Lambda)$ and $(\mathbb{V}_E, E)$, respectively.

Finite energy. There exists a constant $c = c(J,U) > 0$ such that the following holds. For any finite subgraph $G = (V,E)$, any $e \in E$ and any $a, b \in \{0,1\}^E$ with $a \subseteq b$

$$\mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}}\big((\omega_\tau(e), \omega_{\tau\tau'}(e)) = (a_e, b_e) \,\big|\, (\omega_\tau, \omega_{\tau\tau'}) = (a,b) \text{ on } E \setminus \{e\}\big) > c. \tag{FE}$$

"DLR" property. Given a subgraph $G' = (V', E')$ of G and boundary conditions $\eta_\tau, \eta_{\tau\tau'}, \eta'_\tau, \eta'_{\tau\tau'}$ with $\eta_\tau = \eta'_\tau$ on $\mathbb{E} \setminus E'$ and $\eta_{\tau\tau'} = \eta'_{\tau\tau'}$ on $\mathbb{E} \setminus E'$,

$$\mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}}(\,\cdot \mid (\omega_\tau, \omega_{\tau\tau'}) = (\eta'_\tau, \eta'_{\tau\tau'}) \text{ on } \mathbb{E} \setminus E') = \mathtt{ATRC}_{G';J,U}^{\eta'_\tau,\eta'_{\tau\tau'}}(\cdot). \tag{DLR}$$

Stochastic domination and positive association. We first introduce these notions in a general setting. Given a partially ordered set $\mathcal{S}$ (states) and some index set I (eg. a set of edges or vertices), consider S^I: the set of functions from I to $\mathcal{S}$, i.e. configurations on I with values in $\mathcal{S}$. A subset $A \subseteq \mathcal{S}^I$ is called *increasing* if $\mathbb{1}_A$ is increasing with respect to the pointwise order. Given a σ-algebra $\mathcal{A}$ on $\mathcal{S}^I$ and two probability measures μ and ν on $\mathcal{A}$, we say that ν stochastically dominates μ, and write $\mu \leq_{\text{st}} \nu$ (or $\nu \geq_{\text{st}} \mu$) if $\mu(A) \leq \nu(A)$ for any increasing event $A \in \mathcal{A}$. We say that μ is *positively associated* or satisfies the *FKG property* (named after the seminal work of Fortuin, Kasteleyn and Ginibre [FKG71]) if, for all increasing events $A, B \in \mathcal{A}$, we have

$$\mu(A \cap B) \geq \mu(A)\mu(B). \tag{FKG}$$

If I is finite and $\mu(\omega) > 0$ for any $\omega \in \mathcal{S}^I$, we say that μ satisfies the *strong FKG* property if

$$\forall\, J \subset I \;\forall\, \omega_1 \leq \omega_2: \quad \mu(\,\cdot \mid \omega = \omega_1 \text{ on } J) \leq_{\text{st}} \mu(\,\cdot \mid \omega = \omega_2 \text{ on } J), \tag{strong-FKG}$$

where we wrote $\omega = \omega_i$ on J for the event that $\omega(j) = \omega_i(j)$ for all $j \in J$. We note that this is equivalent to the *FKG lattice condition*; see [Gri06, Section 2] and [GHM01, Section 4] for the background.

In order to incorporate $\mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}}$ into this general framework, note that this is a positive measure on $\{(0,0),(0,1),(1,1)\}^E$. Indeed, ω_τ and $\omega_{\tau\tau'}$ that coincide respectively with η_τ and $\eta_{\tau\tau'}$ on $\mathbb{E}^\bullet \setminus E$ can be viewed as elements of $\{0,1\}^E$, and moreover $\omega_\tau(e) \leq \omega_{\tau\tau'}(e)$ for every $e \in E$ by (4.5).

Lemma 4.3.1 ([PV97]). *For any $U > J > 0$, any subgraph $G = (V,E)$ of $\mathbb{G}$ and all boundary conditions $\eta_\tau, \eta_{\tau\tau'}$, the measure $\mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}}$ satisfies the strong FKG property.*

This is an easy consequence of [PV97, Proposition 4.1] and its proof.

By Lemma 4.3.1 and (DLR), for any increasing sequence of subgraphs $G_k \nearrow \mathbb{G}$, the measures $\mathtt{ATRC}_{G_k;J,U}^{1,1}$ form a decreasing sequence in the sense of stochastic domination. Consequently, the weak limit exists and is independent (G_k). We denote it by $\mathtt{ATRC}_{J,U}^{1,1}$. Analogously, we define $\mathtt{ATRC}_{J,U}^{0,0}$ as the (increasing) limit of $\mathtt{ATRC}_{G_k;J,U}^{0,0}$.

Furthermore, Lemma 4.3.1 and (DLR) can be used to compare different boundary conditions: if $U > J > 0$, $\eta_\tau \leq \eta'_\tau$ and $\eta_{\tau\tau'} \leq \eta'_{\tau\tau'}$, then

$$\mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}} \leq_{\text{st}} \mathtt{ATRC}_{G;J,U}^{\eta'_\tau,\eta'_{\tau\tau'}}. \tag{CBC}$$

FK-type relations on the square lattice. Recall the infinite-volume AT expectation operators $\langle\,\cdot\,\rangle^{\mathrm{f,f}}$ and $\langle\,\cdot\,\rangle^{+,+}$ defined in Section 4.1.3 and the notation $\overset{\omega}{\longleftrightarrow}$ for a connectivity event introduced in Section 4.2. The essential feature of the ATRC is that its connection probabilities are related to the correlations in the AT model [PV97, Proposition 3.1]: for $U > J > 0$ and any $i, j \in \mathbb{L}_\bullet$,

$$\langle \tau_i \tau_j \rangle^{+,+} = \mathtt{ATRC}^{1,1}_{J,U}(i \overset{\omega_\tau}{\longleftrightarrow} j), \qquad \langle \tau_i \tau'_i \tau_j \tau'_j \rangle^{+,+} = \mathtt{ATRC}^{1,1}_{J,U}(i \overset{\omega_{\tau\tau'}}{\longleftrightarrow} j). \tag{4.7}$$

The analogous relations for $\langle\cdot\rangle^{\mathrm{f,f}}$ and $\mathtt{ATRC}^{0,0}_{J,U}$ are also valid.

Duality on the square lattice. Recall the primal and dual square lattices $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ and $(\mathbb{L}_\circ, \mathbb{E}^\circ)$ (Section 4.2) and the notion of a dual percolation configuration $\eta^* \in \{0,1\}^{\mathbb{E}^\circ}$ for each $\eta \in \{0,1\}^{\mathbb{E}^\bullet}$ (Section 4.1). Choose J, U as in (4.3). Then, for any finite $E \subset \mathbb{E}^\bullet$ and any $\eta_\tau, \eta_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$, by [PV97, Proposition 3.2],

$$(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{ATRC}^{\eta_\tau, \eta_{\tau\tau'}}_{E;J,U} \quad \text{implies} \quad (\omega^*_{\tau\tau'}, \omega^*_\tau) \sim \mathtt{ATRC}^{\eta^*_{\tau\tau'}, \eta^*_\tau}_{*E;J,U}. \tag{4.8}$$

Notice the different order in the dual pair.

4.3.2 Results on the ATRC

In this section, we present our main results concerning the self-dual ATRC model on the square lattice $\mathbb{L}_\bullet$.

Mixing properties. We establish exponential relaxation for the single edges, which together with (strong-FKG) implies *exponential weak mixing* (see Section 4.5).

Proposition 4.3.2. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists $c > 0$ such that, for any edge $e \in \mathbb{E}^\bullet$ with $0 \in e$, and any $n \geq 1$,*

$$\max_{\sigma \in \{\tau, \tau\tau'\}} \Big| \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\big(\omega_\sigma(e) = 1\big) - \mathtt{ATRC}^{0,0}_{\Lambda_n;J,U}\big(\omega_\sigma(e) = 1\big) \Big| \leq e^{-cn}.$$

As a consequence, the measures $\mathtt{ATRC}^{\eta_\tau, \eta_{\tau\tau'}}_{G;J,U}$ converge (as $G \nearrow \mathbb{L}_\bullet$) to a limit measure $\mathtt{ATRC}_{J,U}$, which is independent of the choice of boundary conditions $\eta_\tau, \eta_{\tau\tau'}$. Furthermore, the limit $\mathtt{ATRC}_{J,U}$ is the unique ATRC *Gibbs measure*. Moreover, applying the classical work of Alexander [Ale98], we derive *ratio* weak mixing.

Theorem 16. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exist $C \geq 0, c > 0$ such that, for any finite subgraph $G = (V, E)$ of $\mathbb{L}_\bullet$, any $F \subset E$, and any F-measurable event A having positive probability,*

$$\sup_{\eta_\tau, \eta_{\tau\tau'}, \eta'_\tau, \eta'_{\tau\tau'}} \left| \frac{\mathtt{ATRC}^{\eta_\tau, \eta_{\tau\tau'}}_{G;J,U}(A)}{\mathtt{ATRC}^{\eta'_\tau, \eta'_{\tau\tau'}}_{G;J,U}(A)} - 1 \right| \leq C \sum_{f \in F} \sum_{e \in E^c} e^{-c\,\mathrm{d}_\infty(e,f)},$$

whenever the right side is strictly less than 1.

The notion of ATRC Gibbs measures can be defined in an analogous way to the FK model; see [Gri06, Chapter 4.4]. The following is a direct consequence of Theorem 16.

Corollary 4.3.3. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. It holds that*

$$\mathtt{ATRC}^{0,0}_{J,U} = \mathtt{ATRC}^{1,1}_{J,U} =: \mathtt{ATRC}_{J,U}.$$

In particular, $\mathtt{ATRC}_{J,U}$ is the unique ATRC Gibbs measure, and for any choice of boundary conditions $\eta_\tau, \eta_{\tau\tau'}$, the measures $\mathtt{ATRC}^{\eta_\tau, \eta_{\tau\tau'}}_{G;J,U}$ converge to $\mathtt{ATRC}_{J,U}$ as $G \nearrow \mathbb{L}_\bullet$.

We refer to Section 4.6, for our results on *strong* mixing properties of the ATRC.

Uniform exponential decay. In Section 4.5, we prove exponential weak mixing (Theorem 19). It allows to apply the work of Alexander [Ale04] to derive the following uniform decay of connection probabilities:

Theorem 17. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists a constant $c > 0$ such that, for any finite simply lattice-connected $E \subset \mathbb{E}^\bullet$, and any $i, j \in \mathbb{V}_E$,*

$$\mathtt{ATRC}^{1,1}_{E;J,U}\big(i \xleftrightarrow{\omega_\tau \cap E} j\big) \leq e^{-c\,\mathrm{d}_\infty(i,j)},$$

where d_∞ is the distance induced by the L^∞ norm.

Proof. It is enough to check the conditions stated in [Ale04, Theorem 1.1]: the push-forward of $\mathtt{ATRC}_{J,U}$ by ω_τ is translation invariant, has finite-energy for closing edges, exponential decay of connectivity probabilities [ADG24, Proposition 1.1], and is exponentially weak mixing by Theorem 19. Finally, for n with $E \subseteq \mathbb{E}_{\Lambda_{n-1}}$, we have by (DLR) that

$$\mathtt{ATRC}_{J,U}(\omega_\tau \in \cdot \mid \omega_\tau(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda_n} \setminus E) = \mathtt{ATRC}^{1,1}_{E;J,U}. \qquad \square$$

Ornstein–Zernike asymptotics. Recall the definition of the inverse correlation length $\nu_{J,U}$ in the AT model in Section 4.1.3. By the FK-type relations (4.7) and by Corollary 4.3.3,

$$\nu_{J,U}(x) = -\lim_{n\to\infty} \tfrac{1}{n} \ln \mathtt{ATRC}_{J,U}(0 \xleftrightarrow{\omega_\tau} \lfloor nx \rfloor).$$

The following theorem is the analogue of Theorem 15 for the ATRC model.

Theorem 18. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. The inverse correlation length $\nu_{J,U}$ is a norm on $\mathbb{R}^2$. Furthermore, uniformly in $|x| \to \infty$,*

$$\mathtt{ATRC}_{J,U}(0 \xleftrightarrow{\omega_\tau} x) = \tfrac{g(x/|x|)}{\sqrt{|x|}}\, e^{-\nu(x)}\,(1 + o(1)),$$

where $\nu = \nu_{J,U}$, and $g = g_{J,U}$ is a strictly positive analytic function on $\mathbb{S}^1$.

4.4 Couplings and interfaces

This section is concerned with the construction of a coupling of the FK-percolation and the Ashkin–Teller models, via the six-vertex model, and the derivation of its basic properties. The coupling of the FK and six-vertex measures is an adaptation of the Baxter–Kelland–Wu (BKW) coupling to the Dobrushin boundary conditions [BKW76]. The relation of the six-vertex and AT measures has first been noticed in [Fan72a] comparing their critical properties, and it was made explicit in [Fan72b, Weg72] on a level of partition functions. We build on [GP23], where a coupling of the six-vertex model and a graphical representation of the AT model (a marginal of ATRC) was constructed.

4.4.1 Different models and combinatorial mappings

This section provides an overview of the combinatorial objects that will be encountered, as well as a description of their relations. We first discuss oriented loop configurations, which serve as an intermediate step in the BKW coupling of the FK-percolation and the six-vertex models. This is followed by a description of two of the representations of the six-vertex model: edge orientations and spin configurations. The height function representation appears only in Section 4.5, where

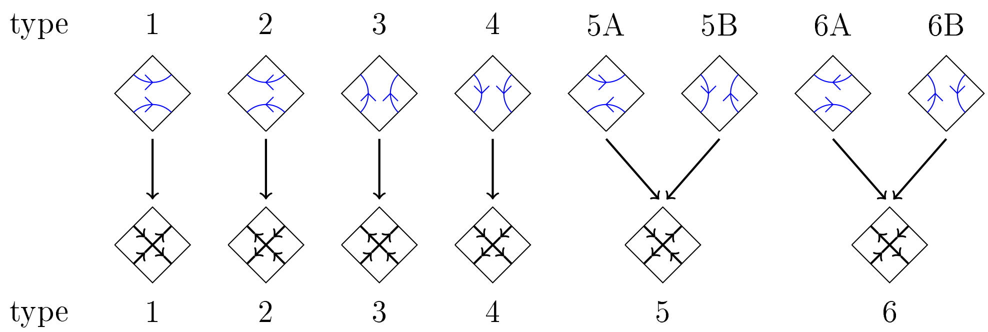


Figure 4.7: Tiles of the oriented loop model and their types and weights, and the mapping from oriented loop arcs to six-vertex edge orientations.

its monotonicity properties are crucial to derive mixing and relaxation properties of the ATRC measures.

In Section 4.1, we saw that percolation configurations are in bijection with (unoriented) loop configurations. To make the correspondence $\omega \leftrightarrow \omega^* \leftrightarrow \ell = \mathrm{loop}(\omega)$ explicit, we can regard each of these models as an assignment of a local piece of drawing of an edge and two *arcs* to lozenge tiles centred at the mid-edges as depicted in Fig. 4.3. Clearly, retaining only either the primal or dual edges, or the arcs, provides complete information about all three.

Oriented loop configurations are obtained from unoriented ones by assigning an orientation to each loop; formally, this is done by means of a sequence of independent uniform random variables on $[0, 1]$ indexed by the set $\mathcal{L}$ of all loops. Alternatively, one can assign orientations to the loop arcs on each tile, subject to the constraint that the orientations of neighbouring arcs match. The eight local configurations that can occur at a tile are referred to as *types*, see Fig. 4.7.

The edge orientations of the six-vertex model [Pau35, Rys63] are assignments of orientations to the edges in $\mathbb{E}^\diamond$, obtained from oriented loop configurations via the natural surjection; see Fig. 4.7. These edge orientations satisfy the *ice rule*: at every vertex, there are two incoming and two outgoing edges of $\mathbb{E}^\diamond$. This constraint permits six possible local configurations at a tile, which are also called types; see Fig. 4.7. The local inverse operation can be considered as *splitting* the oriented edges into two oriented loop arcs. While tiles of types 1-4 permit a unique reconstruction of the loop arcs based on the edge orientations, there are two possibilities for tiles of types 5-6, giving the latter a special role.

The six-vertex spin representation. The six-vertex model may be represented by pairs of spin-configurations $(\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$ [Wu71, KW71, Lis22, GP23], obtained from the edge orientations via the following two-valued mapping. Fix the value of $\sigma_\bullet$ or $\sigma_\circ$ at some arbitrary fixed vertex and proceed iteratively: given an edge $e \in \mathbb{E}^\diamond$, denote the vertices that it separates by $i \in \mathbb{L}_\bullet$ from a vertex $u \in \mathbb{L}_\circ$; we impose $\sigma_\bullet(i) = \sigma_\circ(u)$ if i is to the left side of e (with respect to its assigned orientation) and we impose $\sigma_\bullet(i) = -\sigma_\circ(u)$ otherwise; see Fig. 4.8. The ice rule ensures that this mapping is well-defined.

This mapping is two-valued due to the liberty to choose the value of $\sigma_\bullet$ or $\sigma_\circ$ at one vertex, and the two images are related to each other by a global spin flip. Furthermore, it is injective, meaning that the edge orientations can be reconstructed from the spins. The type of a tile with respect to $(\sigma_\bullet, \sigma_\circ)$ is given by the type of the corresponding edge orientations; see Fig. 4.8. It should also be noted that the ice rule can be translated as follows: for any tile $t \in \mathbb{L}_\diamond$, either $\sigma_\bullet$ is constant on the endpoints of e_t or $\sigma_\circ$ is constant on the endpoints of e_t^*. Formally,

$$\big(\sigma_\bullet(i) - \sigma_\bullet(j)\big)\big(\sigma_\circ(u) - \sigma_\circ(v)\big) = 0 \qquad \text{for any } t \in \mathbb{L}_\diamond \text{ with } e_t = ij,\ e_t^* = uv. \tag{4.9}$$

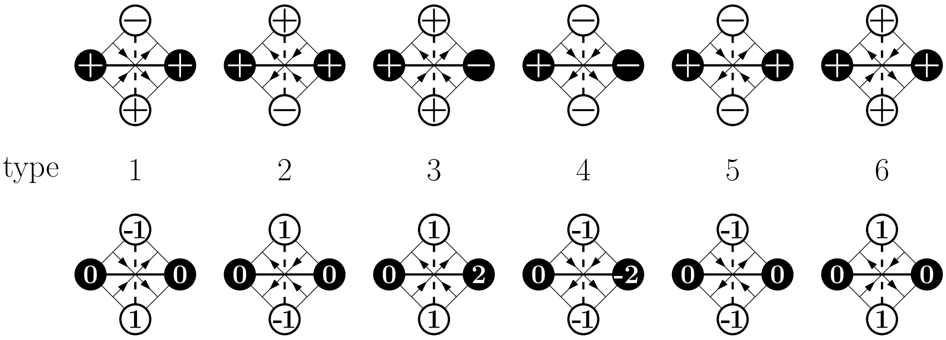


Figure 4.8: The six-vertex types for all representations at a tile corresponding to a horizontal primal edge e. Top: the spin at the left endpoint of e is fixed to be $+$. Bottom: the height at the left endpoint of e is fixed to be 0.

In the context of spins, this property will henceforth be referred to as the ice rule.

Baxter–Kelland–Wu correspondence [BKW76]. Given a measure on oriented loops, taking its push-forwards with respect to the above mappings, one obtains measures on the six-vertex edge orientations and spin configurations.

4.4.2 Coupling under Dobrushin conditions

The idea is to take the FK measure with Dobrushin boundary conditions and construct from it first a measure on pairs of spin configurations on $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$ that satisfy the ice rule and then a measure on ATRC configurations. We will then identify these two measures as the six-vertex and the ATRC measures, respectively, under suitable versions of Dobrushin boundary conditions. This gives a coupling between the FK and a modified ATRC, which will allow us to transfer the study of the former to the study of the latter. See also Section 4.5.1 for the BKW coupling [BKW76] without Dobrushin boundary conditions and in the context of height functions.

Recall the subgraphs $G_{n,m} = (V_{n,m}, E_{n,m})$ of $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ given in Section 4.2. As n, m will be fixed in this section, we will omit them in the notation and simply write $G = (V, E)$. Consider $\mathtt{FK}_G^{1/0}$: the FK measure on G under Dobrushin boundary conditions defined in Section 4.1; see Fig. 4.2.

Coupling measure. To couple $\mathtt{FK}_G^{1/0}$ with both a six-vertex spin measure and a modified version of the ATRC measure, we augment our probability space to incorporate independent uniform $[0,1]$ random variables assigned to every loop and every tile. Formally, let $\mathcal{L}$ be the set of all unoriented loops drawn on the tiles of $\mathbb{L}_\diamond$ (see the right of Fig. 4.3). Define $\Omega^{1/0} := \{0,1\}^{\mathbb{E}^\bullet} \times [0,1]^{\mathcal{L}} \times [0,1]^{\mathbb{L}_\diamond}$ equipped with the product of Borel sigma algebras. Define Q and Q' respectively as the product measures on $[0,1]^{\mathcal{L}}$ and $[0,1]^{\mathbb{L}_\diamond}$. Finally, the coupling measure is defined by

$$\Psi_G^{1/0} := \mathtt{FK}_G^{1/0} \otimes Q \otimes Q'.$$

In Lemmata 4.4.1 and 4.4.2 below, we describe how to obtain the following two measures as push-forwards of $\Psi_G^{1/0}$: the six-vertex spin measure and the modified ATRC measure, both under suitable Dobrushin boundary conditions (Definitions 2 and 3).

From FK to six-vertex. Recall the parameters $\mathbf{c}$, $\mathbf{c}_{\mathrm{b}}$, and λ from (4.3) and the standard rectangular domains from Section 4.2. Below we omit n, m everywhere.

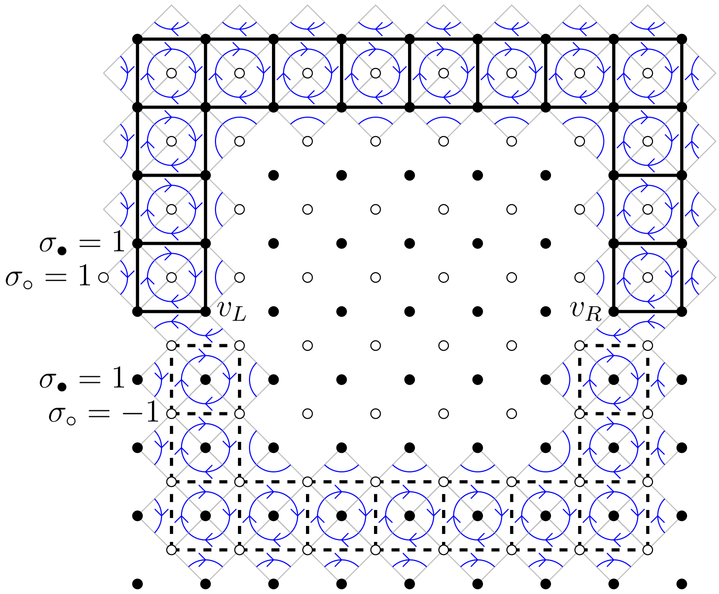


Figure 4.9: Boundary conditions on oriented loops and on six-vertex spins. The edges drawn are those induced by the oriented loop boundary conditions.

Definition 2. The six-vertex spin model under Dobrushin conditions is a probability measure on $\sigma = (\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$ defined by

$$\mathtt{Spin}_{\mathcal{D}}^{+,+-}(\sigma) \propto \mathbf{c}^{|T^{\mathrm{i}}_{5,6}(\sigma)|}\, \mathbf{c}_{\mathrm{b}}^{|T^{\mathrm{b}}_{5,6}(\sigma)|}\, \mathbb{1}_{\Sigma^+_\Lambda \times \Sigma^{+-}_{\Lambda'}}(\sigma)\, \mathbb{1}_{\mathrm{ice}}(\sigma), \tag{4.10}$$

where $\mathbf{c}, \mathbf{c}_{\mathrm{b}}$ are defined by (4.3)-(4.4), $T^{\mathrm{i}}_{5,6}$ and $T^{\mathrm{b}}_{5,6}$ are the sets of tiles of types 5-6 in A^{i} and ∂A, respectively (see Fig. 4.8), $\mathbb{1}_{\mathrm{ice}}$ is the indicator imposing the ice rule (4.9), and Σ^+_Λ and $\Sigma^{+-}_{\Lambda'}$ are the boundary conditions defined by

$$\Sigma^+_\Lambda := \{\sigma_\bullet \in \{\pm 1\}^{\mathbb{L}_\bullet} : \sigma_\bullet(i) = 1\, \forall i \in \mathbb{L}_\bullet \setminus \Lambda\},$$
$$\Sigma^{+-}_{\Lambda'} := \{\sigma_\circ \in \{\pm 1\}^{\mathbb{L}_\circ} : \sigma_\circ(u) = \mathbb{1}_{\mathbb{H}^+}(u) - \mathbb{1}_{\mathbb{H}^-}(u)\, \forall u \in \mathbb{L}_\circ \setminus \Lambda'\}.$$

We now describe how to obtain $\mathtt{Spin}_{\mathcal{D}}^{+,+-}$ as a push-forward of the coupling measure $\Psi_G^{1/0}$.

Lemma 4.4.1. *Let (ω, U, U') be distributed according to $\Psi_G^{1/0}$ and let ℓ be the unoriented loop configuration associated to ω. Orient the loops of ℓ as follows (see Fig. 4.9):*

- *each loop $l \in \ell$ outside of G (i.e. surrounding a vertex $(\mathbb{L}_\circ \cap \mathbb{H}^+) \setminus \Lambda'$ or in $(\mathbb{L}_\bullet \cap \mathbb{H}^-) \setminus \Lambda$ is oriented clockwise;*
- *the unique bi-infinite path is oriented from right to left;*
- *each loop $l \in \ell$ inside of G (i.e. surrounding a vertex in $\Lambda \cup \Lambda'$) is oriented clockwise if $U_l < e^\lambda/\mathbf{c}$ and counter-clockwise otherwise.*

Denote by $\ell_\rightarrow$ the obtained oriented loop configuration. Recall the combinatorial mappings introduced above and let $(\sigma_\bullet, \sigma_\circ)$ be the associated six-vertex spin configurations with $\sigma_\bullet((n+1, 0)) = +1$. Then, the law of $(\sigma_\bullet, \sigma_\circ)$ is given by $\mathtt{Spin}_{\mathcal{D}}^{+,+-}$.

Proof. We follow the ideas of [BKW76]. One has to examine which values of (ω, U) result in a given $(\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$. The probability to obtain $(\sigma_\bullet, \sigma_\circ) \in \Sigma^+_\Lambda \times \Sigma^{+-}_{\Lambda'}$ is the sum of the probabilities of all oriented loop configurations $\ell_\to$ that induce the edge orientations corresponding to $(\sigma_\bullet, \sigma_\circ)$. The probability of a given oriented loop configuration $\ell_\to$ satisyfing the boundary conditions in Fig. 4.9 is proportional to

$$\sqrt{q}^{|\mathrm{loop}(\ell_\to)|} \cdot \left(\tfrac{e^\lambda}{e^\lambda + e^{-\lambda}}\right)^{|\mathrm{loop}_{\circlearrowright}(\ell_\to)| - |\mathrm{loop}_{\circlearrowleft}(\ell_\to)|} \propto e^{\lambda(|\mathrm{loop}_{\circlearrowright}(\ell_\to)| - |\mathrm{loop}_{\circlearrowleft}(\ell_\to)|)}, \tag{4.11}$$

where $\mathrm{loop}_{\circlearrowright}(\ell_\to)$ and $\mathrm{loop}_{\circlearrowleft}(\ell_\to)$ are respectively the sets of clockwise and counter-clockwise oriented loops in $\ell_\to$ not imposed by boundary conditions, and $\mathrm{loop}(\ell_\to)$ is their union. Indeed, by 4.1, the first factor on the left side is proportional to $\mathtt{FK}_G^{1/0}(\omega(\ell))$, where $\omega(\ell) \in \{0,1\}^{\mathbb{E}^\bullet}$ is the percolation configuration associated to the unoriented loop configuration ℓ corresponding to $\ell_\to$. The second factor comes from the values of the uniforms necessary to obtain correct orientations of loops in $\mathrm{loop}(\ell_\to)$. The equality holds since $\sqrt{q} = e^\lambda + e^{-\lambda}$ due to the choice of λ.

Notice that each loop which is oriented clockwise does 4 more right quarter-turns than left quarter-turns, that the converse holds for counter-clockwise oriented loops, and that the number of left and right quarter-turns in the left-right interface differ by a universal constant. Thus, the expression on the right side of (4.11) is proportional to

$$\exp(\lambda(\#_{\frown}(\ell_\to) - \#_{\frown}(\ell_\to))/4),$$

where $\#_{\frown}(\ell_\to)$ and $\#_{\frown}(\ell_\to)$ are respectively the number of right and left quarter-turns in $\ell_\to$. The key idea is to count these oriented loop arcs locally at each tile in $A = A^{\mathrm{i}} \cup A^{\mathrm{b}}$. Observe that, for tiles in A^{i}, types 5B,6A correspond to a pair of right-oriented loop arcs and types 5A,6B to a pair of left-oriented loop arcs, whereas types 1-4 correspond to one right-oriented and one left-oriented loop arc each. Moreover, due to the boundary conditions (see Fig. 4.9), a tile in A^{b} contains a right turn precisely if it is of type 5,6, and it contains a left turn otherwise. We deduce that the probability of $\ell_\to$ is proportional to

$$\prod_{t \in T_{5,6}(\ell_\to) \cap A^{\mathrm{i}}} \left(e^{\lambda/2}\, \mathbb{1}_{T_{5B,6A}(\ell_\to)}(t) + e^{-\lambda/2}\, \mathbb{1}_{T_{5A,6B}(\ell_\to)}(t)\right) \cdot \left(e^{\lambda/2}\right)^{|T_{5,6}(\ell_\to) \cap \partial A|}, \tag{4.12}$$

where $T_{5B,6A}(\ell_\to)$ and $T_{5A,6B}(\ell_\to)$ are respectively the sets of tiles of types 5B and 6A and the set of tiles of types 5A and 6B in $\ell_\to$, and $T_{5,6}(\ell_\to)$ is their union.

Finally, fix a pair $(\sigma_\bullet, \sigma_\circ) \in \Sigma_\Lambda^{+} \times \Sigma_{\Lambda'}^{+-}$ that satisfies the ice-rule, and consider its associated edge orientations. It remains to identify all oriented loop configurations $\ell_\to$ that satisfy the boundary conditions in Fig. 4.9 and that induce these edge orientations. Observe that the boundary conditions and the spins $(\sigma_\bullet, \sigma_\circ)$ uniquely determine the oriented loop arcs at tiles in $(\mathbb{L}_\diamond \setminus A) \cup A^{\mathrm{b}}$ and at tiles in $A^{\mathrm{i}} \setminus T_{5,6}^{\mathrm{i}}(\sigma_\bullet, \sigma_\circ)$. For a tile in $T_{5,6}^{\mathrm{i}}(\sigma_\bullet, \sigma_\circ)$, one can split the oriented edges either into a pair of right-oriented loops arcs (types 5B,6A) or into a pair of left-oriented loop arcs (types 5A,6B). Summing the probabilities (4.12) of all oriented loop configurations obtained in that way, we obtain that the probability of $(\sigma_\bullet, \sigma_\circ)$ is proportional to

$$\left(e^{\lambda/2} + e^{-\lambda/2}\right)^{|T_{5,6}^{\mathrm{i}}(\sigma_\bullet, \sigma_\circ)|} \cdot \left(e^{\lambda/2}\right)^{|T_{5,6}^{\mathrm{b}}(\sigma_\bullet, \sigma_\circ)|}.$$

Recalling that $\mathbf{c} = e^{\lambda/2} + e^{-\lambda/2}$ and $\mathbf{c}_{\mathrm{b}} = e^{\lambda/2}$ finishes the proof. □

From six-vertex to modified ATRC. We now adapt the coupling described in [GP23] to the Dobrushin boundary conditions. In particular, we need to define the associated modified ATRC measure. Recall the parameters from (4.3) and the (augmented) rectangular domains from Section 4.2. Again, we omit n, m from the notation.

Definition 3. The modified ATRC model $\mathtt{mATRC}_{n,m} \equiv \mathtt{mATRC}_K$ is a probability measure on $\{0,1\}^{\bar{E}} \times \{0,1\}^{\bar{E}}$ defined by

$$\mathtt{mATRC}_K(\omega_\tau, \omega_{\tau\tau'}) \propto \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq E} 2^{|\omega_\tau \cap E|} \big(\tfrac{2}{\mathbf{c}_\mathrm{b}-1}\big)^{|\omega_\tau \cap E_\mathrm{b}^+|} (\mathbf{c}-2)^{|\omega_{\tau\tau'} \setminus \omega_\tau|} \cdot 2^{\kappa_K(\omega_\tau)} \prod_{\mathcal{C} \in \mathrm{cl}_\Lambda(\omega_{\tau\tau'})} \big(\mathbb{1}_{\mathcal{C} \subseteq \Lambda} + \mathbf{c}_\mathrm{b}^{I(\mathcal{C})}\big),$$

where $k_K(\omega_\tau)$ is the number of clusters of ω_τ, $\mathrm{cl}_\Lambda(\omega_{\tau\tau'})$ is the set of clusters of $\omega_{\tau\tau'}$ that intersect Λ (both ω_τ and $\omega_{\tau\tau'}$ are viewed as spanning subgraph of K), and $I(\mathcal{C}) := |E_\mathrm{b}^- \cap \partial^{\mathrm{edge}}_{\mathbb{L}_\bullet} \mathcal{C}|$. We say that a cluster of ω_τ or $\omega_{\tau\tau'}$ is an *inner cluster* if it is entirely contained in Λ and a *boundary cluster* otherwise.

We now describe how to obtain $\mathtt{mATRC}_K$ as a push-forward of the coupling measure $\Psi_G^{1/0}$.

Lemma 4.4.2. *Let (ω, U, U') be distributed according to $\Psi^{1/0}$. Define $(\sigma_\bullet, \sigma_\circ)$ as described in Lemma 4.4.1. Now define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\bar{E}}$ as follows using the notation $e_t = ij \in \bar{E}$ and $e_t^* = uv$ for each tile $t \in A^\mathrm{i} \cup \partial^+ A$:*

- *if $\sigma_\circ(u) \neq \sigma_\circ(v)$, set $\omega_\tau(e_t) = \omega_{\tau\tau'}(e_t) = 1$;*
- *if $\sigma_\bullet(i) \neq \sigma_\bullet(j)$, set $\omega_\tau(e_t) = \omega_{\tau\tau'}(e_t) = 0$ (never happens for $t \in \partial^+ A$);*
- *if both $\sigma_\circ(u) = \sigma_\circ(v)$ and $\sigma_\bullet(i) = \sigma_\bullet(j)$ hold (types 5-6), let*

$$(\omega_\tau(e_t), \omega_{\tau\tau'}(e_t)) = \begin{cases} \mathbb{1}_{[0,\frac{1}{\mathbf{c}})}(U'_t) \cdot (1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}},\frac{2}{\mathbf{c}})}(U'_t) \cdot (0,0) + \mathbb{1}_{[\frac{2}{\mathbf{c}},1]}(U'_t) \cdot (0,1) & \text{if } t \in A^\mathrm{i}, \\ \mathbb{1}_{[0,\frac{1}{\mathbf{c}_\mathrm{b}})}(U'_t) \cdot (1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}_\mathrm{b}},1]}(U'_t) \cdot (0,0) & \text{if } t \in \partial^+ A. \end{cases}$$

Then, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is $\mathtt{mATRC}_K(\cdot \,|\, v_L \xleftrightarrow{\omega_\tau} v_R)$. In particular, we have $\omega_\tau \subseteq \omega_{\tau\tau'}$ and $\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \{e_t : t \in A^\mathrm{i}\} = E$; also $\sigma_\bullet \sim \omega_{\tau\tau'}$ and $\sigma_\circ \sim \omega_\tau^*$.

Remark 4.4.3. The above sampling rule for $t \in A^\mathrm{i}$ applied to a six-vertex spin or height measure without modified boundary weight $\mathbf{c}_\mathrm{b}$ yields an ATRC measure, as defined in Secton 4.3. See Section 4.5.1 for details.

Proof. We build on [GP23, Proof of Lemma 7.1]. One has to examine which values of $(\sigma_\bullet, \sigma_\circ, U)$ result in a given pair $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\bar{E}}$. Recall that $\sigma_\bullet \sim \omega_{\tau\tau'}$ and $\sigma_\circ \sim \omega_\tau^*$. By (4.10), the probability of a quadruplet $(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'})$ is

$$\begin{aligned} &\mathtt{Spin}_\mathcal{D}^{+,+-}(\sigma_\bullet, \sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq E} \\ &\quad \cdot \prod_{t \in T_{5,6}^\mathrm{i}} \big(\tfrac{1}{\mathbf{c}} (\mathbb{1}_{t \in \omega_\tau} + \mathbb{1}_{t \in \omega_{\tau\tau'}^*}) + \tfrac{\mathbf{c}-2}{\mathbf{c}} \mathbb{1}_{t \in \omega_{\tau\tau'} \setminus \omega_\tau}\big) \prod_{t \in T_{5,6}^{\mathrm{b},+}} \big(\tfrac{1}{\mathbf{c}_\mathrm{b}} \mathbb{1}_{t \in \omega_\tau} + \tfrac{\mathbf{c}_\mathrm{b}-1}{\mathbf{c}_\mathrm{b}} \mathbb{1}_{t \in \omega_\tau^*}\big) \\ &\propto \mathbb{1}_{\Sigma_\Lambda^+}(\sigma_\bullet) \mathbb{1}_{\Sigma_{\Lambda'}^{+-}}(\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq E}\, \mathbf{c}_\mathrm{b}^{|T_{5,6}^{\mathrm{b},-}|} \\ &\quad \cdot \prod_{t \in T_{5,6}^\mathrm{i}} \big(\mathbb{1}_{t \in \omega_\tau} + \mathbb{1}_{t \in \omega_{\tau\tau'}^*} + (\mathbf{c}-2) \mathbb{1}_{t \in \omega_{\tau\tau'} \setminus \omega_\tau}\big) \prod_{t \in T_{5,6}^{\mathrm{b},+}} \big(\mathbb{1}_{t \in \omega_\tau} + (\mathbf{c}_\mathrm{b}-1) \mathbb{1}_{t \in \omega_\tau^*}\big), \end{aligned} \tag{4.13}$$

where $T_{5,6}^{\mathrm{b},\pm}(\sigma_\bullet, \sigma_\circ)$ is the set of tiles of types 5-6 in $\partial^\pm A$, and where we used the shorthand $T_{5,6}^\# \equiv T_{5,6}^\#(\sigma_\bullet, \sigma_\circ)$ and the fact that, for $\sigma_\bullet \in \Sigma_\Lambda^+$ and $\sigma_\circ \in \Sigma_{\Lambda'}^{+-}$,

$$\mathbb{1}_{\mathrm{ice}}(\sigma_\bullet, \sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} = \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}.$$

Observe that, on the event $\{\sigma_\bullet \sim \omega_{\tau\tau'},\, \sigma_\circ \sim \omega^*_\tau,\, \omega_\tau \subseteq \omega_{\tau\tau'}\}$,

$$\mathbb{1}_{T^{\mathrm{i}}_{5,6}}(t) = \mathbb{1}_{\omega_\tau}(t)\mathbb{1}_{\sigma_\circ\sim e^*_t} + \mathbb{1}_{\omega^*_{\tau\tau'}}(t)\mathbb{1}_{\sigma_\bullet\sim e_t} + \mathbb{1}_{\omega_{\tau\tau'}\setminus\omega_\tau}(t).$$

Moreover, on the same event intersected with $\{\sigma_\bullet \in \Sigma^+_\Lambda,\, \sigma_\circ \in \Sigma^{+-}_{\Lambda'},\, \omega_{\tau\tau'} \setminus \omega_\tau \subseteq E\}$,

$$\mathbb{1}_{T^{\mathrm{b},-}_{5,6}}(t) = \mathbb{1}_{\sigma_\bullet\sim e_t} \qquad \text{and} \qquad \mathbb{1}_{T^{\mathrm{b}+}_{5,6}}(t) = \mathbb{1}_{\omega_\tau}(t)\mathbb{1}_{\sigma_\circ\sim e^*_t} + \mathbb{1}_{\omega^*_\tau}(t).$$

Using this, (4.13) becomes

$$\begin{aligned}
&\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\Lambda'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet\sim\omega_{\tau\tau'}}\mathbb{1}_{\sigma_\circ\sim\omega^*_\tau}\mathbb{1}_{\omega_\tau\subseteq\omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'}\setminus\omega_\tau\subseteq E}\prod_{t\in\partial^- A}\mathbf{c}_{\mathrm{b}}^{\mathbb{1}_{T^{\mathrm{b},-}_{5,6}}(t)}\\
&\cdot\prod_{t\in A^{\mathrm{i}}}\left(\mathbb{1}_{t\in\omega_\tau} + \mathbb{1}_{t\in\omega^*_{\tau\tau'}} + (\mathbf{c}-2)\mathbb{1}_{t\in\omega_{\tau\tau'}\setminus\omega_\tau}\right)^{\mathbb{1}_{T^{\mathrm{i}}_{5,6}}(t)}\prod_{t\in\partial^+ A}\left(\mathbb{1}_{t\in\omega_\tau} + (\mathbf{c}_{\mathrm{b}}-1)\mathbb{1}_{t\in\omega^*_\tau}\right)^{\mathbb{1}_{T^{\mathrm{b},+}_{5,6}}(t)}\\
=\,&\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\Lambda'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet\sim\omega_{\tau\tau'}}\mathbb{1}_{\sigma_\circ\sim\omega^*_\tau}\mathbb{1}_{\omega_\tau\subseteq\omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'}\setminus\omega_\tau\subseteq E}\prod_{e\in E^-_{\mathrm{b}}}\mathbf{c}_{\mathrm{b}}^{\mathbb{1}_{\sigma_\bullet\sim e}}\\
&\cdot(\mathbf{c}-2)^{|\omega_{\tau\tau'}\setminus\omega_\tau|}(\mathbf{c}_{\mathrm{b}}-1)^{|E^+_{\mathrm{b}}\setminus\omega_\tau|}.
\end{aligned}$$

Observe that $\sigma_\circ \in \Sigma^{+-}_{\Lambda'}$ and $\sigma_\circ \sim \omega^*_\tau$ imply that v_L and v_R are connected in ω_τ (see Fig. 4.9). To obtain the probability of a pair $(\omega_\tau, \omega_{\tau\tau'})$ satisfying $v_L \xleftrightarrow{\omega_\tau} v_R$, we need to sum the last expression over $(\sigma_\bullet, \sigma_\circ)$.

