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Variants of the Haag-Kastler-Axioms and how to connect them

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Abstract

Over the years there have been many iterations and versions of the Haag-Kastler Axioms. Given their role as foundations for algebraic quantum field theory one would like to see the connection between different versions of the Haag-Kastler Axioms. The aim of this thesis is to connect a selection of various Haag-Kastler Axioms in the hope of clearing up possible confusions which may arise when entering this field. The main ingredient in this endeavor will be the inductive limit, a tool in the theory of C^*-algebras. Along the way we will venture in the field of Lorentzian geometry, where we will explore the concept of causal completion, which will prove useful to us in building a bridge between the different versions of the Haag-Kastler Axioms.

Kurzfassung

Im Laufe der Jahre gab es viele Iterationen und Versionen der Haag-Kastler Axiome. Angesichts ihrer Rolle als Grundlagen der algebraischen Quantenfeldtheorie würde man gerne die Verbindung zwischen verschiedenen Versionen der Haag-Kastler Axiome sehen. Das Ziel dieser Arbeit ist es, eine Auswahl verschiedener Haag-Kastler Axiome miteinander zu verbinden, in der Hoffnung, mögliche Verwirrungen aufzuklären, welche beim Betreten dieses Gebiets auftreten können. Der Hauptbestandteil dieses Unterfangens wird der induktive Limes sein, ein Werkzeug in der Theorie der C^*-Algebren. Unterwegs werden wir uns auf dem Gebiet der Lorentzschen Geometrie wagen und das Konzept der kausalen Vervollständigung erforschen, welches sich für uns beim Schlagen einer Brücke zwischen den verschiedenen Versionen des Haag-Kastler Axiome als nützlich erweisen wird.

Contents

Introduction

Algebraic quantum field theory is an axiomatic approach to quantum field theory. It, contrary to quantum field theories built up from the Wightman Axioms, focuses on the algebra of observables rather than the space of states.
This approach has been around for about 60 years at the time of writing this thesis. Its foundation, the Haag-Kastler Axioms, were first introduced in a paper by Rudolf Haag and Daniel Kastler in 1964. In fact the Haag-Kastler Axioms seem to be a translation of the Wightman Axioms to the algebra of observables.
The Haag-Kastler Axioms have seen many iterations and variants over the years. This is somewhat unsettling, since one expects a certain stability from"axioms".

The aim of this thesis is to shine some light on different versions of the Haag-Kastler Axioms (HKA). Specifically we choose three different versions, the original one by Haag and Kastler, by Haag alone and by Klaus Fredenhagen. The goal is to understand the mathematical connection between those three versions. We will not go into detail about the physical justification for the axioms.[1]

The thesis starts in Chapter 1 by introducing the necessary mathematical structures to understand the Haag-Kastler Axioms. Chapter 2 then looks at the three sets of Haag-Kastler Axioms in depth, first separately and then points out the differences and the overlap between the three versions. Following that, we will introduce the main mathematical tool to bridge the gap between the different HKA in Chapter 3 and use it to connect all but one of the axioms. For the last axiom we need to make a quite substantial excursion into Lorentzian geometry in Chapter 4 and then resolve the last part in Chapter 5.

A solid understanding of both the theory of C^*-algebras and Lorentzian geometry is advised, since this thesis' aim is not to introduce these subjects. For the theory of C^*-algebras we point to the first few chapters of [Con00], chapter 4 in [KR83] or the lecture notes on "C^*-algebras with Aspects of Quantum Physic" [Hö24] and for an introduction to Lorentzian geometry the chapters 1-3, 5 and 14 of [O'N83] are recommended.

We hope to clean up some of the confusions one might face when first encountering algebraic quantum field theory and the (different versions of) Haag-Kastler Axioms.

[1] For this we refer to [HK64, Haa96].

1. Preliminaries

This chapter introduces the concepts necessary to understand the Haag-Kastler Axioms in the next chapter.
This will consist predominantly of references to existing definitions and theorems. The purpose is mostly fixing the notation since a deeper understanding of the ideas presented here needs additional study outside the scope of this thesis.
We will first introduce the definitions and theorems coming from the theory of C^*-algebras and then switch to concepts from Lorentzian geometry.

1.1. From C*-algebra theory

Let us start with the basic definitions in the theory of C^*-algebras.

Definition 1.1.1 ($*$-, Banach-, C^*-algebra)
A *normed algebra* $\mathcal{A}$ is a complex algebra with a norm $\|\,.\,\|$ such that $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathcal{A}$ and $\mathcal{A}$.
We call a normed algebra a *Banach algebra* if it is complete under its norm.
A $*$-*(normed/Banach) algebra* is a complex (normed/Banach) algebra with the addition of an *involution*, which means a map $* : \mathcal{A} \to \mathcal{A}$ such that

(i) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$

(ii) $(AB)^* = B^* A^*$

(iii) $A^{**} = (A^*)^* = A$

for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{A}$.
A C^**-algebra* $\mathcal{A}$ is a Banach $*$-algebra fulfilling the C^**-relation*

$$\|A^* A\| = \|A\|^2 \quad \forall A \in \mathcal{A}\,.$$

If a C^*-algebra $\mathcal{A}$ has a unit $\mathbb{1}$ and $\|\mathbb{1}\| = 1$ then we call $\mathcal{A}$ a *unital* C^**-algebra*. [Con00, 1.1, 1.2][Hö24, 0.1, 0.2, 1.1, 1.2]

Definition 1.1.2 ($*$-homo-,$*$-iso-, $*$-automorphism)
A map $\varphi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called $*$-*homomoprhism* if it is an algebra-homomorphism and it is compatible with the involution, i.e., $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{A}$.
φ is called *unital* if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\varphi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$.
A $*$-*isomorphism* is a bijective $*$-homomorphism and a $*$-*automorphism* is a $*$-isomorphism where $\mathcal{B} = \mathcal{A}$. [Con00, p.2]

Remark. Note that $*$-isomorphsims between unital C^*-algebras are automatically unital.

The one theorem in this section we want to introduce is the following. Its main takeaway for us it that injective $*$-homomorphisms are isometric.

Theorem 1.1.3 (Properties of *-homomorphisms)
Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism between C^*-algebras. Then the kernel $\ker \varphi = \{A \in \mathcal{A} \mid \varphi(A) = 0\}$ is a closed self-adjoint ideal in $\mathcal{A}$ and the image $\operatorname{ran} \varphi = \{\varphi(A) \mid A \in \mathcal{A}\}$ is a C^*-subalgebra of $\mathcal{B}$. If φ is injective, then φ is an isometry.[KR83, 4.1.8][Hö24, 2.12]

We will now introduce a fairly simple but more advanced definition, which might not be introduced in a standard course. As a preparation for that we need the following.

Definition 1.1.4 (Directed set)
A *directed set* Λ is a set Λ with a reflexive, transitive relation[1] $\leq$ with the additional property that for any $\alpha, \beta \in \Lambda$ there exists a $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.[Hö24, 7.1]

The last definition in this section is *nets of C^*-algebras*. It is a summary of properties scattered around in [Haa96, ch. III.1].

Definition 1.1.5 (Net of C^*-algebras)
Let Λ be a directed set and $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ be a family of C^*-algebras.
We call $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ a *net of C^*-algebras* if $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ for all $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$.
Moreover, we call $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ a *net of unital C^*-algebras* if in addition every $\mathcal{A}_\alpha$ is a unital C^*-algebra and they all have a common unit, i.e., $\mathbb{1} \in \mathcal{A}_\alpha \, \forall \alpha \in \Lambda$.

A more advanced version of this is found in [BW92, 5.1.8] and it is also part of [BR87, 2.6.3].

Remark. In [Haa96, ch. III.1] the net of C^*-algebras is also denoted by $\alpha \mapsto \mathcal{A}_\alpha$ with $\alpha \in \Lambda$.

1.2. From Lorentzian geometry

We introduce a few definitions on the topic of Lorentzian geometry in this section, which we need to understand the Haag-Kastler Axioms. We will dive more into Lorentzian geometry in a later chapter.

We begin with the central definitions.

Definition 1.2.1 (Space-time)
We call M a *space-time* if it is a time-oriented, connected Lorentzian manifold.

There are some variations of this definition throughout the literature. For example [O'N83] adds that the manifold should be four-dimensional or [Bee96, Min19] add a restriction of the signature.

[1] Sometimes $\leq$ is assumed to be a partial order relation, i.e., additionally antisymmetric. This thesis would also work with a partial order.

A prominent case of a space-time is of course *Minkwoski space*.

Definition 1.2.2 (Minkowski space)
We denote with $\mathbb{R}^{1,d}$ the $d+1$-dimensional *Minkwoski space(-time)* with metric signature $(+,-,-,-)$.

The next two definitions are mostly to fix notation. They make it easier to talk about the causal relation between points or sets.

Definition 1.2.3 (Causal relations)
Let M be a space-time and $p, q \in M$.

1. $p \ll q$ means there exists a future-pointing timelike curve in M from p to q.

2. $p < q$ means there exists a future-pointing causal curve in M from p to q.

3. $p \leq q$ means $p < q$ or $p = q$.

[O'N83, p.402]

Definition 1.2.4 (Timelike/Causal past/future)
Let M be a space-time and $A \subseteq M$ any subset.
We call

$$I^+(A) := \{q \in M \mid \exists p \in A : p \ll q\}$$

the *timelike future* of A and

$$I^-(A) := \{q \in M \mid \exists p \in A : q \ll p\}$$

the *timelike past* of A. Moreover we write

$$I(A) := I^-(A) \cup I^+(A) .$$

Similarly we call

$$J^+(A) := \{q \in M \mid \exists p \in A : p \leq q\}$$

the *causal future* of A. Analogous for $J^-(A)$, the *causal past* of A, and $J(A)$. [O'N83, p.402]

We see, that there is a lot of notation to talk about timelike/causally-separated points/sets. The next two definitions expand this by introducing some notation for *spacelike seperated* sets.

Definition 1.2.5 (spacelike separated)
Let M be a space-time and $V, W \subseteq M$.
We say V and W are *spacelike separated* if every $x \in V$ is spacelike to every $y \in W$, i.e., there exists no causal curve between x and y.

Definition 1.2.6 (Causal complement/completion in Minkowski space)
Let V be any subset of Minkowski space.
The *causal complement* of V is the set of all points which lie spacelike to all points of V. Denoted by V'.
$V'' := (V')'$ is called the *causal completion* of V. V is called *causally complete* if $V'' = V$.[Haa96, III.4.1.1]

Contrary to the rest of the definitions in this section we have introduced the causal completion/complement only for Minkowski space. We will revisit this in Chapter 4.

We conclude this chapter with a very central concept in Lorentzian geometry.

Definition 1.2.7 (Cauchy hypersurface)
Let M be a space-time.
A subset $S \subseteq M$ is called *Cauchy hypersurface of* M if every inextendible timelike curve in M intersects S exactly once.
If $U \subseteq M$ is an open subset, then we call $S \subseteq U \subseteq M$ a *Cauchy hypersurface of the open set* U if S is a Cauchy surface in the above sense of the induced manifold U. [O'N83, 14.28]

2. Haag-Kastler Axioms

With the preliminaries covered, we can start introducing and comparing different sets of Haag-Kastler Axioms (HKA).
We will look at the set of HKA first introduced in a paper by Haag and Kastler in 1964 [HK64], the refined version listed in *Local Quantum Physics* by Haag [Haa96] and the set of HKA given by Fredenhagen in the first chapter of *Advances in Algebraic Quantum Field Theory* [Fre15]. Since the goal is a comparison between different versions, it seemed fitting to include the initial version by Haag and Kastler. The other two sets have been chosen due to their similarity to the original.

There are of course more variants of the Haag-Kastler Axioms out there. For example Horuzhy [Hor90] uses the axioms in the context of von Neumann algebras, an approach first introduced by Araki [Ara99]. Fewster and Verch [FV15] wrote them using category theory and curved spacetimes. An in-depth comparison to those two versions goes beyond the scope of this thesis.

2.1. Different versions of the Haag-Kastler Axioms

We start by stating the respective versions of the HKA and gaining familiarity before we compare them. As stated in the introduction, we do not intend to justify the axioms on a physical basis. Instead this investigation is concerned with mathematical consistency and connection between variants.
The different sets of axioms, as we will state below, are already slightly rewritten and shortened but are still faithful to the mathematical content as we understand them.

2.1.1. HKA by Haag-Kastler (1964)

Let us begin with the original version, the one by Haag and Kastler [HK64].

1. Let $\mathcal{B}$ be the set of open, relatively compact subsets of Minkowksi space. There is an assignment C^*-Algebra $\mathcal{A}(B)$ for every $B \in \mathcal{B}$.

2. Isotony: If $B_1, B_2 \in \mathcal{B}$, such that $B_1 \subseteq B_2$ then $\mathcal{A}(B_1) \subseteq \mathcal{A}(B_2)$. Additionally neither algebra has a unit or both $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$ have the same unit.

3. Local Causality: If $B_1, B_2 \in \mathcal{B}$, such that B_1 and B_2 are spacelike separated, then $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$ commute.

4. The set-theoretic union $\mathcal{A}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{A}(B)$ is a normed * -algebra and its completion $\mathcal{A}$ is a C^*-Algebra. $\mathcal{A}$ is called *algebra of quasilocal observables*[1].

5. The inhomogeneous Lorentz group[2] is represented on $\mathcal{A}$, i.e., there exists a group homomorphism from the inhomogeneous Lorentz group to the group of $*$-automorphisms on $\mathcal{A}$, denoted by $A \mapsto A^L$ for every Lorentz transform L and every $A \in \mathcal{A}$, such that $\mathcal{A}(B)^L = \mathcal{A}(LB)$ for every $B \in \mathcal{B}$.

6. $\mathcal{A}$ is primitive, i.e. if there exists a faithful, algebraically irreducible representation of $\mathcal{A}$.

Remark. If one is familiar with the Wightman Axioms, one might wonder why the fifth axiom does not require continuity of the representation, since the Wightman Axioms state that the representation should be strongly continuous. We cannot do that here because in abstract C^*-algebras we only have the norm-topology and continuity in the norm-topology would be a much too strong requirement. But this is still no issue, since the HKA are not equivalent to the Wightman Axioms without additional conditions, one of which is the strongly continuous, unitary representation of the Poincaré-group in some representation of the C^*-algebra $\mathcal{A}$.[Haa96, p.106, p.110, p.132]
This comment extends to the other two sets of HKA we will go through.

Most of these axioms are very clear statements. The only axiom which needs a comment is the third one. A priori it is not clear whether the algebras $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$ are related in any way and if we can multiply two elements of different algebras. Hence also the notion of commutator makes no sense. But in light of the fourth axiom, we see that $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$ are sub-algebras of $\mathcal{A}(\mathfrak{B})$ and hence we can assume that the commutativity is calculated in $\mathcal{A}(\mathfrak{B})$ (or $\mathcal{A}$ after completion).

There is also a somewhat hidden seventh axiom in [HK64, p.852]. Written up in similar fashion as the others, it reads

7. $\mathcal{A}(B_2) \subseteq \mathcal{A}(B_1)$ if B_2 is contained in the *causal shadow* of B_1.

Unfortunately, a definition of causal shadow is not given in [HK64]. In [Hor90, p.15] the definition of the causal shadow coincides with the definition of the Cauchy development (Definition 4.3.1) for Minkowski space. But since we could not find an earlier definition for the causal shadow and [Hor90] came out significantly later than the paper by Haag and Kastler, it is not clear to us if they used this definition as well.
Also, looking at the HKA in Haag's book [Haa96] (or alternatively in the section below) and compare it with the axiom, which is the most similar to this one, then it suggests that the causal shadow is meant to be the causal completion. So the uncertainty still persists.

