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Triality in W-algebras from an analysis of OPEs obtained by the quantum Miura transformation

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Abstract

The context of this thesis is the field of 2-dimensional conformal field theories. More specifically, the subject at hand consists of a certain class of W-algebras that appear in 2-d conformal field theories of which the well-known quantum Virasoro algebra is a certain type, the so-called W_2-algebra. In this thesis the explicit form of the normalization independent parameter $C(N,c)$ characterizing the quantum W_N-algebra is proven. Two different methods are presented. The first method uses the generalized Wick theorem. The second one uses already existing knowledge about the W_{1+N}-algebra. More concretely, a transformation between the W_N-algebra and the W_{1+N}-algebra is found. Currents implicitly defined by the quantum Miura transformation in the W_N-algebra are then expressed through currents implicitly defined by the quantum Miura transformation in the W_{1+N}-algebra. Already known operator product expansions between currents in the W_{1+N}-algebra as well as a specific `Mathematica` package are then used to successfully calculate those singularity terms of operator product expansions between primary currents in the W_N-algebra that contain factors that make up the parameter $C(N,c)$.

Zusammenfassung

Die vorliegende Arbeit wurde im Kontext 2-dimensionaler konformer Feldtheorien verfasst. Konkret wird eine spezielle Klasse der W-Algebren, die in diesem Bereich auftauchen, thematisiert, zu denen auch die bekannte Quanten-Virasoro-Algebra als W_2-Algebra zählt. In dieser Arbeit wird die explizite Form des normalisierungs-unabhängigen Parameters $C(N,c)$, welcher die Quanten-W_N-Algebra charakterisiert, bewiesen. Dabei werden zwei unterschiedliche Methoden präsentiert. Die erste Methode stützt sich auf die Anwendung des generalisierten Wick-Theorems. Bei der zweiten wird hingegen bereits existierendes Wissen bezüglich der W_{1+N}-Algebra angewandt. Konkret wird eine Transformation zwischen Strömen der W_N-Algebra und Strömen der W_{1+N}-Algebra gefunden, wobei folglich Ströme, die implizit durch die Quanten-Miura-Transformation in der W_N-Algebra definiert sind, durch Ströme, die implizit durch die Quanten-Miura-Transformation in der W_{1+N}-Algebra definiert sind, ausgedrückt werden. Mithilfe bereits bekannter Operator-Produkt-Expansionen zwischen Strömen der W_{1+N}-Algebra sowie einem spezifischen `Mathematica`-Paket können dann jene singulären Terme, welche Faktoren des Parameters $C(N,c)$ beinhalten, in den Operator-Produkt-Expansionen zwischen primären Strömen der W_N-Algebra berechnet werden.

Contents

1 Introduction

This Master thesis is written in the context of a certain class of algebras that appear in 2-dimensional conformal field theories (CFTs), namely the so-called W-algebras of which there exist different types. The primary focus lies on W_N-algebras as well as the related W_∞-algebra. The W_N-algebra possesses a normalization independent parameter $C(N, c)$ which will be the focus of this thesis.

This parameter, after N is replaced by the real variable λ appears again as $C(\lambda, c)$ in the closed W_∞-algebra which possesses a certain characteristic called the triality, see e.g. [GG12] and [CF24]. This term refers to a threefold symmetry that is satisfied by three parameters (in actual fact only two since the third parameter is entirely dependent on the first two) of the algebra, namely the real parameter λ, the central charge c of the quantum Virasoro algebra W_2 and a certain parameter $C(\lambda, c)$ which is related to the highest singularity terms of the operator product expansions (OPEs) of $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$. More concretely, W_∞-algebras which have the same parameters C and c are isomorphic and it turns out that generically, three different values of λ, i.e. λ_1, λ_2 and λ_3, lead – for fixed central charge c – to the same value of C, i.e. $C(\lambda_1, c) = C(\lambda_2, c) = C(\lambda_3, c)$. The triality then consists of three separate identities which are satisfied by $\lambda_1, \lambda_2, \lambda_3, c$ and C. Apart from the isomorphic relation between W_∞-algebras, the triality symmetry which will be discussed in more detail in chapter 3.1.3 might have various consequences for the representation theory of this algebra that are yet to be understood.

This thesis aims to prove the explicit form of $C(N, c)$ by determining the concrete form of the highest singularity terms of the OPEs $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$. This is done using two different methods. In chapter 4.1 the highest singularity term of the OPE $W_3(z)W_3(w)$ is calculated using the generalized Wick theorem. It is shown that although this method is successful, it is also very tedious.

In chapter 4.2 on the other hand the same problem is solved using already known expressions of currents that make up the $W_{1+\infty}$-algebra and OPEs thereof. We first find a suitable transformation between spin-1 currents of the W_N- and the W_{1+N}-algebra. Subsequently we find and prove a general expression between the currents $\tilde{U}_j$ and U_i that are implicitely defined through the quantum Miura transformation in chapter 4.2.1. With this formula it is then possible to find concrete expressions of the currents $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ of the W_N-algebra in dependence of the currents U_1, U_2, U_3 and U_4 of the W_{1+N}-algebra. Lastly, we present in chapter 4.2.5 the `Mathematica` code that is used to find the highest singularity terms of the OPEs $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$.

Apart from this we present in chapter 2 relevant theory regarding 2-dimensional CFT and the concept of OPEs, the generalized Wick theorem as well as the quantization of classical W-algebras. In chapter 3 concrete versions of the W_N-algebra in the primary basis and W_{1+N}-algebra in the quadratic basis that are both constructed by the quantum Miura transformation will be discussed. Especially in the latter case, where there exists a formula for the general construction of OPEs between currents that make up the W_{1+N}-algebra in [Pro15], we show details of the calculations necessary to find those OPEs.

2 Theoretical and Mathematical Framework

In this chapter, we will discuss the theoretical and mathematical framework in which W-algebras arise. The aim is to provide sufficient information for the reader to be able to follow and understand all further chapters of this thesis.

As has been described e.g. in [Dic97], W-algebras arise in CFT as extensions of the so-called Virasoro algebra which is integral to 2-dimensional CFT. It turned out that the mathematical framework for further extension of Virasoro algebra existed prior to its discovery within CFT as the theory of integrable systems, see e.g. [LF90] and [Luk88]. More concretely, the Virasoro algebra and Zamolodchikov's W_3-algebra were identified with the quantized versions of the second Poisson structure of the equations of the second and third Korteweg–de Vries (KdV) type. Classical W-algebras also appear as Poisson algebras in the form of asymptotic symmetries of higher-spin gauge theories on AdS_3[1] backgrounds which has been thoroughly discussed e.g. in [CFP11].

In this chapter we will recall certain characteristics of CFTs and present the 2-dimensional case. We will also discuss how W-algebras arise in this theory and shortly present how they appear in the classical case. Furthermore, the concepts of radial quantization, radial ordering and subsequently the concept of OPEs, which are all vitally important for the objective of this thesis, will be discussed.

2.1 Classical W-Algebras

W-algebras do not only appear in CFT as we will see in the next section, but classical versions also can be found in the form of asymptotic symmetries of higher-spin gauge theories on AdS_3 backgrounds where they arise as classical Poisson algebras.

By a procedure called the Drinfeld-Sokolov reduction which is applied to the dual space of the affinization $\hat{\mathfrak{g}}$ of a semisimple Lie algebra $\mathfrak{g}$ a corresponding reduced space is retrieved, see [DS85]. The Poisson algebra of functionals on this particular space is called the classical W-algebra, i.e. to any arbitrary semisimple Lie algebra $\mathfrak{g}$ there can be associated a classical W-algebra. In particular, the classical Virasoro algebra,

$$i\left\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n\right\} = (m-n)\hat{\mathcal{L}}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad , \tag{2.1}$$

is the classical W-algebra $W(\hat{\mathfrak{sl}_2})$, or simply $W^{(2)}$, associated to $\mathfrak{sl}_2$ and the classical $W^{(3)}$-algebra,

[1] Anti-de-Sitter spacetime in three dimensions, see e.g.[Gib11]

$$
\begin{aligned}
i\left\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n\right\} &= (m-n)\hat{\mathcal{L}}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
i\left\{\hat{\mathcal{L}}_m, \mathcal{W}_n\right\} &= (2m-n)\mathcal{W}_{m+n} \\
i\left\{\mathcal{W}_m, \mathcal{W}_n\right\} &= \frac{1}{12}\Big[(m-n)(2m^2+2n^2-mn-8)\hat{\mathcal{L}}_{m+n} + \frac{96}{c}(m-n)\Lambda_{m+n} \\
&\quad + \frac{c}{12}m(m^2-1)(m^2-4)\delta_{m+n,0}\Big] \quad ,
\end{aligned}
\tag{2.2}
$$

where

$$\Lambda_p \coloneqq \sum_{q\in\mathbb{Z}} \hat{\mathcal{L}}_{p+q}\hat{\mathcal{L}}_{-q} \tag{2.3}$$

is the classical W-algebra $W(\mathfrak{sl}_3)$ associated to $\mathfrak{sl}_3$. For details see, e.g. chapter 3 in [CF24].

In this thesis we consider the corresponding quantum W-algebras. To prepare for their description we first need to introduce concepts of 2-dimensional CFTs which we will do in the following.

2.2 Conformal Field Theory

Before presenting the special case of 2-dimensional CFT in which W-algebras arise, we first recall a couple of important aspects of general CFTs, where we will in large part closely follow chapter 2 of [BP09].

By definition, CFTs are quantum field theories that are invariant under the conformal group. These theories are able to describe systems at critical points and furthermore play a huge role in string theory, see e.g. [Gab00]. To be able to proceed a couple of definitions are in order. These align with the description provided in [BP09].

Definition 1. ***(Conformal Group)***
The ***conformal group*** *is defined as the group consisting of finite conformal transformations that are globally defined and invertible.*

Definition 2. ***(Conformal Algebra)***
The Lie algebra which corresponds to the conformal group is called the ***conformal algebra****.*

Definition 3. ***(Conformal Transformation, Scale Factor)***
Let M and M' be manifolds and $U \subset M$ and $V \subset M'$ open subsets thereof. Then a differentiable map $\varphi : U \to V$ is called a ***conformal transformation*** *if the metric tensors g and g' satisfy*

$$\varphi^* g' = \Lambda g \quad . \tag{2.4}$$

*$\varphi^* g'$ denotes the pullback of g' under φ and Λ is a positive function called the **scale factor**.*

Equivalently, when denoting $x' := \varphi(x)$ and understanding Einstein's summation convention the condition (2.4) can be expressed as

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) \quad . \tag{2.5}$$

In the following we will exclusively focus on the case where $M = M'$ and $g = g'$. Additionally, we will only consider flat space equipped with the flat metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, ..., +1, ...)$.

Up to first order in the small parameter $\varepsilon(x) \ll 1$ infinitesimal coordinate transformations read

$$x'^{\rho} = x^{\rho} + \varepsilon^{\rho}(x) + \mathcal{O}(\varepsilon^2) \quad , \tag{2.6}$$

and we arrive for (2.5) at the following conditions:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} \left(\partial \cdot \varepsilon\right) \eta_{\mu\nu} \quad , \tag{2.7}$$

and

$$(d-1) \, \Box \, (\partial \cdot \varepsilon) = 0 \quad , \tag{2.8}$$

where $(\partial \cdot \varepsilon) = \partial^\mu \varepsilon_\mu$ and the d'Alembert operator as usual is defined as $\Box := \partial^\mu \partial_\mu$. The details of these calculations can be found in chapter 2.1 of [BP09].

From (2.8) one can derive four different conformal transformations and their corresponding generators:

translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_\mu = -i\partial_\mu$
dilation	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix^\mu \partial_\mu$
rotation	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
SCT[2]	$x'^{\mu} = \frac{x^\mu - (x\cdot x) b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x \cdot x)}$	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x\cdot x)\partial_\mu)$.

The conformal group for $d \geq 3$ can be found by concentrating on its corresponding algebra first. Observing that the rotation generator $L_{\mu\nu}$ is anti-symmetric, the total number of generators amounts to

$$N = \frac{(d+2)(d+1)}{2} \quad . \tag{2.9}$$

After defining

[2]Special Conformal Transformation

$$\begin{aligned} J_{\mu,\nu} &= L_{\mu\nu} \quad , \\ J_{-1,0} &= D \quad , \\ J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \quad , \\ J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{2.10}$$

for $\mu \in \{1, ..., d\}$ one can verify that the $J_{m,n} = J_{mn}$ for $m, n \in \{-1, 0, 1, ..., d\}$ satisfy the commutation relations

$$[J_{mn}, J_{rs}] = i(\eta_{ms} J_{nr} + \eta_{nr} J_{ms} - \eta_{mr} J_{ns} - \eta_{ns} J_{mr}) \quad . \tag{2.11}$$

For a space $\mathbb{R}^{p,q}$ these can be identified with the commutation relations of the Lie algebra $\mathfrak{so}(p+1, q+1)$. Thus the general result is that for $d = p + q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is the special orthogonal group $SO(p + 1, q + 1)$. In this thesis, however, we are interested in the special case $d = 2$, which we will discuss in more detail in the following.

2.3 Conformal Field Theory in two Dimensions

Following chapter 2 of [BP09] further, in two dimensions the condition (2.7) for conformal invariance under infinitesimal conformal transformations reads

$$\partial_0 \varepsilon_0 = \partial_1 \varepsilon_1 \quad , \quad \partial_0 \varepsilon_1 = -\partial_1 \varepsilon_0 \quad . \tag{2.12}$$

We recognize them as the Cauchy–Riemann equations appearing in complex analy-sis which hold (on some open set) for complex differentiable functions of the form $\varepsilon(z) = \varepsilon_0(z) + i\varepsilon_1(z)$ which are then called analytic or holomorphic. This motivates us to introduce complex variables

$$\varepsilon := \varepsilon^0 + i\varepsilon^1 \quad , \quad \bar{\varepsilon} := \varepsilon^0 - i\varepsilon^1 \quad . \tag{2.13}$$

Furthermore we define

$$z := x^0 + ix^1 \quad , \quad \bar{z} := x^0 - ix^1 \tag{2.14}$$

and accordingly

$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \quad , \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \quad . \tag{2.15}$$

Since $\varepsilon(z)$ and z are both holomorphic functions, their sum $f(z) := \varepsilon(z) + z$ is holomorphic as well. This function in turn gives rise to an infinitesimal 2-dimensional conformal transformation $z \mapsto f(z)$, $\varepsilon(z) \ll 1$. The metric tensor then transforms as

$$\begin{aligned} g &= ds^2 \\ &= (dx^0)^2 + (dx^1)^2 = dz d\bar{z} \mapsto \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dz d\bar{z} \end{aligned} \tag{2.16}$$

from which we can read off the scale factor as

$$\Lambda(z, \bar{z}) = \left|\frac{\partial f}{\partial z}\right|^2 \quad . \tag{2.17}$$

Since up until now we only determined that $\varepsilon(z)$ is holomorphic on some open set we have to deal with the possibility that it is indeed a meromorphic function, i.e. a function on a subset of the complex plane that is holomorphic on the entire subset except for those points that are the poles of the function. Therefore we perform a Laurent expansion of $\varepsilon(z)$ around $z = 0$ and write a general infinitesimal conformal transformation as

$$f(z) = z + \varepsilon(z) = z + \sum_{n \in \mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \quad , \tag{2.18}$$

$$\bar{f}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right) \quad , \tag{2.19}$$

where the infinitesimal parameters $\varepsilon_n, \bar{\varepsilon}_n \in \mathbb{Z}$ are constant. Those generators that correspond to the transformation for a particular n are

$$l_n := -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n := -\bar{z}^{n+1}\partial_{\bar{z}} \quad . \tag{2.20}$$

Note that we have an infinite number of generators due to $n \in \mathbb{Z}$. The commutation relations of all possible generators then are

$$[l_n, l_m] = (m - n) l_{m+n} \tag{2.21}$$

which actually defines a copy of the Witt algebra,

$$[\bar{l}_n, \bar{l}_m] = (m - n) \bar{l}_{m+n} \tag{2.22}$$

and

$$[l_n, \bar{l}_m] = 0 \quad . \tag{2.23}$$

In summary we find that in Euclidean 2-dimensional space the algebra of infinitesimal conformal transformations is infinite dimensional.

Moreover, one can convince themselves that globally defined conformal transformations on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \infty$ are generated by l_{-1}, l_0 and l_1 and that the conformal group of S^2 is the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

2.3.1 Virasoro Algebra

We will now discuss the fact that the Witt algebra of infinitesimal conformal transformations admits a central extension following chapter 2.1 in [BP09]. In general, the central extension $\hat{\mathfrak{g}} = \mathfrak{g} \bigoplus \mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by $\mathbb{C}$ is characterized by the commutation relations

$$\begin{aligned} [\hat{x}, \hat{y}]_{\hat{\mathfrak{g}}} &= [x, y] + cf(x, y) \quad , \\ [\hat{x}, c]_{\hat{\mathfrak{g}}} &= 0 \quad , \\ [c, c]_{\hat{\mathfrak{g}}} &= 0 \quad , \end{aligned} \tag{2.24}$$

which hold $\forall \hat{x}, \hat{y} \in \hat{\mathfrak{g}}$ and $x, y \in \mathfrak{g}$, and where $c \in \mathbb{C}$ and $f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

The so-called the Virasoro algebra Vir_c is in fact the central extension of the Witt algebra. It is characterized by the commutation relation

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0} \quad , \tag{2.25}$$

denoting its elements by L_n and naming c the central charge. In 1984 an extension of the Virasoro algebra (which in that context was denoted by W_2) has been found [BPZ84] which was called the W_3-algebra.

Before discussing W-algebras in more detail we introduce in the following some essential concepts.

2.3.2 Primary and quasi-primary Fields

In the previous sections we arrived at two commuting copies of the Witt algebra. We also introduced complex variables in (2.14). The generators of the Witt algebras are expressed in terms of z and $\bar{z}$. Therefore, it is convenient to consider them as independent variables. The complexification $\mathbb{R}^2 \to \mathbb{C}^2$ means for the fields ϕ in this theory

$$\phi(x^0, x^1) \to \phi(z, \bar{z}) \quad . \tag{2.26}$$

In the following we will in accordance with [BP09] define a couple of characteristic properties of some fields.

Definition 4. ***(Chiral Fields and Anti-Chiral Fields)***
A chiral (or holomorphic) field is a field $\phi(z)$ which only depends on z. An anti-chiral (or anti-holomorphic) field is a field $\phi(\bar{z})$ which only depends on $\bar{z}$.

Definition 5. ***(Conformal Dimensions)***
The conformal dimensions $(h, \bar{h})$ are properties of a field $\phi_{h,\bar{h}}(z, \bar{z})$ which under any scaling transformation $z \mapsto \lambda z$ transforms according to

$$\phi_{h,\bar{h}}(z, \bar{z}) \mapsto \phi'_{h,\bar{h}}(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}) \quad . \tag{2.27}$$

Definition 6. ***(Primary Filed (1))***
A primary field is a field $\phi_{h,\bar{h}}(z, \bar{z})$ that transforms under any conformal transformation $z \mapsto f(z)$ according to

$$\phi_{h,\bar{h}}(z,\bar{z}) \mapsto \phi'_{h,\bar{h}}(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi\left(f(z), \bar{f}(\bar{z})\right) \quad . \tag{2.28}$$

Definition 7. ***(Quasi-Primary Filed (1))***
A quasi-primary field is a field $\phi(z,\bar{z})$ that transforms according to (2.28) under global conformal transformations.

Note that a field is quasi-primary if it is primary. As described by the definitions, primary and quasi-primary fields are characterized by their conformal weights h and $\bar{h}$. An equivalent characterization can occur by the **spin** $s := h - \bar{h}$ or the **scaling dimension** $\Delta := h + \bar{h}$. By taking derivatives of quasi-primary field, one can obtain all other fields of the theory [CF24].

Definition 8. ***(Secondary Filed)***
A secondary field is a field $\phi(z,\bar{z})$ that is neither primary nor quasi-primary.

We now look at how a primary field $\phi_{h,\bar{h}}(z,\bar{z})$ behaves under infinitesimal conformal transformations in more detail. For such a mapping $z \mapsto f(z) = z + \varepsilon(z)$, $\varepsilon(z) \ll 1$ we have the following behaviour of parts of (2.28):

$$\begin{aligned}\left(\frac{\partial f}{\partial z}\right)^h &= \left(\frac{\partial(z+\varepsilon(z))}{\partial z}\right)^h \\ &= \left(1 + \frac{\partial \varepsilon(z)}{\partial z}\right)^h \\ &= 1 + h\partial_z\varepsilon(z) + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\phi\left(f(z),\bar{z}\right) &= \phi(z+\varepsilon(z),\bar{z}) \\ &= \phi(z) + \varepsilon(z)\partial_z\phi(z,\bar{z}) + \mathcal{O}(\varepsilon^2) \quad .\end{aligned} \tag{2.30}$$

Using these linear approximations, the infinitesimal conformal transformation of a primary field amounts to

$$\phi_{h,\bar{h}}(z,\bar{z}) \mapsto \phi'_{h,\bar{h}}(z,\bar{z}) \approx \Bigg(1 + h\partial_z\varepsilon(z) + \bar{h}\partial_{\bar{z}}\varepsilon(\bar{z}) + \varepsilon(z)\partial_z + \bar{\varepsilon}(\bar{z})\partial_{\bar{z}0}\Bigg)\phi_{h,\bar{h}}(z,\bar{z}) \quad , \tag{2.31}$$

i.e.

$$\delta\phi_{h,\bar{h}}(z,\bar{z}) = \Bigg(h\partial_z\varepsilon(z) + \varepsilon(z)\partial_z + \bar{h}\partial_{\bar{z}}\varepsilon(\bar{z}) + \bar{\varepsilon}(\bar{z})\partial_{\bar{z}}\Bigg)\phi_{h,\bar{h}}(z,\bar{z}) \quad . \tag{2.32}$$

We recall that a field theory is usually defined in terms of a Lagrangian action. From this action various important properties and objects of the theory are derived. One of those objects, the energy-momentum tensor always is of special interest. It can be determined from the variation of the action w.r.t. the metric, i.e. it encodes the behavior of the theory under infinitesimal transformations of the form

$$g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu} \quad , \quad \delta g_{\mu\nu} \ll 1 \quad . \tag{2.33}$$

First, we recall Noether's theorem, see e.g. [BR16]. It states that for every continuous symmetry of a field theory there exists a conserved current j_μ, i.e.

$$\partial^\mu j_\mu = 0 \quad . \tag{2.34}$$

In the case of CFT with a conformal symmetry $x^\mu \mapsto x^\mu + \varepsilon^\mu(x)$, this current is of the form

$$j_\mu = T_{\mu\nu}\varepsilon^\nu \quad . \tag{2.35}$$

Here, $T_{\mu\nu}$ is symmetric and called the energy-momentum or stress-energy tensor. In the special case of constant ε^μ, it follows from the conservation condition (2.34) and (2.35) that

$$0 = \partial^\mu j_\mu = (\partial^\mu T_{\mu\nu})\,\varepsilon^\nu \quad \Rightarrow \quad \partial^\mu T_{\mu\nu} = 0 \quad . \tag{2.36}$$

For a general transformation on the other hand we have

$$\begin{aligned} 0 &= \partial^\mu j_\mu \\ &= (\partial^\mu T_{\mu\nu})\,\varepsilon^\nu + T_{\mu\nu}\partial^\mu\varepsilon^\nu \\ &= \frac{1}{2}T_{\mu\nu}\,(\partial^\mu\varepsilon^\nu + \partial^\nu\varepsilon^\mu) \\ &= \frac{1}{2}T_{\mu\nu}\frac{2}{d}\eta^{\mu\nu}\,(\partial\cdot\varepsilon) \\ &= \frac{1}{d}T_\mu{}^\mu\,(\partial\cdot\varepsilon) \quad , \end{aligned} \tag{2.37}$$

having used (2.36) and (2.7) in step 3 and 4, respectively.

The conclusion from this equation is that since it has to hold for arbitrary conformal transformations, it follows that

$$T_\mu{}^\mu = 0 \quad , \tag{2.38}$$

i.e. the energy-momentum tensor is traceless in any CFT.

Since our special interest is in 2-dimensional Euclidean space, we will investigate now the consequences of a traceless energy-momentum tensor in this particular case. As before, we first change the coordinates from real to complex, i.e.

$$x^0 = \frac{1}{2}(z + \bar{z}) \quad , \quad x^1 = \frac{1}{2i}(z - \bar{z}) \quad . \tag{2.39}$$

Using

$$T_{\mu\nu}(z,\bar{z}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(x^0,x^1) \quad , \tag{2.40}$$

where $\tilde{x}^0 \coloneqq z$ and $\tilde{x}^1 \coloneqq \bar{z}$, we find that

$$\begin{aligned} T_{zz} &= \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right) \quad , \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) \end{aligned} \tag{2.41}$$

and

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}\left(T_{00} + T_{11}\right) = \frac{1}{4}T_\mu{}^\mu = 0 \quad . \tag{2.42}$$

The last relation implies that

$$T_{00} = -T_{11} \quad , \tag{2.43}$$

from which it directly follows that

$$\begin{aligned} T_{zz} &= \frac{1}{2}\left(T_{00} - iT_{10}\right) \quad , \\ T_{\bar{z}\bar{z}} &= \frac{1}{2}\left(T_{00} + iT_{10}\right) \quad . \end{aligned} \tag{2.44}$$

Lastly, we use the condition for translational invariance (2.36),

$$\partial_0 T_{00} + \partial_1 T_{10} = 0 \quad , \quad \partial_0 T_{01} + \partial_1 T_{11} = 0 \quad , \tag{2.45}$$

to find

$$\begin{aligned} \partial_{\bar{z}} T_{zz} &= \frac{1}{2}\left(\partial_0 + i\partial_1\right)\left(T_{00} - iT_{10}\right) \\ &= \frac{1}{2}\left(\partial_0 T_{00} + i\partial_1 \underbrace{T_{00}}_{=-T_{11}} - i\partial_0 \underbrace{T_{10}}_{=T_{01}} + \partial_1 T_{10}\right) \\ &= 0 \end{aligned} \tag{2.46}$$

as well as

$$\partial_z T_{\bar{z}\bar{z}} = 0 \quad . \tag{2.47}$$

We conclude that the chiral field

$$T(z) \coloneqq T_{zz}(z,\bar{z}) \tag{2.48}$$

and the anti-chiral field

$$\bar{T}(\bar{z}) \coloneqq T_{\bar{z}\bar{z}}(z, \bar{z}) \quad . \tag{2.49}$$

are the only nonvanishing components of the energy-momentum tensor.