The configurations $\sigma_\circ \in \Sigma^{+-}_{\Lambda'}$ with $\sigma_\circ \sim \omega^*_\tau$ are in bijective correspondence with assignments of ±1 to the clusters of ω^*_τ in K^* that are contained in Λ', whence there exist $2^{k_{K^*}(\omega^*_\tau)-1}$ of them. By Euler's formula,

$$k_{K^*}(\omega^*_\tau) - 1 = k_K(\omega_\tau) + |\omega_\tau| - |\mathbb{V}_{\bar{E}}|.$$

In particular, the probability of a triplet $(\omega_\tau, \omega_{\tau\tau'}, \sigma_\bullet)$ with $v_L \xleftrightarrow{\omega_\tau} v_R$ is proportional to

$$\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\sigma_\bullet\sim\omega_{\tau\tau'}}\mathbb{1}_{\omega_\tau\subseteq\omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'}\setminus\omega_\tau\subseteq E}2^{k_K(\omega_\tau)}2^{|\omega_\tau|}(\mathbf{c}-2)^{|\omega_{\tau\tau'}\setminus\omega_\tau|}(\mathbf{c}_{\mathrm{b}}-1)^{|E^+_{\mathrm{b}}\setminus\omega_\tau|}\prod_{e\in E^-_{\mathrm{b}}}\mathbf{c}_{\mathrm{b}}^{\mathbb{1}_{\sigma_\bullet\sim e}}. \tag{4.14}$$

We now sum this expression over $\sigma_\bullet$: since $\sigma_\bullet \sim \omega_{\tau\tau'}$, the value of $\sigma_\bullet$ is constant at each cluster of $\omega_{\tau\tau'}$; since $\sigma_\bullet \in \Sigma^+_\Lambda$, this value is fixed to be $+1$ on boundary clusters. Denote the sets of inner and boundary clusters of $\omega_{\tau\tau'}$ by $\mathrm{cl}^{\mathrm{i}}(\omega_{\tau\tau'})$ and $\mathrm{cl}^{\mathrm{b}}(\omega_{\tau\tau'})$, respectively. If $e \in E^-_{\mathrm{b}} \setminus \omega_{\tau\tau'}$, then $\sigma_\bullet \sim e$ in precisely two cases: (i) if $e \in \partial^{\mathrm{edge}}_{\mathbb{L}_\bullet}\mathcal{C}$ for some $\mathcal{C} \in \mathrm{cl}^{\mathrm{b}}(\omega_{\tau\tau'})$; (ii) if $e \in \partial^{\mathrm{edge}}_{\mathbb{L}_\bullet}\mathcal{C}$ for some $\mathcal{C} \in \mathrm{cl}^{\mathrm{i}}(\omega_{\tau\tau'})$ on which $\sigma_\bullet = +1$. Therefore,

$$\sum_{\substack{\sigma_\bullet\in\Sigma^+_\Lambda\\ \sigma_\bullet\sim\omega_{\tau\tau'}}}\prod_{e\in E^-_{\mathrm{b}}}\mathbf{c}_{\mathrm{b}}^{\mathbb{1}_{\sigma_\bullet\sim e}} = \prod_{\mathcal{C}\in\mathrm{cl}^{\mathrm{b}}(\omega_{\tau\tau'})}\mathbf{c}_{\mathrm{b}}^{|E^-_{\mathrm{b}}\cap\partial^{\mathrm{edge}}_{\mathbb{L}_\bullet}\mathcal{C}|}\cdot\prod_{\mathcal{C}\in\mathrm{cl}^{\mathrm{i}}(\omega_{\tau\tau'})}\left(1+\mathbf{c}_{\mathrm{b}}^{|E^-_{\mathrm{b}}\cap\partial^{\mathrm{edge}}_{\mathbb{L}_\bullet}\mathcal{C}|}\right).$$

Substituting the last display in the sum of (4.14) over $\sigma_\bullet$, we get $\texttt{mATRC}_K$. □

4.4.3 Properties of the modified ATRC

In this section, we will derive basic properties of the modified ATRC measure that will be instrumental in its analysis in Section 4.8 and in the proof of Theorem 14. We continue in the setting of the previous section.

Positive association. The motivation for sampling the modified ATRC only on $\bar{E} = E \cup E_{\mathrm{b}}^{+}$ (rather than on $E \cup E_{\mathrm{b}}$) is that this (marginal) distribution satisfies (strong-FKG). This allows to "sandwich" the associated modified ATRC measure between unmodified ATRC measures and deduce convergence to the unique infinite-volume *Gibbs measure*; see Sections 4.3 and 4.5.

Lemma 4.4.4. *The measure* $\mathtt{mATRC}_K$ *satisfies* (strong-FKG).

Proof. Recall that $\mathbf{c}_{\mathrm{b}} = e^{\lambda}$ with $\lambda > 0$. We verify the Holley criterion [Hol74], see also [GHM01, Section 4]. To shorten notation, given $a, b \in \{0,1\}^{\bar{E}}$ with $a \subseteq b$ and $a = b$ on E_b^{+}, we write $\mathtt{mATRC}_K(a_e, b_e \mid a_{e^c}, b_{e^c})$ for the conditional probability

$$\mathtt{mATRC}_K\big((\omega_\tau(e), \omega_{\tau\tau'}(e)) = (a_e, b_e) \,\big|\, (\omega_\tau, \omega_{\tau\tau'}) = (a, b) \text{ on } (\bar{E} \setminus \{e\}\big).$$

Case 1: $e = ij \in E$. Denote by C_i^a, C_j^a respectively the clusters of a_{e^c} containing i, j, and by $\widetilde{C}_i^b, \widetilde{C}_j^b$ the clusters of $b_{e^c} \cup E_{\mathrm{b}}^{+}$ containing i, j. Observe that $\widetilde{C}_i^b \neq \widetilde{C}_j^b$ implies that $\widetilde{C}_i^b \subseteq \Lambda$ or $\widetilde{C}_j^b \subseteq \Lambda$. Then, for any $a, b \in \{0,1\}^{E \cup E_{\mathrm{b}}^{+}}$ with $a \subseteq b$ and $a = b$ on E_b^{+},

$$\mathtt{mATRC}_K(a_e, b_e \mid a_{e^c}, b_{e^c}) = \big(2^{a_e} \tfrac{\mathbf{c}}{\mathbf{c}+1}\big)^{\mathbb{1}_{C_i^a = C_j^a}} (\mathbf{c} - 2)^{b_e - a_e} \tfrac{f(b)^{b_e}}{1 + (\mathbf{c}-1) f(b)}, \tag{4.15}$$

where

$$f(b) = \mathbb{1}_{\widetilde{C}_i^b = \widetilde{C}_j^b} + \alpha\big(\mathbb{1}_{\widetilde{C}_i^b \neq \widetilde{C}_j^b \not\subseteq \Lambda} + \mathbb{1}_{\Lambda \not\supseteq \widetilde{C}_i^b \neq \widetilde{C}_j^b}\big) + \beta \mathbb{1}_{\Lambda \supseteq \widetilde{C}_i^b \neq \widetilde{C}_j^b \subseteq \Lambda}$$

and

$$\alpha = \frac{e^{\lambda I(\widetilde{C}_i^b)}}{1 + e^{\lambda I(\widetilde{C}_i^b)}}, \qquad \beta = \frac{1 + e^{\lambda I(\widetilde{C}_i^b) + \lambda I(\widetilde{C}_j^b)}}{(1 + e^{\lambda I(\widetilde{C}_i^b)})(1 + e^{\lambda I(\widetilde{C}_j^b)})}.$$

Note first that $\mathbf{c} \geq 1$ (in fact $\mathbf{c} > 2$) and $\mathbb{1}_{C_i^a = C_j^a}$ is increasing in a. Moreover, $\beta \leq \alpha \leq 1$ and both α and β are increasing in $I(\widetilde{C}_i^b)$ and $I(\widetilde{C}_j^b)$ and hence in b. Therefore $f \geq 1$ is increasing in b. This implies that $\mathtt{mATRC}_K(1, 1 \mid a_{e^c}, b_{e^c})$ is increasing in a, b and $\mathtt{mATRC}_K(0, 0 \mid a_{e^c}, b_{e^c})$ is decreasing in a, b.

Case 2: $e = \{i, j\} \in E_b^{+}$. If $a_e = b_e$, one has

$$\mathtt{mATRC}_K(a_e, b_e \mid a_{e^c}, b_{e^c}) = \big(2^{a_e} \tfrac{\mathbf{c}_{\mathrm{b}}}{\mathbf{c}_{\mathrm{b}}+1}\big)^{\mathbb{1}_{C_i^a = C_j^a}} \Big(\tfrac{f(b)}{\mathbf{c}_{\mathrm{b}} - 1}\Big)^{a_e} \tfrac{\mathbf{c}_{\mathrm{b}} - 1}{\mathbf{c}_{\mathrm{b}} - 1 + f(b)}, \tag{4.16}$$

where $f(b)$ is as above. Recall that $\mathbf{c}_{\mathrm{b}} = e^{\lambda} > 1$ and that $\mathbb{1}_{C_i^a = C_j^a}$ and f are increasing in a and b, respectively. This implies that $\mathtt{mATRC}_K(1, 1 \mid a_{e^c}, b_{e^c})$ is increasing in a, b, and hence $\mathtt{mATRC}_K(0, 0 \mid a_{e^c}, b_{e^c})$ is decreasing in a, b, and the proof is complete. $\square$

Finite energy. The explicit expressions (4.15)-(4.16) of the conditional probabilities readily imply that $\mathtt{mATRC}_K$ satisfies the finite-energy property:

Lemma 4.4.5. *The exists a constant $c > 0$ such that, for any $e \in E$ and any $a, b \in \{0,1\}^{E \cup E_{\mathrm{b}}^{+}}$ with $a \subseteq b$ and $b \setminus a \subseteq E$,*

$$\mathtt{mATRC}_K\big((\omega_\tau(e), \omega_{\tau\tau'}(e)) = (a_e, b_e) \,\big|\, (\omega_\tau, \omega_{\tau\tau'}) = (a, b) \text{ on } \bar{E} \setminus \{e\}\big) > c.$$

The statement also holds for $e \in E_{\mathrm{b}}^{+}$, provided that $a_e = b_e$.

Decoupling property. The following decoupling property (and analogues of it), which also holds for the standard ATRC measures, will be crucial in the derivation of mixing properties in Sections 4.5-4.6 and the invariance principle in Section 4.8.

Let us first introduce some terminology.

Definition 4. We identify circuits in $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$ with their sets of edges as well as with their planar embedding given by the union of line-segments between consecutive vertices. Given a simple circuit γ in $\mathbb{L}_\bullet$ or $\mathbb{L}_\circ$, we say that a subset $R \subset \mathbb{R}^2$ is *within* γ if it is contained in the topological closure of the bounded connected component of $\mathbb{R}^2 \setminus \gamma$. In this case, γ is said to *surround* R, or R is surrounded by γ.

Lemma 4.4.6. *Let $(\omega_\tau, \omega_{\tau\tau'})$ be distributed according to $\mathtt{mATRC}_K$. Let $F_1 \subset F_2 \subset \bar{E}$, and let $a, b \in \{0,1\}^{F_2 \setminus F_1}$ with $a \subseteq b$ and $b \setminus a \subseteq E$ be such that there exist circuits in a^* and in b that both surround the edges in F_1. Then, conditionally on $(\omega_\tau|_{F_2\setminus F_1}, \omega_{\tau\tau'}|_{F_2\setminus F_1}) = (a,b)$, the restrictions*

$$(\omega_\tau|_{F_1}, \omega_{\tau\tau'}|_{F_1}) \quad \text{and} \quad (\omega_\tau|_{\bar{E}\setminus F_2}, \omega_{\tau\tau'}|_{\bar{E}\setminus F_2})$$

are independent.

Proof. Take any $a_1, b_1 \in \{0,1\}^{F_1}$ and $a_2, b_2 \in \{0,1\}^{\bar{E}\setminus F_2}$, and write

$$\bar{a} = a_1 \sqcup a \sqcup a_2 \quad \text{and} \quad \bar{b} = b_1 \sqcup b \sqcup b_2.$$

By Definition 3,

$$\begin{aligned} &\mathtt{mATRC}_K\big((\bar{a}, \bar{b}) \,\big|\, (\omega_\tau|_{F_2\setminus F_1}, \omega_{\tau\tau'}|_{F_2\setminus F_1}) = (a,b)\big) \\ &\propto \Big(\prod_{i=1}^{2} \mathbb{1}_{a_i \subseteq b_i} \mathbb{1}_{b_i \setminus a_i \subseteq E} 2^{|a_i \cap E|} \big(\tfrac{2}{\mathbf{c}_\mathrm{b}-1}\big)^{|a_i \cap E_\mathrm{b}^+|} (\mathbf{c}-2)^{|b_i \setminus a_i|}\Big) 2^{\kappa_K(\bar{a})} \prod_{\mathcal{C} \in \mathrm{cl}_\Lambda(\bar{b})} \big(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + \mathbf{c}_\mathrm{b}^{I(\mathcal{C})}\big), \end{aligned} \tag{4.17}$$

where the constant of proportionality depends on a, b.

Let $\gamma' \subset a^* \subset \mathbb{E}^\circ$ be a circuit that surrounds F_1. Let $E^{\mathrm{in}}_{\gamma'} \subset \bar{E}$ be the set of edges within γ', and set $E^{\mathrm{ex}}_{\gamma'} = \bar{E} \setminus (E^{\mathrm{in}}_{\gamma'} \cup *\gamma')$. For $\# \in \{\mathrm{in}, \mathrm{ex}\}$, let $K_a^{\#}$ be the graph obtained from $(\mathbb{V}_{E^{\#}_{\gamma'}}, E^{\#}_{\gamma'})$ by identifying endpoints of edges in a. As the circuit γ' is open in a^* and separates K_a^{in} from K_a^{ex}, we clearly have

$$\kappa_K(\bar{a}) = \kappa_{K_a^{\mathrm{in}}}(a_1) + \kappa_{K_a^{\mathrm{ex}}}(a_2).$$

Similarly, let $\gamma \subset b \subset \mathbb{E}^\bullet$ be a circuit that surrounds F_1. Let $E^{\mathrm{in}}_\gamma \subset \bar{E}$ be the set of edges within γ, and set $E^{\mathrm{ex}}_\gamma = \bar{E} \setminus E^{\mathrm{in}}_\gamma$. For $\# \in \{\mathrm{in}, \mathrm{ex}\}$, define $K_b^{\#}$ as the graph obtained from $(\mathbb{V}_{E^{\#}_\gamma}, E^{\#}_\gamma)$ by identifying endpoints of edges in b. Let $\mathrm{cl}_{K_b^{\mathrm{in}}}(b_1)$ and $\mathrm{cl}_{K_b^{\mathrm{ex}}}(b_2)$ be the sets of clusters of b_1 in K_b^{in} and b_2 in K_b^{ex}, respectively. Then, every cluster in $\mathrm{cl}_\Lambda(\bar{b})$ that does not contain $\mathbb{V}_\gamma$ belongs to either $\mathrm{cl}_{K_b^{\mathrm{in}}}(b_1)$ or $\mathrm{cl}_{K_b^{\mathrm{ex}}}(b_2)$. Moreover, every cluster $\mathcal{C} \in \mathrm{cl}_{K_b^{\mathrm{in}}}(b_1)$ except that of the vertex corresponding to γ satisfies $\mathcal{C} \subseteq \Lambda$ and $I(\mathcal{C}) = 0$. Therefore,

$$\prod_{\mathcal{C} \in \mathrm{cl}_\Lambda(\bar{b})} \big(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + \mathbf{c}_\mathrm{b}^{I(\mathcal{C})}\big) = 2^{|\mathrm{cl}_{K_b^{\mathrm{in}}}(b_1)|-1} \prod_{\mathcal{C} \in \mathrm{cl}_{K_b^{\mathrm{ex}}}(b_2)} \big(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + \mathbf{c}_\mathrm{b}^{I(\mathcal{C})}\big).$$

Therefore, the expression in (4.17) factorises, and the proof is complete. $\square$

4.4.4 Interface in the modified ATRC

Recall the coupling measure $\Psi^{1/0}$ of the six-vertex spin random variable $(\sigma_\bullet, \sigma_\circ) \sim \mathtt{Spin}_{\mathcal{D}}^{+,+-}$ and the modified ATRC pair $(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{mATRC}_K(\cdot \mid v_L \overset{\omega_\tau}{\longleftrightarrow} v_R)$ (Lemmata 4.4.1 and 4.4.2).

Interface. Since $\sigma_\circ$ is constant on edges in ω_τ^*, the Dobrushin boundary conditions ($\sigma_\circ \in \Sigma_{\Lambda'}^{+-}$) impose the existence of a path in ω_τ that connects v_L and v_R.

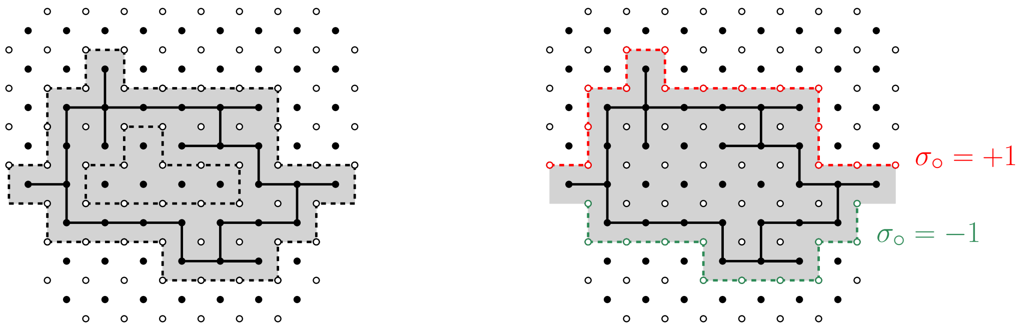


Figure 4.10: Left: A realisation of $\mathcal{C} = \mathcal{C}_{v_L}$ (endpoints of solid edges) in K for $n = m = 3$, the dual $*\partial^{\text{edge}}_{\mathbb{L}_\bullet}\mathcal{C}$ of its edge-boundary in $\mathbb{L}_\bullet$ (dashed edges), and the surrounding polygon $\mathcal{P}$ (grey). Right: a path in $*\partial^{\text{edge}}_K\mathcal{C}$ connecting $(-n-\frac{3}{2}, \frac{1}{2})$ and $(n+\frac{3}{2}, \frac{1}{2})$ on which $\sigma_\circ = +1$ (red dashed edges), and a path in $*\partial^{\text{edge}}_K\mathcal{C}$ connecting $(-n-\frac{1}{2}, -\frac{1}{2})$ and $(n+\frac{1}{2}, -\frac{1}{2})$ on which $\sigma_\circ = -1$ (green dashed edges).

Definition 5. Denote by $\mathcal{C} = \mathcal{C}_{v_L} \subseteq \bar{V}$ the cluster of v_L in ω_τ; see Fig. 4.10. Let $\gamma \subset \mathbb{R}^2$ be the unique closed curve formed by the line-segments between endpoints of edges in $*\partial^{\text{edge}}_{\mathbb{L}_\bullet}\mathcal{C} \subset \mathbb{E}^\circ$ such that $\mathcal{C}$ is contained in the bounded connected component of $\mathbb{R}^2 \setminus \gamma$. Denote the closure of this component by $\mathcal{P}$, and define the *interface* $\Gamma = \Gamma^{n,m}_{\mathtt{ATRC}}$ by

$$\Gamma := \mathcal{P} \cap \mathbb{L}_\bullet \subseteq \bar{V}.$$

4.5 Weak mixing in the ATRC

The main results proved in this section are the single-edge relaxation of the ATRC, Proposition 4.3.2, and the *ratio* weak mixing property for the ATRC, Theorem 16.

In Subsections 4.5.1 and 4.5.2 we define the height function of the six-vertex model and use [GP23] to show that it relaxes exponentially to its infinite-volume limit. In Subsection 4.5.3, we show that this implies exponential relaxation for the ATRC measure with “0, 1” boundary conditions, stated in Proposition 4.5.9. In Subsection 4.5.4, we derive Proposition 4.3.2 from Proposition 4.5.9 and an input from [ADG24], stated in Proposition 4.5.11. In Subsection 4.5.5, we show how to derive from Proposition 4.3.2 the *exponential* weak mixing property of the ATRC, stated in Theorem 19. Finally, we prove Theorem 16 using the classical work of Alexander [Ale98], Theorem 19 and the imput, Proposition 4.5.11, from [ADG24].

4.5.1 The six-vertex height function

The proof of Proposition 4.3.2 relies on exponential relaxation of the ATRC measures with “0, 1” boundary conditions. The latter will be established via the height function representation of the six-vertex model and the BKW coupling [BKW76] with FK-percolation, while it also requires some input from [GP23]. We first introduce the six-vertex height function and provide the necessary graph definitions.

Six-vertex height function

Recall the edge orientations and spin representation of the six-vertex model introduced in Section 4.4.1. A six-vertex *height function* is an assignment of integers, called *heights*, to the vertices in both $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$. Given edge orientations that satisfy the ice rule, define h on $\mathbb{L}_\bullet \cup \mathbb{L}_\circ$ as follows. Fix an integer height at some arbitrary fixed vertex. Then iteratively define the heights at other vertices by increasing the height by 1 when traversing an edge $e \in \mathbb{E}^{\diamond}$ from its left to its right (with respect to its assigned orientation) and decreasing the height by 1 when traversing an arrow from its right to its left; see Fig. 4.8. As with the spins, the procedure is self-consistent due to the ice rule. Note that the heights on $\mathbb{L}_\bullet$ and on $\mathbb{L}_\circ$ automatically have different parity. By convention, we set the parity on $\mathbb{L}_\bullet$ to be even. The gradient of h is in bijective correspondence with the edge orientations and hence with the spin representation, up to a global spin flip, as demonstrated by the following relation (see Fig. 4.8):

$$h(u) - h(i) = \sigma_\bullet(i)\sigma_\circ(u) \qquad \text{for any } t \in \mathbb{L}_\diamond \text{ with } i \in e_t,\, u \in e_t^*.$$

On the other hand, up to a global spin flip, the spins are obtained from the height function by setting, for all $i \in \mathbb{L}_\bullet$ and $u \in \mathbb{L}_\circ$,

$$\sigma_\bullet(i) = \begin{cases} +1 & \text{if } h(i) = 0 \mod 4 \\ -1 & \text{if } h(i) = 2 \mod 4 \end{cases}, \quad \sigma_\circ(u) = \begin{cases} +1 & \text{if } h(u) = 1 \mod 4 \\ -1 & \text{if } h(u) = 3 \mod 4 \end{cases}. \tag{4.18}$$

This motivates the following definition.

Definition 6. A function $h : \mathbb{L}_\bullet \cup \mathbb{L}_\circ \to \mathbb{Z}$ is called a (six-vertex) height function if it satisfies the following:

- for any $t \in \mathbb{L}_\diamond$, $i \in t \cap \mathbb{L}_\bullet$, and $u \in t \cap \mathbb{L}_\circ$, one has $|h(i) - h(u)| = 1$,
- for any $i \in \mathbb{L}_\bullet$, one has $h(i) \in 2\mathbb{Z}$.

Denote the set of all height functions by Ω_{hf}. The type of a tile $t \in \mathbb{L}_\diamond$ in a height function h is given by the type of its gradient function; see Fig. 4.8. Fix $g \in \Omega_{\mathrm{hf}}$, parameters $\mathbf{c}, \mathbf{c}_\mathrm{b} > 0$ and a finite subset $\Delta \subset \mathbb{L}_\bullet \cup \mathbb{L}_\circ$. The set $A := A_\Delta \subset \mathbb{L}_\diamond$ of tiles of Δ is given by the tiles with at least one corner in Δ. The set $\partial A \subseteq A$ of boundary tiles of Δ is given by the tiles with precisely one corner in Δ. Let $\delta \subseteq \partial A$. The corresponding height function measure on $\mathbb{Z}^{\mathbb{L}_\bullet \cup \mathbb{L}_\circ}$ is defined by

$$\mathtt{HF}^{g}_{\Delta,\delta;\mathbf{c},\mathbf{c}_\mathrm{b}}(h) \propto \mathbf{c}^{|T_{5,6}^{A\setminus\delta}(h)|}\, \mathbf{c}_\mathrm{b}^{|T_{5,6}^{\delta}(h)|}\, \mathbb{1}_{\Omega_{\mathrm{hf}}}(h)\, \mathbb{1}_{h(x)=g(x)\, \forall x \in (\mathbb{L}_\bullet \cup \mathbb{L}_\circ)\setminus\Delta}\,, \tag{4.19}$$

where $T_{5,6}^{A\setminus\delta}(h)$ (resp. $T_{5,6}^{\delta}(h)$) is the set of tiles $t \in A \setminus \delta$ (resp. $t \in \delta$) of type 5-6 in h.

If g is constant n on $\mathbb{L}_\bullet$ and constant m on $\mathbb{L}_\circ$ (n even and $m = n \pm 1$), we simply write $\mathtt{HF}^{n,m}_{\Delta,\delta;\mathbf{c},\mathbf{c}_\mathrm{b}}$. When $\delta = \partial A$, we omit δ from the subscript. When $\delta = \varnothing$, we omit δ and $\mathbf{c}_\mathrm{b}$ from the subscript.

Rotated lattice and domains

Consider the rotated square lattice $\mathbb{L}$ with vertex-set $\mathbb{L}_\bullet \cup \mathbb{L}_\circ$ and edges between nearest neighbours, that is, between vertices of Euclidean distance $1/\sqrt{2}$. Given $\Delta \subseteq \mathbb{L}$, we write $\Delta_\bullet = \Delta \cap \mathbb{L}_\bullet$ and $\Delta_\circ = \Delta \cap \mathbb{L}_\circ$. The augmented graph $\overline{\mathbb{L}}$ has the same vertex-set as $\mathbb{L}$ and all edges of $\mathbb{L}$, $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$, see Fig. 4.11. We restrict the notion of simple circuits in $\overline{\mathbb{L}}$ to those that do not traverse both e and e^* for any $e \in \mathbb{E}^\bullet$, so that they can be embedded in $\mathbb{R}^2$. We identify such circuits with their planar embedding.

Definition 7 ($\mathbb{L}$-domains)**.** A finite subset $\mathcal{D} \subset \mathbb{L}$ is called an $\mathbb{L}$-domain if there exists a simple circuit C in $\overline{\mathbb{L}}$ such that $\mathcal{D}$ is given by the vertices of $\mathbb{L}$ strictly within C, that is, in the bounded connected component of $\mathbb{R}^2 \setminus C$. It is called *even* (respectively, *odd*) if $\partial^{\mathrm{ex}}_{\mathbb{L}}\mathcal{D} \subset \mathbb{L}_\bullet$ (respectively, $\partial^{\mathrm{ex}}_{\mathbb{L}}\mathcal{D} \subset \mathbb{L}_\circ$); see Fig. 4.11.

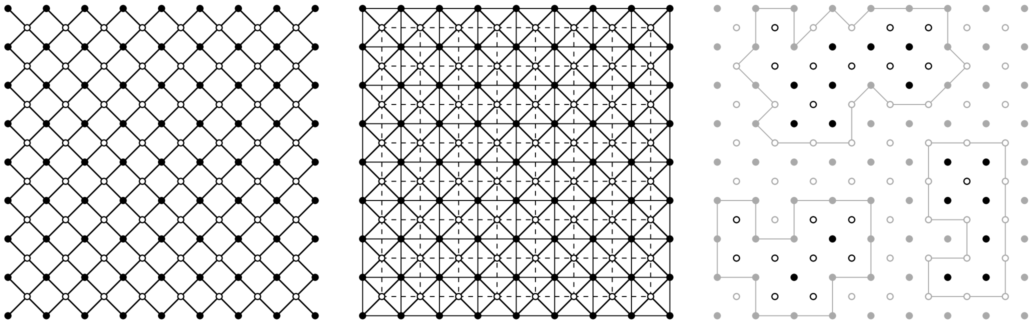


Figure 4.11: Left: part of the rotated square lattice $\mathbb{L}$. Its faces are the tiles in $\mathbb{L}_\diamond$. Center: part of the augmented lattice $\overline{\mathbb{L}}$. Right: $\mathbb{L}$-domains given by the vertices strictly within simple circuits in $\overline{\mathbb{L}}$, $\mathbb{L}_\bullet$, $\mathbb{L}_\circ$, respectively. The lower left and right $\mathbb{L}$-domains are even and odd, respectively.

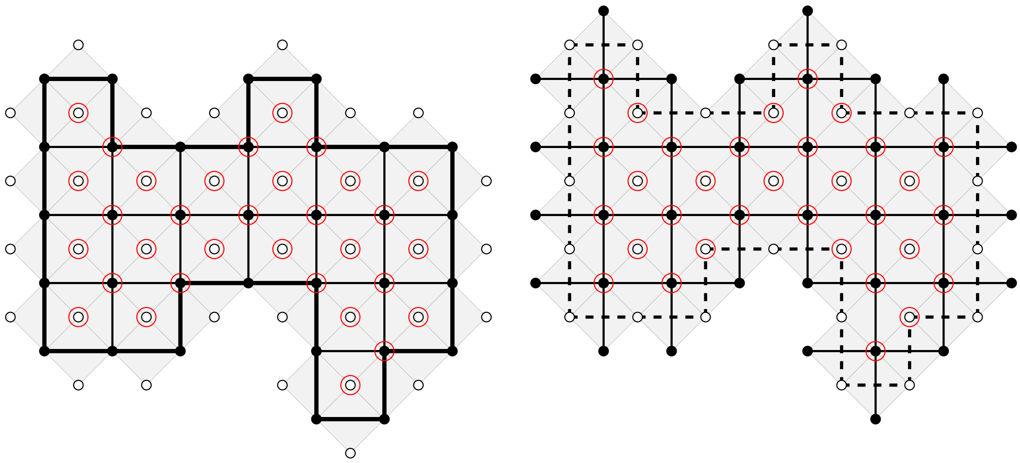


Figure 4.12: Left: a circuit in $\mathbb{L}_\bullet$ (thick black edges) and the corresponding $\mathbb{L}_\bullet$-domain of the first kind (thin and thick black edges). Right: a circuit in $\mathbb{L}_\circ$ (dashed edges) obtained by shifting the left one by $(1/2, 1/2)$ and the corresponding $\mathbb{L}_\bullet$-domain of the second kind (black edges). The tiles corresponding to the edges of the domains are shaded grey, and the vertices of the corresponding $\mathbb{L}$-domains are surrounded by red circles.

Definition 8 ($\mathbb{L}_\bullet$- and $\mathbb{L}_\circ$-domains)**.** A subgraph $G = (V, E)$ of $\mathbb{L}_\bullet$ is called an $\mathbb{L}_\bullet$-domain of the *first kind* if $E = \mathbb{E}_V$ and V is given by the vertices within (see Definition 4) a simple circuit in $\mathbb{L}_\bullet$. We define $\mathbb{L}_\circ$-domains of the first kind in the same manner. We say that a subgraph $G = (V, E)$ of $\mathbb{L}_\bullet$ is a domain of the *second kind* if its dual $(\mathbb{V}_{*E}, *E)$ is an $\mathbb{L}_\circ$-domain of the first kind. It is called an $\mathbb{L}_\bullet$-domain if it is an $\mathbb{L}_\bullet$-domain of the first or second kind. We define $\mathbb{L}_\circ$-domains (of the second kind) analogously; see Fig. 4.12.

Given a subgraph $G = (V, E)$ of $\mathbb{L}_\bullet$ with dual $G^* = (\mathbb{V}_{*E}, *E)$, define Δ_G as the set of vertices of degree 4 in G and G^*, that is,

$$\Delta_G := \{i \in V : |\partial_G^{\text{edge}}\{i\}| = 4\} \cup \{u \in \mathbb{V}_{*E} : |\partial_{G^*}^{\text{edge}}\{u\}| = 4\}.$$

Observe that, if G is an $\mathbb{L}_\bullet$-domain, then $\mathcal{D}_G := \Delta_G$ is an $\mathbb{L}$-domain. In this case, any tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}_G$ precisely if its associated edge $e_t \in \mathbb{E}^\bullet$ belongs to E, that is, $\{e_t : t \in A_{\mathcal{D}_G}\} = E$; see Section 4.5.1 and Fig. 4.12.

Duality coupling with the AT model

The ATRC measures can be sampled locally from the HF measures, which is the content of the following lemma. We build on [GP23], where a marginal of the ATRC measure was sampled. Given $\mathbf{c} > 2$, a height function $h \in \mathbb{Z}^{\mathbb{L}}$, an edge $e = ij \in \mathbb{E}^\bullet$ with $e^* = uv$, and $x \in [0,1]$, define $\mathcal{X}_{\mathbf{c}}(h,e,x) \in \{(0,0),(0,1),(1,1)\}$ as follows:

$$
\begin{aligned}
&\bullet\ \text{if } h(u) \neq h(v),\ \text{set } \mathcal{X}_{\mathbf{c}}(h,e,x) = (1,1),\\
&\bullet\ \text{if } h(i) \neq h(j),\ \text{set } \mathcal{X}_{\mathbf{c}}(h,e,x) = (0,0),\\
&\bullet\ \text{if } h(u) = h(v) \text{ and } h(i) = h(j),\ \text{set}\\
&\quad \mathcal{X}_{\mathbf{c}}(h,e,x) = \mathbb{1}_{[0,\frac{1}{\mathbf{c}})}(x)\cdot(1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}},\frac{2}{\mathbf{c}})}(x)\cdot(0,0) + \mathbb{1}_{[\frac{2}{\mathbf{c}},1]}(x)\cdot(0,1).
\end{aligned}
\tag{4.20}
$$

Observe that, in order to define $\mathcal{X}_{\mathbf{c}}(h,e,x)$, it suffices to know the values of the corresponding six-vertex spins $\sigma_\bullet(i), \sigma_\bullet(j), \sigma_\circ(u), \sigma_\circ(v)$, defined in (4.18).

Lemma 4.5.1. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$ and take $\mathbf{c} := \coth 2J$. Let $G = (V,E)$ be an $\mathbb{L}_\bullet$-domain, and let $\mathcal{D} := \mathcal{D}_G$ be its associated $\mathbb{L}$-domain. Let h be distributed according to $\mathtt{HF}^{0,1}_{\mathcal{D};\mathbf{c}}$, and let $(U_e)_{e\in E}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of h. Define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$ by setting, for $e \in \mathbb{E}^\bullet$,*

$$
(\omega_\tau(e), \omega_{\tau\tau'}(e)) := \begin{cases} (0,1) & \text{if } e \in \mathbb{E}^\bullet \setminus E,\\ \mathcal{X}_{\mathbf{c}}(h,e,U_e) & \text{if } e \in E. \end{cases} \tag{4.21}
$$

Then, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is given by $\mathtt{ATRC}^{0,1}_{G;J,U}$.

Proof. Denote the underlying probability measure by $\mathbf{P}$. Recall that a tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}$ precisely if its associated edge $e_t \in \mathbb{E}^\bullet$ belongs to E, see Fig. 4.12. Let $(\sigma_\bullet, \sigma_\circ)$ with values in $\{\pm1\}^{\mathbb{L}_\bullet} \times \{\pm1\}^{\mathbb{L}_\circ}$ be the spin configurations obtained from h by (4.18). For ease of notation, we use the same symbols for the random variables and their realisations. For $\sigma_\bullet \in \{\pm1\}^{\mathbb{L}_\bullet}$, $\sigma_\circ \in \{\pm1\}^{\mathbb{L}_\circ}$, $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ and $\# \in \{0,1\}$, define

$$
\Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet) := \mathbb{1}_{\sigma_\bullet \equiv 1 \text{ on } \mathbb{L}_\bullet \setminus \mathcal{D}_\bullet}, \quad \Sigma^+_{\mathcal{D}_\circ}(\sigma_\circ) := \mathbb{1}_{\sigma_\circ \equiv 1 \text{ on } \mathbb{L}_\circ \setminus \mathcal{D}_\circ}, \quad \Omega^\#_E(\omega) = \mathbb{1}_{\omega \equiv \# \text{ on } \mathbb{E}^\bullet \setminus E}.
$$

Then, the law of the quadruple $(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'})$ can be written as follows:

$$
\begin{aligned}
\mathbf{P}(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'}) \propto\ & \mathbb{1}_{\mathrm{ice}}(\sigma_\bullet, \sigma_\circ)\, \Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet)\, \Sigma^+_{\mathcal{D}_\circ}(\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega^*_\tau}\\
& \cdot (\mathbf{c}-2)^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, \Omega^0_E(\omega_\tau)\, \Omega^1_E(\omega_{\tau\tau'}).
\end{aligned}
$$

Observe that $\sigma_\bullet \sim \omega_{\tau\tau'}$, $\sigma_\circ \sim \omega^*_\tau$ and $\omega_\tau \subseteq \omega_{\tau\tau'}$ imply that $(\sigma_\bullet, \sigma_\circ)$ satisfies the ice rule. Summing over $\sigma_\bullet$ and $\sigma_\circ$, we obtain the law of $(\omega_\tau, \omega_{\tau\tau'})$:

$$
\mathbf{P}(\omega_\tau, \omega_{\tau\tau'}) \propto 2^{k_V(\omega_{\tau\tau'}) + k_{V'}(\omega^*_\tau)}\, \Omega^0_E(\omega_\tau)\, \Omega^1_E(\omega_{\tau\tau'})\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, (\mathbf{c}-2)^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|},
$$

where $V' := \mathbb{V}_{*E}$, and where we used that there exist $2^{k_V(\omega_{\tau\tau'})-1}$ spin configurations $\sigma_\bullet \sim \omega_{\tau\tau'}$ with $\Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet) = 1$, and similarly for $\sigma_\circ \sim \omega^*_\tau$. By planar duality, $k_{V'}(\omega^*_\tau) = k_V(\omega_\tau) + |\omega_\tau \cap E| + \mathrm{const}(G)$. Since $\mathrm{w}_\tau, \mathrm{w}_{\tau\tau'}$ from (4.6) satisfy $\mathrm{w}_\tau = 2$ and $\mathrm{w}_{\tau\tau'} = \mathbf{c} - 2$, the proof is complete. $\square$

Baxter–Kelland–Wu coupling

In Section 4.4.2, we described the BKW coupling by sampling the FK-percolation from the six-vertex model. We now focus on the reverse direction of the BKW. Fix $q > 4$ and $\beta =$

$\beta_c(q) = \ln(1+\sqrt{q})$, and let $\lambda, \mathbf{c}, \mathbf{c}_\mathrm{b}$ be as in (4.3) and (4.4). Recall the loop representation of FK-percolation and the representations of the six-vertex model introduced in Section 4.4.1.

Let $\mathcal{D}$ be an even $\mathbb{L}$-domain, and set

$$E := *\mathbb{E}_{\mathcal{D}_\circ} \subset \mathbb{E}^\bullet, \quad \Lambda := \mathbb{V}_E \subset \mathbb{L}_\bullet, \quad \text{and} \quad G := (\Lambda, E). \tag{4.22}$$

The following result [BKW76] is classical. It may be proved through the arguments presented in the proof of Lemma 4.4.1; see [GP23, Theorem 7] for a proof in a similar setting. We will construct a height function from a loop configuration by increasing or decreasing the height whenever we cross a loop, which is the same as orienting the loops as in Section 4.4.2. We chose the first option for the sake of brevity.

Proposition 4.5.2. *Let ω be a random element of $\{0,1\}^{\mathbb{E}^\bullet}$ distributed according to $\mathtt{FK}^{\mathrm{w}}_{G;\beta,q}$. Consider the corresponding loop configuration* $\mathrm{loop}(\omega)$*, and define a height function h as follows:*

- ***H1*** *Set $h(i) = 0$ for $i \in \mathbb{L}_\bullet \setminus \mathcal{D}$, and $h(u) = 1$ for $u \in \mathbb{L}_\circ \setminus \mathcal{D}$.*
- ***H2*** *Assign constant values to clusters of ω and ω^* in $\mathcal{D}$ by* ***decreasing*** *the height by 1 with probability $e^{\lambda}/\sqrt{q}$ and* ***increasing*** *the height by 1 with probability $e^{-\lambda}/\sqrt{q}$ when crossing a loop from outside, independently for every loop.*

The height function h is distributed according to $\mathtt{HF}^{0,1}_{\mathcal{D};\mathbf{c},\mathbf{c}_\mathrm{b}}$.

The above procedure also works for odd $\mathbb{L}$-domains with the difference that one has to take $\omega \sim \mathtt{FK}^{\mathrm{f}}_{\mathcal{D}_\bullet;p,q}$, and **H2** must be replaced by

- **H2'** Assign constant values to clusters of ω and ω^* in $\mathcal{D}$ by **decreasing** the height by 1 with probability $e^{\lambda}/\sqrt{q}$ and **increasing** the height by 1 with probability $e^{-\lambda}/\sqrt{q}$ when crossing a loop from outside, independently for every loop.

Input from [GP23]

For the whole section, fix $q > 4$ and $\beta = \beta_c(q) = \ln(1+\sqrt{q})$, and let $\lambda, \mathbf{c}, \mathbf{c}_\mathrm{b}$ be as in (4.3) and (4.4). The following proposition is a consequence of [GP23, Proposition 6.1 and Lemma 6.2] and their proofs.

Proposition 4.5.3. *For any sequence of even or odd $\mathbb{L}$-domains $\mathcal{D}_k \nearrow \mathbb{L}$, the measures $\mathtt{HF}^{0,1}_{\mathcal{D}_k;\mathbf{c},\mathbf{c}_\mathrm{b}}$ converge to some $\mathtt{HF}^{0,1}_{\mathbf{c}}$ which is independent of the sequence $(\mathcal{D}_k)$. Moreover, the limiting measure $\mathtt{HF}^{0,1}_{\mathbf{c}}$ can be constructed in either of the following two ways:*

- *(i) Sample $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ according to $\mathtt{FK}^{\mathrm{w}}_{\beta,q}$. Set $h = 0$ on the unique infinite cluster of ω, and sample h elsewhere according to* **H2** *in Section 4.5.1.*
- *(ii) Sample $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ according to $\mathtt{FK}^{\mathrm{f}}_{\beta,q}$. Set $h = 1$ on the unique infinite cluster of ω^*, and sample h elsewhere according to* **H2'** *in Section 4.5.1.*

The following lemma is a slight generalisation of [GP23, Eq. (28)] and can be proved in exactly the same manner.

Lemma 4.5.4. *Let $\mathcal{D}$ be an $\mathbb{L}$-domain, and let δ be a set of boundary tiles of $\mathcal{D}$. Define $\mathcal{D}^{\mathrm{even}} = \mathcal{D} \setminus (\partial^{\mathrm{in}}\mathcal{D})_\bullet$ and $\mathcal{D}^{\mathrm{odd}} = \mathcal{D} \setminus (\partial^{\mathrm{in}}\mathcal{D})_\circ$, where the boundaries are taken in $\mathbb{L}$. Then, $\mathcal{D}^{\mathrm{even}}$ and $\mathcal{D}^{\mathrm{odd}}$ are disjoint unions of even and odd domains in $\mathbb{L}$, respectively. Moreover, the following stochastic ordering of measures holds:*

$$\mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_\mathrm{b}} \leq_{\mathrm{st}} \mathtt{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_\mathrm{b}} \leq_{\mathrm{st}} \mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{odd}};\mathbf{c},\mathbf{c}_\mathrm{b}}.$$

This lemma readily implies that $\mathtt{HF}^{0,1}_{\mathcal{D}_k,\delta_k;\mathbf{c},\mathbf{c}_\mathrm{b}}$ (in particular $\mathtt{HF}^{0,1}_{\mathcal{D}_k;\mathbf{c}}$) converges to the same limit, no matter which sequences of $\mathcal{D}_k$ and δ_k we chose.

4.5.2 Relaxation of height function measures

The aim of this section is to prove a relaxation statement for the six-vertex height function measures with modified weight $\mathbf{c}_\mathrm{b}$ on arbitrary boundary tiles, in particular for the measure with unmodified weight $\mathbf{c}$ on all tiles. For the whole section, fix $q > 4$ and $\beta = \beta_c(q) = \ln(1 + \sqrt{q})$, and let $\lambda, \mathbf{c}, \mathbf{c}_\mathrm{b}$ be as in (4.3) and (4.4).

Given a measure $\mathtt{HF}$ on $\mathbb{Z}^{\mathbb{L}}$ and $\Delta \subset \mathbb{L}$, define $\mathtt{HF}|_\Delta$ as its marginal on $\mathbb{Z}^\Delta$.

