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Family of Weyl-algebras and the Carrollian limit

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Robert Tiefenbacher BSc BSc MSc

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Abstract

In the past few years, Carroll theories attracted some renewed interest, due to the newly discovered connection to holography, cosmology and dark energy. Those theories stem from the question what happens if the speed of light goes to zero. Such limits are usually taken before quantizing the theory. The aim of this thesis is to explore another approach where we consider the limiting process in the quantum description of a free scalar field. We continuously deform the fields, while fixing the metric, in order to obtain a Carrollian theory in an appropriate limit. Further, we acquire a Carrollian Weyl-algebra in the process where we study possible states that arise from the limit procedure.

Kurzfassung

In den letzten Jahren haben Carroll-Theorien aufgrund der neu entdeckten Verbindung zu Holographie, Kosmologie und dunkler Energie erneutes Interesse bekommen. Diese Theorien basieren auf der Frage, was passiert, wenn die Lichtgeschwindigkeit zu Null tendiert. Solche Limites werden normalerweise vor der Quantisierung der Theorie durchgeführt. Das Ziel dieser Arbeit ist es, einen anderen Ansatz zu untersuchen, bei dem wir den Grenzwertprozess in der Quantenbeschreibung eines freien Skalarfeldes berücksichtigen. Wir verformen die Felder kontinuierlich, während wir gleichzeitig die Metrik fest halten, um eine Carrollsche Theorie in einem geeigneten Limes zu erhalten. Darüber hinaus bekommen wir durch diesen Prozess eine Carrollsche Weyl-Algebra, in der wir mögliche Zustände untersuchen, die sich aus dem Grenzwertverfahren ergeben.

Contents

Introduction

Carroll theories, quantum field theories which the Carroll transformations [LL65, SG66, dBHO$^+$22] as symmetries, have seen renewed interest in recent years. The reason for this is their connection to flat space holography and the BMS symmetries [FOGP19, DGH14] as well as to dark energy and inflation [dBHO$^+$22]. There has also been some connection made to algebraic quantum field theories (AQFT) under the name ultralocal theories [Kla00, Kla70].
The idea in the construction of Carroll symmetries is to take the Lorentz transformation, specifically the Lorentz boost, and letting the speed of light c tend to 0. This is then a conceptual analog to Galilei symmetries, which one obtains by letting $c \to \infty$. Morally, changing c means changing the underlying metric.

Recent works, which look at Carroll theories, start from some action where the appropriate Carrollian limit has already been taken, for example [dBHO$^+$23], which then is turned in a quantum theory.
We, on the other hand, already start with a quantum field theory and then "contract", to borrow the term from group contractions, the theory to get back a Carroll theory. To be more precise, we start with a regular free complex scalar theory and then deform the fields, and therefore the equations of motion, in a way that reproduces the Carroll equations of motion in the titular *Carrollian limit*. We can see this as taking the limit "inside" the quantum theory, which allows the use of standard tools for a free scalar theory. Deforming the fields has the added benefit of leaving the metric as is.
Further, we aim to show that we can translate the net of fields, which we gained through the deformation, to a family of *Weyl-algebras* and that there exists another Weyl-algebra, which we can consider the Carrollian limit of said family. This opens up further connections to AQFT.

An introduction to the necessary mathematical tools is made in Chapter 1 of this thesis. In Chapter 2 we motivate the scaling ansatz of the free fields and show how to construct the family of Weyl-algebras and the limit algebra. Chapter 3 focuses on the existence of a special state on the limit algebra, which is necessary to give an important connection to the usual way of handling quantum field theories. This chapter also proves that this special state satisfies Carroll symmetries and hence allows us to really consider this a *electric* Carroll limit of the free complex scalar theory. The purpose of Chapter 4 is to fill in technical gaps in the arguments used in the preceding chapters and to better classify the constructed limit Weyl-algebra. Finally, Chapter 5 shows that the entire construction can be repeated for the conjugated momenta as well, leading to the similar *magnetic* Carroll limit. We finish this thesis with a brief discussion in Chapter 6.

A solid understanding of the theory of operators and C^*-algebras as well as quantum field theory is assumed, since this thesis' aim is not to introduce these subjects.
For the theory of C^*-algebras we point to the first few chapters of [Con00], chapter 4 in [KR83] or the lecture notes on "C^*-algebras with Aspects of Quantum Physic" [Hö24].
An introduction to operator theory can be found in [Con90] and the lecture notes on "Advanced functional analysis" [Hö25]. For quantum field theory we recommend [NNB82].

1. Preliminary

This first chapter introduces the mathematical concepts necessary to understand the rest of the thesis. It will consist predominantly of references to existing definitions and theorems. The purpose is mostly fixing the notation since a deeper understanding of the ideas presented here needs additional study outside the scope of this thesis.

1.1. Basic Notation and Fourier transform

A key tool for us will be the *Fourier transform.* In this section we are going to introduce our notation and convention for the Fourier transform and also some properties, which we will use throughout the thesis.

This thesis only deals with $d+1$-dimensional Minkowski space, i.e., with $\mathbb{R}^{d+1}$ together with the metric

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)\,.$$

We will for the most part only use $\mathbb{R}^{d+1}$ for the notation, since the pseudo-Riemannian structure of Minkowski space does not play much of a role for us. However, if we want to specifically point to Minkowski spaces as a pseudo-Riemannian manifold we denote it as $\mathbb{R}^{1,d}$.
We denote elements of $\mathbb{R}^{d+1}$ as x^μ. If we need to decompose those elements we write it as $x^\mu = (x^0, x)$, where $x^0 \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Hence the inner product of $x^\mu, y^\mu \in \mathbb{R}^{d+1}$ can be written as $x^\mu y_\mu = x_0 y_0 - xy$.

To talk about the Fourier transform, we need to first introduce the *Schwartz space.*

Definition 1.1.1 (Space of Schwartz-functions)
Let $n \in \mathbb{N}$. We call

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta \in \mathbb{N}^n : \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| < \infty \right\}$$

the *space of Schwartz functions* or *Schwartz space.* [Ste03, p.180]

We denote with $C_c^\infty(\mathbb{R}^n)$ the real-valued smooth functions with compact support on $\mathbb{R}^n$. Clearly, we have the following chain of inclusions $C_c^\infty(\mathbb{R}^n) \subset C_c^\infty(\mathbb{R}^n, \mathbb{C}) \subset \mathcal{S}(\mathbb{R}^n)$.

We can now introduce the convention we are going to use for the Fourier transform.

Definition 1.1.2 (Fourier transform on Minkowski space)
Let $f \in \mathcal{S}(\mathbb{R}^{d+1})$, $g \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^d)$.
Then

$$\mathcal{F}f(p^\mu) := \int_{\mathbb{R}^{d+1}} d^{d+1}x^\mu \, f(x^\mu) e^{-ix^\mu p_\mu}$$

is called the *Fourier transform* of f evaluated at $p^\mu \in \mathbb{R}^{d+1}$.
We call

$$\mathcal{F}_t g(p^0) := \int_{\mathbb{R}} dx^0 \, g(x^0) e^{-ix^0 p^0}$$

the *time Fourier transform* of g evaluated at $p^0 \in \mathbb{R}$, and we call

$$\mathcal{F}_x h(p) := \int_{\mathbb{R}^d} d^d x \, h(x) e^{ixp}$$

the *space Fourier transform* of h evaluated at $p \in \mathbb{R}^d$.

As a shorthand, we will often denote $\hat{f} := \mathcal{F}f$.
The time and space Fourier transform have been introduced because at some point we need to only Fourier transform part of the variables and it allows as to write $\mathcal{F} = \mathcal{F}_t \mathcal{F}_x$ as operators for functions on $\mathbb{R}^{d+1}$, since

$$\begin{aligned}\mathcal{F}_t \mathcal{F}_x f(p^0, p) &= \int_{\mathbb{R}} dx^0 \int_{\mathbb{R}^d} d^d x \, f(x^0, x) e^{ixp} e^{-ix^0 p^0} \\ &= \int_{\mathbb{R}^{d+1}} d^{d+1}x^\mu \, f(x^0, x) e^{-ix^0 p^0 + ixp} = \mathcal{F}f(p^0, p) \, .\end{aligned}$$

We make sure to always apply $\mathcal{F}_t$ to the 0^{th}-variable and $\mathcal{F}_x$ to the other d variables.

The Fourier transform can be defined on a broader class of functions then Schwartz functions. However, as it turns out, the Fourier transform is a well-defined, linear and bijective map from the Schwartz space to itself [Ste03, 6.2.2]. The inverse maps are given as follows.

Definition 1.1.3 (Inverse Fourier transform on Minkowski space)
Let $f \in \mathcal{S}(\mathbb{R}^{d+1})$, $g \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^d)$.
Then

$$\mathcal{F}^{-1} f(x^\mu) := \int_{\mathbb{R}^{d+1}} \frac{d^{d+1}p^\mu}{(2\pi)^{d+1}} \, f(p^\mu) e^{ix^\mu p_\mu}$$

is called the *inverse Fourier transform* of f evaluated at $x^\mu \in \mathbb{R}^{d+1}$.
We call

$$\mathcal{F}_t^{-1} g(x^0) := \int_{\mathbb{R}} \frac{dp^0}{2\pi} \, g(p^0) e^{ix^0 p^0}$$

the *inverse time Fourier transform* of g evaluated at $x^0 \in \mathbb{R}$, and we call

$$\mathcal{F}_x^{-1} h(x) := \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d}\, h(p) e^{-ixp}$$

the *inverse space Fourier transform* of h evaluated at $x \in \mathbb{R}^d$.

We finish this section by bringing some properties of the Fourier transform to mind, which we are going to use extensively.
Let $f \in \mathcal{S}(\mathbb{R}^{d+1})$.

(i)

$$\frac{\partial}{\partial p^0} \mathcal{F} f(p^\mu) = -i\mathcal{F}[x_0 f](p^\mu) \quad \text{and} \quad \frac{\partial}{\partial p^j} \mathcal{F} f(p^\mu) = i\mathcal{F}[x_j f](p^\mu) \quad 1 \leq j \leq d\,.$$

(ii)

$$\mathcal{F}\left[\frac{\partial}{\partial x^0} f\right](p^\mu) = -ip_0 \mathcal{F} f(p^\mu) \quad \text{and} \quad \mathcal{F}\left[\frac{\partial}{\partial x^j} f\right](p^\mu) = ip_j \mathcal{F} f(p^\mu) \quad 1 \leq j \leq d\,.$$

(iii) If f is real-valued, then $\overline{\mathcal{F} f(p^\mu)} = \mathcal{F} f(-p^\mu)$.

Those properties hold not only for Schwartz functions, instead they hold whenever both sides of an equation make sense.

From this point forward, whenever we write only $\int$ we mean $\int_{\mathbb{R}^n}$, where n depends on the context.

1.2. C*-algebra theory

As the title of this thesis suggests, we are going to encounter *Weyl-algebras*, which are a special type of C^**-algebras*. Therefore, in this section, we introduce the basic notions of C^*-algebra theory that we will encounter.

We begin with the basic definition of what a C^*-algebra is and some special elements.

Definition 1.2.1 ($*$-, Banach-, C^*-algebra)
A *normed algebra* $\mathcal{A}$ is a complex algebra with a norm $\|\,.\,\|$ such that $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathcal{A}$.
We call a normed algebra a *Banach algebra* if it is complete under its norm.
A (normed/Banach) $*$*-algebra* is a complex (normed/Banach) algebra with the addition of an *involution*, which means a map $* : \mathcal{A} \to \mathcal{A}$ such that

(i) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$

(ii) $(AB)^* = B^* A^*$

(iii) $A^{**} = (A^*)^* = A$

for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}$.
A *C^*-algebra* $\mathcal{A}$ is a Banach $*$-algebra fulfilling the *C^*-relation*

$$\|A^*A\| = \|A\|^2 \quad \forall A \in \mathcal{A}\,.$$

If a C^*-algebra $\mathcal{A}$ has a unit $\mathbb{1}$ and $\|\mathbb{1}\| = 1$, we call $\mathcal{A}$ a *unital C^*-algebra.* [Con00, 1.1, 1.2][Hö24, 0.1, 0.2, 1.1, 1.2]

Remark. As an example, for every Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$, is a C^*-algebra.

Definition 1.2.2 (Self-adjoint, normal, unitary and positive elements)
Let $\mathcal{A}$ be a unital C^*-algebra and $A \in \mathcal{A}$.

(i) A is *self-adjoint*, if $A^* = A$.

(ii) A is *normal*, if $AA^* = A^*A$.

(iii) A is *unitary*, if $A^*A = AA^* = \mathbb{1}$.

(iv) A is *positive*, denoted $A \geq 0$, if there exists a $C \in \mathcal{A}$ such that $A = C^*C$.

[Con00, 1.6, 3.4][Hö24, 1.6, 1.8, 2.11]

Next, we are also interested in how different C^*-algebras are connected. The following definitions give one of the most general type of maps between C^*-algebras as well as two special kinds of maps.

Definition 1.2.3 ($*$-homo-,$*$-iso-, $*$-automorphism)
A map $\varphi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called *$*$-homomorphism* if it is an algebra-homomorphism and it is compatible with the involution, i.e., $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{A}$.
φ is called *unital* if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\varphi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$.
A *$*$-isomorphism* is a bijective $*$-homomorphism and a *$*$-automorphism* is a $*$-isomorphism where $\mathcal{B} = \mathcal{A}$. [Con00, p.2]

Remark. Note that $*$-isomorphisms between unital C^*-algebras are automatically unital.

Definition 1.2.4 (Representations)
Let $\mathcal{A}$ be a unital C^*-algebra.

(i) A *representation* of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a unital $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$.

(ii) A representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is called *cyclic* if there exists an $x \in \mathcal{H}$, called the *cyclic vector*, such that $\pi(\mathcal{A})x := \{\pi(A)x \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

(iii) A representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ invariant under all operators from $\pi(\mathcal{A})$.

[Con00, 6.1, 6.7, 16.10][Hö24, 3.1, 3.14]

Remark. This definition works without the assumption of $\mathcal{A}$ being unital. In that case, any representation of $\mathcal{A}$ also does not need to be unital.

Definition 1.2.5 (States)
Let $\mathcal{A}$ be a unital C^*-algebra and $\rho : \mathcal{A} \longrightarrow \mathbb{C}$ a linear functional.

(i) ρ is called *positive* if $\rho(A) \geq 0$ for all $A \geq 0$.

(ii) ρ is called *unital* if $\rho(\mathbb{1}) = 1$.

(iii) ρ is called a *state* if ρ is positive and unital.

(iv) A state ρ is called *pure* if for any two states ρ_1, ρ_2 and $t \in (0,1)$ with $\rho = t\rho_1 + (1-t)\rho_2$ we have $\rho_1 = \rho_2 = \rho$.

[Con00, 7.1][Hö24, 1.10, 1.12]

Remark. For any unit vector x in a Hilbert space $\mathcal{H}$ we can define a state $\rho_x : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ as $\rho_x(A) = \langle x | \, A \, | x \rangle$.

One useful property of states is that they satisfy an analog of the Cauchy-Schwarz inequality.

Proposition 1.2.6 (Cauchy-Schwarz-inequality for positive linear functionals)
Let ρ be a positive linear functional on a C^*-algebra $\mathcal{A}$. Then for all $A, B \in \mathcal{A}$ we have

$$|\rho(A^*B)|^2 \leq \rho(A^*A)\rho(B^*B)\,.$$

[Con00, 7.2][Hö24, 1.10]

We finish this chapter by referencing a theorem, which gives a practical connection between states and representations. This theorem also guarantees the existence of representations for unital C^*-algebras. It is also the reason, why one can for the most part think of unital C^*-algebras as subalgebras of bounded operators.

Theorem 1.2.7 (GNS-representation)
Let ρ be a state on a unital C^*-algebra $\mathcal{A}$. Then there exists a cyclic representation π_ρ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\rho$ with cyclic unit vector x_ρ such that

$$\rho(A) = \langle x_\rho, \pi_\rho(A)x_\rho \rangle \quad \forall A \in \mathcal{A}\,.$$

[Con00, 7.7][Hö24, 3.4]

If we use pure states in the above, we can even say more about the GNS-representation.

Theorem 1.2.8
Let ρ be a state on a unital C^*-algebra $\mathcal{A}$. Then the corresponding GNS-representation π_ρ is irreducible if and only if ρ is pure. [Con00, 32.6][Hö24, 3.17]

2. Family of Weyl-algebras

With the preliminaries out of the way, we can start the main construction of this thesis. In this chapter, we will first explain the main ansatz of this thesis and then construct a family of operators which in turn will lead naturally to the construction of a family of Weyl-algebras. Lastly, we will construct what we call the *electric Carrollian limit* of the said family of Weyl-algebras and discuss how we can make sense of this limit.

2.1. Motivating the ϕ_ϵ-ansatz

Consider the free complex scalar fields ϕ and $\phi^\dagger$ with mass m, which both solve the Klein-Gordon-equation

$$P\psi := (\partial_t^2 - \nabla^2 + m^2)\psi = 0\,. \tag{2.1.1}$$

When moving to a quantum theory one replaces ϕ and $\phi^\dagger$ with operator-valued distributions, which we denote with the same symbol, and supposes that the equal-time commutator relation hold. In our case, this leads to the following relation,[1]

$$\Big[\phi(t,x), \phi^\dagger(s,y)\Big] = \Delta(t-s, x-y) \tag{2.1.2}$$

where

$$\Delta(t,x) := \int \frac{d^dp}{(2\pi)^d} \int dp_0 \,\mathrm{sgn}(p_0)\, \delta(p_0^2 - p^2 - m^2)\, e^{-ip_0 t + ipx}\,. \tag{2.1.3}$$

We understand (2.1.3), and therefore (2.1.2), in the distributional sense. The Hilbert space, on which ϕ and $\phi^\dagger$ act, is at this point not really relevant to us, as we are only interested in the algebraic relations at the beginning. Thus, we consider the computations involving ϕ and $\phi^\dagger$ in this chapter to be formal. This is why we let the Hilbert space and the domain of ϕ and $\phi^\dagger$ remain ambiguous. In Chapter 3 we will construct a representation on an appropriate Hilbert space.

The conceptual idea of Carroll theories is that we let $c \to 0$, as outlined in [dBHO+22, dBHO+23]. We take a different route. Instead of changing c, and therefore the underlying metric, we scale the coordinates. Only we want to scale the time-coordinate differently from the space-coordinate in a way that "time goes faster to 0 than space", which resembles the idea of $c \to 0$. This leads to the following definition,

$$\phi_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\phi(\epsilon t, x) \quad \text{and} \quad \phi_\epsilon^\dagger(t,x) := \epsilon^{-\frac{1}{2}}\phi^\dagger(\epsilon t, x)\,. \tag{2.1.4}$$

[1] If one is not familiar with this commutator relation, it can alternatively be taken as an assumption at this point. A calculation showing why this is the right commutator relation is contained in Chapter 4.

At this point, there is still some arbitrariness here on how fast to scale the time-coordinate compared to the space-coordinate and the pre-factor. Some of this will be cleared up in the next section, but some freedom of choice will remain.

The fields ϕ_ϵ and $\phi_\epsilon^\dagger$ now fulfill a different equation.
Set $t' := \epsilon t$. Then

$$\begin{aligned} 0 &\overset{(2.1.1)}{=} P\psi(t',x) = (\partial_{t'}^2 - \nabla^2 + m^2)\psi(t',x) = (\frac{1}{\epsilon^2}\partial_t^2 - \nabla^2 + m^2)\psi(\epsilon t, x) \\ &= \frac{1}{\epsilon^2}(\partial_t^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2)\psi(\epsilon t,x) = \epsilon^{-\frac{3}{2}}(\partial_t^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2)\epsilon^{-\frac{1}{2}}\psi(\epsilon t,x)\,. \end{aligned}$$

The prefactor $\epsilon^{-\frac{3}{2}}$ is irrelevant for the solutions of this equation, hence ϕ_ϵ and $\phi_\epsilon^\dagger$ satisfy the *scaled Klein-Gordon-equation*

$$P_\epsilon\psi := (\partial_t^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2)\psi = 0\,. \tag{2.1.5}$$

We also refer to P_ϵ as the *scaled Klein-Gordon-operator*. Note that for $\epsilon = 1$ we get the original Klein-Gordon equation, i.e., $P_1 = P$.

2.2. Limit of Commutators

In this section, we start by converting the ϕ_ϵ-field, which is an operator-valued distribution at this point, to an operator and we will show how the commutator (2.1.2) translates. This will complete the motivation for the ansatz.

Now for the next few calculations, consider

$$\tilde{\phi}_\epsilon(t,x) := \epsilon^{n(\alpha,\beta)}\phi(\epsilon^\alpha t, \epsilon^\beta x) \tag{2.2.1}$$

and analogously for $\tilde{\phi}_\epsilon^\dagger$, where $\alpha > \beta \geq 0$ and $n : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. This still captures the idea of letting the time-coordinate going faster to zero than the space-coordinate by the restrictions on α and β. We will use this general approach to address the arbitrariness of (2.1.4) mentioned in Section 2.1.

We will now smear out the fields using

$$\tilde{\phi}_\epsilon(f) := \int dtd^dx\,\tilde{\phi}_\epsilon(t,x)\,f(t,x) \tag{2.2.2}$$

and analogously for $\tilde{\phi}^\dagger(g)$, for any $f,g \in C_c^\infty(\mathbb{R}^{d+1})$, and calculate the commutator. As a notation, we are going to use $E_p := \sqrt{p^2+m^2}$.

$$\left[\tilde{\phi}_\epsilon(f), \tilde{\phi}_\epsilon^\dagger(g)\right] \overset{(2.2.2)}{=} \int dtd^dx\,f(t,x)\int dsd^dy\,g(s,y)\left[\tilde{\phi}_\epsilon(t,x), \tilde{\phi}_\epsilon^\dagger(s,y)\right]$$

$$
\begin{aligned}
&\overset{(2.2.1)}{=} \int dt d^d x\, f(t,x) \int ds d^d y\, g(s,y) \epsilon^{2n(\alpha,\beta)} \Big[\phi(\epsilon^\alpha t, \epsilon^\beta x), \phi^\dagger(\epsilon^\alpha s, \epsilon^\beta y)\Big] \\
&\overset{(2.1.2)}{=} \epsilon^{2n(\alpha,\beta)} \int dt d^d x\, f(t,x) \int ds d^d y\, g(s,y) \int \frac{d^d p}{(2\pi)^d} \\
&\quad \cdot \int dp_0 \operatorname{sgn}(p_0)\, \delta(p_0^2 - p^2 - m^2)\, e^{-ip_0 \epsilon^\alpha (t-s) + ip\epsilon^\beta (x-y)} \\
&= \epsilon^{2n(\alpha,\beta)} \int \frac{d^d p}{(2\pi)^d} \int dp_0 \operatorname{sgn}(p_0)\, \delta(p_0^2 - p^2 - m^2) \hat{f}(\epsilon^\alpha p_0, \epsilon^\beta p) \hat{g}(-\epsilon^\alpha p_0, -\epsilon^\beta p) \\
&= \epsilon^{2n(\alpha,\beta)} \int \frac{d^d p}{(2\pi)^d} \int dp_0 \operatorname{sgn}(p_0) \left(\frac{\delta(p_0 - E_p)}{2E_p} + \frac{\delta(p_0 + E_p)}{2E_p} \right) \\
&\quad \cdot \hat{f}(\epsilon^\alpha p_0, \epsilon^\beta p) \hat{g}(-\epsilon^\alpha p_0, -\epsilon^\beta p) \\
&= \frac{1}{2} \epsilon^{2n(\alpha,\beta)} \int \frac{d^d p}{(2\pi)^d E_p} \Big(\hat{f}(\epsilon^\alpha E_p, \epsilon^\beta p) \hat{g}(-\epsilon^\alpha E_p, -\epsilon^\beta p) \\
&\quad - \hat{f}(-\epsilon^\alpha E_p, \epsilon^\beta p) \hat{g}(\epsilon^\alpha E_p, -\epsilon^\beta p) \Big) \\
&\overset{p \mapsto -\epsilon^{-\beta} p}{=} \frac{1}{2} \int \frac{d^d p}{(2\pi)^d \epsilon^{d\beta - 2n(\alpha,\beta)} E_{p/\epsilon^\beta}} \Big(\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p) \hat{g}(-\epsilon^\alpha E_{p/\epsilon^\beta}, -p) \\
&\quad - \hat{f}(-\epsilon^\alpha E_{p/\epsilon^\beta}, -p) \hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p) \Big) \\
&= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d \epsilon^{d\beta - 2n(\alpha,\beta)} E_{p/\epsilon^\beta}} \Big(\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p) \overline{\hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)} \\
&\quad - \overline{\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)} \hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p) \Big)\,.
\end{aligned}
$$

The swapping of integrals in this calculation is in a sense part of the definition of the distributional representation of Equation (2.1.3). Another argument for why we can exchange the integrals is because we only integrate smooth functions with compact support. This also holds for any subsequent calculations. Note also that $E_{-p} = E_p$.

Looking at the expression, $\epsilon^\alpha E_{p/\epsilon^\beta}$ we see that

$$
\epsilon^\alpha E_{p/\epsilon^\beta} = \epsilon^\alpha \sqrt{(p/\epsilon^\beta)^2 + m^2} = \sqrt{\epsilon^{2(\alpha-\beta)} p^2 + \epsilon^{2\alpha} m^2} \underset{\epsilon \to 0}{\longrightarrow} 0\,,
$$

since $\alpha, \alpha - \beta > 0$. Similarly, for $\epsilon^{d\beta - 2n(\alpha,\beta)} E_{p/\epsilon^\beta}$ we can distinguish four cases.

$$
\begin{aligned}
\epsilon^{d\beta - 2n(\alpha,\beta)} E_{p/\epsilon^\beta} =& \sqrt{\epsilon^{(2d-2)\beta - 4n(\alpha,\beta)} p^2 + \epsilon^{2d\beta - 4n(\alpha,\beta)} m^2} \\
&\underset{\epsilon \to 0}{\longrightarrow} \begin{cases} 0 & \text{, if } (d-1)\beta - 2n(\alpha,\beta) > 0 \\ E_p & \text{, if } d\beta - 2n(\alpha,\beta) = (d-1)\beta - 2n(\alpha,\beta) = 0 \\ |p| & \text{, if } d\beta - 2n(\alpha,\beta) > (d-1)\beta - 2n(\alpha,\beta) = 0 \\ \infty & \text{, if } d\beta - 2n(\alpha,\beta) < 0 \end{cases}\,.
\end{aligned}
$$

It may seem that those are not all cases, but considering that we always have $(d-1)\beta - 2n(\alpha,\beta) \leq d\beta - 2n(\alpha,\beta)$ we can conclude that we have covert every possibility. Note that the case $d\beta - 2n(\alpha,\beta) = (d-1)\beta - 2n(\alpha,\beta) = 0$ can only happen if $\beta = 0$ and $n(\alpha, 0) = 0$.

We can now look case by case, how the commutator $\left[\tilde{\phi}_\epsilon(f), \tilde{\phi}^\dagger_\epsilon(g)\right]$ behaves in the limit $\epsilon \longrightarrow 0$. Note that swapping the limit with the integral is justified if the resulting integral converges, since $\hat{f}$ and $\hat{g}$ are Schwartz-functions.

Case 1:
If $\epsilon^{d\beta-2n(\alpha,\beta)} E_{p/\epsilon^\beta} \underset{\epsilon\to 0}{\longrightarrow} \infty$, then

$$\frac{1}{2}\int \frac{d^d p}{(2\pi)^d \epsilon^{d\beta-2n(\alpha,\beta)} E_{p/\epsilon^\beta}} \left(\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)\overline{\hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)} - \overline{\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)}\hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)\right) \underset{\epsilon\to 0}{\longrightarrow} 0\,,$$

since

$$\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)\overline{\hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)} - \overline{\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)}\hat{g}(\epsilon^\alpha E_{p/\epsilon^\beta}, p) \underset{\epsilon\to 0}{\longrightarrow} \hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\hat{g}(0,p)\,.$$

Of course, this is not a useful commutator for us.

Case 2:
Similarly, if $\epsilon^{d\beta-2n(\alpha,\beta)} E_{p/\epsilon^\beta} \underset{\epsilon\to 0}{\longrightarrow} |p|$, then

$$\left[\tilde{\phi}_\epsilon(f), \tilde{\phi}^\dagger_\epsilon(g)\right] \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{2}\int \frac{d^d p}{(2\pi)^d\, |p|} \left(\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\hat{g}(0,p)\right)\,.$$

But

$$\begin{aligned}
&\frac{1}{2}\int \frac{d^d p}{(2\pi)^d\, |p|} \left(\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\hat{g}(0,p)\right) \\
&\overset{p\mapsto -p}{=} \frac{1}{2}\int \frac{d^d p}{(2\pi)^d\, |p|} \left(\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,-p)}\hat{g}(0,-p)\right) \\
&= \frac{1}{2}\int \frac{d^d p}{(2\pi)^d\, |p|} \left(\hat{f}(0,p)\overline{\hat{g}(0,p)} - \hat{f}(0,p)\overline{\hat{g}(0,p)}\right) = 0\,,
\end{aligned}$$

which again is not a useful commutator.

Case 3:
Analogously to Case 2, if $\epsilon^{d\beta-2n(\alpha,\beta)} E_{p/\epsilon^\beta} \underset{\epsilon\to 0}{\longrightarrow} E_p$, then

$$\left[\tilde{\phi}_\epsilon(f), \tilde{\phi}^\dagger_\epsilon(g)\right] \underset{\epsilon\to 0}{\longrightarrow} 0\,.$$

Case 4:
A priori, if $\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta} \underset{\epsilon\to 0}{\longrightarrow} 0$, it looks like the commutator diverges. But there is a way to remedy this.

$$\begin{aligned}
&\frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \left(\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)} - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)\right)\\
&\quad = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \Big(\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)} - \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\\
&\qquad - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p) + \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\Big)\\
&\quad \overset{p\mapsto -p}{=} \frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \Big(\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)} - \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\\
&\qquad - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p) + \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},-p)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},-p)}\Big)\\
&\quad = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \left[\left(\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p) - \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\right)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\right.\\
&\qquad \left. - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\left(\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p) - \hat g(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\right)\right].
\end{aligned}$$

Using Taylor's theorem on any smooth function $h:\mathbb{R}\to\mathbb{R}$, we get

$$\begin{aligned}
h(x) &= h(0) + \frac{\partial h}{\partial x}(0)x + \frac{\partial^2 h}{\partial x^2}(0)x^2 + \mathcal{O}(x^3)\\
h(-x) &= h(0) - \frac{\partial h}{\partial x}(0)x + \frac{\partial^2 h}{\partial x^2}(0)x^2 + \mathcal{O}(x^3)
\end{aligned}$$

for x sufficiently small. Hence,

$$h(x) - h(-x) = 2\frac{\partial h}{\partial x}(0)x + \mathcal{O}(x^3)\,.$$

Since any smooth function with compact support is Schwartz and the Fourier-transform of any Schwartz-function is again Schwartz, and therefore smooth, we can apply this to $\hat f(\,.\,,p)$ and $\hat g(\,.\,,p)$. We will use the notion $\frac{\partial}{\partial p_0}\hat f(0,p)$ for $\frac{\partial}{\partial p_0}\hat f(p_0,p)\Big|_{p_0=0}$.

$$\begin{aligned}
&\frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \left[\left(\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p) - \hat f(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\right)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\right.\\
&\qquad \left. - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\left(\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p) - \hat g(-\epsilon^\alpha E_{p/\epsilon^\beta},p)\right)\right]\\
&\quad = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d\epsilon^{d\beta-2n(\alpha,\beta)}E_{p/\epsilon^\beta}} \left[\left(2\epsilon^\alpha E_{p/\epsilon^\beta}\frac{\partial}{\partial p_0}\hat f(0,p) + \mathcal{O}((\epsilon^\alpha E_{p/\epsilon^\beta})^3)\right)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\right.\\
&\qquad \left. - \overline{\hat f(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\left(2\epsilon^\alpha E_{p/\epsilon^\beta}\frac{\partial}{\partial p_0}\hat g(0,p) + \mathcal{O}((\epsilon^\alpha E_{p/\epsilon^\beta})^3)\right)\right]\\
&\quad = \epsilon^{\alpha-d\beta+2n(\alpha,\beta)}\int \frac{d^dp}{(2\pi)^d}\left[\left(\frac{\partial}{\partial p_0}\hat f(0,p) + \mathcal{O}((\epsilon^\alpha E_{p/\epsilon^\beta})^2)\right)\overline{\hat g(\epsilon^\alpha E_{p/\epsilon^\beta},p)}\right.
\end{aligned}$$

$$- \overline{\hat{f}(\epsilon^\alpha E_{p/\epsilon^\beta}, p)} \left(\frac{\partial}{\partial p_0} \hat{g}(0,p) + \mathcal{O}((\epsilon^\alpha E_{p/\epsilon^\beta})^2) \right) \Bigg] \,.$$

This converges to a non-trivial result if $\alpha - d\beta + 2n(\alpha,\beta) = 0$. Hence, if

$$n(\alpha,\beta) = -\frac{\alpha - d\beta}{2} \tag{2.2.3}$$

we get

$$\left[\tilde{\phi}_\epsilon(f), \tilde{\phi}_\epsilon^\dagger(g)\right] \underset{\epsilon\to 0}{\longrightarrow} \int \frac{d^d p}{(2\pi)^d} \left(\frac{\partial}{\partial p_0} \hat{f}(0,p) \overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)} \frac{\partial}{\partial p_0} \hat{g}(0,p) \right) .$$

The condition $(d-1)\beta - 2n(\alpha,\beta) > 0$ is automatically fulfilled with (2.2.3) since

$$(d-1)\beta - 2n(\alpha,\beta) \overset{(2.2.3)}{=} (d-1)\beta + (\alpha - d\beta) = \alpha - \beta > 0\,,$$

by definition of α and β.

For the special case $\alpha = 1$ and $\beta = 0$ we get $n(1,0) = -\frac{1}{2}$, which is exactly what we choose in (2.1.4). This also yields for the commutator

$$\begin{aligned} \left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right] &= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d \epsilon E_p} \left(\hat{f}(\epsilon E_p, p) \overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)} \hat{g}(\epsilon E_p, p) \right) && (2.2.4) \\ &\underset{\epsilon\to 0}{\longrightarrow} \int \frac{d^d p}{(2\pi)^d} \left(\frac{\partial}{\partial p_0} \hat{f}(0,p) \overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)} \frac{\partial}{\partial p_0} \hat{g}(0,p) \right) . && (2.2.5) \end{aligned}$$

2.3. Family of Weyl-algebras

With the ansatz sufficiently motivated, we can move on to the main construction of this chapter. We will convert the operator $\phi_\epsilon(f)$ from the previous section into an element of, what we will argue to be, a *Weyl-algebra*.

Consider the integral

$$\langle f, g \rangle_\epsilon := \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \hat{f}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} \tag{2.3.1}$$

for $\epsilon > 0$, where $E_{\epsilon,p} := \epsilon E_p$, which is part of (2.2.4). This is a positive semi-definite hermitian ($\mathbb{R}$-)bilinear map $\langle\,.\,,\,.\,\rangle_\epsilon : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{C}$.
Indeed, bilinearity follows from the linearity of the Fourier transform and the integral. It is also hermitian since we only deal with integrals over $\mathbb{R}^d$, which implies

$$\begin{aligned} \overline{\langle f, g \rangle_\epsilon} &= \overline{\int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \hat{f}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)}} = \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \overline{\hat{f}(E_{\epsilon,p}, p)} \hat{g}(E_{\epsilon,p}, p) \\ &= \langle g, f \rangle_\epsilon \,. \end{aligned}$$

Positive semi-definiteness holds also, since $E_p \geq 0$ for all $p \in \mathbb{R}^d$ and therefore

$$\langle f, f \rangle_\epsilon = \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left| \hat{f}(E_{\epsilon,p}, p) \right|^2 \geq 0 \,.$$

To justify the notation, we want to mention at this point that we can develop this into a complex scalar product. A detailed construction will be done in the next chapter.

We can however construct a (real) symplectic vector space by defining

$$\sigma_\epsilon(f, g) := -\frac{1}{2} \mathrm{Im} \, \langle f, g \rangle_\epsilon \tag{2.3.2}$$

for $f, g \in C_c^\infty(\mathbb{R}^{d+1})$. The prefactor $-\frac{1}{2}$ will make sense later on.
If we move to $V_\epsilon := C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker}\, \sigma_\epsilon$, where

$$\mathrm{Ker}\, \sigma_\epsilon := \{ f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \sigma_\epsilon(f, g) = 0 \, \forall g \in C_c^\infty(\mathbb{R}^{d+1}) \} \,, \tag{2.3.3}$$

we can infer a symplectic form on V_ϵ as follows.
Linearity and antisymmetry of σ_ϵ follow quickly, since $C_c^\infty(\mathbb{R}^{d+1})$ is a real vector space and $\mathrm{Im}\, \bar{c} = -\mathrm{Im}\, c$ for any $c \in \mathbb{C}$. Nondegeneracy follows by construction of V_ϵ since $\sigma(f, g)_\epsilon = 0 \, \forall g \in C_c^\infty(\mathbb{R}^{d+1})$ implies $f \in \mathrm{Ker}\, \sigma_\epsilon$, that is, $[f] = [0] \in V_\epsilon$.
Lastly, assume $f, g \in C_c^\infty(\mathbb{R}^{d+1})$ and $f', g' \in \mathrm{Ker}\, \sigma_\epsilon$. We then get

$$\begin{aligned}
\sigma_\epsilon(f + f', g + g') &= -\frac{1}{2} \mathrm{Im} \, \langle f + f', g + g' \rangle_\epsilon = -\frac{1}{2} \mathrm{Im} \left[\langle f, g \rangle_\epsilon + \langle f', g \rangle_\epsilon + \langle f, g' \rangle_\epsilon + \langle f', g' \rangle_\epsilon \right] \\
&\overset{(2.3.3)}{=} -\frac{1}{2} \mathrm{Im} \, \langle f, g \rangle_\epsilon = \sigma_\epsilon(f, g) \,.
\end{aligned}$$

Hence, we can regard σ_ϵ as a well-defined symplectic form on V_ϵ.
From this point onward, we are going to abuse some notation and interpret $f \in V_\epsilon$ as either the equivalence class of f in V_ϵ or the function in $C_c^\infty(\mathbb{R}^{d+1})$, depending on the context. The well-definedness of various statements will be shown regardless if needed.