2.3.3 Radial Quantization

Since we focus on CFTs defined on Euclidean 2-dimensional flat space, we will now discuss its compactification in accordance with [BP09]. As usual, we denote the Euclidean time direction by the coordinate x^0 and its space direction by x^1. This Euclidean space can be compactified in the space direction x^1 on a circle of radius R which is mostly chosen to be $R = 1$. The resulting CFT is defined on an infinitely long cylinder described by the complex coordinate

$$w = x^0 + ix^1 \quad , \tag{2.50}$$

for which we have the periodicity

$$w \sim w + 2\pi i \quad . \tag{2.51}$$

This is the starting point for the following concept of **radial quantization**. By a change of variables we now map this cylinder to a complex plane,

$$z = e^w = e^{x^0} e^{ix^1} \quad . \tag{2.52}$$

Through this, former time translations $x^0 \mapsto x^0 + a$ are mapped to complex dilations, i.e. $z \mapsto e^a z$, and former space translations $x^1 \mapsto x^1 + b$ are mapped to complex rotations, i.e. $z \mapsto e^{ib} z$.

Performing a Laurent expansion around $z_0 = 0 = \bar{z}_0$ for a field $\phi_{h,\bar{h}}(z, \bar{z})$, we write using (2.52)

$$\phi_{h,\bar{h}}(z, \bar{z}) = \sum_{m,\bar{m} \in \mathbb{Z}} z^{-m-h} z^{-\bar{m}-\bar{h}} \phi_{m,\bar{m}} \quad , \tag{2.53}$$

where we find $(m, \bar{m})$ to be the scaling dimensions of the Laurent modes $\phi_{m,\bar{m}}$ and its quantization is achieved by promoting these modes to operators. Considering that through the change of variables (2.52) the infinite past $x^0 = -\infty$ is mapped to $z = 0 = \bar{z}$ one is naturally motivated to define an asymptotic *in*-state $|\phi\rangle$ as

$$|\phi\rangle \coloneqq \lim_{z,\bar{z} \to 0} \phi_{h,\bar{h}}(z, \bar{z}) |0\rangle \quad . \tag{2.54}$$

In order for this state to be well-defined, i.e. to be nonsingular at $z = 0$, in accordance with (2.53) it is required that

$$\phi_{m,\bar{m}} |0\rangle = 0 \quad \text{for} \quad m > -h, \quad \bar{m} > -\bar{h} \quad . \tag{2.55}$$

Thus, (2.54) can be simplified to read

$$|\phi\rangle = \phi_{-h,-\bar{h}}\,|0\rangle \quad . \tag{2.56}$$

Regarding the Hermitian conjugate of ϕ denoted by $\phi^\dagger$ it is the case that in Euclidean space the time $x^0 = it$ upon Hermitian conjugation transforms like $x^0 \to -x^0$ while the space coordinate x^1 stays invariant. It follows that for the complex coordinate $z = e^{x^0+ix^1}$ Hermitian conjugation translates to $z \to \frac{1}{\bar{z}}$, identifying $\bar{z}$ with the complex conjugate of z. In summary, one thus can define the Hermitian conjugate of $\phi_{h,\bar{h}}(z,\bar{z})$ as

$$\phi^\dagger_{h,\bar{h}}(z,\bar{z}) \coloneqq \bar{z}^{-2h} z^{-2\bar{h}} \phi(\bar{z}^{-1}, z^{-1}) \quad . \tag{2.57}$$

Its Laurent expansion then reads

$$\begin{aligned} \phi^\dagger_{h,\bar{h}}(z,\bar{z}) &= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,\bar{m}\in\mathbb{Z}} \bar{z}^{m+h} z^{\bar{m}+\bar{h}} \phi_{m,\bar{m}} \\ &= \sum_{m,\bar{m}\in\mathbb{Z}} \bar{z}^{m-h} z^{\bar{m}-\bar{h}} \phi_{m,\bar{m}} \quad , \end{aligned} \tag{2.58}$$

where we can in comparison to (2.53) see that the Laurent modes transform as

$$(\phi_{m,\bar{m}})^\dagger = \phi_{-m,-\bar{m}} \tag{2.59}$$

under Hermitian conjugation. With these ingredients it is natural to define an asymptotic *out*-state as

$$\begin{aligned} \langle\phi| &\coloneqq \lim_{z,\bar{z}\to 0} \langle 0|\, \phi_{h,\bar{h}}(z,\bar{z}) \\ &= \lim_{w,\bar{w}\to 0} w^{2h}\bar{w}^{2\bar{h}} \langle 0|\, \phi(w,\bar{w}) \quad , \end{aligned} \tag{2.60}$$

where $\bar{z} \coloneqq w^{-1}$ and $z \coloneqq \bar{w}^{-1}$ and (2.57) was employed in the second step. By the same reasoning as above this state is well-defined only if

$$\langle 0|\, \phi_{m,\bar{m}} = 0 \quad \text{for} \quad m < h, \quad \bar{m} < \bar{h} \quad . \tag{2.61}$$

The *out*-state too can be simplified using (2.58) and (2.60) to

$$\langle\phi| = \langle 0|\, \phi_{+h,+\bar{h}} \quad . \tag{2.62}$$

2.3.4 Operator Product Expansion

In this section we will make sense of the notion of two local fields approaching each other. We will motivate the necessary introduction of radial ordering through conserved charges of the theory and again follow chapter 2 of [BP09].

To begin, we recall that since the current (2.35) is conserved there exists a conserved charge Q at $x_0 = \text{const.}$,

$$Q = \int dx^1 j_0 \quad , \tag{2.63}$$

which generates a symmetry transformation

$$\delta A = [Q, A] \tag{2.64}$$

for an operator A. The commutator in (2.64) is evaluated at equal times. We infer from (2.39) that

$$x_0 = \text{const.} \quad \Rightarrow \quad |z| = \text{const.} \quad , \tag{2.65}$$

and thus the integral in (2.63) is transformed into a contour integral $\oint$. According to the convention that contour integrals are counter clockwise, we get the generalization of the conserved charge (2.63) to complex coordinates,

$$Q = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left(dz\ T(z)\varepsilon(z) + d\bar{z}\ \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}) \right) \quad . \tag{2.66}$$

Next, we calculate the infinitesimal transformation of a field $\phi_{h,\bar{h}}(z,\bar{z})$ which is generated by Q by calculating $\delta\phi_{h,\bar{h}} = [Q, \phi_{h,\bar{h}}]$,

$$\delta\phi_{h,\bar{h}}(w,\bar{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz [T(z)\varepsilon(z), \phi_{h,\bar{h}}(w,\bar{w})] + \frac{1}{2\pi i} \oint_{\mathcal{C}} d\bar{z} [\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}), \phi_{h,\bar{h}}(w,\bar{w})] \quad . \tag{2.67}$$

Considering this last equation we see that we have to make the decision whether w is inside or outside the contour. We recall from quantum field theory that correlation functions are defined as time ordered products. Analogously to chapter 2.5 in [BP09], we define the radial ordering $\mathcal{R}(-)$ of two local operators A and B as

$$\mathcal{R}(A(z)B(w)) \coloneqq \begin{cases} A(z)B(w) & \text{if} \quad |z| > |w| \\ B(w)A(z) & \text{if} \quad |w| > |z| \end{cases} \quad . \tag{2.68}$$

According to this definition an equation of the form of (2.67) is interpreted as

$$\begin{aligned} \oint_{\mathcal{C}} dz [A(z), B(w)] &= \oint_{|z|>|w|} dz\ A(z)B(w) - \oint_{|w|>|z|} dz\ B(w)A(z) \\ &= \oint_{\mathcal{C}(w)} dz\ \mathcal{R}(A(z)B(w)) \quad . \end{aligned} \tag{2.69}$$

Thus (2.67) becomes

$$\begin{aligned} \delta\phi_{h,\bar{h}}(w,\bar{w}) = &\frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\ \varepsilon(z) \mathcal{R}(T(z)\phi_{h,\bar{h}}(w,\bar{w})) \\ &+ \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\ \bar{\varepsilon}(\bar{z}) \mathcal{R}(\bar{T}(\bar{z})\phi_{h,\bar{h}}(w,\bar{w})) \quad . \end{aligned} \tag{2.70}$$

Comparing this to (2.32) and using the identities

$$\begin{aligned} h\partial_w \varepsilon(w)\phi_{h,\bar{h}}(w,\bar{w}) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\ h\frac{\varepsilon(z)}{(z-w)^2} \\ \varepsilon(w)\partial_w\phi_{h,\bar{h}}(w,\bar{w}) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} dz\ \frac{\varepsilon(z)}{(z-w)^1}\partial_w\phi_{h,\bar{h}}(w,\bar{w}) \end{aligned} \tag{2.71}$$

we find for a biholomorphic[3] field $\phi_{h,\bar{h}}(w,\bar{w})$ that

$$\mathcal{R}(T(z)\phi_{h,\bar{h}}(w,\bar{w})) \sim \frac{h}{(z-w)^2}\phi_{h,\bar{h}}(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi_{h,\bar{h}}(w,\bar{w}) \quad . \tag{2.72}$$

Here $\sim$ denotes that we explicitly write only the singular terms. This expression is our first example of a so-called operator product expansion (OPE) which we define in the following explicitly. From now onwards radial ordering will be assumed for any product of fields and therefore $\mathcal{R}$ will be omitted in our notation. In CFT the term "field" refers to local operators in general. OPEs provide an algebraic way of computing correlation functions, see e.g. [WZ72] and [Thi95].

For the following formal definition of OPEs we adhere to the one given in [Gin88].

Definition 9. ***(Operator Product Expansion)***
Let $O_i(z,\bar{z})$ and $O_j(w,\bar{w})$ be two local operators. Their operator product expansion (OPE) is defined as

$$O_i(z,\bar{z})O_j(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})O_k(w,\bar{w}) \quad . \tag{2.73}$$

The expression $C_{ij}^k(z-w,\bar{z}-\bar{w})$ denotes functions which depend only on the distance between the two local operators.

From (2.73) it becomes clear that an OPE defines an algebraic product structure on the space of quantum fields or, in other terms, it is a statement about what happens when those fields approach each other. It is important to note that OPEs have singular behavior as $z \to w$. Whenever we are only interested in the singular part of the equation we will denote this by $\sim$ instead of $=$ and simply omit the nonsingular terms.

We now have all the ingredients to write down an alternative definition of primary fields and quasi-primary fields in terms of their OPE with the energy-momentum tensor, see e.g. [CF24].

Definition 10. ***(Primary Field (2))***
Let $T(z)$ the energy-momentum tensor and $\phi_{h,\bar{h}}(w,\bar{w})$ be a field. $\phi(w,\bar{w})$ is called a primary field with dimensions $(h,\bar{h})$ if the OPEs

$$T(z)\phi_{h,\bar{h}}(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi_{h,\bar{h}}(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi_{h,\bar{h}}(w,\bar{w}) \tag{2.74}$$

[3] A biholomorphic field is a bijective holomorphic function whose inverse is also holomorphic.

and

$$\bar{T}(\bar{z})\phi_{h,\bar{h}}(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi_{h,\bar{h}}(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}}\phi_{h,\bar{h}}(w,\bar{w}) \tag{2.75}$$

hold.

Definition 11. ***(Quasi-Primary Field (2))***
Let $T(z)$ the energy-momentum tensor and $\phi_{h,\bar{h}}(w,\bar{w})$ be a field. $\phi(w,\bar{w})$ is called a quasi-primary field with dimensions $(h,\bar{h})$ if the OPEs

$$\begin{aligned} T(z)\phi_{h,\bar{h}}(w,\bar{w}) \sim\ & \textit{higher order poles} \\ & + \frac{0}{(z-w)^3} + \frac{h}{(z-w)^2}\phi_{h,\bar{h}}(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi_{h,\bar{h}}(w,\bar{w}) \end{aligned} \tag{2.76}$$

and

$$\begin{aligned} \bar{T}(\bar{z})\phi_{h,\bar{h}}(w,\bar{w}) \sim\ & \textit{higher order poles} \\ & + \frac{0}{(z-w)^3} + \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi_{h,\bar{h}}(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_w\phi_{h,\bar{h}}(w,\bar{w}) \end{aligned} \tag{2.77}$$

hold.

Let us now show that the OPE of the energy-momentum tensor with itself reads

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) \quad , \tag{2.78}$$

where c denotes the central charge, revealing that $T(z)$ itself is according to (2.76) a quasi-primary field. To do so, following [BP09], we perform a Laurent expansion of $T(z)$,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \quad , \tag{2.79}$$

with modes

$$L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1}T(z) \quad . \tag{2.80}$$

For a particular conformal transformation $-\varepsilon_n z^{n+1}$ we find that the chiral part of the associated conserved charge is

$$\begin{aligned} Q_n &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz\ T(z)(-\varepsilon_n z^{n+1}) \\ &= -\varepsilon_n \sum_{m\in\mathbb{Z}} \oint_{\mathcal{C}} \frac{dz}{2\pi i} L_m z^{n-m+1} \\ &= -\varepsilon_n L_n \quad . \end{aligned} \tag{2.81}$$

When compared to the previous sections, we see that the Laurent modes L_n of the energy-momentum tensor can be identified with the generators l_n of infinitesimal conformal transformations. Indeed, using (2.80) and radial ordering (2.69) as well as the OPE of the energy-momentum tensor with itself (2.78) we find

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad . \tag{2.82}$$

For more details see chapter 2.5 in [BP09].

2.3.5 Quantum W-Algebras

We have seen in section 2.3.2 that in 2-dimensional CFT the conservation of the energy-momentum tensor implies that both its z- and its $\bar{z}$-component are separately conserved. This implication holds in any unitary 2-dimensional scale-invariant theory, see e.g. [BP09] and [DMS97]. Those separately conserved currents then can be split into a set of holomorphic currents $W^{(s_i)}(z)$ of spin s_i and a set of anti-holomorphic currents $\bar{W}^{(s_i)}(\bar{z})$ of spin $-s_i$. Note that in the holomorphic case the scaling dimension is $\Delta_i = h_i = s_i$ and in the anti-holomorphic case $\Delta_i = \bar{h}_i = s_i$. In the following we will explicitly focus on these holomorphic currents. It turns out that their set is closed under OPE, meaning that if one takes the operator product of two holomorphic currents, in the short-distance expansion there appear only holomorphic fields which in turn define the quantum W-algebra [CF24].

Following [CF24], the holomorphic field operators can analogously to (2.79) be expanded in modes,

$$W^{(s)}(z) = \sum_{n\in\mathbb{Z}} z^{-n-s} W_n^{(s)} \quad , \tag{2.83}$$

where the modes $W_n^{(s)}$ are operators that are defined via contour integrals,

$$W_n^{(s)} = \frac{1}{2\pi i}\oint dz\ z^{n+s-1} W^{(s)}(z) \quad . \tag{2.84}$$

The commutation relations $[W_m^{(s)}, W_n^{(t)}]$ they satisfy are determined by the OPE $W^{(s)}(z)W^{(t)}(w)$ of the fields. Indeed,

$$\begin{aligned}
[W_m^{(s)}, W_n^{(t)}] &= \left[\frac{1}{2\pi i}\oint dz\ z^{m+s-1}W^{(s)}(z), \frac{1}{2\pi i}\oint dw\ w^{n+t-1}W^{(t)}(w)\right] \\
&= \frac{1}{(2\pi i)^2}\left\{\oint\oint_{|z|>|w|} - \oint\oint_{|z|<|w|}\right\} dzdw\ z^{m+s-1}w^{n+t-1}\mathcal{R}\left(W^{(s)}(z)W^{(t)}(w)\right) \\
&= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz\ z^{m+s-1}w^{n+t-1}\mathcal{R}\left(W^{(s)}(z)W^{(t)}(w)\right) \quad ,
\end{aligned} \tag{2.85}$$

where we assumed analyticity of the radially ordered product in the last step. We see that the commutator of the modes, $[W_m^{(s)}, W_n^{(t)}]$, is determined by the poles in the OPE of the currents $W^{(s)}$ and $W^{(t)}$.

Corollary 12. ***(Quasi-primary holomorphic currents)***
A holomorphic current of the form (2.83) of spin s is quasi-primary if and only if its modes satisfy

$$\left[L_m, W_n^{(s)}\right] = ((s-1)m - n)W_{m+n}^{(s)} \tag{2.86}$$

for $m \in \{-1, 0, 1\}$.

Corollary 13. ***(Primary holomorphic currents)***
A holomorphic current of the form (2.83) of spin s is primary if and only if its modes satisfy (2.86) for all $m \in \mathbb{Z}$.

As has been briefly mentioned in section 2.3.1 the quantum $W^{(2)}$-algebra is generated by the modes L_n, $n \in \mathbb{Z}$, see (2.25). The quantization of the Virasoro algebra is a straight forward procedure, see chapter 4.1 in [CF24], where the classical modes $\hat{\mathcal{L}}_n$ in (2.1) are replaced by operators L_m and the Poisson brackets $i\{\_,\_\}$ by commutators $[\_,\_]$, which amounts to the quantum Virasoro algebra and its characterizing commutation relations,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad , \tag{2.87}$$

that appears in 2-dimensional CFT.

It is, however, not as straight forward to quantize (2.2) since in contrast to the Virasoro algebra, its Poisson algebra is not linear. This fact leads to a certain freedom in choosing an ordering in the nonlinear terms. Additionally, the classical Jacobi identity does not guarantee that the quantum version satisfies the Jacobi identity as well for some chosen ordering. Following chapter 4.1 in [CF24], we sketch how one can find the quantum version of the classical $W^{(3)}$-algebra (2.2).

The quantum $W^{(3)}$-algebra is generated by the modes L_m and modes of the form (2.83) for $s = 3$, i.e. $W_n^{(3)}$. It is possible to find this algebra by using the principle of replacing the classical modes by operators and the Poisson brackets $i\{\_,\_\}$ by $[\_,\_]$ commutators in (2.2) and making the ansatz

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad , \\
\left[L_m, W_n^{(3)}\right] &= (2m-n)W_{m+n} \quad , \\
\left[W_m^{(3)}, W_n^{(3)}\right] &= \frac{1}{12}\Big[(m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{96}{c}\alpha(m-n)\Lambda_{m+n} \\
&\quad + \frac{c}{12}m(m^2-1)(m^2-4)\delta_{m+n,0}\Big] \quad ,
\end{aligned} \tag{2.88}$$

as can be seen in chapter 4.3 in [CF24] in detail. This tactic works fine as long as the Poisson bracket relations are linear, which stops being the case in the third commutator relation in (2.88) due to (2.3). It can be shown that the corresponding Jacobi identities are satisfied for

$$\alpha := \frac{c}{c + \frac{22}{5}} \quad , \tag{2.89}$$

where c is the central charge, and that the modes Λ_m are the modes of a quasi-primary field of spin 4, namely

$$\Lambda = (TT) - \frac{3}{10} T^{(2)} \quad , \tag{2.90}$$

where T is the energy-momentum tensor and $T^{(2)}$ denotes the second derivative thereof, i.e. $T^{(2)}(z) := \partial_z^2 T(z)$. For details see chapter 4.3 in [CF24].

One can see from these examples that it is a highly nontrivial question on the one hand whether a given classical W-algebra possesses a quantum deformation and, on the other hand, how to find it. Fortunately, there is a more practical approach than taking the classical Poisson brackets of a given classical W-algebra, deforming them into commutators and checking the Jacobi identities, namely by applying the quantum Miura transformation. By this construction, the quantum W_N-algebra is realised in terms of free fields as we will see in more detail in chapter 3.1.

In addition to W_N-algebras there exist the W_∞-algebra, W_{1+N}-algebras, $W_{1+\infty}$-algebra and their super-extensions, see e.g. [Pop91] or [CF24]. In the following we introduce them very quickly according to those sources.

W_N-algebras are higher-spin ($N < \infty$) extensions of the Virasoro algebra. For $N > 2$ they are nonlinear algebras due to the fact that the OPE of currents with spins s and t produces terms that have spin $s + t - 2$ at leading order. Except in the case of the Virasoro algebra W_2, where $2 + 2 - 2 = 2$, it will follow from every attempt of building an algebra with fundamental currents of spins $2 \leq s \leq N$ that there will inevitabely appear currents from OPEs that have spin $t \geq N$. What makes those algebras nonlinear is the interpretation of those currents as composite fields that are themselves built from products of the fundamental currents. In the exception of $N = \infty$, where $\infty + \infty - 2 = \infty$ no spin that is too high to be represented by the fundamental currents can be produced in an OPE. In this case it is indeed possible to construct a linear W_∞-algebra that is generated by currents of spin $2 \leq s \leq \infty$, see [PRS90b] and [PRS90a]. Another related algebra which contains currents of spin $1 \leq s \leq \infty$ is the $W_{1+\infty}$-algebra [PRS91].

Apart from those, there exist further classes of W-algebras which we will not discuss in more detail since those are not relevant for this thesis. For details see e.g. [BS93].

2.3.6 OPEs and the Generalized Wick Theorem

In alternative to (2.73), following [Pro15], the OPE of two local operators $A(z)$ and $B(w)$ can be captured in the following general form:

$$A(z)B(w) = \sum_{k=-\infty}^{\Delta_A+\Delta_B} \frac{\{AB\}_k(w)}{(z-w)^k} \ , \tag{2.91}$$

where Δ_A and Δ_B denote the scaling dimensions of $A(z)$ and $B(w)$, respectively, and the contour integral representation of $\{AB\}_k$ is

$$\{AB\}_k(w) = \frac{1}{2\pi i}\oint_w dz\ A(z)B(w)(z-w)^{k-1} \quad . \tag{2.92}$$

For $k = 0$ we find

$$\{AB\}_0(w) = \frac{1}{2\pi i}\oint_w dz \frac{A(z)B(w)}{(z-w)} =: (AB)(w) =: \ :A(w)B(w): \quad , \tag{2.93}$$

which is the **normal ordered product** of A and B. The singular part of the r.h.s. of (2.91), i.e. where $k \geq 1$, will be denoted by a contraction:

$$\overbracket{A(z)B}(w) = \sum_{k=1}^{\Delta_A+\Delta_B} \frac{\{AB\}_k(w)}{(z-w)^k} \quad . \tag{2.94}$$

Comparing with (2.91) and (2.94) we see that the latter can also be written as

$$:A(z)B(w): \ = \lim_{z\to w}\left[A(z)B(w) - \overbracket{A(z)B}(w)\right] \tag{2.95}$$

since in this case every term of (2.91) that is proportional to a positive power of $(z-w)$ goes to zero.

Moreover, regarding derivatives, one can derive from (2.91) that the following holds:

$$\begin{aligned} \partial\{AB\}_k &= \{A^{(1)}B\}_k + \{AB^{(1)}\}_k \quad , \\ \{A^{(1)}B\}_{k+1} &= -k\{AB\}_k \quad . \end{aligned} \tag{2.96}$$

The second one of these equations then can be used to relate the regular terms of an OPE to the normal ordered product, namely

$$\{AB\}_{-k} = \frac{1}{k!}(A^{(k)}B)(w) \quad \text{for } k \geq 0 \quad . \tag{2.97}$$

Furthermore, using a formal Taylor series expansion as well as (2.97), we define an operator of the form

$$
\begin{aligned}
(A(z)B(w)) &\coloneqq \sum_{k=0}^{\infty} \frac{(z-w)^k}{k!} (A^{(k)}B)(w) \\
&= \sum_{k\leq 0} \frac{\{AB\}_k(w)}{(z-w)^k} \\
&= A(z)B(w) - \overbracket{A(z)B}(w) \quad .
\end{aligned}
\tag{2.98}
$$

As one can see, this bilocal operator exclusively represents the regular terms of an OPE.

To be able to make sense of OPEs of the form $A(z)(BC)(w)$ and $(AB)(z)C(w)$ we make use of a reformulation of Wick's theorem for interacting fields using generalization of the concept of normal ordering as has been shown in [DMS97], [Tak17] and [Bai+88]:

$$
\overbracket{A(z)(B}C)(w) = \frac{1}{2\pi i} \oint_w \frac{dx}{x-w} \left\{ \overbracket{A(z)B}(x)C(w) + B(x)\overbracket{A(z)C}(w) \right\} \quad . \tag{2.99}
$$

Indeed, the contractions on the r.h.s. extract all singular terms of the integral coming from the OPEs of $A(z)$ with $B(x)$ as well as $A(z)$ with $C(w)$ as $z \to w$. This can be seen by replacing the contractions on the r.h.s. of (2.99) by expressions of the form (2.94),

$$
\overbracket{A(z)(B}C)(w) = \frac{1}{2\pi i} \oint_w \frac{dx}{x-w} \left\{ \sum_{k=1}^{\Delta_A+\Delta_B} \frac{\{AB\}_k(x)C(w)}{(z-x)^k} + \sum_{k=1}^{\Delta_A+\Delta_C} \frac{B(x)\{AC\}_k(w)}{(z-w)^k} \right\} \quad . \tag{2.100}
$$

By analyzing this equation we see that since $k \geq 1$ all inverse powers of $(z-x)$ and $(z-w)$ in the integrand yield inverse powers of $(z-w)$ after integration.

Equation (2.99) will play an integral role in chapter 4.1, where a lot of OPE manipulations will be made. We will also define multiple normal ordered products of operators $A, B, C, ...O, P$ as

$$
(ABC...OP) \coloneqq (A(B(C...(OP)...))) \quad . \tag{2.101}
$$

2.4 Useful Identities

For direct calculations we will also make use of two particular identities, namely

$$
\frac{1}{2\pi i} \oint_w \frac{dx}{(x-w)^n} \frac{F(w)}{(z-x)^m} = \frac{(n+m-2)!}{(n-1)!(m-1)!} \frac{F(w)}{(z-w)^{n+m-1}} \quad , \tag{2.102}
$$

where $x, w, z \in \mathbb{C}$, $n, m \in \mathbb{N}$ and $F(w)$ some complex function [DMS97], and Cauchy's integral formula,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_w dx \, \frac{f(x)}{(x-w)^{n+1}} \quad , \tag{2.103}$$

where $f^{(n)}$ denotes the n-th derivative of the infinitely differentiable function f.

