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Zeros of Gaussian Random Functions

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Zusammenfassung

Das Ziel dieser Arbeit ist es, ein besseres Verständnis einiger Konzepte in der Theorie bestimmter Punktprozesse zu bieten, die aus Nullmengen von Gaußschen Zufallsfunktionen stammen.

In Kapitel 1 fassen wir die Definitionen und Resultate der Wahrscheinlichkeitstheorie und der reellen Analysis zusammen, die zum Verständnis der Inhalte der folgenden Kapitel erforderlich sind. Wir werden einen Überblick über Zufallsvektoren und Konvergenz von Zufallsvariablen geben, einige der wichtigen Aspekte der mehrdimensionalen Gaußschen Verteilung behandeln und den Abschnitt mit einer kurzen Erinnerung an Faltungen beenden.

In Kapitel 2 werden wir zunächst den Begriff des Punktprozesses mit technischen Annahmen einführen, die einen geeigneten Rahmen für die Behandlung von Nullmengen von Zufallsfunktionen als einen bestimmten Typ von Punktprozessen gewährleisten. Wir definieren erste Intensitätsfunktionen, um einen Einblick in die Verteilung eines Punktprozesses zu geben. Anschließend werden wir den Begriff einer Gaußschen analytischen Funktion (GAF) einführen und zeigen, dass diese aus einer Koeffizientenfolge von gaußschen Zufallsvariablen und einer geeigneten Folge deterministischer holomorpher Funktionen konstruiert werden kann. Nachdem wir die Nullmengen von Gaußschen analytischen Funktionen definiert haben, schließen wir Kapitel 2, indem wir zeigen, dass die erste Intensitätsfunktion solcher Nullmengen eine explizite Formel zulässt – die sogenannte Edelman-Kostlan-Formel. Die hier präsentierten Beweise stammen aus [1] (unserer Hauptreferenz für diesen Abschnitt) und wir werden gelegentlich auf einige Argumente näher eingehen.

In Kapitel 3 betrachten wir genauer zwei spezielle Typen von Gaußschen analytischen Funktionen, die aufgrund ihrer Isometrie-Invarianz von besonderem Interesse sind. Genauer ausgedrückt werden wir zeigen (wiederum [1] folgend), dass die entsprechenden Nullmengen der beiden Typen invariant in Verteilung unter Isometriegruppen ihrer jeweiligen Domänen sind. Das zentrale Resultat dieses Abschnitts betrifft die Ergodizität der Nullmengen isometrie-invarianter GAFs. Für dieses Resultat folgen wir der Idee des in [1] für planare GAFs vorgestellten Beweises und passen ihn an den hyperbolischen Fall an. In einem Versuch, die Argumente näher zu erläutern, werden wir die zugrunde liegende Wahrscheinlichkeitsstruktur einer GAF in Bezug auf seine zugehörigen Koeffizienten untersuchen, was uns zu einem Approximationsresultat führt, das im oben genannten Beweis verwendet wird. Der Übersichtlichkeit halber fügen wir auch einige technische Zwischenschritte mit Verweisen auf das im Kapitel 1 besprochene Hintergrundmaterial hinzu.

Schließlich wenden wir uns in Kapitel 4 dem Fall einer bestimmten Klasse

nicht-analytischer Gaußscher Zufallsfunktionen zu, die in [12] eingeführt wurden – den Gaußschen Weyl-Heisenberg-Funktionen (GWHF). Nachdem wir die Rahmenbedingungen und die notwendigen Definitionen eingeführt haben, folgen wir dem Aufbau des analytischen Falls und geben einige der Resultate aus [12] an, einschließlich der Invarianz unter den sogenannten *twisted shifts*, sowie der Formel für die erste Intensitätsfunktion ihrer Nullmengen. Anschließend zeigen wir die Ergodizität der Nullmengen von GWHFs. Während die Idee des Beweises dieselbe bleibt wie in Kapitel 3, erfordert die Nicht-Analytizität dieser Funktionen eine andere Vorgehensweise für einige der entscheidenden Argumente. Dieses Kapitel soll erläutern, wie diese Hindernisse umgangen werden können, und die notwendigen Anpassungen aufzeigen, die man im Beweis für eine analytische Gaußsche Funktion vornehmen müsste. Motiviert durch die Karhunen-Loève-Entwicklung für einen stochastischen Prozess, der auf einem kompakten reellen Intervall indiziert ist, betrachten wir in diesem Zusammenhang eine Entwicklung einer GWHF in Bezug auf mit ihrem Kovarianzkernel verbundenen Funktionen und entsprechend definierten Gaußschen Koeffizienten. Wir werden sehen, dass dies möglich ist, indem wir eine entsprechende Mercer-ähnliche Entwicklung für den Kovarianzkernel mithilfe der Faltungstheorie und approximativer Identitäten aufstellen.

Abstract

The aim of this thesis is to offer a better understanding of some of the concepts in the theory of particular point processes stemming from zero sets of Gaussian random functions.

In Chapter 1 we revisit the definitions and results from probability theory and real analysis that are necessary in order to understand the contents of subsequent chapters. We will review random vectors and convergence of random variables, cover some of the important aspects of multivariate Gaussian distribution and end the section with a brief reminder on convolutions.

In Chapter 2, we will first introduce the notion of point processes with technical assumptions that guarantee an appropriate setting for tackling zeros of random functions as a particular type of point processes. We define first intensity functions to provide some insight into the distribution of a point process. We will then introduce the notion of a Gaussian analytic function (GAF) and show that these can be constructed from a coefficient sequence of Gaussian random variables and a suitable sequence of non-random holomorphic functions. After defining the zero sets of Gaussian analytic functions, we close off Chapter 2 by showing that the first intensity function of such zero sets allows for an explicit formula - the so-called Edelman-Kostlan formula. The proofs we present here are from [1] (our main reference for this section) and we will occasionally elaborate on some of the arguments.

In Chapter 3, we look more closely at two distinguished types of Gaussian analytic functions, which are of particular interest due to their isometry-invariance property. More precisely, we will show (again, following [1]) that the corresponding zero sets of the two types are invariant in distribution under isometry groups of their respective domains. The central result of this section is concerned with ergodicity of the zero sets of isometry-invariant GAFs. For this result we follow the idea of the proof presented in [1] for planar GAFs, adjusting it to the hyperbolic case. In an attempt to elaborate on the arguments, we will explore the underlying probabilistic structure of a GAF in terms of its associated coefficients, which leads us to an approximation result used in the aforementioned proof. For the sake of clarity, we also add some intermediate technical steps with references to the background material reviewed in Chapter 1.

Lastly, in Chapter 4, we turn to the case of a specific class of non-analytic Gaussian random functions introduced in [12] - the Gaussian Weyl-Heisenberg functions (GWHF). After introducing the setting and necessary definitions, we follow the layout of the analytic case and state some of the

results from [12], including the invariance under the so-called twisted shifts, as well as the formula for the first intensity function of their zero sets. We then proceed to show the ergodicity of the zero sets of GWHFs. While the idea of the proof remains the same as in Chapter 3, non-analyticity of these functions requires a different approach to some of the decisive arguments. This chapter should clarify how to sidestep these obstacles and indicate the necessary adjustments one would need to make in the proof for an analytic Gaussian function. In this context, motivated by the Karhunen-Loève expansion for a stochastic processes indexed on a compact real interval, we consider an expansion of a GWHF with respect to functions associated to its covariance kernel and suitably defined Gaussian coefficients. We will see that this is possible upon establishing a corresponding Mercer-like expansion for the covariance kernel, which we will show by making use of convolution theory and approximate identities.

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Contents

Notation glossary

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . . . the unit disk

$\mathcal{B}(\Lambda)$ . . . the Borel σ-algebra of a topological space Λ

$\mathcal{H}(\Lambda)$ . . . the space of holomorphic functions on Λ

$C_b(\Lambda)$ . . . the space of bounded continuous functions on Λ

$C_c(\Lambda)$ . . . the space of continuous functions on Λ with compact support

$C^k(\Lambda)$ . . . the space of k-times continuously differentiable functions on Λ

$C_c^\infty(\Lambda)$ . . . the space of smooth functions on Λ with compact support

$\operatorname{supp}(f)$ . . . support of a function f

2^X . . . power set of a set X, i.e. $2^X = \{A \mid A \subseteq X\}$

dm . . . Lebesgue measure on $\mathbb{C}$

I_n . . . n-dimensional identity matrix

$A^\top$ . . . transpose of a matrix (or vector) A

A^* . . . conjugate transpose of a matrix A

$\#B$ . . . cardinality of a set B

δ_x . . . Dirac measure at a point x

$\stackrel{d}{=}$ . . . equality in distribution

$\mathbb{1}_A$ . . . the indicator function of a set A

$|\Lambda|$ . . . Lebesgue measure of $\Lambda \subseteq \mathbb{C}$

Δf . . . Laplacian of a function f

$A \triangle B$ . . . symmetric difference of the sets A and B

$\mathrm{D}(0, R) = \{z \in \mathbb{C} \ : \ |z| < R\}$ . . . open disk of radius R centered at 0

$\partial\mathrm{D}(0, R) = \{z \in \mathbb{C} \ : \ |z| = R\}$ . . . circle of radius R centered at 0

$\mathbb{C}^\mathbb{C}$ . . . the set of functions $\mathbb{C} \to \mathbb{C}$

$DF(x)$ . . . Jacobian matrix of a function $F : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$

0 Introduction

Without diving into precise mathematical formulations, one can think of a point process as a random allocation of points in some mathematical space. They are well-suited for modelling "physical phenomena characterized by highly localized events distributed randomly in a continuum" ([2], 1.1 Introduction), which is why their study has taken various directions and its applications are found among many different scientific fields, e.g. in electrical and communication engineering, counting problems, astronomy, nuclear medicine, seismology, and many other (see [2] or [3]).

One way to produce such a random configuration of points would be to take a polynomial whose coefficients are random variables and then extract its zeros. The study of zeros of random polynomials has a long history, dating back to Mark Kac (1959) and even Paley and Wiener (1934)[1], with the topic evolving in numerous directions over the years. Gaussian random polynomials (i.e. those whose coefficients corresponding to normally distributed random variables) and their limits, i.e. Gaussian power series, turned out to be quite useful in physical applications relating to chaotic quantum systems, gases of interacting particles and random matrices (see [3], [4], [5] and references therein).

Random polynomials and analytic functions are particularly useful because they can be used (upon suitable assumptions) to generate a point process whose occurrences exhibit mutual repulsion (see Section 1.1 in [1]). In the context of physical applications, this kind of anti-clumping behaviour is often desired in modelling certain phenomenona - for example, think of a system of like-charged particles (e.g. electrons) subjected to an external potential (closely related to the theory of Coulomb gases, an active area of mathematical and physical research). It was Macchi who introduced "a process whose points tend to exclude one another" ([6], 1975) to describe the statistical distribution of a fermion system in thermal equilibrium. He called this a "fermion process" - today these are known as determinantal point processes, a special class of repulsive point processes which, apart from quantum mechanics, are also used in matrix theory, combinatorics and machine learning and which a large portion of our main reference ([1]) is dedicated to.

A related question to that of zeros of random polynomials concerns the zeros of infinite linear combinations of the form

$$\sum_{n=0}^{\infty} a_n \psi_n(z) \tag{0.1}$$

[1]see [1], *2.6. Notes* and references therein

for suitable analytic functions ψ_n on some domain in $\mathbb{C}$ and i.i.d. Gaussians a_n. We will see that under certain technical conditions these define the so-called Gaussian analytic functions (GAFs). The main object of interest in the first part of this thesis are the so-called isometry-invariant GAFs, whose zero sets give rise to a point process which is invariant in distribution under the corresponding transformations of their domains. The planar GAFs (defined on $\mathbb{C}$) were partly introduced in the 1990s by Bogomolny-Bohigas-Leboeuf, Kostlan and Shub-Smale (see [1] and references therein). Since Bogomolny-Bohigas-Leboeuf investigated these functions in the context of quantum chaotic systems ([7]), they are sometimes referred to as "chaotic analytic functions" (see [8],[9]). The other class we will examine here are the hyperbolic GAF (defined on the unit disk), which are of the form

$$\sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\cdots(L+n-1)}}{\sqrt{n!}} z^n \tag{0.2}$$

for i.i.d. standard complex Gaussians a_n and $L > 0$. Setting the parameter $L = 1$, the expression (0.2) simplifies to the above mentioned Gaussian power series

$$\sum_{n=0}^{\infty} a_n \, z^n$$

which has proven to be of particular interest since it defines a determinantal point process, as shown by Peres and Virág in [3] (for a thorough discussion see Section 5 in [1]).

The question (0.1) goes back to Paley and Wiener, as well as Kac and Rice (see [5] and references therein). In his contributions to this problem, Rice "systematically treated both theoretical and applied aspects of random noises in radio signals" ([5]). The connection of the theory of Gaussian analytic functions to signal processing and harmonic analysis are still being explored in the ongoing research (for example, see [10], [11] and works cited therein). However, a less restrictive setting than analyticity is often wished for in applications, as explained in [12], the setting of which we will adopt for the second part of this thesis. We will see that a specific class of non-analytic Gaussian random function also provides us with an a point process stochastically invariant under the transformations of their domain. This is achieved by imposing stochastic invariance under a certain representation of the Weyl-Heisenberg group on the function, which is why they are referred to as Gaussian Weyl-Heisenberg functions. The stochastic invariance of their zero sets - as in the case of isometry-invariant GAFs - allows us to carry over the arguments from the results presented in this thesis to the non-analytic setting.

1 Preliminaries

We start by recalling and collecting some notions and results from probability theory and real analysis that will be used throughout the thesis. As they are not of separate interest for the thesis, we will mostly omit the proofs and refer to the relevant sources.

1.1 Random vectors and convergence of random variables

Before going into the specifics of multivariate normal distribution, we use this section to revisit some of the essential definitions and results from probability theory concerning random vectors and different notions of convergence.

We briefly recall a few basic facts about random vectors. A (n-dimensional) real random vector $X = (X_1, \dots, X_n)^\top$ is a measurable mapping from a probability space into $\mathbb{R}^n$, or, equivalently, a n-tuple of random variables $X_1, \dots, X_n$. Accordingly, a complex random vector is a tuple of complex random variables. The distribution function of a real random vector $X = (X_1, \dots, X_n)^\top$ is a real-valued function F_X defined on $\mathbb{R}^n$ via

$$F_X(x_1, \dots, x_n) = \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n)$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The law or probability distribution Q_X of X is the probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ defined via

$$Q_X(B) := \mathbb{P}(X \in B), \quad B \in \mathcal{B}(\mathbb{R}^n),$$

where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra of $\mathbb{R}^n$. The expectation $\mathbb{E}[Z] \in \mathbb{C}^n$ of a complex random vector $Z = (Z_1, \dots, Z_n)^\top$ is defined as

$$\mathbb{E}[Z] = (\, \mathbb{E}[Z_1], \dots, \mathbb{E}[Z_n] \,)^\top$$

and the notion of variance for complex random variables is replaced by the covariance (matrix) K_{ZZ} of Z given by

$$\begin{aligned} K_{ZZ} &:= \operatorname{cov}(Z) := \mathbb{E}[(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^*] \in \mathbb{C}^{n \times n} \\ \text{i.e. } (K_{ZZ})_{ij} &= \mathbb{E}[(Z_i - \mathbb{E}[Z_i])\overline{(Z_j - \mathbb{E}[Z_j])}\,], \end{aligned}$$

where * denotes the conjugate transpose. Note that a covariance matrix is necessarily a positive semidefinite matrix, since

$$\begin{aligned} a^*\, \mathbb{E}[(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^*]\, a &= \mathbb{E}[\, a^*(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^* a \,] \\ &= \mathbb{E}[\, |a^*(Z - \mathbb{E}[Z])|^2 \,] \geq 0 \end{aligned}$$

for any $a \in \mathbb{C}^n$ and similarly for the real case. Hence, the covariance matrix of a random vector is non-singular (i.e. invertible), if and only if it is positive definite.

Two real random vectors (of possibly different dimension) X and Y are said to be uncorrelated, if their cross-covariance matrix K_{XY} is a zero matrix, where

$$(K_{XY})_{i,j} = \mathbb{E}\left[\,(X_i - \mathbb{E}[X_i])(Y_j - \mathbb{E}[Y_j])\,\right]. \tag{1.1}$$

and one can easily see that $K_{XY} = 0$ is equivalent to $\mathbb{E}\left[XY^\top\right] = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]^\top$. In the case of complex random variables Z and W, uncorrelatedness means that their covariance and pseudo-covariance are 0, i.e.

$$\mathbb{E}\left[Z\overline{W}\right] = \mathbb{E}\left[ZW\right] = 0. \tag{1.2}$$

Note that identifying $Z = Z_1 + iZ_2$ and $W = W_1 + iW_2$ with 2-dimensional real random vectors consisting of their real and imaginary parts we have the compatibility of these two definitions:

$$\begin{aligned} &Z, W \text{ uncorrelated in the sense of (1.2)} \\ \iff\ &(Z_1, Z_2)^\top, (W_1, W_2)^\top \text{uncorrelated in the sense of (1.1)}, \end{aligned}$$

as one can easily verify.

Recall that a n-dimensional complex random vector $Z = (Z_1, \ldots, Z_n)^\top$ with $\mathrm{Re}(Z_i) = X_i$ and $\mathrm{Im}(Z_i) = Y_i$ can be identified with the $2n$-dimensional real random vector by decomposing the corresponding coordinates into their real and imaginary parts:

$$(X, Y)^\top = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^\top,$$

so that the cumulative distribution function of Z is defined as the joint cumulative distribution function of X and Y, i.e.

$$F_Z(z) = \mathbb{P}(X_1 \leq x_1, Y_1 \leq y_1, \ldots, X_n \leq x_n, Y_n \leq y_n)$$

for $\mathbb{C}^n \ni z = (z_1, \ldots, z_n) \cong (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$, where $z = x + i \cdot y$.

We state the following definition of different convergence notions for the case of real random vectors. Note that the same definitions hold in the case of random vectors with complex coordinates by the above identification with real random vectors consisting of their real and imaginary parts. It obviously also includes (complex) random variables as a special case. We use the notation $|\cdot|$ for both the Euclidean norm in $\mathbb{R}^d$ ($d \geq 2$) and the absolute value on the real line (i.e. the Euclidean norm on $\mathbb{R}$ in the case $d = 1$).

Definition 1.1. *Let X and $(X_n)_{n\in\mathbb{N}}$ be random vectors defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$.*

(i) Denote with Q_X and Q_{X_n} the probability distributions of X and X_n, respectively. We say that X_n converges to X in distribution[2]*, if Q_{X_n} converges weakly to Q_x, meaning*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f \, dQ_{X_n} = \int_{\mathbb{R}^d} f \, dQ_X$$

for all $f \in C_b(\mathbb{R}^d)$. We write $X_n \xrightarrow{d} X$ in this case. This type of convergence is also referred to as convergence in law or weak convergence.

(ii) We say that X_n converges to X in probability, if

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$$

for every $\varepsilon > 0$ and write $X_n \xrightarrow{\mathbb{P}} X$.

(iii) We say that X_n converges to X almost surely, if

$$\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1,$$

i.e., more precisely,

$$\mathbb{P}(\{\omega \in \Omega : \lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0\}) = 1,$$

and we write $X_n \xrightarrow{a.s.} X$ in this case.

(iv) Let $r \geq 1$. Provided that X_n have finite r-th moments, meaning $\mathbb{E}[\,|X_n|^r\,] < \infty$, we say that X_n converges to X in r-th mean, if

$$\lim_{n\to\infty} \mathbb{E}[\,|X_n - X|^r\,] = 0$$

and write $X_n \xrightarrow{L^r} X$.

Remark. *(i) It is not difficult to see that the convergence in probability of random vectors is characterized by convergence in probability of their corresponding vector components. More precisely, if $(X_n)_{n\in\mathbb{N}}$ is a sequence of $\mathbb{R}^d$-valued random vectors with $X_n = (X_n^{(1)}, \ldots, X_n^{(d)})^\top$, then*

$$X_n \xrightarrow{\mathbb{P}} X \quad \Longleftrightarrow \quad X_n^{(i)} \xrightarrow{\mathbb{P}} X^{(i)} \;\; \forall\, i \in \{1, \ldots, d\},$$

[2] The definition of convergence in distribution does not in fact require X_n and X to be defined on the same probability space.

where $X = (X^{(1)}, \dots, X^{(d)})^\top$.
(ii) There is a more intuitive way of introducing convergence in distribution for real random vectors (following Section 23.9 in [13] and Sections 20, 29 in [14]), namely

$$\lim_{n\to\infty} F_{X_n}(x_1, \dots, x_d) = F_X(x_1, \dots, x_d)$$

for all continuity points $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ *of* F_X*, i.e. all* $x \in \mathbb{R}^d$*, such that*

$$\lim_{\varepsilon\downarrow 0} F_X(x_1 - \varepsilon, \dots, x_d - \varepsilon) = F_X(x).$$

However, since we will not be dealing with distributions of random vectors explicitly, we chose this approach (which is frequently seen in a more general setting) from [15] to be consistent with the proof of Lemma 1.4.