With our new notation, we are now able to rewrite the commutator (2.2.4).

$$\begin{aligned}
\left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g) \right] &= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left(\hat{f}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} - \overline{\hat{f}(E_{\epsilon,p}, p)} \hat{g}(E_{\epsilon,p}, p) \right) \\
&\overset{(2.3.1)}{=} \frac{1}{2} \left(\langle f, g \rangle_\epsilon - \langle g, f \rangle_\epsilon \right) = \frac{1}{2} \left(\langle f, g \rangle_\epsilon - \overline{\langle f, g \rangle_\epsilon} \right) = i \, \mathrm{Im} \, \langle f, g \rangle_\epsilon \\
&= -2i \sigma_\epsilon(f, g) \,.
\end{aligned} \tag{2.3.4}$$

To proceed, we define two new (families of) operators. The first one is

$$\psi_\epsilon(f) := \frac{1}{\sqrt{2}} \left(\phi_\epsilon(f) + \phi_\epsilon^\dagger(f) \right) \,. \tag{2.3.5}$$

The factor $\frac{1}{\sqrt{2}}$ is only for normalization, as we will see shortly. The commutator of this new operator with itself comes up as

$$\begin{aligned}[\psi_\epsilon(f), \psi_\epsilon(g)] &\overset{(2.3.5)}{=} \frac{1}{2}\left[\phi_\epsilon(f) + \phi_\epsilon^\dagger(f), \phi_\epsilon(g) + \phi_\epsilon^\dagger(g)\right] = \frac{1}{2}\left(\left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right] + \left[\phi_\epsilon^\dagger(f), \phi_\epsilon(g)\right]\right) \\ &\overset{(2.3.4)}{=} \frac{1}{2}\left(-2i\sigma_\epsilon(f,g) + 2i\sigma_\epsilon(g,f)\right) = -i\left(\sigma_\epsilon(f,g) - \sigma_\epsilon(g,f)\right) = \\ &= -i\left(\sigma_\epsilon(f,g) + \sigma_\epsilon(f,g)\right) = -2i\sigma_\epsilon(f,g)\,. \end{aligned} \tag{2.3.6}$$

The second family of operators we define is

$$W_\epsilon(f) := e^{i\psi_\epsilon(f)} \tag{2.3.7}$$

for $f \in C_c^\infty(\mathbb{R}^{d+1})$. Since this is a formal computation we do not need to worry about the convergence of (2.3.7). For more a more explicit calculation, where we have specified $\psi_\epsilon(f)$ as a densely defined, unbounded and self-adjoint operator, since $(\phi_\epsilon(f))^* = \phi_\epsilon^\dagger(f)$ and $(\phi_\epsilon^\dagger(f))^* = \phi_\epsilon(f)$, we need to use functional calculus [RS80, VIII.5] to define (2.3.7). Omitting the details for now, we will revisit them in Chapter 4, but we can use [BGP07, Theorem 3.4.7] to imply

$$\operatorname{Im} P_\epsilon = \operatorname{Ker} \sigma_\epsilon\,, \tag{2.3.8}$$

which in turn implies that for any $h \in \operatorname{Ker}\sigma_\epsilon$ we can find a $\tilde{h} \in C_c^\infty(\mathbb{R}^{d+1})$ such that $P_\epsilon \tilde{h} = h$. This together with the fact that

$$\phi_\epsilon(P_\epsilon f) = \int dt d^d x\, \phi_\epsilon(t,x)\, P_\epsilon f(t,x) = \int dt d^d x\, P_\epsilon \phi_\epsilon(t,x)\, f(t,x) = 0\,,$$

since ϕ_ϵ is a solution of P_ϵ, which also implies $\psi_\epsilon(P_\epsilon f) = 0$, yields

$$W_\epsilon(h) = e^{i\psi_\epsilon(h)} = e^{i\psi_\epsilon(P_\epsilon \tilde{h})} = e^0 = \mathbb{1} \quad \forall h \in \operatorname{Ker}\sigma_\epsilon\,. \tag{2.3.9}$$

We then apply the following theorem to $(V_\epsilon, \sigma_\epsilon)$.

Theorem 2.3.1 (Existence of Weyl-algebras)
Let (V, σ) be a symplectic vector space. There exists a C^*-algebra $\mathcal{W}(V,\sigma)$, uniquely determined up to $*$-isomorphism, that is generated by a family of elements $W(u)$, $u \in V$, such that for all $u, v \in V$,

$$W(0) = \mathbb{1}, \quad W(-u) = W(u)^*, \quad \text{and} \quad W(u)W(v) = e^{i\sigma(u,v)}W(u+v). \tag{2.3.10}$$

We call $\mathcal{W}(V,\sigma)$ the *Weyl algebra* over (V,σ).[Pet90, 2.1][Hö24, 5.5]

This yields the Weyl-algebra $\mathcal{W}_\epsilon := \mathcal{W}(V_\epsilon, \sigma_\epsilon)$. But as it stands, we only get an abstract C^*-algebra from the theorem, which for our purposes is not easy to work with. But luckily for us, the family $\{W_\epsilon(f) \mid f \in V_\epsilon\}$ also fulfills the *Weyl-relations* (2.3.10). Indeed, the first Weyl-relation is satisfied using (2.3.9) and since $f \mapsto \psi_\epsilon(f)$ is $\mathbb{R}$-linear, by (2.2.2) and (2.3.5), we get

$$W_\epsilon(f)^* = e^{-i\psi_\epsilon^*(f)} = e^{-i\psi_\epsilon(f)} = e^{i\psi_\epsilon(-f)} = W_\epsilon(-f)\,.$$

To show that the last of the Weyl-relations hold we use the Baker-Campell-Hausdorff-formula

$$e^X e^Y = e^{\frac{1}{2}[X,Y]} e^{X+Y} \,,$$

where X, Y are operators such that $[X,[X,Y]] = [Y,[Y,X]] = 0$.
This can be applied to $W_\epsilon(f)W_\epsilon(g)$ since $[\psi_\epsilon(f), \psi_\epsilon(g)] \propto \mathbb{1}$, which yields

$$\begin{aligned} W_\epsilon(f)W_\epsilon(g) &= e^{i\psi_\epsilon(f)} e^{i\psi_\epsilon(g)} = e^{\frac{1}{2}[i\psi_\epsilon(f), i\psi_\epsilon(g)]} e^{i\psi_\epsilon(f)+i\psi_\epsilon(g)} = e^{-i\frac{1}{2}\mathrm{Im}\langle f,g\rangle_\epsilon} e^{i\psi_\epsilon(f+g)} \\ &= e^{i\sigma_\epsilon(f,g)} W_\epsilon(f+g) \,. \end{aligned}$$

This also explains the prefactor in (2.3.2).

Remark. At this point, we need to say, that we can apply the Baker-Campell-Hausdorff-formula in this context because this is a formal calculation. In a specific case things become more subtle, as discussed in [RS80, VIII.5]. We would need to consider *analytic vectors* [RS75, X.6] for $\psi_\epsilon(f)$ and $\psi_\epsilon(g)$ to make pointwise sense of the Baker-Campell-Hausdorff-formula.
However, as we already indicated in Section 2.1, we will show in Chapter 3 that we can find a Fock space representation of $\psi_\epsilon(f)$, where it takes the form $\psi_\epsilon(f) = \frac{1}{\sqrt{2}} \left(a_\epsilon(f) + a_\epsilon^\dagger(f) \right)$, where $a_\epsilon(f)$ and $a_\epsilon^\dagger(f)$ are annihilation and creation operators. In this case, the Weyl-relations are implied by [BR97, 5.2.4].

Finally, we address the well-definedness of $W_\epsilon(f)$ if $f \in V_\epsilon$. Assume $f \in C_c^\infty(\mathbb{R}^{d+1})$ and $h \in \operatorname{Ker} \sigma_\epsilon$. Then

$$W_\epsilon(f+h) = e^{-i\sigma_\epsilon(f,h)} W_\epsilon(f)W_\epsilon(h) = W_\epsilon(f)W_\epsilon(h) \overset{(2.3.9)}{=} W_\epsilon(f) \,.$$

Therefore, by the uniqueness part of Theorem 2.3.1, we get that the C^*-algebra generated from $\{W_\epsilon(f) \mid f \in V_\epsilon\}$ is $*$-isomorphic to $\mathcal{W}_\epsilon$. Hence, without loss of generality, we can assume that $\mathcal{W}_\epsilon$ is the C^*-algebra generated from $\{W_\epsilon(f) \mid f \in V_\epsilon\}$.
This construction works for every $\epsilon > 0$, so we gain a family of Weyl-algebras $\{\mathcal{W}_\epsilon \mid e > 0\}$.

2.4. Electric Carrollian limit

In the previous section, we learned that the commutator $\left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right]$ is proportional to the symplectic form $\sigma_\epsilon(f,g)$ (Equation (2.3.4)). We have also seen in (2.2.5) that the commutators have a well defined limit. So similarly to what we did in the last section, we define

$$\sigma_0(f,g) := \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \left(\frac{\partial}{\partial p_0} \hat{f}(0,p) \overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)} \frac{\partial}{\partial p_0} \hat{g}(0,p) \right) . \tag{2.4.1}$$

It is easy to see that this is a ($\mathbb{R}$-)bilinear antisymmetric map $\sigma_0(\,.\,,\,.\,) : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{R}$. In an analogous process to the previous section, we can turn σ_0 into a symplectic form on $V_0 := C_c^\infty(\mathbb{R}^{d+1}) / \operatorname{Ker} \sigma_0$. Applying Theorem 2.3.1 to (V_0, σ_0) yields a Weyl-algebra $\mathcal{W}_0 := \mathcal{W}(V_0, \sigma_0)$.

Now clearly, by (2.2.5) we have the pointwise convergence $\sigma_\epsilon \to \sigma_0$ as $\epsilon \to 0$. This certainly suggests that there is some connection between $\mathcal{W}_0$ and $\mathcal{W}_\epsilon$, but $\mathcal{W}_0$ is not a proper limit of the family $\{\mathcal{W}_\epsilon \mid \epsilon > 0\}$, since we do not have a meaningful way of comparing different abstract C^*-algebras. We will revisit this question in the next chapter. Regardless, we call $\mathcal{W}_0$ the *electric Carrollian (limit) Weyl-algebra* of $\{\mathcal{W}_\epsilon \mid \epsilon > 0\}$.

3. States and symmetries

In the previous chapter we have constructed a C^*-algebra $\mathcal{W}_0$, which can be understood as some sort of limit of the family $\{\mathcal{W}_\epsilon \mid \epsilon > 0\}$. So far $\mathcal{W}_0$ is an abstract C^*-algebra, which is not very useful for standard physics. The purpose of this chapter is to show that $\mathcal{W}_0$ can be understood as some operator algebra and to add another reason why we call $\mathcal{W}_0$ the limit of $\{\mathcal{W}_\epsilon \mid \epsilon > 0\}$. We also show in what sense the Carroll symmetries [LL65, SG66] hold.

3.1. State of the electric Carrolian limit

As stated above, we want to represent $\mathcal{W}_0$ as an operator algebra. For that purpose we want to use the GNS-representation (Theorem 1.2.7), which requires us to have a state on the C^*-algebra.

3.1.1. Constructing a Fock space representations

Our first step in constructing such a state requires us to look again at the maps $\langle\,.\,,\,.\,\rangle_\epsilon : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{C}$ from Equation (2.3.1). We are going to extend them to proper inner products.
Consider the map $\langle\,.\,,\,.\,\rangle_\epsilon^{\mathbb{C}} : C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \times C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \to \mathbb{C}$ defined by

$$\langle f, g\rangle_\epsilon^{\mathbb{C}} := \langle f_1, g_1\rangle_\epsilon + \langle f_2, g_2\rangle_\epsilon + i\left(\langle f_2, g_1\rangle_\epsilon - \langle f_1, g_2\rangle_\epsilon\right)\,, \tag{3.1.1}$$

where f_1 and g_1 are the real part and f_2 and g_2 are the imaginary part of f and g, respectively. This map is positive semi-definite, sesquilinear and hermitian. Indeed, linearity in the first component follows from

$$\begin{aligned}
\langle f + h, g\rangle_\epsilon^{\mathbb{C}} &= \langle f_1 + h_1, g_1\rangle_\epsilon + \langle f_2 + h_2, g_2\rangle_\epsilon + i\left(\langle f_2 + h_2, g_1\rangle_\epsilon - \langle f_1 + h_1, g_2\rangle_\epsilon\right) \\
&= \langle f_1, g_1\rangle_\epsilon + \langle h_1, g_1\rangle_\epsilon + \langle f_2, g_2\rangle_\epsilon + \langle h_2, g_2\rangle_\epsilon \\
&\quad + i\left(\langle f_2, g_1\rangle_\epsilon + \langle h_2, g_1\rangle_\epsilon - \langle f_1, g_2\rangle_\epsilon - \langle h_1, g_2\rangle_\epsilon\right) \\
&= \langle f, g\rangle_\epsilon^{\mathbb{C}} + \langle h, g\rangle_\epsilon^{\mathbb{C}}
\end{aligned}$$

and

$$\begin{aligned}
\langle \alpha f, g\rangle_\epsilon^{\mathbb{C}} &= \langle (\alpha_1 + i\alpha_2)(f_1 + if_2), g\rangle_\epsilon^{\mathbb{C}} = \langle (\alpha_1 f_1 - \alpha_2 f_2) + i(\alpha_2 f_1 + \alpha_1 f_2), g\rangle_\epsilon^{\mathbb{C}} \\
&= \langle \alpha_1 f_1 - \alpha_2 f_2, g_1\rangle_\epsilon + \langle \alpha_2 f_1 + \alpha_1 f_2, g_2\rangle_\epsilon \\
&\quad + i\left(\langle \alpha_2 f_1 + \alpha_1 f_2, g_1\rangle_\epsilon - \langle \alpha_1 f_1 - \alpha_2 f_2, g_2\rangle_\epsilon\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_1 \left(\langle f_1, g_1 \rangle_\epsilon + \langle f_2, g_2 \rangle_\epsilon + i \left(\langle f_2, g_1 \rangle_\epsilon - \langle f_1, g_2 \rangle_\epsilon \right) \right) \\
&\quad + \alpha_2 \left(\langle f_1, g_2 \rangle_\epsilon - \langle f_2, g_1 \rangle_\epsilon + i \left(\langle f_1, g_1 \rangle_\epsilon + \langle f_2, g_2 \rangle_\epsilon \right) \right) \\
&= \alpha_1 \langle f, g \rangle_\epsilon^{\mathbb{C}} + \alpha_2 i \langle f, g \rangle_\epsilon^{\mathbb{C}} = \alpha \langle f, g \rangle_\epsilon^{\mathbb{C}} \ ,
\end{aligned}
$$

for $\alpha \in \mathbb{C}$ and $f, g, h \in C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C})$.
$\langle \, . \, , \, . \, \rangle_\epsilon^{\mathbb{C}}$ is hermitian since

$$
\begin{aligned}
\langle f, g \rangle_\epsilon^{\mathbb{C}} &= \langle f_1, g_1 \rangle_\epsilon + \langle f_2, g_2 \rangle_\epsilon + i \left(\langle f_2, g_1 \rangle_\epsilon - \langle f_1, g_2 \rangle_\epsilon \right) \\
&= \overline{\langle g_1, f_1 \rangle_\epsilon + \langle g_2, f_2 \rangle_\epsilon} - i \, \overline{\left(\langle g_2, f_1 \rangle_\epsilon - \langle g_1, f_2 \rangle_\epsilon \right)} = \overline{\langle g, f \rangle_\epsilon^{\mathbb{C}}} \, .
\end{aligned}
$$

Finally, positive-semi-definitness is implied by

$$
\begin{aligned}
\langle f, f \rangle_\epsilon^{\mathbb{C}} &= \langle f_1, f_1 \rangle_\epsilon + \langle f_2, f_2 \rangle_\epsilon + i \left(\langle f_2, f_1 \rangle_\epsilon - \langle f_1, f_2 \rangle_\epsilon \right) = \langle f_1, f_1 \rangle_\epsilon + \langle f_2, f_2 \rangle_\epsilon - 2 \mathrm{Im} \, \langle f_2, f_1 \rangle_\epsilon \\
&\geq \langle f_1, f_1 \rangle_\epsilon + \langle f_2, f_2 \rangle_\epsilon - 2 \sqrt{\langle f_1, f_1 \rangle_\epsilon \langle f_2, f_2 \rangle_\epsilon} = \left(\sqrt{\langle f_1, f_1 \rangle_\epsilon} - \sqrt{\langle f_2, f_2 \rangle_\epsilon} \right)^2 \geq 0 \, ,
\end{aligned}
$$

where we used the Cauchy-Schwarz-inequality for positive semi-definite hermitian sesquilinear forms [Con90, 1.4] to get

$$
\mathrm{Im} \, \langle f_2, f_1 \rangle_\epsilon \leq |\langle f_2, f_1 \rangle_\epsilon| \leq \sqrt{\langle f_1, f_1 \rangle_\epsilon \langle f_2, f_2 \rangle_\epsilon} \, .
$$

All the square roots in the above computation are real-valued due to the positivity of $\langle \, . \, , \, . \, \rangle_\epsilon$.
By construction we have the additional property that $\langle f, g \rangle_\epsilon^{\mathbb{C}} = \langle f, g \rangle_\epsilon$ whenever f and g are real-valued.
We are now able to extend $\langle \, . \, , \, . \, \rangle_\epsilon^{\mathbb{C}}$ to a proper inner product on $C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \Big/ \mathrm{Ker}_\epsilon^{\mathbb{C}}$, where

$$
\mathrm{Ker}_\epsilon^{\mathbb{C}} := \left\{ f \in C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \mid \langle f, f \rangle_\epsilon^{\mathbb{C}} = 0 \right\} \, .
$$

We denote the completion of $C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \Big/ \mathrm{Ker}_\epsilon^{\mathbb{C}}$ with respect to $\langle \, . \, , \, . \, \rangle_\epsilon^{\mathbb{C}}$ by $\mathcal{H}_\epsilon$.

The question is now whether $V_\epsilon = C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker} \, \sigma_\epsilon$ is somehow part of $\mathcal{H}_\epsilon$. The answer is yes but first we need the two following results.
First, for any positive semi-definite, hermitian, sesquilinear map b on a vector space V we have

$$
\{ f \in V \mid b(f, f) = 0 \} = \{ f \in V \mid b(f, g) = 0 \, \forall g \in V \} \ . \tag{3.1.2}
$$

This holds since, on the one hand, for $f \in \{ f \in V \mid b(f, g) = 0 \, \forall g \in V \}$ we get $b(f, f) = 0$ as the special case $g = f$, hence $f \in \{ f \in V \mid b(f, f) = 0 \}$. On the other hand, for $f \in \{ f \in V \mid b(f, f) = 0 \}$ and $g \in V$ we get

$$
|b(f, g)|^2 \leq b(f, f) b(g, g) = 0 \, ,
$$

by the Cauchy-Schwarz-inequality for positive semi-definite hermitian sesquilinear forms [Con90, 1.4], which implies $b(f, g) = 0$.

Second, we have

$$\operatorname{Ker}\sigma_\epsilon = \left\{ f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \langle f, g\rangle_\epsilon = 0\, \forall g \in C_c^\infty(\mathbb{R}^{d+1}) \right\} . \tag{3.1.3}$$

This holds because on the one hand, if $f \in \left\{ f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \langle f, g\rangle_\epsilon = 0\, \forall g \in C_c^\infty(\mathbb{R}^{d+1}) \right\}$, then $\sigma_\epsilon(f,g) := -\frac{1}{2}\mathrm{Im}\,\langle f, g\rangle_\epsilon = 0$ for all $g \in C_c^\infty(\mathbb{R}^{d+1})$ by (2.3.2), i.e., $f \in \operatorname{Ker}\sigma_\epsilon$. On the other hand, by Equation (2.3.8) we can find for any $f \in \operatorname{Ker}\sigma_\epsilon$ a $\tilde{f} \in C_c^\infty(\mathbb{R}^{d+1})$ such that $f = P_\epsilon \tilde{f}$. Then, for any $g \in C_c^\infty(\mathbb{R}^{d+1})$, we get

$$\begin{aligned}
\langle f, g\rangle_\epsilon = \left\langle P_\epsilon \tilde{f}, g \right\rangle_\epsilon &\overset{(2.3.1)}{=} \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \widehat{P_\epsilon \tilde{f}}(E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,p}, p)} \\
&= \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \mathcal{F}\left[(\partial_t^2 - \epsilon^2 \nabla^2 + \epsilon^2 m^2)\tilde{f} \right] (E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,p}, p)} \\
&= \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} (-p_0^2 + \epsilon^2 p + \epsilon^2 m^2)|_{p_0 = E_{\epsilon,p}} \hat{\tilde{f}}(E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,p}, p)} = 0\,.
\end{aligned}$$

Now we are ready to show that the map $V_\epsilon \to C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \Big/ \operatorname{Ker}_\epsilon^{\mathbb{C}} : [f] \mapsto [[f]]$ is an embedding, i.e., injective, linear and well-defined. Here, we temporarily differentiate between the function f and the equivalence classes $[f] \in V_\epsilon$ and $[[f]] \in C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \Big/ \operatorname{Ker}_\epsilon^{\mathbb{C}}$.
Linearity is easy to see and for well-definedness we need to show that $[[f+h]] = [[f]]$ whenever $h \in \operatorname{Ker}\sigma_\epsilon$. This is equivalent to showing that $h \in \operatorname{Ker}\sigma_\epsilon$ implies $h \in \operatorname{Ker}_\epsilon^{\mathbb{C}}$. By (3.1.3) and (3.1.2) we know that $h \in \operatorname{Ker}\sigma_\epsilon$ implies $\langle h, h\rangle_\epsilon = 0$ and since $h \in C_c^\infty(\mathbb{R}^{d+1})$, this also implies $\langle h, h\rangle_\epsilon^{\mathbb{C}} = 0$.
Lastly, for injectivity we need to show that if $[[f]] = [[0]]$, then $[f] = [0]$. In other words, if $f \in \operatorname{Ker}_\epsilon^{\mathbb{C}}$, then $f \in \operatorname{Ker}\sigma_\epsilon$ for any $f \in C_c^\infty(\mathbb{R}^{d+1})$. But this follows quickly since $f \in \operatorname{Ker}_\epsilon^{\mathbb{C}}$ means $0 = \langle f, f\rangle_\epsilon^{\mathbb{C}} = \langle f, f\rangle_\epsilon$. By (3.1.2) and (3.1.3) this implies $f \in \operatorname{Ker}\sigma_\epsilon$.
Hence, we can consider V_ϵ a subset of $C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \Big/ \operatorname{Ker}_\epsilon^{\mathbb{C}}$ and thus also of $\mathcal{H}_\epsilon$.

We shift our focus to the fields ϕ and $\phi^\dagger$ now. In physics, one usually writes the solution of the Klein-Gordon equation as

$$\phi(t,x) = \int \frac{d^d p}{(2\pi)^d \sqrt{2E_p}} \left(A(p) e^{-itE_p + ipx} + B^\dagger(p) e^{itE_p - ipx} \right) \tag{3.1.4}$$

$$\phi^\dagger(t,x) = \int \frac{d^d p}{(2\pi)^d \sqrt{2E_p}} \left(A^\dagger(p) e^{itE_p - ipx} + B(p) e^{-itE_p + ipx} \right) . \tag{3.1.5}$$

When moving to a quantum theory, the functions $A, A^\dagger, B$ and $B^\dagger$ are also replaced by operator-valued distributions with commutator relations

$$\left[A(p), A^\dagger(q) \right] = \left[B(p), B^\dagger(q) \right] = (2\pi)^d \delta^d(p-q) \tag{3.1.6}$$

and all other being 0.

We define now the operators

$$A_\epsilon(f) := \int \frac{d^dp}{(2\pi)^d\sqrt{E_{\epsilon,p}}}\,A(p)\hat{f}(E_{\epsilon,p},p) \quad \text{and} \quad A_\epsilon^\dagger(f) := \int \frac{d^dp}{(2\pi)^d\sqrt{E_{\epsilon,p}}}\,A^\dagger(p)\overline{\hat{f}(E_{\epsilon,p},p)} \tag{3.1.7}$$

for $f \in C_c^\infty(\mathbb{R}^{d+1})$ and analogously for $B_\epsilon(f)$ and $B_\epsilon^\dagger(f)$. This definition is motivated by the following calculation.

$$\begin{aligned}
\phi_\epsilon(f) &= \int dtd^dx\,\phi_\epsilon(t,x)f(t,x) \overset{(2.1.4)}{=} \int dtd^dx\,\epsilon^{-\frac{1}{2}}\phi(\epsilon t,x)f(t,x) \\
&\overset{(3.1.4)}{=} \epsilon^{-\frac{1}{2}}\int dtd^dx\int\frac{d^dp}{(2\pi)^d\sqrt{2E_p}}\left(A(p)e^{-i\epsilon tE_p+ipx}+B^\dagger(p)e^{i\epsilon tE_p-ipx}\right)f(t,x) \\
&= \int\frac{d^dp}{(2\pi)^d\sqrt{2E_{\epsilon,p}}}\left(A(p)\hat{f}(E_{\epsilon,p},p)+B^\dagger(p)\hat{f}(-E_{\epsilon,p},-p)\right) \\
&\overset{(3.1.7)}{=} \frac{1}{\sqrt{2}}\left(A_\epsilon(f)+B_\epsilon^\dagger(f)\right)
\end{aligned} \tag{3.1.8}$$

and similarly

$$\phi_\epsilon^\dagger(f) = \frac{1}{\sqrt{2}}\left(A_\epsilon^\dagger(f)+B_\epsilon(f)\right)\,.$$

Thus we can rewrite $\psi_\epsilon(f)$ as

$$\begin{aligned}
\psi_\epsilon(f) &\overset{(2.3.5)}{=} \frac{1}{\sqrt{2}}\left[\phi_\epsilon(f)+\phi_\epsilon^\dagger(f)\right] \overset{(3.1.8)}{=} \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(A_\epsilon(f)+B_\epsilon^\dagger(f)\right)+\frac{1}{\sqrt{2}}\left(A_\epsilon^\dagger(f)+B_\epsilon(f)\right)\right] \\
&= \frac{1}{\sqrt{2}}\left(a_\epsilon(f)+a_\epsilon^\dagger(f)\right)\,,
\end{aligned} \tag{3.1.9}$$

where for the last step we defined

$$a_\epsilon(f) := \frac{1}{\sqrt{2}}\left(A_\epsilon(f)+B_\epsilon(f)\right) \quad \text{and} \quad a_\epsilon^\dagger(f) := \frac{1}{\sqrt{2}}\left(A_\epsilon^\dagger(f)+B_\epsilon^\dagger(f)\right)\,. \tag{3.1.10}$$

Moving to the commutators, we can compute for any $f,g \in C_c^\infty(\mathbb{R}^{d+1})$

$$\begin{aligned}
\left[A_\epsilon(f),A_\epsilon^\dagger(g)\right] &\overset{(3.1.7)}{=} \int\frac{d^dp}{(2\pi)^d\sqrt{E_{\epsilon,p}}}\int\frac{d^dq}{(2\pi)^d\sqrt{E_{\epsilon,q}}}\left[A(p),A^\dagger(q)\right]\hat{f}(E_{\epsilon,p},p)\overline{\hat{g}(E_{\epsilon,q},q)} \\
&\overset{(3.1.6)}{=} \int\frac{d^dp}{(2\pi)^d\sqrt{E_{\epsilon,p}}}\int\frac{d^dq}{(2\pi)^d\sqrt{E_{\epsilon,q}}}(2\pi)^d\delta^d(p-q)\hat{f}(E_{\epsilon,p},p)\overline{\hat{g}(E_{\epsilon,q},q)} \\
&= \int\frac{d^dp}{(2\pi)^dE_{\epsilon,p}}\hat{f}(E_{\epsilon,p},p)\overline{\hat{g}(E_{\epsilon,p},p)} \qquad (3.1.11) \\
&\overset{(2.3.1)}{=} \langle f,g\rangle_\epsilon \qquad (3.1.12)
\end{aligned}$$

and similarly $\left[B_\epsilon(f),B_\epsilon^\dagger(g)\right] = \langle f,g\rangle_\epsilon$ and every other commutator between $A_\epsilon, A_\epsilon^\dagger, B_\epsilon$ and $B_\epsilon^\dagger$ being 0.

This then quickly implies that

$$
\begin{aligned}
\left[a_\epsilon(f), a_\epsilon^\dagger(g)\right] &\overset{(3.1.10)}{=} \left[\frac{1}{\sqrt{2}}\left(A_\epsilon(f)+B_\epsilon(f)\right), \frac{1}{\sqrt{2}}\left(A_\epsilon^\dagger(g)+B_\epsilon^\dagger(g)\right)\right] \\
&= \frac{1}{2}\left(\left[A_\epsilon(f), A_\epsilon^\dagger(g)\right] + \left[B_\epsilon(f), A_\epsilon^\dagger(g)\right] + \left[A_\epsilon(f), B_\epsilon^\dagger(g)\right] + \left[B_\epsilon(f), B_\epsilon^\dagger(g)\right]\right) \\
&\overset{(3.1.12)}{=} \frac{1}{2}\left(\langle f, g\rangle_\epsilon + \langle f, g\rangle_\epsilon\right) = \langle f, g\rangle_\epsilon
\end{aligned}
\tag{3.1.13}
$$

and analogously $[a_\epsilon(f), a_\epsilon(g)] = \left[a_\epsilon^\dagger(f), a_\epsilon^\dagger(g)\right] = 0$.

What this result allows us to do is to represent a_ϵ and $a_\epsilon^\dagger$ as annihilation and creation operators on some appropriate Fock space. Given what we computed at the beginning of this section, the obvious candidate for such a Fock space is $\mathcal{F}(\mathcal{H}_\epsilon)$, the *Fock space over the one-particle Hilbert space* $\mathcal{H}_\epsilon$.

We denote with $a_\epsilon(f)$ and $a_\epsilon^\dagger(g)$, for $f, g \in \mathcal{H}_\epsilon$, the *annihilation and creation operator* and with $|0_\epsilon\rangle$ the normalized *vacuum vector* of $\mathcal{F}(\mathcal{H}_\epsilon)$. The annihilation and creation operator fulfill

$$
\left[a_\epsilon(f), a_\epsilon^\dagger(g)\right] = \langle f, g\rangle_\epsilon^{\mathbb{C}} \quad \forall f, g \in \mathcal{H}_\epsilon \quad \text{and} \tag{3.1.14}
$$

$$
a_\epsilon(f)\,|0_\epsilon\rangle = 0 \qquad \forall f \in \mathcal{H}_\epsilon\,. \tag{3.1.15}
$$

Since we have

$$
\left[a_\epsilon(f), a_\epsilon^\dagger(g)\right] \overset{(3.1.14)}{=} \langle f, g\rangle_\epsilon^{\mathbb{C}} = \langle f, g\rangle_\epsilon
$$

whenever $f, g \in C_c^\infty(\mathbb{R}^{d+1})$, we omit a distinction between the annihilation and creation operator of $\mathcal{F}(\mathcal{H}_\epsilon)$ and $a_\epsilon(f)$ and $a_\epsilon^\dagger(f)$ used in (3.1.9).

Remark. Using Equation (3.1.9) as a definition together with [BR97, Proposition 5.2.3] gives an abstract argument for why $\psi_\epsilon(f)$ is essentially self-adjoint and has (2.3.6) as commutator relation.

3.1.2. Limit of vacuum expectation values

Now that we have constructed Fock spaces on which we can represent ψ_ϵ, a very natural question that arises is what the vacuum expectation value of $W_\epsilon(f)$ is and what happens in the limit as $\epsilon \to 0$.

So what we want to compute now is $\langle 0_\epsilon|\, W_\epsilon(f)\,|0_\epsilon\rangle$. Remembering the power series for the exponential we conclude that we need to compute $\langle 0_\epsilon|\, \psi_\epsilon^n(f)\,|0_\epsilon\rangle$, for $n \in \mathbb{N}$, as a first step. For this we require Wick's theorem.[1]

[1] To be precise, we use here is a slightly different version of Wick's theorem. The original is written with fields and time-ordering. However, the proof of the version we use here follows the exact same way as in [Wic50], as this is essentially a combinatorics problem.

The theorem states that for $A_1, A_2, \cdots, A_n$ operators on a Fock space, where each operator A_i is made up of sums and products of annihilaton and creation operators[2], we can rewrite their product as

$$
\begin{aligned}
A_1 A_2 A_3 \cdots A_n =&: A_1 A_2 A_3 \cdots A_n : + \sum_{\text{1 contraction}} : \wick{\c1 A_1 \c1 A_2} A_3 \cdots A_n : \\
&+ \sum_{\text{2 contractions}} : \wick{\c1 A_1 \c2 A_2 \c1 A_3 \cdots \c2 A_n} : + \cdots + \sum_{\lfloor \frac{n}{2} \rfloor \text{ contractions}} : \wick{\c1 A_1 \c2 A_2 \c1 A_3 \cdots \c2 A_n} :
\end{aligned}
$$

where a *contraction* of two operators is defined as

$$
\wick{\c1 A \c1 B} := AB - : AB : \tag{3.1.16}
$$

with $:(\,.\,):$ being the *normal ordering*, which in turn is defined as a linear map from operators to operators such that "all creation operators are to the left of all annihilation operators". As examples we show four simple cases.

$$
: a_i a_j : = a_i a_j, \quad : a_i^\dagger a_j : = a_i^\dagger a_j, \quad : a_i a_j^\dagger : = a_j^\dagger a_i, \quad : a_i^\dagger a_j^\dagger : = a_i^\dagger a_j^\dagger . \tag{3.1.17}
$$

We want to compute $\psi_\epsilon^n(f)$ using Wick's theorem for which we need to compute the contraction $\wick{\c1 \psi_\epsilon(f) \c1 \psi_\epsilon(f)}$. This can be straightforwardly calculated. Let $f \in V_\epsilon$. Then

$$
\begin{aligned}
\wick{\c1 \psi_\epsilon(f) \c1 \psi_\epsilon(f)} &\overset{(3.1.16)}{=} \psi_\epsilon(f)^2 - : \psi_\epsilon(f)^2 : \\
&\overset{(3.1.9)}{=} \frac{1}{2}(a_\epsilon(f) + a_\epsilon^\dagger(f))(a_\epsilon(f) + a_\epsilon^\dagger(f)) - \frac{1}{2} : (a_\epsilon(f) + a_\epsilon^\dagger(f))(a_\epsilon(f) + a_\epsilon^\dagger(f)) : \\
&= \frac{1}{2}(a_\epsilon(f)^2 + a_\epsilon(f) a_\epsilon^\dagger(f) + a_\epsilon^\dagger(f) a_\epsilon(f) + a_\epsilon^\dagger(f)^2) \\
&\quad - \frac{1}{2} : (a_\epsilon(f)^2 + a_\epsilon(f) a_\epsilon^\dagger(f) + a_\epsilon^\dagger(f) a_\epsilon(f) + a_\epsilon^\dagger(f)^2) : \\
&\overset{(3.1.17)}{=} \frac{1}{2}(a_\epsilon(f)^2 + a_\epsilon(f) a_\epsilon^\dagger(f) + a_\epsilon^\dagger(f) a_\epsilon(f) + a_\epsilon^\dagger(f)^2) \\
&\quad - \frac{1}{2}(a_\epsilon(f)^2 + 2 a_\epsilon^\dagger(f) a_\epsilon(f) + a_\epsilon^\dagger(f)^2) \\
&= \frac{1}{2}(a_\epsilon(f) a_\epsilon^\dagger(f) - a_\epsilon^\dagger(f) a_\epsilon(f)) = \frac{1}{2}\left[a_\epsilon(f), a_\epsilon^\dagger(f)\right] \\
&\overset{(3.1.13)}{=} \frac{1}{2} \langle f, f \rangle_\epsilon = \frac{1}{2} \| f \|_\epsilon^2 .
\end{aligned} \tag{3.1.18}
$$

An important point to note here is that $\wick{\c1 \psi_\epsilon(f) \c1 \psi_\epsilon(f)} \propto \mathbb{1}$.

[2]By the construction of the Fock space, the annihilation and creation operators all have the same dense domain and map said domain to itself [Hö24, 5.14-5.15]. Hence we have no problem taking sums and products of annihilation and creation operators.