3 Quantum W_N-Algebras and W_{1+N}-Algebras

3.1 Quantum W_N-Algebra

3.1.1 W_N-Algebra and the Quantum Miura Transformation

One way to realize the quantum W_N-algebra in terms of free fields is via the quantum Miura transformation. First, a set of $(N-1)$ currents K_a, $a \in \{1, ..., N-1\}$ is chosen that satisfy the standard OPE,

$$K_a(z)K_b(w) \sim \frac{\delta_{ab}}{(z-w)^2} \ , \tag{3.1}$$

where $\sim$ denotes that only the singular term is written out, yet the added regular term is understood as well. The next ingredient is a set of N vectors $\epsilon_k \in \mathbb{R}^{N-1}$ with $k \in \{1, ..., N\}$ which satisfy

$$\epsilon_i \cdot \epsilon_k = \delta_{ij} - \frac{1}{N} \tag{3.2}$$

with respect to the canonical scalar product. Obviously it holds that

$$\sum_{i=1}^{N} \epsilon_i = 0 \ , \tag{3.3}$$

since for all ϵ_j it must hold that

$$\begin{aligned} \sum_{i=1}^{N} \epsilon_i \cdot \epsilon_j &= \sum_{i=1}^{N} \left(\delta_{ij} - \frac{1}{N}\right) \\ &= 1 - \frac{N}{N} \\ &= 0 \quad . \end{aligned} \tag{3.4}$$

If we now combine the currents K_a into a vector $K = (K_1, ..., K_{N-1})$, we can define N spin-1 currents J_i as linear combinations of its components:

$$\tilde{J}_i(z) = \epsilon_i \cdot K(z) \ . \tag{3.5}$$

Those satisfy the OPE

$$\tilde{J}_i(z)\tilde{J}_j(w) \sim \frac{\delta_{ij} - \frac{1}{N}}{(z-w)^2} \tag{3.6}$$

since by (3.1) and (3.2)

$$\begin{aligned}\tilde{J}_i(z)\tilde{J}_j(w) &= \delta_{ab}\epsilon_i^a K_b(z)\delta_{cd}\epsilon_j^c K_d(w) \\ &\sim \delta_{ab}\delta_{cd}\epsilon_i^a\epsilon_j^c \frac{\delta_{bd}}{(z-w)^2} \\ &= \delta_{ac}\epsilon_i^a\epsilon_j^c \frac{1}{(z-w)^2} \\ &= \frac{\delta_{ij}-\frac{1}{N}}{(z-w)^2} \ .\end{aligned} \tag{3.7}$$

Moreover, due to (3.3)

$$\sum_{i=1}^{N} \tilde{J}_i = 0 \ . \tag{3.8}$$

Finally, spin-s fields U_s are defined implicitly via the quantum Miura transformation using the previously defined spin-1 currents J_i,

$$:\prod_{i=1}^{N}\left(\alpha_0\partial - \tilde{J}_i\right): = (\alpha_0\partial)^N - \sum_{s=1}^{N}\tilde{U}_s\,(\alpha_0\partial)^{N-s} \ , \tag{3.9}$$

where α_0 denotes some parameter and the l.h.s. is a normally ordered product. By a quick observation we see that for $s=0$ we need to find the respective term on the l.h.s. that is proportional to $(\alpha_0\partial)^N$ to be

$$\tilde{U}_0 = \mathbb{1} \ , \tag{3.10}$$

i.e. the identity current. Regarding $\tilde{U}_1$, we see that all terms that are proportional to $(\alpha_0\partial)^{N-1}$ on the l.h.s. of (3.9) are as follows:

$$-\tilde{J}_i(\alpha_0\partial)^{N-1} \ . \tag{3.11}$$

But since the sum of the spin-1 currents $\tilde{J}_i$ is zero as we have seen in (3.8) it follows that

$$\tilde{U}_1 = 0 \ . \tag{3.12}$$

$\tilde{U}_2$ which arises from the coefficient $(\alpha_0\partial)^{N-2}$ is much more interesting. Its contributions on the one hand are products of two currents, $\tilde{J}_i$ and $\tilde{J}_j$, and on the other hand derivatives of each of the currents except the first one. This amounts to

$$\tilde{U}_2 = -\sum_{i<j}\tilde{J}_i\tilde{J}_j + \alpha_0\sum_j (j-1)\tilde{J}_j^{(1)} \ . \tag{3.13}$$

As it happens, $\tilde{U}_2$ also satisfies the OPE of the energy-momentum tensor and thus from now on we will write $\tilde{U}_2 =: T$. Indeed, as can be retraced in the OPE of $\tilde{U}_2$ is

$$\tilde{U}_2(z)\tilde{U}_2(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2}{(z-w)^3}\tilde{U}_2(w) + \frac{1}{(z-w)}\partial_w \tilde{U}_2(w) \tag{3.14}$$

with central charge $c = c(N)$,

$$c = (N-1)\left[1 - N(N+1)\alpha_0^2\right] \ . \tag{3.15}$$

Thus the algebra W_2 generated by the modes of T is the Virasoro algebra and we define

$$W_2 := \tilde{U}_2 \quad . \tag{3.16}$$

The spin-3 current $\tilde{U}_3$ defined via (3.9) is

$$\tilde{U}_3 = \sum_{i<j<k} \tilde{J}_i \tilde{J}_j \tilde{J}_k - \alpha_0 \sum_{i<j} \left((i-1)\tilde{J}_i^{(1)} \tilde{J}_j + (j-2)\tilde{J}_i \tilde{J}_j^{(1)} \right) \quad . \tag{3.17}$$

This current, however, is not quasi-primary. This is why we define the W_3-algebra via its quasi-primary projection W_3,

$$W_3 := \tilde{U}_3 - \alpha_0 \frac{(N-2)}{2} T^{(1)} \quad . \tag{3.18}$$

W_3 is also primary. The algebra W_3 is generated by the modes of T and W_3.
At last we are also interested in the current $\tilde{U}_4$, given by

$$\begin{aligned}
\tilde{U}_4 := & \sum_{i<j<k<l} \tilde{J}_i \tilde{J}_j \tilde{J}_k \tilde{J}_l + \alpha_0 \sum_{i<j<k} \left[(i-1)\tilde{J}_i^{(1)} \tilde{J}_j \tilde{J}_k + (j-2)\tilde{J}_i \tilde{J}_j^{(1)} \tilde{J}_k + (k-3)\tilde{J}_i \tilde{J}_j \tilde{J}_k^{(1)} \right] \\
& - \alpha_0^2 \sum_{i<j} \left[\binom{i-1}{2} \tilde{J}_i^{(2)} \tilde{J}_j + \binom{j-2}{2} \tilde{J}_i \tilde{J}_j^{(2)} + (i-1)(j-1)\tilde{J}_i^{(1)} \tilde{J}_j^{(1)} \right] \\
& + \alpha_0^3 \sum_{i=4}^{N} \binom{i-1}{3} \tilde{J}_i^{(3)} \quad .
\end{aligned} \tag{3.19}$$

This current is neither primary nor quasi-primary. Its quasi-primary projection, however, is

$$V_4 := \tilde{U}_4 - \alpha_0 \frac{(N-3)}{2} \tilde{U}_3^{(1)} + \alpha_0^2 \frac{(N-2)(N-3)}{10} \tilde{U}_2^{(2)} \quad , \tag{3.20}$$

which is not yet a primary field. Up to normalization there exists exactly one primary field at spin-4, namely

$$W_4 := V_4 + \frac{(N-2)(N-3)}{2N} \frac{[5 - N(5N+7)\alpha_0^2]}{[17 + 5N - 5N(N^2-1)\alpha_0^2]} \Lambda \quad , \tag{3.21}$$

where

$$\Lambda := \tilde{U}_2\tilde{U}_2 - \frac{3}{10}\tilde{U}_2^{(2)} \quad . \tag{3.22}$$

3.1.2 Expectation of highest singularity Terms

In the review [CF24], for $N > 3$ the following convention of normalization of primary fields W_s is used:

$$W_s(z)W_s(w) = \frac{n_s c}{(z-w)^{2s}} + \text{subleading poles} \quad . \tag{3.23}$$

It also is expected and proven for $N \in \{1, 2, ..., 10\}$ that the central charge of primary fields W_3 and W_4 are n_3c and n_4c, respectively, where

$$n_3 = \frac{(N-2)\left[4 - N(N+2)\alpha_0^2\right]}{6N} \tag{3.24}$$

and

$$n_4 = n_3\frac{3(N+1)(N-3)\left[1 - N(N-1)\alpha_0^2\right]\left[9 - N(N+3)\alpha_0^2\right]}{2N(17 + 5N - 5N(N^2-1)\alpha_0^2)} \quad . \tag{3.25}$$

In chapter 4.1 we will prove by brute force that it is indeed the case that

$$W_3(z)W_3(w) = \frac{n_3 c}{(z-w)^6} + \text{subleading poles} \quad \forall\, N \geq 3 \quad . \tag{3.26}$$

The illustration of this tedious calculation then should serve as a motivation to find a more elegant way of finding the exact expression of central charges for higher spins.

3.1.3 The Triality Motivation

Knowing the exact form of the central charges $n_s c$ in (3.23) is motivated by a certain characteristic of the related W_∞-algebra, the so-called triality. We will describe this characteristic, closely following chapter 4.4 and 4.5 in [CF24].

Apart from (3.23), in [CF24] it is expected that the leading term of the OPE $W_3(z)W_4(w)$ is proportional to $W_3(w)$, concretely

$$W_3(z)W_4(w) \sim \frac{c_{34}^3}{(z-w)^4}W_3(w) + \text{subleading poles} \quad , \tag{3.27}$$

and that the coefficient relates to the central charges n_3c and n_4c via

$$c_{34}^3 = 4\frac{n_4}{n_3} \quad . \tag{3.28}$$

Furthermore the W_N-algebra is characterized by the normalisation independent quantity

$$C(N,c) = \frac{(c_{34}^3)^2}{n_4} \quad . \tag{3.29}$$

The structure constants n_3, n_4 and c_{34}^3 which are all rational functions in N can be continued to real parameters, replacing $N \in \mathbb{N}$ by $\lambda \in \mathbb{R}$. Through that the OPEs of the quantum W_∞-algebra by those of the W_N-algebra are obtained. Its characteristic normalization independent ratio can be found to be

$$C(\lambda,c) = \frac{144(c+2)}{22+5c} \frac{(\lambda-3)\,[2(\lambda-1)(4\lambda+3)+c(\lambda+3)]}{(\lambda-2)\,[(\lambda-1)(3\lambda+2)+c(\lambda+2)]} \quad , \tag{3.30}$$

i.e. W_∞-algebras are isomorphic if they have the same characteristic parameter $C(\lambda,c)$, see [CF24]. Furthermore it can be observed that generically only three different values of $\lambda \in \{\lambda_1,\lambda_2,\lambda_3\}$ lead to the same value of $C(\lambda,c)$ for fixed central charge, i.e.

$$C(\lambda_1,c) = C(\lambda_2,c) = C(\lambda_3,c) \quad . \tag{3.31}$$

It can be shown that these three solutions of λ satisfy the threefold symmetry

$$\begin{aligned} 0 &= \lambda_1\lambda_2 + \lambda_2 + \lambda_3 + \lambda_3\lambda_1 \\ c &= (\lambda_1-1)(\lambda_2-2)(\lambda_3-3) \\ C &= \frac{144(c+2)}{22+5c}\frac{(\lambda_1-3)(\lambda_2-3)(\lambda_3-3)}{(\lambda_1-2)(\lambda_2-2)(\lambda_3-2)} \end{aligned} \tag{3.32}$$

for a given C and c. Together the three equations (3.32) are called **triality**.

In the literature it has been shown in [Lin21] and has also been observed in [GG12] that the W_∞-algebra is fully characterized by its central charge and a normalization independent parameter which is fully defined by the OPEs in the algebra. In this thesis we prove that indeed $C(N,c)$ is of the form (3.29) by determining the highest singularity terms of the OPE of $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$.

3.2 W_{1+N}-algebra in the quadratic Basis

This chapter is motivated by a goal which we will persue in the last part of this thesis, namely, of defining primary currents in the W_N-algebra via currents in the $W_{1+\infty}$ algebra of which the exact OPEs are already known. We will closely follow chapter 3 of [Pro15].

To get to a free field representation of W_{1+N}, similarly to the W_N case we define N spin-1 currents J_j in terms of N free bosons ϕ_j, $j \in \{1,...,N\}$,

$$J_j \coloneqq i\partial\phi_j \quad . \tag{3.33}$$

Those satisfy the OPE

$$J_i(z)J_j(w) \sim \frac{\delta_{ij}}{(z-w)^2} \; . \tag{3.34}$$

The goal now is to represent $(N+1)$ generators of $W_{1+\infty}$ in terms of the free fields (3.33), which then form the quadratic (but nonprimary) basis of $W_{1+\infty}$. Again, we define the quantum Miura transformation as an operator of the form

$$:\prod_{i=1}^{N}(\alpha_0\partial + J_i): = \sum_{s=0}^{N} U_s\,(\alpha_0\partial)^{N-s} \quad , \tag{3.35}$$

which represents the generators U_i of the W_{1+N}-algebra in terms of the free fields J_j. Explicitly, the generators U_0, U_1, U_2, U_3 and U_4 read

$$U_0 = \mathbb{1} \quad , \tag{3.36}$$

$$U_1 = \sum_{j=1}^{N} J_j =: J \quad , \tag{3.37}$$

$$U_2 = \sum_{j<k}(J_j J_k) + \alpha_0 \sum_{j=1}^{N}(j-1)J_j^{(1)} \quad , \tag{3.38}$$

$$\begin{aligned} U_3 = &\sum_{j<k<l} J_j J_k J_l + \alpha_0 \sum_{j<k}(j-1)J_j^{(1)} J_k + \alpha_0 \sum_{j<k}(k-2)J_j J_k^{(1)} \\ &+ \frac{\alpha_0^2}{2}\sum_{j=1}^{N}(j-1)(j-2)J_j^{(2)} \quad , \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} U_4 = &\sum_{j<k<l<m} J_j J_k J_l J_m + \frac{\alpha_0^3}{6}\sum_{j=1}^{N}(j-1)(j-2)(j-3)J_j^{(3)} \\ &+ \alpha_0 \sum_{j<k<l}\left[(j-1)J_j^{(1)} J_k J_l + (k-2)J_j J_k^{(1)} J_l + (l-3)J_j J_k J_l^{(1)}\right] \\ &+ \frac{\alpha_0^2}{2}\sum_{j<k}\Big[(j-1)(j-2)J_j^{(2)} J_k + 2(j-1)(k-3)J_j^{(1)} J_k^{(1)} \\ &\qquad + (k-2)(k-3)J_j J_k^{(2)}\Big] \quad . \end{aligned} \tag{3.40}$$

It will become clear in chapter 4.1 of this thesis why U_1 gets a special notation, i.e. $U_1 \coloneqq J$. In chapter 4.2 we will express generators in W_N in terms of the previously shown generators in W_{1+N} and normally ordered products thereof, concretely

$$\tilde{U}_i = \tilde{U}_i(U_j) \;, \quad j \le i \quad . \tag{3.41}$$

It is then possible to calculate any OPE of the form $\tilde{U}_i(z)\tilde{U}_n(w)$
where $\tilde{U}_i$ as above and

$$\tilde{U}_n = \tilde{U}_n(U_l) \ , \quad l \leq n \tag{3.42}$$

provided we find all OPEs of the form

$$U_j(z)U_l(w) \ . \tag{3.43}$$

In principle one can calculate any OPE between of the form $U_i(z)U_j(w)$ using generalized Wick theorem (2.99). However, these calculations become exceedingly tedious as i, j become larger integer numbers. In the following we write down explicitly the OPEs that will be used in further calculations. They can partially be adopted from chapter 3.2 in [Pro15]:

$$J(z)J(w) \sim \frac{N}{(z-w)^2} \quad , \tag{3.44}$$

$$J(z)U_2(w) \sim \frac{N(N-1)\alpha_0}{(z-w)^3} + \frac{(N-1)J(w)}{(z-w)^2} \quad , \tag{3.45}$$

$$\begin{aligned} J(z)U_3(w) \sim &\frac{N(N-1)(N-2)\alpha_0^2}{(z-w)^4} + \frac{(N-1)(N-2)\alpha_0 J(w)}{(z-w)^3} \\ &+ \frac{(N-2)U_2(w)}{(z-w)^2} \quad , \end{aligned} \tag{3.46}$$

$$\begin{aligned} J(z)U_4(w) \sim &\frac{N(N-1)(N-2)(N-3)\alpha_0^3}{(z-w)^5} + \frac{(N-1)(N-2)(N-3)\alpha_0^2 J(w)}{(z-w)^4} \\ &+ \frac{(N-2)(N-3)\alpha_0 U_2(w)}{(z-w)^3} + \frac{(N-3)U_3(w)}{(z-w)^2} \quad . \end{aligned} \tag{3.47}$$

Note that in (3.47) the term $\sim \frac{1}{(z-w)^3}$ has been corrected, compare [Pro15]. Moreover,

$$\begin{aligned} U_2(z)U_2(w) \sim &\frac{\frac{1}{2}N(N-1)\left[1+2(1-2N)\alpha_0^2\right]}{(z-w)^4} - \frac{2U_2(w)}{(z-w)^2} + \frac{(N-1)(JJ)(w)}{(z-w)^2} \\ &+ \frac{N(N-1)\alpha_0 J^{(1)}(w)}{(z-w)^2} - \frac{U_2^{(1)}(w)}{(z-w)} + \frac{(N-1)(J^{(1)}J)(w)}{(z-w)} \\ &+ \frac{\frac{1}{2}N(N-1)\alpha_0 J^{(2)}}{(z-w)} \quad , \end{aligned} \tag{3.48}$$

$$
\begin{aligned}
U_2(z)U_3(w) \sim\ & \frac{N(N-1)(N-2)(1+\alpha_0^2-3N\alpha_0^2)\alpha_0}{(z-w)^5} + \frac{\frac{1}{2}(N-1)(N-2)(1-4N\alpha_0^2)J(w)}{(z-w)^4} \\
& + N(N-1)(N-2)\alpha_0^2\left[\frac{J(w)}{(z-w)^4} + \frac{J^{(1)}(w)}{(z-w)^3} + \frac{\frac{1}{2}J^{(2)}(w)}{(z-w)^2} + \frac{\frac{1}{6}J^{(3)}(w)}{(z-w)}\right] \\
& + (N-1)(N-2)\alpha_0\left[\frac{(JJ)(w)}{(z-w)^3} + \frac{(J^{(1)}J)(w)}{(z-w)^2} + \frac{\frac{1}{2}(J^{(2)}J)(w)}{(z-w)}\right] \\
& - \frac{(N-2)(N+1)\alpha_0 U_2(w)}{(z-w)^3} - \frac{3U_3(w)}{(z-w)^2} - \frac{U_3^{(1)}(w)}{(z-w)} \\
& + (N-2)\left[\frac{(JU_2)(w)}{(z-w)^2} + \frac{(J^{(1)}U_2)(w)}{(z-w)}\right] \quad ,
\end{aligned}
\tag{3.49}
$$

and

$$
\begin{aligned}
U_2(z)U_4(w) \sim\ & \frac{\frac{1}{2}N(N-1)(N-2)(N-3)\alpha_0^2(3+2\alpha_0^2-8N\alpha_0^2)}{(z-w)^6} \\
& + \frac{(N-1)(N-2)(N-3)\alpha_0(1-3N\alpha_0^2)J(w)}{(z-w)^5} \\
& + N(N-1)(N-2)(N-3)\alpha_0^3\left[\frac{J(w)}{(z-w)^5} + \frac{J^{(1)}(w)}{(z-w)^4} + \frac{\frac{1}{2}J^{(2)}(w)}{(z-w)^3}\right. \\
& \qquad\qquad \left. + \frac{\frac{1}{6}J^{(3)}}{(z-w)^2} + \frac{\frac{1}{24}J^{(4)}}{(z-w)}\right] \\
& + \frac{\frac{1}{2}(N-2)(N-3)(1-2\alpha_0^2-4N\alpha_0^2)U_2(w)}{(z-w)^4} - \frac{(N-3)(N+2)\alpha_0 U_3(w)}{(z-w)^3} \\
& + (N-1)(N-2)(N-3)\alpha_0^2\left[\frac{(JJ)(w)}{(z-w)^4} + \frac{(J^{(1)}J)(w)}{(z-w)^3} + \frac{\frac{1}{2}(J^{(2)}J)(w)}{(z-w)^2}\right. \\
& \qquad\qquad \left. + \frac{\frac{1}{6}(J^{(3)}J)(w)}{(z-w)}\right] \\
& + (N-2)(N-3)\alpha_0\left[\frac{(JU_2)(w)}{(z-w)^3} + \frac{(J^{(1)}U_2)(w)}{(z-w)^2} + \frac{(J^{(2)}U_2)(w)}{(z-w)}\right] \\
& + (N-3)\left[\frac{(JU_3)(w)}{(z-w)^2} + \frac{(J^{(1)}U_3)(w)}{(z-w)}\right] - \frac{4U_4(w)}{(z-w)^2} - \frac{U_4^{(1)}(w)}{(z-w)} \quad .
\end{aligned}
\tag{3.50}
$$

More generally, it has been observed in [Pro15] that the formula

$$U_j(z)U_k(w) = \sum_{l+m\leq j+k} \frac{C^{l,m}_{j,k}(\alpha_0,N)U_{l,m}(z,w)}{(z-w)^{j+k-l-m}} \quad , \tag{3.51}$$

which describes the full operator expansion of two generators, is consistent with explicitly computed OPEs using Jacobi identities for coefficients $C^{l,m}_{j,k}$ up to $j+k\leq 15$. Here, $U_{l,m}(z,w)$ is a set of bilocal operators that are regular as $z\to w$. $C^{l,m}_{j,k}(\alpha_0,N)$ are polynomials in two variables (structure constants) which satisfy a symmetry relation of the form

$$C^{l,m}_{j,k} = (-1)^{j+k-l-m}\, C^{m,l}_{k,j} \tag{3.52}$$

as well as the shift symmetry

$$C^{l+n,m+n}_{j+n,k+n}(\alpha_0,N+n) = C^{l,m}_{j,k}(\alpha_0,N) \quad \text{for } n\in\mathbb{N} \quad . \tag{3.53}$$

Through these two symmetries the computation of any such structure constant is reduced to the computation of

$$C^{l,m}_{0,k}(\alpha_0,N) \quad \text{and} \quad C^{l,0}_{j,k}(\alpha_0,N) \quad . \tag{3.54}$$

The first one of those simply reads

$$C^{l,m}_{0,k}(\alpha_0,N) = \delta^l_0\delta^m_k \quad . \tag{3.55}$$

In [Pro15] we see that unless $j=k\neq 0$ the two point function coefficients $C^{0,0}_{j,k}$ can be expressed through polynomials $P_l(\alpha_0,N)$,

$$C^{0,0}_{j,k} = (-1)^{j+1}\binom{N}{j}\binom{N}{k} j!k! \sum_{l=0}^{\infty}\binom{j-1}{l}\binom{k-1}{l}\frac{(-1)\alpha_0^{j+k-2l-2}P_l(\alpha_0,N)}{\binom{N}{l+1}(l+1)!^2} \quad , \tag{3.56}$$

where provided $N\leq l$,

$$P_l(\alpha_0,N) = \sum_{j=0}^{l}(-1)^j\alpha_0^{2j}\frac{l!(l+1)!}{(l-j)!(l-j+1)!}\binom{N-l-1+j}{j}\prod_{k=1}^{l-j}(1-k(k+1)\alpha_0^2) \quad . \tag{3.57}$$

The first four explicitly read

$$\begin{aligned}
P_0(\alpha_0,N) &= 1\\
P_1(\alpha_0,N) &= 1-2N\alpha_0^2\\
P_2(\alpha_0,N) &= 1+4\alpha_0^2-6N\alpha_0^2-6N\alpha_0^4+6N^2\alpha_0^4\\
P_3(\alpha_0,N) &= 1+16\alpha_0^2-12N\alpha_0^2+36\alpha_0^4-84N\alpha_0^4+36N^2\alpha_0^4-48N\alpha_0^6-24N^3\alpha_0^6 \quad .
\end{aligned} \tag{3.58}$$

Together, (3.56) and (3.57) can be expressed as

$$
\begin{aligned}
C^{0,0}_{j,k} = {} & \frac{(-1)^{j+1}\Gamma(N+1)\Gamma(N)(j+k-2)!\alpha_0^{j+k-2}}{(j-1)!(k-1)!\Gamma(N-j+1)\Gamma(N-k+1)}\times \\
& \times\ {}_4F_3\left(1-j, 1-k, \frac{3}{2}+\frac{1}{2\alpha_0}\sqrt{4+\alpha_0^2}, \frac{3}{2}-\frac{1}{2\alpha_0}\sqrt{4+\alpha_0^2}; 2, 2-j-k, 1-N; 1\right) \quad ,
\end{aligned} \tag{3.59}
$$

where ${}_4F_3(\_,\_,\_,\_;\_,\_,\_;\_)$ denotes the generalized hypergeometric function.

Additionally, in [Pro15] the general formulas for the second kind of structure constants of (3.54) for $l > 0$ are expected to be

$$
\begin{aligned}
C^{l,0}_{j,k} = \delta^l_j\delta^0_k + {} & \frac{(-1)^{j-1}\Gamma(N-l+1)\Gamma(N+1)\alpha_0^{j+k-l-2}}{\Gamma(N-j+1)\Gamma(N-k+1)}\times \\
\times \Bigg[& \sum_{a=0}^{\frac{l+1}{2}} \sum_{b=a-1}^{l-a-1} \frac{(-1)^{a+b+1}}{\Gamma(N-a+1)} \left(\binom{l-a-1}{a}\binom{l-2a-1}{b-a} \right. \\
& \left. + \binom{l-a-1}{a-1}\binom{l-2a}{b-a+1} \right) \phi^{j+k-l-1,N-a}_{j-b-1,k-l+b+1} \\
& + \sum_{a=0}^{\frac{l+1}{2}} \sum_{b=a-1}^{l-a} \frac{(-1)^{a+b}}{\Gamma(N-a+1)} \left(\binom{l-a-1}{a}\binom{l-2a-1}{b-a} \right. \\
& \left. + \binom{l-a}{a-1}\binom{l-2a+1}{b-a+1} \right) \phi^{j+k-l-2,N-a}_{j-b-1,k-l+b} \Bigg] \quad ,
\end{aligned} \tag{3.60}
$$

where

$$
\begin{aligned}
\phi^{a,b}_{c,d}(\alpha_0) &= \sum_{\alpha=0}^{min(c-1,d-1)} \frac{\Gamma(a-\alpha)\Gamma(b-\alpha)}{\Gamma(c-\alpha)\Gamma(d-\alpha)} \prod_{\beta=1}^{\alpha}\left(1-\frac{1}{\beta(\beta+1)\alpha_0^2}\right) \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}\times \\
&\quad \times\ {}_4F_3\left(1-c, 1-d, \frac{3}{2}+\frac{1}{2\alpha_0}\sqrt{4+\alpha_0^2}, \frac{3}{2}-\frac{1}{2\alpha_0}\sqrt{4+\alpha_0^2}; 2, 1-a, 1-b; 1\right) \quad .
\end{aligned} \tag{3.61}
$$

Notice that in the case of $a, b \leq \alpha = 0$ this function becomes indeterminate unless either $c \leq 0$ or $d \leq 0$ which in that case amounts to an empty sum and thus zero. This fact will be taken into account when doing calculations using `Mathematica`, where we will define $\phi^{a,b}_{c,d}$ to be zero for these cases.