The special case $r = 2$ of Definition 1.1 (iv) is commonly referred to as *convergence in mean square* (and abbreviated with m.s.), which is the type of convergence we will mostly work with in Section 4. Note that mean square convergence is in fact convergence in the vector space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of (equivalence classes of a.s. equal) random variables with finite second moments (i.e. $\mathbb{E}[\,|X|^2\,] < \infty$) equipped with

$$\langle X, Y\rangle := \mathbb{E}[\,X\overline{Y}\,] \text{ and } \|X\|_{L^2} = \mathbb{E}[\,|X|^2\,]^{1/2}.$$

We know that L^2 defined on an arbitrary measure space defines a Hilbert space; consequently, this is also true for any given probability space. We state this as a result and refer to the proof in [16] for the sake of completeness:

Proposition 1.2. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Then* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *is a Hilbert space.*

Proof. See Proposition 4, p. 399 in [16]. □

Two consequences are immediate: due to completeness, showing that a sequence (X_n) of random variables with finite second moments is Cauchy in m.s., meaning

$$\forall\varepsilon > 0\ \exists\, N \in \mathbb{N} \text{ such that } \mathbb{E}[\,|X_m - X_n|^2\,] < \varepsilon \text{ for } m, n > N,$$

suffices to show (X_n) is convergent in m.s. to some random variable X with a finite second moment. Secondly, we can use the (reverse) triangle inequality of the norm expressed in terms of the expectation:

$$|\mathbb{E}[\,|X|^2\,]^{1/2} - \mathbb{E}[\,|Y|^2\,]^{1/2}| \le \mathbb{E}[\,|X - Y|^2\,]^{1/2}.$$

We now take note of the hierarchy between the above defined notions of convergence. Almost-sure convergence is a stronger notion than convergence in probability:

Lemma 1.3. *Let* $(X_n)_{n\in\mathbb{N}}$ *and* X *be (complex) random variables defined on a common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. If* $X_n \xrightarrow{a.s.} X$*, then* $X_n \xrightarrow{\mathbb{P}} X$*.*

Proof. Theorem 4.5.1. A in [15]. □

Convergence in probability implies convergence in distribution:

Lemma 1.4. *Let* X *and* $(X_n)_{n\in\mathbb{N}}$ *be random vectors defined on a common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in* $\mathbb{R}^d$*. If* $X_n \xrightarrow{\mathbb{P}} X$*, then* $X_n \xrightarrow{d} X$*.*

Proof. See Theorem 4.5.5 in [15]. □

Observation 1.5. *Suppose we are given a sequence of* $\mathbb{C}^d$*-valued random vectors* $(X_n)_n$ *such that* $X_n^{(i)} \xrightarrow{a.s.} X^{(i)}$ *for all* $i = 1, \dots, d$*. Then we know by Lemma 1.3 that* $X_n^{(i)} \xrightarrow{\mathbb{P}} X^{(i)}$ *and hence by Remark (i) after Definition 1.1 we obtain convergence of random vectors* $X_n \to X$ *in probability for* $X = (X^{(1)}, \dots, X^{(d)})^\top$*. Applying Lemma 1.4 yields* $X_n \xrightarrow{d} X$*. Therefore, almost sure convergence of corresponding components ensures the convergence of random vectors in distribution.*

Convergence in mean square implies convergence in probability:

Lemma 1.6. *Let* $(X_n)_{n\in\mathbb{N}}$ *and* X *be (complex) random variables. If* $X_n \xrightarrow{L^2} X$*, then* $X_n \xrightarrow{\mathbb{P}} X$*.*

Proof. This is an immediate consequence of Markov's inequality:

$$\mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{E}\,[\,|X_n - X|^2\,]/\varepsilon^2 \to 0.$$

□

Note that an analogous conclusion to Observation 1.5 holds if we replace almost-sure convergence by mean square convergence:

Observation 1.7. *Suppose we are given a sequence of* $\mathbb{C}^d$*-valued random vectors* $(X_n)_n$ *such that* $X_n^{(i)} \xrightarrow{L^2} X^{(i)}$ *for all* $i = 1, \dots, d$*. Then we know by Lemma 1.6 that* $X_n^{(i)} \xrightarrow{\mathbb{P}} X^{(i)}$ *and hence* $X_n \xrightarrow{d} X$ *by the same arguments from Observation 1.5. Therefore, mean square convergence of corresponding components ensures the convergence of random vectors in distribution.*

We make a brief note of a few basic facts that we will also employ:

(1) The convergence of a sequence of random variables in $L^1(\Omega)$ obviously implies the convergence of the corresponding expectation values, i.e.

$$X_n \xrightarrow{L^1} X \implies \mathbb{E}[X_n] \to \mathbb{E}[X].$$

This trivially also holds for convergence in $L^2(\Omega)$, which implies convergence in $L^1(\Omega)$ by the Cauchy-Schwartz inequality.

(2) Similarly, the convergence in mean square implies the convergence of the corresponding variances:

$$X_n \xrightarrow{L^2} X \implies \mathrm{Var}(X_n) \to \mathrm{Var}(X),$$

since the reverse triangle inequality for the norm $\| \; \|_{L^2(\Omega)}$ implies

$$|\mathbb{E}[\,|X|^2\,]^{1/2} - \mathbb{E}[\,|X_n|^2\,]^{1/2}| \le \mathbb{E}[\,|X - X_n|^2\,]^{1/2} \to 0$$

and hence

$$\mathrm{Var}(X_n) = \mathbb{E}[\,|X_n|^2\,] - |\,\mathbb{E}[X_n]\,|^2 \underset{n\to\infty}{\longrightarrow} \mathbb{E}[\,|X|^2\,] - |\,\mathbb{E}[X]\,|^2 = \mathrm{Var}(X).$$

(3) Given two sequences of random variables (X_n), (Y_n) converging to their limits X, Y in mean square (i.e. in $L^2(\Omega)$), we obtain convergence of their product in $L^1(\Omega)$ by the Cauchy-Schwartz inequality:

$$\begin{aligned} \mathbb{E}[\,|X_nY_n - XY|\,] &\le \mathbb{E}[\,|X_nY_n - X_nY|\,] + \mathbb{E}[\,|X_nY - XY|\,] \\ &\le \underbrace{\mathbb{E}[\,|X_n|^2\,]^{1/2}}_{\to \mathbb{E}[\,|X|^2\,]^{1/2}} \underbrace{\mathbb{E}[\,|Y_n - Y|^2\,]^{1/2}}_{\to 0} + \mathbb{E}[\,|Y|^2\,]^{1/2} \underbrace{\mathbb{E}[\,|X_n - X|^2\,]^{1/2}}_{\to 0} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

so that $\mathbb{E}[X_nY_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[XY]$ by the item (1).

1.2 Multivariate Gaussian distribution

We can now lay the groundwork for understanding Gaussian random functions (and their zero sets) by reviewing some essential facts about multivariate Gaussianity and related notions.

Recall that a (real-valued) random variable X is called a Gaussian, if it is normally distributed, i.e. its probability density function f_X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{1.3}$$

where μ is the expectation and σ^2 the variance of X. For such a random variable we use the standard notation $X \sim \mathcal{N}(\mu, \sigma^2)$.

The random variables we will be interested in are complex-valued, so we need the notion of a complex Gaussian:

Definition 1.8. *A standard complex Gaussian $Z = X + i\,Y$ is a complex-valued random variable whose real part X and imaginary part Y are i.i.d. random variables such that $X, Y \sim \mathcal{N}(0, \frac{1}{2})$.*

For the expectation and the variance of a standard complex Gaussian Z we obtain:

$$\mathbb{E}[Z] = \mathbb{E}[X] + i\,\mathbb{E}[Y] = 0 \quad \text{and}$$
$$\mathrm{Var}[Z] = \mathbb{E}[\,|Z - \mathbb{E}[Z]|^2\,] = \mathbb{E}[\,|Z|^2\,] = \mathbb{E}[X^2 + Y^2] = \mathrm{Var}(X) + \mathrm{Var}(Y) = 1,$$

which justifies calling it “standard”. We use the notation $Z \sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

Definition 1.9. *A complex random vector $X = (X_1, \dots, X_n)^\top$ is called a standard Gaussian vector, if its coordinates $X_1, \dots, X_n$ are i.i.d. standard complex Gaussians, i.e. if X_i are pairwise independent and*

$$X_i \sim \mathcal{N}_{\mathbb{C}}(0, 1) \;\; \forall i \in \{1, \dots, n\}.$$

Note that if $X = (X_1, \dots, X_n)^\top$ is a standard Gaussian vector, we have $\mathbb{E}[X] = (\,\mathbb{E}[X_1], \dots, \mathbb{E}[X_n]\,)^\top = 0$ and

$$(K_{XX})_{ij} = \mathbb{E}[X_i \overline{X_j}] = \mathbb{E}[X_i]\,\mathbb{E}[\overline{X_j}] = \delta_{ij} \quad \Longrightarrow \quad K_{XX} = I_n$$

and hence we use the notation

$$X \sim \mathcal{N}_{\mathbb{C}}^n(0, I_n).$$

Remark/Definition. *A standard (n-dimensional) Gaussian X is defined analogously in the real case - it is a real random vector whose coordinates are i.i.d. standard real Gaussians $X_i \sim \mathcal{N}(0, 1)$. In this case we use the notation $X \sim \mathcal{N}_{\mathbb{R}}^n(0, I_n)$.*
A real random vector X is called a Gaussian vector, if there exist $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$ such that $X \overset{d}{=} AY + b$, where Y is a m-dimensional standard real Gaussian vector. We denote such a vector with $X \sim \mathcal{N}_{\mathbb{R}}^n(\mu, \Sigma)$, where μ is the expectation and Σ the covariance matrix. If $X \overset{d}{=} AY + b$, then $\mu = b$ and $\Sigma = AA^\top$.

As different authors often start with a different definition for a (non-standard) complex Gaussian vector, we will state a characterization from [13] and note that one could take any of the following conditions as a definition, since they are equivalent (as shown in [13]):

Proposition 1.10. *Let X be a n-dimensional complex random vector. Each of the following conditions is equivalent to $X = (X_1, \dots, X_n)^\top$ being a complex Gaussian:*

(i) The real $2n$-dimensional random vector

$$(\mathrm{Re}(X_1), \dots, \mathrm{Re}(X_n), \mathrm{Im}(X_1), \dots, \mathrm{Im}(X_n))^\top$$

is a real Gaussian vector;

(ii) $a^\top Z$ is a complex Gaussian random variable for every $a \in \mathbb{C}^n$;

(iii) there exist $A, B \in \mathbb{C}^{n\times m}$ and $\mu \in \mathbb{C}^n$ such that

$$X \overset{d}{=} AY + B\overline{Y} + \mu,$$

where Y is a m-dimensional standard complex Gaussian and $\overline{Y}$ denotes componentwise complex conjugation.

Proof. See Proposition 24.3.10 in [13]. □

Remark. *We write $X \sim \mathcal{N}_{\mathbb{C}}^n(\mu, \Sigma)$ to indicate that X is a complex Gaussian vector of dimension n, or simply $X \sim \mathcal{N}_{\mathbb{C}}^n$ without specifying the expectation and covariance.*

Note that in the case of a general real random vector X the probability density function f_X can only exist if the covariance matrix K_{XX} is non-singular (see [13], Section 23.4.3). An invertible covariance matrix still does not necessarily imply the existence of f_X in general. However, if the random vector in question is a Gaussian, then invertibility suffices to ensure the existence of a probability density function:

Lemma 1.11. *Let $X \sim \mathcal{N}_{\mathbb{R}}^n(\mu, \Sigma)$ be a real random vector with a non-singular covariance matrix $\Sigma = K_{XX}$. Then the probability density function f_X of X exists and is given by*

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}}\, e^{-\frac{1}{2}(X-\mu)^\top \Sigma^{-1}(X-\mu)} \tag{1.4}$$

for $x \in \mathbb{R}^n$.

Proof. See Section 23.6.4 in [13]. □

Recall that the density function f_Z (given that it exists) of a complex random variable Z is defined as the joint intensity $f_{X,Y}$ of its real and imaginary parts $\mathrm{Re}(Z) = X$ and $\mathrm{Im}(Z) = Y$. Therefore, in the case of a standard complex Gaussian $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$, we obtain

$$f_Z(z) = f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{\sqrt{\pi}}e^{-x^2}\frac{1}{\sqrt{\pi}}e^{-y^2} = \frac{1}{\pi}e^{-|z|^2} \tag{1.5}$$

for $\mathbb{C} \ni z = x + i\,y$, where we have used the independence of $X, Y \sim \mathcal{N}(0, \frac{1}{2})$. Alternatively (and slightly redundantly), identifying $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$ with $(X,Y)^\top \sim \mathcal{N}_{\mathbb{R}}^2$, we notice that the corresponding covariance matrix

$$K_{ZZ} = \begin{bmatrix} \mathbb{E}\left[X^2\right] & \mathbb{E}\left[XY\right] \\ \mathbb{E}\left[YX\right] & \mathbb{E}\left[Y^2\right] \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{1.6}$$

is non-singular, so that Lemma 1.11 guarantees the existence of a f_Z and (1.4) indeed yields (1.5).

Knowing that two real or complex random vectors are Gaussians, it suffices to show that they have the same expectation and covariance to conclude that they are equal in distribution:

Lemma 1.12. *The expectation and the covariance of a Gaussian random vector determine its distribution.*

Proof. See the solution to the Exercise 2.1.1(ii) in Section 2.7 of [1]. □

A linear transformation of a Gaussian remains a Gaussian:

Lemma 1.13. *Let $X \sim \mathcal{N}_{\mathbb{C}}^n(\mu, \Sigma)$ be a complex Gaussian and $M \in \mathbb{C}^{m\times n}$ a (deterministic) matrix. Then $MX \sim \mathcal{N}_{\mathbb{C}}^m(M\mu, M\Sigma M^*)$.*

Proof. That MX is again a complex Gaussian is clear from Proposition 1.10 (iii). One easily verifies that $\mathbb{E}[MX] = M\mathbb{E}[X] = M\mu$ and

$$\begin{aligned} \mathrm{cov}(MX) &= \mathbb{E}[(MX - \mathbb{E}[MX])(MX - \mathbb{E}[MX])^*] \\ &= \mathbb{E}[M(X - \mathbb{E}[X])(X - \mathbb{E}[X])^* M^*] = M\mathrm{cov}(X)M^*. \end{aligned}$$

□

Another fact we will use later on is that weak limits of Gaussians are still Gaussians. For the proof of the following result we refer to [17] (Theorem 4, p.17). Note that the proof is done for the real-valued case; however, since complex Gaussian vectors can be characterized in terms of real Gaussian vectors (by Proposition 1.10), the analogous result holds in the complex case:

Lemma 1.14. *Let (X_k) be a sequence of (n-dimensional) complex Gaussians with $X_k \sim \mathcal{N}_{\mathbb{C}}^n(\mu_k, B_k)$. If $X_k \xrightarrow{d} X$, then there exist $\mu := \lim \mu_k$ and $B = \lim B_k$ such that $X \sim \mathcal{N}_{\mathbb{C}}^n(\mu, B)$.*

We briefly recall the notion of jointly Gaussian random vectors. Instead of dealing with this notion in terms of joint characteristic functions or joint density functions, we appeal to a definition that will be the most convenient to make use of in the proofs of subsequent results:

Definition 1.15. *Two random vectors X_1, X_2 are said to be jointly Gaussian, if the random vector $(X_1, X_2)^\top$ is a Gaussian.*

Remark. *(i) A special case of the definition is for one-dimensional X_1, X_2, i.e. complex random variables - they are said to be jointly Gaussian, if the random vector whose coordinates correspond to these random variables is Gaussian.*
(ii) Due to Proposition 1.10, two random vectors $X = (X_1, \ldots, X_n)^\top, Y = (Y_1, \ldots, Y_m)^\top$ are jointly Gaussian, iff the linear combination of their components

$$a_1 X_1 + \cdots + a_n X_n + a_{n+1} Y_1 + \cdots a_{n+m} Y_m$$

is a Gaussian (random variable) for all $a = (a_1, \ldots, a_{n+m})^\top \in \mathbb{C}^{n+m}$.

Knowing that two random vectors (or variables) are jointly Gaussian reduces the question of independence to uncorrelatedness:

Lemma 1.16. *If two jointly Gaussian random vectors are uncorrelated, then they are independent.*

Proof. See Proposition 23.7.3. in [13]. □

We recall the definitions of circular symmetry and properness for a complex random variable and state a result from [13] which relates these two notions for complex Gaussians:

Definition 1.17. *A complex random variable X is said to be circularly symmetric, if*

$$X \stackrel{d}{=} e^{i\theta} X$$

for any deterministic $\theta \in \mathbb{R}$.

Definition 1.18. *Let X be a complex random variable. We say that X is proper, if*

$$(i)\ \mathbb{E}[X] = 0, \quad (ii)\ \mathrm{Var}[X] = \mathbb{E}[\,|X - \mathbb{E}[X]|^2\,] < \infty, \quad (iii)\ \mathbb{E}[X^2] = 0.$$

In case of a Gaussian complex random variable, properness is equivalent to circular symmetry:

Lemma 1.19. *A complex Gaussian random variable is circularly-symmetric if, and only if, it is proper.*

Proof. See Proposition 24.2.12 in [13]. □

Lastly, we state a result that claims the properness of a mean square limit of a sequence of proper random variables and refer to [13] for the proof.

Lemma 1.20. *If* $(X_n)_{n\in\mathbb{N}}$ *and* X *are complex random variables such that*

(i) X_n *is proper* $\forall\, n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \mathbb{E}\,[\,|X_n - X|^2\,] = 0$, *i.e.* $X_n \xrightarrow{L^2} X$,

then X *must be proper.*

Proof. See Proposition 17.6.1 in [13]. □

1.3 Convolutions

Given that some of the arguments in Section 4 rely on convolutions and their associated properties, we mention a few of the important results relating to a convolution of two complex-valued functions f and g defined on $\mathbb{C}$, which is defined pointwise via

$$(f * g)(z) = \int_{\mathbb{C}} f(z-w)g(w)dm(w) \tag{1.7}$$

under appropriate conditions, where $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}$, i.e. $dm(z) = dx\,dy$ for $z = x + iy$. The expression (1.7) obviously corresponds to the convolution of complex-valued functions defined on $\mathbb{R}^2$

$$(f * g)(z_1 + iz_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z_1 + iz_2 - w_1 - iw_2)g(w_1 + iw_2)\, dw_1\, dw_2$$

by the usual identification, so that the following results also hold when $\mathbb{R}^n$ is replaced by $\mathbb{C}$.

Regarding the conditions under which the convolution is well-defined, we focus on the ones relevant for the discussions in Section 4 and refer to [18] for the proofs of these results. The first one we mention is Young's inequality:

Lemma 1.21. *Let $1 \leq p \leq \infty$. If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g\,(x)$ exists for almost every $x \in \mathbb{R}^n$, $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Proof. See Proposition 8.7 in [18]. $\square$

Convolving two functions from (appropriately chosen) L^p spaces with conjugate exponents yields a continuous function vanishing at infinity:

Lemma 1.22. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g\,(x)$ exists for almost every $x \in \mathbb{R}^n$, $f * g \in C_0(\mathbb{R}^n)$ and*

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q.$$

Proof. See Proposition 8.8 in [18]. $\square$

The convolution of two compactly supported continuous functions is again a continuous function with compact support:

Lemma 1.23. *If $f \in C_c(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n)$, then $f * g \in C_c(\mathbb{R}^n)$.*

Proof. Choose any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Since $f \in C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, by Lemma 1.22 we know that $f * g$ is continuous and

$$f * g\,(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy \tag{1.8}$$

is well-defined for almost every $x \in \mathbb{R}^n$; that the support of $f * g$ is compact is now obvious from (1.8). $\square$

Remark. *We briefly remind of the fact that continuous functions with compact support are dense in the space of continuous functions vanishing at 0, i.e. $\overline{C_c(\mathbb{R}^n)}^{\|\ \|_\infty} = C_0(\mathbb{R}^n)$ (abusing the notation here by using $\|\ \|_\infty$ for the supremum norm). For the proof of this fact (in a more general setting), see Proposition 4.35 in [18].*

Lemma 1.24. *If $f \in L^1(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$, then $f * g\,(x)$ exists for almost every $x \in \mathbb{R}^n$ and $f * g \in C_0(\mathbb{R}^n)$.*

Proof. Let $f \in L^1(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$. Since $C_0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, Lemma 1.21 guarantees that $f * g$ exists a.e. and that $f * g \in L^\infty(\mathbb{R}^n)$.

We now show that the convolution map $(f,g) \mapsto f * g$ is continuous $L^1(\mathbb{R}^n) \times C_0(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$. Fix $f_0 \in L^1(\mathbb{R}^n), g_0 \in C_0(\mathbb{R}^n)$, let $\varepsilon > 0$ be arbitrary and take $f \in L^1(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$ such that

$$\|f - f_0\|_1 < \frac{\varepsilon}{\|g_0\|_\infty}, \quad \|g - g_0\|_\infty < \frac{\varepsilon}{\varepsilon/\|g_0\|_\infty + \|f_0\|_1}.$$

Then $\|f\|_1 < \varepsilon/\|g_0\|_\infty + \|f_0\|_1$ and therefore

$$\begin{aligned}\|f * g - f_0 * g_0\|_\infty &\leq \|(f - f_0) * g_0\|_\infty + \|f * (g - g_0)\|_\infty \\ &\leq \|f - f_0\|_1 \|g_0\|_\infty + \|f\|_1 \|g - g_0\|_\infty < 2\varepsilon,\end{aligned}$$

implying continuity as claimed.

To see that $f * g \in C_0(\mathbb{R}^n)$ for $f \in L^1(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$, choose $(f_n), (g_n) \subseteq C_c(\mathbb{R}^n)$ with $f_n \to f$ and $g_n \to g$. We know by Lemma 1.23 that $f_n * g_n \in C_c(\mathbb{R}^n)$; by the above argued continuity of the convolution map, we obtain an approximating sequence $(f_n * g_n) \subset C_c(\mathbb{R}^n)$ with $f_n * g_n \to f * g$ in $L^\infty(\mathbb{R}^n)$. Therefore,

$$f * g \in \overline{C_c(\mathbb{R}^n)}^{\|\ \|_\infty} = C_0(\mathbb{R}^n).$$

$\square$

2 Background

Since the formal setting for studying zero sets of Gaussian random functions is framed within the point processes theory in our main references ([1] and [12]), we introduce some of the concepts from this framework that we will specifically apply to zero sets introduced in the chapters to follow.

Intuitively, one should think of a point process as a random discrete subset of a given space Λ. In order to study the theory of such objects, we need a suitable "space of point processes". The usual setting in probability theory is to have a complete separable metric space (c.s.m.s.) and to take care of this, we have the following technical assumptions:

Λ . . . a locally compact Polish space, i.e. a locally compact topological space that can be topologized by a complete and separable metric;

μ . . . a Borel measure on Λ that is finite on the compact subsets of Λ, i.e. a Radon measure

Taking the set of all discrete subsets of Λ would not yield a c.s.m.s. in general; however, due to the above assumptions on Λ, one can show that

$$\mathcal{M}(\Lambda) := \text{the space of all } \sigma\text{-finite Borel measures on } \Lambda$$

defines a c.s.m.s of its own (see [1] and references therein).