Proposition 4.5.5. *There exist constants $c, \alpha > 0$ such that, for every finite $\Delta \subset \mathbb{L}$, every $\mathbb{L}$-domain $\mathcal{D}$ that contains Δ, and every set $\delta \subset \mathbb{L}_\diamond$ of boundary tiles of $\mathcal{D}$,*

$$\mathrm{d}_{\mathrm{TV}}\left(\mathtt{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta, \mathtt{HF}^{0,1}_{\mathbf{c}}|_\Delta\right) < c\,|\Delta|\,(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))^2\, e^{-\alpha\,\mathrm{d}_\infty(\Delta,\mathcal{D}^c)}.$$

The statement on even or odd $\mathbb{L}$-domains and with modified weight $\mathbf{c}_\mathrm{b}$ on all boundary tiles can be proven in the same way as [ADG24, Proposition 4.2], which is slightly less general. We provide only the statement.

Lemma 4.5.6. *There exist constants $c, \alpha > 0$ such that, for every finite $\Delta \subset \mathbb{L}$ and every even or odd $\mathbb{L}$-domain $\mathcal{D}$ that contains Δ,*

$$\mathrm{d}_{\mathrm{TV}}\left(\mathtt{HF}^{0,1}_{\mathcal{D};\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta, \mathtt{HF}^{0,1}_{\mathbf{c}}|_\Delta\right) < c\,(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))\, e^{-\alpha\,\mathrm{d}_\infty(\Delta,\mathcal{D}^c)}.$$

Before deducing Proposition 4.5.5 from the above and the stochastic ordering of height function measures, Lemma 4.5.4, we need a general lemma.

Lemma 4.5.7. *[HS22, Lemma 2.16] Let μ and ν be two probability measures on a finite* totally ordered *set S with $\mu \leq_{\mathrm{st}} \nu$. Let $\mathbf{P}$ be a monotone coupling of $X \sim \mu$ and $Y \sim \nu$. Then,*

$$\mathbf{P}(X \neq Y) \leq (|S| - 1)\,\mathrm{d}_{\mathrm{TV}}(\mu, \nu).$$

Proof. Identify S with $\{0, \dots, |S| - 1\}$. Let $\mathbf{P}_{\mathrm{op}}$ be an optimal coupling of X and Y, that is, $\mathbf{P}_{\mathrm{op}}(X \neq Y) = \mathrm{d}_{\mathrm{TV}}(\mu, \nu)$. Since $\mathbf{P}(X \leq Y) = 1$, we have

$$\begin{aligned}\mathbf{P}(X \neq Y) = \mathbf{P}(Y - X \geq 1) &\leq \mathbf{E}[Y - X] \\ &= \mathbf{E}_{\mathrm{op}}[Y - X] \leq (|S| - 1)\,\mathbf{P}_{\mathrm{op}}(X \neq Y),\end{aligned}$$

where we applied Markov's inequality. □

Proof of Proposition 4.5.5. Let $\Delta \subset \mathbb{L}$ be finite, let $\mathcal{D}$ be an $\mathbb{L}$-domain, and let $\delta \subset \mathbb{L}_\diamond$ be a set of boundary tiles of $\mathcal{D}$. Assume without loss of generality that $\Delta \subseteq \mathcal{D} \setminus \partial^{\mathrm{in}}\mathcal{D}$. Let $\mathcal{D}^{\mathrm{even}}$ and $\mathcal{D}^{\mathrm{odd}}$ be as in Lemma 4.5.4, and recall the stochastic domination statement therein. Consider a monotone coupling $\mathbf{P}$ of $h^- \sim \mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_\mathrm{b}}$, $h \sim \mathtt{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_\mathrm{b}}$ and $h^+ \sim \mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{odd}};\mathbf{c},\mathbf{c}_\mathrm{b}}$, that is,

$$\mathbf{P}(h^- \leq h \leq h^+) = 1.$$

Take any vertex $v \in \Delta$ and note that, deterministically, the height functions at v take values in $[-m, m]$, where $m := 2(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))$. By Lemma 4.5.7, we have

$$\begin{aligned}\mathbf{P}(h^-(v) \neq h(v)) \leq \mathbf{P}(h^-(v) \neq h^+(v)) &\leq 2m\,\mathrm{d}_{\mathrm{TV}}\big(\mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta, \mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{odd}};\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta\big) \\ &\leq 2m\,\left(\mathrm{d}_{\mathrm{TV}}\big(\mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta, \mathtt{HF}^{0,1}_{\mathbf{c}}|_\Delta\big) + \mathrm{d}_{\mathrm{TV}}\big(\mathtt{HF}^{0,1}_{\mathbf{c}}|_\Delta, \mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{odd}};\mathbf{c},\mathbf{c}_\mathrm{b}}|_\Delta\big)\right).\end{aligned} \tag{4.23}$$

Now, $\mathcal{D}^{\mathrm{even}}$ is a disjoint union of even $\mathbb{L}$-domains $\mathcal{D}_1, \dots, \mathcal{D}_n$ with $\partial^{\mathrm{ex}}\mathcal{D}_i \cap \mathcal{D}_j = \varnothing$ for any $i \neq j$. There exists an i with $\Delta \subseteq \mathcal{D}_i$ and $\mathrm{d}_\infty(\Delta, \mathcal{D}_i^c) \geq \mathrm{d}_\infty(\Delta, \mathcal{D}^c) - 1$. Moreover, the definition of the height function measures (4.19) readily implies that, for any event $A \subset \mathbb{Z}^{\mathbb{L}}$ depending only on values in Δ, we have $\mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_{\mathrm{b}}}(A) = \mathtt{HF}^{0,1}_{\mathcal{D}_i;\mathbf{c},\mathbf{c}_{\mathrm{b}}}(A)$. Similar reasoning applies to $\mathcal{D}^{\mathrm{odd}}$. By Lemma 4.5.6, there exist $c, \alpha > 0$, such that the right-hand side of (4.23) is bounded by $c\, m^2\, e^{-\alpha\, \mathrm{d}_\infty(\Delta, \mathcal{D}^c)}$. The union bound gives

$$\mathrm{d}_{\mathrm{TV}}\left(\mathtt{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_{\mathrm{b}}}|_\Delta, \mathtt{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_{\mathrm{b}}}|_\Delta\right) \le \mathbf{P}(h^-|_\Delta \neq h|_\Delta) \le \sum_{v\in\Delta} \mathbf{P}(h^-(v) \neq h(v)).$$

Another application of the triangle inequality and of Lemma 4.5.6 finishes the proof. $\square$

Remark 4.5.8. Using that h^-, h^+ have uniformly bounded second moment (localisation) and adapting Lemma 4.5.7 allow to remove the square of $(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))^2$ in the bound in Proposition 4.5.5.

4.5.3 Relaxation of ATRC measures

The proof of Proposition 4.3.2 is based on Proposition 4.5.11 and another one below, which is a consequence of Proposition 4.5.5.

The following proposition establishes exponential relaxation of the ATRC measures on domains and with "0, 1" boundary conditions. Once Proposition 4.3.2 is proven, it can be concluded that the limit coincides with the unique ATRC Gibbs measure. Recall the definition (4.20) of $\mathcal{X}_{\mathbf{c}}$.

Proposition 4.5.9. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists a measure $\mathtt{ATRC}_{J,U}$ on $\{0,1\}^{\mathbb{E}^\bullet} \times \{0,1\}^{\mathbb{E}^\bullet}$ and constants $c, \alpha > 0$ such that, for any $\mathbb{L}_\bullet$-domain $G = (V, E)$ and any $F \subseteq E$,*

$$\mathrm{d}_{\mathrm{TV}}\left(\mathtt{ATRC}^{0,1}_{G;J,U}[(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \in \cdot\,], \mathtt{ATRC}_{J,U}[(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \in \cdot\,]\right) \\ < c\, |\mathbb{V}_F|\, (\mathrm{diam}(\mathbb{V}_F) + \mathrm{d}_\infty(\mathbb{V}_F, V^c))^2\, e^{-\alpha\, \mathrm{d}_\infty(\mathbb{V}_F, V^c)},$$

where diam *denotes the diameter with respect to d_∞. Furthermore, the measure $\mathtt{ATRC}_{J,U}$ is constructed as follows. Let $\mathbf{c} = \coth 2J$, let h be distributed according to $\mathtt{HF}^{0,1}_{\mathbf{c}}$, and let $(U_e)_{e\in\mathbb{E}^\bullet}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of h. The measure $\mathtt{ATRC}_{J,U}$ is given by the law of $(\omega_\tau, \omega_{\tau\tau'}) \in \{0,1\}^{\mathbb{E}^\bullet} \times \{0,1\}^{\mathbb{E}^\bullet}$ defined by*

$$(\omega_\tau(e), \omega_{\tau\tau'}(e)) := \mathcal{X}_{\mathbf{c}}(h, e, U_e), \;\; e \in \mathbb{E}^\bullet.$$

Remark 4.5.10. It is possible to improve the bound in Proposition 4.5.9 and remove the square in $(\mathrm{diam}(\mathbb{V}_F) + \mathrm{d}_\infty(\mathbb{V}_F, V^c))^2$, see Remark 4.5.8.

The above proposition follows from the analogous statement for the height function measures, Proposition 4.5.5, and the fact that the ATRC measures can be sampled locally from these measures, which is the content of Lemma 4.5.1.

Proof of Proposition 4.5.9. Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$ and $\mathbf{c} = \coth 2J$. Let $G = (V, E)$ be an $\mathbb{L}_\bullet$-domain and $\mathcal{D} := \mathcal{D}_G$ be its associated $\mathbb{L}$-domain. Recall that a tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}$ precisely if its associated primal edge $e_t \in \mathbb{E}^\bullet$ belongs to E, see Fig. 4.12. Let $F \subset E$, and set $\Delta := \mathbb{V}_F \cup \mathbb{V}_{*F}$.

Take $h \sim \mathtt{HF}^{0,1}_{\mathcal{D};\mathbf{c}}$ and $h' \sim \mathtt{HF}^{0,1}_{\mathbf{c}}$, and consider an optimal coupling of $h|_\Delta$ and $h'|_\Delta$. Let $(U_e)_{e\in\mathbb{E}^\bullet}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of (h, h'). Denote the corresponding probability measure by $\mathbf{P}$. Define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$ from h and

(U_e) as in (4.21). Define $\omega'_\tau, \omega'_{\tau\tau'}$ from h' and (U_e) by setting $(\omega'_\tau(e), \omega'_{\tau\tau'}(e)) = \mathcal{X}_{\mathbf{c}}(h', e, U_e)$ for $e \in \mathbb{E}^\bullet$. Then, clearly

$$(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \neq (\omega'_\tau|_F, \omega'_{\tau\tau'}|_F) \quad \text{implies} \quad h|_\Delta \neq h'|_\Delta.$$

Therefore, by the definition of an optimal coupling and by Proposition 4.5.5,

$$\begin{aligned}\mathbf{P}\big((\omega_\tau|_F, \omega_{\tau\tau'}|_F) \neq (\omega'_\tau|_F, \omega'_{\tau\tau'}|_F)\big) &< c\,|\Delta|\,(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))^2\, e^{-\alpha\, \mathrm{d}_\infty(\Delta, \mathcal{D}^c)} \\ &< c'\,|F|\,(\mathrm{diam}(\mathbb{V}_F) + \mathrm{d}_\infty(\mathbb{V}_F, V^c))^2\, e^{-\alpha\, \mathrm{d}_\infty(\mathbb{V}_F, V^c)},\end{aligned}$$

for some $c, c', \alpha > 0$. By Lemma 4.5.1, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is given by $\mathtt{ATRC}^{0,1}_{G;J,U}$. Finally, denote the law of $(\omega'_\tau, \omega'_{\tau\tau'})$ by $\mathtt{ATRC}_{J,U}$, and the proof is complete. □

4.5.4 Proof of Proposition 4.3.2

Finally, we derive Proposition 4.3.2 from Proposition 4.5.9 and the following one that we import from [ADG24].

Proposition 4.5.11. *[ADG24, Proposition 1.1] Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists $c > 0$ such that, for every $n \geq 1$,*

$$\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}(0 \xleftrightarrow{\omega_\tau} \partial^{\mathrm{ex}}\Lambda_n) < e^{-cn}.$$

Proof of Proposition 4.3.2. Let $\sigma \in \{\tau, \tau\tau'\}$. Assume without loss of generality that $n = 2k$ is even (the other case is treated similarly). We proceed in two steps.
Claim 1. There exists $\alpha > 0$ such that

$$\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] - \mathtt{ATRC}^{0,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] \leq e^{-\alpha n}.$$

Let $(\omega_\tau, \omega_{\tau\tau'})$ be distributed according to $\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}$. Define $\mathcal{C}$ to be the outermost (dual) circuit in ω^*_τ that surrounds Λ_k and is contained in Λ_{2k} if it exists, otherwise set $\mathcal{C} = \varnothing$. By exponential decay of connection probabilities in ω_τ, Proposition 4.5.11, there exist $c, \alpha > 0$ such that

$$\begin{aligned}\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] &\leq \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} \neq \varnothing\right] + \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}[\Lambda_k \xleftrightarrow{\omega_\tau} \partial^{\mathrm{in}}\Lambda_{2k}] \\ &\leq \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} \neq \varnothing\right] + cke^{-\alpha k},\end{aligned}$$

where we also used the strong FKG and DLR properties of the ATRC measures (Lemma 4.3.1 and (DLR) in Section 4.3). Now, again by the strong FKG and (DLR) and by exponential relaxation, Proposition 4.5.9, there exist $c', \alpha' > 0$ such that

$$\begin{aligned}&\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} \neq \varnothing\right] \\ &\qquad = \sum_C \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} = C\right]\mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\mathcal{C} = C\,\middle|\,\mathcal{C} \neq \varnothing\right] \\ &\qquad \leq \sum_C \mathtt{ATRC}^{0,1}_{G_C;J,U}[\omega_\sigma(e)]\ \mathtt{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\mathcal{C} = C\,\middle|\,\mathcal{C} \neq \varnothing\right] \\ &\qquad \leq \mathtt{ATRC}^{0,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\right] + c'k^2e^{-\alpha' k},\end{aligned}$$

where the summation is over all realisations C of $\mathcal{C}$, and where G_C is the largest $\mathbb{L}_\bullet$-domain of the first kind within C that contains Λ_k. This proves Claim 1.

Claim 2. There exists $\alpha > 0$ such that

$$\mathtt{ATRC}^{0,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] - \mathtt{ATRC}^{0,0}_{\Lambda_n;J,U}[\omega_\sigma(e)] \leq e^{-\alpha n}.$$

We use the same strategy as in Claim 1. Indeed, let $(\tilde{\omega}_\tau, \tilde{\omega}_{\tau\tau'})$ be distributed according to $\mathtt{ATRC}^{0,0}_{\Lambda_n;J,U}$. Since, by self-duality (4.8), $\tilde{\omega}^*_{\tau\tau'}$ has the same law as ω_τ (but on the dual graph of Λ_n) from Claim 1, Proposition 4.5.11 allows to find a circuit in $\tilde{\omega}_{\tau\tau'}$ that surrounds Λ_k and is contained in Λ_{2k}. For a realisation C of an outermost such circuit, consider the largest $\mathbb{L}_\bullet$-domain of the second kind within C that contains Λ_k. The remainder of the argument is analogous to Claim 1. □

4.5.5 Ratio weak mixing: proof of Theorem 16

We first derive the *exponential* weak mixing property from the single edge relaxation, Proposition 4.3.2. Given $F \subset \mathbb{E}^\bullet$ and a measure $\mathtt{ATRC}$ on $\{0,1\}^{\mathbb{E}^\bullet} \times \{0,1\}^{\mathbb{E}^\bullet}$, we write $\mathtt{ATRC}|_F$ for its marginal on $\{0,1\}^F \times \{0,1\}^F$.

Theorem 19. *Let* $0 < J < U$ *satisfy* $\sinh 2J = e^{-2U}$*. There exists* $c > 0$ *such that, for any finite subgraph* $G = (V, E)$ *of* $\mathbb{L}_\bullet$ *and any* $F \subset E$*,*

$$\sup_{\eta_\tau, \eta_{\tau\tau'}, \eta'_\tau, \eta'_{\tau\tau'}} \mathrm{d}_{\mathrm{TV}}\Big(\mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{G;J,U}|_F, \mathtt{ATRC}^{\eta'_\tau,\eta'_{\tau\tau'}}_{G;J,U}|_F\Big) \leq 2\sum_{e\in F} e^{-c\,\mathrm{d}_\infty(e,V^c)},$$

where d_∞ *is the distance induced by the* L^∞ *norm.*

As a consequence, the measures $\mathtt{ATRC}^{\eta_\tau,\eta_{\tau\tau'}}_{G;J,U}$ converge (as $G \nearrow \mathbb{L}_\bullet$) to a limit measure $\mathtt{ATRC}_{J,U}$, which is independent of the choice of boundary conditions $\eta_\tau, \eta_{\tau\tau'}$. Furthermore, the limit $\mathtt{ATRC}_{J,U}$ is the unique ATRC *Gibbs* measure.

Proof. By the strong FKG property, Lemma 4.3.1, the supremum is attained for $\eta_\tau = \eta_{\tau\tau'} = 1$ and $\eta'_\tau = \eta'_{\tau\tau'} = 0$. Let Ψ be a monotone coupling of $\mathtt{ATRC}^{1,1}_{G;J,U}$ and $\mathtt{ATRC}^{0,0}_{G;J,U}$, and let $(X,Y) = \big((X_\tau, X_{\tau\tau'}), (Y_\tau, Y_{\tau\tau'})\big) \sim \Psi$. Then,

$$\begin{aligned}\mathrm{d}_{\mathrm{TV}}\Big(\mathtt{ATRC}^{1,1}_{G;J,U}|_F, \mathtt{ATRC}^{0,0}_{G;J,U}|_F\Big) &\leq \Psi(X|_F \neq Y|_F)\\ &\leq \Psi(X_\tau|_F \neq Y_\tau|_F) + \Psi(X_{\tau\tau'}|_F \neq Y_{\tau\tau'}|_F).\end{aligned}$$

Now, for $\sigma \in \{\tau, \tau\tau'\}$,

$$\begin{aligned}\Psi(X_\sigma|_F \neq Y_\sigma|_F) &\leq \sum_{e\in F} \Psi(X_\sigma(e) \neq Y_\sigma(e)) = \sum_{e\in F} \Psi(X_\sigma(e) > Y_\sigma(e))\\ &= \sum_{e\in F} \big(\Psi(X_\sigma(e) = 1) - \Psi(Y_\sigma(e) = 1)\big) \leq \sum_{e\in F} e^{-c\mathrm{d}_\infty(e,V^c)}\end{aligned}$$

for some $c > 0$ by Proposition 4.3.2, where we also used strong positive association and the DLR property of the ATRC measures. □

A less trivial consequence is the *ratio* weak mixing property, Theorem 16. This follows from the study performed in [Ale98], relying on Theorem 19 and Proposition 4.5.11.

Proof of Theorem 16. This is a direct application of [Ale98, Section 5]. The reasoning there requires two properties of the measure. The first is exponential weak mixing, which is the content of Theorem 19. The second is the admittance of exponentially bounded controlling regions in the sense of [Ale98], and we argue its validity as follows.

Let $(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{ATRC}_{G;J,U}^{\eta_\tau,\eta_{\tau\tau'}}$, fix $F \subset E$, and define

$$H(F) = \{e \in E \setminus F : e \overset{\omega_\tau}{\longleftrightarrow} \mathbb{V}_F \text{ or } e^* \overset{\omega^*_{\tau\tau'}}{\longleftrightarrow} \mathbb{V}_{*F}\}.$$

Conditionally on the states of the edges in $H(F)$, the values of the field in F and $E\setminus(F \cup H(F))$ are independent. Moreover, the probability that $e \in E\setminus F$ belongs to $H(F)$ decays exponentially in $\mathrm{d}_\infty(e, \mathbb{V}_F)$ by uniform exponential decay in ω_τ and in the dual of $\omega_{\tau\tau'}$. The former is the content of Proposition 4.5.11 and the latter follows from self-duality. One can then apply [Ale98, Section 5] to obtain the desired claim. □

4.6 Strong mixing

In this section, we use [Ott25] to push the mixing properties derived in the previous section to strong mixing properties for finite volume ATRC measures.

4.6.1 Setup

For $k \geq 1$, let $\Gamma_k = ((2k+1)\mathbb{Z})^2$. Given $x \in \mathbb{Z}^2$ or $V \subset \mathbb{Z}^2$, denote $[x]_k := x + \{-k, \dots, k\}^2$ or $[V]_k := \cup_{x \in V}(x + \{-k, \dots, k\}^2)$ respectively.
Path decoupling property. Let $k \geq 1$. For a simple closed nearest-neighbour path γ on Γ_k, denote by $F_{\text{in}}(\gamma)$, $F_{\text{out}}(\gamma)$ the sets of edges in $\mathbb{E}^\bullet$ that are, respectively, in the finite connected component of $\mathbb{E}^\bullet \setminus \mathbb{E}_{[\gamma]_k}$, in the infinite connected component of $\mathbb{E}^\bullet \setminus \mathbb{E}_{[\gamma]_k}$.

Let $(a, b) \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}^\bullet}$, and $F \subset \mathbb{E}^\bullet$ finite. Say that $\mathtt{ATRC}_F^{a,b}$ has the *k-decoupling path property* if for any simple closed nearest-neighbour path γ in Γ_k, and any configuration $(\xi, \xi') \in \{(0,0),(0,1),(1,1)\}^{F \cap \mathbb{E}_{[\gamma]_k}}$ such that

- the dual of the configuration $\xi_F 0_{F^c}$ (ξ on F and constant 0 on F^c) contains a simple closed path of open dual edges surrounding $[\mathring{\gamma}]_k$, where $\mathring{\gamma}$ is the set of sites in Γ_k surrounded by γ,
- the configuration $\xi'_F 1_{F^c}$ (ξ' on F and constant 1 on F^c) contains a simple closed path of open edges surrounding $[\mathring{\gamma}]_k$,

one has that if

$$(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{ATRC}_F^{a,b}\big(\ \big|\, \omega_\tau(e) = \xi(e),\, \omega_{\tau\tau'}(e) = \xi'(e)\ \forall e \in \mathbb{E}_{[\gamma]_k}\big),$$

then the restriction of $(\omega_\tau, \omega_{\tau\tau'})$ to $F \cap F_{\text{in}}(\gamma)$ is independent from the restriction of $(\omega_\tau, \omega_{\tau\tau'})$ to $F \cap F_{\text{out}}(\gamma)$. A configuration (ξ, ξ') as above is said to contain *a pair of decoupling paths for γ in F*.

One has this property in, for example, the case of constant boundary conditions $(0,1)$, or in the case of F simply connected and constant $(0,0)$ or constant $(1,1)$ boundary conditions. More generally, to check if the property holds, it suffices to check that cluster count is factorized by the presence of the required paths in $\mathbb{E}_{[\gamma]_k}$, as in Lemma 4.4.6.
ATRC as a model on sites. The setup of [Ott25] is a collection of site models indexed by finite volumes and some index set T. Cast ATRC as a model on sites by taking the spin space at site x to be

$$\Omega_x = \{(0,0),(0,1),(1,1)\}^{\{\{x,x+\mathrm{e}_1\},\{x,x+\mathrm{e}_2\}\}}.$$

I.e. store the value of the field at a given edge in its left/bottom endpoint. Let

$$\psi : \{(0,0),(0,1),(1,1)\}^{\mathbb{E}^\bullet} \to \bigtimes_{x \in \mathbb{Z}^2} \Omega_x$$

be the natural bijection induced by the above. Let

$$\mathbb{E}_{BL}(\Lambda) = \big\{\{x, x+\mathrm{e}_1\}, \{x, x+\mathrm{e}_2\} :\ x \in \Lambda\big\}.$$

Bloc Percolation. For $p, q \in [0,1]$, let X_i, Y_i, $i \in \mathbb{Z}^2$ be an independent family of Bernoulli random variables with parameters

$$P(X_i = 1) = p, \quad P(Y_i = 1) = q \quad \forall i \in \mathbb{Z}^2.$$

For $k \geq 1$, $F \subset \mathbb{E}^\bullet$, $x \in \mathbb{Z}^2$, define $i_x \in \Gamma_k$ to be the unique point with $x \in [i_x]_k$, and

$$\eta_x = \begin{cases} X_{i_x} & \text{if } \mathbb{E}_{BL}([i_x]_{3k+1}) \subset F, \\ Y_{i_x} & \text{if } x \in \mathbb{V}_F, \text{ and } \mathbb{E}_{BL}([i_x]_{3k+1}) \cap F^c \neq \varnothing, \\ 0 & \text{else.} \end{cases}$$

Let $P_{F;k,p,q}$ be the law of $(\eta_x)_{x\in\mathbb{Z}^2}$.

4.6.2 Strong mixing: main Theorem

Then, the main result of the section is the next Theorem. To get it, we will apply the main theorem of [Ott25] (see Section 4.6.3). For $F_-, F_+ \subset F \subset \mathbb{E}^\bullet$, $(a,b) \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}^\bullet}$, define

$$\nu^{a,b}_{F,F_-,F_+}(\cdot) = \mathtt{ATRC}^{a,b}_F\big(\cdot \mid \omega_\tau(e) = 0\ \forall e \in F_-,\ \omega_{\tau\tau'}(e) = 1\ \forall e \in F_+\big). \tag{4.24}$$

Theorem 20. *Let $0 \leq J < U$ be such that $\sinh(2J) = e^{-2U}$. There is $l \geq 1$ such that: for any $p > 0$, there are $L < \infty$ and $q < 1$ such that for any $F \subset \mathbb{E}^\bullet$, any $(a,b) \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}^\bullet}$ such that $\mathtt{ATRC}^{a,b}_F$ has the l-decoupling path property, any $F_-, F_+ \subset \partial^{\mathrm{in}} F$, any $F_1, F_2 \subset F$ disjoint, and any $(f,g),(f',g') \in \{(0,0),(0,1),(1,1)\}^{F_1}$ having positive probability under $\nu^{a,b}_{F,F_-,F_+}$ (defined in (4.24))*

$$\begin{aligned} \mathrm{d_{TV}}\big(\nu^{a,b}_{F,F_-,F_+}\big(W|_{F_2} \in \cdot \,\big|\, W|_{F_1} = (f,g)\big), \nu^{a,b}_{F,F_-,F_+}\big(W|_{F_2} \in \cdot \,\big|\, W|_{F_1} = (f',g')\big)\big) \\ \leq P_{F\setminus(F_+\cup F_-);L,p,q}\big([\mathbb{V}_{F_1}]_L \leftrightarrow_* [\mathbb{V}_{F_2}]_L\big), \end{aligned}$$

where $W = (\omega_\tau, \omega_{\tau\tau'})$, and $\leftrightarrow_*$ means $*$-connections (connections with $\|\ \|_\infty$ nearest-neighbours).

Remark 4.6.1. We will systematically apply this result in cases where $\{x \in \mathbb{V}_F :\ \mathbb{E}_{BL}([i_x]_{3L+1}) \cap F^c \neq \varnothing\}$ is a very elongated one dimensional object, so exponential decay of the connection probability can be obtained by a coarse-graining argument as in [OV18, Lemma 3.2].

4.6.3 Verifications of the mixing and decoupling hypotheses

In this section, we apply [Ott25] to obtain Theorem 20. The setup of [Ott25] is a family of site models satisfying a collection of hypotheses, denoted Mix1, Mix2, Mar1, Mar2, Mar3. We cast our model in that setup, and check that these hypotheses hold. Theorem 20 then follows from [Ott25, Theorem 1.1]

Family of site measures Let $l \geq 1$ be some number to be fixed large later (it will depend only on J, U). In [Ott25], a family $(\nu^t_\Lambda)_{\Lambda\subset\mathbb{Z}^2, t\in T}$ is considered, with T some index set. We take T to be the set of quintuplets (F, a, b, F_-, F_+) such that $(a,b) \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}^\bullet}$, $F \subset \mathbb{E}^\bullet$ finite such that $\mathtt{ATRC}^{a,b}_F$ has the l-decoupling path property, and $F_-, F_+ \subset \partial^{\mathrm{in}} F$.

The family $(\nu^t_\Lambda)_{\Lambda\subset\mathbb{Z}^2, t\in T}$ is then given by

$$\nu_\Lambda^{a,b,F,F_+,F_-}(\cdot) := \begin{cases} \mathtt{ATRC}_F^{a,b}\big(\psi^{-1}(\cdot)\,\big|\,\omega_\tau(e)=0\,\forall e\in F_-,\ \omega_{\tau\tau'}(e)=1\,\forall e\in F_+\big) & \text{if } \Lambda=\mathbb{V}_F, \\ \mathtt{ATRC}_{\mathbb{E}_\Lambda}^{0,1}\big(\psi^{-1}(\cdot)\big) & \text{else .}\end{cases} \tag{4.25}$$

Note the small abuse of notation: we used the same notation as in (4.24) for the measure on sites rather than on edges. We are only interested in the first case, the second is only there to fit the setup of [Ott25].

Mixing hypotheses. The hypotheses Mix1 and Mix2 are exponential mixing hypotheses which are weak versions of the ratio weak mixing property (Theorem 16).

Markov-type hypotheses. To avoid lengthy displays, introduce the shorthand

$$F' = \begin{cases} F & \text{if } \mathbb{V}_F=\Lambda, \\ \mathbb{E}_\Lambda & \text{else.}\end{cases}$$

Hypotheses Mar1, Mar2, Mar3 ask for

- a family of local events, Ma_i, $i\in\Gamma_l$, supported on $\mathbb{E}_{BL}([i]_{3l+1})$;
- a family of local events, $\mathrm{Ma}^t_{i;\Lambda}$, $i\in\Gamma_l$, $t=(F,a,b,F_-,F_+)\in T$, $\Lambda\subset\mathbb{Z}^2$, supported on $\mathbb{E}_{BL}([i]_{3l+1})\cap F'$;
- configurations $\mathrm{a}_i\in\{(0,0),(0,1),(1,1)\}^{\mathbb{E}_{BL}([i]_l)}$, $i\in\Gamma_l$;
- configurations $\mathrm{a}^t_{i,\Lambda}\in\{(0,0),(0,1),(1,1)\}^{\mathbb{E}_{BL}([i]_l)\cap F'}$, $i\in\Gamma_l$.

These should satisfy:

1. if γ is a simple closed nearest-neighbour path in Γ_l, and (ξ,ξ') is a configuration on $F'\cap\mathbb{E}_{[\gamma]_{3l+1}}$ such that for any $i\in\gamma$ with $\mathbb{E}_{[i]_{5l+2}}\subset F'$ one has
$$(\xi,\xi')|_{\mathbb{E}_{[i]_{3l+1}}}\in\mathrm{Ma}_i,$$
and for any $i\in\gamma$ with $\mathbb{E}_{[i]_{5l+2}}\not\subset F'$, $\mathbb{E}_{[i]_{3l+1}}\cap F'\neq\varnothing$, one has
$$(\xi,\xi')|_{\mathbb{E}_{[i]_{3l+1}}\cap F'}\in\mathrm{Ma}^t_{i;\Lambda},$$
then (ξ,ξ') contains a pair of decoupling paths for γ in F'. This is Mar1.

2. The probability of Ma_i conditionally on the configuration outside of $\mathbb{E}_{[i]_{3l+1}}$ is lower bounded by $p_0<1$ fixed universal, uniformly over the configuration outside of $\mathbb{E}_{[i]_{3l+1}}$. This is Mar2.

3. If $w\in\mathrm{Ma}_i$, and $j\in\Gamma_l$ with $\|i-j\|_\infty\leq 2l+1$, then the configuration obtained by swapping the value of $w|_{\mathbb{E}_{BL}([j]_l)}$ to a_j is still in Ma_i, and similarly for $\mathrm{a}^t_{i,\Lambda}$. Moreover, the probability of a_i, $\mathrm{a}^t_{i,\Lambda}$ conditionally on the configuration outside of their respective supports is greater than $\theta>0$ for some $\theta>0$ uniform over the configuration outside of the support. This is Mar3.

Take the events/configurations (l is taken large enough):

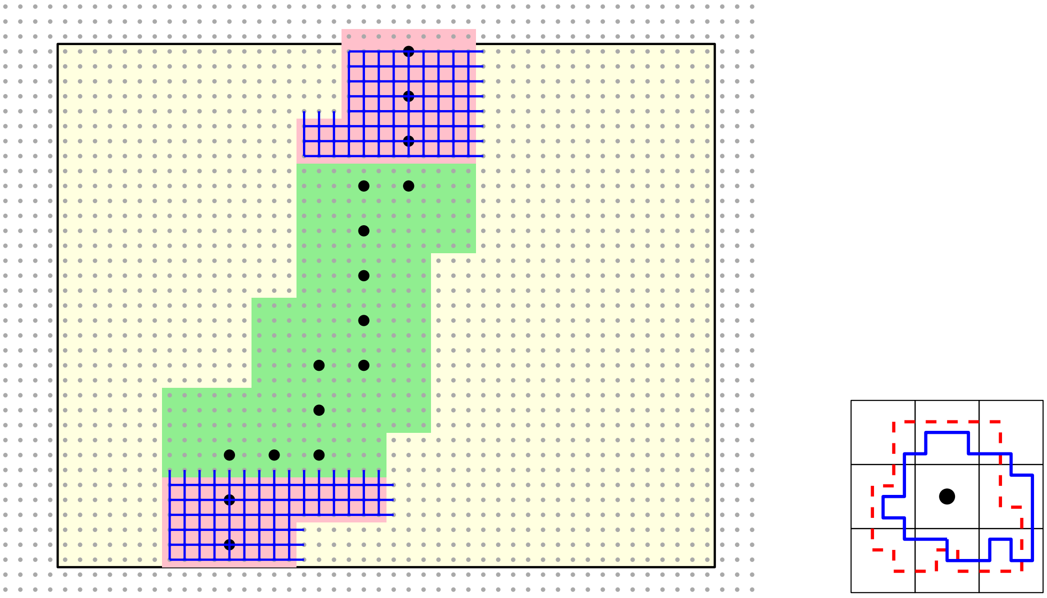


Figure 4.13: Left: a path in Γ_1 (black disks), the blue edges have their state forced to be $(0,1)$, the black disks in a green square imply the realization of Ma_i at that place. Right: a realization of Ma_i (blue edges are open in $\omega_{\tau\tau'}$, while the dashed red path represent the dual of edges closed in ω_τ).

- Ma_i is the event that $\mathbb{E}_{[i]_{3l/2}} \setminus \mathbb{E}_{[i]_{l+1}}$ contains both a simple closed path of edges open in $\omega_{\tau\tau'}$ surrounding $[i]_{l+1}$, and a simple closed path of dual edges open in ω_τ^* surrounding $[i]_{l+1}$.
- $\mathrm{a}_{i,\Lambda}^t$, and a_i are the constant $(0,1)$ configurations.
- $\mathrm{Ma}_{i,\Lambda}^t$ contains only the constant $(0,1)$ configuration.

See Fig. 4.13. Mar1 follows by Jordan's curve Theorem. Mar2 follows (once l is fixed large enough) from exponential mixing (Theorem 19) and uniform exponential decay of connectivities in ω_τ, and $\omega_{\tau\tau'}^*$ (Theorem 17). Finally, Mar3 follows from finite energy (FE).

4.7 Random Walk picture in the ATRC

We start this section by introducing the objects necessary to the development of the "Ornstein-Zernike theory" (renewal picture of connection probabilities). We follow the general strategy used in [CI02, CIV03, CIV08, OV18, AOV24], most results will be imported when their proof is a repetition of existing arguments. To keep notations readable, we will use the following short-hand throughout this section:

$$\Phi \equiv \mathtt{ATRC}_{J,U}, \qquad \Phi_\Lambda^{a,b} \equiv \mathtt{ATRC}_{\Lambda;J,U}^{a,b},$$

where $\mathtt{ATRC}_{J,U}$ is the unique infinite-volume measure for the ATRC model (recall Proposition 4.5.9). The theorems that will be used in other sections will be stated with the notation matching the rest of the chapter. Moreover, again to lighten notations,

connections are in ω_τ unless explicitly stated otherwise.

We will denote $\mathcal{C}_0$ the cluster of 0 in ω_τ.

4.7.1 Decay rate, Cones, and Diamonds

Start by introducing some objects.

Norm induced by the decay rates and associated sets. For $s \in \mathbb{S}^1$, define

$$\nu(s) = -\lim_{n\to\infty} \tfrac{1}{n} \ln \Phi(0 \leftrightarrow ns). \tag{4.26}$$

Existence of the limit follows in the standard fashion by Fekete's Lemma, using sub-additivity which follows from FKG inequality. We now extend ν to $\mathbb{R}^2$ by positive homogeneity of order one: for any $s \in \mathbb{S}^1$ and $r \geq 0$, define

$$\nu(rs) := r\nu(s).$$

Using existence of the limit and the FKG inequality, one can show that ν is a non-degenerate norm as soon as it is non-zero: ν is positive homogeneous by definition; exponential decay of connection probabilities in Φ (the ATRC model) implies positivity; the FKG inequality for Φ implies the triangular inequality for ν. See [Ale01, Section 2] for details. Clearly, ν inherits the symmetries of the lattice, that is axial and diagonal reflections and rotations by $\pi/2$. From these symmetry considerations, one has that for any $s \in \mathbb{S}^1$,

$$\tfrac{1}{\sqrt{2}}\nu(\mathrm{e}_1) \leq \nu(s) \leq \sqrt{2}\nu(\mathrm{e}_1),$$

with $\mathrm{e}_1 = (1, 0)$. Moreover, one has

$$\Phi(0 \leftrightarrow x) = e^{-\nu(x)(1+o(1))}. \tag{4.27}$$

As ν is a norm, there are two convex sets naturally associated to it: the equi-decay set $\mathcal{U}$, and the convex set of which ν is the support function, $\mathcal{W}$.

$$\mathcal{U} = \{x \in \mathbb{R}^d : \ \nu(x) \leq 1\}, \qquad \mathcal{W} = \bigcap_{s\in\mathbb{S}^1} \{x \in \mathbb{R}^d : \ x \cdot s \leq \nu(s)\}. \tag{4.28}$$

It is easy to see that $\mathcal{U}$ and $\mathcal{W}$ are dual to each other, that is: $\mathcal{W}$ is the equi-decay set of the norm dual to ν, and the norm dual to ν is the support function of $\mathcal{U}$. We say that $(s,t) \in \mathbb{S}^1 \times \partial\mathcal{W}$ is a *dual pair* if $\nu(s) = t \cdot s$. From the definitions, one has, for any $x \in \mathbb{R}^d$,

$$\nu(x) = \max_{t\in\partial\mathcal{W}} t \cdot x. \tag{4.29}$$

We refer to [Roc70] for details on convex duality. From the last display, it is also easy to see that $\mathcal{W}$ is the closure of the convergence domain of

$$\mathbb{G}(t) = \sum_{x\in\mathbb{Z}^2} \Phi(0 \leftrightarrow x) e^{t\cdot x}. \tag{4.30}$$

Cones and Diamonds. Let $t \in \partial\mathcal{W}, \delta \in (0,1)$. Let us introduce the geometric objects used in the interface study. We first define the cones and the associated diamonds (see Fig. 4.14):

$$\mathcal{Y}^{\blacktriangleleft}_{t,\delta} := \big\{x \in \mathbb{R}^2 : \ x \cdot t \geq (1-\delta)\nu(x)\big\}, \quad \mathcal{Y}^{\blacktriangleright}_{t,\delta} := -\mathcal{Y}^{\blacktriangleleft}_{t,\delta},$$
$$\mathrm{Diamond}_{t,\delta}(u,v) := (u + \mathcal{Y}^{\blacktriangleleft}_{t,\delta}) \cap (v + \mathcal{Y}^{\blacktriangleright}_{t,\delta}).$$

As δ goes to 1, the cone $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ converges to the half-plane that contains t and whose boundary is orthogonal to t. As δ goes to 0, the cone $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ converges to the convex cone generated by the directions dual to t. The latter set is a line when ν is strictly convex.

Let $V \subset \mathbb{Z}^2$. We will say that V is:

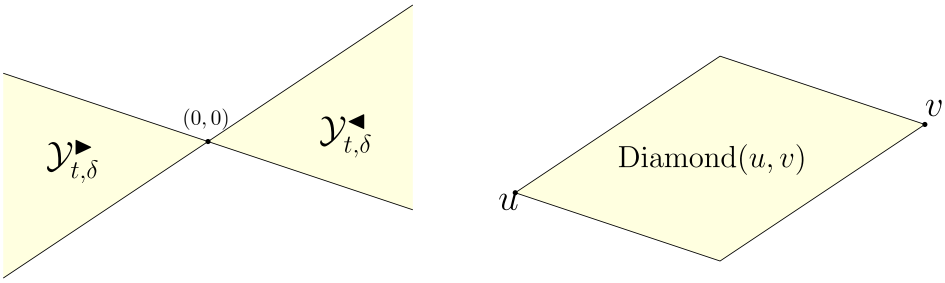


Figure 4.14: Forward and backward cones, and the associated diamond.

- (t,δ)-*forward-confined* if there exists $u \in V$ such that $V \subset u + \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$. When it exists, such a u is unique; we denote it by $\mathbf{f}(V)$.
- (t,δ)-*backward-confined* if there exists $v \in V$ such that $V \subset v + \mathcal{Y}^{\blacktriangleright}_{t,\delta}$. When it exists, such a v is unique; we denote it by $\mathbf{b}(V)$.
- (t,δ)-*diamond-confined* if it is both forward- and backward-confined.

We will say that $v \in V$ is a (t,δ)-*cone-point* of V if

$$V \subset v + (\mathcal{Y}^{\blacktriangleright}_{t,\delta} \cup \mathcal{Y}^{\blacktriangleleft}_{t,\delta}).$$

We denote $\mathrm{CPts}_{t,\delta}(V)$ the set of cone-points of V. When speaking about cone-points of graphs, we mean cone-points of their vertex set.

We call a graph with a distinguished vertex a *marked graph*. The distinguished vertex is denoted v^*. Define (see Fig. 4.15)

- The sets of confined pieces (all are sets of finite connected sub-graphs of $(\mathbb{Z}^2, \mathbb{E})$):

$$\begin{aligned}\mathfrak{B}_L(t,\delta) &= \{\gamma \text{ marked backward-confined with } v^* = 0\},\\ \mathfrak{B}_R(t,\delta) &= \{\gamma \text{ marked forward-confined with } \mathbf{f}(\gamma) = 0\},\\ \mathfrak{A}(t,\delta) &= \{\gamma \text{ diamond-confined with } \mathbf{f}(\gamma) = 0\}.\end{aligned}$$

  To fix ideas we shall, unless stated otherwise, think of $\mathfrak{A}$ as of a subset of $\mathfrak{B}_L$, that is, by default the vertex $\mathbf{f}(\gamma) = 0$ is marked for any $\gamma \in \mathfrak{A}$. Note that $\mathfrak{A}$ can alternatively be viewed as subset of $\mathfrak{B}_R$ by marking $\mathbf{b}(\gamma)$.

- The displacement along a piece:

$$\mathrm{X}(\gamma) := \begin{cases}\mathbf{b}(\gamma) & \text{if } \gamma \in \mathfrak{B}_L, \text{ in particular, if } \gamma \in \mathfrak{A},\\ v^* & \text{if } \gamma \in \mathfrak{B}_R.\end{cases} \tag{4.31}$$

- The *concatenation* operation (see Fig. 4.16): for $\gamma_1 \in \mathfrak{B}_L$ and $\gamma_2 \in \mathfrak{B}_R$ define the concatenation of γ_2 to γ_1 as

$$\gamma_1 \circ \gamma_2 = \gamma_1 \cup (\mathrm{X}(\gamma_1) + \gamma_2).$$

  The concatenation of two graphs in $\mathfrak{A}$ is an element of $\mathfrak{A}$, the concatenation of a graph in $\mathfrak{A}$ to an element of $\mathfrak{B}_L$ is an element of $\mathfrak{B}_L$, and the concatenation of a element in $\mathfrak{B}_R$ to an graph in $\mathfrak{A}$ is an element of $\mathfrak{B}_R$. The displacement along a concatenation is the sum of the displacements along the pieces.