The bilocal fields $U_{l,m}(z, w)$ can be inductively computed using (3.51), assuming the knowledge of the two point function coefficients $C^{l,m}_{j,k}$ as well as the singular part of the concerning OPE. We illustrate the necessary calculations in the cases of $U_{0,j}(z,w)$, $U_{j,0}(z,w)$, $U_{1,1}(z,w)$, $U_{2,1}(z,w)$ and $U_{2,2}(z,w)$ which will be used in later calculations. First, we find from (3.51) using (3.36) as well as (3.55) that

$$\begin{aligned} U_j(w) = U_0(z)U_j(w) &= \sum_{l+m\leq j} \frac{C^{l,m}_{0,j} U_{l,m}(z,w)}{(z-w)^{j-l-m}} \\ &= \sum_{l+m\leq j} \frac{\delta^l_0 \delta^m_j U_{l,m}(z,w)}{(z-w)^{j-l-m}} \\ &= U_{0,j}(z,w) \quad . \end{aligned} \tag{3.62}$$

Analogously, using (3.52) one finds that

$$\begin{aligned} U_j(z) = U_j(z)U_0(w) &= \sum_{j+m\leq j} \frac{C^{l,m}_{j,0} U_{l,m}}{(z-w)^{j-l-m}} \\ &= \sum_{j+m\leq j} \frac{(-1)^{j-l-m}\delta^m_0 \delta^l_j U_{l,m}(z,w)}{(z-w)^{j-l-m}} \\ &= U_{j,0}(z,w) \quad . \end{aligned} \tag{3.63}$$

Regarding $U_{1,1}(z,w)$ we see that

$$\begin{aligned} U_1(z)U_1(w) &= \sum_{l+m\leq 2} \frac{C^{l,m}_{1,1} U_{l,m}(z,w)}{(z-w)^{2-l-m}} \\ &= \frac{C^{0,0}_{1,1} U_{0,0}(z,w)}{(z-w)^2} + \frac{C^{1,0}_{1,1} U_{1,0}(z,w)}{(z-w)^1} + \frac{C^{0,1}_{1,1} U_{0,1}(z,w)}{(z-w)^1} + C^{1,1}_{1,1} U_{1,1}(z,w) \\ &= \frac{N}{(z-w)^2} + U_{1,1}(z,w) \quad , \end{aligned} \tag{3.64}$$

having used (3.59), (3.60) and the symmetry relation (3.52) to find the two point function coefficients

$$\begin{aligned} C^{0,0}_{1,1} &= N \\ C^{1,0}_{1,1} &= 0 \\ C^{0,1}_{1,1} &= 0 \\ C^{1,1}_{1,1} &= 1 \quad . \end{aligned} \tag{3.65}$$

After rearrangement, identifying the singular term in (3.44) and making use of (2.98) in the second step we conclude that

$$
\begin{aligned}
U_{1,1}(z,w) &= U_1(z)U_1(w) - \frac{N}{(z-w)^2} \\
&= U_1(z)U_1(w) - \overbrace{U_1(z)U_1(w)} \\
&=: (U_1(z)U_1(w)) \quad .
\end{aligned}
\tag{3.66}
$$

Analogously, we find that

$$
\begin{aligned}
U_1(z)U_2(w) &= \sum_{l+m\leq 3} \frac{C^{l,m}_{1,2}U_{l,m}(z,w)}{(z-w)^{3-l-m}} \\
&= \frac{C^{0,0}_{1,2}U_{0,0}(z,w)}{(z-w)^3} + \frac{C^{1,0}_{1,2}U_{1,0}(z,w)}{(z-w)^2} + \frac{C^{0,1}_{1,2}U_{0,1}(z,w)}{(z-w)^2} + \frac{C^{1,1}_{1,2}U_{1,1}(z,w)}{(z-w)^1} \\
&\quad + \frac{C^{2,0}_{1,2}U_{2,0}(z,w)}{(z-w)^1} + \frac{C^{0,2}_{1,2}U_{0,2}(z,w)}{(z-w)^1} + C^{2,1}_{1,2}U_{2,1}(z,w) + C^{1,2}_{1,2}U_{1,2}(z,w) \\
&= \frac{N(N-1)\alpha_0}{(z-w)^3} + \frac{(N-1)U_{0,1}(z,w)}{(z-w)^2} + U_{1,2}(z,w) \quad ,
\end{aligned}
\tag{3.67}
$$

where

$$
\begin{aligned}
C^{0,0}_{1,2} &= N(N-1)\alpha_0 \\
C^{1,0}_{1,2} &= 0 \\
C^{0,1}_{1,2} &= N-1 \\
C^{1,1}_{1,2} &= 0 \\
C^{2,0}_{1,2} &= 0 \\
C^{0,2}_{1,2} &= 0 \\
C^{2,1}_{1,2} &= 0 \\
C^{1,2}_{1,2} &= 1 \quad ,
\end{aligned}
\tag{3.68}
$$

and thus

$$
\begin{aligned}
U_{1,2} &= U_1(z)U_2(w) - \frac{N(N-1)\alpha_0}{(z-w)^3} - \frac{(N-1)J(w)}{(z-w)^2} \\
&= U_1(z)U_2(w) - \overbrace{U_1(z)U_2(w)} \\
&= (J(z)U_2)(w) \quad .
\end{aligned}
\tag{3.69}
$$

Furthermore,

$$
\begin{aligned}
U_2(z)U_1(w) &= \sum_{l+m\leq 3} \frac{C^{l,m}_{2,1} U_{l,m}(z,w)}{(z-w)^{3-l-m}} \\
&= \frac{C^{0,0}_{2,1} U_{0,0}(z,w)}{(z-w)^3} + \frac{C^{1,0}_{2,1} U_{1,0}(z,w)}{(z-w)^2} + \frac{C^{0,1}_{2,2} U_{0,1}(z,w)}{(z-w)^2} + \frac{C^{1,1}_{2,1} U_{1,1}(z,w)}{(z-w)^1} \\
&\quad + \frac{C^{2,0}_{2,1} U_{2,0}(z,w)}{(z-w)^1} + \frac{C^{0,2}_{2,1} U_{0,2}(z,w)}{(z-w)^1} + C^{2,1}_{2,1} U_{2,1}(z,w) + C^{1,2}_{2,1} U_{1,2}(z,w) \\
&= -\frac{N(N-1)\alpha_0}{(z-w)^3} + \frac{(N-1)U_{1,0}(z,w)}{(z-w)^2} + U_{2,1}(z,w) \quad ,
\end{aligned}
\tag{3.70}
$$

where

$$
\begin{aligned}
C^{0,0}_{2,1} &= -N(N-1)\alpha_0 \\
C^{1,0}_{2,1} &= N-1 \\
C^{0,1}_{2,1} &= 0 \\
C^{1,1}_{2,1} &= 0 \\
C^{2,0}_{2,1} &= 0 \\
C^{0,2}_{2,1} &= 0 \\
C^{2,1}_{2,1} &= 1 \\
C^{1,2}_{2,1} &= 0 \quad ,
\end{aligned}
\tag{3.71}
$$

resulting in

$$
\begin{aligned}
U_{2,1}(z,w) &= U_2(z)U_1(w) + \frac{N(N-1)\alpha_0}{(z-w)^3} - \frac{(N-1)J^{(1)}(w)}{(z-w)^2} - \frac{(N-1)J(w)}{(z-w)} \\
&= U_2(z)U_1(w) - U_2(z)U_1(w) \\
&= (U_2(z)U_1(w)) \quad .
\end{aligned}
\tag{3.72}
$$

Regarding $U_{2,2}(z,w)$ we find that

$$
\begin{aligned}
U_2(z)U_2(w) &= \sum_{l+m\leq 4} \frac{C_{2,2}^{l,m} U_{l,m}(z,w)}{(z-w)^{4-l-m}} \\
&= \frac{C_{2,2}^{0,0} U_{0,0}(z,w)}{(z-w)^4} + \frac{C_{2,2}^{1,0} U_{1,0}(z,w)}{(z-w)^3} + \frac{C_{2,2}^{0,1} U_{0,1}(z,w)}{(z-w)^3} + \frac{C_{2,2}^{1,1} U_{1,1}(z,w)}{(z-w)^2} \\
&\quad + \frac{C_{2,2}^{2,0} U_{2,0}(z,w)}{(z-w)^2} + \frac{C_{2,2}^{0,2} U_{0,2}(z,w)}{(z-w)^2} + \frac{C_{2,2}^{2,1} U_{2,1}(z,w)}{(z-w)^1} + \frac{C_{2,2}^{1,2} U_{1,2}(z,w)}{(z-w)^1} \\
&\quad + \frac{C_{2,2}^{3,0} U_{3,0}(z,w)}{(z-w)^1} + \frac{C_{2,2}^{0,3} U_{0,3}(z,w)}{(z-w)^1} + C_{2,2}^{3,1} U_{3,1}(z,w) + C_{2,2}^{1,3} U_{1,3}(z,w) \\
&\quad + C_{2,2}^{4,0} U_{4,0}(z,w) + C_{2,2}^{0,4} U_{0,4}(z,w) + C_{2,2}^{2,2} U_{2,2}(z,w) \\
&= \frac{\frac{1}{2}N(N-1)\left[1+(2-4N)\alpha_0^2\right]}{(z-w)^4} + \frac{N(N-1)\alpha_0 U_{1,0}(z,w)}{(z-w)^3} \\
&\quad - \frac{N(N-1)\alpha_0 U_{0,1}(z,w)}{(z-w)^3} + \frac{(N-1)U_{1,1}(z,w)}{(z-w)^2} - \frac{U_{2,0}(z,w)}{(z-w)^2} - \frac{U_{0,2}(z,w)}{(z-w)^2} \\
&\quad + U_{2,2}(z,w) \quad ,
\end{aligned}
\tag{3.73}
$$

where the two point function coefficients are

$$
\begin{aligned}
C_{2,2}^{0,0} &= \frac{1}{2}N(N-1)\left[1+(2-4N)\alpha_0^2\right] \\
C_{2,2}^{1,0} &= N(N-1)\alpha_0 \\
C_{2,2}^{0,1} &= (-1)C_{2,2}^{1,0} \\
C_{2,2}^{1,1} &= N-1 \\
C_{2,2}^{2,0} &= -1 \\
C_{2,2}^{0,2} &= C_{2,2}^{2,0} \\
C_{2,2}^{2,1} &= 0 \\
C_{2,2}^{1,2} &= (-1)C_{2,2}^{2,1} \\
C_{2,2}^{3,0} &= 0 \\
C_{2,2}^{0,3} &= (-1)C_{2,2}^{3,0} \\
C_{2,2}^{3,1} &= 0 \\
C_{2,2}^{1,3} &= (-1)C_{2,2}^{3,1} \\
C_{2,2}^{4,0} &= 0 \\
C_{2,2}^{0,4} &= C_{2,2}^{4,0} \\
C_{2,2}^{2,2} &= 1 \quad .
\end{aligned}
\tag{3.74}
$$

We proceed by rearranging (3.73). After making use of previous results and Taylor expansions indicated by "...",

$$\begin{aligned}
U_{2,2}(z,w) &= U_2(z)U_2(w) - \frac{\frac{1}{2}N(N-1)\left[1+(2-4N)\alpha_0^2\right]}{(z-w)^4} - \frac{N(N-1)\alpha_0 J(w)}{(z-w)^3} \\
&\quad - \frac{N(N-1)\alpha_0 J^{(1)}(w)}{(z-w)^2} - \frac{\frac{1}{2}N(N-1)\alpha_0 J^{(2)}(w)}{(z-w)} + \frac{N(N-1)\alpha_0 J(w)}{(z-w)^3} \\
&\quad - \frac{(N-1)(JJ)(w)}{(z-w)^2} - \frac{(N-1)(J^{(1)}J)(w)}{(z-w)} + \frac{U_2(w)}{(z-w)^2} \\
&\quad + \frac{U_2^{(1)}(w)}{(z-w)} + \frac{U_2(w)}{(z-w)^2} \\
&\quad - \frac{1}{3!}N(N-1)\alpha_0 J^{(3)}(w) - ... - \frac{1}{2!}(N-1)(J^{(2)}J)(w) - ... + \frac{1}{2!}U_2^{(2)} + ... \\
&= U_2(z)U_2(w) - \overline{U_2(z)U_2(w)} \\
&\quad - \frac{1}{3!}N(N-1)\alpha_0 J^{(3)}(w) - ... - \frac{1}{2!}(N-1)(J^{(2)}J)(w) - ... + \frac{1}{2!}U_2^{(2)} + ... \quad ,
\end{aligned} \tag{3.75}$$

we arrive at the bilocal field

$$\begin{aligned}
U_{2,2}(z,w) &= (U_2(z)U_2(w)) - \frac{1}{3!}N(N-1)\alpha_0 J^{(3)}(w) - ... - \frac{1}{2!}(N-1)(J^{(2)}J)(w) - ... \\
&\quad + \frac{1}{2!}U_2^{(2)} + ... \\
&= (U_2U_2)(w) - \frac{1}{6}N(N-1)\alpha_0 J^{(3)}(w) - \frac{1}{2}(N-1)(J^{(2)}J)(w) + \frac{1}{2}U_2^{(2)} \\
&\quad + (z-w)\Big[(U_2^{(1)}U_2)(w) - \frac{1}{24}N(N-1)\alpha_0 J^{(4)}(w) - \frac{1}{6}(N-1)(J^{(3)}J)(w) \\
&\quad + \frac{1}{6}U_2^{(3)}\Big] + ... \quad .
\end{aligned} \tag{3.76}$$

For later calculations, we additionally need to know certain singular terms of the OPEs $U_3(z)U_3(w)$, $U_3(z)U_4(w)$ and $U_4(z)U_4(w)$. By making use of (3.59) and (3.60) as well as the symmetry relations (3.52) and (3.53), we find that

$$\begin{aligned}
U_3(z)U_3(w) &= \frac{C_{3,3}^{0,0}}{(z-w)^6} + \frac{C_{3,3}^{1,0}U_{10}(z,w)}{(z-w)^5} + \frac{C_{3,3}^{0,1}U_{01(z,w)}}{(z-w)^5} + \frac{C_{3,3}^{2,0}U_{20}(z,w)}{(z-w)^4} + \frac{C_{3,3}^{0,2}U_{02}(z,w)}{(z-w)^4} \\
&\quad + \frac{C_{3,3}^{1,1}U_{11}(z,w)}{(z-w)^4} + \text{(lower singularity terms)} \quad ,
\end{aligned} \tag{3.77}$$

where

$$
\begin{aligned}
C_{3,3}^{0,0} &= \frac{1}{6}N(N-1)(N-2)\left[1+2\alpha_0^2\left(14-9N+6(1+3(N-2)N)\alpha_0^2\right)\right] \\
C_{3,3}^{1,0} &= (N-1)(N-2)\left[N-1+N(5-3N)\alpha_0^2\right]\alpha_0 \\
C_{3,3}^{0,1} &= (-1)C_{3,3}^{1,0} \\
C_{3,3}^{2,0} &= (N-2)\left[(N-1)N\alpha_0^2-1\right] \\
C_{3,3}^{02} &= C_{3,3}^{2,0} \\
C_{3,3}^{11} &= \frac{1}{2}(N-1)(N-2)\left[1-(4N-6)\alpha_0^2\right] \quad .
\end{aligned}
\tag{3.78}
$$

Moreover,

$$
\begin{aligned}
U_3(z)U_4(w) \sim{}& \frac{C_{3,4}^{0,0}}{(z-w)^7} + \frac{C_{3,4}^{1,0}U_{10}}{(z-w)^6} + \frac{C_{3,4}^{0,1}U_{01}}{(z-w)^6} + \frac{C_{3,4}^{2,0}U_{20}}{(z-w)^5} + \frac{C_{3,4}^{0,2}U_{02}}{(z-w)^5} + \frac{C_{3,4}^{1,1}U_{11}}{(z-w)^5} \\
&+ \frac{C_{3,4}^{3,0}U_{30}}{(z-w)^4} + \frac{C_{3,4}^{0,3}U_{03}}{(z-w)^4} + \frac{C_{3,4}^{1,2}U_{12}}{(z-w)^4} + \frac{C_{3,4}^{2,1}U_{21}}{(z-w)^4} \\
&+ (\text{lower singularity terms}) \quad ,
\end{aligned}
\tag{3.79}
$$

where

$$
\begin{aligned}
C_{3,4}^{0,0} &= \frac{1}{2}N(N-1)(N-2)(N-3)\left[1+4\alpha_0^2(4-3N+(1+N(5N-9))\alpha_0^2\right] \\
C_{3,4}^{1,0} &= \frac{1}{2}(N-1)(N-2)(N-3)\left[3N-2+4(3-2N)\alpha_0^2\right]\alpha_0^2 \\
C_{3,4}^{0,1} &= (-1)C_{3,4}^{1,0} \\
C_{3,4}^{2,0} &= (N-2)(N-3)\left[(N-1)N\alpha_0^2-1\right]\alpha_0 \\
C_{3,4}^{0,2} &= (N-2)(N-3)\left[3(N-1)N\alpha_0^2-1-N\right]\alpha_0 \\
C_{3,4}^{1,1} &= (N-1)(N-2)(N-3)\left[(3N-4)\alpha_0^2-1\right]\alpha_0 \\
C_{3,4}^{3,0} &= N-3 \\
C_{3,4}^{0,3} &= (N-3)\left[(N^2+2)\alpha_0^2-2\right] \\
C_{3,4}^{1,2} &= \frac{1}{2}(N-2)(N-3)\left[1-4(N-1)\alpha_0^2\right] \\
C_{3,4}^{2,1} &= (N-1)(N-2)(N-3)\alpha_0^2 \quad .
\end{aligned}
\tag{3.80}
$$

Lastly,

$$
U_4(z)U_4(w) \sim \frac{C_{4,4}^{0,0}}{(z-w)^8} \quad + (\text{lower singularity terms}) \quad , \tag{3.81}
$$

where

$$C_{4,4}^{0,0} = \frac{1}{24}N(N-1)(N-2)(N-3)\big[1+4\alpha_0^2\big[31-12n+3(93+2n(15n-59))\alpha_0^2 \\ -12(2n-3)(1+5(n-3)n)\alpha_0^4\big]\big] \quad . \tag{3.82}$$

Note that for all further calculations done in this thesis it suffices to know all singular terms in (3.79) and (3.81) of singularity $\sim \frac{1}{(z-w)^4}$ and higher.

With these ingredients we are able to continue with the next chapter.

4 Methods of proving the highest singularity Terms

In this chapter two methods are presented that can be used to calculate the highest singularity terms of the OPEs $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$. The first method which is discussed in chapter 4.1 uses the generalized Wick theorem. The second method employs knowledge about the W_{1+N}-algebra and is discussed in chapter 4.2.

4.1 Brute Force using the generalized Wick Theorem

The goal of this section is to prove that the coefficient of the highest singularity term in the OPE $W_3(z)W_3(w)$ is of the form

$$cn_3 \quad , \tag{4.1}$$

where c is the central charge of the Virasoro algebra (3.15) and

$$n_3 = \frac{1}{6N}(N-2)\left[4 - N(N+2)\alpha_0^2\right] \quad . \tag{4.2}$$

Recalling the definition of the W_3 current (3.18) we can write the OPE

$$\begin{aligned} W_3(z)W_3(w) = \tilde{U}_3(z)\tilde{U}_3(w) - \alpha_0\frac{(N-2)}{2}\left[T^{(1)}(z)\tilde{U}_3(w) + \tilde{U}_3(z)T^{(1)}(w)\right] \\ + \alpha_0^2\frac{(N-2)^2}{2^2}T^{(1)}(z)T^{(1)}(w) \quad . \end{aligned} \tag{4.3}$$

This gives us again four different OPEs which we have to determine (in part):

$$\tilde{U}_3(z)\tilde{U}_3(w) \quad , \tag{4.4}$$

$$T^{(1)}(z)\tilde{U}_3(w) = \partial_z T(z)\tilde{U}_3(w) \quad , \tag{4.5}$$

$$\tilde{U}_3(z)T^{(1)}(w) = \partial_w \tilde{U}_3(z)T(w) \quad , \tag{4.6}$$

$$T^{(1)}(z)T^{(1)}(w) = \partial_z\partial_w T(z)T(w) \quad . \tag{4.7}$$

We start with the OPE (4.4). Using (3.17) we find

$$
\begin{aligned}
\tilde{U}_3(z)\tilde{U}_3(w) = & \sum_{i<j<k}\sum_{l<m<n}(J_i(J_jJ_k))(z)(J_l(J_mJ_n))(w) \\
& -\alpha_0\sum_{i<j}(i-1)\sum_{l<m<n}(J_i^{(1)}J_j)(z)(J_l(J_mJ_n))(w) \\
& -\alpha_0\sum_{i<j}(j-2)\sum_{l<m<n}(J_iJ_j^{(1)})(z)(J_l(J_mJ_n))(w) \\
& +\alpha_0^2\sum_{i>2}\binom{i-1}{2}\sum_{l<m<n}(J_i^{(2)})(z)(J_l(J_mJ_n))(w) \\
& -\alpha_0\sum_{i<j<k}\sum_{l<m}(l-1)(J_i(J_jJ_k))(z)(J_l^{(1)}J_m)(w) \\
& +\alpha_0^2\sum_{i<j}(i-1)\sum_{l<m}(l-1)(J_i^{(1)}J_j)(z)(J_l^{(1)}J_m)(w) \\
& +\alpha_0^2\sum_{i<j}(j-2)\sum_{l<m}(l-1)(J_iJ_j^{(1)})(z)(J_l^{(1)}J_m)(w) \\
& -\alpha_0^3\sum_{i>2}\binom{i-2}{2}\sum_{l<m}(l-1)(l-1)(J_i^{(2)})(z)(J_l^{(1)}J_m)(w) \\
& -\alpha_0\sum_{i<j<k}\sum_{l<m}(m-1)(J_i(J_jJ_k))(z)(J_lJ_m^{(1)})(w) \\
& +\alpha_0^2\sum_{i<j}(i-1)\sum_{l<m}(m-2)(J_i^{(1)}J_j)(z)(J_lJ_m^{(1)})(w) \\
& +\alpha_0^2\sum_{i<j}(j-2)\sum_{l<m}(m-2)(J_iJ_j^{(1)})(z)(J_lJ_m^{(1)})(w) \\
& -\alpha_0^3\sum_{i>2}\binom{i-1}{2}\sum_{l<m}(m-2)(J_i^{(2)})(z)(J_lJ_m^{(1)})(w) \\
& +\alpha_0^2\sum_{i<j<k}\sum_{l>2}(J_i(J_jJ_k))(z)(J_l^{(2)})(w) \\
& -\alpha_0^3\sum_{i<j}(i-1)\sum_{l>2}\binom{l-1}{2}(J_i^{(1)}J_j)(z)(J_l^{(2)})(w) \\
& -\alpha_0^3\sum_{i<j}(j-2)\sum_{l>2}\binom{l-1}{2}(J_iJ_j^{(1)})(z)(J_l^{(2)})(w) \\
& +\alpha_0^4\sum_{i>2}\binom{i-1}{2}\sum_{l>2}\sum_{l>2}\binom{l-1}{2}(J_i^{(2)})(z)(J_l^{(2)})(w) \quad ,
\end{aligned}
\tag{4.8}
$$

and in particular

$$
\begin{aligned}
(J_i(J_jJ_k))(z)(J_l(J_mJ_n))(w) \sim \frac{1}{(z-w)^6}\Bigg\{ &\left(\delta_{li}-\frac{1}{N}\right)\Bigg[\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\Bigg] \\
&+ \left(\delta_{lj}-\frac{1}{N}\right)\Bigg[\left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right)\Bigg] \\
&+ \left(\delta_{lk}-\frac{1}{N}\right)\Bigg[\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{jn}-\frac{1}{N}\right)\Bigg]\Bigg\} \\
(\partial J_iJ_j)(z)(J_l(J_mJ_n))(w) \sim{}& (\text{lower singularity terms}) \\
(J_i\partial J_j)(z)(J_l(J_mJ_n))(w) \sim{}& (\text{lower singularity terms}) \\
(\partial^2 J_i)(z)(J_l(J_mJ_n))(w) \sim{}& (\text{lower singularity terms}) \\
(J_i(J_jJ_k))(z)(\partial J_lJ_m)(w) \sim{}& (\text{lower singularity terms}) \\
(\partial J_iJ_j)(z)(\partial J_lJ_m)(w) \sim{}& \frac{1}{(z-w)^6}\left[-6\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)-4\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)\right] \\
(J_i\partial J_j)(z)(\partial J_lJ_m)(w) \sim{}& \frac{1}{(z-w)^6}\left[-6\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)-4\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\right] \\
(\partial^2 J_i)(z)(\partial J_lJ_m)(w) \sim{}& (\text{lower singularity terms}) \\
(J_i(J_jJ_k))(z)(J_l\partial J_m)(w) \sim{}& (\text{lower singularity terms}) \\
(\partial J_iJ_j)(z)(J_l\partial J_m)(w) \sim{}& \frac{1}{(z-w)^6}\left[-4\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)-6\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)\right] \\
(J_i\partial J_j)(z)(J_l\partial J_m)(w) \sim{}& \frac{1}{(z-w)^6}\left[-4\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)-6\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\right] \\
(\partial^2 J_i)(z)(J_l\partial J_m)(w) \sim{}& (\text{lower singularity terms}) \\
(J_i(J_jJ_k))(z)(\partial^2 J_l)(w) \sim{}& (\text{lower singularity terms}) \\
(\partial J_iJ_j)(z)(\partial^2 J_l)(w) \sim{}& (\text{lower singularity terms}) \\
(J_i\partial J_j)(z)(\partial^2 J_l)(w) \sim{}& (\text{lower singularity terms}) \\
(\partial^2 J_i)(z)(\partial^2 J_l)(w) \sim{}& \frac{120}{(z-w)^6}\left(\delta_{il}-\frac{1}{N}\right) \quad .
\end{aligned}
\tag{4.9}
$$

The respective calculations can be done either by hand or using `Mathematica` and the relevant package which will be described in 4.2.5. For a more thorough understanding, we illustrate the calculations for $(\partial J_i J_j)(z)(\partial J_l J_m)(w)$.