[1]The C^*-algebra $\mathcal{A}$ in (4) is often referred to as *quasi-local algebra*. This can lead to confusion since for example [BR87, 2.6.3] defines a much more elaborate object as the quasi-local algebra. This thesis will use the former.

[2]Inhomogeneous Lorentz group is another name for the Poincaré-group.

2.1.2. HKA by Haag (1996)

Next up are the Haag-Kastler Axioms as written by Haag [Haa96, III.1].

1. There is exists a net of C^*-Algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ are the open, finitely extended region in Minkowski space.

2. The Poincaré group is realized by a group of automorphisms $g \mapsto \alpha_g$ s.t. $\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O})$.

3. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike, then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute.

4. If $\mathcal{O}''$ is the causal completion of $\mathcal{O}$, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$.

According to a footnote by Haag, finitely extended means compact closure, i.e. relative compactness.

We can see that, similarly to the analogous axiom by Haag-Kastler in Section 2.1.1, the commutator is again a bit problematic. Here we do not even have an axiom which guarantees the existence of a quasilocal algebra on which the commutator makes sense. In [Haa96, III.1] Haag comments on the existence of a quasilocal algebra $\mathcal{A}$, which supposedly follows from the definition of nets of C^*-algebras. Therefore we assume that the commutator is taken in $\mathcal{A}$. We will address this in more detail in the next chapter.

In the axiom about the Poincaré group we think it is implicitly assumed that the automorphisms are meant to be on the above mentioned quasilocal algebra $\mathcal{A}$. Otherwise we encounter some issues.

The last axiom on this list will be referred to as Causal Completion Axiom (CCA). Note that from the definition of the causal completion (Definition 1.2.6) it is not obvious whether $\mathcal{O}''$ is open or bounded. Therefore it is not clear whether we can even associate an algebra to $\mathcal{O}''$. This will be discussed in the later chapters.

2.1.3. HKA by Fredenhagen (2015)

Lastly, we include the list of HKA as written up by Fredenhagen [Fre15].

1. For every open and bounded region $\mathcal{O}$ in Minkowski space there exists a unital C^*-Algebra $\mathcal{A}(\mathcal{O})$.

2. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$ then there exists an embedding, i.e. unital injective $*$-homomorphism, $i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$, s.t.

$$i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3\,.$$

3. The connected component of the identity in the Poincaré group is represented by $*$-automorphisms α_L with the properties

$$\alpha_L \circ \mathcal{A} = \mathcal{A} \circ L \qquad \text{where } \mathcal{A} : \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \tag{2.1}$$

$$\alpha_L \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{L\mathcal{O}_2 L\mathcal{O}_1} \circ \alpha_L \tag{2.2}$$

$$\alpha_{L_1} \circ \alpha_{L_2} = \alpha_{L_1 L_2} \tag{2.3}$$

4. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated and $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{O}$, then

$$[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0 \quad \forall A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2)\,.$$

5. Time-Slice Axiom (TSA): If $\mathcal{O}_1 \subset \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism.

The third axiom on this list needs to be addressed.
Equation (2.1) states that the image $\alpha_L(\mathcal{A}(\mathcal{O})) = \mathcal{A}(L\mathcal{O})$ for every $\mathcal{O}$. Therefore α_L can not be a $*$-automorphism on every $\mathcal{A}(\mathcal{O})$ (unless $L = id$). Also, the notation suggests that there is only one α_L for every L and since we do not know the domain or codomain of α_L we cannot call α_L a $*$-automorphism.
Note that these are similar problems we would run into Section 2.1.2 if there was no quasilocal algebra in the analogous axiom. This suggests that there should also be a quasilocal algebra on which α_L is a $*$-automorphism but as is stated above the existence is not obvious.

Furthermore this issue extends to Equation (2.2), since the confusion about domains does not make it clear if Equation (2.2) is sensible or not.
Equation (2.3) would be a straightforward statement of the group homomorphism property of representations, yet the domain issue makes it unclear onto which set the map $L \mapsto \alpha_L$ actually maps.
All these issues will be addressed in the next chapter.

2.2. Comparison of the various Haag-Kastler-Axioms

In this section we start to compare the various sets of axioms. To make the comparison easier, at least visually, we adjusted the notation of the three versions above, grouped them into categories and listed them in the following table 2.1.

	HKA by Haag-Kastler	HKA by Haag	HKA by Fredenhagen
Algebra existence	$\mathcal{O}$ open, bound region in Minkowski space, $\mathcal{A}(\mathcal{O})$ associated C^*-algebra	Net of C^*-algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ for $\mathcal{O}$ an open, relatively compact region in Minkowski space	$\mathcal{O}$ open, bounded region in Minkowski space, $\mathcal{A}(\mathcal{O})$ associated *unital* C^*-algebra
Quasi-local algebra	$\mathcal{A}$ is the completion of the set-theoretic union $\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$.		
Isotony	If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. Additionally either no unit or same unit	Included in the definition of net of C^*-algebra	If $\mathcal{O}_1 \subset \mathcal{O}_2$ then there exists an embedding $i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$, s.t. $i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1}$ if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$.
Spacelike separation	If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute.	If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute.	If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated and $\mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{O}$, then $[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0$ $\forall A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2)$.
Symmetries	The Poincaré group is represented by $A \mapsto A^L$ s.t. $\mathcal{A}(\mathcal{O})^L = \mathcal{A}(L\mathcal{O})$.	The Poincaré group is realized by a group of $*$-automorphisms $L \mapsto \alpha_L$ s.t. $\alpha_L(\mathcal{A}(\mathcal{O})) = \mathcal{A}(L\mathcal{O})$.	The component of the identity in the Poincaré group is represented by $*$-automorphisms α_L s.t. $\alpha_L \circ \mathcal{A} = \mathcal{A} \circ L$, $\alpha_L \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{L\mathcal{O}_2 L\mathcal{O}_1} \circ \alpha_L$ and $\alpha_{L_1} \circ \alpha_{L_2} = \alpha_{L_1 L_2}$.
Dynamics	$\mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1)$, if $\mathcal{O}_2$ is contained in the causal shadow of $\mathcal{O}_1$.	CCA: If $\mathcal{O}''$ is the causal completion of $\mathcal{O}$, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$.	TSA: If $\mathcal{O}_1 \subset \mathcal{O}_2$ contains a Cauchy surface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism.
Representation	$\mathcal{A}$ is primitive		

Table 2.1.: Comparison of three different sets of Haag-Kastler Axioms

Let us conclude this chapter by highlighting the differences in these three versions. For the remainder of this thesis we will refer to the HKA by Haag and Kastler as *Version Haag-Kastler (Version HK)*, the HKA by Haag as *Version Haag (Version H)* and the HKA by Fredenhagen as *Version Fredenhagen (Version F)*.

Algebra existence
Looking at the definition of nets of C^*-algebras (Definition 1.1.5), we can see that Version H contains the same information as Version HK, since relatively compact and bounded are equivalent in Minkowsky-space. Version F only differs from the other two in the fact that it uses unital C^*-algebras.

Quasi-local algebra
As mentioned in Section 2.1.2 it is implied that in the Haag-Version, and by similarity in the Fredenhagen-Version, there exists a quasilocal algebra. This is further discussed in the next chapter.

Isotony
Besides that additional condition with the unit, Version HK and Version H are still the same (using Definition 1.1.5).
Version F, in essence, states the same thing, but replaces the subset-embedding with more abstract embeddings.

Spacelike separation
Clearly the Haag-Kastler-Version and Haag-Version say the same thing here. It is implied in both versions that the commutator is taken in the respective quasi-local algebra.
This is exactly why Version F needs more conditions. Since there exists a priori no quasi-local algebra, it requires the technicality of going to $\mathcal{A}(\mathcal{O})$ to be able to take the commutator.

Symmetries
Besides notation, Version HK and Version H are again identical since any group representation on a C^*-algebra (in both cases the respective quasi-local algebra) leads to a group of $*$-automorphisms.
Version F again, tries to deal with the fact that the algebras in this version are not necessarily subsets of each other and there is no explicit quasi-local algebra, where each original algebra is a sub-algebra. The various problems of this attempt to write it down have already been mentioned in Section 2.1.3.

Dynamics
In this category, all three version differ drastically at first sight. All versions deal with what the effect of causality should be on the algebras. Their connection will be explored later on.

Representation

Since the Haag-Kastler-Version is the only one with such an axiom it cannot be compared to the other versions.

Remark. Since the Axiom of Representation is unique in the regard that it has no counterpart in other versions of the HKA, we wanted to at least give some comments.

Maybe the most unintuitive part of the definition of a primitive algebra is the notion of being *algebraically irreducible.* Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{H}$ a Hilbert space. A representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is called algebraically irreducible if $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces of $\mathcal{H}$. A subspace $K \subseteq \mathcal{H}$ is called invariant if $\pi(A)K \subseteq K$ for all $A \in \mathcal{A}$.
If one usually talks about irreducible representation one means *topologically irreducible*, i.e., if $\{0\}$ and $\mathcal{H}$ are the only closed (!) invariant subspaces of $\mathcal{H}$.[HK64, Appendix I]
It is fairly obvious that algebraically irreducible implies topologically irreducible. But it turns out, if $\mathcal{A}$ is a C^*-algebra, then the converse is also true.[Mur90, 5.2.3]

Very often one finds the condition that the algebra of quasi-local observables $\mathcal{A}$ should be *simple*. A simple algebra is an algebra which contains only the trivial ideals $\{0\}$ and $\mathcal{A}$.[HK64, Appendix I] This implies that any non-trivial representation of $\mathcal{A}$ is faithful, since the kernel of any representation is an ideal, hence can only be $\{0\}$.
For C^*-algebras in particular, simplicity then implies primitivity, since every C^*-algebra has at least one irreducible representation.[Mur90, 5.1.12]

Since physically one can only measure the local algebras $\mathcal{A}(\mathcal{O})$, it seems reasonable to avoid conditions on the quasi-local algebra $\mathcal{A}$. Luckily, we can pass the simplicity of $\mathcal{A}$ to the algebras $\mathcal{A}(\mathcal{O})$. If every $\mathcal{A}(\mathcal{O})$ is simple, then $\mathcal{A}$ is simple.[Sak71, 1.23.8]

3. Building a connection

In this chapter we start connecting the different sets of Haag-Kastler Axioms presented in the previous chapter.
Our starting point will be the HKA by Fredenhagen ([Fre15] or Section 2.1.3) as those allow for the most general approach out of the three versions presented in the previous chapter. Each axiom will be stated in a mathematically precise and consistent way, as to avoid the problems mentioned in Section 2.1.3. Then, if possible, the conditions and proofs to go from the Fredenhagen-version of the HKA to the other sets and back are given.

3.1. Inductive Limit

In this section we are going to introduce the mathematical tool, which allows us to connected the different variants of the HKA - the *inductive limit*.
Almost every definition and proof in this section follows chapter 1.23 in [Sak71]. Some notational adaptations have been made and some details have been added.

Theorem 3.1.1 (Existence of inductive limit)
Let Λ be a directed set and $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ be a family of unital C^*-algebras. Suppose for $\alpha, \beta \in \Lambda$, with $\alpha \leq \beta$, there exists an injective unital $*$-homomorphism $\Phi_{\beta,\alpha} : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ such that

$$\Phi_{\gamma,\beta}\Phi_{\beta,\alpha} = \Phi_{\gamma,\alpha} \qquad (\alpha, \beta, \gamma \in \Lambda,\, \alpha \leq \beta \leq \gamma)\,. \tag{3.1}$$

Then there exists a unital C^*-algebra $\mathcal{A}$, a family $\left\{\tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\right\}$ of unital C^*-subalgebras of $\mathcal{A}$ and a family of $*$-isomorphisms $\left\{\Phi_\alpha : \mathcal{A}_\alpha \to \tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\right\}$ such that

(i) $\Phi_\beta\Phi_{\beta,\alpha} = \Phi_\alpha$ if $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$,

(ii) $\tilde{\mathcal{A}}_\alpha \subseteq \tilde{\mathcal{A}}_\beta$ if $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$,

(iii) $\mathcal{A}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \tilde{\mathcal{A}}_\alpha$.

Proof. Let $\mathfrak{F}$ be the set of all maps $x = \{x_\alpha\}_\alpha : \Lambda \to \bigcup_{\alpha\in\Lambda} A_\alpha$, such that $x_\alpha := x(\alpha) \in \mathcal{A}_\alpha$ for all $\alpha \in \Lambda$, which is a $*$-algebra under pointwise addition/multiplication and $(\{x_\alpha\}_\alpha)^* := \{x_\alpha^*\}_\alpha$.

Define

$$\mathfrak{L} := \{x = \{x_\alpha\}_\alpha \in \mathfrak{F} \mid \exists\gamma_x \in \Lambda : x_\alpha = \Phi_{\alpha,\gamma_x}(x_{\gamma_x})\,, \forall\alpha \geq \gamma_x\}\,.$$

We claim that $\mathfrak{L}$ is a unital $*$-algebra:

- If $x, y \in \mathfrak{L}$ then there exists $\gamma_x, \gamma_y \in \Lambda$ such that $x_\alpha = \Phi_{\alpha,\gamma_x}(x_{\gamma_x})\,, \forall \alpha \geq \gamma_x$ and $y_\alpha = \Phi_{\alpha,\gamma_y}(y_{\gamma_y})\,, \forall \alpha \geq \gamma_y$. Take $\gamma \geq \gamma_x, \gamma_y$. We then have $\Phi_{\alpha,\gamma}(x_\gamma) = \Phi_{\alpha,\gamma}(\Phi_{\gamma,\gamma_x}(x_{\gamma_x})) = \Phi_{\alpha,\gamma_x}(x_{\gamma_x}) = x_\alpha\,, \forall \alpha \geq \gamma \geq \gamma_x$ by (3.1) and analogously $y_\alpha = \Phi_{\alpha,\gamma}(y_\gamma)\,, \forall \alpha \geq \gamma$. Therefore $\Phi_{\alpha,\gamma}(x_\gamma + y_\gamma) = \Phi_{\alpha,\gamma}(x_\gamma) + \Phi_{\alpha,\gamma}(y_\gamma) = x_\alpha + y_\alpha\,, \forall \alpha \geq \gamma$ and hence $\{x_\alpha + y_\alpha\}_\alpha \in \mathfrak{L}$.
- Similarly, $\{x_\alpha\}_\alpha, \{y_\alpha\}_\alpha \in \mathfrak{L}$ imply $\{x_\alpha y_\alpha\}_\alpha \in \mathfrak{L}$ with $\{\mathbb{1}_\alpha\}_\alpha$ ($\mathbb{1}_\alpha$ the unit in $\mathcal{A}_\alpha$) being the unit.
- If $x = \{x_\alpha\}_\alpha \in \mathfrak{L}$ then $\{x^*_\alpha\}_\alpha \in \mathfrak{L}$ since $x^*_\alpha = (\Phi_{\alpha,\gamma_x}(x_{\gamma_x}))^* = \Phi_{\alpha,\gamma_x}(x^*_{\gamma_x})\,, \forall \alpha \geq \gamma_x$.
- Similarly, $\{x_\alpha\}_\alpha \in \mathfrak{L}$ and $\lambda \in \mathbb{C}$ imply $\{\lambda x_\alpha\}_\alpha \in \mathfrak{L}$.