Applying Wick's theorem to $\psi_\epsilon(f)^n$ now yields

$$\begin{aligned}
\psi_\epsilon(f)^n =&: \psi_\epsilon(f)^n : + \frac{n(n-1)}{2}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right) : \psi_\epsilon(f)^{n-2} : \\
&+ \frac{1}{2!}\frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^2 : \psi_\epsilon(f)^{n-4} : + \cdots \\
&+ \frac{1}{\left\lfloor\frac{n}{2}\right\rfloor !}\frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}\cdots\frac{(n-2\left\lfloor\frac{n}{2}\right\rfloor+2)(n-2\left\lfloor\frac{n}{2}\right\rfloor+1)}{2} \\
&\cdot\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^{\left\lfloor\frac{n}{2}\right\rfloor} : \psi_\epsilon(f)^{n-2\left\lfloor\frac{n}{2}\right\rfloor} : \\
=& \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{1}{k!}\frac{n!}{2^k(n-2k)!}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^k : \psi_\epsilon(f)^{n-2k} : . \qquad (3.1.19)
\end{aligned}$$

We can pull any power of $\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)$ out of the normal ordering precisely because $\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f) \propto \mathbb{1}$.
Equation (3.1.15) in particular implies $a_\epsilon(f)\left|0_\epsilon\right\rangle = 0$ and $\left\langle 0_\epsilon\right| a_\epsilon^\dagger(f) = 0$ for all $f \in V_\epsilon$. It follows then that $\left\langle 0_\epsilon\right| : \psi_\epsilon(f)^k : \left|0_\epsilon\right\rangle = 0$ for all $k \geq 1$ since $\psi_\epsilon(f)$, and therefore $: \psi_\epsilon(f)^k :$, has no term without either $a_\epsilon(f)$ or $a_\epsilon^\dagger(f)$.
Differentiating between even and odd powers of $\left\langle 0_\epsilon\right| \psi_\epsilon(f)^n \left|0_\epsilon\right\rangle$ we get

$$\begin{aligned}
\left\langle 0_\epsilon\right| \psi_\epsilon(f)^{2n+1} \left|0_\epsilon\right\rangle &\overset{(3.1.19)}{=} \sum_{k=0}^{n}\frac{1}{k!}\frac{(2n+1)!}{2^k(2n+1-2k)!}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^k \left\langle 0_\epsilon\right| : \psi_\epsilon(f)^{2n+1-2k} : \left|0_\epsilon\right\rangle \\
&= 0\,, \qquad (3.1.20)
\end{aligned}$$

since $n-2k \geq 1$ for all $k = 0, \ldots, \left\lfloor\frac{n}{2}\right\rfloor$, and similarly

$$\begin{aligned}
\left\langle 0_\epsilon\right| \psi_\epsilon(f)^{2n} \left|0_\epsilon\right\rangle &\overset{(3.1.19)}{=} \sum_{k=0}^{n}\frac{1}{k!}\frac{(2n)!}{2^k(2n-2k)!}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^k \left\langle 0_\epsilon\right| : \psi_\epsilon(f)^{2n-2k} : \left|0_\epsilon\right\rangle \\
&= \frac{1}{n!}\frac{(2n)!}{2^n(2n-2n)!}\left(\overbrace{\psi_\epsilon(f)\psi_\epsilon}(f)\right)^n \left\langle 0_\epsilon\right| : \psi_\epsilon(f)^{2n-2n} : \left|0_\epsilon\right\rangle \\
&\overset{(3.1.18)}{=} \frac{1}{n!}\frac{(2n)!}{2^n}\left(\frac{1}{2}\left\|f\right\|_\epsilon^2\right)^n = \frac{(2n)!}{n!}\left(\frac{1}{4}\left\|f\right\|_\epsilon^2\right)^n . \qquad (3.1.21)
\end{aligned}$$

This now allows us to compute the vacuum expectation value of $W_\epsilon(f)$.

$$\begin{aligned}
\left\langle 0_\epsilon\right| W_\epsilon(f) \left|0_\epsilon\right\rangle &= \left\langle 0_\epsilon\right| e^{i\psi_\epsilon(f)} \left|0_\epsilon\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}i^n \left\langle 0_\epsilon\right| \psi_\epsilon(f)^n \left|0_\epsilon\right\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{(2n)!}i^{2n} \left\langle 0_\epsilon\right| \psi_\epsilon(f)^{2n} \left|0_\epsilon\right\rangle + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}i^{2n+1} \left\langle 0_\epsilon\right| \psi_\epsilon(f)^{2n+1} \left|0_\epsilon\right\rangle
\end{aligned}$$

$$\overset{(3.1.20)+(3.1.21)}{=} \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n \frac{(2n)!}{n!} \left(\frac{1}{4} \|f\|_\epsilon^2\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{4} \|f\|_\epsilon^2\right)^n$$
$$= e^{-\frac{1}{4}\|f\|_\epsilon^2} . \tag{3.1.22}$$

Finally we can define the linear functional $\rho_0 : \mathcal{W}_0 \longrightarrow \mathbb{C}$ by

$$\rho_0(W_0(f)) := \lim_{\epsilon \to 0} \langle 0_\epsilon | \, W_\epsilon(f) \, | 0_\epsilon \rangle \tag{3.1.23}$$

for all $f \in V_0 = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker} \sigma_0$. We now need to show that ρ_0 really is a well-defined state on the C^*-algebra $\mathcal{W}_0$, i.e, positive and normalized.
The first question is whether the limit exists. Since

$$\rho_0(W_0(f)) \overset{(3.1.23)}{=} \lim_{\epsilon \to 0} \langle 0_\epsilon | \, W_\epsilon(f) \, | 0_\epsilon \rangle \overset{(3.1.22)}{=} \lim_{\epsilon \to 0} e^{-\frac{1}{4}\|f\|_\epsilon^2} , \tag{3.1.24}$$

we need to understand what exactly happens with $\|f\|_\epsilon^2$ as $\epsilon \to 0$.

$$\|f\|_\epsilon^2 = \langle f, f \rangle_\epsilon \overset{(2.3.1)}{=} \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left| \hat{f}(E_{\epsilon,p}, p) \right|^2$$

Since $x^\mu \mapsto |g(x^\mu)|^2 = g(x^\mu)\overline{g(x^\mu)}$ is smooth if g is smooth, we can apply Taylor's theorem to $\left|\hat{f}(E_{\epsilon,p}, p)\right|$, which yields

$$\begin{aligned}
\|f\|_\epsilon^2 &= \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left| \hat{f}(E_{\epsilon,p}, p) \right|^2 \\
&= \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left(\left|\hat{f}(0,p)\right|^2 + \partial_{p_0} \left[\left| \hat{f}(p_0,p) \right|^2 \right] \Big|_{p_0=0} E_{\epsilon,p} + \mathcal{O}(E_{\epsilon,p}^2) \right) \\
&= \int \frac{d^d p}{(2\pi)^d E_{\epsilon,p}} \left|\hat{f}(0,p)\right|^2 + \int \frac{d^d p}{(2\pi)^d} \left(\partial_{p_0} \hat{f}(0,p) \overline{\hat{f}(0,p)} + \hat{f}(0,p) \partial_{p_0} \overline{\hat{f}(0,p)} + \mathcal{O}(E_{\epsilon,p}) \right) \\
&\overset{p \mapsto -p}{=} \frac{1}{\epsilon} \int \frac{d^d p}{(2\pi)^d E_p} \left|\hat{f}(0,p)\right|^2 + \int \frac{d^d p}{(2\pi)^d} \left(\partial_{p_0} \hat{f}(0,p) \overline{\hat{f}(0,p)} + \hat{f}(0,-p) \underbrace{\partial_{p_0} \overline{\hat{f}(0,-p)}}_{\partial_{p_0} \overline{\hat{f}(p_0,-p)}\big|_{p_0=0}} \right) \\
&\quad + \int \frac{d^d p}{(2\pi)^d} \mathcal{O}(E_{\epsilon,p}) \\
&= \frac{1}{\epsilon} \int \frac{d^d p}{(2\pi)^d E_p} \left|\hat{f}(0,p)\right|^2 + \int \frac{d^d p}{(2\pi)^d} \left(\partial_{p_0} \hat{f}(0,p) \overline{\hat{f}(0,p)} + \overline{\hat{f}(0,p)} \, \partial_{p_0} \hat{f}(-p_0,p) \Big|_{p_0=0} \right) \\
&\quad + \int \frac{d^d p}{(2\pi)^d} \mathcal{O}(E_{\epsilon,p}) \\
&= \frac{1}{\epsilon} \int \frac{d^d p}{(2\pi)^d E_p} \left|\hat{f}(0,p)\right|^2 + \int \frac{d^d p}{(2\pi)^d} \left(\partial_{p_0} \hat{f}(0,p) \overline{\hat{f}(0,p)} - \overline{\hat{f}(0,p)} \partial_{p_0} \hat{f}(0,p) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\int \frac{d^dp}{(2\pi)^d}\,\mathcal{O}(E_{\epsilon,p}) \\
&=\frac{1}{\epsilon}\int \frac{d^dp}{(2\pi)^d E_p}\,\left|\hat{f}(0,p)\right|^2+\int \frac{d^dp}{(2\pi)^d}\,\mathcal{O}(E_{\epsilon,p})\,.
\end{aligned}
$$

The integral $\int \frac{d^dp}{(2\pi)^d}\,\mathcal{O}(E_{\epsilon,p})$ converges to 0, since $E_{\epsilon,p}\to 0$ as $\epsilon\to 0$[3]. The only divergent part is $\frac{1}{\epsilon}$ in front of the first integral. So the only way in which $\|f\|_\epsilon^2$ does not diverge is, if $\int \frac{d^dp}{(2\pi)^d E_p}\,|\hat{f}(0,p)|^2=0$. This however is only possible if $\hat{f}(0,p)=0$ for all $p\in\mathbb{R}^d$ by continuity and since $\frac{1}{(2\pi)^d E_p}\,|\hat{f}(0,p)|^2\geq 0$ for all $p\in\mathbb{R}^d$. Hence, we conclude that

$$
\lim_{\epsilon\to 0}\|f\|_\epsilon^2=\begin{cases} 0 & \text{if } \hat{f}(0,\,.\,)=0\,,\\ \infty & \text{else}\,.\end{cases}
$$

But this result immediately translates to

$$
\rho_0(W_0(f))=\lim_{\epsilon\to 0}e^{-\frac{1}{4}\|f\|_\epsilon^2}=\begin{cases} 1 & \text{if } \hat{f}(0,\,.\,)=0\,,\\ 0 & \text{else}\end{cases} \tag{3.1.25}
$$

by the continuity of the exponential function.

The next question is whether ρ_0 is well-defined. So we need to show that $\rho_0(W_0(f+h))=\rho_0(W_0(f))$ whenever $h\in\operatorname{Ker}\sigma_0$. By our evaluation of ρ_0 in Equation (3.1.25) this is equivalent to showing that $\hat{h}(0,\,.\,)=0$ if $h\in\operatorname{Ker}\sigma_0$. Now, let $h\in\operatorname{Ker}\sigma_0$. This means that

$$
\sigma_0(h,g)\overset{(2.4.1)}{=}\frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{h}(0,p)\overline{\hat{g}(0,p)}-\overline{\hat{h}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right)=0
$$

for all $g\in C_c^\infty(\mathbb{R}^{d+1})$. We set $g(t,x)=i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t h(0,x)$, where

$$
\varphi(t)=\frac{\tilde{\varphi}(t)}{\int\tilde{\varphi}(s)\,ds} \tag{3.1.26}
$$

and $\tilde{\varphi}:\mathbb{R}\to\mathbb{R}$ is any bump-function, i.e., an even, smooth function with compact support such that $\tilde{\varphi}|_{B_1(0)}=1$ and $\operatorname{Im}\tilde{\varphi}\subseteq[0,1]$. g is indeed an element of $C_c^\infty(\mathbb{R}^{d+1})$ since

$$
\begin{aligned}
g(t,x)&=i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t h(0,x)=\frac{\partial}{\partial t}\mathcal{F}_t^{-1}[\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t h(0,x)\\
&=\varphi'(t)\mathcal{F}_t h(0,x)=\varphi'(t)\int h(s,x)\,ds
\end{aligned}
$$

and everything in the last line is real-valued and has compact support.

[3]Some caution is advised here. The integral $\int\frac{d^dp}{(2\pi)^d}\,E_{\epsilon,p}$ diverges, yet in our case every term in $\mathcal{O}(E_{\epsilon,p})$ contains some derivative of $\hat{f}(0,p)$ (and $\overline{\hat{f}(0,p)}$). Since f is Schwartz so is $\hat{f}$, which implies that $E_{\epsilon,p}$ multiplied by any derivative of $\hat{f}(0,p)$, and therefore also $\mathcal{O}(E_{\epsilon,p})$, is integrable.

To go on further, we need to show some properties of φ.

$$\mathcal{F}_t\varphi(0) = \int \varphi(t)\,dt = \frac{\int \tilde{\varphi}(t)\,dt}{\int \tilde{\varphi}(s)\,ds} = 1 \tag{3.1.27}$$

and, although not needed now but important for later,

$$\begin{aligned} \frac{d}{dp_0}\mathcal{F}_t\varphi(0) &= \frac{d}{dp_0}\Big|_{p_0=0} \int \varphi(t)\,e^{-ip_0 t}\,dt = -i\int t\varphi(t)\,e^{-ip_0 t}\,dt\Big|_{p_0=0} \\ &= -i\int t\varphi(t)\,dt = 0\,, \end{aligned} \tag{3.1.28}$$

where we used the even-ness of φ. Calculating the Fourier transform of g yields

$$\begin{aligned} \mathcal{F}g(p_0,p) &= \mathcal{F}\left(i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t h(0,x)\right)(p_0,p) \\ &= \mathcal{F}_t\mathcal{F}_x\left(i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t h(0,x)\right)(p_0,p) \\ &= \mathcal{F}_t\left[i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\right](p_0)\mathcal{F}_x\left[\mathcal{F}_t h(0,x)\right](p) \\ &= ip_0\,\mathcal{F}_t\varphi(p_0)\hat{h}(0,p)\,, \end{aligned}$$

which implies

$$\hat{g}(0,p) = 0 \quad \text{and} \tag{3.1.29}$$

$$\begin{aligned} \frac{\partial}{\partial p_0}\hat{g}(0,p) &= \frac{\partial}{\partial p_0}\left[ip_0\,\mathcal{F}_t\varphi(p_0)\hat{h}(0,p)\right]\Big|_{p_0=0} \\ &= i\left[\mathcal{F}_t\varphi(p_0) + p_0\frac{\partial}{\partial p_0}\mathcal{F}_t\varphi(p_0)\right]\Big|_{p_0=0}\hat{h}(0,p) \\ &\overset{(3.1.27)}{=} i\hat{h}(0,p) \end{aligned} \tag{3.1.30}$$

for any $p \in \mathbb{R}^d$. Hence

$$\begin{aligned} 0 = \sigma_0(h,g) &\overset{(2.4.1)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{h}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{h}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right) \\ &\overset{(3.1.29)+(3.1.30)}{=} \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\left|\hat{h}(0,p)\right|^2\,. \end{aligned} \tag{3.1.31}$$

By the non-degeneracy of the L^2-inner product we get that $\hat{h}(0,p) = 0$ for all $p \in \mathbb{R}^d$, which is what we wanted to show.

ρ_0 being normalized follows from the Weyl-relations since

$$\rho_0(\mathbb{1}) = \rho_0(W_0(0)) = 1\,.$$

So the last question is whether ρ_0 is positive, i.e., $\rho_0(A) \geq 0$ whenever $A \geq 0$.
First up, consider the relation $f \sim g :\Longleftrightarrow \hat{f}(0, .) = \hat{g}(0, .)$ on V_0. It is easy to see that this is an equivalence relation on V_0 and it is well-defined, since for $f', g' \in \operatorname{Ker}\sigma_0$ we have by (3.1.31) that $\widehat{f'}(0, .) = \widehat{g'}(0, .) = 0$ and therefore $f + f' \sim g + g' \iff f \sim g$. We denote the equivalence class of f with respect to $\sim$ as $[f] \in V_0/_\sim$ and a representative of that equivalence class as $g \in [f]$.
By Definition 1.2.2, any element A of a C^*-algebra is positive if and only if it can be written as $A = C^*C$ for some C. Hence the question becomes if $\rho_0(C^*C) \geq 0$ for all $C \in \mathcal{W}_0$. Any element in the Weyl-algebra $\mathcal{W}_0$ can be written as $C = \sum_{f\in V_0} c_f W_0(f)$, with $c_f \in \mathbb{C}\,\forall f \in V_0$ and $\sum_{f\in V_0} |c_f| < \infty$. Hence

$$
\begin{aligned}
\rho_0(C^*C) &= \rho_0\left(\left(\sum_{f\in V_0} c_f W_0(f)\right)^* \left(\sum_{g\in V_0} c_g W_0(g)\right)\right) = \rho_0\left(\sum_{f,g\in V_0} \overline{c_f} c_g W_0(-f) W_0(g)\right)\\
&= \sum_{f,g\in V_0} \overline{c_f} c_g e^{-i\sigma_0(f,g)} \rho_0\left(W_0(g-f)\right) = \sum_{\substack{f,g\in V_0\\ \hat{f}(0,.)-\hat{g}(0,.)=0}} \overline{c_f} c_g e^{-i\sigma_0(f,g)}\\
&= \sum_{[h]\in V_0/_\sim} \sum_{f,g\in[h]} \overline{c_f} c_g e^{-i\sigma_0(f,g)}\,.
\end{aligned}
$$

We can now rewrite $\sigma_0(f,g)$ as

$$
\begin{aligned}
\sigma_0(f,g) &= \sigma_0((f-h)+h,(g-h)+h) = \sigma_0(f-h,g-h) + \sigma_0(f-h,h) + \sigma_0(h,g-h) + \sigma_0(h,h)\\
&= \sigma_0(f,h) - \sigma_0(g,h)\,,
\end{aligned}
$$

where we mostly used bilinearity and anti-symmetry of σ_0 and the fact that by $f, g \in [h]$ the functions $f' := f - h$ and $g' := g - h$ satisfy $\widehat{f'}(0, .) = \hat{f}(0, .) - \hat{h}(0, .) = 0$ similarly $\widehat{g'}(0, .) = 0$, which implies

$$
\sigma_0(f',g') \overset{(2.4.1)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\widehat{f'}(0,p)\overline{\widehat{g'}(0,p)} - \overline{\widehat{f'}(0,p)}\frac{\partial}{\partial p_0}\widehat{g'}(0,p)\right) = 0\,.
$$

So we can further write

$$
\begin{aligned}
\rho_0(C^*C) &= \sum_{[h]\in V_0/_\sim} \sum_{f,g\in[h]} \overline{c_f} c_g e^{-i\sigma_0(f,g)} = \sum_{[h]\in V_0/_\sim} \sum_{f,g\in[h]} \overline{c_f} c_g e^{-i\sigma_0(f,h)} e^{i\sigma_0(g,h)}\\
&= \sum_{[h]\in V_0/_\sim} \sum_{f,g\in[h]} \overline{c_f e^{i\sigma_0(f,h)}} c_g e^{i\sigma_0(g,h)} \sum_{[h]\in V_0/_\sim} \overline{\left(\sum_{f\in[h]} c_f e^{i\sigma_0(f,h)}\right)} \left(\sum_{g\in[h]} c_g e^{i\sigma_0(g,h)}\right)\\
&\geq 0\,.
\end{aligned}
$$

Therefore ρ_0 truly is a state on $\mathcal{W}_0$.
We refer to ρ_0 as the *electric Carrollian limit state.*

We can even go one step further and show that ρ_0 is a pure state.
Assume we have states ρ_1 and ρ_2 on $\mathcal{W}_0$ such that $\rho_0 = \lambda\rho_1 + (1-\lambda)\rho_2$ for some $\lambda \in (0,1)$. What we want to show is that $\rho_1 = \rho_2 = \rho_0$ which by linearity and the Weyl-relations means that we only need to check if they agree on every generating element $W_0(f)$.
So assume $f \in V_0$ with $\hat{f}(0,\,.\,) = 0$. We have

$$1 = \rho_0(W_0(f)) = \lambda\rho_1(W_0(f)) + (1-\lambda)\rho_2(W_0(f))\,.$$

Writing $\rho_1(W_0(f)) = a + ib$ and $\rho_2(W_0(f)) = c + id$, with $a,b,c,d \in \mathbb{R}$, we get two equations,

$$\lambda a + (1-\lambda)c = 1 \tag{3.1.32}$$

$$\lambda b + (1-\lambda)d = 0\,. \tag{3.1.33}$$

Since the absolute value of every state evaluated on a unitary element is bounded by 1[4] we find that $a \leq |\rho_1(W_0(f))| \leq 1$ and similarly $c \leq 1$. But then (3.1.32) implies $a = c = 1$, otherwise there is a contradiction to $1 = \lambda a + (1-\lambda)c < \lambda + (1-\lambda) = 1$. But then $|\rho_1(W_0(f))| = a^2 + b^2 \leq 1$ implies $b = 0$ and similarly $d = 0$. Hence $\rho_1(W_0(f)) = \rho_2(W_0(f)) = \rho_0(W_0(f)) = 1$.
Now, let $f \in V_0$ with $\hat{f}(0,\,.\,) \neq 0$. For any element $A \in \mathcal{W}_0$ and $g \in V_0$ with $\hat{g}(0,\,.\,) = 0$ the Cauchy-Schwarz-inequality for states (Proposition 1.2.6) implies

$$|\rho_1(A(W_0(g) - \mathbb{1}))|^2 \leq \rho_1(A^*A)\rho_1((W_0(g) - \mathbb{1})^*(W_0(g) - \mathbb{1})) = 0\,,$$

since $\rho_1((W_0(g) - \mathbb{1})^*(W_0(g) - \mathbb{1})) = \rho_1(2 - W_0(-g) - W_0(g)) = 2 - \rho_1(W_0(-g)) - \rho_1(W_0(g)) = 0$. Hence $\rho_1(A(W_0(g) - \mathbb{1})) = 0$ and $\rho_1(AW_0(g)) = \rho_1(A)$ for all $A \in \mathcal{W}_0$. Using that property twice yields

$$\begin{aligned} \rho_1(W_0(f)) = \rho_1(W_0(g)W_0(f)W_0(-g)) &= e^{i\sigma_0(f,g)}\rho_1(W_0(g)W_0(f-g)) \\ &= e^{i2\sigma_0(f,g)}\rho_1(W_0(f))\,. \end{aligned} \tag{3.1.34}$$

We can now choose a $g \in V_0$ with $\hat{g}(0,\,.\,) = 0$ and $\sigma_0(f,g) \neq 0$. Indeed, analogously to the calculation for the well-definedness of ρ_0, $g(t,x) = i\mathcal{F}_t^{-1}[p_0\,\mathcal{F}_t\varphi(p_0)](t)\mathcal{F}_t f(0,x)$ has $\hat{g}(0,p) = 0$ and $\frac{\partial}{\partial p_0}\hat{g}(0,p) = i\hat{f}(0,p)$ which further yields

$$\begin{aligned} \sigma_0(f,g) \overset{(2.4.1)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\left|\hat{f}(0,p)\right|^2 \\ \neq 0\,. \end{aligned}$$

Hence, by possibly rescaling g we get $e^{i2\sigma_0(f,g)} \neq 1$ and therefore by (3.1.34) we must have $\rho_1(W_0(f)) = 0$. An analogous computation for ρ_2 then implies $\rho_1(W_0(f)) = \rho_2(W_0(f)) = \rho_0(W_0(f)) = 0$.
Thus ρ_1, ρ_2 and ρ_0 agree on every $W_0(f)$, hence on every element in $\mathcal{W}_0$, which makes ρ_0 by Definition 1.2.5 pure.

---

[4] For any state ρ and any unitary element U we have, by the Cauchy-Schwarz-inequality for states (Proposition 1.2.6), $|\rho(U)|^2 = |\rho(\mathbb{1}^*U)|^2 \leq \rho(\mathbb{1}^*\mathbb{1})\rho(U^*U) = \rho(\mathbb{1})^2 = 1$.

Remark. The computation for positivity and purity of ρ_0 follows [Fre13, 2.4.2], which contains a more general discussion of states ω_K on $\mathcal{W}(V,\sigma)$ of the form

$$\omega_K(W(x)) = \begin{cases} 1 & \text{for } x \in K \\ 0 & \text{for } x \notin K \end{cases},$$

where K is maximal subset of V such that $\sigma(x,y) = 0\,\forall x,y \in K$. In our case, $K = \left\{ f \in V_0 \mid \hat{f}(0,\,.\,) = 0 \right\}$, which is indeed maximal.
We chose to do the calculation explicitly in order to give a more physically relevant intuition of the state than referencing some abstract result.

By the GNS-representation (Theorem 1.2.7) we now get a Hilbert space $\mathcal{H}_{\rho_0}$ and a cyclic representation $\pi_{\rho_0} : \mathcal{W}_0 \to \mathcal{B}(\mathcal{H}_{\rho_0})$ with cyclic vector x_{ρ_0} such that $\rho_0(A) = \langle x_{\rho_0}, \pi_{\rho_0}(A)x_{\rho_0}\rangle$. This means that we finally reconnected $\mathcal{W}_0$ to a more common notion, in physics terms.
Moreover, since ρ_0 is pure and by Theorem 1.2.8, we also find that π_{ρ_0} is even an irreducible representation.

3.2. Symmetries of the Limit-State

Usually in a quantum field theory, we have a symmetry group of the underlying space-time, which is implemented by a unitary representation on the Hilbert space of the theory. For example, in the Wightman axioms the Poincaré group is represented as a family of strongly continuous unitary operators. We want to show now that the Carroll group [LL65, SG66] is similarly implemented.
The main mathematical backbone for this section will be the following theorem.

Theorem 3.2.1 (Unitary representation of automorphisms)
Let α be a $*$-automorphism of a unital $*$-algebra $\mathcal{A}$. If a state ω on $\mathcal{A}$ is invariant under α, i.e., $\alpha^*\omega = \omega$, then α is implemented in the GNS representation of ω by a unitary operator U_α on $\mathcal{H}_\omega$ such that U_α leaves the GNS vector invariant. Any group of $*$-automorphsims leaving ω invariant is unitarly represented on $\mathcal{H}_\omega$.[FR19, Theorem 23]

In our case this shifts the question to which $*$-automorphisms leave the state ρ_0 invariant. We will refer to $*$-automorphisms, which leave ρ_0 invariant as *symmetries of* ρ_0.

We consider a special class of $*$-automorphism on $\mathcal{W}_0$. Let $\alpha : C_c^\infty(\mathbb{R}^{d+1}) \to C_c^\infty(\mathbb{R}^{d+1})$ be an automorphism. Then we define the multiplicative linear map $\tilde{\alpha} : \mathcal{W}_0 \to \mathcal{W}_0$ by $\tilde{\alpha}(W_0(f)) := W_0(\alpha(f))$. We now try to find out under which conditions $\tilde{\alpha}$ is a well-defined $*$-automorphism.
We begin by looking at the multiplicativity of $\tilde{\alpha}$. Let $f,g \in C_c^\infty(\mathbb{R}^{d+1})$. On the one hand we have

$$\tilde{\alpha}(W_0(f)W_0(g)) = e^{i\sigma_0(f,g)}\tilde{\alpha}(W_0(f+g)) = e^{i\sigma_0(f,g)}\mathcal{W}_0(\alpha(f+g)) = e^{i\sigma_0(f,g)}\mathcal{W}_0(\alpha(f)+\alpha(g))$$

and on the other hand we find

$$\begin{aligned}\tilde{\alpha}(W_0(f)W_0(g)) &= \tilde{\alpha}(W_0(f))\tilde{\alpha}(W_0(g)) = W_0(\alpha(f))W_0(\alpha(g)) \\ &= e^{i\sigma_0(\alpha(f),\alpha(g))}\mathcal{W}_0(\alpha(f)+\alpha(g))\,.\end{aligned}$$

Therefore, for $\tilde{\alpha}$ to be an $*$-automorphism, we require

$$\sigma_0(\alpha(f),\alpha(g)) = \sigma_0(f,g) \quad \forall f,g \in C_c^\infty(\mathbb{R}^{d+1})\,. \tag{3.2.1}$$

This condition has an additional benefit. It implies that $\alpha(h) \in \operatorname{Ker}\sigma_0$ if $h \in \operatorname{Ker}\sigma_0$. Indeed, since α is an automorphism it has an inverse α^{-1}. It follows then that

$$\sigma_0(\alpha(h),g) = \sigma_0(\alpha(h),\alpha(\alpha^{-1}(g))) \overset{(3.2.1)}{=} \sigma_0(h,\alpha^{-1}(g)) = 0 \quad \forall g \in C_c^\infty(\mathbb{R}^{d+1})\,,$$

provided $h \in \operatorname{Ker}\sigma_0$. In turn this implies that $\tilde{\alpha}$ is well-defined, as we have

$$\tilde{\alpha}(W_0(f+h)) = W_0(\alpha(f)+\alpha(h)) = e^{i\sigma_0(\alpha(f),\alpha(h))}W_0(\alpha(f))W_0(\alpha(h)) = W_0(\alpha(f))\,.$$

Compatibility with the adjoint follows from linearity of α and the Weyl-relations.

$$\tilde{\alpha}(W_0(f)^*) = \tilde{\alpha}(W_0(-f)) = W_0(\alpha(-f)) = W_0(-\alpha(f)) = W_0(\alpha(f))^* = (\tilde{\alpha}(W_0(f)))^*\,.$$

Hence the only condition for $\tilde{\alpha}$ to be a well-defined $*$-homomorphism is Equation (3.2.1). Bijectivity is then implied by the existence of an inverse, which we can explicitly give with $\tilde{\alpha}^{-1} = \widetilde{\alpha^{-1}}$. However, we need to make sure that the analog of (3.2.1) for α^{-1} holds as well. But this holds indeed, since

$$\sigma_0(\alpha^{-1}(f),\alpha^{-1}(g)) \overset{(3.2.1)}{=} \sigma_0(\alpha(\alpha^{-1}(f)),\alpha(\alpha^{-1}(g))) = \sigma_0(f,g) \quad \forall f,g \in C_c^\infty(\mathbb{R}^{d+1})\,.$$

Hence $\tilde{\alpha}$ is indeed an $*$-automorphism, given (3.2.1) holds.

If ρ_0 should be invariant under $\tilde{\alpha}$, i.e.,

$$\tilde{\alpha}^*\rho_0(W_0(f)) = \rho_0(\tilde{\alpha}(W_0(f))) = \rho_0(W_0(\alpha(f))) = \rho_0(W_0(f))$$

then, by Equation (3.1.25) we find the condition

$$\widehat{\alpha(f)}(0,\,.\,) = 0 \iff \hat{f}(0,\,.\,) = 0 \quad \forall f \in C_c^\infty(\mathbb{R}^{d+1})\,. \tag{3.2.2}$$

Therefore, for any automorphism $\alpha : C_c^\infty(\mathbb{R}^{d+1}) \to C_c^\infty(\mathbb{R}^{d+1})$ which satisfies (3.2.1) and (3.2.2), we have found a symmetry of the state ρ_0.

3.2.1. Carroll symmetries

We go on to consider an even more specific case. Let $\alpha : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be a diffeomorphism, i.e., a smooth bijection with smooth inverse. We claim that the pull back $\alpha^* f := f \circ \alpha$ defines an automorphism on $C_c^\infty(\mathbb{R}^{d+1})$.
First, α^* is linear since precomposition is linear. Next, we need to show that $\alpha^* f \in C_c^\infty(\mathbb{R}^{d+1})$ whenever $f \in C_c^\infty(\mathbb{R}^{d+1})$. By the smoothness of α we get smoothness of $\alpha^* f$ and it is easy to see that $\operatorname{supp} \alpha^* f = \operatorname{supp} f$. Hence $\alpha^* f \in C_c^\infty(\mathbb{R}^{d+1})$. Lastly, we need to show that $\alpha^* : C_c^\infty(\mathbb{R}^{d+1}) \to C_c^\infty(\mathbb{R}^{d+1})$ is bijective. Since α^{-1} exists by assumption and $(\alpha^*)^{-1} = (\alpha^{-1})^*$ we have explicitly constructed an inverse. Hence α^* is an automorphism on $C_c^\infty(\mathbb{R}^{d+1})$.

One example of such a diffeomorphism is given by the following map. Let $h \in C^\infty(\mathbb{R}^d)$, $R \in SO(d)$ and $a \in \mathbb{R}^d$. We then define

$$\alpha_{h,R,a} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} : (t,x) \mapsto (t + h(x), Rx + a)\,. \tag{3.2.3}$$

This is a smooth map since it can be seen as a composition of smooth maps. The inverse is then given by $\alpha_{\tilde{h},R^{-1},-R^{-1}a}$, with $\tilde{h}(x) := -h(R^{-1}(x-a))$. Indeed

$$\begin{aligned}
\alpha_{\tilde{h},R^{-1},-R^{-1}a} \circ \alpha_{h,R,a}(t,x) &= \alpha_{\tilde{h},R^{-1},-R^{-1}a}(t + h(x), Rx + a) \\
&= (t + h(x) + \tilde{h}(Rx + a), R^{-1}(Rx + a) - R^{-1}a) \\
&= (t + h(x) - h(R^{-1}((Rx + a) - a), R^{-1}Rx + R^{-1}a - R^{-1}a) = (t,x)
\end{aligned}$$

and similarly for $\alpha_{h,R,a} \circ \alpha_{\tilde{h},R^{-1},-R^{-1}a}$.
To show conditions (3.2.2) and (3.2.1) we first look at what happens with the Fourier transform of the pullback $\widehat{\alpha^*_{h,R,a} f}(p_0,p)$.

$$\begin{aligned}
\widehat{\alpha^*_{h,R,a} f}(p_0,p) &= \int dt d^d x\, \alpha^*_{h,R,a} f(t,x) e^{-itp_0 + ixp} = \int dt d^d x\, f(t + h(x), Rx + a) e^{-itp_0 + ixp} \\
&\overset{x \mapsto R^{-1}x - R^{-1}a}{=} \int dt d^d x\, f(t + h(R^{-1}x - R^{-1}a), R(R^{-1}x - R^{-1}a) + a) \\
&\quad \cdot e^{-itp_0 + i(R^{-1}x - R^{-1}a)p} \\
&= e^{-iR^{-1}ap} \int dt d^d x\, f(t + h(R^{-1}x - R^{-1}a), x) e^{-itp_0 + iR^{-1}xp} \\
&= e^{-iR^{-1}ap} \int dt d^d x\, f(t + h(R^{-1}x - R^{-1}a), x) e^{-itp_0 + ixRp} \\
&\overset{t \mapsto t - h(x)}{=} e^{-iR^{-1}ap} \int dt d^d x\, f(t,x) e^{-i(t - h(R^{-1}x - R^{-1}a))p_0 + ixRp}\,.
\end{aligned}$$

We define $\tilde{h}(x) := h(R^{-1}x - R^{-1}a)$ to make the next few computations more compact. This leads to

$$\widehat{\alpha^*_{h,R,a} f}(0,p) = e^{-iR^{-1}ap} \int dt d^d x\, f(t,x) e^{ixRp} = e^{-iR^{-1}ap} \hat{f}(0, Rp)\,, \tag{3.2.4}$$

which automatically implies (3.2.2) since $e^{-iR^{-1}ap} \neq 0$. Furthermore we get

$$\begin{aligned}
\frac{\partial}{\partial p_0}\widehat{\alpha^*_{h,R,a}f}(0,p) &= \frac{\partial}{\partial p_0}\left[e^{-iR^{-1}ap}\int dtd^dx\, f(t,x)e^{-i(t-h(R^{-1}x-R^{-1}a))p_0+ixRp}\right]\Big|_{p_0=0} \\
&= \frac{\partial}{\partial p_0}\left[e^{-iR^{-1}ap}\int dtd^dx\, f(t,x)e^{-i(t-\tilde{h}(x))p_0+ixRp}\right]\Big|_{p_0=0} \\
&= -ie^{-iR^{-1}ap}\int dtd^dx\,(t-\tilde{h}(x))f(t,x)e^{ixRp} \\
&= e^{-iR^{-1}ap}\left(\int dtd^dx\,(-it)f(t,x)e^{ixRp} + i\int dtd^dx\,\tilde{h}(x)f(t,x)e^{ixRp}\right) \\
&= e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{f}(0,Rp) + i\widehat{\tilde{h}f}(0,Rp)\right) . \qquad (3.2.5)
\end{aligned}$$

We can finally compute

$$\begin{aligned}
&\sigma_0(\alpha^*{}_{h,R,a}f, \alpha^*_{h,R,a}g) \\
&\overset{(2.4.1)}{=} \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\widehat{\alpha^*_{h,R,a}f}(0,p)\overline{\widehat{\alpha^*_{h,R,a}g}(0,p)} - \overline{\widehat{\alpha^*_{h,R,a}f}(0,p)}\frac{\partial}{\partial p_0}\widehat{\alpha^*_{h,R,a}g}(0,p)\right) \\
&= \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\widehat{\alpha^*_{h,R,a}f}(0,p)\widehat{\alpha^*_{h,R,a}g}(0,-p) - \widehat{\alpha^*_{h,R,a}f}(0,-p)\frac{\partial}{\partial p_0}\widehat{\alpha^*_{h,R,a}g}(0,p)\right) \\
&\overset{(3.2.4)+(3.2.5)}{=} \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left[e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{f}(0,Rp)+i\widehat{\tilde{h}f}(0,Rp)\right)e^{iR^{-1}ap}\hat{g}(0,-Rp)\right. \\
&\qquad \left. - e^{iR^{-1}ap}\hat{f}(0,-Rp)e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{g}(0,Rp)+i\widehat{\tilde{h}g}(0,Rp)\right)\right] \\
&= \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left[\frac{\partial}{\partial p_0}\hat{f}(0,Rp)\hat{g}(0,-Rp) + i\widehat{\tilde{h}f}(0,Rp)\hat{g}(0,-Rp)\right. \\
&\qquad \left. - \hat{f}(0,-Rp)\frac{\partial}{\partial p_0}\hat{g}(0,Rp) - i\hat{f}(0,-Rp)\widehat{\tilde{h}g}(0,Rp)\right] \\
&\overset{p\mapsto R^{-1}p}{=} \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left[\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right. \\
&\qquad \left. + i\widehat{\tilde{h}f}(0,p)\hat{g}(0,-p) - i\hat{f}(0,-p)\widehat{\tilde{h}g}(0,p)\right] \\
&\overset{(2.4.1)}{=} \sigma_0(f,g) - \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\left(\widehat{\tilde{h}f}(0,p)\hat{g}(0,-p) - \hat{f}(0,-p)\widehat{\tilde{h}g}(0,p)\right) .
\end{aligned}$$

The second term of the above equation is 0, since

$$\begin{aligned}
\int\frac{d^dp}{(2\pi)^d}\widehat{\tilde{h}f}(0,p)\hat{g}(0,-p) &= \int\frac{d^dp}{(2\pi)^d}\int dtd^dx\,\tilde{h}(x)f(t,x)e^{ixp}\int dsd^dy\, g(s,y)e^{-iyp} \\
&= \int dtd^dx\int dsd^dy\,\tilde{h}(x)f(t,x)g(s,y)\int\frac{d^dp}{(2\pi)^d}e^{i(x-y)p}
\end{aligned}$$

$$\begin{aligned}
&= \int dtd^dx \int dsd^dy\, \tilde{h}(x)f(t,x)g(s,y)\delta^d(x-y) \\
&= \int dtd^dx \int dsd^dy\, f(t,x)\tilde{h}(y)g(s,y)\delta^d(y-x) \\
&= \int dtd^dx \int dsd^dy\, f(t,x)\tilde{h}(y)g(s,y) \int \frac{d^dp}{(2\pi)^d}\, e^{i(y-x)p} \\
&= \int \frac{d^dp}{(2\pi)^d} \int dtd^dx\, f(t,x)e^{-ixp} \int dsd^dy\, \tilde{h}(y)g(s,y)e^{iyp} \\
&= \int \frac{d^dp}{(2\pi)^d}\, \hat{f}(0,-p)\widehat{\tilde{h}g}(0,p)\,. \qquad (3.2.6)
\end{aligned}$$

Hence (3.2.1) holds true and we have found a family of symmetries for the state ρ_0. In particular, the family $\left\{\alpha_{h,R,a} \mid h \in C^\infty(\mathbb{R}^d), R \in SO(d), a \in \mathbb{R}^d\right\}$ includes the Carroll group [LL65, SG66] and what is referred to as "supertranslations" [DGH14].
Moreover, by Theorem 3.2.1, the GNS-vector x_{ρ_0} is invariant under the unitary representations of $\left\{\alpha_{h,R,a} \mid h \in C^\infty(\mathbb{R}^d), R \in SO(d), a \in \mathbb{R}^d\right\}$. Hence x_{ρ_0} acts like an invariant vacuum state, which gives more credit to the notion of $\mathcal{W}_0$ as the Carrollian limit and the interpretation of ρ_0 as vacuum state of this Carrollian theory.