Using the generalized Wick theorem (2.99) we find

$$
\begin{aligned}
(\partial J_i J_j)(z)(\partial J_l J_m)(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{(x-w)} \Big\{ & (\partial J_i J_j)(z)(\partial J_l)(x) J_m(w) \\
& + \partial J_l(x)(\partial J_i J_j)(z) J_m(w) \Big\} \quad .
\end{aligned}
\tag{4.10}
$$

Then we calculate $(\partial J_i J_j)(z)(\partial J_l)(x)$ and $(\partial J_i J_j)(z) J_m(w)$ using (2.102) and (2.103). Starting with $(\partial J_i J_j)(z)(\partial J_l)(x)$, we first determine the OPE $(\partial J_l)(z)(\partial J_i J_j)(x)$,

$$
\begin{aligned}
(\partial J_l)(z)(\partial J_i J_j)(x) &= \frac{1}{2\pi i}\oint_x \frac{dy}{(y-x)} \Big\{ (\partial J_l)(z)(\partial J_i)(y) J_j(x) + (\partial J_i)(y)(\partial J_l)(z) J_j(x) \Big\} \\
&= \frac{1}{2\pi i}\oint_x \frac{dy}{(y-x)} \left\{ \partial_z \partial_y \left(\frac{\left(\delta_{li} - \frac{1}{N}\right)}{(z-y)^2} \right) J_j(x) + J_i^{(1)}(y) \partial_z \left(\frac{\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^2} \right) \right\} \\
&= \frac{1}{2\pi i}\oint_x \frac{dy}{(y-x)} \left\{ \frac{(-6)\left(\delta_{li} - \frac{1}{N}\right)}{(z-y)^4} J_j(x) - \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(y) \right\} \\
&= \frac{(1+4-2)!}{(1-1)!(4-1)!} \frac{(-6)\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^4} J_j(x) - \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(y) \\
&= \frac{(-6)\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^4} J_j(x) - \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(x) \quad ,
\end{aligned}
$$

before switching x and z,

$$
(\partial J_l)(x)(\partial J_i J_j)(z) = \frac{(-6)\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^4} J_j(z) + \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(z) \quad ,
$$

and Taylor expanding the currents on the r.h.s. of the OPE around the point x,

$$
\begin{aligned}
(\partial J_i J_j)(z)(\partial J_l)(x) = & -\frac{6\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^4} J_j(x) - \frac{6\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^3} J_j^{(1)}(x) - \frac{3\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^2} J_j^{(2)}(x) \\
& - \frac{\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)} J_j^{(3)}(x) + \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(x) + \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^2} J_i^{(2)}(x) \\
& + \frac{\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)} J_i^{(3)}(x)
\end{aligned}
\tag{4.11}
$$

The exact same procedure is used for $(\partial J_i J_j)(z) J_m(w)$,

$$
\begin{aligned}
J_m(z)(\partial J_i J_j)(w) &= \frac{1}{2\pi i}\oint \frac{dy}{(y-w)}\left\{ J_m(z)\partial J_i(y) J_j(w) + \partial J_i(y) J_m(z) J_j(w) \right\} \\
&= \frac{1}{2\pi i}\oint \frac{dy}{(y-w)}\left\{ \partial_y \left(\frac{\left(\delta_{mi} - \frac{1}{N}\right)}{(z-y)^2} \right) J_j(w) + \partial J_i(y) \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} \right\} \\
&= \frac{1}{2\pi i}\oint \frac{dy}{(y-w)}\left\{ \frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-y)^3} J_j(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} \partial J_i(y) \right\} \\
&= \frac{(1+3-2)!}{(1-1)!(3-1)!} \frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^3} J_j(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} J_i^{(1)}(w) \\
&= \frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^3} J_j(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} J_i^{(1)}(w) \quad ,
\end{aligned}
$$

$$
J_m(w)(\partial J_i J_j)(z) = -\frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^3} J_j(z) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} J_i^{(1)}(z) \quad ,
$$

and

$$
\begin{aligned}
(\partial J_i J_j)(z) J_m(w) = &-\frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^3} J_j(w) - \frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^2} J_j^{(1)}(w) - \frac{\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)} J_j^{(2)}(w) \\
&+ \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} J_i^{(1)}(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)} J_i^{(2)}(w) \quad .
\end{aligned}
\tag{4.12}
$$

Now we insert (4.11) and (4.12) into (4.10),

$$
\begin{aligned}
&(\partial J_i J_j)(z)(\partial J_l J_m)(w) = \\
&= \frac{1}{2\pi i}\oint_w \frac{dx}{(x-w)}\left\{ (\partial J_i J_j)(z)(\partial J_l)(x) J_m(w) + \partial J_l(x)(\partial J_i J_j)(z) J_m(w) \right\} \\
&= \frac{1}{2\pi i}\oint_w \frac{dx}{(x-w)}\left\{ \left[-\frac{6\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^4} J_j(x) - \frac{6\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^3} J_j^{(1)}(x) - \frac{3\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)^2} J_j^{(2)}(x) \right.\right. \\
&\quad - \frac{\left(\delta_{li} - \frac{1}{N}\right)}{(z-x)} J_j^{(3)}(x) + \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^3} J_i^{(1)}(x) + \frac{2\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)^2} J_i^{(2)}(x) \\
&\quad \left. + \frac{\left(\delta_{lj} - \frac{1}{N}\right)}{(z-x)} J_i^{(3)}(x) \right] J_m(w) + J_l^{(1)}(x) \left[-\frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^3} J_j(w) - \frac{2\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)^2} J_j^{(1)}(w) \right. \\
&\quad \left.\left. - \frac{\left(\delta_{mi} - \frac{1}{N}\right)}{(z-w)} J_j^{(2)}(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)^2} J_i^{(1)}(w) + \frac{\left(\delta_{mj} - \frac{1}{N}\right)}{(z-w)} J_i^{(2)}(w) \right] \right\} \quad ,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&(\partial J_i J_j)(z)(\partial J_l J_m)(w) = \\
&= \frac{1}{2\pi i}\oint_w \frac{dx}{(x-w)}\Bigg\{ -\frac{6\left(\delta_{li}-\frac{1}{N}\right)}{(z-x)^4}\frac{\left(\delta_{jm}-\frac{1}{N}\right)}{(x-w)^2} - \frac{6\left(\delta_{li}-\frac{1}{N}\right)}{(z-x)^3}\frac{(-2)\left(\delta_{jm}-\frac{1}{N}\right)}{(x-w)^3} \\
&\quad - \frac{3\left(\delta_{li}-\frac{1}{N}\right)}{(z-x)^2}\frac{6\left(\delta_{jm}-\frac{1}{N}\right)}{(x-w)^4} - \frac{\left(\delta_{li}-\frac{1}{N}\right)}{(z-x)}\frac{(-24)\left(\delta_{jm}-\frac{1}{N}\right)}{(x-w)^5} \\
&\quad + \frac{2\left(\delta_{lj}-\frac{1}{N}\right)}{(z-x)^3}\frac{(-2)\left(\delta_{im}-\frac{1}{N}\right)}{(x-w)^3} + \frac{2\left(\delta_{lj}-\frac{1}{N}\right)}{(z-x)^2}\frac{6\left(\delta_{im}-\frac{1}{N}\right)}{(x-w)^4} + \frac{\left(\delta_{lj}-\frac{1}{N}\right)}{(z-x)}\frac{(-24)\left(\delta_{im}-\frac{1}{N}\right)}{(x-w)^5} \\
&\quad - \frac{2\left(\delta_{mi}-\frac{1}{N}\right)}{(z-w)^3}\frac{(-2)\left(\delta_{lj}-\frac{1}{N}\right)}{(x-w)^3} - \frac{2\left(\delta_{mi}-\frac{1}{N}\right)}{(z-w)^2}\frac{(-6)\left(\delta_{lj}-\frac{1}{N}\right)}{(x-w)^4} - \frac{\left(\delta_{mi}-\frac{1}{N}\right)}{(z-w)}\frac{(-24)\left(\delta_{lj}-\frac{1}{N}\right)}{(x-w)^5} \\
&\quad + \frac{\left(\delta_{mj}-\frac{1}{N}\right)}{(z-w)^2}\frac{(-6)\left(\delta_{li}-\frac{1}{N}\right)}{(x-w)^4} + \frac{\left(\delta_{mj}-\frac{1}{N}\right)}{(z-w)}\frac{(-24)\left(\delta_{li}-\frac{1}{N}\right)}{(x-w)^5}\Bigg\} \quad .
\end{aligned}
\tag{4.13}
$$

Again, we make use of (2.102) and (2.103) to determine that

$$
\begin{aligned}
(\partial J_i J_j)(z)(\partial J_l J_m)(w) &= \frac{(4+3-2)!}{(4-1)!(3-1)!}\frac{(-6)}{(z-w)^6}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right) \\
&+ \frac{(3+4-2)!}{(3-1)!(4-1)!}\frac{12}{(z-w)^6}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right) \\
&+ \frac{(2+5-2)!}{(2-1)!(5-1)!}\frac{(-18)}{(z-w)^6}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right) \\
&+ \frac{(1+6-2)!}{(1-1)!(6-1)!}\frac{24}{(z-w)^6}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right) \\
&+ \frac{(3+4-2)!}{(3-1)!(4-1)!}\frac{(-4)}{(z-w)^6}\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right) \\
&+ \frac{(2+5-2)!}{(2-1)!(5-1)!}\frac{12}{(z-w)^6}\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right) \\
&+ \frac{(1+6-2)!}{(1-1)!(6-1)!}\frac{(-24)}{(z-w)^6}\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right) \\
&+ 0+0+0+0+0 \quad .
\end{aligned}
$$

Notice that the last five terms in (4.13) cancel due to (2.103) which is the entire second summand on the r.h.s. of (4.10). Thus

$$
\begin{aligned}
&(\partial J_i J_j)(z)(\partial J_l J_m)(w) \\
&\quad = \frac{1}{(z-w)^6}\Bigg[(-60+120-90+24)\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + (-40+60-24)\left(\delta_{mi}-\frac{1}{N}\right)\left(\delta_{lj}-\frac{1}{N}\right)\Bigg] \\
&= \frac{1}{(z-w)^6}\left[(-6)\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)+(-4)\left(\delta_{mi}-\frac{1}{N}\right)\left(\delta_{lj}-\frac{1}{N}\right)\right] \quad ,
\end{aligned}
$$

as has been proposed in (4.9).

After having determined the highest singularity terms, we take the respective sums of the OPEs in (4.9) omitting $\frac{1}{(z-w)^6}$. We denote such a term that stems from a certain OPE and in that OPE is proportional to $\frac{1}{(z-w)^6}$, e.g. $(J_i(J_jJ_k))(z)(J_l(J_mJ_n)(w)|_{\frac{1}{(z-w)^6}}$.

$$
\begin{aligned}
\sum_{i<j<k}\sum_{l<m<n}(J_i(J_jJ_k))(z)(J_l(J_mJ_n))(w)|_{\frac{1}{(z-w)^6}} &= \frac{2N}{3}-2+\frac{4}{3N} \\
\alpha_0^2\sum_{i<j}(i-1)\sum_{l<m}(l-1)(\partial J_iJ_j)(z)(\partial J_lJ_m)(w)|_{\frac{1}{(z-w)^6}} &= \alpha_0^2\Bigg[-\frac{7N^4}{90}-\frac{N^3}{6} \\
&\qquad +\frac{25N^2}{18}-\frac{11N}{6}+\frac{31}{45}\Bigg] \\
\alpha_0^2\sum_{i<j}(j-2)\sum_{l<m}(l-1)(J_i\partial J_j)(z)(\partial J_lJ_m)(w)|_{\frac{1}{(z-w)^6}} &= \alpha_0^2\Bigg[\frac{7N^4}{90}-\frac{7N^3}{6} \\
&\qquad +\frac{71N^2}{18}-\frac{29N}{6}+\frac{89}{45}\Bigg] \\
\alpha_0^2\sum_{i<j}(i-1)\sum_{l<m}(m-2)(\partial J_iJ_j)(z)(J_l\partial J_m)(w)|_{\frac{1}{(z-w)^6}} &= \alpha_0^2\Bigg[\frac{7N^4}{90}-\frac{7N^3}{6} \\
&\qquad +\frac{71N^2}{18}-\frac{29N}{6}+\frac{89}{45}\Bigg] \\
\alpha_0^2\sum_{i<j}(j-2)\sum_{l<m}(m-2)(J_i\partial J_j)(z)(J_l\partial J_m)(w)|_{\frac{1}{(z-w)^6}} &= \alpha_0^2\Bigg[-\frac{7N^4}{90}-\frac{5N^3}{6} \\
&\qquad +\frac{85N^2}{18}-\frac{43N}{6}+\frac{151}{45}\Bigg] \\
\alpha_0^4\sum_{i>2}\binom{i-1}{2}\sum_{l>2}\binom{l-1}{2}(\partial^2J_i)(z)(\partial^2J_l)(w)|_{\frac{1}{(z-w)^6}} &= \alpha_0^4\Bigg[\frac{8N^5}{3}-10N^4 \\
&\qquad +\frac{20N^3}{3}+10N^2-\frac{28N}{3}\Bigg]
\end{aligned} \tag{4.14}
$$

To illustrate how we arrive at these terms we will in part write out the the calculation of the first term explicitly

$$
\begin{aligned}
&\sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}(J_i(J_jJ_k))(z)(J_l(J_mJ_n))(w)|_{\frac{1}{(z-w)^6}} = \\
&\quad = \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\Bigg\{\left(\delta_{li}-\frac{1}{N}\right)\Bigg[\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\Bigg] \\
&\qquad + \left(\delta_{lj}-\frac{1}{N}\right)\Bigg[\left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right)\Bigg] \\
&\qquad + \left(\delta_{lk}-\frac{1}{N}\right)\Bigg[\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \\
&\qquad\qquad + \left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{jn}-\frac{1}{N}\right)\Bigg]\Bigg\} \quad .
\end{aligned}
\tag{4.15}
$$

Since it is not possible to calculate this sum for a general N with `Mathematica` it has been done by hand. First need to get rid of all Kronecker-δ symbols, so we define

$$(A) := \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right) \quad , \tag{4.16}$$

$$(B) := \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{li}-\frac{1}{N}\right)\left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{jn}-\frac{1}{N}\right) \quad , \tag{4.17}$$

$$(C) := \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{km}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \quad , \tag{4.18}$$

$$(D) := \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{lj}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{kn}-\frac{1}{N}\right) \quad , \tag{4.19}$$

$$(E) := \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{lk}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\left(\delta_{in}-\frac{1}{N}\right) \quad , \tag{4.20}$$

and

$$(F) \coloneqq \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(\delta_{lk}-\frac{1}{N}\right)\left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{jn}-\frac{1}{N}\right) \quad . \tag{4.21}$$

In the following we will repeatedly use that

$$\sum_{j=2}^{N}\sum_{i=1}^{j-1} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \quad . \tag{4.22}$$

Further detailing of (A) gives the following terms:

$$\begin{aligned}(A1) \coloneqq \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\delta_{li}\delta_{jm}\delta_{kn} &= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}1\\ &= \frac{N^3}{6}-\frac{N^2}{2}+\frac{N}{3} \quad ,\end{aligned} \tag{4.23}$$

$$\begin{aligned}(A2) &\coloneqq \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)\delta_{li}\delta_{kn}\\ &= \sum_{k=3}^{N}\sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{l=1}^{k-2}\sum_{m=l+1}^{k-1}\left(-\frac{1}{N}\right)\delta_{li}\delta_{kn}\\ &= \sum_{k=3}^{N}\sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{m=i+1}^{k-1}\left(-\frac{1}{N}\right)\\ &= -\frac{N^3}{12}+\frac{N^2}{3}-\frac{5N}{12}+\frac{1}{6} \quad ,\end{aligned} \tag{4.24}$$

$$\begin{aligned}(A3) &\coloneqq \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)\delta_{jm}\delta_{kn}\\ &= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{l=1}^{j-1}\left(-\frac{1}{N}\right)\\ &= -\frac{N^3}{12}+\frac{N^2}{3}-\frac{5N}{12}+\frac{1}{6} \quad ,\end{aligned} \tag{4.25}$$

$$
\begin{aligned}
(A4) &:= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)\delta_{li}\delta_{jm} \\
&= \sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\sum_{i=1}^{j-1}\sum_{n=j+1}^{N}\sum_{l=1}^{j-1}\delta_{li}\left(-\frac{1}{N}\right) \\
&= \sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\sum_{i=1}^{j-1}\sum_{n=j+1}^{N}\left(-\frac{1}{N}\right) \\
&= -\frac{N^3}{12}+\frac{N^2}{3}-\frac{5N}{12}+\frac{1}{6} \quad ,
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
(A5) &:= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)^2\delta_{li} \\
&= \sum_{i=1}^{N-2}\sum_{k=i+2}^{N}\sum_{j=i+1}^{k-1}\sum_{l=1}^{N-2}\sum_{l+2}^{N}\sum_{m=l+1}^{n-1}\frac{1}{N^2}\delta_{li} \\
&= \sum_{i=1}^{N-2}\sum_{k=i+2}^{N}\sum_{j=i+1}^{k-1}\sum_{i+2}^{N}\sum_{m=i+1}^{n-1}\frac{1}{N^2} \\
&= \frac{N^3}{20}-\frac{N^2}{4}+\frac{5N}{12}-\frac{1}{4}+\frac{1}{30N} \quad ,
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
(A6) &:= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)^2\delta_{jm} \\
&= \sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\sum_{i=1}^{j-1}\sum_{n=j+1}^{N}\sum_{l=1}^{j-1}\left(-\frac{1}{N}\right)^2 \\
&= \frac{N^3}{30}-\frac{N^2}{6}+\frac{N}{3}-\frac{1}{3}+\frac{2}{15N} \quad ,
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
(A7) &:= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{n=3}^{N}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)^2\delta_{kn} \\
&= \sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\sum_{m=2}^{k-1}\sum_{l=1}^{m-1}\left(-\frac{1}{N}\right)^2 \\
&= \frac{N^3}{20}-\frac{N^2}{4}+\frac{5N}{12}-\frac{1}{4}+\frac{1}{30N} \quad ,
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
(A8) &:= \sum_{k=3}^{N} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \sum_{n=3}^{N} \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} \left(-\frac{1}{N}\right)^3 \\
&= -\frac{N^3}{36} + \frac{N^2}{6} - \frac{13N}{36} + \frac{1}{3} - \frac{1}{9N} \quad .
\end{aligned}
\tag{4.30}
$$

In total these terms amount to

$$
\begin{aligned}
(A) &= (A1) + (A2) + (A3) + (A4) + (A5) + (A6) + (A7) + (A8) \\
&= \frac{N^3}{45} - \frac{N}{9} + \frac{4}{45N} \quad .
\end{aligned}
\tag{4.31}
$$

In the same way we get the rest of the terms,

$$
(B) = -\frac{N^3}{90} + \frac{N^2}{12} - \frac{N}{9} - \frac{1}{12} + \frac{11}{90N} \quad , \tag{4.32}
$$

$$
(C) = -\frac{N^3}{90} + \frac{2N}{9} - \frac{1}{2} + \frac{13}{45N} \quad , \tag{4.33}
$$

$$
(D) = -\frac{N^3}{90} + \frac{N^2}{12} - \frac{N}{9} - \frac{1}{12} + \frac{11}{90N} \quad , \tag{4.34}
$$

$$
(E) = \frac{N^3}{45} - \frac{N^2}{6} + \frac{5N}{9} - \frac{5}{6} + \frac{19}{45N} \quad , \tag{4.35}
$$

and

$$
(F) = -\frac{N^3}{90} + \frac{2N}{9} - \frac{1}{2} + \frac{13}{45N} \quad . \tag{4.36}
$$

One can check that indeed

$$
\begin{aligned}
&\sum_{k=3}^{N} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \sum_{n=3}^{N} \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} (J_i(J_jJ_k))(z)(J_l(J_mJ_n))(w)|_{\frac{1}{(z-w)^6}} = \\
&\quad = \frac{2N}{3} - 2 + \frac{4}{3N} \\
&\quad = (A) + (B) + (C) + (D) + (E) + (F) \quad .
\end{aligned}
\tag{4.37}
$$

Similar sums appear in the computation of the other OPEs, where we stated the results without giving details of the derivation.

We continue now with the rest of the OPEs. To calculate the highest singularity term in the partial derivative of either $T(z)\tilde{U}_3(w)$ or $\tilde{U}_3(z)T(w)$, we calculate first the OPE (or rather the highest singularity term thereof) starting from (3.13) and (3.17)

$$
\begin{aligned}
T(z)\tilde{U}_3(w) = &-\sum_{i<j}\sum_{l<m<n}(J_iJ_j)(z)(J_l(J_mJ_n))(w) \\
&+\alpha_0\sum_{j}\sum_{l<m<n}(j-1)\left(\partial J_j\right)(z)(J_l(J_mJ_n))(w) \\
&+\alpha_0\sum_{i<j}\sum_{l<m}(l-1)(J_iJ_j)(z)(\partial J_lJ_m)(w) \\
&-\alpha_0^2\sum_{j}\sum_{l<m}(j-1)(l-1)\left(\partial J_j\right)(z)(\partial J_lJ_m)(w) \\
&+\alpha_0\sum_{i<j}\sum_{l<m}(m-2)(J_iJ_j)(z)(J_l\partial J_m)(w) \\
&-\alpha_0^2\sum_{j}\sum_{l<m}(j-1)(m-2)\left(\partial J_j\right)(z)(J_l\partial J_m)(w) \\
&-\alpha_0^2\sum_{i<j}\sum_{l>2}^{N}\binom{l-1}{2}(J_iJ_j)(z)J_l^{(2)}(w) \\
&+\alpha_0^3\sum_{j}\sum_{l>2}^{N}(j-1)\binom{l-1}{2}\left(\partial J_j\right)(z)J_l^{(2)}(w) \quad .
\end{aligned} \tag{4.38}
$$

From that we easily get the highest singularity term of $U_3(z)T(w)$ by means of first switching z and w in $T(z)U_3(w)$ and Taylor expand all currents around w. All that is left to do then is to partially differentiate the highest singularity term.

More concretely, we give a detailed sketch of the necessary calculations in the following: Again, we first calculate the necessary OPEs and then take the sum of the highest singularity terms. This gives us

$$
\begin{aligned}
(J_iJ_j)(z)(J_l(J_mJ_n))(w) &\sim \text{(lower singularity terms)} \\
J_j^{(1)}(z)(J_l(J_mJ_n))(w) &\sim \text{(lower singularity terms)} \\
(J_iJ_j)(z)(J_l^{(1)}J_m)(w) &\sim \frac{2}{(z-w)^5}\left[\left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{jl}-\frac{1}{N}\right)+\left(\delta_{il}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\right] \\
J_j^{(1)}(z)(J_l^{(1)}J_m)(w) &\sim \text{(lower singularity terms)} \\
(J_iJ_j)(z)(J_lJ_m^{(1)})(w) &\sim \frac{2}{(z-w)^5}\left[\left(\delta_{im}-\frac{1}{N}\right)\left(\delta_{jl}-\frac{1}{N}\right)+\left(\delta_{il}-\frac{1}{N}\right)\left(\delta_{jm}-\frac{1}{N}\right)\right] \\
J_j^{(1)}(z)(J_lJ_m^{(1)})(w) &\sim \text{(lower singularity terms)} \\
(J_iJ_j)(z)J_l^{(2)}(w) &\sim \text{(lower singularity terms)} \\
J_j^{(1)}(z)J_l^{(2)}(w) &\sim -\frac{24}{(z-w)^5}\left(\delta_{jl}-\frac{1}{N}\right)
\end{aligned} \tag{4.39}
$$

To get the contribution to the highest singularity term of (4.3) we first partially differentiate by z and then take the respective sums. This amounts to

$$\begin{aligned}
\alpha_0 \sum_{i<j} \sum_{l<m} (l-1) \left[\partial_z (J_i J_j)_{(z)} (\partial J_l J_m)_{(w)} \right] |_{\frac{1}{(z-w)^6}} &= -5\alpha_0 \left[\frac{N^2}{3} - N + \frac{2}{3} \right] \\
\alpha_0 \sum_{i<j} \sum_{l<m} (m-2) \left[\partial_z (J_i J_j)_{(z)} (J_l \partial J_m)_{(w)} \right] |_{\frac{1}{(z-w)^6}} &= -5\alpha_0 \left[\frac{2N^2}{3} - 2N + \frac{4}{3} \right] \\
\alpha_0^3 \sum_j \sum_{l>2}^N (j-1) \binom{l-1}{2} \left[\partial_z J_j^{(1)}(z) J_l^{(2)}(w) \right] |_{\frac{1}{(z-w)^6}} &= -5\alpha_0^3 \left[-N^4 + 2N^3 + N^2 - 2N \right] \quad .
\end{aligned} \tag{4.40}$$

We notice that if for some $a(\alpha_0, N)$,

$$T(z)\tilde{U}_3(w) \sim \frac{a(\alpha_0, N)}{(z-w)^5} + \text{(lower singularity terms)} \quad , \tag{4.41}$$

then

$$\tilde{U}_3(z)T(w) \sim -\frac{a(\alpha_0, N)}{(z-w)^5} + \text{(lower singularity terms)} \quad . \tag{4.42}$$

However, since in the former case we take the derivative of the highest singularity term with respect to z and in the latter case the derivative of the highest singularity term with respect to w, we get in each case the exact same contributing term, i.e.

$$-\frac{5a(\alpha_0, N)}{(z-w)^6} \quad . \tag{4.43}$$

The highest singularity term of (4.7) can easily be found by differentiating the highest singularity term in (3.14):

$$\partial_z \partial_w \frac{\frac{c}{2}}{(z-w)^4} = -\frac{10c}{(z-w)^6} \quad . \tag{4.44}$$

Adding all highest singularity terms multiplied by their respective factor in (4.3) we get

$$\begin{aligned}
\left[W_3(z) W_3(w) \right] |_{\frac{1}{(z-w)^6}} = \alpha_0^4 &\left(\frac{N^5}{6} - \frac{5N^3}{6} + \frac{2N}{3} \right) \\
&+ \alpha_0^2 \left(-\frac{5N^3}{6} + \frac{3N^2}{2} + \frac{4N}{3} - 2 \right) \\
&+ \frac{2N}{3} - 2 + \frac{4}{3N} \quad ,
\end{aligned} \tag{4.45}$$

which is exactly the expected cn_3.