Hence, $\mathfrak{L}$ is indeed a unital $*$-algebra.
After defining

$$\|x\| = \|\{x_\alpha\}_\alpha\| := \lim_{\alpha \in \Lambda} \|x_\alpha\|_\alpha \,, \tag{3.2}$$

where $\|\,.\,\|_\alpha$ is the norm in $\mathcal{A}_\alpha$, the next claim is that (3.2) is a well-defined C^*-semi-norm on $\mathfrak{L}$. Indeed, $\|x_\alpha\| = \|\Phi_{\alpha,\gamma_x}(x_{\gamma_x})\| = \|x_{\gamma_x}\|\ \forall \alpha \geq \gamma_x$, by the definition of $\mathfrak{L}$ and Theorem 1.1.3, guarantees convergence. The rest of the properties necessary for $\|\,.\,\|$ to be a C^*-semi-norm follow from the fact that every $\|\,.\,\|_\alpha$ is a C^*-norm and properties of limits. Note that $\|\{x_\alpha\}_\alpha\| = 0$ implies $\|x_\alpha\|_\alpha = 0$ and hence $x_\alpha = 0$ for all $\alpha \geq \gamma_x$.

Next, set

$$\mathfrak{J} := \{\{x_\alpha\}_\alpha \in \mathfrak{L} \mid \|\{x_\alpha\}_\alpha\| = 0\}\,.$$

This is a self-adjoint two-sided ideal of $\mathfrak{L}$:

- $\mathfrak{J}$ being a vector-space follows easily from properties of semi-norms.
- $\|\{x_\alpha\}^*_\alpha\| := \lim_{\alpha\in\Lambda} \|x^*_\alpha\|_\alpha = \lim_{\alpha\in\Lambda} \|x_\alpha\|_\alpha = \|\{x_\alpha\}_\alpha\|$ proves the self-adjointness.
- If $\{x_\alpha\}_\alpha \in \mathfrak{J}$ and $\{y_\alpha\}_\alpha \in \mathfrak{L}$, then $\|\{x_\alpha y_\alpha\}_\alpha\| \leq \|\{x_\alpha\}_\alpha\|\,\|\{y_\alpha\}_\alpha\| = 0$. Analogously for $\|\{y_\alpha x_\alpha\}_\alpha\|$.

Hence, $\|x + \mathfrak{J}\| := \|x\|$ defines a C^*-norm on the $*$-algebra $\mathfrak{L}/\mathfrak{J}$.

Define $\mathcal{A}$ as the completion of $\mathfrak{L}/\mathfrak{J}$ with respect to $\|\,.\,\|$. Set

$$\mathfrak{L}_\beta := \{\{x_\alpha\}_\alpha \in \mathfrak{L} \mid x_\alpha = \Phi_{\alpha,\beta}(x_\beta)\,\forall \alpha \geq \beta\}\,.$$

Similar calculations as in the claim that $\mathfrak{L}$ is a unital $*$-algebra show that $\mathfrak{L}_\beta$ is a unital $*$-subalgebra of $\mathfrak{L}$ for all $\beta \in \Lambda$.
Define a map $\mathcal{A}_\beta \to \mathfrak{L}_\beta$ for all $\beta \in \Lambda$ by

$$x_\beta \mapsto \{x_{x_\beta,\alpha}\}_\alpha := \begin{cases} \Phi_{\alpha,\beta}(x_\beta) & , \quad \alpha \geq \beta\,, \\ 0 & , \quad \alpha \not\geq \beta \end{cases} \,.$$

Such a map induces a $*$-isomorphism $\Phi_\beta : \mathcal{A}_\beta \to (\mathfrak{L}_\beta + \mathfrak{J})/\mathfrak{J} =: \tilde{\mathcal{A}}_\beta$, $x_\beta \mapsto \{x_{x_\beta,\alpha}\}_\alpha + \mathfrak{J}$:

- Φ_β being a $*$-homomorphism follows from the definition of $\{x_{x_\beta,\alpha}\}_\alpha$ and $\Phi_{\alpha,\beta}$ being a $*$-homomorphisms.
- If $\{x_{x_\beta,\alpha}\}_\alpha - \{y_{y_\beta,\alpha}\}_\alpha \in \mathfrak{J}$ then there exists a $\gamma \in \Lambda$ such that $x_{x_\beta,\alpha} - y_{y_\beta,\alpha} = 0, \forall \alpha \geq \gamma$, by the definition of $\mathfrak{J}$ and (3.2). Injectivity of $\Phi_{\alpha,\beta}$ then implies $x_\beta = y_\beta$. Hence, Φ_β is injective.
- Any element in $\tilde{A}_\beta = (\mathfrak{L}_\beta + \mathfrak{J})/\mathfrak{J}$ has a representative in the form of $\{x_\alpha\}_\alpha + \mathfrak{J}$ with $\{x_\alpha\}_\alpha \in \mathfrak{L}_\beta$. By the definition of $\mathfrak{L}_\beta$ we have that $x_\alpha = \Phi_{\alpha,\beta}(x_\beta) = x_{x_\beta,\alpha}, \forall \alpha \geq \beta$ which yields $\{x_\alpha\}_\alpha - \{x_{x_\beta,\alpha}\}_\alpha \in \mathfrak{J}$, hence $\Phi_\beta(x_\beta) = \{x_\alpha\}_\alpha$ and Φ_β is surjective.

Note that $\tilde{\mathcal{A}}_\beta$ is a unital normed $*$-algebra and Φ_β being a $*$-isomorphism guarantees completeness, hence $\tilde{\mathcal{A}}_\beta$ is a unital C^*-algebra.

Finally, we can show the three properties in the statement.

(i) Take $\beta, \gamma \in \Lambda$ with $\gamma \geq \beta$ and $x_\beta \in \mathcal{A}_\beta$. On the one hand, we get $\Phi_\gamma \Phi_{\gamma,\beta}(x_\beta) = \{x_{x_\gamma,\alpha}\}_\alpha + \mathfrak{J}$, where $x_\gamma := \Phi_{\gamma,\beta}(x_\beta)$. The defining property of $\{x_{x_\gamma,\alpha}\}_\alpha$ yields $x_{x_\gamma,\alpha} = \Phi_{\alpha,\gamma}(x_\gamma) = \Phi_{\alpha,\gamma}(\Phi_{\gamma,\beta}(x_\beta)) = \Phi_{\alpha,\beta}(x_\beta), \forall \alpha \geq \gamma$, using (3.1).
On the other hand, $\Phi_\beta(x_\beta) = \{x'_{x_\beta,\alpha}\}_\alpha + \mathfrak{J}$, where $x'_{x_\beta,\alpha} = \Phi_{\alpha,\beta}(x_\beta), \forall \alpha \geq \beta$. Hence, for all $\alpha \geq \gamma \geq \beta$, we have $x'_{x_\beta,\alpha} = \Phi_{\alpha,\beta}(x_\beta) = x_{x_\gamma,\alpha}$. This implies $\{x_{x_\gamma,\alpha}\}_\alpha - \{x'_{x_\beta,\alpha}\}_\alpha \in \mathfrak{J}$ and therefore $\Phi_\gamma \Phi_{\gamma,\beta}(x_\beta) = \Phi_\beta(x_\beta)$.

(ii) $\mathfrak{L}_\alpha \subseteq \mathfrak{L}_\beta$ if $\alpha \leq \beta$, hence $\tilde{\mathcal{A}}_\alpha \subseteq \tilde{\mathcal{A}}_\beta$.

(iii) Clearly $\bigcup_{\alpha\in\Lambda} \mathfrak{L}_\alpha = \mathfrak{L}$ which implies $\bigcup_{\alpha\in\Lambda} \tilde{\mathcal{A}}_\alpha = \mathfrak{L}/\mathfrak{J}$, hence $\mathcal{A}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \tilde{\mathcal{A}}_\alpha$.

□

Definition 3.1.2 (Inductive Limit)
Let Λ be a directed set and $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ be a family of unital C^*-algebras. Suppose for $\alpha, \beta \in \Lambda$, with $\alpha \leq \beta$, there exists an injective unital $*$-homomorphism $\Phi_{\beta,\alpha} : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ such that

$$\Phi_{\gamma,\beta}\Phi_{\beta,\alpha} = \Phi_{\gamma,\alpha} \qquad (\alpha, \beta, \gamma \in \Lambda,\ \alpha \leq \beta \leq \gamma).$$

The C^*-algebra $\mathcal{A}$ constructed in (the proof of) Theorem 3.1.1, which we denote by

$$\lim_{\alpha\in\Lambda} \{\mathcal{A}_\alpha; \Phi_{\beta,\alpha} \mid (\beta,\alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\} := \mathcal{A},$$

is called the *inductive limit* of $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ defined by the family of injective unital $*$-homomorphisms $\{\Phi_{\beta,\alpha}\}$.[Sak71, 1.23.1]

As we will show shortly, the inductive limit is unique up to $*$-isomorphism under certain assumptions. This then justifies the introduction of the notation in Definition 3.1.2. Usually one would introduce the definition after the uniqueness result, but we decided to introduce the notation at this point to make it easier to reference the inductive limit going forward.

Next let us show the connection between a directed set of subalgebras and the inductive limit.

Corollary 3.1.3 (Inductive limit of subalgebras)
Let $\mathcal{A}$ be a unital C^*-algebra, Λ a directed set and $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ a family of unital C^*-subalgebras of $\mathcal{A}$ such that $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ for $\alpha, \beta \in \Lambda$ with $\beta \geq \alpha$ and $\mathcal{A}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha$.
Then

$$\mathcal{A} \cong \lim_{\alpha\in\Lambda} \{\mathcal{A}_\alpha; i_{\beta,\alpha} \mid (\beta, \alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\} ,$$

where $i_{\beta,\alpha} : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ is the subalgbra embedding.[Sak71, 1.23.3]

Remark. The statement is justified, since $i_{\gamma,\beta}\, i_{\beta,\alpha} = i_{\gamma,\alpha}$ with $\alpha, \beta, \gamma \in \Lambda$, $\alpha \leq \beta \leq \gamma$ for the subalgebra embeddings due to $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$.

Proof. By Theorem 3.1.1 we know that there exists $\{\tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\}$ and $\{i_\alpha : \mathcal{A}_\alpha \to \tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\}$ such that

$$i_\beta\, i_{\beta,\alpha} = i_\alpha \tag{3.3}$$

for $\alpha, \beta \in \Lambda$ with $\beta \geq \alpha$ and $\lim_{\alpha\in\Lambda}\{\mathcal{A}_\alpha; i_{\beta,\alpha} \mid (\beta, \alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \tilde{\mathcal{A}}_\alpha$.
The map $i_\alpha : \mathcal{A}_\alpha \to \tilde{\mathcal{A}}_\alpha$ is a $*$-isomorphism and if we omit the subalgebra embedding $i_{\beta,\alpha}$ in (3.3), we get $i_\beta = i_\alpha$ on $\mathcal{A}_\alpha$ for all $\beta \geq \alpha$.
Therefore, we have the well-defined $*$-isomorphism $i : \mathcal{A} \to \lim_{\alpha\in\Lambda}\{\mathcal{A}_\alpha; i_{\beta,\alpha} \mid (\beta, \alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$ given by $i = i_\alpha$ on $\mathcal{A}_\alpha$ for all $\alpha \in \Lambda$. □

We will finish this section by showing how to construct $*$-isomorphisms between inductive limits. This will also imply the uniqueness of the inductive limit.

Proposition 3.1.4 (Isomorphism between inductive limits)
Let $\mathcal{A} = \lim_{\alpha\in\Lambda}\{\mathcal{A}_\alpha; \Phi_{\beta,\alpha} \mid (\beta, \alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$ and
$\mathcal{B} = \lim_{\alpha\in\Lambda}\{\mathcal{B}_\alpha; \Psi_{\beta,\alpha} \mid (\beta, \alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$. Suppose that there exists a $*$-isomorphism $\varphi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ for every $\alpha \in \Lambda$ such that

$$\varphi_\beta \Phi_{\beta,\alpha} = \Psi_{\beta,\alpha} \varphi_\alpha \qquad \forall \alpha, \beta \in \Lambda \text{ with } \beta \geq \alpha\,. \tag{3.4}$$

Then there exists a unique $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi|_{\tilde{\mathcal{A}}_\alpha} = \Psi_\alpha \varphi_\alpha \Phi_\alpha^{-1}\, \forall \alpha \in \Lambda$, where Φ_α and Ψ_α come from Theorem 3.1.1 for $\mathcal{A}$ and $\mathcal{B}$ respectively.[Sak71, 1.23.4]

Remark. This proof is an extension of the proof in Corollary 3.1.3.

Proof. By Theorem 3.1.1 we know that there exists $\{\tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\}$ and $\{\Phi_\alpha \mid \alpha \in \Lambda\}$ such that $\Phi_\beta \Phi_{\beta,\alpha} = \Phi_\alpha$ for $\alpha, \beta \in \Lambda$ with $\beta \geq \alpha$ (respectively $\{\tilde{\mathcal{B}}_\alpha \mid \alpha \in \Lambda\}$,$\{\Psi_\alpha \mid \alpha \in \Lambda\}$).

Then the map $\Psi_\alpha \varphi_\alpha \Phi_\alpha^{-1} : \tilde{\mathcal{A}}_\alpha \to \tilde{\mathcal{B}}_\alpha$ is a $*$-isomorphism as a composition of three $*$-isomorphisms.
Let $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ and $x_\alpha \in \tilde{\mathcal{A}}_\alpha$. We have

$$\begin{aligned}\Psi_\beta \varphi_\beta \Phi_\beta^{-1}(x_\alpha) =& \Psi_\beta \varphi_\beta \Phi_\beta^{-1} \Phi_\alpha \Phi_\alpha^{-1}(x_\alpha) \overset{3.1.1\text{ i)}}{=} \Psi_\beta \varphi_\beta \Phi_{\beta,\alpha} \Phi_\alpha^{-1}(x_\alpha) \overset{(3.4)}{=} \Psi_\beta \Psi_{\beta,\alpha} \varphi_\alpha \Phi_\alpha^{-1}(x_\alpha) \\ \overset{3.1.1\text{ i)}}{=}& \Psi_\alpha \varphi_\alpha \Phi_\alpha^{-1}(x_\alpha)\,.\end{aligned}$$

So $\Psi_\beta \varphi_\beta \Phi_\beta^{-1} = \Psi_\alpha \varphi_\alpha \Phi_\alpha^{-1}$ on $\tilde{\mathcal{A}}_\alpha$ for all $\beta \geq \alpha$.
Therefore we have the well-defined $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ given by $\varphi = \Psi_\alpha \varphi_\alpha \Phi_\alpha^{-1}$ on $\tilde{\mathcal{A}}_\alpha$ for all $\alpha \in \Lambda$. □

Corollary 3.1.5 (Uniqueness of inductive limits)
The inductive limit $\mathcal{A} = \lim_{\alpha\in\Lambda}\{\mathcal{A}_\alpha; \Phi_{\beta,\alpha} \mid (\beta,\alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$ is unique up to $*$-isomorphism for fixed $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ and $\{\Phi_{\beta,\alpha} | \alpha, \beta \in \Lambda, \alpha \leq \beta\}$ (satisfying the hypothesis of Theorem 3.1.1).
More precisely, for any unital C^*-algebra $\mathcal{B}$ with a family of unital C^*-subalgebras $\left\{\tilde{\mathcal{B}}_\alpha \mid \alpha \in \Lambda\right\}$ and a family of $*$-isomorphisms $\left\{\Psi_\alpha : \mathcal{A}_\alpha \to \tilde{\mathcal{B}}_\alpha \mid \alpha \in \Lambda\right\}$ such that

$$\Psi_\beta \Phi_{\beta,\alpha} = \Psi_\alpha \tag{3.5}$$

and $\tilde{\mathcal{B}}_\alpha \subseteq \tilde{\mathcal{B}}_\beta$ for $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ and $\mathcal{B}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \tilde{\mathcal{B}}_\alpha$, i.e. the conclusion of Theorem 3.1.1, we get $\mathcal{A} \cong \mathcal{B}$.

