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1. Introduction

"If men define situations as real, they are real in their consequences." This result known in sociology as the Thomas Theorem stresses the fact that a situation is determined by the actions undertaken by agents which are however based on the subjective perceptions about this situation, implying that expectations can become self - fulfilling. Since this should be true for any social interaction, it should in particular hold for economic interactions and indeed a classical example for a so called self - fulfilling prophecy given by the sociologist Robert Merton describes the bankruptcy of a bank caused by the expectations about this event which led the costumers to withdraw their money.

However, although the effect of uncertainty on economic outcomes has been the subject of extensive research throughout economic history, it was not until the early 1980's that economists also considered the effect of uncertainty entering the analysis solely through agents' expectations, but not affecting any fundamentals of the economy.

Azariadis (1981), who was the first to introduce the concept of so called extraneous or extrinsic uncertainty to a simple macroeconomic model, motivated his analysis by historic examples for speculative bubbles, such as the Dutch tulip mania, occurring in the first half of the 17th century. During this event, prices for tulip bulbs exploded without any objective reason. Examples like that led Azariadis to the hypothesis that the dependence of a rational expectations equilibrium on the evolution of some extrinsic variable, which has no effect on the fundamentals of the economy, is possible if expectations which are conditioned on this variable become self - fulfilling. Azariadis (1981) refers to such equilibria as "replicating equilibria" since due to the self - fulfilling aspect of expectations it becomes indeed rational for future generations to have these expectations themselves. Moreover, he shows that in a simple overlapping generations model up to one half of all rational expectations equilibria are influenced by self - fulfilling beliefs.

Another pioneering contribution to the literature on extrinsic uncertainty is given by Cass and Shell (1983), who also coined the term "sunspot equilibrium" for a rational expectations equilibrium in which extrinsic uncertainty matters. They identify the natural restrictions on market participation in economies with agents who can only live for a finite time, i.e. the fact that people can only participate in markets while they are alive, as one cause for the existence of these equilibria. However, Cass and Shell base their analysis mainly on microeconomic considerations, such as the analysis of so called contingent claim markets and unlike Azariadis they do not consider

a dynamic model and are thus not able to take the replicating or self - perpetuating aspect of expectation formation into account. Therefore, the focus of this thesis will be on the strand of literature based on Azariadis (1981) and the subsequent contributions by Azariadis and Guesnerie (1982) and (1986), who developed more rigorous conditions for the existence of sunspot equilibria in a simple overlapping generations model. However, the analysis of Cass and Shell (1983) shows that extrinsic variables can affect an economy even if there exist markets for insuring against the risk associated with the extrinsic uncertainty. This argument was also used by Woodford (1987) to explain the relevance of the concept of sunspot equilibria.

Although the result that extrinsic uncertainty can affect a rational expectations equilibrium might at first glance seem to be a satisfying explanation for speculative bubbles, this is not necessarily true: As there will in each case also exist rational expectations equilibria for which extrinsic uncertainty does not matter, it is not clear if agents can actually coordinate on a sunspot equilibrium. This problem can be solved by assuming that agents use an adaptive learning process in order to form their expectations about the economic variables of interest. Ideally, only one rational expectations equilibrium can be attained asymptotically by such a learning rule, at least when it is taken into account that agents might initially misspecify the law of motion for the relevant economic variable.

Since the assumption of rational expectations relies in general on great insight of the agents into the economic relationships, such as on the knowledge of all relevant parameter values of the model, it is at least in the short run rather unplausible to assume that agents possess the ability to form rational expectations. Therefore, the implications of adaptive learning behaviour for the outcome of various linear expectations models has been widely studied. In particular this analysis is based on a mathematical technique known as stochastic approximation, as discussed by Ljung (1977) and as further developed by Evans and Honkapohja (1998a).

The remainder of this thesis is organized as follows: Section 2 discusses the existence of sunspot equilibria under the assumption of rational expectations in a simple overlapping generations model. Therefore, I will first describe the perfect foresight dynamics of this model, before discussing the concept of extrinsic uncertainty more rigorously. After that, a link between the perfect foresight dynamics and the existence of sunspot equilibria will be established.

Section 3 will then provide an introduction to the analysis of so called stochastic recursive algorithms which are frequently used in economic models to describe

adaptive learning processes. Therefore, I will first introduce different specifications for the learning behaviour of agents. After that, I will describe a mathematical technique known as stochastic approximation which is frequently used in order to analyse the qualitative behaviour of such learning algorithms. After illustrating this technique within the well known Cobweb model, I will proceed to describe a simpler stability concept for rational expectations equilibria under agents' learning behaviour due to Evans (1989) and discuss the robustness of the stability results obtained for more general univariate expectations models than the Cobweb model with respect to different specifications of the learning rule, as done for example by Evans and Honkapohja (1992).

Section 4 will then introduce two plausible specifications of the learning behaviour of agents in the overlapping generations model already discussed in section 2 and will use the results given in section 3 in order to analyse whether sunspot equilibria can actually be regarded as realistic outcomes. In order to answer this question, I will concentrate on the results obtained by Woodford (1990), who was the first to discuss the stability of sunspot equilibria under adaptive learning processes. However, additionally I will also discuss results for a specific parametric class of utility functions obtained by other authors.

Section 5 contains concluding remarks.

2. Sunspot Equilibria

2.1. Perfect Foresight Equilibria in an OLG Model

Before introducing the concept of extrinsic uncertainty within the framework of a basic overlapping generations model, I will first discuss the nature of perfect foresight equilibria in this model. This analysis will later on serve as a benchmark case for the existence of sunspot equilibria. For this purpose, I am going to introduce the Samuelson overlapping generations model with money, as done for example in Azariadis (1981), Azariadis and Guesnerie (1986) or Woodford (1990).

Therefore, it is assumed that time is discrete and extends to infinity. In every period a constant number of agents is born and lives for two periods. Hence, there is no population growth in the economy under consideration. Furthermore, all agents in one generation are assumed to be identical, which implies that it is only necessary to consider one representative agent of each generation in order to analyse the model. A member of the first generation born in period zero is endowed with one unit of

fiat money¹, which is used as the unique store of value.² It is assumed that this money supply is stationary and that the nominal interest rate is zero.

For all agents not born in period zero the following applies: An agent born in period t is young in this period and is endowed with a certain amount of time $\hat{n}$ which can either be consumed as leisure or can be used as an input in a constant returns to scale technology in order to produce a single consumption good³ denoted as y_t . In other words, production of the consumption good y_t is a linear function of n_t , the labour supply of an agent born in period t . Hence, by choosing measurement units for y_t appropriately, it can be assumed without loss of generality that the production function is given by $y_t = n_t$.

Moreover, it is assumed that agents cannot consume when they are young and that they cannot transfer any production directly into their old age. Hence, agents who are young in period t must sell their entire production to those who are old in this period (i.e. those who were born in period $t-1$) at the market price p_t . Therefore, at the end of period t each member of the generation born in this period holds $p_t y_t$ units of fiat money which he can spend in his old age on consumption. Furthermore, it is assumed that all agents take prices as exogenously given and that both the money and the goods market are perfectly competitive.⁴ Therefore, prices adjust in such a way that markets clear, meaning that for all periods t it holds that $p_t n_t = p_t c_t = 1$, where c_t denotes the consumption in period t of an agent born in period $t-1$. Since the money supply of an old agent is constant at one, this condition requires that each period the stock of money is used entirely in transactions between old agents (consumers) and young agents (producers) and that the price level is such that the optimal labour supply choice of young agents and the optimal consumption choice of old agents coincide.

Agents are furthermore assumed to have intertemporal preferences over leisure and consumption which can be represented by an additively separable utility function⁵

¹The term fiat money refers to some medium of exchange, the value of which is only institutionally regulated and accepted, but not intrinsically given.

²The use of fiat money as the store of value is not crucial to the following analysis. As Azariadis (1981) shows all results of the model, especially the existence of stationary sunspot equilibria, would remain unchanged if a durable commodity (e.g. land) was used instead of fiat money.

³This consumption good can be interpreted as a consumer basket, where its price corresponds to the respective consumer price index. Guesnerie (1986) considers the more realistic situation in which there is a finite number of consumption goods. However, this extension does not change the results on the existence of stationary sunspot equilibria given here much.

⁴Note that this can be justified by thinking of each generation as a continuum of identical individuals.

⁵The assumption of an additively separable utility function is not crucial, but made in order to keep the notation as simple as possible. For a more general specification of agents' utility see

$u(c_{t+1}) - v(n_t)$, where $v(n_t)$ denotes the disutility associated with working. It is assumed that $u(\cdot)$ and $v(\cdot)$ are strictly increasing, twice continuously differentiable and that $u''(\cdot) < 0$, whereas $v''(\cdot) > 0$ on $\mathbb{R}_+$ and $[0, \hat{n}]$, respectively. This implies that the utility function is strictly concave in both arguments c_{t+1} and n_t .

For future reference, the following assumptions on the limiting behaviour of the utility function are made:

Assumption: The utility function $u(c) - v(n)$ satisfies:

$$(i) \lim_{c \rightarrow 0} u'(c) = \infty$$

$$(ii) \lim_{n \rightarrow \hat{n}} v'(n) = \infty$$

Note that these assumptions made by Woodford (1990) together with the assumption of strict concavity guarantee that utility maximization yields a maximum with positive amounts of leisure and consumption, respectively and that thus the first order condition of the maximization problem is both necessary and sufficient for a global maximum.

Furthermore, it is assumed that preferences are such that both leisure and consumption are normal goods, i.e. such that an increase in the income of an agent results in an increased demand for both goods.

As already noted above, money market clearing imposes that demand and supply for the consumption good in period t satisfy $p_t c_t = p_t n_t = 1$ and thus $c_t = n_t = \frac{1}{p_t}$. Therefore, a perfect foresight equilibrium in this model can be defined as follows:

Definition: A perfect foresight equilibrium corresponds to a sequence of non-negative prices $(p_1, p_2, \dots)$, which is such that in each period all markets clear and for which $n_t = \frac{1}{p_t}$, i.e. the labour supply in any period t , is chosen optimally.

More precisely, an agent born in period t chooses his labour supply and his old - age consumption such that it maximizes his lifetime utility given that his intertemporal budget restriction is not violated.

Hence, an agent born in period t tries to solve the following problem:

$$\begin{array}{ll} \max_{n_t, c_{t+1}} & u(c_{t+1}) - v(n_t) \\ s.t. & p_t n_t \geq p_{t+1} c_{t+1} \end{array}$$

for example Azariadis and Guesnerie (1986)

where the constraint requires that not more than the income earned through production in period t is spent on consumption in period $t + 1$. Since utility is strictly increasing in c_{t+1} (i.e. preferences are locally non - satiated), this constraint must be binding. If this were not the case, it would be possible to increase consumption in period $t + 1$ without increasing n_t which under the assumptions made on preferences would increase utility.

Substituting the constraint into the objective function shows that the supply of labour in period t , and therefore also the supply of the consumption good in this period, is given by a function $s(R_t)$ which fulfils:

$$s(R_t) = \arg \max_{n_t} u(R_t n_t) - v(n_t)$$

where since the nominal interest rate is zero, $R_t := \frac{p_t}{p_{t+1}}$ denotes the real return on money holdings from period t to period $t + 1$. Note that since p_t corresponds to the nominal wage earned by an agent born in period t and p_{t+1} corresponds to the price of consumption such an individual will face, R_t can also be interpreted as the real wage of an agent born in period t .

Goods market clearing in period t implies that the excess demand for the consumption good in this period is zero. Using the money market clearing condition to express demand for consumption in period t through p_t , this implies that equilibrium prices must fulfil the following condition:⁶

$$D(p_t, p_{t+1}) = \frac{1}{p_t} - s\left(\frac{p_t}{p_{t+1}}\right) = 0 \quad (1)$$

which corresponds to an implicit first order difference equation that can be used in order to determine the equilibrium price level sequence given any initial value.

However, since the economy described by this model is stationary, it is natural to analyse the existence of perfect foresight equilibria for which the price level is constant over time first.⁷

In order to do this, note that since $u(\cdot) - v(\cdot)$ is strictly concave, $s(R_t)$ is a single valued function, meaning that given the real wage only one level of labour supply maximizes lifetime utility. Therefore, in particular the equation $\frac{1}{p} = s(1)$ must have

⁶Note that this condition can also be described by noting that the equilibrium prices are determined as the intersection of the supply curve $s(\cdot)$ and the demand curve.

⁷Note that stationary equilibria are not the only form of equilibria which can occur in this model. As Azariadis (1981) illustrates, there is also the possibility of cycles.

a unique solution p^* . Since $\frac{1}{p_t}$ also has the interpretation of the commodity price of money, i.e. of how much consumption has to be given up in order to buy one unit of money, it can therefore be concluded that $p_t = p^*$ for all t is the only stationary equilibrium in which money has a positive value. Hence, this equilibrium is often referred to as the monetary steady state.

However, the sequence $\mathbf{p}^* := (p^*, p^*, \dots)$ does not have to be the unique stationary perfect foresight equilibrium. In order to see this, note that optimal labour supply $s\left(\frac{p_t}{p_{t+1}}\right)$ has to fulfil the first order condition:

$$\frac{p_t}{p_{t+1}} u' \left(\frac{p_t}{p_{t+1}} s \left(\frac{p_t}{p_{t+1}} \right) \right) - v' \left(s \left(\frac{p_t}{p_{t+1}} \right) \right) = 0$$

Using the goods market clearing condition (1) this can be rewritten as:

$$U \left(\frac{1}{p_{t+1}} \right) = V \left(\frac{1}{p_t} \right)$$

where $U(y)$ and $V(y)$ are defined as $yu'(y)$ and $yv'(y)$, respectively.

Since by the assumptions made on the utility function $V\left(\frac{1}{p_t}\right)$ converges to zero as p_t goes to infinity, it follows that if $U\left(\frac{1}{p_{t+1}}\right)$ also converges to zero as p_{t+1} goes to infinity, $p_t = \infty \forall t$ also constitutes a stationary perfect foresight equilibrium. However, in this stationary equilibrium the commodity price of money is zero in all periods and thus money does not have any value. Such a situation is often referred to as an autarchy equilibrium. (Azariadis, 1981)

However, since any economy is subject to small random shocks, an equilibrium can only be regarded as a realistic outcome if small perturbations from the equilibrium result in an adjustment process of the economic variables leading back to the equilibrium. Therefore, in any dynamic model it is also necessary to undertake an analysis of the stability of equilibria with respect to such small shocks.

A useful tool for this stability analysis of perfect foresight equilibria in this model is the elasticity of labour supply with respect to the real wage, which can be obtained by differentiating the first order condition used to determine the labour supply function $s(R_t)$ with respect to R_t :

$$\frac{\partial}{\partial R_t} [R_t u'(R_t s(R_t)) - v'(s(R_t))] = 0$$

$$\begin{aligned}
u'(R_t s(R_t)) + R_t \left[s(R_t) u''(R_t s(R_t)) + R_t \frac{ds}{dR_t} u''(R_t s(R_t)) \right] - \frac{ds}{dR_t} v''(s(R_t)) &= 0 \\
\Rightarrow \frac{ds(R_t)}{dR_t} &= \frac{u'(R_t s(R_t)) + R_t s(R_t) u''(R_t s(R_t))}{-R_t^2 u''(R_t s(R_t)) + v''(s(R_t))} \\
\Rightarrow \frac{ds(R_t)}{dR_t} \Big|_{R_t=1} &= \frac{u'(s(1)) + s(1) u''(s(1))}{-u''(s(1)) + v''(s(1))}
\end{aligned}$$

Therefore, the elasticity of labour supply evaluated at the monetary steady state is given by:

$$\varepsilon(1) = \frac{ds(R_t)}{dR_t} \frac{R_t}{s(R_t)} \Big|_{R_t=1} \quad (2)$$

$$\varepsilon(1) = \left[\frac{u'(s(1)) + s(1) u''(s(1))}{-u''(s(1)) + v''(s(1))} \right] \frac{1}{s(1)} \quad (3)$$

Note that the sign of this expression is not determined by the assumptions made so far. Since the denominator is by assumption strictly positive, it only depends on the sign of the numerator $u'(s(1)) + s(1) u''(s(1)) = U'(s(1))$, where $U(\cdot)$ is defined as above.

In general, an increase in the real wage has two opposing effects on labour supply: On the one hand, given the labour supply choice it increases income, and since leisure and consumption are both assumed to be normal goods, this should result in an increase in the demand for leisure and old - age consumption, respectively.

On the other hand, an increase in the real wage implies that leisure has become relatively more expensive compared to old - age consumption and that thus agents will substitute leisure by old - age consumption and increase their labour supply.

If the income effect is dominating at the monetary steady state, leisure and consumption are locally gross complements and the elasticity of labour supply at the monetary steady state is negative.

If however the substitution effect is dominating at the monetary steady state, leisure and consumption are locally gross substitutes and the elasticity of labour supply is positive.

If the elasticity of labour supply is zero, the income and the substitution effect exactly offset each other.

However, the assumptions made so far imply the following property of $\varepsilon(1)$:

Lemma 1 *If leisure and consumption are locally at the monetary steady state gross*

complements, the elasticity of labour supply with respect to the real wage is such that

$$|\varepsilon(1)| < 1.$$

Proof: Since it is assumed that $-u''(\cdot) > 0$, $u'(\cdot) > 0$, $v''(\cdot) > 0$ and since $s(1)$ is a positive constant, the following holds:

$$-s(1)u''(s(1)) - u'(s(1)) < -s(1)u''(s(1)) + s(1)v''(s(1))$$

Moreover, since under the assumption of leisure and consumption being locally gross complements it holds that $|\varepsilon(1)| = -\varepsilon(1)$, it follows from (3) that:

$$|\varepsilon(1)| = \frac{-s(1)u''(s(1)) - u'(s(1))}{-s(1)u''(s(1)) + s(1)v''(s(1))} < 1.$$

With this technical background it is possible to establish the following result on the local stability of the monetary steady state:

Theorem 1 *The monetary steady state is locally asymptotically stable under the dynamics of the implicit first order difference equation (1) if $\varepsilon(1) < -\frac{1}{2}$.*

Proof: Following Azariadis and Guesnerie (1986), consider the first order Taylor expansion of equation (1), i.e. the excess demand, about the monetary steady state:

$$D(p_t, p_{t+1}) \approx \underbrace{\frac{1}{p^*} - s(1)}_{=0} + \left(-\frac{1}{p^{*2}} - \frac{1}{p^*}s'(1) \quad \frac{p^*}{p^{*2}}s'(1) \right) \begin{pmatrix} p_t - p^* \\ p_{t+1} - p^* \end{pmatrix}$$

Denoting deviations of actual prices from their monetary steady state levels as x_t and x_{t+1} , respectively and using that in an equilibrium excess demand must be zero leads to the following first order difference equation:

$$\begin{aligned} \left(-\frac{1}{p^{*2}} - \frac{1}{p^*}s'(1) \right) x_t + \frac{p^*}{p^{*2}}s'(1) x_{t+1} &= 0 \\ -x_t - p^*s'(1) x_t + p^*s'(1) x_{t+1} &= 0 \end{aligned}$$

Using the money market clearing condition at the monetary steady state (i.e. that $p^* = \frac{1}{s(1)}$) and the definition of $\varepsilon(1)$, i.e. equation (2), then yields:

$$\begin{aligned} (-1 - \varepsilon(1))x_t + \varepsilon(1)x_{t+1} &= 0 \\ x_{t+1} &= \frac{1 + \varepsilon(1)}{\varepsilon(1)}x_t \end{aligned}$$

Since this difference equation is the linearisation of the implicit difference equation (1) about the monetary steady state, it can be concluded that the monetary steady state is locally asymptotically stable if $\left| \frac{1+\varepsilon(1)}{\varepsilon(1)} \right| < 1$

Case 1: Leisure and consumption are locally at the monetary steady state gross complements ($\varepsilon(1) < 0$)

- (a) $\frac{1+\varepsilon(1)}{\varepsilon(1)} \geq 0 \Leftrightarrow \varepsilon(1) \leq -1$, in which case it always holds that $\frac{1+\varepsilon(1)}{\varepsilon(1)} < 1$
- (b) $\frac{1+\varepsilon(1)}{\varepsilon(1)} < 0 \Leftrightarrow \varepsilon(1) > -1$, in which case it has to hold that $-\frac{1+\varepsilon(1)}{\varepsilon(1)} < 1 \Leftrightarrow 1 + \varepsilon(1) < -\varepsilon(1) \Leftrightarrow \varepsilon(1) < -\frac{1}{2}$

To summarize: if leisure and consumption are locally at the monetary steady state gross complements, it has to hold that $\varepsilon(1) < -\frac{1}{2}$ for the monetary steady state to be locally asymptotically stable.

Case 2: Leisure and consumption are locally at the monetary steady state gross substitutes ($\varepsilon(1) > 0$)

In this case it can never hold that $\frac{1+\varepsilon(1)}{\varepsilon(1)} < 1$.

Note that this also implies that the monetary steady state can only be locally stable if the negative income effect of an increase in the real wage on labour supply outweighs the positive substitution effect at the monetary steady state by a sufficient margin and that conversely the monetary steady state is unstable if leisure and consumption are gross substitutes at the monetary steady state.

Azariadis (1981) and Azariadis and Guesnerie (1982) provide an intuitive argument for this latter result:

Suppose that the current price level is higher than the monetary steady state price level. Then, since demand for consumption is lower than in the monetary steady state, goods market clearing requires that labour supply is also lower than in the monetary steady state. Since leisure and consumption are assumed to be gross substitutes, this is only possible if the real wage is lower than in the monetary steady state (i.e. lower than 1), which implies that the price level in the next period must be higher than in the current period. This however implies that the price level di-

verges further away from its monetary steady state level. An analogous argument holds if the current price level is lower than the monetary steady state price level. If the monetary steady state is however locally asymptotically stable, it is always indeterminate, meaning that there are infinitely many price sequences all converging to the monetary steady state regardless of their initial value. As Woodford (1990) points out and as will be seen later, this feature is essential for the existence of sunspot equilibria.

2.2. Extrinsic Uncertainty in an OLG Model

By departing from the perfect foresight assumption made so far, it is necessary to specify how agents form their expectations about future prices. Therefore, it is assumed that the appearance of the world agents live in changes over time according to some stochastic process, meaning that agents do not know with certainty how their environment will be like in the next period. However, it is still assumed that the fundamentals of the economy, i.e. the endowments, preferences and the production technology, are stationary over time, meaning that they are not subject to the purely extrinsic uncertainty.

For the moment it is also assumed that agents expect a perfect correlation between the price level and the state of nature⁸, meaning that they expect prices to be the same in two periods if in these periods the world around them is the same, whereas they expect prices to be different if the world around them is different.

Loosely speaking, a sunspot equilibrium is then defined as a situation in which these particular expectations are self - fulfilling and hence become rational, in the sense that given the available information and the expectations of the other agents the expectations of a single agent are chosen optimally, meaning that they correspond to the objective mathematical expectations.⁹

For simplicity, it is assumed that the appearance of the world or state of nature can be captured by a Markov chain $(S_t)_{t \geq 0}$, which can take on values in the state space $I = \{a, b\}$.¹⁰ This extrinsic stochastic process is usually referred to as a "sunspot

⁸This assumption will be relaxed when learning dynamics are taken into account. However, it will be shown that even then there are situations in which the state of nature matters asymptotically for the economic outcomes.

⁹Here it is again implicitly assumed that there exists a continuum of identical individuals since it has to hold that the expectations of a single agent cannot affect economic outcomes for the optimal expectations of this agent corresponding to the mathematical expectations.

¹⁰In a later section it will also be taken into account that the world is more complex and can have more than two states.

process". This terminology was coined by Cass and Shell (1983) and is based on the English economist William Jevons, who used the occurrence of sunspots to explain business cycle fluctuations. Although Jevons assumed that sunspots have a real effect on agricultural production possibilities, the occurrence of sunspots is still a valid example for extrinsic uncertainty if this possibility is not taken into account since it cannot be assumed that it affects any other fundamentals of an economy in any way.¹¹ State a of the Markov chain could then represent a situation in which sunspots occur, whereas state b could correspond to a situation in which sunspots do not occur.

The transition probability matrix of this process is assumed to be stationary over time and is given by:

$$\Pi = \begin{pmatrix} \pi_{aa} & 1 - \pi_{aa} \\ 1 - \pi_{bb} & \pi_{bb} \end{pmatrix}$$

where π_{aa} denotes the probability that given the current state of nature is a , the state of nature in the next period will also be a , whereas $1 - \pi_{aa} =: \pi_{ab}$ denotes the probability that given the current state of nature is a , the state of nature in the following period will be b .

It is furthermore assumed that it is common knowledge that the state of nature evolves according to a Markov process with transition probability matrix Π , so that all agents can update their price expectations for period $t + 1$ based on the observed state of nature in period t .

A young agent in period t then chooses his labour supply and his old - age consumption such that they maximize his expected lifetime utility given the information available in period t and given that his intertemporal budget restriction is not violated. In other words, a rational agent born in period t tries to solve the following problem:

$$\begin{aligned} \max_{n_t, c_{t+1}} \quad & \mathbb{E}[u(c_{t+1}) | \Omega_t] - v(n_t) \\ \text{s.t.} \quad & p_t n_t = p_{t+1} c_{t+1} \end{aligned}$$

where $\mathbb{E}[\cdot]$ denotes the mathematical expectations operator and where Ω_t captures the information available in period t which in particular includes the state of nature

¹¹Woodford (1990) also discusses the possibility that the sunspot process describes the evolution of a variable which has an effect on the fundamentals of the economy. In this case, he argues, is the observed effect of this variable on economic outcomes greater than would simply follow from the direct effect on fundamentals, since there is also the effect via expectations described here.

in period t and the transition probability matrix Π .