These sets can be seen as equivalence classes of general marked forward/backward/diamond confined graphs modulo translations. A general such graph, $\gamma'_L, \gamma'_R, \gamma'$, can then be recovered uniquely from an element $\gamma_L, \gamma_R, \gamma$ of $\mathfrak{B}_L, \mathfrak{B}_R, \mathfrak{A}$ by specifying the translation vector.

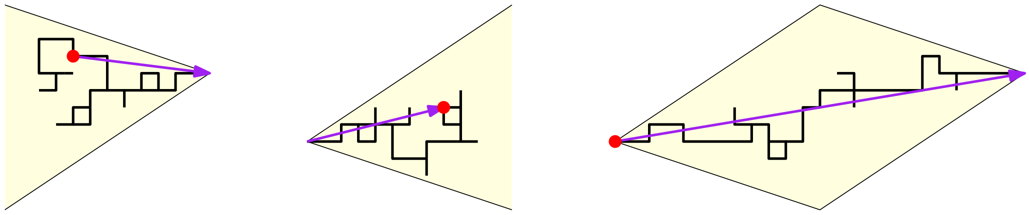


Figure 4.15: Backward, forward, and diamond confined marked graphs,with their displacement.

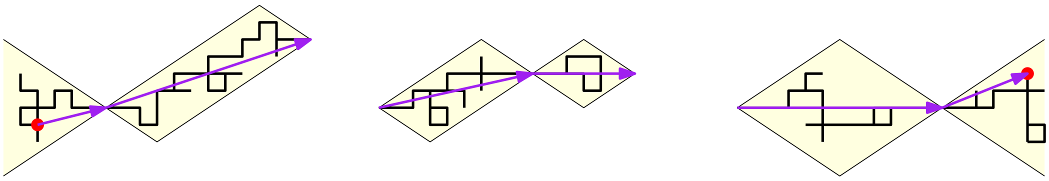


Figure 4.16: Concatenation of confined graphs. Red dots are the marked vertices of the marked graphs.

4.7.2 Main result of the section

Our main goal is to prove the following "coupling with random walk" in infinite volume for long connections in the ATRC model. Recall that $\mathrm{e}_1 = (1,0), \mathrm{e}_2 = (0,1)$ are the canonical basis vectors.

Theorem 21. *Let $0 \leq J < U$ be such that $\sinh(2J) = e^{-2U}$. Let $s_0 \in \mathbb{S}^1$, $t \in \partial\mathcal{W}$ dual to s_0 and $\delta \in (0,1)$ be such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm\mathrm{e}_1, \pm\mathrm{e}_2\}$. Then, there exist constants $\mathrm{C}, C_1, C_2 \geq 0, c_1, c_2 > 0, \epsilon > 0$ and probability measures p_L, p, p_R on $\mathfrak{B}_L(t,\delta)$, $\mathfrak{B}_R(t,\delta)$, $\mathfrak{A}(t,\delta)$ respectively such that*

1. renewal structure: *for any $x \in \mathbb{Z}^2$ such that $x \cdot s_0 \geq (1-\epsilon)|x|$, and any f real valued function of the cluster of 0,*

$$\Big|\mathrm{C}\sum_{\gamma_L,\gamma_R} p_L(\gamma_L)p_R(\gamma_R)\sum_{k\geq 0}\sum_{\gamma_1,\dots,\gamma_k} f(\bar{\gamma})\mathbb{1}_{\mathrm{X}(\bar{\gamma})=x}\prod_{i=1}^{k} p(\gamma_k) \\ - e^{t\cdot x}\mathtt{ATRC}_{J,U}\big(f(\mathcal{C}_0)\mathbb{1}_{0\xleftrightarrow{\omega_\tau}x}\big)\Big| \leq C_1 \|f\|_\infty e^{-c_1|x|}, \quad (4.32)$$

where $\bar{\gamma} = \gamma_L \circ \gamma_1 \circ \ldots \circ \gamma_k \circ \gamma_R$, and the sums are over $\gamma_L \in \mathfrak{B}_L(t,\delta)$, $\gamma_R \in \mathfrak{B}_R(t,\delta)$, and $\gamma_1,\dots,\gamma_k \in \mathfrak{A}(t,\delta)$;

2. exponential tails and "finite energy" for the steps: *for $(p_*, D_*) \in \{(p_L, \mathfrak{B}_L(t,\delta)), (p_R, \mathfrak{B}_R(t,\delta)), (p, \mathfrak{A}(t,\delta))\}$,*

$$\sum_{\gamma\in D_*} p_*(\gamma)\mathbb{1}_{\|\mathrm{X}(\gamma)\|\geq \ell} \leq C_2 e^{-c_2\ell} \quad \forall \ell \geq 0, \quad (4.33)$$

$$p_*(\gamma) \geq \alpha^{|\gamma|+1} \quad \forall \gamma \in D_*, \quad (4.34)$$

for some $\alpha > 0$ depending on J, U only,

3. mean value of a step is proportional to s_0: *there is $a > 0$ such that*

$$\sum_{\gamma\in\mathfrak{A}(t,\delta)} p(\gamma)\mathrm{X}(\gamma) = a s_0. \quad (4.35)$$

Moreover, $\partial\mathcal{U}$, $\partial\mathcal{W}$ are analytic manifolds and the norm ν is uniformly strictly convex, that is $\mathcal{U}$ is strictly convex and $\partial\mathcal{U}$ has uniformly lower bounded curvature. In particular, each direction $s \in \mathbb{S}^1$ has a unique dual vector $t_s \in \partial\mathcal{W}$ and there exist $\kappa > 0$ such that the following sharp triangle inequality holds:

$$\nu(x) + \nu(y) - \nu(x+y) \geq \kappa(|x| + |y| - |x+y|).$$

The rest of the section is devoted to the proof of Theorem 21. Subsections 4.7.3, 4.7.4, and 4.7.5 are preparations for the core of the proof. The proof itself is then divided in two main steps, presented respectively in Subsections 4.7.6, and 4.7.7.

Remark 4.7.1. We use the value of J, U through only two inputs: the exponential decay in ω_τ and $\omega^*_{\tau\tau'}$, and the mixing of Theorems 16 and 20 (which follows from edge relaxation, Proposition 4.3.2, and exponential decay in ω_τ and $\omega^*_{\tau\tau'}$). In particular, if one can extend these properties beyond the self-dual line (part of which is done in [ADG24]), Theorem 21 extends directly.

4.7.3 The coarse-graining procedure

The first step is to analyse the typical geometry of long connections. The analysis of [CIV08, AOV24] is based on a coarse-graining of the cluster of 0. Introduce the cells: for $K, k \geq 1$ and $A \subset \mathbb{Z}^2$,

$$[A]_k = \bigcup_{x \in A} (x + \{-k, \dots, k\}^2),$$
$$\Delta_K = K\mathcal{U} \cap \mathbb{Z}^2, \quad \Delta'_K = [\Delta_K]_{\ln(K)^2}.$$

The scale parameter K will be picked large enough in the course of the proof.

We then coarse-grain the cluster of 0 using the same algorithm as in [CIV08]. Denote $\Delta \equiv \Delta_K$ and $\Delta' \equiv \Delta'_K$. For C a realization of $\mathcal{C}_0$ define $\mathrm{Sk}(C)$ via the next algorithm (see also Fig. 4.17).

Set $v_0 = 0$, $\mathrm{Sk}_V = \{v_0\}$, $\mathrm{Sk}_E = \varnothing$, $V = \Delta'$, $i = 1$;

while $A = \{z \in \partial^{\mathrm{ex}} V : z \xleftrightarrow{(z+\Delta)\setminus V} \partial^{\mathrm{in}}(z + \Delta)\} \neq \varnothing$ **do**

Set $v_i = \min A$;

Let v^* be the smallest $v \in \mathrm{Sk}_V$ such that $v_i \in \partial^{\mathrm{in}}(\Delta + v^*)$;

Update $\mathrm{Sk}_V = \mathrm{Sk}_V \cup \{v_i\}$, $\mathrm{Sk}_E = \mathrm{Sk}_E \cup \{\{v^*, v_i\}\}$, $V = V \cup (v_i + \Delta')$, $i = i + 1$;

end

return $(\mathrm{Sk}_V, \mathrm{Sk}_E)$;

Algorithm 1: Coarse graining of a cluster containing 0.

The output $\mathrm{Sk}(C) = (\mathrm{Sk}_V(C), \mathrm{Sk}_E(C))$ of Algorithm 1 is a tree with $\mathrm{Sk}_V(C) \subset \mathbb{Z}^2$. use the shorthand $|\mathrm{Sk}(C)| := |\mathrm{Sk}_V(C)|$ for its size.

4.7.4 Energy-Entropy estimates

The next step is to establish energy bounds which control the probability to see a given tree as the skeleton of the cluster.

Lemma 4.7.2. *For any $0 \in A \subset \mathbb{Z}^2$, and $K \geq 1$,*

$$\Phi^{1,1}_{[\Delta_K \cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K \cap A} \partial^{\mathrm{in}} \Delta_K) \leq e^{-K(1+o_K(1))}, \tag{4.36}$$

where $o_K(1)$ is a quantity that goes to 0 as K goes to ∞.

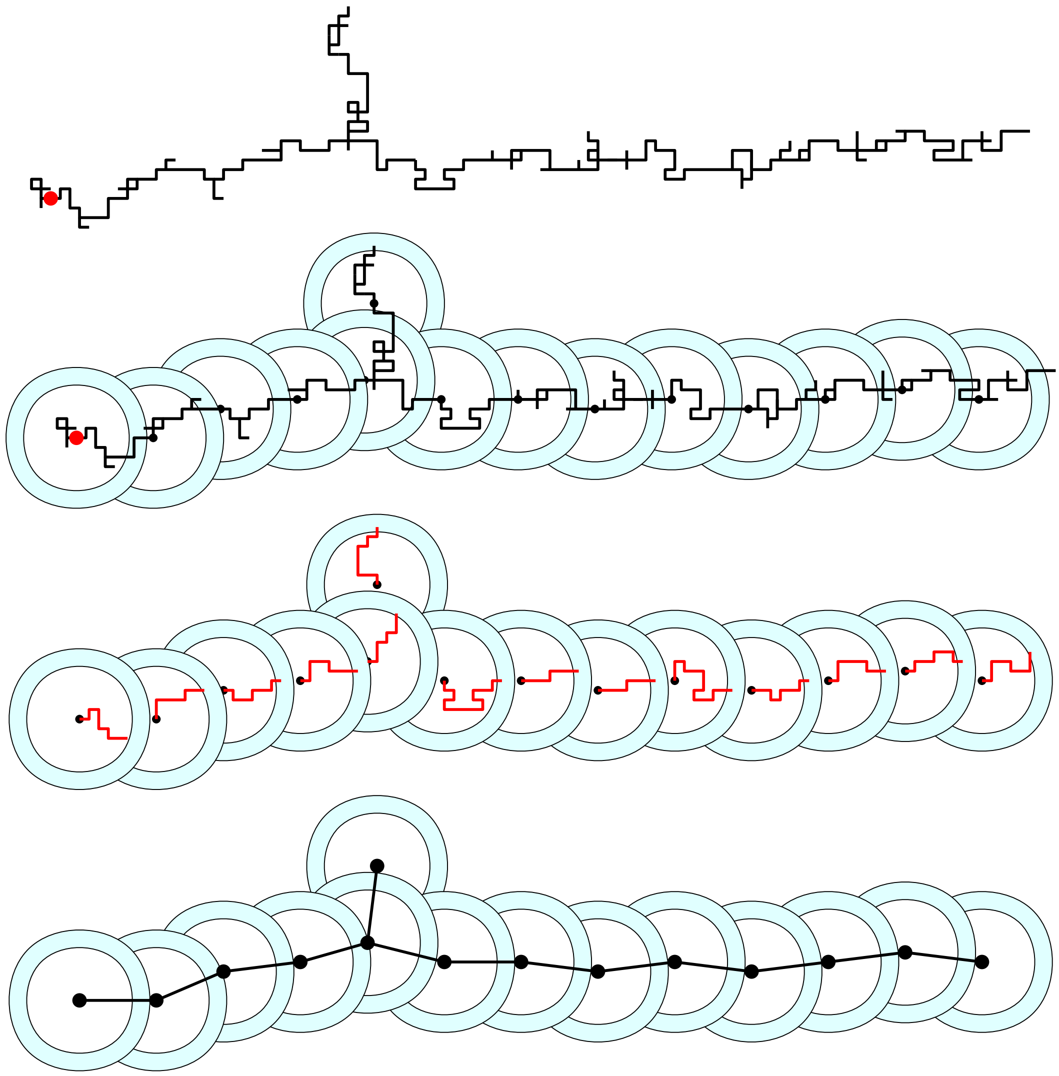


Figure 4.17: From top to bottom: a realization of $\mathcal{C}_0$ (0 is in red); the cells of the associated coarse-graining; the connections implied by the presence of the cells; the skeleton (in black).

Note that the wanted probabilities are zero whenever $A \cap \Delta_K$ does not contain a path going from 0 to $\partial^{\mathrm{in}}\Delta_K$.

Proof. By the definition of $\mathcal{U}$, the ratio weak mixing property (Theorem 16) and monotonicity:

$$\begin{aligned}\Phi^{1,1}_{[\Delta_K\cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K\cap A} \partial^{\mathrm{in}}\Delta_K) &\leq (1+CK^2e^{-c'\ln(K)^2})\Phi^{0,0}_{[\Delta_K\cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K\cap A} \partial^{\mathrm{in}}\Delta_K)\\ &\leq 2\Phi(0 \xleftrightarrow{\Delta_K} \partial^{\mathrm{in}}\Delta_K)\\ &\leq C\sum_{x\in\partial^{\mathrm{in}}\Delta_K} e^{-\nu(x)(1+o_K(1))} = C|\partial^{\mathrm{in}}\Delta_K|e^{-K(1+o_K(1))}\end{aligned}$$

as soon as K is large enough. □

As a direct consequence of Lemma 4.7.2 and of the definition of the coarse-graining procedure, one gets the following:

Lemma 4.7.3 (Energy bound). *There exists $K_0 \geq 0$ such that, for any $K \geq K_0$, and any $\mathcal{T} \subset \mathbb{Z}^2$,*

$$\Phi\big(\mathrm{Sk}_V(\mathcal{C}_0) = \mathcal{T}\big) \leq e^{-K|\mathcal{T}|(1+o_K(1))},$$

where the o_K is uniform over $\mathcal{T}$

Indeed, every new vertex of the tree away from the boundary induces a connection of the form (4.36) in the complement of the neighbourhood of the previously explored vertices (See Fig. 4.17). We do not provide further details, see for example [CIV08, Display (2.2)] for the implementation of the bound.

Recall that $\mathrm{Sk}(C)$ is a tree rooted at 0. Denote by Tree_N the set of all possible values of $\mathrm{Sk}(C)$ if $|\mathrm{Sk}(C)| = N$. We now state a general combinatorial lemma that bounds the size of Tree_N.

Lemma 4.7.4 (Entropy bound). *There exists a universal $c > 0$ such that*

$$|\mathrm{Tree}_N| \leq e^{c\ln(K)N} = K^{cN}.$$

This Lemma follows from the fact that, for some $C \geq 0$, the size of Tree_N is smaller or equal to the number of N-vertex connected sub-trees of the CK^2-regular tree containing 0. The latter is bounded by $e^{c\ln(K)N}$ for some $c > 0$ by Kesten's argument [Kes82, page 85].

4.7.5 Input from [CIV08, CIV03]: skeleton and cluster cone-points

The main result that we import from [CIV08, CIV03] is [CIV08, Theorem 2.1] that describes a typical geometry of skeletons ([CIV08] builds on [CIV03]). The result in [CIV08] is stated for the FK-percolation, but the proof is general and relies only on Lemmata 4.7.3 and 4.7.4.

Lemma 4.7.5 (Skeleton Cone-points). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $v \in \mathbb{Z}^2$, and let $\mathcal{C}$ be a random finite connected subset of $\mathbb{Z}^2$ containing v defined on $(\Omega, \mathcal{F}, P)$. Suppose that for any tree $\mathcal{T}$ in the image of Sk, $K \geq 0$,*

$$P(\mathrm{Sk}(\mathcal{C} - v) = \mathcal{T}) \leq e^{-K|\mathcal{T}|(1+o_K(1))}.$$

Then, for any $\delta > 0$, there are $c_1, c_2 > 0$, $K_0 \geq 0$ such that, for any $K \geq K_0$, $t \in \partial\mathcal{W}$, $w \in \mathbb{Z}^2$,

$$e^{t\cdot(w-v)}P\big(w \in \mathcal{C}, |\mathrm{CPts}_{t,\delta}(\mathrm{Sk}(\mathcal{C} - v))| \leq c_1|w - v|/K\big) \leq e^{-c_2|w-v|}.$$

Moreover, by monotonicity, c_1, c_2, K_0 can be taken uniform over $\delta \geq \delta_0 > 0$.

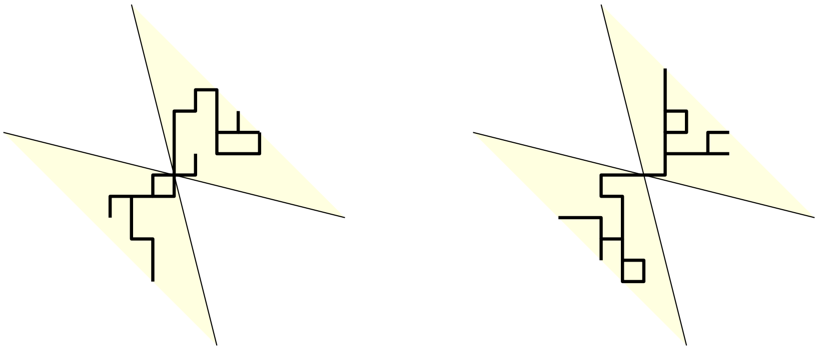


Figure 4.18: Left: a cone-point that is *not* regular. Right: a regular cone-point.

Remark 4.7.6. Note that the Lemma holds directly when w is not in $v+\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$: then $P(w\in\mathcal{C})\leq e^{-\nu(w-v)(1+o(1))}$ (as the event $w\in\mathcal{C}$ implies that $|\mathrm{Sk}(\mathcal{C})|\geq \nu(w-v)/K$) but $t\cdot(w-v)-\nu(w-v)\leq -\delta\nu(w-v)$ by definition of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$.

The second result we import is a simple but notationally heavy use of finite energy. One can find two different implementations of this argument in [CIV08, Section 2.9], and [AOV24, Section 6.1]. Introduce a small variation on the notion of cone-points which will be convenient later (one could work directly with cone-points, but the equations become a bit heavier).

Recall $\mathrm{e}_1=(1,0)$, $\mathrm{e}_2=(0,1)$.

Definition 9 (Regular Cone-points)**.** Let $C=(V,E)$ be a connected sub-graph of $(\mathbb{Z}^2,\mathbb{E})$. Say that $v\in V$ is a *regular* (t,δ)*-cone-point* of C (see Fig. 4.18) if it is a (t,δ)-cone-point of V and, if $\{\pm\mathrm{e}_1\}\cap\mathcal{Y}^{\blacktriangleleft}_{t,\delta}\neq\varnothing$, $\{v,v+\mathrm{e}_1\},\{v,v-\mathrm{e}_1\}\in E$ and $\{v,v+\mathrm{e}_2\},\{v,v-\mathrm{e}_2\}\notin E$. Denote $\mathrm{rCPts}_{t,\delta}(C)$ the set of (t,δ)-regular cone-points of C.

Note than when $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains exactly one element of $\{\pm\mathrm{e}_1,\pm\mathrm{e}_2\}$, all cone-points are necessarily regular.

The idea for going from Lemma 4.7.5 to the next lemma is simple: when $v\leftrightarrow w$, up to an exponentially small error, there must be at least $c|v-w|/K$ cone-points of the skeleton by Lemma 4.7.5. Up to anther exponentially small error, a positive fraction of these cone-points must then be regular cone-points of $\mathcal{C}$ by uniform finite energy.

Lemma 4.7.7 (Cluster Cone-points)**.** *Let $(\Omega,\mathcal{F},P)$ be probability space, $v\in\mathbb{Z}^2$, and $\omega:\Omega\to\{0,1\}^{\mathbb{E}}$ be a bond percolation random variable. The cluster of v in ω is denoted by $\mathcal{C}=\mathcal{C}(\omega)$ Suppose that*

- *ω has uniform finite energy (for opening and closing edges): for $e\in\mathbb{E}$, let $\mathcal{F}_{e^c}$ be the sigma-algebra generated by $(\omega_f)_{f\neq e}$. Then, there is $\epsilon\in(0,1)$ such that for every $e\in\mathbb{E}$,*

$$\epsilon<E(\omega_e\mid\mathcal{F}_{e^c})<1-\epsilon,$$

  P-almost surely.

- *the conclusion of Lemma 4.7.5 holds for $\mathcal{C}$.*

Then, for any $\delta>0$, there are $c_1',c_2'>0$, $L_0\geq 1$ such that, for any $t\in\partial\mathcal{W}$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm\mathrm{e}_1,\pm\mathrm{e}_2\}$, and any $w\in\mathbb{Z}^2\cap\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ with $|w-v|\geq L_0$,

$$e^{t\cdot(w-v)}P\big(w\in\mathcal{C},\ |\mathrm{rCPts}_{t,\delta}(\mathcal{C})|\leq c_1'|w-v|\big)\leq e^{-c_2'|v-w|}.$$

By monotonicity, c_1',c_2',L_0 can be taken uniform over $\delta\geq\delta_0>0$.

See [CIV08, Section 2.9] or [AOV24, Section 6.1] for two different proofs of Lemma 4.7.7 via local surgery arguments.

For these two lemmas we can deduce our main cone-point estimate.

Theorem 22. *Let $\delta \in (0,1)$. There exist $C \geq 0, c_1, c_2 > 0$ such that for any $t \in \partial\mathcal{W}$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm e_1, \pm e_2\}$, and any $x \in \mathbb{Z}^2$,*

$$e^{t\cdot x}\Phi\big(0 \leftrightarrow x, |\mathrm{rCPts}_{t,\delta}(\mathcal{C}_0)| \leq c_1|x|\big) \leq Ce^{-c_2|x|}.$$

Proof. When $x \notin \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$, the claim is trivial, see Remark 4.7.6. For $x \in \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$, the result follows from Lemmata 4.7.5, and 4.7.7: Lemma 4.7.3 provides the required bound on the probability of a given skeleton, and the model has finite energy, which are the needed hypotheses for Lemmata 4.7.5, and 4.7.7. □

This concludes our preparations. We are now ready to attack the proof of Theorem 21.

4.7.6 Proof of Theorem 21 part I: Cone-points and pre-renewal structure

We now fix $s_0 \in \mathbb{S}^1$, $t_0 \in \partial\mathcal{W}$ dual to s_0, $\delta \in (0,1)$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t_0,\delta}$ contains and element of $\{\pm e_1, \pm e_2\}$, and we set

$$\begin{aligned} \mathrm{rCPts} \equiv \mathrm{rCPts}_{t_0,\delta}, \quad \mathcal{Y}^{\blacktriangleleft} \equiv \mathcal{Y}^{\blacktriangleleft}_{t_0,\delta}, \quad \mathcal{Y}^{\blacktriangleright} \equiv \mathcal{Y}^{\blacktriangleright}_{t_0,\delta}, \\ \mathfrak{B}_L \equiv \mathfrak{B}_L(t_0,\delta), \quad \mathfrak{B}_R \equiv \mathfrak{B}_R(t_0,\delta), \quad \mathfrak{A} \equiv \mathfrak{A}(t_0,\delta). \end{aligned} \tag{4.37}$$

From Theorem 22, typical long clusters have many cone-points: having a non-linear (in $|x|$), number of cone-points is exponentially more unlikely than $\{x \in \mathcal{C}_0\}$. The idea is now to write a realization of $\mathcal{C}_0$ containing many cone-points as a concatenation of "irreducible graphs" by splitting it at its regular cone-points. This will lead to a structure that, graphically, looks like a renewal structure. This is what we will do in this subsection. The last step will then be to extract a *real* renewal structure (at the level of the measure) from this graphical one. This will be Subsection 4.7.7. To this end, introduce the notion of *irreducible graphs*. Let $e = \{\pm e_1\} \cap \mathcal{Y}^{\blacktriangleleft}$ if the intersection is not empty, and $e = \{\pm e_2\} \cap \mathcal{Y}^{\blacktriangleleft}$ else. Say that

- A marked backward-confined graph (γ_L, v^*) is *irreducible* if the diamond $\mathrm{Diamond}(v^*, \mathbf{b}(\gamma_L))$ does not contains a regular cone-point of γ_L, and $\mathbf{b}(\gamma_L)$ is a regular cone-point of the graph $\gamma_L \cup \{\mathbf{b}(\gamma_L), \mathbf{b}(\gamma_L) + e\}$ (see Fig. 4.19).
- A marked forward-confined graph (γ_R, v^*) is *irreducible* if the diamond $\mathrm{Diamond}(\mathbf{f}(\gamma_R), v^*)$ does not contains a regular cone-point of γ_R, and $\mathbf{f}(\gamma_R)$ is a regular cone-point of the graph $\gamma_R \cup \{\mathbf{f}(\gamma_R), \mathbf{f}(\gamma_R) - e\}$.
- A diamond-confined graph γ is *irreducible* if it does not contains a regular cone-point, and if $\mathbf{f}(\gamma), \mathbf{b}(\gamma)$ are regular cone-points of $\gamma \cup \{\{\mathbf{f}(\gamma), \mathbf{f}(\gamma) - e\}, \{\mathbf{b}(\gamma), \mathbf{b}(\gamma) + e\}\}$ (see Fig. 4.19).

In words: irreducible graphs are those that cannot be written as the concatenation of two non-trivial graphs, with the concatenation point being a regular cone-point, and so that the concatenation of two irreducible graphs leads to a regular cone-point. We will denote the sets of irreducible marked graphs in, respectively, $\mathfrak{B}_L$, $\mathfrak{B}_R$, $\mathfrak{A}$ by

$$\mathfrak{B}^{\mathrm{ir}}_L, \quad \mathfrak{B}^{\mathrm{ir}}_R, \quad \mathfrak{A}^{\mathrm{ir}}.$$

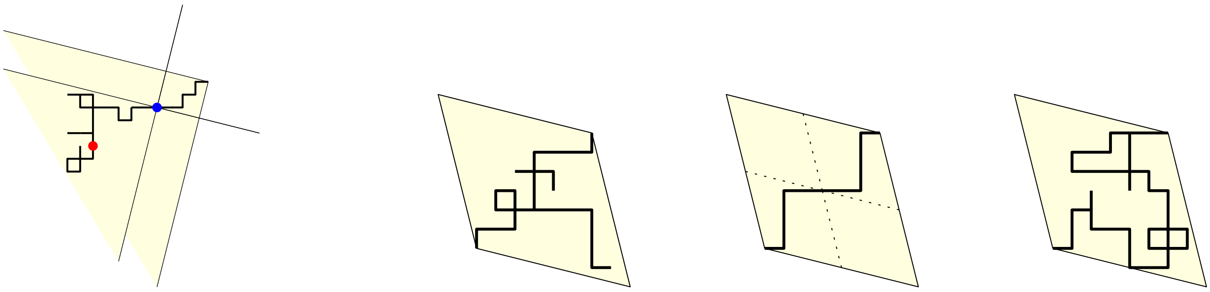


Figure 4.19: From left to right. 1) Backward confined graph which is not irreducible: the graph can be written as the concatenation of two graphs so that the concatenation point is a regular cone-point. 2) The graph is not irreducible as its endpoints will not generate regular cone-points when concatenated. 3) The graph is not irreducible as it contains a regular cone-point. 4) An irreducible graph.

For $x \in \mathcal{Y}^{\blacktriangleleft} \cap \mathbb{Z}^2$, and $C_0 \ni x$ a realization of $\mathcal{C}_0$ containing at least two regular cone points v_1, v_2 with $0 \in v_i + \mathcal{Y}^{\blacktriangleright}$, $x \in v_i + \mathcal{Y}^{\blacktriangleleft}$, we can introduce the splitting into irreducible components:

$$C_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R,$$

where $M \geq 1$, $(\eta_L, 0)$, (η_R, x), $\eta_1, \ldots, \eta_M$ are all irreducible, confined, (marked) graphs, and $\sqcup$ means disjoint union of edges (there are sites overlap at cone-points). Now, as mentioned in the end of Section 4.7.1, there is a bijection between pairs $(\tilde{\gamma}, v) \in \mathfrak{A} \times \mathbb{Z}^2$ and diamond-confined connected graphs (translate $\tilde{\gamma}$ by v to obtained the graph $\gamma = v + \tilde{\gamma}$). Similar considerations hold for marked forward/backward confined connected graphs. In particular, for η_i in the above decomposition, there is a unique $w_i \in \mathbb{Z}^2$ and a unique $\tilde{\eta}_i \in \mathfrak{A}^{\mathrm{ir}}$ such that $\eta_i = w_i + \tilde{\eta}_i$. Similarly for η_L, η_R. As the marked point of η_L is 0, one has directly

$$w_L = 0, \quad w_1 = \mathrm{X}(\tilde{\eta}_L), \quad w_2 = \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1), \ldots \quad w_R = \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_M),$$

and the equivalent writing of C_0:

$$\eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R = \tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_M \circ \tilde{\eta}_R, \qquad \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_R) = x,$$

with $\tilde{\eta}_1, \ldots, \tilde{\eta}_M \in \mathfrak{A}^{\mathrm{ir}}$, $\tilde{\eta}_L \in \mathfrak{B}_L^{\mathrm{ir}}$, and $\tilde{\eta}_R \in \mathfrak{B}_R^{\mathrm{ir}}$.

By our definition of "regular cone points", each $\tilde{\eta} \in \mathfrak{A}^{\mathrm{ir}}$ contains a unique edge incident to $\mathbf{f}(\tilde{\eta})$, and the same for $\mathbf{b}(\tilde{\eta})$. Denote these edges $\mathbf{f}_e(\tilde{\eta}), \mathbf{b}_e(\tilde{\eta})$.

For $\tilde{\eta}, \tilde{\eta}_L, \tilde{\eta}_R$ as above, and $v \in \mathbb{Z}^2$, introduce the percolation events $A_v(\tilde{\eta})$ to be the event that

- the edges in $(v + \tilde{\eta} \setminus \{\mathbf{f}_e(\tilde{\eta}), \mathbf{b}_e(\tilde{\eta})\})$ are open in ω_τ,
- $v + \mathbf{f}_e(\tilde{\eta})$, and $v + \mathbf{b}_e(\tilde{\eta})$ are open in $\omega_{\tau\tau'}$ and closed in ω_τ,
- $\partial^{\mathrm{ex}}(v + \tilde{\eta})$ are closed in ω_τ.

Define the edges $\mathbf{b}_e(\tilde{\eta}_L)$, $\mathbf{f}_e(\tilde{\eta}_R)$, and the events $A_v(\tilde{\eta}_L)$, $A_v(\tilde{\eta}_R)$ similarly. See Fig. 4.20.

Claim 1. *One has*

$$\Phi(\mathcal{C}_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R) = \Big(\frac{\sinh(2J)}{e^{-2J} - e^{-2U}}\Big)^{2(M+1)} \Phi\big(A_{w_L}(\tilde{\eta}_L), A_{w_1}(\tilde{\eta}_1), \ldots, A_{w_R}(\tilde{\eta}_R)\big).$$

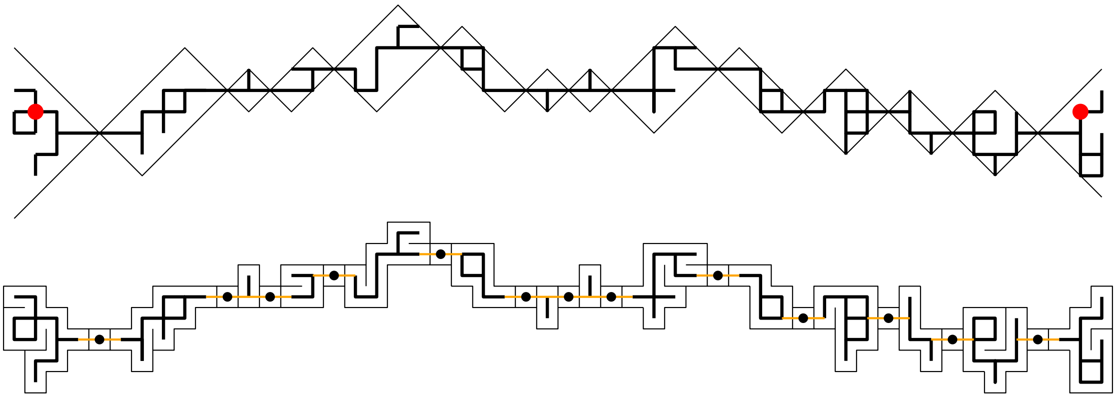


Figure 4.20: Top: A cluster with its irreducible decomposition. Bottom: the corresponding percolation events. Bold black edges are open in ω_τ (hence also in $\omega_{\tau\tau'}$); golden edges are closed in ω_τ but open in $\omega_{\tau\tau'}$; thin lines are the dual of edges that are closed in ω_τ, but for which the state of $\omega_{\tau\tau'}$ is not prescribed.

Proof. Under $\mathcal{C}_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R$, every edge of the form $\mathbf{b}_e, \mathbf{f}_e$ always separates two distinct clusters of ω_τ. Swapping the state of such an edge from open to close in ω_τ (while keeping $\omega_{\tau\tau'}$ open) brings a factor

$$\frac{\mathrm{w}_\tau}{2\mathrm{w}_{\tau\tau'}} = \frac{2e^{2U}\sinh(2J)}{2(e^{2(U-J)}-1)},$$

by definition of the measure: closing e in ω_τ creates a cluster in ω_τ, hence the factor 2 in the denominator, and swaps the value of $(\omega_\tau(e), \omega'_{\tau\tau}(e))$ from $(1,1)$, which has weight w_τ, to $(0,1)$, which has weight $\mathrm{w}_{\tau\tau'}$. The claim follows from the definition of $\mathrm{w}_\tau, \mathrm{w}_{\tau\tau'}$, see (4.6), the fact that there are $2(M+1)$ edges of the form $\mathbf{b}_e, \mathbf{f}_e$ (2 per regular cone-point, $M+1$ regular cone-points), and that the event obtained from $\{\mathcal{C}_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R\}$ by closing the edges of the form $\mathbf{b}_e, \mathbf{f}_e$ is precisely $A_{w_L}(\tilde\eta_L) \cap A_{w_1}(\tilde\eta_1) \cap \cdots \cap A_{w_R}(\tilde\eta_R)$. □

For $\tilde\eta_L \in \mathfrak{B}_L^{\mathrm{ir}}$, and $\tilde\eta \in \mathfrak{A}^{\mathrm{ir}} \cup \mathfrak{B}_R^{\mathrm{ir}}$, introduce the conditional weights

$$\begin{aligned} q_L(\tilde\eta_L) &:= \Phi\big(A_{w_L}(\tilde\eta_L)\big), \\ q(\tilde\eta \mid \tilde\eta_L, \tilde\eta_1, \dots, \tilde\eta_k) &:= c_{J,U}\Phi\big(A_{\mathrm{X}(\bar\eta)}(\tilde\eta) \,\big|\, A_{w_L}(\tilde\eta_L), A_{w_1}(\tilde\eta_1), \dots, A_{w_k}(\tilde\eta_k)\big), \end{aligned} \tag{4.38}$$

where $\bar\eta = \tilde\eta_L \circ \tilde\eta_1 \circ \ldots \circ \tilde\eta_k$, and $c_{J,U} = \big(\frac{\sinh(2J)}{e^{-2J}-e^{-2U}}\big)^2$.
Using Claim 1, this leads to the following conditional weight decomposition:

$$\begin{aligned} &\Phi(\mathcal{C}_0 = \tilde\eta_L \circ \tilde\eta_1 \circ \ldots \circ \tilde\eta_M \circ \tilde\eta_R) \\ &\qquad = q_L(\tilde\eta_L) q(\tilde\eta_1 \mid \tilde\eta_L) q(\tilde\eta_2 \mid \tilde\eta_L, \tilde\eta_1) \ldots q(\tilde\eta_M \mid \tilde\eta_L, \ldots, \tilde\eta_{M-1}). \end{aligned} \tag{4.39}$$

Let us describe informally the second part of the proof: one has represented the probability of a given cluster participating to the event $0 \leftrightarrow x$ as a product of *dependent* weights, which one can see as a product of conditional kernels (which are *not* probability kernels). The idea is now to represent this product of dependent kernels as a mixture of *independent* (factorized) kernels defined on diamond-confined graphs, which will then be suitably normalize by a factor $e^{t_0\cdot \mathrm{X}(\gamma)}$ to obtain probability kernels.

4.7.7 Proof of Theorem 21 part II: Mixing of weights and renewal structure

We will use the same procedure as [AOV24, Section 7] with the tricks from [OV18, Appendix C] to compensate for the lack of monotonicity in the conditional kernels. We only describe the needed inputs, and will refer to [AOV24, Section 7] once we arrive at a stage where the remaining arguments is a copy-pasting of [AOV24, Section 7].

The first step is to prove that the conditional kernels of (4.39) have good mixing properties. This is the content of the next lemma.

Lemma 4.7.8. *With the notations and definitions of Subsection 4.7.6, there exist $\rho > 0, C \geq 0, c > 0$, such that*

- *for every $\tilde{\eta} \in \mathfrak{B}_R^{\mathrm{ir}} \cup \mathfrak{B}_R^{\mathrm{ir}}$, one has that*

$$\inf_{n\geq 0} \inf_{\tilde{\eta}_0 \in \mathfrak{B}_L^{\mathrm{ir}}} \inf_{\tilde{\eta}_1,\dots,\tilde{\eta}_n \in \mathfrak{A}^{\mathrm{ir}}} q(\tilde{\eta} \mid \tilde{\eta}_0, \dots, \tilde{\eta}_n) \geq \rho^{|\tilde{\eta}|}; \tag{4.40}$$

- *for any $n, k, k' \geq 0$, any $\tilde{\eta}_0, \tilde{\eta}_0' \in \mathfrak{B}_L^{\mathrm{ir}}$, any $\tilde{\eta} \in \mathfrak{B}_R^{\mathrm{ir}} \cup \mathfrak{A}^{\mathrm{ir}}$, and any $\tilde{\eta}_1, \dots, \tilde{\eta}_{k+n}, \tilde{\eta}_1', \dots, \tilde{\eta}_{k'} \in \mathfrak{A}^{\mathrm{ir}}$*

$$\Big| \frac{q\big(\tilde{\eta} \,\big|\, \tilde{\eta}_0, \tilde{\eta}_1, \dots, \tilde{\eta}_k, \tilde{\eta}_{k+1}, \dots, \tilde{\eta}_{k+n}\big)}{q\big(\tilde{\eta} \,\big|\, \tilde{\eta}_0', \tilde{\eta}_1', \dots, \tilde{\eta}_{k'}', \tilde{\eta}_{k+1}, \dots, \tilde{\eta}_{k+n}\big)} - 1 \Big| \leq C e^{-cn}. \tag{4.41}$$

Proof. The first point is by definition of q, see (4.38), and finite energy for Φ: the support of the event $A_{\mathrm{X}(\tilde{\eta})}(\tilde{\eta})$ in (4.38) is of size at most $4|\tilde{\eta}|$. Focus on the second point. We implicitly always work in a large finite volume, $\Lambda_N = \{-N, \dots, N\}^2$, with $0, 1$ boundary conditions and take limits (everything being uniform over the large enough volume). Let $\tilde{\eta}_{k'+i}' \equiv \tilde{\eta}_{k+i}$ for $i = 1, \dots n$. For $l \geq 0$, let

$$v_l = \mathrm{X}(\tilde{\eta}_l), \quad v_l' = \mathrm{X}(\tilde{\eta}_l'), \quad w_l = \mathrm{X}(\tilde{\eta}_0 \circ \tilde{\eta}_1 \dots \circ \tilde{\eta}_l), \quad w_l' = \mathrm{X}(\tilde{\eta}_0' \circ \tilde{\eta}_1' \dots \circ \tilde{\eta}_l').$$

Let also

$$u_l = w_{k+l} - w_k = \mathrm{X}(\tilde{\eta}_{k+1} \circ \dots \circ \tilde{\eta}_{k+l}) = w_{k'+l}' - w_{k'}'.$$

For $v \in \mathbb{Z}^2$, introduce the (translations of) the events corresponding to a given chain of irreducible graphs:

$$B_{L,v} = A_v(\tilde{\eta}_0) \cap \bigcap_{i=1}^{k} A_{v+w_{i-1}}(\tilde{\eta}_i), \quad B_{L,v}' = A_v(\tilde{\eta}_0') \cap \bigcap_{i=1}^{k'} A_{v+w_{i-1}'}(\tilde{\eta}_i'),$$
$$B_v = \bigcap_{l=1}^{n} A_{v+u_l}(\tilde{\eta}_{k+l}), \quad B_{R,v} = A_{v+u_n}(\tilde{\eta}).$$

Now, that by definition of q (see (4.38)),

$$\frac{q\big(\tilde{\eta} \,\big|\, \tilde{\eta}_0, \tilde{\eta}_1, \dots, \tilde{\eta}_k, \tilde{\eta}_{k+1}, \dots, \tilde{\eta}_{k+n}\big)}{q\big(\tilde{\eta} \,\big|\, \tilde{\eta}_0', \tilde{\eta}_1', \dots, \tilde{\eta}_{k'}', \tilde{\eta}_{k+1}, \dots, \tilde{\eta}_{k+n}\big)}$$
$$= \frac{\Phi\big(A_{w_{k+n}}(\tilde{\eta}) \,\big|\, A_0(\tilde{\eta}_0), A_{w_0}(\tilde{\eta}_1), \dots, A_{w_{k-1}}(\tilde{\eta}_k)\big)}{\Phi\big(A_{w_{k'+n}'}(\tilde{\eta}) \,\big|\, A_0(\tilde{\eta}_0), A_{w_0'}(\tilde{\eta}_1), \dots, A_{w_{n+k'-1}'}(\tilde{\eta}_{n+k'}')\big)} = \frac{\Phi\big(B_{R,w_k} \,\big|\, B_{L,0} \cap B_{w_k}\big)}{\Phi\big(B_{R,w_k'} \,\big|\, B_{L,0}' \cap B_{w_{k'}'}\big)}.$$

Then, using translation invariance of Φ,

$$\frac{\Phi\big(B_{R,w_k}\,\big|\,B_{L,0}\cap B_{w_k}\big)}{\Phi\big(B_{R,w'_k}\,\big|\,B'_{L,0}\cap B_{w'_{k'}}\big)}=\frac{\Phi\big(B_{R,0}\,\big|\,B_{L,-w_k}\cap B_0\big)}{\Phi\big(B_{R,0}\,\big|\,B'_{L,-w'_{k'}}\cap B_0\big)}$$
$$=\frac{\Phi\big(B_{R,0}\cap B_{L,-w_k}\,\big|\,B_0\big)}{\Phi\big(B_{R,0}\,\big|\,B_0\big)\Phi\big(B_{L,-w_k}\,\big|\,B_0\big)}\frac{\Phi\big(B_{R,0}\,\big|\,B_0\big)\Phi\big(B'_{L,-w'_{k'}}\,\big|\,B_0\big)}{\Phi\big(B'_{L,-w'_{k'}}\cap B_{R,0}\,\big|\,B_0\big)}. \tag{4.42}$$

Now, $B_{R,0}$ is supported on the edges with at least one endpoint in $u_n+\mathcal{Y}^{\blacktriangleleft}$, denoted $E_{\triangleleft}(u_n)$, whilst $B_{L,-w_k}, B'_{L,-w'_{k'}}$ are supported on the edges with at least one endpoint in $-\mathcal{Y}^{\blacktriangleleft}$, denoted $E_{\triangleright}$. Looking at (4.42), (4.41) will follow from

1. uniform upper and lower bounds on (4.42) (which deals with the small n case),
2. a suitable form of ratio mixing for $P(\cdot):=\Phi(\cdot\mid B_0)$ (which deals with the case n large enough).