4.2 Expressing W_N -Currents via W_{1+N} -Currents

4.2.1 Transformation between W_{1+N} and W_N

In the following we will attempt to express the currents $\tilde{U}_j^{(-)}$ of the W_N-algebra implicitly defined by the quantum Miura transformation

$$: \prod_{s=1}^{N} \left(\alpha_0 \partial - \tilde{J}_s \right) : = \sum_{j=0}^{N} \tilde{U}_j^{(-)} \left(\alpha_0 \partial \right)^{N-j} \tag{4.46}$$

in terms of the W_{1+N} currents U_i implicitly defined analogously by

$$: \prod_{k=1}^{N} \left(\alpha_0 \partial + J_k \right) : = \sum_{i=0}^{N} U_i \left(\alpha_0 \partial \right)^{N-i} \quad , \tag{4.47}$$

where : ... : denote the usual normal ordering. Notice the slightly different definition and notation in (4.46) when compared to (3.9). This is a matter of convenience. It will become apparent in the following why it is useful to define the currents of spin j slightly different to (3.9). They relate like

$$\tilde{U}_j^{(-)} = \tilde{U}_j \quad , \; j \in \{0, 1\} \quad , \tag{4.48}$$

and

$$\tilde{U}_j^{(-)} = -\tilde{U}_j \quad , \; j \in \mathbb{N}_0 / \{0, 1\} \quad . \tag{4.49}$$

We have to keep that in mind when we express currents in the W_N-algebra in terms of currents in the W_{1+N}-algebra according to (3.41).

If we define

$$J \coloneqq \sum_{j=0}^{N} J_j \tag{4.50}$$

$$-\tilde{J}_i \coloneqq J_i - \frac{1}{N} J \tag{4.51}$$

the necessary condition (3.8) holds.

Furthermore, we define the transformation[1]

$$e^{\alpha \phi} \partial e^{-\alpha \phi} \coloneqq \partial - \alpha J \tag{4.52}$$

for some $\alpha \in \mathbb{R}$ and $J \coloneqq \partial \phi$ as has been done in [Pro19].

From this definition it follows that

[1] Special thanks to Tomáš Procházka who provided this idea!

$$\begin{aligned} e^{\frac{1}{\alpha_0 N}\phi}\left(\alpha_0\partial + J_i\right)e^{-\frac{1}{\alpha_0 N}\phi} &= \alpha_0\partial - \frac{1}{N}J + J_i \\ &= \alpha_0\partial - \tilde{J}_i \quad . \end{aligned} \tag{4.53}$$

By inserting (4.53) into (4.46) and observing that

$$e^{-\frac{1}{\alpha_0 N}\phi}e^{\frac{1}{\alpha_0 N}\phi} = 1 \quad , \tag{4.54}$$

we arrive at an implicit expression of the currents $\tilde{U}_j$ of the W_N-algebra through the U_i currents of the $W_{1+\infty}$-algebra,

$$\begin{aligned} \sum_{j=0}^{N}\tilde{U}_j^{(-)}\left(\alpha_0\partial\right)^{N-j} &= :e^{\frac{1}{\alpha_0 N}\phi}\left(\alpha_0\partial + J_1\right)\left(\alpha_0\partial + J_2\right)\ \dots\ \left(\alpha_0\partial + J_N\right)e^{-\frac{1}{\alpha_0 N}\phi}: \\ &= :e^{\frac{1}{\alpha_0 N}\phi}\sum_{i=0}^{N}U_i\left(\alpha_0\partial\right)^{N-i}e^{\frac{1}{\alpha_0 N}\phi}: \\ &= :\sum_{i=0}^{N}U_i\alpha_0^{N-i}e^{\frac{1}{\alpha_0 N}\phi}\partial^{N-i}e^{-\frac{1}{\alpha_0 N}\phi}: \quad . \end{aligned} \tag{4.55}$$

Using (4.52) we observe that

$$\begin{aligned} e^{\frac{1}{\alpha_0 N}\phi}\partial^k e^{-\frac{1}{\alpha_0 N}\phi} &= \underbrace{e^{\frac{1}{\alpha_0 N}\phi}\partial\underbrace{e^{-\frac{1}{\alpha_0 N}\phi}e^{\frac{1}{\alpha_0 N}\phi}}_{=1}\partial e^{-\frac{1}{\alpha_0 N}\phi}\ \dots e^{\frac{1}{\alpha_0 N}\phi}\partial e^{-\frac{1}{\alpha_0 N}\phi}}_{k-times} \\ &= \left(\partial - \frac{1}{\alpha_0 N}J\right)^k \\ &= \left(\partial - B\right)^k \quad , \end{aligned} \tag{4.56}$$

where

$$B := \frac{1}{\alpha_0 N}J \quad . \tag{4.57}$$

Using

$$\partial^i B = \sum_{j=0}^{i}\binom{i}{j}B^{(j)}\partial^{i-j} \quad , \tag{4.58}$$

the explicit expression for $\left(\partial - B\right)^k$ is suspected to be

$$\left(\partial - B\right)^k \overset{(I.A.)}{=} \partial^k + \sum_{j=0}^{k-1}(-1)^{k-j}\prod_{m=1}^{k-j}\left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\partial^{j-\sum_{n=1}^{k-j}i_n} \quad . \tag{4.59}$$

It is proven by complete induction in the following. For a better understanding of (4.59) we note that another way to write the induction assumption is

$$(\partial - B)^k \overset{(I.A.)}{=} \partial^k + \sum_{j=0}^{k-1} (-1)^{k-j} \sum_{\substack{i_1, i_2, \dots i_{k-j} \\ i_1+i_2+\dots+i_{k-j} \leq j}} \prod_{m=1}^{k-j} \left(\partial^{i_m} B\right) \partial^{j-\sum_{n=1}^{k-j} i_n} \quad . \tag{4.60}$$

Proof. **Base Case:** $k = 1$

For $k = 1$ the r.h.s. of (4.59) amounts to

$$\begin{aligned} \partial^1 + \sum_{j=0}^{0} (-1)^{1-j} \prod_{m=1}^{1-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{j-\sum_{n=1}^{1-j} i_n} &= \partial - \prod_{m=1}^{1} \left(\sum_{i_m=0}^{0-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{0-\sum_{n=1}^{1-j} i_n} \\ &= \partial - \sum_{i_1=0}^{0} \partial^{i_1} B \partial^{0-i_n} \\ &= \partial - B \quad . \end{aligned} \tag{4.61}$$

Since obviously

$$(\partial - B)^k = \partial - B \tag{4.62}$$

for $k = 1$ the Base Case is thus proven.

Induction Step: $k \to k + 1$

The induction assumption (4.59) holds if and only if

$$\begin{aligned} &\partial^{k+1} + \underbrace{\sum_{j=0}^{k} (-1)^{k+1-j} \prod_{m=1}^{k+1-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{j-\sum_{n=1}^{k+1-j} i_n}}_{=:(*)} = \\ &\overset{(I.A.)}{=} \left[\partial^k + \sum_{j=0}^{k-1} (-1)^{k-j} \prod_{m=1}^{k-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{j-\sum_{n=1}^{k-j} i_n} \right] (\partial - B) \\ &= \partial^{k+1} + \underbrace{\sum_{j=0}^{k-1} (-1)^{k-j} \prod_{m=1}^{k-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{j+1-\sum_{n=1}^{k-j} i_n}}_{=:(\triangle)} \end{aligned}$$

$$\underbrace{-\sum_{j=0}^{k-1}(-1)^{k-j}\prod_{m=1}^{k-j}\left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\partial^{j-\sum_{n=1}^{k-j}i_n}B}_{=:(\triangle\triangle)}-\partial^k B \quad .$$

We observe that

$$\begin{aligned}(*) &= \sum_{j=0}^{k}(-1)^{k+1-j}\prod_{m=1}^{k-j}\left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\sum_{i_{k+1-j}=0}^{j-\sum_{n=1}^{k-j}i_n}\partial^{i_{k+1-j}}B\partial^{j-\sum_{n=1}^{k+1-j}i_n} \\ &= \sum_{j=0}^{k}(-1)^{k+1-j}\prod_{m=1}^{k-j}\left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\left[\partial^{j-\sum_{n=1}^{k-j}i_n}B\right. \\ &\quad \left.+\sum_{i_{k+1-j}=0}^{j-1-\sum_{n=1}^{k-j}i_n}\partial^{i_{k+1-j}}B\partial^{j-\sum_{n=1}^{k+1-j}i_n}\right] \quad .\end{aligned} \tag{4.63}$$

Since

$$\partial^{j-\sum_{n=1}^{k+1-j}i_n}=\partial^{j-\sum_{n=1}^{k-j}i_n-i_{k+1-j}}=1 \tag{4.64}$$

for $i_{k+1-j}=j-\sum_{n=1}^{k-j}i_n$.

Moreover,

$$(\triangle\triangle)-\partial^k B=-\sum_{j=0}^{k}(-1)^{k-j}\prod_{m=1}^{k-j}\left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\partial^{j-\sum_{n=1}^{k-j}i_n}B \tag{4.65}$$

which equals the first summand in (4.63).

Lastly, by redefining $j:=j'+1$ the second summand in (4.63) can be rewritten as

$$\sum_{j'=-1}^{k-1}(-1)^{k-j'}\prod_{m=1}^{k-1-j'}\left(\sum_{i_m=0}^{j'+1-\sum_{n=1}^{m-1}i_n}\partial^{i_m}B\right)\sum_{i_{k-j'}=0}^{j'-\sum_{n=1}^{k-1-j'}i_n}\partial^{i_{k-j'}}B\partial^{j'+1-\sum_{n=1}^{k-j'}i_n} \quad , \tag{4.66}$$

where the term for $j'=-1$ cancels since in this case

$$\sum_{i_{k-j'}=0}^{j'-\sum_{n=1}^{k-1-j'}i_n}\partial^{i_{k-j'}}B|_{j'=-1}=\sum_{i_{k-j'}=0}^{<-1}\partial^{i_{k-j'}}B=0 \quad . \tag{4.67}$$

Moreover, all terms where $i_m=j'+1-\sum_{n=1}^{m-1}$ cancel as well since in this case

$$i_m = j' + 1 - \sum_{n=1}^{m-1} \quad \Leftrightarrow \quad \sum_{n=1}^{m} i_n = j' + 1 \quad \text{for} \quad m \leq k - 1 - j' \tag{4.68}$$

and thus

$$\begin{aligned} \sum_{i_{k-j'}=0}^{j'-\sum_{n=1}^{k-1-j'} i_n} \partial^{i_{k-j'}} B\big|_{i_m = j'+1-\sum_{n=1}^{m-1} i_n} &= \sum_{i_{k-j'}}^{j'-(j'+1)} \partial^{i_{k-j'}} B \\ &= \sum_{i_{k-j'}=0}^{<-1} \partial^{i_{k-j'}} B \\ &= 0 \quad . \end{aligned} \tag{4.69}$$

We finish the proof by observing that

$$\begin{aligned} (\triangle) &= \sum_{j=0}^{k-1} (-1)^{k-j} \prod_{m=1}^{k-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \partial^{j+1-\sum_{n=1}^{k-j} i_n} \\ &= \sum_{j=0}^{k-1} (-1)^{k-j} \prod_{m=1}^{k-1-j} \left(\sum_{i_m=0}^{j-\sum_{n=1}^{m-1} i_n} \partial^{i_m} B \right) \sum_{i_{k-j}=0}^{j-\sum_{n=1}^{k-1-j} i_n} \partial^{i_{k-j}} B \partial^{j+1-\sum_{n=1}^{k-j} i_n} \quad . \end{aligned} \tag{4.70}$$

□

4.2.2 Finding the Currents

Now we illustrate how we find an expression of the currents $\tilde{U}_0$, $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ through the currents U_0, U_2, U_2, U_3 and U_4 by making use of

$$\sum_{j=0}^{N} \tilde{U}_j^{(-)} (\alpha_0 \partial)^{N-j} = \sum_{i=0}^{N} U_i \alpha_0^{N-i} (\partial - B)^{N-i} \quad . \tag{4.71}$$

Trivially,

$$\tilde{U}_0 (\alpha_0 \partial)^N = U_0 \alpha_0^N \partial^N \quad , \tag{4.72}$$

since the only contribution on the r.h.s. of (4.71) is the one where $i = 0$ and which is proportional to ∂^N.

Thus

$$\boxed{\tilde{U}_0 = U_0 = \mathbb{1}} \quad . \tag{4.73}$$

To express the currents $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ we need all terms in (4.59) that are linear, quadratic, cubic and quartic in B. We will give them explicitly.

Terms that are *linear* in B.

$$
\begin{aligned}
(\partial - B)^k \,|_{\text{linear in } B} &= -\sum_{i_1=0}^{k-1} \partial^{i_1} B \partial^{k-1-i_1} \\
&= -\sum_{i_1=0}^{k-1} \sum_{j_1=0}^{i_1} \binom{i_1}{j_1} B^{(j_1)} \partial^{k-1-j_1}
\end{aligned} \tag{4.74}
$$

Terms that are *quadratic* in B.

$$
\begin{aligned}
(\partial - B)^k \,|_{\text{quadratic in } B} &= \sum_{i_1=0}^{k-2} \sum_{i_2=0}^{k-2-i_1} \partial^{i_1} B \partial^{i_2} B \partial^{k-2-i_1-i_2} \\
&= \sum_{i_1=0}^{k-2} \sum_{i_2=0}^{k-2-i_1} \sum_{j_1=0}^{i_1} \sum_{j_2=0}^{i_2} \binom{i_1}{j_1} \binom{i_2}{j_2} \sum_{l=0}^{i_1-j_1} \binom{i_1-j_1}{l} \\
&\quad \times B^{(j_1)} B^{(j_2+l)} \partial^{k-2-j_1-j_2-l}
\end{aligned} \tag{4.75}
$$

Terms that are *cubic* in B.

$$
\begin{aligned}
(\partial - B)^k \,|_{\text{cubic in } B} &= -\sum_{i_1=0}^{k-3} \sum_{i_2=0}^{k-3-i_1} \sum_{i_3=0}^{k-3-i_1-i_2} \partial^{i_1} B \partial^{i_2} B \partial^{i_3} B \partial^{k-3-i_1-i_2-i_3} \\
&= -\sum_{i_1=0}^{k-3} \sum_{i_2=0}^{k-3-i_1} \sum_{i_3=0}^{k-3-i_1-i_2} \sum_{j_1=0}^{i_1} \sum_{j_2=0}^{i_2} \sum_{j_3=0}^{i_3} \binom{i_1}{j_1} \binom{i_2}{j_2} \binom{i_3}{j_3} \\
&\quad \times \sum_{l=0}^{i_2-j_2} \sum_{m=0}^{i_1-j_1} \binom{i_2-j_2}{l} \binom{i_1-j_1}{m} \sum_{n=0}^{i_1-j_1-m} \binom{i_1-j_1-m}{n} \\
&\quad \times B^{(j_1)} B^{(j_2+m)} B^{(j_3+l+n)} \partial^{k-3-j_1-j_2-j_3-l-m-n}
\end{aligned} \tag{4.76}
$$

Terms that are *quartic* in B.

$$
\begin{aligned}
&(\partial - B)^k|_{\text{quartic in } B} = \\
&\quad = \sum_{i_1=0}^{k-4} \sum_{i_2=0}^{k-4-i_1} \sum_{i_3=0}^{k-4-i_1-i_2} \sum_{i_4=0}^{k-4-i_1-i_2-i_3} \partial^{i_1} B \partial^{i_2} B \partial^{i_3} B \partial^{i_4} B \partial^{k-4-i_1-i_2-i_3-i_4} \\
&\quad = \sum_{i_1=0}^{k-4} \sum_{i_2=0}^{k-4-i_1} \sum_{i_3=0}^{k-4-i_1-i_2} \sum_{i_4=0}^{k-4-i_1-i_2-i_3} \sum_{j_1=0}^{i_1} \sum_{j_2=0}^{i_2} \sum_{j_3=0}^{i_3} \sum_{j_4=0}^{i_4} \binom{i_1}{j_1}\binom{i_2}{j_2}\binom{i_3}{j_3}\binom{i_4}{j_4} \\
&\qquad \times \sum_{l=0}^{i_3-j_3} \sum_{m=0}^{i_2-j_2} \sum_{\alpha=0}^{i_1-j_1} \binom{i_3-j_3}{l}\binom{i_2-j_2}{m}\binom{i_1-j_2}{\alpha} \\
&\qquad \times \sum_{n=0}^{i_2-j_2-m} \sum_{\beta=0}^{i_1-j_2-\alpha} \binom{i_2-j_2-m}{n}\binom{i_1-j_2-\alpha}{\beta} \sum_{\gamma=0}^{i_1-i_2-\alpha-\beta} \binom{i_1-i_2-\alpha-\beta}{\gamma} \\
&\qquad \times B^{(j_1)} B^{(j_2+\alpha)} B^{(j_3+m+\beta)} B^{(j_4+l+n+\gamma)} \partial^{k-4-j_1-j_2-j_3-j_4-l-m-n-\alpha-\beta-\gamma}
\end{aligned}
\tag{4.77}
$$

We observe that on the r.h.s. of (4.71) terms where $i = 1$ as well as terms where $i = 0$ contribute to $\tilde{U}_1$. For $i = 1$ the only term that contributes is

$$
U_1 \alpha_0^{N-1} \partial^{N-1} \quad . \tag{4.78}
$$

For $i = 0$, i.e. $k = N$, all linear terms in (4.74) for which it holds that

$$
N - 1 - j_1 \stackrel{!}{=} N - 1 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 0
$$

contribute, i.e.

$$
\begin{aligned}
U_0 \alpha_0^N (-1) \sum_{i_1=0}^{N-1} \binom{i_1}{0} B^{(0)} \partial^{N-1} &= -\alpha_0^N N B \partial^{N-1} \\
&= -J \alpha_0^{N-1} \partial^{N-1} \quad ,
\end{aligned}
\tag{4.79}
$$

since $U_1 = J$ and (4.57) holds.
Thus

$$
\boxed{\tilde{U}_1 = 0} \quad . \tag{4.80}
$$

Regarding $\tilde{U}_2$ the following terms on the r.h.s. of (4.71) contribute.
The first one trivially is

$$
U_2 \alpha_0^{N-2} \partial^{N-2} \, . \tag{4.81}
$$

For $i = 1$, i.e. $k = N - 1$, all linear terms in (4.74) for which it holds that

$$N-2-j_1 \stackrel{!}{=} N-2 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 0$$

contribute, i.e.

$$\begin{aligned} U_1\alpha_0^{N-1}(-1)\sum_{i_1=0}^{N-2}\binom{i_1}{0}B^{(0)}\partial^{N-2} &= U_1\alpha_0^{N-1}(-1)(N-1)B\partial^{N-2} \\ &= -\frac{(N-1)}{N}J^2\alpha_0^{N-2}\partial^{N-2} \quad . \end{aligned} \tag{4.82}$$

For $i=0$, i.e. $k=N$, all linear terms in (4.74) for which it holds that

$$N-1-j_1 \stackrel{!}{=} N-2 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 1$$

and all quadratic terms in (4.75) for which it holds that

$$\begin{aligned} N-2-j_1-j_2-l \stackrel{!}{=} N-2 \quad &\Leftrightarrow \quad j_1+j_2+l \stackrel{!}{=} 0 \\ &\Leftrightarrow \quad j_1 \stackrel{!}{=} j_2 \stackrel{!}{=} l \stackrel{!}{=} 0 \end{aligned}$$

contribute, i.e.

$$\begin{aligned} U_0\alpha_0^N(-1)\sum_{i_1=0}^{N-1}\binom{i_1}{1}B^{(1)}\partial^{N-2} &= U_0\alpha_0^N(-1)\frac{N(N-1)}{2}B^{(1)}\partial^{N-2} \\ &= -\frac{(N-1)\alpha_0}{2}J^{(1)}\alpha_0^{N-2}\partial^{N-2} \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} U_0\alpha_0^N\sum_{i_1=0}^{N-2}\sum_{i_2=0}^{N-2-i_1}\binom{i_1}{0}^2\binom{i_2}{0}B^{(0)}B^{(0)}\partial^{N-2} &= U_0\alpha_0^N\frac{N(N-1)}{2}B^2\partial^{N-2} \\ &= \frac{(N-1)}{2N}J^2\alpha_0^{N-2}\partial^{N-2} \quad . \end{aligned} \tag{4.84}$$

Thus

$$\boxed{\tilde{U}_2 = -U_2 + \frac{(N-1)}{2N}J^2 + \frac{(N-1)\alpha_0}{2}J^{(1)}} \quad . \tag{4.85}$$

Regarding $\tilde{U}_3$ the following terms on the r.h.s. of (4.71) contribute.
The first one is

$$U_3\alpha_0^{N-3}\partial^{N-3} \quad . \tag{4.86}$$

For $i=2$, i.e. $k=N-2$, all linear terms in (4.74) for which it holds that

$$N-3-j_1 \stackrel{!}{=} N-3 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 0$$

contribute, i.e.

$$\begin{aligned} U_2\alpha_0^{N-2}(-1)\sum_{i_1=0}^{N-3}\binom{i_1}{0}B^{(0)}\partial^{N-3} &= U_2\alpha_0^{N-2}(-1)(N-2)B^{(0)}\partial^{N-3} \\ &= -\frac{(N-2)}{N}U_2J\alpha_0^{N-3}\partial^{N-3} \quad . \end{aligned} \tag{4.87}$$

For $i=1$, i.e. $k=N-1$, all linear terms in (4.74) for which it holds that

$$N-2-j_1 \stackrel{!}{=} N-3 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 1$$

and all quadratic terms in (4.75) for which it holds that

$$\begin{aligned} N-3-j_1-j_2-l \stackrel{!}{=} N-3 \quad &\Leftrightarrow \quad j_1+j_2+l \stackrel{!}{=} 0 \\ &\Leftrightarrow \quad j_1 \stackrel{!}{=} j_2 \stackrel{!}{=} l \stackrel{!}{=} 0 \end{aligned}$$

contribute, i.e.

$$\begin{aligned} U_1\alpha_0^{N-1}(-1)\sum_{i_1=0}^{N-2}\binom{i_1}{1}B^{(1)}\partial^{N-3} &= U_1\alpha_0^{N-1}(-1)(\frac{(N-1)(N-2)}{2})B^{(1)}\partial^{N-3} \\ &= -\frac{(N-1)(N-2)\alpha_0}{2N}JJ^{(1)}\alpha_0^{N-3}\partial^{N-3} \end{aligned} \tag{4.88}$$

and

$$\begin{aligned} U_1\alpha_0^{N-1}\sum_{i_1=0}^{N-3}\sum_{i_2=0}^{N-3-i_1}\binom{i_1}{0}^2\binom{i_2}{0}B^{(0)}B^{(0)}\partial^{N-3} &= U_1\alpha_0^{N-1}\frac{(N-1)(N-2)}{2}B^{(0)}B^{(0)}\partial^{N-3} \\ &= \frac{(N-1)(N-2)}{2N^2}J^3\alpha_0^{N-3}\partial^{N-3} \quad . \end{aligned} \tag{4.89}$$

For $i=0$, i.e. $k=N$, all linear terms in (4.74) for which it holds that

$$N-1-j_1 \stackrel{!}{=} N-3 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 2 \quad ,$$

all quadratic terms for which it holds that

$$N-2-j_1-j_2-l \overset{!}{=} N-3 \quad \Leftrightarrow \quad j_1+j_2+l \overset{!}{=} 1$$
$$\Rightarrow \quad \text{3 cases:}$$

- $j_1 = 1,\ j_2 = l = 0$
- $j_1 = 0,\ j_2 = 1,\ l = 0$
- $j_1 = j_2 = 0,\ l = 1$

and all cubic terms in (4.76) for which it holds that

$$\begin{aligned} N-3-j_1-j_2-j_3-l-m-n \overset{!}{=} N-3 \quad &\Leftrightarrow \quad j_1+j_2+j_3+l+m+n \overset{!}{=} 0 \\ &\Leftrightarrow \quad j_1 \overset{!}{=} j_2 \overset{!}{=} j_3 \overset{!}{=} l \overset{!}{=} m \overset{!}{=} n \overset{!}{=} 0 \end{aligned}$$

contribute, i.e.