3.2.2. Conformal symmetries

We finish this section by investigating whether ρ_0 has the conformal transformations [DFMS97, 4.1][Sch08, 1.9] as symmetries.

First, consider the map $\alpha_{\kappa,\lambda} : C_c^\infty(\mathbb{R}^{d+1}) \to C_c^\infty(\mathbb{R}^{d+1})$ with $\kappa, \lambda \in \mathbb{R}\,, \kappa, \lambda \neq 0$, defined as $\alpha_{\kappa,\lambda}(f)(t,x) := \kappa^{\frac{3}{2}}\lambda^{\frac{d}{2}} f(\kappa t, \lambda x)$, which represents the dilations. This map is clearly linear and bijective with inverse $(\alpha_{\kappa,\lambda})^{-1} = \alpha_{\kappa^{-1},\lambda^{-1}}$, hence an automorphism. The calculations below will show why we need the prefactor $\kappa^{\frac{3}{2}}\lambda^{\frac{d}{2}}$.

To show (3.2.1) and (3.2.2), similarly to the previous subsection, we first look at the Fourier transform of $\alpha_{\kappa,\lambda}(f)$ for $f \in C_c^\infty(\mathbb{R}^{d+1})$.

$$\begin{aligned}
\widehat{\alpha_{\kappa,\lambda}(f)}(p_0,p) &= \int dtd^dx\, \alpha_{\kappa,\lambda}(f)(t,x)e^{-itp_0+ixp} = \int dtd^dx\, \kappa^{\frac{3}{2}}\lambda^{\frac{d}{2}} f(\kappa t, \lambda x)e^{-itp_0+ixp} \\
&\overset{(t,x)\mapsto(\kappa^{-1}t,\lambda^{-1}x)}{=} \frac{\kappa^{\frac{3}{2}}\lambda^{\frac{d}{2}}}{\kappa\lambda^d} \int dtd^dx\, f(t,x)e^{-i\kappa^{-1}tp_0+i\lambda^{-1}xp} = \kappa^{\frac{1}{2}}\lambda^{-\frac{d}{2}}\hat{f}(\kappa^{-1}p_0, \lambda^{-1}p)\,.
\end{aligned}$$

We immediately get that condition (3.2.2) is satisfied since

$$\widehat{\alpha_{\kappa,\lambda}(f)}(0,p) = \kappa^{\frac{1}{2}}\lambda^{-\frac{d}{2}}\hat{f}(0,\lambda^{-1}p)\,.$$

Further we get

$$\frac{\partial}{\partial p_0}\widehat{\alpha_{\kappa,\lambda}(f)}(0,p) = \kappa^{\frac{1}{2}}\lambda^{-\frac{d}{2}}\, \frac{\partial}{\partial p_0}\hat{f}(\kappa^{-1}p_0,\lambda^{-1}p)\bigg|_{p_0=0} = \kappa^{-\frac{1}{2}}\lambda^{-\frac{d}{2}}\frac{\partial}{\partial p_0}\hat{f}(0,\lambda^{-1}p)\,.$$

Now we simply check if (3.2.1) holds.

$$
\begin{aligned}
&\sigma_0(\alpha_{\kappa,\lambda}(f), \alpha_{\kappa,\lambda}(g)) \\
&\overset{(2.4.1)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\widehat{\alpha_{\kappa,\lambda}(f)}(0,p)\overline{\widehat{\alpha_{\kappa,\lambda}(g)}(0,p)} - \overline{\widehat{\alpha_{\kappa,\lambda}(f)}(0,p)}\frac{\partial}{\partial p_0}\widehat{\alpha_{\kappa,\lambda}(g)}(0,p)\right) \\
&= \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\widehat{\alpha_{\kappa,\lambda}(f)}(0,p)\widehat{\alpha_{\kappa,\lambda}(g)}(0,-p) - \widehat{\alpha_{\kappa,\lambda}(f)}(0,-p)\frac{\partial}{\partial p_0}\widehat{\alpha_{\kappa,\lambda}(g)}(0,p)\right) \\
&= \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\Bigg(\kappa^{-\frac{1}{2}}\lambda^{-\frac{d}{2}}\frac{\partial}{\partial p_0}\hat{f}(0,\lambda^{-1}p)\kappa^{\frac{1}{2}}\lambda^{-\frac{d}{2}}\hat{g}(0,-\lambda^{-1}p) \\
&\quad - \kappa^{\frac{1}{2}}\lambda^{-\frac{d}{2}}\hat{f}(0,-\lambda^{-1}p)\kappa^{-\frac{1}{2}}\lambda^{-\frac{d}{2}}\frac{\partial}{\partial p_0}\hat{g}(0,\lambda^{-1}p)\Bigg) \\
&= \frac{i}{2}\lambda^{-d}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,\lambda^{-1}p)\hat{g}(0,-\lambda^{-1}p) - \hat{f}(0,-\lambda^{-1}p)\frac{\partial}{\partial p_0}\hat{g}(0,\lambda^{-1}p)\right) \\
&\overset{p\mapsto\lambda p}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\hat{g}(0,-p) - \hat{f}(0,-p)\frac{\partial}{\partial p_0}\hat{g}(0,p)\right) \\
&\overset{(2.4.1)}{=} \sigma_0(f,g)\,.
\end{aligned}
$$

Hence, the dilations $\alpha_{\kappa,\lambda}$ satisfy both (3.2.1) and (3.2.2) and therefore we found that $\{\alpha_{\kappa,\lambda} \mid \kappa,\lambda \in \mathbb{R}\,,\kappa,\lambda \neq 0\}$ is another symmetry group of ρ_0.

With the rotations and shifts from the previous subsection and now dilations, this completes the conformal transformations on $\mathbb{R}^{1,d}$. What is left in the conformal group are the special conformal transformations,

$$x^\mu \longmapsto x'^\mu := \frac{x^\mu - b^\mu x^\nu x_\nu}{1 - 2b^\nu x_\nu + b^\nu b_\nu x^\delta x_\delta}$$

for $b^\mu \in \mathbb{R}^{d+1}$. However, such maps are not bijective on $\mathbb{R}^{d+1}$ [Sch08, 1.9]. For example, if $x^\mu = \frac{b^\mu}{b^\delta b_\delta}$ then

$$1 - 2b^\nu x_\nu + b^\nu b_\nu x^\delta x_\delta = 1 - 2b^\nu\frac{b_\nu}{b^\delta b_\delta} + b^\nu b_\nu\frac{b^\delta}{b^\mu b_\mu}\frac{b_\delta}{b^\rho b_\rho} = 1-2+1 = 0\,,$$

which makes x'^μ is not defined.
If we take a look at what is called the *conformal Carroll group* [BMN19, DGH14, DFMS97], we can see that there is a decoupling between the temporal and the spacial special conformal transformations in the Carroll limit. The temporal special conformal transformations can be written as $(t,x) \mapsto (t + ax^2, x)$, for $a \in \mathbb{R}$, which in our case is just a super translation and therefore part of the symmetries of ρ_0.
The spacial special conformal transformations on the other hand still have the singularity problem. However, special conformal transformations can be defined on compactifications of $\mathbb{R}^n$ [Sch08, 2.1]. Hence, if we consider spacially compact manifold instead of $\mathbb{R}^{1,d}$ as our underlying manifold, this would not be an issue.
As a last note, although we have found many symmetries of ρ_0, we do not claim that we have found all possible symmetries.

4. Green Operators and the Solution Space

In this final chapter our goal is to better understand those somewhat artificially constructed spaces V_ϵ from the previous chapters, as they play a central role in the construction of the Weyl-algebras and the Carrollian limit. For this we will take a look at the Green function for the scaled Klein-Gordon operators. At first this seems like a detour, but we will explain the connection between the Green function and V_ϵ subsequently.

4.1. Green function of the scaled Klein-Gordon operator

In this section, we are going to calculate the Green function for the scaled Klein-Gordon operator P_ϵ from Equation (2.1.5). The calculations for the Green function in this section will not be very different from the usual procedure for the normal Klein-Gordon operator ($\epsilon = 1$), but it is interesting to see where exactly we need to mind the parameter ϵ.

The Green function G_ϵ of the operator P_ϵ is defined as a solution of

$$P_\epsilon G_\epsilon(x^\mu, y^\mu) = (\partial_t^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2)G_\epsilon(x^\mu, y^\mu) = \delta^{d+1}(x^\mu - y^\mu)\,, \tag{4.1.1}$$

where P_ϵ uses the $x^\mu \in \mathbb{R}^{d+1}$ coordinates. We consider $y^\mu \in \mathbb{R}^{d+1}$ a parameter for the most part.
Fourier transforming both sides, i.e., applying $\int d^{d+1}x^\mu\, e^{-ix_\mu p^\mu}$, yields

$$\int d^{d+1}x^\mu\, e^{-ix_\mu p^\mu}(\partial_t^2 - \epsilon^2\partial_x^2 + \epsilon^2 m^2)G_\epsilon(x^\mu, y^\mu) = \int d^{d+1}x^\mu\, e^{-ix_\mu p^\mu}\delta^{d+1}(x^\mu - y^\mu) = e^{-iy_\mu p^\mu}\,,$$

which further gives

$$\begin{aligned} e^{-iy_\mu p^\mu} &= \int d^{d+1}x^\mu\, (\partial_t^2 - \epsilon^2\partial_x^2 + \epsilon^2 m^2)e^{-ix_\mu p^\mu}G_\epsilon(x^\mu, y^\mu) \\ &= \int d^{d+1}x^\mu\, (-p_0^2 + \epsilon^2 p^2 + \epsilon^2 m^2)e^{-ix_\mu p^\mu}G_\epsilon(x^\mu, y^\mu) \\ &= (-p_0^2 + \epsilon^2 p^2 + \epsilon^2 m^2)\hat{G}_\epsilon(p^\mu, y^\mu) \end{aligned} \tag{4.1.2}$$

using integration by parts. At this point we need to be careful. We want to find a solution $\hat{G}_\epsilon(p^\mu, y^\mu)$ to the above equation for all $p^\mu \in \mathbb{R}^{d+1}$. If $-p_0^2 + \epsilon^2 p^2 + \epsilon^2 m^2 \neq 0$ we easily find the solution,

$$\hat{G}_\epsilon(p^\mu, y^\mu) = -\frac{1}{p_0^2 - \epsilon^2 p^2 - \epsilon^2 m^2}e^{-iy_\mu p^\mu}\,. \tag{4.1.3}$$

But if $-p_0^2+\epsilon^2 p^2+\epsilon^2 m^2=0$, we have a problem. Luckily, a workaround can be found. However, there are multiple ways to do this, so we only illustrate the two standard versions to get the *retarded* and *advanced* solution here.
We consider the equation

$$e^{-iy_\mu p^\mu} = (-(p_0-i\tilde\eta)^2+\epsilon^2p^2+\epsilon^2m^2)\hat G_{\epsilon,\tilde\eta}(p^\mu,y^\mu) \tag{4.1.4}$$

for $\tilde\eta\in\mathbb{R},\tilde\eta\neq 0$. To be consistent, we require that the limit $\tilde\eta\to 0$ of $G_{\epsilon,\tilde\eta}(x^\mu,y^\mu)$ exists as a distribution and that the original equation is solved in the distributional limit, i.e.,

$$\int d^{d+1}p^\mu\, e^{-iy_\mu p^\mu} f(p^\mu) = \lim_{\tilde\eta\to 0}\int d^{d+1}p^\mu\,(-p_0^2+\epsilon^2p^2+\epsilon^2m^2)\hat G_{\epsilon,\tilde\eta}(p^\mu,y^\mu)f(p^\mu)\,, \tag{4.1.5}$$

for $f\in C_c^\infty(\mathbb{R}^{d+1})$.[1] The limit should be take either from above or below. We will address this later.
Now $(-(p_0-i\tilde\eta)^2+\epsilon^2p^2+\epsilon^2m^2)\neq 0$ for all $p^\mu\in\mathbb{R}^{d+1}$, hence we can find a solution to (4.1.4),

$$\hat G_{\epsilon,\tilde\eta}(p^\mu,y^\mu) = -\frac{1}{(p_0-i\tilde\eta)^2-\epsilon^2p^2-\epsilon^2m^2}e^{-iy_\mu p^\mu} = -\frac{1}{(p_0-i\tilde\eta)^2-E_{\epsilon,p}^2}e^{-iy_\mu p^\mu}\,, \tag{4.1.6}$$

where $E_{\epsilon,p}:=\epsilon E_p=\sqrt{\epsilon^2p^2+\epsilon^2m^2}$. Hence (4.1.6) is a well-defined solution and we can apply the inverse Fourier transform, which yields

$$\begin{aligned}
G_{\epsilon,\tilde\eta}(x^\mu,y^\mu) &= \int\frac{d^{d+1}p^\mu}{(2\pi)^{d+1}}\,e^{ip_\mu x^\mu}\hat G_{\epsilon,\tilde\eta}(p^\mu,y^\mu) = \int\frac{d^{d+1}p^\mu}{(2\pi)^{d+1}}\,e^{ip_\mu x^\mu}\frac{-1}{(p_0-i\tilde\eta)^2-E_{\epsilon,p}^2}e^{-iy_\mu p^\mu}\\
&= -\int\frac{d^{d+1}p^\mu}{(2\pi)^{d+1}}\,\frac{1}{(p_0-i\tilde\eta)^2-E_{\epsilon,p}^2}e^{i(x_\mu-y_\mu)p^\mu}\\
&\overset{p^\mu\mapsto -p^\mu}{=} -\int\frac{d^{d+1}p^\mu}{(2\pi)^{d+1}}\,\frac{1}{(p_0+i\tilde\eta)^2-E_{\epsilon,p}^2}e^{-i(x_\mu-y_\mu)p^\mu}\\
&= -\int\frac{d^dp}{(2\pi)^d}\,e^{i(x-y)p}\int\frac{dp_0}{2\pi}\,\frac{1}{(p_0+i\tilde\eta)^2-E_{\epsilon,p}^2}e^{-i(x^0-y^0)p_0}\,.
\end{aligned} \tag{4.1.7}$$

Remark. For $\tilde\eta\to 0$ we formally get

$$G_\epsilon(x^\mu,y^\mu)=G_{\epsilon,0}(x^\mu,y^\mu) = -\int\frac{d^dp}{(2\pi)^d}\,e^{i(x-y)p}\int\frac{dp_0}{2\pi}\,\frac{1}{p_0^2-E_{\epsilon,p}^2}e^{-i(x^0-y^0)p_0}\,.$$

This integrand has of course poles at $\pm E_{\epsilon,p}$ and can therefore not be (directly) evaluated. However, note that for $\epsilon\neq 1$ we do not seem to have Lorentz invariance of G_ϵ, i.e., $G_\epsilon(\Lambda^\mu_{\ \nu}x^\nu,\Lambda^\mu_{\ \nu}y^\nu)\neq G_\epsilon(x^\mu,y^\mu)$ for any orthochronous Lorentz transform Λ. This comes from the observation that $p_0^2-E_{\epsilon,p}^2=p_0^2-\epsilon^2p^2-\epsilon^2m^2$ is not Lorentz invariant for $\epsilon\neq 1$. But there is a deformed Lorentz invariance.

[1] By linearity of the integral it suffices to only consider real valued test function.

To progress further we need to evaluate the p_0-integral. The usual way is to consider a sequence of complex integrals along a sequence of contours, which converge to the original integral.
We also distinguish the two cases $\tilde{\eta} > 0$ and $\tilde{\eta} < 0$. We write $\eta = |\tilde{\eta}|$ and the superscript $+$ or $-$ denotes the sign of $\tilde{\eta}$.
So, consider the two complex integrals

$$\begin{aligned} J^{\pm}_{\epsilon,\eta,R} : = \theta(x^0 - y^0) \int_{\gamma_R^-} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)z} \\ + \theta(y^0 - x^0) \int_{\gamma_R^+} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)z} , \end{aligned} \tag{4.1.8}$$

where $\gamma_R^{\pm} := [-R, R] \dotplus \{Re^{\pm it} | \, t \in [0, \pi]\}$[2] and $\eta > 0$.
The integrals

$$I^{\pm}_{\epsilon,\eta,R} := \int_{[-R,R]} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)z} \tag{4.1.9}$$

have the limit

$$\lim_{R\to\infty} I^{\pm}_{\epsilon,\eta,R} = \int \frac{dp_0}{2\pi} \frac{1}{(p_0 \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)p_0} , \tag{4.1.10}$$

which is exactly the integral that we want. The difference of the two integrals $J^{\pm}_{\epsilon,\eta,R}$ and $I^{\pm}_{\epsilon,\eta,R}$ can be rewritten, using properties of the contour integral, as

$$\begin{aligned} J^{\pm}_{\epsilon,\eta,R} - I^{\pm}_{\epsilon,\eta,R} &\overset{(4.1.8)+(4.1.9)}{=} \theta(x^0 - y^0) \int_{\{Re^{-it} | \, t\in[0,\pi]\}} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)z} \\ &+ \theta(y^0 - x^0) \int_{\{Re^{+it} | \, t\in[0,\pi]\}} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)z} \\ &= \theta(x^0 - y^0) \int_0^{\pi} \frac{1}{2\pi} \frac{-iRe^{-it}}{(Re^{-it} \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)R\cos(t)} e^{-(x^0-y^0)R\sin(t)} dt \\ &+ \theta(y^0 - x^0) \int_0^{\pi} \frac{1}{2\pi} \frac{iRe^{it}}{(Re^{it} \mp i\eta)^2 - E^2_{\epsilon,p}} e^{-i(x^0-y^0)R\cos(t)} e^{(x^0-y^0)R\sin(t)} dt . \end{aligned}$$

If we have $x^0 - y^0 > 0$ then $e^{-(x^0-y^0)R\sin(t)} \leq 1$ for R large enough and similarly if $x^0 - y^0 < 0$ then $e^{(x^0-y^0)R\sin(t)} \leq 1$ for large R. In either case the dominating term is

$$\frac{iRe^{it}}{(Re^{it} \mp i\eta)^2 - E^2_{\epsilon,p}}$$

which tends to 0 as $R \to \infty$, since the denominator of the integrands contains a higher power of R then the numerator.

[2] $\dotplus$ is meant as the concatenation of contours, in our case $[-R, R]$, the straight line from $-R$ to R, and $\{Re^{\pm it} | \, t \in [0, \pi]\}$, the half circle from R to $-R$ either in the upper or lower half plane, depending on the sign of the exponent.

For $x^0 - y^0 = 0$ we find that

$$\int_0^\pi \frac{1}{2\pi} \frac{-iRe^{-it}}{(Re^{-it} \mp i\eta)^2 - E_{\epsilon,p}^2} dt + \int_0^\pi \frac{1}{2\pi} \frac{iRe^{it}}{(Re^{it} \mp i\eta)^2 - E_{\epsilon,p}^2} dt$$

vanishes as $R \to \infty$ by the same argument. Hence

$$J^\pm_{\epsilon,\eta,R} - I^\pm_{\epsilon,\eta,R} \underset{R\to\infty}{\longrightarrow} 0\,. \tag{4.1.11}$$

To compute $J^\pm_{\epsilon,\eta,R}$ we use the residue theorem. The theorem states that for any meromorphic function f and closed curve γ in $\mathbb{C}$ we have

$$\int_\gamma f(z)dz = 2\pi i \sum_{a \text{ pole of } f} \mathrm{Ind}_\gamma(a)\mathrm{Res}(f,a)\,. \tag{4.1.12}$$

In our case

$$f^\pm(z) = \frac{1}{2\pi} \frac{1}{(z \mp i\eta)^2 - E_{\epsilon,p}^2} e^{-i(x^0-y^0)z}\,,$$

are indeed meromorphic functions with poles at $a^+_{1,2} = \pm E_{\epsilon,p} + i\eta$ for f^+ and $a^-_{1,2} = \pm E_{\epsilon,p} - i\eta$ for f^-. Those poles are inside the contour γ_R^+ and γ_R^- respectively, if R is large enough, and outside the respective other contour. How large depends on η, with the threshold decreasing if η gets smaller. This means we do not run into conflict if we take both the limit $R \to \infty$ and $\eta \to 0$.
Hence, for R large enough, we get

$$\mathrm{Ind}_{\gamma_R^\pm}(a^\pm_{1,2}) = \pm 1 \quad \text{and} \quad \mathrm{Ind}_{\gamma_R^\mp}(a^\pm_{1,2}) = 0 \tag{4.1.13}$$

for the winding number. We can calculate the residues as follows.

$$\begin{aligned}\mathrm{Res}(f^\pm, a_1^\pm) &= \lim_{z\to a_1^\pm}(z-a_1^\pm)f^\pm(z) = \lim_{z\to a_1^\pm}(z-a_1^\pm)\frac{1}{2\pi}\frac{1}{(z-a_1^\pm)(z-a_2^\pm)}e^{-i(x^0-y^0)z} \\ &= \frac{1}{2\pi}\frac{1}{(a_1^\pm - a_2^\pm)}e^{-i(x^0-y^0)a_1^\pm} = \frac{1}{2\pi}\frac{1}{2E_{\epsilon,p}}e^{-i(x^0-y^0)(E_{\epsilon,p}\pm i\eta)}\,,\end{aligned} \tag{4.1.14}$$

since $a_1^\pm$ is a simple pole, and similarly

$$\mathrm{Res}(f^\pm, a_2^\pm) = \frac{1}{2\pi}\frac{1}{-2E_{\epsilon,p}}e^{-i(x^0-y^0)(-E_{\epsilon,p}\pm i\eta)}\,. \tag{4.1.15}$$

Therefore, for R large enough, we can evaluate $J^\pm_{\epsilon,\eta,R}$.

$$\begin{aligned}J^+_{\epsilon,\eta,R} &\overset{(4.1.8)}{=} \theta(x^0-y^0)\int_{\gamma_R^-} f^+(z)dz + \theta(y^0-x^0)\int_{\gamma_R^+} f^+(z)dz \\ &\overset{(4.1.12)+(4.1.13)}{=} 2\pi i\theta(y^0-x^0)(\mathrm{Res}(f^+,a_1^+) + \mathrm{Res}(f^+,a_2^+)) \\ &\overset{(4.1.14)+(4.1.15)}{=} \theta(y^0-x^0)2\pi i\left(\frac{1}{4\pi E_{\epsilon,p}}e^{-i(x^0-y^0)(E_{\epsilon,p}+i\eta)} - \frac{1}{4\pi E_{\epsilon,p}}e^{-i(x^0-y^0)(-E_{\epsilon,p}+i\eta)}\right) \\ &= \theta(-(x^0-y^0))\frac{1}{2iE_{\epsilon,p}}\left(e^{i(x^0-y^0)(E_{\epsilon,p}-i\eta)} - e^{-i(x^0-y^0)(E_{\epsilon,p}+i\eta)}\right)\end{aligned}$$

and analogously for $J^-_{\epsilon,\eta,R}$. Compactly written we get

$$J^{\pm}_{\epsilon,\eta,R} = \pm\theta(\mp(x^0-y^0))\frac{1}{2iE_{\epsilon,p}}\left(e^{i(x^0-y^0)(E_{\epsilon,p}\mp i\eta)} - e^{-i(x^0-y^0)(E_{\epsilon,p}\pm i\eta)}\right). \tag{4.1.16}$$

In our case, there are two choices, $I^+_{\epsilon,\eta,R}$ or $I^-_{\epsilon,\eta,R}$, to compute the Green functions, using Equation (4.1.10). So we define

$$G^{\pm}_{\epsilon}(x^\mu,y^\mu) := -\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} I^{\pm}_{\epsilon,\eta,R}\,. \tag{4.1.17}$$

Remark. A crucial detail here is that we do not know a priori if $G^{\pm}_{\epsilon}(x^\mu,y^\mu)$ satisfies the defining equation (4.1.1). Even though

$$G^{\pm}_{\epsilon,\eta}(x^\mu,y^\mu) := -\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{R\to\infty} I^{\pm}_{\epsilon,\eta,R}$$

satisfies Equation (4.1.4), it is not clear at this point whether $G^{\pm}_{\epsilon,\eta}(x^\mu,y^\mu)$ fulfills the consistency condition (4.1.5). We will show in Subsection 4.3.1 that both Equation (4.1.1) as well as Equation (4.1.5) hold for $G^{\pm}_{\epsilon}(x^\mu,y^\mu)$.

As a final step we compute the limit explicitly.

$$\begin{aligned}
G^{\pm}_{\epsilon}(x^\mu,y^\mu) &\overset{(4.1.17)}{=} -\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} I^{\pm}_{\epsilon,\eta,R}\\
&\overset{(4.1.11)}{=} -\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} J^{\pm}_{\epsilon,\eta,R}\\
&\overset{(4.1.16)}{=} -\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty}\Bigg[\pm\theta(\mp(x^0-y^0))\frac{1}{2iE_{\epsilon,p}}\Big(e^{i(x^0-y^0)(E_{\epsilon,p}\mp i\eta)}\\
&\qquad - e^{-i(x^0-y^0)(E_{\epsilon,p}\pm i\eta)}\Big)\Bigg]\\
&= \mp\theta(\mp(x^0-y^0))\int \frac{d^dp}{(2\pi)^d}\, e^{i(x-y)p}\frac{1}{2iE_{\epsilon,p}}\left(e^{i(x^0-y^0)E_{\epsilon,p}} - e^{-i(x^0-y^0)E_{\epsilon,p}}\right)\\
&\overset{p\mapsto -p}{=} \mp\theta(\mp(x^0-y^0))\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{2iE_{\epsilon,p}}\left(e^{i(x^\mu-y^\mu)p_\mu} - e^{-i(x^\mu-y^\mu)p_\mu}\right)\Big|_{p_0=E_{\epsilon,p}}\\
&= \mp\theta(\mp(x^0-y^0))\int \frac{d^dp}{(2\pi)^d}\,\frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Bigg|_{p_0=E_{\epsilon,p}}.
\end{aligned} \tag{4.1.18}$$

We call $G^+_{\epsilon}(x^\mu,y^\mu)$ the *scaled advanced Green function* and $G^-_{\epsilon}(x^\mu,y^\mu)$ the *scaled retarded Green function.* Note that this result is consistent with the unscaled case ($\epsilon=1$).

Let us finish this section by mentioning two properties of the Green functions. The first one is

$$G^{\pm}_{\epsilon}(x^\mu,y^\mu) = G^{\mp}_{\epsilon}(y^\mu,x^\mu)\,. \tag{4.1.19}$$

This holds since

$$
\begin{aligned}
G^{\pm}_{\epsilon}(x^\mu, y^\mu) &\overset{(4.1.18)}{=} \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0 = E_{\epsilon,p}} \\
&= \mp\theta(\pm(y^0 - x^0)) \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin(-(y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0 = E_{\epsilon,p}} \\
&= \pm\theta(\pm(y^0 - x^0)) \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0 = E_{\epsilon,p}} \overset{(4.1.18)}{=} G^{\mp}_{\epsilon}(y^\mu, x^\mu)\,.
\end{aligned}
$$

We already mentioned in the remark after Equation (4.1.7) that for $\epsilon = 1$ we have Lorentz invariance of $G_1(x^\mu, y^\mu)$. Since we will need this property later, we quickly show it. We first rewrite $G^{\pm}_1(x^\mu, y^\mu)$.

$$
\begin{aligned}
G^{\pm}_1(x^\mu, y^\mu) &\overset{(4.1.18)}{=} \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((x^\mu - y^\mu)p_\mu)}{E_p}\right|_{p_0 = E_p} \\
&= \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} \frac{1}{2iE_p} \left.\left(e^{i(x^0-y^0)p_0} e^{-i(x-y)p} - e^{-i(x^0-y^0)p_0} e^{i(x-y)p}\right)\right|_{p_0 = E_p} \\
&\overset{p \mapsto -p}{=} \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} e^{i(x-y)p} \frac{1}{2iE_p} \left.\left(e^{i(x^0-y^0)p_0} - e^{-i(x^0-y^0)p_0}\right)\right|_{p_0 = E_p} \\
&= \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} e^{i(x-y)p} \int dp_0\, e^{-i(x^0-y^0)p_0} \frac{1}{2iE_p} \left(\delta(p_0 + E_p) - \delta(p_0 - E_p)\right) \\
&= \mp i\theta(\mp(x^0 - y^0)) \int \frac{d^{d+1} p^\mu}{(2\pi)^d} e^{-i(x^\mu - y^\mu)p_\mu} \operatorname{sgn}(p_0)\delta(p_0^2 - E_p^2)\,.
\end{aligned}
$$

Remark. In this form, we can see that $G^{\pm}_1(x^\mu, y^\mu) \propto \Delta(x^0 - y^0, x - y)$ where Δ is the same as Equation (2.1.3). This is not a coincidence and we will touch on that later on.

Now, let Λ be any orthochronous Lorentz transform. We write shorthand $(\Lambda x)^\mu$ for $\Lambda^\mu_\nu x^\nu$. Then by applying the calculation above we get

$$
\begin{aligned}
G^{\pm}_1((\Lambda x)^\mu, (\Lambda y)^\mu) &= \mp i\theta(\mp((\Lambda x)^0 - (\Lambda y)^0)) \int \frac{d^{d+1} p^\mu}{(2\pi)^d} e^{-i((\Lambda x)^\mu - (\Lambda y)^\mu)p_\mu} \operatorname{sgn}(p_0)\delta(p_0^2 - E_p^2) \\
&\overset{p^\mu \mapsto (\Lambda p)^\mu}{=} \mp i\theta(\mp(\Lambda(x - y))^0) \int \frac{d^{d+1} p^\mu}{(2\pi)^d} e^{-i(\Lambda(x-y))^\mu (\Lambda p)_\mu} \operatorname{sgn}((\Lambda p)_0)\delta((\Lambda p)_\mu(\Lambda p)^\mu - m^2) \\
&\mp i\theta(\mp(\Lambda(x - y))^0) \int \frac{d^{d+1} p^\mu}{(2\pi)^d} e^{-i(x^\mu - y^\mu)p_\mu} \operatorname{sgn}((\Lambda p)_0)\delta(p_\mu p^\mu - m^2)\,.
\end{aligned}
$$

Since Λ is orthochronous, it respects the sign of the 0-component, i.e., $(\Lambda u)^0 > 0$ if $u^0 > 0$ and $(\Lambda u)^0 < 0$ if $u^0 < 0$. Hence $\operatorname{sgn}((\Lambda p)_0) = \operatorname{sgn}(p_0)$ and $\theta(\mp(\Lambda(x-y))^0) = \theta(\mp(x^0 - y^0))$ and therefore

$$
G^{\pm}_1((\Lambda x)^\mu, (\Lambda y)^\mu) = G^{\pm}_1(x^\mu, y^\mu)\,. \tag{4.1.20}
$$

4.2. Green function of the Carrollian operator

Throughout this thesis, we have always dealt with the limit as ϵ approaches 0. So naturally the next question which arises is what happens to the Green functions in the limit. To do this, we introduce in this section the *electric Carrollian operator*, for which we are again calculating the Green function and compare it to the result of the previous section.

The scaled Klein-Gordon operator P_ϵ from Equation (2.1.5) has, at least formally, the limit

$$P_\epsilon = \partial_t^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2 \quad \text{“} \underset{\epsilon\to 0}{\longrightarrow} \text{”} \quad \partial_t^2 \,.$$

Even though we are not really interested if the operators converge in a proper sense, we want to note that this can indeed be made more rigorous.
As a sketch of the proof we start by realizing that $P_\epsilon : \mathcal{S}(\mathbb{R}^{d+1}) \longrightarrow \mathcal{S}(\mathbb{R}^{d+1})$ can be extended to, for example, $P_\epsilon : H^2(\mathbb{R}^{d+1}) \longrightarrow L^2(\mathbb{R}^{d+1})$, where $H^2(\mathbb{R}^{d+1})$ denotes the Sobolev space over $\mathbb{R}^{d+1}$ [Bre11, 8.2].
Then, for $f \in \mathcal{S}(\mathbb{R}^{d+1})$, we have

$$\left\|P_\epsilon f - \partial_t^2 f\right\|_{L^2(\mathbb{R}^{d+1})} = \left\|(-\epsilon^2\nabla^2 + \epsilon^2 m^2) f\right\|_{L^2(\mathbb{R}^{d+1})} = \epsilon^2 \left\|(\nabla^2 - m^2) f\right\|_{L^2(\mathbb{R}^{d+1})} \,.$$

Since $\left\|(\nabla^2 - m^2) f\right\|_{L^2(\mathbb{R}^{d+1})} \leq 2\max\{1, m^2\} \left\|f\right\|_{H^2(\mathbb{R}^{d+1})} < \infty$ and by density of $\mathcal{S}(\mathbb{R}^{d+1})$ in $L^2(\mathbb{R}^{d+1})$ and $H^2(\mathbb{R}^{d+1})$, P_ϵ converges to ∂_t^2 in the strong operator topology.
We call

$$P_0 := \partial_t^2 : \mathcal{S}(\mathbb{R}^{d+1}) \longrightarrow \mathcal{S}(\mathbb{R}^{d+1}) \tag{4.2.1}$$

the *electric Carrollian operator*.

Remark. The electric Carrollian equation $P_0\psi = 0$ is the same equation of motion which one gets from the action of the electric scalar theory used in [dBHO+23] for $m = 0$. Or, if we think the other way around, we could counter scale $m \mapsto \epsilon^{-1} m$ and we would get back the equations of motion $\tilde{P}_0\psi = (\partial_t^2 + m^2)\psi = 0$ for the electric scalar theory.

We proceed, as mentioned before, by computing the Green function of the electric Carrollian operator. This will follow almost exactly the same steps as in the previous chapter, which is why we only highlight the key differences here. The complete computation can be found in Appendix A.

The computation starts identically with the defining equation

$$P_0 G_0(x^\mu, y^\mu) = \partial_t^2 G_0(x^\mu, y^\mu) = \delta^{d+1}(x^\mu - y^\mu) \,.$$

Using Fourier transform, shifting the poles and inverse Fourier transform we get

$$G^\pm_{0,\eta}(x^\mu, y^\mu) = -\int \frac{d^d p}{(2\pi)^d}\, e^{i(x-y)p} \int \frac{dp_0}{2\pi} \frac{1}{(p_0 \mp i\eta)^2} e^{-i(x^0 - y^0)p_0} \,.$$

To evaluate the second integral in $G^{\pm}_{0,\eta}(x^\mu, y^\mu)$ we again define

$$J^{\pm}_{0,\eta,R} := \theta(x^0 - y^0) \int_{\gamma_R^-} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} + \theta(y^0 - x^0) \int_{\gamma_R^+} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} ,$$

where $\gamma_R^{\pm} := [-R, R] \dotplus \{Re^{\pm it} | t \in [0, \pi]\}$ and $\eta > 0$, and

$$I^{\pm}_{0,\eta,R} := \int_{[-R,R]} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x_0-y_0)z} .$$

By the same argument as in the previous section we conclude that the difference between $J^{\pm}_{0,\eta,R}$ and $I^{\pm}_{0,\eta,R}$ vanishes as $\mathbb{R} \to \infty$, depending on the sign of $x^0 - y^0$, and the latter converges to the desired integral as $\eta \to 0$ and $\mathbb{R} \to \infty$.
To evaluate the former, we use the residue theorem (Equation (4.1.12)) with the integrand

$$f^{\pm}(z) = \frac{1}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} .$$

The main difference to Section 4.1 is that the poles $a^+ = -i\eta$ for f^+ and $a^- = i\eta$ for f^- are now second order poles, which means that we have to use a different formula to calculate their residue. In the end we get

$$\mathrm{Ind}_{\gamma_R^{\pm}}(a^{\pm}) = \pm 1 \quad \text{and} \quad \mathrm{Ind}_{\gamma_R^{\mp}}(a^{\pm}) = 0$$

for the winding number and

$$\mathrm{Res}(f^{\pm}, a^{\pm}) = \frac{1}{2\pi} (-i(x^0 - y^0)) e^{\pm(x^0-y^0)\eta}$$

for the residue. This still requires R to be large enough. This leads to $J^{\pm}_{0,\eta,R}$ being evaluated as

$$J^{\pm}_{0,\eta,R} = \pm\theta(\mp(x^0 - y^0))(x_0 - y_0) e^{\pm(x_0-y_0)\eta} .$$

Defining

$$G^{\pm}_0(x^\mu, y^\mu) := - \int \frac{d^d p}{(2\pi)^d} \, e^{i(x-y)p} \lim_{\eta \to 0} \lim_{R \to \infty} I^{\pm}_{0,\eta,R} .$$

yields

$$G^{\pm}_0(x^\mu, y^\mu) = \mp\theta(\mp(x^0 - y^0))(x^0 - y^0)\delta^d(x - y) . \tag{4.2.2}$$

We call $G^+_0(x^\mu, y^\mu)$ the *advanced Carrollian Green function* and $G^-_0(x^\mu, y^\mu)$ the *retarded Carrollian Green function.*

Lastly, in this section, we show that we also have an analogous symmetry property to Equation (4.1.19).

$$\begin{aligned} G^{\pm}_0(x^\mu, y^\mu) &\overset{(4.2.2)}{=} \mp\theta(\mp(x^0 - y^0))(x^0 - y^0)\delta^d(x - y) \\ &= \pm\theta(\pm(y^0 - x^0))(y^0 - x^0)\delta^d(y - x) \\ &\overset{(4.2.2)}{=} G^{\mp}_0(y^\mu, x^\mu) . \end{aligned} \tag{4.2.3}$$

4.3. Green operators and the Solution Space

In this section, we transition from the Green functions to the *Green operators* and show an interesting connection between the Green operators, the vector spaces V_ϵ and the *Solution spaces* for P_ϵ.