Start with the uniform bound on the ratios.

Claim 2. *There exists $c\in[1,\infty)$ such that for any $n\geq 0$, any $v\in\mathbb{Z}^2$, any $\tilde{\eta}_1,\dots,\tilde{\eta}_n\in\mathfrak{A}^{\mathrm{ir}}$ with $\mathrm{X}(\tilde{\eta}_1\circ\ldots\circ\tilde{\eta}_n)=v$, any $F_{\triangleright}\subset E_{\triangleright}$, $F_{\triangleleft}\subset E_{\triangleleft}(v)$ finite, and any $\alpha\in\{(0,0),(0,1),(1,1)\}^{F_{\triangleleft}}$,*

$$\frac{1}{c}\leq\inf_{\xi,\xi'}\frac{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi)}{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi')}\leq\sup_{\xi,\xi'}\frac{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi)}{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi')}\leq c,$$

where the sup/inf are over ξ,ξ' having positive probability, $P=\Phi(\ \mid B_0)$, and $Y_F=(\omega_\tau|_F,\omega_{\tau\tau'}|_F)$.

Proof. By symmetry of the expression, it is sufficient to prove the upper bound. Let $M\geq 1$ be a large enough integer and $\Lambda_M=\{-M,\dots,M\}^2$, $E_M=\mathbb{E}_{\Lambda_M}$. Let ξ,ξ' have positive P-probability. Then, there are non-empty events D_L,D_{L0}, D'_L,D'_{L0}, D_{B0},D_{BR}, D_{R0},D_R with D_L,D'_L supported on $E_{\triangleright}\setminus\mathbb{E}_{\Lambda_M}$, D_{RB},D_R supported on $E_{\triangleleft}\setminus\mathbb{E}_{\Lambda_M}$, and $D_{L0},D'_{L0},D_{B0},D_{R0}$ supported on $\mathbb{E}_{\Lambda_M}$, such that

$$B_0=D_{B0}\cap D_{BR},\quad\{Y_{F_{\triangleleft}}=\alpha\}=D_{R0}\cap D_R,$$
$$\{Y_{F_{\triangleright}}=\xi\}=D_L\cap D_{L0},\quad\{Y_{F_{\triangleright}}=\xi'\}=D'_L\cap D'_{L0}.$$

Now, by finite energy applied to lower bound the probability of the events with supported in $\mathbb{E}_{\Lambda_M}$ conditionally on the others, one has that for some $c\in(0,\infty)$,

$$\begin{aligned}\frac{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi)}{P(Y_{F_{\triangleleft}}=\alpha\mid Y_{F_{\triangleright}}=\xi')}&\\ &=\frac{\Phi(D_{R0}\cap D_R\cap D_{L0}\cap D_L\cap D_{B0}\cap D_{BR})\Phi(D'_{L0}\cap D'_L\cap D_{B0}\cap D_{BR})}{\Phi(D_{L0}\cap D_L\cap D_{B0}\cap D_{BR})\Phi(D_{R0}\cap D_R\cap D'_{L0}\cap D'_L\cap D_{B0}\cap D_{BR})}\\ &\leq\frac{\Phi(D_R\cap D_L\cap D_{BR})\Phi(D'_L\cap D_{BR})}{\Phi(D_L\cap D_{BR})\Phi(D_R\cap D'_L\cap D_{BR})}e^{cM^2}.\end{aligned}$$

But, by exponential ratio weak mixing (Theorem 16), for M large enough (depending only on the angular aperture of $\mathcal{Y}^{\blacktriangleleft}$),

$$\frac{\Phi(D_L\cap D_R\cap D_{BR})}{\Phi(D'_L\cap D_R\cap D_{BR})}\leq\frac{\Phi(D_L)\Phi(D_R\cap D_{BR})}{\Phi(D'_L)\Phi(D_R\cap D_{BR})}\frac{1+e^{-cM}}{1-e^{-cM}}\leq 2\frac{\Phi(D_L)}{\Phi(D'_L)},$$

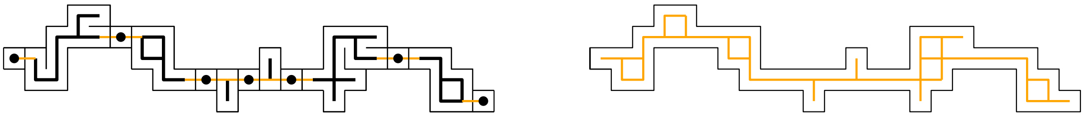


Figure 4.21: Black edges have state $(1,1)$, gold ones have state $(0,1)$, thin lines are dual to edges where ω_τ is 0. Left: the percolation event one conditions on. Right: the boundary conditions it effectively imposes.

and

$$\frac{\Phi(D'_L\cap D_{BR})}{\Phi(D_L\cap D_{BR})}\leq\frac{\Phi(D'_L)\Phi(D_{BR})}{\Phi(D_L)\Phi(D_{BR})}\frac{1+e^{-cM}}{1-e^{-cM}}\leq 2\frac{\Phi(D'_L)}{\Phi(D_L)}.$$

This concludes the proof of the claim. □

Divide the proof of ratio mixing into two claims. We start by proving a mixing bound for P. We will use the following observation: let γ be the simple closed dual path surrounding the connected graph $\tilde{\eta}_1\circ\ldots\circ\tilde{\eta}_n$. Let $*\gamma$ be the set of primal edges that are crossed by γ, and denote V_γ the set of sites surrounded by γ. Then,

$$P|_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}=\mathtt{ATRC}^{1,1}_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}\big(\ \big|\omega_\tau|_{*\gamma}=0\big)=\mathtt{ATRC}^{0,1}_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}\big(\ \big|\omega_\tau|_{*\gamma}=0\big),\tag{4.43}$$

where $\omega_\tau|_{*\gamma}$ is the restriction of ω_τ to $*\gamma$, see Fig. 4.21.

For $\tilde{\eta}_1,\ldots,\tilde{\eta}_n$, $u_1,\ldots,u_n$ as before, let $E_{\epsilon,\times}(\tilde{\eta}_1,\ldots,\tilde{\eta}_n)$ be the edges with both endpoints in $\mathbb{R}^2\setminus\big((u_{\lceil n/3\rceil}+\mathcal{Y}^{\blacktriangleright}_{t_0,\delta+\epsilon})\cup(u_{\lfloor 2n/3\rfloor}+\mathcal{Y}^{\blacktriangleleft}_{t_0,\delta+\epsilon})\big)$ (t_0,δ are fixed in (4.37)).

Claim 3. *For any $\epsilon>0$, there exist $n_0\geq 1$, $C\geq 0,c>0$ such that, for any $n\geq n_0$, any $\tilde{\eta}_1,\ldots,\tilde{\eta}_n\in\mathfrak{A}^{\mathrm{ir}}$, any finite sets of edges $F_\triangleright\subset E_\triangleright$, $F_\triangleleft\subset E_\triangleleft$, $F_\times\subset E_{\epsilon,\times}(\tilde{\eta}_1,\ldots,\tilde{\eta}_n)$, any $F_1\in\{F_\triangleright,F_\triangleleft,F_\times\}$, and any $\xi,\xi'\in\{(0,0),(0,1),(1,1)\}^{F_2}$, where $F_2=(F_\triangleright\cup F_\triangleleft\cup F_\times)\setminus F$,*

$$\mathrm{d_{TV}}\big(P(X_{F_1}\in\cdot\mid X_{F_2}=\xi),P(X_{F_1}\in\cdot\mid X_{F_2}=\xi')\big)\leq Ce^{-cn},$$

where P,X are defined as in Claim 2.

Proof. The idea is simple but the details of the percolation estimate are a bit tedious to write down, so we only sketch the percolation argument. Looking at (4.43), $P|_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}$ has the l-path decoupling property of Section 4.6.1 (it falls in the family of ATRC measure for which Theorem 20 applies). One can therefore use Theorem 20 to obtain the claim. Indeed, one can bound the wanted total variation distance with a connection probability in a Bernoulli block percolation model. The model one compares with is at worse of the type depicted on Fig. 4.22. The $n/3$ cone-points between the different regions each provide an opportunity to disconnect F_1 from F_2: each induce a bottleneck in the orange chain illustrated on Fig. 4.23, and each bottleneck gives a positive probability to disconnect F_1 from F_2. □

We then turn this into ratio mixing using Appendix 4.A.1.

Claim 4. *There exists $C\geq 0,c>0,n_0\geq 1$ such that for any $n\geq n_0$, any $v\in\mathbb{Z}^2$, any $\tilde{\eta}_1,\ldots,\tilde{\eta}_n\in\mathfrak{A}^{\mathrm{ir}}$ with $\mathrm{X}(\tilde{\eta}_1\circ\ldots\circ\tilde{\eta}_n)=v$, any $F_\triangleright\subset E_\triangleright$, $F_\triangleleft\subset E_\triangleleft(v)$ finite, and any $\alpha\in\{(0,0),(0,1),(1,1)\}^{F_\triangleleft}$,*

$$\sup_{\xi,\xi'}\left|\frac{P(X_{F_\triangleleft}=\alpha\mid X_{F_\triangleright}=\xi)}{P(X_{F_\triangleleft}=\alpha\mid X_{F_\triangleright}=\xi')}-1\right|\leq Ce^{-cn},$$

where the sup is over ξ,ξ' having positive probability, P,X are defined as in Claims 2, and 3.

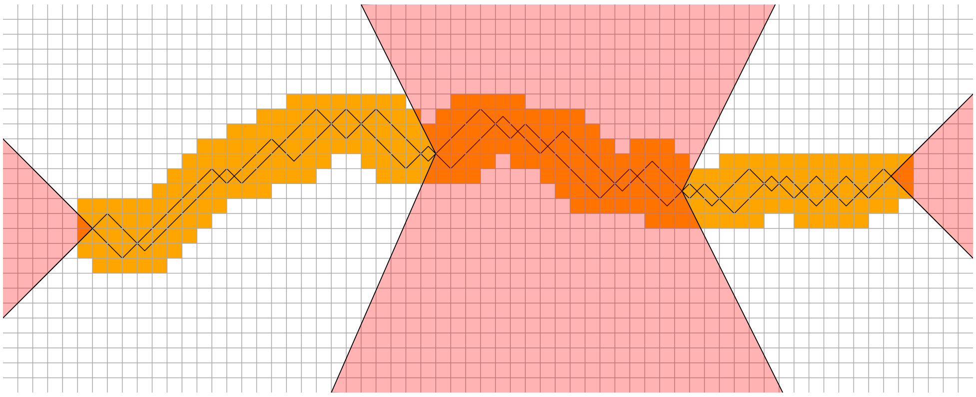


Figure 4.22: The grey grid represent the blocks of the block-percolation of Theorem 20. The orange squares have a strictly smaller than one probability to be open. The white squares have a probability as small as wanted to be open. The red regions represent $E_{\triangleright}, E_{\epsilon,\times}, E_{\triangleleft}$.

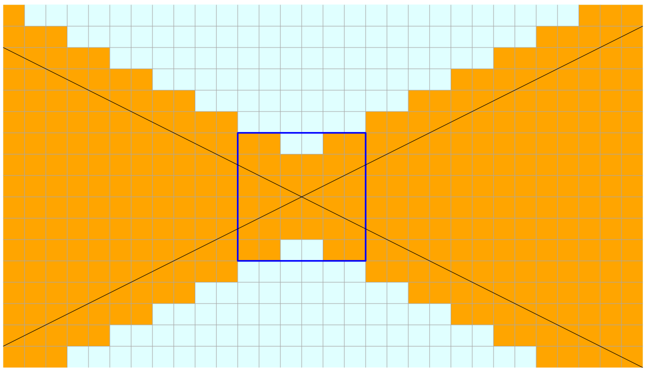


Figure 4.23: The grey grid represent the blocks of the block-percolation of Theorem 20. The orange squares have a strictly smaller than one probability to be open. The light blue squares have a probability as small as wanted to be open. By exponential decay in the blue blocks, a positive density of bottleneck have the property that the two orange sides of the blue square are not connected in the blue blocks. Then, with positive probability, the blocks in the square are closed, cutting the orange chain at that bottleneck.

Proof. The proof will be an application of Lemma 4.A.1. Recall (4.43).

Let $1-\delta > \epsilon > 0$, and let $E_\times = E_{\epsilon,\times}(\tilde{\eta}_1,\dots,\tilde{\eta}_n)$. We apply Lemma 4.A.1 with

- $\Omega_1 = \{(0,0),(0,1),(1,1)\}^{F_\triangleleft}$, $\Omega_2 = \{(0,0),(0,1),(1,1)\}^{E_\times}$,
- $\mu = P(\ \mid X_{F_\triangleright} = \xi)|_{F_\triangleleft \sqcup E_\times}$, $\nu = P(\ \mid X_{F_\triangleright} = \xi')|_{F_\triangleleft \sqcup E_\times}$,
- $D \subset \Omega_2$ is the event that there is an open path in $\omega_{\tau\tau'}$ from the top boundary of Λ_N (the implicit finite volume we work in) to its bottom boundary staying in $E_\times$, and that there is a dual path of open edges in ω_τ^* from the top boundary of Λ_N to its bottom boundary using only edges dual to edges in $E_\times$.

The hypotheses of Lemma 4.A.1 hold with $\epsilon = e^{-cn}$ (when n is large enough) by Claim 3, and the uniform exponential decay of $\omega_{\tau\tau'}^*, \omega_\tau$ (Theorem 17). □

This concludes the proof of (4.41). □

This Lemma is the needed input to put ourself in the framework of [AOV24, Section 7]. Let us describe the needed modifications (which all take place in the beginning of the argument). The main difference is that our conditional weights q are not monotonic, we thus need to define the p_k's used there in a slightly different way.

For $k > l \geq 1$, $\tilde{\eta}_0 \in \mathfrak{B}_L^{\mathrm{ir}}$, $\tilde{\eta}_1,\dots,\tilde{\eta}_{k-1} \in \mathfrak{A}^{\mathrm{ir}}$, and $\tilde{\eta} \in \mathfrak{A}^{\mathrm{ir}} \cup \mathfrak{B}_R^{\mathrm{ir}}$, set

$$a_0(\tilde{\eta} \mid \tilde{\eta}_0,\dots,\tilde{\eta}_{k-1}) \equiv a_0(\tilde{\eta}) = \inf_{n\geq 0} \inf_{\zeta_0 \in \mathfrak{B}_L^{\mathrm{ir}}} \inf_{\zeta_1,\dots,\zeta_n \in \mathfrak{A}^{\mathrm{ir}}} q\big(\tilde{\eta} \,\big|\, \zeta_0,\zeta_1,\dots,\zeta_n\big),$$

$$a_k(\tilde{\eta} \mid \tilde{\eta}_0,\dots,\tilde{\eta}_{k-1}) = q(\tilde{\eta} \mid \tilde{\eta}_0,\dots,\tilde{\eta}_{k-1}),$$

and

$$\begin{aligned} a_l(\tilde{\eta} \mid \tilde{\eta}_0,\dots,\tilde{\eta}_{k-1}) &\equiv a_l(\tilde{\eta} \mid \tilde{\eta}_{k-l},\dots,\tilde{\eta}_{k-1}) \\ &= \inf_{n\geq 0} \inf_{\zeta_0 \in \mathfrak{B}_L^{\mathrm{ir}}} \inf_{\zeta_1,\dots,\zeta_n \in \mathfrak{A}^{\mathrm{ir}}} q\big(\tilde{\eta} \,\big|\, \zeta_0,\dots,\zeta_n,\tilde{\eta}_{k-l},\dots,\tilde{\eta}_{k-1}\big) \end{aligned}$$

In worlds: a_k records the minimal mass given by a conditional probability to an irreducible graph for a fixed frozen "recent" past, and any "less recent" possible past. Introduce then the mass increments (p_k has the same definition domain as a_k)

$$p_0 = a_0, \quad p_k = a_k - a_{k-1}.$$

Note that by definition of the a_k's, the p_k's are always non-negative. This property is what compensates the lack of monotonicity of $q(\tilde{\eta} \mid \tilde{\eta}_0,\dots,\tilde{\eta}_{k-1})$ (monotonicity is property P6 there). One can now duplicate [AOV24, Section 7] to conclude the proof of Theorem 21 with the following adaptations: use the p_k's defined above in place of the ones defined in [AOV24], and use Lemma 4.7.8 as a replacement for [AOV24, properties P5, P7] (these are the only properties used in the argument).

4.8 Invariance principle for the modified ATRC cluster

4.8.1 Notations and main result of the section

As in Section 4.7, we will describe the geometry of the cluster of v_L under $\mathtt{mATRC}_{n,m}(\ \mid v_L \leftrightarrow v_R)$ (defined in Section 4.4) using a coupling with a random walk bridge in such a way that the cluster is included in diamonds with endpoints at the random walk steps. We will focus on direction e_1

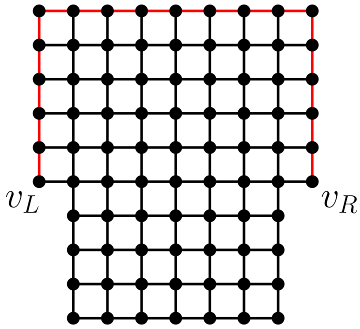


Figure 4.24: $\bar{V}_{n,m}$ (black circles), $E_{n,m}$ (black lines), and E_b^+ (red lines).

for simplicity, but the analysis can easily be adapted to other directions at the cost of slightly heavier notations. For notational convenience, we define the relevant objects with explicit n, m dependency, but will only stress the m dependence, as the n one is obvious (n with therefore usually be omitted from the notation).

Recall the setup of Section 4.4: let $\Lambda_{n,m} = \{-n, \dots, n\} \times \{-m, \dots, m\}$,

$$\begin{aligned} E_{n,m} &= \mathbb{E}_\Lambda \cup \{e \in \partial^{\mathrm{edge}}\mathbb{E}_\Lambda :\ e \subset \mathbb{H}_+\}, \\ \bar{V}_{n,m} &= \mathbb{V}_{E_{n,m}} \cup \{(-n-1, m+1), (n+1, m+1)\}, \\ E_b^+ &= \big\{\{i,j\} \in \mathbb{E} :\ i, j \notin \Lambda, i, j \in \bar{V}_{n,m}\big\}, \end{aligned}$$

see Fig. 4.24.

Recall also

$$\partial_-\Lambda = \{i \in \partial^{\mathrm{in}}\Lambda :\ i \in \mathbb{H}_-\}.$$

As in the previous section, $\Phi \equiv \mathtt{ATRC}$. We will denote

$$\Phi_m \equiv \Phi_{n,m} = \mathtt{mATRC}_{n,m}.$$

Φ_m is supported on $\Omega_m \equiv \Omega_{n,m} = \{(0,0), (0,1), (1,1)\}^{E_m} \times \{(0,0), (1,1)\}^{E_b^+}$. Denote also for $F \subset E_m \cup E_b^+$, and $(a, b) \in \Omega_m$,

$$\Phi^{a,b}_{m;F} := \Phi_m\big(\ \mid \omega_\tau(e) = a_e, \omega_{\tau\tau'}(e) = b_e \,\forall e \in F^c\big)|_F \tag{4.44}$$

its conditional version. Recall its main properties (we will use them without referring to a precise statement each time):

1. The probability of a given pair $(a, b) \in \Omega_m$ is given by

$$\Phi_m(a, b) \propto \prod_{e \in E} 2^{a_e} (\mathbf{c} - 2)^{b_e - a_e} \prod_{e \in E_b^+} \Big(\frac{2}{\mathbf{c}_b - 1}\Big)^{a_e} \cdot 2^{\kappa_+(a)} \prod_{C \in \mathrm{cl}(b)} \big(\mathbb{1}_{C \subset \Lambda_m} + \mathbf{c}_b^{\sum_{i \in \partial_-\Lambda_m \cap C} d_i}\big)$$

   where:

   - $d_i = 1$ for all i in $\partial_-\Lambda_m$ but the lower corners, and equals 2 at the corners;
   - $\kappa_+(a)$ is the number of connected components in the graph $(\bar{V}_m, a)$;
   - $\mathrm{cl}(b)$ is the set of connected components in the graph $(\bar{V}_m, b)$.

2. Φ_m is strong-FKG (satisfies the FKG-lattice condition).

3. $\Phi^{a,b}_{m;E_m} \preccurlyeq \mathtt{ATRC}^{1,1}_{E_m}$.

In particular, for any $F \subset \mathbb{E}_{\Lambda_{n-1,m-1}}$, $F' \subset E$, and any $(a,b) \in \Omega_m$ one has

$$\mathtt{ATRC}^{0,0}_{F} \preccurlyeq \Phi^{a,b}_{m;F}, \quad \Phi^{a,b}_{m;F'} \preccurlyeq \mathtt{ATRC}^{1,1}_{F'}. \tag{4.45}$$

For the lower comparison: use the monotonicity for Φ_m to closed edges in F^c and note that when all edges of F^c are closed, the measure on F is $\mathtt{ATRC}^{0,0}_{F}$. The upper comparison follows from strong-FKG and $\Phi_m \preccurlyeq \mathtt{ATRC}^{1,1}_{E}$.

Recall that $v_L = (-n-1, 0)$ and $v_R = (n+1, 0)$, and that we wish to study the cluster of v_L under $\Phi_m(\ \mid v_L \xleftrightarrow{\omega_\tau} v_R)$. For the sake of readability, all connections will be understood to take place in ω_τ if not specified otherwise. We denote $(\omega_\tau, \omega_{\tau\tau'})$ a sample from Φ_m, and $\mathcal{C}$ the cluster of v_L in the ω_τ marginal (see Section 4.4.4).

We will use the probabilistic description of long clusters from Theorem 21 with a suitable choice of cones. First, note that by symmetry the (unique by Theorem 21) $t \in \partial\mathcal{W}$ dual to e_1 is $t = \nu_1\mathrm{e}_1$. Then, by strict convexity and symmetry, the cones $\mathcal{Y}^{◀}_{\nu_1\mathrm{e}_1,\delta}$ are invariant under reflection through $\{x:\ x_2 = 0\}$, and have an angular aperture continuous in δ, strictly increasing, which converges to 0 as $\delta \to 0$ and to π as $\delta \to 1$. In particular, for any $\theta \in (0, \pi/2)$ there is a unique $\delta \in (0,1)$ such that $\mathcal{Y}^{◀}_{\theta} = \mathcal{Y}^{◀}_{\nu_1\mathrm{e}_1,\delta}$ where

$$\mathcal{Y}^{◀}_{\theta} = \{x \in \mathbb{R}^2 :\ \tan(\theta)x_1 \geq |x_2|\}$$

is the symmetric cone with angular aperture 2θ. Similarly to Section 4.7, say that x is a θ-cone-point of A if $A \subset x + (\mathcal{Y}^{◀}_{\theta} \cup \mathcal{Y}^{▶}_{\theta})$. We (re-)introduce

$$\mathcal{Y}^{▶}_{\theta} = -\mathcal{Y}^{◀}_{\theta}, \quad \mathrm{Diamond}_\theta(u,v) = (u + \mathcal{Y}^{◀}_{\theta}) \cap (v + \mathcal{Y}^{▶}_{\theta}),$$
$$\mathrm{CPts}_\theta(A) = \{x \in A :\ x \text{ is a } \theta\text{-cone-point of } A\}.$$

When omitted from the notation, θ is set to be $\pi/4$. Denote $\mathfrak{B}_L, \mathfrak{B}_R, \mathfrak{A}$ the sets of backward-confined, forward-confined, diamond-confined graphs (see Section 4.7.1) for $\theta = \pi/4$. Let then p_L, p_R, p be the measures given by Theorem 21 for $(\delta, \nu_1\mathrm{e}_1)$ with $\mathcal{Y}^{◀} = \mathcal{Y}^{◀}_{\nu_1\mathrm{e}_1,\delta}$. Denote $\mathcal{Q}$ the positive measure (*not* probability measure) on pairs length + sequences of graphs given by

$$\mathcal{Q}(M; \gamma_L, \gamma_1, \dots, \gamma_M, \gamma_R) = \mathrm{C} p_L(\gamma_L) p_R(\gamma_R) \prod_{i=1}^{M} p(\gamma_i)$$

where C is the constant given by Theorem 21. If is worth stressing that whenever f vanishes when the displacement of $\gamma_L \circ \gamma_1 \circ \dots \circ \gamma_M \circ \gamma_R$ becomes larger than some constant, integrating f against $\mathcal{Q}$ is just a finite sum. We will write

$$\int d\mathcal{Q} f(M; \gamma_L, \gamma_1, \dots, \gamma_M, \gamma_R)$$
$$= \mathrm{C} \sum_{M \geq 0} \sum_{\gamma_L \in \mathfrak{B}_L} \sum_{\gamma_R \in \mathfrak{B}_R} \sum_{\gamma_1, \dots, \gamma_M \in \mathfrak{A}} f(M; \gamma_L, \dots, \gamma_R) p_L(\gamma_L) p_R(\gamma_R) \prod_{i=1}^{M} p(\gamma_i). \tag{4.46}$$

Let $\gamma_1, \gamma_2, \dots$ be an i.i.d. sequence of random diamond-confined graphs with law p, and define

$$T_v = \inf\{k \geq 1 :\ \mathrm{X}(\gamma_1 \circ \dots \circ \gamma_k) = v\},$$

where the inf of an empty set is set to be $+\infty$ by convention. Let Q_v be the law of $(\gamma_1, \gamma_2, \dots, \gamma_{T_v})$ conditioned on $\{T_v < \infty\}$.

The main Theorem of this section is the following "representation as a mixture of random walk bridges". From this result, it is mostly an exercise to prove the invariance principle, and it will be done in Section 4.8.6.

Theorem 23. *There are $c > 0, C_0 \geq 0, c_0 \geq 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq C_0 n$, there is a probability measure $\overline{\mathrm{Me}}_{n,m}$ on $\mathfrak{B}_L \times \mathfrak{B}_R$ such that*

1. *$\overline{\mathrm{Me}}_{n,m}$ is supported on pairs of graphs with displacement sup-norm at most $c_0 \ln^2(n)$;*
2. *for any f function of $\mathcal{C}$,*

$$\Big| \sum_{\zeta_L,\zeta_R} \overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R) E_{Q_{v(\zeta_L,\zeta_R)}}\big(f(v_L + \zeta_L \circ \gamma_1 \circ \ldots \circ \gamma_M \circ \zeta_R)\big) \\ - \Phi_m\big(f(\mathcal{C}) \,\big|\, v_L \leftrightarrow v_R\big)\Big| \leq \|f\|_\infty e^{-c\ln(n)}$$

where $(\gamma_1, \gamma_2, \ldots, \gamma_M) \sim Q_{v(\zeta_L,\zeta_R)}$, E_{Q_v} denotes expectation with respect to Q_v, and $v(\zeta_L, \zeta_R) = v_R - \mathrm{X}(\zeta_R) - v_L - \mathrm{X}(\zeta_L)$.

Remark 4.8.1. This theorem gives the existence of a coupling between the law of $\mathcal{C}$ under $\Phi_m(\ \,|\, v_L \leftrightarrow v_R)$ and a product law on chains of diamond confined graphs, such that the chain and $\mathcal{C}$ are equal with a probability going to 1 as $n \to \infty$. The law of the sequence of cone-points of this chain is then a random walk bridge, and the distance (e.g. Hausdorff) between the linear interpolation of this bridge and the actual cluster is at most a constant times the norm of the largest step. In Subsection 4.8.6 (Lemma 4.8.11), we prove that this linear interpolation converges to a Brownian Bridge under diffusive scaling and that the maximal step size is $C\ln^2(n)$ with probability going to 1 as $n \to \infty$. This therefore implies an invariance principle for the upper/lower envelopes of $\mathcal{C}$.

4.8.2 Basic properties of the measure

We start with some general observations on Φ_m before going to the heart of the argument. In the course of the proof, we will introduce several constants, denoted

$$c_0, c_1, \ldots$$

They depend only on J, U. As this quantity appear several times, it is worthwhile to define it in advance

$$\bar{n}_1 = n - \lfloor c_1 \ln(n) \rfloor. \tag{4.47}$$

First, from Theorem 17, we can deduce the exponential decay for Φ_m.

Lemma 4.8.2. *There is $c > 0$ such that for any $n, m \geq 1$, any simply connected $F \subset E \cup E_b^+$, and any $x, y \in \mathbb{V}_F$,*

$$\sup_\eta \Phi_m\big(x \overset{F}{\leftrightarrow} y \,\big|\, \omega_\tau(e) = \eta(e)\, \forall e \in F^c\big) \leq e^{-c|x-y|}.$$

In particular, there is an integer $c_0 \geq 0$ such that for any $m \geq c_0 n$,

$$\Phi_m(\exists x \in \mathcal{C} :\ |x_2| \geq c_0 n) \leq e^{-3\nu_1 n}.$$

Proof. The second point follows from the first and a union bound. We focus on the first. The claim with $\Phi = \mathtt{ATRC}$ replacing Φ_m is Theorem 17. In particular, by the stochastic domination (4.45), for any $F \subset E$,

$$\Phi_m\big(\cdot \,\big|\, \omega_\tau(e) = \eta(e)\, \forall e \in F^c\big) \preccurlyeq \mathtt{ATRC}_F^{1,1} = \Phi(\ \,|\, \omega_\tau(e) = 1\, \forall e \in F^c),$$

so the first equation holds for Φ_m when $F \cap E_b^+ = \varnothing$. The case where $F \cap E_b^+ \neq \varnothing$ is a simplification of the arguments of [Ale04, Section 2] ($F \cap E_b^+$ having a simpler geometry than what is treated in [Ale04, Section 2]). $\square$

Then, from 16, we obtain fast relaxation of Φ_m towards Φ.

Lemma 4.8.3. *There are $c > 0, C \geq 0$ such that for every $n, m \geq 2$, any $F \subset \mathbb{E}_{\Lambda_{n-1,m-1}}$, and any event A supported on F with $\Phi(A) > 0$,*

$$\Big|\frac{\Phi_m(A)}{\Phi(A)} - 1\Big| \leq C \sum_{e \in F} e^{-\mathrm{cd}_\infty(e, \Lambda^c_{n-1,m-1})}.$$

In particular, there is $\mathrm{c}_1 \geq 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq \mathrm{c}_0 n + \ln^2(n)$, and any event B supported in $\mathbb{E}_{\Lambda_{\bar{n}_1, \mathrm{c}_0 n}}$, one has

$$\frac{1}{2} \leq \frac{\Phi_m(B)}{\Phi(B)} \leq 2.$$

Proof. This follows from the stochastic domination (4.45) and from the exponential ratio weak mixing of ATRC (Theorem 16). □

Recall that $\Lambda_{k,l} = \{-k, \dots, k\} \times \{-l, \dots, l\}$. Define

- $\mathrm{Cross}^{\geq 2}_{k,l}$ the event that there is at least two disjoint clusters in $(\Lambda_{l,k}, \omega_\tau \cap \mathbb{E}_{\Lambda_{l,k}})$ crossing $\Lambda_{k,l}$ from left to right;
- $\mathrm{Cross}^{1}_{k,l}$ the event that there is exactly one cluster in $(\Lambda_{l,k}, \omega_\tau \cap \mathbb{E}_{\Lambda_{l,k}})$ crossing $\Lambda_{k,l}$ from left to right.

Lemma 4.8.4. *There are $c > 0, n_0 \geq 1$, such that for any $n \geq n_0$, $m \geq \ln^2(n)$, $k \leq \min(m - \ln^2(n), n^2)$, one has*

$$\Phi_m\big(\mathrm{Cross}^{\geq 2}_{\bar{n}_1,k}\big) \leq 2(2k+1)^2 e^{-\nu_1 2n - cn},$$
$$\Phi_m\big(\mathrm{Cross}^{1}_{\bar{n}_1,k}\big) \leq 2(2k+1)^2 e^{-\nu_1 2\bar{n}_1},$$

where $\bar{n}_1$ is defined in (4.47) (and c_1 is given by Lemma 4.8.3).

Proof. Note that the events $\mathrm{Cross}^{\geq 2}_{\bar{n}_1,k}$, $\mathrm{Cross}^{1}_{\bar{n}_1,k}$ are supported on $F_k \equiv \mathbb{E}_{\Lambda_{\bar{n}_1,k}}$. Given the constraints on k, one can apply Lemma 4.8.3 to reduce the problem to proving the same bound for Φ instead of Φ_m, (the cost for passing from Φ_m to Φ is the factor 2 in the claim). Now, by a union bound,

$$\Phi(\mathrm{Cross}^{\geq 2}_{\bar{n}_1,k}) \leq \sum_{-k \leq x_2 < u_2 \leq k} \sum_{-k \leq y_2 < v_2 \leq k} \Phi(x \xleftrightarrow{F_k} y \overset{F_k}{\not\leftrightarrow} u \xleftrightarrow{F_k} v)$$

where connections are understood in ω_τ, and $x = (-\bar{n}_1, x_2)$, $u = (-\bar{n}_1, u_2)$, $y = (\bar{n}_1, y_2)$, and $v = (\bar{n}_1, v_2)$. Denote then $\mathcal{C}_{k,x}$ the cluster of x in the restriction of ω_τ to F_k, and, for C a realisation of $\mathcal{C}_{k,x}$ with $y \in C$, $\mathcal{F}^+_k(C)$ the set of edges in F_k above the union of C and its boundary. $\mathcal{F}^+_k(C)$ is then a simply connected set, so by Theorem 17,

$$\Phi(x \xleftrightarrow{F_k} y \overset{F_k}{\not\leftrightarrow} u \xleftrightarrow{F_k} v) \leq \sum_{\substack{C \subset R_{\bar{n}_1,k} \\ x,y \in C}} \Phi(\mathcal{C}_{k,x} = C) \mathtt{ATRC}^{1,1}_{\mathcal{F}^+_k(C)}(u \leftrightarrow v)$$
$$\leq e^{-c|u-v|} \Phi(x \leftrightarrow y) \leq e^{-cn} e^{-\nu(y-x)}.$$

Now, by convexity and symmetry, $\nu(y-x) \geq 2\bar{n}_1 \nu((y-x)/|y-x|) \geq 2\bar{n}_1 \nu_1$, so for n large enough,

$$\Phi(\mathrm{Cross}^{\geq 2}_{\bar{n}_1,k}) \leq (2k+1)^2 e^{-cn} e^{-2n\nu_1},$$

which is the first half of the claim. The second follows from the same argument:

$$\Phi(\mathrm{Cross}^1_{\bar{n}_1,k}) \leq 2 \sum_{-k\leq x_2,y_2\leq k} \Phi(x \overset{F_k}{\longleftrightarrow} y) \leq 2(2k+1)^2 \Phi(x \leftrightarrow y) \leq 2(2k+1)^2 e^{-2\bar{n}_1\nu_1}.$$

□

Lemma 4.8.5. *There is $c \geq 0$ such that for any $\epsilon > 0$ there is $n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq \epsilon n + \ln^2(n)$,*

$$\Phi_m(v_L \leftrightarrow v_R) \geq e^{-c\ln(n)} e^{-2n\nu_1}.$$

Proof. Let $N = n - \lceil C\ln(n)\rceil$ with C large enough. Let $w_L = (-N, 0)$, $w_R = (N, 0)$ Then, by inclusion of events and FKG inequality,

$$\begin{aligned}\Phi_m(v_L \leftrightarrow v_R) &\geq \Phi_m(w_L \leftrightarrow v_L)\Phi_m(w_R \leftrightarrow v_R)\Phi_m(w_L \overset{R_{N,\lfloor\epsilon n\rfloor}}{\longleftrightarrow} w_R)\\ &\geq 2\Phi(w_L \overset{R_{N,\lfloor\epsilon n\rfloor}}{\longleftrightarrow} w_R)\Phi_m(w_L \leftrightarrow v_L)\Phi_m(w_R \leftrightarrow v_R)\end{aligned}$$

where we used Lemma 4.8.3, C and n large enough in the second line.

We can then use Theorem 21 to lower bound the first probability: $\Phi(w_L \overset{R_{N,\lfloor\epsilon n\rfloor}}{\longleftrightarrow} w_R)$ is equal to $e^{-2\nu_1 N}$ times the probability for a random walk with step distribution supported on the whole of $\mathcal{Y}^{\blacktriangleleft} \cap \mathbb{Z}^2$, exponential tails, and mean proportional to e_1 to hit $w_R - w_L$ while having second component less (in modulus) than ϵn. The probability of hitting $w_R - w_L$ is greater than $C/\sqrt{n}$ by the Local Limit Theorem in dimension 2 (see for example [AOV24, Section 8.1]). The probability that the second component less than ϵn is greater than $1 - e^{-c_\epsilon n}$ by standard large deviation bounds.

Finally, we use finite energy to bound the last two probabilities: let $F_L = \big\{\{x, x + \mathrm{e}_1\} : x_1 = -n-1, \dots, -N-1\big\}$. Opening the edges of F_L has probability at least $e^{-c|F_L|} = e^{-c\ln(n)}$ for some $c > 0$. So $\Phi_m(w_L \leftrightarrow v_L) \geq e^{-c\ln(n)}$. Same with $w_R \leftrightarrow v_R$. □

This leads us to a first notion of "good configurations":

$$\mathrm{Good}^1_n = \{v_L \leftrightarrow v_R\} \cap \mathrm{Cross}^1_{\bar{n}_1,\mathrm{c}_0 n} \cap \{\forall x \in \mathcal{C} : \ |x_2| < \mathrm{c}_0 n\}, \tag{4.48}$$

where c_0 is given by Lemma 4.8.2. Then, by Lemmas 4.8.5, 4.8.4, and 4.8.2, there are $c > 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq \mathrm{c}_0 n + \ln^2(n)$,

$$\Phi_m(\mathrm{Good}^1_n \mid v_L \leftrightarrow v_R) \geq 1 - e^{-cn}. \tag{4.49}$$

Define $\mathcal{C}_{\mathrm{cr}}$ to be the unique crossing cluster of $\Lambda_{\bar{n}_1,\mathrm{c}_0 n}$ when there is a unique cluster, and $\mathcal{C}_{\mathrm{cr}} = \Lambda_{\bar{n}_1,\mathrm{c}_0 n}$ else. Under Good^1_n, one has $\mathcal{C}_{\mathrm{cr}} \subset \mathcal{C}$. The general plan for the next subsection will be to study $\mathcal{C}$ by first proving that it is close (in Hausdorff distance) to $\mathcal{C}_{\mathrm{cr}}$, and then by studying $\mathcal{C}_{\mathrm{cr}}$ using the infinite volume study of Section 4.7.

4.8.3 Geometry of crossing cluster

We start with the proximity between $\mathcal{C}_{\mathrm{cr}}$ and $\mathcal{C}$. Introduce

$$\mathcal{L}_L(C) = C \cap \{-\bar{n}_1\} \times \mathbb{Z}, \quad \mathcal{L}_R(C) = C \cap \{\bar{n}_1\} \times \mathbb{Z}, \quad \mathcal{L}(C) = \mathcal{L}_L(C) \cup \mathcal{L}_R(C).$$

Lemma 4.8.6. *Let c_0 be given by Lemma 4.8.2. There are $\mathrm{c}_2 \geq 0, c > 0, n_0 \geq 1$, such that for any $n \geq n_0$, $m \geq \mathrm{c}_0 n + \ln^2(n)$, one has that for any $K \geq \mathrm{c}_2 \ln(n)$,*

$$\Phi_m\big(\exists x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}} : \ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \geq K \,\big|\, \mathrm{Good}^1_n\big) \leq e^{-cK}.$$

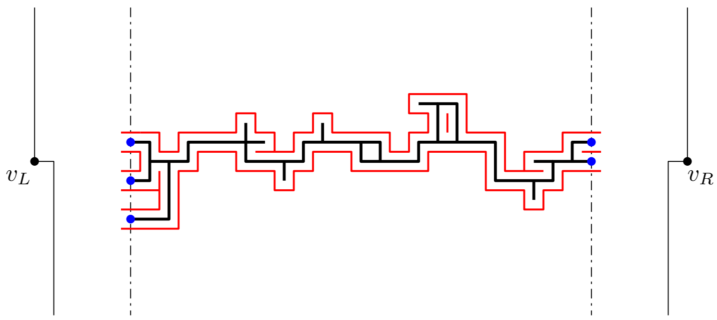


Figure 4.25: The open cluster $\mathcal{C}_{\mathrm{cr}}$ (in black) and the edge forced to be closed by its presence (dual of the red). The blue points are where connections to $\mathcal{C}_{\mathrm{cr}}$ have to arrive.

Proof. Let c_2 be large enough and $K \geq \mathrm{c}_2 \ln(n)$. Let $C \neq \varnothing$ be a realisation of $\mathcal{C}_{\mathrm{cr}}$. Now, under the event $\mathcal{C}_{\mathrm{cr}} = C$, $x \notin C$ is connected to C only if x is connected to $\mathcal{L}_L(C) \cup \mathcal{L}_R(C)$ (see Fig. 4.25). So, by Lemma 4.8.2,

$$\Phi_m(x \leftrightarrow C \mid \mathcal{C}_{\mathrm{cr}} = C) \leq \sum_{y \in \mathcal{L}(C)} e^{-c|\mathrm{d}_\infty(x,y)|}.$$

In particular, for C a realisation of $\mathcal{C}_{\mathrm{cr}}$ such that $C \subset \Lambda_{\bar{n}_1, \mathrm{c}_0 n}$, and n large enough,

$$\Phi_m(\exists x:\ \mathrm{d}(x,C) \geq K, x \leftrightarrow C \mid \mathcal{C}_{\mathrm{cr}} = C) \\ \leq \sum_{y \in \mathcal{L}(C)} \sum_{x \in \mathbb{Z}^2 : \mathrm{d}_\infty(x,y) \geq K} e^{-c|\mathrm{d}_\infty(x,y)|} \leq e^{-cK/2},$$

as $|\mathcal{L}(C)| \leq 2\mathrm{c}_0 n$, and c_2 is large enough. In particular,

$$\Phi_m\big(\exists x:\ \mathrm{d}(x,C) \geq K, x \leftrightarrow C \mid \mathrm{Cross}^1_{\bar{n}_1, \mathrm{c}_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}}:\ |x_2| < \mathrm{c}_0 n\}\big) \\ \leq \sum_C^{*} \Phi_m\big(\mathcal{C}_{\mathrm{cr}} = C \mid \mathrm{Cross}^1_{\bar{n}_1, \mathrm{c}_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}}:\ |x_2| < \mathrm{c}_0 n\}\big) e^{-cK/2} = e^{-cK/2}.$$

where $\sum_C^*$ is over realisation of $\mathcal{C}_{\mathrm{cr}}$ such that $C \subset \Lambda_{\bar{n}_1, \mathrm{c}_0 n}$. To conclude,

$$\frac{\Phi_m\big(\exists x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}}:\ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \geq K, \mathrm{Good}^1_n\big)}{\Phi_m(\mathrm{Good}^1_n)} \\ \leq \frac{e^{-cK}\Phi_m(\mathrm{Cross}^1_{\bar{n}_1, \mathrm{c}_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}}:\ |x_2| < \mathrm{c}_0 n\})}{\Phi_m(\mathrm{Good}^1_n)} \leq e^{-cK} e^{c' \ln(n)} \leq e^{-cK/2}$$

as c_2 is taken large enough, where the numerator is bounded using the bound we just derived, inclusion of events, and Lemma 4.8.4, and the denominator is bounded using Lemma 4.8.5, and (4.49). □

We then turn to the geometry of $\mathcal{C}_{\mathrm{cr}}$. Introduce, with c_2 given by Lemma 4.8.6,

$$\mathrm{Good}^2_n = \{\forall x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}}:\ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \leq \mathrm{c}_2 \ln(n)\} \cap \mathrm{Good}^1_n.$$

By Lemma 4.8.6, and (4.49),

$$\Phi_m(\mathrm{Good}^2_n \mid v_L \leftrightarrow v_R) \geq 1 - e^{-c\ln(n)}, \tag{4.50}$$

for some $c > 0$ and n large enough.