Substituting the intertemporal budget constraint into the objective function yields then the following simplified problem:

$$\max_{n_t} \mathbb{E} \left[u \left(\frac{p_t}{p_{t+1}} n_t \right) | \Omega_t \right] - v(n_t) \quad (4)$$

Since in a sunspot equilibrium the expectations about a perfect correlation between the price level and the state of nature are self - fulfilling, it is rational to conclude from observing any particular state $i \in I$ in the current period that prices in the next period will be p_i with probability π_{ii} and p_j with the complementary probability, meaning that in equilibrium these probabilities are not just subjective perceptions, but indeed the objective mathematical probabilities for these two events. Therefore, given that in period t state $i \in I$ is observed, this maximization problem can be simplified further by explicitly calculating the mathematical or rational expectations about utility in order to obtain the following maximization problem:

$$\max_{n_t} \pi_{ii} u \left(\frac{p_i}{p_i} n_t \right) + (1 - \pi_{ii}) u \left(\frac{p_i}{p_j} n_t \right) - v(n_t)$$

where it can be seen that with probability π_{ii} agents expect prices to be equal to the characteristic prices associated with state i since the state of nature does not change from period t to period $t + 1$ with this probability. In other words, agents expect to receive a real wage of one with probability π_{ii} . However, with the complementary probability $1 - \pi_{ii}$ agents expect a real wage of $\frac{p_i}{p_j}$ since the state of nature changes with this probability from i to j .

Denoting $\frac{p_i}{p_j}$ as R_t , the optimal labour supply of an agent born in period t in which state $i \in I$ is observed can be expressed as the following function depending on the real wage this agent would face if states were to change during his lifetime and the probability of the state of nature remaining unchanged from period t to period $t + 1$:

$$z(R_t, \pi_{ii}) = \arg \max_{n_t} [\pi_{ii} u(n_t) + (1 - \pi_{ii}) u(R_t n_t) - v(n_t)]$$

As in the case of perfect foresight, it must also hold under extrinsic uncertainty that in an equilibrium the goods and the money market clear. Before using the market clearing conditions in order to define a sunspot equilibrium formally, I will however briefly state some links between the optimal labour supply function under extrinsic uncertainty and under perfect foresight, which will be of use in the next section where sufficient conditions for the existence of sunspot equilibria are developed.

First note that if the probability of remaining in the current state is zero, the maximization problem under extrinsic uncertainty coincides with the problem considered under perfect foresight and thus $z(R_t, 0)$ must coincide with $s(R_t)$.

If however agents expect prices to be equal in all states of nature, i.e. if R_t is equal to one, the situation under extrinsic uncertainty is not different from the situation under perfect foresight with stationary prices. Therefore it must hold that $z(1, \pi_{ii})$ is constant at $s(1)$, i.e. the optimal labour supply at the monetary steady state, regardless of π_{ii} .

Moreover, since the objective function under extrinsic uncertainty is a convex combination of $u(n_t) - v(n_t)$ and $u(R_t n_t) - v(n_t)$, it has to hold that $z(R_t, \pi_{ii})$ lies between $s(1)$ (the argument, which maximizes the first function) and $s(R_t)$ (the argument, which maximizes the second function).

With these properties of the optimal labour supply function under extrinsic uncertainty established, I will now proceed to define the concept of a sunspot equilibrium. As already noted before, the optimal labour supply choice $z(R_t, \pi_{ii})$ can only be part of a sunspot equilibrium if the prices defining R_t are such that the goods and money market clear. Analogously to condition (1), this requires that in any period t in which state a is observed it holds that:

$$D(p_t, p_{t+1} | S_t = a) := \frac{1}{p_a} - z\left(\frac{p_a}{p_b}, \pi_{aa}\right) = 0 \quad (5)$$

Whereas, in any period t' in which state b is observed it imposes:

$$D(p_{t'}, p_{t'+1} | S_{t'} = b) := \frac{1}{p_b} - z\left(\frac{p_b}{p_a}, \pi_{bb}\right) = 0 \quad (6)$$

A stationary two - state sunspot equilibrium is then formally defined as follows:

Definition: A stationary two - state sunspot equilibrium is given by prices p_a and p_b , each corresponding to a different realization of S_t , which constitute a solution to (5) - (6) for transition probabilities π_{aa} and π_{bb} , that lie strictly between 0 and 1, and for which it holds that $p_a \neq p_b$.

Note that the sunspot equilibrium concept described here is stationary since prices associated with a certain state are not supposed to change over time which would be the case if agents expected that the sunspot process and the evolution of prices were only weakly correlated.

Also note that (5) and (6) require that labour supply maximizes expected lifetime

utility given that the allocation lies within the budget set and that both the goods market and the money market clear in period t .

Moreover, it is imposed by this definition that prices associated with realizations of the sunspot process are indeed different from each other. If this were not the case, this would imply that the extrinsic uncertainty did not matter for the equilibrium allocation. Clearly, the monetary steady state found under perfect foresight, which can now be expressed as $(p_a \ p_b) = (p^* \ p^*) =: \mathbf{p}^*$, is such an equilibrium since by a previous remark on the optimal labour supply it holds that $z(1, \pi_{ii}) = s(1)$ which implies that equations (5) and (6) are equivalent to equation (1) evaluated at the monetary steady state. Hence, even under extrinsic uncertainty it is still rational for agents to believe that prices are the same in all periods.

However, the question of interest here is under which circumstances equilibria exist for which the extrinsic uncertainty matters. This will be analysed in the next section.

Another restriction, which is imposed by this definition of Azariadis and Guesnerie (1986), is that transition probabilities must lie strictly between zero and one in order to implement a sunspot equilibrium.

A possible motivation for this is the following: If for example the diagonal elements of the transition probability matrix Π approach zero, uncertainty vanishes since then the state of nature necessarily changes from one period to the next. Therefore, also price levels must alternate which results in an equilibrium price sequence that is periodic of order 2. Hence, agents can perfectly foresee prices in the next period which implies that their behaviour is not subject to any uncertainty.¹²

Extrinsic uncertainty also vanishes from the economy if π_{aa} and π_{bb} approach one. If there nevertheless exist distinct solutions for p_a and p_b to equations (5) and (6), it follows that also the perfect foresight system allows for multiple steady states¹³, in which case true sunspot equilibria (according to the definition given above) can be obtained as randomizations of these multiple perfect foresight equilibria.

If however for example only π_{bb} is one, whereas π_{aa} lies strictly between zero and one, extrinsic uncertainty is indeed governing the behaviour of agents as long as the state of nature is a and would only vanish when state b is reached for the first time which with probability one occurs in finite time. This would, according to the definition given above, not be enough to implement a sunspot equilibrium. However, as will be

¹²Azariadis and Guesnerie (1986) however point out that such a deterministic cycle can be interpreted as the limit of a sunspot equilibrium as the diagonal elements of the transition probability matrix approach zero.

¹³Recall that by a previous remark on optimal labour supply it holds that $z(R_t, 1) = s(R_t)$

seen later, stationary sunspot equilibria cannot exist for such transition probability matrices in the simple model under consideration if leisure and consumption are gross complements.

Therefore, Guesnerie (1986) notes that requiring that transition probabilities to lie strictly between zero and one is merely a matter of taste. Moreover, it should also be noted that there are authors who allow for the transition probabilities to be zero or one (see for example Azariadis (1981)).

2.3. The Existence of Sunspot Equilibria

Before analysing the existence of sunspot equilibria, it is convenient to establish the following relationship between the elasticity of labour supply in the case of extrinsic uncertainty and in the case of perfect foresight, as done in Azariadis and Guesnerie (1986):

Lemma 2 $\eta(1, \pi_{ii})$, the elasticity of labour supply under extrinsic uncertainty with respect to R_t , evaluated at the monetary steady state, fulfils:

$$\eta(1, \pi_{ii}) = (1 - \pi_{ii}) \varepsilon(1)$$

Proof: This can be seen by differentiating the first order condition used to obtain the optimal labour supply under extrinsic uncertainty $z(R_t, \pi_{ii})$ with respect to R_t :

$$\begin{aligned} \frac{\partial}{\partial R_t} [\pi_{ii} u'(z) + (1 - \pi_{ii}) R_t u'(R_t z) - v'(z)] &= 0 \\ \pi_{ii} \frac{\partial z}{\partial R_t} u''(z) + (1 - \pi_{ii}) \left[u'(R_t z) + R_t \left[z u''(R_t z) + \frac{\partial z}{\partial R_t} u''(R_t z) \right] \right] - \frac{\partial z}{\partial R_t} v''(z) &= 0 \\ \Rightarrow \frac{\partial z(R_t, \pi_{ii})}{\partial R_t} &= \frac{(1 - \pi_{ii}) R_t z u''(R_t z) + (1 - \pi_{ii}) u'(R_t z)}{-\pi_{ii} u''(z) - (1 - \pi_{ii}) R_t^2 u''(R_t z) + v''(z)} \\ \Rightarrow \frac{\partial z(R_t, \pi_{ii})}{\partial R_t} \Big|_{R_t=1} &= \frac{(1 - \pi_{ii}) z(1, \pi_{ii}) u''(z(1, \pi_{ii})) + (1 - \pi_{ii}) u'(z(1, \pi_{ii}))}{-\pi_{ii} u''(z(1, \pi_{ii})) - (1 - \pi_{ii}) u''(z(1, \pi_{ii})) + v''(z(1, \pi_{ii}))} \end{aligned}$$

Since by a previous remark $z(1, \pi_{ii}) = s(1)$, it follows that:

$$\frac{\partial z(R_t, \pi_{ii})}{\partial R_t} \Big|_{R_t=1} = (1 - \pi_{ii}) \frac{s(1) u''(s(1)) + u'(s(1))}{-u''(s(1)) + v''(s(1))}$$

and that thus the elasticity of labour supply under extrinsic uncertainty, evaluated at the monetary steady state, is given by.

$$\eta(1, \pi_{ii}) = \frac{\partial z(R_t, \pi_{ii})}{\partial R_t} \frac{R_t}{z(R_t, \pi)} \Big|_{R_t=1} \quad (7)$$

$$\eta(1, \pi_{ii}) = (1 - \pi_{ii}) \frac{s(1) u''(s(1)) + u'(s(1))}{-u''(s(1)) + v''(s(1))} \frac{1}{s(1)} \quad (8)$$

Comparing (3) and (8) shows that:

$$\eta(1, \pi_{ii}) = (1 - \pi_{ii}) \varepsilon(1) \quad (9)$$

This Lemma essentially states that the elasticity of labour supply under extrinsic uncertainty, evaluated at the monetary steady state, is proportional to the corresponding elasticity under perfect foresight and that it is in absolute terms always smaller than the perfect foresight elasticity.

Moreover, this Lemma shows that if the probability of remaining in the current state approaches one, $\eta(1, \pi_{ii})$ approaches zero. This is intuitive since then states would never change and therefore the price level is always expected to be constant by rational agents. Hence, they know that their real wage will always be one and thus, changes in $\frac{p_i}{p_j}$, the fictitious real wage earned if states of nature changed, do not induce any changes in their optimal behaviour.

Furthermore, it can be seen from this result that the smaller the probability of remaining in the current state, the higher is $\eta(1, \pi_{ii})$ in absolute terms. This is the case since an increase in the probability of states of nature changing from one period to the next results in agents regarding it more likely that prices will change and that therefore the real wage they will face will actually be given by $\frac{p_i}{p_j}$. Thus, for transition probabilities π_{ij} high enough, changes in this ratio have similar effects on optimal behaviour as changes of the real wage under perfect foresight.

With this relationship between the elasticity of labour supply under extrinsic uncertainty and under perfect foresight it is now finally possible to establish a connection between the local stability or more precisely the indeterminacy of the monetary steady state and the existence of sunspot equilibria, which can also be found in Azariadis (1981).

However, in order to make the analysis as easy as possible it is convenient to reduce the two dimensional problem considered so far to a one dimensional problem by

introducing the variable $w = \frac{p_a}{p_b}$, as done by Azariadis and Guesnerie (1986).

In order to state the goods market clearing conditions (5) and (6) in terms of this new variable, it is necessary to rewrite these conditions as $\frac{1}{p_a} = z(w, \pi_{aa})$ and $\frac{1}{p_b} = z\left(\frac{1}{w}, \pi_{bb}\right)$, respectively. Then the first equation can be divided by the second equation in order to obtain:

$$\frac{1}{w} = \frac{z(w, \pi_{aa})}{z\left(\frac{1}{w}, \pi_{bb}\right)}$$

By rearranging this condition, it can be seen that the equilibrium prices p_a and p_b , which are such that the goods and money market clear, correspond to values of w which are roots of the following function:

$$G(w, \pi_{aa}, \pi_{bb}) = wz(w, \pi_{aa}) - z\left(\frac{1}{w}, \pi_{bb}\right)$$

Since in a sunspot equilibrium prices associated with different states of nature must be different from each other, a stationary two - state sunspot equilibrium exists if, and only if, G has a root w which is positive and different from 1.

Moreover, note here that $w = 1$ implies that p_a and p_b are equal. Therefore, this case corresponds to a situation in which agents know with certainty that prices will always be equal¹⁴. However, it has been shown in section 2.1 that then the only equilibrium in which money has a positive value is the monetary steady state. Hence, $w = 1$ corresponds to this equilibrium and must always constitute a root of G since at the monetary steady state markets clear and the labour supply maximizes the expected lifetime utility of agents.

In order to analyse whether $G(w, \pi_{aa}, \pi_{bb})$ has also roots different from $w = 1$, Azariadis and Guesnerie (1986) establish the following properties of this function for $w > 0$:

Lemma 3 *The following properties of $G(w, \pi_{aa}, \pi_{bb})$ hold for every $w > 0$ and for all $\pi_{aa}, \pi_{bb} \in (0, 1)$:*

- (i) $G(w, \pi_{aa}, \pi_{bb})$ is continuous for all (w, π_{aa}, π_{bb})
- (ii) $G(1, \pi_{aa}, \pi_{bb}) = 0$
- (iii) $G(w, \pi_{aa}, \pi_{bb}) \rightarrow \infty$ for $w \rightarrow \infty$
- (iv) $G(w, \pi_{aa}, \pi_{bb}) < 0$ for w small enough

¹⁴Note that possible shocks which would cause the actual price to deviate unsystematically from the rational expectations about this price are not part of the model.

(v) If $\hat{w}$ is a root of $G(w, \pi_{aa}, \pi_{bb})$, then $\frac{1}{\hat{w}}$ is a root of $G(w, \pi_{bb}, \pi_{aa})$

This Lemma is stated without proof. However, note that property (i) follows from the fact that optimal labour supply must be a continuous function as the maximand of a function which is continuous in w , π_{aa} and π_{bb} . The content of property (ii) has already been explained above, where it has been argued that the monetary steady state always corresponds to a root of G . Properties (iii) and (iv) essentially require that optimal labour supply is bounded, and property (v) can be proved by setting $G(w, \pi_{aa}, \pi_{bb})$ to zero and rearranging the resulting equation.

The boundary properties of $G(w, \pi_{aa}, \pi_{bb})$ (namely properties (iii) and (iv)) and the fact that the monetary steady state corresponds to a root of G (property (ii)) imply that G having a negative slope at the monetary steady state is sufficient for the existence of at least two other distinct roots of G , as can easily be seen from Figure 1. If this condition holds for transition probabilities which lie strictly between zero and one, these roots satisfy all requirements stated in the definition of a stationary two - state sunspot equilibrium given in the previous section since, as has been argued above, roots of G satisfy the market clearing conditions (5) and (6) by construction and since moreover $w \neq 1$ implies that $p_a \neq p_b$.

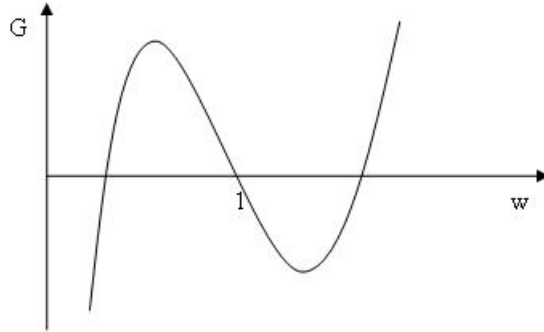


Figure 1: Stationary Rational Expectations Equilibria

Thus, the following theorem, established by Azariadis and Guesnerie (1986), holds:

Theorem 2 *If preferences are such that Lemma 3 holds, sufficient conditions for the existence of stationary two - state sunspot equilibria are:*

$$\varepsilon(1) < 0 \quad \text{and} \quad \pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|}$$

Proof: The partial derivative of G with respect to w is given by:

$$\begin{aligned}
\frac{\partial G(w, \pi_{aa}, \pi_{bb})}{\partial w} &= z(w, \pi_{aa}) + w z_1(w, \pi_{aa}) + \frac{1}{w^2} z_1\left(\frac{1}{w}, \pi_{bb}\right) \\
\frac{\partial G(w, \pi_{aa}, \pi_{bb})}{\partial w} \Big|_{w=1} &= \underbrace{z(1, \pi_{aa})}_{=s(1)} \left[1 + z_1(1, \pi_{aa}) \frac{1}{z(1, \pi_{aa})} + z_1(1, \pi_{bb}) \underbrace{\frac{1}{z(1, \pi_{aa})}}_{=z(1, \pi_{bb})} \right] \\
&= s(1) [1 + \eta(1, \pi_{aa}) + \eta(1, \pi_{bb})] \\
&= s(1) [1 + (1 - \pi_{aa}) \varepsilon(1) + (1 - \pi_{bb}) \varepsilon(1)]
\end{aligned}$$

where $z_1(\cdot, \cdot)$ denotes the partial derivative of z with respect to its first argument. Since $s(1)$, the optimal labour supply at the monetary steady state, is strictly positive, the slope of G with respect to w is negative at the monetary steady state if, and only if:

$$\begin{aligned}
1 + (1 - \pi_{aa}) \varepsilon(1) + (1 - \pi_{bb}) \varepsilon(1) &< 0 \\
(2 - \pi_{aa} - \pi_{bb}) \varepsilon(1) &< -1
\end{aligned}$$

Since by the definition given in the previous section both π_{aa} and π_{bb} must be strictly smaller than 1 in order to implement a stationary two - state sunspot equilibrium, $\varepsilon(1)$ must be negative for this inequality to be fulfilled. Therefore, this expression can be rearranged to yield:

$$\begin{aligned}
-(2 - \pi_{aa} - \pi_{bb}) &< -\frac{1}{|\varepsilon(1)|} \\
\pi_{aa} + \pi_{bb} &< 2 - \frac{1}{|\varepsilon(1)|}
\end{aligned}$$

Note that this condition can be interpreted in two ways:

On the one hand, given certain preferences, this relationship identifies a set of transition probability matrices for which stationary two - state sunspot equilibria necessarily exist. This is depicted in Figure 2, where stationary sunspot equilibria necessarily exist for pairs of transition probabilities (π_{aa}, π_{bb}) lying within the shaded area.

However, since the condition $\pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|}$ is only sufficient, but not necessary for the existence of sunspot equilibria, it is possible that actually the shaded area only constitutes a subset of the total set of transition probability matrices for which stationary sunspot equilibria exist. This point was made by Guesnerie (1986) and

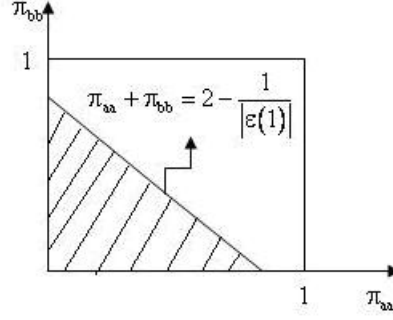


Figure 2: Set of Two - State Sunspot Equilibria

will be discussed in greater detail in a later section.

On the other hand, given a stochastic process that models the state of nature, preferences which fulfil this condition allow for the existence of stationary two - state sunspot equilibria.

In order to describe these two ways of thinking more precisely, note that since according to Lemma 1 $\frac{1}{|\varepsilon(1)|} > 1$ for $\varepsilon(1) < 0$, and that since according to the definition of a stationary sunspot equilibrium used here all transition probabilities must be strictly positive in order to implement a sunspot equilibrium, the sufficient condition for the existence of stationary two - state sunspot equilibria stated in Theorem 2 can also be expressed as:

$$0 < \pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|} < 1$$

Therefore, if on the one hand the monetary steady state is locally asymptotically stable or indeterminate and thus the negative income effect of a change in the real wage outweighs the positive substitution effect by a sufficient margin (i.e. if $0 < 2 - \frac{1}{|\varepsilon(1)|}$), it is possible to construct transition probabilities for which stationary two - state sunspot equilibria necessarily exist for the given preferences and which are such that $\pi_{aa} + \pi_{bb} < 1$.

Whereas, if on the other hand the extrinsic uncertainty can be captured by a Markov process with transition probabilities satisfying $\pi_{aa} + \pi_{bb} < 1$, it is possible to find preferences for which stationary sunspot equilibria exist (i.e. which fulfil $\pi_{aa} + \pi_{bb} < 2 - \frac{1}{|\varepsilon(1)|}$). The utility function representing these preferences must be such that leisure and consumption are locally at the monetary steady state gross complements and such that the monetary steady state is locally asymptotically stable or indeterminate under the perfect foresight dynamics.

So far, I have only been concerned with sufficient conditions for the existence of stationary sunspot equilibria. Thus, it has for example been shown that $\pi_{aa} + \pi_{bb} < 1$ necessarily implies the existence of stationary two - state sunspot equilibria for appropriately chosen preferences. However, Azariadis (1981) shows that if leisure and consumption are gross complements, this condition is even necessary for the existence of sunspot equilibria in the simple model considered here.

In order to see this, note that the optimal labour supply functions under extrinsic uncertainty are given by:

$$z\left(\frac{p_a}{p_b}, \pi_{aa}\right) = \arg \max_{n_t} \pi_{aa} u(n_t) + (1 - \pi_{aa}) u\left(\frac{p_a}{p_b} n_t\right) - v(n_t)$$

and

$$z\left(\frac{p_b}{p_a}, \pi_{bb}\right) = \arg \max_{n_t} \pi_{bb} u(n_t) + (1 - \pi_{bb}) u\left(\frac{p_b}{p_a} n_t\right) - v(n_t).$$

Therefore, labour supply in a stationary two - state sunspot equilibrium must fulfil the following first order conditions:

$$\begin{aligned} \pi_{aa} u' \left(z \left(\frac{p_a}{p_b}, \pi_{aa} \right) \right) + (1 - \pi_{aa}) \frac{p_a}{p_b} u' \left(\frac{p_a}{p_b} z \left(\frac{p_a}{p_b}, \pi_{aa} \right) \right) - v' \left(z \left(\frac{p_a}{p_b}, \pi_{aa} \right) \right) &= 0 \\ \pi_{bb} u' \left(z \left(\frac{p_b}{p_a}, \pi_{bb} \right) \right) + (1 - \pi_{bb}) \frac{p_b}{p_a} u' \left(\frac{p_b}{p_a} z \left(\frac{p_b}{p_a}, \pi_{bb} \right) \right) - v' \left(z \left(\frac{p_b}{p_a}, \pi_{bb} \right) \right) &= 0 \end{aligned}$$

Using the goods and money market clearing conditions (i.e. equations (5) and (6)) to express the equilibrium labour supplies through the market clearing prices p_a and p_b , these first order conditions can be written as:

$$\pi_{aa} U \left(\frac{1}{p_a} \right) + (1 - \pi_{aa}) U \left(\frac{1}{p_b} \right) = V \left(\frac{1}{p_a} \right) \quad (10)$$

$$\pi_{bb} U \left(\frac{1}{p_b} \right) + (1 - \pi_{bb}) U \left(\frac{1}{p_a} \right) = V \left(\frac{1}{p_b} \right) \quad (11)$$

where $U(\cdot)$ and $V(\cdot)$ are defined as above.

Now, first consider the case $\pi_{aa} + \pi_{bb} > 1$, i.e. the case $\pi_{aa} > \pi_{ba}$, which corresponds to a situation in which the states of nature in two consecutive periods are positively correlated.

The first order conditions (10) and (11) can then be rewritten to yield:

$$\pi_{aa} \left[U \left(\frac{1}{p_a} \right) - U \left(\frac{1}{p_b} \right) \right] = V \left(\frac{1}{p_a} \right) - U \left(\frac{1}{p_b} \right)$$

$$\pi_{ba} \left[U \left(\frac{1}{p_a} \right) - U \left(\frac{1}{p_b} \right) \right] = V \left(\frac{1}{p_b} \right) - U \left(\frac{1}{p_b} \right)$$

Without loss of generality it can be assumed that a solution to this system of equations corresponding to a stationary two - state sunspot equilibrium satisfies $p_a > p_b$ (if this were not the case, the states of nature could be relabelled). Since leisure and consumption are assumed to be gross complements, $U(\cdot)$ must be strictly decreasing, as has been discussed in section 2.1. Therefore, it must hold that $U\left(\frac{1}{p_a}\right) - U\left(\frac{1}{p_b}\right)$ is positive, and hence if $\pi_{aa} > \pi_{ab}$, it follows from the rewritten first order conditions stated above that $V\left(\frac{1}{p_a}\right) > V\left(\frac{1}{p_b}\right)$. Since $V(\cdot)$ is by assumption a strictly increasing function, this would require that $p_a < p_b$ which however yields a contradiction. Therefore, the only solution satisfying the first order conditions must be such that prices are always constant, i.e. $p_a = p_b$.

Next, consider the case $\pi_{aa} + \pi_{bb} = 1$, i.e. the case $\pi_{aa} = \pi_{ba}$. This implies that the state of nature in the next period does not depend on the current state since the probability of reaching state a in the following period is the same regardless of the current state being a or b . Therefore, agents cannot obtain any new information on prices in the next period by observing the state of nature in the current period.

In this case the left hand sides of the first order conditions (10) and (11) are identical and thus $V\left(\frac{1}{p_a}\right)$ must equal $V\left(\frac{1}{p_b}\right)$ in order for the system of equations to be consistent. However, since $V(\cdot)$ is a strictly increasing function, this can only be the case if $p_a = p_b$.

Therefore, in the model considered here a negative correlation between the states of nature in two consecutive periods is indeed necessary and not only sufficient for stationary two - state sunspot equilibria to exist.