One might wonder why would $\mathcal{M}(\Lambda)$ be an appropriate choice for the "space of discrete random point configurations". Vaguely speaking, a discrete subset A of Λ can be identified with a counting measure on that subset, i.e., we can identify $A = \{a_i \mid i \in I\}$ with the measure $\mu = \sum_{i\in I} \delta_{a_i}$. Therefore, it should make sense to define a point process as a random measure on Λ.

Let $\mathcal{B}(\Lambda)$ be the Borel σ-algebra of Λ as introduced above. A random measure $\mathcal{X}$ on $(\Lambda, \mathcal{B}(\Lambda))$ can be thought of as a measure $\mathcal{X}_\omega$ depending on a parameter ω from some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$; the technical conditions on measurability ensure that the integral of the form

$$\int f d\mathcal{X}$$

is a well-defined random variable $\omega \mapsto \int f d\mathcal{X}_\omega$ for suitable functions f (see [19], [20] for more details). Motivated by the fact that a random measure of a set should represent the number of points falling into that set, one requires its realisation to be a *counting measure*, i.e. a positive integer-valued Borel measure. Furthermore, to ensure that we have a discrete random subset, we impose that the corresponding random measure is locally finite, so that there are no accumulation points. In fact, due to assumptions on Λ, such measures can be represented in the form

$$\sum_{i \in I} m_i \delta_{\lambda_i} \tag{2.1}$$

for $m_i \in \mathbb{N}$ and countably many $\lambda_i \in \Lambda$ (see [19], [20]), which justifies the above intuition. We refer to [19] and [20] as excellent sources for the interplay between point processes and random measures and now focus on the approach from [1], where the following definition is from:

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Λ a locally compact Polish space. A point process $\mathcal{P}$ on Λ is a random integer-valued positive Radon measure $\Omega \to \mathcal{M}(\Lambda)$. A point process $\mathcal{P}$ is called simple, if $\mathbb{P}(\mathcal{P}(\{\lambda\}) \leq 1) = 1$ for any $\lambda \in \Lambda$, i.e. if $\mathcal{P}$ almost surely assigns at most measure 1 to any singleton.*

Remark. *By assuming that Λ is a locally compact Polish space, we know*[3] *that it is exhaustible by (countably many) compact subsets; since the realisation $\mathcal{P}_\omega$ of a point process $\mathcal{P}$ is defined to be a Radon measure, i.e. finite on compact subsets of Λ, $\mathcal{P}_\omega$ is indeed an element of $\mathcal{M}(\Lambda)$.*

Since a point process is by the above definition an integer-valued positive measure, the property of being simple reduces to

$$\mathbb{P}(\, \mathcal{P}(\{\lambda\}) \in \{0, 1\}\,) = 1,$$

so that a simple point process can indeed be identified with a discrete subset of Λ. In this case

$$\mathcal{P}(B) \text{ represents the number of points of } \mathcal{P} \text{ falling into } B \tag{2.2}$$

for a measurable set $B \subseteq \Lambda$. In terms of the above mentioned decomposition of locally finite counting measures into sums of point measures, for a simple point process we must have $m_i = 1$ ([19], [20]), so that (2.1) simplifies to $\sum_{i \in I} \delta_{\lambda_i}$.

Being a random object, a point process should be assigned a suitable notion of distribution. A distribution of a point process is the probability measure it induces on $\mathcal{M}(\Lambda)$ - this will be discussed in Section 3.1 in detail.

[3] Recall that for metric spaces separability is equivalent to second countability and further note that second countable locally compact Hausdorff spaces are exhaustible by (countably many) compact sets.

However, it is more useful to think of the distribution of a point process in terms of the so-called joint intensities (of arbitrary order $k \in \mathbb{N}$). Since we will not be interested in joint intensities of higher order, we state here the definition from [1] (Definition 1.2.2) only for the first intensity:

Definition 2.2. *Let $\mathcal{P}$ be a simple point process on a locally compact Polish space Λ with Radon measure μ. The first intensity function of $\mathcal{P}$ w.r.t μ is a function $\rho_1 : \Lambda \to [0, \infty)$ that satisfies*

$$\mathbb{E}\left[\mathcal{P}(B)\right] = \int\limits_B \rho_1(x)\, d\mu(x)$$

for every Borel measurable $B \subseteq \Lambda$.

Remark. *(i) The existence of a first intensity function (and those of higher order) is not guaranteed and one can discuss the conditions under which a point process does admit joint intensities (see Remark 1.2.2 in [21] and references therein). However, this discussion is not particularly relevant to the arguments we present later, since we are only interested in the first intensity functions, which do exist for the particular point processes we tackle in this thesis.*
(ii) As explained in (2.2), given a measurable set $B \subseteq \Lambda$, $\mathcal{P}(B)$ represents the number of points from $\mathcal{P}$ falling into B; the expected number of these points defines the first intensity measure μ_1 of B:

$$\mu_1(B) := \mathbb{E}\left[\,\mathcal{P}(B)\,\right] \tag{2.3}$$

and if μ_1 happens to be absolutely continuous w.r.t. μ, then the Radon-Nikodym derivative ρ_1 is called the first intensity function.

In particular, in the context of Gaussian analytic functions, it suffices to consider only the first intensity function - it is shown in Theorem 2.5.2 in [1] that the zero sets of two Gaussian analytic functions (which are particular point processes we are interested in) have the same distribution, if their first intensity functions coincide.

We present another approach from [1] (compatible with Definition 2.2) to define the first intensity, namely via the so-called linear statistic. We will make use of this definition in deriving an explicit formula for the first intensity function of the zero set of a given GAF in Section 2.2.

Definition 2.3. *Let $\mathcal{P}$ be a point process on Λ and $\phi : \Lambda \to \mathbb{R}$ a measurable function. The random variable defined by*

$$\mathcal{P}(\phi) = \int_\Lambda \phi\, d\mathcal{P}$$

is called a linear statistic. We define a positve linear functional T_1 on $C_c(\Lambda)$ via $T_1(\phi) := \mathbb{E}(\mathcal{P}(\phi))$ for $\phi \in C_c(\Lambda)$. By the Riesz representation theorem, there exists a unique positive regular Borel measure μ_1 such that

$$T_1(\phi) = \int_\Lambda \phi \, d\mu_1.$$

The measure μ_1 is called the first intensity measure of $\mathcal{P}$. If μ_1 is absolutely continuous to μ, then the Radon-Nikodym derivative of μ_1 with respect to μ is called the first intensity function of $\mathcal{P}$ and denoted by ρ_1. In this case, we have $d\mu_1 = \rho_1 \, d\mu$ and

$$\mathbb{E}(\mathcal{P}(\phi)) = \int_\Lambda \phi \, \rho_1 \, d\mu. \tag{2.4}$$

2.1 Gaussian analytic functions and their zero sets

The study of random analytic functions (with the primary focus on their zero sets) has a long history, dating back to 1930s and is associated with some well-known names such as Borel, Paley and Wiener, Polya and others (see [22] and Section 0 above). Imposing Gaussianity on the random coefficients of such functions has proven to be very useful in applications and the investigations of quantum mechanical chaotic systems in the 1990s have sparked a particular interest in a special class of Gaussian analytic functions, whose zero sets invariant under the isometries of their corresponding domains. Before focusing on isometry-invariant GAFs in Section 3, we lay the groundwork for understanding Gaussian analytic functions in a more general context, starting with the following definition:

Definition 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote with $\mathcal{H}(\Lambda)$ the space of analytic functions on a region $\Lambda \subseteq \mathbb{C}$. A random variable $f : \Omega \to \mathcal{H}(\Lambda)$ is said to be a Gaussian analytic function (GAF), if*

$$(f(z_1), \ldots, f(z_n))^\top \sim \mathcal{N}_{\mathbb{C}}^n$$

with zero mean for every $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \Lambda$.

This means that for any $z_1, \ldots, z_n \in \Lambda$, the random vector $(f(z_1), \ldots, f(z_n))^\top$ is complex Gaussian with zero expectation and covariance matrix

$$(K(z_i, z_j))_{1 \le i,j \le n} = (\, \mathbb{E}[f(z_i)\overline{f(z_j)}] \,)_{1 \le i,j \le n}.$$

We denote the corresponding covariance kernel with $K_f(z, w) = \mathbb{E}[f(z)\overline{f(w)}]$ and drop the index f and write $K(z, w)$ when no ambiguity can arise.

In particular, given a $z \in \Lambda$, we know that

$$f(z) \sim \mathcal{N}_{\mathbb{C}}(0, K(z,z)).$$

We state the following result on how to construct a GAF given a sequence of i.i.d. standard Gaussian coefficients and an appropriate sequence of holomorphic functions. We will follow the proof from [1] and elaborate on the argument that addresses Gaussianity of the functions constructed in this way.

Lemma 2.5. *If $(\psi_n)_{n\in\mathbb{N}_0} \subseteq \mathcal{H}(\Lambda)$ are such that $\sum\limits_{n=0}^{\infty} |\psi_n(z)|^2$ converges uniformly on compact subsets of Λ and $(a_n)_{n\in\mathbb{N}_0}$ are i.i.d. random variables with $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$, then $\sum\limits_{n=0}^{\infty} a_n\psi_n(z)$ converges uniformly on compact subsets of Λ almost surely and defines a Gaussian analytic function with covariance kernel $K(z,w) = \sum\limits_{n=0}^{\infty} \psi_n(z)\overline{\psi_n}(w)$.*

Proof. We follow the proof of Lemma 2.2.3 in [1]. Let $K \subseteq \Lambda$ be a compact set. For $n \in \mathbb{N}$, denote the corresponding partial sum with

$$X_n := \sum_{k=0}^{n} a_k\psi_k$$

and consider X_n as an element of $L^2(K, dm)$, where dm denotes the restriction of Lebesgue measure on $\mathbb{C}$ to K. Simplifying the notation by writing $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ instead of $\|\cdot\|_{L^2(K,\,dm)}$ and $\langle\cdot,\cdot\rangle_{L^2(K,\,dm)}$, we obtain for $k < n$

$$\begin{aligned}
\mathbb{E}\left[\, \|X_n\|^2 \mid a_1,\dots,a_k \,\right] &= \mathbb{E}\left[\, \|X_k + \sum_{j=k+1}^{n} a_j\psi_j\|^2 \mid a_1,\dots,a_k \,\right] \\
&= \mathbb{E}\left[\, \|X_k\|^2 \mid a_1,\dots,a_k \,\right] + \mathbb{E}\left[\, \|\sum_{j=k+1}^{n} a_j\psi_j\|^2 \mid a_1,\dots,a_k \,\right] \\
&= \|X_k\|^2 + \mathbb{E}\left[\, \|\sum_{j=k+1}^{n} a_j\psi_j\|^2 \,\right] \\
&= \|X_k\|^2 + \sum_{\substack{i\neq j \\ k+1\leq i,j\leq n}} \underbrace{\mathbb{E}\left[a_i\overline{a_j}\right]}_{=0}\langle\psi_i,\psi_j\rangle + \sum_{j=k+1}^{n} \underbrace{\mathbb{E}\left[|a_j|^2\right]}_{=1} \|\psi_j\|^2 \\
&= \|X_k\|^2 + \sum_{j=k+1}^{n} \|\psi_j\|^2 \geq \|X_k\|^2. \qquad (2.5)
\end{aligned}$$

For a fixed $\varepsilon > 0$, we define the stopping time $\tau_\varepsilon := \inf\{j \in \mathbb{N} : \|X_j\| \geq \varepsilon\}$ (and write τ to simplify the notation) and obtain

$$\begin{aligned}
\mathbb{E}\,[\,\|X_n\|^2\,] \geq \mathbb{E}\,[\,\|X_n\|^2 \mathbb{1}_{\{\tau \leq n\}}\,] &= \sum_{k=1}^{n} \mathbb{E}\,[\,\|X_n\|^2 \mathbb{1}_{\{\tau = k\}}\,] \\
&= \sum_{k=1}^{n} \mathbb{E}\,[\,\mathbb{E}\,[\,\|X_n\|^2 \mathbb{1}_{\{\tau = k\}} \,|\, a_1, \ldots, a_k\,]\,] \\
&= \sum_{k=1}^{n} \mathbb{E}\,[\,\mathbb{1}_{\{\tau = k\}} \underbrace{\mathbb{E}\,[\,\|X_n\|^2 \,|\, a_1, \ldots, a_k\,]}_{\geq \|X_k\|^2 \text{ by (2.5)}}\,] \\
&\geq \sum_{k=1}^{n} \mathbb{E}\,[\,\mathbb{1}_{\{\tau = k\}} \|X_k\|^2\,] \geq \varepsilon^2\, \mathbb{E}\,[\,\underbrace{\sum_{k=1}^{n} \mathbb{1}_{\{\tau = k\}}}_{\mathbb{1}_{\{\tau \leq n\}}}\,] \\
&= \varepsilon^2\, \mathbb{P}(\tau \leq n) \geq \varepsilon^2 \mathbb{P}(\,\sup_{j \leq n} \|X_j\| \geq \varepsilon\,)
\end{aligned} \tag{2.6}$$

and since $\mathbb{E}\,[\,\|X_n\|^2\,] = \sum_{j=1}^{n} \|\psi_j\|^2$, (2.6) further implies

$$\mathbb{P}(\,\sup_{j \leq n} \|X_j\| \geq \varepsilon\,) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \|\psi_j\|^2. \tag{2.7}$$

Applying (2.7) to the sequence $(X_{N+n} - X_N)_{n \in \mathbb{N}}$ yields

$$\mathbb{P}(\,\sup_{j \leq n} \|X_{N+j} - X_N\| \geq \varepsilon\,) \leq \frac{1}{\varepsilon^2} \sum_{j=N+1}^{N+n} \|\psi_j\|^2 \tag{2.8}$$

and by letting $n \to \infty$ in (2.8), we obtain

$$\begin{aligned}
\mathbb{P}(\,\sup_{m,n \geq N} \|X_m - X_n\| \geq 2\varepsilon\,) &\leq \mathbb{P}(\,\sup_{n \geq 1} \|X_{N+n} - X_N\| \geq \varepsilon\,) \\
&\leq \frac{1}{\varepsilon^2} \sum_{j=N+1}^{\infty} \|\psi_j\|^2 \xrightarrow[N \to \infty]{} 0,
\end{aligned}$$

which implies

$$\mathbb{P}(\exists N \in \mathbb{N} :\ \|X_{N+n} - X_N\| < \varepsilon \ \ \forall\, n) = 1,$$

so that (X_n) is almost surely a Cauchy sequence in $L^2(K)$.

Now fix $z_0 \in \Lambda$ and $R > 0$ such that $\mathrm{D}(z_0, 4R) \subseteq \Lambda$. Being a finite sum of analytic functions on Λ, X_n is itself analytic and we can apply Cauchy's formula to obtain

$$X_n(z) = \frac{1}{2\pi i} \int_{C_r} \frac{X_n(w)}{w - z}\, dw, \tag{2.9}$$

for every $z \in \mathrm{D}(z_0, r)$, where $C_r(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$. Integrating both sides of (2.9) over $[2R, 3R]$ with respect to the radius r we obtain

$$\begin{aligned} X_n(z) &= \frac{1}{2\pi i} \int_{2R}^{3R} \int_{C_r} \frac{X_n(w)}{w - z} \, dw \, dr \\ &= \frac{1}{2\pi} \int_{2R}^{3R} \int_0^{2\pi} \frac{X_n(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z} r \, e^{i\theta} \, d\theta \, dr \\ &= \frac{1}{2\pi} \int_A X_n(\zeta) \varphi_z(\zeta) \, dm(\zeta), \end{aligned} \tag{2.10}$$

for $A := \{\zeta : 2R < |z_0 - \zeta| < 3R\}$ and note that φ_z defined by (2.10) satisfies $\|\varphi_z\|_{L^2(A)} \leq c_R$, where the constant c_R does not depend on $z \in \mathrm{D}(z_0, r)$.

We have shown above that X_n is almost surely a Cauchy sequence in $L^2(K)$ for $K := \overline{\mathrm{D}(z_0, 4R)}$, hence $\exists \, X \in L^2(K)$ such that $X_n \to X$ in $L^2(K)$ and we obtain

$$\begin{aligned} &\Big| \int_A X(\zeta) \varphi_z(\zeta) \, dm(\zeta) - \int_A X_n(\zeta) \varphi_z(\zeta) \, dm(\zeta) \Big| \leq \int_A |X(\zeta) - X_n(\zeta)| |\varphi_z(\zeta)| dm(\zeta) \\ &\qquad \leq c_R \cdot \|X - X_n\|_{L^2(A)} \xrightarrow[n \to \infty]{} 0 \end{aligned}$$

uniformly in $z \in \mathrm{D}(z_0, r)$. Therefore, we can conclude that almost surely $X_n \to X$ uniformly on compact subsets of Λ and thus X defines an analytic function on Λ. The formula for the covariance is easily verified using

$$X(z) = \sum_{n=0}^{\infty} a_n \psi_n(z), \ \mathbb{E}\,[\, a_i \overline{a_j} \,] = 0 \ \text{ for } \ i \neq j \ \text{ and } \ \mathbb{E}\,[\, |a_n|^2] = 1.$$

We now argue why X indeed defines a GAF. For any given $z_1, \dots, z_N \in \Lambda$ and $b \in \mathbb{C}^n$, the random variable

$$b^\top (X_n(z_1), \dots, X_n(z_N))^\top$$

corresponds to a linear combination of i.i.d. $a_k \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, hence defines a complex Gaussian random variable with zero expectation. This implies $(X_n(z_1), \dots, X_n(z_N))^\top \sim \mathcal{N}_{\mathbb{C}}^N$ with zero mean by Proposition 1.10 (ii). As the components $X_n(z_i)$ converge almost surely for every z_i, we obtain

$$(X_n(z_1), \dots, X_n(z_N))^\top \xrightarrow{d} (X(z_1), \dots, X(z_N))^\top \ \text{ as } n \to \infty$$

by Observation 1.5. Appealing to Lemma 1.14, $(X(z_1), \dots, X(z_N))^\top \sim \mathcal{N}_{\mathbb{C}}^N$ with zero expectation and hence $X = \sum_{k=0}^{\infty} a_k \, \psi_k$ defines a Gaussian analytic function. $\square$

We now introduce one of the main objects of interest in this thesis arising from a given Gaussian analytic function:

Definition 2.6. *Let f be a GAF. The zero set n_f of f is the random counting measure (with multiplicities) on $f^{-1}\{0\} \subseteq \Lambda$ given by*

$$n_f = \sum_{z \in f^{-1}\{0\}} m_z \delta_z,$$

where m_z denotes the multiplicity of z and δ_z stands for the point measure concentrated at $z \in f^{-1}\{0\}$.

Remark. *As noted before, due to assumptions on the space Λ, we know it is exhaustible by (countably many) compact sets, hence the realisation n_{f_ω} is indeed an element of $\mathcal{M}(\Lambda)$, since a nontrivial holomorphic function must have a finite number of zeros on any compact set, so n_{f_ω} defines a σ-finite Borel measure.*

Note that n_f is defined on the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ of f and takes values in the space of σ-finite Borel measures that we denote with $\mathcal{M}(\Lambda)$, i.e.

$$\begin{aligned} n_f : \Omega &\to \mathcal{M}(\Lambda) \\ \omega &\mapsto n_{f_\omega} = \sum_{z \in f_\omega^{-1}\{0\}} m_z \delta_z \end{aligned}$$

where $f_\omega := f(\omega) \in \mathcal{H}(\Lambda)$ is a realisation of the GAF f. We will introduce an appropriate sigma-algebra on $\mathcal{M}(\Lambda)$ to make it into an measurable space in Section 3.1 for further considerations.

Incorporating multiplicities into the definition of n_f is somewhat redundant, since it turns out that the zero set of an arbitrary Gaussian analytic function is a simple point process - we state the corresponding result from [1] and refer to the same source for its proof.

Proposition 2.7. *A Gaussian analytic function has no non-deterministic zeros of multiplicity greater than* 1*.*

Proof. See 2.4.1 in [1]. □

For the zero set n_f of a GAF f, the observation in (2.2) now reads

$$n_f(B) = \#(B \cap f^{-1}\{0\})$$

for a Borel measurable set $B \subseteq \Lambda$, so the random measure n_f gives the number of zeros of f that fall into the set B. Recalling (2.3) from the previous section, we have

$$\mu_1(B) = \mathbb{E}\,[\, n_f(B)\,] = \mathbb{E}\,[\,\#(B \cap f^{-1}\{0\})\,] \tag{2.11}$$

so that the first intensity measure of the set B is simply the expected number of zeros of f that fall into it. Note that n_f is a simple point process by Proposition 2.7, so it indeed fits into the setting of Definition 2.2.

It is easy to see that it suffices to show that the covariance kernels of two GAFs coincide, if we want to show that the corresponding zero sets are identically distributed:

Corollary 2.8. *Let f and g be two GAFs with corresponding covariance kernels K_f and K_g. If $K_f = K_g$, then the zero sets of f and g have the same distribution.*

Proof. As the expectation of a GAF f is 0 by definition, the covariance kernel K_f determines the distribution of f by Lemma 1.12; hence $K_f = K_g$ implies $f \overset{d}{=} g$, so the distributions of the corresponding zero sets must be the same. □

2.2 Edelman-Kostlan Formula

In this section we demonstrate that the first intensity function for the zero set of a Gaussian analytic function f allows for a very simple and elegant explicit formula, namely

$$\rho_1(z) = \frac{1}{4\pi}\Delta \log K(z,z) \tag{2.12}$$

where $K = K_f$ is the covariance kernel of f. The formula (2.12) is commonly referred to as Edelman-Kostlan formula (see [23]) and we present the proof from [1] (Section 2.4.1), which relies on Definition 2.4, i.e. on the formula

$$\mathbb{E}\left[\int_\Lambda \varphi \, d\mathcal{P}\right] = \mathbb{E}(\mathcal{P}(\varphi)) = \int_\Lambda \varphi\, \rho_1 \, d\mu \quad (\, \varphi \in C_c(\Lambda)\,)$$

for $\mathcal{P} = n_f$. The main ingredient for this proof is the following lemma which demonstrates that the counting measure corresponding to the zero set of a deterministic holomorphic function f can be expressed in terms of $\log|f(z)|$ in distributional sense:

Lemma 2.9. *Let $f \in \mathcal{H}(\Lambda)$, $f \not\equiv 0$. Then*

$$dn_f = \frac{1}{2\pi} \Delta \log|f(z)|$$

holds in distributional sense, i.e.

$$\int_\Lambda \varphi(z) dn_f(z) = \int_\Lambda \Delta\varphi(z) \frac{1}{2\pi} \log|f(z)| dm(z),$$

where $\varphi \in C_c^\infty(\Lambda)$ is an arbitrary test function.