Introduce the slabs: for $l \in \mathbb{Z}, k \geq 0$,

$$\mathcal{S}_{l,k} = \{l, l+1, \dots, l+k\} \times \mathbb{Z}.$$

Lemma 4.8.7. *There are* $c_3 \geq 0, c > 0, n_0 \geq 1, \rho' > 0$ *such that for any* $n \geq n_0$*,* $m \geq c_0 n + \ln^2(n)$*,* $K \geq c_3 \ln(n)$*,* $l \in \{-\bar{n}_1, \ldots, \bar{n}_1 - K\}$

$$\Phi_m\big(|\mathcal{S}_{l,K} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}})| \leq \rho' K, \mathrm{Good}_n^2\big) \leq e^{-cK} e^{-2\nu_1 n}.$$

Proof. Under Good_n^2, $\mathcal{C}_{\mathrm{cr}}$ contains both a point in $\Lambda_{\bar{n}_1, c_0 n} \cap (v_L + \{-\lceil c_2 \ln(n)\rceil, \lceil c_2 \ln(n)\rceil\})$ and a point in $\Lambda_{\bar{n}_1, c_0 n} \cap (v_R + \{-\lceil c_2 \ln(n)\rceil, \lceil c_2 \ln(n)\rceil\})$. So, by a union bound and Lemma 4.8.3, we have the following upper bound on the target probability

$$2 \sum_x \sum_y \Phi\big(x \leftrightarrow y,\ |\mathcal{S}_{l,K} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_x)| \leq \rho' K\big)$$

where the sum is over $x \in \Lambda_{\bar{n}_1, c_0 n} \cap (v_L + \{-\lceil c_2 \ln(n)\rceil, \lceil c_2 \ln(n)\rceil\})$, $y \in \Lambda_{\bar{n}_1, c_0 n} \cap (v_R + \{-\lceil c_2 \ln(n)\rceil, \lceil c_2 \ln(n)\rceil\})$, and $\mathcal{C}_x$ is the connected component of x. By Theorem 21, the probability inside the sum multiplied by $e^{\nu(x-y)}$ is less than e^{-cK} as the distances between cone-points have exponential tails. In particular, by symmetry of ν, for c_3 fixed large enough,

$$\Phi\big(x \leftrightarrow y,\ |\mathcal{S}_{l,K} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_x)| \leq \rho' K\big) \leq e^{-2\nu_1 n} e^{-cK},$$

which gives the claim. □

The next Lemma combines Lemma 4.8.7 and a deterministic observation to control the cone-points of $\mathcal{C}$.

Lemma 4.8.8. *There are* $c_4 \geq 0, \rho > 0, c > 0, n_0 \geq 1$ *such that for any* $n \geq n_0$*,* $m \geq c_0 n + \ln^2(n)$*,* $K \geq c_4 \ln(n)$*,* $l \in \{-\bar{n}_1, \ldots, \bar{n}_1 - K\}$

$$\Phi_m\big(|\mathcal{S}_{l,K} \cap \mathrm{CPts}(\mathcal{C})| \leq \rho K, \mathrm{Good}_n^2\big) \leq e^{-cK} e^{-2\nu_1 n}.$$

Proof. First note that under Good_n^2, one has that

$$\mathcal{C} \subset \Big(\mathcal{C}_{\mathrm{cr}} \cup \bigcup_{x \in \mathcal{L}(\mathcal{C}_{\mathrm{cr}})} (x + [-c_2 \ln(n), c_2 \ln(n)]^2)\Big).$$

In particular, there is $C \geq 0$ large enough such that if there are $x, y, z \in \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}})$, and

$$x \cdot \mathrm{e}_1 + C \ln(n) \leq y \cdot \mathrm{e}_1 \leq z \cdot \mathrm{e}_1 - C \ln(n),$$

then $y \in \mathrm{CPts}(\mathcal{C})$ (see Fig. 4.26). In particular, if $K \geq 5C \ln(n)$, then by the above observation and Lemma 4.8.7, one has, with $\rho = \rho'/5$,

$$\begin{aligned}
&\Phi_m\big(|\mathcal{S}_{l,K} \cap \mathrm{CPts}(\mathcal{C})| \leq \rho K, \mathrm{Good}_n^2\big) \\
&\quad \leq \Phi_m\big(\mathcal{S}_{l,K/5} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}}) = \varnothing, \mathrm{Good}_n^2\big) \\
&\qquad + \Phi_m\big(\mathcal{S}_{l+4K/5,K/5} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}}) = \varnothing, \mathrm{Good}_n^2\big) \\
&\qquad + \Phi_m\big(\mathcal{S}_{l,K/5} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}}) \neq \varnothing, \mathcal{S}_{l+4K/5,K/5} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}}) \neq \varnothing, \\
&\qquad\qquad |\mathcal{S}_{l+2K/5,K/5} \cap \mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}})| \leq \rho' K/5, \mathrm{Good}_n^2\big) \\
&\quad \leq 3 e^{-cK} e^{-2\nu_1 n}
\end{aligned}$$

when n is large enough. □

Let $\ell_n \equiv \ell_n(r) = \lceil r \ln(n)\rceil$. We are now in position to introduce the set of good clusters: let $\mathrm{GCl}(r) = \mathrm{GCl}_{n,m}(r)$ be the set of realizations C of $\mathcal{C}$ such that (see Fig. 4.27)

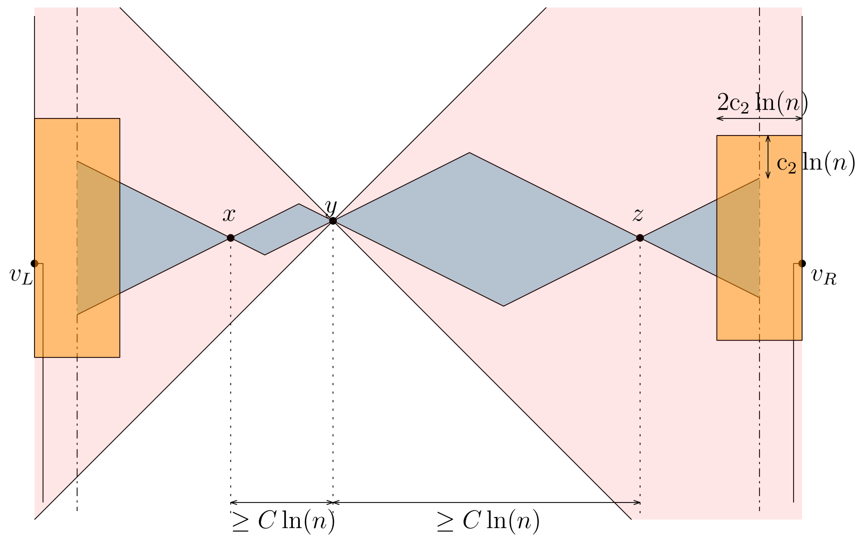


Figure 4.26: The presence of x, y, z in $\mathrm{CPts}_{\pi/8}(\mathcal{C}_{\mathrm{cr}})$ forces $\mathcal{C}$ to be included in the union of the blue diamonds/cones and of the orange rectangles. Enlarging the aperture of the cones at y allows to ensure that the orange rectangles are included in the cones started from y.

- $\mathcal{S}_{-n+\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$, $\mathcal{S}_{n-2\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$;
- $|\mathcal{S}_{-n+2\ell_n+1,\ell_n} \cap \mathrm{CPts}(C)| \geq \ln(n)$, $|\mathcal{S}_{n-3\ell_n+1,\ell_n} \cap \mathrm{CPts}(C)| \geq \ln(n)$;
- $|\mathcal{S}_{-n+4\ell_n+1,\ell_n} \cap \mathrm{CPts}(C)| \geq \ln(n)$, $|\mathcal{S}_{n-5\ell_n+1,\ell_n} \cap \mathrm{CPts}(C)| \geq \ln(n)$;
- $\mathcal{S}_{-n+5\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$, $\mathcal{S}_{n-6\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$;
- $\mathcal{S}_{-n+6\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$, $\mathcal{S}_{n-7\ell_n+1,\ell_n} \cap \mathrm{CPts}(C) \neq \varnothing$.

Lemma 4.8.9. *There are* $\mathrm{c}_5 > 0, c > 0$ *such that for any* $r \geq \mathrm{c}_5$*, there is* $n_0 \geq 0$ *such that for any* $n \geq n_0$*,* $m \geq \mathrm{c}_0 n + \ln^2(n)$*,*

$$\Phi_m\big(\mathcal{C} \in \mathrm{GCl}(r) \mid v_L \leftrightarrow v_R\big) \geq 1 - e^{-c\ln(n)}.$$

Proof. By Lemma 4.8.8 and a union bound, for r large enough (and n large enough),

$$\Phi_m\big(\mathcal{C} \notin \mathrm{GCl}(r), \mathrm{Good}_n^2\big) \leq 8e^{-c\ell_n(r)}e^{-2\nu_1 n}.$$

Then, fixing r large enough, for n large enough,

$$\begin{aligned}\Phi_m\big(\mathcal{C} \notin \mathrm{GCl}(r) \mid v_L \leftrightarrow v_R\big) = \frac{\Phi_m\big(\mathcal{C} \notin \mathrm{GCl}(r), \mathrm{Good}_n^2\big)}{\Phi_m\big(v_L \leftrightarrow v_R\big)} + \Phi_m\big((\mathrm{Good}_n^2)^c \mid v_L \leftrightarrow v_R\big)\\ \leq e^{-c\ln(n)} + e^{-c'\ln(n)},\end{aligned}$$

by the previous display, Lemma 4.8.5, and (4.50). □

Lemma 4.8.9 gives that for any fixed $r \geq \mathrm{c}_5$,

$$\Phi_m(\mathcal{C} \in \mathrm{GCl}(r) \mid v_L \leftrightarrow v_R) \geq 1 - e^{-c\ln(n)}, \tag{4.51}$$

for some $c > 0$ and any n large enough, $m \geq \mathrm{c}_0 n + \ln^2(n)$.

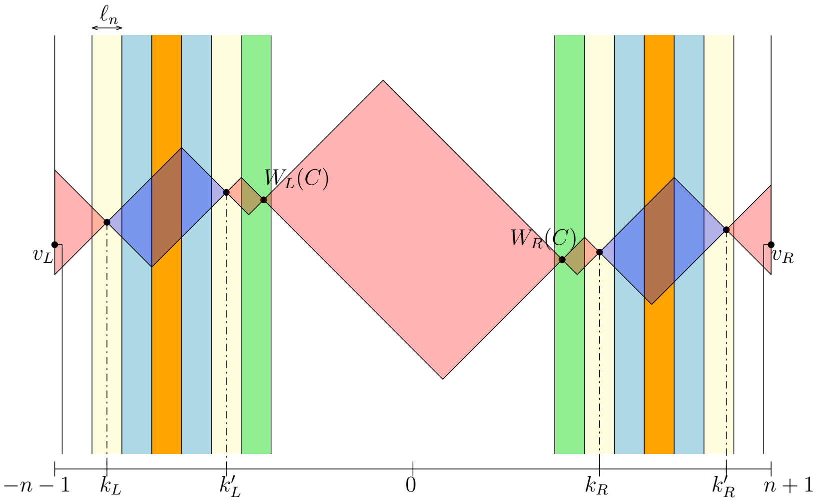


Figure 4.27: For $C \in \mathrm{GCl}$, the blue stipes contain both $\ln(n)$ cone-points, the yellow stripes each contain at last one cone-point, $W_L(C), W_R(C)$ are in the green stipes. The decoupling event D lives in the orange part.

4.8.4 Cluster decomposition and density swapping

Now that we have a good control over the typical geometry of $\mathcal{C}$, we can turn to the precise study of its law. Recall $\mathrm{GCl} = \mathrm{GCl}(r)$, $\ell_n = \ell_n(r)$ where r is some parameter than we can adjust. For any $C \in \mathrm{GCl}$, the slabs $\mathcal{S}_{-n+6\ell_n+1,\ell_n}$, $\mathcal{S}_{n-7\ell_n+1,\ell_n}$ both contain a cone point of C. Let

- $W_L = W_L(\mathcal{C})$ be the leftmost cone-point of $\mathcal{C}$ in the slab $\mathcal{S}_{-n+6\ell_n+1,\ell_n}$,
- $W_R = W_R(\mathcal{C})$ be the rightmost cone-point of $\mathcal{C}$ in the slab $\mathcal{S}_{n-7\ell_n+1,\ell_n}$.

Let $\mathfrak{B}_L^1$, $\mathfrak{B}_R^1$ be the subsets (depending on r) of $\mathfrak{B}_L$, $\mathfrak{B}_R$ respectively such that for every $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, there is $\eta \in \mathfrak{A}$ such that

$$C = (v_L + \eta_L \circ \eta \circ \eta_R) \in \mathrm{GCl},$$
$$W_L(C) = v_L + \mathrm{X}(\eta_L), \quad W_R(C) = v_R - \mathrm{X}(\eta_R).$$

One can then uniquely decompose a cluster $C \in \mathrm{GCl}$ as

$$C = v_L + \eta_L \circ \eta \circ \eta_R$$

with $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, $\eta \in \mathfrak{A}$, $\mathrm{X}(\eta_L) = W_L(C) - v_L$, $\mathrm{X}(\eta_R) = v_R - W_R(C)$. Now, notice that by the cone constraint, $\mathbf{b}(\eta_L)$ has degree one in η_L, $\mathbf{b}(\eta_R)$ has degree one in η_R, and $\mathbf{f}(\eta), \mathbf{b}(\eta)$ both have degree 1 in η. Let

$$\mathbf{f}_e(\eta) = \{\mathbf{f}(\eta), \mathbf{f}(\eta) + \mathrm{e}_1\}, \quad \mathbf{b}_e(\eta) = \{\mathbf{b}(\eta), \mathbf{b}(\eta) - \mathrm{e}_1\},$$
$$\mathbf{f}_e(\eta_R) = \{\mathbf{f}(\eta_R), \mathbf{f}(\eta_R) + \mathrm{e}_1\}, \quad \mathbf{b}_e(\eta_L) = \{\mathbf{b}(\eta_L), \mathbf{b}(\eta_L) - \mathrm{e}_1\}.$$

Introduce percolation events associated with $\eta_L \in \mathfrak{B}_L^1, \eta_R \in \mathfrak{B}_R^1, \eta \in \mathfrak{A}$: let $u_R = v_R - \mathrm{X}(\eta_R)$ and define (these are the finite volume versions of the events defined in Section 4.7.6)

- $A_L(\eta_L)$ is the event that edges of $v_L + \eta_L$ are open in $\omega_{\tau\tau'}$, edges of $(v_L + \eta_L) \setminus \mathbf{b}_e(v_L + \eta_L)$ are open in ω_τ, edges of $(v_L + \partial^{\mathrm{ex}}\eta_L) \cap (E \cup E_b^+)$ are closed in ω_τ, $\mathbf{b}_e(v + \eta_L)$ is closed in ω_τ;

- $A_R(\eta_R)$ is the event that edges of $u_R+\eta_R$ are open in $\omega_{\tau\tau'}$, edges of $(u_R+\eta_R)\setminus\mathbf{f}_e(u_R+\eta_R)$ are open in ω_τ, edges of $(u_R+\partial^{\mathrm{ex}}\eta_R)\cap(E\cup E_b^+)$ are closed in ω_τ, $\mathbf{f}_e(u_R+\eta_R)$ is closed in ω_τ;

- $A_v(\eta)$ is the event that edges of $v+\eta$ are open in $\omega_{\tau\tau'}$, edges of $(v+\eta)\setminus\{\mathbf{b}_e(v+\eta),\mathbf{f}_e(v+\eta)\}$ are open in ω_τ, edges of $\partial^{\mathrm{ex}}(v+\eta)$ are closed in ω_τ, $\mathbf{b}_e(v+\eta),\mathbf{f}_e(v+\eta)$ are closed in ω_τ.

Using the same trick as in Section 4.7.6: swapping the state of $\mathbf{f}_e(\eta),\mathbf{b}_e(\eta),\mathbf{f}_e(\eta_R),\mathbf{b}_e(\eta_L)$ from open to close in ω_τ brings a weight $C_{J,U}=\big(\frac{\sinh(2J)}{e^{-2J}-e^{-2U}}\big)^4$ (see Claim 1), one gets

$$\Phi_m(\mathcal{C}=v_L+\eta_L\circ\eta\circ\eta_R)=C_{J,U}\Phi_m\big(A_L(\eta_L)\cap A_{v_L+\mathrm{X}(\eta_L)}(\eta)\cap A_R(\eta_R)\big). \tag{4.52}$$

Now, define the infinite volume events corresponding to the pieces η_L,η_R:

- $\tilde{A}_L(\eta_L)$ is the event that edges of $v_L+\eta_L$ are open in $\omega_{\tau\tau'}$, edges of $(v_L+\eta_L)\setminus\mathbf{b}_e(v_L+\eta_L)$ are open in ω_τ, edges of $v_L+\partial^{\mathrm{ex}}\eta_L$ are closed in ω_τ, $\mathbf{b}_e(v+\eta_L)$ is closed in ω_τ;

- $\tilde{A}_R(\eta_R)$ is the event that edges of $u_R+\eta_R$ are open in $\omega_{\tau\tau'}$, edges of $(u_R+\eta_R)\setminus\mathbf{f}_e(u_R+\eta_R)$ are open in ω_τ, edges of $u_R+\partial^{\mathrm{ex}}\eta_R$ are closed in ω_τ, $\mathbf{f}_e(u_R+\eta_R)$ is closed in ω_τ.

As for Φ_m (see (4.52)),

$$\Phi(\mathcal{C}_{v_L}=v_L+\eta_L\circ\eta\circ\eta_R)=C_{J,U}\Phi\big(\tilde{A}_L(\eta_L)\cap A_{v_L+\mathrm{X}(\eta_L)}(\eta)\cap\tilde{A}_R(\eta_R)\big). \tag{4.53}$$

Define, for $i=1,\dots,7$ (the volumes corresponding to the slabs of Fig. 4.27)

$$\Delta_n^i=\{-n+i\ell_n+1,\dots n-i\ell_n-1\}\times\{-\mathrm{c}_0n+i\lceil\ln^2(n)\rceil,\dots\mathrm{c}_0n-i\lceil\ln^2(n)\rceil\}.$$

In particular, for any $C=v_L+\eta_L\circ\eta\circ\eta_R\in\mathrm{GCl}$, $W_L(C),W_R(C)\in\Delta_n^6\setminus\Delta_n^7$, and $W_L(C)+\eta$ is a subset of the edges in $\mathbb{E}_{\Delta_n^6}$.

Lemma 4.8.10 (Density swapping)**.** *There are $c>0,\mathrm{c}_6\geq 0$ such that for any $r\geq\mathrm{c}_6$, there is $n_0\geq 1$ such that for any $n\geq n_0$, $m\geq\mathrm{c}_0n+\ln^2(n)$, any $\eta_L\in\mathfrak{B}_L^1$, $\eta_R\in\mathfrak{B}_R^1$, and any event B with support in $\mathbb{E}_{\Delta_n^6}$, one has*

$$\Big|\frac{\Phi_m\big(B\mid A_L(\eta_L)\cap A_R(\eta_R)\big)}{\Phi\big(B\mid\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)}-1\Big|\leq e^{-c\ln(n)}. \tag{4.54}$$

Proof. We will use Lemma 4.A.1 with the following inputs:

- Ω_1 is the set of elements in $\{(0,0),(0,1),(1,1)\}^{\mathbb{E}_{\Delta_n^6}}$ having positive probability under $\Phi_m\big(\ \mid A_L(\eta_L)\cap A_R(\eta_R)\big)$ restricted to $\mathbb{E}_{\Delta_n^6}$, Ω_2 is the set of elements in $\{(0,0),(0,1),(1,1)\}^{\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}}$ having positive probability under $\Phi_m\big(\ \mid A_L(\eta_L)\cap A_R(\eta_R)\big)$ restricted to $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$;

- μ is the restriction of $\Phi_m\big(\ \mid A_L(\eta_L)\cap A_R(\eta_R)\big)$ to $\mathbb{E}_{\Delta_n^6}\cup\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$;

- ν is the restriction of $\Phi\big(\ \mid\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)$ to $\mathbb{E}_{\Delta_n^6}\cup\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$;

- D is the event that there is a path of open edges in $\omega_{\tau\tau'}$ in $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$ surrounding Δ_n^3, and two paths of edges dual to edges in $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$, open in ω_τ^*, one joining the upper boundary of η_L in $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$ to the upper boundary of η_R in $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$, and one joining the lower boundary of η_L to the lower boundary of η_R (in $\mathbb{E}_{\Delta_n^3\setminus\Delta_n^4}$). See Fig. 4.28.

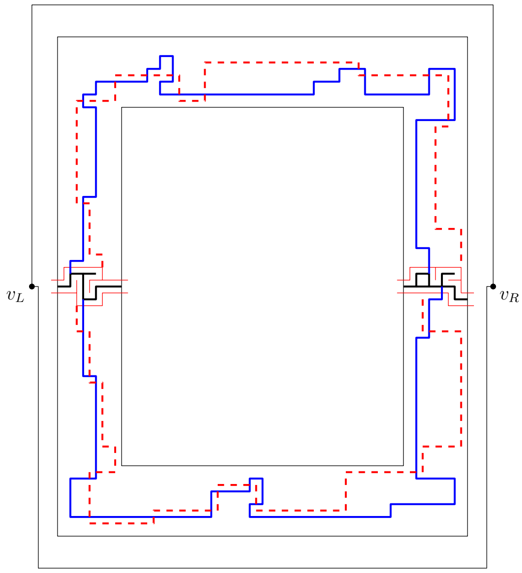


Figure 4.28: Decoupling paths imposed by the event D.

We check the required bounds. First, note that the paths imposed by D split the cluster weights (as in Lemma 4.4.6) and therefore the third condition of Lemma 4.A.1 is fulfilled with $\epsilon_3 = e^{-c\ln(n)}$ by uniform exponential decay in $\omega_\tau, \omega^*_{\tau\tau'}$ (Theorem 17), once r is taken large enough (c being uniform over η_L, η_R).

Let then $\gamma_{L/R}$ be the closest dual path surrounding $\eta_{L/R}$, and let $F_{L/R}$ be the sets of edges that are surrounded by $\gamma_{L/R}$. Let $k_L \equiv k_L(\eta_L)$ be the first coordinate of the leftmost cone-point of η_L in Δ_n^1, and $k_R \equiv k_R(\eta_R)$ be the first coordinate of the leftmost cone-point of η_R in Δ_n^5. Let $k'_L \equiv k'_L(\eta_L)$ be the first coordinate of the rightmost cone-point of η_L in Δ_n^5, and $k'_R \equiv k'_R(\eta_R)$ be the first coordinate of the rightmost cone-point of η_R in Δ_n^1, see Fig. 4.27. Let

$$\mathcal{R}_1 = \{k_L, \dots, k'_R\} \times \{-c_0 n + \lceil \ln^2(n) \rceil, \dots c_0 n - \lceil \ln^2(n) \rceil\},$$
$$\mathcal{R}_2 = \{k'_L, \dots, k_R\} \times \{-c_0 n + 5\lceil \ln^2(n) \rceil, \dots c_0 n - 5\lceil \ln^2(n) \rceil\},$$
$$\mathcal{E} = \mathbb{E}_{\mathcal{R}_1 \setminus \mathcal{R}_2} \setminus (F_L \cup F_R), \quad F = (*\gamma_L \cup *\gamma_R) \cap \mathcal{E}.$$

Let then $(a_-, b_-), (a_+, b_+) \in \{(0,0), (0,1), (1,1)\}^{\mathbb{E}}$ be given by

$$(a_-, b_-)_e = \begin{cases} (0,1) & \text{if } e \in (F_L \cup F_R) \cap \mathbb{E}_{\mathcal{R}_1}, \\ (0,0) & \text{else,} \end{cases}$$

and

$$(a_+, b_+)_e = \begin{cases} (0,1) & \text{if } e \in (F_L \cup F_R) \cap \mathbb{E}_{\mathcal{R}_1}, \\ (1,1) & \text{else.} \end{cases}$$

By the lattice-FKG property of Φ_m, Φ, the first point of Lemma 4.A.1 is true with $\epsilon_1 = e^{-c\ln(n)}$ if one has

$$\mathrm{d_{TV}}\Big(\mathtt{ATRC}_{\mathcal{E}}^{a_+,b_+}(\ \mid \omega_\tau(e) = 0\, \forall e \in F)|_{\mathbb{E}_{\Delta_n^3 \setminus \Delta_n^4}},$$
$$\mathtt{ATRC}_{\mathcal{E}}^{a_-,b_-}(\ \mid \omega_\tau(e) = 0\, \forall e \in F)|_{\mathbb{E}_{\Delta_n^3 \setminus \Delta_n^4}}\Big) \leq e^{-c\ln(n)}, \quad (4.55)$$

which follows from the strong mixing property of ATRC (Theorem 20) once r is fixed large enough as in the proof of Lemma 4.7.8 (recall that η_L, η_R are forced to have at least $\ln(n)$ cone-points in the slabs $\mathcal{S}_{-n+2\ell_n+1,\ell_n}$, $\mathcal{S}_{-n+4\ell_n+1,\ell_n}$, and $\mathcal{S}_{n-3\ell_n+1,\ell_n}$, $\mathcal{S}_{n-5\ell_n+1,\ell_n}$ respectively).

The second point follows in a similar fashion with $\epsilon_2 = \epsilon_1$: by monotonicity (FKG-lattice) the distance between μ_2 and ν_2 (notations of Lemma 4.A.1) is again bounded by (4.55).

For r fixed large enough, and n large enough, one can therefore apply Lemma 4.A.1 with $\epsilon = e^{-c\ln(n)}$, $c > 0$ to obtain the result. $\square$

4.8.5 Proof of Theorem 23

From now on, let

$$r = \mathrm{c}_6, \quad \ell_n = \lceil \mathrm{c}_6 \ln(n) \rceil,$$

where c_6 is the one given by Lemma 4.8.10. In particular, one has that for any $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$,

$$\|\mathrm{X}(\eta_L)\|_\infty \leq 8\lceil \mathrm{c}_6 \ln(n)\rceil, \quad \|\mathrm{X}(\eta_R)\|_\infty \leq 8\lceil \mathrm{c}_6 \ln(n)\rceil. \tag{4.56}$$

For $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, and C a realization of $\mathcal{C}$ containing v_R, define $\mathbb{1}_{C\sim\eta_L,\eta_R}$ to be 1 if $C = v_L + \eta_L \circ \eta \circ \eta_R$ for some diamond-confined η, and 0 else. Also, introduce

$$r_m(\eta_L, \eta_R) = \frac{\Phi_m\big(A_L(\eta_L) \cap A_R(\eta_R)\big)}{\Phi\big(\tilde{A}_L(\eta_L) \cap \tilde{A}_R(\eta_R)\big)}.$$

From Lemma 4.8.10 (and using (4.52) and (4.53)), we get that for any f function of $\mathcal{C}$ with $\|f\|_\infty \leq 1$, we have

$$\Big|\Phi_m\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big) - \sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)\Big| \\ \leq e^{-c\ln(n)}\Phi_m\big(\mathcal{C} \in \mathrm{GCl}\big), \tag{4.57}$$

where the computation goes by 1) partitioning over the realisation of $\mathcal{C} = \eta_L \circ \eta \circ \eta_R$ in both expectations of the L.H.S., 2) using (4.52) and (4.53) on each of the obtained summand, 3) apply Lemma 4.8.10 and

$$\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R} \mid \tilde{A}_L(\eta_L) \cap \tilde{A}_R(\eta_R)\big) = \frac{\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)}{\Phi\big(\tilde{A}_L(\eta_L) \cap \tilde{A}_R(\eta_R)\big)},$$

to each summand, 4) re-sum to obtain the R.H.S..

Now, by Theorem 21 and translation invariance of Φ, there is $c > 0$ independent of n such that (recall (4.46)) for any $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, and f with $\|f\|_\infty \leq 1$,

$$\Big|\int d\mathcal{Q} f(v_L + \bar{\gamma})\mathbb{1}_{(v_L+\bar{\gamma})\sim\eta_L,\eta_R} - e^{2\nu_1 n}\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)\Big| \leq e^{-cn}, \tag{4.58}$$

where $\bar{\gamma} = \gamma_L \circ \gamma_1 \circ \ldots \circ \gamma_M \circ \gamma_R$ (we used that if $\mathbb{1}_{(v_L+\bar{\gamma})\sim\eta_L,\eta_R} = 1$, then $\mathrm{X}(\bar{\gamma}) = v_R - v_L$).

For $s \geq 0$, introduce

$$T_L(s) = \min\big\{k \geq 0 : \ \|\mathrm{X}(\gamma_L \circ \gamma_1 \circ \ldots \circ \gamma_k)\|_\infty \geq s\ln^2(n)\big\},$$

and

$$T_R(s) = \min\big\{k \geq 0 : \ \|\mathrm{X}(\gamma_{M-k+1} \circ \ldots \circ \gamma_R)\|_\infty \geq s\ln^2(n)\big\},$$

where $\gamma_0 \equiv \gamma_L$, $\gamma_{M+1} \equiv \gamma_R$, and we put $+\infty$ for the min of an empty set by convention.

From the exponential tails of p_L, p_R, p, one obtains that ($\bar{\gamma} = \gamma_L \circ \ldots \circ \gamma_R$)

$$\int d\mathcal{Q}\mathbb{1}_{T_L=+\infty}\mathbb{1}_{\mathrm{X}(\bar{\gamma})=v_R-v_L} \leq e^{-cn}, \tag{4.59}$$

and the same for T_R, and that, for c_7 fixed large enough, any $K \geq c_7 \ln(n)$, and any $\square \in \{L, R\} \cup \{1, \ldots, M\}$,

$$\begin{aligned} \int dQ \mathbb{1}_{\|\mathrm{X}(\gamma_L \circ \ldots \circ \gamma_{T_L})\|_\infty \geq K} \mathbb{1}_{\mathrm{X}(\bar{\gamma}) = v_R - v_L} &\leq e^{-cK}, \\ \int dQ \mathbb{1}_{\|\mathrm{X}(\gamma_\square)\|_\infty \geq K} \mathbb{1}_{\mathrm{X}(\bar{\gamma}) = v_R - v_L} &\leq e^{-cK}. \end{aligned} \tag{4.60}$$

The first bound also holds for $\gamma_{M-T_R+1} \circ \ldots \circ \gamma_R$.

Let then $B_n(s)$ be the set of sequences $(\gamma_L, \gamma_1, \ldots, \gamma_M, \gamma_R)$ such that

- $0 < T_R(s) < \infty$, $0 < T_L(s) < \infty$,
- $\mathrm{X}(\gamma_L \circ \ldots \circ \gamma_R) = v_R - v_L$,
- $\max_{\square \in \{L,R\} \cup \{1,\ldots,M\}} \|\mathrm{X}(\gamma_\square)\|_\infty \leq s \ln^2(n)$,
- $\|\mathrm{X}(\gamma_L \circ \ldots \circ \gamma_{T_L})\|_\infty \leq 2s \ln^2(n)$, and $\|\mathrm{X}(\gamma_{M-T_R+1} \circ \ldots \circ \gamma_R)\|_\infty \leq 2s \ln^2(n)$.

From (4.59), (4.60), and the definition of $B_n(s)$, one directly obtains the next claim.

Claim 5. *There is $c_8 \geq 0, c > 0, n_0 \geq 1$ such that for any $s \geq c_8$, $n \geq n_0$,*

1. *for any $(\gamma_L, \gamma_1, \ldots, \gamma_M, \gamma_R) \in B_n(s)$, $\|\mathrm{X}(\gamma_L \circ \ldots \circ \gamma_{T_L})\|_\infty \geq 8\lceil c_6 \ln(n)\rceil$, and $\|\mathrm{X}(\gamma_{M-T_R+1} \circ \ldots \circ \gamma_R)\|_\infty \geq 8\lceil c_6 \ln(n)\rceil$;*
2. *$\int dQ \mathbb{1}_{\bar{\gamma} \notin B_n(s)} \leq e^{-cs \ln^2(n)}$.*

Let then

$$\begin{aligned} \mathfrak{B}_L^2(s) &= \big\{\zeta_L \in \mathfrak{B}_L :\ s \ln^2(n) \leq \|\mathrm{X}(\zeta_L)\|_\infty \leq 2s \ln^2(n) \text{ and } \exists \eta_L \in \mathfrak{B}_L^1, \eta \in \mathfrak{A},\ \zeta_L = \eta_L \circ \eta\big\}, \\ \mathfrak{B}_R^2(s) &= \big\{\zeta_R \in \mathfrak{B}_R :\ s \ln^2(n) \leq \|\mathrm{X}(\zeta_R)\|_\infty \leq 2s \ln^2(n) \text{ and } \exists \eta_R \in \mathfrak{B}_R^1, \eta \in \mathfrak{A},\ \zeta_R = \eta \circ \eta_R\big\}. \end{aligned}$$

Introduce two weights that will be our new boundary weights: for $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, $\zeta_L \in \mathfrak{B}_L^2(s)$, and $\zeta_R \in \mathfrak{B}_R^2(s)$, define (where C is the normalization constant in the definition of $\mathcal{Q}$)

$$\begin{aligned} b_L(\zeta_L; \eta_L) = \sqrt{\mathrm{C}} \sum_{\gamma_L} \sum_{k \geq 1} \sum_{\gamma_1, \ldots, \gamma_k} \mathbb{1}_{\gamma_L \circ \ldots \circ \gamma_k = \zeta_L} \mathbb{1}_{\exists l: \gamma_L \circ \ldots \circ \gamma_l = \eta_L} \\ \cdot \mathbb{1}_{\|\mathrm{X}(\gamma_L \circ \ldots \circ \gamma_{k-1})\|_\infty < s \ln^2(n)} \cdot p_L(\gamma_L) \prod_{i=1}^k p(\gamma_i), \end{aligned} \tag{4.61}$$

and

$$\begin{aligned} b_R(\zeta_R; \eta_R) = \sqrt{\mathrm{C}} \sum_{\gamma_R} \sum_{k \geq 1} \sum_{\gamma_k, \ldots, \gamma_1} \mathbb{1}_{\exists l: \gamma_l \circ \ldots \circ \gamma_1 \circ \gamma_R = \eta_R} \mathbb{1}_{\gamma_k \circ \ldots \circ \gamma_1 \circ \gamma_R = \zeta_R} \\ \cdot \mathbb{1}_{\|\mathrm{X}(\gamma_{k-1} \circ \ldots \circ \gamma_R)\|_\infty < s \ln^2(n)} \cdot p_L(\gamma_L) \prod_{i=1}^k p(\gamma_i), \end{aligned} \tag{4.62}$$

where the sums are over $\gamma_L \in \mathfrak{B}_L$, $\gamma_1, \cdots \in \mathfrak{A}$, $\gamma_R \in \mathfrak{B}_R$.

Also introduce one measure (the normalized version of it is our Random Walk bridge measure): for $v \in \mathbb{Z}^2 \setminus 0$, $M \geq 1$,

$$\tilde{Q}_v(M; \gamma_1, \dots, \gamma_M) = \mathbb{1}_{\mathrm{X}(\gamma_1 \circ \dots \circ \gamma_M) = v} \prod_{i=1}^{M} p(\gamma_i), \quad Z_v = \int \tilde{Q}_v, \quad Q_v = \tilde{Q}_v / Z_v. \tag{4.63}$$

From the above considerations, partitioning over the realizations ζ_L, ζ_R of, respectively, $\gamma_L \circ \dots \circ \gamma_{T_L}$, and $\gamma_{M-T_R+1} \circ \dots \circ \gamma_R$, one has that for any $s \geq \mathrm{c}_8$, and function f,

$$\begin{aligned}\sum_{\zeta_L \in \mathfrak{B}_L^2(s)} \sum_{\zeta_R \in \mathfrak{B}_R^2(s)} b_L(\zeta_L; \eta_L) b_R(\zeta_R; \eta_R) \int d\tilde{Q}_{v(\zeta_L, \zeta_R)} f(v_L + \zeta_L \circ \gamma_1 \circ \dots \circ \gamma_M \circ \zeta_R) \\ = \int dQ f(v_L + \bar{\gamma}) \mathbb{1}_{(v_L + \bar{\gamma}) \sim \eta_L, \eta_R} \mathbb{1}_{B_n(s)}(\gamma_L, \dots, \gamma_R),\end{aligned} \tag{4.64}$$

where $v(\zeta_L, \zeta_R) = v_R - \mathrm{X}(\zeta_R) - v_L - \mathrm{X}(\zeta_L)$.

Claim 6. *With the notations above, there are $c > 0, \mathrm{c}_9 \geq 0, n_0 \geq 1$ such that for any $n \geq n_0$, and any f with $\|f\|_\infty \leq 1$,*

$$\begin{aligned}\Big| \sum_{\eta_L, \eta_R} \sum_{\zeta_L, \zeta_R} r_m(\eta_L, \eta_R) e^{-2\nu_1 n} b_L(\zeta_L; \eta_L) b_R(\zeta_R; \eta_R) \int d\tilde{Q}_{v(\zeta_L, \zeta_R)} f(v_L + \zeta_L \circ \gamma_1 \circ \dots \circ \gamma_M \circ \zeta_R) \\ - \Phi_m\big(f(\mathcal{C}) \mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big) \Big| \leq \Phi_m\big(\mathcal{C} \in \mathrm{GCl}\big) e^{-c \ln(n)}\end{aligned} \tag{4.65}$$

where the sums are over $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$, and $\zeta_L \in \mathfrak{B}_L^2(\mathrm{c}_9)$, $\zeta_R \in \mathfrak{B}_R^2(\mathrm{c}_9)$.

Proof. Start by observing that by (4.51), and Lemma 4.8.5, one has

$$\Phi_m(\mathcal{C} \in \mathrm{GCl}) \geq e^{-2\nu_1 n} e^{-c \ln(n)},$$

for some $c > 0$. First, from (4.57), one has, for some $c > 0$,

$$\Big| \Phi_m\big(f(\mathcal{C}) \mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big) - \sum_{\eta_L, \eta_R} r_m(\eta_L, \eta_R) \Phi\big(f(\mathcal{C}) \mathbb{1}_{\mathcal{C} \sim \eta_L, \eta_R}\big) \Big| \leq e^{-c \ln(n)} \Phi_m\big(\mathcal{C} \in \mathrm{GCl}\big),$$

where the summation over $\eta_L \in \mathfrak{B}_L^1$, $\eta_R \in \mathfrak{B}_R^1$. Then, using (4.58), and

$$\sum_{\eta_L, \eta_R} r_m(\eta_L, \eta_R) = \sum_{\eta_L, \eta_R} \frac{\Phi_m\big(A_L(\eta_L) \cap A_R(\eta_R)\big)}{\Phi\big(\tilde{A}_L(\eta_L) \cap \tilde{A}_R(\eta_R)\big)} \leq e^{c \ln^2(n)},$$

for some $c > 0$, as $\Phi\big(\tilde{A}_L(\eta_L) \cap \tilde{A}_R(\eta_R)\big) \geq e^{-c \ln^2(n)}$ by finite energy (and control over the maximal size of η_L, η_R: they contain at most $c \ln^2(n)$ edges each), one has

$$\begin{aligned}\Big| \sum_{\eta_L, \eta_R} r_m(\eta_L, \eta_R) \Phi\big(f(\mathcal{C}) \mathbb{1}_{\mathcal{C} \sim \eta_L, \eta_R}\big) \\ - \sum_{\eta_L, \eta_R} r_m(\eta_L, \eta_R) e^{-2\nu_1 n} \int dQ f(v_L + \bar{\gamma}) \mathbb{1}_{v_L + \bar{\gamma} \sim \eta_L, \eta_R} \Big| \\ \leq e^{-2\nu_1 n - cn} \sum_{\eta_L, \eta_R} r_m(\eta_L, \eta_R) \leq e^{-cn/2} \Phi_m(\mathcal{C} \in \mathrm{GCl}),\end{aligned}$$

for n large enough. Then, using Claim 5, one obtains that for s large enough,

$$\Big|\sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)e^{-2\nu_1 n}\int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{(v_L+\bar{\gamma})\sim\eta_L,\eta_R}\mathbb{1}_{B_n}(\gamma_L,\dots,\gamma_R) \\ - \sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)e^{-2\nu_1 n}\int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{(v_L+\bar{\gamma})\sim\eta_L,\eta_R}\Big| \\ \leq e^{-2\nu_1 n}\sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)e^{-cs\ln^2(n)} \leq e^{-cs\ln(n)}\Phi_m(\mathcal{C}\in \mathrm{GCl}),$$

for n large enough, once s has been fixed large enough. A look at (4.64) and triangular inequality finishes the proof. □

We are now in position to conclude the proof of Theorem 23. Define a measure $\mathrm{Me}_{n,m}$ on $\mathfrak{B}_L\times\mathfrak{B}_R$ via

$$\mathrm{Me}_{n,m}(\zeta_L,\zeta_R) = \mathbb{1}_{\mathfrak{B}_L^2(c_9)}(\zeta_L)\mathbb{1}_{\mathfrak{B}_R^2(c_9)}(\zeta_R) \\ \cdot\frac{e^{-2\nu_1 n}}{\Phi_m(\mathcal{C}\in\mathrm{GCl})}\sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)b_L(\zeta_L;\eta_L)b_R(\zeta_R;\eta_R)Z_{v(\zeta_L,\zeta_R)} \tag{4.66}$$

where the sum is over $\eta_L\in\mathfrak{B}_L^1$, $\eta_R\in\mathfrak{B}_R^1$, and $v(\zeta_L,\zeta_R)=v_R-\mathrm{X}(\zeta_R)-v_L-\mathrm{X}(\zeta_L)$ (Z_v is defined in (4.63)). From (4.65) with $f=1$, one obtains

$$\Big|\sum_{\zeta_L,\zeta_R}\mathrm{Me}_{n,m}(\zeta_L,\zeta_R)-1\Big|\leq e^{-c\ln(n)}. \tag{4.67}$$

Letting

$$\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R)=\frac{1}{\sum_{\zeta_L,\zeta_R}\mathrm{Me}_{n,m}(\zeta_L,\zeta_R)}\mathrm{Me}_{n,m}(\zeta_L,\zeta_R), \tag{4.68}$$

one has the final “mixture of random walk bridges” representation: for f a function of the cluster of 0,

$$\Big|\sum_{\zeta_L,\zeta_R}\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R)\int dQ_{v(\zeta_L,\zeta_R)}f(v_L+\zeta_L\circ\gamma_1\circ\dots\circ\gamma_M\circ\zeta_R) \\ -\Phi_m\big(f(\mathcal{C})\,\big|\,\mathcal{C}\in\mathrm{GCl}\big)\Big|\leq\|f\|_\infty e^{-c\ln(n)}.$$

In particular, as $\Phi_m(\mathcal{C}\in\mathrm{GCl}\mid v_L\leftrightarrow v_R)\geq 1-e^{-c\ln(n)}$ (see (4.51)), for any f,

$$\Big|\sum_{\zeta_L,\zeta_R}\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R)\int dQ_{v(\zeta_L,\zeta_R)}f(v_L+\zeta_L\circ\gamma_1\circ\dots\circ\gamma_M\circ\zeta_R) \\ -\Phi_m\big(f(\mathcal{C})\,\big|\,v_L\leftrightarrow v_R\big)\Big|\leq\|f\|_\infty e^{-c\ln(n)}, \tag{4.69}$$

which is the statement of Theorem 23.