It is however also important to note that this latter result strongly depends on the simple model considered here and not at all on the non-informativeness of the extrinsic stochastic process concerning future economic outcomes. Guesnerie (1986) shows that it is indeed possible that stationary sunspot equilibria exist even if the states of nature in two consecutive periods are independent.

In order to show this, he introduces the concept of backward looking equilibria:

For any price p_{t+1} , a backward looking equilibrium determines a price p_t which is such that markets clear and for which the labour supply choice of individuals is optimal. A sufficient condition for the existence of several backward looking equilibria associated with a perfect foresight steady state is then derived as follows:

Fix p_{t+1} at the monetary steady state level p^* , and consider excess demand as a func-

tion only depending on p_t . As pointed out above, it must hold that $D(p^*, p^*) = 0$. Moreover, it can be shown that there exists an interval $[\underline{p}, \bar{p}]$, which contains p^* and which is such that for $p_t = \underline{p}$ excess demand is positive, whereas it is negative for $p_t = \bar{p}$. Hence, if the excess demand function $D(p_t, p^*)$ is upward sloping at $p_t = p^*$ (i.e. if $\frac{\partial D(p_t, p^*)}{\partial p_t}|_{p_t=p^*} > 0$) there necessarily exist two other roots of $D(p_t, p^*)$ in $(\underline{p}, \bar{p})$, meaning that the monetary steady state is associated with several backward looking equilibria.¹⁵

Guesnerie (1986) then demonstrates that if there either exist several stationary steady states which are all associated with a unique backward looking equilibrium under perfect foresight dynamics or if there exists a unique stationary steady state under perfect foresight dynamics which fulfils the sufficient condition for the existence of several backward looking equilibria stated above, sunspot equilibria for transition probabilities satisfying $\pi_{aa} + \pi_{bb} = 1$ necessarily exist.¹⁶ Guesnerie (1986) refers to this class of equilibria as "non-informative" sunspot equilibria.

In the model considered here the monetary steady state is however the only perfect foresight equilibrium for which the partial derivative of excess demand with respect to the current price level is different from zero and thus also the only perfect foresight equilibrium at which the argument above, i.e. the Poincaré - Hopf index theorem, can be applied. However, at the monetary steady state it is the case that:

$$\begin{aligned} \frac{\partial D(p_t, p^*)}{\partial p_t}|_{p_t=p^*} &= -\frac{1}{p^{*2}} - \frac{1}{p^*} s'(1) \\ &= -\frac{1}{p^{*2}} [1 + p^* s'(1)] \\ &= -\frac{1}{p^{*2}} [1 + \varepsilon(1)] \end{aligned}$$

which is negative since by Lemma 1 $\varepsilon(1)$ is strictly greater than -1 . In other words, this implies that the sufficient condition for the existence of several backward looking equilibria is not fulfilled at the monetary steady state.

A model in which "non-informative" sunspot equilibria might however occur is an

¹⁵Guesnerie (1986) extends this condition to cases in which the excess demand under perfect foresight is a vector valued function (which is the case if there are more than one consumption good). In order to do this, he uses the Poincaré - Hopf Index Theorem to show that a sufficient condition for the existence of more than one backward looking equilibrium is that $\det \left[\frac{\partial D(p_t, p^*)}{\partial p_t} \right]_{p_t=p^*}$ has the same sign as $(-1)^{n+1}$, where n denotes the number of components of $D(\cdot, p^*)$. This mathematical technique is frequently used in order to derive sufficient conditions for the existence of sunspot equilibria and will be described in greater detail further below.

¹⁶Guesnerie shows this by using an argument based on the Poincaré - Hopf Index Theorem.

overlapping generations model with more than one commodity as it has been analysed by Guesnerie (1986). But since the sufficient condition for the existence of "non-informative" sunspot equilibria only depends on the perfect foresight dynamical system, changes in the number of possible states of nature can have no effect on the existence of "non-informative" sunspot equilibria since, contrary to changes in the number of commodities, this change has no effect under perfect foresight.

However, also note that although the necessary condition $\pi_{aa} + \pi_{bb} < 1$ does not automatically translate into a more complex model, also in an overlapping generations model with more than one commodity $\pi_{aa} + \pi_{bb}$ must be smaller than a certain constant for stationary two - state sunspot equilibria to necessarily exist. In other words, also in more complex models it is still sufficient for the existence of sunspot equilibria that the probability of the state of nature changing from one period to the next is high enough.

In more complex models it might however not be possible to restrict the analysis to a one dimensional problem, as done above. Therefore, I will now briefly discuss an alternative technique which is frequently used in order to obtain sufficient conditions for the existence of sunspot equilibria:

An early discussion of this technique is given by Azariadis and Guesnerie (1982), who first show that there exists a compact set for prices p_a and p_b which is such that at its boundary all trajectories of prices point inwards (e.g. if p_a is equal to a certain level $\underline{p}$, there is excess demand on the goods market in periods in which state a is observed, causing the price in these periods to increase) and which is such that it contains all solutions to (5) and (6) except the autarchy equilibrium $p_a = p_b = 0$. Azariadis and Guesnerie then arrive at the conditions stated in Theorem 2 by employing the Poincaré - Hopf Index Theorem¹⁷ to show that a sufficient condition for the existence of (at least two) stationary two - state sunspot equilibria is that the determinant of the Jacobi matrix of (5) and (6) evaluated at the monetary steady state is negative. Through simple rearrangements of this condition, Azariadis and Guesnerie demonstrate that it is indeed equivalent to the conditions stated in Theorem 2, which however allow for a better interpretation.

¹⁷Chiappori and Guesnerie (1989) give a very simplified description of this theorem: under the conditions stated above, a root of the (vector valued) function F is associated with an index of +1 (resp. -1) if the determinant of the Jacobi matrix of F evaluated at this root is positive (resp. negative). In the present case, the sum of these indices must be equal to $(-1)^2$. Hence, if the index at a root of F is (-1) , there must exist at least two other roots of F associated with an index of $(+1)$.

This mathematical technique is also used by Woodford (1990) in order to derive a sufficient condition for the existence of sunspot equilibria. However, Woodford uses a somewhat different approach from Azariadis and Guesnerie (1982):

Since money and goods market clearing require that $p_t n_t = 1$ and thus determine a one-to-one relationship between the price level and the labour supply in a certain period, an equilibrium can either be described by a sequence of prices, as done for example by Azariadis and Guesnerie (1982) or (1986), or by a sequence of labour supplies. This latter approach is used by Woodford (1990).

Using this market clearing condition it becomes apparent that the real wage $R_t = \frac{p_t}{p_{t+1}}$ can also be written as $\frac{n_{t+1}}{n_t}$. Denoting the market clearing labour supplies in state a and state b as n_a and n_b , respectively in order to distinguish them from the labour supply functions $z\left(\frac{p_a}{p_b}, \pi_{aa}\right)$ and $z\left(\frac{p_b}{p_a}, \pi_{bb}\right)$, the first order conditions for optimal labour supply in a market equilibrium under extrinsic uncertainty can therefore be written as:

$$F_a := F(n_a, n_b, \Pi | S_t = a) = \frac{1}{n_a} [\pi_{aa} n_a u'(n_a) + \pi_{ab} n_b u'(n_b)] - v'(n_a) = 0 \quad (12)$$

$$F_b := F(n_a, n_b, \Pi | S_k = b) = \frac{1}{n_b} [\pi_{bb} n_b u'(n_b) + \pi_{ba} n_a u'(n_a)] - v'(n_b) = 0 \quad (13)$$

These conditions simply state that all markets must clear and that the marginal utility of a change in the current labour supply given that the economy is in either state a or state b must be zero, meaning that labour supply is chosen optimally. Hence, if there exist distinct values for n_a and n_b fulfilling these conditions for transition probabilities lying strictly between zero and one, all requirements for a stationary two - state sunspot equilibrium stated in the definition given in the previous section are fulfilled.

In order to prove the existence of such solutions to (12) and (13), Woodford follows Azariadis and Guesnerie (1982) and shows that there exists a compact set $[\underline{n}, \bar{n}]^2$ containing all roots of $F = (F_a, F_b)'$, except the possible autarchy equilibrium $p_a = p_b = 0$, and which is such that at its boundary all trajectories of labour supply are pointing inwards. Hence, like Azariadis and Guesnerie (1982), Woodford can employ the Poincaré - Hopf Index Theorem¹⁸ to derive a sufficient condition for $(F_a, F_b)'$ having at least two other roots than the monetary steady state which, as already

¹⁸Woodford (1990) notes that the application of the Poincaré - Hopf Index Theorem does actually not only require the existence of a compact set as described above, but also that this set has a smooth boundary. However, he argues that it is possible to smooth out the corners of $[\underline{n}, \bar{n}]^2$, such that indeed all requirements for the application of the Poincaré - Hopf Index Theorem are satisfied.

argued for the approach of Azariadis and Guesnerie (1986), implies the existence of two distinct stationary sunspot equilibria.

Hence, Woodford (1990) arrives at the following theorem, where $\mathbf{n}^* := (n^* \ n^*)'$ denotes the vector of optimal labour supplies for which both components are at the monetary steady state levels, and where $DF(\cdot)$ denotes the Jacobi matrix of the vector valued function $(F_a, F_b)'$ with respect to (n_a, n_b) .¹⁹

Theorem 3 *If $\Delta(\mathbf{n}^*, \Pi) := \det DF(\mathbf{n}^*, \Pi) < 0$, stationary two - state sunspot equilibria necessarily exist.*

As Guesnerie (1986) points out and as can be easily seen from this result, the set of transition probability matrices which are necessarily associated with stationary sunspot equilibria might be disconnected. This is necessarily the case if $\Delta(\mathbf{n}^*, \Pi)$ has at least 3 distinct roots and is visualized in Figure 3, where stationary two - state sunspot equilibria necessarily exist for transition probabilities within the shaded area, and where the diagonal boundaries correspond to those combinations of π_{aa} and π_{bb} for which $\Delta(\mathbf{n}^*, \Pi)$ is zero.

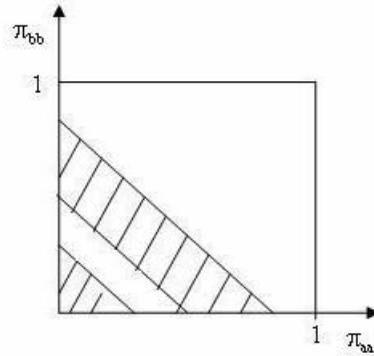


Figure 3: Disconnected Set of Two - State Sunspot Equilibria

Before extending the concept of a stationary sunspot equilibrium discussed so far to sunspot processes which can take on more than two values, I will now briefly focus on a special class of sunspot equilibria which receives much attention in the literature on extrinsic uncertainty.

¹⁹Actually, this is a special case of Woodford's theorem which applies to sunspot processes with arbitrarily, but finitely many states. This more general result will be discussed in greater detail in a later section.

2.4. Sunspot Equilibria around the Monetary Steady State

In the previous section it has been shown that if the monetary steady state is locally asymptotically stable and thus indeterminate under perfect foresight dynamics, it is possible to construct transition probabilities for which stationary two - state sunspot equilibria exist. As will be seen in this section, it is then even possible to construct stationary two - state sunspot equilibria arbitrarily close to the monetary steady state.

Theorem 4 *Stationary two - state sunspot equilibria exist in an arbitrarily small neighbourhood of the monetary steady state if, and only if, it is indeterminate under perfect foresight dynamics.*

Proof: In a small neighbourhood of the monetary steady state the system (5) - (6) can be approximated by its first order Taylor expansion about this point. Therefore, consider the first order Taylor expansion of excess demand (5) about the monetary steady state, as done in Azariadis and Guesnerie (1986):

$$D(p_t, p_{t+1} | S_t = a) \approx \underbrace{\frac{1}{p^*} - \underbrace{z(1, \pi_{aa})}_{=s(1)}}_{=0} + \left(-\frac{1}{p^{*2}} - \frac{1}{p^*} z_1(1, \pi_{aa}) \quad \frac{p^*}{p^{*2}} z_1(1, \pi_{bb}) \right) \begin{pmatrix} p_a - p^* \\ p_b - p^* \end{pmatrix}$$

where the partial derivative of $z\left(\frac{p_a}{p_b}, \pi_{aa}\right)$ with respect to its first argument evaluated at the monetary state is denoted as $z_1(1, \pi_{aa})$.

Denoting deviations of actual prices from the monetary steady state prices in state a and b as x_a and x_b , respectively and using the fact that in an equilibrium excess demand must be zero yields:

$$\begin{aligned} \left(-\frac{1}{p^{*2}} - \frac{1}{p^*} z_1(1, \pi_{aa}) \right) x_a + \frac{1}{p^*} z_1(1, \pi_{aa}) x_b &= 0 \\ -x_a - p^* z_1(1, \pi_{aa}) x_a + p^* z_1(1, \pi_{aa}) x_b &= 0 \end{aligned}$$

Using the goods market clearing condition at the monetary steady state (i.e. that $p^* = \frac{1}{s(1)} = \frac{1}{z(1, \pi_{aa})}$) and the definition of the elasticity of labour supply under extrinsic uncertainty, equation (7), then yields:

$$-x_a - \eta(1, \pi_{aa}) x_a + \eta(1, \pi_{aa}) x_b = 0$$

In order to establish a connection with the situation under perfect foresight, Lemma 2 can be used to obtain:

$$\left(-1 - \underbrace{(1 - \pi_{aa})}_{=\pi_{ab}} \varepsilon(1) \right) x_a + \underbrace{(1 - \pi_{aa})}_{=\pi_{ab}} \varepsilon(1) x_b = 0$$

or equivalently:

$$(1 + \pi_{ab}\varepsilon(1)) x_a - \pi_{ab}\varepsilon(1) x_b = 0 \quad (14)$$

In a similar manner a Taylor expansion of (6) about the monetary steady state leads to:

$$\pi_{ba}\varepsilon(1) x_a - (1 + \pi_{ba}\varepsilon(1)) x_b = 0 \quad (15)$$

It is obvious that $x_a = x_b = 0$ (the monetary steady state) is a solution to the system of equations (14) - (15) and that since $1 + \pi_{ij}\varepsilon(1) \neq \pi_{ij}\varepsilon(1)$, there is no other solution for which $p_a = p_b$. Thus, a necessary and sufficient condition for a sunspot equilibrium $p_a \neq p_b$ to exist close to the monetary steady state is that the coefficient matrix of the linear system of equations (14) - (15) is singular and thus maps a vector different from the zero vector to the zero vector.

Hence, there exist sunspot equilibria arbitrarily close to the monetary steady state if, and only if:

$$\begin{aligned} \det \begin{pmatrix} 1 + \pi_{ab}\varepsilon(1) & -\pi_{ab}\varepsilon(1) \\ \pi_{ba}\varepsilon(1) & -(1 + \pi_{ba}\varepsilon(1)) \end{pmatrix} &= 0 \\ -(1 + \pi_{ab}\varepsilon(1))(1 + \pi_{ba}\varepsilon(1)) + \pi_{ab}\pi_{ba}\varepsilon(1)^2 &= 0 \\ -1 - \pi_{ba}\varepsilon(1) - \pi_{ab}\varepsilon(1) &= 0 \\ \pi_{ab} + \pi_{ba} &= -\frac{1}{\varepsilon(1)} \end{aligned}$$

Note that this condition requires that the elasticity of labour supply at the monetary state is negative since otherwise $\pi_{ab} + \pi_{ba}$ would be negative, which is however clearly impossible. In other words, the income effect has to be dominating at the monetary steady state in order for a stationary two - state sunspot equilibrium to exist close to it.

Note further that since the definition of a sunspot equilibrium requires that the diagonal elements of the transition probability matrix are positive, $\pi_{ab} + \pi_{ba}$ has to be strictly smaller than 2. This implies that the condition $\pi_{ab} + \pi_{ba} = -\frac{1}{\varepsilon(1)}$ ²⁰ reduces

²⁰Note that this condition can only be regarded as an approximation, which is worse the further away a sunspot equilibrium is from the monetary steady state.

to $-\frac{1}{\varepsilon(1)} < 2$ or $\varepsilon(1) < -\frac{1}{2}$. Thus, if the monetary steady state is locally asymptotically stable and therefore indeterminate, it is possible to construct a transition probability matrix for which stationary two - state sunspot equilibria exist in an arbitrarily small neighbourhood of the monetary steady state.

From the previous section it is clear that transition probabilities fulfilling the condition $\pi_{ab} + \pi_{ba} = -\frac{1}{\varepsilon(1)}$ are roots of the partial derivative of G with respect to w evaluated at the monetary steady state and lie therefore at the boundary of the set of transition probabilities depicted in Figure 2.

If the transition probabilities change in such a way that they pass this boundary, stationary sunspot equilibria do not necessarily exist for (π_{aa}, π_{bb}) lying above it, whereas two stationary sunspot equilibria necessarily exist for (π_{aa}, π_{bb}) lying below this boundary.

To study what happens when transition probabilities change from the first to the latter situation it is assumed that they only depend on a real number α and that there exists a continuous mapping φ which is given by:

$$\begin{aligned}\varphi : [\alpha_1, \alpha_2] &\rightarrow (0, 1)^2 \\ \alpha &\mapsto (\pi_{aa}(\alpha), \pi_{bb}(\alpha))\end{aligned}$$

Furthermore it is assumed that there exists a value α^* , which is such that:

$$\pi_{ab}(\alpha^*) + \pi_{ba}(\alpha^*) = -\frac{1}{\varepsilon(1)}$$

and that ²¹

$$\frac{d}{d\alpha} [\pi_{aa}(\alpha) + \pi_{bb}(\alpha)]|_{\alpha=\alpha^*} \neq 0$$

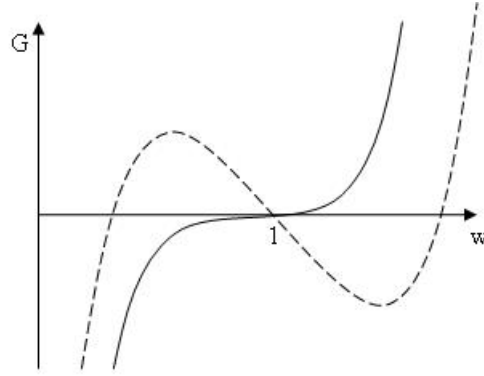
This means that in a small neighbourhood around the boundary it must hold that α is smaller than α^* on one side and greater than α^* on the other side of this boundary. Therefore, a passage of the boundary can be regarded as an increase (or respectively a decrease) of α . Without loss of generality it can be assumed that $\frac{d}{d\alpha} [\pi_{aa}(\alpha) + \pi_{bb}(\alpha)]|_{\alpha=\alpha^*} < 0$, such that changing from a situation without stationary sunspot equilibria to a situation in which stationary sunspot equilibria necessarily exist corresponds to an increase of α .

The change of the qualitative behaviour of the dynamical system implied by a change of α is referred to as a (local) bifurcation, α is then called the bifurcation parameter.

²¹Guesnerie (1986) refers to these assumptions as "regular crossing"

Guesnerie (1986) notes that in a neighbourhood around the line $\pi_{ab} + \pi_{ba} = -\frac{1}{\varepsilon(1)}$ stationary sunspot equilibria only depend on the sum of π_{ab} and π_{ba} and therefore, as Azariadis and Guesnerie (1986) point out, it does not matter whether transition probabilities cross $\pi_{aa} + \pi_{bb} = 2 - \frac{1}{|\varepsilon(1)|}$ transversally (i.e. along a line on which $\pi_{aa} \neq \pi_{bb}$) or diagonally (i.e. along a path for which $\pi_{aa} = \pi_{bb}$) for a qualitative analysis.

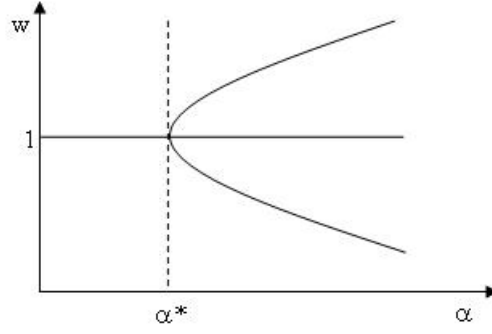
Along this latter path the boundary properties of G and the fact that the partial derivative of G with respect to w is zero at the monetary state for α^{*22} require that G has a saddle point at $w = 1$ and $\alpha = \alpha^*$ (otherwise properties (iii) and (iv) of Lemma 3 require that the number of roots of G which are different from 1 is odd, however for $\pi_{aa} = \pi_{bb}$ both $\hat{w}$ and $\frac{1}{\hat{w}}$ must be roots of G (see Lemma 3 (v)) and thus the number of roots different from 1 must necessarily be even). Changes in α do however result in changes of the shape of G . This is illustrated in the following figure, where the broken line corresponds to an $\alpha > \alpha^*$ and the black line depicts $G(w, \alpha^*)$.



As pointed out above, for $\alpha < \alpha^*$, G does not necessarily have any roots other than the monetary steady state: Since G is continuous with respect to α , it follows that a small decrease in α below α^* cannot lead to a big deviation of $G(w, \alpha)$ from $G(w, \alpha^*)$. This decrease in α however implies that it must be the case that $\pi_{aa}(\alpha) + \pi_{bb}(\alpha) > \pi_{aa}(\alpha^*) + \pi_{bb}(\alpha^*) = 2 - \frac{1}{|\varepsilon(1)|}$, implying that $G(w, \alpha)$ is upward slopping at the monetary steady state. Hence, it follows that $G(w, \alpha^*)$ must rotate anti-clockwise around the monetary steady state if α decreases.

The following bifurcation diagram shows the steady states (i.e. the roots of G) as a function of the bifurcation parameter α :

²²Note that G can now be expressed as a function of w and α (which determines the transition probabilities).



Note that the necessary condition for the existence of stationary sunspot equilibria arbitrarily close to the monetary steady state found in the proof of Theorem 4 approximately characterizes sunspot equilibria which correspond to an α close to α^* since for these values it must hold that $\pi_{aa}(\alpha) + \pi_{bb}(\alpha) \approx \pi_{aa}(\alpha^*) + \pi_{bb}(\alpha^*) = 2 - \frac{1}{|\varepsilon(1)|}$.

More precisely, Guesnerie (1986) shows that a necessary condition for the existence of sunspot equilibria close to the monetary steady state is the following:

If a sequence of prices (p_a^n, p_b^n) associated with a sunspot equilibrium converges to the monetary steady state, it must be the case that the associated transition probabilities converge to values which fulfil $\pi_{ab} + \pi_{ba} = -\frac{1}{\varepsilon(1)}$.

This can also be seen from the bifurcation diagram by considering only one branch of sunspot equilibria, e.g. those for which $w > 1$. Then clearly, w decreasing toward the monetary steady state is associated with α converging to α^* , meaning that in the limit the transition probabilities fulfil $\pi_{ab} + \pi_{ba} = -\frac{1}{\varepsilon(1)}$.

After this brief attempt to describe the nature of certain stationary two - state sunspot equilibria, I will now discuss sufficient conditions for the existence of stationary sunspot equilibria when the sunspot process capturing the extrinsic uncertainty in the economy can take on more than two values.

2.5. Sunspot Equilibria of Order k

In many situation it might not seem reasonable to assume that agents perceive the world around them to be in one of only two possible states. If, as in section 2.2, sunspot activity is taken as an example for extrinsic uncertainty, agents could very well believe that not just the occurrence of sunspots, but also the number of observed

sunspots has an effect on economic outcomes.²³

In order to account for situations like this as well, it will be assumed that the sunspot variable capturing the extrinsic uncertainty in the economy follows a Markov process $(S_t)_{t \geq 0}$, which can now take on values in a finite state space $I = \{1, \dots, k\}$. The transition probability matrix is hence given by the $k \times k$ matrix $\Pi = (\pi_{ij})_{1 \leq i \leq k; 1 \leq j \leq k}$, where, as above, $\pi_{ij} := \text{Prob}(S_{t+1} = j | S_t = i)$ denotes the probability of the state of nature in the next period being j given that the state of nature in the current period is i . It is still assumed that the transition probability matrix is stationary over time and common knowledge.

Therefore, if in period t state i is observed, the optimization problem of an agent born in period t who has rational expectations about future economic outcomes is given by:

$$\max_{n_t} \mathbb{E} \left[u \left(\frac{p_t}{p_{t+1}} n_t \right) | S_t = i \right] - v(n_t)$$

where, as above, the intertemporal budget constraint of an agent born in period t has been used to express old - age consumption in terms of the real wage and the labour supply, and where $\mathbb{E}[\cdot]$ denotes the (conditional) mathematical expectations operator.

Since in a sunspot equilibrium the subjective expectations about prices in periods in which any state $j \in \{1, \dots, k\}$ is observed being p_j are self - fulfilling, the mathematical expectation about future utility, and thus also the maximization problem of a rational agent born in period t , can be simplified as²⁴:

$$\max_{n_t} \sum_{j=1}^k \pi_{ij} u \left(\frac{p_i}{p_j} n_t \right) - v(n_t)$$

²³Guesnerie (1986) mentions that indeed the cyclical fluctuations in the number of sunspots, and not just the occurrence of them, led Jevons to the hypothesis that business cycles are caused by sunspot activity.

²⁴The argument used here is analogous to the one presented in section 2.2 and requires again that a single agents' expectations cannot influence economic outcomes (otherwise the optimal expectations for this agent need not necessarily correspond to the mathematical expectations) and that all agents believe in a perfect correlation between prices and the sunspot process (otherwise these expectations need not be self - fulfilling).

Moreover, if the existence of sunspot equilibria is taken as an indicator for their attainability, it is implicitly assumed that it is common knowledge that the other agents believe in a perfect correlation between prices and the sunspot process. Otherwise, it could be that agents coordinate their beliefs on a perfect foresight equilibrium. The issue whether agents can actually coordinate on the beliefs associated with a given rational expectations equilibrium will however be discussed in greater detail in the next section.