Proof. Assume $\mathcal{B}$, $\left\{\tilde{\mathcal{B}}_\alpha \mid \alpha \in \Lambda\right\}$ and $\left\{\Psi_\alpha : \mathcal{A}_\alpha \to \tilde{\mathcal{B}}_\alpha \mid \alpha \in \Lambda\right\}$ are as in the hypothesis.
By Corollary 3.1.3 we know $\mathcal{B} \cong \lim_{\alpha\in\Lambda}\{\tilde{\mathcal{B}}_\alpha; i_{\beta,\alpha} \mid (\beta,\alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\}$, where
$i_{\beta,\alpha} : \tilde{\mathcal{B}}_\alpha \to \tilde{\mathcal{B}}_\beta$ is the subalgebra embedding.

Since Ψ_α fulfills

$$\Psi_\beta \Phi_{\beta,\alpha} = i_{\beta,\alpha} \Psi_\alpha\,,$$

where we just added the subalgebra embedding to (3.5), we can apply Proposition 3.1.4, which yields

$$\mathcal{A} \cong \lim_{\alpha\in\Lambda}\{\tilde{\mathcal{B}}_\alpha; i_{\beta,\alpha} \mid (\beta,\alpha) \in \Lambda \times \Lambda, \beta \geq \alpha\} \cong \mathcal{B}\,.$$

□

Remark. This also shows that $\left\{\tilde{\mathcal{A}}_\alpha \mid \alpha \in \Lambda\right\}$ is unique up to $*$-isomorphisms.

3.2. Net of C*-algebras

With the first mathematical tool in place we can now start building the connections between the different HKA.

First a bit of notation.

Definition 3.2.1 (Set of open, relatively compact subsets)
Let M be a manifold. Then we denote

$$\mathfrak{B}(M) := \{\mathcal{O} \subseteq M \mid \mathcal{O} \text{ open and relatively compact}\}.$$

Moreover, we write the shorthand $\mathfrak{B} := \mathfrak{B}(\mathbb{R}^{1,d})$, where $\mathbb{R}^{1,d}$ is the $d+1$-dimensional Minkwoski space.

Remark. (i) For our purposes the dimension of Minkowski space does not matter, which is why we omit a distinction in the notation of $\mathfrak{B}$.

(ii) As mentioned in the previous chapter, the properties of subsets of being relatively compact and bounded are equivalent in Minkwoski space.

(iii) Note that $\mathfrak{B}(M)$ is a directed set by inclusion.

The first three axioms, the Axiom of Algebra existence, Quasi-local algebra and Isotony, all deal with the basic object in the HKA. Without those, talking about the other axioms makes little sense. This is also indicated in definition of nets of C^*-algebras (1.1.5). So we will tackle them together.
Luckily for us, we already did the heavy lifting in the previous section.

If we take another look at Theorem 3.1.1, we see that the family of unital C^*-(sub)-algebras $\left\{\tilde{\mathcal{A}_\alpha} \mid \alpha \in \Lambda\right\}$ fulfill Definition 1.1.5. This motivates the next definition.

Definition 3.2.2 (Induced net of C^*-algebras)
Let Λ be a directed set and $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ be a family of unital C^*-algebras. Suppose for $\alpha, \beta \in \Lambda$, with $\alpha \leq \beta$, there exists an injective unital $*$-homomorphism $\Phi_{\beta,\alpha} : \mathcal{A}_\alpha \to \mathcal{A}_\beta$ such that

$$\Phi_{\gamma,\beta}\Phi_{\beta,\alpha} = \Phi_{\gamma,\alpha} \qquad (\alpha, \beta, \gamma \in \Lambda,\, \alpha \leq \beta \leq \gamma).$$

The family of unital C^*-algebras $\left\{\tilde{\mathcal{A}_\alpha} \mid \alpha \in \Lambda\right\}$, constructed in (the proof of) Theorem 3.1.1, is called the *induced net of C^*-algebras* of $\{\mathcal{A}_\alpha | \alpha \in \Lambda\}$ defined by the family of injective unital $*$-homomorphisms $\{\Phi_{\beta,\alpha}\}$.

With this definition, the connection of the Axiom of Algebra existence, Quasi-local algebra and Isotony in the three variants of the HKA becomes a mere technicality.
From the Fredenhagen-Version of the axioms, namely

F1) There exists a unital C^*-Algebra $\mathcal{A}(\mathcal{O})$ for every $\mathcal{O} \in \mathfrak{B}$.

F2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then there exists a unital injective $*$-homomorphism, $i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$, s.t.

$$i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3.$$

we can immediately get the induced net $\left\{\widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\right\}$. Moreover, the inductive limit $\lim_{\mathcal{O}_1 \in \mathfrak{B}} \{\mathcal{A}(\mathcal{O}_1); i_{\mathcal{O}_2,\mathcal{O}_1} \mid (\mathcal{O}_2, \mathcal{O}_1) \in \mathfrak{B} \times \mathfrak{B}, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$ is then the quasi-local algebra of the induced net.
This means we can always construct a net of C^*-algebras, which fulfills the Haag-Kastler-Version

HK1) There exists a unital C^*-Algebra $\mathcal{A}(\mathcal{O})$ for every $\mathcal{O} \in \mathfrak{B}$.

HK2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$, such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$. Additionally, both $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ have the same unit.

HK3) The set-theoretic union $\mathcal{A}(\mathfrak{B}) = \bigcup_{\mathcal{O} \in \mathfrak{B}} \mathcal{A}(\mathcal{O})$ is a normed * -algebra and its completion $\mathcal{A}$ is a C^*-Algebra.

and the Haag-Version

H1) There exists a net of unital C^*-Algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, where we take $\mathcal{O} \in \mathfrak{B}$.

of the axioms.

Conversely, we can go from Version HK and Version H to Version F just setting $i_{\mathcal{O}_2,\mathcal{O}_1}$ to the subalgebra embedding for $\mathcal{O}_2, \mathcal{O}_1 \in \mathfrak{B}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, since $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$.
In the Haag-Version we have that the C^*-algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathfrak{B}$, are not subalgebras of the inductive limit $\lim_{\mathcal{O}_1 \in \mathfrak{B}} \{\mathcal{A}(\mathcal{O}_1); e_{\mathcal{O}_2,\mathcal{O}_1} \mid (\mathcal{O}_2, \mathcal{O}_1) \in \mathfrak{B} \times \mathfrak{B}, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$, where $e_{\mathcal{O}_2,\mathcal{O}_1}$ is the corresponding subalgebra embedding. But they are $*$-isomorphic to a subalgebra by Theorem 3.1.1.

Remark. We restricted ourselves here to only using unital C^*-algebras in the Haag-Version and the Haag-Kastler-Version. For most applications this should not be much of a hindrance.

Obviously one can also go from Version HK to Version H directly, since this is just Definition 1.1.5. Note however, that the inductive limit of the net in H1) is only $*$-isomorphic to $\mathcal{A}$ from HK3) , by Corollary 3.1.3.

So from this point onward we will use the following conventions.

In the Fredenhagen-Version, we mean with the induced net of F1)-F2) the family $\left\{\widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\right\}$, the induced net of $\{\mathcal{A}(\mathcal{O})\}$ defined by $\{i_{\mathcal{O}_1,\mathcal{O}_2}\}$.
We call $\lim_{\mathcal{O}_1 \in \mathfrak{B}} \{\mathcal{A}(\mathcal{O}_1); i_{\mathcal{O}_2,\mathcal{O}_1} \mid (\mathcal{O}_2, \mathcal{O}_1) \in \mathfrak{B} \times \mathfrak{B}, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$ the induced limit of F1)-F2).

In the Haag-Version, we call $\lim_{\mathcal{O}_1 \in \mathfrak{B}} \{\mathcal{A}(\mathcal{O}_1); i_{\mathcal{O}_2,\mathcal{O}_1} \mid (\mathcal{O}_2, \mathcal{O}_1) \in \mathfrak{B} \times \mathfrak{B}, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$ the inductive limit of H1), where $i_{\mathcal{O}_2,\mathcal{O}_1}$ is the corresponding subalgebra embedding.
Additionally, if we talk about $\mathcal{A}(\mathcal{O})$ as a subalgebra of the inductive limt of H1) we mean the $*$-isomorphic C^*-subalgebra $\widetilde{\mathcal{A}}(\mathcal{O})$.

3.3. Causality

Next up is the Axiom of Causality. All the preparations are already done, so this also becomes a technicality.

Consider the Axiom of Causality in Version F

F3) If $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O} \in \mathfrak{B}$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated and $\mathcal{O}_1 \cup \mathcal{O}_2 \subseteq \mathcal{O}$, then
$$[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0 \quad \forall A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2)\,,$$

where we assume that F1)-F2) hold, and the corresponding one in Version H

H2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute in $\mathcal{A}$,

where we assume H1) holds and $\mathcal{A}$ being the inductive limit of H1).

To translate F3) to the induced net of F1-F2), we need to show that the commutator relation in F3) implies commutativity in the induced limit of F1-F2).
Indeed, by Theorem 3.1.1, which we used in the construction of the induced net, we get a family of $*$-isomorphisms $\{i_{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \to \widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ with the condition $i_{\mathcal{O}_2} i_{\mathcal{O}_2,\mathcal{O}_1} = i_{\mathcal{O}_1}$ if $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ and $\mathcal{O}_1 \subseteq \mathcal{O}_2$.
Taking now arbitrary $A \in \widetilde{\mathcal{A}}(\mathcal{O}_1)$ and $B \in \widetilde{\mathcal{A}}(\mathcal{O}_2)$ we get

$$\left[i_{\mathcal{O}\mathcal{O}_1}(i_{\mathcal{O}_1}^{-1}(A)), i_{\mathcal{O}\mathcal{O}_2}(i_{\mathcal{O}_2}^{-1}(B))\right] = 0$$

since $i_{\mathcal{O}_1}^{-1}(A) \in \mathcal{A}(\mathcal{O}_1)$ and $i_{\mathcal{O}_2}^{-1}(B) \in \mathcal{A}(\mathcal{O}_2)$ by F3).
We can calculate

$$0 = \left[i_{\mathcal{O}\mathcal{O}_1}(i_{\mathcal{O}_1}^{-1}(A)), i_{\mathcal{O}\mathcal{O}_2}(i_{\mathcal{O}_2}^{-1}(B))\right] = \left[i_{\mathcal{O}}^{-1}(A), i_{\mathcal{O}}^{-1}(B)\right] = i_{\mathcal{O}}^{-1}([A, B])$$

where in the last step we used that $i_{\mathcal{O}}$ and therefore $i_{\mathcal{O}}^{-1}$ are $*$-isomorphisms, which then also implies that $[A, B] = 0$ for all $A \in \widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}) \subseteq \widetilde{\mathcal{A}}$ and $B \in \widetilde{\mathcal{A}}(\mathcal{O}_2) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}) \subseteq \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is the inductive limit of F1)-F2).

Translating H2) into the notation of the Fredenhagen-Version of this axiom is managed very quickly.
By convention, the statement actually involves the $*$-isomorphic C^*-subalgebras $\widetilde{\mathcal{A}}(\mathcal{O}_1)$ and $\widetilde{\mathcal{A}}(\mathcal{O}_2)$. Since $\{\widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ is also net of C^*-algebras, we know that $\widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O})$ and $\widetilde{\mathcal{A}}(\mathcal{O}_2) \subseteq \widetilde{\mathcal{A}}(\mathcal{O})$ for any $\mathcal{O}_1 \cup \mathcal{O}_2 \subseteq \mathcal{O}$.
Since commutativity of $\widetilde{\mathcal{A}}(\mathcal{O}_1)$ and $\widetilde{\mathcal{A}}(\mathcal{O}_2)$ also hold in any subalgebra of $\mathcal{A}$ containing both $\widetilde{\mathcal{A}}(\mathcal{O}_1)$ and $\widetilde{\mathcal{A}}(\mathcal{O}_2)$, we can see that the rest is just rewriting using the subalgebra embedding.

Remark. As we have discussed in Section 2.2, the Axiom of Causality is identical in Version H and Version HK, at least in how it is written. The difference being context. This is then clarified by Section 3.2 as the setup for both axioms is equivalent. Therefore, the explanation in this section extends to the Version HK.

3.4. Symmetries

The last axiom, which we are going to discuss in this chapter, is the Axiom of Symmetries. We will denote the Poincaré group with $\mathfrak{P}$ from this point onward.

As mentioned in Section 2.1.3, there is the notational issue with the Axiom of Symmetries in the Fredenhagen-Version of the HKA. We remedy this by dropping the requirement that α_L is a $*$-automorphism and instead imposing that there is a family of $*$-isomorphisms $\alpha_L^{\mathcal{O}}$. The existence of a $*$-automorphism with the properties mentioned by Fredenhagen is then implicitly shown in the proposition below.

Proposition 3.4.1 (Equivalence of the Axiom of Symmetries)
The axiom

F4) For every $L \in \mathfrak{P}$ there is a family of $*$-isomorphisms $\{\alpha_L^{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(L\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ such that for all $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and $L, L_1, L_2 \in \mathfrak{P}$ we have

$$\alpha_L^{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{L\mathcal{O}_2 L\mathcal{O}_1} \circ \alpha_L^{\mathcal{O}_1} \tag{3.6}$$

$$\alpha_{L_2}^{L_1\mathcal{O}} \circ \alpha_{L_1}^{\mathcal{O}} = \alpha_{L_2 L_1}^{\mathcal{O}}\,, \tag{3.7}$$

where we assume that F1)-F2) hold, is equivalent to

H3) There exists a group homomorphism $L \mapsto \alpha_L$ from $\mathfrak{P}$ to the $*$-automorphisms on $\mathcal{A}$ such that $\alpha_L(\mathcal{A}(\mathcal{O})) = \mathcal{A}(L\mathcal{O})$ for all $L \in \mathfrak{P}$,

where we assume H1) holds and $\mathcal{A}$ is the inductive limit of H1).
More precisely, when translating F4) to the induced net/inductive limit of F1)-F2), it becomes H3) and conversely H3) can be translated to look like F4).

Remark.
- Similarly to Section 3.3, the discussion in Section 2.2 and Section 3.2 show that the Version H and Version HK of this axiom are already the same. Hence, this proposition is an equivalence for all three versions.
- Note also that we extended the statment of Version F (Section 2.1.3) from "connected component of the identity in the Poincaré group" to the entire $\mathfrak{P}$. It makes no difference in the proof and the restriction can always be made.