$$\begin{aligned} U_0\alpha_0^N(-1)\sum_{i_1=2}^{N-1}\binom{i_1}{2}B^{(2)}\partial^{N-3} &= U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)}{6}B^{(2)}\partial^{N-3} \\ &= -\frac{(N-1)(N-2)\alpha_0^2}{6}J^{(2)}\alpha_0^{N-3}\partial^{N-3} \quad , \end{aligned} \tag{4.90}$$

$$\begin{aligned} U_0\alpha_0^N\sum_{i_1=1}^{N-2}\sum_{i_2=0}^{N-2-i_1}\binom{i_1}{1}\binom{i_2}{0}\binom{i_1-1}{0}B^{(1)}B^{(0)}\partial^{N-3} &= U_0\alpha_0^N\frac{N(N-1)(N-2)}{6}B^{(0)}B^{(1)}\partial^{N-3} \\ &= \frac{(N-1)(N-2)\alpha_0}{6N}JJ^{(1)}\alpha_0^{N-3}\partial^{N-3} \quad , \end{aligned} \tag{4.91}$$

$$\begin{aligned} U_0\alpha_0^N\sum_{i_1=0}^{N-2}\sum_{i_2=1}^{N-2-i_1}\binom{i_1}{0}\binom{i_2}{1}\binom{i_1}{0}B^{(0)}B^{(1)}\partial^{N-3} &= U_0\alpha_0^N\frac{N(N-1)(N-2)}{6}B^{(0)}B^{(1)}\partial^{N-3} \\ &= \frac{(N-1)(N-2)\alpha_0}{6N}JJ^{(1)}\alpha_0^{N-3}\partial^{N-3} \quad , \end{aligned} \tag{4.92}$$

$$\begin{aligned} U_0\alpha_0^N\sum_{i_1=1}^{N-2}\sum_{i_2=0}^{N-2-i_1}\binom{i_1}{0}\binom{i_2}{0}\binom{i_1}{1}B^{(0)}B^{(1)}\partial^{N-3} &= U_0\alpha_0^N\frac{N(N-1)(N-2)}{6}B^{(0)}B^{(1)}\partial^{N-3} \\ &= \frac{(N-1)(N-2)\alpha_0}{6N}JJ^{(1)}\alpha_0^{N-3}\partial^{N-3} \quad , \end{aligned} \tag{4.93}$$

and

$$\begin{aligned} & U_0\alpha_0^N(-1)\sum_{i_1=0}^{N-3}\sum_{i_2=0}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{0}^3\binom{i_2}{0}^2\binom{i_3}{0}B^{(0)}B^{(0)}B^{(0)}\partial^{N-3} = \\ & = U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)}{6}B^{(0)}B^{(0)}B^{(0)}\partial^{N-3} \\ & = -\frac{(N-1)(N-2)}{6N^2}J^3\alpha_0^{N-3}\partial^{N-3} \quad . \end{aligned} \tag{4.94}$$

Thus

$$\boxed{\tilde{U}_3 = -U_3 + \frac{(N-2)}{N}U_2J - \frac{(N-1)(N-2)}{3N^2}J^3 + \frac{(N-1)(N-2)\alpha_0^2}{6}J^{(2)}} \quad . \tag{4.95}$$

Regarding $\tilde{U}_4$ the following terms on the r.h.s. of (4.71) contribute.
The first on is

$$U_4\alpha_0^{N-4}\partial^{N-4} \quad . \tag{4.96}$$

For $i = 3$, i.e. $k = N-3$, all linear terms in (4.74) for which it holds that

$$N-4-j_1 \stackrel{!}{=} N-4 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 0$$

contribute, i.e.

$$\begin{aligned} U_3\alpha_0^{N-3}(-1)\sum_{i_1=0}^{N-4}\binom{i_1}{0}B^{(0)}\partial^{N-4} &= U_3\alpha_0^{N-3}(-1)(N-3)B^{(0)}\partial^{N-4} \\ &= -\frac{(N-3)}{N}U_3J\alpha_0^{N-4}\partial^{N-4} \quad . \end{aligned} \tag{4.97}$$

For $i = 2$, i.e. $k = N-2$, all linear terms in (4.74) for which it holds that

$$N-3-j_1 \stackrel{!}{=} N-4 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 1$$

and all quadratic terms in (4.75) for which it holds that

$$\begin{aligned} N-4-j_1-j_2-l \stackrel{!}{=} N-4 \quad &\Leftrightarrow \quad j_1+j_2+l \stackrel{!}{=} 0 \\ &\Leftrightarrow \quad j_1 \stackrel{!}{=} j_2 \stackrel{!}{=} l \stackrel{!}{=} 0 \end{aligned}$$

contribute, i.e.

$$
\begin{aligned}
U_2\alpha_0^{N-2}(-1)\sum_{i_1=1}^{N-3}\binom{i_1}{1}B^{(1)}\partial^{N-4} &= U_2\alpha_0^{N-2}(-1)\frac{(N-2)(N-3)}{2}B^{(1)}\partial^{N-4} \\
&= -\frac{(N-2)(N-3)\alpha_0}{2N}U_2J^{(1)}\alpha_0^{N-4}\partial^{N-4}
\end{aligned}
\tag{4.98}
$$

and

$$
\begin{aligned}
U_2\alpha_0^{N-1}\sum_{i_1=0}^{N-4}\sum_{i_2=0}^{N-4-i_1}\binom{i_1}{0}^2\binom{i_2}{0}B^{(0)}B^{(0)}\partial^{N-3} &= U_2\alpha_0^{N-2}\frac{(N-2)(N-3)}{2}B^{(0)}B^{(0)}\partial^{N-4} \\
&= \frac{(N-2)(N-3)}{2N^2}U_2J^2\alpha_0^{N-4}\partial^{N-4} \quad .
\end{aligned}
\tag{4.99}
$$

For $i=1$, i.e. $k=N-1$, all linear terms in (4.74) for which it holds that

$$
N-2-j_1 \stackrel{!}{=} N-4 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 2 \quad ,
$$

all quadratic terms in (4.75) for which it holds that

$$
\begin{aligned}
N-3-j_1-j_2-l \stackrel{!}{=} N-4 \quad &\Leftrightarrow \quad j_1+j_2+l \stackrel{!}{=} 1 \\
&\Rightarrow \quad \text{3 cases:} \quad \bullet\ j_1=1,\ j_2=l=0 \\
&\qquad\qquad\qquad\quad \bullet\ j_1=0,\ j_2=1,\ l=0 \\
&\qquad\qquad\qquad\quad \bullet\ j_1=j_2=0,\ l=1
\end{aligned}
$$

and all cubic terms in (4.76) for which it holds that

$$
\begin{aligned}
N-4-j_1-j_2-j_3-l-m-n \stackrel{!}{=} N-4 \quad &\Leftrightarrow \quad j_1+j_2+j_3+l+m+n \stackrel{!}{=} 0 \\
&\Leftrightarrow \quad j_1 \stackrel{!}{=} j_2 \stackrel{!}{=} j_3 \stackrel{!}{=} l \stackrel{!}{=} m \stackrel{!}{=} n \stackrel{!}{=} 0
\end{aligned}
$$

contribute, i.e.

$$
\begin{aligned}
U_1\alpha_0^{N-1}(-1)\sum_{i_1=2}^{N-2}\binom{i_1}{2}B^{(2)}\partial^{N-4} &= U_1\alpha_0^{N-1}(-1)\frac{(N-1)(N-2)(N-3)}{6}B^{(2)}\partial^{N-4} \\
&= -\frac{(N-1)(N-2)(N-3)\alpha_0^2}{6N}JJ^{(2)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.100}
$$

$$
\begin{aligned}
U_1\alpha_0^{N-1}\sum_{i_1=1}^{N-3}\sum_{i_2=0}^{N-3-i_1}\binom{i_1}{1}\binom{i_2}{0}\binom{i_1-1}{0}B^{(1)}B^{(0)}\partial^{N-4} = \\
&= U_1\alpha_0^{N-1}\frac{(N-1)(N-2)(N-3)}{6}B^{(1)}B^{(0)}\partial^{N-4} \\
&= \frac{(N-1)(N-2)(N-3)\alpha_0}{6N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.101}
$$

$$
\begin{aligned}
U_1\alpha_0^{N-1}\sum_{i_1=0}^{N-3}\sum_{i_2=1}^{N-3-i_1}\binom{i_1}{0}\binom{i_2}{1}\binom{i_1}{0}B^{(0)}B^{(1)}\partial^{N-4} = \\
&= U_1\alpha_0^{N-1}\frac{(N-1)(N-2)(N-3)}{6}B^{(0)}B^{(1)}\partial^{N-4} \\
&= \frac{(N-1)(N-2)(N-3)\alpha_0}{6N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.102}
$$

$$
\begin{aligned}
U_1\alpha_0^{N-1}\sum_{i_1=1}^{N-3}\sum_{i_2=0}^{N-3-i_1}\binom{i_1}{0}\binom{i_2}{0}\binom{i_1}{1}B^{(0)}B^{(1)}\partial^{N-4} = \\
&= U_1\alpha_0^{N-1}\frac{(N-1)(N-2)(N-3)}{6}B^{(0)}B^{(1)}\partial^{N-4} \\
&= \frac{(N-1)(N-2)(N-3)\alpha_0}{6N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}
\end{aligned}
\tag{4.103}
$$

and

$$
\begin{aligned}
U_1\alpha_0^{N-1}(-1)\sum_{i_1=0}^{N-4}\sum_{i_2=0}^{N-4-i_1}\sum_{i_3=0}^{N-4-i_1-i_2}\binom{i_1}{0}^3\binom{i_2}{0}^2\binom{i_3}{0}B^{(0)}B^{(0)}B^{(0)}\partial^{N-4} = \\
&= U_1\alpha_0^{N-1}(-1)\frac{(N-1)(N-2)(N-3)}{6}B^{(0)}B^{(0)}B^{(0)}\partial^{N-4} \\
&= -\frac{(N-1)(N-2)(N-3)}{6N^3}J^4\alpha_0^{N-4}\partial^{N-4} \quad .
\end{aligned}
\tag{4.104}
$$

For $i=0$, i.3. $k=N$, all linear terms in (4.74) for which it holds that

$$
N-1-j_1 \stackrel{!}{=} N-4 \quad \Leftrightarrow \quad j_1 \stackrel{!}{=} 3 \quad ,
$$

all cubic terms in (4.76) for which it holds that

$$
\begin{aligned}
N-2-j_1-j_2-l \stackrel{!}{=} N-4 \quad &\Leftrightarrow \quad j_1+j_2+l \stackrel{!}{=} 2 \\
&\Rightarrow \quad \text{6 cases:} \quad \bullet\ j_1 = 1 = j_2\ ,\ l=0 \\
&\qquad\qquad\qquad\quad \bullet\ j_1 = 1\ ,\ j_2 = 0\ ,\ l=1 \\
&\qquad\qquad\qquad\quad \bullet\ j_1 = 0\ ,\ j_2 = 1 = l
\end{aligned}
$$

- $j_1 = 2$, $j_2 = 0$, $l = 0$
- $j_1 = 0$, $j_2 = 2$, $l = 0$
- $j_1 = 0$, $j_2 = 0$, $l = 2$,

all cubic terms in (4.76) for which it holds that

$$N - 3 - j_1 - j_2 - j_3 - l - m - n \overset{!}{=} N - 4 \quad \Leftrightarrow \quad j_1 + j_2 + j_3 + l + m + n \overset{!}{=} 1$$

$\Rightarrow$ 6 cases:

- $j_1 = 1$, $j_2 = j_3 = l = m = n = 0$
- $j_1 = 0$, $j_2 = 1$, $j_3 = l = m = n = 0$
- $j_1 = j_2 = 0$, $j_3 = 1$, $l = m = n = 0$
- $j_1 = j_2 = j_3 = 0$, $l = 1$, $m = n = 0$
- $j_1 = j_2 = j_3 = l = 0$, $m = 1$, $n = 0$
- $j_1 = j_2 = j_3 = l = m = 0$, $n = 1$

and all quartic terms in (4.75) for which it holds that

$$\begin{aligned} & N - 4 - j_1 - j_2 - j_3 - j_4 - l - m - n - \alpha - \beta - \gamma \overset{!}{=} N - 4 \\ \Leftrightarrow \quad & j_1 + j_2 + j_3 + j_4 + l + m + n + \alpha + \beta + \gamma \overset{!}{=} 0 \\ \Leftrightarrow \, & j_1 \overset{!}{=} j_2 \overset{!}{=} j_3 \overset{!}{=} j_4 \overset{!}{=} l \overset{!}{=} m \overset{!}{=} n \overset{!}{=} \alpha \overset{!}{=} \beta \overset{!}{=} \gamma \overset{!}{=} 0 \end{aligned}$$

contribute, i.e.

$$\begin{aligned} U_0 \alpha_0^N (-1) \sum_{i_1=3}^{N-1} \binom{i_1}{3} B^{(3)} \partial^{N-4} &= U_0 \alpha_0^N (-1) \frac{N(N-1)(N-2)(N-3)}{24} B^{(3)} \partial^{N-4} \\ &= -\frac{(N-1)(N-2)(N-3)\alpha_0^3}{24} J^{(3)} \alpha_0^{N-4} \partial^{N-4} \quad , \end{aligned} \tag{4.105}$$

$$\begin{aligned} & U_0 \alpha_0^N \sum_{i_1=1}^{N-2} \sum_{i_2=1}^{N-2-i_1} \binom{i_1}{1} \binom{i_2}{1} \binom{i_1 - 1}{0} B^{(1)} B^{(1)} \partial^{N-4} = \\ & = U_0 \alpha_0^N \frac{N(N-1)(N-2)(N-3)}{24} B^{(1)} B^{(1)} \partial^{N-4} \\ & = \frac{(N-1)(N-2)(N-3)\alpha_0^2}{24N} J^{(1)} J^{(1)} \alpha_0^{N-4} \partial^{N-4} \quad , \end{aligned} \tag{4.106}$$

$$
\begin{aligned}
U_0\alpha_0^N \sum_{i_1=2}^{N-2}\sum_{i_2=0}^{N-2-i_1} \binom{i_1}{1}\binom{i_2}{0}\binom{i_1-1}{1} B^{(1)}B^{(1)}\partial^{N-4} = \\
= U_0\alpha_0^N \frac{N(N-1)(N-2)(N-3)}{12} B^{(1)}B^{(1)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)\alpha_0^2}{12N} J^{(1)}J^{(1)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.107}
$$

$$
\begin{aligned}
U_0\alpha_0^N \sum_{i_1=1}^{N-2}\sum_{i_2=1}^{N-2-i_1} \binom{i_1}{0}\binom{i_2}{1}\binom{i_1}{1} B^{(0)}B^{(2)}\partial^{N-4} = \\
= U_0\alpha_0^N \frac{N(N-1)(N-2)(N-3)}{24} B^{(0)}B^{(2)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)\alpha_0^2}{24N} JJ^{(2)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.108}
$$

$$
\begin{aligned}
U_0\alpha_0^N \sum_{i_1=2}^{N-2}\sum_{i_2=0}^{N-2-i_1} \binom{i_1}{2}\binom{i_2}{0}\binom{i_1-2}{0} B^{(2)}B^{(0)}\partial^{N-4} = \\
= U_0\alpha_0^N \frac{N(N-1)(N-2)(N-3)}{24} B^{(0)}B^{(2)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)\alpha_0^2}{24N} JJ^{(2)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.109}
$$

$$
\begin{aligned}
U_0\alpha_0^N \sum_{i_1=0}^{N-2}\sum_{i_2=2}^{N-2-i_1} \binom{i_1}{0}\binom{i_2}{2}\binom{i_1}{0} B^{(0)}B^{(2)}\partial^{N-4} = \\
= U_0\alpha_0^N \frac{N(N-1)(N-2)(N-3)}{24} B^{(0)}B^{(2)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)\alpha_0^2}{24N} JJ^{(2)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.110}
$$

$$
\begin{aligned}
U_0\alpha_0^N \sum_{i_1=2}^{N-2}\sum_{i_2=0}^{N-2-i_1} \binom{i_1}{0}\binom{i_2}{0}\binom{i_1}{2} B^{(0)}B^{(2)}\partial^{N-4} = \\
= U_0\alpha_0^N \frac{N(N-1)(N-2)(N-3)}{24} B^{(0)}B^{(2)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)\alpha_0^2}{24N} JJ^{(2)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.111}
$$

$$
\begin{aligned}
&U_0\alpha_0^N(-1)\sum_{i_1=1}^{N-3}\sum_{i_2=0}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{1}\binom{i_1-1}{0}^2\binom{i_2}{0}^2\binom{i_3}{0}B^{(1)}B^{(0)}B^{(0)}\partial^{N-4}=\\
&=U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(1)}B^{(0)}B^{(0)}\partial^{N-4}\\
&=-\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}\quad ,
\end{aligned}
\tag{4.112}
$$

$$
\begin{aligned}
&U_0\alpha_0^N(-1)\sum_{i_1=0}^{N-3}\sum_{i_2=1}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{0}^3\binom{i_2}{1}\binom{i_2-1}{0}\binom{i_3}{0}B^{(0)}B^{(1)}B^{(0)}\partial^{N-4}=\\
&=U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(1)}B^{(0)}\partial^{N-4}\\
&=-\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}\quad ,
\end{aligned}
\tag{4.113}
$$

$$
\begin{aligned}
&U_0\alpha_0^N(-1)\sum_{i_1=0}^{N-3}\sum_{i_2=0}^{N-3-i_1}\sum_{i_3=1}^{N-3-i_1-i_2}\binom{i_1}{0}^3\binom{i_2}{0}^2\binom{i_3}{1}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4}=\\
&=U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4}\\
&=-\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}\quad ,
\end{aligned}
\tag{4.114}
$$

$$
\begin{aligned}
&U_0\alpha_0^N(-1)\sum_{i_1=0}^{N-3}\sum_{i_2=1}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{0}^3\binom{i_2}{0}\binom{i_2}{1}\binom{i_3}{0}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4}=\\
&=U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4}\\
&=-\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}\quad ,
\end{aligned}
\tag{4.115}
$$

$$
\begin{aligned}
&U_0\alpha_0^N(-1)\sum_{i_1=1}^{N-3}\sum_{i_2=0}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{0}\binom{i_1}{1}\binom{i_1-1}{0}\binom{i_2}{0}^2\binom{i_3}{0}B^{(0)}B^{(1)}B^{(0)}\partial^{N-4}=\\
&=U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(1)}B^{(1)}\partial^{N-4}\\
&=-\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4}\quad ,
\end{aligned}
\tag{4.116}
$$

$$
\begin{aligned}
U_0\alpha_0^N(-1)\sum_{i_1=1}^{N-3}\sum_{i_2=0}^{N-3-i_1}\sum_{i_3=0}^{N-3-i_1-i_2}\binom{i_1}{0}^2\binom{i_1}{1}\binom{i_2}{0}^2\binom{i_3}{0}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4} = \\
= U_0\alpha_0^N(-1)\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(0)}B^{(1)}\partial^{N-4} \\
= -\frac{(N-1)(N-2)(N-3)\alpha_0}{24N^2}J^2J^{(1)}\alpha_0^{N-4}\partial^{N-4} \quad ,
\end{aligned}
\tag{4.117}
$$

and

$$
\begin{aligned}
U_0\alpha_0^N\sum_{i_1=0}^{N-4}\sum_{i_2=0}^{N-4-i_1}\sum_{i_3=0}^{N-4-i_1-i_2}\sum_{i_4=0}^{N-4-i_1-i_2-i_3}\binom{i_1}{0}^4\binom{i_2}{0}^3\binom{i_3}{0}^2\binom{i_4}{0}B^{(0)}B^{(0)}B^{(0)}B^{(0)}\partial^{N-4} = \\
= U_0\alpha_0^N\frac{N(N-1)(N-2)(N-3)}{24}B^{(0)}B^{(0)}B^{(0)}B^{(0)}\partial^{N-4} \\
= \frac{(N-1)(N-2)(N-3)}{24N^3}J^4\alpha_0^{N-4}\partial^{N-4} \quad .
\end{aligned}
\tag{4.118}
$$

Thus

$$
\boxed{
\begin{aligned}
\tilde{U}_4 = -U_4 + \frac{(N-3)}{N}U_3J + \frac{(N-2)(N-3)\alpha_0}{2N}U_2J^{(1)} - \frac{(N-2)(N-3)}{2N^2}U_2J^2 \\
+ \frac{(N-1)(N-2)(N-3)\alpha_0^3}{24}J^{(3)} - \frac{(N-1)(N-2)(N-3)\alpha_0^2}{8N}J^{(1)}J^{(1)} \\
- \frac{(N-1)(N-2)(N-3)\alpha_0}{4N^2}J^2J^{(1)} + \frac{(N-1)(N-2)(N-3)}{8N^3}J^4
\end{aligned}
}
\tag{4.119}
$$

4.2.3 Mathematica and Normal Ordering

Special care has to be taken regarding the normal ordering of expressions like U_2J, especially when dealing with a `Mathematica` code. First we notice that the normal ordering of free fields J_i that satisfy the OPE (3.34) is commutative, i.e. an expression of the form

$$
:J_{i_1}^{(n_1)}J_{i_2}^{(n_2)}....J_{i_k}^{(n_k)}: \quad , \tag{4.120}
$$

where the $J_{i_j}^{(n_j)}$ denotes the n_j-th derivative of J_{i_j}, $j \in \{1, 2...k\}$ is invariant under normal ordering. ": ... :" denotes a right-nested normal ordering, which translates to the code

...`NO[...,NO[...,NO[...]]]`... . (4.121)

Concretely, in our computation this means that a term of the form $:U_2J:$ is being read as follows: Any U_k is some expression of normal ordered products of free fields and derivatives thereof. So, if analogous to (4.50) and (3.38),

$$J = \sum_i J_i \quad , \tag{4.122}$$

and

$$U_2 = \sum_{j<k}^{N} a_{jk} J_j J_k + \sum_j b_j J_j^{(1)} \quad , \tag{4.123}$$

with a_{jk} and b_j constants, then the normal ordering $:U_2J:$ translates to the code

$$\begin{aligned} :U_2J: &= \sum_{\mathtt{j<k}}^{\mathtt{N}} \sum_{\mathtt{i}} \mathtt{a_{jk}NO[J_j,NO[J_k,J_i]]} \sum_{\mathtt{j}} \sum_{\mathtt{i}} \mathtt{b_jNO[J_j^{(1)},J_i]} \\ &= \sum_{\mathtt{j<k}}^{\mathtt{N}} \mathtt{a_{jk}NO[J_j,NO[J_k,J]]} \sum_{\mathtt{j}} \mathtt{b_jNO[J_j^{(1)},J]} \quad . \end{aligned} \tag{4.124}$$

This, however is noticeably different from an expression of the form

$$\mathtt{NO[U_2,J]} = \mathtt{NO[}\sum_{\mathtt{j<k}}^{\mathtt{N}} \mathtt{a_{jk}NO[J_j,J_k]} \sum_{\mathtt{j}} \mathtt{b_j,J_k^{(1)},J]} \tag{4.125}$$

and gives different results. Due to the invariance of $:...:$ under the reordering of free fields and the definition of J, we have the equality

$$\mathtt{NO[J_j,NO[J_k,J]]]} = \mathtt{NO[J,NO[J_j,J_j]]} \tag{4.126}$$

and thus

$$\begin{aligned} :U_2J: &= \mathtt{NO[J,} \sum_{\mathtt{j<k}}^{\mathtt{N}} \mathtt{a_{jk}NO[J_j,J_k]} \sum_{\mathtt{j}} \mathtt{b_jJ_k^{(1)}]]} \\ &= \mathtt{NO[J,U_2]} \quad . \end{aligned} \tag{4.127}$$

4.2.4 Mathematica Calculations

With the help of `Mathematica` we are able to calculate the highest singularity terms of $W_2(z)W_2(w)$, $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ and $W_3(z)W_4(w)$ with much less effort than calculating it by hand.

Fortunately, we find the explicit form (up to some typos) of almost all OPEs we need in [Pro15]. The only ones we do not find there are $U_3(z)U_4(w)$ and $U_4(z)U_4(w)$. We will now justify why for the latter one it suffices to calculate the highest singularity term using (3.51), (3.56) and (3.57). Similarly, it can be justified that for the calculations done in this thesis it suffices to calculate singular terms of singularity $\frac{1}{(z-w)^4}$ and higher of the OPEs $U_3(z)U_3(w)$ and $U_3(z)U_4(w)$.

The OPES $U_3(z)U_4(w)$ and $U_4(z)U_4(w)$ both occur in $W_4(z)W_4(w)$. According to its definitions in (3.20) and (3.21) we have

$$\begin{aligned}
W_4(z)W_4(w) &= V_4(z)V_4(w) + c_1V_4(z)\Lambda(w) + c_1\Lambda(z)V_4(w) + c_1^2\Lambda(z)\Lambda(w)\\
&= \tilde{U}_4(z)\tilde{U}_4(w) + c_2\tilde{U}_4(z)\tilde{U}_3^{(1)}(w) + c_3\tilde{U}_4(z)\tilde{U}_2(w)\\
&\quad + c_2\tilde{U}_3^{(1)}(z)\tilde{U}_4(w) + c_2^2\tilde{U}_3^{(1)}(z)\tilde{U}_3^{(1)}(w) + c_2c_3\tilde{U}_3^{(1)}(z)\tilde{U}_2(w)\\
&\quad + c_3\tilde{U}_2(z)\tilde{U}_4(w) + c_2c_3\tilde{U}_3^{(1)}(z)\tilde{U}_2(w) + c_3^2\tilde{U}_2(z)\tilde{U}_2(w)\\
&\quad + c_1\tilde{U}_4(z)\left(\tilde{U}_2\tilde{U}_2\right)(w) - c_1\frac{3}{10}\tilde{U}_4(z)\tilde{U}_2^{(2)}(w) + c_1c_2\tilde{U}_3^{(1)}(z)\left(\tilde{U}_2\tilde{U}_2\right)(w)\\
&\quad - c_1c_2\frac{3}{10}\tilde{U}_3^{(1)}(z)\tilde{U}_2^{(2)}(w) + c_1c_3\tilde{U}_2(z)\left(\tilde{U}_2\tilde{U}_2\right)(w) - c_1c_3\frac{3}{10}\tilde{U}_2(z)\tilde{U}_2^{(2)}(w)\\
&\quad + c_1\left(\tilde{U}_2\tilde{U}_2\right)(z)\tilde{U}_4(w) - c_1\frac{3}{10}\tilde{U}_2^{(2)}(z)\tilde{U}_4(w) + c_1c_2\left(\tilde{U}_2\tilde{U}_2\right)(z)\tilde{U}_3^{(1)}(w)\\
&\quad - c_1c_2\frac{3}{10}\tilde{U}_2^{(2)}(z)\tilde{U}_3^{(1)}(w) + c_1c_3\left(\tilde{U}_2\tilde{U}_2\right)(z)\tilde{U}_2(w) - c_1c_3\frac{3}{10}\tilde{U}_2^{(2)}(z)\tilde{U}_2(w)\\
&\quad + c_1^2\left(\tilde{U}_2\tilde{U}_2\right)(z)\left(\tilde{U}_2\tilde{U}_2\right)(w) - c_1^2\frac{3}{10}\left(\tilde{U}_2\tilde{U}_2\right)(z)\tilde{U}_2^{(2)}(w)\\
&\quad - c_1^2\frac{3}{10}\tilde{U}_2^{(2)}(z)\left(\tilde{U}_2\tilde{U}_2\right)(w) + c_1^2\frac{9}{100}\tilde{U}_2^{(2)}(z)\tilde{U}_2^{(2)}(w)
\end{aligned} \tag{4.128}$$

where

$$c_1 := \frac{(N-2)(N-3)}{2N}\frac{\left[5 - N(5N+7)\alpha_0^2\right]}{\left[17 + 5N - 5N(N^2-1)\alpha_0^2\right]} \quad , \tag{4.129}$$

$$c_2 := -\alpha_0\frac{(N-3)}{2} \quad , \tag{4.130}$$

and

$$c_3 := \alpha_0^2\frac{(N-2)(N-3)}{10} \quad . \tag{4.131}$$

However, when comparing each term with the definitions of $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ in (4.85), (4.95) and (4.119), respectively, we see that the only terms entailing OPEs of the form $U_3(z)U_4(w)$, $U_4(z)U_3(w)$ or $U_4(z)U_4(w)$ are

$$\begin{aligned}
\tilde{U}_4(z)\tilde{U}_4(w) = U_4(z)U_4(w) &- \frac{(N-3)}{N}U_4(z)\left(JU_3\right)(w)\\
&- \frac{(N-3)}{N}\left(JU_3\right)(z)U_4(w) + \ldots \quad ,
\end{aligned} \tag{4.132}$$

$$c_2\tilde{U}_4(z)\tilde{U}_3^{(1)}(w) = \partial_w U_4(z)U_3(w) + \ldots \quad , \tag{4.133}$$

and

$$c_2\tilde{U}_3^{(1)}(z)\tilde{U}_4(w) = \partial_z U_3(z)U_4(w) + ... \quad . \tag{4.134}$$

We also know from the generalized Wick theorem (2.99) that terms of the form $U_4(z)\,(JU_3)\,(w)$ and $(JU_3)\,(z)U_4(w)$ result in expressions proportional to currents and thus do not contribute to the highest singularity term. In conclusion, since we are only interested in the highest singularity term of $W_4(z)W_4(w)$ it suffices to know the highest singularity term of $U_3(z)U_4(w)$ and $U_4(z)U_4(w)$.