Thus, whenever agents observe sunspot state i , the optimal labour supply is, analogously to section 2.2, given as the following function depending on the probabilities of the state of nature changing from i to any other state $j \in \{1, \dots, k\}$ and the hypothetical real wages which would be earned by an agent born in a period in which state i was observed in the case such a change actually occurred:

$$z\left(\frac{p_i}{p_1}, \dots, \frac{p_i}{p_k}, \pi_{i1}, \dots, \pi_{ik}\right) = \arg \max_{n_t} \sum_{j=1}^k \pi_{ij} u\left(\frac{p_i}{p_j} n_t\right) - v(n_t)$$

which implies that the optimal labour supply has to fulfil the following first order condition:

$$\sum_{j=1}^k \pi_{ij} \frac{p_i}{p_j} u'\left(\frac{p_i}{p_j} z\left(\frac{p_i}{p_1}, \dots, \frac{p_i}{p_k}, \pi_{i1}, \dots, \pi_{ik}\right)\right) - v'\left(z\left(\frac{p_i}{p_1}, \dots, \frac{p_i}{p_k}, \pi_{i1}, \dots, \pi_{ik}\right)\right) = 0$$

In order to analyse the existence of sunspot equilibria, I will proceed as done in Woodford (1990) by using goods and money market clearing to describe the market equilibrium by a sequence of labour supplies rather than by a sequence of prices. As indicated in section 2.3, it is however equivalent to use the latter approach and employ the techniques used in the following directly on the excess demand functions, rather than on the first order condition for optimal labour supply.

Using the goods and money market clearing condition $p_t n_t = 1$, the first order conditions for $(n_1, \dots, n_k)^{25}$ can be rewritten as:

$$F_i(n_1, \dots, n_k, \Pi) := \frac{1}{n_i} \sum_{j=1}^k \pi_{ij} n_j u'(n_j) - v'(n_i) = 0 \quad \forall i \in \{1, \dots, k\} \quad (16)$$

In order to establish a sufficient condition for the existence of stationary sunspot equilibria, Woodford (1990) first shows that the Poincaré - Hopf Index Theorem, introduced in Section 2.3, applies to the present problem. Therefore, he demonstrates that there exists a compact set $[\underline{n}, \bar{n}]^k$, which contains all roots of $F(n_1, \dots, n_k, \Pi) := (F_1, \dots, F_k)'$ and which is such that at its boundary the trajectories of the vector $(n_1, \dots, n_k)'$ are pointing inwards, meaning that if the labour supply in sunspot state l were equal to $\underline{n}$, $F_l(n_1, \dots, n_k, \Pi)$, the marginal utility which could be obtained from increasing n_l beyond $\underline{n}$, would be positive.

²⁵Note that, as above, n_i refers to the equilibrium labour supply associated with state i , i.e. the specific value of the labour supply function for state i , $z\left(\frac{p_i}{p_1}, \dots, \frac{p_i}{p_k}, \pi_{i1}, \dots, \pi_{ik}\right)$, for which markets in periods in which state i is observed clear

Woodford (1990) formally establishes this in the following two lemmas:

Suppose that n_l is the smallest value in the vector $(n_1, \dots, n_k)'$ and that $\hat{n}$ is defined as in section 2.1 as agents' endowment of time, then the following lemma holds:

Lemma 4 *There exists a $\underline{n} > 0$, such that for $0 < n_l \leq \underline{n}$ and for $n_l \leq n_j < \hat{n}$ for all $j \neq l$ it holds that $F_l(n_1, \dots, n_k, \Pi) > 0$.*

Proof: Since it is assumed that $\hat{n} > n_j \geq n_l$ for all $j \neq l$ and since $u'(\cdot)$ is strictly decreasing, it holds that $n_j u'(n_j) \geq n_l u'(\hat{n})$ for all $j \neq l$. Therefore, a lower bound for the first part of the sum $F_l(n_1, \dots, n_k, \Pi) = n_l^{-1} \sum_{j \neq l} \pi_{lj} n_j u'(n_j) + \pi_{ll} u'(n_l) - v'(n_l)$ can be obtained through the following rearrangements:

$$\begin{aligned} \sum_{j \neq l} \pi_{lj} n_j u'(n_j) &\geq \sum_{j \neq l} \pi_{lj} n_l u'(\hat{n}) \\ \sum_{j \neq l} \pi_{lj} \frac{n_j}{n_l} u'(n_j) &\geq u'(\hat{n}) \sum_{j \neq l} \pi_{lj} \\ &= u'(\hat{n}) (1 - \pi_{ll}) \end{aligned}$$

Furthermore, since it is assumed that $n_l \leq \underline{n}$ and since $u''(\cdot) < 0$ and $v''(\cdot) > 0$, it holds that $u'(n_l) \geq u'(\underline{n})$ and that $-v'(n_l) \geq -v'(\underline{n})$. Hence, a lower bound for $F_l(n_1, \dots, n_k, \Pi)$ is given by:

$$F_l(n_1, \dots, n_k, \Pi) \geq (1 - \pi_{ll}) u'(\hat{n}) + \pi_{ll} u'(\underline{n}) - v'(\underline{n})$$

Assumption (i) made on the limiting behaviour of the utility function in section 2.1 implies that $u'(\underline{n}) \rightarrow \infty$ for $\underline{n} \rightarrow 0$. Furthermore, the assumptions on the utility function imply that $v'(\underline{n})$ is finite. Therefore, there must exist a value of $\underline{n}$ small enough, but strictly greater than zero, such that the following holds:

$$(1 - \pi_{ll}) u'(\hat{n}) + \pi_{ll} u'(\underline{n}) - v'(\underline{n}) > 0$$

In other words, $F_l(n_1, \dots, n_k, \Pi)$ is bounded above zero if the smallest component of the vector $(n_1, \dots, n_k)'$ is smaller than an appropriately chosen strictly positive (but possibly very small) constant $\underline{n}$.

This immediately implies that roots of $F(n_1, \dots, n_k, \Pi)$ can only lie within the set $[\underline{n}, \hat{n}]^k$. Furthermore, it is clear from this lemma that whenever the smallest component of the vector $(n_1, \dots, n_k)'$ is equal to $\underline{n}$ and thus the vector of labour supplies lies on the boundary of the set $[\underline{n}, \hat{n}]^k$, the marginal utility from increasing

the smallest component is positive. In other words, this means that at this boundary the trajectories of labour supply are pointing inwards.

In order to rule out very high levels of equilibrium labour supply as well, suppose that n_h is the greatest component of the vector of labour supplies $(n_1, \dots, n_k)'$. Then the following lemma holds:

Lemma 5 *There exists a $\bar{n} < \hat{n}$, such that for $\bar{n} \leq n_h < \hat{n}$ and for $\underline{n} \leq n_j \leq n_h$ for all $j \neq h$ it holds that $F_h(n_1, \dots, n_k, \Pi) < 0$.*

The proof of this lemma is analogous to the previous proof and therefore omitted.

Together with the previous lemma this result implies that roots of $F(n_1, \dots, n_k, \Pi)$ can only lie within the compact set $[\underline{n}, \bar{n}]^k$. Furthermore, it also holds that whenever the greatest component of the vector $(n_1, \dots, n_k)'$ is equal to $\bar{n}$, the marginal utility from decreasing this component is positive.

Therefore, it has indeed been shown that the trajectories of prices are pointing inward on every boundary of $[\underline{n}, \bar{n}]^k$.

As explained above, Woodford (1990) notes that it follows from this result that all requirements for the application of the Poincaré - Hopf Index Theorem are satisfied. According to this theorem the sum of the indices at the roots of $F(n_1, \dots, n_k, \Pi)$ must be equal to $(-1)^k$. Furthermore, as has already been argued for the case of a two state Markov process, the monetary steady state, which can be expressed as the vector $\mathbf{n}^* := (n^*, \dots, n^*) \in \mathbb{R}^k$, clearly constitutes a root of $F(n_1, \dots, n_k, \Pi)$. If this root is now associated with an index of opposite sign than $(-1)^k$, it follows therefore that there must exist at least two other roots of $F(n_1, \dots, n_k, \Pi)$ which are associated with an index of $(-1)^k$. Since the monetary steady state is the unique root of F associated with equal labour supply in all states, it follows that at least two components of these roots must be different from each other and that thus the extrinsic uncertainty matters for these market equilibria if indeed all transition probabilities lie strictly between zero and one.²⁶

Therefore, denoting the Jacobi matrix of $F(n_1, \dots, n_k, \Pi)$ evaluated at the monetary steady state as $DF(\mathbf{n}^*, \Pi)$, and supposing that the transition probability matrix Π fulfils this last requirement, the following theorem holds:

Theorem 5 *If $\Delta(\mathbf{n}^*, \Pi) := (-1)^k \det DF(\mathbf{n}^*, \Pi) < 0$, two stationary sunspot equi-*

²⁶As in the case of a two - state sunspot process, ruling out transition probabilities which are zero or one is a matter of taste, but can be justified by the same arguments used in section 2.2.

libria associated with an index of $(-1)^k$ necessarily exist.

Furthermore, Woodford (1990) and Chiappori and Guesnerie (1989) point out that, as in the case of a two state Markov process, the indeterminacy of the monetary steady state under perfect foresight is also in this more general case sufficient for the existence of stationary sunspot equilibria. This means that if the monetary steady state is indetermined under perfect foresight dynamics, it is also in this more general case possible to find transition probabilities such that the necessary condition stated in this theorem is fulfilled.

However, note that this condition does not per se guarantee that all components of $(n_1, \dots, n_k)'$ are different. This means that if the extrinsic uncertainty is described by a stochastic process with more than two possible states, it could very well be the case that in some states prices and therefore labour supplies in a sunspot equilibrium, i.e. a market equilibrium on which extrinsic uncertainty has some effect, are the same.

Therefore, it is necessary to make the following distinction, introduced by Chiappori and Guesnerie (1989):

Definition: A stationary sunspot equilibrium of cardinality k is given by a $k \times k$ transition probability matrix Π , the elements of which lie strictly between zero and one, and a vector of labour supplies $(n_1, \dots, n_k)'$, where n_i is associated with sunspot state i , which is such that there exist indices i and j for which $n_i \neq n_j$, and which solves (16), meaning that all markets clear and that rational agents maximize their expected lifetime utility.

Definition: A stationary sunspot equilibrium of order k is given by a $l \times l$ transition probability matrix Π , the elements of which lie strictly between zero and one, and a vector of labour supplies $(n_1, \dots, n_l)'$, which is such that there exist k indices $i_1, \dots, i_k$ for which $n_j \neq n_m$ for all $j, m \in \{i_1, \dots, i_k\}$ and $j \neq m$, and which solves (16).

In other words, the cardinality of a sunspot equilibrium basically describes the number of states of the sunspot process influencing agents' expectations, whereas the order of a sunspot equilibrium describes the number of states which are actually associated with different behaviour by the economic agents. Of course, as is clear from the definition of a two state sunspot equilibrium in section 2.2 and as is pointed out by Chiappori and Guesnerie (1989), both the cardinality and the order of a sunspot

equilibrium must be at least two, since otherwise extrinsic uncertainty would have no effect on the economy.

It is also important to note that the probability of a change in the state of nature only matters for the economic outcome since prices are expected to change with this probability. Therefore, it might in some cases be justified to focus on sunspot equilibria for which each state of nature is associated with a different behaviour of the agents. Hence, the following distinction is necessary:

Definition: A stationary sunspot equilibria is referred to as nondegenerate if its cardinality and order are equal. Conversely, it is referred to as degenerate if its order is smaller than its cardinality.

Chiappori and Guesnerie (1989) argue that any degenerate sunspot equilibrium can also be expressed as a nondegenerate sunspot equilibrium of smaller cardinality, which justifies the focus on nondegenerate sunspot equilibria further.

In order to see this, suppose for instance, as done in Chiappori and Guesnerie (1989), that in a stationary sunspot equilibrium of cardinality k and order $k - 1$ the states 1 and k are associated with the same labour supply n_1 . Then, by combining states 1 and k in the new state $1'$ this equilibrium can also be expressed as a certain nondegenerate stationary sunspot equilibrium of cardinality $k - 1$ by defining the transition probabilities as follows: The transition probabilities between any states i and j with $i, j \neq 1, k$ are still given by the old transition probabilities π_{ij} , whereas the transition probabilities from any state i to the newly created state $1'$ are given by $\pi_{i1'} = \pi_{i1} + \pi_{ik}$. In order to define the transition probabilities from state $1'$ to any state i there are several possibilities: One possibility is to define $\pi_{1'1'}$ as $\frac{1}{2}(\pi_{11} + \pi_{1k}) + \frac{1}{2}(\pi_{kk} + \pi_{k1})$ and accordingly $\pi_{1'i}$ as $\frac{1}{2}\pi_{1i} + \frac{1}{2}\pi_{ki}$ for any $i \in \{2, \dots, k - 1\}$, where the weights of $\frac{1}{2}$ guarantee that the entries in the first row of the new transition probability matrix add up to one. An intuitive argument for this way to define the new transition probability matrix is that if the labour supply choice n_1 is observed, it is equally likely that the state of nature is 1 or k . However, it is clear that any other two weights which add up to one would work as well.²⁷

Chiappori and Guesnerie (1989) use this result in order to show that the set of nondegenerate sunspot equilibria of cardinality k is an open and dense subset of the total set of sunspot equilibria of cardinality k . This means that there exists a nondegenerate sunspot equilibrium in any arbitrarily small neighbourhood around

²⁷Chiappori and Guesnerie (1989) mention the possibilities of either defining $\pi_{1'1'}$ as $\pi_{11} + \pi_{1k}$ and $\pi_{1'i}$ as π_{1i} , or of defining $\pi_{1'1'}$ as $\pi_{k1} + \pi_{kk}$ and $\pi_{1'i}$ as π_{ki} .

any (degenerate or nondegenerate) sunspot equilibrium of cardinality k .

Moreover, they argue that this result implies that if an economy allows for sunspot equilibria of order 2, it also allows for nondegenerate sunspot equilibria of any order. This result was already indicated by Guesnerie (1986), who illustrated how a sunspot equilibrium of order $k - 1$ can be transformed into a (nondegenerate) sunspot equilibrium of order k .

After briefly introducing these more refined concepts of sunspot equilibria, I will now turn to the question whether sunspot equilibria can actually be regarded as realistic outcomes. However, before being able to address this question it is first necessary to describe some deficiencies of the rational expectations assumption and to discuss alternative ways to model the process of agents' expectation formation.

3. Adaptive Learning Dynamics

3.1. Introduction and Preliminaries

3.1.1. Coordination on Rational Expectations Equilibria

The previous analysis showed that in a very simple overlapping generations model extrinsic uncertainty can have an effect on economic outcomes even despite the fact that agents are fully rational in the sense that they know all relevant characteristics of the model and form their expectations optimally given this information and the expectations of the other agents.²⁸

However, this does not at all imply that it is indeed plausible to observe sunspot equilibria in reality. The first and most obvious drawback is that although it is fully rational to believe that an extrinsic sunspot process affects the economy, it is also fully rational to believe that it does not affect the economy and that instead prices and labour supplies are either at their monetary steady state levels or at the levels corresponding to the autarchic equilibrium (if it exists). Thus, although it is optimal or rational for one agent to believe that the economic outcome will vary with the state of nature if all other agents think so, there is no guarantee that this particular equilibrium will actually be reached.

Following Lucas (1986) rational expectations equilibria can only be regarded as the description of an economy in the long run revealing that agents have learned the

²⁸As Evans and Honkapohja (2001) briefly note, rational expectations equilibria can be regarded as Nash equilibria in which agents form their expectations using forecast rules which are best responses to the forecast rules used by the other agents.

relevant characteristics of the economy by adaptively updating their expectations as new observations on actual economic outcomes and thus new information about the structure of the economy became available. More precisely, Lucas (1986, p. S402) states the following:

"Technically, I think of economics as studying decision rules that are steady states of some adaptive process, decision rules that are found to work over a range of situations and hence are no longer revised as more experience accumulates."

Most importantly, this interpretation indicates a possibility to reduce the number of rational expectations equilibria to those which can actually be regarded as realistic economic outcomes: If there is no adaptive learning process which will eventually lead to a certain rational expectations equilibrium, this equilibrium is merely a theoretical concept, but has no practical implications. Evans and Honkapohja (2001) argue that in many situations it is even possible to select a unique equilibrium which is stable under adaptive learning rules if it is taken into account that agents might initially overparametrize the rational expectations equilibrium under consideration. This particular stability concept for rational expectations equilibria known as strong expectational stability will be discussed in greater detail in a later section.

However, even despite the potential multiplicity of rational expectations equilibria, the formation of rational expectations relies on rather strong assumption on agents' knowledge about the economic environment they live in. For example, it is assumed that agents perfectly know the numerical values of all parameters of the model which describes their economy. However, as Evans and Honkapohja (2001) point out, even economists have to rely on econometric techniques in order to estimate the correct parameter values and forecast economic variables, and it is in this sense not realistic, they argue, to assume that agents actually possess greater insight into economic relationships than economists.

Another problem agents face in the short run, is that they might not know the stochastic properties of possible random shocks to the economy. To illustrate the potential complications due to random shocks, consider the overlapping generations model discussed in the previous sections, but augmented by a random preference shock, as done in Woodford (1990). It is assumed that the preference shock can only be observed after agents have decided on their labour supply and can therefore also be interpreted as a random measurement error. More precisely, the utility of an agent born in period t is no longer given by $u(c_{t+1}) - v(n_t)$, but instead by $u(c_{t+1}) - v(n_t) + n_t \nu_t$, where the ν_t 's are the realizations of random variables which

are identically and independently distributed over time and have mean zero and a constant and finite variance. Using this utility function the maximization problem (4) an agent born in period t has to solve is given by:

$$\max_{n_t} \mathbb{E} \left[u \left(\frac{p_t}{p_{t+1}} n_t \right) \right] - v(n_t) + n_t \mathbb{E} [\nu_t]$$

In a rational expectations equilibrium agents are supposed to realize that the random shock has mean zero which implies that there is actually no change to problem (4) and the resulting rational expectations equilibria. However, if the agents for some reason perceived the disturbance term to contain some structural information about the economy, their behaviour would of course be different. Especially, if agents do not know the properties of the random disturbances to the economy, but have to infer them from a limited number of observations, there is no reason to believe that they will actually obtain the correct estimate for their mean. Instead they will possibly interpret the disturbance term as a randomly fluctuating, but persistent measurement error.

This complication also arises in linear rational expectations models such as the Cobweb model which would allow for a unique rational expectations equilibrium. This model, has been the subject of extensive research in the literature on adaptive learning. Already DeCanio (1979) noted that the existence of a unique rational expectations equilibrium does not imply that it can be attained by agents who form their expectations adaptively. However, it turns out that if agents form their estimates by using ordinary least squares or similar learning rules, the condition for stability of the rational expectations equilibrium in the Cobweb model is always satisfied when demand and supply curves fulfil standard assumptions in economics. Hence, regarding agents as learning individuals can in many cases actually justify the prevalent focus on rational expectations equilibria in the long run (as Lucas (1986) pointed out), but helps to select equilibria which can be attained by real life people.

Another problem, which will however also have an effect on the attainability of rational expectations equilibria, is that agents might interpret large random shocks as recurring structural change to their economic environment. This perception might lead agents to adopt learning rules which cannot converge to any rational expectations equilibrium. The question of how adaptive learning processes which can at least under certain circumstances converge to rational expectations equilibria must be designed will be briefly discussed in a later section.

The next section will however introduce the univariate Cobweb model which will be used later to illustrate some results on adaptive learning.

3.1.2. The Cobweb Model

As already indicated in the previous section, it is more realistic to replace the assumption of rational expectations by assuming that agents behave like econometricians and use for example ordinary least squares estimation in order to form their expectations. The ordinary least squares algorithm is often the central learning algorithm analysed since it constitutes a consistent and unbiased estimator in a well specified model, and since it is, due to its simplicity, frequently used for econometric analysis. Therefore, even if agents were not able to apply this econometric technique themselves, they might have access to professional forecasts based on it.²⁹

The most prominent model in which expectations formation through ordinary least squares estimation has been investigated is the Cobweb model.³⁰ In order to illustrate how agents can recursively update their forecasts as new data becomes available and which problems they will face doing so, I will briefly introduce this model, as done by Evans and Honkapohja (2001), before discussing the ordinary least squares and other adaptive learning algorithms in greater detail.

The Cobweb model studies the equilibrium of supply and demand on an isolated goods market. Therefore, it is assumed that consumers can base their demand decisions on the actual price of the good, whereas producers can base their supply decisions only on the anticipated price since due to a production lag they already have to decide on their supply for period t in period $t - 1$. Thus, the demand curve can be expressed as:

$$d_t = \alpha_1 - \alpha_2 p_t + \nu_{1t}$$

where it is usually assumed that α_2 is positive, such that the demand curve is downward sloping, and where ν_{1t} is the realization of a random demand shock at time t with mean zero.

Whereas the supply curve is given by:

$$s_t = \beta_1 + \beta_2 p_t^e + w_{t-1} \beta_3 + \nu_{2t}$$

²⁹In this sense, Evans and Honkapohja (2001) argue that a realistic learning rule should not be based on techniques that were not available at the time which should be described by the model.

³⁰See for example Bray and Savin (1986) or Evans and Honkapohja (2001).

where it is usually assumed that β_2 is positive, such that the supply curve is upward sloping. Furthermore, p_t^e denotes the producers' subjective expectations about the price in period t formed in period $t - 1$ and w_{t-1} denotes a $1 \times K$ vector of exogenous variables which are observable at the time producers must decide on their supply and which may influence their decision. Accordingly, β_3 is a $K \times 1$ vector of coefficients. ν_{2t} is assumed to be the realization of a random supply shock with mean zero, observed at time t . For later reference, it is also assumed that for each point in time w_{t-1} is a realization of a random vector with mean zero and positive definite variance - covariance matrix Ω .

Using the market clearing condition the reduced form equation for the equilibrium price level can be obtained as:

$$\alpha_2 p_t = \alpha_1 - \beta_1 - \beta_2 p_t^e - w_{t-1} \beta_3 + \nu_{1t} - \nu_{2t}$$

Denoting $\frac{\alpha_1 - \beta_1}{\alpha_2}$ as μ , $\frac{-\beta_2}{\alpha_2}$ as α , $\frac{-\beta_3}{\alpha_2}$ as δ and $\frac{\nu_{1t} - \nu_{2t}}{\alpha_2}$ as ν_t yields that equilibrium prices are generated by the following law of motion:³¹

$$p_t = \mu + \alpha p_t^e + w_{t-1} \delta + \nu_t$$

where clearly, ν_t is the realization of a random variable with mean zero.

Since in a rational expectations equilibrium the subjective expectations about the price level correspond to the correct mathematical expectations given the available information, the unique rational expectations solution for prices can be obtained by using the mathematical expectations operator, conditional on the information available in period $t - 1$, on both sides of this equation:

$$\begin{aligned} \mathbb{E}_{t-1} [p_t] &= \mu + \alpha \mathbb{E}_{t-1} [p_t] + w_{t-1} \delta \\ \mathbb{E}_{t-1} [p_t] &= \frac{\mu}{1 - \alpha} + w_{t-1} \frac{1}{1 - \alpha} \delta \end{aligned}$$

Substituting this expression for the subjective expectations p_t^e in the reduced form for prices shows that in the unique rational expectations equilibrium of the Cobweb model prices are actually generated by the following process:

$$p_t = \underbrace{\frac{\mu}{1 - \alpha}}_{=\bar{a}} + w_{t-1} \underbrace{\frac{1}{1 - \alpha}}_{=\bar{b}} \delta + \nu_t \quad (17)$$

³¹Evans and Honkapohja (2001) point out that this reduced form also describes the equilibrium of other models, such as the Lucas Aggregate Supply Model.

With these characteristics of the Cobweb model under rational expectations established, it is now possible to consider deviations from the rational expectations assumption. In particular, some deficiencies of adaptive learning algorithms become apparent without even referring to a concrete algorithm. One of these problems will be discussed in the next section.

3.1.3. Rational vs. Reasonable Learning Algorithms

In order to study whether any rational expectations equilibrium can be attained under the least squares or other adaptive learning algorithms, it is necessary to specify which beliefs agents have about the data generating process they face. These initial beliefs are usually referred to as the perceived law of motion.

In order to illustrate which complications arise for the analysis of the convergence of some adaptive learning algorithm to a certain rational expectations equilibrium even if the specification of the perceived law of motion is consistent with this given equilibrium, consider again the simple Cobweb model introduced in the previous section. For simplicity, it is assumed that although agents cannot actually calculate the numerical values $\bar{a}$ and $\bar{b}$ of the coefficients of the price generating process in the unique rational expectations equilibrium, they believe that prices are generated by the following process:

$$p_t = a + w_{t-1}b + \nu_t \quad (18)$$

where a and b are unknown parameters which need to be estimated. In other words, the functional form of agents' perceived law of motion coincides with the functional form of the law of motion for prices in the unique rational expectations equilibrium, meaning that agents are assumed to understand the relevant structural relationships in their economy, but do not know the concrete parameter values.

Clearly, it is also plausible to assume that agents lack some structural knowledge of the economy as well and that thus they believe that the law of motion for prices depends on certain variables in their information set which however do not affect the rational expectations equilibrium. The issue whether a rational expectations equilibrium can be attained despite such overparametrized perceived law of motions will be taken up in a later section. However, since this section only aims at demonstrating the deficiencies of adaptive learning algorithms and the need for a thorough analysis of the circumstances under which an adaptive learning process will eventually lead to a rational expectations equilibrium, I will, for the moment, restrict attention to situations in which agents only try to estimate the parameter values a

and b , but know the structural form of the rational expectations equilibrium under consideration.