Remark. *There is a slight abuse of notation here: as introduced in Definition 2.6, f denoted a random variable (being a GAF) taking values in the space of holomorphic functions $\mathcal{H}(\Lambda)$, so n_f in Definition 2.6 is a random object, whose realisation $n_{f(\omega)}$ is a measure on Λ. Here, on the other hand, f is a deterministic holomorphic function and hence n_f is a non-random object, i.e. a measure on Λ.*

Proof. We follow the proof presented in [1], Section 2.4.1. For $\{\alpha_1, \dots, \alpha_n\} := f^{-1}\{0\} \cap \operatorname{supp}(\varphi)$, we can express f as

$$f(z) = g(z) \prod_{k=1}^n (z - \alpha_k)^{m_k}$$

on $\operatorname{supp}(\varphi)$, which implies

$$\log|f(z)| = \log|g(z)| + \sum_{k=1}^n m_k \log|z - \alpha_k|$$

where m_k is the multiplicity of the zero α_k and g is an analytic function with no zeros in $\operatorname{supp}(\varphi)$. Note that $\Delta \log|g| = 0$ on $\operatorname{supp}(\varphi)$, because $\log|g|$ is locally the real part of an analytic function (any continuous choice of $\log(g)$), hence harmonic. Applying Δ to the above expression, multiplying it with φ and taking the integral over Λ we arrive at

$$\begin{aligned}\int_\Lambda \Delta \log|f(z)| \varphi(z) dm(z) &= \sum_{k=1}^n m_k \int_\Lambda \Delta \log|z - \alpha_k| \varphi(z) dm(z) \\ &= \sum_{k=1}^n m_k \int_\Lambda \log|z - \alpha_k| \Delta\varphi(z) dm(z)\end{aligned}$$

and we recognize Green's function (fundamental solution to the planar Laplace equation) $G(\alpha_k, z) = \frac{1}{2\pi}\log|z-\alpha_k|$, such that

$$\begin{aligned}\int\limits_{\Lambda} \frac{1}{2\pi}\Delta \log|f(z)|\varphi(z)dm(z) &= \sum_{k=1}^{n} m_k \int\limits_{\Lambda} G(\alpha_k, z)\Delta\varphi(z)dm(z) \\ &= \sum_{k=1}^{n} m_k\varphi(\alpha_k) = \int\limits_{\Lambda} \varphi(z)dn_f(z)\end{aligned}$$

so the claim holds true. □

Proposition 2.10. *Let f be a GAF. Then the first intensity function ρ_1 of the corresponding zero set n_f is given by the so-called Edelman-Kostlan formula:*

$$\rho_1(z) = \frac{1}{4\pi}\Delta \log K(z,z)$$

with respect to Lebesgue measure $dm(z)$.

Proof. We quote the proof from [1], Section 2.4.1. We know by definition of f that $f(z) \sim \mathcal{N}_{\mathbb{C}}(0, K(z,z))$. Using the previous lemma for f, now a Gaussian analytic function, we get

$$\begin{aligned}\mathbb{E}\left(\int\limits_{\Lambda} \varphi(z)dn_f(z)\right) &= \mathbb{E}\left(\int\limits_{\Lambda} \Delta\varphi(z)\frac{1}{2\pi}\log|f(z)|dm(z)\right) \\ &= \int\limits_{\Lambda} \Delta\varphi(z)\frac{1}{2\pi}\mathbb{E}\left(\log|f(z)|\right)dm(z), \qquad (2.13)\end{aligned}$$

where the second equality follows by applying Fubini-Tonelli's theorem. To justify this, note that $a := f(z)\,/\sqrt{K(z,z)}$ is a standard complex Gaussian, i.e. $a \sim \mathcal{N}_{\mathbb{C}}(0,1)$, so we get

$$\begin{aligned}\mathbb{E}\left(\,|\log|f(z)|\,|\,\right) &= \mathbb{E}\left(\left|\log\left|\frac{f(z)}{\sqrt{K(z,z)}}\right| + \log\sqrt{K(z,z)}\right|\right) \\ &\le \mathbb{E}(\,|\log|a|\,|\,) + \frac{1}{2}\,|\log K(z,z)\,| \\ &= \frac{1}{2}\int\limits_{0}^{\infty} |\log r\,|\,e^{-r}dr + \frac{1}{2}\,|\log K(z,z)\,|,\end{aligned}$$

where $|\log K(z,z)|$ is locally everywhere integrable, except possibly at $z = z_0$, such that $K(z_0, z_0) = 0$; in this case we can express $K(z,z)$ locally around z_0 as

$$K(z,z) = |z - z_0|^{2p} L(z,z),$$

where $L(z_0, z_0) \neq 0$, so locally around z_0 we see that $\log K(z,z)$ grows like $\log|z - z_0|$ as $z \to z_0$, hence integrable. Overall, we conclude that

$$\int\limits_{\Lambda} |\Delta\varphi(z)| \frac{1}{2\pi} \mathbb{E}\left[\,|\log|f(z)||\,\right] dm(z) < \infty,$$

so applying Fubini-Tonelli's theorem is justified. We continue with (2.13):

$$\mathbb{E}\left(\int\limits_{\Lambda} \varphi(z) dn_f(z)\right) = \int\limits_{\Lambda} \varphi(z) \frac{1}{2\pi} \Delta\mathbb{E}\left(\log|f(z)|\right) dm(z),$$

and since $\mathbb{E}\left(\log|f(z)|\right) = \mathbb{E}\left(\log|a|\right) + \frac{1}{2}\log K(z,z)$, we have that $\Delta\mathbb{E}\left(\log|f(z)|\right) = \frac{1}{2}\Delta\log K(z,z)$, hence we obtain:

$$\mathbb{E}\left(\int\limits_{\Lambda} \varphi(z) dn_f(z)\right) = \int\limits_{\Lambda} \varphi(z) \frac{1}{4\pi} \Delta \log K(z,z) dm(z),$$

which implies the so-called Edelman-Kostlan formula for the first intensity function:

$$\rho_1(z) = \frac{1}{4\pi} \Delta \log K(z,z).$$

$\square$

3 Isometry-invariant zero sets and ergodicity

Following [1] (Chapter 2.3), we look at two particular classes of Gaussian analytic functions given by the following expansions:

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n \qquad (\Lambda = \mathbb{C}) \tag{3.1}$$

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\cdots(L+n-1)}}{\sqrt{n!}} z^n \qquad (\Lambda = \mathbb{D}) \tag{3.2}$$

for $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d. and $L > 0$. Note that these indeed define Gaussian analytic functions due to Lemma 2.5, since the conditions on the corresponding holomorphic function sequence (ψ_n)

$$(\Lambda = \mathbb{C}) \quad \sum_{n=0}^{\infty} \left| \frac{\sqrt{L^n}}{\sqrt{n!}} z^n \right|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} (L|z|^2)^n = e^{L|z|^2} < \infty,$$

$$(\Lambda = \mathbb{D}) \quad \sum_{n=0}^{\infty} \left| \frac{\sqrt{L(L+1)\cdots(L+n-1)}}{\sqrt{n!}} z^n \right|^2 = \sum_{n=0}^{\infty} \binom{L+n-1}{n} (|z|^2)^n = (1-|z|^2)^{-L} < \infty$$

are satisfied. Recalling the notation Λ and μ from the introduction, we have

(1) $\Lambda = \mathbb{C}$ equipped with Lebesgue measure dm(z),

(2) $\Lambda = \mathbb{D}$ equipped with hyperbolic measure $\dfrac{dm(z)}{(1-|z|^2)^2}$.

In particular, one gets an explicit expression for the covariance kernels of these GAFs by appealing to Lemma 2.5:

(1) in the planar case $\Lambda = \mathbb{C}$ we have $\psi_n(z) = (\sqrt{L^n}/\sqrt{n!}) z^n$ for which we obtain:

$$K(z,w) = \sum_{n=0}^{\infty} \psi_n(z)\overline{\psi_n(w)} = \sum_{n=0}^{\infty} \frac{\sqrt{L^n}}{\sqrt{n!}} z^n \overline{\left(\frac{\sqrt{L^n}}{\sqrt{n!}} w^n\right)} = \sum_{n=0}^{\infty} \frac{1}{n!} (Lz\overline{w})^n = \exp(Lz\overline{w});$$

(2) in the hyperbolic case $\Lambda = \mathbb{D}$ for $\psi_n(z) = \left(\sqrt{L(L+1)\cdots(L+n-1)}/\sqrt{n!}\right) z^n$

we get

$$K(z,w) = \sum_{n=0}^{\infty} \psi_n(z)\overline{\psi_n(w)} = \sum_{n=0}^{\infty} \frac{L(L+1)\cdots(L+n-1)}{n!}(z\overline{w})^n$$
$$= \sum_{n=0}^{\infty} \binom{L+n-1}{n}(z\overline{w})^n = (1-z\overline{w})^{-L}.$$

Knowing their covariance kernels explicitly, one can now easily compute the first intensity functions for the planar and hyperbolic GAFs using Proposition 2.10 and obtain:

$$\text{(1)} \quad \rho_1(z) = \frac{L}{\pi} \quad \text{w.r.t. Lebesgue measure } dm(z) \text{ on } \mathbb{C},$$
$$\text{(2)} \quad \rho_1(z) = \frac{L}{\pi} \quad \text{w.r.t. hyperbolic measure } \frac{dm(z)}{(1-|z|^2)^2} \text{ on } \mathbb{D}.$$

We consider the two domains $\Lambda = \mathbb{C}$ and $\Lambda = \mathbb{D}$ due to their property of being invariant under the following transformations:

1) For the complex plane $\mathbb{C}$ we consider the Euclidian motions given by

$$\varphi_{\lambda,\beta}(z) = \lambda z + \beta \qquad (z \in \mathbb{C})$$

for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\beta \in \mathbb{C}$, where λ corresponds to a rotation and β to a translation. We denote the corresponding group of such transformations by

$$\tau_{\mathbb{C}} := \{\, \varphi_{\lambda,\beta} \;\mid\; \lambda \in \mathbb{C}, |\lambda| = 1 \text{ and } \beta \in \mathbb{C}\}$$

and note that every $\varphi_{\lambda,\beta}$ obviously preserves the Euclidian metric, hence $\tau_{\mathbb{C}}$ is an isometry group on $\mathbb{C}$.

2) For the unit disk $\mathbb{D}$ we consider the linear fractional transformations given by

$$\varphi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad (z \in \mathbb{D})$$

for $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$. We denote the corresponding set of such transformations by $\tau_{\mathbb{D}}$. Note that every $\varphi_{\alpha,\beta} : \mathbb{D} \to \mathbb{D}$ is bijective (the inverse function being $\varphi_{\overline{\alpha},-\beta}$) and that it preserves the hyperbolic metric, since a direct calculation shows

$$\begin{aligned}
\varphi'_{\alpha,\beta}(z) &= \frac{|\alpha|^2 - |\beta|^2}{(\overline{\beta}z + \overline{\alpha})^2} = \frac{1}{(\overline{\beta}z + \overline{\alpha})^2} \\
\text{and}\quad 1 - |\varphi_{\alpha,\beta}(z)|^2 &= \frac{|\overline{\beta}z + \overline{\alpha}|^2 - |\alpha z + \beta|^2}{|\overline{\beta}z + \overline{\alpha}|^2} = \frac{1 - |z|^2}{|\overline{\beta}z + \overline{\alpha}|^2} \\
\implies \quad \frac{|\varphi'_{\alpha,\beta}(z)|}{1 - |\varphi_{\alpha,\beta}(z)|^2} &= \frac{1}{1 - |z|^2}
\end{aligned}$$

and given $\varphi_{\alpha,\beta}, \varphi_{\zeta,\eta} \in \tau_{\mathbb{D}}$, one can easily verify that

$$\varphi_{\alpha,\beta} \circ \varphi_{\zeta,\eta} = \varphi_{\alpha\zeta + \beta\overline{\eta}\,,\,\alpha\eta + \beta\overline{\zeta}} \in \tau_{\mathbb{D}},$$

since a straightforward calculation shows $|\alpha\zeta + \beta\overline{\eta}|^2 - |\alpha\eta + \beta\overline{\zeta}| = 1$, so again we obtain an isometry group $\tau_{\mathbb{D}}$ on $\mathbb{D}$.

We now show that the zero sets of the Gaussian analytic functions defined in (3.1), (3.2) are invariant under the corresponding isometry groups:

Lemma 3.1. *The zero sets of the GAFs given by (3.1) and (3.2) are invariant in distribution under the transformations from the corresponding isometry groups $\tau_{\mathbb{C}}$ and $\tau_{\mathbb{D}}$, respectively.*

Proof. For the proof of the case $\Lambda = \mathbb{C}$ we refer to [1] (Proposition 2.3.4) and here we will handle the hyperbolic case $\Lambda = \mathbb{D}$. We set

$$g(z) := f(\varphi_{\alpha,\beta}(z)) = f\left(\frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}}\right)$$

and by recalling that $K_f(z,w) = (1 - z\overline{w})^{-L}$, we get that the covariance kernel of g is given by

$$\begin{aligned}
K_g(z,w) &= \mathbb{E}\left[f(\varphi_{\alpha,\beta}(z))\overline{f(\varphi_{\alpha,\beta}(w))} \right] = K_f(\varphi_{\alpha,\beta}(z), \varphi_{\alpha,\beta}(w)) \\
&= \left(1 - \frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}}\frac{\overline{\alpha}\overline{w} + \overline{\beta}}{\beta\overline{w} + \alpha}\right)^{-L} = \left(\frac{1 - z\overline{w}}{(\overline{\beta}z + \overline{\alpha})(\beta\overline{w} + \alpha)}\right)^{-L}.
\end{aligned}$$

Now by setting

$$h(z) := f(z)(\overline{\beta}z + \overline{\alpha})^L,$$

we notice that the covariance kernels of g and h coincide:

$$\begin{aligned}
K_h(z,w) &= \mathbb{E}\left[h(z)\overline{h(w)}\right] = (\overline{\beta}z + \overline{\alpha})^L(\beta\overline{w} + \alpha)^L\, \mathbb{E}\left[f(z)\overline{f(w)}\right] \\
&= \left(\frac{1 - z\overline{w}}{(\overline{\beta}z + \overline{\alpha})(\beta\overline{w} + \alpha)}\right)^{-L} = K_g(z,w)
\end{aligned}$$

and note that $\forall z_1, \ldots, z_n \in \Lambda$

$$(g(z_1), \ldots, g(z_n))^\top = (f(\varphi_{\alpha,\beta}(z_1)), \ldots, f(\varphi_{\alpha,\beta}(z_1)))^\top$$

$$(h(z_1), \ldots, h(z_n))^\top = \begin{bmatrix} (\overline{\beta} z_1 + \overline{\alpha})^L & & 0 \\ & \ddots & \\ 0 & & (\overline{\beta} z_n + \overline{\alpha})^L \end{bmatrix} (f(z_1), \ldots, f(z_n))^\top,$$

so g and h are both Gaussian[4] and have zero expectation; hence, by Lemma 1.12, we can conclude that g and h have the same distribution, i.e.

$$(f \circ \varphi_{\alpha,\beta})(z) = g(z) \overset{d}{=} h(z) = f(z)(\overline{\beta} z + \overline{\alpha})^L$$

and since $|\overline{\beta} z + \overline{\alpha}| \geq |\alpha| - |\beta||z| > |\alpha| - |\beta| = \frac{1}{|\alpha|+|\beta|} \geq 0$, the function $(\overline{\beta} z + \overline{\alpha})^L$ has no zeros, which implies that the distribution of the zeros of $(f \circ \varphi_{\alpha,\beta})(z)$ is the same as that of $f(z)$. $\square$

3.1 Ergodicity of the zero set

Suppose we are given a planar or hyperbolic GAF f as introduced above. The mere invariance under the isometry groups does not exclude the possibility that with certain probability the GAF in question will have no zeros at all. To show that this cannot be the case, we will prove that the zero set of such GAFs is ergodic (precise definition will follow). Since the event that f has no zeros is invariant under the corresponding transformation of its domain (see Definition 3.3 below), ergodicity then implies that the probability of this event is either 0 or 1. Knowing the first intensity function $\rho_1(z) = \frac{L}{\pi}$ and recalling (2.11) from Section 2.1, we conclude that the expected number of zeros $\mu_1(D) = \mathbb{E}\left[n_f(D) \right]$ is positive for any Borel subset $D \subseteq \Lambda$, whence the aforementioned probability cannot be 1, i.e. the probability of f having no zeros must be equal to 0.

The ergodicity of the zero sets of isometry-invariant GAFs is a known fact and is done e.g. in [1] (Proposition 2.3.7) - we will follow the idea of the proof therein. The proof in [1] is done for the planar case - we will make the necessary adjustments and carry out the proof for the hyperbolic case, elaborate the arguments and add some intermediate steps. For this we also need to fit the setting of point processes into the formal definition of ergodicity (see Definition 3.3 below), and take a closer look at the structure of the probability space underlying f in terms of its corresponding random coefficients and prove an auxiliary result based on this structure which will

[4] g by the very definition of f and h by also appealing to Lemma 1.13

be used in the proof of ergodicity.

The point process n_f associated with the zero set of a GAF f is a map $\Omega \to \mathcal{M}(\Lambda)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space of f. By equipping the space $\mathcal{M}(\Lambda)$ with an appropriate sigma-algebra, we see that n_f is a map between two measurable spaces and hence we can use it as a push-forward to define a probability measure $\mathbb{P}_f$ on $\mathcal{M}(\Lambda)$ by setting

$$\mathbb{P}_f(A) := \mathbb{P}(\, n_f^{-1}(A)\,) = \mathbb{P}(\, n_f \in A\,)$$

for measurable $A \subseteq \mathcal{M}(\Lambda)$. By the transformation formula for image measures, we have

$$\mathbb{P}_f(A) = \int_{\mathcal{M}(\Lambda)} \mathbb{1}_A \, d\,\mathbb{P}_f = \int_{\Omega} \mathbb{1}_A \circ n_f \, d\,\mathbb{P} = \mathbb{E}\left[\mathbb{1}_A \circ n_f\right],$$

which clarifies the interchangeable use of $\mathbb{P}_f(A)$ and $\mathbb{E}\left[\mathbb{1}_A(n_f)\right]$ in Proposition 3.4 below.

To define an appropriate sigma-algebra $\mathcal{F}$ on $\mathcal{M}(\Lambda)$, we first need to get some insight into the structure of the measurable space (Ω, Σ), which then suggests a natural definition of $\mathcal{F}$.

Given the coefficient functions $a_n : \Omega_n \to \mathbb{C}$ from the power series expansion of a GAF f, we consider the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ of f as a product space of $(\Omega_n)_{n \in \mathbb{N}_0}$ in the following way:

$$\Omega = \prod_{n \geq 0} \Omega_n$$

with the sigma-algebra Σ generated by the sequence of the cylinder sets

$$\prod_{k=0}^{n} E_k \times \prod_{k>n} \Omega_k$$

for $n \in \mathbb{N}_0$, where E_k is a measurable subset of Ω_k. For $N \in \mathbb{N}_0$ we further define Σ_N as the sigma-algebra generated by the cylinder sets of the form

$$\prod_{k=0}^{N} E_k \times \prod_{k>N} \Omega_k$$

Note that obviously

$$\Sigma = \sigma(\{\Sigma_N \mid N \in \mathbb{N}_0\}) = \sigma(\{A \mid A \in \Sigma_N \text{ for some } N \in \mathbb{N}_0\}),$$

i.e. Σ is the smallest sigma-algebra containing all the sigma-algebras Σ_N. We introduce the corresponding sigma-algebras on $\mathcal{M}(\Lambda)$ as push-forwards of Σ and Σ_N:

$$\mathcal{F} := \{ \, A \subseteq \mathcal{M}(\Lambda) \mid n_f^{-1}(A) \in \Sigma \, \}$$
$$\mathcal{F}_N := \{ \, A \subseteq \mathcal{M}(\Lambda) \mid n_f^{-1}(A) \in \Sigma_N \}$$

for $N \in \mathbb{N}_0$. Note that $\mathcal{F}$ is indeed a sigma-algebra:

1. $n_f^{-1}(\mathcal{M}(\Lambda)) = \Omega \in \Sigma$, since Σ is a sigma-algebra on Ω;
2. if $A \in \mathcal{F}$, then $n_f^{-1}(A) \in \Sigma$, hence $(n_f^{-1}(A))^c \in \Sigma$ and as $n_f^{-1}(A^c) = (n_f^{-1}(A))^c \in \Sigma$, we obatin that $A^c \in \mathcal{F}$;
3. lastly, if $(A_n) \subseteq \mathcal{F}$, then $n_f^{-1}(A_n) \in \Sigma$ and hence $n_f^{-1}(\cup A_n) = \cup\, n_f^{-1}(A_n) \in \Sigma$, therefore $\cup A_n \in \mathcal{F}$,

and analogously one can see that $\mathcal{F}_N$ is a sigma-algebra as well. Since $\Sigma_N \subseteq \Sigma$, we observe that $\mathcal{F}_N \subseteq \mathcal{F}$ for every $N \in \mathbb{N}_0$. We now want to show that $\mathcal{F}$ is in fact the smallest sigma-algebra containing all $\mathcal{F}_N$.