Proof. Assume that F1)-F2) and F4) hold.
Take L in the Poincaré group and define $\mathcal{B}(\mathcal{O}) := \mathcal{A}(L\mathcal{O})$ and $\tilde{i}_{\mathcal{O}_2\mathcal{O}_1} := i_{L\mathcal{O}_2L\mathcal{O}_1}$. Since L is continuous and has a continuous inverse, it maps open and bounded sets to open and bounded set. Therefore, for $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \in \mathfrak{B}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3$ we get $L\mathcal{O}_1, L\mathcal{O}_2, L\mathcal{O}_3 \in \mathfrak{B}$ and $L\mathcal{O}_1 \subseteq L\mathcal{O}_2 \subseteq L\mathcal{O}_3$. With this we get the equation

$$\tilde{i}_{\mathcal{O}_3\mathcal{O}_2} \circ \tilde{i}_{\mathcal{O}_2\mathcal{O}_1} = i_{L\mathcal{O}_3L\mathcal{O}_2} \circ i_{L\mathcal{O}_2L\mathcal{O}_1} = i_{L\mathcal{O}_3L\mathcal{O}_1} = \tilde{i}_{\mathcal{O}_3\mathcal{O}_1}$$

from the definition of $\tilde{i}$. Hence $\{\mathcal{B}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ and $\{\tilde{i}_{\mathcal{O}_2\mathcal{O}_1} \mid \mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}\,, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$ fulfill the conditions of Theorem 3.1.1 which yields the unital C^*-algebra $\mathfrak{B}$ as the inductive limit.
If we rewrite (3.6) with the new notation, we get

$$\alpha_L^{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = \tilde{i}_{\mathcal{O}_2\mathcal{O}_1} \circ \alpha_L^{\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subseteq \mathcal{O}_2\,.$$

This is the condition needed to apply Proposition 3.1.4 to $\{\alpha_L^{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(L\mathcal{O}) = \mathcal{B}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$. Hence, we get a unique $*$-isomorphism $\alpha_L : \mathcal{A} \to \mathfrak{B}$ with

$$\alpha_L|_{\widetilde{\mathcal{A}}(\mathcal{O})} = \tilde{i}_{\mathcal{O}} \circ \alpha_L^{\mathcal{O}} \circ i_{\mathcal{O}}^{-1} : \widetilde{\mathcal{A}}(\mathcal{O}) \to \widetilde{\mathcal{B}}(\mathcal{O})\,, \tag{3.8}$$

where $\tilde{i}_{\mathcal{O}} : \mathcal{B}(\mathcal{O}) \to \widetilde{\mathcal{B}}(\mathcal{O})$, and similarly $i_{\mathcal{O}}$, is from the construction of the inductive limit. $\alpha_L|_{\widetilde{\mathcal{A}}(\mathcal{O})}$ is a $*$-isomorphism as composition of $*$-isomorphisms.
Again, since L is a homeomorphism we get $\{\mathcal{B}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\} = \{\mathcal{A}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ and $\{\tilde{i}_{\mathcal{O}_2\mathcal{O}_1} \mid \mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}\,, \mathcal{O}_1 \subseteq \mathcal{O}_2\} = \{i_{\mathcal{O}_2\mathcal{O}_1} \mid \mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}\,, \mathcal{O}_1 \subseteq \mathcal{O}_2\}$. Since we constructed both $\mathcal{A}$ and $\mathcal{B}$ from the same families, using Theorem 3.1.1, we get $\mathcal{A} = \mathcal{B}$.[1] Therefore $\alpha_L : \mathcal{A} \to \mathcal{A}$ is a $*$-automorphism with

$$\alpha_L(\widetilde{\mathcal{A}}(\mathcal{O})) = \tilde{\mathcal{B}}(\mathcal{O}) = \widetilde{\mathcal{A}}(L\mathcal{O})\,. \tag{3.9}$$

Lastly, for any L_1, L_2 in the Poincaré group and $\mathcal{O} \in \mathfrak{B}$ we have

$$\begin{aligned}\alpha_{L_2} \circ \alpha_{L_1}|_{\widetilde{\mathcal{A}}(\mathcal{O})} &\overset{(3.9)}{=} \alpha_{L_2}|_{\widetilde{\mathcal{A}}(L_1\mathcal{O})} \circ \alpha_{L_1}|_{\widetilde{\mathcal{A}}(\mathcal{O})} \overset{(3.8)}{=} i_{L_2L_1\mathcal{O}} \circ \alpha_{L_2}^{L_1\mathcal{O}} \circ i_{L_1\mathcal{O}}^{-1} \circ i_{L_1\mathcal{O}} \circ \alpha_{L_1}^{\mathcal{O}} \circ i_{\mathcal{O}}^{-1} \\ &\overset{(3.7)}{=} i_{L_2L_1\mathcal{O}} \circ \alpha_{L_2L_1}^{\mathcal{O}} \circ i_{\mathcal{O}}^{-1} \overset{(3.8)}{=} \alpha_{L_2L_1}|_{\widetilde{\mathcal{A}}(\mathcal{O})}\,.\end{aligned}$$

This implies $\alpha_{L_2} \circ \alpha_{L_1} = \alpha_{L_2L_1}$, since $\bigcup_{\mathcal{O}\in\mathfrak{B}} \widetilde{\mathcal{A}}(\mathcal{O})$ is dense in $\mathcal{A}$, and that $L \mapsto \alpha_L$ is a group homomorphism. All together this means that H3) holds.

Conversely, assume now that H1) and H3) hold.
Then for every L in the Poincaré group we define $\alpha_L^{\mathcal{O}} := \alpha_L|_{\widetilde{\mathcal{A}}(\mathcal{O})}$. Remember that we use the convention to identify $\mathcal{A}(\mathcal{O})$ and $\widetilde{\mathcal{A}}(\mathcal{O})$.
Since α_L is a $*$-automorphism $\alpha_L^{\mathcal{O}}$ is a $*$-isomorphism onto its image, which is $\mathcal{A}(L\mathcal{O})$.

[1]Using Corollary 3.1.5 would only imply $\mathcal{A} \cong \mathcal{B}$. But we could compose α_L with the $*$-isomoprhism $\mathcal{B} \to \mathcal{A}$ in the next step to still get a $*$-automorphism.

The group structure of $L \mapsto \alpha_L$ yields (3.7).
Since $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $L\mathcal{O}_1, L\mathcal{O}_2 \in \mathfrak{B}$ and $L\mathcal{O}_1 \subseteq L\mathcal{O}_2$, we get $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ and $\mathcal{A}(L\mathcal{O}_1) \subseteq \mathcal{A}(L\mathcal{O}_2)$. From Section 3.2 we know that $i_{\mathcal{O}_2\mathcal{O}_1}$ and $i_{L\mathcal{O}_2 L\mathcal{O}_1}$ are the natural subalgebra embeddings. Together with the definition of $\alpha_L^{\mathcal{O}_1}$ and $\alpha_L^{\mathcal{O}_2}$ they imply (3.6). Therefore we have F4). $\square$

4. Interlude of causal concepts

The Axiom of Dynamics deals with some more evolved concepts in Lorentzian geometry. Mainly the causal completion (Definition 1.2.6) is an object which does not get mentioned in standard Lorentzian geometry literature [O'N83, Bee96, Min19].
Though we only worked in Minkowski space so far, we are going to general Lorentzian manifolds as much as possible in this chapter. Since generalizations of the Haag-Kastler Axioms to curved spacetime have already been done, for example [FV15], we hope that working here in arbitrary Lorentzian manifolds adds to knowledge in this field.

In this chapter we analyze some properties of the causal completion and the *Cauchy development for general subsets*. Furthermore, we will build relations between those and more known objects in Lorentzian geometry.

4.1. Even more definitions

In this first section we are going to expand on Section 1.2 with some additional definitions as well as some more advanced statements in Lorentzian geometry, which we will use later on. We will not prove any of the statements in this section, as they can be found in more suited literature and may be already know to the familiar reader.

A surprisingly important property of subsets for this thesis, as it will turn out, will be the next definition.

Definition 4.1.1 (Causally convex sets)
Let M be a space-time.
A subset $V \subseteq M$ is called *causally convex* if every causal curve between two points $p, q \in V$ is fully contained in V, i.e. $J^+(p) \cap J^-(q) \subseteq V$ for all $p, q \in V$.[Min19, 1.32]

Remark. Some authors even define causally convex sets as open sets with the additional property. For example [Bee96, p.59].

We have already introduced Cauchy hypersurfaces in Chapter 1. Related to them are of course *globally hyperbolic* space-times.

Definition 4.1.2 (Globally hyperbolic)
A space-time M is called *globally hyperbolic* if one of the following equivalent conditions hold

(a) Non-total imprisonment and $\overline{J^+(p) \cap J^-(q)}$ is compact for all $p, q \in M$.

(b) Causality and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

[Min19, 4.117]

Remark. (i) In the given reference, there are actually more equivalent conditions given for globally hyperbolic space-times. We omitted them to not bloat the definition unnecessarily.

(ii) Note that any open, causally convex subset of a globally hyperbolic manifold is itself globally hyperbolic, since causality is inherited and causal convexity guarantees the compactness of the causal diamonds in the subset topology.

The most relevant characteristics of globally hyperbolic space-times for out purposes will be the compactness of the *causal diamonds* $J^+(p) \cap J^-(q)$ and the non-total imprisonment. For clarity, we will reference the definition of the latter.

Definition 4.1.3 (Non-totally imprisoning)
A space-time is called *non-totally imprisoning* or *non-imprisoning* if no future (past) inextendible causal curve is contained in a compact set. [Min19, 4.36]

The aforementioned relation to Cauchy hypersurfaces comes with the following theorem.

Theorem 4.1.4 (Globally hyperbolic and Cauchy hypersurfaces)
Let M be a spacetime. Then the following are equivalent:

(a) M is globally hyperbolic.

(b) There exists a Cauchy hypersurface in M.

(c) M is isometric to $R \times S$ with metric $-\beta dt^2 + g_t$, where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and ech $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.

[BGP07, 1.3.10]

4.2. Causal complement and causal completion

Even though we already know the causal complement for the special case of Minkowski space we will start by rewriting said definition so it lends itself to a more general approach. Note that by writing the definition of the causal complement (Definition 1.2.6) in notation from Lorentz geometry, we get

$$V' = \{x \in \mathbb{R}^{1,d} \mid x - y \text{ spacelike}, \forall y \in V\} = \mathbb{R}^{1,d} \setminus J(V)$$

for $V \subseteq \mathbb{R}^{1,d}$. This leads naturally to a definition of a more general notion.

Definition 4.2.1 (Causal complement/completion)
Let M be a space-time and $V \subseteq M$ be any subset.
The set

$$V' := M \setminus J(V)$$

is called *causal complement* of V.
$V'' := (V')'$ is called the *causal completion* of V. V is called *causally complete* if $V'' = V$.

Remark. The first two properties of the causal complement follow now very easily from this general definition:

(i) If $V \subseteq W \subseteq M$ then $J(V) \subseteq J(W)$ and hence

$$W' = M \setminus J(W) \subseteq M \setminus J(V) = V' .$$

(ii) Assume $V \cap J(V') \neq \emptyset$. Then for $p \in V \cap J(V')$ there exists a $q \in V'$ which is connected by a causal curve to $p \in V$. This implies that $q \in J(p) \subseteq J(V)$ which is a contradiction to $q \in V' = M \setminus J(V)$. Hence, $V \cap J(V') = \emptyset$ and $V \subseteq V'' = M \setminus J(V')$.

In the context of the previous chapters, the first question which arises is whether or not $\mathcal{O}''$ is open and bounded if $\mathcal{O}$ is open and bounded. We start with the following lemma.

Lemma 4.2.2 (Causal future of open sets)
Let M be a space-time and $O \subseteq M$ be open. Then $J^{\pm}(O)$ is open.

Proof. Let $p \in J^+(O)$. There exists a $q \in O$ such that $q \leq p$. Since O is open, there exists a neighborhood U of q such that $q \in U \subseteq O$. Furthermore $U \cap I^-(q) \neq \emptyset$, since $q \in \partial I^-(q)$. Choosing $\tilde{q} \in U \cap I^-(q) \subseteq O$, we get $\tilde{q} \ll q \leq p$. Hence $\tilde{q} \ll p$, i.e. $p \in I^+(O)$. Therefore, we have $J^+(O) \subseteq I^+(O)$ and further

$$J^+(O) \subseteq I^+(O) = \text{int } J^+(O) \subseteq J^+(O) .$$

Hence, $J^+(O) = \text{int } J^+(O)$ which implies that $J^+(O)$ is open.

Analogously for $J^-(O)$ which together yield $J(O) = J^+(O) \cup J^-(O)$ open. □

Corollary 4.2.3 (Causal complement of open sets)
Let M be a space-time and $O \subseteq M$ be open. Then O' is closed.

Proof. Follows immediately from Lemma 4.2.2 and Definition 4.2.1. □

This means that $\mathcal{O}' \notin \mathfrak{B}$ even if $\mathcal{O} \in \mathfrak{B}$ and therefore we can not associate an C^*-algebra to $\mathcal{O}'$ within the HKA we investigated so far. Gladly we will never try to associate an algebra to $\mathcal{O}'$ in this thesis, only to $\mathcal{O}''$. [1]

[1] Some works do in fact use $\mathcal{O}'$ directly. For example Horuzhy [Hor90, p.14] uses the definition $\mathcal{O}' = \text{int}\{x \in \mathbb{R}^{1,d} | x - y \text{ spacelike}, \forall y \in M\}$ and Fewster and Verch [FV15, p.133] use $\mathcal{O}' = M \setminus \overline{J(\mathcal{O})}$ to guarantee that $\mathcal{O}'$ is open. Those two notions are in fact equivalent. This follows from general topology, since $(A^\circ)^c = \overline{A^c}$ for any subset A of a topological space.

Unfortunately a similar statement as in Lemma 4.2.2 for closed subsets, i.e. $J^{\pm}(A)$ closed if A closed, is not true in general (see Figure 4.1).
So we need to postpone the question, whether $\mathcal{O}'' \in \mathfrak{B}(M)$ if $\mathcal{O} \in \mathfrak{B}(M)$.

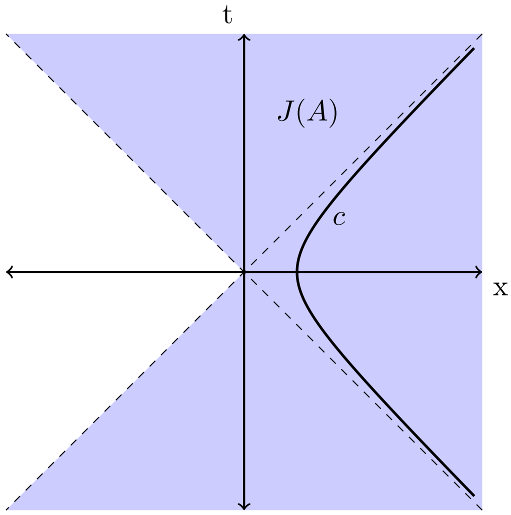

Figure 4.1.: Let $c : \mathbb{R} \to \mathbb{R}^{1,1}$ be a causal curve, which approximates the future/past lightcone from the outside as $t \mapsto \pm\infty$, in this case a hyperbola. Then $A := c(\mathbb{R})$ is closed but $J^{\pm}(A)$ and therefore $J(A)$ (shaded area) is open. [Kun24]

The definition of the causal completion, though easy to write down, is not very handy. Therefore we present an alternative description.

Lemma 4.2.4 (Alternative definition for causal completion)
Let M be a space-time and $V \subseteq M$ be any subset. Then

$$V'' = \{q \in M \mid J(q) \subseteq J(V)\}$$

Remark. A proof of this lemma for Minkowski space was found in [Noi15].