As pointed out above, the simplest procedure to obtain these estimates is to use ordinary least squares estimation which would usually constitute a consistent estimator if each new observation was indeed generated by a process of the form agents try to estimate. Hence, if producers believe that there exists a true model of the form of the perceived law of motion (18), i.e. true coefficients a and b , generating each observation of prices, their expectation formation process can be described as follows:

At period $t - 1$ producers will use all available data on past and current prices and past exogenous variables w to obtain ordinary least squares estimates $\hat{a}_{t-1}$ and $\hat{b}_{t-1}$ for the hypothesized true coefficients a and b of the perceived law of motion. Since producers' maintained hypothesis is that prices are generated by (18), where the error term ν_t is assumed to be white noise (meaning that it fulfils the usual assumptions for ordinary least squares estimation), these parameter estimates will lead to price expectations $p_t^e = \hat{a}_{t-1} + w_{t-1}\hat{b}_{t-1}$. These subjective expectations will then determine the supply choice of producers for period t and thus the price level in period t through the reduced form equation $p_t = \mu + \alpha p_t^e + w_{t-1}\delta + \nu_t$. Inserting the subjective expectations of producers formed in period $t - 1$ in this reduced form equation shows that the price level in period t is actually generated by the following process which is usually referred to as the actual law of motion implied by the perceived law of motion (18):

$$p_t = (\mu + \alpha\hat{a}_{t-1}) + w_{t-1}(\delta + \alpha\hat{b}_{t-1}) + \nu_t \quad (19)$$

From this formulation, it becomes obvious that each observation of prices is actually generated by a different model, or more precisely by a model with time varying parameters, contradicting producers' maintained hypothesis of a true data generating process of the form of their perceived law of motion. This means that during the learning process producers base their estimation on a misspecified model, implying that one central assumption needed for the consistency of the ordinary least squares estimator is violated.

In other words, producers' learning behaviour induces a feedback on the actual data generating process resulting in time varying parameters. Since agents fail to account for this feedback in their estimation method, the model they are estimating during their learning process is misspecified and thus the ordinary least squares estimates for a and b need not necessarily converge to the corresponding coefficients

of the price generating process under rational expectations although agents could correctly specify this process.

This deficiency of adaptive learning algorithms was first discussed by Bray (1982), who investigated a model of informed and informed traders, and is not restricted to ordinary least squares learning rules.

Bray and Savin (1986) argue however that although using ordinary least squares estimation in order to form subjective expectations is not "rational" since agents do not fully understand the data generating process in the sense that they do not detect the misspecification of their econometric model induced by the feedback of their own behaviour on the data generating process, it is nevertheless "reasonable". They justify this expression with the fact that the specification of the econometric model producers use to make their forecasts would be correct if the economy were already in the rational expectations equilibrium. In order to see this, consider for example a situation in which producers knew how to form rational expectations, but in which consumers lacked the knowledge to do so and tried to estimate the coefficients of the price generating process adaptively from past observations. Since consumers can nevertheless base their demand decisions on the currently observed price level, their expectations about the coefficients of their perceived law of motion for prices do not enter the actual data generating process. Therefore, each period the price level is indeed generated by equation (17), meaning that if consumers were using the perceived law of motion (18) as an econometric model to obtain their ordinary least squares estimates, these estimates would be consistent estimators for $\bar{a}$ and $\bar{b}$, the coefficients of the true data generating model. Therefore, these estimates would eventually converge to the coefficients under rational expectations as more and more observations can be incorporated into the estimation.

Although it is a fairly intuitive argument that adaptive learning algorithms which provide consistent estimates at least when agents' maintained hypothesis about the model specification is correct should be preferred, it must also be taken into account that agents who behave like econometricians will constantly evaluate the model specification using various econometric tests. Therefore, Bray and Savin (1986) use computer simulations of the ordinary least squares algorithm in the Cobweb model³² to show that at least in certain cases the misspecification of the econometric model cannot be detected by employing standard econometric tests which justifies the assumption that agents base their forecasts on ordinary least squares estimation

³²Note that Bray and Savin (1986) also consider expectation formation based on Bayesian techniques. However, the results are similar to the results obtained through the analysis of the ordinary least squares algorithm.

further.

More precisely, they based this conclusion both on the Chow Breakpoint Test and a more informal argument based on the inspection of the estimates for non-overlapping subperiods. More precisely, this latter argument can be described as follows: If the evolution of the estimates for these non-overlapping subperiods showed a significant trend, it should lead agents to the conclusion that the true model had time varying parameters. If there were no significant trend in the subperiod estimates, agents would probably attribute the variation in the estimates to random fluctuations.

Using this argument, Bray and Savin (1986) found that if agents' initial beliefs about the coefficients of their perceived law of motion were at the rational expectations levels and the ordinary least squares estimates for these coefficients converged to the rational expectations equilibrium, the misspecification was not detectable. This conclusion remained also valid if the analysis was based on the Chow Breakpoint test rather than on this intuitive argument. However, if the initial beliefs were not at their rational expectations levels, the evolution of the estimates for consecutive subperiods showed a systematic trend and thus suggested that the true model had indeed time - varying coefficients.

Bray and Savin also used the Durbin - Watson statistic to test for a general misspecification of the econometric model and found that in their simulations, it never detected the misspecification due to the feedback of agents' learning behaviour on the actual data generating process, except when the exogenous variables influencing supply were generated by an AR(1) process and the estimates were converging slowly to their rational expectations levels. The question how agents will respond to detecting the misspecification of their econometric model will be briefly taken up in a later section. However, it could lead to unpredictable changes in agents' model specification.

More importantly, the analysis of Bray and Savin (1986) shows that it is indeed plausible that agents who evaluate their model specification using econometric testing stick to using ordinary least squares estimation despite the fact that their model is misspecified. It is however also important to note that the analysis of Bray and Savin (1986) and others showed that convergence of agents' expectations to a rational expectations equilibrium is far from obvious. In order to analyse under which circumstances rational expectations equilibria can actually be attained by adaptive learning rules, I will next give some examples of so called stochastic recursive algorithms, which are frequently used in order to model adaptive learning processes. After that, these techniques will be applied to the overlapping generations model

with taste shocks in order to analyse whether it is realistic that an economy ends up in a stationary sunspot equilibrium.

3.2. Stochastic Recursive Algorithms

3.2.1. Recursive Least Squares Learning

As has already been pointed out above, due to its simplicity, ordinary least squares estimation constitutes a widely used example for how agents who are not able to form rational expectations can nevertheless obtain subjective expectations about future realizations of economic variables. In order to analyse under which circumstances the subjective expectations obtained through this technique can converge to rational expectations, consider again the simple Cobweb model introduced in section 3.1.2. Denoting the $1 \times (K + 1)$ vector $(1 \quad w_{t-1})$ as x_{t-1} and the $(1 + K) \times 1$ coefficient vector $(a \quad b)'$ as β , the ordinary least squares estimates for the coefficients of the perceived law of motion (18) formed in period $t - 1$, $\hat{\beta}_{t-1} = (\hat{a}_{t-1} \quad \hat{b}'_{t-1})'$, can be obtained through the usual formula:

$$\hat{\beta}_{t-1} = (X'^{(t-1)} X^{(t-1)})^{-1} (X'^{(t-1)} p^{(t-1)})$$

where $X^{(t-1)}$ denotes the $(t - 1) \times (K + 1)$ matrix whose i 'th row is given by x_{i-1} , and where $p^{(t-1)}$ denotes the $(t - 1) \times 1$ vector whose i 'th element is given by p_i . However, a major disadvantage of this so called "off-line" estimation is that the amount of data needed to be stored increases each period which, over a long time, may result in great practical difficulties. Therefore, a more efficient calculation of the estimates would be to use a recursive or "on-line" procedure which only requires to store the last periods estimates and to use the currently observed data to update them. Following Ljung and Söderström (1983), it is convenient to rewrite the ordinary least squares estimator in order to obtain a recursive formula for it:

$$\begin{aligned} \hat{\beta}_t &= \left(\sum_{i=1}^t x'_{i-1} x_{i-1} \right)^{-1} \left(\sum_{i=1}^t x'_{i-1} p_i \right) \\ &= \left(\sum_{i=1}^t x'_{i-1} x_{i-1} \right)^{-1} \left(\sum_{i=1}^{t-1} x'_{i-1} p_i + x'_{t-1} p_t \right) \end{aligned}$$

Using an analogous definition for $\hat{\beta}_{t-1}$ to express $\sum_{i=1}^{t-1} x'_{i-1} p_i$ as $(\sum_{i=1}^{t-1} x'_{i-1} x_{i-1}) \hat{\beta}_{t-1}$ and defining the estimate for the variance - covariance matrix of the random vector

x , i.e. $\frac{1}{t} \left(\sum_{i=1}^t x'_{i-1} x_{i-1} \right)$, using observations up to period t , as R_t yields:

$$\begin{aligned}
\hat{\beta}_t &= \frac{1}{t} R_t^{-1} \left[\left(\sum_{i=1}^{t-1} x'_{i-1} x_{i-1} \right) \hat{\beta}_{t-1} + x'_{t-1} p_t \right] \\
&= \frac{1}{t} R_t^{-1} \left[\left(\sum_{i=1}^t x'_{i-1} x_{i-1} \right) \hat{\beta}_{t-1} - x'_{t-1} x_{t-1} \hat{\beta}_{t-1} + x'_{t-1} p_t \right] \\
&= \frac{1}{t} R_t^{-1} \left[t R_t \hat{\beta}_{t-1} - x'_{t-1} x_{t-1} \hat{\beta}_{t-1} + x'_{t-1} p_t \right] \\
&= \hat{\beta}_{t-1} + \frac{1}{t} R_t^{-1} x'_{t-1} \left(p_t - x_{t-1} \hat{\beta}_{t-1} \right)
\end{aligned}$$

However, since R_t still depends on all past observations of the exogenous variables $w_0, \dots, w_{t-1}$, this expression is not yet a recursion for $\hat{\beta}_t$. Therefore, it is also necessary to express R_t recursively as:

$$\begin{aligned}
R_t &= \frac{1}{t} \left(\sum_{i=1}^{t-1} x'_{i-1} x_{i-1} + x'_{t-1} x_{t-1} \right) \\
&= \frac{t-1}{t(t-1)} \sum_{i=1}^{t-1} x'_{i-1} x_{i-1} + \frac{1}{t} x'_{t-1} x_{t-1} \\
&= \frac{1}{t-1} \sum_{i=1}^{t-1} x'_{i-1} x_{i-1} - \frac{1}{t(t-1)} \sum_{i=1}^{t-1} x'_{i-1} x_{i-1} + \frac{1}{t} x'_{t-1} x_{t-1} \\
&= R_{t-1} + \frac{1}{t} (x'_{t-1} x_{t-1} - R_{t-1})
\end{aligned}$$

Hence, a recursive formulation of the ordinary least squares estimator is given by the following algorithm:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \frac{1}{t} R_t^{-1} x'_{t-1} \left(p_t - x_{t-1} \hat{\beta}_{t-1} \right) \quad (20)$$

$$R_t = R_{t-1} + \frac{1}{t} (x'_{t-1} x_{t-1} - R_{t-1}) \quad (21)$$

However, as Evans and Honkapohja (2001) point out, it is necessary to specify appropriate initial conditions for the estimates $\hat{\beta}$ and R in order for the recursive system to be well specified. The only initial conditions guaranteeing that the estimates calculated according to the recursion (20) - (21) are indeed equivalent to the ordinary least squares estimates are the following:

$$\begin{aligned}
\hat{\beta}_{K+1} &= \left(X^{(K+1)'} X^{(K+1)} \right)^{-1} \left(X^{(K+1)'} p^{(K+1)} \right) \\
&= X^{(K+1)} p^{(K+1)}
\end{aligned}$$

$$\begin{aligned}
R_{K+1} &= \frac{1}{K+1} \sum_{i=1}^{K+1} x_i' x_i \\
&= \frac{1}{K+1} X'^{(K+1)} X^{(K+1)}
\end{aligned}$$

It is clear, that for $t < K + 1$ the matrix $X^{(t)}$ is singular and that thus the ordinary least squares estimate $\hat{\beta}_t$ is not defined for such t . To deal with this problem, it is assumed that until agents have observed enough data, their estimates are arbitrary, but once it is possible to calculate the ordinary least squares estimates they do so. Another possible problem is, that agents are required to invert the $(K + 1) \times (K + 1)$ matrix R_t in order to calculate their estimates according to the recursive algorithm (20) - (21). For a large number of explanatory variables this might result in computational difficulties. Therefore, Ljung and Söderström (1983) suggest a different form of this algorithm which does only require the inversion of a scalar, however at the cost of introducing a further variable.

A simpler alternative to avoid the inversion of a potentially very large matrix is the so called stochastic gradient algorithm, although the estimates calculated according to this algorithm are less efficient than the ordinary least squares estimates. The next section will introduce the necessary technique to derive this algorithm. This technique developed by Robbins and Monro (1951) can in general be a useful learning rule for agents who are faced with an optimization problem and can thus also be applied to the overlapping generations model discussed in section 2.

3.2.2. The Robbins - Monro Algorithm

The assumption that agents form their expectations by using ordinary least squares estimation implies that agents can specify a perceived data generating process (although they do not take the feedback of their learning behaviour into account) and try to estimate the coefficients of this perceived law of motion.

A more general problem agents might face is that they need to find the unique solution θ^* to the equation $M(\theta) = \alpha$, where the nature of $M(\cdot)$ is unknown. However, if agents have an estimate θ_t for θ^* , they can observe the realization of a random variable $Y(\theta_t)$ which is an unbiased estimator for $M(\theta_t)$, conditional on θ_t . This problem was first analysed by Robbins and Monro (1951), who highlighted its relevance for the analysis of a series of experiments in technical applications.

A relevant application of this problem in economics is however pointed out by Ljung (1977): If agents try to minimize the expected value of a function $J(\theta, \phi)$ which de-

depends on some choice variable θ (this variable could for example stand for agents' labour supply or consumption choice) and the realization of a random variable ϕ , they have to solve the problem:

$$\mathbb{E} \left[\frac{\partial}{\partial \theta} J(\theta, \phi) \right] = 0$$

In the notation of Robbins and Monro, $M(\theta)$ corresponds to the left hand side of this equation, $\alpha = 0$ and $Y(\theta) = \frac{\partial}{\partial \theta} J(\theta, \phi)$.

If the distribution function of ϕ is unknown, then also the functional form of $M(\theta)$ is not known and thus the optimization problem agents face cannot be solved analytically. However, given that agents have some initial estimate for the solution θ^* , they can in any period t observe an estimate $y_t = \frac{\partial}{\partial \theta} J(\theta_{t-1}, \phi_t)$ for the marginal benefit which could be obtained from changing the estimate made for θ^* one period ago, given ϕ_t , the currently observed realization of the random variable ϕ . The agents can then adjust the estimate made in the previous period in the direction $\alpha - y_t$ given by this observation which is assumed to constitute an unbiased estimator for the direction of steepest decent of $\mathbb{E}[J(\theta_{t-1}, \phi)]$, conditional on the estimate θ_{t-1} . In other words, this implies that agents will on average adjust their estimates, such that the distance between α and $M(\theta_t)$ is reduced most rapidly. Hence, agents will be able to find the minimum of $\mathbb{E}[J(\theta, \phi)]$ numerically by gradually adjusting their estimates for this minimum according to the mechanism just described.

More precisely, Robbins and Monro (1951) show that the algorithm

$$\theta_t = \theta_{t-1} + \gamma_t (\alpha - y_t) \tag{22}$$

converges in mean square to the correct solution θ^* , i.e. that $\lim_{t \rightarrow \infty} \mathbb{E}[(\theta_t - \theta^*)^2] = 0$, if certain assumptions are fulfilled. In particular, it is required that the sequence $\{\gamma_t\}_{t=1}^{\infty}$ which determines by how much the previous estimates respond to new observations converges "slowly" (in a sense which will be discussed in greater detail below) to zero. Moreover, Robbins and Monro (1951) show that the estimates converge to θ^* especially if $M(\cdot)$ is not decreasing. Hence, the Robbins - Monro algorithm can indeed be applied to find the global minimum of a convex function.

However, it is not necessarily the case that the latest observation on the direction of steepest descent $\alpha - y_t$ does indeed constitute a conditionally unbiased estimator for the direction which decreases the distance between $M(\theta)$ and α most rapidly, as Robbins and Monro (1951) assume. An example for which this is not satisfied will be given in the next section in terms of the Cobweb model discussed above. This

problem will however also arise when the Robbins - Monro algorithm is applied to the overlapping generations model discussed in section 2. Nevertheless, if agents' estimates had already converged to some constant value θ^* , this problem would not arise. Thus similar to ordinary least squares learning, the Robbins - Monro algorithm can be regarded as a "reasonable", but not a completely "rational" learning rule. This point will also be illustrated further below. It implies however that the convergence of this algorithm has to be analysed using a different technique.

3.2.3. Stochastic Gradient Learning

Although it is plausible that agents who are confronted with the dynamics of the Cobweb model base their expectations on ordinary least squares estimation, it is also possible that they choose another estimator even despite the fact that OLS has, under certain assumptions, the smallest variance among all linear unbiased estimators. It is especially plausible that instead of choosing the estimates such that past observations can in a certain way be described best, agents who are only interested in forming expectations about the future, but not in explaining the economic structure try to make a forecast error which is as close to zero as possible. In other words, it is plausible to assume that agents' sole interest is to obtain the best forecasts given their current information. This can be modelled by assuming that agents try to minimize the expected squared forecast error.

Evans and Honkapohja (1998b) give a description of this behaviour in terms of a multivariate Cobweb model which describes the simultaneous equilibrium on several unrelated goods markets. However, to keep the analysis as simple as possible, I will stick to the univariate model introduced in Section 3.1.2, where it has been assumed that market clearing prices are generated through the reduced form equation:

$$p_t = \mu + \alpha p_t^e + w_{t-1}\delta + \nu_t$$

and where the expectation of producers about the price in period t is given by:

$$p_t^e = x_{t-1}\hat{\beta}_{t-1}$$

where x_{t-1} is, as before, given by the $1 \times (K + 1)$ vector $(1 \quad w_{t-1})$, and where $\hat{\beta}_{t-1}$ denotes now the estimate for β chosen in period $t - 1$ such that it minimizes the expected squared forecast error in period t which is given by:

$$\mathbb{E} [(p_t - p_t^e)^2] = \mathbb{E} [(p_t - x_{t-1}\beta)^2]$$

Hence, in period $t - 1$ agents try to find those values for the coefficients of their perceived law motion which satisfy the following first order condition:

$$\begin{aligned} \mathbb{E} [-2x'_{t-1} (p_t - x_{t-1}\beta)] &= 0 \\ \Rightarrow \underbrace{\mathbb{E} [-x'_{t-1} (p_t - x_{t-1}\beta)]}_{=M(\beta)} &= \underbrace{0}_{=\alpha} \end{aligned}$$

Clearly, this problem can be solved using the Robbins - Monro algorithm discussed in the previous section. However, since agents have not decided on $\hat{\beta}_{t-1}$ yet, the latest observation on the marginal forecast error available in period $t - 1$ is given by:

$$y_{t-1} = -x'_{t-2} [p_{t-1} - x_{t-2}\hat{\beta}_{t-2}]$$

Inserting this into the Robbins Monro algorithm given by equation (22) yields the following learning rule:

$$\hat{\beta}_{t-1} = \hat{\beta}_{t-2} + \gamma_{t-1} \left[0 + x'_{t-2} (p_{t-1} - x_{t-2}\hat{\beta}_{t-2}) \right]$$

or equivalently:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \gamma_t x'_{t-1} (p_t - x_{t-1}\hat{\beta}_{t-1}) \quad (23)$$

As Evans and Honkapohja (1998b) point out this algorithm is a gradient rule which adjusts the estimates made in the previous period in the direction in which the squared forecast error is expected to decline most rapidly. However, due to the time varying parameter estimates for β , this expectation is biased. This can be seen by noting that, through the reduced form equation, the expected value of realized prices (conditional on the estimates) in period $t + 1$ depends on $\hat{\beta}_t$. However, when agents try to minimize the expected squared forecast error made in period $t + 1$, they only possess information on the forecast error made in period t . The expected value of the actual prices entering this forecast error however depend on $\hat{\beta}_{t-1}$, and thus $x'_{t-1} (p_t - x_{t-1}\hat{\beta}_{t-1})$ does not constitute an unbiased estimator for $\mathbb{E} [x'_t (p_{t+1} - x_t\hat{\beta}_t)]$, the direction of the steepest descent for the expected squared forecast error made in period $t + 1$ which should be minimized. Therefore, in any period, agents adjust their estimates formed in the previous period in a direction which does not most rapidly decrease the expected squared forecast error made in the next period, meaning that it is not clear whether the estimates will eventually converge to a minimum or not without analysing this algorithm further. However, it is also clear that if the estimates did not vary over time, but were equal to some

value β^* in all periods, this problem would not arise since then any observation on the marginal forecast error would constitute an unbiased estimator for the direction of steepest ascent of the expected squared forecast error made in any other period.

Comparing the stochastic gradient algorithm (23) with the recursive formulation of ordinary least squares (20) - (21), it is apparent that this algorithm does not include an estimator for the variance - covariance matrix of the explanatory variables, as does the recursive least squares algorithm. This implies that on the one hand, the estimates according to the stochastic gradient algorithm are easier to compute than the estimates calculated according to the more complicated recursive least squares algorithm since only the recursion for one variable has to be considered and additionally it is not necessary to invert a matrix. On the other hand, it could also be the case that neglecting this information has negative implications as well. However, Evans and Honkapohja (1998b) show that this is not the case and that in a Cobweb model the stochastic gradient algorithm and the recursive least squares algorithm converge to the unique rational expectations equilibrium under exactly the same circumstances. This result will be demonstrated in greater detail in a later section using so called expectational stability conditions. As will be shown below, expectational stability is in the Cobweb model equivalent to the more complicated, but direct analysis of the recursive least squares and stochastic gradient algorithms. As this result does however not hold for all models and all possible adaptive learning algorithms, I will introduce the direct analysis of stochastic recursive algorithms first. In order to do so, it is convenient to state all recursive algorithms introduced so far in a general form. This will be done in the next section.

3.2.4. General Form of Stochastic Recursive Algorithms

Following Evans and Honkapohja (1998a) or (2001), a stochastic recursive algorithm can in general be expressed as:³³

$$\theta_t = \theta_{t-1} + \gamma_t H(\theta_{t-1}, X_t) \quad (24)$$

This algorithm determines the evolution of a $d \times 1$ vector of parameter estimates θ_t which, through a function $H(\theta_{t-1}, X_t)$, depends on an $l \times 1$ vector of exogenous

³³Evans and Honkapohja actually allow for an even more general representation. However for the algorithms considered here this is not necessary, although it should be noted that in some of the examples discussed here the function used to update the estimates will explicitly depend on time.

observable state variables X_t and on the estimates made in previous period. The so called "gain parameter" $\gamma_t \in \mathbb{R}$ determines how responsive the estimates are with respect to $H(\theta_{t-1}, X_t)$.

In order to see that the stochastic gradient algorithm discussed in the previous section can indeed be written in this form, it is only necessary to note that by inserting producers expectations, which are still assumed to be based on the perceived law of motion (18), into the reduced form equation, the actual prices in period t can be expressed as:

$$\begin{aligned} p_t &= \mu + \alpha \left(x_{t-1} \hat{\beta}_{t-1} \right) + w_{t-1} \delta + \nu_t \\ &= x_{t-1} \left[\alpha \hat{\beta}_{t-1} + (\mu \quad \delta)' \right] + \nu_t \end{aligned}$$

Inserting this into the stochastic gradient algorithm (23) shows that the estimates for β are updated as:

$$\begin{aligned} \hat{\beta}_t &= \hat{\beta}_{t-1} + \gamma_t x'_{t-1} \left(x_{t-1} \left[\alpha \hat{\beta}_{t-1} + (\mu \quad \delta)' \right] + \nu_t - x_{t-1} \hat{\beta}_{t-1} \right) \\ &= \hat{\beta}_{t-1} + \gamma_t x'_{t-1} \left(x_{t-1} \left[(\alpha - 1) \hat{\beta}_{t-1} + (\mu \quad \delta)' \right] + \nu_t \right) \end{aligned}$$

Clearly, the estimates $\hat{\beta}_t$ correspond to θ_t in the general notation. The exogenous state variables are in this case given by $X_t = (x_{t-1} \quad \nu_t)'$, and thus the function $H(\theta_{t-1}, X_t)$ is just given by the direction of steepest descent for the squared forecast error observed in period t . For this algorithm the gain parameter γ_t is not further specified. However, in order to allow for the convergence of the estimates it is necessary to impose certain assumptions which will be stated in the next section. A usual choice for the sequence of gain parameters is $\{\gamma_t\}_{t=1}^{\infty} = \{\frac{1}{t}\}_{t=1}^{\infty}$, which is also used in the recursive least squares algorithm and which is therefore also chosen by Evans and Honkapohja (1998b) for the stochastic gradient algorithm.

As pointed out above, for the recursive least squares algorithm R , the for the variance - covariance matrix of the explanatory variables, also needs to be estimated. Therefore, for this algorithm θ_t is in principal given by $vec(\beta_t \quad R_t)$, where $vec(\cdot)$ denotes the vectorization of a matrix.

However, as pointed out by Evans and Honkapohja (2001), the recursive algorithm (20) - (21) for least squares learning is not yet in the general form (24) since it is the case that the current estimate R_t influences the current estimate $\hat{\beta}_t$, which is

ruled out by the formulation (24) in which only θ_{t-1} can influence θ_t . To solve this problem, it is necessary to define a new variable $S_{t-1} := R_t$. From (21) it follows that a recursion for S_t is given by:

$$\begin{aligned} S_t &= S_{t-1} + \frac{1}{t+1} (x'_t x_t - S_{t-1}) \\ &= S_{t-1} + \frac{1}{t} \left(\frac{t}{t+1} \right) (x'_t x_t - S_{t-1}) \end{aligned}$$

Substituting the actual law of motion of prices into the learning algorithm, as done for the stochastic gradient algorithm, yields the general formulation:

$$\begin{aligned} \hat{\beta}_t &= \hat{\beta}_{t-1} + \frac{1}{t} S_{t-1}^{-1} x'_{t-1} \left(x_{t-1} \left[(\alpha - 1) \hat{\beta}_{t-1} + (\mu \quad \delta)' \right] + \nu_t \right) \\ S_t &= S_{t-1} + \frac{1}{t} \left(\frac{t}{t+1} \right) (x'_t x_t - S_{t-1}) \end{aligned}$$

where the vector of estimates θ_t is now given by $\text{vec}(\beta_t \ S_t)$, the vector of state variables is given by $X_t = (x_{t-1} \ x_t \ \nu_t)'$ and the sequence of gain parameters is given by $\{\gamma_t\}_{t=1}^\infty = \{\frac{1}{t}\}_{t=1}^\infty$.