4.3.1. From Green functions to Green operators

We start by smearing out the Green functions, by integrating it with a test function. To be more precise, we define the maps $G^\pm_\epsilon : C^\infty_c(\mathbb{R}^{d+1}) \longrightarrow C^\infty(\mathbb{R}^{d+1})$ by

$$G^\pm_\epsilon f(x^\mu) := \int d^{d+1}y^\mu\, G^\pm_\epsilon(x^\mu, y^\mu) f(y^\mu) \tag{4.3.1}$$

for any $f \in C^\infty_c(\mathbb{R}^{d+1})$ and $x^\mu \in \mathbb{R}^{d+1}$. We call $G^+_\epsilon : C^\infty_c(\mathbb{R}^{d+1}) \longrightarrow C^\infty(\mathbb{R}^{d+1})$ the *advanced scaled Green operator* and $G^-_\epsilon : C^\infty_c(\mathbb{R}^{d+1}) \longrightarrow C^\infty(\mathbb{R}^{d+1})$ the *retarded scaled Green operator*.
Next up, we want to show that the advanced and retarded scaled Green operators are in fact Green operators according to the following definition.

Definition 4.3.1 (Green operator)
Let M be a time-oriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. A linear map $G_+ : \mathcal{D}(M,E) \to C^\infty(M,E)$ satisfying

(i) $P \circ G_+ = \mathrm{id}_{\mathcal{D}(M,E)}$,

(ii) $G_+ \circ P|_{\mathcal{D}(M,E)} = \mathrm{id}_{\mathcal{D}(M,E)}$,

(iii) $\mathrm{supp}\,(G_+\phi) \subseteq J^M_+(\mathrm{supp}\,\phi)$ for all $\phi \in \mathcal{D}(M,E)$,

is called an *advanced Green operator* for P. Similarly, a linear map $G_- : \mathcal{D}(M,E) \to C^\infty(M,E)$ satisfying (i), (ii) and

(iii') $\mathrm{supp}\,(G_-\phi) \subseteq J^M_-(\mathrm{supp}\,\phi)$ for all $\phi \in \mathcal{D}(M,E)$

is called a *retarded Green operator* for P.[BGP07, Definition 3.4.1]

A normally hyperbolic operator is defined as follows.

Definition 4.3.2 (Normally hyperbolic operator)
Let (M,g) be a Lorentzian manifold of dimension n and let $E \to M$ be a real or complex vector bundle.
A linear differential operator $P : C^\infty(M,E) \longrightarrow C^\infty(M,E)$ of second order will be called *normally hyperbolic* if P has the form

$$P = g^{\mu\nu}(x)\frac{\partial^2}{\partial x^\mu \partial x^\nu} + A^\mu(x)\frac{\partial}{\partial x^\mu} + B(x)\,, \tag{4.3.2}$$

where $A^\mu(x)$ and $B(x)$ are matrix-valued coefficients depending smoothly on x, after choosing local coordinates x^μ on M and a local trivialization of E. [BGP07, p.33-34]

Remark. This definition uses the metric signature $(+,-,\ldots,-)$. If $(-,+,\ldots,+)$ is used, then the term $g^{\mu\nu}(x)\frac{\partial^2}{\partial x^\mu \partial x^\nu}$ in (4.3.2) gets a minus in front.

In our case, we have $M = \mathbb{R}^{1,d}$ and $E = \mathbb{R} \times \mathbb{R}^{1,d}$. The smooth section $C^\infty(M,E)$ then becomes the set of smooth real-valued functions $C^\infty(\mathbb{R}^{d+1})$. Similarly $\mathcal{D}(M,E)$ becomes $C_c^\infty(\mathbb{R}^{d+1})$.
From our derivation of the scaled Klein-Gordon-operator in Section 2.1, we know that, after rescaling, $\frac{1}{\epsilon^2}P_\epsilon$ is just the original Klein-Gordon-operator P in coordinates $(\tau,\xi) = (\epsilon t, x)$. Alternatively, we can consider P_ϵ as a normal Klein-Gordon-operator with mass ϵm in coordinates $(\tau,\xi) = (t, \frac{1}{\epsilon}x)$,

$$P_\epsilon = \partial_t^2 - \epsilon^2 \nabla_x^2 + \epsilon^2 m^2 = \partial_\tau^2 - \nabla_\xi^2 + (\epsilon m)^2 \,.$$

Hence $P_\epsilon : C^\infty(\mathbb{R}^{d+1}) \longrightarrow C^\infty(\mathbb{R}^{d+1})$ is a normally hyperbolic operator. Note, however, that it does not matter for our purposes whether P_ϵ is or is not normally hyperbolic, as we will see shortly.

So we proceed by showing that our Green operators $G_\epsilon^\pm$ from Equation (4.3.1) for $\epsilon \in (0,1]$ fulfill the rest of Definition 4.3.1, i.e., are linear and satisfy (i)-(iii). The restriction to $\epsilon \in (0,1]$ will make sense later on. Since we are mostly interested in the limit $\epsilon \to 0$ this does not hinder us.

Linearity:
Linearity follows from the linearity of the integral.

(i):
To make the next computation a little more compact, we quickly note that

$$\int \frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p},x^0=y^0} = \int \frac{d^dp}{(2\pi)^d}\ \frac{\sin(-(x-y)p)}{E_{\epsilon,p}} = 0\,, \tag{4.3.3}$$

and hence $G_\epsilon^\pm(x^\mu,y^\mu)|_{x^0=y^0} = 0$, since this is an odd integrand over a symmetric domain. Thus, a direct calculation shows

$$\begin{aligned}
P_\epsilon G_\epsilon^\pm f(x^\mu) &\overset{(4.3.1)}{=} P_\epsilon \int d^{d+1}y^\mu\, G_\epsilon^\pm(x^\mu,y^\mu) f(y^\mu) \\
&\overset{(4.1.18)}{=} \mp P_\epsilon \int d^{d+1}y^\mu\, \theta(\mp(x^0-y^0)) \int \frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p}} f(y^\mu) \\
&= \mp(\partial_{x^0}^2 - \epsilon^2\nabla^2 + \epsilon^2 m^2) \int d^{d+1}y^\mu\, \theta(\mp(x^0-y^0)) \int \frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p}} f(y^\mu) \\
&= \partial_{x^0}^2 \int d^dy \int_{\pm\infty}^{x^0} dy^0 \int \frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p}} f(y^\mu) \\
&\quad \mp \int d^{d+1}y^\mu\, \theta(\mp(x^0-y^0)) \int \frac{d^dp}{(2\pi)^d}\ (\epsilon^2p^2+\epsilon^2m^2)\frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p}} f(y^\mu)
\end{aligned}$$

$$
\begin{aligned}
&= \partial_{x^0}\left(\int d^dy \int \frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}} f(y^\mu)\Big|_{y^0=x^0}\right.\\
&\quad \left.+\int d^dy\int_{\pm\infty}^{x^0} dy^0\,\partial_{x^0}\int\frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}} f(y^\mu)\right)\\
&\quad \mp\int d^{d+1}y^\mu\,\theta(\mp(x^0-y^0))\int\frac{d^dp}{(2\pi)^d}\ E_{\epsilon,p}\sin((x^\mu-y^\mu)p_\mu)|_{p_0=E_{\epsilon,p}} f(y^\mu)\\
&\overset{(4.3.3)}{=} \int d^dy\int\frac{d^dp}{(2\pi)^d}\ \partial_{x^0}\frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}} f(y^\mu)\Big|_{y^0=x^0}\\
&\quad +\int d^dy\int_{\pm\infty}^{x^0} dy^0\,\partial_{x^0}^2\int\frac{d^dp}{(2\pi)^d}\ \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}} f(y^\mu)\\
&\quad \mp\int d^{d+1}y^\mu\,\theta(\mp(x^0-y^0))\int\frac{d^dp}{(2\pi)^d}\ E_{\epsilon,p}\sin((x^\mu-y^\mu)p_\mu)|_{p_0=E_{\epsilon,p}} f(y^\mu)\\
&= \int d^dy\int\frac{d^dp}{(2\pi)^d}\ \cos(-(x-y)p)f(x^0,y)\\
&\quad \mp\int d^{d+1}y^\mu\,\theta(\mp(x^0-y^0))\int\frac{d^dp}{(2\pi)^d}\ -E_{\epsilon,p}^2\frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}} f(y^\mu)\\
&\quad \mp\int d^{d+1}y^\mu\,\theta(\mp(x^0-y^0))\int\frac{d^dp}{(2\pi)^d}\ E_{\epsilon,p}\sin((x^\mu-y^\mu)p_\mu)|_{p_0=E_{\epsilon,p}} f(y^\mu)\\
&= \int d^dy\int\frac{d^dp}{(2\pi)^d}\ \cos(-(x-y)p)f(x^0,y)\\
&= \int d^dy\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\left(e^{-i(x-y)p)}+e^{i(x-y)p)}\right)f(x^0,y)\\
&= \int d^dy\frac{1}{2}\left(\delta^d(-(x-y))+\delta^d(x-y)\right)f(x^0,y)=f(x^\mu)\,,
\end{aligned}
$$

for any $f\in C_c^\infty(\mathbb{R}^{d+1})$, which implies (i). This finally implies that $G_\epsilon^\pm(x^\mu,y^\mu)$ satisfies Equation (4.1.1), which we commented on in the remark after Equation (4.1.17).

(ii):

For the next calculation we need do distinguish on which variables the operator P_ϵ acts. If it acts on the variables x^μ will denote this by P_ϵ^x. So let $f\in C_c^\infty(\mathbb{R}^{d+1})$. Then

$$
\begin{aligned}
G_\epsilon^\pm P_\epsilon f(x^\mu) &\overset{(4.3.1)}{=} \int d^{d+1}y^\mu\,G_\epsilon^\pm(x^\mu,y^\mu)(P_\epsilon^y f)(y^\mu) \overset{(4.1.19)}{=} \int d^{d+1}y^\mu\,G_\epsilon^\mp(y^\mu,x^\mu)(P_\epsilon^y f)(y^\mu)\\
&\overset{(4.1.18)}{=} \pm\int d^{d+1}y^\mu\,\theta(\pm(y^0-x^0))\int\frac{d^dp}{(2\pi)^d}\ \frac{\sin((y^\mu-x^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}}(P_\epsilon^y f)(y^\mu)\\
&= \int d^dy\int_{x^0}^{\pm\infty}dy^0\int\frac{d^dp}{(2\pi)^d}\ \frac{\sin((y^\mu-x^\mu)p_\mu)}{E_{\epsilon,p}}\Big|_{p_0=E_{\epsilon,p}}((\partial_{y^0}^2-\epsilon^2\nabla_y^2+\epsilon^2m^2)f)(y^\mu)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int d^d y \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \left.\partial_{y^0} f(y^\mu)\right|_{y^0=x^0} \\
&\quad - \int d^d y \int_{x^0}^{\pm\infty} dy^0\, \partial_{y^0} \left(\int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \right) \partial_{y^0} f(y^\mu) \\
&\quad + \int d^d y \int_{x^0}^{\pm\infty} dy^0\, (-\epsilon^2 \nabla_y^2 + \epsilon^2 m^2) \left(\int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \right) f(y^\mu) \\
&\overset{(4.3.3)}{=} \int d^d y\, \partial_{y^0} \left.\left(\int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \right) f(y^\mu)\right|_{y^0=x^0} \\
&\quad + \int d^d y \int_{x^0}^{\pm\infty} dy^0\, \partial_{y^0}^2 \left(\int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \right) f(y^\mu) \\
&\quad + \int d^d y \int_{x^0}^{\pm\infty} dy^0\, (-\epsilon^2 \nabla_y^2 + \epsilon^2 m^2) \left(\int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((y^\mu - x^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \right) f(y^\mu)
\end{aligned}
$$

From this point onward it is the same calculation as for (i). Hence (ii) holds.
With that (ii) we can imply that $G^\pm_{\epsilon,\eta}(x^\mu, y^\mu)$ satisfies Equation (4.1.5), by useing Plancherel's theorem [Ste03, 6.2.4] on $G^\pm_\epsilon P_\epsilon f(x^\mu)$ as a distributional limit and therefore completing the remark after Equation (4.1.17).

(iii) and (iii'):
To show (iii) (or (iii')) we first show that $G^\pm_\epsilon(x^\mu, y^\mu) = 0$ whenever x^μ and y^μ are spacelike to each other, i.e., $x^\mu - y^\mu$ is spacelike.
For the next calculation we want to keep track of the parameter m. So we denote $G^\pm_\epsilon(x^\mu, y^\mu)$ as $G^\pm_{\epsilon,m}(x^\mu, y^\mu)$. We rewrite

$$
\begin{aligned}
G^\pm_{\epsilon,m}(x^\mu, y^\mu) &\overset{(4.1.18)}{=} \mp\theta(\mp(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \\
&\overset{p\mapsto \frac{1}{\epsilon}p}{=} \mp\theta(\mp(x^0 - y^0)) \frac{1}{\epsilon^d} \int \frac{d^d p}{(2\pi)^d} \left.\frac{\sin((x^0 - y^0)p_0 - \frac{1}{\epsilon}(x-y)p)}{\sqrt{p^2 + \epsilon^2 m^2}}\right|_{p_0=\sqrt{p^2+\epsilon^2 m^2}} \\
&= \frac{1}{\epsilon^d} G^\pm_{1,\epsilon m}((x^0, \frac{1}{\epsilon}x), (y^0, \frac{1}{\epsilon}y))\,.
\end{aligned}
$$

Whenever now u^μ and v^μ are spacelike to each other we know we can find a Lorentz transform Λ_{uv} such that $(\Lambda_{uv}(u - v))^0 = 0$.[3] This then implies $G^\pm_1(u^\mu, v^\mu) = 0$ since

$$
G^\pm_1(u^\mu, v^\mu) \overset{(4.1.20)}{=} G^\pm_1((\Lambda_{uv}u)^\mu, (\Lambda_{uv}v)^\mu)
$$

[3]Sketch: For any spacelike $x^\mu \in \mathbb{R}^{d+1}$ we can first find a Lorentz transform Λ_1 such that $(\Lambda_1 x)^\mu = (x^0, |x|, 0, \ldots, 0)$. Since $\frac{|x_0|}{|x|} < 1$ for x^μ is spacelike we can then find a boost Λ_2 in the 1-direction with velocity $v = \frac{x_0}{|x|}$. $\Lambda_2\Lambda_1$ then has the desired property.

$$
\begin{aligned}
&\overset{(4.1.18)}{=} \mp\theta(\mp(\Lambda_{uv}(u-v))^0) \int \frac{d^dp}{(2\pi)^d} \left.\frac{\sin((\Lambda_{uv}(u-v))^\mu p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \\
&= \mp\theta(0) \int \frac{d^dp}{(2\pi)^d} \left.\frac{\sin((\Lambda_{uv}(u-v))^j p_j)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} \\
&= \mp \int \frac{d^dp}{(2\pi)^d} \frac{\sin(\Lambda_{uv}(u-v)p)}{E_{\epsilon,p}} = 0\,, \qquad (4.3.4)
\end{aligned}
$$

where in the last step we used that the integral has an odd integrand and is over a symmetric domain.
Hence, what remains to show is that $(x^0, \frac{1}{\epsilon}x)$ is spacelike to $(y^0, \frac{1}{\epsilon}y)$ whenever x^μ is spacelike to y^μ. If that is the case, then $G^\pm_{\epsilon,m}(x^\mu, y^\mu) = \frac{1}{\epsilon^d} G^\pm_{1,\epsilon m}((x^0, \frac{1}{\epsilon}x), (y^0, \frac{1}{\epsilon}y)) = 0$ by Equation (4.3.4).
Since we restricted to $\epsilon \in (0,1]$ we have that $\frac{1}{\epsilon} \geq 1$. Hence

$$
(x^0 - y^0)^2 - (\frac{1}{\epsilon}x - \frac{1}{\epsilon}y)^2 = (x^0 - y^0)^2 - \frac{1}{\epsilon^2}(x-y)^2 \leq (x^0 - y^0)^2 - (x-y)^2 < 0\,,
$$

since x^μ is spacelike to y^μ. Therefore $(x^0, \frac{1}{\epsilon}x) - (y^0, \frac{1}{\epsilon}y)$ is spacelike.
We have now shown that $G^\pm_\epsilon(x^\mu, y^\mu) = 0$ whenever x^μ and y^μ are spacelike to each other.
Let now $f \in C^\infty_c(\mathbb{R}^{d+1})$ and $x^\mu \notin J^+(\operatorname{supp} f)$. This means that that there does not exist a future directed causal curve from any $y^\mu \in \operatorname{supp} f$ to x^μ. Since we are in Minkwoski space, this means that for all $y^\mu \in \operatorname{supp} f$ we do not have both $(x^0 - y^0)^2 - (x-y)^2 \geq 0$ and $x^0 - y^0 \geq 0$. We write

$$
\begin{aligned}
G^-_\epsilon f(x^\mu) &\overset{(4.3.1)}{=} \int d^{d+1}y^\mu\, G^-_\epsilon(x^\mu, y^\mu) f(y^\mu) = \int_{\operatorname{supp} f} d^{d+1}y^\mu\, G^-_\epsilon(x^\mu, y^\mu) f(y^\mu) \\
&= \int_{\{y^\mu \in \operatorname{supp} f | x^\mu - y^\mu \text{ spacelike}\}} d^{d+1}y^\mu\, G^-_\epsilon(x^\mu, y^\mu) f(y^\mu) \\
&\quad + \int_{\{y^\mu \in \operatorname{supp} f | x^\mu - y^\mu \text{ not spacelike}\}} d^{d+1}y^\mu\, G^-_\epsilon(x^\mu, y^\mu) f(y^\mu)\,.
\end{aligned}
$$

The first integral vanishes, since $G^-_\epsilon(x^\mu, y^\mu) = 0$ by the above calculation, and the second one is also 0 because if $x^\mu - y^\mu$ is not spacelike, we must have $x^0 - y^0 < 0$ by the assumption and

$$
G^-_\epsilon(x^\mu, y^\mu) \overset{(4.1.18)}{=} \theta(x^0 - y^0) \int \frac{d^dp}{(2\pi)^d} \left.\frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}}\right|_{p_0=E_{\epsilon,p}} = 0
$$

whenever $x^0 - y^0 < 0$.
Therefore (iii) holds for G^-_ϵ and analogously (iii') holds for G^+_ϵ.

This means that G^-_ϵ is an advanced Green operator for P_ϵ according to Definition 4.3.1 and similarly G^+_ϵ is a retarded Green operator for P_ϵ. This small discrepancy arises simply because [BGP07] uses a different naming convention as is usual in physics. We shall carefully ignore that difference and remain with the convention that G^+_ϵ and G^-_ϵ are the advanced and retarded scaled Green operator respectively.

4.3.2. Connection to the Solution Space

So we have finally shown that $G_\epsilon^\pm$ are Green operators according to Definition 4.3.1. We want to use the following theorem.

Theorem 4.3.3
Let M be a connected time-oriented Lorentzian manifold. Let P a normally hyperbolic operator acting on sections in a vector bundle E over M. Let G_+ and G_- be the advanced and retarded Green operators for P respectively.
Then the sequence of linear maps

$$0 \longrightarrow \mathcal{D}(M,E) \xrightarrow{P} \mathcal{D}(M,E) \xrightarrow{G} C^\infty_{sc}(M,E) \xrightarrow{P} C^\infty_{sc}(M,E)$$

is a complex, i.e., the composition of any two subsequent maps is zero. The complex is exact at the first $\mathcal{D}(M,E)$. If M is globally hyperbolic, then the complex is exact everywhere. [BGP07, Theorem 3.4.7]

Remark. For example $\cdots \xrightarrow{\phi} V \xrightarrow{\psi} \cdots$ being a complex means that $\operatorname{Im}\phi \subseteq \operatorname{Ker}\psi$. To be exact at V means $\operatorname{Im}\phi = \operatorname{Ker}\psi$ [Spa66, 4.5].

Some clarifications are in order. $C^\infty_{sc}(M,E)$ are the smooth sections, which are "spacially compact", meaning that for every $\phi \in C^\infty_{sc}(M,E)$ there exists a $K \subseteq M$ compact such that $\operatorname{supp}\phi \subseteq J^M(K)$. The map G is defined as $G := G_+ - G_- : \mathcal{D}(M,E) \longrightarrow C^\infty_{sc}(M,E)$. Note that by property (iii) and (iii') of $G_\pm$ we can also write $G_\pm : \mathcal{D}(M,E) \longrightarrow C^\infty_{sc}(M,E)$ for the Green operators.

We can now apply Theorem 4.3.3 to P_ϵ. To avoid confusion we denote
$\Delta G_\epsilon := G_\epsilon^+ - G_\epsilon^- : C^\infty_c(\mathbb{R}^{d+1}) \longrightarrow C^\infty_{sc}(\mathbb{R}^{d+1})$. This yields the exact complex

$$0 \longrightarrow C^\infty_c(\mathbb{R}^{d+1}) \xrightarrow{P_\epsilon} C^\infty_c(\mathbb{R}^{d+1}) \xrightarrow{\Delta G_\epsilon} C^\infty_{sc}(\mathbb{R}^{d+1}) \xrightarrow{P_\epsilon} C^\infty_{sc}(\mathbb{R}^{d+1})\,. \tag{4.3.5}$$

Remark. We mentioned before, that contrary to our naming convetion G_ϵ^+ is the retarded and G_ϵ^- is the advanced Green operator for P_ϵ according to Definition 4.3.1. This means, strictly speaking, that we should use $\Delta G'_\epsilon := G_\epsilon^- - G_\epsilon^+$ for the exact complex 4.3.5. But by linearity of $\Delta G'_\epsilon = -\Delta G_\epsilon$ we have $\operatorname{Ker}\Delta G'_\epsilon = \operatorname{Ker}\Delta G_\epsilon$ and $\operatorname{Im}\Delta G'_\epsilon = \operatorname{Im}\Delta G_\epsilon$, which means for the complex it makes no difference.

Since there are two instances of P_ϵ in this complex with different domains, we denote with $\operatorname{Im}_c P_\epsilon$ the image of $P_\epsilon : C^\infty_c(\mathbb{R}^{d+1}) \longrightarrow C^\infty_c(\mathbb{R}^{d+1})$ and with $\operatorname{Ker}_{sc} P_\epsilon$ the kernel of $P_\epsilon : C^\infty_{sc}(\mathbb{R}^{d+1}) \longrightarrow C^\infty_{sc}(\mathbb{R}^{d+1})$.
The exact complex (4.3.5) then implies that $\operatorname{Im}_c P_\epsilon = \operatorname{Ker}\Delta G_\epsilon$ and $\operatorname{Im}\Delta G_\epsilon = \operatorname{Ker}_{sc} P_\epsilon$. This allows us to do the following.

$$C^\infty_c(\mathbb{R}^{d+1}) \big/ \operatorname{Im}_c P_\epsilon = C^\infty_c(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\Delta G_\epsilon \cong \operatorname{Im}\Delta G_\epsilon = \operatorname{Ker}_{sc} P_\epsilon\,, \tag{4.3.6}$$

where we used the isomorphism theorem for vector spaces.

We call $\mathrm{Sol}(P_\epsilon) := \mathrm{Ker}_{sc} P_\epsilon$ the *solution space* of P_ϵ as it contains all spatially compact functions ϕ such that $P_\epsilon \phi = 0$. Since $C_c^\infty(\mathbb{R}^{d+1}) \subseteq C_{sc}^\infty(\mathbb{R}^{d+1})$, $\mathrm{Sol}(P_\epsilon)$ also contains all solutions with compact support. For the case $\epsilon = 1$ this has already been done in [FR19].

As the final piece of the puzzle and the last part of this section, we show the connection to the spaces $V_\epsilon = C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker}\,\sigma_\epsilon$ from Section 2.3.
We begin by setting $\Delta G_\epsilon(x^\mu, y^\mu) := G_\epsilon^+(x^\mu, y^\mu) - G_\epsilon^-(x^\mu, y^\mu)$ and calculating

$$\begin{aligned}\Delta G_\epsilon(x^\mu, y^\mu) &= G_\epsilon^+(x^\mu, y^\mu) - G_\epsilon^-(x^\mu, y^\mu) = \\ &= -\theta(-(x^0 - y^0)) \int \frac{d^d p}{(2\pi)^d} \left. \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}} \right|_{p_0 = E_{\epsilon,p}} \\ &\quad - \theta(x^0 - y^0) \int \frac{d^d p}{(2\pi)^d} \left. \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}} \right|_{p_0 = E_{\epsilon,p}} \\ &= - \int \frac{d^d p}{(2\pi)^d} \left. \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}} \right|_{p_0 = E_{\epsilon,p}} . \end{aligned} \tag{4.3.7}$$

Clearly $\Delta G_\epsilon f(x^\mu) = \int d^{d+1}y^\mu \, \Delta G_\epsilon(x^\mu, y^\mu) f(y^\mu)$. We quickly show three properties of $\Delta G_\epsilon(x^\mu, y^\mu)$. First

$$\Delta G_\epsilon(x^\mu, y^\mu)|_{x^0 = y^0} = 0 \tag{4.3.8}$$

follows from (4.3.7) and (4.3.3). The second one is

$$\partial_{x^0} \Delta G_\epsilon(x^\mu, y^\mu)|_{x^0 = y^0} = -\delta^d(x - y)\,. \tag{4.3.9}$$

We can see that this holds by a small calculation.

$$\begin{aligned}\partial_{x^0} \Delta G_\epsilon(x^\mu, y^\mu)|_{x^0 = y^0} &\overset{(4.3.7)}{=} - \int \frac{d^d p}{(2\pi)^d} \, \partial_{x^0} \left. \left. \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}} \right|_{p_0 = E_{\epsilon,p}} \right|_{x^0 = y^0} \\ &= - \int \frac{d^d p}{(2\pi)^d} \cos(-(x-y)p) = - \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \left(e^{-i(x-y)p} + e^{i(x-y)p} \right) \\ &= -\frac{1}{2} \left(\delta^d(-(x-y)) + \delta^d(x-y) \right) = -\delta^d(x-y)\,. \end{aligned}$$

Lastly, $\Delta G_\epsilon(x^\mu, y^\mu)$ is antisymmetric.

$$\begin{aligned}\Delta G_\epsilon(x^\mu, y^\mu) &= G_\epsilon^+(x^\mu, y^\mu) - G_\epsilon^-(x^\mu, y^\mu) \overset{(4.1.19)}{=} G_\epsilon^-(y^\mu, x^\mu) - G_\epsilon^+(y^\mu, x^\mu) \\ &= -\Delta G_\epsilon(y^\mu, x^\mu)\,. \end{aligned} \tag{4.3.10}$$

Consider now the map

$$\begin{aligned}\Delta \tilde{G}_\epsilon(f, g) &:= \int d^{d+1}x^\mu \, f(x^\mu) \Delta G_\epsilon g(x^\mu) \\ &= \int d^{d+1}x^\mu \int d^{d+1}y^\mu \, \Delta G_\epsilon(x^\mu, y^\mu) f(x^\mu) g(y^\mu)\,. \end{aligned} \tag{4.3.11}$$

We note immediately, that $\Delta\tilde{G}_\epsilon(f,g)$ is antisymmetric since

$$\begin{aligned}\Delta\tilde{G}_\epsilon(f,g) &\overset{(4.3.11)}{=} \int d^{d+1}x^\mu \int d^{d+1}y^\mu\, \Delta G_\epsilon(x^\mu,y^\mu) f(x^\mu) g(y^\mu) \\ &\overset{(4.3.10)}{=} -\int d^{d+1}x^\mu \int d^{d+1}y^\mu\, \Delta G_\epsilon(y^\mu,x^\mu) f(x^\mu) g(y^\mu) \overset{(4.3.11)}{=} -\Delta\tilde{G}_\epsilon(g,f)\,.\end{aligned}$$

Equation (4.3.11) can be rewritten as

$$\begin{aligned}\Delta\tilde{G}_\epsilon(f,g) &\overset{(4.3.11)}{=} \int d^{d+1}x^\mu \int d^{d+1}y^\mu\, \Delta G_\epsilon(x^\mu,y^\mu) f(x^\mu) g(y^\mu) \\ &\overset{(4.3.7)}{=} -\int d^{d+1}x^\mu \int d^{d+1}y^\mu \int \frac{d^dp}{(2\pi)^d}\, \frac{\sin((x^\mu-y^\mu)p_\mu)}{E_{\epsilon,p}}\bigg|_{p_0=E_{\epsilon,p}} f(x^\mu)g(y^\mu) \\ &= -\int \frac{d^dp}{(2\pi)^d} \int d^{d+1}x^\mu \int d^{d+1}y^\mu\, \frac{1}{2iE_{\epsilon,p}} \big(e^{ix^\mu p_\mu} e^{-iy^\mu p_\mu} \\ &\quad - e^{-ix^\mu p_\mu} e^{iy^\mu p_\mu}\big)\big|_{p_0=E_{\epsilon,p}} f(x^\mu)g(y^\mu) \\ &= -i\frac{1}{2}\int \frac{d^dp}{(2\pi)^d E_{\epsilon,p}} \Big(\hat{f}(E_{\epsilon,p},p)\overline{\hat{g}(E_{\epsilon,p},p)} - \overline{\hat{f}(E_{\epsilon,p},p)}\hat{g}(E_{\epsilon,p},p)\Big)\,,\end{aligned}$$

which is up to a factor of $-i$ the same as $\left[\phi_\epsilon(f),\phi_\epsilon^\dagger(g)\right]$ by Equation (2.2.4). As we have already announced during the calculation of Equation (4.1.20), this is not a coincidence. The integral kernel of $\left[\phi_\epsilon(f),\phi_\epsilon^\dagger(g)\right]$, i.e., $\left[\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right]$ is a solution of the scaled Klein-Gordon equation since ϕ_ϵ is also a solution.

$$P_e^x\left[\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right] = \left[P_e^x\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right] = 0\,.$$

By the equal-time commutation relations we have

$$\begin{aligned}\left[\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right]\Big|_{x^0=y^0} &= 0 \quad \text{and} \\ \partial_{x^0}\left[\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right]\Big|_{x^0=y^0} &= \left[\partial_{x^0}\phi_\epsilon(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right]\Big|_{x^0=y^0} = \left[\Pi_\epsilon^\dagger(x^\mu),\phi_\epsilon^\dagger(y^\mu)\right]\Big|_{x^0=y^0} \\ &= -i\delta^d(x-y)\,.\end{aligned}$$

Similarly, $i\Delta G_\epsilon(x^\mu,y^\mu)$ is a solution since Equation (4.3.5) implies

$$0 = iP_e^x\Delta G_\epsilon f(x^\mu) = \int d^{d+1}y^\mu\, P_e^x i\Delta G_\epsilon(x^\mu,y^\mu)f(y^\mu)$$

for all $f\in C_c^\infty(\mathbb{R}^{d+1})$ and all $x^\mu\in\mathbb{R}^{d+1}$, which in turn implies that $P_e^x\Delta G_\epsilon(x^\mu,y^\mu)=0$ by the nondegeneracy of the L^2-inner product. We also have the initial conditions

$$i\Delta G_\epsilon(x^\mu,y^\mu)|_{x^0=y^0} = 0 \quad \text{and} \quad \partial_{x^0} i\Delta G_\epsilon(x^\mu,y^\mu)|_{x^0=y^0} = -i\delta^d(x-y)$$

by (4.3.8) and (4.3.9).

Hence, $i\Delta G_\epsilon(x^\mu, y^\mu) = \left[\phi_\epsilon(x^\mu), \phi_\epsilon^\dagger(y^\mu)\right]$ by the uniqueness of solutions for second order partial differential equations, which implies $i\Delta\tilde{G}_\epsilon(f,g) = \left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right]$. The $\epsilon = 1$ case then explains our choice for Equation (2.1.3), which was not really a choice for that reason.

So we know that $\sigma_\epsilon(f,g) \propto \left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right] \propto \Delta\tilde{G}_\epsilon(f,g)$ for all $f, g \in C_c^\infty(\mathbb{R}^{d+1})$, which implies $\operatorname{Ker}\sigma_\epsilon = \operatorname{Ker}\Delta\tilde{G}_\epsilon := \{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \Delta\tilde{G}_\epsilon(f,g) = 0\,\forall g \in C_c^\infty(\mathbb{R}^{d+1})\}$. The claim is now that $\operatorname{Ker}\Delta\tilde{G}_\epsilon = \operatorname{Ker}\Delta G_\epsilon$.
If $f \in \operatorname{Ker}\Delta G_\epsilon$ then $\Delta G_\epsilon f(x^\mu) = 0$ for all $x^\mu \in \mathbb{R}^{d+1}$ which yields $\Delta\tilde{G}_\epsilon(f,g) = -\Delta\tilde{G}_\epsilon(g,f) = -\int d^{d+1}x^\mu\, g(x^\mu)\Delta G_\epsilon f(x^\mu) = 0$ for any $g \in C_c^\infty(\mathbb{R}^{d+1})$. Hence $f \in \operatorname{Ker}\Delta\tilde{G}_\epsilon$.
Conversely, if $f \in \operatorname{Ker}\Delta\tilde{G}_\epsilon$ then $0 = \Delta\tilde{G}_\epsilon(f,g) = -\Delta\tilde{G}_\epsilon(g,f) = -\int d^{d+1}x^\mu\, g(x^\mu)\Delta G_\epsilon f(x^\mu)$ for all $g \in C_c^\infty(\mathbb{R}^{d+1})$, which implies $\Delta G_\epsilon f = 0$ by the nondegeneracy of the L^2-inner product. Hence $f \in \operatorname{Ker}\Delta G_\epsilon$ and the claim holds.
Therefore, $\operatorname{Ker}\sigma_\epsilon = \operatorname{Ker}\Delta\tilde{G}_\epsilon = \operatorname{Ker}\Delta G_\epsilon$ and finally

$$V_\epsilon = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_\epsilon = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\Delta G_\epsilon \overset{(4.3.6)}{\cong} \operatorname{Sol}(P_\epsilon)\,.$$

This, at long last, finishes the argument for Equation (2.3.8).

4.4. Solution space for the Carrollian operator

The same argument from the previous section can be repeated for P_0 and $G_0^\pm$, provided that $G_0^\pm$ satisfy Definition 4.3.1. In this final section of Chapter 4 we want to show that $G_0^\pm$ really do satisfy Definition 4.3.1 and show explicitly what the isomorphism between V_0 and $\operatorname{Sol}(P_0)$ is.

We start directly by checking Definition 4.3.1 for $G_0^\pm : C_c^\infty(\mathbb{R}^{d+1}) \longrightarrow C^\infty(\mathbb{R}^{d+1})$, defined by

$$G_0^\pm f(x^\mu) := \int d^{d+1}y^\mu\, G_0^\pm(x^\mu, y^\mu) f(y^\mu)\,. \tag{4.4.1}$$

Linearity:
Linearity follows again from the linearity of the integral.

(i):
For any $f \in C_c^\infty(\mathbb{R}^{d+1})$, we can imply (i) by

$$\begin{aligned} P_0 G_0^\pm f(x^\mu) &\overset{(4.4.1)}{=} P_0 \int d^{d+1}y^\mu\, G_0^\pm(x^\mu, y^\mu) f(y^\mu) \\ &\overset{(4.2.2)}{=} \mp P_0 \int d^{d+1}y^\mu\, \theta(\mp(x^0 - y^0))(x^0 - y^0)\delta^d(x-y) f(y^\mu) \\ &= \partial_{x^0}^2 \int_{\pm\infty}^{x^0} dy^0\,(x^0 - y^0) f(y^0, x) \end{aligned}$$

$$= \partial_{x^0} \left((x^0 - y^0) f(y^0, x)\big|_{y^0 = x^0} + \int_{\pm\infty}^{x^0} dy^0\, \partial_{x^0} (x^0 - y^0) f(y^0, x) \right)$$
$$= \partial_{x^0} \int_{\pm\infty}^{x^0} dy^0\, f(y^0, x) = f(x^0, x)\,.$$

(ii):
Similarly, we can compute

$$G_0^{\pm} P_0 f(x^\mu) \overset{(4.4.1)}{=} \int d^{d+1}y^\mu\, G_0^{\pm}(x^\mu, y^\mu)(P_0^y f)(y^\mu) \overset{(4.2.3)}{=} \int d^{d+1}y^\mu\, G_0^{\mp}(y^\mu, x^\mu)(P_0^y f)(y^\mu)$$
$$\overset{(4.2.2)}{=} \pm \int d^{d+1}y^\mu\, \theta(\pm(y^0 - x^0))(y^0 - x^0)\delta^d(y - x)(P_\epsilon^y f)(y^\mu)$$
$$= \int_{x^0}^{\pm\infty} dy^0\, (y^0 - x^0)(\partial_{y^0}^2 f)(y^0, x)$$
$$= -\,(y^0 - x^0)(\partial_{y^0} f)(y^0, x)\big|_{y^0 = x^0} - \int_{x^0}^{\pm\infty} dy^0\, \partial_{y^0}[(y^0 - x^0)](\partial_{y^0} f)(y^0, x)$$
$$= -\int_{x^0}^{\pm\infty} dy^0\, (\partial_{y^0} f)(y^0, x) = f(x^0, x)$$

for any $f \in C_c^\infty(\mathbb{R}^{d+1})$, which implies (ii).