Claim: $\mathcal{F} = \sigma(\{\mathcal{F}_N \mid N \in \mathbb{N}_0\}) = \sigma(\{A \mid A \in \mathcal{F}_N \text{ for some } N \in \mathbb{N}_0\})$

To see this, we use the following measure-theoretical fact: if $f : X \to Y$ is a map and $\mathcal{C} \subseteq 2^Y$ a collection of subsets of Y, then (e.g. see Corollary 1.2.9 in [24])

$$f^{-1}(\sigma(\mathcal{C})) = \sigma(f^{-1}(\mathcal{C}))$$

Applying this to the map $n_f : \Omega \to \mathcal{M}(\Lambda)$ and $\mathcal{C} := \{\mathcal{F}_N \mid N \in \mathbb{N}_0\}$, we obtain:

$$n_f^{-1}(\sigma(\{\mathcal{F}_N\})) = \sigma(n_f^{-1}(\{\mathcal{F}_N\}))$$

and since $A \in \mathcal{F}_N \iff n_f^{-1}(A) \in \Sigma_N$, we observe that

$$n_f^{-1}(\{\mathcal{F}_N\}) = \{n_f^{-1}(A) \mid A \in \{\mathcal{F}_N\}\} \subseteq \{\Sigma_N\} \implies \sigma(n_f^{-1}(\{\mathcal{F}_N\})) \subseteq \sigma(\{\Sigma_N\})$$

and hence we obtain

$$n_f^{-1}(\sigma(\{\mathcal{F}_N\})) = \sigma(n_f^{-1}(\{\mathcal{F}_N\})) \subseteq \sigma(\{\Sigma_N\}) = \Sigma. \tag{3.3}$$

Now let $A \in \sigma(\{\mathcal{F}_N\})$; then, by (3.3), we know that $n_f^{-1}(A) \in \Sigma$, which further implies $A \in \mathcal{F}$, hence $\sigma(\{\mathcal{F}_N\}) \subseteq \mathcal{F}$, proving the above claim.

Note that the condition

$$n_f^{-1}(A) \in \Sigma \;\; \forall A \in \mathcal{F}$$

is automatically satisfied by construction, hence the sigma-algebra $\mathcal{F}$ is compatible with the definition of $\mathbb{P}_f$ as an image measure, i.e. choosing $\mathcal{F}$ as a sigma-algebra on $\mathcal{M}(\Lambda)$, $\mathbb{P}_f$ is well-defined.

We can now prove the above mentioned approximation property of measurable sets in $\mathcal{M}(\Lambda)$. Note that this result is not a distinctive property of $(\,\mathcal{M}(\Lambda), \mathcal{F}, \mathbb{P}_f\,)$ and holds in more generality for sigma-finite measures (e.g. see Theorem D, §13 in [25]):

Lemma 3.2. *Let $(\,\mathcal{M}(\Lambda), \mathcal{F}, \mathbb{P}_f\,)$ be the probability space as introduced above with $\mathcal{F} = \sigma(\{\mathcal{F}_N \mid N \in \mathbb{N}_0\})$. Then for any $A \in \mathcal{F}$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}_0$ and $A_N \in \mathcal{F}_N$, such that*

$$\mathbb{P}_f(\,A \triangle A_N\,) \le \varepsilon$$

Remark. *Recalling how we defined $\mathcal{F}_N$, this means that given any measurable subset A of $\mathcal{M}(\Lambda)$ and $\varepsilon > 0$, there is a measurable set A_N that depends only on the first N power series coefficients of f that approximates A with accuracy of ε.*

Proof of Lemma 3.2. We set

$$\widetilde{\mathcal{F}} := \{A \in \mathcal{F} \mid \forall\, \varepsilon > 0 \;\exists\, A_N \in \mathcal{F}_N : \mathbb{P}_f(\,A \triangle A_N\,) \le \varepsilon\}$$

and show that $\widetilde{\mathcal{F}}$ is a sigma-algebra; then since $\mathcal{F}_N \subseteq \widetilde{\mathcal{F}} \;\; \forall N \in \mathbb{N}_0$ implies $\sigma(\{\mathcal{F}_N \mid N \in \mathbb{N}_0\}) \subseteq \widetilde{\mathcal{F}}$, we can conclude $\mathcal{F} = \widetilde{\mathcal{F}}$, which then proves the claim. Indeed,

1. since $\mathcal{F}_N$ is a sigma-algebra, we know that $\Omega \in \mathcal{F}_N \;\; \forall N \in \mathbb{N}_0$, hence for $A_N := \Omega$ we obtain $\mathbb{P}_f(\,\Omega \triangle \Omega\,) = 0 \le \varepsilon$ for any $\varepsilon > 0$, hence $\Omega \in \widetilde{\mathcal{F}}$;

2. let $A \in \widetilde{\mathcal{F}}$; since $A_N \in \mathcal{F}_N \implies A_N^c \in \mathcal{F}_N$, the set-theoretical identity $A \triangle A_N = A^c \triangle A_N^c$ implies $\mathbb{P}_f(\,A^c \triangle A_N^c\,) = \mathbb{P}_f(\,A \triangle A_N\,) \le \varepsilon$, hence $A^c \in \widetilde{\mathcal{F}}$;

3. let $(A_k) \subseteq \widetilde{\mathcal{F}}$ and $\varepsilon > 0$, set $A := \cup A_k$; then for any $k \in \mathbb{N}_0$ we can find some $N_k \in \mathbb{N}_0$, such that $\mathbb{P}_f(\,A_k \triangle A_{N_k}\,) \le \varepsilon 2^{-k}$; since $\mathbb{P}_f(A) = \mathbb{P}_f(\cup_{k=1}^{\infty} A_k) = \lim_{n\to\infty} \mathbb{P}_f(\cup_{k=1}^{n} A_k)$, there is a $M \in \mathbb{N}_0$ such

that $\mathbb{P}_f(A) - \mathbb{P}_f(\cup_{k=1}^M A_k) < \varepsilon/2$; we then obtain

$$
\begin{aligned}
&\widetilde{A}_M := \cup_{k=1}^M A_{N_k} \in \mathcal{F}_{N_M} \\
&\quad\implies \mathbb{P}_f(\, A \triangle \widetilde{A}_M \,) \leq \mathbb{P}_f(\, A \setminus \cup_{k=1}^M A_k \,) + \mathbb{P}_f((\cup_{k=1}^M A_k) \triangle (\cup_{k=1}^M A_{N_k})) \\
&\qquad\qquad\qquad\qquad < \varepsilon/2 + \mathbb{P}_f(\cup_{k=1}^M (A_k \triangle A_{N_k})) \\
&\qquad\qquad\qquad\qquad \leq \varepsilon/2 + \sum_{k=1}^M \mathbb{P}_f(A_k \triangle A_{N_k}) \\
&\qquad\qquad\qquad\qquad \leq \varepsilon/2 + \sum_{k=1}^M \varepsilon 2^{-k} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon;
\end{aligned}
$$

where in the first step we have used $C \subseteq A \implies A \triangle B \subseteq (A \setminus C) \cup (C \triangle B)$ for any sets A, B, C, and in the second step we have used the following property of the symmetric difference:

$$
(\cup_{j \in \mathcal{J}} A_j) \triangle (\cup_{j \in \mathcal{J}} B_j) \subseteq \cup_{j \in \mathcal{J}} (A_j \triangle B_j)
$$

for arbitrary sets A_j, B_j and an index set $\mathcal{J}$. We have therefore found $M \in \mathbb{N}_0$ such that $\mathbb{P}_f(\, A \triangle \widetilde{A}_M \,) \leq \varepsilon$ for $\widetilde{A}_M \in \mathcal{F}_{N_M}$, hence $A \in \widetilde{\mathcal{F}}$.

$\square$

We now briefly recall the definition of ergodicity (from [1] or [26], Example 6.1.4):

Definition 3.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*

(i) A measurable map $\varphi : \Omega \to \Omega$ is said to be measure-preserving, if $\mathbb{P}(\varphi^{-1}(A)) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$.

(ii) Let τ be a group of measure-preserving transformations $\varphi : \Omega \to \Omega$. A set $A \in \mathcal{F}$ is said to be invariant (under the action of τ), if $\varphi(A) = A$ for every $\varphi \in \tau$.

(iii) The action of τ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be ***ergodic****, if for any invariant set $A \in \mathcal{F}$ we have $\mathbb{P}(A) \in \{0, 1\}$. In this case, we will also say that $\mathbb{P}$ is ergodic under the action of τ.*

We want to apply this definition to the probability space $(\mathcal{M}(\Lambda), \mathcal{F}, \mathbb{P}_f)$ to show that the zero sets of the GAFs described above are ergodic, or, more precisely - that $\mathbb{P}_f$ is ergodic under the action of the corresponding measure-preserving transformations. The action of an isometry $\varphi_{\alpha,\beta} \in \tau_{\mathbb{D}}$ on a measure $\xi \in \mathcal{M}(\mathbb{D})$ is given by

$$
(\varphi_{\alpha,\beta}(\xi))(B) := \xi(\varphi_{\alpha,\beta}(B))
$$

for any measurable $B \subseteq \mathbb{D}$ and analogously for $\varphi_{\lambda,\beta} \in \tau_{\mathbb{C}}$ and $\Lambda = \mathbb{C}$. In the case of a hyperbolic GAF f defined in (3.2) and $\Lambda = \mathbb{D}$ we have

$$\begin{aligned}(\varphi_{\alpha,\beta}(n_f))(B) = n_f(\varphi_{\alpha,\beta}(B)) &= \#\{z \in \varphi_{\alpha,\beta}(B) : f(z) = 0\} \\ &= \#\{z \in B : f(\varphi_{\alpha,\beta}(z)) = 0\} \\ &= \#\{z \in B : f_{\alpha,\beta}(z) = 0\} = n_{f_{\alpha,\beta}}(B),\end{aligned}$$

where $f_{\alpha,\beta}(z) := (f \circ \varphi_{\alpha,\beta})(z)(\overline{\beta}z + \overline{\alpha})^{-L}$ and the second to last equality holds because $(\overline{\beta}z + \overline{\alpha})^{-L} \neq 0$ for any $z \in \mathbb{D}$ (as argued in the proof of Lemma 3.1). Note that the transformations $\varphi_{\alpha,\beta}$ from the isometry group $\tau_{\mathbb{D}}$ are indeed measure-preserving, since

$$\begin{aligned}\mathbb{P}_f(A) &= \mathbb{P}(n_f \in A) = \mathbb{P}(\,\varphi_{\alpha,\beta}(n_f) \in \varphi_{\alpha,\beta}(A)\,) \\ &= \mathbb{P}(n_{f_{\alpha,\beta}} \in \varphi_{\alpha,\beta}(A)) = \mathbb{P}(n_f \in \varphi_{\alpha,\beta}(A)) = \mathbb{P}_f(\varphi_{\alpha,\beta}(A))\end{aligned}$$

holds for any $A \in \mathcal{F}$ and the second to last equality holds because the proof of Lemma 3.1 implies $f_{\alpha,\beta} \stackrel{d}{=} f$ for $f_{\alpha,\beta}(z) := (f \circ \varphi_{\alpha,\beta})(z)(\overline{\beta}z + \overline{\alpha})^{-L}$. Analogous conclusions are of course true in the planar case (one argues analogously using the equality in distribution of f with the suitable modification of f by means of Euclidian motions - this is done [1], Proposition 2.3.4 which corresponds to Lemma 3.1 in this thesis).

We are now ready to prove the main result of this chapter:

Proposition 3.4. *The zero sets of the GAFs f given by*

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L^n}}{\sqrt{n!}} z^n \qquad (\Lambda = \mathbb{C})$$

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\cdots(L+n-1)}}{\sqrt{n!}} z^n \qquad (\Lambda = \mathbb{D})$$

(where $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d. and $L > 0$) are ergodic under the action of the corresponding isometry groups: $\tau_{\mathbb{C}}$ for (1) and $\tau_{\mathbb{D}}$ for (2).

Proof of the case (2). For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$ we set

$$f_{\alpha,\beta}(z) := (f \circ \varphi_{\alpha,\beta})(z)(\overline{\beta}z + \overline{\alpha})^{-L}$$

and we know from the proof of Lemma 3.1 that $f_{\alpha,\beta} \stackrel{d}{=} f$.

Step 1. Asymptotic uncorrelatedness of the coefficients of f and $f_{\alpha,\beta}$.

Fix $0 < r < 1$ and let $z, w \in \partial \mathrm{D}(0,r) := \{\zeta \in \mathbb{D} : |\zeta| = r\}$. The covariance of $f_{\alpha,\beta}(z)$ and $f(w)$ is given by

$$
\begin{aligned}
\mathbb{E}\left[f_{\alpha,\beta}(z)\overline{f(w)}\right] &= \mathbb{E}\left[f(\varphi_{\alpha,\beta}(z))(\overline{\beta}z+\overline{\alpha})^{-L}\overline{f(w))}\right] \\
&= (\overline{\beta}z+\overline{\alpha})^{-L}\,\mathbb{E}\left[f(\varphi_{\alpha,\beta}(z))\overline{f(w))}\right] \\
&= (\overline{\beta}z+\overline{\alpha})^{-L}K_f(\varphi_{\alpha,\beta}(z), w) \\
&= (\overline{\beta}z+\overline{\alpha})^{-L}\,(1-\varphi_{\alpha,\beta}(z)\,\overline{w})^{-L}\,.
\end{aligned}
\tag{3.4}
$$

Since $|\varphi_{\alpha,\beta}(z)| < 1$ and $|z| = |w| = r$, we have

$$
\begin{aligned}
&|1-\varphi_{\alpha,\beta}(z)\,\overline{w}| \geq 1 - |\varphi_{\alpha,\beta}(z)||w| > 1-r \\
\text{and}\quad &|\overline{\beta}z+\overline{\alpha}| \geq |\,|\beta|r-|\alpha|\,| = |\beta|\cdot\left|r-\sqrt{1+\frac{1}{|\beta|^2}}\right|
\end{aligned}
$$

for $|\alpha|^2 - |\beta|^2 = 1, \beta \neq 0$, so that (3.4) implies

$$
\left|\mathbb{E}\left[f_{\alpha,\beta}(z)\overline{f(w)}\right]\right| < \frac{1}{|\beta|^L(1-r)^L\left|r-\sqrt{1+\frac{1}{|\beta|^2}}\right|^L} \to 0
$$

as $|\beta| \to \infty$, uniformly in $z, w \in \partial\mathrm{D}(0,r)$. Note that by Cauchy's formula we can express the power series coeffients a_n of f and $b_m^{\alpha,\beta}$ of $f_{\alpha,\beta}$ as

$$
a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\,\zeta, \quad b_m^{\alpha,\beta} = \frac{1}{2\pi i}\int_C \frac{f_{\alpha,\beta}(\eta)}{\eta^{m+1}}d\,\eta, \tag{3.5}
$$

where C denotes the contour parametrised by $C(t) := re^{it}$, which then yields

$$
\begin{aligned}
\mathbb{E}[\,b_m^{\alpha,\beta}\overline{a_n}\,] &= \mathbb{E}\left[\frac{1}{(2\pi)^2}\int_C\int_C \frac{f_{\alpha,\beta}(\eta)}{\eta^{m+1}}\frac{\overline{f(\zeta)}}{\overline{\zeta}^{n+1}}d\eta\overline{d\zeta}\right] \qquad (3.6)\\
&= \frac{1}{(2\pi)^2}\int_C\int_C \frac{\mathbb{E}[f_{\alpha,\beta}(\eta)\overline{f(\zeta)}]}{\eta^{m+1}\overline{\zeta}^{n+1}}\,d\eta\,\overline{d\zeta} \to 0
\end{aligned}
$$

as $|\alpha|, |\beta| \to \infty$, so we conclude that the power series coefficients a_n and $b_m^{\alpha,\beta}$ of f and $f_{\alpha,\beta}$ are asymptotically uncorrelated as $|\alpha|, |\beta| \to \infty$.

We briefly clarify why bringing the expectation inside the integral in (3.6) is allowed. Writing out the above expressions for contour integrals with the parametrising curve $C(t) = re^{it}$ with $t \in [0, 2\pi)$, we obtain

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(C(t))}{C(t)^{n+1}}C'(t)\,dt, \\
b_m^{\alpha,\beta} &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\overline{\beta}C(s)+\overline{\alpha})^{-L}f(\varphi_{\alpha,\beta}(C(s)))}{C(s)^{m+1}}C'(s)\,ds,
\end{aligned}
$$

so that

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\mathbb{E}\,[\,|f(C(t))||f(\varphi_{\alpha,\beta}(C(s)))|\,]}{|C(t)|^{n+1}|C(s)|^{m+1}}\,|C'(t)||C'(s)|\,dt\,ds, \qquad (3.7)$$
$$=\frac{1}{r^{n+m}}\int_0^{2\pi}\int_0^{2\pi}\mathbb{E}\,[\,|f(C(t))||f(\varphi_{\alpha,\beta}(C(s)))|\,]\,dt\,ds,$$

since $|C(t)| = |re^{it}| = r$ and $|C'(t)| = |r\cdot i\cdot e^{it}| = r$. Since $K_f(z,w) = (1-z\overline{w})^{-L}$, we obtain

$$\begin{aligned}\mathbb{E}\,[\,|f(C(t))||f(\varphi_{\alpha,\beta}(C(s)))|\,] &\le \mathbb{E}\,[\,|f(C(t))|^2\,]^{1/2}\,\mathbb{E}\,[\,|f(\varphi_{\alpha,\beta}(C(s)))|^2\,]^{1/2}\\ &= |\overline{\beta}\,C(s)+\overline{\alpha}|^{-L}\sqrt{K_f(C(t),C(t))}\sqrt{K_f(\varphi_{\alpha,\beta}(C(s)),\varphi_{\alpha,\beta}(C(s)))}\\ &= |\overline{\beta}\,C(s)+\overline{\alpha}|^{-L}(1-\underbrace{|C(t)|^2}_{=r^2})^{-L/2}(1-|\varphi_{\alpha,\beta}(C(s))|^2)^{-L/2}\\ &\le c_{r,L,\alpha,\beta}\cdot|\overline{\beta}\,C(s)+\overline{\alpha}|^{-L}(1-r^2)^{-L/2} = \tilde{c}_{r,L,\alpha,\beta}\cdot|\overline{\beta}\,C(s)+\overline{\alpha}|^{-L},\end{aligned}$$

where $\tilde{c}_{r,L,\alpha,\beta}$ is defined by the last equality and $c_{r,L,\alpha,\beta}$ comes from the fact that $(1-|\varphi_{\alpha,\beta}(\eta)|^2)^{-L/2}$ is uniformly bounded in $\eta\in\partial\mathrm{D}(0,r)$, since $\varphi_{\alpha,\beta}$ is a continuous function and $\partial\mathrm{D}(0,r)$ a compact subset of $\mathbb{D}$. This ensures that the integral in (3.7) is bounded above by

$$\begin{aligned}\frac{\tilde{c}_{r,L,\alpha,\beta}}{r^{n+m}}\int_0^{2\pi}\int_0^{2\pi}|\overline{\beta}\,C(s)+\overline{\alpha}|^{-L}\,dt\,ds &= \frac{2\pi\tilde{c}_{r,L,\alpha,\beta}}{r^{n+m}}\int_0^{2\pi}\frac{1}{|\overline{\beta}\,C(s)+\overline{\alpha}|^{L}}\,ds\\ &\le \frac{2\pi\tilde{c}_{r,L,\alpha,\beta}}{r^{n+m}}\int_0^{2\pi}\frac{1}{(|\alpha|-|\beta||C(s)|)^L}\,ds = \frac{4\pi^2\tilde{c}_{r,L,\alpha,\beta}}{r^{n+m}(|\alpha|-r|\beta|)^L},\end{aligned}$$

so that Fubini-Tonelli justifies exchanging the integrals with the expected value in (3.6).

Step 2. Joint Gaussianity of coefficients of f and $f_{\alpha,\beta}$.

By the remark after Definition 1.15, it suffices to show that

$$I_{p_1,p_2} := p_1 a_n + p_2 b_m^{\alpha,\beta}$$

is Gaussian for every $p_1,p_2\in\mathbb{C}$. Writing out I_{p_1,p_2} as the limit of the Riemann integral, where $0=t_0<t_1<\cdots<t_N=2\pi$ denotes a partition of the interval $[0,2\pi)$ (and similarly for s_j), we have

$$I_{p_1,p_2} = \lim_{N\to\infty}\sum_{j=0}^{N}\left[\,c_j\cdot f(C(t_j)) + d_j\cdot f(\varphi_{\alpha,\beta}(C(s_j))\,\right],$$

where

$$c_j := \frac{p_1}{2\pi i}\frac{C'(t_j)}{C(t_j)^{n+1}},\, d_j := \frac{p_2}{2\pi i}\frac{(\overline{\beta}C(s_j)+\overline{\alpha})^{-L}C'(s_j)}{C(s_j)^{m+1}},$$

so we observe that I_{p_1,p_2} is a limit of an expression of the form $a^\top Z$ for

$$\begin{aligned}
&a = (c_1, d_1, \dots, c_N, d_N)^\top \in \mathbb{C}^{2N}\\
&Z = (\, f(C(t_1)), f(\varphi_{\alpha,\beta}(C(s_1))), \dots, f(C(t_N)), f(\varphi_{\alpha,\beta}(C(s_N)))\,)^\top
\end{aligned}$$

and by the definition of f, we know that Z is a ($2N$-)dimensional Gaussian random vector; therefore, by Proposition 1.10, $a^\top Z$ is a Gaussian as well, and finally, by applying Lemma 1.14, we conclude that I_{p_1,p_2} is a Gaussian random variable for any $p_1, p_2 \in \mathbb{C}$, which further implies that a_n and $b_m^{\alpha,\beta}$ are jointly Gaussian, so the claim follows.