After introducing this general formulation of adaptive learning algorithms, the next section will illustrate the mathematical techniques which are necessary in order to analyse stochastic recursive algorithms in this general form.

3.3. Stochastic Approximation

Note that since in all applications discussed here the state variables include some random variable, the general form of a recursive learning algorithm discussed in the previous section corresponds to a stochastic difference equation which is in general very hard to analyse.³⁴ Therefore, the basic idea in order to simplify the analysis is to replace the stochastic difference equation with a deterministic differential equation, chosen such that the trajectories of the difference equation are well approximated by the trajectories of the differential equation evaluated at discrete points in time. The advantage of doing so is that the stability of fixed points or steady states under the dynamics of a deterministic differential equation is much easier to analyse than under the dynamics of a stochastic difference equation, and as Evans and Honkapohja (1998a) argue, it will under certain assumptions nevertheless be possible to select plausible equilibria of learning processes based on their stability under the dynamics

³⁴For a direct analysis of the ordinary least squares learning algorithm see for example Bray (1982).

of the associated differential equation.

3.3.1. The Associated Differential Equation

In order to see how it is possible to find an associated differential equation to the stochastic recursive algorithm (24), consider the following heuristic argument provided by Evans and Honkapohja (2001):

Suppose that the gain sequence $\{\gamma_t\}$ is such that it is declining to zero and such that for t large enough the gain parameter is almost constant. Then, θ_{n+N} can be approximated as:

$$\theta_{n+N} \approx \theta_n + \gamma \sum_{i=0}^{N-1} H(\theta_{n+i}, X_{n+1+i})$$

However, since for n large γ must already be very small, meaning that the estimates do not change much from one period to the next, it must approximately hold that:

$$\begin{aligned} \theta_{n+N} &\approx \theta_n + \gamma \sum_{i=0}^{N-1} H(\theta_n, \bar{X}_{n+1+i}) \\ &= \theta_n + N\gamma \frac{1}{N} \sum_{i=0}^{N-1} H(\theta_n, \bar{X}_{n+1+i}) \end{aligned}$$

where $\bar{X}$ indicates that if the state variables in a certain period depend on the estimates for θ made in the previous period³⁵, they are generated by the constant value θ_n .

Since for n large, $\bar{X}_{n+1+i}$ will for all $i = 0, \dots, \infty$ be drawn from distributions with identical expected value³⁶, the law of large numbers implies that for N large $\frac{1}{N} \sum_{i=0}^{N-1} H(\theta_n, \bar{X}_{n+1+i})$ is approximately equal to $\lim_{t \rightarrow \infty} \mathbb{E}[H(\theta_n, \bar{X}_t)]$ which will be denoted as $h(\theta_n)$. Using this notation, it holds that:

$$\theta_{n+N} \approx \theta_n + N\gamma h(\theta_n)$$

Replacing the approximation γ again by the correct gain sequence, it must thus hold that:

$$\theta_{n+1} \approx \theta_n + \gamma_{n+1} h(\theta_n)$$

³⁵A precise assumption on the process generating the state variables will be made below.

³⁶The formal assumption that the process generating X_n is asymptotically stationary will be made below.

which can be regarded as the discretization of the ordinary differential equation:

$$\frac{d\theta}{d\tau} = h(\theta) \quad (25)$$

where the "alternative time" τ is such that for all $n = 1, 2, \dots$ the discrete point in time τ_n is given by $\sum_{i=1}^n \gamma_i$.

In order to see that the approximation for the stochastic recursive algorithm just derived is then indeed a discretization of this differential equation, note that such a discretization is given by:

$$\begin{aligned} \theta_{\tau_{n+1}} &\approx \theta_{\tau_n} + (\tau_{n+1} - \tau_n) h(\theta_{\tau_n}) \\ &= \theta_{\tau_n} + \left(\sum_{i=1}^{n+1} \gamma_i - \sum_{i=1}^n \gamma_i \right) h(\theta_{\tau_n}) \\ &= \theta_{\tau_n} + \gamma_{n+1} h(\theta_{\tau_n}) \end{aligned}$$

This implies that indeed a trajectory of the stochastic difference equation can (at least for large t) be approximated by certain discrete points in a trajectory of the associated differential equation (25).

3.3.2. Assumptions on Stochastic Recursive Algorithms

However, in order to analyse how the stability properties of a steady state of the associated differential equation translate into stability properties of this steady state under the dynamics of the stochastic difference equation, it is necessary to state some assumptions on the stochastic recursive algorithm itself and on the process generating the state variables, as done for example in Ljung (1977), Woodford (1990) or Evans and Honkapohja (1998a) and (2001).

For the following, D is an open subset of $\mathbb{R}^d$ containing θ^* , the rational expectations equilibrium of interest, which is a fixed point both of the stochastic recursive algorithm and the associated differential equation.

The first assumption restricts the gain parameter sequence. Informally, it states that the gain parameter must decrease to zero as time goes to infinity, but that the speed of convergence must not be too fast. This assumption has already been made for the Robbins - Monro algorithm in section 3.2.2. More precisely, it can be stated as:

A.1 $\{\gamma_t\}_{t=0}^{\infty}$ is a nonstochastic and nonincreasing sequence such that:

$$\sum_{t=1}^{\infty} \gamma_t = \infty \text{ and } \sum_{t=1}^{\infty} \gamma_t^2 < \infty$$

Ljung (1977) states a less restrictive assumption: instead of assuming that $\sum_{t=1}^{\infty} \gamma_t^p < \infty$ holds for $p = 2$, he assumes that this holds for some p . Also Evans and Honkapohja (1998a) and (2001) point out that this weaker assumption would actually be sufficient. However, for the usual gain sequence t^{-1} which is for example used in the recursive least squares algorithm and the stochastic gradient algorithm discussed by Evans and Honkapohja (1998b), the stronger assumption is satisfied as well.

An example for which the second part of this assumption is violated is a constant gain sequence, i.e. $\gamma_t = \gamma$ for all t . This would imply that agents take any deviations of the actual outcomes from their previous expectations about these outcomes into account when forming their new estimates, while for decreasing gain sequences agents will eventually stop to revise their estimates. Therefore, in stochastic frameworks in which there will always be random fluctuations a constant gain algorithm cannot converge to a steady state since it implies that agents will always adjust their estimates in response to these random fluctuations.³⁷ This problem is discussed in greater detail by Evans and Honkapohja (2001). However, they also note that since for a constant gain algorithm the weight put on past observations dies out at a geometric rate, while for decreasing gain algorithms, such as recursive least squares learning, all past observations enter with equal weight, a constant gain algorithm can react better to possible structural change in the economy. Thus, they also suggest that if for some reason agents detect the misspecification of their econometric model due to the feedback of their learning behaviour and misinterpret this finding as recurring structural change to the economy, they might change their learning rule to a constant gain algorithm.

In order to investigate whether expectations formation through a constant gain algorithm is a plausible assumption, Branch and Evans (2006) compare the forecast performance of recursive least squares learning, a time varying parameter model and a constant gain algorithm in out-of-sample forecasting of GDP growth and inflation. For this analysis they constructed the constant gain algorithm by replacing the gain sequence t^{-1} in the recursive least squares algorithm by a constant. This constant was chosen such that it minimized the mean squared forecast error in an in-sample

³⁷For non-stochastic frameworks this argument does not hold.

forecasting period. Therefore, it reflected the actual degree of structural change for the variable under consideration.³⁸

Branch and Evans found that this constant gain algorithm had to be preferred on basis of the mean squared forecast error for both variables. Furthermore, they showed that an even simpler model, using the same gain parameter for both variables, resulted in the best fit to the forecasts of the Survey of Professional Forecasters.

Therefore, they conclude that if agents are faced with structural change in the economy and are acting like econometricians, it is indeed plausible to assume that they use a constant gain algorithm, in which case the second part of this assumption would be violated. However, since the models discussed here are not subject to structural change, the assumption of a decreasing gain sequence is justified. Moreover, in situations with structural change, it is not even desired that agents' estimates converge to a stationary rational expectations equilibrium since the rational expectations equilibrium would itself be subject to change. Therefore, the only problem remaining with the assumption of a decreasing gain sequence is that agents might misinterpret some random fluctuations or the misspecification due to learning as structural change and thus wrongly adopt a constant gain algorithm. However, as the simulations of Bray and Savin (1986) showed, this is not very likely.

Furthermore, Evans and Honkapohja (1998a) and (2001) note that if the first part of this assumption were not met, i.e. if the gain sequence decreased too fast to zero, the recursive algorithm might converge to a non - equilibrium point since fluctuations which still contain information that could lead to better forecasts are not properly taken into account.

Moreover, Ljung (1977) assumes that the gain sequence satisfies:

$$\limsup_{t \rightarrow \infty} \left[\frac{1}{\gamma_t} - \frac{1}{\gamma_{t-1}} \right] < \infty$$

This is also an unnumbered assumption in Evans and Honkapohja (1998a) and (2001), and it is in particular satisfied for the sequence $\gamma_t = t^{-1}$.

The next assumptions made by Evans and Honkapohja (1998a) and (2001) concern the function $H(\theta, x)$. The first states that $H(\theta, x)$ is bounded by a polynomial depending on the state variables:

A.2 For any compact subset $Q \subset D$, there exist constants C and q , such that for

³⁸A higher gain parameter implies that past observations are discounted more. Thus, the optimal constant gain parameter should be higher in economies with a high degree of structural change.

all $\theta \in Q$ it holds that:

$$|H(\theta, x)| \leq C(1 + |x|^q)$$

As Evans and Honkapohja (2001) show, this assumption assures that the right hand side of the differential equation (25), $h(\theta) = \lim_{t \rightarrow \infty} \mathbb{E}[H(\theta, \bar{X}_t(\theta))]$, is well defined which is however directly assumed by Ljung (1977) or Woodford (1990).

Furthermore, Evans and Honkapohja (1998a) and (2001) assume that $H(\theta, x)$ is twice continuously differentiable and that its derivatives are bounded, which implies that $H(\theta, x)$ and its derivative with respect to x are Lipschitz continuous. More precisely, they assume the following:

A.3 For any compact subset $Q \subset D$ and for all θ and $\theta' \in Q$ and x_1 and $x_2 \in \mathbb{R}^l$ there exist constants L_1 and L_2 , such that the function $H(\theta, x)$ satisfies:

- (i) $|H(\theta, x_1) - H(\theta, x_2)| \leq L_1 |x_1 - x_2|$
- (ii) $|H(\theta, 0) - H(\theta', 0)| \leq L_2 |\theta - \theta'|$
- (iii) $\left| \frac{\partial H(\theta, x)}{\partial x} - \frac{\partial H(\theta', x)}{\partial x} \right| \leq L_2 |\theta - \theta'|$

Evans and Honkapohja (2001) show that this assumption, together with one of the following assumptions, implies that $h(\theta)$ is Lipschitz continuous and that therefore given any initial value, the ordinary differential equation (25) has a unique solution which approximates the trajectory of the given stochastic recursive algorithm.

For the evolution of the state variables Ljung (1977), Evans and Honkapohja (1998a) and (2001) assume that they follow conditionally linear dynamics. Evans and Honkapohja (1998a) and (2001) argue that this is the relevant case for most economic applications. Formally, this assumption can be stated as:

B.1 $X_t = A(\theta_{t-1})X_{t-1} + B(\theta_{t-1})e_t$

where $A(\theta_{t-1})$ and $B(\theta_{t-1})$ are matrix valued functions of θ_{t-1} , and where e_t is as described by the next assumption.

Evans and Honkapohja (2001) also discuss a more general case in which the state variables follow a Markov process. However, assumption B.1 covers all models which will be discussed here.

Furthermore, Ljung (1977), Evans and Honkapohja (1998a) and (2001) assume the following on e_t :

B.2 e_t is the realization of a sequence of independent, identically distributed random variables with finite absolute moments, i.e. $\mathbb{E}[|e_t|^p] < \infty$ for all p .

Ljung (1977) notes that instead of the second part of this assumption it is also possible to assume that e_t is bounded with probability one for all t . Moreover, Woodford (1990) shows that also the first part of this assumption is not necessary if other assumptions are strengthened. Thus, although this assumption might appear too restrictive, it is not crucial for the following results.

The last assumption made on the state dynamics by Evans and Honkapohja (1998a) and (2001) is the following:

B.3 For any compact subset $Q \subset D$ and for some matrix norm $|\cdot|$ it holds that:

$$\sup_{\theta \in Q} |B(\theta)| \leq M \text{ and } \sup_{\theta \in Q} |A(\theta)| \leq \rho < 1$$

Furthermore, $A(\theta)$ and $B(\theta)$ satisfy Lipschitz conditions on Q .

Evans and Honkapohja (2001) note that this assumption is stronger than asymptotic stationarity of the process generating X_t , which assures that if some constant value $\bar{\theta}$ is used to generate the state variables, their mean will be asymptotically constant. Therefore, it is indeed possible to approximate $\frac{1}{N} \sum_{i=0}^{N-1} H(\theta_n, \bar{X}_{n+1+i})$ with $\lim_{t \rightarrow \infty} \mathbb{E}[H(\theta_n, \bar{X}_t)]$ for N large, as done in the heuristic argument given above. Furthermore Evans and Honkapohja (2001) use this assumption together with assumption A.3 to show that $h(\theta)$ is Lipschitz continuous.

With these assumptions, it is possible to establish the following theoretical results on the stability of rational expectations equilibria, or more generally of fixed points, under the dynamics of adaptive learning rules which can be expressed in the general form of stochastic recursive algorithms.

3.3.3. Main Results in Stochastic Approximation

The first result which will be used in later sections to translate the qualitative properties of the associated differential equation into the qualitative properties of the learning algorithm is due to Ljung (1977) and states to which points the stochastic recursive algorithm can converge:

Theorem 6 *Consider a stochastic recursive algorithm of the general form (24) which satisfies the assumptions stated in the previous section.*

Furthermore, suppose that the probability of θ_t converging to an arbitrarily small

open ball around $\theta^* \in D$ is strictly positive, and assume that the variance - covariance matrix of $H(\theta^*, \bar{X}_t)$ is bounded from below by a strictly positive definite matrix. Additionally, assume that in a neighbourhood around θ^* , $\mathbb{E}[H(\theta, \bar{X}_t)]$ is continuously differentiable with respect to θ and that this derivative converges uniformly as t goes to infinity.

Then it must hold that $h(\theta^*) = 0$ and that all eigenvalues of $\frac{d}{d\theta}h(\theta^*)$ have negative (or zero) real part.

In other words, this theorem states that a stochastic recursive algorithm can only converge to locally asymptotically stable stationary steady states of the associated ordinary differential equation (25). Conversely, it also follows from this theorem that if a certain point is no steady state, an unstable, or only a saddle path stable steady state of the associated ordinary differential equation, the stochastic recursive algorithm cannot converge to this fixed point. Ljung (1977) argues that actually this contraposition of the theorem is its most important application. This will also become obvious in the applications of this result in the following sections where this result will be used to rule out implausible equilibria which cannot be attained by an adaptive learning process in the case of multiple rational expectations equilibria.

Another important result which will be used in the following sections to show convergence of the trajectories of a stochastic recursive algorithm to some particular steady state is the following theorem due to Ljung (1977) which is also stated in Woodford (1990):

Theorem 7 *Consider a stochastic recursive algorithm of the general form (24) which satisfies the assumption stated in the previous section.*

Furthermore, assume that e_t is bounded with probability one for all t , that there exists a compact set $Q \subset D$ which is such that $\theta_t \in Q$ infinitely often with probability one, and assume that the associated differential equation (25) has an invariant set I whose domain of attraction is $D_I \supset Q$.

Then it follows that θ_t converges with probability one to I as time goes to infinity.

As Ljung (1977) points out, an important special case of this theorem is the following: If the invariant set I were a singleton containing only one steady state of the associated differential equation, it would follow from this theorem that the estimates θ_t converge to this stationary point with probability one provided that indeed all requirements stated in the theorem (i.e. the "visiting property") are satisfied. This is in particular the case if the associated differential equation is defined on a

compact set and if the steady state under consideration is globally stable on this domain under the dynamics of the associated differential equation. In order to see this, note that in this case the compact set Q can be chosen to be the whole domain of the associated differential equation, meaning that the "visiting property" is trivially satisfied since moreover D_I , the domain of attraction of the steady state under the dynamics of the associated differential equation, coincides with Q . Note that his argument, although in a more general form allowing the set I to contain more than one steady state, will be used in a later section to show convergence of an adaptive learning algorithm to the set of stationary two - state sunspot equilibria.

However, the assumption that the trajectories visit a subset of the domain of attraction of the invariant set I under the dynamics of the associated differential equation infinitely often seems rather strong. Therefore, Ljung (1977) shows that the conclusion of this theorem remains valid if this condition is not satisfied, but if instead a so called projection facility is employed. A projection facility consists of two sets $D_1 \subset D_2$ which are such that D_2 is a subset of D_I . Furthermore, whenever the estimate θ_{t-1} lies within the interior of D_2 , θ_t is calculated according to usual stochastic recursive algorithm, whereas otherwise it is projected to some point in D_1 .

Evans and Honkapohja (1998a) note however that also this assumption is very restrictive since it might constrain the estimates to lie within a very small neighbourhood of the steady state under consideration. Therefore, they show that even without this condition, it is possible to determine a lower bound for the probability of convergence of the estimates calculated according to a stochastic recursive algorithm to a locally asymptotically stable steady state of the associated differential equation. More precisely, they state the following theorem:

Theorem 8 *Suppose that θ^* is a locally asymptotically stable steady state of the associated differential equation (25) and that the assumptions stated in the previous section are met. Furthermore, denote the probability distribution of (X_t, θ_t) for $t \geq n$ conditional on $X_n = x$ and $\theta_n = a$ as $P_{n,x,a}$.*

Then for any compact subset Q of D , there exist constants F and s which are independent of the gain sequence $\{\gamma_t\}_{t=1}^\infty$ and which are such that for all $n \geq 0$, $a \in Q$ and for all x it holds that:

$$P_{n,x,a} [\theta_t \rightarrow \theta^*] \geq 1 - F (1 + |x|^s) J(n)$$

where $J(n)$ is a decreasing sequence with $\lim_{n \rightarrow \infty} J(n) = 0$, which is given by

$$J(n) = \left(1 + \sum_{k=n+1}^{\infty} \gamma_k^2\right) \left(\sum_{k=n+1}^{\infty} \gamma_k^2\right)$$

In other words, this theorem shows that when the "infinitely visiting" property or Ljung's assumption of a projection facility cannot be justified, the estimate calculated according to a stochastic recursive algorithm must lie within a compact neighbourhood of a locally asymptotically stable steady state of the associated differential equation at a point in time which is large enough in order to obtain convergence of the corresponding trajectory of the stochastic recursive algorithm to this steady state with probability arbitrarily close to one.

Evans and Honkapohja (1998a) explain this result with the fact that the trajectories of the stochastic recursive algorithm is not very well approximated by trajectories of the associated differential equation early in time. Therefore, it might be the case, they argue, that although the trajectory of the stochastic recursive algorithm is within the domain of attraction of the steady state θ^* under the associated differential equation, large random shocks which are not incorporated into the approximation through the deterministic dynamical system cause the trajectory to leave this domain of attraction and to diverge from θ^* . However, when a trajectory is within the domain of attraction under the dynamics of the associated differential equation at a point in time large enough, this will not happen (or more precisely, it will only happen with arbitrarily small probability) since then the trajectory of the stochastic recursive algorithm is very well approximated by a trajectory of the associated differential equation which converges to the steady state.

However, it also follows from this result that since $J(n)$ tends to zero as $\sum_{k=n+1}^{\infty} \gamma_k^2 \rightarrow 0$, it is possible to modify the gain sequence by multiplying it with a small constant such that even for $n = 0$, that is even for trajectories starting within a compact set containing the steady state θ^* , convergence to θ^* with arbitrarily high probability is possible. Evans and Honkapohja (1998a) refer to this situation as "slow adaption". In this case, the rate of adaption to random shocks is sufficiently small such that their effect on the trajectories of the stochastic recursive algorithm will never be strong enough to cause them to leave the domain of attraction under the dynamics of the associated differential equation. For general gain sequences, Evans and Honkapohja (1998a) show that convergence to the steady state will take place with positive probability if the initial estimate for θ lies within a compact set containing

the steady state θ^* .³⁹

Therefore, they argue that it is indeed possible to select plausible rational expectations equilibria based on conditions which determine whether a given rational expectations equilibrium is locally asymptotically stable under the associated differential equation, as has already been argued in Theorem 6.⁴⁰

Ljung (1977) shows a similar result: Loosely speaking, he provides an upper bound for the probability of the largest deviation of the trajectory of the associated differential equation from the trajectory of the stochastic recursive algorithm exceeding a given ε . Ljung finds that this upper bound can be made arbitrarily small if the gain sequence converges sufficiently slowly to zero. Therefore, the associated differential equation is in this case a suitable approximation for the stochastic recursive algorithm already for small t implying that the problem indicated above cannot arise.

Evans and Honkapohja (2001) discuss however also one special case in which convergence to a steady state can be obtained with probability one, although only the initial conditions for a trajectory of the stochastic recursive algorithm lie within a compact neighbourhood of this steady state. In order to establish this result concerning globally stable steady states of the associated differential equation, it is however necessary to modify the assumptions made in the previous section slightly. In particular, this is true for the assumptions made on $H(\theta, x)$. Since the following result will however only be used for illustrative purposes in the next section, I will not describe these alternative assumptions in detail. Note however that it is necessary to assume that the process generating the state variables postulated in assumption B.1 must be independent of the estimates for the following result, due to Evans and Honkapohja (2001), to hold.

Theorem 9 *Consider a stochastic recursive algorithm of the general form (24) and suppose that instead of assumptions A.2, A.3 and B.1 the modified assumptions made by Evans and Honkapohja (2001) and all other assumptions made in the previous section hold. Assume furthermore that the associated differential equation (25) has a unique steady state $\theta^* \in \mathbb{R}^d$.*

Suppose additionally that there exists a twice continuously differentiable function $U(\theta)$ defined on $\mathbb{R}^d$ which has bounded second derivatives and which has the following

³⁹Note that in later sections rational expectations equilibria which were found to be possible outcomes of a certain stochastic recursive algorithm based on Theorem 8 will be referred to as locally stable under the dynamics of this learning algorithm.

⁴⁰Note however that this result is stronger than Theorem 6 since it actually shows convergence to a locally stable steady state of the associated differential equation with positive probability.

properties:

$$(i) \ U(\theta^*) = 0 \text{ and } U(\theta) > 0 \text{ for all } \theta \neq \theta^*$$

$$(ii) \ \frac{dU(\theta)}{d\tau} < 0 \text{ for all } \theta \neq \theta^*$$

$$(iii) \ U(\theta) \geq \alpha |\theta|^2 \text{ for some } \alpha \text{ and all } \theta \text{ which are in absolute terms larger than some } \rho_0 > 0$$

Then a trajectory of the stochastic recursive algorithm starting in a compact neighbourhood of the steady state θ^* converges to θ^* with probability one.

Note that assumptions (i) and (ii) made in this theorem imply that $U(\theta)$ is a so called Lyapunov function of the associated differential equation, meaning that under these assumptions the steady state θ^* is globally stable under the dynamics of the associated differential equation. Moreover, note that convergence of a (stochastic) trajectory to θ^* refers to the probability distribution $P_{n,x,a}$ for $n = 0$ introduced in the previous theorem.

3.3.4. Stochastic Approximation in the Cobweb Model

With the technical results introduced in the previous section, it is possible to analyse whether the adaptive learning algorithms introduced so far will actually converge to the unique rational expectations equilibrium of the Cobweb model if it is assumed that agents' perceived law of motion is consistent with the functional form of this equilibrium.

Consider first the simpler stochastic gradient algorithm which, under the assumption of a correctly specified perceived law of motion, has been rewritten in section 3.2.4 in the following general form:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \gamma_t x'_{t-1} \underbrace{\left(x_{t-1} \left[(\alpha - 1) \hat{\beta}_{t-1} + (\mu \ \delta)' \right] + \nu_t \right)}_{=H(\hat{\beta}_{t-1}, X_t)}$$

Hence, for this algorithm $h(\hat{\beta})$, the right hand side of the associated differential equation introduced in section 3.3.1, is given by:

$$\begin{aligned} h(\hat{\beta}) &= \lim_{t \rightarrow \infty} \mathbb{E} \left[x'_{t-1} \left(x_{t-1} \left[(\alpha - 1) \hat{\beta} + (\mu \ \delta)' \right] + \nu_t \right) \right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \Omega \end{pmatrix} \left[(\alpha - 1) \hat{\beta} + (\mu \ \delta)' \right] \end{aligned}$$

where the last line follows from the assumption that x_{t-1} is given by $(1 \ w_{t-1})$ and that the variance - covariance matrix of the random vector w is given by Ω . Furthermore, it has been assumed that x and the random shock ν are uncorrelated. Therefore, the associated ordinary differential equation for the stochastic gradient algorithm in the univariate Cobweb model is given by:

$$\frac{d\hat{\beta}}{d\tau} = \begin{pmatrix} 1 & 0 \\ 0 & \Omega \end{pmatrix} [(\alpha - 1)\hat{\beta} + (\mu \ \delta)']$$

Since in section 3.1.2 Ω has assumed to be positive definite, it can be inverted implying that the unique steady state of this differential equation⁴¹ is given by the coefficient vector of the unique rational expectations equilibrium:

$$\beta^* = \frac{1}{1 - \alpha} \begin{pmatrix} \mu \\ \delta \end{pmatrix}$$

Moreover, this steady state is globally stable under the dynamics of the associated differential equation if all eigenvalues of the matrix $\frac{dh(\hat{\beta})}{d\hat{\beta}}|_{\hat{\beta}=\beta^*}$ have negative real part which is the case if $\alpha < 1$.