(iii) and (iii'):
We again first show that $G_0^{\pm}(x^\mu, y^\mu) = 0$ whenever x^μ and y^μ are spacelike to each other. But this follows quickly since $G_\epsilon^{\pm}(x^\mu, y^\mu) = 0$ whenever x^μ and y^μ are spacelike and $\lim_{\epsilon \to 0} G_\epsilon^{\pm}(x^\mu, y^\mu) = G_0^{\pm}(x^\mu, y^\mu)$.
The rest of the argument follows now identically to the analog for $G_\epsilon^{\pm}$. Let $f \in C_c^\infty(\mathbb{R}^{d+1})$ and $x^\mu \notin J^+(\operatorname{supp} f)$, i.e., for all $y^\mu \in \operatorname{supp} f$ we do not have both $(x^0 - y^0)^2 - (x - y)^2 \geq 0$ and $x^0 - y^0 \geq 0$.
We write

$$G_0^- f(x^\mu) \overset{(4.4.1)}{=} \int d^{d+1}y^\mu\, G_0^-(x^\mu, y^\mu) f(y^\mu) = \int_{\operatorname{supp} f} d^{d+1}y^\mu\, G_0^-(x^\mu, y^\mu) f(y^\mu)$$
$$= \int_{\{y^\mu \in \operatorname{supp} f | x^\mu - y^\mu \text{ spacelike}\}} d^{d+1}y^\mu\, G_\epsilon^-(x^\mu, y^\mu) f(y^\mu)$$
$$+ \int_{\{y^\mu \in \operatorname{supp} f | x^\mu - y^\mu \text{ not spacelike}\}} d^{d+1}y^\mu\, G_0^-(x^\mu, y^\mu) f(y^\mu)\,.$$

The first integral vanishes, since $G_0^-(x^\mu, y^\mu) = 0$ and the second one is also 0 since if $x^\mu - y^\mu$ is not spacelike we must have $x^0 - y^0 < 0$ by the assumption and

$$G_0^-(x^\mu, y^\mu) \overset{(4.2.2)}{=} \theta(x^0 - y^0)(x^0 - y^0)\delta^d(x - y) = 0$$

whenever $x^0 - y^0 < 0$.
Therefore (iii) holds for G_0^- and analogously (iii') holds for G_0^+.

Hence $G_0^\pm$ are Green operators for P_0 according to Definition 4.3.1.

Remark. Here we quickly note, that P_0 is not normally hyperbolic. Clearly $P_0 = \partial_{x^0}^2 \neq \eta^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu}$. However, as far as we can tell, the proof of Theorem 4.3.3 in [BGP07, p. 91] does not rely on P being normally hyperbolic as long as $G_\pm$ otherwise fulfill Definition 4.3.1. Only linearity of P is directly used. Therefore we can still apply this theorem P_0.

So by the same argument as before, we get the exact complex

$$0 \longrightarrow C_c^\infty(\mathbb{R}^{d+1}) \xrightarrow{P_0} C_c^\infty(\mathbb{R}^{d+1}) \xrightarrow{\Delta G_0} C_{sc}^\infty(\mathbb{R}^{d+1}) \xrightarrow{P_0} C_{sc}^\infty(\mathbb{R}^{d+1}) \,, \tag{4.4.2}$$

where $\Delta G_0 := G_0^+ - G_0^-$. We can again write $\Delta G_0 f$ as

$$\Delta G_0 f(x^\mu) = \int d^{d+1}y^\mu \, \Delta G_0(x^\mu, y^\mu) f(y^\mu) \,,$$

where

$$\begin{aligned} \Delta G_0(x^\mu, y^\mu) &= G_0^+(x^\mu, y^\mu) - G_0^-(x^\mu, y^\mu) \\ &= -\theta(-(x^0 - y^0))(x^0 - y^0)\delta^d(x-y) - \theta((x^0 - y^0)(x^0 - y^0)\delta^d(x-y) \\ &= -(x^0 - y^0)\delta^d(x-y) \,. \end{aligned} \tag{4.4.3}$$

We are now able to show that $G_0^\pm f$ is the pointwise limit of $G_\epsilon^\pm f$ as ϵ approaches 0 for every $f \in C_c^\infty(\mathbb{R}^{d+1})$.

$$\begin{aligned} G_\epsilon^\pm f(x^\mu) &\overset{(4.3.1)}{=} \mp \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0)) f(y^\mu) \int \frac{d^d p}{(2\pi)^d} \, \frac{\sin((x^\mu - y^\mu)p_\mu)}{E_{\epsilon,p}} \bigg|_{p_0 = E_{\epsilon,p}} \\ &= \mp \int \frac{d^d p}{(2\pi)^d} \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0)) f(y^\mu) \frac{1}{2iE_{\epsilon,p}} \\ &\quad \cdot \left(e^{i(x^\mu - y^\mu)p_\mu} - e^{-i(x^\mu - y^\mu)p_\mu} \right) \bigg|_{p_0 = E_{\epsilon,p}} \\ &\overset{p \mapsto -p}{=} \mp \int \frac{d^d p}{(2\pi)^d} \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0)) f(y^\mu) \frac{1}{2iE_{\epsilon,p}} e^{i(x-y)p} \\ &\quad \cdot \left(e^{i(x^0 - y^0)E_{\epsilon,p}} - e^{-i(x^0 - y^0)E_{\epsilon,p}} \right) \\ &= \mp \int \frac{d^d p}{(2\pi)^d} \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0)) f(y^\mu) \frac{1}{2iE_{\epsilon,p}} e^{i(x-y)p} \\ &\quad \cdot \left(1 + i(x^0 - y^0)E_{\epsilon,p} + \mathcal{O}(E_{\epsilon,p}^2) - (1 - i(x^0 - y^0)E_{\epsilon,p} + \mathcal{O}(E_{\epsilon,p}^2))\right) \\ &= \mp \int \frac{d^d p}{(2\pi)^d} \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0)) f(y^\mu) e^{i(x-y)p} \left((x^0 - y^0) + \mathcal{O}(E_{\epsilon,p})\right) \\ &\xrightarrow[\epsilon \to 0]{} \mp \int \frac{d^d p}{(2\pi)^d} \int d^{d+1}y^\mu \, \theta(\mp(x^0 - y^0))(x^0 - y^0) f(y^\mu) e^{-iyp} e^{ixp} \\ &= \mp \int \frac{d^d p}{(2\pi)^d} \int dy^0 \, \theta(\mp(x^0 - y^0))(x^0 - y^0) \mathcal{F}_x f(y^0, -p) e^{ixp} \end{aligned}$$

$$
\begin{aligned}
&= \mp \int dy^0 \, \theta(\mp(x^0 - y^0))(x^0 - y^0) f(y^0, x) \\
&= \mp \int d^{d+1} y^\mu \, \theta(\mp(x^0 - y^0))(x^0 - y^0) \delta^d(x - y) f(y^0, y) \\
&\overset{(4.2.2)}{=} \int d^{d+1} y^\mu \, G_0^{\pm}(x^\mu, y^\mu) f(y^\mu) \overset{(4.4.1)}{=} G_0^{\pm} f(x^\mu) \, .
\end{aligned}
\tag{4.4.4}
$$

Note that by (4.4.4) we also have $\lim_{\epsilon \to 0} \Delta G_\epsilon f(x^\mu) = \Delta G_0 f(x^\mu)$ which further implies

$$
\sigma_\epsilon(f, g) \propto \Delta \tilde{G}_\epsilon(f, g) = \int d^{d+1} x^\mu \, f(x^\mu) \Delta G_\epsilon g(x^\mu) \underset{\epsilon \to 0}{\longrightarrow} \int d^{d+1} x^\mu \, f(x^\mu) \Delta G_0 g(x^\mu) \propto \sigma_0(f, g)
$$

by the uniqueness of the limit. The swapping of the integral and the limit is justified because $f \in C_c^\infty(\mathbb{R}^{d+1})$. This further yields

$$
V_0 = C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker}\, \sigma_0 \cong \mathrm{Sol}(P_0) \, , \tag{4.4.5}
$$

by the same arguments as in the previous section.

We finish this section and this chapter by making the isomorphism in Equation (4.4.5) more explicit.
First, we take a closer look at $\mathrm{Sol}(P_0) = \mathrm{Ker}_{sc} P_0$. It is very easy to see, that $P_0 \phi(t, x) = \partial_t^2 \phi(t, x) = 0$, with initial conditions $\phi(0, x) = f(x)$ and $\partial_t \phi(0, x) = g(x)$, is solved by

$$
\phi(t, x) = f(x) + t g(x) \, .
$$

For $\phi \in \mathrm{Ker}_{sc} P_0$ we also require $\phi \in C_{sc}^\infty(\mathbb{R}^{d+1})$. This readily implies that f and g are smooth. By the discussion in [BGP07, 3.4.5] we gather that ϕ is spacially compact if the restriction $\phi|_S$ has compact support for any Cauchy surfaces S of $\mathbb{R}^{1,d}$. In Minkowski space, however, we have the time-slices $\{t = k\}$ for some $k \in \mathbb{R}$ are Cauchy surfaces. Hence $\phi|_{t=k}$ needs to have compact support for all k.
For $k = 0$ this means $\phi(0, x) = f(x)$ needs compact support and $g(x) = \phi(1, x) - \phi(0, x)$ has support contained in $\mathrm{supp}\, \phi(1, x) \cup \mathrm{supp}\, \phi(0, x)$, which is also compact.
Therefore we have

$$
\mathrm{Sol}(P_0) = \left\{ f + tg \mid f, g \in C_c^\infty(\mathbb{R}^d) \right\} \, , \tag{4.4.6}
$$

where tg is the function $(tg)(t, x) = tg(x)$.
We claim now that the $\Delta G_0 : V_0 \to \mathrm{Sol}(P_0)$, defined by $\Delta G_0(f + \mathrm{Ker}\, \sigma_0) := \Delta G_0 f$, is the isomorphism from (4.4.5).[4] We check now if $\Delta G_0 : V_0 \to \mathrm{Sol}(P_0)$ is well-defined and bijective.
For the well-definedness we remember that $\mathrm{Ker}\, \sigma_0 = \mathrm{Ker}\, \Delta G_0$ by the same argument presented in Subsection 4.3.2. Hence, for any $g \in \mathrm{Ker}\, \sigma_0$ we get

$$
\Delta G_0(f + g) := \Delta G_0 f + \Delta G_0 g = \Delta G_0 f \, .
$$

[4] If one is familiar with the proof of the isomorphism theorem then this is obvious.

Injectivity in our context means that whenever $\Delta G_0 f = 0$ we have $f \in \operatorname{Ker}\sigma_0$. By Equation (4.4.2) this means that $f \in \operatorname{Ker}\sigma_0 = \operatorname{Ker}\Delta G_0 = \operatorname{Im} P_0$. So let $f \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\Delta G_0 f = 0$. Since $\Delta G_0 = G_0^+ - G_0^-$ we have $G_0^+ f = G_0^- f$. Setting $g := G_0^+ f = G_0^- f$ yields a smooth function with compact support. Indeed, we have

$$\operatorname{supp} g \subseteq \operatorname{supp} G_0^+ f \cap \operatorname{supp} G_0^- f \subseteq J^+(\operatorname{supp} f) \cap J^-(\operatorname{supp} f)\,,$$

since $G_0^\pm$ are Green operators for P_0. The latter being compact Minkowski space implies that $\operatorname{supp} g$ is compact. $G_0^\pm$ being Green operators for P_0 also then yields $P_0 g = P_0 G_0^+ f = f$, hence $f \in \operatorname{Im} P_0 = \operatorname{Ker}\sigma_0$.

For the surjectivity we need to construct for every $\phi \in \mathrm{Sol}(P_0)$ a $\psi \in V_0$ such that $\Delta G_0 \psi = \phi$. So let $\phi \in \mathrm{Sol}(P_0)$. By Equation (4.4.6) there exists $f, g \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(t,x) = f(x) + t g(x)$. Define now

$$\psi(t,x) := -\mathcal{F}_t^{-1}[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)](t)\,, \tag{4.4.7}$$

where φ is defined the same as (3.1.26). Note that $\mathcal{F}_t\varphi$ is still a Schwartz function and so is $(g(x)+ip_0\, f(x))\mathcal{F}_t\varphi(p_0)$, which means we are allowed to take the inverse Fourier-transform. ψ is indeed real valued and has compact support since

$$\begin{aligned}
\psi(t,x) &= -\mathcal{F}_t^{-1}[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)](t) \\
&= -\mathcal{F}_t^{-1}[g(x)\mathcal{F}_t\varphi(p_0)](t) - \mathcal{F}_t^{-1}[ip_0\, f(x)\mathcal{F}_t\varphi(p_0)](t) \\
&= -g(x)\varphi(t) - \frac{d}{dt}\mathcal{F}_t^{-1}[f(x)\mathcal{F}_t\varphi(p_0)](t) \\
&= -g(x)\varphi(t) - f(x)\varphi'(t)
\end{aligned}$$

and every term in the last line has compact support. We can then calculate

$$\begin{aligned}
\Delta G_0\psi(t,x) &\overset{(4.4)}{=} \int d^{d+1}y^\mu \Delta G_0((t,x),y^\mu)\psi(y^\mu) \overset{(4.4.3)}{=} \int d^{d+1}y^\mu (y^0 - t)\delta^d(x-y)\psi(y^\mu) \\
&= \int dy^0 (y^0 - t)\psi(y^0,x) = \int dy^0\, y^0\psi(y^0,x) - t\int dy^0\, \psi(y^0,x) \\
&= \mathcal{F}_t[(.)\psi(.,x)](0) - t\mathcal{F}_t[\psi(.,x)](0) = i\frac{d}{dp_0}\mathcal{F}_t[\psi(.,x)](0) - t\mathcal{F}_t[\psi(.,x)](0) \\
&\overset{(4.4.7)}{=} -i\frac{d}{dp_0}[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)]|_{p_0=0} + t[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)]|_{p_0=0} \\
&= -i\left[if(x)\mathcal{F}_t\varphi(p_0) + (g(x) + ip_0\, f(x))\frac{d}{dp_0}\mathcal{F}_t\varphi(p_0)\right]\Bigg|_{p_0=0} \\
&\quad + t\,[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)]|_{p_0=0} \\
&\overset{(3.1.27)+(3.1.28)}{=} f(x) + tg(x) = \phi(t,x)\,.
\end{aligned}$$

Hence, ΔG_0 is surjective.

The construction in (4.4.7) also gives a clear method to define the inverse of $\Delta G_0 : V_0 \to \mathrm{Sol}(P_0)$, as the equivalence class of ψ is the preimage of ϕ, i.e., $\Delta G_0^{-1}\phi = \psi \in V_0$. There is also a very nice connection to ρ_0 from Equation (3.1.23). By Equation (3.1.25) we know that $\rho_0(W_0(\psi)) = 1$, for $\psi \in V_0$, if and only if $\widehat{\psi}(0, \,.\,) = 0$.
Taking $\phi = f + tg \in \mathrm{Sol}(P_0)$ and constructing ψ according to (4.4.7) we can compute

$$\begin{aligned}\widehat{\psi}(0,p) &= \mathcal{F}_x\mathcal{F}_t\psi(0,p) \overset{(4.4.7)}{=} -\mathcal{F}_x[(g(x) + ip_0\, f(x))\mathcal{F}_t\varphi(p_0)](p)|_{p_0=0} \\ &= -\mathcal{F}_x[g(x)\mathcal{F}_t\varphi(0)](p) \overset{(3.1.27)}{=} \hat{g}(p)\,.\end{aligned}$$

Hence $\rho_0(W_0(\psi)) = 1$ if and only if $g = 0$, since the Fourier transform is unitary on Schwartz functions and therefore $\hat{g} = 0 \iff g = 0$.

5. Magnetic Carrollian limit

For the entirety of this thesis we have focused on the fields ϕ and have never considerd what happens to the conjugate momenta π. In this short chapter we repeat the whole process and show that we can obtain a different limit algebra.
The bulk of the calculations is done in Appendix B, as they are essentially identical to what we have done before, and we will only present the main results.

5.1. Magnetic Carrollian limit algebra

We consider the scaled momenta

$$\pi_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\pi(\epsilon t, x) \quad \text{and} \quad \pi_\epsilon^\dagger(t,x) := \epsilon^{-\frac{1}{2}}\pi^\dagger(\epsilon t, x)\,.$$

Immediately something interesting arises. The conjugate momentum of ϕ_ϵ and $\phi_\epsilon^\dagger$ can be computed, for $\epsilon > 0$, via[1]

$$\begin{aligned}\tilde{\pi}_\epsilon(t,x) &= \partial_t\phi_\epsilon^\dagger(t,x) = \epsilon^{-\frac{1}{2}}\partial_t\phi^\dagger(\epsilon t,x) = \epsilon^{\frac{1}{2}}\pi(\epsilon t,x)\\ \tilde{\pi}_\epsilon^\dagger(t,x) &= \partial_t\phi_\epsilon(t,x) = \epsilon^{-\frac{1}{2}}\partial_t\phi(\epsilon t,x) = \epsilon^{\frac{1}{2}}\pi^\dagger(\epsilon t,x)\,.\end{aligned}$$

The necessary prefactor of $\tilde{\pi}_\epsilon(t,x)$ (and $\tilde{\pi}_\epsilon^\dagger(t,x)$) can also be deduced from the equal-time canonical commutator relations since

$$[\phi(t,x),\pi(t,y)] = i\delta^d(x-y) = [\phi_\epsilon(t,x),\tilde{\pi}_\epsilon(t,y)]\,.$$

Hence $\tilde{\pi}_\epsilon \neq \pi_\epsilon$ and $\tilde{\pi}_\epsilon^\dagger \neq \pi_\epsilon^\dagger$, yet we need π_ϵ and $\pi_\epsilon^\dagger$ for a convergent commutator, as we will see shortly. This is an early sign that the results will be different from the electric case.

The commutator is then given by

$$\left[\pi(t,x),\pi^\dagger(s,y)\right] = -\partial_t\partial_s\left[\phi(t,x),\phi^\dagger(s,y)\right] \overset{(2.1.2)}{=} -\partial_t\partial_s\Delta(t-s,x-y)\,.$$

Smearing the fields with $f,g \in C_c^\infty(\mathbb{R}^{d+1})$ yields

$$\left[\pi_\epsilon(f),\pi_\epsilon^\dagger(g)\right] = \frac{1}{2}\epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}E_p\left(\hat{f}(\epsilon E_p,p)\overline{\hat{g}(\epsilon E_p,p)} - \overline{\hat{f}(\epsilon E_p,p)}\hat{g}(\epsilon E_p,p)\right)\,,$$

[1] We can consider the Lagrangian $\mathcal{L} = \partial_0\phi_\epsilon\partial_0\phi_\epsilon^\dagger - \epsilon^2\nabla\phi_\epsilon\nabla\phi_\epsilon^\dagger - \epsilon^2m^2\phi_\epsilon\phi_\epsilon^\dagger$, which yields the scaled Klein-Gordon equation $P_\epsilon\psi = 0$ as equations of motion.

which we can then turn into a proto-symplectic form, i.e., a bilinear antisymmetric map on $C_c^\infty(\mathbb{R}^{d+1})$, by defining

$$\sigma_{\pi,\epsilon}(f,g) := \frac{i}{2}\left[\pi_\epsilon(f), \pi_\epsilon^\dagger(g)\right] = \frac{i}{4}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d} E_p \left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right). \tag{5.1.1}$$

Hence, if we go to $V_{\pi,\epsilon} := C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_{\pi,\epsilon}$, where

$$\operatorname{Ker}\sigma_{\pi,\epsilon} := \{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \sigma_{\pi,\epsilon}(f,g) = 0 \,\forall g \in C_c^\infty(\mathbb{R}^{d+1})\},$$

we get the symplectic space $(V_{\pi,\epsilon}, \sigma_{\pi,\epsilon})$. Similarly, we can turn

$$\sigma_{\pi,0}(f,g) := \frac{i}{2}\lim_{\epsilon\to 0}\left[\pi_\epsilon(f), \pi_\epsilon^\dagger(g)\right] = \frac{i}{2}\int \frac{d^dp}{(2\pi)^d} E_p^2 \left(\frac{\partial \hat{f}}{\partial p_0}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial \hat{g}}{\partial p_0}(0,p)\right)$$

into a symplectic form on $V_{\pi,0} := C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_{\pi,0}$.

Applying Theorem 2.3.1 yields the Weyl-algebras $\mathcal{W}_{\pi,\epsilon} := \mathcal{W}(V_{\pi,\epsilon}, \sigma_{\pi,\epsilon})$ and $\mathcal{W}_{\pi,0} := \mathcal{W}(V_{\pi,0}, \sigma_{\pi,0})$. We call $\mathcal{W}_{\pi,0}$ the *magnetic Carrollian (limit) Weyl-algebra* of $\{\mathcal{W}_{\pi,\epsilon} \mid \epsilon > 0\}$.

5.2. Connection between underling vector spaces

To construct an analog of ρ_0 we first need to represent $\mathcal{W}_{\pi,\epsilon}$ on a Fock space. For this reason we define the operators

$$\Pi_\epsilon(f) := \frac{1}{\sqrt{2}}\left(\pi_\epsilon(f) + \pi_\epsilon^\dagger(f)\right) \quad \text{and}$$
$$W_{\pi,\epsilon}(f) := e^{i\Pi_\epsilon(f)}$$

for $f \in C_c^\infty(\mathbb{R}^{d+1})$. To apply the uniqueness clause of Theorem 2.3.1 we need to show that the family $\{W_{\pi,\epsilon}(f) \mid f \in V_{\pi,\epsilon} \in\}$ satisfies Weyl-relations. A crucial ingredient for this when showing the same for $\{W_\epsilon(f) \mid f \in V_\epsilon \in\}$ was Equation (2.3.8), i.e., $\operatorname{Im} P_\epsilon = \operatorname{Ker}\sigma_\epsilon$, which we got as a result from Chapter 4. It was used in showing that $W_\epsilon(f+h) = W_\epsilon(f)$, whenever $h \in \operatorname{Ker}\sigma_\epsilon$.
What we are going to show is that $\operatorname{Ker}\sigma_\epsilon = \operatorname{Ker}\sigma_{\pi,\epsilon}$. This then implies that $\operatorname{Im} P_\epsilon = \operatorname{Ker}\sigma_{\pi,\epsilon}$ which allows us to proceed with constructing a representation of $\mathcal{W}_{\pi,\epsilon}$ in a similar manner as we did for $\mathcal{W}_\epsilon$ in the next section. Moreover this implies that $V_{\pi,\epsilon} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_{\pi,\epsilon} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_\epsilon = V_\epsilon$.
We are also going to show $\operatorname{Ker}\sigma_0 = \operatorname{Ker}\sigma_{\pi,0}$, which implies that $V_{\pi,0} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_{\pi,0} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \operatorname{Ker}\sigma_0 = V_0$. Hence the classification of V_0, which we did in Section 4.4 applies to $V_{\pi,0}$ as well.

Let $z, w \in \mathbb{C}$ and we write $z = a + ib$ and $w = c + id$ with $a, b, c, d \in \mathbb{R}$. We can then write

$$\begin{aligned} z\bar{w} - \bar{z}w &= (a+ib)(c-id) - (a-ib)(c+id) = ac + bd + ibc - iad - ac - bd + ibc - iad \\ &= 2i(bc - ad)\,. \end{aligned}$$

Thus we can write for $f, g \in C_c^\infty(\mathbb{R}^{d+1})$

$$\begin{aligned} \sigma_\epsilon(f,g) &\overset{(2.3.4)+(2.2.4)}{=} \frac{i}{4}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right) \\ &= \frac{i}{4}\int \frac{d^dp}{(2\pi)^d \epsilon E_p} 2i\left(\operatorname{Im}\hat{f}(\epsilon E_p, p)\operatorname{Re}\hat{g}(\epsilon E_p, p) - \operatorname{Re}\hat{f}(\epsilon E_p, p)\operatorname{Im}\hat{g}(\epsilon E_p, p)\right) \\ &= -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\operatorname{Im}\hat{f}(\epsilon E_p, p)\operatorname{Re}\hat{g}(\epsilon E_p, p) - \operatorname{Re}\hat{f}(\epsilon E_p, p)\operatorname{Im}\hat{g}(\epsilon E_p, p)\right)\,. \end{aligned} \tag{5.2.1}$$

For a fixed f we define

$$g_1(t,x) := \mathcal{F}^{-1}\left[\operatorname{Im}\hat{f}\right](t,x) = \frac{1}{2i}\mathcal{F}^{-1}\left[\hat{f} - \overline{\hat{f}}\right](t,x) = \frac{1}{2i}\left(f(t,x) - f(-t,-x)\right) \tag{5.2.2}$$

$$g_2(t,x) := i\mathcal{F}^{-1}\left[\operatorname{Re}\hat{f}\right](t,x) = \frac{i}{2}\mathcal{F}^{-1}\left[\hat{f} + \overline{\hat{f}}\right](t,x) = \frac{i}{2}\left(f(t,x) + f(-t,-x)\right)\,, \tag{5.2.3}$$

which are both elements of $C_c^\infty(\mathbb{R}^{d+1})$ since f is as well. So for $f \in \operatorname{Ker}\sigma_\epsilon$ we get

$$\begin{aligned} 0 = \sigma_\epsilon(f, g_1) &\overset{(5.2.1)}{=} -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\operatorname{Im}\hat{f}(\epsilon E_p, p)\operatorname{Re}\widehat{g_1}(\epsilon E_p, p) - \operatorname{Re}\hat{f}(\epsilon E_p, p)\operatorname{Im}\widehat{g_1}(\epsilon E_p, p)\right) \\ &\overset{(5.2.2)}{=} -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\operatorname{Im}\hat{f}(\epsilon E_p, p)\right)^2 \end{aligned}$$

and

$$\begin{aligned} 0 = \sigma_\epsilon(f, g_2) &\overset{(5.2.1)}{=} -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\operatorname{Im}\hat{f}(\epsilon E_p, p)\operatorname{Re}\widehat{g_2}(\epsilon E_p, p) - \operatorname{Re}\hat{f}(\epsilon E_p, p)\operatorname{Im}\widehat{g_2}(\epsilon E_p, p)\right) \\ &\overset{(5.2.3)}{=} \frac{1}{2}\int \frac{d^dp}{(2\pi)^d \epsilon E_p}\left(\operatorname{Re}\hat{f}(\epsilon E_p, p)\right)^2\,. \end{aligned}$$

Those equations imply that $\hat{f}(\epsilon E_p, p) = \operatorname{Re}\hat{f}(\epsilon E_p, p) + i\operatorname{Im}\hat{f}(\epsilon E_p, p) = 0$ for all $p \in \mathbb{R}^d$. Hence we find that $\operatorname{Ker}\sigma_\epsilon \subseteq \left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \hat{f}(\epsilon E_p, p) = 0\,\forall p \in \mathbb{R}^d\right\}$. But clearly $\left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \hat{f}(\epsilon E_p, p) = 0\,\forall p \in \mathbb{R}^d\right\} \subseteq \operatorname{Ker}\sigma_\epsilon$ since if $\hat{f}(\epsilon E_p, p) = 0\,\forall p \in \mathbb{R}^d$ then $\sigma_\epsilon(f,g) = 0$ for all $g \in C_c^\infty(\mathbb{R}^{d+1})$ by 5.2.1. Therefore

$$\operatorname{Ker}\sigma_\epsilon = \left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \hat{f}(\epsilon E_p, p) = 0\,\forall p \in \mathbb{R}^d\right\}\,.$$

With the same trick as before we can rewrite

$$\sigma_{\pi,\epsilon}(f,g) = -\frac{1}{2}\int \frac{d^d p}{(2\pi)^d \epsilon} E_p \left(\mathrm{Im}\,\hat{f}(\epsilon E_p,p)\mathrm{Re}\,\hat{g}(\epsilon E_p,p) - \mathrm{Re}\,\hat{f}(\epsilon E_p,p)\mathrm{Im}\,\hat{g}(\epsilon E_p,p)\right)$$

and by the same arguments as above, even with he same test functions g_1 and g_2, we find

$$\mathrm{Ker}\,\sigma_{\pi,\epsilon} = \left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \hat{f}(\epsilon E_p,p) = 0\,\forall p \in \mathbb{R}^d\right\} = \mathrm{Ker}\,\sigma_\epsilon\,. \tag{5.2.4}$$

For the analogous result for σ_0 and $\sigma_{\pi,0}$ we consider the special functions

$$h_1(t,x) := \frac{\partial}{\partial p_0}\mathcal{F}_t f(0,x)\varphi(t) = -i\varphi(t)\int s f(s,x)\,ds \tag{5.2.5}$$

$$h_2(t,x) := -i\mathcal{F}_t f(0,x)\mathcal{F}_t^{-1}[p_0\mathcal{F}_t\varphi(p_0)](t) = \mathcal{F}_t f(0,x)\varphi'(t) = \varphi'(t)\int f(s,x)\,ds\,, \tag{5.2.6}$$

for some fixed $f \in C_c^\infty(\mathbb{R}^{d+1})$ and φ defined as in (3.1.26). h_1 and h_2 are again elements of $C_c^\infty(\mathbb{R}^{d+1})$. Their Fourier transform is

$$\widehat{h_1}(p_0,p) = \mathcal{F}_t\mathcal{F}_x h_1(p_0,p) \overset{(5.2.5)}{=} \mathcal{F}_x\left[\frac{\partial}{\partial p_0}\mathcal{F}_t f(0,x)\right](p)\mathcal{F}_t\varphi(p_0) = \frac{\partial}{\partial p_0}\hat{f}(0,p)\mathcal{F}_t\varphi(p_0)$$

$$\widehat{h_2}(p_0,p) = \mathcal{F}_t\mathcal{F}_x h_2(p_0,p) \overset{(5.2.6)}{=} -i\mathcal{F}_x\left[\mathcal{F}_t f(0,x)\right](p)p_0\mathcal{F}_t\varphi(p_0) = -ip_0\hat{f}(0,p)\mathcal{F}_t\varphi(p_0)$$

which yields

$$\widehat{h_1}(0,p) = \frac{\partial}{\partial p_0}\hat{f}(0,p)\mathcal{F}_t\varphi(0) \overset{(3.1.27)}{=} \frac{\partial}{\partial p_0}\hat{f}(0,p) \tag{5.2.7}$$

$$\frac{\partial}{\partial p_0}\widehat{h_1}(0,p) = \frac{\partial}{\partial p_0}\hat{f}(0,p)\frac{\partial}{\partial p_0}\mathcal{F}_t\varphi(0) \overset{(3.1.28)}{=} 0 \tag{5.2.8}$$

and

$$\widehat{h_2}(0,p) = 0 \tag{5.2.9}$$

$$\begin{aligned}\frac{\partial}{\partial p_0}\widehat{h_2}(0,p) &= \frac{\partial}{\partial p_0}\left[-ip_0\hat{f}(0,p)\mathcal{F}_t\varphi(p_0)\right]\Big|_{p_0=0}\\ &= -i\left[\hat{f}(0,p)\mathcal{F}_t\varphi(p_0) + p_0\hat{f}(0,p)\frac{\partial}{\partial p_0}\mathcal{F}_t\varphi(p_0)\right]\Bigg|_{p_0=0}\\ &\overset{(3.1.27)+(3.1.28)}{=} -i\hat{f}(0,p)\,.\end{aligned} \tag{5.2.10}$$

Similarly to before we get, for $f \in \mathrm{Ker}\,\sigma_0$,

$$\begin{aligned}0 = \sigma_0(f,h_1) &\overset{(2.4.1)}{=} \frac{i}{2}\int\frac{d^d p}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\widehat{h_1}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\widehat{h_1}(0,p)\right)\\ &\overset{(5.2.7)+(5.2.8)}{=} \frac{i}{2}\int\frac{d^d p}{(2\pi)^d}\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\frac{\partial}{\partial p_0}\hat{f}(0,p)} = \frac{i}{2}\int\frac{d^d p}{(2\pi)^d}\left|\frac{\partial}{\partial p_0}\hat{f}(0,p)\right|^2\end{aligned}$$

and

$$0=\sigma_0(f,h_2)\overset{(2.4.1)}{=}\frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\widehat{h_2}(0,p)}-\overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\widehat{h_2}(0,p)\right)$$
$$\overset{(5.2.9)+(5.2.10)}{=}-\frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\,\overline{\hat{f}(0,p)}\hat{f}(0,p)=-\frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\,\left|\hat{f}(0,p)\right|^2$$

which imply that $\frac{\partial}{\partial p_0}\hat{f}(0,p)=\hat{f}(0,p)=0$ for all $p\in\mathbb{R}^d$. Hence we conclude that $\operatorname{Ker}\sigma_0\subseteq\left\{f\in C_c^\infty(\mathbb{R}^{d+1})\mid\frac{\partial}{\partial p_0}\hat{f}(0,p)=\hat{f}(0,p)=0\,\forall p\in\mathbb{R}^d\right\}$. Again, it is quite obvious from Equation (2.4.1) that $\left\{f\in C_c^\infty(\mathbb{R}^{d+1})\mid\frac{\partial}{\partial p_0}\hat{f}(0,p)=\hat{f}(0,p)=0\,\forall p\in\mathbb{R}^d\right\}\subseteq\operatorname{Ker}\sigma_0$. Therefore

$$\operatorname{Ker}\sigma_0=\left\{f\in C_c^\infty(\mathbb{R}^{d+1})\mid\frac{\partial}{\partial p_0}\hat{f}(0,p)=\hat{f}(0,p)=0\,\forall p\in\mathbb{R}^d\right\}.$$

Repeating that argument for $\sigma_{\pi,0}$ yields

$$\operatorname{Ker}\sigma_{\pi,0}=\left\{f\in C_c^\infty(\mathbb{R}^{d+1})\mid\frac{\partial}{\partial p_0}\hat{f}(0,p)=\hat{f}(0,p)=0\,\forall p\in\mathbb{R}^d\right\}=\operatorname{Ker}\sigma_0\,. \tag{5.2.11}$$

5.3. Magnetic Carollian limit state

With the result from the previous section we can conclude that $\{W_{\pi,\epsilon}(f)\mid f\in V_{\pi,\epsilon}\}$ satisfies the Weyl-relations. Hence, by Theorem 2.3.1, we can assume that $\mathcal{W}_{\pi,\epsilon}$ is the C^*-algebra generated by $\{W_{\pi,\epsilon}(f)\mid f\in V_{\pi,\epsilon}\}$.

The next step is to represent $W_{\pi,\epsilon}(f)$ on a Fock space. To follow the same path as in the electric case, we define the map $\langle\,.\,,\,.\,\rangle_{\pi,\epsilon}:C_c^\infty(\mathbb{R}^{d+1})\times C_c^\infty(\mathbb{R}^{d+1})\to\mathbb{C}$ as

$$\langle f,g\rangle_{\pi,\epsilon}:=\epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}E_p\,\hat{f}(\epsilon E_p,p)\overline{\hat{g}(\epsilon E_p,p)}\,.$$

This map is a $\mathbb{R}$-bilinear, hermitian, positive semi-definite map, which we can extend to a sesquilinear, hermitian, positive semi-definite map $\langle\,.\,,\,.\,\rangle_{\pi,\epsilon}^{\mathbb{C}}:C_c^\infty(\mathbb{R}^{d+1},\mathbb{C})\times C_c^\infty(\mathbb{R}^{d+1},\mathbb{C})\to\mathbb{C}$, defined by

$$\langle f,g\rangle_{\pi,\epsilon}^{\mathbb{C}}:=\langle f_1,g_1\rangle_{\pi,\epsilon}+\langle f_2,g_2\rangle_{\pi,\epsilon}+i\left(\langle f_2,g_1\rangle_{\pi,\epsilon}-\langle f_1,g_2\rangle_{\pi,\epsilon}\right),$$

where f_1 and g_1 are the real part and f_2 and g_2 are the imaginary part of f and g, respectively. Therefore $\langle\,.\,,\,.\,\rangle_{\pi,\epsilon}^{\mathbb{C}}$ is a scalar product on $C_c^\infty(\mathbb{R}^{d+1},\mathbb{C})\big/\operatorname{Ker}_{\pi,\epsilon}^{\mathbb{C}}$, where

$$\operatorname{Ker}_{\pi,\epsilon}^{\mathbb{C}}:=\left\{f\in C_c^\infty(\mathbb{R}^{d+1},\mathbb{C})\mid\langle f,f\rangle_{\pi,\epsilon}^{\mathbb{C}}=0\right\},$$

and we can complete $C_c^\infty(\mathbb{R}^{d+1},\mathbb{C}) \big/ \mathrm{Ker}^{\mathbb{C}}_{\pi,\epsilon}$ to a Hilbert space $\mathcal{H}_{\pi,\epsilon}$. Hence, we can construct a Fock space $\mathcal{F}(\mathcal{H}_{\pi,\epsilon})$. Again, $V_{\pi,\epsilon}$ is embedded in $\mathcal{H}_{\pi,\epsilon}$.