Proof. Let $q \in M$. The statement can be rewritten as

$$J(q) \subseteq J(V) \iff q \in V'' ,$$

which can be further rewritten (using the definition of V'') as

$$(\forall p \in M : p \in J(q) \implies p \in J(V)) \iff (\forall p \in M : p \in V' \implies q \notin J(p)) .$$

" $\implies$ ": For all $p \in M$ we either have $p \in \{q\}' = M \setminus J(q)$ or $p \in J(q) \subseteq J(V)$. So if $p \in V' = M \setminus J(V)$ then we are in the case $p \in M \setminus J(q)$, i.e. $p \notin J(q)$. This is equivalent to $q \notin J(p)$.

" $\Longleftarrow$ ": The contraposition of the RHS reads

$$q \in J(p) \implies p \notin V' = M \setminus J(V)\,,$$

by the definition of V', which already is the LHS since $q \in J(p) \iff p \in J(q)$. □

We finish this section by showing an interesting property of the causal completion.

Corollary 4.2.5 (Causal completion is causally convex)
Let M be a space-time and $V \subseteq M$ be any subset.
Then V'' is causally convex.

Proof. Take $p, q \in V''$. If $J^+(p) \cap J^-(q) \neq \emptyset$ take $x \in J^+(p) \cap J^-(q)$. By Lemma 4.2.4 we have

$$J^+(x) \subseteq J^+(J^+(p)) = J^+(p) \subseteq J(V)$$

and similarly $J^-(x) \subseteq J(V)$. Hence, again by Lemma 4.2.4 $x \in V''$.
Therefore $J^+(p) \cap J^-(q) \subseteq V''$ and V'' is causally convex by Definition 4.1.1. □

4.3. Cauchy development

In this section we are going to look at the *Cauchy development* for arbitrary subsets. We do this, because, as discussed in Section 2.1.1, the mentioned uncertainty hints at a connection between the causal completion and the Cauchy development.

Let us start with the definition.

Definition 4.3.1 (Cauchy Development)
Let M be a space-time and $V \subseteq M$ any subset.
We call

$$D^+(V) := \{p \in M \mid \text{every past-directed inextendible causal curve through } p \text{ meets } V\}$$

the *future Cauchy development* of V. Similarly

$$D^-(V) := \{p \in M \mid \text{every future-directed inextendible causal curve through } p \text{ meets } V\}$$

is called the *past Cauchy development* of V.

$$D(V) := D^+(V) \cup D^-(V) = \{p \in M \mid \text{every inextendible causal curve through } p \text{ meets } V\}$$

is called the *Cauchy development* of V. [Min19, 3.1, 3.29]

Remark. (i) Different sources define the Cauchy development for different classes of subsets. For example [O'N83] defines it only for achronal subsets and [Bee96] for closed subsets. The Cauchy development gains some desirable properties automatically in those cases. Even [Min19] only introduces the Cauchy development for arbitrary sets and moves to closed and/or achronal sets very quickly.

(ii) Note also it follows immediately that $D^{\pm}(V) \subseteq D^{\pm}(W)$ if $V \subseteq W \subseteq M$, since every causal curve which meets V also meets W.

A notable relation between the Cauchy development and Cauchy hypersurfaces, which we will not prove, comes from the following lemma.

Lemma 4.3.2 (Cauchy development of Cauchy hypersurfaces)
Let M be a space-time and $A \subseteq M$ be an achronal subset.
A is a Cauchy hypersurface if and only if $D(A) = M$.[O'N83, p.420]

As a quick interlude, we are going to introduce the *limit curve* and a proposition associated to it.

Definition 4.3.3 (Limit curve)
Let M be a space-time.
A curve γ in M is called a *limit curve of the sequence* $\{\gamma_n\}_{n\in\mathbb{N}}$ of curves in M if there exists a subsequence $\{\gamma_{n_m}\}_{m\in\mathbb{N}}$ such that for all $p \in \gamma$ we have that every neighborhood of p intersects all but finitely many curves of $\{\gamma_{n_m}\}_{m\in\mathbb{N}}$. The subsequence $\{\gamma_{n_m}\}_{m\in\mathbb{N}}$ is said to *distinguish* the limit curve γ. [Bee96, 3.28]

Proposition 4.3.4 (Existence of limit curves)
Let M be a space-time and $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of inextendible causal curves.
If p is an accumulation point of the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, i.e., every neighborhood of p intersects infinitely many γ_n, then there exists a causal limit curve γ of the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $p \in \gamma$ and γ is inextendible. [Bee96, 3.31][Haw23, 6.2.1]

We can now show some properties which the Cauchy development inherits form the set it is associated to.

Lemma 4.3.5 (Cauchy development of open sets)
Let M be a space-time and $O \subseteq M$ be an open subset. Then $D(O)$ is open.

Proof. Consider a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq M \setminus D(O)$ which converges to $p \in M$.
Since every p_n is not in $D(O)$, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of inextendible causal curves s.t. $p_n \in \gamma_n$, which do not meet O for all $n \in \mathbb{N}$. Furthermore p is an accumulation point of the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ since every neighborhood of p contains infinitely many p_n and therefore intersects infinitely many γ_n.

By the Proposition 4.3.4 we know there exists a inextendible causal limit curve γ with $p \in \gamma$. Assume γ intersects O, say at $q \in \gamma$. Since O is open, there exists a neighborhood U of q with $U \subseteq O$. By Definition 4.3.3, U intersects at least one γ_N, which means that γ_N intersects O. This contradicts the construction of γ_N. Therefore γ does not meet O which implies $p \notin D(O)$.
Hence $M \setminus D(O)$ is closed and $D(O)$ is open. □

Lemma 4.3.6 (Cauchy development of compact sets)
Let M be Minkowski space and $K \subseteq M$ be a compact set. Then $D(K)$ is relatively compact, i.e., bounded.

Proof. We assume M is $d+1$-dimensional and denote points in M with $(t,x) \in M$ where $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Since K is bounded it is contained in the Ball $B_R := \{(t,x) \in M \mid t^2 + |x|^2 < R^2\}$. We claim that $D(B_R)$ is contained in $B_{\sqrt{2}R}$.

Let $p = (t_p, x_p) \in M \setminus B_{\sqrt{2}R}$. By the rotational symmetry in x and the $t \mapsto -t$ symmetry, we can assume without loss of generality that $t_p \geq 0$ and $x_p = (x_1, 0, \dots, 0)$, with $x_1 \geq 0$. We therefore can restrict to a 1+1-dimensional setting. So $x_p \in \mathbb{R}$ and $x_p \geq 0$.

Choose

$$\lambda_p(t) := (t_p + t, x_p - t) \quad t \in \mathbb{R}\,,$$

which is a future directed inextendible null curve through p, since $\lambda_p(0) = (t_p, x_p) = p$ and

$$\dot{\lambda}_p(t) = (1, -1)\,.$$

We show that $\lambda_p(t) \notin B_R$ for all $t \in \mathbb{R}$, which implies that λ_p never meets K.
The equation

$$\begin{aligned}|\lambda_p(t)|^2 =&(t_p + t)^2 + (x_p - t)^2 = t_p^2 + 2tt_p + t^2 + x_p^2 - 2tx_p + t^2\\ =&2t^2 + 2t(t_p - x_p) + t_p^2 + x_p^2\end{aligned}$$

has its minimum at $t = -\frac{x_p - t_p}{2}$ since $\frac{d}{dt}|\lambda_p(t)|^2 = 4t + 2(t_p - x_p)$ and $\frac{d^2}{dt^2}|\lambda_p(t)|^2 = 4 > 0$. This yields

$$\begin{aligned}2t^2 + 2t(t_p - x_p) + t_p^2 + x_p^2 \geq& 2\frac{(x_p - t_p)^2}{4} + 2\frac{x_p - t_p}{2}(t_p - x_p) + t_p^2 + x_p^2\\ =&\frac{(x_p - t_p)^2}{2} - (x_p - t_p)^2 + t_p^2 + x_p^2 = -\frac{(x_p - t_p)^2}{2} + t_p^2 + x_p^2\\ =&\frac{x_p^2 + t_p^2}{2} + x_p t_p\,.\end{aligned}$$

Since $p \notin B_{\sqrt{2}R}$, we have $t_p^2 + x_p^2 \geq 2R^2$. Hence,

$$|\lambda_p(t)|^2 \geq \frac{x_p^2 + t_p^2}{2} + x_p t_p \geq R^2 + x_p t_p \geq R^2\,,$$

since $x_p, t_p \geq 0$. Therefore $\lambda_p(t) \notin B_R$ for all $t \in \mathbb{R}$.

This means we can always find an inextendible causal curve starting at $p \in M \setminus B_{\sqrt{2}R}$, which does not meet B_R.
Hence, $D(K) \subseteq D(B_R) \subseteq B_{\sqrt{2}R}$. □

Remark. Note that the relative compactness of $D(K)$ seems to be a property which is strongly related to Minkowski space, since this statement does not hold for general globally

hyperbolic space-times. Consider the Lorentz cylinder $M := \mathbb{R} \times S^1$ and $K := [-1, 1] \times S^1$. M is indeed globally hyperbolic by Theorem 4.1.4. Since $S := \{0\} \times S^1$ is a Cauchy hypersurfaces for M we have $M = D(S) \subseteq D(K) \subseteq M$ by Lemma 4.3.2. Hence, even though K is compact, $D(K)$ clearly is not relatively compact.
One ingredient for this counterexample is that M possesses a compact Cauchy hypersurface. It is only a hunch but the statement might still work in globally hyperbolic space-times without compact Cauchy hypersurfaces.

Lemma 4.3.7 (Existence of Cauchy hypersurfaces in Cauchy developments)
Let M be a globally hyperbolic space-time and $O \subseteq M$ be an open causally convex subset. Then $D(O)$ has a Cauchy hypersurface, which is contained in O.

Proof. Since O is open and causally convex it is also globally hyperbolic, as remarked after Definition 4.1.2. Hence O also has a Cauchy hypersurface $S \subseteq O$ by Theorem 4.1.4. the claim is that S is also a Cauchy hypersurface of $D(O)$.

Let c be an inextendible timelike curve in $D(O)$. Without loss of generality c has a point $p \in D^+(O)$. We can parametrize c such that $p = c(0)$ and c is past-directed. Then, since $p \in D^+(O)$, c meets O. We call the intersection $\tilde{c} := c \cap O$.

$\tilde{c}$ has only one connected component.
Otherwise we could choose $c(t_1) \in O$ and $c(t_2) \in O$ with $t_1 < t_2$ from different connected components of $\tilde{c}$. But since O is causally convex we have that $c|_{[t_1,t_2]} \subseteq O$, contradicting the assumption that $c(t_1)$ and $c(t_2)$ are from different connected components.

Next, we notice that $\tilde{c}$ is an inextendible timelike curve in O. Hence it has exactly one intersection with the Cauchy hypersurface S. Therefore c has exactly one intersection with S which implies that S is a Cauchy hypersurface of $D(O)$. □

To proceed we require a fact about Cauchy hypersurfaces.

Lemma 4.3.8 (Cauchy hypersurfaces are achronal)
Let M be a space-time.
A Cauchy hypersurface S (of M) is a closed achronal topological hypersurface and is met by every inextendible causal curve.[O'N83, 14.29]

We are now equipped to tackle the last lemma in this section.

Lemma 4.3.9 (Cauchy development of Cauchy hypersurfaces of open subset)
Let M be a space-time and $O \subseteq M$ an open subset. Suppose O has a Cauchy hypersurface S. Then $O \subseteq D(S)$.

Remark. In light of Lemma 4.3.2 one might think this statement is trivial since S is a Cauchy hypersurfaces in the manifold O. But $D(S)$ taken in O is not the same as $D(S)$ taken in M. This is why from now on, if we need to emphasize when we take $D(S)$ in a subset, we denote it with an index, i.e., $D_O(S)$ if we take the Cauchy development with respect to O.

Proof. Let $p \in O$. By Lemma 4.3.2 we know $O = D_O(S)$. Assume $p \in D_O^+(S) \subseteq J^+(S)$ and consider a past-directed inextendible causal curve c in M starting at p.
Since O is open, it follows from continuity of c that c stays in O, at least for some small parameter ϵ. Therefore the maximal extension of c in O starting from p, denoted $\tilde{c}$, can be considered an inextendible causal curve in O. By Lemma 4.3.8 we know that $\tilde{c}$ intersects S, hence c intersects S.
Therefore p lies in the future Cauchy development $D^+(S)$. Similarly, if $p \in D_O^-(S) \subseteq J^-(S)$ then $p \in D^-(S)$. So $p \in D^-(S) \cup D^+(S) = D(S)$, hence $O \subseteq D(S)$. □

4.4. Relations to causally convex sets

As the title of this section suggests our next goal is to show how the Cauchy development and the causal complement are related to causally convex sets.
Causally convex sets are very much related to the next definition, as we will show shortly.

Definition 4.4.1 (Causal Hull)
Let M be a space-time and $V \subseteq M$ a subset. Then

$$\mathrm{ch}(V) := J^+(V) \cap J^-(V)$$

is called the *causal hull* of V.[Bru24]

Remark. We can immediately deduce two properties of the causal hull.

(i) $V \subseteq \mathrm{ch}(V)$ since $V \subseteq J^\pm(V)$.

(ii) $\mathrm{ch}(V)$ is causally convex, since for any $x, y \in \mathrm{ch}(V)$ we have $p, q \in V$ such that $x \in J^+(p)$ and $y \in J^-(q)$. Hence

$$J^+(x) \cap J^-(y) \subseteq J^+(J^+(p)) \cap J^-(J^-(q)) = J^+(p) \cap J^-(q) \subseteq \mathrm{ch}(V)\,.$$

The next statement is actually a corollary to Lemma 4.2.2. It was just postponed due to introducing the causal hull only now.

Corollary 4.4.2 (Causal hull of open set)
Let M be a space-time and $O \subseteq M$ be open. Then $\mathrm{ch}(O)$ is open.

Proof. By Lemma 4.2.2 we get $J^\pm(O)$ open and therefore $\mathrm{ch}(O) = J^+(O) \cap J^-(O)$ is open. □

To progress further we need another property of globally hyperbolic space-times.

Lemma 4.4.3 (Causal diamonds are compact)
Let M be a globally hyperbolic space-time.
Then, for any two compact $K_1, K_2 \subseteq M$, we have that $J^+(K_1) \cap J^-(K_2)$ is compact.[BGP07, A.5.7]

This is generally a very useful lemma which we are going to use multiple times.

To start with, we can use it to prove the next corollary.

Corollary 4.4.4 (Causal hull of open, relatively compact set)
Let M be a globally hyperbolic space-time. Then

$$\mathcal{O} \in \mathfrak{B}(M) \implies \mathrm{ch}(\mathcal{O}) \in \mathfrak{B}(M)$$

Proof. Openness follows form Corollary 4.4.2 and

$$\overline{\mathrm{ch}(\mathcal{O})} = \overline{J^+(\mathcal{O}) \cap J^-(\mathcal{O})} \subseteq \overline{J^+(\overline{\mathcal{O}}) \cap J^-(\overline{\mathcal{O}})}$$

implies relative compactness since $J^+(\overline{\mathcal{O}}) \cap J^-(\overline{\mathcal{O}})$ is compact by Lemma 4.4.3. □

The connection between causally convex sets and causal hull is due to the next lemma.