Therefore, it can be concluded by Theorem 6 that whenever $\alpha > 1$, the rational expectations equilibrium cannot be the outcome of the stochastic gradient algorithm, and that thus agents' estimates must diverge since there do not exist any other steady states of the associated differential equation. Moreover, it can be concluded by Theorem 9 that whenever $\alpha < 1$, the stochastic gradient algorithm converges to the rational expectations equilibrium with probability one regardless of the initial estimate since the rational expectations equilibrium β^* is a globally stable steady state of the associated differential equation. Note that Evans and Honkapohja (2001) explicitly verify the existence of a Lyapunov function satisfying the assumptions made in this theorem. However, in order to keep the argument as simple as possible, I will skip this technicality.

As Evans and Honkapohja (2001) point out, $\alpha < 1$ is a plausible assumption for the Cobweb model since it requires that the supply curve has a greater slope than the demand curve which is always satisfied for the usually assumed downward sloping demand and upward sloping supply curves. Therefore, it follows that under economically reasonable assumptions agents in the Cobweb model can learn to form rational expectations by using their observations in order to adaptively update their

⁴¹Recall that at a steady state or fixed point $\frac{d\hat{\beta}}{d\tau}$ must be zero.

expectations about future prices using a plausible learning algorithm. As pointed out above, this result justifies the use of the rational expectations approach to model expectations formation in the long run.

In a similar manner it can be shown that the stability of the unique rational expectations equilibrium of the Cobweb model under recursive least squares learning is determined by the same condition as under stochastic gradient learning. The argument for this result follows Evans and Honkapohja (2001). As could be seen in section 3.2.4, the stochastic recursive algorithm for recursive least squares learning under the assumption that agents' perceptions about the functional form of the law of motion for prices coincide with the unique rational expectations equilibrium is given by:

$$\begin{aligned}\hat{\beta}_t &= \hat{\beta}_{t-1} + \frac{1}{t} \underbrace{S_{t-1}^{-1} x'_{t-1} \left(x_{t-1} \left[(\alpha - 1) \hat{\beta}_{t-1} + (\mu \quad \delta)' \right] + \nu_t \right)}_{=H_{\hat{\beta}}(\theta_{t-1}, X_t)} \\ S_t &= S_{t-1} + \frac{1}{t} \underbrace{\left(\frac{t}{t+1} \right) (x'_t x_t - S_{t-1})}_{=H_S(\theta_{t-1}, X_t)}\end{aligned}$$

Since agents also need to estimate the variance - covariance matrix of the explanatory variables, the vector of estimates θ consists of two parts, one associated with β , the estimates for the coefficients of the perceived law of motion, and one associated with the estimates for this variance - covariance matrix S . Correspondingly, $h(\theta)$ can also be partitioned in this way and is therefore given by the following two components:

$$\begin{aligned}h_{\hat{\beta}}(\hat{\beta}, S) &= \lim_{t \rightarrow \infty} \mathbb{E} \left[S^{-1} x'_{t-1} \left(x_{t-1} \left[(\alpha - 1) \hat{\beta} + (\mu \quad \delta)' \right] + \nu_t \right) \right] \\ h_S(\hat{\beta}, S) &= \lim_{t \rightarrow \infty} \frac{t}{t+1} \mathbb{E} [x'_t x_t - S]\end{aligned}$$

Hence, given the postulated perceived law of motion, the recursive least squares algorithm in the Cobweb model is associated with the following system of ordinary differential equations:

$$\begin{aligned}\frac{d\hat{\beta}}{d\tau} &= S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \Omega \end{pmatrix} \left[(\alpha - 1) \hat{\beta} + (\mu \quad \delta)' \right] \\ \frac{dS}{d\tau} &= \begin{pmatrix} 1 & 0 \\ 0 & \Omega \end{pmatrix} - S\end{aligned}$$

From the differential equation for S , which is independent of $\hat{\beta}$ and can thus be analysed separately from the subsystem of differential equations associated with $\hat{\beta}$, it follows that S converges under all parameter constellations to its steady state level, the variance - covariance matrix of the explanatory variables. Since the only question of interest here is under which parameter constellations trajectories of $\hat{\beta}$ asymptotically approach the rational expectations equilibrium, this limit for S can be inserted into the differential equation for $\hat{\beta}$ which is then given by:

$$\frac{d\hat{\beta}}{d\tau} = (\alpha - 1) \hat{\beta} + (\mu \quad \delta)'$$

Therefore, the unique rational expectations equilibrium is a globally stable steady state of the associated differential equation if all eigenvalues of $(\alpha - 1) I_{K+1}$, where I_{K+1} denotes the $(K + 1) \times (K + 1)$ identity matrix, have negative real part which is the case if $\alpha < 1$. As for the stochastic gradient algorithm, Theorems 6 and 9 apply, from which it follows that for $\alpha < 1$ the unique rational expectations equilibrium is globally stable under the postulated learning behaviour. In other words, although the recursive least squares algorithm uses more information in order to update the estimates, it converges under exactly the same conditions to the unique rational expectations equilibrium in the univariate Cobweb model as the stochastic gradient algorithm. This result also holds in the multivariate Cobweb model analysed by Evans and Honkapohja (1998b).

These two examples illustrate that conditions on the parameters of the model determine whether a given rational expectations equilibrium can be attained by a certain learning algorithm, i.e. whether it is stable under the considered learning dynamics. The next section will focus on a way on how these conditions can be obtained without deriving the sometimes complicated associated differential equation. Moreover, it will be discussed how these conditions are affected by changes in agents' perceived law of motion.

3.4. Alternative Stability Concepts

3.4.1. Expectational Stability

For the Cobweb model, it could already be seen from equation (19) and the discussion in section 3.1.3 that agents' perceptions about the law of motion for prices influence the actual price generating process. Therefore, it is possible to define the following mapping from the coefficients of the perceived law of motion (18) to the coefficients

of the actual law of motion (19):

$$T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix}$$

With this notation, the first part of the recursive least squares estimator for a and b discussed in section 3.2.1 can be reformulated as:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \frac{1}{t} R_t^{-1} x'_{t-1} \left(x_{t-1} \left(T(\hat{\beta}_{t-1}) - \hat{\beta}_{t-1} \right) + \nu_t \right)$$

which shows that essentially the recursive least squares algorithm adjusts the estimates for the coefficients of the perceived law of motion toward the coefficients of the actual law of motion for prices which is implied by these estimates.

Similar to this basic mechanism, Evans (1989) proposes an alternative to the direct analysis of the qualitative behaviour of stochastic recursive algorithms discussed in the previous section. Instead of basing his analysis on an associated differential equation, Evans considers a stylized learning rule which at each point in time reduces the difference between the perceived law of motion and the implied actual law of motion by a certain amount. This stylized learning rule can thus be expressed as follows:

$$\theta_{\tau+\Delta\tau} = \theta_{\tau} + \delta\Delta\tau (T(\theta_{\tau}) - \theta_{\tau})$$

where $\delta > 0$ determines by how much the estimates are altered each unit of time. By rearranging this stylized learning rule, Evans (1989) obtains the difference quotient for θ with respect to τ . Since $\delta > 0$ does not influence the qualitative behaviour of a trajectory for the estimates θ , it can be omitted which makes it possible to obtain the following differential equation as $\Delta\tau \rightarrow 0$:

$$\frac{d\theta}{d\tau} = T(\theta_{\tau}) - \theta_{\tau} \quad (26)$$

Since in a rational expectations equilibrium expectations about the law of motion for the relevant economic variables correspond to the objective mathematical expectations, it must hold that in a rational expectations equilibrium the coefficients of the perceived law of motion, i.e. the vector θ and the coefficients of the actual law of motion, i.e. $T(\theta)$, coincide.⁴² Thus, all rational expectations equilibria clearly constitute steady states of this differential equation.

⁴²Note that here it is implicitly used that the functional form of the perceived law of motion coincides with the functional form of the rational expectations equilibrium.

Evans (1989) then defines a given rational expectations equilibrium as expectationally stable (or E - stable) if it is a (locally) asymptotically stable steady state of this differential equation, meaning that (small) deviations from the given rational expectations equilibrium lead to a gradual readjustment of expectations to their rational expectations levels.

Applying this concept to the Cobweb model with a perceived law of motion of the form $p_t = a + w_{t-1}b + \nu_t$ yields:

$$\frac{d}{d\tau} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore, as for the direct analysis of the recursive least squares and the stochastic gradient algorithm, the E - stability condition for the unique rational expectations equilibrium of this model is given by $\alpha < 1$. In other words, this means that in this model the analysis of E - stability leads to exactly the same conclusions as the direct analysis of the specific recursive algorithm through stochastic approximation. Evans and Honkapohja (2001) argue that for least squares learning this equivalence holds more generally whenever the recursive least squares algorithm satisfies the assumptions made in section 3.3.2. If these assumptions were violated, the stability of the real time learning rule could not be analysed through its associated differential equation. Nevertheless, Evans and Honkapohja (2001) argue that even then the conditions for E - stability determine the stability of a rational expectations equilibrium under least squares learning. They base this conclusion on numerical simulations for a univariate linear expectations model. However, they also indicate that the correspondence between E - stability conditions and the conditions for stability of a rational expectations equilibrium obtained from a direct analysis of the stochastic recursive algorithm need not be identical for all possible learning rules. More precisely, they indicate that although Evans and Honkapohja (1998b) found that in the (multivariate) Cobweb model, E - stability of a rational expectations equilibrium is equivalent to its stability under the stochastic gradient algorithm, it might in some models be the case that these two stability concepts lead to different conclusions.

3.4.2. Weak vs. Strong E - Stability

As Evans (1989) points out, the concept of E - stability discussed so far depends strongly on the assumptions made on the perceived law of motion. In particular, it

has so far always been assumed that agents can correctly specify the functional form of a rational expectations equilibrium, but need to estimate the relevant parameters of the model describing their economy.

Another possible assumption would be that agents believe in the influence of some variables on the economic outcome which however do not affect the rational expectations equilibrium under consideration. If a rational expectations equilibrium is also expectationally stable under such overparametrized perceived laws of motion, it is referred to as strongly E - stable, whereas it is referred to as weakly E - stable if it is only expectationally stable under a correctly specified perceived law of motion.⁴³ The importance of this distinction is stressed by Evans (1985) who argues that since the initial perceptions of agents about the economic relationships constitute a random element, only strongly E - stable rational expectations equilibria can be expected to be persistently observed in reality, whereas weakly E - stable equilibria should only be observed under special circumstances. Therefore, it is indeed necessary to distinguish these two concepts. In the following I will use univariate linear expectations models to illustrate the different conclusions obtained through a focus on weak and strong E - stability conditions.

Univariate linear expectations models constitute a prominent framework for the analysis of the strong E - stability of rational expectations equilibria since if the reduced form equation determining the evolution of the economic variable of interest depends on expected future realizations of this variable, these models allow for multiple equilibria which might all be weakly E - stable, i.e. E - stable under a perceived law of motion coinciding with the functional form of the considered equilibrium. Therefore, it would not be possible to select a unique rational expectations equilibrium as a plausible long run outcome in an economy based on the analysis of weak E - stability discussed so far. Moreover, because of this feature, these models can also be used to analyse whether expectations will converge to a certain rational expectations equilibrium although agents' perceived law of motion coincides with the functional form of a different rational expectations equilibrium.

Furthermore, rational expectations equilibria in these models are given by ARMA processes which makes it indeed plausible that agents who have to decide on an econometric model based on past observations might misspecify the correct order of

⁴³Clearly, also the direct analysis of stochastic recursive algorithms through stochastic approximation depends on the perceived law of motion. Therefore, I will use the terms weakly stable and strongly stable in later sections if the analysis is based on stochastic approximation rather than on the analysis of E - stability.

these ARMA processes.⁴⁴

In order to analyse the differences between weak and strong E - stability, consider the following reduced form equation for y_t , the economic variable of interest, as done in Evans and Honkapohja (2001):⁴⁵

$$y_t = \alpha + \beta_0 y_t^e + \beta_1 y_{t+1}^e + \nu_t \quad (27)$$

where ν_t is the realization of a random variable with mean zero, and where y_t^e and y_{t+1}^e denote the (subjective) expectations formed in period $t - 1$ about y_t and y_{t+1} , respectively.

Evans (1985) shows that all rational expectations equilibria of this reduced form are ARMA(1,1) processes⁴⁶, meaning that they can be expressed as $y_t = a + by_{t-1} + c\nu_{t-1} + \nu_t$. Since in a rational expectations equilibrium y_t^e and y_{t+1}^e correspond to the mathematical expectations of y_t and y_{t+1} , they are given by $a + by_{t-1} + c\nu_{t-1}$ and $a(1 + b) + b^2y_{t-1} + bc\nu_{t-1}$, respectively. Inserting these expectations into the reduced form equation shows that the price generating process in a rational expectations equilibrium is given by:

$$y_t = \alpha + \beta_0 a + \beta_1 a(1 + b) + b(\beta_0 + \beta_1 b)y_{t-1} + (\beta_0 c + \beta_1 bc)\nu_{t-1} + \nu_t$$

Comparing the coefficients of this process with the postulated ARMA(1,1) process then shows that there exist two classes of rational expectations equilibria:

For the first one a is given by $\alpha(1 - \beta_0 - \beta_1)^{-1}$ and b and c are zero. This rational expectations equilibrium is often referred to as the minimal state variable solution since it depends only on a constant and thus on the smallest set of state variables.⁴⁷

⁴⁴In the Cobweb model, which also belongs to the class of univariate linear expectations models, it might be argued that agents know from experience and by good judgement which variables affect supply.

⁴⁵As Evans and Honkapohja (2001) point out, examples of macroeconomic models leading to this reduced form equation are the real balance model of Taylor (1977) and the "ad hoc" model of Sargent and Wallace (1975) in which y_t denotes the logarithm of the price level. Also note that if β_1 were set to zero, this reduced form equation would coincide with the reduced form equation of the univariate Cobweb model discussed above.

⁴⁶Evans and Honkapohja (2001) show that there also exist sunspot equilibria which depend on some variables in the information set of agents which are not correlated with y_t . However for the moment I will concentrate on non - sunspot equilibria in order to avoid confusion with sunspot equilibria represented by Markov processes as they were discussed in the overlapping generations model of section 2.

⁴⁷As indicated by Evans and Honkapohja (2001), it is necessary to employ a subsidiary criterion in order to select a unique minimal state variable solution in cases in which there exist more than one solution which depend on the smallest set of state variables.

The rational expectations equilibria different from the minimal state variable solution are usually referred to as bubble solutions.⁴⁸ In the model considered here, these are given by a continuum of equilibria for which a is given by $-\alpha\beta_1^{-1}$, b is given by $(1 - \beta_0)\beta_1^{-1}$ and c is arbitrary.

However, it is often argued that the minimal state variable solution is the most plausible rational expectations equilibrium since all other solutions depend on variables which only affect the equilibrium because they are expected to do so (in the model considered here, these variables are y_{t-1} and ν_{t-1}). However, as Evans (1985) points out, it is not clear how this argument based on simplicity can actually be enforced by economic mechanisms. Therefore, Evans and Honkapohja (2001) and others fill this gap by showing that in many situations the minimal state variable solution is indeed the unique strongly E - stable rational expectations equilibrium.

In order to see this, consider however first the weak E - stability of the minimal state variable solution. Therefore, it is assumed that agents' perceived law of motion coincides with the functional form of this rational expectations equilibrium, meaning that it is given by $y_t = a + \nu_t$. This implies that agents' expectations about both y_t and y_{t+1} are given by the constant a and that thus the implied actual law of motion can be expressed as $y_t = \alpha + (\beta_0 + \beta_1)a + \nu_t$. Hence, the mapping from the coefficients of the perceived law of motion to the coefficients of the actual law of motion is given by $T(a) = \alpha + (\beta_0 + \beta_1)a$. Accordingly, weak E - stability of the minimal state variable solution is determined through the following differential equation:

$$\frac{da}{d\tau} = \alpha + (\beta_0 + \beta_1)a - a$$

From this, it can be seen that the minimal state variable solution is a locally asymptotically stable steady state of this differential equation and thus weakly E - stable if $\beta_0 + \beta_1 < 1$.

In order to analyse the strong E - stability of this solution, it is taken into account that agents' perceived law of motion might be given by an ARMA process of arbitrary order, meaning that the perceived law of motion for y_t is given by:

$$y_t = a + \sum_{i=1}^s b_i y_{t-i} + \sum_{i=1}^r c_i \nu_{t-i} + \nu_t \quad (28)$$

⁴⁸See for example Evans and Honkapohja (2001).

which implies that agents' subjective expectations based on these perceptions are given by:

$$y_t^e = a + \sum_{i=1}^s b_i y_{t-i} + \sum_{i=1}^r c_i \nu_{t-i}$$

and accordingly by:

$$y_{t+1}^e = a + b_1 y_t^e + \sum_{i=2}^s b_i y_{t+1-i} + \sum_{i=2}^r c_i \nu_{t+1-i}$$

Inserting these expectations into the reduced form equation for y_t shows that the implied actual law of motion is given by the following process:

$$\begin{aligned} y_t = & \alpha + (\beta_0 + \beta_1 + \beta_1 b_1) a + \sum_{i=1}^{s-1} [(\beta_0 + \beta_1 b_1) b_i + \beta_1 b_{i+1}] y_{t-i} + (\beta_0 + \beta_1 b_1) b_s y_{t-s} + \\ & + \sum_{i=1}^{r-1} [(\beta_0 + \beta_1 b_1) c_i + \beta_1 c_{i+1}] \nu_{t-i} + (\beta_0 + \beta_1 b_1) c_r \nu_{t-r} + \nu_t \end{aligned}$$

which results in the following non-linear system of ordinary differential equations based on the mapping from the perceived to the implied actual law of motion for y_t :

$$\begin{aligned} \frac{da}{d\tau} &= \alpha + (\beta_0 + \beta_1 + \beta_1 b_1) a - a \\ \frac{db_i}{d\tau} &= (\beta_0 + \beta_1 b_1) b_i + \beta_1 b_{i+1} - b_i \quad \forall i = 1, \dots, s-1 \\ \frac{db_s}{d\tau} &= (\beta_0 + \beta_1 b_1) b_s - b_s \\ \frac{dc_i}{d\tau} &= (\beta_0 + \beta_1 b_1) c_i + \beta_1 c_{i+1} - c_i \quad \forall i = 1, \dots, r-1 \\ \frac{dc_r}{d\tau} &= (\beta_0 + \beta_1 b_1) c_r - c_r \end{aligned}$$

The minimal state variable solution still constitutes a steady state of this larger system of differential equations and can be expressed as $a^* = \alpha(1 - \beta_0 - \beta_1)^{-1}$, $b_1^* = \dots = b_s^* = 0$ and $c_1^* = \dots = c_r^* = 0$. Stability of this steady state under the dynamics of this larger system of differential equations and therefore also strong E - stability of the minimal state variable solution can be analysed by linearising these differential equations about this steady state:

Since the differential equations for $b_1, \dots, b_s$ constitute an independent subsystem, they can be analysed separately by linearising these differential equations about $b_1^* = \dots = b_s^* = 0$, which corresponds the rational expectations equilibrium under

consideration. Denoting the vector $(b_1 - b_1^*, \dots, b_s - b_s^*)'$ as $(\bar{b}_1, \dots, \bar{b}_s)' =: \bar{b}$, this linearisation can be obtained as:

$$\frac{d\bar{b}_i}{d\tau} = \beta_1 b_i^* \bar{b}_1 + (\beta_0 + \beta_1 b_1^* - 1) \bar{b}_i + \beta_1 \bar{b}_{i+1} \quad \forall i = 1, \dots, s-1 \quad (29)$$

$$\frac{d\bar{b}_s}{d\tau} = \beta_1 b_s^* \bar{b}_1 + (\beta_0 + \beta_1 b_1^* - 1) \bar{b}_s \quad (30)$$

Inserting $b_1^* = \dots = b_s^* = 0$ and rewriting the linearised system of differential equations in matrix notation yields:

$$\frac{d\bar{b}}{d\tau} = \begin{pmatrix} \beta_0 - 1 & \beta_1 & 0 & \dots & 0 \\ 0 & \beta_0 - 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_1 \\ 0 & \dots & \dots & 0 & \beta_0 - 1 \end{pmatrix} \bar{b}$$

From this formulation of the linearised system of differential equations it is apparent that all eigenvalues of the coefficient matrix are given by $\beta_0 - 1$ implying that one condition for the strong E - stability of the minimal state variable solution is given by $\beta_0 < 1$.

Furthermore, the linearisation of the differential equation for a is given by:

$$\frac{d\bar{a}}{d\tau} = (\beta_0 + \beta_1 + \beta_1 b_1^* - 1) \bar{a} + \beta_1 a^* \bar{b}_1$$

where, as above, $\bar{a}$ is defined as $a - a^*$, and where a^* is given by the value corresponding to the minimal state variable solution.

Since for $\beta_0 < 1$ b_1 converges to $b_1^* = 0$ and thus $\bar{b}_1$ converges to zero as τ goes to infinity, the term $\beta_1 a^* \bar{b}_1$ vanishes asymptotically. Therefore, the minimal state variable solution $a^* = \alpha(1 - \beta_0 - \beta_1)^{-1}$ is locally asymptotically stable under the differential equation stated above if $\beta_0 + \beta_1 < 1$.

Similarly, the subsystem of differential equations for $c_1, \dots, c_r$ can be analysed through the following linearised system of differential equations:

$$\begin{aligned} \frac{d\bar{c}_i}{d\tau} &= \beta_1 c_i^* \bar{b}_1 + (\beta_0 + \beta_1 b_1^* - 1) \bar{c}_i + \beta_1 \bar{c}_{i+1} \quad \forall i = 1, \dots, r-1 \\ \frac{d\bar{c}_r}{d\tau} &= \beta_1 c_r^* \bar{b}_1 + (\beta_0 + \beta_1 b_1^* - 1) \bar{c}_r \end{aligned}$$

where, as above, $\bar{c}_i$ is defined as $c_i - c_i^*$, and where c_i^* is given by the level corresponding to the minimal state variable solution, i.e. by $c_1^* = \dots = c_r^* = 0$. As argued above, $\beta_1 c_1^* \bar{b}_1$ and $\beta_1 c_r^* \bar{b}_1$ vanish asymptotically if $\beta_0 < 1$ implying that asymptotically the only difference between the coefficient matrix of this linearised system and that of the linearisation of the differential equations for $b_1, \dots, b_s$ is the dimension of the matrix. Hence, also the last r eigenvalues must all be equal to $\beta_0 - 1$ and correspondingly no new requirements for the strong E - stability of the minimal state variable solution are necessary.

To summarize, it has been shown that if $\beta_0 + \beta_1 < 1$ is fulfilled, the minimal state variable solution of the simple univariate linear expectations model under consideration is weakly E - stable, whereas if additionally $\beta_0 < 1$ were satisfied, it would also be strongly E - stable and thus robust with respect to overparametrizations of the perceived law of motion. Especially, it would hold that even if agents tried to estimate a law of motion consistent with the so called bubble solutions, expectations would eventually converge to the minimal state variable solution.

However, it could still be possible that under different parameter constellations the bubble solutions were weakly or even strongly E - stable. Therefore, consider first a perceived law of motion coinciding with the functional form of the set of bubble solutions, i.e. $y_t = a + by_{t-1} + c\nu_{t-1} + \nu_t$. As already indicated above, this perceived law of motion leads to the expectations $y_t^e = a + by_{t-1} + c\nu_{t-1}$ and $y_{t+1}^e = a(1+b) + b^2y_{t-1} + bc\nu_{t-1}$ and accordingly to the following implied actual law of motion:

$$y_t = \alpha + \beta_0 a + \beta_1 a(1+b) + b(\beta_0 + \beta_1 b)y_{t-1} + (\beta_0 c + \beta_1 bc)\nu_{t-1} + \nu_t$$

Therefore, the mapping from the coefficients of the perceived law of motion to the coefficients of the actual law of motion can be expressed as:

$$T \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \begin{pmatrix} \alpha + \beta_0 a + \beta_1 a(1+b) \\ b(\beta_0 + \beta_1 b) \\ (\beta_0 + \beta_1 b)c \end{pmatrix}$$

Hence, the linearisation of the resulting non-linear system of differential equations about the set of bubble solutions can be written as:

$$\frac{d}{d\tau} \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} = \begin{pmatrix} \beta_0 + \beta_1 (1 + b^*) - 1 & \beta_1 a^* & 0 \\ 0 & \beta_0 + 2\beta_1 b^* - 1 & 0 \\ 0 & \beta_1 c^* & \beta_0 + \beta_1 b^* - 1 \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix}$$

where the notation is as above, but where a^* is now given by $-\alpha\beta_1^{-1}$, b^* is given by $(1 - \beta_0) \beta_1^{-1}$ and c^* is arbitrary.

As before, the weak E -stability of the set of bubble solutions is determined through the sign of the eigenvalues of the coefficient matrix of this linearised system, which are given by $\beta_0 + \beta_1 (1 + b^*) - 1$, $\beta_0 + 2\beta_1 b^* - 1$ and $\beta_0 + \beta_1 b^* - 1$. Therefore, the set of bubble solutions is weakly E -stable if $\beta_1 < 0$ and if $1 - \beta_0 < 0$. Technically, the third eigenvalue corresponding to the differential equation for c is zero. However, following Evans and Honkapohja (1992), this differential equation can be solved explicitly in order to show that whenever b converges to b^* , which is the case if $1 - \beta_0 < 0$ as could already be seen from the linearisation, c must also converge. Evans and Honkapohja (2001) note however that the value to which c converges will depend on the initial conditions. This means that if the economy is initially at a rational expectations equilibrium corresponding to one particular member of the set of bubble solutions and that due to some shock expectations about a , b and c suddenly deviate from their rational expectations levels, they will again converge to a member of the set of bubble solutions, but they need not necessarily converge to the initial equilibrium. This point was also made by Evans (1985) who however used a different stability concept for his analysis.