By the definition of the conjugate momenta $\pi = \partial_t\phi^\dagger$ and $\pi^\dagger = \partial_t\phi$ and Equation (3.1.4) and (3.1.5) we conclude that

$$\begin{aligned}
\pi(t,x) = \partial_t\phi^\dagger(t,x) &\overset{(3.1.5)}{=} \frac{i}{\sqrt{2}}\int \frac{d^dp}{(2\pi)^d}\sqrt{E_p}\left(A^\dagger(p)e^{itE_p-ipx} - B(p)e^{-itE_p+ipx}\right)\\
\pi^\dagger(t,x) = \partial_t\phi(t,x) &\overset{(3.1.4)}{=} \frac{i}{\sqrt{2}}\int \frac{d^dp}{(2\pi)^d}\sqrt{E_p}\left(-A(p)e^{-itE_p+ipx} + B^\dagger(p)e^{itE_p-ipx}\right) .
\end{aligned}$$

This yields the expression

$$\pi_\epsilon(f) = \frac{i}{\sqrt{2}}\left(A^\dagger_{\pi,\epsilon}(f) - B_{\pi,\epsilon}(f)\right) \quad \text{and} \quad \pi^\dagger_\epsilon(f) = \frac{i}{\sqrt{2}}\left(-A_{\pi,\epsilon}(f) + B^\dagger_{\pi,\epsilon}(f)\right)$$

where

$$A_{\pi,\epsilon}(f) := \int \frac{d^dp}{(2\pi)^d}\frac{\sqrt{E_p}}{\sqrt{\epsilon}}A(p)\hat{f}(E_{\epsilon,p},p) \quad \text{and} \quad A^\dagger_{\pi,\epsilon}(f) := \int \frac{d^dp}{(2\pi)^d}\frac{\sqrt{E_p}}{\sqrt{\epsilon}}A^\dagger(p)\overline{\hat{f}(E_{\epsilon,p},p)}$$

for $f \in C_c^\infty(\mathbb{R}^{d+1})$ and analogously for $B_{\pi,\epsilon}(f)$ and $B^\dagger_{\pi,\epsilon}(f)$. Hence $\Pi_\epsilon(f)$ can be rewritten as

$$\Pi_\epsilon(f) = \frac{1}{\sqrt{2}}\left(a_{\pi,\epsilon}(f) + a^\dagger_{\pi,\epsilon}(f)\right) ,$$

with

$$a_{\pi,\epsilon}(f) := \frac{-i}{\sqrt{2}}\left(A_{\pi,\epsilon}(f) + B_{\pi,\epsilon}(f)\right) \quad \text{and} \quad a^\dagger_{\pi,\epsilon}(f) := \frac{i}{\sqrt{2}}\left(A^\dagger_{\pi,\epsilon}(f) + B^\dagger_{\pi,\epsilon}(f)\right) . \tag{5.3.1}$$

$a_{\pi,\epsilon}(f)$ and $a^\dagger_{\pi,\epsilon}(g)$ satisfy the commutator relation

$$\left[a_{\pi,\epsilon}(f), a^\dagger_{\pi,\epsilon}(g)\right] = \langle f,g\rangle_{\pi,\epsilon} \qquad f,g \in C_c^\infty(\mathbb{R}^{d+1})$$

which allows us to represent them on $\mathcal{F}(\mathcal{H}_\epsilon)$. We denote with $a_{\pi,\epsilon}(f)$ and $a^\dagger_{\pi,\epsilon}(g)$ the annihilation and creation operator, for $f,g \in \mathcal{H}_{\pi,\epsilon}$, and with $|0_{\pi,\epsilon}\rangle$ the normalized vacuum vector of $\mathcal{F}(\mathcal{H}_\epsilon)$. The distinction between the annihilation and creation operator and (5.3.1) is omitted, since after the representation they agree for $f \in V_{\pi,\epsilon}$.

We can now define the linear functional $\rho_{\pi,0} : \mathcal{W}_{\pi,0} \longrightarrow \mathbb{C}$ as

$$\rho_{\pi,0}(W_{\pi,0}(f)) := \lim_{\epsilon\to 0}\langle 0_{\pi,\epsilon}|\, W_{\pi,\epsilon}(f)\,|0_{\pi,\epsilon}\rangle = \lim_{\epsilon\to 0} e^{-\frac{1}{4}\|f\|^2_{\pi,\epsilon}}$$

for all $f \in V_{\pi,0} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker}\,\sigma_0$. This is, similarly to its electric counter part, a convergent well-defined pure state on $\mathcal{W}_{\pi,0}$ with

$$\rho_{\pi,0}(W_{\pi,0}(f)) = \begin{cases} 1 & \text{if } \hat{f}(0,\,.\,) = 0\,, \\ 0 & \text{else}\,. \end{cases}$$

We refer to $\rho_{\pi,0}$ as the *magnetic Carrollian limit state*.
This state is invariant under the $*$-automorphisms generated by the family

$$\left\{\alpha_{b,R,a} \mid b \in \mathbb{R}, R \in SO(d), a \in \mathbb{R}^d\right\} , \tag{5.3.2}$$

where we understand generated in the same way as in Section 3.2. Hence (5.3.2) are symmetries of $\rho_{\pi,0}$ which, by Theorem 3.2.1, are unitarily implemented on the GNS Hilbert space $\mathcal{H}_{\rho_{\pi,0}}$ leave the GNS vector $x_{\rho_{\pi,0}}$ invariant.
However, (5.3.2) does not contain the full Carroll symmetry group [LL65, SG66] as the maps $(t, x) \mapsto (t + bx, x)$, for $b \in \mathbb{R}^d$, are missing. This makes calling $\mathcal{W}_{\pi,0}$ the magnetic Carrollian Weyl-algebra and $\rho_{\pi,0}$ the magnetic Carrollian limit state rather misleading. We did this purely as an analogy to the electric case. We will discuss in the next chapter how one might fix that issue.

6. Discussion

Let us summarize and discuss what was done in this thesis. Since this chapter is about interpretations of our results, we omit the details on the mathematical part of the arguments.

After introducing mathematical concepts in the first chapter, we showed in Chapter 2 that if we rescale and dilate a free complex scalar field (Equation (2.1.4)) in the following way

$$\phi_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\phi(\epsilon t, x) \quad \text{and} \quad \phi_\epsilon^\dagger(t,x) := \epsilon^{-\frac{1}{2}}\phi^\dagger(\epsilon t, x)\,, \quad \epsilon > 0\,,$$

which resembles the $c \to 0$ idea of Carrollian theories, we can achieve a non-trivial limit of the commutators (2.2.5), after smearing out the fields with smooth, compactly supported functions,

$$\begin{aligned}\left[\phi_\epsilon(f), \phi_\epsilon^\dagger(g)\right] &= \frac{1}{2}\int \frac{d^d p}{(2\pi)^d \epsilon E_p}\left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right) \\ &\underset{\epsilon\to 0}{\longrightarrow} \int \frac{d^d p}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right)\,.\end{aligned}$$

Rescaling the commutators yields bilinear, antisymmetric maps, which can be made into a symplectic forms (2.3.2),(2.4.1)

$$\begin{aligned}\sigma_\epsilon(f,g) &:= \frac{i}{4}\int \frac{d^d p}{(2\pi)^d \epsilon E_p}\left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right) \\ \sigma_0(f,g) &:= \frac{i}{2}\int \frac{d^d p}{(2\pi)^d}\left(\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p)\right)\end{aligned}$$

on the respective vector spaces (2.3.3)

$$V_\epsilon := C_c^\infty(\mathbb{R}^{d+1})\big/\mathrm{Ker}\,\sigma_\epsilon \quad \text{and} \quad V_0 := C_c^\infty(\mathbb{R}^{d+1})\big/\mathrm{Ker}\,\sigma_0\,.$$

Further, we associated a Weyl-algebra $\mathcal{W}_\epsilon$ to the symplectic space V_ϵ, for every $\epsilon > 0$, as well as the Weyl-algebra $\mathcal{W}_0$ to V_0, using Theorem 2.3.1. We called the latter the electric Carrollian Weyl-algebra $\mathcal{W}_0$.
The main idea why we were interested in the construction of $\mathcal{W}_\epsilon$ and subsequently $\mathcal{W}_0$ was that the operators $e^{i\psi_\epsilon(f)}$, with $\psi_\epsilon(f) := \frac{1}{\sqrt{1}}\left(\phi_\epsilon(f) + \phi_\epsilon^\dagger(f)\right)$, satisfy the Weyl-relations (2.3.10). Since the main ingredient for Weyl-algebras is the sympelctic form, i.e., in our case the commutator, and the symplectic forms σ_ϵ converged, we had a much better chance of finding a "limit" algebra.

In Chapter 3 we then constructed for each $\mathcal{W}_\epsilon$ a representation on a Fock space. This allowed us to ask, what the vacuum expectation value of any generating element is and furthermore what the limit of said vacuum expectation values is. This led to the definition (3.1.23) of the linear functional

$$\rho_0 : \mathcal{W}_0 \to \mathbb{C}, \quad \rho_0(W_0(f)) := \lim_{\epsilon\to 0} \langle 0_\epsilon| W_\epsilon(f) |0_\epsilon\rangle \,.$$

We showed that the limit existed and that ρ_0 was a pure state on $\mathcal{W}_0$ with

$$\rho_0(W_0(f)) = \begin{cases} 1 & \text{if } \hat{f}(0,\,.\,) = 0\,, \\ 0 & \text{else}\,, \end{cases}$$

for which we can find a GNS representation on a Hilbert space.
But what does this state mean physically? Is this even a reasonable state? To answer these questions we consider first the quantum mechanical analog. This will also function as quick a argument why one is interested in Weyl-algebras in the first place.

Let Q_i and P_j, for $1 \le i, j \le n$ be position and momentum operators on $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}(\mathbb{R}^n)$. They are defined as $Q_i f(x) := x_i f(x)$ and $P_j f(x) := -i\partial_j f(x)$. For $a, b \in \mathbb{R}^n$ we write $a \cdot Q := \sum_{i=1}^n a_i Q_i$ and $b \cdot P := \sum_{j=1}^n b_j P_j$. Since $a \cdot Q$ and $b \cdot P$ are self-adjoint operators, we can look at the corresponding strongly continuous unitary groups $U(a) := e^{ia\cdot Q}$ and $V(b) := e^{ib\cdot P}$. It can be shown [Hö24, 5.4] that $U(a)f(x) = e^{iax}f(x)$ and $V(b)f(x) = f(x+b)$ as well as that the operators $W(a,b) := e^{\frac{i}{2}ab}U(a)V(b)$ satisfy the Weyl-relations over the symplectic space $\mathbb{R}^n \times \mathbb{R}^n$ with $\sigma((a,b),(c,d)) := \frac{1}{2}(bc - ad)$. If we would like, we can also use the Baker-Campell-Hausdorff formula to write

$$W(a,b) = e^{\frac{i}{2}ab}e^{ia\cdot Q}e^{ib\cdot P} = e^{\frac{i}{2}ab}e^{-\frac{1}{2}ab}e^{i(a\cdot Q + b\cdot P)}\,.$$

The quantum field theoretical analog of $W(a,b)$ would be

$$W(f,g) := e^{\frac{i}{2}\langle f,g\rangle}e^{-\frac{1}{2}\langle f,g\rangle}e^{i\int f\phi + i\int g\pi} = \epsilon^{\frac{i}{2}\langle f,g\rangle}e^{-\frac{1}{2}\langle f,g\rangle}e^{i\phi(f)+i\pi(g)}\,. \tag{6.1}$$

Let $n = 1$ and consider the state corresponding to the wave function

$$\psi_\epsilon(x) := \frac{1}{(\pi\epsilon)^{\frac{1}{4}}}e^{-\frac{x^2}{2\epsilon}}\,,$$

which is a state for the quantum harmonic oscillator with angular frequency $\omega = \frac{1}{m\epsilon}$. The expectation values of Q and P can be computed and we find $\langle Q\rangle_{\psi_\epsilon} = \langle P\rangle_{\psi_\epsilon} = 0$. However, we find $\left\langle Q^2\right\rangle_{\psi_\epsilon} = \frac{\epsilon}{2}$ and $\left\langle P^2\right\rangle_{\psi_\epsilon} = \frac{1}{2\epsilon}$. This implies that $\Delta Q = \sqrt{\langle Q^2\rangle_{\psi_\epsilon} - \left(\langle Q\rangle_{\psi_\epsilon}\right)^2} \propto \sqrt{\epsilon}$ and similarly $\Delta P \propto \frac{1}{\sqrt{\epsilon}}$. If we let $\epsilon \to 0$ then $\psi_\epsilon^2(x) \to \delta(x)$ in the distributional sense, hence the wave function is localized at $x = 0$. This is also seen as $\Delta Q \to 0$. Simultaneously, the state gets maximally uncertain in the momentum, since $\Delta P \to \infty$. However, if we consider the expectation value of the Weyl-operator $W(a,b)$ we find that

$$\langle W(a,b)\rangle_{\psi_\epsilon} = e^{\frac{i}{2}ab}\int dx\, \psi_\epsilon(x)e^{ia\cdot Q}e^{ib\cdot P}\psi_\epsilon(x) = e^{\frac{i}{2}ab}\int dx\, \psi_\epsilon(x)e^{iax}\psi_\epsilon(x+b)\,,$$

which has the well-defined limit

$$\lim_{\epsilon\to 0} \langle W(a,b)\rangle_{\psi_\epsilon} = \begin{cases} 1 & \text{if } b = 0\,, \\ 0 & \text{else}\,. \end{cases}$$

Therefore, also in quantum mechanics we can encounter limit cases where such singular expectation values arise. This makes ρ_0 a little more reasonable to consider.

If we take another look at (6.1) the question arises why we did not consider Weyl-operators of the form

$$\tilde{W}(f,g) \propto e^{i\phi(f)+i\pi_(g)}\,.$$

Thanks to the equations of motion, we know that $\pi(t,x) = \partial_t\phi(t,x)$. Therefore we can express $\pi(g)$ as

$$\begin{aligned}\pi(g) &= \int dtd^dx\, \pi(t,x)g(t,x) = \int dtd^dx\, \partial_t\phi(t,x)g(t,x) = -\int dtd^dx\, \phi(t,x)\partial_t g(t,x) \\ &= \phi(-\partial_t g)\,,\end{aligned}$$

hence $\tilde{W}(f,g) \propto e^{i\phi(f-\partial_t g)}$. This means that the information about π is already encoded in ϕ. This holds for every ϵ, so every $W_\epsilon(f-\partial_t g) = e^{i\psi_\epsilon(f-\partial_t g)}$ contains the information about the conjugate momentum $\tilde{\Pi}_\epsilon(g)$. The value of $\rho_0(W_0(f-\partial_t g))$ depends on $\widehat{f-\partial_t g}$. More precisely, we have $\widehat{f-\partial_t g}(p_0,p) = \hat{f}(p_0,p) - ip_0\hat{g}(p_0,p)$. In this form it is easy to see, that g does not contribute to the evaluation, since $\widehat{f-\partial_t g}(0,\,.\,) = \hat{f}(0,\,.\,)$. This means, that g and therefore $\tilde{\Pi}_\epsilon(g)$ can be omitted in the Weyl-operator.
If we interpret ρ_0 as a vacuum expectation value, since we have the GNS representation available, we essentially have

$$\rho_0(W_0(f-\partial_t g)) \equiv \langle 0_0|\, e^{i\phi_0(f)+i\pi_0(g)}\, |0_0\rangle\,.$$

This is of course not precise but it gives us a better intuition about the state. Hence, by the above argument, we have

$$\langle 0_0|\, e^{i\pi_0(g)}\, |0_0\rangle \equiv \rho_0(W_0(-\partial_t g)) = 1\,,$$

for any g. The quantum mechanical analog would be $\left\langle e^{ib\cdot P}\right\rangle = 1$ for all $b \in \mathbb{R}^n$. But if we differentiate this result with respect to b multiple times, we can conclude that $\left\langle P^k\right\rangle = 0$ for all $k \in \mathbb{N}^n$. If the P was a random variable, this would imply $P = 0$. We can interpret this as that $P = 0$ for the state we used for the expectation value. However, there is no Schwartz function f with $Pf = 0$ other than $f = 0$. Hence, a state satisfying $\left\langle e^{ib\cdot P}\right\rangle = 1$ exists outside the domain of P, possibly even outside of L^2. In the example from above, the analog would be that

$$\left\langle e^{ia\cdot Q}\right\rangle_{\psi_0} := \lim_{\epsilon\to 0}\left\langle e^{ia\cdot Q}\right\rangle_{\psi_\epsilon} = \lim_{\epsilon\to 0}\langle W(a,0)\rangle_{\psi_\epsilon} = 1\,.$$

The “state” ψ_0 then has $Q = 0$. The idea that ψ_0 as a limit “state” of ψ_ϵ is consistent in a sense with the distributional limit $\psi_\epsilon^2(x) \to \delta(x)$ as δ is also not an element of $\mathcal{S}(\mathbb{R}^n)$.

Hence we interpret ρ_0, or the corresponding GNS vector, as a physical state[1] outside the usual domain of states for free complex scalar fields, with conjugate momentum equal to 0.

We also showed a way to check what symmetries the state ρ_0 has in Chapter 3, using Theorem 3.2.1. We have found that ρ_0 has the Carroll group as well as supertranslations and dilations as symmetries. This is what really allowed us to see ρ_0 as a state for a Carroll theory and subsequently interpret the Weyl-algebra $\mathcal{W}_0$ as the electric Carroll Weyl-algebra.
Those symmetries also include the conformal transformations of $\mathbb{R}^{1,d}$. The special conformal transformations, which are also part of the conformal group, are not conformal transformations of $\mathbb{R}^{1,d}$ and therefore did not yield a symmetry. We already mentioned in Subsection 3.2.2 that we believe for (spacially) compact manifolds we could recover the special conformal transformation as a symmetry.
Two interesting questions for further research, which arise at this point, are what other symmetries are and what the complete symmetry group for ρ_0 is. The framework we outlined in this thesis does not necessarily provide a way to find every symmetry.

Chapter 4 was for the most part a technical chapter. We computed the Green functions for the scaled Klein-Gordon equation (2.1.5) and showed that we can interpret the Green functions as operators. Those operators then satisfied Theorem 4.3.3 which provided an isomorphism between the vector spaces V_ϵ and the solution spaces $\mathrm{Sol}(P_\epsilon) := \mathrm{Ker}_{sc}\, P_\epsilon$. The process was then repeated for the electric Carroll operator (4.2.1), which again yielded an isomorphism between V_0 and $\mathrm{Sol}(P_0)$. The isomorphism in this case was then also explicitly constructed.
This chapter provided a classification for the fairly abstract vector spaces V_ϵ and V_0 used in the construction of the Weyl-algebras.

We repeated the entire process in Chapter 5 for the conjugate momenta

$$\pi_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\pi(\epsilon t, x) \quad \text{and} \quad \pi^\dagger_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\pi^\dagger(\epsilon t, x)\,, \quad \epsilon > 0\,,$$

leading to the family of Weyl-algebras $\{\mathcal{W}_{\pi,\epsilon} \mid \epsilon > 0\}$ and the magnetic Carrollian Weyl-algebra $\mathcal{W}_{\pi,0}$. On the latter, we again were able to define a pure state

$$\rho_{\pi,0} : \mathcal{W}_{\pi,0} \longrightarrow \mathbb{C}, \quad \rho_{\pi,0}(W_{\pi,0}(f)) := \lim_{\epsilon\to 0} \langle 0_{\pi,\epsilon}|\, W_{\pi,\epsilon}(f)\, |0_{\pi,\epsilon}\rangle = \begin{cases} 1 & \text{if } \hat{f}(0,\,.\,) = 0\,, \\ 0 & \text{else}\,. \end{cases}$$

We would like to apply the same argument as above for ρ_0 to get an interpretation for $\rho_{\pi,0}$. The key ingredient for the interpretation of ρ_0 was the insight that we can encode the information about $\pi(g)$ as $\phi(-\partial_t g)$, with help of the equations of motion. For the same idea to work on $\rho_{\pi,0}$, we would need to use the equations of motion for π, i.e., $\partial_t \pi = \nabla^2\phi - m^2\phi$. We would need to find for every g a $\tilde{g}$ such that $g = (\nabla^2 - m^2)\tilde{g}$.

[1] To make a distinction from states of C^*-algebras.

If so, we could encode the information of $\phi(g)$ as

$$\begin{aligned}\phi(g) &= \int dtd^dx\,\phi(t,x)g(t,x) = \int dtd^dx\,\phi(t,x)(\nabla^2-m^2)\tilde{g}(t,x)\\ &= \int dtd^dx\,(\nabla^2-m^2)\phi(t,x)\tilde{g}(t,x) = \int dtd^dx\,\partial_t\pi(t,x)\tilde{g}(t,x)\\ &= -\int dtd^dx\,\pi(t,x)\partial_t\tilde{g}(t,x) = \pi(-\partial_t\tilde{g})\,.\end{aligned}$$

Hence, we have a similar situation as for ρ_0 and we could interpret $\rho_{\pi,0}$ as a physical state outside the usual domain of states for free complex scalar fields, with position equal to 0. However, there is a problem. We only consider smooth functions with compact support as inputs for the fields. But the solution $\tilde{g}$ to $g=(\nabla^2-m^2)\tilde{g}$ will generally not have compact support, even if g has compact support. Therefore, the same argument does not hold and we have lost the analogous interpretation. If we would consider (spacially) compact manifolds as the underlying manifold instead of Minkwoski space, we could find non-trivial solutions to $g=(\nabla^2-m^2)\tilde{g}$ with compact support.

Another strange thing about the magnetic case is that we do not have the full Carroll group as symmetries. The transformations $(t,x)\mapsto(t+bx,x)$ for $b\in\mathbb{R}^d$ are missing. This may be because we did only consider $\pi(f)$ and not $\pi(f)+\phi(g)$. We can find a strong hint in favor of this if we look at the action of a free complex scalar field in Hamiltonian form[2]

$$S[\phi,\pi]=\int\pi\partial_t\phi+\pi^\dagger\partial_t\phi^\dagger-\mathcal{H}(\phi,\phi^\dagger,\pi,\pi^\dagger)=\int\pi\partial_t\phi+\pi^\dagger\partial_t\phi^\dagger-\pi\pi^\dagger-\nabla\phi\nabla\phi^\dagger-m^2\phi\phi^\dagger\,.$$

After deforming the fields to $\pi_\epsilon=\epsilon^{-\frac{1}{2}}\pi(\epsilon t,x)$ and $\pi_\epsilon^\dagger=\epsilon^{-\frac{1}{2}}\pi^\dagger(\epsilon t,x)$, we get

$$S[\tilde{\phi}_\epsilon,\pi_\epsilon]=\frac{1}{\epsilon}\int\pi_\epsilon\partial_t\tilde{\phi}_\epsilon+\pi_\epsilon^\dagger\partial_t\tilde{\phi}_\epsilon^\dagger-\epsilon^2\pi_\epsilon\pi_\epsilon^\dagger-\nabla\tilde{\phi}_\epsilon\nabla\tilde{\phi}_\epsilon^\dagger-m^2\tilde{\phi}_\epsilon\tilde{\phi}_\epsilon^\dagger\,,$$

where $\tilde{\phi}_\epsilon(t,x)=\epsilon^{\frac{1}{2}}\phi(\epsilon t,x)$ and analogously for $\tilde{\phi}_\epsilon^\dagger$. This notation is used to avoid confusion with ϕ_ϵ and $\phi_\epsilon^\dagger$, as they are different. π_ϵ and $\tilde{\phi}_\epsilon$ still satisfy the equal time commutation relations. Rescaling the action and taking the limit $\epsilon\to0$ gives us

$$S[\tilde{\phi}_0,\pi_0]=\int\pi_0\partial_t\tilde{\phi}_0+\pi_0^\dagger\partial_t\tilde{\phi}_0^\dagger-\nabla\tilde{\phi}_0\nabla\tilde{\phi}_0^\dagger-m^2\tilde{\phi}_0\tilde{\phi}_0^\dagger\,,\tag{6.2}$$

which is a typical magnetic Carroll action [dBHO+23]. We can check that the transformation $(t,x)\mapsto(t+h(x),x)$, for some smooth function $h:\mathbb{R}^d\to\mathbb{R}^d$, can only be implemented as a symmetry of the action (6.2) if together with the transformation $\tilde{\phi}_0(t,x)\mapsto\tilde{\phi}_0(t+h(x),x)$ the momentum transforms as

$$\pi_0(t,x)\mapsto\pi_0(t+h(x),x)+\nabla h(x)\nabla\tilde{\phi}_0^\dagger(t+h(x),x)\,.$$

[2]The Euler-Lagrange-equations in this form recover the Hamiltonian equations of motion.

Hence, a mixing of the fields ϕ and the conjugate momenta π seems natural for this transformation to hold, even in the simple case $h(x) = bx$.
So an interesting investigation for the magnetic case would be what state we get if we consider Weyl-algebras generated by $W_\epsilon(f, g) \propto e^{i\Pi_\epsilon(f)+i\tilde{\psi}_e(g)}$.

We have seen throughout this discussion that it would also be interesting to consider this procedure for (spacially) compact manifolds. This would especially be useful for considering one of the motivations for Carroll theories, flat space holography [dBHO+23]. Moreover, an interesting follow-up research would be to apply this procedure to interacting theories.

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A. Appendix: Green function of the Carrollian operator

In this appendix we calculate the Green function G_0 of the operator

$$P_0 = \partial_t^2 \,,$$

which is defined as the solution of

$$P_0 G_0(x^\mu, y^\mu) = \partial_{x^0}^2 G_0(x^\mu, y^\mu) = \delta^{d+1}(x^\mu - y^\mu) \,,$$

where P_0 uses the $x^\mu \in \mathbb{R}^{d+1}$ coordinates. Fourier transforming both sides yields

$$\int d^{d+1}x^\mu \, e^{-ix_\mu p^\mu} \partial_{x^0}^2 G_0(x^\mu, y^\mu) = \int d^{d+1}x^\mu \, e^{-ix_\mu p^\mu} \delta^{d+1}(x^\mu - y^\mu) = e^{-iy_\mu p^\mu} \,,$$

which further gives

$$\begin{aligned} e^{-iy_\mu p^\mu} &= \int d^{d+1}x^\mu \, e^{-ix_\mu p^\mu} \partial_t^2 G_0(x^\mu, y^\mu) = \int d^{d+1}x^\mu \, \partial_t^2 e^{-ix_\mu p^\mu} G_0(x^\mu, y^\mu) \\ &= \int d^{d+1}x^\mu \, (-p_0^2) e^{-ix_\mu p^\mu} G_0(x^\mu, y^\mu) = -p_0^2 \hat{G}_0(p^\mu, y^\mu) \end{aligned}$$

using integration by parts. Similarly to the $\epsilon \neq 0$ case, we consider the shifted equation

$$e^{-iy_\mu p^\mu} = -(p_0 - i\tilde{\eta})^2 \hat{G}_{0,\tilde{\eta}}(p^\mu, y^\mu) \,,$$

with $\tilde{\eta} \in \mathbb{R}, \tilde{\eta} \neq 0$, which is solved by

$$\hat{G}_{0,\tilde{\eta}}(p^\mu, y^\mu) = -\frac{1}{(p_0 - i\tilde{\eta})^2} e^{-iy_\mu p^\mu} \,.$$

Now we just apply the inverse Fourier transform, which yields

$$\begin{aligned} G_{0,\tilde{\eta}}(x^\mu, y^\mu) &= \int \frac{d^{d+1}p^\mu}{(2\pi)^4} \, e^{ip_\mu x^\mu} \hat{G}_{0,\tilde{\eta}}(p^\mu, y^\mu) = \int \frac{d^{d+1}p^\mu}{(2\pi)^{d+1}} \, e^{ip_\mu x^\mu} \frac{-1}{(p_0 - i\tilde{\eta})^2} e^{-iy_\mu p^\mu} \\ &\overset{p^\mu \mapsto -p^\mu}{=} - \int \frac{d^{d+1}p^\mu}{(2\pi)^{d+1}} \, \frac{1}{(p_0 - i\tilde{\eta})^2} e^{-i(x_\mu - y_\mu)p^\mu} \\ &= - \int \frac{dp}{(2\pi)^d} \, e^{i(x-y)p} \int \frac{dp_0}{2\pi} \, \frac{1}{(p_0 - i\tilde{\eta})^2} e^{-i(x_0 - y_0)p_0} \,. \end{aligned}$$

We again distinguish the two possible signs of $\tilde{\eta}$ and consider the four complex integrals

$$J^{\pm}_{0,\eta,R} := \theta(x^0 - y^0) \int_{\gamma_R^-} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} + \theta(y^0 - x^0) \int_{\gamma_R^+} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z}, \tag{A.1}$$

where $\gamma_R^{\pm} := [-R, R] \dot{+} \{Re^{\pm it} | t \in [0, \pi]\}$ and $\eta > 0$, and

$$I^{\pm}_{0,\eta,R} := \int_{[-R,R]} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x_0-y_0)z}.$$

The latter has as the limit the integral

$$\lim_{R\to\infty} I^{\pm}_{0,\eta,R} = \int \frac{dp_0}{2\pi} \frac{1}{(p_0 \mp i\eta)^2} e^{-i(x_0-y_0)p_0}. \tag{A.2}$$

The difference of the two integrals $J^{\pm}_{0,\eta,R}$ and $I^{\pm}_{0,\eta,R}$ can be rewritten as

$$\begin{aligned} J^{\pm}_{0,\eta,R} - I^{\pm}_{0,\eta,R} &= \theta(x^0 - y^0) \int_{\{Re^{-it} | t\in[0,\pi]\}} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} \\ &\quad + \theta(y^0 - x^0) \int_{\{Re^{+it} | t\in[0,\pi]\}} \frac{dz}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x^0-y^0)z} \\ &= \theta(x^0 - y^0) \int_0^{\pi} \frac{1}{2\pi} \frac{-iRe^{-it}}{(Re^{-it} \mp i\eta)^2} e^{-i(x^0-y^0)R\cos(t)} e^{-(x^0-y^0)R\sin(t))} dt \\ &\quad + \theta(y^0 - x^0) \int_0^{\pi} \frac{1}{2\pi} \frac{iRe^{it}}{(Re^{it} \mp i\eta)^2} e^{-i(x^0-y^0)R\cos(t)} e^{(x^0-y^0)R\sin(t))} dt. \end{aligned}$$

If we have $x^0 - y^0 > 0$ then $e^{-(x^0-y^0)R\sin(t)} \leq 1$ for R large enough and similarly if $x^0 - y^0 < 0$ then $e^{(x^0-y^0)R\sin(t)} \leq 1$ for large R. In either case the dominating term is $\frac{-iRe^{-it}}{(Re^{-it} \mp i\eta)^2}$, which tends to 0 as $R \to \infty$, since the denominator of the integrands contains a higher power of R then the numerator. For $x^0 - y^0 = 0$ we find that

$$\int_0^{\pi} \frac{1}{2\pi} \frac{-iRe^{-it}}{(Re^{-it} \mp i\eta)^2} dt + \int_0^{\pi} \frac{1}{2\pi} \frac{iRe^{it}}{(Re^{it} \mp i\eta)^2} dt$$

vanishes as $R \to \infty$ by the same argument. Hence

$$J^{\pm}_{0,\eta,R} - I^{\pm}_{0,\eta,R} \underset{R\to\infty}{\longrightarrow} 0. \tag{A.3}$$

To compute $J^{\pm}_{0,\eta,R}$ we use the residue theorem (Equation (4.1.12)) . In our case

$$f^{\pm}(z) = \frac{1}{2\pi} \frac{1}{(z \mp i\eta)^2} e^{-i(x_0-y_0)z},$$

are indeed meromorphic functions with second order poles at $a^+ = i\eta$ for f^+ and $a^- = -i\eta$ for f^-, which are inside the contour γ_R^+ and γ_R^- respectively, if R is large enough ($R > \eta$).

Hence, for R large enough, we get

$$\mathrm{Ind}_{\gamma_R^\pm}(a^\pm) = \pm 1 \quad \text{and} \quad \mathrm{Ind}_{\gamma_R^\mp}(a^\pm) = 0 \tag{A.4}$$

for the winding number. The residue for second order poles can be calculated as follows.

$$\begin{aligned}\mathrm{Res}(f^\pm, a^\pm) &= \lim_{z\to a^\pm} \frac{d}{dz}((z-a^\pm)^2 f^\pm(z)) = \lim_{z\to a^\pm} \frac{d}{dz}\left((z-a^\pm)^2 \frac{1}{2\pi}\frac{1}{(z-a^\pm)^2} e^{-i(x_0-y_0)z}\right) = \\ &= \lim_{z\to a^\pm} \frac{1}{2\pi}(-i(x_0-y_0))e^{-i(x_0-y_0)z} = \frac{1}{2\pi}(-i(x_0-y_0))e^{\pm(x_0-y_0)\eta}\,. \end{aligned}\tag{A.5}$$

Therefore, for R large enough, we can evaluate $J^\pm_{0,\eta,R}$.

$$\begin{aligned} J^+_{0,\eta,R} &\overset{(A.1)}{=} \theta(x^0-y^0)\int_{\gamma_R^-} f^+(z)dz + \theta(y^0-x^0)\int_{\gamma_R^+} f^+(z)dz \\ &\overset{(4.1.12)}{=} 2\pi i\theta(x^0-y^0)\mathrm{Ind}_{\gamma_R^-}(a_1^+)\mathrm{Res}(f^+,a^+) + 2\pi i\theta(y^0-x^0)\mathrm{Ind}_{\gamma_R^+}(a_1^+)\mathrm{Res}(f^+,a^+) \\ &\overset{(A.4)}{=} 2\pi i\theta(y^0-x^0)\mathrm{Res}(f^+,a^+) \overset{(A.5)}{=} \theta(y^0-x^0)2\pi i\frac{1}{2\pi}(-i(x_0-y_0))e^{(x_0-y_0)\eta} \\ &= \theta(y^0-x^0)(x_0-y_0)e^{(x_0-y_0)\eta}\,. \end{aligned}$$

and analogous for $J^-_{0,\eta,R}$. Compactly written we get

$$J^\pm_{0,\eta,R} = \pm\theta(\mp(x^0-y^0))(x_0-y_0)e^{\pm(x_0-y_0)\eta}\,. \tag{A.6}$$

We define

$$G_0^\pm(x^\mu, y^\mu) := -\int \frac{dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} I^\pm_{0,\eta,R} \tag{A.7}$$

and compute the limit explicitly.

$$\begin{aligned} G_0^\pm(x^\mu,y^\mu) &\overset{(A.7)}{=} -\int \frac{dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} I^\pm_{0,\eta,R} \\ &\overset{(A.3)}{=} -\int \frac{dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} J^\pm_{0,\eta,R} \\ &\overset{(A.6)}{=} -\int \frac{dp}{(2\pi)^d}\, e^{i(x-y)p} \lim_{\eta\to 0}\lim_{R\to\infty} \left[\pm\theta(\mp(x^0-y^0))(x_0-y_0)e^{\pm(x_0-y_0)\eta}\right] \\ &= \mp\theta(\mp(x^0-y^0))(x_0-y_0)\int \frac{dp}{(2\pi)^d}\, e^{i(x-y)p} \\ &= \mp\theta(\mp(x^0-y^0))(x_0-y_0)\delta^d(x-y)\,. \end{aligned}$$

B. Appendix: Magnetic Carrollian limit

In this appendix we give the detailed calculations we omitted in Chapter 5.