4.8.6 Convergence to the Brownian Bridge

From Theorem 23, one can use the result of [Kov04] to obtain the wanted invariance principle. We introduce the notations and state the result which one obtains from [Kov04].

Let m_n be any sequence with $m_n \geq C_0 n$, C_0 given by Theorem 23. Recall the laws p, $\overline{\mathrm{Me}}_{n,m_n}$ of Theorem 23. Let p_* be the push-forward of p by X (it is a probability measure on $\mathbb{Z}^2 \cap \mathcal{Y}^{\blacktriangleleft}$). Let q_*^n be the push-forward of $\overline{\mathrm{Me}}_{n,m_n}$ by X (it is a probability measure on $(\{-\lceil c_0 \ln^2(n) \rceil, \dots, \lceil c_0 \ln^2(n) \rceil\}^2 \cap \mathcal{Y}^{\blacktriangleleft})^2)$.

Define all random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X_1, X_2, \dots$ be an i.i.d. sequence of random variables with law p_*. Let

$$\alpha = E(X_1 \cdot \mathrm{e}_1), \quad \sigma_x^2 = E((X_1 \cdot \mathrm{e}_1)^2) - \alpha^2, \quad \sigma_y^2 = E((Y_1 \cdot \mathrm{e}_2)^2), \quad \chi = \frac{\sigma_y^2}{\alpha}.$$

By reflection symmetry of Φ (and thus of p and p_*), the mean vector and covariance matrix of p_* are given by

$$\bar{\mu} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}. \tag{4.70}$$

Define

$$S_0 = (0,0), \quad S_k = S_{k-1} + X_k. \tag{4.71}$$

Let $(V, W) \sim q_*^n$ be independent of the X_i's. Define

$$T_n = \min\{k \geq 1 : \ S_k = v_R - v_L - V - W\}$$

with $T_n = \infty$ when the set is empty. When $T_n < \infty$, let

$$Z_0 = (0,0), \quad Z_1 = V, \quad Z_k = V + S_{k-1}, \ k = 2, \dots T_n + 1,$$
$$Z_{T_n+2} = Z_{T_n+1} + W = v_R - v_L.$$

For $s \in [0, 2n+2]$, let

$$\tau_-(s) = \max\{k \geq 0 : \ Z_k \cdot \mathrm{e}_1 \leq s\}, \quad \tau_+(s) = \min\{k \geq 0 : \ Z_k \cdot \mathrm{e}_1 \geq s\}.$$

Define the linear interpolation for $s \in [0, 2n+2]$

$$\mathrm{LI}_s(Z) = \frac{Z_{\tau_+(s)} \cdot \mathrm{e}_1 - s}{(Z_{\tau_+(s)} - Z_{\tau_-(s)}) \cdot \mathrm{e}_1} Z_{\tau_-(s)} \cdot \mathrm{e}_2 + \frac{s - Z_{\tau_-(s)} \cdot \mathrm{e}_1}{(Z_{\tau_+(s)} - Z_{\tau_-(s)}) \cdot \mathrm{e}_1} Z_{\tau_+(s)} \cdot \mathrm{e}_2$$

when $T_n < \infty$, and $\mathrm{LI}_s(Z) = 0$ else. Define then the properly re-scaled version of $(\mathrm{LI}_s(Z))_{s \in [0,2n+2]}$:

$$\Gamma_t^n = \Gamma_t^n(Z) = \frac{1}{\sqrt{2n\chi}} \mathrm{LI}_{t(2n+2)}(Z), \quad t \in [0,1].$$

Define

$$\mathbb{P}_n = \mathbb{P}(\ | \ T_n < \infty).$$

The next Lemma concludes our study of the invariance principle for the (modified) ATRC cluster.

Lemma 4.8.11. *The sequence of random functions $(\Gamma_t^n)_{t \in [0,1]}$ under $\mathbb{P}_n$, $n \geq 1$, converges weakly to a standard Brownian Bridge. Moreover, with probability going to 1 as $n \to \infty$, the maximal step of Z has norm less than $C \ln^2(n)$ for some $C \geq 0$.*

Proof. This is the main result of [Kov04] with a slightly simpler setup: the mean of our steps is proportional to a coordinate axis, which is a particular case of the setup of [Kov04]. The estimate on the maximal step size is a simple large deviation estimate combined with the Local Limit Theorem. □

Remark 4.8.12. Note that the value of the diffusivity constant χ can be identified with the curvature of $\partial \mathcal{W}$ at $\nu(\mathrm{e}_1)\mathrm{e}_1$, exactly as in [IOSV22, Appendix B].

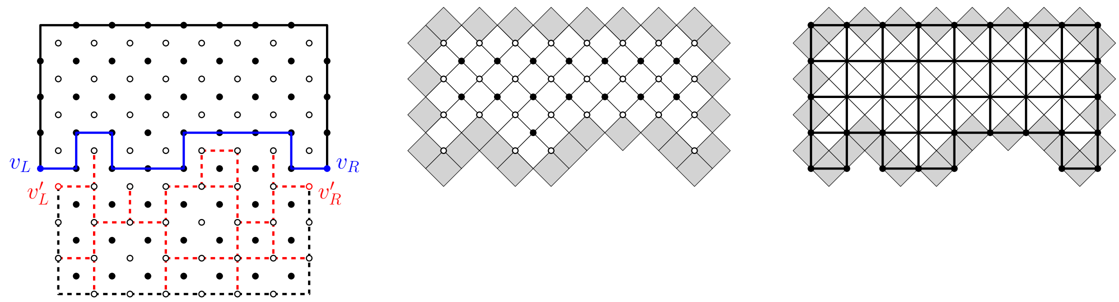


Figure 4.29: For $n = m = 3$. Left: vertices of G (solid) and its dual (hollow) with "1/0" boundary conditions (black solid and dashed edges), the dual cluster $\mathcal{C}' = C'$ (endpoints of dashed edges), and the lowermost path $p^{\uparrow}_{C'}$ (blue solid edges). Center: the set $\mathcal{D}_{C'}$ (solid and hollow vertices), the tiles in $A_{C'}$ that intersect them, with boundary tiles in $A^{\mathrm{b}}_{C'}$ in grey. Right: The associated edges in $\bar{E}^{C'}$, with those in $E^{C'}_{\mathrm{b}}$ contained in grey tiles.

4.9 Proofs of Theorems 13, 14, 15 and 18

Previous sections establish coupling between the FK-percolation and the (modified) ATRC models and several results for the latter: mixing properties, Ornstein–Zernike theory, invariance principle for the interface under suitable Dobrushin boundary conditions. In order to transfer the invariance principle to the FK-percolation, we need to show proximity of interfaces in the two models.

Consider the coupling measure $\Psi^{1/0}$ constructed in Section 4.4. A crucial feature of this coupling is that the respective interfaces stay close to each other with high probability. To measure the distance between interfaces, we will work with the *one-sided Hausdorff distance* defined by

$$\mathrm{d_H}(R, S) := \sup_{x \in R} \inf_{y \in S} \mathrm{d}_\infty(x, y), \qquad R, S \subset \mathbb{R}^2.$$

Let us also formally define the concept of sets being 'above' and 'below' each other.

Definition 10. Given a bi-infinite connected set $C \subset \mathbb{L}_\bullet \cap (\mathbb{R} \times [-c, c])$, we say that a subset $R \subset \mathbb{R}^2$ is (weakly) *above* C if it is contained in (the closure of) the connected component of the point $(0, c+1)$ in $\mathbb{R}^2 \setminus C$, where we identify C with the union of line segments between the endpoints of the edges in $\mathbb{E}_C$. We say that R is (weakly) *below* C if it is contained in (the closure of) the connected component of the point $(0, -c-1)$ in $\mathbb{R}^2 \setminus C$. We make the analogous definitions for finite connected sets by extending them to bi-finite connected sets by attaching left-infinite horizontal lines to all the leftmost points of the finite set and right-infinite horizontal lines to all the rightmost points; and, analogously, for connected subsets $C \subset \mathbb{L}_\circ$.

Recall the upper and lower envelopes $\Gamma^{\pm,n}_{\mathtt{FK}}$ in G_n defined in Section 4.1. Define $\Gamma^{\pm,n,m}_{\mathtt{FK}}$ in $G = G_{n,m}$ analogously. Recall also the definition of $\Gamma^{n,m}_{\mathtt{ATRC}}$ in Section 4.4.4.

Lemma 4.9.1. *There exist constants $C, c > 0$ such that, for any $n, m, k \geq 1$,*

$$\Psi^{1/0}(\mathrm{d_H}(\Gamma^{\pm,n,m}_{\mathtt{FK}}, \Gamma^{n,m}_{\mathtt{ATRC}}) > k) \leq Cnme^{-ck}.$$

Proof. Let us fix $n, m \geq 1$ and omit them from the notation. We present the argument for $\Gamma^{+}_{\mathtt{FK}}$. The statement for $\Gamma^{-}_{\mathtt{FK}}$ is proved in an analogous fashion. Consider the coupling $\Psi^{1/0}$ of $\omega \sim \mathtt{FK}^{1/0}_G$, $(\sigma_\bullet, \sigma_\circ) \sim \mathtt{Spin}^{+,+-}_{\mathcal{D}}$ and $(\omega_\tau, \omega_{\tau\tau'}) \sim \mathtt{mATRC}_K$ described in Section 4.4.2. Let $\mathcal{C}' = \mathcal{C}'_{v'_L}$ be the cluster of $v'_L := (-n - \frac{1}{2}, -\frac{1}{2})$ in ω^*; see Fig. 4.29. Note that

$$\mathrm{d_H}(\Gamma^{+}_{\mathtt{FK}}, \Gamma_{\mathtt{ATRC}}) \leq \mathrm{d_H}(\Gamma^{+}_{\mathtt{FK}}, \mathcal{C}') + \mathrm{d_H}(\mathcal{C}', \Gamma_{\mathtt{ATRC}}) \leq \tfrac{1}{2} + \mathrm{d_H}(\mathcal{C}', \Gamma_{\mathtt{ATRC}}). \tag{4.72}$$

Let $p_- \subset \mathbb{V}_{*E}$ be the uppermost path connecting v'_L to $v'_R := (n+\frac{1}{2}, -\frac{1}{2})$ on which $\sigma_\circ \equiv -1$. Recall the definition of $\mathcal{C} = \mathcal{C}_{v_L}$ in Section 4.4.4. Since $*\partial^{\mathrm{edge}}\mathcal{C}$ contains such a path, as well as a path connecting $(-n-\frac{3}{2}, \frac{1}{2})$ to $(n+\frac{3}{2}, \frac{1}{2})$ on which $\sigma_\circ \equiv 1$ (see Fig. 4.9 and 4.10), we have

$$\mathrm{d_H}(\mathcal{C}', \Gamma_{\mathtt{ATRC}}) \leq \mathrm{d_H}(\mathcal{C}', p_-) + \mathrm{d_H}(p_-, \Gamma_{\mathtt{ATRC}}) \leq \mathrm{d_H}(\mathcal{C}', p_-) + \tfrac{1}{2}. \tag{4.73}$$

Fix a realisation C' of $\mathcal{C}'$. By the coupling, $\sigma_\circ \equiv -1$ on $\mathcal{C}' = C'$ (see Fig. 4.9). Thus, the path p_- cannot contain vertices below C'. Hence, the random variable $\mathrm{d_H}(C', p_-)$ is measurable with respect to $\sigma_\circ$ restricted to the vertices in $\mathcal{D} \cap \mathbb{L}_\circ$ above C'. Let us determine the conditional law of $(\sigma_\bullet, \sigma_\circ)$ restricted to the vertices in $\mathcal{D}$ above C'.

Let $p^\uparrow_{C'}$ be the lowermost path in ω connecting v_L to v_R above C' and $\mathcal{D}_{C'}$ be the subset of vertices in $\mathcal{D}$ above $p^\uparrow_{C'}$. Denote by $A_{C'}$ the set of tiles with at least one corner in $\mathcal{D}_{C'}$, and by $A^{\mathrm{b}}_{C'}$ the set of tiles with precisely one corner in $\mathcal{D}_{C'}$; see Fig. 4.29. Define the associated sets of edges

$$\bar{E}^{C'} = \{e_t : t \in A_{C'}\}, \quad E^{C'}_{\mathrm{b}} = \{e_t : t \in A^{\mathrm{b}}_{C'}\} \quad \text{and} \quad E^{C'} = \bar{E}^{C'} \setminus E^{C'}_{\mathrm{b}}.$$

Set $G_{C'} = (\mathbb{V}_{E^{C'}}, E^{C'})$, $K_{C'} = (\mathbb{V}_{\bar{E}^{C'}}, \bar{E}^{C'})$, and let $K^1_{C'}$ be the graph obtained from $K_{C'}$ by identifying vertices in $\partial^{\mathrm{in}}_{\mathbb{L}_\bullet}\mathbb{V}_{\bar{E}^{C'}}$ and those connected by $p^\uparrow_{C'}$.

By Lemma 4.4.1 and the domain Markov property of FK-percolation,

$$\Psi^{1/0}(\omega|_{E^{C'}} = \omega_0 \mid \mathcal{C}' = C') = \mathtt{FK}^{1/0}_G(\omega|_{E^{C'}} = \omega_0 \mid \mathcal{C}' = C') = \mathtt{FK}^1_{G_{C'}}(\omega|_{E^{C'}} = \omega_0)$$

for any $\omega_0 \in \{0,1\}^{E^{C'}}$. The FK-percolation measure with wired boundary conditions can be coupled to the six-vertex spin measure with $(+,+)$ boundary conditions. This version of the BKW coupling was presented in [GP23, Lis21a] and is very closely related to our Lemma 4.4.1 that deals with the Dobrushin boundary conditions. We provide only the statement.

We first define the six-vertex spin measure on $\mathcal{D}_{C'}$ under $(+,+)$ boundary conditions (compare to (4.10)): for $\sigma = (\sigma_\bullet, \sigma_\circ) \in \{\pm1\}^{\mathbb{L}_\bullet} \times \{\pm1\}^{\mathbb{L}_\circ}$, it is defined by

$$\mathtt{Spin}^{+,+}_{\mathcal{D}_{C'}}(\sigma) \propto \mathbf{c}^{|T^{\mathrm{i}}_{5,6}(\sigma)|}\, \mathbf{c}_{\mathrm{b}}^{|T^{\mathrm{b}}_{5,6}(\sigma)|}\, \mathbb{1}_{\sigma=+1 \text{ on } (\mathbb{L}_\bullet \cup \mathbb{L}_\circ)\setminus \mathcal{D}_{C'}}\, \mathbb{1}_{\mathrm{ice}}(\sigma),$$

where $T^{\mathrm{i}}_{5,6}(\sigma)$ and $T^{\mathrm{b}}_{5,6}(\sigma)$ are respectively the sets of tiles in $A_{C'} \setminus A^{\mathrm{b}}_{C'}$ and $A^{\mathrm{b}}_{C'}$ that are of types 5,6 in σ. Then, as in Lemma 4.4.1,

$$\Psi^{1/0}\big((\sigma_\bullet|_{\mathcal{D}_{C'}\cap\mathbb{L}_\bullet}, \sigma_\circ|_{\mathcal{D}_{C'}\cap\mathbb{L}_\circ}) \in \cdot \,\big|\, \mathcal{C}' = C'\big) = \mathtt{Spin}^{+,+}_{\mathcal{D}_{C'}}|_{\mathcal{D}_{C'}}.$$

In the same vein, we determine the conditional law of $(\omega_\tau, \omega_{\tau\tau'})$ on $\bar{E}^{C'}$ by adapting Lemma 4.4.2 to $+,+$ boundary conditions. We now define a modified ATRC measure on $K_{C'}$ by taking Definition 3 and replacing $(E, E^+_{\mathrm{b}}, E^-_{\mathrm{b}})$ with $(E^{C'}, E^{C'}_{\mathrm{b}}, \varnothing)$ (in particular, $I(\mathcal{C}) \equiv 0$ for all clusters): for any $(a,b) \in \{0,1\}^{\bar{E}^{C'}} \times \{0,1\}^{\bar{E}^{C'}}$, define

$$\mathtt{mATRC}^{\varnothing}_{K_{C'}}(a,b) \propto \mathbb{1}_{a\subseteq b}\mathbb{1}_{b\setminus a\subseteq E^{C'}}\, 2^{k_{K_{C'}}(a) + k_{K^1_{C'}}(b)}\, 2^{|a|}(\mathbf{c}-2)^{|b\setminus a|}(\mathbf{c}_{\mathrm{b}}-1)^{|E^{C'}_{\mathrm{b}}\setminus b|}.$$

Since $\mathcal{C}' = C'$, we have $\sigma_\bullet \equiv 1$ at all endpoints of edges in $E^{C'}_{\mathrm{b}}$. Then, as in Lemma 4.4.2,

$$\Psi^{1/0}\big((\omega_\tau, \omega_{\tau\tau'})|_{\bar{E}^{C'}} \in \cdot \,\big|\, \mathcal{C}' = C'\big) = \mathtt{mATRC}^{\varnothing}_{K_{C'}}.$$

We claim that this measure satisfies (strong-FKG). Indeed, all local terms in the right side of the display trivially satisfy the property and the non-local terms assign weight two to the clusters in a and b. It is standard that such terms satisfy (strong-FKG): see the classical [Gri06, Theorem 3.8] for the case of the random-cluster model; alternatively, the proof of Lemma 4.4.4 applies by setting $\alpha \equiv \beta \equiv \frac{1}{2}$. Thus,

$$\mathtt{mATRC}^{\varnothing}_{K_{C'}} \leq_{\mathrm{st}} \mathtt{mATRC}^{\varnothing}_{K_{C'}}(\cdot \mid \omega_\tau(e) = 1 \text{ for all } e \in E^{C'}_{\mathrm{b}}) = \mathtt{ATRC}^{1,1}_{G_{C'}}.$$

As in the proof of Lemma 4.8.2, uniform exponential decay of connection probabilities in $\mathtt{ATRC}^{1,1}_{G_{C'}}$ (Theorem 17) and the arguments of [Ale04, Section 2] imply that connection probabilities in $\mathtt{mATRC}^{\varnothing}_{K_{C'}}$ decay exponentially as well.

Finally, since $\sigma_\circ \sim \omega^*_\tau$ and $\sigma_\circ = 1$ on $\mathbb{L}_\circ \setminus \mathcal{D}_{C'}$, any connected component of $\{\sigma_\circ = -1\}$ must be surrounded by a circuit in ω_τ, whence

$$\Psi^{1/0}(\mathrm{d_H}(\mathcal{C}', p_-) > k \mid \mathcal{C}' = C') \leq \mathtt{mATRC}^{\varnothing}_{K_{C'}}(\exists\, i \in p^{\uparrow}_{C'} : i \xleftrightarrow{\omega_\tau} i + \partial^{\mathrm{in}}\Lambda_k) \leq Cnmke^{-ck}.$$

Averaging over C' and inserting this in (4.72) and (4.73) completes the proof. □

4.9.1 Proof of Theorem 14

We will derive Theorem 14 from the analogous statement for the cluster $\mathcal{C} = \mathcal{C}_{v_L}$ of v_L in ω_τ with law the first marginal of $\mathtt{mATRC}_{K_{n,m}}(\cdot \mid v_L \xleftrightarrow{\omega_\tau} v_R)$ (Theorem 23 and Lemma 4.8.11) and the proximity statement (Lemma 4.9.1).

Proof of Theorem 14. To lighten the notation, we omit $p_c(q), q$ from the subscripts. Recall the definition of the upper and lower envelopes $\Gamma^{\pm,n}_{\mathtt{FK}}$ in G_n defined in Section 4.1, and of $\Gamma^{\pm,n,m}_{\mathtt{FK}}$ in $G = G_{n,m}$ defined analogously. We will first show the statement for the FK measure $\mathtt{FK}^{1/0}_{G_{n,Cn}}$ for C sufficiently large, and then from it derive the statement for $\mathtt{FK}^{1/0}_{G_n}$.

By Theorem 23 and Lemmata 4.8.11 and 4.9.1, there exists $C > 1$ such that the statement holds for $\Gamma^{\pm,n,Cn}_{\mathtt{FK}}$ under the measure $\mathtt{FK}^{1/0}_{G_{n,Cn}}$. The derivation for $\Gamma^{\pm,n}_{\mathtt{FK}}$ under $\mathtt{FK}^{1/0}_{G_n}$ follows from the FKG and DLR properties of FK percolation, and we only sketch the argument. Define the graphs $G^{\pm}_{n,Cn} = (V^{\pm}_{n,Cn}, E^{\pm}_{n,Cn})$ by

$$\Lambda^-_{n,Cn} := \{-n,\dots,n\} \times \{-Cn,\dots,n\}, \quad \Lambda^+_{n,Cn} := \{-n,\dots,n\} \times \{-n,\dots,Cn\},$$
$$V^{\pm}_{n,Cn} = \Lambda^{\pm}_{n,Cn} \cup (\partial^{\mathrm{ex}}\Lambda^{\pm}_{n,Cn} \cap \mathbb{H}^+), \quad E^{\pm}_{n,Cn} = \mathbb{E}_{V^{\pm}_{n,Cn}} \setminus \mathbb{E}_{(\Lambda^{\pm}_{n,Cn})^c}.$$

Then, by the FKG and DLR properties of the FK measures, it holds that

$$\mathtt{FK}^{1/0}_{G_{n,Cn}} \geq_{\mathrm{st}} \mathtt{FK}^{1/0}_{G^+_{n,Cn}} \leq_{\mathrm{st}} \mathtt{FK}^{1/0}_{G_n} \leq_{\mathrm{st}} \mathtt{FK}^{1/0}_{G^-_{n,Cn}} \geq_{\mathrm{st}} \mathtt{FK}^{1/0}_{G_{n,Cn}}.$$

If ω is distributed according to $\mathtt{FK}^{1/0}_{G_{n,Cn}}$, by the statement for $\Gamma^{\pm,n,Cn}_{\mathtt{FK}}$ under this measure, there exists a left-right crossing above $[-n,n] \times \{0\}$ in ω and one in ω^* below it at arbitrary small linear distance from $[-n,n] \times \{0\}$. A chain of classical monotone coupling arguments finishes the proof. □

4.9.2 Proof of Theorem 13

We will derive Theorem 13 from Theorem 14. Recall the definition of the one-sided Hausdorff distance $\mathrm{d_H}$, and the notion of a subset of $\mathbb{R}^2$ being above or below a connected set in $\mathbb{L}_\bullet$ or $\mathbb{L}_\circ$, introduced above Lemma 4.9.1.

Proof of Theorem 13. To simplify the notation, we omit $n, q, T_c(q), p_c(q)$ from sub and superscripts. Consider the Edwards–Sokal coupling $\mathtt{ES}_\Lambda^{1/0}$ of $\sigma \sim \mathtt{Potts}_\Lambda^{1/\mathrm{f}}$ and $\omega \sim \mathtt{FK}_{G_n}^{1/0}$ described in the introduction; see [Gri06, Section 1.4] for details. Observe that the FK and Potts envelopes deterministically satisfy

$$\Gamma^-_{\mathtt{FK}}(k) \le \Gamma^+_{\mathtt{FK}}(k), \quad \Gamma^-_{\mathtt{Potts}}(k) \le \Gamma^+_{\mathtt{Potts}}(k), \quad \Gamma^\pm_{\mathtt{Potts}}(k) \le \Gamma^\pm_{\mathtt{FK}}(k) \quad \forall k \in \{-n, \dots, n\}.$$

In particular, $\Gamma^+_{\mathtt{Potts}}$ is "sandwiched" between $\Gamma^-_{\mathtt{Potts}}$ and $\Gamma^+_{\mathtt{FK}}$. By Theorem 14, it suffices to show the existence of $c, C > 0$ for which, for any $k \ge 1$,

$$\mathtt{ES}_\Lambda^{1/0}(\mathrm{d_H}(\Gamma^-_{\mathtt{Potts}}, \Gamma^-_{\mathtt{FK}}) > k) < Cn^2 e^{-ck}.$$

Let $\mathcal{C}$ be the cluster of the lower boundary in $\{\sigma \neq 1\}$, that is, the set of $i \in \Lambda$ for which there exists a path $(i_0, \dots, i_\ell)$ in Λ with $i_0 = i$, $i_\ell \in \partial^{\mathrm{in}}\Lambda \cap \mathbb{H}^-$ and $\sigma(i_k) \neq 1$ for $0 \le k \le \ell$. By definition, each point in the lower envelope $\Gamma^-_{\mathtt{Potts}}$ is weakly above $\mathcal{C}$. Therefore, it suffices to show that $\Gamma^-_{\mathtt{FK}}$ is not far above $\mathcal{C}$. Fix a realisation C of $\mathcal{C}$, and define $\Lambda_C = \Lambda \setminus (C \cup \partial^{\mathrm{ex}} C)$. By the coupling and the spatial Markov property of the Potts model, it holds that

$$\mathtt{ES}_\Lambda^{1/0}(\sigma|_{\Lambda_C} \in \cdot \mid \mathcal{C} = C) = \mathtt{Potts}_\Lambda^{1/\mathrm{f}}(\sigma|_{\Lambda_C} \in \cdot \mid \mathcal{C} = C) = \mathtt{Potts}^1_{\Lambda_C}.$$

Let $G_C = (V_C, E_C)$ be defined by $E_C = \mathbb{E}_{\Lambda_C} \cup \partial^{\mathrm{edge}}\Lambda_C$ and $V_C = \mathbb{V}_{E_C}$. Then, by the above and by the coupling,

$$\mathtt{ES}_\Lambda^{1/0}(\omega|_{E_C} \in \cdot \mid \mathcal{C} = C) = \mathtt{FK}^1_{G_C}(\omega|_{E_C} \in \cdot).$$

Now, conditional on $\mathcal{C} = C$, if $\mathrm{d_H}(C, \Gamma^-_{\mathtt{FK}}) > k$, then there exists a dual path of length k in $\omega^*|_{*E_C}$. Since $\mathtt{FK}^1_{G_C}$ stochastically dominates the infinite-volume measure $\mathtt{FK}^1$, which admits exponential decay of connection probabilites in its dual [DGH+21, Theorem 1.2], the proof is complete. □

4.9.3 Proof of Theorems 15 and 18

Finally, the Ornstein-Zernike asymptotics for the two-point function of the AT and ATRC models (Theorems 15 and 18) are direct consequences of Theorem 21.

Proof of Theorems 15 and 18. Theorem 21 gives, in particular, that for $s \in \mathbb{S}^1$, one can find $t \in \partial\mathcal{W}, \delta \in (0,1)$ such that (sums are over pieces confined in the cones/diamonds obtained using $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$)

$$\Big| \mathrm{C} \sum_{\gamma_L, \gamma_R} p_L(\gamma_L) p_R(\gamma_R) \sum_{k \ge 0} \sum_{\gamma_1, \dots, \gamma_k} \mathbb{1}_{\mathrm{X}(\bar{\gamma}) = ns} \prod_{i=1}^k p(\gamma_k) \\ - e^{n\nu(s)} \mathtt{ATRC}_{J,U}(0 \xleftrightarrow{\omega_\tau} ns) \Big| \le C_1 e^{-c_1 n},$$

which is a comparison between $e^{n\nu(s)} \mathtt{ATRC}_{J,U}(0 \leftrightarrow ns)$ and the Green functions of a directed random walk on $\mathbb{Z}^2$ (the push-forward of p_L, p_R, p by X). One can then use the local limit theorem in dimension 2 as in [AOV24, Section 8.1] to obtain

$$e^{n\nu(s)} \mathtt{ATRC}_{J,U}(0 \leftrightarrow ns) = \frac{c(s)}{\sqrt{n}}(1 + o_n(1)), \tag{4.74}$$

which is the wanted asymptotics for the ATRC model. The claim for the AT model follows from the coupling (4.7) between AT and ATRC, . □

4.A Mixing to ratio mixing

We prove here a technical Lemma whose use is recurrent in the proof that mixing implies ratio mixing under suitable conditions. It is a simplified version of the argument in [Ale98, Section 5].

Lemma 4.A.1. *Let Ω_i, $i = 1, 2$ be finite sets. Let $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{F}_i = \{A \subset \Omega_i\}$, $\mathcal{F} = \{A \subset \Omega\}$. Let μ, ν be positive probability measures on $(\Omega, \mathcal{F})$. Let*

$$\pi_i : \Omega \to \Omega_i, \quad \pi_i((\omega_1, \omega_2)) = \omega_i, \quad \mu_i = \mu \circ \pi_i^{-1}.$$

Let $\epsilon_1, \epsilon_2, \epsilon_3 \in [0, 1)$. Suppose that all of the following hold:

1. *Mixing of μ, ν: for every $\xi, \xi' \in \Omega_1$, $A \subset \Omega_2$, $\rho \in \{\mu, \nu\}$*

$$|\rho(\Omega_1 \times A \mid \{\xi\} \times \Omega_2) - \rho(\Omega_1 \times A \mid \{\xi'\} \times \Omega_2)| \leq \epsilon_1$$

2. *Proximity between second marginals: $\mathrm{d_{TV}}(\mu_2, \nu_2) \leq \epsilon_2$;*

3. *Conditional equality: there exists an event $D \subset \Omega_2$ such that for $\rho \in \{\mu_2, \nu_2\}$, $\rho(D) \geq 1 - \epsilon_3$, and for any $y \in D$,*

$$\frac{\mu(x, y)}{\mu_2(y)} = \frac{\nu(x, y)}{\nu_2(y)}, \ \forall x \in \Omega_1.$$

Then, if $\epsilon = \max(\epsilon_1, \sqrt{\epsilon_2}, \epsilon_3) \leq 0.1$, then, for any $x \in \Omega_1$,

$$1 - 9\epsilon \leq \frac{\mu_1(x)}{\nu_1(x)} \leq \frac{1}{1 - 9\epsilon}.$$

Proof. The proof goes by constructing a suitable coupling of μ, ν. Let Q be a maximal coupling of μ_2 and ν_2. Then, sample a random vector $(X, Y) = \big((X_1, X_2), (Y_1, Y_2)\big)$ as follows:

1. sample (X_2, Y_2) using Q;
2. sample X_1 using $\mu(\ \mid \Omega_1 \times \{X_2\}) \circ \pi_1^{-1}$;
3. if $X_2 = Y_2 \in D$, set $Y_1 = X_1$. Else, sample $Y_1 \sim \nu(\ \mid \Omega_1 \times \{Y_2\}) \circ \pi_1^{-1}$ independently of X_1.

Denote Ψ the law of (X, Y).

Claim 7. *All the following points hold.*

1. *Ψ is a coupling of μ and ν.*
2. *$(X_2 = Y_2 \in D) \implies (X_1 = Y_1)$.*
3. *X_1 and Y_2 are independent conditionally on X_2, and Y_1 and X_2 are independent conditionally on Y_2.*

Proof. The second point is by construction. We check the first point. For $x = (x_1, x_2) \in \Omega$,

$$\begin{aligned}\Psi(X = x) &= \sum_{y_2 \in \Omega_2} Q(x_2, y_2) \Psi(X_1 = x_1 \mid X_2 = x_2, Y_2 = y_2) \\ &= \sum_{y_2 \in \Omega_2} Q(x_2, y_2) \frac{\mu(x_1, x_2)}{\mu_2(x_2)} = \mu(x_1, x_2).\end{aligned}$$

Also, for $y = (y_1, y_2) \in \Omega$,

$$\begin{aligned}\Psi(Y = y) &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\Psi(Y_1 = y_1 \mid X_2 = x_2, Y_2 = y_2)\\ &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\Big(\mathbb{1}_{y_2 = x_2 \in D}\frac{\mu(y_1, x_2)}{\mu_2(x_2)} + (1 - \mathbb{1}_{y_2 = x_2 \in D})\frac{\nu(y_1, y_2)}{\nu_2(y_2)}\Big)\\ &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\frac{\nu(y_1, y_2)}{\nu_2(y_2)} = \nu(y_1, y_2),\end{aligned}$$

as, by hypotheses on D, $\mathbb{1}_{y_2=x_2\in D}\frac{\mu(y_1,x_2)}{\mu_2(x_2)} = \mathbb{1}_{y_2=x_2\in D}\frac{\mu(y_1,y_2)}{\mu_2(y_2)} = \mathbb{1}_{y_2=x_2\in D}\frac{\nu(y_1,y_2)}{\nu_2(y_2)}$.

Let us finally check the third point. First, for any $x_1 \in \Omega_1, x_2, y_2 \in \Omega_2$,

$$\Psi(X_1 = x_1 \mid X_2 = x_2, Y_2 = y_2) = \frac{\mu(x_1, x_2)}{\mu_2(x_2)} = \Psi(X_1 = x_1 \mid X_2 = x_2).$$

Then, for any $y_1 \in \Omega_1, x_2, y_2 \in \Omega_2$,

$$\begin{aligned}\Psi(Y_1 = y_1 \mid X_2 = x_2, Y_2 = y_2) &= \mathbb{1}_{y_2=x_2\in D}\frac{\mu(y_1, x_2)}{\mu_2(x_2)} + (1 - \mathbb{1}_{y_2=x_2\in D})\frac{\nu(y_1, y_2)}{\nu_2(y_2)}\\ &= \frac{\nu(y_1, y_2)}{\nu_2(y_2)} = \Psi(Y_1 = y_1 \mid Y_2 = y_2),\end{aligned}$$

by hypotheses on D, as before. □

Introduce then $g, h : \Omega_2 \to \mathbb{R}_+$ defined by

$$\begin{aligned}g(\xi) &= \Psi(X_2 \neq Y_2 \mid X_2 = \xi),\\ h(\xi) &= \Psi(X_2 \neq Y_2 \mid Y_2 = \xi).\end{aligned}$$

Define the event H by

$$H = \{X_2 = Y_2\} \cap \{X_2 \in D\} \cap \{Y_2 \in D\} \cap \{g(X_2) \leq \sqrt{\epsilon_2}\} \cap \{h(Y_2) \leq \sqrt{\epsilon_2}\}.$$

Claim 8. *One has*

$$\begin{aligned}\max_{x_1 \in \Omega_2} \Psi(H^c \mid X_1 = x_1) \leq 9\epsilon,\\ \max_{y_1 \in \Omega_2} \Psi(H^c \mid Y_1 = y_1) \leq 9\epsilon.\end{aligned}$$

Proof. Let $a = \sqrt{\epsilon_2}$. First, let $\{g > a\} = \{x_2 \in \Omega_2 : \ g(x_2) > a\}$, one has

$$\begin{aligned}\mu(\Omega_1 \times \{g > a\}) = \Psi(g(X_2) > a) \leq a^{-1}\Psi(g(X_2)) &= a^{-1}\Psi(\Psi(X_2 \neq Y_2 \mid X_2))\\ &= a^{-1}\Psi(X_2 \neq Y_2) \leq \frac{\epsilon_2}{a},\end{aligned}$$

as $\Psi(X_2 \neq Y_2) = Q(X_2 \neq Y_2) = \mathrm{d_{TV}}(\mu_2, \nu_2) \leq \epsilon_2$, and the same for $\nu(\Omega_1 \times \{h > a\})$. Now, by our first hypotheses,

$$\begin{aligned}\Psi(g(X_2) > a \mid X_1 = x_1) &= \mu(\Omega_1 \times \{g > a\} \mid \{x_1\} \times \Omega_2)\\ &\leq \mu(\Omega_1 \times \{g > a\}) + \epsilon_1 = \frac{\epsilon_2}{a} + \epsilon_1,\end{aligned}$$

and the same for $\Psi(h(Y_2) > a \mid Y_1 = y_1)$. Also,

$$\Psi(X_2 \in D^c \mid X_1 = x_1) = \mu(\Omega_1 \times D^c \mid \{x_1\} \times \Omega_2) \leq \mu(\Omega_1 \times D^c) + \epsilon_1 = \epsilon_3 + \epsilon_1,$$

and the same for $\Psi(Y_2 \in D^c \mid Y_1 = y_1)$.

Then, as X_1 and Y_2 are independent conditionally on X_2 (by Claim 7),

$$\begin{aligned}
&\Psi(X_2 \neq Y_2, g(X_2) \leq a \mid X_1 = x_1) \\
&\quad = \sum_{x_2 : g(x_2) \leq a} \Psi(X_2 = x_2 \mid X_1 = x_1)\Psi(Y_2 \neq x_2 \mid X_2 = x_2) \\
&\quad = \sum_{x_2 : g(x_2) \leq a} \Psi(X_2 = x_2 \mid X_1 = x_1) g(x_2) \leq a.
\end{aligned}$$

In the same fashion, $\Psi(X_2 \neq Y_2, h(Y_2) \leq a \mid Y_1 = y_1) \leq a$. Finally,

$$\begin{aligned}
&\Psi(X_2 = Y_2 \in D, h(Y_2) > a \mid X_1 = x_1) \\
&\quad = \Psi(X_2 = Y_2 \in D \mid X_1 = x_1)\Psi(h(Y_2) > a \mid X_1 = x_1, X_2 = Y_2 \in D) \\
&\quad = \Psi(X_2 = Y_2 \in D \mid X_1 = x_1)\Psi(h(Y_2) > a \mid Y_1 = x_1, X_2 = Y_2 \in D) \\
&\quad = \frac{\Psi(X_2 = Y_2 \in D \mid X_1 = x_1)}{\Psi(X_2 = Y_2 \in D \mid Y_1 = x_1)} \Psi(X_2 = Y_2 \in D, h(Y_2) > a \mid Y_1 = x_1) \\
&\quad \leq \frac{\epsilon_2/a + \epsilon_1}{1 - 2\epsilon_1 - \epsilon_3 - \epsilon_2/a - a},
\end{aligned}$$

where the second equality is because $X_2 = Y_2 \in D \implies X_1 = Y_1$, and the inequality is the previous bounds, and a union bound on $\Psi(\{X_2 = Y_2 \in D\}^c \mid Y_1 = x_1)$. Putting things together,

$$\begin{aligned}
\Psi(H^c \mid X_1 = x_1) &\leq \Psi(X_2 \in D^c \mid X_1 = x_1) + \Psi(g(X_2) > a \mid X_1 = x_1) \\
&\quad + \Psi(X_2 \neq Y_2, g(X_2) \leq a \mid X_1 = x_1) \\
&\quad + \Psi(X_2 = Y_2 \in D, h(Y_2) > a \mid X_1 = x_1) \\
&\leq \epsilon_3 + \epsilon_1 + \frac{\epsilon_2}{a} + \epsilon_1 + a + \frac{\epsilon_2/a + \epsilon_1}{1 - 2\epsilon_1 - \epsilon_3 - \epsilon_2/a - a}.
\end{aligned}$$

One obtains the same bound on $\Psi(H^c \mid Y_1 = y_1)$ in the same way (with $h \leftrightarrow g$ and $Y_2 \leftrightarrow X_2$). Plugging in $a = \sqrt{\epsilon_2}$ and using the definition of ϵ gives that the last display is upper bounded by

$$5\epsilon + \frac{2\epsilon}{1 - 5\epsilon} \leq 9\epsilon,$$

as $\epsilon \leq 0.1$ by hypotheses. □

Let us now see how Claims 7 and 8 imply the wanted result. As Ψ is a coupling,

$$\frac{\mu_1(\xi)}{\nu_1(\xi)} = \frac{\Psi(X_1 = \xi)}{\Psi(Y_1 = \xi)} = \frac{\Psi(X_1 = \xi)\Psi(Y_1 = \xi, H)}{\Psi(Y_1 = \xi)\Psi(X_1 = \xi, H)} = \frac{\Psi(H \mid Y_1 = \xi)}{\Psi(H \mid X_1 = \xi)},$$

where the second equality is because $\{X_1 = Y_1\}$ under H (as $\{X_2 = Y_2 \in D\} \subset H$). But now,

$$1 - 9\epsilon \leq 1 - \Psi(H^c \mid Y_1 = \xi) \leq \frac{\Psi(H \mid Y_1 = \xi)}{\Psi(H \mid X_1 = \xi)} \leq \frac{1}{1 - \Psi(H^c \mid X_1 = \xi)} \leq \frac{1}{1 - 9\epsilon}.$$

□

Chapter 5

Discontinuous transition in 2D Potts: II. Order-order coupling

This chapter considers the FK measures under wired-wired Dobrushin boundary conditions corresponding to the Potts measures under order-order Dobrushin boundary conditions, both with $q > 4$ and at their transition points. We present the analogue of the coupling with the six-vertex spin and the ATRC measures in Section 4.4. On the ATRC side, a pair of 'interface clusters' will naturally appear, and we will show that they behave repulsively towards each other. In a subsequent work [DGO25b], we use this coupling to verify the wetting phenomenon (see Section 1.4.1).

5.1 Setting

5.1.1 Graph definitions

We use the graph definitions from Section 4.2 and modify the respective domains accordingly.

Standard rectangular domains. Define the upper and lower half planes by

$$\mathbb{H}^+ := \mathbb{R} \times \mathbb{R}_{\geq 0}, \quad \mathbb{H}^- := \mathbb{R} \times \mathbb{R}_{<0},$$

and, for $n, m \geq 0$, set

$$\begin{aligned} \mathsf{B}_{n,m} &:= \Lambda_{n,m} = \{-n, \dots, n\} \times \{-m, \dots, m\}, \\ \mathsf{B}'_{n,m} &:= \{-n - \tfrac{1}{2}, \dots, n + \tfrac{1}{2}\} \times \{-m - \tfrac{1}{2}, \dots, m + \tfrac{1}{2}\}. \end{aligned}$$

Define the subgraph $\mathsf{G}_{n,m} = (\mathsf{V}_{n,m}, \mathsf{E}_{n,m})$ of $\mathbb{Z}^2$ by (see Fig. 5.1)

$$\mathsf{V}_{n,m} := \mathsf{B}_{n,m} \cup \partial^{\text{ex}} \mathsf{B}_{n,m} \quad \text{and} \quad \mathsf{E}_{n,m} = \mathbb{E}_{\mathsf{B}_{n,m}} \cup \partial^{\text{edge}} \mathsf{B}_{n,m}.$$

Moreover, set $\mathsf{D}_{n,m} := \mathsf{B}_{n,m} \cup \mathsf{B}'_{n,m}$, and let $\mathsf{A}_{n,m} \subset \mathbb{L}_\diamond$ be the set of tiles with at least one corner in $\mathsf{D}_{n,m}$ and $\partial \mathsf{A}_{n,m} \subset \mathsf{A}_{n,m}$ (boundary tiles) be the set of tiles with precisely one corner in D; see Fig. 5.2. Define also (see Figs. 5.3 and 5.4)

$$\begin{aligned} v_L(n) &:= (-n-1, 0), & v_R(n) &:= (n+1, 0), \\ v'_L(n) &:= (-n - \tfrac{3}{2}, -\tfrac{1}{2}), & v'_R(n) &:= (n + \tfrac{3}{2}, -\tfrac{1}{2}). \end{aligned}$$

Augmented rectangular domains. We also introduce $\mathsf{E}_{n,m}$ augmented by some boundary edges. First, define the set of boundary edges by

$$\mathsf{E}_{\mathrm{b},n,m} := \{e_t : t \in \partial \mathsf{A}_{n,m}\} = \mathbb{E}_{\mathsf{B}_{n+1,m+1}} \setminus \mathsf{E}_{n,m}.$$

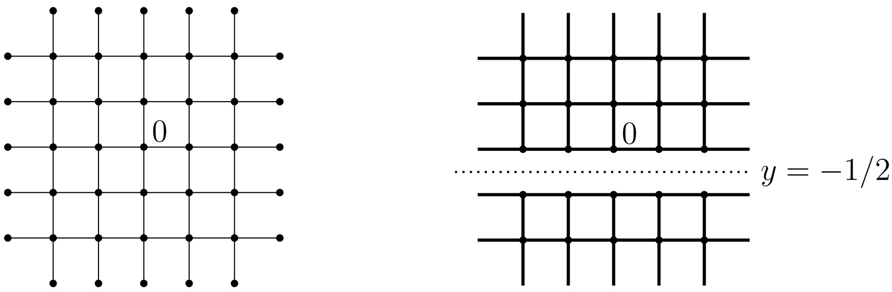


Figure 5.1: Left: the graph $\mathsf{G}_{2,2}$. Right: wired-wired Dobrushin boundary condition $\xi_{1/1}$.