Lemma 4.4.5 (Equivalent definition of causally convex)
Let M be a space-time and $V \subseteq M$ any subset.
Then V is causally convex if and only if $V = \mathrm{ch}(V)$.

Proof. " $\implies$ ":
Let V be causally convex. By the basic properties of the causal hull we know $V \subseteq \mathrm{ch}(V)$. Take $x \in \mathrm{ch}(V)$. Hence, by the Definition 4.4.1, there exits $p, q \in V$ such that $x \in J^+(p) \cap J^-(q)$. Since V is causally convex, we have $x \in J^+(p) \cap J^-(q) \subseteq V$.

" $\impliedby$ ":
Since $\mathrm{ch}(V)$ is causally convex, so is V. □

One last lemma is required before we can show, how the causal completion, the Cauchy development and causally convex sets interact.

Lemma 4.4.6 (Causal curves intersect causal hull)
Let M be a globally hyperbolic space-time and $\mathcal{O} \in \mathfrak{B}(M)$.
Then any inextendible causal curve c, which fully lies in $J(\mathcal{O})$, i.e., $c \subseteq J(\mathcal{O})$, meets the causal hull $\mathrm{ch}(\mathcal{O})$.

Proof. Let c be an inextendible causal curve, which is fully contained in $J(\mathcal{O})$, i.e. $c \subseteq J(\mathcal{O})$. Without loss of generality c is past-directed and has a point in $J^+(\mathcal{O})$, say $p \in J^+(\mathcal{O})$.
We write $\tilde{c}$ for the curve, starting at p and following c. This means that $\tilde{c}$ is a past-directed past-inextendible causal curve. Clearly $\tilde{c} \subseteq J^-(p)$.

Since $\mathcal{O}$ is open and bounded, $\overline{\mathcal{O}}$ is compact and therefore $J^-(p) \cap J^+(\overline{\mathcal{O}})$ is compact by Lemma 4.4.3.
By the Non-total imprisonment of M (Definition 4.1.2, Definition 4.1.3) we know that $\tilde{c}$ needs to leave $J^-(p) \cap J^+(\overline{\mathcal{O}})$. This implies that $\tilde{c}$ leaves $J^+(\mathcal{O}) \subseteq J^+(\overline{\mathcal{O}})$ since we know $\tilde{c} \subseteq J^-(p)$.

Continuity of $\tilde{c}$, and the fact that $\tilde{c}$ leaves $J^+(\mathcal{O})$, imply that $\tilde{c} \cap \partial J^+(\mathcal{O}) \neq \emptyset$.[2] Choose $q \in \tilde{c} \cap \partial J^+(\mathcal{O})$.
By Lemma 4.2.2 $J^+(\mathcal{O})$ is open. Therefore we have $q \notin J^+(\mathcal{O})$. By assumption $\tilde{c} \subseteq J(\mathcal{O}) = J^+(\mathcal{O}) \cup J^-(\mathcal{O})$ and hence $q \in J^-(\mathcal{O})$. Since $J^-(\mathcal{O})$ is also open, again by Lemma 4.2.2, there exists a neighborhood U of q such that $q \in U \subseteq J^-(\mathcal{O})$.

Continuity of $\tilde{c}$ implies that we can find a sequence $\{\tilde{c}(t_n)\} \subseteq J^+(\mathcal{O})$ such that $\tilde{c}(t_n) \to q$. Therefore $\tilde{c}(t_n) \in U \subseteq J^-(\mathcal{O})$ for large enough n and hence $\tilde{c}(t_n) \in J^+(\mathcal{O}) \cap J^-(\mathcal{O}) = \mathrm{ch}(\mathcal{O})$ for large enough n.
Therefore also c intersects $\mathrm{ch}(\mathcal{O})$. □

We can finally show the connection between the causal completion and the Cauchy development.

Theorem 4.4.7 (Cauchy development of causal hull)
Let M be a globally hyperbolic space-time and $\mathcal{O} \in \mathfrak{B}(M)$.
Then the Cauchy development of the causal hull of $\mathcal{O}$ is its causal completion, i.e.

$$\mathcal{O}'' = D(\mathrm{ch}(\mathcal{O}))\,.$$

Proof. "$\supseteq$":
Let $p \in D(\mathrm{ch}(\mathcal{O}))$. Assume $p \in D^+(\mathrm{ch}(\mathcal{O})) \subseteq J^+(\mathrm{ch}(\mathcal{O}))$.
Hence there exists a $q_1 \in \mathrm{ch}(\mathcal{O})$ s.t. $p \in J^+(q_1)$. Further there exists a $q_2 \in \mathcal{O}$ such that $q_1 \in J^+(q_2)$ by Definition 4.4.1. Therefore $J^+(p) \subseteq J^+(J^+(q_1)) \subseteq J^+(J^+(J^+(q_2))) = J^+(q_2) \subseteq J^+(\mathcal{O})$.

Now let $q \in J^-(p)$ be arbitrary. Hence there exists a past-directed causal curve form p to q. Call an inextendible extension of this curve c. Let t_0 be such that $q = c(t_0)$. Since $p \in D^+(\mathrm{ch}(\mathcal{O}))$ and c is a past-directed inextendible causal curve it has an intersection with $\mathrm{ch}(\mathcal{O})$. Call the intersection-point $c(t_1)$. Since $c(t_1) \in \mathrm{ch}(\mathcal{O})$, there exists $x, y \in \mathcal{O}$ s.t. $c(t_1) \in J^+(x) \cap J^-(y)$.
If $t_1 \leq t_0$ then $q \in J^-(c(t_1)) \subseteq J^-(J^-(y)) = J^-(y)$ and if $t_1 \geq t_0$ then $q \in J^+(c(t_1)) \subseteq J^+(J^+(x)) = J^+(x)$. Either way $q \in J(\mathcal{O})$.

Together we get that $J(p) \subseteq J(\mathcal{O})$ which implies $p \in \mathcal{O}''$ by Lemma 4.2.4.

"$\subseteq$":
The statement, which we want to show, would be

$$p \in \mathcal{O}'' \implies p \in D(\mathrm{ch}(\mathcal{O}))\,.$$

The contrapositive of this statement is

$$p \notin D(\mathrm{ch}(\mathcal{O})) \implies p \notin \mathcal{O}'' = M \setminus J(\mathcal{O}')\,.$$

[2]See Appendix, Lemma A.1.

So let $p \notin D(\mathrm{ch}(\mathcal{O}))$, i.e. there exists a causal inextendible curve c, with $p \in c$, which does not meet $\mathrm{ch}(\mathcal{O})$. By Lemma 4.4.6 above, c needs to leave $J(\mathcal{O})$. In other words $c \cap M \setminus J(\mathcal{O}) \neq \emptyset$, hence $c \cap \mathcal{O}' \neq \emptyset$ by Definition 4.2.1.
Therefore $p \in J(\mathcal{O}') = M \setminus \mathcal{O}''$. $\square$

Remark. Note that the first part of the proof, i.e., the statement $D(\mathrm{ch}(\mathcal{O})) \subseteq \mathcal{O}''$ works for any subset and does not require global hyperbolicity.

We finish this section and this chapter by collecting some immediate implications of the previous theorem.

Corollary 4.4.8 (Cauchy development and causal completion)
Let M be a space-time and $V \subseteq M$ a subset.
Then the Cauchy development of V is contained in the causal completion of V, i.e.

$$D(V) \subseteq V''.$$

Proof. By the first part of the proof of Theorem 4.4.7 and the remark above we see that $D(\mathrm{ch}(V)) \subseteq V''$ holds even for arbitrary sets and since $V \subseteq \mathrm{ch}(V)$, we have $D(V) \subseteq D(\mathrm{ch}(V)) \subseteq V''$. $\square$

Corollary 4.4.9 (Causal Development of open causally convex sets)
Let M be a globally hyperbolic space-time and $\mathcal{O} \in \mathfrak{B}(M)$. Additionally assume $\mathcal{O}$ is causally convex.
Then $D(\mathcal{O})$ is causally convex.

Proof. By Lemma 4.4.5 and Theorem 4.4.7 we have $D(\mathcal{O}) = D(\mathrm{ch}(\mathcal{O})) = \mathcal{O}''$. The latter being causally convex by Corollary 4.4.9. $\square$

Corollary 4.4.10 (Causal completion is open)
Let M be a globally hyperbolic space-time and $\mathcal{O} \in \mathfrak{B}(M)$.
Then the causal completion $\mathcal{O}''$ is open.

Proof. Since $\mathcal{O}$ is open, $\mathrm{ch}(\mathcal{O})$ is open by Corollary 4.4.2. It follows then by Lemma 4.3.5 and Theorem 4.4.7 that $\mathcal{O}'' = D(\mathrm{ch}(\mathcal{O}))$ is open. $\square$

We can also finally answer the question whether $\mathcal{O} \in \mathfrak{B}$ implies $\mathcal{O}'' \in \mathfrak{B}$.

Corollary 4.4.11 (Causal completion is open and bounded)
Let M be Minkowski space. Then

$$\mathcal{O} \in \mathfrak{B}(M) \implies \mathcal{O}'' \in \mathfrak{B}(M).$$

Proof. Openness follows from Corollary 4.4.10 since Minkowski space is globally hyperbolic.

Global hyperbolicity also yields that $J^+(\overline{\mathcal{O}}) \cap J^-(\overline{\mathcal{O}})$ is compact (Lemma 4.4.3). The set $\overline{\mathrm{ch}(\mathcal{O})}$ is then compact as a closed subset of a compact set since

$$\overline{\mathrm{ch}(\mathcal{O})} = \overline{J^+(\mathcal{O}) \cap J^-(\mathcal{O})} \subseteq \overline{J^+(\overline{\mathcal{O}}) \cap J^-(\overline{\mathcal{O}})} = J^+(\overline{\mathcal{O}}) \cap J^-(\overline{\mathcal{O}}) \,.$$

Hence, $\overline{D(\overline{\mathrm{ch}(\mathcal{O})})}$ is compact by Lemma 4.3.6 and with Theorem 4.4.7 we then get

$$\overline{\mathcal{O}''} = \overline{D(\mathrm{ch}(\mathcal{O}))} \subseteq \overline{D(\overline{\mathrm{ch}(\mathcal{O})})} \,,$$

which implies $\overline{\mathcal{O}''}$ is compact. $\square$

5. Equivalence of Dynamics - Time-Slice-Axiom and Causal-Completion-Axiom

After we familiarized us in the last chapter with the objects from causality theory present in the Haag-Kastler Axioms, we are now equipped to tackle the axiom of Dynamics.

Our goal in this chapter is to prove an equivalence of the Time-Slice Axiom present in the HKA by Fredenhagen and the Causal Completion Axiom by Haag. We will discuss how this relates to the causal shadow mentioned by Haag and Kastler.
This then lets us finish the question of the equivalence of the Haag-Kastler Axioms.

5.1. Translating the Time-Slice-Axiom

As we have seen in Chapter 3 the key to connecting the Fredenhage-Version of the HKA to the other two was the inductive limit. Every property was translated to an equivalent statement in the inductive limit/ induced net. Hence, it makes sense to do it as well for the TSA.

Lemma 5.1.1 (Time-Slice Axiom II)
Assume F1)-F2) hold. Then the following are equivalent:

TSA: If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is an isomorphism.

TSA v2: If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $\widetilde{\mathcal{A}}(\mathcal{O}_1) = \widetilde{\mathcal{A}}(\mathcal{O}_2)$, where $\{\widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$ is the induced net of F1)-F2).

Proof. Remember that we have $\widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}_2)$, whenever $\mathcal{O}_1 \subseteq \mathcal{O}_2$, and a family of $*$-isomorphisms $\{i_{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \to \widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$, with the condition $i_{\mathcal{O}_2} i_{\mathcal{O}_2,\mathcal{O}_1} = i_{\mathcal{O}_1}$ if $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ and $\mathcal{O}_1 \subseteq \mathcal{O}_2$, from the construction of the induced net of F1)-F2).

"TSA $\Longrightarrow$ TSA v2":
If $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism then so is $\psi := i_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} \circ (i_{\mathcal{O}_1})^{-1} : \widetilde{\mathcal{A}}(\mathcal{O}_1) \to \widetilde{\mathcal{A}}(\mathcal{O}_2)$.
Due to the condition $i_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_1}$, we have $\psi(A) = A \, \forall A \in \widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}_2)$. Or in other words $\psi(\widetilde{\mathcal{A}}(\mathcal{O}_1)) = \widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}_2)$.
ψ being a $*$-isomorphism then yields $\widetilde{\mathcal{A}}(\mathcal{O}_2) = \psi(\widetilde{\mathcal{A}}(\mathcal{O}_1)) = \widetilde{\mathcal{A}}(\mathcal{O}_1)$.

"TSA v2 $\Longrightarrow$ TSA":
We can always rewrite the condition $i_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_1}$ to $i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_2}^{-1} \circ i_{\mathcal{O}_1}$. Even though $i_{\mathcal{O}_1}$ and $i_{\mathcal{O}_2}^{-1}$ are $*$-isomorphism $i_{\mathcal{O}_2\mathcal{O}_1}$ is a priori only a injective $*$-homomorphism, since $\widetilde{\mathcal{A}}(\mathcal{O}_1) \subseteq \widetilde{\mathcal{A}}(\mathcal{O}_2)$.
But the hypothesis in this direction, namely $\widetilde{\mathcal{A}}(\mathcal{O}_1) = \widetilde{\mathcal{A}}(\mathcal{O}_2)$, then makes $i_{\mathcal{O}_2\mathcal{O}_1}$ also surjective, hence a $*$-isomorphism. □

5.2. Dynamics

We are now able to prove one of the implications between the CCA and TSA.

Lemma 5.2.1 (CCA implies TSA)
The axiom

CCA: If $\mathcal{O}''$ is the causal completion of $\mathcal{O} \in \mathfrak{B}$, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$,

where we assume H1) holds, implies

TSA: If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism,

where we assume F1)-F2) hold.

Note that with Corollary 4.4.11 it is now justified to associate a C^*-algebra to $\mathcal{O}''$.

Proof. Let $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contain a Cauchy hypersurface S of $\mathcal{O}_2$ and assume that H1) holds. By Lemma 4.3.9 we have that $\mathcal{O}_2 \subseteq D(S)$ and by Corollary 4.4.8 we know $D(S) \subseteq S''$. This yields the chain $S \subseteq \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq D(S) \subseteq S'' \subseteq \mathcal{O}_1''$ by the basic properties of the causal complement, hence,

$$\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1'') \tag{5.1}$$

by the isotony of H1).

By the discussion in Section 3.2 we know F1)-F2) hold with $i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$ being the corresponding subalgebra embedding.
The CCA then implies $\mathcal{A}(\mathcal{O}_1) = \mathcal{A}(\mathcal{O}_2)$ by (5.1) which means that the "subalgebra" embedding, i.e., $i_{\mathcal{O}_2\mathcal{O}_1}$, is a $*$-isomorphism. □

The next step of course would be to show the other direction. However we believe that the converse is not true in general. This is because in essence we would like to prove this by showing that the causal completion $\mathcal{O}''$, for $\mathcal{O} \in \mathfrak{B}$, always has a Cauchy hypersurface S and S is contained in $\mathcal{O}$, so we can then apply the TSA.
The existence of a Cauchy hypersurface can be shown[1] but that it is contained in $\mathcal{O}$ seems not to hold in general (See Figure 5.1).