Knowing that a_n and $b_m^{\alpha,\beta}$ are jointly Gaussian and asymptotically uncorrelated $|\alpha|, |\beta| \to \infty$, Lemma 1.16 implies that a_n and $b_m^{\alpha,\beta}$ are asymptotically independent as $|\alpha|, |\beta| \to \infty$.

Step 3. Approximation of invariant events.

Let $A \in \mathcal{F}$ be an invariant event and $\varepsilon > 0$ arbitrary. By Lemma 3.2, we can choose an event $A_n \in \mathcal{F}_n$ that depends on the first n power series coefficients and satisfies $\mathbb{P}_f(A \triangle A_n) \le \varepsilon$. Note that

$$|\mathbb{1}_A(n_f) - \mathbb{1}_{A_n}(n_f)| \le \mathbb{1}_{A\setminus A_n}(n_f) + \mathbb{1}_{A_n\setminus A}(n_f) = \mathbb{1}_{A\triangle A_n}(n_f),$$

so by monotonicity of expectation we obtain

$$\mathbb{E}\left[\,|\mathbb{1}_A(n_f) - \mathbb{1}_{A_n}(n_f)|\,\right] \le \mathbb{E}\left[\,\mathbb{1}_{A\triangle A_n}(n_f)\,\right] = \mathbb{P}_f(A \triangle A_n) \le \varepsilon$$

and the same inequality holds with $n_{f_{\alpha,\beta}}$ in place of n_f, since $f_{\alpha,\beta} \overset{d}{=} f$ implies $\mathbb{P}(n_{f_{\alpha,\beta}} \in B) = \mathbb{P}(n_f \in B)$ for any measurable $B \subset \mathcal{M}(\Lambda)$. We can use these to estimate the following expression:

$$\begin{aligned}
&|\,\mathbb{E}[\mathbb{1}_A(n_f)\mathbb{1}_A(n_{f_{\alpha,\beta}}) - \mathbb{1}_{A_n}(n_f)\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})]\,|\\
&\quad = \left|\,\mathbb{E}\left[\mathbb{1}_A(n_f)(\mathbb{1}_A(n_{f_{\alpha,\beta}}) - \mathbb{1}_{A_n}(n_{f_{\alpha,\beta}}))\right] - \mathbb{E}\left[\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})(\mathbb{1}_{A_n}(n_f) - \mathbb{1}_A(n_f))\right]\right|\\
&\quad \le \mathbb{E}\left[|\mathbb{1}_{A_n}(n_f)||\mathbb{1}_A(n_{f_{\alpha,\beta}}) - \mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})|\right] + \mathbb{E}\left[|\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})||\mathbb{1}_{A_n}(n_f) - \mathbb{1}_A(n_f)|\right]\\
&\quad \le \mathbb{E}\left[|\mathbb{1}_A(n_{f_{\alpha,\beta}}) - \mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})|\right] + \mathbb{E}\left[|\mathbb{1}_{A_n}(n_f) - \mathbb{1}_A(n_f)|\right] \le \varepsilon + \varepsilon = 2\varepsilon
\end{aligned} \tag{3.8}$$

Step 4. Finalizing the argument.

By the asymptotic independence of the coefficients of f and $f_{\alpha,\beta}$, we have

$$\mathbb{E}\,[\mathbb{1}_{A_n}(n_f)\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})\,] - \mathbb{E}\,[\mathbb{1}_{A_n}(n_f)\,]\cdot\mathbb{E}\,[\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})\,] \to 0$$

as $|\alpha|, |\beta| \to \infty$, implying

$$\mathbb{E}\,[\mathbb{1}_{A_n}(n_f)\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})\,] \to (\,\mathbb{E}\,[\mathbb{1}_{A_n}(n_f)]\,)^2 \tag{3.9}$$

as $|\alpha|, |\beta| \to \infty$, since $\mathbb{E}\,[\mathbb{1}_{A_n}(n_f)] = \mathbb{E}\,[\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})]$. Furthermore,

$$\begin{aligned} |(\mathbb{E}[\mathbb{1}_A(n_f)]\,)^2 - (\mathbb{E}[\mathbb{1}_{A_n}(n_f)]\,)^2| &\leq |\mathbb{E}[\mathbb{1}_A(n_f)]|\cdot|\mathbb{E}[\mathbb{1}_A(n_f)] - \mathbb{E}[\mathbb{1}_{A_n}(n_f)]| \\ &\quad + |\mathbb{E}[\mathbb{1}_{A_n}(n_f)]|\cdot|\mathbb{E}[\mathbb{1}_A(n_f)] - \mathbb{E}[\mathbb{1}_{A_n}(n_f)]| \\ &\leq 2\,\mathbb{E}\,[\,|\mathbb{1}_A(n_f) - \mathbb{1}_{A_n}(n_f)|\,] \\ &\leq 2\varepsilon. \end{aligned} \tag{3.10}$$

Finally,

$$\begin{aligned} &|\,\mathbb{E}[\mathbb{1}_A(n_f)\mathbb{1}_A(n_{f_{\alpha,\beta}})] - (\,\mathbb{E}[\mathbb{1}_A(n_f)]\,)^2\,| \\ &\qquad \leq |\,\mathbb{E}[\mathbb{1}_A(n_f)\mathbb{1}_A(n_{f_{\alpha,\beta}})] - \mathbb{E}[\mathbb{1}_{A_n}(n_f)\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})]| \\ &\qquad\quad + |\mathbb{E}[\mathbb{1}_{A_n}(n_f)\mathbb{1}_{A_n}(n_{f_{\alpha,\beta}})] - (\,\mathbb{E}[\mathbb{1}_{A_n}(n_f)]\,)^2| \\ &\qquad\quad + |(\mathbb{E}[\mathbb{1}_A(n_f)]\,)^2 - (\mathbb{E}[\mathbb{1}_{A_n}(n_f)]\,)^2| \end{aligned}$$

and combining (3.8), (3.9) and (3.10) we arrive at

$$\limsup_{|\alpha|,|\beta|\to\infty} |\,\mathbb{E}[\mathbb{1}_A(n_f)\mathbb{1}_A(n_{f_{\alpha,\beta}})] - (\,\mathbb{E}[\mathbb{1}_A(n_f)]\,)^2\,| \leq 4\varepsilon. \tag{3.11}$$

Since the invariance of A implies $\mathbb{1}_A(n_f)\mathbb{1}_A(n_{f_{\alpha,\beta}}) = \mathbb{1}_A(n_f)$ and $\varepsilon > 0$ was chosen arbitrarily, we obtain

$$\mathbb{E}[\mathbb{1}_A(n_f)] = (\,\mathbb{E}[\mathbb{1}_A(n_f)]\,)^2$$

from (3.11), which implies

$$\mathbb{P}_f(A) = \mathbb{E}[\mathbb{1}_A(n_f)] \in \{0, 1\}.$$

The proof of the case (1) is analogous. $\square$

4 Gaussian Weyl-Heisenberg functions and their zero sets

We have seen in the context of isometry-invariant GAFs that even though the functions themselves are not invariant in distribution under the corresponding transformations, incorporating a zero-free non-random factor into the transformation of the function guarantees stochastic invariance of their zero sets. Concretely, for the hyperbolic GAF f and a linear fractional transformation $\varphi_{\alpha,\beta}$, we made use of the stochastic invariance with respect to

$$f(z) \mapsto (\overline{\beta}z + \overline{\alpha})^{-L}(f \circ \varphi_{\alpha,\beta})(z) =: f_{\alpha,\beta}(z),$$

which followed from the equality of covariance kernels of f and $f_{\alpha,\beta}$. We can also exploit this kind of stationarity in a non-analytic setting for a class of Gaussian random functions called Gaussian Weyl-Heisenberg functions (GWHF), introduced in [12]. For a GWHF F, the non-random zero-free factor will be a complex exponential, so that the so-called twisted shifts

$$F(z) \mapsto e^{i\,\mathrm{Im}(z\overline{\zeta})}F(z-\zeta) =: F_\zeta(z) \tag{4.1}$$

guarantee stochastic invariance of F and its twisted shifts F_ζ and further imply invariance in distribution under translations in $\mathbb{C}$ for the zero set of a GWHF (see Lemma 4.5). The attribute *Weyl-Heisenberg* should indicate the connection of the transformations in (4.1) to the (reduced) Weyl-Heisenberg groups (see [12] and references therein).

Definition 4.1. *A Gaussian Weyl-Heisenberg function F is a random function $F : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{C}^{\mathbb{C}}$ such that*

- *F is circularly symmetric, i.e. $F(z) \sim e^{i\theta}F(z)$ for all $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$*
- *$(F(z_1), \ldots, F(z_n))^\top \sim \mathcal{N}_{\mathbb{C}}^n$ for every $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{C}$*
- *the covariance kernel of F is given by twisted convolution:*

  $$K_F(z,w) = \mathbb{E}[F(z)\overline{F(w)}] = H(z-w)\, e^{i\,\mathrm{Im}(z\overline{w})}$$

  where $H : \mathbb{C} \to \mathbb{C}$ is called the twisted kernel.

Remark. *Circular symmetry of F already implies*

$$\mathbb{E}[F(z)] = \mathbb{E}[e^{i\theta}F(z)] = e^{i\theta}\,\mathbb{E}[F(z)] \;\; \forall\theta \in \mathbb{R} \implies \mathbb{E}[F(z)] = 0,$$

so since f is a Gaussian, by Lemma 1.12 the twisted kernel indeed determines the distribution of f.

We restate standing assumptions on the twisted kernel H and the Gaussian Weyl-Heisenberg function F from [12]. The relevance of these assumptions is only partly obvious in the arguments presented in this thesis - nevertheless, they are necessary for the proofs of the results from [12] we refer to and do not carry out here. We also make one extra assumption on integrability of the twisted kernel, which will be of use to us in Section 4.2.

Standing assumptions:

(i) to ensure that K_F given in Definition 4.1 defines a covariance kernel, we assume that H is such that

$$(H(z_k - z_j)\, e^{i\,\mathrm{Im}(z_k \overline{z_j})})_{1 \leq j,k \leq n} \geq 0 \quad \text{for all } z_1, \ldots, z_n \in \mathbb{C},$$

which then implies that $H(-z) = \overline{H(z)}$ and $H(0) \geq 0$;

(ii) $H \in L^1(\mathbb{C})$, $H(0) = 1$, $|H(z)| < 1, z \in \mathbb{C} \setminus \{0\}$;

(iii) H is C^2 in the real sense;

(iv) almost every realization of F is a $C^2(\mathbb{R}^2)$ function.

Note that the above assumption $H(0) = 1$ implies $K(z,z) = 1$ for the covariance kernel $K = K_F$, which we use frequently in computations to follow, as well as $K(z,w) = \overline{K(w,z)}$ following directly from $K(z,w) = \mathbb{E}\left[F(z)\overline{F(w)}\right]$. In the view of item (iii), the following notation for partial derivatives of H is used:

$$\begin{aligned}
&H^{(1,0)}(x+iy) = \partial_x H(x+iy), H^{(2,0)}(x+iy) = \partial_{xx} H(x+iy),\\
&H^{(1,1)}(x+iy) = \partial_x \partial_y H(x+iy), \text{ etc.}
\end{aligned}$$

4.1 Zero sets and the first intensity function for a GWHF

In comparison to deriving the Edelman-Kostlan formula for a Gaussian analytic function, computing the first intensity function ρ_1 for a GWHF is far more involved. We briefly indicate the approach from [12] and state the result with the explicit formula for ρ_1 in terms of the twisted kernel H.

The existence of the first intensity function is guaranteed by a result we quote from [27], which gives the formula for the expected number of parameter points in the level sets of a given Gaussian random field Z, i.e. the sets of the form

$$\{t \in E : Z(t) = u\}.$$

Note that a random field is simply a stochastic process with a multidimensional parameter set - more precisely, in the context of the following result, it is a family $\{Z(t) : t \in U\}$ of $\mathbb{R}^d$-valued random vectors $Z(t)$ indexed on an open subset U of $\mathbb{R}^d$. The random field being Gaussian means that for any set of points $t_1, \ldots, t_n \in U$, the random vectors $Z(t_1), \ldots, Z(t_n)$ are jointly Gaussian. Following [12], we now quote the aforementioned result from [27] (Theorem 6.2), an instance of the so-called Kac-Rice formulae:

Proposition 4.2. *Let $U \subseteq \mathbb{R}^d$ be open, $Z : U \to \mathbb{R}^d$ a Gaussian random field, $u \in \mathbb{R}^d$ a fixed point and assume that*

(i) the function $t \mapsto Z(t)$ is almost surely C^1,

(ii) $Z(t)$ has a non-degenerate distribution for every $t \in U$, i.e. $\operatorname{cov}(Z(t))$ is positive definite for every $t \in U$,

(iii) $\mathbb{P}(\exists\, t \in U : Z(t) = u \text{ and } \det DZ(t) = 0) = 0$.

Then for every Borel measurable set $E \subseteq U$:

$$\mathbb{E}\left[\,\#(t \in E : Z(t) = u)\,\right] = \int_E \mathbb{E}\left[\,|\det DZ(t)| \;\middle|\; Z(t) = u\right] f_{Z(t)}(u)dt \tag{4.2}$$

where $f_{Z(t)}$ denotes the probability density function of $Z(t)$. If E is compact, both sides of (4.2) are finite.

Remark. *Note that the second condition ensures the existence of a probability density function $f_{Z(t)}$ according to Lemma 1.11 (since $Z(t)$ is a Gaussian for every $t \in U$) and the observation from Section 1 stating that a covariance matrix is non-singular iff it is positive defnite.*

It is clear that for $u = 0$ the expression on the left hand side is the first intensity measure of the set E, so one can use (4.2) to compute the first intensity function if the assumptions from Proposition 4.2 are applicable to a GWHF F. We verify that this is indeed the case:

(i) the item (i) from Proposition 4.2 is obviously satisfied by the standing assumptions we imposed on F;

(ii) given a GWHF F and $z \in \mathbb{C}$, $F(z)$ is a Gaussian random variable with $\mathbb{E}[F(z)] = 0$ and $\operatorname{Var}(F(z)) = K(z,z) = H(0) = 1$, so $F(z) \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and hence its covariance matrix is obviously (see (1.6) in Section 1.2) a positive definite matrix, so the condition (ii) from Proposition 4.2 is satisfied;

(iii) the third item follows from Proposition 4.3 below.

We take note of the following fact from Proposition 3.2 in [12], since apart from ensuring that Proposition 4.2 (iii) is satisfied for a GWHF, it also demonstrates that the zero set of a GWHF is indeed a simple point process:

Proposition 4.3. *Let F be a GWHF satisfying the standing assumptions. Then zeros of F are almost surely simple and thus isolated. More precisely,*

$$\mathbb{P}(\,\exists\, z \in \mathbb{C} : F(z) = 0 \ \text{ and } \ \det DF(z) = 0\,) = 0.$$

Proof. See Proposition 3.2 in [12]. □

Remark. *Another insight from this result is that the realisations of n_F are indeed elements of $\mathcal{M}(\mathbb{C})$: the closed set of isolated zeros can only have finitely many points in a compact subset of $\mathbb{C}$, whence the realisations of n_F indeed define a Radon measure and are therefore elements of $\mathcal{M}(\mathbb{C})$ by Remark after Definition 2.1.*

Since $F(z) \sim \mathcal{N}_{\mathbb{C}}(0,1)$, we know from (1.5) that $f_{F(z)}(0) = 1/\pi$, so the formula (4.2) implies

$$\rho_1(z) = \frac{1}{\pi}\mathbb{E}\left[\,|\det DF(z)|\,\Big|\,F(z) = 0\right] \tag{4.3}$$

for the first intensity function of a Gaussian Weyl-Heisenberg function F. It turns out that the right hand side of (4.3) is independent of z and can be expressed in terms of partial derivatives of the twisted kernel H evaluated at 0. We omit these computations and quote the final result from [12] (Theorem 1.6), referring to the same source for a detailed proof:

Theorem 4.4. *Let F be a GWHF with twisted kernel H satisfying the standing assumptions. Then n_F is a stationary random measure with first intensity function*

$$\rho_1 = \frac{1}{2\pi}\frac{\triangle_H + 2}{\sqrt{\triangle_H + 1}} \tag{4.4}$$

where

$$\triangle_H := \det\begin{bmatrix} -H^{(2,0)}(0) - |H^{(1,0)}(0)|^2 & -H^{(1,1)}(0) - i - H^{(1,0)}(0)H^{(0,1)}(0) \\ \overline{-H^{(1,1)}(0) + i - H^{(1,0)}(0)H^{(0,1)}(0)} & -H^{(0,2)}(0) - |H^{(0,1)}(0)|^2 \end{bmatrix} \tag{4.5}$$

In addition, $\triangle_H \geq 0$ and therefore $\rho_1 \geq 1/\pi$.

We now introduce the corresponding transformations of the zero set and the function itself in the following result as an analogue of Lemma 3.1. Note that the stationarity of the zero set is already claimed in the previous result, however we isolate this fact and carry out the proof due to its analogy to Lemma 3.1.

Lemma 4.5. *The zero set of a GWHF is invariant in distribution under translations T_ζ ($\zeta \in \mathbb{C}$) in $\mathbb{C}$.*

Proof. We follow the proof of Proposition 3.2. (a) from [12] and the idea of the proof of Proposition 2.3.4 in [1]. Define

$$G_\zeta(z) := F(T_\zeta(z)) = F(z - \zeta),$$

then the covariance kernel of G_ζ is given by

$$\begin{aligned} K_{G_\zeta}(z, w) &= \mathbb{E}\left[F(z - \zeta)\overline{F(w - \zeta)}\right] = K_F(z - \zeta, w - \zeta) \\ &= e^{i\,\mathrm{Im}(z-\zeta)\overline{(w-\zeta)}} H(z - w) = e^{i\,\mathrm{Im}(-z\overline{\zeta} - \overline{w}\zeta)}\, e^{i\,\mathrm{Im}(z\overline{w})} H(z - w) \\ &= e^{i\,\mathrm{Im}(-z\overline{\zeta} - \overline{w}\zeta)}\, K_F(z, w) \end{aligned}$$

and by setting $H_\zeta(z) := e^{-i\,\mathrm{Im}(z\overline{\zeta})}\, F(z)$, we notice that $K_{H_\zeta}(z, w) = K_{G_\zeta}(z, w)$. By Gaussianity of H_ζ and G_ζ (and the obvious fact that the expectation of both $G_\zeta(z)$ and $H_\zeta(z)$ is 0), Lemma 1.12 implies that $G_\zeta(z)$ and $H_\zeta(z)$ are identically distributed, so we obtain $F(z - \zeta) = G_\zeta(z) \overset{d}{=} H_\zeta(z) = e^{-i\,\mathrm{Im}(z\overline{\zeta})}\, F(z)$; in particular,

$$F(z) \overset{d}{=} e^{i\,\mathrm{Im}(z\overline{\zeta})} F(z - \zeta) =: F_\zeta(z)$$

and since the factor $e^{i\,\mathrm{Im}(z\overline{\zeta})}$ has no zeros, the distribution of the zero set of F_ζ is the same as that of F. $\square$

Remark. *(i) One insight of the proof is that the twisted shift of a GWHF*

$$F(z) \mapsto e^{i\,\mathrm{Im}(z\overline{\zeta})} F(z - \zeta) = F_\zeta(z),$$

leaves its distribution invariant, as indicated at the very beginning of this section.

(ii) The other insight is that $F(z) \overset{d}{=} F_\zeta(z)$ ensures we have a measure-preserving transformation:

$$\begin{aligned} \mathbb{P}_F(A) &= \mathbb{P}(n_F \in A) = \mathbb{P}(\,T_\zeta(n_f) \in T_\zeta(A)\,) \\ &= \mathbb{P}(n_{F_\zeta} \in T_\zeta(A)) = \mathbb{P}(n_F \in T_\zeta(A)) = \mathbb{P}_F(T_\zeta(A)). \end{aligned} \tag{4.6}$$

4.2 Ergodicity of the zero set of a GWHF

We now go on to proving that the zero set of a Gaussian Weyl-Heisenberg function is ergodic as in the case of isometry-invariant GAFs. We will demonstrate that one can follow the same idea for the proof of Proposition 3.4 for the corresponding zero sets upon making suitable adjustments in some of the arguments.

Note that the reasoning from the beginning of Section 3.1 remains valid here, since the first intensity function of the zero set corresponding to a GWHF satisfies $\rho_1 \geq 1/\pi$ (as claimed in Theorem 4.4), so the ergodicity again implies that the probability of a GWHF having no zeros is equal to 0.

Given a GWHF with the underlying probability space Ω, the first thing we want to take care of is to establish a suitable expansion of F, so that the structure of Ω in terms of the probability spaces Ω_n underlying the random coefficients of this expansion carries over from Section 3.1. For this construction, we will follow the idea of the so-called Karhunen-Loève expansion for stochastic processes indexed on a compact real interval.