The following `Mathematica` code proves (3.23) for $s \in \{2, 3, 4\}$.

4.2.5 Mathematica Code

In this section we present all integral parts of the `Mathematica` Code used for the proofs of (3.23) for $s \in \{2, 3, 4\}$ and (3.27). Notice the slightly different notations in the code compared to the rest of the text of the master thesis, e.g. N is denoted by n, $\tilde{U}_j$ is denoted by Vj and V_4 in (3.20) is denoted by $VV4$.

First, we call the package `OPEdefs_new.m`, more concretely OPEdefs Version 3.1 (beta 4) by Kris Thielemans, which is a `Mathematica` package for OPEs. This package is described in detail in [Thi91] and [Thi95].

```
SetDirectory[NotebookDirectory[]];
«OPEdefs_new.m
?OPEdefsHelp
```

Then, we define the functions (3.59), (3.61) and (3.60) in that order,

```
Czerozero[j_,k_] := ((-1)^(j+1) Gamma[n+1]×Gamma[n](j+k-2)!α0^(j+k-2)) / ((j-1)!(k-1)!Gamma[n-j+1]×Gamma[n-k+1]) ;
× HypergeometricPFQ[{1-j,1-k,3/2+1/(2α0) √(4+α0^2),3/2-1/(2α0) √(4+α0^2)},{2,2-j-k,1-n},1];

pSC[o_,p_,q_,r_] :=If [q<1||r,1,0,(Gamma[o]×Gamma[p])/(Gamma[q]×Gamma[r])
× HypergeometricPFQ[{1-q,1-r,3/2+1/(2α0) √(4+α0^2),3/2-1/(2α0) √(4+α0^2)},{2,1-o,1-p},1];

C0sc[l_,j_,k_] := KroneckerDelta[l,j]×KroneckerDelta[0,k]
+ ((-1)^(j-1) Gamma[n-l+1]×Gamma[n+1])/(Gamma[n-j+1]×Gamma[n-k+1]) × ( Sum_{a=0}^{(l+1)/2} Sum_{b=a-1}^{l-a-1} (-1)^(a+b+1)/Gamma[n-a+1] × (Binomial[l-a-1,a]
× Binomial[l-2a-1,b-1]+Binomial[l-a-1,a-1] × Binomial[l-2a,b-a+1])
× pSC[j+k-l-1,n-a,j-b-1,k-l+b+1]
+ Sum_{a=0}^{(l+1)/2} Sum_{b=a-1}^{l-a} (-1)^(a+b)/Gamma[n-a+1] × (Binomial[l-a-1,a]  ×  Binomial[l-2a-1,b-a]
+ Binomial[l-a,a-1] × Binomial[l-2a+1,b-a+1])
```

$$\times\ \texttt{pSC[j+k-l-2,n-a,j-b-1,k-l+b]}\Big)\texttt{;}$$

before calculating all relevant constants given in chapter 3.2.

The next goal is to define all necessary OPEs which we will later use for our calculations. To do so, we first must define J, U_2, U_3 and U_4 as bosonic currents.

```
Bosonic[U2];
Bosonic[U3];
Bosonic[U4];
```

Then, we define all relevant bilocal fields up to the necessary terms, e.g.

$$\begin{aligned}&\texttt{U01 = J;}\\ &\texttt{U10 = J + (z-w)J' + }\tfrac{\texttt{(z-w)}^2\texttt{J''}}{2}\texttt{;}\\ &\texttt{U11 = NO[J,J + (z-w)NO[J',J]];}\end{aligned}$$

After that we define all OPEs written in chapter 3.2, e.g.

```
OPE[J,J] = MakeOPE[{n One,0}];
```

where the inputs of the `MakeOPE[_,_,_]` function consist of the singular terms of the OPE. For example, a singular term of the form

$$\frac{N}{(z-w)^2} \tag{4.135}$$

appears as

```
n One
```

in the second slot from the right. All other OPEs are defined analogously.

Finally, we are able to define the currents $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ in the W_N-algebra – denoted by `V2`, `V3` and `V4` – in terms of currents in the W_{1+N}-algebra according to (4.85), (4.95) and (4.119).

$$\texttt{V2 = - }\left(\texttt{U2 - }\tfrac{\texttt{(n-1)}}{\texttt{2n}}\texttt{NO[J,J] - }\tfrac{\texttt{(n-1)}\alpha\texttt{0}}{\texttt{2}}\texttt{J'}\right)\texttt{;}$$

$$\texttt{V3 = - }\left(\texttt{U3 - }\tfrac{\texttt{(n-2)}}{\texttt{n}}\texttt{NO[J,U2] + }\tfrac{(n-1)(n-2)}{\texttt{3n}^2}\texttt{NO[J,NO[J,J]] - }\tfrac{\texttt{(n-1)(n-2)}\alpha\texttt{0}^2}{\texttt{6}}\texttt{J''}\right)\texttt{;}$$

$$\begin{aligned}\texttt{V4 = - }\Bigg(&\texttt{U4 - }\tfrac{\texttt{(n-3)}}{\texttt{n}}\texttt{NO[J,U3] - }\tfrac{\texttt{(n-2)(n-3)}\alpha\texttt{0}}{\texttt{2n}}\texttt{NO[J',U2] + }\tfrac{\texttt{(n-2)(n-3}}{\texttt{2n}^2}\texttt{NO[J,NO[J,U2]]}\\ &\texttt{- }\tfrac{\texttt{(n-1)(n-2)(n-3)}\alpha\texttt{0}^3}{\texttt{24}}\texttt{J''' + }\tfrac{\texttt{(n-1)(n-2)(n-3)}\alpha\texttt{0}^2}{\texttt{8n}}\texttt{NO[J',J']}\\ &\texttt{+ }\tfrac{\texttt{(n-1)(n-2)(n-3)}\alpha\texttt{0}}{\texttt{4n}^2}\texttt{NO[J,NO[J,J']] - }\tfrac{(n-1)(n-2)(n-3)}{\texttt{8n}^3}\texttt{NO[J,NO[J,J]]}\Bigg)\texttt{;}\end{aligned}$$

To later check the results we define (3.15), (3.24), (3.25) as well as (3.28).

$$\texttt{c = (n-1)(1-n(n+1)}\alpha\texttt{0}^2\texttt{;}$$
$$\texttt{n3 = }\frac{\texttt{(n-2)(4-n(n+2))}\alpha\texttt{0}^2}{\texttt{6n}}\texttt{;}$$
$$\texttt{n4 = n3}\frac{\texttt{3(n+1)(n-3)(1-n(n-1)}\alpha\texttt{0}^2\texttt{)(9-n(n+3)}\alpha\texttt{0}^2\texttt{)}}{\texttt{2n(17+5n-5n(n}^2\texttt{-1)}\alpha\texttt{0}^2\texttt{)}}\texttt{;}$$
$$\texttt{c334 = 4}\frac{\texttt{n4}}{\texttt{n3}}\texttt{;}$$

What is left to define are the quasi-primary current V_4 from (3.20) – which will be denoted by `VV4` – , Λ from (3.22) and of course the primary currents W_3 and W_4.

$$\texttt{VV4 = V4 - }\alpha\texttt{0}\frac{\texttt{(n-3)}}{\texttt{2}}\texttt{V3' + }\alpha\texttt{0}^2\frac{\texttt{(n-2)(n-3)}}{\texttt{10}}\texttt{V2'';}$$
$$\texttt{W3 = V3 - }\alpha\texttt{0}\frac{\texttt{(n-2)}}{\texttt{2}}\texttt{V2';}$$
$$\texttt{LAM = NO[V2,V2] - }\frac{\texttt{3}}{\texttt{10}}\texttt{V2'';}$$
$$\texttt{W4 = VV4 + }\frac{\texttt{(n-2)(n-3)}}{\texttt{2n}}\times\frac{\texttt{(5-n(5n+7)}\alpha\texttt{0}^2\texttt{)}}{\texttt{17+5n-5n(n}^2\texttt{-1)}\alpha\texttt{0}^2}\texttt{LAM;}$$

Finally, we check the highest singularity terms by using the `OPEPole[...][OPE[...]]` function.

```
OPEPole[4][OPE[V2,V2]] // Simplify
OPEPole[6][OPE[W3,W3]] // Simplify
OPEPole[8][OPE[W4,W4]] // Simplify
OPEPole[7][OPE[W3,W4]] // FullSimplify
OPEPole[6][OPE[W3,W4]] // FullSimplify
OPEPole[5][OPE[W3,W4]] // FullSimplify
OPEPole[7][OPE[W3,W4]] - c334 W3 // FullSimplify
```

These yield the results

$$\texttt{OPEPole[4][OPE[V2,V2]] = }\frac{1}{2}\texttt{One(-1 + n + n}\alpha\texttt{0}^2\texttt{ - n}^3\alpha\texttt{0}^2\texttt{)} \quad , \tag{4.136}$$

$$\texttt{OPEPole[6][OPE[W3,W3]] = }\frac{\texttt{(2-3n+n}^2\texttt{)One(4-6n}\alpha\texttt{0}^2\texttt{+3n}^3\alpha\texttt{0}^4+\texttt{n}\alpha\texttt{0}^4\texttt{+n}^2\alpha\texttt{0}^2\texttt{(-5+2}\alpha\texttt{0}^2\texttt{))}}{\texttt{6n}} \quad , \tag{4.137}$$

and

$$\texttt{OPEPole[8][OPE[W4,W4]] = } -\frac{\texttt{(-6+5n+5n}^2\texttt{-5n}^3\texttt{+n}^4\texttt{)One}}{\texttt{4n}^2\texttt{(-17+5n}^3\alpha\texttt{0}^2\texttt{-5n(1+}\alpha\texttt{0}^2\texttt{))}}$$
$$\times\Big(\texttt{36-30n}\alpha\texttt{0}^2\texttt{+5n}^7\alpha\texttt{0}^8\texttt{+n}^8\alpha\texttt{0}^8\texttt{+5n}^6\alpha\texttt{0}^6\texttt{(-3+}\alpha\texttt{0}^2\texttt{)}$$
$$\texttt{-5n}^5\alpha\texttt{0}^6\texttt{(8+}\alpha\texttt{0}^2\texttt{)+5n}^3\alpha\texttt{0}^4\texttt{(13+6}\alpha\texttt{0}^2\texttt{)-5n}^2\alpha\texttt{0}^2\texttt{(17+6}\alpha\texttt{0}^2\texttt{)+n}^4\texttt{(63}\alpha\texttt{0}^4\texttt{+}\alpha\texttt{0}^6\texttt{-6}\alpha\texttt{0}^8\texttt{)}\Big) \quad . \tag{4.138}$$

These are exactly the expected constants $\frac{c}{2}$, n_3c and n_4c, respectively. Furthermore, as expected we get that

$$\texttt{OPEPole[7][OPE[W3,W4]] = 0} \quad , \tag{4.139}$$

$$\texttt{OPEPole[6][OPE[W3,W4]] = 0} \quad , \tag{4.140}$$

$$\texttt{OPEPole[5][OPE[W3,W4]] = 0} \quad , \tag{4.141}$$

and

$$\texttt{OPEPole[4][OPE[W3,W4]] - c334 W3 = 0} \quad , \tag{4.142}$$

Apart from these results, one can verify by using `Mathematica` that the currents W_3 (`V3`) and W_4 (`V4`) are primary. In each case the commands

```
OPE[V2,W3] // OPEToSeries // Simplify
OPE[V2,W4] // OPEToSeries // Simplify
```

amount to currents of the form (2.74). Additionaly, in the same way one can check that $\tilde{U}_2$ (`V2`), Λ (`LAM`) and V_4 (`VV4`) are quasi-primary currents since in each case the commands

```
OPE[V2,V2] // OPEToSeries // Simplify
OPE[V2,LAM] // OPEToSeries // Simplify
OPE[V2,VV4] // OPEToSeries // Simplify
```

result in a current of the form (2.76).

5 Discussion

In summary, we proved that the explicit forms of n_3, n_4 and c_{34}^3 indeed are those that were expected in [CF24]. In this thesis we listed them in (3.24), (3.25) and (3.28), respectively. Additionally, we have found the formula (4.59) which together with (4.55) expresses any current implicitly defined via the quantum Miura transformation in the W_N-algebra (4.46) through currents implicitly defined via the quantum Miura transformation in the W_{1+N}-algebra (4.47). Using this formula we found expressions for the currents $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$ in the W_N-algebra and thus for expressions for the quasi-primary current V_4 as well as the primary currents W_3 and W_4. Since due to [Pro15] it was possible to find all necessary OPEs between currents in the $W_{1+\infty}$-algebra we were able to prove the form of the highest singularity terms of the OPEs $W_3(z)W_3(w)$, $W_4(z)W_4(w)$ as well as the fact that the highest nonzero singularity term of $W_3(z)W_4(w)$ is $c_{34}^3 W_3$.

The procedure shown in chapter 4.2 could be used for further calculations. Notice that the currents $\tilde{U}_i(U_j)$ that can be found using this method are neither necessarily primary nor quasi-primary. Another fact that has to be kept in mind is that formula (3.51) that is presented in chapter 3.2 is yet to be proven. More precisely, using the Jacobi Identities (3.51) has been shown to hold up to $j + k \leq 15$, see [Pro15]. Nonetheless, any OPE that has been used for the calculations and is written explicitly in chapter 3.2 is correct.

Another thing that must be discussed in this context is the fact that in [Pro15] expressions of W_2, W_3 and W_4 have been found using different means. In short, up to an overall normalization for each dimension (spin) one can find a unique linear combination of fields of the same dimension. This resulting field then has a zero two-point function $C_{j,k}^{0,0}$, see (3.60) for $l = 0$, with all fields that have a lower dimension and with every composite field of the same dimension. Applying the normal ordering condition and normalization that he has, Procházka finds the same exact fields as W_2, W_3 and W_4 that are found in this thesis in chapter 4.2. For more details see [Pro15] chapter 3.9.2.

Additionaly, Procházka provides concrete expressions of the highest singularity terms of the same OPEs we are interested. Those terms are written in dependence of a parameter S_k, where

$$S_k := (N-k)(k^2 - Nk\alpha_0^2 - N^2 k) \quad . \tag{5.1}$$

It turns out that these OPEs coincide with the OPEs found in this thesis. Indeed, since

$$\begin{aligned} S_1 &= (N-1)(1 - N\alpha_0^2 - N^2\alpha_0^2) \\ &= (N-1)\left[1 - N(1+N)\alpha_0^2\right] \\ &= c \quad , \end{aligned} \tag{5.2}$$

obviously

$$W_2(z)W_2(w) \sim \frac{S_1}{2(z-w)^4} + \textit{subleading poles}$$
$$\sim \frac{\frac{c}{2}}{(z-w)^4} + \textit{subleading poles} \quad . \tag{5.3}$$

Additionally, since

$$\begin{aligned} S_2 &= (N-2)(4 - 2N\alpha_0^2 - N^2\alpha_0^2) \\ &= (N-2)\left[4 - N(2+N)\alpha_0^2\right] \end{aligned} \tag{5.4}$$

and

$$n_3 = \frac{S_2}{6N} \quad , \tag{5.5}$$

it can be easily checked that

$$\begin{aligned} W_3(z)W_3(w) &\sim \frac{S_1 S_2}{6N(z-w)^6} + \textit{subleading poles} \\ &= \frac{n_3 c}{(z-w)^6} + \textit{subleading poles} \quad . \end{aligned} \tag{5.6}$$

Finally, since

$$\begin{aligned} S_3 &= (N-3)(9 - 3N\alpha_0^2 - N^2\alpha_0^2) \\ &\sim (N-3)\left[9 - N(3+N)\alpha_0^2\right] \quad , \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} S_{-1} &= (N+1)(1 + N\alpha_0 - N^2\alpha_0^2) \\ &= (N+1)\left[1 + N(1-N)\alpha_0^2\right] \quad , \end{aligned} \tag{5.8}$$

and since

$$n_4 = \frac{S_2 S_3 S_{-1}}{4N^2(17 + 5N - 5N(N^2-1)\alpha_0^2)} \quad , \tag{5.9}$$

as well as

$$\begin{aligned} 4N^2(5c+22) &= 4N^2\left[5(N-1)(1 - N\alpha_0^2 - N^2\alpha_0^2) + 22\right] \\ &= 4N^2\left[5N - 5 - 5N^2\alpha_0^2 + 5N\alpha_0^2 - 5N^3\alpha_0^2 + 5N^2\alpha_0^2 + 22\right] \\ &= 4N^2\left[17 + 5N - 5N(N^2-1)\alpha_0^2\right] \quad , \end{aligned} \tag{5.10}$$

it is indeed the case that

$$
\begin{aligned}
W_4(z)W_4(w) &\sim \frac{S_1 S_2 S_3 S_{-1}}{4N^2(5c+22)(z-w)^8} + \textit{subleading poles} \\
&\sim \frac{n_4 c}{(z-w)^8} + \textit{subleading poles} \quad .
\end{aligned}
\tag{5.11}
$$

In summary, the currents W_3 and W_4 in the W_{1+N}-algebra in [Pro15] coincide exactly with the currents W_3 and W_4 in the W_N-algebra in [CF24] and we also find maching results for the OPEs $W_3(z)W_3(w)$ and $W_4(z)W_4(w)$.

6 Conclusion

In conclusion we have definitively proven that the highest singularity terms of the OPEs $W_3(z)W_3(w)$ and $W_4(z)W_4(w)$ as well as $W_3(z)W_4(w)$ have the same exact form as has been suspected in [CF24].

Furthermore we found a relevant general formula of the connection between the currents $\tilde{U}_j$ of the W_N-algebra and the currents U_i of the W_{1+N}-algebra which are implicitly defined by the quantum Miura transformation. This formula can be used to find expressions beyond the spin-4 current $\tilde{U}_4$ for further calculations of any kind. Since this method strongly depends on the knowledge of an increasing amount of OPEs between currents in the $W_{1+\infty}$-algebra a future algebraic proof of the general formula for these OPEs proposed by Procházka in [Pro15] would be of great value.

Additionally, it would be interesting to explore the relation between the method of finding W_3 and W_4 in [Pro15] as shortly mentioned in the discussion of this thesis and the method described in chapter 4.

Bibliography

[Bai+88] F.A. Bais et al. "Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants". In: *Nuclear Physics B* 304 (1988), pp. 348–370. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(88)90631-1.

[BP09] R. Blumenhagen and E. Plauschinn. *Introduction to Conformal Field Theory: With Applications to String Theory*. Lecture Notes in Physics. Springer Berlin Heidelberg, 2009. ISBN: 9783642004490. DOI: https://doi.org/10.1007/978-3-642-00450-6.

[BPZ84] A.A. Belavin, A.M. Polyakov, and A. B. Zamolodchikov. "Infinite conformal symmetry in two-dimensional quantum field theory". In: *Nuclear Physics B* 241.2 (1984), pp. 333–380. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(84)90052-X.

[BR16] M. Bañados and I. Reyes. "A short review on Noether's theorems, gauge symmetries and boundary terms". In: *International Journal of Modern Physics D* 25.10 (Aug. 2016), p. 1630021. ISSN: 1793-6594. DOI: 10.1142/s0218271816300214.

[BS93] P. Bouwknegt and K. Schoutens. "W symmetry in conformal field theory". In: *Physics Reports* 223.4 (Feb. 1993), pp. 183–276. ISSN: 0370-1573. DOI: 10.1016/0370-1573(93)90111-p.

[CF24] A. Campoleoni and S. Fredenhagen. "Higher-Spin Gauge Theories in Three Spacetime Dimensions". In: *Introductory Lectures on Higher-Spin Theories*. Springer International Publishing, 2024, pp. 121–267. ISBN: 9783031596568. DOI: 10.1007/978-3-031-59656-8_2.

[CFP11] A. Campoleoni, S. Fredenhagen, and S. Pfenninger. "Asymptotic $\mathcal{W}$-symmetries in three-dimensional higher-spin gauge theories". In: *Journal of High Energy Physics* 2011.9 (Sept. 2011). ISSN: 1029-8479. DOI: 10.1007/jhep09(2011)113.

[Dic97] L.A. Dickey. "Lectures on Classical W-Algebras". In: *Acta Appl. Math.* 47.3 (1997), pp. 243–321. DOI: 10.1023/A:1017903416906.

[DMS97] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3, 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9.

[DS85] V.G. Drinfeld and V.V. Sokolov. "Lie algebras and equations of Korteweg-de Vries type". In: *J. Sov. Math.* 30 (1985), pp. 1975–2036. DOI: 10.1007/BF02105860.

[Gab00] M.R. Gaberdiel. "An introduction to conformal field theory". In: *Reports on Progress in Physics* 63.4 (Mar. 2000), pp. 607–667. ISSN: 1361-6633. DOI: 10.1088/0034-4885/63/4/203.

[GG12] M.R. Gaberdiel and R. Gopakumar. "Triality in minimal model holography". In: *Journal of High Energy Physics* 2012.7 (July 2012). ISSN: 1029-8479. DOI: 10.1007/jhep07(2012)127.

[Gib11] G.W. Gibbons. *Anti-de-Sitter spacetime and its uses.* (2011). DOI: https://doi.org/10.48550/arXiv.1110.1206.

[Gin88] Paul Ginsparg. *Applied Conformal Field Theory.* 1988. arXiv: hep-th/9108028 [hep-th].

[LF90] S. Lukyanov and V.A. Fateev. "Additional Symmetries and Exactly-Soluble Models in Two-Dimensional Conformal Field Theory". In: *Soviet Scientific Reviews* 15 (July 1990), pp. 1–117.

[Lin21] A.R. Linshaw. "Universal two-parameter W_∞-algebra and vertex algebras of type $W(2, 3, \dots, \mathrm{N})$". In: *Compositio Mathematica* 157.1 (Jan. 2021), pp. 12–82. ISSN: 1570-5846. DOI: 10.1112/s0010437x20007514.

[Luk88] S.L. Lukyanov. "Quantization of the Gel'fand-Dikii brackets". English (US). In: *Functional Analysis and Its Applications* 22.4 (Oct. 1988), pp. 255–262. ISSN: 0016-2663. DOI: 10.1007/BF01077417.

[Pop91] C.N. Pope. *Lectures on W algebras and W gravity.* 1991. arXiv: hep-th/9112076 [hep-th].

[Pro15] T. Procházka. "Exploring $\mathcal{W}_\infty$ in the quadratic basis". In: *Journal of High Energy Physics* 2015.9 (Sept. 2015). ISSN: 1029-8479. DOI: 10.1007/jhep09(2015)116.

[Pro19] T. Procházka. "Instanton R-matrix and W-symmetry". In: *Journal of High Energy Physics* 2019.12 (Dec. 2019). ISSN: 1029-8479. DOI: 10.1007/jhep12(2019)099.

[PRS90a] C. N. Pope, L. J. Romans, and X. Shen. "W_∞ and the Racah-Wigner Algebra". In: *Nucl. Phys. B* 339 (1990), pp. 191–221. DOI: 10.1016/0550-3213(90)90539-P.

[PRS90b] C. N. Pope, L. J. Romans, and X. Shen. "The Complete Structure of W_∞". In: *Phys. Lett. B* 236 (1990), pp. 173–178. DOI: 10.1016/0370-2693(90)90822-N.

[PRS91] C.N. Pope, L.J. Romans, and X. Shen. "Conditions for anomaly-free W and super-W algebras". In: *Phys. Lett. B* 254 (1991), pp. 401–410. DOI: 10.1016/0370-2693(91)91175-U.

[Tak17] T. Takagi. "A new generalized Wick theorem in conformal field theory". In: *Theoretical and Mathematical Physics* 192.2 (Aug. 2017), pp. 1230–1241. ISSN: 1573-9333. DOI: 10.1134/s0040577917080116.

[Thi91] K. Thielemans. “A Mathematica package for computing operator product expansions”. In: *Int. J. Mod. Phys. C* 2 (1991), pp. 787–798. DOI: `10.1142/S0129183191001001`.

[Thi95] K. Thielemans. *An Algorithmic Approach to Operator Product Expansions, W-Algebras and W-Strings.* (1995). DOI: `https://doi.org/10.48550/arXiv.hep-th/9506159`.

[WZ72] K.G. Wilson and W. Zimmermann. “Operator product expansions and composite field operators in the general framework of quantum field theory”. In: *Commun. Math. Phys.* 24 (1972), pp. 87–106. DOI: `10.1007/BF01878448`.