[1] A sketch of a proof: Since $\mathcal{O}''$ is open and causally convex(Corollary 4.4.10, Corollary 4.2.5) it also globally hyperbolic by definition of globally hyperbolic subsets [BGP07, 1.3.9] (or the remark after Definition 4.1.2). The induced manifold then has a Cauchy hypersurface by 4.1.4.

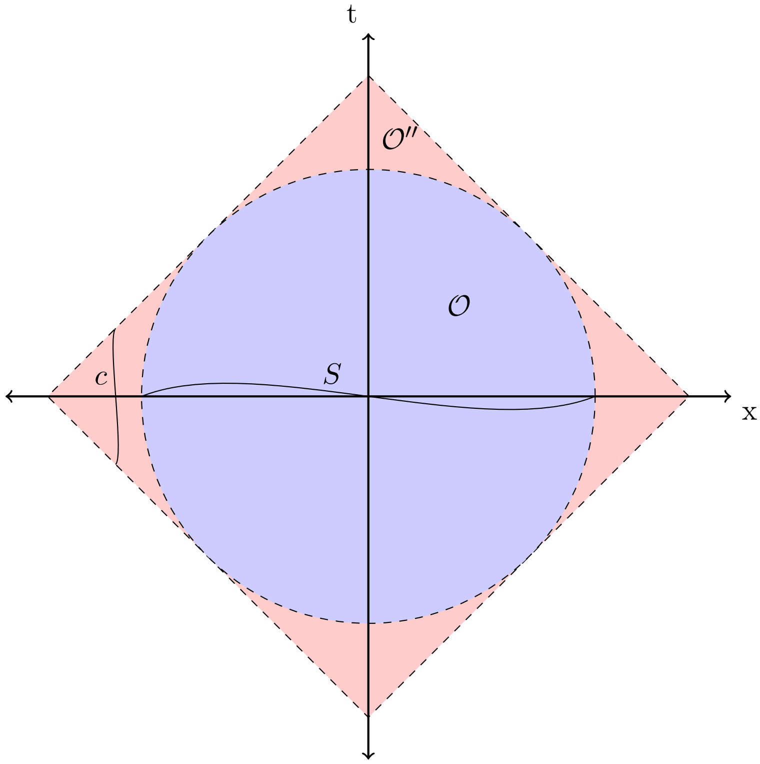

Figure 5.1.: Consider $\mathbb{R}^{1,1}$ and $\mathcal{O}$ any open ball (blue shaded area). Let S be a Cauchy hypersufrace of $\mathcal{O}''$ (red shaded area) and assume that S is contained in $\mathcal{O}$. Then we can construct a timelike inextendible curve c in $\mathcal{O}''$ which does not meet S, contradicting that S is a Cauchy hypersurface of $\mathcal{O}''$.

Even if it is not true that $\mathcal{O}$ contains a Cauchy hypersurfaces of $\mathcal{O}''$ in general, this does not disprove the implication "TSA $\Longrightarrow$ CCA", since the actual statement is about the associated algebras.

Luckily, if we restrict ourselves to only looking at $\mathcal{O} \in \mathfrak{B}$ which are in addition causally convex a proof can be found.

Lemma 5.2.2 (TSA implies CCA)
The axiom

TSA: If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism,

where we assume F1)-F2) hold, implies

CCA: If $\mathcal{O}''$ is the causal completion of $\mathcal{O} \in \mathfrak{B}$, $\mathcal{O}$ causally convex, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$,

where we assume H1) holds.

Proof. Let $\mathcal{O} \in \mathfrak{B}$ and causally convex and assume that F1)-F2) hold.
Then $\mathcal{O} = \mathrm{ch}(\mathcal{O})$ by Lemma 4.4.5. By Theorem 4.4.7 we know that $\mathcal{O}'' = D(\mathrm{ch}(\mathcal{O})) = D(\mathcal{O})$. Lemma 4.3.7 then guarantees the existence of a Cauchy hypersurface S of $D(\mathcal{O})$ which is contained in $\mathcal{O}$.
Hence, S is a Cauchy hypersurface of $\mathcal{O}''$ and $S \subseteq \mathcal{O} \subseteq \mathcal{O}''$.

We know form Section 3.2 that H1) hold on the induced net of F1)-F2) $\{\widetilde{\mathcal{A}}(\mathcal{O}) \mid \mathcal{O} \in \mathfrak{B}\}$. By 5.1.1 we know that TSA v2 holds which then implies CCA, i.e., $\widetilde{\mathcal{A}}(\mathcal{O}) = \widetilde{\mathcal{A}}(\mathcal{O}'')$. □

The last thing we need to adress is the axiom of Dynamics in the Version HK of the HKA.
As discussed in Section 2.1.1, we could find two possible definitions for the mentioned causal shadow: Either as the Cauchy development or the causal completion. Fortunately for us, we can now rapidly conclude that those are actually the same in the case of open, bounded and causally convex sets. Indeed by 4.4.7 and 4.4.5 we get $\mathcal{O}'' = D(\mathrm{ch}(\mathcal{O})) = D(\mathcal{O})$.

Hence if we agree upon using, for instance, the Cauchy development as the definition for the causal shadow we can summarise this chapter with the following proposition:

Proposition 5.2.3 (Equivalence of the Axiom of Dynamics)
The following three axioms are equivalent:

F5) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism,

where we assume F1)-F2) hold.

H4) If $\mathcal{O}''$ is the causal completion of $\mathcal{O} \in \mathfrak{B}$, with $\mathcal{O}$ causally convex, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$,

where we assume H1) holds.

HK6) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ and $\mathcal{O}_1$ causally convex such that $\mathcal{O}_2 \subseteq D(\mathcal{O}_1)$, then $\mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1)$,

where we assume HK1)-HK3) hold.
More precisely we mean that F5) implies H4) and HK6) on the induced net of F1)-F2) and conversly H4) and HK6) can be translated to F5).

Proof. The equivalence of F5) and H4) is given by Lemma 5.2.1 and Lemma 5.2.2.

Let H1) and H4) hold. Then by Section 3.2 we know HK1)-HK3) hold as well.
If we assume now we have $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{B}$ and $\mathcal{O}_1$ causally convex such that $\mathcal{O}_2 \subseteq D(\mathcal{O}_1)$ we get, by 4.4.7 and the causal convexity of $\mathcal{O}_1$, that $\mathcal{O}_2 \subseteq D(\mathcal{O}_1) = \mathcal{O}_1''$.
Hence, by the isotony of H1) and H4) we get $\mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1'') = \mathcal{A}(\mathcal{O}_1)$.

Assume now HK1)-HK3) and HK6) hold. Again by Section 3.2 we know H1) holds as well.
Let $\mathcal{O} \in \mathfrak{B}$ with $\mathcal{O}$ being causally convex. Clearly $D(\mathcal{O}) \subseteq D(\mathcal{O})$. HK6 then implies $\mathcal{A}(D(\mathcal{O})) \subseteq \mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}'')$, where the last inclusion comes form $\mathcal{O} \subseteq \mathcal{O}''$ and HK3).
Hence, $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'')$, since again $D(\mathcal{O}) = \mathcal{O}''$ by 4.4.7 and the causal convexity. □

5.3. Equivalence of the Haag-Kastler Axioms

Let us recap what has been shown so far.
After clearing some notation we saw that most of the axioms in the Fredenhagen-, Haag- and Haag-Kastler-Versions of the Haag-Kastler Axioms are equivalent. For this we used the inductive limit and its properties.
The equivalence of the axiom of Dynamics required us to restrict to using open, bounded and causally convex sets, instead of only open and bounded. This should not be an issue for most applications since the usual "nice" sets one thinks of are "ball-like" and causally convex. Additionally one can always associate a causally convex set to any $\mathcal{O} \in \mathfrak{B}$ by taking $\mathcal{O} \subseteq \mathrm{ch}(\mathcal{O})$.

So for the restriction we introduce the last bit of notation.

Definition 5.3.1 (Set of open, relatively compact and causally convex subsets)
Let M be a space-time. Then we denote

$$\mathfrak{C}(M) := \{\mathcal{O} \subseteq M \mid \mathcal{O} \text{ open, relatively compact and causally convex}\}.$$

Moreover, we write the shorthand $\mathfrak{C} := \mathfrak{C}(\mathbb{R}^{1,d})$, where $\mathbb{R}^{1,d}$ is the $d+1$-dimensional Minkwoski space.

Remark. This definition can also be found in [FV15, p.131], though with a different notation.

If M is globally hyperbolic then $\mathfrak{C}(M)$ is a directed set with inclusion. Indeed, $\mathfrak{C}(M)$ inherits the inclusion relation from $\mathfrak{B}(M)$ and for any $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}(M)$ we have $\mathcal{O}_1 \cup \mathcal{O}_2 \subseteq \mathrm{ch}(\mathcal{O}_1 \cup \mathcal{O}_2)$, where $\mathrm{ch}(\mathcal{O}_1 \cup \mathcal{O}_2) \in \mathfrak{C}(M)$ by 4.4.4 and $\mathrm{ch}(\mathcal{O}_1 \cup \mathcal{O}_2)$ always being causally convex.

Hence we can summarize the equivalence of the different sets of the Haag-Kastler Axioms.

Theorem 5.3.2 (Equivalence of HKA)
The following three sets of Haag-Kastler Axioms are equivalent:
HKA by Fredenhagen:

F1) There exists a unital C^*-Algebra $\mathcal{A}(\mathcal{O})$ for all $\mathcal{O} \in \mathfrak{C}$.

F2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then there exists an unital injective $*$-homomorphism, $i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$, s.t.

$$i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3 \,.$$

F3) If $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O} \in \mathfrak{C}$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated and $\mathcal{O}_1 \cup \mathcal{O}_2 \subseteq \mathcal{O}$, then

$$[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0 \quad \forall A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2) \,.$$

F4) For every $L \in \mathfrak{P}$, the Poincaré group, there is a family of $*$-isomorphisms $\{\alpha_L^{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(L\mathcal{O}) \mid \mathcal{O} \in \mathfrak{C}\}$ such that for all $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and $L, L_1, L_2 \in \mathfrak{P}$ we have

$$\begin{aligned} \alpha_L^{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} &= i_{L\mathcal{O}_2 L\mathcal{O}_1} \circ \alpha_L^{\mathcal{O}_1} \,, \\ \alpha_{L_2}^{L_1\mathcal{O}} \circ \alpha_{L_1}^{\mathcal{O}} &= \alpha_{L_2 L_1}^{\mathcal{O}} \,. \end{aligned}$$

F5) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $i_{\mathcal{O}_2\mathcal{O}_1}$ is a $*$-isomorphism.

HKA by Haag:

H1) There exists a net of unital C^*-Algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ for every $\mathcal{O} \in \mathfrak{C}$.

H2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike seperated, then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute in $\mathcal{A}$, where $\mathcal{A}$ is the inductive limit which we can associate to H1) (by 3.1.1).

H3) There exists a group homomorphism $L \mapsto \alpha_L$ from $\mathfrak{P}$ to the $*$-automorphisms on $\mathcal{A}$ such that $\alpha_L(\mathcal{A}(\mathcal{O})) = \mathcal{A}(L\mathcal{O})$ for all $L \in \mathfrak{P}$.

H4) If $\mathcal{O}''$ is the causal completion of $\mathcal{O} \in \mathfrak{C}$, with $\mathcal{O}$ causally convex, then $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})$.

HKA by Haag-Kastler:

HK1) There exists a unital C^*-Algebra $\mathcal{A}(\mathcal{O})$ for every $\mathcal{O} \in \mathfrak{C}$.

HK2) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$, such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$. Additionally, both $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ have the same unit.

HK3) The set-theoretic union $\mathcal{A}(\mathfrak{C}) = \bigcup_{\mathcal{O} \in \mathfrak{C}} \mathcal{A}(\mathcal{O})$ is a normed * -algebra and its completion $\mathcal{A}$ is a C^*-Algebra.

HK4) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ such that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute in $\mathcal{A}$.

HK5) There exists a group homomorphism $L \mapsto \alpha_L$ from $\mathfrak{P}$ to the $*$-automorphisms on $\mathcal{A}$ such that $\alpha_L(\mathcal{A}(\mathcal{O})) = \mathcal{A}(L\mathcal{O})$ for all $L \in \mathfrak{P}$.

HK6) If $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{C}$ and $\mathcal{O}_1$ causally convex such that $\mathcal{O}_2 \subseteq D(\mathcal{O}_1)$, then $\mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1)$.

Proof. The discussion in Section 3.2 and Section 3.2 only required that $\mathfrak{B}$ was a directed set. Hence, we can still apply it with $\mathfrak{C}$. So Section 3.2 implies the equivalence of F1)-F2), H1) and HK1)-HK3) and Section 3.2 implies the equivalence of F3), H2) and HK4).
For Proposition 3.4.1 to be applicable we additionally need that $L\mathcal{O}$ is open, bounded and causally convex for any $L \in \mathfrak{P}$ and $\mathcal{O} \in \mathfrak{C}$. Luckily this is true, since L is continuous, has a continuous inverse and respects causal structure. Therefore Proposition 3.4.1 implies the equivalence of F4), H3) and HK5).
The equivalence of F5), H4) and HK6) is straight forwardly given by Proposition 5.2.3. □

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Acronyms

CCA Causal Completion Axiom. 9, 11, 41–44

HKA Haag-Kastler Axioms. iii, 1, 3, 4, 7–9, 11–13, 15, 19, 20, 23, 27, 29, 41, 44–46

TSA Time-Slice Axiom. 10, 11, 41–44

Version F Version Fredenhagen. 12, 21–23

Version H Version Haag. 12, 21–23

Version HK Version Haag-Kastler. 12, 21, 23, 44

A. Appendix

Lemma A.1
Let X be any topological space, $A \subseteq X$ any subset and $c : [0,1] \to X$ a continuous curve with $c(0) \in A$ and $c(1) \in X \setminus A$.
Then $c([0,1]) \cap \partial A \neq \emptyset$.

Proof. Case 1:
If $c(0) \notin A^\circ$ then $c(0) \in A \setminus A^\circ \subseteq \overline{A} \setminus A^\circ = \partial A$.

Case 2:
If $c(0) \in A^\circ$ and $c(1) \in \overline{A}$ then $c(1) \notin A^\circ$, since $c(1) \in X \setminus A \subseteq X \setminus A^\circ$, and hence $c(1) \in \overline{A} \setminus A^\circ = \partial A$.

Case 3:
If $c(0) \in A^\circ$ and $c(1) \notin \overline{A}$ then Define $C_0 := c^{-1}(A^\circ) = \{t \in [0,1] \mid c(t) \in A^\circ\}$ and $C_1 := c^{-1}((X \setminus A)^\circ) = \{t \in [0,1] \mid c(t) \in (X \setminus A)^\circ\}$.
By the assumption of this case we get $C_0, C_1 \neq \emptyset$, since $X \setminus \overline{A} = (X \setminus A)^\circ$, and by continuity of c we find that C_0 and C_1 are open. Additionally, since A° and $(X \setminus A)^\circ$ are disjoint, so are C_0 and C_1.
Assume now $c([0,1]) \cap \partial A = \emptyset$. Rewritten this means $c(t) \notin \overline{A} \setminus A^\circ$ for all $t \in [0,1]$. So

$$\forall t \in [0,1] : c(t) \in A^\circ \quad \text{or} \quad c(t) \in X \setminus \overline{A} = (X \setminus A)^\circ\,.$$

This yield that $C_0 \cup C_1 = [0,1]$ which contradicts the connectedness of $[0,1]$. □