For the covariance kernel $K(z,w) = e^{i\,\mathrm{Im}(z\overline{w})}H(z-w)$ on $\mathbb{C}\times\mathbb{C}$ we define the corresponding integral operator via

$$\mathcal{K}(f)(z) := \int\limits_{\mathbb{C}} K(z,w)f(w)\,dm(w), \qquad f \in L^2(\mathbb{C}, dm). \tag{4.7}$$

Assumption: $\mathcal{K}$ is an orthogonal projection on $L^2(\mathbb{C}, dm)$

Being an orthogonal projection, we know that $\operatorname{ran}\mathcal{K}$ is a closed subspace of $L^2(\mathbb{C}, dm)$, hence it possesses a complete orthonormal system $\{\, e_n \mid n \in \mathbb{N}\,\}$, such that the action of $\mathcal{K}$ on $f \in L^2(\mathbb{C}, dm)$ can be expressed as

$$\mathcal{K}(f) = \sum_{n=0}^{\infty} \langle f, e_n\rangle\, e_n$$

For $N \in \mathbb{N}$ we define the associated trunctation $\mathcal{K}_N$ of the ortogonal projection $\mathcal{K}$ via

$$\mathcal{K}_N(f) = \sum_{n=0}^{N} \langle f, e_n\rangle\, e_n$$

and setting $K_N(z,w) := \sum\limits_{n=0}^{N} e_n(z)\overline{e_n(w)}$, notice that

$$\begin{aligned}\mathcal{K}_N(f)(z) &= \sum_{n=0}^{N} \langle f, e_n\rangle\, e_n(z) = \int\limits_{\mathbb{C}} \left(\sum_{n=0}^{N} e_n(z)\overline{e_n(w)}\right) f(w)\, dm(w)\\ &= \int\limits_{\mathbb{C}} \left(\sum_{n=0}^{N} e_n(z)\overline{e_n(w)}\right) f(w)\, dm(w),\end{aligned}$$

so we conclude that $\mathcal{K}_N$ is the integral operator corresponding to the truncated covariance kernel K_N:

$$\mathcal{K}_N(f)(z) = \int\limits_{\mathbb{C}} K_N(z,w) f(w)\, dm(w).$$

Since $\mathcal{K}_N$ is an orthogonal projection onto $\overline{\operatorname{span}\{e_n\}_{n=1}^{N}} \subseteq \operatorname{ran}\mathcal{K}$, we know that $\langle \mathcal{K}_N f, f\rangle \leq \langle \mathcal{K} f, f\rangle$ for all $f \in L^2(\mathbb{C})$, or, explicitly:

$$\int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}} K_N(z,w) f(w)\overline{f(z)}\, dm(w)\, dm(z) \leq \int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}} K(z,w) f(w)\overline{f(z)}\, dm(w)\, dm(z). \tag{4.8}$$

For a fixed $z_0 \in \mathbb{C}$ we choose the approximate identity φ_ε given by

$$\varphi_\varepsilon(\cdot) = \varepsilon^{-d} \mathbb{1}_{[-\varepsilon/2,\, \varepsilon/2]^d}(z_0 - \cdot)$$

and obtain

$$\begin{aligned}\varphi_\varepsilon(w)\overline{\varphi_\varepsilon(z)} &= \varepsilon^{-2d} \mathbb{1}_{[-\varepsilon/2,\, \varepsilon/2]^d}(z_0 - w) \cdot \mathbb{1}_{[-\varepsilon/2,\, \varepsilon/2]^d}(z_0 - z)\\ &= \varepsilon^{-2d} \mathbb{1}_{[-\varepsilon/2,\, \varepsilon/2]^{2d}}((z_0, z_0) - (z,w))\\ &=: \psi_\varepsilon((z_0,z_0) - (z,w)),\end{aligned}$$

so that ψ_ε is again an approximate identity and choosing $f = \varphi_\varepsilon$ in (4.8) yields

$$\begin{aligned}&\int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}} K_N(z,w)\psi_\varepsilon((z_0,z_0) - (z,w))\, dm(w)\, dm(z)\\ &\qquad\qquad \leq \int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}} K(z,w)\psi_\varepsilon((z_0,z_0) - (z,w))\, dm(w)\, dm(z),\end{aligned}$$

which translates to

$$(K_N * \psi_\varepsilon)(z_0, z_0) \leq (K * \psi_\varepsilon)(z_0, z_0). \tag{4.9}$$

Since K_N is locally integrable and K is continuous and bounded by standing assumptions on H, we have[5]

$$\lim_{\varepsilon\downarrow 0}(K_N * \psi_\varepsilon)(z) = K_N(z) \text{ for a.e. } z \in \mathbb{C}^2$$
$$\text{and } \lim_{\varepsilon\downarrow 0}(K * \psi_\varepsilon)(z) = K(z) \text{ for all } z \in \mathbb{C}^2, \tag{4.10}$$

so that (4.9) implies

$$\sum_{n=0}^{N} |e_n(z)|^2 = K_N(z,z) \leq K(z,z) = H(0) = 1$$

for almost every $z \in \mathbb{C}$ and therefore

$$\sum_{n=0}^{N} |e_n(z)|\,|e_n(w)| \leq \left(\sum_{n=0}^{N} |e_n(z)|^2\right)^{1/2} \left(\sum_{n=0}^{N} |e_n(w)|^2\right)^{1/2} \leq 1 \tag{4.11}$$

for almost every $z, w \in \mathbb{C}$ and arbitrary $N \in \mathbb{N}$. Note that $(e_n) \subseteq \operatorname{ran} \mathcal{K}$, so $e_n = \mathcal{K} e_n$ for every $n \in \mathbb{N}$ and

$$e_n(z) = \int_{\mathbb{C}} K(z,w) e_n(w)\, dm(w) \tag{4.12}$$

holds pointwise for every $z \in \mathbb{C}$ upon modifying $e_k \in L^2(\mathbb{C})$ on a set of measure zero such that (4.7) holds for every $z \in \mathbb{C}$. Using (4.11) and (4.12) we obtain

$$\begin{aligned}
\sum_{n=0}^{N} |e_n(z)|^2 &\leq \sum_{n=0}^{N} \Big| \int_{\mathbb{C}}\int_{\mathbb{C}} K(z,w)\, e_n(w)\, \overline{K(z,w')e_n(w')}\, dm(w)\, dm(w') \Big| \\
&\leq \int_{\mathbb{C}}\int_{\mathbb{C}} \underbrace{\left(\sum_{n=0}^{N} |e_n(w)|\,|e_n(w')|\right)}_{\leq 1 \text{ almost everywhere}} |K(z,w)|\,|K(z,w')|\, dm(w)\, dm(w') \\
&\leq \int_{\mathbb{C}} |H(z-w)|\, dm(w) \int_{\mathbb{C}} |H(z-w')|\, dm(w') = \|H\|_1^2
\end{aligned}$$

for any $N \in \mathbb{N}$ and $z \in \mathbb{C}$, since $|K(z,w)| = |e^{i\operatorname{Im}(z\overline{w})} H(z-w)| = |H(z-w)|$. This now implies

$$\sum_{n=0}^{\infty} |e_n(z)|^2 < \infty \quad \text{for every } z \in \mathbb{C}. \tag{4.13}$$

[5]Note that the first conclusion in (4.10) holds upon passing to a subsequence of (ε); we abuse the notation by using the same notation for the subsequence.

Note that for any $f \in L^2(\mathbb{C})$ we have

$$\|\mathcal{K}f - \mathcal{K}_N f\|_2^2 = \|\sum_{n=N+1}^{\infty} \langle f, e_n\rangle\, e_n\| = \sum_{n=N+1}^{\infty} |\langle f, e_n\rangle|^2 \underset{N\ \to\infty}{\longrightarrow} 0,$$

showing convergence $\mathcal{K}_N \to \mathcal{K}$ in strong operator topology as $N \to \infty$, which further implies convergence in weak operator topology, i.e.

$$\langle \mathcal{K}_N f, g\rangle \to \langle \mathcal{K}f, g\rangle \quad \text{for all } f, g \in L^2(\mathbb{C}).$$

In particular, for bounded $f, g \in L^2(\mathbb{C})$ with compact support we obtain

$$\begin{aligned}
&\int_{\mathbb{C}}\int_{\mathbb{C}} K(z,w) f(w)\, \overline{g(z)}\, dm(w)\, dm(z)\\
&\qquad = \lim_{N\to\infty} \int_{\mathbb{C}}\int_{\mathbb{C}} K_N(z,w) f(w)\, \overline{g(z)}\, dm(w)\, dm(z)\\
&\qquad = \lim_{N\to\infty} \int_{\mathbb{C}}\int_{\mathbb{C}} \left(\sum_{n=0}^{N} e_n(z)\overline{e_n(w)}\right) f(w)\, \overline{g(z)}\, dm(w)\, dm(z)\\
&\qquad = \int_{\mathbb{C}}\int_{\mathbb{C}} \left(\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}\right) f(w)\, \overline{g(z)}\, dm(w)\, dm(z), \qquad (4.14)
\end{aligned}$$

where bringing the limit inside the integral is allowed by the dominated convergence theorem, since

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \underbrace{\sum_{n=0}^{\infty} |e_n(z)||e_n(w)|}_{\leq 1 \text{ almost everywhere}} |f(w)|\, |g(z)|\, dm(w)\, dm(z) \leq \|f\|_1 \|g\|_1.$$

For fixed $z_0, w_0 \in \mathbb{C}$, we define

$$\begin{aligned}
\varphi_\varepsilon(\cdot) &= \varepsilon^{-d}\mathbb{1}_{[-\varepsilon/2,\ \varepsilon/2]^d}(w_0 - \cdot)\\
\text{and}\quad \psi_\varepsilon(\cdot) &= \varepsilon^{-d}\mathbb{1}_{[-\varepsilon/2,\ \varepsilon/2]^d}(z_0 - \cdot),
\end{aligned}$$

so that

$$\varphi_\varepsilon(w)\overline{\psi_\varepsilon(z)} = \varepsilon^{-2d}\mathbb{1}_{[-\varepsilon/2,\ \varepsilon/2]^{2d}}((z_0,w_0) - (z,w)) =: \phi_\varepsilon((z_0,w_0) - (z,w)),$$

and thus plugging $f = \varphi_\varepsilon$ and $g = \psi_\varepsilon$ into (4.14) yields

$$(K * \phi_\varepsilon)(z_0, w_0) = (\widetilde{K} * \phi_\varepsilon)(z_0, w_0) \qquad (4.15)$$

for $\widetilde{K}(z,w) := \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$. By the same arguments as in (4.10), letting $\varepsilon \to 0^+$ in (4.15) implies

$$K(z,w) = \sum_{n=0}^{\infty} e_n(z)\, \overline{e_n(w)}$$

for almost every $z, w \in \mathbb{C}$.

Expansion of a GWHF in terms of orthonormal basis functions corresponding to its covariance kernel

For $R > 0$ and $n \in \mathbb{N}$, we define truncated versions of prospective coefficients in the expansion of a GWHF F:

$$a_n^R := \int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_n(z)}\, dm(z).$$

Recall that by our standing assumption F almost surely defines a C^2 function, hence

$$\mathbb{P}\big(\sup_{z \in \mathrm{D}(0,R)} |F(z)| < \infty \,\big) = 1,$$

so since

$$\begin{aligned}
|a_n^R| &\leq \int\limits_{\mathrm{D}(0,R)} |F(z)||e_n(z)|\, dm(z) \\
&\leq \left(\int\limits_{\mathbb{C}} |e_n(z)|^2\, dm(z) \right)^{1/2} \left(\int\limits_{\mathbb{C}} |\mathbb{1}_{\mathrm{D}(0,R)}(z)|^2 |F(z)|^2\, dm(z) \right)^{1/2} \\
&\leq \|e_n\|_2\, |\mathrm{D}(0,R)|^{1/2} \sup_{\mathrm{D}(0,R)} |F|,
\end{aligned}$$

we have that $\mathbb{P}(|a_n^R| < \infty) = 1$ for every $R > 0$, i.e.

$$a_n^R \text{ is almost surely well-defined for every } R > 0.$$

To show that $(a_n^R)_R$ is a Cauchy sequence, note that for w.l.o.g. $R > S$ we obtain

$$\begin{aligned}
\mathbb{E}[\, |a_n^R - a_n^S|^2\,] &= \mathbb{E}[(a_n^R - a_n^S)\overline{(a_n^R - a_n^S)}] \\
&= \mathbb{E}[|a_n^R|^2] - \mathbb{E}[a_n^R \overline{a_n^S}] - \mathbb{E}[a_n^S \overline{a_n^R}] + \mathbb{E}[|a_n^S|^2].
\end{aligned} \tag{4.16}$$

Claim 1: $\mathbb{E}[a_n^R \overline{a_n^S}] \underset{R,S\to\infty}{\longrightarrow} 1$

We first carry out the main computation and then justify the intermediate steps:

$$\begin{aligned}
\mathbb{E}[a_n^R \overline{a_n^S}] &= \mathbb{E}[\int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_n(z)}\,dm(z) \int\limits_{\mathrm{D}(0,S)} \overline{F(w)}e_n(w)\,dm(w)] \\
&= \int\limits_{\mathrm{D}(0,R)} \int\limits_{\mathrm{D}(0,S)} K(z,w)e_n(w)\overline{e_n(z)}\,dm(w)dm(z) \\
&\underset{R,S\to\infty}{\longrightarrow} \int\limits_{\mathbb{C}} \left(\int\limits_{\mathbb{C}} K(z,w)e_n(w)dm(w) \right) \overline{e_n(z)}dm(z) \\
&= \int\limits_{\mathbb{C}} \mathcal{K}(e_n)(z)\overline{e_n(z)}dm(z) = \int\limits_{\mathbb{C}} |e_n(z)|^2 dm(z) = \|e_n\|^2 = 1,
\end{aligned}$$

where we have used $\mathbb{E}[F(z)\overline{F(w)}] = K(z,w)$ upon interchanging expectation and integration; in the last line we use $e_n \in \operatorname{ran}\mathcal{K}$, so $\mathcal{K}(e_n) = e_n$. To see why bringing the expectation inside the integral is allowed, note that Cauchy-Schwartz implies

$$\mathbb{E}[\,|F(z)\overline{F(w)}|\,] \le (\mathbb{E}[|F(z)|^2])^{1/2}(\mathbb{E}[|F(w)|^2])^{1/2} = \sqrt{K(z,z)}\sqrt{K(w,w)} = 1,$$

so we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} \mathbb{E}[|F(z)||F(w)|]\,|e_n(w)||\mathbb{1}_{\mathrm{D}(0,S)}(w)||e_n(z)||\mathbb{1}_{\mathrm{D}(0,R)}(z)|\,dm(w)dm(z) \\
&\qquad \le \int\limits_{\mathbb{C}} |e_n(w)||\mathbb{1}_{\mathrm{D}(0,S)}(w)|\,dm(w) \int\limits_{\mathbb{C}} |e_n(z)||\mathbb{1}_{\mathrm{D}(0,R)}(z)|\,dm(z) \\
&\qquad \le \|e_n\|_2^2\,|\mathrm{D}(0,S)|^{1/2}|\mathrm{D}(0,R)|^{1/2} < \infty,
\end{aligned}$$

so Fubini-Tonelli justifies interchanging the integrals and the expectation. The second intermediate step that needs justifying is taking the limit w.r.t. radii of the disks we are integrating over. Note that

$$|K(z,w)e_n(w)\mathbb{1}_{\mathrm{D}(0,S)}(w)\overline{e_n(z)}\mathbb{1}_{\mathrm{D}(0,R)}(z)| \le |K(z,w)||e_n(w)||e_n(z)|$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} |K(z,w)||e_n(w)||e_n(z)|\,dm(z)dm(w) \\
&\qquad = \int\limits_{\mathbb{C}} \left(\int\limits_{\mathbb{C}} |H(z-w)||e_n(z)|dm(z) \right) |e_n(w)|\,dm(w) \\
&\qquad = \int\limits_{\mathbb{C}} (|\widetilde{H}| * |e_n|)(w)\,|e_n(w)|\,dm(w) = \langle\, |\widetilde{H}| * |e_n|, |e_n|\,\rangle < \infty,
\end{aligned}$$

where $\widetilde{H}(z) := H(-z)$, so $\widetilde{H} \in L^1(\mathbb{C})$ and $e_n \in L^2(\mathbb{C})$ imply $|\widetilde{H}|*|e_n| \in L^2(\mathbb{C})$ by Young's convolution inequality. We can thus use the dominated convergence theorem to interchange the limit with the integrals for the pointwise limit

$$K(z,w)e_n(w)\mathbb{1}_{\mathrm{D}(0,S)}(w)\overline{e_n(z)}\mathbb{1}_{\mathrm{D}(0,R)}(z) \underset{R,S\to\infty}{\longrightarrow} K(z,w)e_n(w)\overline{e_n(z)},$$

so that this step is justified as well and the claim follows.

By the same reasoning we easily see that all the other terms in (4.16) converge to 1 and hence

$$\mathbb{E}[\,|a_n^R - a_n^S|^2\,] \underset{R,S\to\infty}{\longrightarrow} 0,$$

so we obtain a Cauchy sequence $(a_n^R)_R$ in $L^2(\Omega)$ and we define the n-th coefficient a_n as the limit of this sequence:

$$a_n := \lim_{R\to\infty} a_n^R \qquad (n \in \mathbb{N}),$$

where the limit is taken in the mean-square sense, i.e. $\mathbb{E}[\,|a_n - a_n^R|^2\,] \underset{R\to\infty}{\longrightarrow} 0$.

Coefficients defined in this way have zero mean and unit variance:

Claim: $\mathbb{E}[a_n] = 0$ and $\mathrm{Var}[a_n] = 1$.

Recall that $\mathbb{E}\left[F(z)\right] = 0$, so

$$\mathbb{E}\left[a_n^R\right] = \mathbb{E}\left[\int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_n(z)}\,dm(z)\right] = \int\limits_{\mathrm{D}(0,R)} \mathbb{E}\left[F(z)\right]\overline{e_n(z)}\,dm(z) = 0,$$

where taking the expectation inside the integral is allowed, since

$$\begin{aligned}
&\int\limits_{\mathbb{C}} \mathbb{E}\left[|F(z)|\right] |\mathbb{1}_{\mathrm{D}(0,R)}(z)||e_n(z)|\,dm(z) \\
&\quad\le \left(\int\limits_{\mathbb{C}} \mathbb{E}\left[|F(z)|\right]^2 |\mathbb{1}_{\mathrm{D}(0,R)}(z)|^2\,dm(z)\right)^{1/2} \left(\int\limits_{\mathbb{C}} |e_n(z)|^2\,dm(z)\right)^{1/2} \\
&\quad\le \|e_n\|_2\,|\mathrm{D}(0,R)|^{1/2}
\end{aligned}$$

since $\mathbb{E}\left[|F(z)|\right]^2 \le \mathbb{E}\left[|F(z)|^2\right] = K(z,z) = 1$, so again by Fubini-Tonelli we can bring the expectation inside. Convegence in mean square now implies

$$\mathbb{E}\left[a_n\right] = \lim_{R\to\infty} \mathbb{E}\left[a_n^R\right] = 0.$$

We have shown previously that $\mathrm{Var}[a_n^R] = \mathbb{E}\,[\,|a_n^R|^2\,] \underset{R\to\infty}{\longrightarrow} 1$, so again convergence in mean square yields

$$\mathrm{Var}\,[a_n] = \lim_{R\to\infty} \mathrm{Var}\,[a_n^R] = 1,$$

so the claim holds true.

We now want to show that (a_n) define a sequence of i.i.d. random variables with $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ - the only thing we are missing is the independence of a_n and a_m for $n \neq m$.

Although it would be cumbersome to write down, it is not difficult to see that given $R > 0$ and $n \neq m$, the truncated coefficients a_n^R and a_m^R are jointly Gaussian. One can use the same arguments as in the proof of Proposition 3.4: instead of dealing with contour integrals, we are integrating over disks in $\mathbb{C}$, but these are (e.g. upon passing to polar coordinates) again limits of linear combinations of evaluations of F, so the arguments carry over by Gaussianity of F. By Definition 1.15, Observation 1.7 and Lemma 1.14, the joint Gaussianity of a_n^R and a_m^R implies that of a_n and a_m.

Now that we know that $(a_n)_{n\in\mathbb{N}}$ are pairwise jointly Gaussian, Lemma 1.16 reduces the question of independence to their uncorrelatedness. To see that a_n and a_m $(n \neq m)$ are uncorrelated, one can employ the same arguments from the proof of Claim 1 (with the only difference of now having two different e_j's), so that we obtain

$$\begin{aligned}\mathbb{E}[a_n^R\overline{a_m^R}] &= \mathbb{E}[\int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_n(z)}\,dm(z) \int\limits_{\mathrm{D}(0,R)} \overline{F(w)}e_m(w)\,dm(w)] \\ &\underset{R\to\infty}{\longrightarrow} \int\limits_{\mathbb{C}} \mathcal{K}(e_m)(z)\overline{e_n(z)}dm(z) = \int\limits_{\mathbb{C}} e_m(z)\overline{e_n(z)}\,dm(z) = \langle\, e_m, e_n\,\rangle = 0\end{aligned}$$

for $n \neq m$, since $(e_n)_n$ are pairwise orthogonal. The convergence of both a_n^R and $\overline{a_m^R}$ in $L^2(\Omega)$ implies the convergence of their product in $L^1(\Omega)$, i.e.

$$\mathbb{E}\,[\,|a_n^R\overline{a_m^R} - a_n\overline{a_m}|\,] \underset{R\to\infty}{\longrightarrow} 0,$$

which further implies

$$\mathbb{E}\,[\,a_n\overline{a_m}\,] = \lim_{R\to\infty} \mathbb{E}\,[\,a_n^R\overline{a_m^R}\,] = 0,$$

such that a_n and a_m are indeed uncorrelated.

We have thus shown:

Given a GWHF F with covariance kernel K satisfying the standing assumptions and the corresponding integral operator $\mathcal{K}$, one can define a sequence of coefficients $(a_n)_{n\in\mathbb{N}}$ as the mean square limits of

$$a_n^R := \int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_n(z)}\,dm(z) \quad \text{as } R \to \infty,$$

where $\{e_n\}$ is the orthonormal basis of $\operatorname{ran}\mathcal{K}$. The coefficient sequence (a_n) consists of i.i.d. random variables with $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$.