Consider the scaled conjugate momenta $\pi_\epsilon(t,x)$ and $\pi_e^\dagger(t,x)$, for $\epsilon > 0$, with

$$\pi_\epsilon(t,x) := \epsilon^{-\frac{1}{2}}\pi(\epsilon t,x) \quad \text{and} \quad \pi_\epsilon^\dagger(t,x) := \epsilon^{-\frac{1}{2}}\pi^\dagger(\epsilon t,x)\,. \tag{B.1}$$

We have the commutator relation

$$\Big[\pi(t,x),\pi^\dagger(s,y)\Big] = -\partial_t\partial_s\Delta(t-s,x-y)\,, \tag{B.2}$$

where

$$\Delta(t,x) = \int \frac{d^dp}{(2\pi)^d}\int dp_0\,\mathrm{sgn}(p_0)\,\delta(p_0^2-p^2-m^2)\,e^{-ip_0t+ipx}\,. \tag{B.3}$$

Smearing out the fields with $f,g\in C_c^\infty(\mathbb{R}^{d+1})$ allows us to calculate

$$\begin{aligned}
\Big[\pi_\epsilon(f),\pi_\epsilon^\dagger(g)\Big] &= \int dtd^dx\,f(t,x)\int dsd^dy\,g(s,y)\,\Big[\pi_\epsilon(t,x),\pi_\epsilon^\dagger(s,y)\Big]\\
&\overset{(B.1)}{=} \int dtd^dx\,f(t,x)\int dsd^dy\,g(s,y)\epsilon^{-1}\,\Big[\pi(\epsilon t,x),\pi^\dagger(\epsilon s,y)\Big]\\
&\overset{(B.2)+(B.3)}{=} \epsilon^{-1}\int dtd^dx\,f(t,x)\int dsd^dy\,g(s,y)\frac{1}{\epsilon^2}\partial_t\partial_s\int\frac{d^dp}{(2\pi)^d}\\
&\quad\cdot\int dp_0\,\mathrm{sgn}(p_0)\,\delta(p_0^2-p^2-m^2)\,e^{-ip_0\epsilon(t-s)+ip(x-y)}\\
&= \epsilon^{-1}\int dtd^dx\,f(t,x)\int dsd^dy\,g(s,y)\int\frac{d^dp}{(2\pi)^d}\\
&\quad\cdot\int dp_0\,\mathrm{sgn}(p_0)\,\delta(p_0^2-p^2-m^2)\,p_0^2e^{-ip_0\epsilon(t-s)+ip(x-y)}\\
&= \epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}\int dp_0\,\mathrm{sgn}(p_0)\,\delta(p_0^2-p^2-m^2)p_0^2\,\hat f(\epsilon p_0,p)\hat g(-\epsilon p_0,p)\\
&= \epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}\int dp_0\,\mathrm{sgn}(p_0)\,\left(\frac{\delta(p_0-E_p)}{2E_p}+\frac{\delta(p_0+E_p)}{2E_p}\right)p_0^2\,\hat f(\epsilon p_0,p)\hat g(-\epsilon p_0,-p)\\
&= \frac{1}{2}\epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}E_p\,\Big(\hat f(\epsilon E_p,p)\hat g(-\epsilon E_p,-p)-\hat f(-\epsilon E_p,p)\hat g(\epsilon E_p,-p)\Big)\\
&\overset{p\mapsto -p}{=} \frac{1}{2}\epsilon^{-1}\int\frac{d^dp}{(2\pi)^d}E_p\,\Big(\hat f(\epsilon E_p,p)\overline{\hat g(\epsilon E_p,p)}-\overline{\hat f(\epsilon E_p,p)}\hat g(\epsilon E_p,p)\Big)\,.
\end{aligned}$$

Using again the trick, that for smooth functions $h : \mathbb{R} \to \mathbb{R}$ we can apply Taylor's theorem to get

$$h(x) - h(-x) = 2\frac{\partial h}{\partial x}(0)x + \mathcal{O}(x^3)\,,$$

allows us to compute the $\epsilon \to 0$ limit of the commutator.

$$\begin{aligned}
\left[\pi_\epsilon(f), \pi_\epsilon^\dagger(g)\right] &= \frac{1}{2}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right) \\
&= \frac{1}{2}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\left(\hat{f}(\epsilon E_p, p)\hat{g}(-\epsilon E_p, -p) - \hat{f}(-\epsilon E_p, -p)\hat{g}(\epsilon E_p, p)\right) \\
&= \frac{1}{2}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\Big[\left(\hat{f}(\epsilon E_p, p) - \hat{f}(-\epsilon E_p, p)\right)\hat{g}(-\epsilon E_p, -p) \\
&\quad - \hat{f}(-\epsilon E_p, -p)\left(\hat{g}(\epsilon E_p, p) - \hat{g}(-\epsilon E_p, p)\right)\Big] \\
&= \frac{1}{2}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\left[\left(2\epsilon E_p\frac{\partial \hat{f}}{\partial p_0}(0,p) + \mathcal{O}((\epsilon E_p)^3)\right)\hat{g}(-\epsilon E_p, -p)\right. \\
&\quad \left. - \hat{f}(-\epsilon E_p, -p)\left(2\epsilon E_p\frac{\partial \hat{g}}{\partial p_0}(0,p) + \mathcal{O}((\epsilon E_p)^3)\right)\right] \\
&= \int \frac{d^dp}{(2\pi)^d}\,E_p^2\left[\left(\frac{\partial \hat{f}}{\partial p_0}(0,p) + \mathcal{O}((\epsilon E_p)^2)\right)\overline{\hat{g}(\epsilon E_p, p)}\right. \\
&\quad \left. - \overline{\hat{f}(\epsilon E_p, p)}\left(\frac{\partial \hat{g}}{\partial p_0}(0,p) + \mathcal{O}((\epsilon E_p)^2)\right)\right] \\
&\underset{\epsilon\to 0}{\longrightarrow} \int \frac{d^dp}{(2\pi)^d}\,E_p^2\left(\frac{\partial \hat{f}}{\partial p_0}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial \hat{g}}{\partial p_0}(0,p)\right)\,.
\end{aligned}$$

We define

$$\langle f, g\rangle_{\pi,\epsilon} := \epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} \quad \text{and} \tag{B.4}$$

$$\sigma_{\pi,\epsilon}(f,g) := \frac{i}{2}\left[\pi_\epsilon(f), \pi_\epsilon^\dagger(g)\right]\,. \tag{B.5}$$

By the same arguments as in Section 2.3 we conclude that $\langle\,.\,,\,.\,\rangle_{\pi,\epsilon} : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{C}$ is an $\mathbb{R}$-bilinear, hermitian, positive semi-definite map and $\sigma_{\pi,\epsilon}(\,.\,,\,.\,) : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{R}$ is $\mathbb{R}$-bilinear antisymmetric map since

$$\begin{aligned}
\sigma_{\pi,\epsilon}(f,g) &= \frac{i}{2}\left[\pi_\epsilon(f), \pi_\epsilon^\dagger(g)\right] = \frac{i}{4}\epsilon^{-1}\int \frac{d^dp}{(2\pi)^d}E_p\,\left(\hat{f}(\epsilon E_p, p)\overline{\hat{g}(\epsilon E_p, p)} - \overline{\hat{f}(\epsilon E_p, p)}\hat{g}(\epsilon E_p, p)\right) \\
&= \frac{i}{4}\left(\langle f, g\rangle - \langle g, f\rangle\right) = \frac{i}{4}2i\mathrm{Im}\,\langle f, g\rangle_{\pi,\epsilon} = -\frac{1}{2}\mathrm{Im}\,\langle f, g\rangle_{\pi,\epsilon}\,.
\end{aligned} \tag{B.6}$$

Similarly

$$\sigma_{\pi,0}(f,g) := \lim_{\epsilon\to 0}\frac{i}{2}\left[\pi_\epsilon(f),\pi_\epsilon^\dagger(g)\right] = \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\,E_p^2\left(\frac{\partial\hat{f}}{\partial p_0}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial\hat{g}}{\partial p_0}(0,p)\right) \tag{B.7}$$

is a $\mathbb{R}$-bilinear antisymmetric map from $C_c^\infty(\mathbb{R}^{d+1})\times C_c^\infty(\mathbb{R}^{d+1})$ to $\mathbb{R}$. Hence $\sigma_{\pi,\epsilon}$ and $\sigma_{\pi,0}$ are symplectic forms on $V_{\pi,\epsilon} := C_c^\infty(\mathbb{R}^{d+1})\big/\mathrm{Ker}\,\sigma_{\pi,\epsilon}$ and $V_{\pi,0} := C_c^\infty(\mathbb{R}^{d+1})\big/\mathrm{Ker}\,\sigma_{\pi,0}$, respectively.

Next we define the operator

$$\Pi_\epsilon(f) := \frac{1}{\sqrt{2}}\left(\pi_\epsilon(f)+\pi_\epsilon^\dagger(f)\right) \tag{B.8}$$

for $f\in C_c^\infty(\mathbb{R}^{d+1})$. It has a the following commutator with itself.

$$\begin{aligned}
[\Pi_\epsilon(f),\Pi_\epsilon(g)] &\overset{(B.8)}{=} \frac{1}{2}\left[\pi_\epsilon(f)+\pi_\epsilon^\dagger(f),\pi_\epsilon(g)+\pi_\epsilon^\dagger(g)\right] = \frac{1}{2}\left(\left[\pi_\epsilon(f),\pi_\epsilon^\dagger(g)\right]+\left[\pi_\epsilon^\dagger(f),\pi_\epsilon(g)\right]\right)\\
&\overset{(B.6)}{=} \frac{1}{2}\left(-2i\sigma_{\pi,\epsilon}(f,g)+2i\sigma_{\pi,\epsilon}(g,f)\right) = -i\left(\sigma_{\pi,\epsilon}(f,g)-\sigma_{\pi,\epsilon}(g,f)\right) =\\
&= -i\left(\sigma_{\pi,\epsilon}(f,g)+\sigma_{\pi,\epsilon}(f,g)\right) = -2i\sigma_{\pi,\epsilon}(f,g)\,.
\end{aligned} \tag{B.9}$$

Similar to the electric case, we are only interested in the formal calculations, which means we can just define

$$W_{\pi,\epsilon}(f) := e^{i\Pi_\epsilon(f)},\quad f\in C_c^\infty(\mathbb{R}^{d+1})\,. \tag{B.10}$$

In concrete applications, where $\Pi(f)$ is a densely-defined self-adjoint unbounded operator, we use functional calculus [RS80, VIII.5]. The family $\{W_{\pi,\epsilon}(f)\mid f\in V_{\pi,\epsilon}\}$ is well-defined and satisfies the Weyl-relations in Theorem 2.3.1.
To show this, we first need to recognize that π_ϵ and $\pi_\epsilon^\dagger$ satisfy the scaled Klein-Gordon equation. Indeed, we have

$$\begin{aligned}
P_\epsilon\pi_\epsilon(t,x) &\overset{(B.1)}{=} (\partial_t^2-\epsilon^2\nabla_x^2-\epsilon^2m^2)\epsilon^{-\frac{1}{2}}\pi(\epsilon t,x) = (\partial_t^2-\epsilon^2\nabla_x^2-\epsilon^2m^2)\epsilon^{-\frac{1}{2}}\frac{1}{\epsilon}\partial_t\phi^\dagger(\epsilon t,x)\\
&= \frac{1}{\epsilon}\partial_t(\partial_t^2-\epsilon^2\nabla_x^2-\epsilon^2m^2)\epsilon^{-1/2}\phi^\dagger(\epsilon t,x) \overset{(2.1.4)}{=} \frac{1}{\epsilon}\partial_t P_\epsilon\phi_\epsilon^\dagger(t,x) \overset{(2.1.5)}{=} 0
\end{aligned}$$

and analogously for $\pi^\dagger$. We also find, by (2.3.8) and (5.2.4), that $\mathrm{Im}\,P_\epsilon = \mathrm{Ker}\,\sigma_\epsilon = \mathrm{Ker}\,\sigma_{\pi,\epsilon}$, which means for every $h\in\mathrm{Ker}\,\sigma_{\pi,\epsilon}$ we find a $\tilde{h}\in C_c^\infty(\mathbb{R}^{d+1})$ such that $h=P_\epsilon\tilde{h}$. Hence $\pi_\epsilon(h)=0$ since

$$\pi_\epsilon(h) = \pi_\epsilon(P_\epsilon\tilde{h}) = \int dtd^dx\,\pi_\epsilon(t,x)P_\epsilon\tilde{h}(t,x) = \int dtd^dx\,P_\epsilon\pi_\epsilon(t,x)\tilde{h}(t,x) = 0\,.$$

Of course, this holds for $\pi_\epsilon^\dagger(h)$ as well. This implies that $\Pi(h)=0\,\forall h\in\mathrm{Ker}\,\sigma_{\pi,\epsilon}$ and

$$W_{\pi,\epsilon}(h) = e^{i\Pi_\epsilon(h)} = e^0 = \mathbb{1},\quad \forall h\in\mathrm{Ker}\,\sigma_{\pi,\epsilon}\,. \tag{B.11}$$

This immediately implies the first of the Weyl-relation, i.e., $W_{\pi,\epsilon}(0) = \mathbb{1}$. The second and third Weyl-relations hold too, since

$$\begin{aligned} W_{\pi,\epsilon}(-f) &= e^{i\Pi_\epsilon(-f)} = e^{-i\Pi_\epsilon(f)} = \left(e^{i\Pi_\epsilon(f)}\right)^* = W_{\pi,\epsilon}(f)^* \\ W_{\pi,\epsilon}(f)W_{\pi,\epsilon}(g) &= e^{i\Pi_\epsilon(f)}e^{i\Pi_\epsilon(g)} = e^{\frac{1}{2}[i\Pi_\epsilon(f),i\Pi_\epsilon(g)]}e^{i\Pi_\epsilon(f)+i\Pi_\epsilon(g)} \\ &= e^{-\frac{1}{2}[\Pi_\epsilon(f),\Pi_\epsilon(g)]}e^{i\Pi_\epsilon(f+g)} \overset{\text{(B.9)}}{=} e^{i\sigma_{\pi,\epsilon}(f,g)}W_{\pi,\epsilon}(f+g)\,, \end{aligned}$$

for $f, g \in C_c^\infty(\mathbb{R}^{d+1})$, where we used linearity and self-adjointess of Π_ϵ and the Baker-Campell-Hausdorff-formula. Note that this works as a formal calculation. We discussed the issues and solutions for concrete applications in Section 2.3.
The third Weyl-relation then also implies well-definedness, since for $f \in C_c^\infty(\mathbb{R}^{d+1})$ and $h \in \operatorname{Ker}\sigma_{\pi,\epsilon}$ we have

$$W_{\pi,\epsilon}(f+h) = e^{-i\sigma_{\pi,\epsilon}(f,h)}W_{\pi,\epsilon}(f)W_{\pi,\epsilon}(h) \overset{\text{(B.11)}}{=} W_{\pi,\epsilon}(f)\,.$$

Therefore, by Theorem 2.3.1, we get a Weyl-algebra $\mathcal{W}_{\pi,\epsilon} := \mathcal{W}(V_{\pi,\epsilon}, \sigma_{\pi,\epsilon})$ and by the uniqueness part of that theorem we can assume that $\mathcal{W}_{\pi,\epsilon}$ is the C^*-algebra generated by the family $\{W_{\pi,\epsilon}(f) \mid f \in V_{\pi,\epsilon}\}$. From the same theorem we get the Weyl-algebra $\mathcal{W}_{\pi,0} := \mathcal{W}(V_{\pi,0}, \sigma_{\pi,0})$.

The extension of $\langle\,.\,,\,.\,\rangle_{\pi,\epsilon} : C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \to \mathbb{C}$ to $\langle\,.\,,\,.\,\rangle^{\mathbb{C}}_{\pi,\epsilon} : C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \times C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \to \mathbb{C}$, defined by

$$\langle f, g\rangle^{\mathbb{C}}_{\pi,\epsilon} := \langle f_1, g_1\rangle_{\pi,\epsilon} + \langle f_2, g_2\rangle_{\pi,\epsilon} + i\left(\langle f_2, g_1\rangle_{\pi,\epsilon} - \langle f_1, g_2\rangle_{\pi,\epsilon}\right)\,,$$

where f_1 and g_1 are the real part and f_2 and g_2 are the imaginary part of f and g respectively, follows exactly the same steps as in Subsection 3.1.1, which is why we will not repeat them again. We gain a Hilbert space $\mathcal{H}_{\pi,\epsilon}$ as the completion of $C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \big/ \operatorname{Ker}^{\mathbb{C}}_{\pi,\epsilon}$, where $\operatorname{Ker}^{\mathbb{C}}_{\pi,\epsilon} := \left\{f \in C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \mid \langle f, f\rangle^{\mathbb{C}}_{\pi,\epsilon} = 0\right\}$, with respect to $\langle\,.\,,\,.\,\rangle^{\mathbb{C}}_{\pi,\epsilon}$. It remains to show that $V_{\pi,\epsilon}$ is embedded in $\mathcal{H}_{\pi,\epsilon}$.

We can follow the same steps as we did for V_ϵ. Again, we temporarily differentiate between the function f and the equivalence classes $[f] \in V_{\pi,\epsilon}$ and $[[f]] \in C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \big/ \operatorname{Ker}^{\mathbb{C}}_{\pi,\epsilon}$ and consider the map $V_{\pi,\epsilon} \to C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \big/ \operatorname{Ker}^{\mathbb{C}}_{\pi,\epsilon} : [f] \mapsto [[f]]$.
Linearity is obvious and for well-definedness we need to show that $[[f+h]] = [[f]]$ whenever $h \in \operatorname{Ker}\sigma_{\pi,\epsilon}$, which is equivalent to showing that $h \in \operatorname{Ker}\sigma_{\pi,\epsilon}$ implies $h \in \operatorname{Ker}^{\mathbb{C}}_{\pi,\epsilon}$.
As an intermediate result we need to show

$$\operatorname{Ker}\sigma_{\pi,\epsilon} = \left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \langle f, g\rangle_{\pi,\epsilon} = 0\,\forall g \in C_c^\infty(\mathbb{R}^{d+1})\right\}\,, \tag{B.12}$$

which is the analog to Equation (3.1.3). $\left\{f \in C_c^\infty(\mathbb{R}^{d+1}) \mid \langle f, g\rangle_{\pi,\epsilon} = 0\,\forall g \in C_c^\infty(\mathbb{R}^{d+1})\right\} \subseteq \operatorname{Ker}\sigma_{\pi,\epsilon}$ holds because of (B.6), i.e., $\sigma_{\pi,\epsilon}(f, g) = -\frac{1}{2}\operatorname{Im}\langle f, g\rangle_{\pi,\epsilon}$.

The other direction holds since by (2.3.8) and (5.2.4) we can find for any $f \in \operatorname{Ker} \sigma_{\pi,\epsilon}$ a $\tilde{f} \in C_c^\infty(\mathbb{R}^{d+1})$ such that $f = P_\epsilon \tilde{f}$. Then, for any $g \in C_c^\infty(\mathbb{R}^{d+1})$, we get

$$\begin{aligned}
\langle f, g \rangle_{\pi,\epsilon} = \left\langle P_\epsilon \tilde{f}, g \right\rangle_{\pi,\epsilon} &\overset{(2.3.1)}{=} \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} \widehat{P_\epsilon \tilde{f}}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} \\
&= \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} \left[(\partial_t^2 - \epsilon^2 \nabla^2 + \epsilon^2 m^2) \tilde{f} \right]^\wedge (E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} \\
&= \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} (-p_0^2 + \epsilon^2 p + \epsilon^2 m^2)|_{p_0 = E_{\epsilon,p}} \hat{\tilde{f}}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} \\
&= \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} (-p_0^2 + (E_{\epsilon,p}))|_{p_0 = E_{\epsilon,p}} \hat{\tilde{f}}(E_{\epsilon,p}, p) \overline{\hat{g}(E_{\epsilon,p}, p)} = 0 \,.
\end{aligned}$$

By (B.12) and (3.1.2) we know that $h \in \operatorname{Ker} \sigma_{\pi,\epsilon}$ implies $\langle h, h \rangle_{\pi,\epsilon} = 0$ and since $h \in C_c^\infty(\mathbb{R}^{d+1})$ this also implies $\langle h, h \rangle_{\pi,\epsilon}^{\mathbb{C}} = 0$.
For injectivity we need to show that if $[[f]] = [[0]]$ then $[f] = [0]$, i.e., if $f \in \operatorname{Ker}_{\pi,\epsilon}^{\mathbb{C}}$ then $f \in \operatorname{Ker} \sigma_{\pi,\epsilon}$ for any $f \in C_c^\infty(\mathbb{R}^{d+1})$, which is true by (3.1.2) and (B.12) and $0 = \langle f, f \rangle_{\pi,\epsilon}^{\mathbb{C}} = \langle f, f \rangle_{\pi,\epsilon}$.
Hence, we can consider $V_{\pi,\epsilon}$ a subset of $C_c^\infty(\mathbb{R}^{d+1}, \mathbb{C}) \big/ \operatorname{Ker}_{\pi,\epsilon}^{\mathbb{C}}$ and therefore also of $\mathcal{H}_{\pi,\epsilon}$.

To further represent $\mathcal{W}_{\pi,\epsilon}$ on the Fock space $\mathcal{F}(\mathcal{H}_{\pi,\epsilon})$ we need to rewrite π_ϵ and $\pi_\epsilon^\dagger$. We use the definition of the conjugate momenta $\pi = \partial_t \phi^\dagger$ and $\pi^\dagger = \partial_t \phi$ and Equation (3.1.4) and (3.1.5) to get

$$\begin{aligned}
\pi(t,x) = \partial_t \phi^\dagger(t,x) &\overset{(3.1.5)}{=} \frac{i}{\sqrt{2}} \int \frac{d^dp}{(2\pi)^d} \sqrt{E_p} \left(A^\dagger(p) e^{itE_p - ipx} - B(p) e^{-itE_p + ipx} \right) \\
\pi^\dagger(t,x) = \partial_t \phi(t,x) &\overset{(3.1.4)}{=} \frac{i}{\sqrt{2}} \int \frac{d^dp}{(2\pi)^d} \sqrt{E_p} \left(-A(p) e^{-itE_p + ipx} + B^\dagger(p) e^{itE_p - ipx} \right) .
\end{aligned} \tag{B.13}$$

If we define

$$A_{\pi,\epsilon}(f) := \int \frac{d^dp}{(2\pi)^d} \frac{\sqrt{E_p}}{\sqrt{\epsilon}} A(p) \hat{f}(E_{\epsilon,p}, p) \quad \text{and} \quad A_{\pi,\epsilon}^\dagger(f) := \int \frac{d^dp}{(2\pi)^d} \frac{\sqrt{E_p}}{\sqrt{\epsilon}} A^\dagger(p) \overline{\hat{f}(E_{\epsilon,p}, p)} \,, \tag{B.14}$$

for $f \in C_c^\infty(\mathbb{R}^{d+1})$ and analogously for $B_{\pi,\epsilon}(f)$ and $B_{\pi,\epsilon}^\dagger(f)$, we can rewrite $\pi_\epsilon(f)$ as

$$\begin{aligned}
\pi_\epsilon(f) &= \int dt d^dx \, \pi_\epsilon(t,x) f(t,x) \overset{(B.1)}{=} \int dt d^dx \, \epsilon^{-\frac{1}{2}} \pi(\epsilon t, x) f(t,x) \\
&\overset{(B.13)}{=} \epsilon^{-\frac{1}{2}} \frac{i}{\sqrt{2}} \int dt d^dx \int \frac{d^dp}{(2\pi)^d} \sqrt{E_p} \left(A^\dagger(p) e^{i\epsilon t E_p - ipx} - B(p) e^{-i\epsilon t E_p + ipx} \right) f(t,x) \\
&= \epsilon^{-\frac{1}{2}} \frac{i}{\sqrt{2}} \int \frac{d^dp}{(2\pi)^d} \sqrt{E_p} \int dt d^dx \left(A^\dagger(p) f(t,x) e^{i\epsilon t E_p - ipx} - B(p) f(t,x) e^{-i\epsilon t E_p + ipx} \right) \\
&\overset{(B.14)}{=} \frac{i}{\sqrt{2}} \left(A_{\pi,\epsilon}^\dagger(f) - B_{\pi,\epsilon}(f) \right) .
\end{aligned} \tag{B.15}$$

Analogously we get

$$\pi_\epsilon^\dagger(f) = \frac{i}{\sqrt{2}}\left(-A_{\pi,\epsilon}(f) + B_{\pi,\epsilon}^\dagger(f)\right) . \tag{B.16}$$

We can then rewrite

$$\begin{aligned}\Pi_\epsilon(f) &\overset{(B.8)}{=} \frac{1}{\sqrt{2}}\left(\pi_\epsilon(f) + \pi_\epsilon^\dagger(f)\right)\\ &\overset{(B.15)+(B.16)}{=} \left(\frac{i}{\sqrt{2}}\left(A_{\pi,\epsilon}^\dagger(f) - B_{\pi,\epsilon}(f)\right) + \frac{i}{\sqrt{2}}\left(-A_{\pi,\epsilon}(f) + B_{\pi,\epsilon}^\dagger(f)\right)\right)\\ &= \frac{1}{\sqrt{2}}\left(a_{\pi,\epsilon}(f) + a_{\pi,\epsilon}^\dagger(f)\right) ,\end{aligned}$$

with

$$a_{\pi,\epsilon}(f) := \frac{-i}{\sqrt{2}}\left(A_{\pi,\epsilon}(f) + B_{\pi,\epsilon}(f)\right) \quad \text{and} \quad a_{\pi,\epsilon}^\dagger(f) := \frac{i}{\sqrt{2}}\left(A_{\pi,\epsilon}^\dagger(f) + B_{\pi,\epsilon}^\dagger(f)\right) . \tag{B.17}$$

For the commutators we compute, for $f, g \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\begin{aligned}\left[A_{\pi,\epsilon}(f), A_{\pi,\epsilon}^\dagger(g)\right] &\overset{(B.14)}{=} \int \frac{d^dp}{(2\pi)^d}\frac{\sqrt{E_p}}{\sqrt{\epsilon}} \int \frac{d^dq}{(2\pi)^d}\frac{\sqrt{E_q}}{\sqrt{\epsilon}} \left[A(p), A^\dagger(q)\right] \hat{f}(E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,q}, q)}\\ &\overset{(3.1.6)}{=} \int \frac{d^dp}{(2\pi)^d}\frac{\sqrt{E_p}}{\sqrt{\epsilon}} \int \frac{d^dq}{(2\pi)^d}\frac{\sqrt{E_q}}{\sqrt{\epsilon}} (2\pi)^d\delta^d(p-q)\hat{f}(E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,q}, q)}\\ &= \int \frac{d^dp}{(2\pi)^d}\frac{E_p}{\epsilon}\hat{f}(E_{\epsilon,p}, p)\overline{\hat{g}(E_{\epsilon,p}, p)} \overset{(B.4)}{=} \langle f, g\rangle_{\pi,\epsilon}\end{aligned} \tag{B.18}$$

and similarly $\left[B_{\pi,\epsilon}(f), B_{\pi,\epsilon}^\dagger(g)\right] = \langle f, g\rangle_{\pi,\epsilon}$. Every other commutator between $A_{\pi,\epsilon}$, $A_{\pi,\epsilon}^\dagger$, $B_{\pi,\epsilon}$ and $B_{\pi,\epsilon}^\dagger$ is 0. This then implies that

$$\begin{aligned}\left[a_{\pi,\epsilon}(f), a_{\pi,\epsilon}^\dagger(g)\right] &\overset{(B.17)}{=} \left[\frac{-i}{\sqrt{2}}\left(A_{\pi,\epsilon}(f) + B_{\pi,\epsilon}(f)\right), \frac{i}{\sqrt{2}}\left(A_{\pi,\epsilon}^\dagger(g) + B_{\pi,\epsilon}^\dagger(g)\right)\right]\\ &= \frac{1}{2}\left(\left[A_{\pi,\epsilon}(f), A_{\pi,\epsilon}^\dagger(g)\right] + \left[B_{\pi,\epsilon}(f), A_{\pi,\epsilon}^\dagger(g)\right]\right.\\ &\quad \left. + \left[A_{\pi,\epsilon}(f), B_{\pi,\epsilon}^\dagger(g)\right] + \left[B_{\pi,\epsilon}(f), B_{\pi,\epsilon}^\dagger(g)\right]\right)\\ &\overset{(B.18)}{=} \frac{1}{2}\left(\langle f, g\rangle_{\pi,\epsilon} + \langle f, g\rangle_{\pi,\epsilon}\right) = \langle f, g\rangle_{\pi,\epsilon}\end{aligned}$$

and analogously $[a_{\pi,\epsilon}(f), a_{\pi,\epsilon}(g)] = \left[a_{\pi,\epsilon}^\dagger(f), a_{\pi,\epsilon}^\dagger(g)\right] = 0$.

Hence we can represent $a_{\pi,\epsilon}$ and $a_{\pi,\epsilon}^\dagger$ as the annihilation and creation operators on the Fock space $\mathcal{F}(\mathcal{H}_{\pi,\epsilon})$. We denote with $a_{\pi,\epsilon}(f)$ and $a_{\pi,\epsilon}^\dagger(g)$, for $f, g \in \mathcal{H}_{\pi,\epsilon}$, the annihilation and creation operator and with $|0_{\pi,\epsilon}\rangle$ the normalized vacuum vector of $\mathcal{F}(\mathcal{H}_{\pi,\epsilon})$. A distinction between the annihilation and creation operators of $\mathcal{F}(\mathcal{H}_{\pi,\epsilon})$ and $a_{\pi,\epsilon}(f)$ and $a_{\pi,\epsilon}^\dagger(f)$ from Equation (B.17) is again omitted, since they agree on $f \in C_c^\infty(\mathbb{R}^{d+1})$.

Due to the similar structure of $\psi_\epsilon(f)$ and $\Pi_\epsilon(f)$, the computation of the vacuum expectation value for $W_{\pi,\epsilon}(f)$ is exactly the same as for $W_\epsilon(f)$ in Subsection 3.1.2. Hence we quickly conclude that

$$\langle 0_{\pi,\epsilon}| \, W_{\pi,\epsilon}(f) \, |0_{\pi,\epsilon}\rangle = e^{-\frac{1}{4}\|f\|^2_{\pi,\epsilon}} \, .$$

This means that in order to calculate the value of the linear functional $\rho_{\pi,0} : \mathcal{W}_{\pi,0} \longrightarrow \mathbb{C}$, defined as

$$\rho_{\pi,0}(W_{\pi,0}(f)) := \lim_{\epsilon \to 0} \langle 0_{\pi,\epsilon}| \, W_{\pi,\epsilon}(f) \, |0_{\pi,\epsilon}\rangle \ ,$$

for all $f \in V_{\pi,0} = C_c^\infty(\mathbb{R}^{d+1}) \big/ \mathrm{Ker}\,\sigma_0$, we need to compute $\lim_{\epsilon\to 0} e^{-\frac{1}{4}\|f\|^2_{\pi,\epsilon}}$.

First, we look at $\|f\|^2_{\pi,\epsilon} = \langle f, f\rangle_{\pi,\epsilon}$. Here we use Taylor's theorem on $\left|\hat{f}(p_0,p)\right|^2 = \hat{f}(p_0,p)\overline{\hat{f}(p_0,p)}$, since it is smooth.

$$\begin{aligned}
\langle f, f\rangle_{\pi,\epsilon} &= \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} \left|\hat{f}(\epsilon E_p, p)\right|^2 \\
&= \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{\epsilon} \left(\left|\hat{f}(0,p)\right|^2 + \partial_{p_0} \left[\left|\hat{f}(p_0,p)\right|^2 \right] \Big|_{p_0=0} \epsilon E_p + \mathcal{O}((\epsilon E_p)^2) \right) \\
&= \frac{1}{\epsilon} \int \frac{d^dp}{(2\pi)^d} E_p \left|\hat{f}(0,p)\right|^2 + \int \frac{d^dp}{(2\pi)^d} E_p^2 \left(\partial_{p_0}\hat{f}(0,p)\overline{\hat{f}(0,p)} + \hat{f}(0,p)\partial_{p_0}\overline{\hat{f}(0,p)} \right) \\
&\quad + \int \frac{d^dp}{(2\pi)^d} E_p^2 \, \mathcal{O}(\epsilon E_p) \\
&\overset{p\mapsto -p}{=} \frac{1}{\epsilon} \int \frac{d^dp}{(2\pi)^d} E_p \left|\hat{f}(0,p)\right|^2 + \int \frac{d^dp}{(2\pi)^d} E_p^2 \left(\partial_{p_0}\hat{f}(0,p)\overline{\hat{f}(0,p)} + \hat{f}(0,-p)\partial_{p_0}\overline{\hat{f}(0,-p)} \right) \\
&\quad + \int \frac{d^dp}{(2\pi)^d} E_p^2 \, \mathcal{O}(\epsilon E_p) \\
&= \frac{1}{\epsilon} \int \frac{d^dp}{(2\pi)^d} E_p \left|\hat{f}(0,p)\right|^2 + \int \frac{d^dp}{(2\pi)^d} E_p^2 \left(\partial_{p_0}\hat{f}(0,p)\overline{\hat{f}(0,p)} - \overline{\hat{f}(0,p)}\partial_{p_0}\hat{f}(0,p) \right) \\
&\quad + \int \frac{d^dp}{(2\pi)^d} E_p^2 \, \mathcal{O}(\epsilon E_p) \\
&= \frac{1}{\epsilon} \int \frac{d^dp}{(2\pi)^d} E_p \left|\hat{f}(0,p)\right|^2 + \int \frac{d^dp}{(2\pi)^d} E_p^2 \, \mathcal{O}(\epsilon E_p) \, .
\end{aligned}$$

The integral $\int \frac{d^dp}{(2\pi)^d} E_p^2 \, \mathcal{O}(\epsilon E_p)$ converges to 0, since $\epsilon E_p \to 0$ as $\epsilon \to 0$[1]. Similarly to the electric case, the only divergent part is $\frac{1}{\epsilon}$ in front of the first integral. So the only way in

[1]Again, similarly to the electric case, the integral $\int \frac{d^dp}{(2\pi)^d} E^3_{\epsilon,p}$ diverges, yet in our case every term in $E_p^2 \, \mathcal{O}(E_{\epsilon,p})$ contains some derivative of $\hat{f}(0,p)$ (and $\overline{\hat{f}(0,p)}$). Since f is Schwartz so is $\hat{f}$, which implies that $E^3_{\epsilon,p}$ multiplied by any derivative of $\hat{f}(0,p)$, and therefore also $\mathcal{O}(E^3_{\epsilon,p})$, is integrable.

which $\|f\|^2_{\pi,\epsilon}$ does not diverge is, if $\int \frac{d^dp}{(2\pi)^d}\, E_p \left|\hat{f}(0,p)\right|^2 = 0$, hence if $\hat{f}(0,p) = 0$ for all $p \in \mathbb{R}^d$. So

$$\lim_{\epsilon\to 0} \|f\|^2_{\pi,\epsilon} = \begin{cases} 0 & \text{if } \hat{f}(0,\,.\,) = 0\,, \\ \infty & \text{else}\,, \end{cases}$$

which yields

$$\rho_{\pi,0}(W_{\pi,0}(f)) = \lim_{\epsilon\to 0} \langle 0_{\pi,\epsilon}|\, W_{\pi,\epsilon}(f)\, |0_{\pi,\epsilon}\rangle = \lim_{\epsilon\to 0} e^{-\frac{1}{4}\|f\|^2_{\pi,\epsilon}} = \begin{cases} 1 & \text{if } \hat{f}(0,\,.\,) = 0\,, \\ 0 & \text{else}\,. \end{cases}$$

Next we show that $\rho_{\pi,0}$ is well-defined, i.e., $\rho_{\pi,0}(W_{\pi,0}(f+h)) = \rho_{\pi,0}(W_{\pi,0}(f))$ for all $h \in \operatorname{Ker}\sigma_{\pi,0}$. By the same argument as in the electric case, this means we need to show that $\hat{h}(0,\,.\,) = 0$ for all $h \in \operatorname{Ker}\sigma_{\pi,0}$. But this holds by the classification of $\operatorname{Ker}\sigma_{\pi,0}$ in Equation (5.2.11).
The proofs for normalization, positivity and pureness are also the same as for the electric case, hence $\rho_{\pi,0}$ is a well-defined pure state on $\mathcal{W}_{\pi,0}$.

Lastly, we show that the $*$-automorphism generated by

$$\left\{\alpha_{b,R,a} \mid b \in \mathbb{R}, R \in SO(d), a \in \mathbb{R}^d\right\}\,,$$

where $\alpha_{b,R,a}(t,x) = (t+b, Rx+a)$, are indeed symmetries of $\rho_{\pi,0}$. By the same analysis done in Section 3.2 we need to check two conditions

$$\sigma_{\pi,0}(\alpha^*_{b,R,a}f, \alpha^*_{b,R,a}g) = \sigma_{\pi,0}(f,g) \quad \forall f,g \in V_0 \tag{B.19}$$

$$\widehat{\alpha^*_{b,R,a}f}(0,\,.\,) = 0 \iff \hat{f}(0,\,.\,) = 0 \quad \forall f \in V_0\,. \tag{B.20}$$

Since a constant can be seen as a smooth function, we can conclude that Equation (3.2.4) and (3.2.5) still hold. Hence

$$\widehat{\alpha^*_{b,R,a}f}(0,p) = e^{-iR^{-1}ap}\hat{f}(0,Rp) \quad \text{and} \tag{B.21}$$

$$\frac{\partial}{\partial p_0}\widehat{\alpha^*_{b,R,a}f}(0,p) = e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{f}(0,Rp) + ib\hat{f}(0,Rp)\right) \tag{B.22}$$

and the first one implies Equation (B.20). For (B.19) we compute

$$\begin{aligned}
&\sigma_{\pi,0}(\alpha^*_{b,R,a}f, \alpha^*_{b,R,a}g) \\
&\overset{(B.7)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\, E_p^2 \left(\frac{\partial}{\partial p_0}\widehat{\alpha^*_{b,R,a}f}(0,p)\overline{\widehat{\alpha^*_{b,R,a}g}(0,p)} - \overline{\widehat{\alpha^*_{b,R,a}f}(0,p)}\frac{\partial}{\partial p_0}\widehat{\alpha^*_{b,R,a}g}(0,p)\right) \\
&\overset{(B.21)+(B.22)}{=} \frac{i}{2}\int \frac{d^dp}{(2\pi)^d}\, E_p^2 \Bigg[e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{f}(0,Rp) + ib\hat{f}(0,Rp)\right)e^{iR^{-1}ap}\hat{g}(0,-Rp) \\
&\qquad - e^{iR^{-1}ap}\hat{f}(0,-Rp)e^{-iR^{-1}ap}\left(\frac{\partial}{\partial p_0}\hat{g}(0,Rp) + ib\hat{g}(0,Rp)\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2}\int \frac{d^d p}{(2\pi)^d}\, E_p^2 \bigg[\frac{\partial}{\partial p_0}\hat{f}(0,Rp)\hat{g}(0,-Rp) + ib\widehat{f}(0,Rp)\hat{g}(0,-Rp) \\
&\qquad - \hat{f}(0,-Rp)\frac{\partial}{\partial p_0}\hat{g}(0,Rp) - i\hat{f}(0,-Rp)b\widehat{g}(0,Rp)\bigg] \\
&\overset{p\,\mapsto\, R^{-1}p}{=} \frac{i}{2}\int \frac{d^d p}{(2\pi)^d}\, E_p^2 \bigg[\frac{\partial}{\partial p_0}\hat{f}(0,p)\overline{\hat{g}(0,p)} - \overline{\hat{f}(0,p)}\frac{\partial}{\partial p_0}\hat{g}(0,p) \\
&\qquad + ib\widehat{f}(0,p)\hat{g}(0,-p) - i\hat{f}(0,-p)b\widehat{g}(0,p)\bigg] \\
&\overset{(B.7)}{=} \sigma_{\pi,0}(f,g) - \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, E_p^2 \Big(b\widehat{f}(0,p)\hat{g}(0,-p) - \hat{f}(0,-p)b\widehat{g}(0,p)\Big) \\
&\overset{p\,\mapsto\,-p}{=} \sigma_{\pi,0}(f,g) - \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, E_p^2 \Big(b\widehat{f}(0,p)\hat{g}(0,-p) - \hat{f}(0,p)b\widehat{g}(0,-p)\Big) \\
&= \sigma_{\pi,0}(f,g)\,.
\end{aligned}
$$