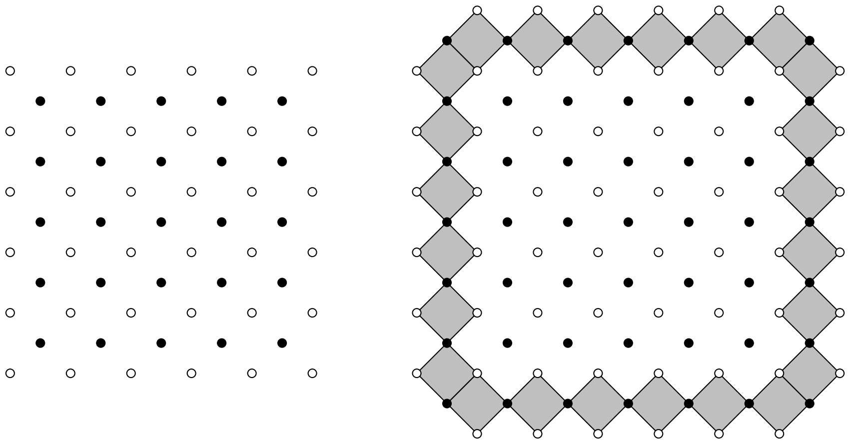


Figure 5.2: Left: the sets $\mathsf{B}_{2,2}$ and $\mathsf{B}'_{2,2}$. Right: the boundary tiles $\partial\mathsf{A}_{2,2}$.

Then define the augmented sets of edges and vertices by

$$\bar{\mathsf{E}}_{n,m} := \mathsf{E}_{n,m} \cup \mathsf{E}_{\mathrm{b},n,m} = \mathbb{E}_{\mathsf{B}_{n+1,m+1}} \qquad \text{and} \qquad \bar{\mathsf{V}}_{n,m} := \mathbb{V}_{\bar{\mathsf{E}}_{n,m}} = \mathsf{B}_{n+1,m+1},$$

and the corresponding augmented graphs (see Fig. 5.3) as follows:

- set $\mathsf{K}_{n,m} := (\bar{\mathsf{V}}_{n,m}, \bar{\mathsf{E}}_{n,m})$, and let $\mathsf{K}^1_{n,m}$ be the graph obtained from $\mathsf{K}_{n,m}$ by identifying the vertices in $\partial^{\mathrm{in}}\bar{\mathsf{V}}_{n,m} \cap \mathbb{H}^+$ and those in $\partial^{\mathrm{in}}\bar{\mathsf{V}}_{n,m} \cap \mathbb{H}^-$,
- set $\mathsf{K}'_{n,m} := (\mathbb{V}_{*\bar{\mathsf{E}}_{n,m}}, *\bar{\mathsf{E}}_{n,m})$, and let $(\mathsf{K}'_{n,m})^1$ be the graph obtained from $\mathsf{K}'_{n,m}$ by identifying the vertices in $\partial^{\mathrm{in}}\mathbb{V}_{*\bar{\mathsf{E}}_{n,m}} \cap \mathbb{H}^+$ and those in $\partial^{\mathrm{in}}\mathbb{V}_{*\bar{\mathsf{E}}_{n,m}} \cap \mathbb{H}^-$.

5.1.2 FK percolation with wired-wired Dobrushin boundary conditions.

Recall the definition of the FK percolation measures in Section 4.1. Consider the subgraph $\mathsf{G}_{n,m} = (\mathsf{V}_{n,m}, \mathsf{E}_{n,m})$ defined in Section 5.1.1 above. Define the wired-wired Dobrushin boundary condition $\xi_{1/1} \in \mathbb{E}_{\mathbb{Z}^2}$ by (see Figure 5.1)

$$\xi_{1/1}(e) = \mathbb{1}_{e \subset \mathbb{H}^+} + \mathbb{1}_{e \subset \mathbb{H}^-}, \qquad \text{and set} \qquad \mathsf{FK}^{1/1}_{\mathsf{G}_{n,m};p,q} := \mathsf{FK}^{\xi_{1/1}}_{\mathsf{G}_{n,m};p,q}.$$

Analogously to the order-disorder case in Section 4.1, the Edwards–Sokal coupling relates this measure to the Potts measure $\mathtt{Potts}^{1/2}_{\mathsf{B}_{n,m};T,q}$ with $p = 1 - e^{1/T}$ and where the boundary condition is given by $\eta_{1/2}(x) = \mathbb{1}_{x \in \mathbb{H}^+} + 2\mathbb{1}_{x \in \mathbb{H}^-}$ (see also Section 1.4.1 and Figure 1.8).

We will use the following representation. For $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$, define $\mathrm{loop}_{\mathsf{G}_{n,m}}(\omega)$ as the set of loops surrounding vertices in $\mathsf{V}_{n,m}$ obtained from ω by the mapping in Figure 4.3.

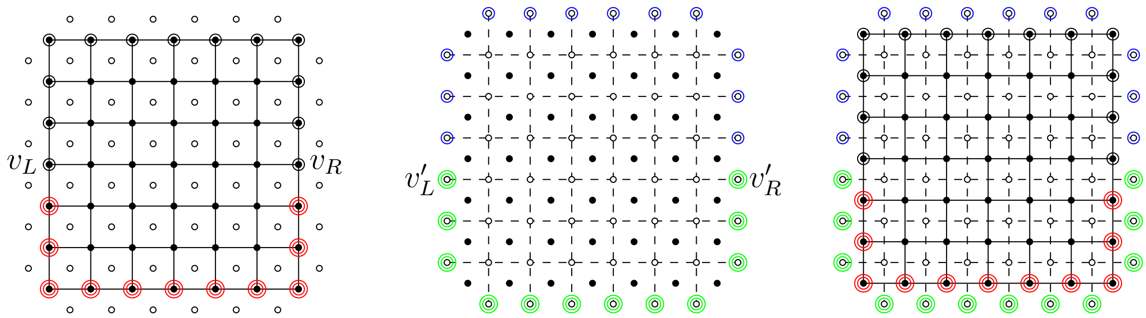


Figure 5.3: The graphs $\mathsf{K}_{2,2}$ (left), $\mathsf{K}'_{2,2}$ (middle), and the planar duality relation between their edges (right). Solid vertices surrounded by a black circle and solid vertices surrounded by two red circles are identified in $\mathsf{K}^1_{2,2}$, respectively. Hollow vertices surrounded by a blue circle and hollow vertices surrounded by two green circles are identified in $(\mathsf{K}'_{2,2})^1$, respectively.

Lemma 5.1.1. *If $T = T_c(q) = \ln(1+\sqrt{q})^{-1}$, one has that*

$$\mathsf{FK}^{1/1}_{\mathsf{G}_{n,m};q,T,\beta}(\omega) \propto \sqrt{q}^{|\mathrm{loop}_{\mathsf{G}_{n,m}}(\omega)|}\, \mathbb{1}_{\omega=\eta^{1/1} \text{ on } \mathbb{E}^\bullet \setminus \mathsf{E}_{n,m}}.$$

5.2 Coupling under wired-wired Dobrushin conditions

The idea is analogous to the order-disorder case in Section 4.4.2. Take the FK measure with wired-wired Dobrushin boundary conditions and construct from it first a measure on pairs of spin configurations on $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$ that satisfy the ice rule and then a measure on ATRC configurations. We will then identify these two measures as the six-vertex and the ATRC measures, respectively, under suitable versions of Dobrushin boundary conditions. This gives a coupling between the FK and a modified ATRC, which will allow us to transfer the study of the former to the study of the latter.

Recall the definition of the subgraphs $\mathsf{G}_{n,m} = (\mathsf{V}_{n,m}, \mathsf{E}_{n,m})$ of $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ given in Section 5.1.1. As n, m will be fixed in this section, we will omit them in the notation and simply write $\mathsf{G} = (\mathsf{V}, \mathsf{E})$. Recall also the definition of the FK measure $\mathsf{FK}^{1/1}_{\mathsf{G},p_c(q),q}$ on G with $q > 4$ and under wired-wired Dobrushin boundary conditions in Section 5.1.2 above (see Fig. 5.1). Since q and $p_c(q)$ will also be fixed, we will simply refer to it as $\mathsf{FK}^{1/1}_{\mathsf{G}}$.

Coupling measure. To couple $\mathsf{FK}^{1/1}_{\mathsf{G}}$ with both a six-vertex spin measure and a modified version of the ATRC measure, we augment our probability space to incorporate independent uniform $[0,1]$ random variables assigned to every loop and every tile. Formally, let $\mathcal{L}$ be the set of all unoriented loops drawn on the tiles of $\mathbb{L}_\diamond$ (see the right of Fig. 4.3). Define $\Omega^{1/1} := \{0,1\}^{\mathbb{E}^\bullet} \times [0,1]^{\mathcal{L}} \times [0,1]^{\mathbb{L}_\diamond}$ equipped with the product of Borel sigma algebras. Define Q and Q' respectively as the product measures on $[0,1]^{\mathcal{L}}$ and $[0,1]^{\mathbb{L}_\diamond}$. Finally, the coupling measure is defined by

$$\Psi^{1/1}_{\mathsf{G}} := \mathsf{FK}^{1/1}_{\mathsf{G}} \otimes Q \otimes Q'.$$

In Lemmata 5.2.1 and 5.2.2 below, we describe how to obtain the following two measures as push-forwards of $\Psi^{1/1}_{\mathsf{G}}$: the six-vertex spin measure and the modified ATRC measure, both under suitable Dobrushin boundary conditions (Definitions 11 and 12).

From FK to six-vertex. Recall the parameters $\mathbf{c}$, $\mathbf{c}_\mathrm{b}$, and λ from (4.3) and the standard rectangular domains from Section 5.1. Below we omit n, m everywhere.

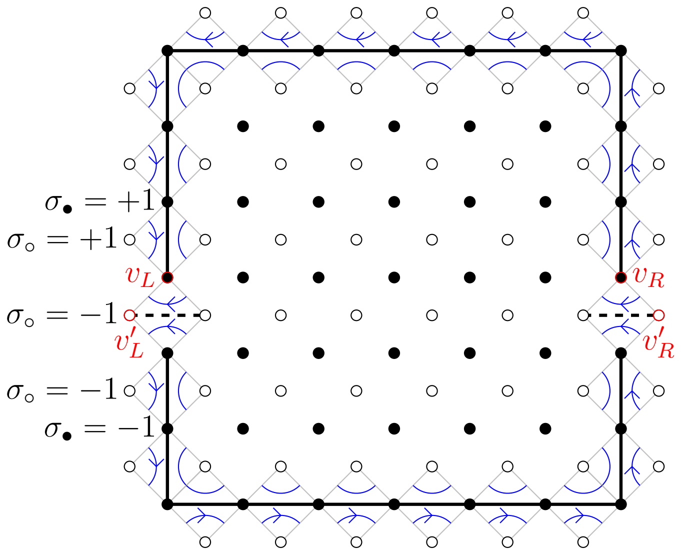


Figure 5.4: Boundary conditions on oriented loops and on spins.

Definition 11. The six-vertex spin model under *double* Dobrushin conditions is a probability measure on $\sigma = (\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$ defined by

$$\mathtt{Spin}_{\mathsf{D}}^{+-,+-}(\sigma) \propto \mathbf{c}^{|T^{\mathrm{i}}_{5,6}(\sigma)|}\, \mathbf{c}_{\mathrm{b}}^{|T^{\mathrm{b}}_{5,6}(\sigma)|}\, \mathbb{1}_{\Sigma^{+-}_{\mathsf{B}} \times \Sigma^{+-}_{\mathsf{B}'}}(\sigma)\, \mathbb{1}_{\mathrm{ice}}(\sigma), \tag{5.1}$$

where $\mathbf{c}, \mathbf{c}_{\mathrm{b}}$ are defined by (4.3)-(4.4), $T^{\mathrm{i}}_{5,6}$ and $T^{\mathrm{b}}_{5,6}$ are the sets of tiles of types 5-6 in A^{i} and $\partial\mathsf{A}$, respectively (see 4.8 and Fig. 5.2), $\mathbb{1}_{\mathrm{ice}}$ is the indicator imposing the ice rule (4.9), and Σ^{+-}_{B} and $\Sigma^{+-}_{\mathsf{B}'}$ are the boundary conditions defined by

$$\Sigma^{+-}_{\mathsf{B}} := \{\sigma_\bullet \in \{\pm 1\}^{\mathbb{L}_\bullet} : \sigma_\bullet(i) = \mathbb{1}_{\mathbb{H}^+}(i) - \mathbb{1}_{\mathbb{H}^-}(i)\, \forall i \in \mathbb{L}_\bullet \setminus \mathsf{B}\},$$
$$\Sigma^{+-}_{\mathsf{B}'} := \{\sigma_\circ \in \{\pm 1\}^{\mathbb{L}_\circ} : \sigma_\circ(u) = \mathbb{1}_{\mathbb{H}^+}(u) - \mathbb{1}_{\mathbb{H}^-}(u)\, \forall u \in \mathbb{L}_\circ \setminus \mathsf{B}'\}.$$

We now describe how to obtain $\mathtt{Spin}_{\mathsf{D}}^{+-,+-}$ as a push-forward of the coupling measure $\Psi^{1/1}_{\mathsf{G}}$.

Lemma 5.2.1. *Let (ω, U, U') be distributed according to $\Psi^{1/1}_{\mathsf{G}}$ and let ℓ be the unoriented loop configuration associated to ω. Orient the loops of ℓ as follows (see Fig. 5.4):*

- *each loop $l \in \ell$ outside of G (i.e. surrounding a vertex in $\mathbb{L}_\circ \setminus \mathsf{B}'$) is oriented clockwise;*
- *the two bi-infinite paths are oriented from right to left;*
- *each loop $l \in \ell$ inside of G (i.e. surrounding a vertex in $\mathsf{B} \cup \mathsf{B}'$) is oriented clockwise if $U_l < e^\lambda/\mathbf{c}$ and counter-clockwise otherwise.*

Denote by $\ell_\to$ the obtained oriented loop configuration. Recall the combinatorial mappings introduced above and let $(\sigma_\bullet, \sigma_\circ)$ be the associated six-vertex spin configurations with $\sigma_\bullet((n+1, 0)) = +1$. Then, the law of $(\sigma_\bullet, \sigma_\circ)$ is given by $\mathtt{Spin}_{\mathsf{D}}^{+-,+-}$.

The proof is very similar to that of Lemma 4.4.1, but we provide it for completeness.

Proof. We adapt the proof of the order-disorder case, following the ideas of [BKW76]. One has to examine which values of (ω, U) result in a given $(\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$. The probability to obtain $(\sigma_\bullet, \sigma_\circ) \in \Sigma^{+-}_{\mathsf{B}} \times \Sigma^{+-}_{\mathsf{B}'}$ is the sum of the probabilities of all oriented loop configurations $\ell_\to$

that induce the edge orientations corresponding to $(\sigma_\bullet, \sigma_\circ)$. The probability of a given oriented loop configuration $\ell_\to$ satisyfing the boundary conditions in Fig. 5.4 is proportional to

$$\sqrt{q}^{\,|\mathrm{loop}(\ell_\to)|} \cdot \left(\tfrac{e^\lambda}{e^\lambda+e^{-\lambda}}\right)^{|\mathrm{loop}_{\circlearrowright}(\ell_\to)|-|\mathrm{loop}_{\circlearrowleft}(\ell_\to)|} \propto e^{\lambda(|\mathrm{loop}_{\circlearrowright}(\ell_\to)|-|\mathrm{loop}_{\circlearrowleft}(\ell_\to)|)}, \tag{5.2}$$

where $\mathrm{loop}_{\circlearrowright}(\ell_\to)$ and $\mathrm{loop}_{\circlearrowleft}(\ell_\to)$ are respectively the sets of clockwise and counter-clockwise oriented loops in $\ell_\to$ not imposed by boundary conditions, and $\mathrm{loop}(\ell_\to)$ is their union. Indeed, by Lemma 5.1.1, the first factor on the left side is proportional to $\mathsf{FK}_{\mathsf{G}}^{1/1}(\omega(\ell))$, where $\omega(\ell) \in \{0,1\}^{\mathbb{E}^\bullet}$ is the percolation configuration associated to the unoriented loop configuration ℓ corresponding to $\ell_\to$. The second factor comes from the values of the uniforms necessary to obtain the correct orientations of loops in $\mathrm{loop}(\ell_\to)$. The proportionality holds since $\sqrt{q} = e^\lambda + e^{-\lambda}$ due to the choice of λ.

Notice that each loop which is oriented clockwise does 4 more right quarter-turns than left quarter-turns, that the converse holds for counter-clockwise oriented loops, and that the numbers of left and right quarter-turns in the two bi-infinite paths differ by a universal constant. Thus, the expression on the right side of (5.2) is proportional to

$$\exp(\lambda(\#_{\frown}(\ell_\to) - \#_{\frown}(\ell_\to))/4),$$

where $\#_{\frown}(\ell_\to)$ and $\#_{\frown}(\ell_\to)$ are respectively the number of right and left quarter-turns in $\ell_\to$. The key idea is to count these oriented loop arcs locally at each tile in $\mathsf{A} = \mathsf{A}^{\mathrm{i}} \cup \mathsf{A}^{\mathrm{b}}$. Observe that, for tiles in A^{i}, types 5B,6A correspond to a pair of right-oriented loop arcs and types 5A,6B to a pair of left-oriented loop arcs, whereas types 1-4 correspond to one right-oriented and one left-oriented loop arc each. Moreover, due to the boundary conditions (see Fig. 5.4), a tile in A^{b} contains a right turn precisely if it is of type 5,6, and it contains a left turn otherwise. We deduce that the probability of $\ell_\to$ is proportional to

$$\prod_{t \in T_{5,6}(\ell_\to)\cap \mathsf{A}^{\mathrm{i}}} \left(e^{\lambda/2}\,\mathbb{1}_{T_{5B,6A}(\ell_\to)}(t) + e^{-\lambda/2}\,\mathbb{1}_{T_{5A,6B}(\ell_\to)}(t)\right) \cdot \left(e^{\lambda/2}\right)^{|T_{5,6}(\ell_\to)\cap \partial \mathsf{A}|}, \tag{5.3}$$

where $T_{5B,6A}(\ell_\to)$ and $T_{5A,6B}(\ell_\to)$ are respectively the sets of tiles of types 5B and 6A and the set of tiles of types 5A and 6B in $\ell_\to$, and $T_{5,6}(\ell_\to)$ is their union.

Finally, fix a pair $(\sigma_\bullet, \sigma_\circ) \in \Sigma_{\mathsf{B}}^{+-} \times \Sigma_{\mathsf{B}'}^{+-}$ that satisfies the ice-rule, and consider its associated edge orientations. It remains to identify all oriented loop configurations $\ell_\to$ that satisfy the boundary conditions in Fig. 5.4 and that induce these edge orientations. Observe that the boundary conditions and the spins $(\sigma_\bullet, \sigma_\circ)$ uniquely determine the oriented loop arcs at tiles in $(\mathbb{L}_\diamond \setminus \mathsf{A}) \cup \mathsf{A}^{\mathrm{b}}$ and at tiles in $\mathsf{A}^{\mathrm{i}} \setminus T^{\mathrm{i}}_{5,6}(\sigma_\bullet, \sigma_\circ)$. For a tile in $T^{\mathrm{i}}_{5,6}(\sigma_\bullet, \sigma_\circ)$, one can split the oriented edges either into a pair of right-oriented loops arcs (types 5B,6A) or into a pair of left-oriented loop arcs (types 5A,6B). Summing the probabilities (5.3) of all oriented loop configurations obtained in that way, we obtain that the probability of $(\sigma_\bullet, \sigma_\circ)$ is proportional to

$$\left(e^{\lambda/2} + e^{-\lambda/2}\right)^{|T^{\mathrm{i}}_{5,6}(\sigma_\bullet,\sigma_\circ)|} \cdot \left(e^{\lambda/2}\right)^{|T^{\mathrm{b}}_{5,6}(\sigma_\bullet,\sigma_\circ)|}.$$

Recalling that $\mathbf{c} = e^{\lambda/2} + e^{-\lambda/2}$ and $\mathbf{c}_{\mathrm{b}} = e^{\lambda/2}$ finishes the proof. □

From six-vertex to modified ATRC. We now adapt the coupling described in [GP23] to the double Dobrushin boundary conditions. In particular, we need to define the associated modified ATRC measure. Recall the parameters from (4.3) and the (augmented) rectangular domains from Section 5.1. Again, we omit n, m from the notation.

Definition 12. The modified ATRC model $\mathtt{mATRC}_{n,m} \equiv \mathtt{mATRC}_{\mathsf{K}}$ is a probability measure on $\{0,1\}^{\bar{\mathsf{E}}} \times \{0,1\}^{\bar{\mathsf{E}}}$ defined by

$$\begin{aligned}&\mathtt{mATRC}_{\mathsf{K}}(\omega_\tau, \omega_{\tau\tau'}) \\ &\quad \propto \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}} \, 2^{|\omega_\tau \cap \mathsf{E}|} \left(\tfrac{2}{\mathbf{c}_{\mathrm{b}}-1}\right)^{|\omega_\tau \cap \mathsf{E}_{\mathrm{b}}|} (\mathbf{c}-2)^{|\omega_{\tau\tau'} \setminus \omega_\tau|} \, 2^{k_{\mathsf{K}}(\omega_\tau) + k_{\mathsf{K}^1}(\omega_{\tau\tau'})}.\end{aligned} \tag{5.4}$$

where $k_{\mathsf{K}}(\omega_\tau)$ and $k_{\mathsf{K}^1}(\omega_{\tau\tau'})$ are the numbers of clusters of ω_τ and $\omega_{\tau\tau'}$ viewed as spanning subgraphs of K and K^1, respectively. We say that a cluster of ω_τ or $\omega_{\tau\tau'}$ is an *inner cluster* if it is entirely contained in B and a *boundary cluster* otherwise.

We now describe how to obtain $\mathtt{mATRC}_{\mathsf{K}}$ as a push-forward of the coupling measure $\Psi_{\mathsf{G}}^{1/1}$.

Lemma 5.2.2. *Let (ω, U, U') be distributed according to $\Psi^{1/1}$. Define $(\sigma_\bullet, \sigma_\circ)$ as described in Lemma 5.2.1. Now define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\bar{\mathsf{E}}}$ as follows using the notation $e_t = ij \in \bar{\mathsf{E}}$ and $e_t^* = uv$ for each tile $t \in \mathsf{A} = \mathsf{A}^{\mathrm{i}} \sqcup \partial\mathsf{A}$:*

- *if $\sigma_\circ(u) \neq \sigma_\circ(v)$, set $\omega_\tau(e_t) = \omega_{\tau\tau'}(e_t) = 1$;*
- *if $\sigma_\bullet(i) \neq \sigma_\bullet(j)$, set $\omega_\tau(e_t) = \omega_{\tau\tau'}(e_t) = 0$;*
- *if both $\sigma_\circ(u) = \sigma_\circ(v)$ and $\sigma_\bullet(i) = \sigma_\bullet(j)$ hold (types 5-6), let*

$$(\omega_\tau(e_t), \omega_{\tau\tau'}(e_t)) = \begin{cases} \mathbb{1}_{[0,\frac{1}{\mathbf{c}})}(U_t') \cdot (1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}},\frac{2}{\mathbf{c}})}(U_t') \cdot (0,0) + \mathbb{1}_{[\frac{2}{\mathbf{c}},1]}(U_t') \cdot (0,1) & \text{if } t \in \mathsf{A}^{\mathrm{i}}, \\ \mathbb{1}_{[0,\frac{1}{\mathbf{c}_{\mathrm{b}}})}(U_t') \cdot (1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}_{\mathrm{b}}},1]}(U_t') \cdot (0,0) & \text{if } t \in \partial\mathsf{A}. \end{cases}$$

Then, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is $\mathtt{mATRC}_{\mathsf{K}}(\cdot \mid v_L \xleftrightarrow{\omega_\tau} v_R, v_L' \xleftrightarrow{\omega_{\tau\tau'}^*} v_R')$. In particular, we have $\omega_\tau \subseteq \omega_{\tau\tau'}$ and $\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \{e_t : t \in \mathsf{A}^{\mathrm{i}}\} = \mathsf{E}$; also $\sigma_\bullet \sim \omega_{\tau\tau'}$ and $\sigma_\circ \sim \omega_\tau^*$.

Remark 5.2.3. The above sampling rule for $t \in \mathsf{A}^{\mathrm{i}}$ applied to a six-vertex spin or height measure without modified boundary weight $\mathbf{c}_{\mathrm{b}}$ yields an ATRC measure, as defined in Secton 4.3.

The proof is again very similar to that of Lemma 4.4.2, but we provide it for completeness.

Proof. We adapt the proof of the order-disorder case, building on [GP23, Proof of Lemma 7.1]. One has to examine which values of $(\sigma_\bullet, \sigma_\circ, U)$ result in a given pair $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\bar{\mathsf{E}}}$. Recall that $\sigma_\bullet \sim \omega_{\tau\tau'}$ and $\sigma_\circ \sim \omega_\tau^*$. By (5.1), the probability of a quadruplet $(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'})$ is

$$\begin{aligned}&\mathtt{Spin}_{\mathsf{D}}^{+-,+-}(\sigma_\bullet, \sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}} \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*} \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}} \\ &\qquad \cdot \prod_{t \in T_{5,6}^{\mathrm{i}}} \left(\tfrac{1}{\mathbf{c}} (\mathbb{1}_{t \in \omega_\tau} + \mathbb{1}_{t \in \omega_{\tau\tau'}^*}) + \tfrac{\mathbf{c}-2}{\mathbf{c}} \mathbb{1}_{t \in \omega_{\tau\tau'} \setminus \omega_\tau}\right) \prod_{t \in T_{5,6}^{\mathrm{b}}} \left(\tfrac{1}{\mathbf{c}_{\mathrm{b}}} \mathbb{1}_{t \in \omega_\tau} + \tfrac{\mathbf{c}_{\mathrm{b}}-1}{\mathbf{c}_{\mathrm{b}}} \mathbb{1}_{t \in \omega_\tau^*}\right) \\ &\propto \mathbb{1}_{\Sigma_{\mathsf{B}}^{+-}}(\sigma_\bullet) \mathbb{1}_{\Sigma_{\mathsf{B}'}^{+-}}(\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}} \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*} \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}} \\ &\qquad \cdot \prod_{t \in T_{5,6}^{\mathrm{i}}} \left(\mathbb{1}_{t \in \omega_\tau} + \mathbb{1}_{t \in \omega_{\tau\tau'}^*} + (\mathbf{c}-2) \mathbb{1}_{t \in \omega_{\tau\tau'} \setminus \omega_\tau}\right) \prod_{t \in T_{5,6}^{\mathrm{b}}} \left(\mathbb{1}_{t \in \omega_\tau} + (\mathbf{c}_{\mathrm{b}}-1) \mathbb{1}_{t \in \omega_\tau^*}\right),\end{aligned} \tag{5.5}$$

where we used the shorthand $T_{5,6}^{\#} \equiv T_{5,6}^{\#}(\sigma_\bullet, \sigma_\circ)$ and the fact that, for $\sigma_\bullet \in \Sigma_{\mathsf{B}}^{+-}$ and $\sigma_\circ \in \Sigma_{\mathsf{B}'}^{+-}$,

$$\mathbb{1}_{\mathrm{ice}}(\sigma_\bullet, \sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}} \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*} \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} = \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}} \mathbb{1}_{\sigma_\circ \sim \omega_\tau^*} \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}.$$

Observe that, on the event $\{\sigma_\bullet \sim \omega_{\tau\tau'},\, \sigma_\circ \sim \omega_\tau^*,\, \omega_\tau \subseteq \omega_{\tau\tau'}\}$, for any tile $t \in \mathsf{A}^{\mathrm{i}}$,

$$\mathbb{1}_{T_{5,6}^{\mathrm{i}}}(t) = \mathbb{1}_{\omega_\tau}(t) \mathbb{1}_{\sigma_\circ \sim e_t^*} + \mathbb{1}_{\omega_{\tau\tau'}^*}(t) \mathbb{1}_{\sigma_\bullet \sim e_t} + \mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau}(t).$$

Let $t_1, t_2 \in \partial\mathsf{A}$ be the two boundary tiles that intersect both $\mathbb{H}^+$ and $\mathbb{H}^-$. Then, on the same event as above intersected with $\{\sigma_\bullet \in \Sigma^{+-}_{\mathsf{B}},\ \sigma_\circ \in \Sigma^{+-}_{\mathsf{B}'},\ \omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}\}$, for any tile $t \in \partial\mathsf{A} \setminus \{t_1, t_2\}$,

$$\mathbb{1}_{T^{\mathrm{b}}_{5,6}}(t) = \mathbb{1}_{\omega_\tau}(t)\mathbb{1}_{\sigma_\circ \sim e^*_t} + \mathbb{1}_{\omega^*_\tau}(t).$$

Notice also that, due to the boundary condition $\sigma_\bullet \in \Sigma^{+-}_{\mathsf{B}}$, the two boundary tiles t_1, t_2 cannot be of type 5,6. Moreover, since $\sigma_\bullet \sim \omega_{\tau\tau'}$, the associated edges $e_{t_1}, e_{t_2} \in \mathsf{E}_{\mathrm{b}}$ most be closed in $\omega_{\tau\tau'}$ (and ω_τ, as they coincide on E_{b}). Altogether, (5.5) becomes

$$\begin{aligned}
&\mathbb{1}_{\Sigma^{+-}_{\mathsf{B}}}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\mathsf{B}'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\mathbb{1}_{\sigma_\circ \sim \omega^*_\tau}\mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}} \\
&\cdot \prod_{t \in \mathsf{A}^{\mathrm{i}}} \left(\mathbb{1}_{t \in \omega_\tau} + \mathbb{1}_{t \in \omega^*_{\tau\tau'}} + (\mathbf{c}-2)\mathbb{1}_{t \in \omega_{\tau\tau'} \setminus \omega_\tau}\right)^{\mathbb{1}_{T^{\mathrm{i}}_{5,6}}(t)} \prod_{t \in \partial\mathsf{A}} \left(\mathbb{1}_{t \in \omega_\tau} + (\mathbf{c}_{\mathrm{b}}-1)\mathbb{1}_{t \in \omega^*_\tau}\right)^{\mathbb{1}_{T^{\mathrm{b}}_{5,6}}(t)} \\
= {}&\mathbb{1}_{\Sigma^{+-}_{\mathsf{B}}}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\mathsf{B}'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\mathbb{1}_{\sigma_\circ \sim \omega^*_\tau}\mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}}\,(\mathbf{c}-2)^{|\omega_{\tau\tau'} \setminus \omega_\tau|}\,(\mathbf{c}_{\mathrm{b}}-1)^{|\mathsf{E}_{\mathrm{b}} \setminus \omega_\tau|-2}.
\end{aligned}$$

Observe that $\sigma_\circ \in \Sigma^{+-}_{\mathsf{B}'}$ and $\sigma_\circ \sim \omega^*_\tau$ imply that v_L and v_R are connected in ω_τ (see Fig. 5.4). To obtain the probability of a pair $(\omega_\tau, \omega_{\tau\tau'})$ satisfying $v_L \xleftrightarrow{\omega_\tau} v_R$, we need to sum the last expression over $(\sigma_\bullet, \sigma_\circ)$. The configurations $\sigma_\circ \in \Sigma^{+-}_{\mathsf{B}'}$ with $\sigma_\circ \sim \omega^*_\tau$ are in bijective correspondence with assignments of ± 1 to the clusters of ω^*_τ in $(\mathsf{K}')^1$ that are contained in B', whence there exist $2^{k_{(\mathsf{K}')^1}(\omega^*_\tau)-2}$ of them. By Euler's formula,

$$k_{(\mathsf{K}')^1}(\omega^*_\tau) = k_{\mathsf{K}}(\omega_\tau) + |\omega_\tau| - \mathrm{const}(\mathrm{K}').$$

Similarly, if v'_L and v'_R are connected in $\omega^*_{\tau\tau'}$, then there exist $2^{k_{\mathsf{K}^1}(\omega_{\tau\tau'})-2}$ spin configurations $\sigma_\bullet \in \Sigma^{+-}_{\mathsf{B}}$ with $\sigma_\bullet \sim \omega_{\tau\tau'}$. Therefore, the probability of a tuple $(\omega_\tau, \omega_{\tau\tau'})$ with $v_L \leftrightarrow v_R$ in ω_τ and $v'_L \leftrightarrow v'_R$ in $\omega^*_{\tau\tau'}$ is proportional to

$$\mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}}\,(\mathbf{c}-2)^{|\omega_{\tau\tau'} \setminus \omega_\tau|}(\mathbf{c}_{\mathrm{b}}-1)^{|\mathsf{E}_{\mathrm{b}} \setminus \omega_\tau|}\,2^{k_{\mathsf{K}}(\omega_\tau)+|\omega_\tau|}\,2^{k_{\mathsf{K}^1}(\omega_{\tau\tau'})},$$

which is proportional to $\mathtt{mATRC}_{\mathsf{K}}(\omega_\tau, \omega_{\tau\tau'})$ as defined in (5.4), and the proof is complete. $\square$

5.2.1 Positive association and repulsiveness

We continue in the setting of the previous section. The measure $\mathtt{mATRC}_{\mathsf{K}}$ is strongly positively associated.

Lemma 5.2.4. $\mathtt{mATRC}_{\mathsf{K}}$ *satisfies the FKG-lattice condition: for any* $a, b, a', b' \in \{0,1\}^{\bar{\mathsf{E}}}$,

$$\mathtt{mATRC}_{\mathsf{K}}(a \cup a', b \cup b')\,\mathtt{mATRC}_{\mathsf{K}}(a \cap a', b \cap b') \geq \mathtt{mATRC}_{\mathsf{K}}(a, b)\,\mathtt{mATRC}_{\mathsf{K}}(a', b'). \tag{5.6}$$

Proof. First observe that the indicators $\mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}$, $\mathbb{1}_{\omega_{\tau\tau'} \setminus \omega_\tau \subseteq \mathsf{E}}$ satisfy (5.6). Indeed,

- $a \subseteq b$ and $a' \subseteq b'$ imply $a \cup a' \subseteq b \cup b'$ and $a \cap a' \subseteq b \cap b'$,
- $b \setminus a \subseteq \mathsf{E}$ and $b' \setminus a' \subseteq \mathsf{E}$ imply $(b \cup b') \setminus (a \cup a') \subseteq \mathsf{E}$ and $(b \cap b') \setminus (a \cap a') \subseteq \mathsf{E}$.

From now on, assume that $a \subseteq b$ and $a' \subseteq b'$ (otherwise, the right side of (5.6) is zero). We now check (5.6) for each factor of (5.4). For the factor $2^{|\omega_\tau \cap \mathsf{E}|}$ in (5.4), the exponent on the left side of (5.6) is

$$\begin{aligned}
|(a \cup a') \cap \mathsf{E}| + |(a \cap a') \cap \mathsf{E}| &= |(a \cap \mathsf{E}) \cup (a' \cap \mathsf{E})| + |(a \cap \mathsf{E}) \cap (a' \cap \mathsf{E})| \\
&= |a \cap \mathsf{E}| + |a' \cap \mathsf{E}|,
\end{aligned}$$

which is the exponent on the right side. Replacing E by E_b in the above computation gives the same for the factor $(2/(\mathbf{c}_\mathrm{b}-1))^{|\omega_\tau\cap\mathsf{E}_\mathrm{b}|}$ in (5.4). For the factor $(\mathbf{c}-2)^{|\omega_{\tau\tau'}\setminus\omega_\tau|}$ in (5.4), the exponent on the left side of (5.6) is the sum of the following two:

$$|(b\cup b')\setminus(a\cup a')| = |b\setminus(a\cup b')| + |b'\setminus(b\cup a')| + |(b\cap b')\setminus(a\cup a')|,$$
$$|(b\cap b')\setminus(a\cap a')| = |(b\cap b')\setminus(a\cup a')| + |(a\cap b')\setminus a'| + |(b\cap a')\setminus a|.$$

On the other hand, the exponent on the right side is the sum of the following two

$$|b\setminus a| = |b\setminus(a\cup b')| + |(b\cap b')\setminus(a\cup a')| + |(b\cap a')\setminus a|,$$
$$|b'\setminus a'| = |b'\setminus(b\cup a')| + |(b\cap b')\setminus(a\cup a')| + |(a\cap b')\setminus a'|.$$

Clearly, these sums coincide. It remains to check the inequality for the last factor $2^{\kappa_{\mathsf{K}}(\omega_\tau)+\kappa_{\mathsf{K}^1}(\omega_{\tau\tau'})}$ in (5.4). It is classical (see, e.g., [Gri06, Proof of Theorem 3.8]) that

$$\kappa_{\mathsf{K}}(a\cup a') + \kappa_{\mathsf{K}}(a\cap a') \geq \kappa_{\mathsf{K}}(a) + \kappa_{\mathsf{K}}(a'),$$

and analogously for $\kappa_{\mathsf{K}^1}(\omega_{\tau\tau'})$. This completes the proof. □

This lemma implies that connections in ω_τ induce 'maximal boundary conditions'. Recall first the notion of sets being above and below each other, introduced in Definition 10.

Corollary 5.2.5. *Let p be a deterministic set of edges forming a simple path that connects v_L to v_R. Let $p_\uparrow$ and $p_\downarrow$ be the sets of edges in $\bar{\mathsf{E}}\setminus p$ that are above and below p, respectively. For any $a_0, b_0 \in \{0,1\}^p$, $a_\uparrow, b_\uparrow \in \{0,1\}^{p_\uparrow}$ with $a_0\cup a_\uparrow \subseteq b_0\cup b_\uparrow$,*

$$\begin{aligned}
&\mathtt{mATRC}_{\mathsf{K}}\big((\omega_\tau|_{p_\downarrow},\omega_{\tau\tau'}|_{p_\downarrow})\in\cdot \;\big|\; (\omega_\tau|_p,\omega_{\tau\tau'}|_p)=(a_0,b_0),\,(\omega_\tau|_{p_\uparrow},\omega_{\tau\tau'}|_{p_\uparrow})=(a_\uparrow,b_\uparrow)\big)\\
\leq_{\mathrm{st}}\,&\mathtt{mATRC}_{\mathsf{K}}\big((\omega_\tau|_{p_\downarrow},\omega_{\tau\tau'}|_{p_\downarrow})\in\cdot \;\big|\; \omega_\tau|_p\equiv 1,\,(\omega_\tau|_{p_\uparrow},\omega_{\tau\tau'}|_{p_\uparrow})=(a_\uparrow,b_\uparrow)\big)\\
=\,&\mathtt{mATRC}_{\mathsf{K}}\big((\omega_\tau|_{p_\downarrow},\omega_{\tau\tau'}|_{p_\downarrow})\in\cdot \;\big|\; \omega_\tau|_p\equiv 1\big).
\end{aligned}$$

Remark 5.2.6. In regard to Lemma 5.2.2, the corollary implies that the connection imposed by $v_L \overset{\omega_\tau}{\longleftrightarrow} v_R$ has a repulsive effect on the dual connection imposed by $v_L' \overset{\omega^*_{\tau\tau'}}{\longleftrightarrow} v_R'$.

Proof. Since $\mathtt{mATRC}_{\mathsf{K}}(\omega_\tau\subseteq\omega_{\tau\tau'})=1$, the stochastic domination statement is a consequence of Lemma 5.2.4 and [Gri06, Theorem 2.27]. The equality follows if, for $a_\uparrow, b_\uparrow\in\{0,1\}^{p_\uparrow}$ and $a_\downarrow, b_\downarrow\in\{0,1\}^{p_\downarrow}$, the probability

$$\mathtt{mATRC}_{\mathsf{K}}\big((\omega_\tau|_{p_\downarrow},\omega_{\tau\tau'}|_{p_\downarrow})=(a_\downarrow,b_\downarrow),\,\omega_\tau|_p\equiv 1,\,(\omega_\tau|_{p_\uparrow},\omega_{\tau\tau'}|_{p_\uparrow})=(a_\uparrow,b_\uparrow)\big)$$

factorises into two parts that depend only on $(a_\downarrow,b_\downarrow)$ and $(a_\uparrow,b_\uparrow)$, respectively.

Recall that $\mathtt{mATRC}_{\mathsf{K}}(\omega_\tau\subseteq\omega_{\tau\tau'})=1$, and set

$$(a,b) = (a_\downarrow\sqcup p\sqcup a_\uparrow,\, b_\downarrow\sqcup p\sqcup b_\uparrow).$$

Then, by the definition (5.4) of the measure, the above probability equals

$$\begin{aligned}
\mathtt{mATRC}_{\mathsf{K}}\big((\omega_\tau,\omega_{\tau\tau'})=(a,b)\big) &\propto 2^{|a\cap\mathsf{E}|}\big(\tfrac{2}{\mathbf{c}_\mathrm{b}-1}\big)^{|a\cap\mathsf{E}_\mathrm{b}|}(\mathbf{c}-2)^{|b\setminus a|}2^{\kappa_{\mathsf{K}}(a)+\kappa_{\mathsf{K}^1}(b)}\\
&= 2^{|a_\downarrow\cap\mathsf{E}|+|p\cap\mathsf{E}|+|a_\uparrow\cap\mathsf{E}|}\big(\tfrac{2}{\mathbf{c}_\mathrm{b}-1}\big)^{|a_\downarrow\cap\mathsf{E}_\mathrm{b}|+|p\cap\mathsf{E}_\mathrm{b}|+|a_\uparrow\cap\mathsf{E}_\mathrm{b}|}(\mathbf{c}-2)^{|b_\downarrow\setminus a_\downarrow|+|b_\uparrow\setminus a_\uparrow|}2^{\kappa_{\mathsf{K}}(a)+\kappa_{\mathsf{K}^1}(b)}.
\end{aligned}$$

For $\# \in\{\varnothing,1\}$, let $\mathsf{K}^\#_\uparrow$ (respectively, $\mathsf{K}^\#_\downarrow$) be the graph obtained from $\mathsf{K}^\#$ by identifying all vertices on and below p (respectively, on and above p). Since p divides $\mathsf{K}^\#$ into two disjoint parts and since $p\subseteq a,b$, we clearly have

$$\kappa_{\mathsf{K}}(a) = \kappa_{\mathsf{K}_\downarrow}(a_\downarrow)+\kappa_{\mathsf{K}_\uparrow}(a_\uparrow)-1 \qquad\text{and}\qquad \kappa_{\mathsf{K}^1}(b)=\kappa_{\mathsf{K}^1_\downarrow}(b_\downarrow)+\kappa_{\mathsf{K}^1_\uparrow}(b_\uparrow)-1.$$

Therefore, the probability factorises, and the proof is complete. □

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