We can now start working towards an appropriate expansion of F using the coefficients a_n constructed above. We define the partial sum corresponding to the prospective expansion:

$$S_N(z) := \sum_{n=0}^{N} a_n e_n(z)$$

and show that $S_N(z) \to F(z)$ in mean square, i.e. $\mathbb{E}\,[\,|F(z) - S_N(z)|^2\,] \to 0$ as $N \to \infty$. Observe that

$$|S_N(z)|^2 = \left(\sum_{n=0}^{N} a_n e_n(z)\right)\left(\sum_{k=0}^{N} \overline{a_k}\overline{e_k(z)}\right) = \sum_{k=0}^{N}\sum_{l=0}^{k} a_l\,\overline{a_{k-l}}\,e_l(z)\,\overline{e_{k-l}(z)}$$

and since $\mathbb{E}\,[a_l\,\overline{a_{k-l}}] = \delta_{l,k-l}$, we obtain

$$\mathbb{E}\,[\,|S_N(z)|^2\,] = \sum_{k=0}^{N} |e_k(z)|^2.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\,[\,|F(z) - S_N(z)|^2\,] &= \mathbb{E}\,[\,|F(z)|^2\,] + \mathbb{E}\,[\,|S_N(z)|^2\,] - 2\operatorname{Re}\mathbb{E}\,[S_N(z)\overline{F(z)}] \\
&= K(z,z) + \sum_{n=0}^{N} |e_n(z)|^2 - 2\operatorname{Re}\mathbb{E}\,[S_N(z)\overline{F(z)}] \\
&= \sum_{n=0}^{\infty} |e_n(z)|^2 + \sum_{n=0}^{N} |e_n(z)|^2 - 2\operatorname{Re}\mathbb{E}\,[S_N(z)\overline{F(z)}]
\end{aligned}$$

and observe that

$$\mathbb{E}\,[S_N(z)\overline{F(z)}] = \sum_{n=0}^{N} e_n(z)\,\mathbb{E}\,[a_n\overline{F(z)}].$$

Now by mean square convergence of a_n^R, we obtain

$$\begin{aligned}
\mathbb{E}\,[a_n\overline{F(z)}] &= \lim_{R\to\infty}\mathbb{E}\,[a_n^R\overline{F(z)}] = \lim_{R\to\infty}\mathbb{E}\left[\,\int\limits_{\mathrm{D}(0,R)} F(w)\overline{e_n(w)}\,\overline{F(z)}\,dm(w)\right]\\
&= \lim_{R\to\infty}\int\limits_{\mathrm{D}(0,R)}\mathbb{E}\,[F(w)\overline{F(z)}]\overline{e_n(w)}\,dm(w)\\
&= \lim_{R\to\infty}\int\limits_{\mathrm{D}(0,R)}\overline{K(z,w)e_n(w)}\,dm(w) = \int\limits_{\mathbb{C}}\overline{K(z,w)e_n(w)}\,dm(w)\\
&= \overline{\mathcal{K}(e_n)(z)} = \overline{e_n(z)}, \qquad (4.17)
\end{aligned}$$

where we have used Fubini-Tonelli to bring the expectation inside the integral, since $\mathbb{E}\,[|F(w)||F(z)|] \le \mathbb{E}\,[|F(w)|^2]^{1/2}\mathbb{E}\,[|F(z)|^2]^{1/2} = 1$ implies

$$\int\limits_{\mathbb{C}}\mathbb{E}\,[|F(w)||F(z)|]|e_n(z)||\mathbb{1}_{\mathrm{D}(0,R)}(z)|\,dm(z) \le \int\limits_{\mathrm{D}(0,R)}|e_n(z)|\,dm(z) < \infty;$$

taking the limit $R\to\infty$ inside the integral is possible by the dominated convergence theorem, since

$$|K(z,w)||e_n(w)||\mathbb{1}_{\mathrm{D}(0,R)}| \le |H(z-w)||e_n(w)|$$

and

$$\int\limits_{\mathbb{C}}|H(z-w)||e_n(w)|\,dm(w) = (|H| * |e_n|)(z)$$

which is well-defined, since $H\in L^1(\mathbb{C})$ and $e_n\in L^2(\mathbb{C})$. The equality from (4.17) now implies

$$\mathbb{E}\,[S_N(z)\overline{F(z)}] = \sum_{n=0}^{N} e_n(z)\,\mathbb{E}\,[a_n\overline{F(z)}] = \sum_{n=0}^{N}|e_n(z)|^2.$$

Finally, we see that

$$\begin{aligned}
\mathbb{E}\,[\,|F(z)-S_N(z)|^2\,] &= \sum_{n=0}^{\infty}|e_n(z)|^2 + \sum_{n=0}^{N}|e_n(z)|^2 - 2\,\mathrm{Re}\,\mathbb{E}\,[S_N(z)\overline{F(z)}]\\
&= \sum_{n=0}^{\infty}|e_n(z)|^2 - \sum_{n=0}^{N}|e_n(z)|^2 \underset{N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

so that we obtain the expansion

$$F(z) = \sum_{n=0}^{\infty} a_n\,e_n(z)$$

with convergence in mean square for a.e. $z \in \mathbb{C}$.

In order to interpret Ω as the infinite product of probability spaces Ω_n associated with random coefficients a_n, we also need to show the converse statement - given predescribed i.i.d. Gaussian random variables a_n as coefficients, the expansion still makes sense and defines a GWHF.

Suppose we are given a sequence of i.i.d. $(a_n)_{n\in\mathbb{N}}$ with $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and an orthonormal system $\{e_n\}$ of $\operatorname{ran}\mathcal{K} \subset L^2(\mathbb{C})$ induced by $K(z,w) = e^{i\,\mathrm{Im}(z\overline{w})}\,H(z-w)$.

Claim: $F(z) := \sum_{n=0}^{\infty} a_n\, e_n(z)$ is a GWHF.

1. Denote the partial sum with $S_N := \Sigma_{n=0}^{N} a_n\, e_n(z)$ for $N \in \mathbb{N}$, write $a_n = x_n + i \cdot y_n$ and $e_n(z) = r_n + i \cdot p_n$ (and drop the dependence on z for simpler notation); then we have for $M > N$:

$$\begin{aligned}
\mathbb{E}\left[\,|S_M(z) - S_N(z)|^2\,\right] &= \mathbb{E}\left[\left|\Sigma_{n=N+1}^{M} a_n\, e_n(z)\right|^2\right] \\
&= \mathbb{E}\left[\left|\Sigma_{n=N+1}^{M}(x_n + i\cdot y_n)\cdot(r_n + i\cdot p_n)\right|^2\right] \\
&= \mathbb{E}\left[\left|\Sigma_{n=N+1}^{M}(x_n r_n - y_n p_n) + i\cdot \Sigma_{n=N+1}^{M}(x_n p_n + y_n r_n)\right|^2\right] \\
&= \mathbb{E}\left[\,[\Sigma_{n=N+1}^{M}(x_n r_n - y_n p_n)]^2 + [\Sigma_{n=N+1}^{M}(x_n p_n + y_n r_n)]^2\,\right]
\end{aligned}$$

and one easily notices that all the mixed terms have zero expectation by the independence of the corresponding real and imaginary parts of independent coefficients a_n:

$$\begin{aligned}
\mathbb{E}[x_n x_j] = \mathbb{E}[y_n y_j] &= 0 \;\text{ for }\; j \neq n, \\
\mathbb{E}[x_n y_j] &= 0 \;\text{ for any }\; j, n,
\end{aligned}$$

so that we are left with the following:

$$\begin{aligned}
\mathbb{E}\left[\,|S_M(z) - S_N(z)|^2\,\right] &= \mathbb{E}\left[\left|\Sigma_{n=N+1}^{M} a_n\, e_n(z)\right|^2\right] \\
&= \mathbb{E}[\Sigma_{n=N+1}^{M} x_n^2 r_n^2 + y_n^2 p_n^2 + x_n^2 p_n^2 + y_n^2 r_n^2] \\
&= \mathbb{E}[\Sigma_{n=N+1}^{M} |a_n|^2 r_n^2 + |a_n|^2 p_n^2] \\
&= \Sigma_{n=N+1}^{M} \mathbb{E}[|a_n|^2] |e_n(z)|^2 \\
&= \Sigma_{n=N+1}^{M} |e_n(z)|^2 \underset{M,N\to\infty}{\longrightarrow} 0
\end{aligned}$$

for every $z \in \mathbb{C}$ by (4.13). Hence we get a Cauchy sequence in $L^2(\Omega)$ and therefore the convergence of

$$F(z) = \sum_{n=0}^{\infty} a_n \, e_n(z)$$

in mean square for every $z \in \mathbb{C}$.

2. $K(z, w)$ is indeed the covariance kernel of F:

$$\begin{aligned} K_F(z, w) &= \mathbb{E}\left[F(z)\,\overline{F(w)}\right] = \lim_{N\to\infty} \mathbb{E}\left[S_N(z)\,\overline{S_N(w)}\right] \\ &= \lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{n=0}^{N} a_n\, e_n(z)\right)\left(\sum_{k=0}^{N} \overline{a_k}\,\overline{e_k(w)}\right)\right] \\ &= \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{k=0}^{n} \mathbb{E}\left[a_k\,\overline{a_{n-k}}\right] e_k(z)\,\overline{e_{n-k}(w)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} e_n(z)\,\overline{e_n(w)} = K(z, w). \end{aligned}$$

3. Let $m \in \mathbb{N}$ and $b = (b_1, \ldots, b_m)^\top \in \mathbb{C}^m$ be arbitrary; then

$$b^\top \begin{pmatrix} S_N(z_1) \\ \vdots \\ S_N(z_m) \end{pmatrix} = \sum_{n=0}^{N} a_n (b_1\, e_n(z_1) + \cdots + b_m\, e_n(z_m))$$

is a linear combination of i.i.d. Gaussians a_n, so it is itself a Gaussian random variable, hence $(S_N(z_1), \ldots, S_N(z_m))^\top \sim \mathcal{N}_{\mathbb{C}}^m$ by Proposition 1.10 (ii). Since the components $S_N(z_i)$ of $(S_N(z_1), \ldots, S_N(z_m))^\top$ converge in mean square to $F(z_i)$, by Observation 1.7 we obtain

$$(S_N(z_1), \ldots, S_N(z_m))^\top \xrightarrow{d} (F(z_1), \ldots, F(z_m))^\top \;\text{ as } N \to \infty,$$

so by Lemma 1.14 $(F(z_1), \ldots, F(z_m))^\top \sim \mathcal{N}_{\mathbb{C}}^m$, i.e. we obtain Gaussianity of F.

4. Since $a_n = x_n + i \cdot y_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ with i.i.d. $x_n, y_n \sim \mathcal{N}_{\mathbb{R}}(0, 1/2)$ we have $\mathbb{E}[a_n] = 0$ and

$$\mathbb{E}\left[a_n^2\right] = \mathbb{E}\left[x_n^2\right] + 2i\,\mathbb{E}\left[x_n\, y_n\right] - \mathbb{E}\left[y_n^2\right] = 0,$$

since $\mathbb{E}[x_n y_n] = \mathbb{E}[x_n] \cdot \mathbb{E}[y_n] = 0$. Therefore,

$$\begin{aligned}
\mathbb{E}[S_N(z)] &= \sum_{n=0}^{N} \mathbb{E}[a_n] e_n(z) = 0, \\
\mathbb{E}[S_N(z)^2] &= \mathbb{E}\left[\left(\sum_{n=0}^{N} a_n\, e_n(z)\right)\left(\sum_{j=0}^{N} a_j\, e_j(z)\right)\right] \\
&= \sum_{n=0}^{N}\sum_{k=0}^{n} \mathbb{E}[a_k\, a_{n-k}]\, e_k(z)\, e_{n-k}(z) = \sum_{n=0}^{N} \mathbb{E}[a_n^2]\, e_n(z)^2 = 0, \\
\mathrm{Var}[S_N(z)] &= \mathbb{E}[|S_N(z)|^2] - \underbrace{|\mathbb{E}[S_N(z)]|^2}_{=0} = \sum_{n=0}^{N} |e_n(z)|^2 < \infty,
\end{aligned}$$

so recalling Definition 1.18, we see that $S_N(z)$ is proper. Since $S_N(z) \to F(z)$ in mean square, Lemma 1.20 implies that $F(z)$ is proper as well. Knowing that F is Gaussian from the previous step, Lemma 1.19 now implies that F is circularly symmetric.

Asymptotic independence of coefficients

To apply analogous reasoning from the proof of Proposition 3.4 in the case of a GWHF, we need to show the asymptotic independence of the coefficients a_k of F and the coefficients b_m^ζ of its twisted shift F_ζ, which are the mean square limits of

$$a_k^R = \int\limits_{\mathrm{D}(0,R)} F(z)\overline{e_k(z)}\, dm(z) \quad \text{and} \quad b_m^{\zeta,R} = \int\limits_{\mathrm{D}(0,R)} F_\zeta(w)\overline{e_m(w)}\, dm(w). \tag{4.18}$$

The joint Gaussianity of a_k^R and $b_m^{\zeta,R}$ (and of a_k and b_m^ζ) can be argued in the same way as done before for a_n^R and a_m^R (and of a_n and a_m). The more interesting part is about their uncorrelatedness: in the case of a GAF we used analyticity and Cauchy's formula to express the coefficients as contour integrals and estimated the covariance of $f_{\alpha,\beta}(z)$ and $f(w)$ by a concrete expression which we have seen converges to 0. Here, however, the covariance kernel is given in terms of the twisted kernel H, for which we cannot produce an explicit upper bound (we only know that its modulus is bounded by 1 - this we cannot use, since it would annul the dependence on the translation parameter). A way around this obstacle is via convolution. We take a closer look at the covariance of the coefficients to see where convolution comes in:

$$\mathbb{E}\,[a_k^R\,\overline{b_m^{\zeta,R}}] = \mathbb{E}\left[\int\limits_{\mathrm{D}(0,R)}\int\limits_{\mathrm{D}(0,R)} F(z)\overline{F_\zeta(w)e_k(z)}\,e_m(w)\,dm(z)\,dm(w)\right]$$

$$= \int\limits_{\mathrm{D}(0,R)}\int\limits_{\mathrm{D}(0,R)} \mathbb{E}\,[F(z)\overline{F_\zeta(w)}]\,\overline{e_k(z)}\,e_m(w)\,dm(z)\,dm(w)$$

$$= \int\limits_{\mathrm{D}(0,R)}\int\limits_{\mathrm{D}(0,R)} e^{i\,\mathrm{Im}(w\overline{\zeta})}\,e^{i\,\mathrm{Im}(z(\overline{w}-\overline{\zeta}))}\,H(z-w+\zeta)\,\overline{e_k(z)}\,e_m(w)\,dm(z)\,dm(w),$$

where in the last step we have used $\mathbb{E}\,[F(z)\overline{F_\zeta(w)}] = e^{i\,\mathrm{Im}(w\overline{\zeta})}\,K(z,w-\zeta)$. The interchanging of the expectation and the integrals is justified by Fubini-Tonelli, since

$$\mathbb{E}\,[\,|F(z)|\,|F_\zeta(w)|\,] = |e^{i\,\mathrm{Im}(w\overline{\zeta})}|\cdot\mathbb{E}\,[\,|F(z)|\,|F(w-\zeta)|\,]$$

and $|e^{i\,\mathrm{Im}(w\overline{\zeta})}| = 1$, so by applying Cauchy-Schwartz we obtain

$$\int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}}\mathbb{E}\,[\,|F(z)|\,|F_\zeta(w)|\,]\,|e_k(z)\mathbb{1}_{\mathrm{D}(0,R)}(z)|\,|e_m(w)\mathbb{1}_{\mathrm{D}(0,R)}(w)|\,dm(z)\,dm(w)$$

$$\leq \int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}}\mathbb{E}\,[\,|F(z)|^2\,]^{1/2}\cdot\mathbb{E}\,[\,|F(w-\zeta)|^2\,]^{1/2}|e_k^R(z)|\,|e_m^R(w)|\,dm(z)\,dm(w)$$

$$\leq \int\limits_{\mathbb{C}}|e_k^R(z)|\,dm(z)\int\limits_{\mathbb{C}}|e_m^R(w)|\,dm(w) < \infty \tag{4.19}$$

for $e_k^R := e_k\cdot\mathbb{1}_{\mathrm{D}(0,R)}$ (similarly for e_m), where we have used $\mathbb{E}\,[\,|F(z)|^2\,] = K(z,z) = 1 = K(w-\zeta,w-\zeta) = \mathbb{E}\,[\,|F(w-\zeta)|^2\,]$. Note that the expression in (4.19) is finite by the same arguments as in the proof of Claim 1. We now estimate:

$$|\mathbb{E}\,[a_k^R\,\overline{b_m^{\zeta,R}}]| \leq \int\limits_{\mathrm{D}(0,R)}\int\limits_{\mathrm{D}(0,R)} |H(z-w+\zeta)|\,|e_k(z)|\,|e_m(w)|\,dm(z)dm(w)$$

$$\leq \int\limits_{\mathbb{C}}\int\limits_{\mathbb{C}}|H(\zeta-w-z)|\,|\widetilde{e_k}(z)|\,|e_m(w)|\,dm(z)dm(w),$$

where $\widetilde{e_k}(z) := e_k(-z)$. Now since

$$(|H|*|\widetilde{e_k}|)(\zeta-w) = \int\limits_{\mathbb{C}}|H(\zeta-w-z)|\,|\widetilde{e_k}(z)|\,dm(z)$$

we obtain

$$
\begin{aligned}
((|H| * |\widetilde{e_k}|) * |e_m|)(\zeta) &= \int\limits_{\mathbb{C}} (|H| * |\widetilde{e_k}|)(\zeta - w)\, |e_m(w)|\, dm(w) \\
&= \int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}} |H(\zeta - w - z)|\, |\widetilde{e_k}(z)|\, |e_m(w)|\, dm(z) dm(w)
\end{aligned}
$$

and hence

$$
|\mathbb{E}\,[a_k^R\, \overline{b_m^{\zeta,R}}]| \leq ((|H| * |\widetilde{e_k}|) * |e_m|)(\zeta).
$$

Recall that $(e_n) \subseteq L^2(\mathbb{C})$, so we know that $|\widetilde{e_k}|, |e_m| \in L^2(\mathbb{C})$ and hence $|\widetilde{e_k}| * |e_m| \in C_0(\mathbb{C})$ by Lemma 1.22. We also know that $|H| \in L^1(\mathbb{C})$, so by Lemma 1.24,

$$
|H| * |\widetilde{e_k}| * |e_m| \in C_0(\mathbb{C})
$$

and hence

$$
|\mathbb{E}\,[a_k^R\, \overline{b_m^{\zeta,R}}]| \leq (\, |H| * |\widetilde{e_k}| * |e_m|\,)(\zeta) \underset{|\zeta|\to\infty}{\longrightarrow} 0, \tag{4.20}
$$

implying that the truncated coefficients a_k^R and $b_m^{\zeta,R}$ are indeed asymptotically uncorrelated as $|\zeta| \to \infty$ for every $R > 0$. Note that (4.20) implies

$$
\mathbb{E}\,[a_k^R\, \overline{b_m^{\zeta,R}}] \underset{|\zeta|\to\infty}{\longrightarrow} 0 \quad \text{uniformly in R,}
$$

so that we obtain

$$
\lim_{|\zeta|\to\infty} \mathbb{E}\,[a_k\, \overline{b_m^{\zeta}}] = \lim_{|\zeta|\to\infty} \lim_{R\to\infty} \mathbb{E}\,[a_k^R\, \overline{b_m^{\zeta,R}}] = \lim_{R\to\infty} \lim_{|\zeta|\to\infty} \mathbb{E}\,[a_k^R\, \overline{b_m^{\zeta,R}}] = 0.
$$

We have thus shown:

Claim 2: a_k and b_m^{ζ} are asymptotically uncorrelated and hence asymptotically independent as $|\zeta| \to \infty$ by their joint Gaussianity.

For the sake of completeness, we state the result claiming the ergodicity of the zero set of a Gaussian Weyl-Heisenberg function analogous to Proposition 3.4 and clarify the necessary adjustments.

As before, we denote with n_F the corresponding zero set of the GWHF F. With the established existence of coefficients associated with a GWHF, we can use the same construction from Section 3.1 for the map $n_F : \Omega \to \mathcal{M}(\mathbb{C})$, so that Lemma 3.2 holds with $\mathbb{P}_F, \mathcal{F}, \mathcal{F}_N$[6] defined in the obvious way. As noted in (4.6), we also have a measure-preserving transformation in this case, so that this setting indeed also corresponds to Definition 3.3.

[6] we use the same notation for the sigma-algebras here for obvious reasons

Proposition 4.6. *The zero set of a Gaussian Weyl-Heisenberg function is ergodic under the action of translations in* $\mathbb{C}$*.*

Proof. Denote the translation parameter with $\zeta \in \mathbb{C}$ and define

$$F_\zeta(z) := e^{i\,\mathrm{Im}(z\bar{\zeta})} F(z-\zeta);$$

we know by Lemma 4.5 that $F \overset{d}{=} F_\zeta$. For the coefficients a_k and b_m^ζ of F and F_ζ as defined in (4.18), we know from Claim 2 that

$$a_k \text{ and } b_m^\zeta \text{ are asymptotically independent as } |\zeta| \to \infty.$$

By Lemma 3.2, we obtain

$$\mathbb{E}\left[\,|\mathbb{1}_A(n_F) - \mathbb{1}_{A_n}(n_F)|\,\right] \le \varepsilon \;\text{ and }\; \mathbb{E}\left[\,|\mathbb{1}_A(n_{F_\zeta}) - \mathbb{1}_{A_n}(n_{F_\zeta})|\,\right] \le \varepsilon$$

for an invariant event $A \in \mathcal{F}$ and $A_n \in \mathcal{F}_n$ depending only on the first n coefficients of F, which further implies

$$|\,\mathbb{E}[\mathbb{1}_A(n_F)\mathbb{1}_A(n_{F_\zeta}) - \mathbb{1}_{A_n}(n_F)\mathbb{1}_{A_n}(n_{F_\zeta})]\,| \le 2\varepsilon.$$

By the asymptotic independence of the coefficients of F and F_ζ, we have

$$\mathbb{E}[\mathbb{1}_{A_n}(n_F)\mathbb{1}_{A_n}(n_{F_\zeta})] \to \left(\,\mathbb{E}[\mathbb{1}_{A_n}(n_F)]\,\right)^2 \tag{4.21}$$

as $|\zeta| \to \infty$, since $\mathbb{E}\,[\mathbb{1}_{A_n}(n_F)] = \mathbb{E}\,[\mathbb{1}_{A_n}(n_{F_\zeta})]$. With analogous estimations to those from the proof of Proposition 3.4, we arrive at

$$\mathbb{E}[\mathbb{1}_A(n_F)] = \left(\,\mathbb{E}[\mathbb{1}_A(n_F)]\,\right)^2 \quad\Longrightarrow\quad \mathbb{P}_F(A) = \mathbb{E}[\mathbb{1}_A(n_F)] \in \{0,1\}.$$

$\square$

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