The analysis of strong E - stability of the set of bubble solution is technically more demanding: As for the minimal state variable solution, the overparametrized perceived law of motion is assumed to be given by (28). Therefore, the system of ordinary differential equations is not affected. However, if $b_1^* = (1 - \beta_0) \beta_1^{-1}$ and $b_2^* = \dots = b_s^* = 0$, i.e. the values for $b_1, \dots, b_s$ corresponding to the set of bubble solutions, are inserted into the linearised subsystem (29) - (30), it is given by:

$$\frac{d\bar{b}}{d\tau} = \begin{pmatrix} 1 - \beta_0 & \beta_1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \beta_1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \bar{b}$$

From this, it can easily be seen that $s - 1$ eigenvalues of the coefficient matrix are equal to zero, meaning that it is not possible to decide on the stability of the steady state under consideration based on this linearisation. However, Evans and Honkapohja (1992) show in a similar model that by employing the centre manifold technique it follows that the set of bubble solution cannot be stable under the differential equation considered here and that thus it can never be strongly E - stable. Therefore, although bubble solutions might sometimes be observed, it is more realistic to expect to observe the minimal state variable solution or to observe divergence of the estimates.

Clearly, this result is however only valid for models of the specific form discussed here. There are indeed more general univariate linear expectations models, as for example the model considered by Evans and Honkapohja (1992) which also allows for a direct dependence of y_t on y_{t-1} .⁴⁹ However, the main conclusions obtained from an analysis of this model do not change compared to the model considered here. Also in this more general model, the set of bubble solutions can under certain circumstances be weakly E - stable, but can never be strongly E - stable. One difference to the model discussed here is however that the more general model allows for two rational expectations equilibria which both depend on a minimum set of explanatory variables. Evans and Honkapohja (1992) show that either one of these solutions can be strongly E - stable. In particular they argue that also the equilibrium which is not selected by a subsidiary criterion as the minimal state variable solution can be strongly E - stable, thus contradicting the opinion that the minimal state variable solution is in each case the most plausible rational expectations equilibrium.

Finally, it should be noted that the models discussed in this section also illustrate the advantage of using E - stability conditions compared to the direct analysis of the learning algorithm through stochastic approximation. As Evans and Honkapohja (2001) point out, the techniques discussed in section 3.3 cannot be applied in cases in which there is a continuum of rational expectations equilibria, as for example the set of bubble solutions in the model considered here. However, based on simulations they argue that even in this case E - stability seems to determine the circumstances under which recursive least squares learning converges to a certain rational expectations equilibrium. Hence, it becomes clear that since the analysis of E - stability does not rely on any assumptions it can be even applied in situations in which stochastic approximation fails and that it will even in these cases yield the

⁴⁹As Evans and Honkapohja (2001) point out, one example for a macroeconomic model which is of this form is the overlapping contract model of Taylor in which it is assumed that each period one part of the firms in the economy set their wages for the two consecutive periods.

correct conclusions concerning the qualitative behaviour of recursive least squares algorithms.

After illustrating the analysis of various adaptive learning algorithms in rather simplified models, I will now return to the more complicated overlapping generations model discussed in section 2 and address whether sunspot equilibria can actually be realistic economic outcomes.

4. The Stability of Sunspot Equilibria under Adaptive Learning Rules

4.1. Possible Specifications of the Learning Process

With the methods introduced in the previous sections, it is finally possible to analyse whether stationary sunspot equilibria can actually be expected to be observed as long run phenomena in an economy. As explained above, this requires to take into account that in the short run agents will not be able to form rational expectations as has been assumed in section 2.2. Instead they must base their labour supply decision on their subjective expectations about the utility they will receive from their old - age consumption. This means that agents who are young in period t try to choose their labour supply such that it solves the following problem corresponding to (4), but where the mathematical expectations operator has been replaced by the subjective expectations:

$$\max_{n_t} u^e \left(\frac{p_t}{p_{t+1}} n_t \right) - v(n_t)$$

In other words, this means that when agents try to decide on their labour supply in period t , they would like to base this decision on the utility obtained through old - age consumption implied by the real wage $R_t = \frac{p_t}{p_{t+1}}$. But since neither p_{t+1} nor p_t (which through market clearing depends on the labour supply decision made by agents in period t) are known at the time agents have to decide on their labour supply in period t , agents need to find a way in which they can still decide optimally given this lack of knowledge.

In particular, there are two possible and equally plausible descriptions of how agents decide on their labour supply when faced with uncertainty regarding their real wage: The first possibility introduced by Woodford (1990) is based on the Robbins - Monro algorithm which has been discussed in section 3.2.2. Woodford (1990) basically assumes that agents believe that each new observation of the real wage is drawn from

a certain, but unknown distribution function, meaning that they cannot explicitly calculate the labour supply maximizing their subjectively expected lifetime utility. Instead, Woodford assumes that the labour supply choice of agents who are young in period $t - 1$ results in an observation on the marginal utility which could be obtained by increasing this labour supply. This observation can then be used by agents who are young in period t to improve the labour supply choice of their predecessors. In other words, agents do not estimate prices or labour supplies in order to obtain an estimate for the real wages they could possibly face and accordingly also subjective expectations about $u\left(\frac{p_t}{p_{t+1}}n_t\right)$, the utility obtained through old - age consumption, as assumed in the specification which will be discussed next, but they rather use past observations on the real wage and on labour supplies in order to adjust the labour supply of their predecessors gradually toward the level maximizing their subjectively expected utility.

Another approach discussed by Evans and Honkapohja (2003) and implicitly assumed by Evans (1989) is probably more intuitive: They assume that agents who are young in any period $t - 1$ simply estimate the possible real wages they could face during their lifetime and, together with the probability attributed to each of these possible outcomes, obtain a subjective expectation about the utility they will receive from their old - age consumption. Therefore, they can then decide optimally on their labour supply in period $t - 1$ given these subjective expectations. This observed decision can then be used by agents born in period t in order to update the forecast for the possible real wages they could face.

The difference between these two approaches is thus clear: While in Woodford's setting agents do not act optimally during the transition to the equilibrium, but rather try to find the optimum by numerical considerations, agents in the latter setting also act optimally during the learning process given the subjective expectations about their lifetime utility.

Both specifications have advantages and disadvantages, depending on the particular question which should be analysed. For example, the specification of Woodford (1990) results in a very simple associated differential equation which for the case of a two - state sunspot process immediately allows to establish a direct link between the sufficient condition for the existence of stationary sunspot equilibria noted in Theorem 5 and their stability under the dynamics of adaptive learning algorithms. However, it does not allow to draw conclusions for the stability of sunspot equilibria whenever the extrinsic sunspot process describing the state of nature has more than

two states.⁵⁰ Another disadvantage of Woodford's formulation is that even in the case of two - state sunspot processes it is only possible to show convergence to some sunspot equilibrium and that it is not possible to obtain more precise information like the location of this equilibrium. This problem can however be solved partly by focusing on the local stability of certain sunspot equilibria, as already indicated by Woodford (1990).

By contrast, the approach of Evans and Honkapohja (2003) which is based on least squares learning and thus on the E - stability of sunspot equilibria also allows to investigate the circumstances under which expectationally stable sunspot equilibria exist arbitrarily close to the monetary steady state. Unfortunately, in order to use this approach it is necessary to specify a particular form for the utility function since for the analysis of the learning algorithm it is necessary to explicitly solve agents' utility maximization problem. Therefore, it was for example pointed out by Woodford (1990) that the results on strong expectational stability obtained by Evans (1989) strongly depend on the particular utility function which has been assumed.

In order to address all the issues brought up so far, it is however necessary to formally define both possible specifications of agents' learning algorithms. After that, I will proceed by addressing the more general question raised by Woodford (1990). After that I will move on to discuss the results obtained by Evans (1989) and Evans and Honkapohja (2003) for a particular class of utility functions.

4.1.1. The Robbins - Monro Algorithm in an OLG Model

In order to motivate his analysis, Woodford (1990) considers the basic overlapping generations model discussed in section 2, but augmented by random preferences shocks. More precisely, Woodford assumes that the utility of a representative agent born in period t is given by $u(c_{t+1}) - v(n_t) + n_t \nu_t$, where ν_t captures the random disturbance to preferences which can only be observed by agents after they have decided on their labour supply in period t and which is assumed to be a realization of a random variable with mean zero and bounded support. Therefore, as already pointed out in section 3.1.1, this modification to the basic model discussed in section 2 has no effect on the analysis of sunspot equilibria under the assumption of rational expectations, meaning that all results discussed in section 2 remain valid.

However, Woodford (1990) notes that this modification can actually justify the

⁵⁰More precisely, in this situation conclusions are only possible in special cases which will be discussed in a later section.

assumption of agents believing in an influence of the extrinsic sunspot process on economic outcomes: He motivates this with the fact that at the beginning of time it will not be obvious to agents that the disturbances have mean zero. Instead, agents who try to learn the characteristics of the distribution of these disturbances adaptively from past observations will alter their estimates for the mean of the disturbances each period which (given the real wage) will result in changes of their optimal behaviour. Moreover, Woodford argues that these fluctuations in agents' labour supply will show a weak and accidental sample correlation with the observed sunspot process which might be enough for agents to believe that the extrinsic sunspot variable is responsible for the observed fluctuations in economic outcomes.⁵¹ More precisely, he argues that agents will believe that the distribution function of the disturbances and thus also of the real wage depends on the observed state of nature. Therefore, under this hypothesis, the optimal labour supply in any period t in which state i is observed is given by the following expression corresponding to the solution of (4), where preference shocks are taken into account and where the expectation is calculated according to the subjective belief that R_t and ν_t are distributed according to some unknown function G_i :

$$n_i = \arg \max_{n_t} \int [u(R_t n_t) - v(n_t) + \nu_t n_t] dG_i(R_t, \nu_t)$$

where R_t is the market clearing real wage $\frac{n_{t+1}}{n_t}$, and where G_i denotes the unknown common distribution function of the real wage and the disturbances when the current state of nature is $i \in \{1, \dots, k\}$. Thus, n_i , the optimal labour supply of an agent born in period t associated with state i ⁵², has to fulfil the first order condition:

$$\int [R_t u'(R_t n_i) - v'(n_i) + \nu_t] dG_i(R_t, \nu_t) = 0$$

which simply says that if labour supply is chosen optimally, the expected marginal utility from increasing it must be zero.

However, since agents do not know the distribution function G_i , it is impossible for them to solve this first order condition analytically for n_i . Instead, Woodford

⁵¹As Woodford notes, it would be enough if some influential media broadcast an accidental but striking relationship between economic outcomes and the extrinsic process for this belief to arise among agents.

⁵²Note that since the market clearing real wage has already been inserted into the optimization problem, n_i denotes, as in section 2.3, the labour supply in the market equilibrium and not just the optimal labour supply function. However, for brevity I will in the following only use the term optimal labour supply for n_i .

(1990) assumes that each period agents revise their estimate or approximation for the optimal labour supply in any state of nature using the Robbins - Monro algorithm discussed in section 3.2.2 and hence solve the problem of finding their optimal labour supply numerically. That is, if in period $t - 1$ state i has been observed, agents use the corresponding observed marginal utility in order to update their estimate for n_i in period t . The estimates for the optimal labour supply in the other states of nature are left unchanged since no new observation from the distribution of the real wage depending on this state of nature which could be used to improve the estimate has been made from period $t - 1$ to period t . Finally, if the state of nature in period t is i , agents choose the estimate for n_i which has been formed in period t as their labour supply.

Since the Robbins - Monro algorithm is designed to find the (global) minimum of a function, it is necessary to rewrite the maximization problem of agents as a minimization problem, i.e. as:

$$n_i = \arg \min_{n_t} \int -[u(R_t n_t) - v(n_t) + \nu_t n_t] dG_i(R_t, \nu_t)$$

which in turn results in the first order condition:

$$\int -[R_t u'(R_t n_i) - v'(n_i) + \nu_t] dG_i(R_t, \nu_t) = 0 \quad (31)$$

where, in the notation from section 3.2.2, θ^* is given by $(n_1, \dots, n_k)'$, the integral $\int -[R_t u'(R_t n_i) - v'(n_i) + \nu_t] dG_i(R_t, \nu_t)$ corresponds to $M(\theta^*)$, and where $\alpha = 0$. Denoting the estimate for n_i made at time $t - 1$ as $\hat{n}_{it-1}$, the Robbins - Monro algorithm (22) for this problem is given by:

$$\hat{n}_{it} = \hat{n}_{it-1} + \mathbb{1}_{\{S_{t-1}=i\}} \frac{h}{M_{it}} [0 + R_{t-1} u'(R_{t-1} \hat{n}_{it-1}) - v'(\hat{n}_{it-1}) + \nu_{t-1}] \quad \forall i = 1, \dots, k$$

where M_{it} denotes the number of times state i has been observed up to period t and h is a positive constant determining the rate of adaption of the estimates to newly observed data.⁵³ Furthermore, $\mathbb{1}_{\{S_{t-1}=i\}}$ denotes the indicator function which is needed to describe that the last observation on the marginal utility is only taken into account for the estimate of optimal labour supply associated with the particular state of nature which has been observed in period $t - 1$. Formally, the

⁵³In other words, the term $\frac{h}{M_{it}}$ replaces the gain parameter $\gamma_t = \frac{1}{t}$ used throughout section 3.

indicator function is defined as:

$$\mathbb{1}_{\{S_{t-1}=i\}} = \begin{cases} 1 & \text{if } S_{t-1} = i \\ 0 & \text{otherwise} \end{cases}$$

Unfortunately, there are however several problems with the learning algorithm formulated above:

As already discussed in section 3.2.2, this estimation technique for n_i would be consistent provided that $R_{t-1}u'(R_{t-1}\hat{n}_{it-1}) - v'(\hat{n}_{it-1}) + \nu_{t-1}$, the observation on the direction of steepest decent made in period $t - 1$, is unbiased for the direction of steepest decent of the subjectively expected lifetime utility of an agent born in period t , conditional on the estimated parameters, that is on R_{t-1} and $\hat{n}_{it-1}$. Since however, due to the learning process, these estimates change from one period to the next this observation from period $t - 1$ is a biased estimator for the expected direction of steepest descent in period t . In other words, agents on average adjust their estimates in the "wrong" way, that is in a direction which does not most rapidly decrease the value of the objective function. However, as discussed earlier, the learning process is still reasonable since in a given rational expectations equilibrium the parameter estimates do not change and thus the observed direction of steepest decent in one period is of course an unbiased estimator for the expected direction of steepest decent in any other period. Therefore, the circumstances under which such a rational expectations equilibrium can be attained by this learning algorithm despite the misspecification during the learning process must be analysed using stochastic approximation.

Another complication for the analysis, although it is not apparent from the formulation of the learning algorithm given above, is that in certain cases a problem of simultaneity will arise. To see this, note that the real wage agents born in period $t - 1$ face can be expressed as $R_{t-1} = \frac{n_t}{n_{t-1}}$. Thus, if state i is observed both in period $t - 1$ and in period t , R_{t-1} not only determines $\hat{n}_{it}$, but also depends on this estimate since agents will choose it as their actual labour supply n_t in period t . In order to solve this problem which would complicate the analysis, Woodford (1990) does not use the correct value for R_{t-1} in his analysis, but rather assumes that not the latest estimates for the labour supply enter the real wage. More precisely, he assumes that the real wage in the learning rule of agents is given by:

$$R_{t-1} = \frac{\sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_{jt-1}}{\sum_{j=1}^k \mathbb{1}_{\{S_{t-1}=j\}} \hat{n}_{jt-2}} \quad (32)$$

In other words, if in period $t - 1$ state i and in period t state l are observed, the real wage agents born in period $t - 1$ would face is not as usually calculated by $R_{t-1} = \frac{n_t}{n_{t-1}} = \frac{\hat{n}_{lt}}{\hat{n}_{it-1}}$, but instead estimates for n_l and n_i which are one period older are used for the analysis of the learning algorithm. If the states of nature changed from one period to the next and thus the problem of simultaneity did not arise, there would be no difference to the correct form of the real wage since then also the estimate for n_l could not be updated from period t to period $t - 1$. If however the states of nature did not change from period t to period $t - 1$, the problem of simultaneity between $\hat{n}_{jt}$ and R_{t-1} would be solved. Moreover, since the estimates converge, there is no big difference between the correct form and the simplification for t sufficiently large. Since it will anyhow be necessary to analyse the recursive algorithm in terms of an associated differential equation which only approximates the learning algorithm for large points in time, this simplification can therefore not affect the results concerning the qualitative behaviour of the learning rule.

Another problem for the analysis of this learning algorithm is that M_{it} , the number of times state i has been observed until period t , depends on the whole history of the sunspot process. Thus, the learning algorithm stated above is clearly not in the general form of a stochastic recursive algorithm which could be analysed using the techniques introduced in section 3.3. In order to solve this problem it is necessary to determine a recursion for M_{it} . Therefore, it is convenient to rewrite this quantity as tq_{it} , where q_{it} denotes the proportion of periods up to t in which sunspot state i has been observed, which will also make the formulation of the learning algorithm in the general form of a stochastic recursive algorithm possible. Multiplying this fraction by the total number of periods up to t clearly results in the total number of times state i has been observed. Nevertheless, also q_{it} depends on the entire history of the sunspot process. However, it is possible to determine the following recursion for q_{it} , as done in Woodford (1990):

$$\begin{aligned}
q_{it} &:= \frac{M_{it}}{t} \\
&= \frac{M_{it-1} + \mathbb{1}_{\{S_t=i\}}}{t} \\
&= \frac{tM_{it-1} - M_{it-1}}{t(t-1)} + \frac{1}{t}\mathbb{1}_{\{S_t=i\}} \\
&= \frac{M_{it-1}}{t-1} + \frac{1}{t} \left(\mathbb{1}_{\{S_t=i\}} - \frac{M_{it-1}}{t-1} \right) \\
&= q_{it-1} + \frac{1}{t} (\mathbb{1}_{\{S_t=i\}} - q_{it-1})
\end{aligned}$$

where it has been used that the number of times state i has been observed up to period t can be decomposed into the number of times state i has been observed up to period $t - 1$ and an indicator describing whether state i has also been observed in period t .

Thus, the recursive learning algorithm for agents using the Robbins - Monro algorithm in order to find the labour supply maximizing their subjectively expected utility numerically is given by:

$$\begin{aligned}\hat{n}_{it} &= \hat{n}_{it-1} + \frac{h}{t} q_{it}^{-1} \mathbb{1}_{\{S_{t-1}=i\}} [R_{t-1} u' (R_{t-1} \hat{n}_{it-1}) - v' (\hat{n}_{it-1}) + \nu_{t-1}] \quad \forall i = 1, \dots, k \\ q_{it} &= q_{it-1} + \frac{1}{t} (\mathbb{1}_{\{S_t=i\}} - q_{it-1}) \quad \forall i = 1, \dots, k\end{aligned}$$

Another problem which becomes apparent from this formulation of the algorithm, is that q_{it} , which is determined in period t , influences the estimate for n_i made in period t . Since such a simultaneity has been ruled out by the general form of a stochastic recursive algorithm described in section 3.2.4, it is necessary to alter the algorithm slightly in order to make the approximation through an associated differential equation possible. Therefore, Woodford (1990) argues that for t large q_{it} and q_{it-1} are approximately equal, meaning that it is possible to replace q_{it} by q_{it-1} in the recursive equation for the labour supply without affecting the results of the qualitative analysis of the associated differential equation.

Furthermore, Woodford (1990) assumes that although agents misspecify the model during their learning process in the sense that they believe in a unique optimum, they nevertheless believe that this optimal labour supply lies within the compact set $[\underline{n}, \bar{n}]^k$, for which it has been shown in section 2.5 that it indeed contains all rational expectations equilibria of the model except the autarchy equilibrium. Thus, no possible rational expectations equilibrium is ruled out per se as the outcome of the learning algorithm by this assumption⁵⁴, but it will ensure that Theorem 7 can be used in order to analyse the stability of rational expectations equilibria under the dynamics of this adaptive learning process.

Hence, when the estimate for the labour supply associated with state i formed in period t lies within the compact set $[\underline{n}, \bar{n}]$, the stochastic recursive algorithm can

⁵⁴The autarchy steady state which is ruled out by this assumption cannot be stable under adaptive learning, as will be shown further below.

finally be written as:

$$\hat{n}_{it} = \hat{n}_{it-1} + \frac{h}{t} q_{it-1}^{-1} \mathbb{1}_{\{S_{t-1}=i\}} [R_{t-1} u'(R_{t-1} \hat{n}_{it-1}) - v'(\hat{n}_{it-1}) + \nu_{t-1}] \quad (33)$$

$$q_{it} = q_{it-1} + \frac{1}{t} (\mathbb{1}_{\{S_t=i\}} - q_{it-1}) \quad (34)$$

where R_{t-1} is defined as in (32). Whereas if the estimate for the labour supply calculated according to this algorithm lies outside this set, equation (33) is replaced by:

$$\hat{n}_{it} = \begin{cases} \underline{n} & \text{if } \hat{n}_{it}^{alg} < \underline{n} \\ \bar{n} & \text{if } \hat{n}_{it}^{alg} > \bar{n} \end{cases} \quad (35)$$

where $\hat{n}_{it}^{alg}$ denotes the estimate calculated according to the recursive algorithm (33) - (34).

Before answering the question whether there exist circumstances under which extrinsic uncertainty has an effect on the economy although agents try to find their optimal labour supply numerically through this algorithm, I will first discuss the second possible specification of agents' learning behaviour in greater detail.

4.1.2. Recursive Least Squares Learning in an OLG Model

In order to analyse the implications of the assumption that agents use a recursive least squares algorithm to form their expectations, Evans and Honkapohja (2003) assume that agents believe in a perfect correlation between the sunspot process and economic outcomes. It is however assumed that despite this influence agents do not believe in the dependence of prices on any other variable. Hence, in period t agents will estimate prices associated with state i as the average price in periods in which state i has been observed. In order to avoid simultaneity, it is assumed that in doing so agents only use observations on prices until period $t - 1$. This is necessary since the price in period t will be determined through goods market clearing after agents have decided on their optimal labour supply which however depends on the expectations about prices formed in period t .

Moreover, the specification of Evans and Honkapohja (2003) differs from Woodford's (1990) assumptions in the sense that Evans and Honkapohja assume that the random disturbances do not enter the utility function, but that these shocks enter directly the learning rule of agents in the sense that the data available to agents is subject to a random measurement error. Evans and Honkapohja argue further that these disturbances are only necessary in order to make an analysis of the learning

algorithm, as discussed in section 3.3, possible.⁵⁵

Since markets are assumed to clear in all periods, meaning that there is a one-to-one relationship between prices and labour supply, it is also possible to express the learning algorithm agents use in order to form expectations about their potential real wages not in terms of prices, but in terms of labour supplies. Thus, agents' expectations about labour supply in any state of nature i are given by the ordinary least squares estimate for n_i in the following regression equation:

$$n_t = n_i \quad \text{if } S_t = i$$

In period t agents can thus use $(t-1)q_{it-1}$ observations on labour supplies from periods in which sunspot state i has been observed, where the fraction q_{it-1} is defined as above. Therefore, the recursive least squares estimates (20) - (21) are in this case given by:

$$\hat{n}_{it} = \hat{n}_{it-1} + \frac{1}{t-1} \mathbb{1}_{\{S_{t-1}=i\}} q_{it-1}^{-1} (n_{t-1} + \nu_{t-1} - \hat{n}_{it-1}) \quad (36)$$

where it is assumed that instead of the correct data on the realized labour supply n_{t-1} , agents only have access to noisy data, meaning that they only observe $n_{t-1} + \nu_{t-1}$, where the ν_t 's are assumed to be realizations of random variables which are identically and independently distributed over time and have mean zero and bounded support.

Note that in this case $(\sum_{i=1}^t x'_i x_i)^{-1}$ is simply given by t^{-1} since the only explanatory variable used by agents is equal to the number one. Thus, it is not necessary to provide a recursion for R_t , as done for the Cobweb model analysed in the previous sections.

Given these estimate for $i = 1, \dots, k$ and the belief that economic outcomes are perfectly correlated with the extrinsic sunspot process, the subjective expectation agents born in any period t in which state i is observed have about the utility obtained through old - age consumption is given by:

$$u^e(c_{t+1}) = u^e\left(\frac{p_t}{p_{t+1}} n_t\right) = \sum_{j=1}^k \pi_{ij} u\left(\frac{\hat{n}_{jt}}{\hat{n}_{it}} n_t\right) \quad (37)$$

Clearly, since agents have not decided yet on their labour supply in period t , they cannot use n_t for their expectations about the price in period t , but since they

⁵⁵In particular, they argue that this disturbance term is only necessary for the instability result, i.e. the contraposition of Theorem 6.

already know that the state of nature in period t is i , they can simply replace it by their expectation $\hat{n}_{it}$ and condition their subjective expectations on the state in period t being i . Since agents do however not know which state of nature will prevail in period $t + 1$, they must also take the corresponding transition probabilities from state i to any other state of nature into account when forming their expectation about utility obtained through old - age consumption in period $t + 1$.

The actual labour supply in period t , n_t , is then chosen such that it maximizes the subjective expectations about lifetime utility of an agent born in period t . Therefore, although the learning process described here can in principal be analysed using the simple E - stability conditions discussed in section 3.4.1, the analysis requires the knowledge of the result of this maximization problem since otherwise, the learning algorithm which depends through n_{t-1} on this solution could not be correctly determined. Hence, it is necessary to specify a precise form for the utility function, as done for example in Evans (1989). However, this implies that the results obtained from the analysis of this algorithm strongly depend on this choice and that thus no general statements are possible if it is assumed that agents in an overlapping generations model use a recursive least squares algorithm in order to form their expectations.

Therefore, I will first show that if agents use the Robbins - Monro algorithm described in the previous section, extrinsic uncertainty has an effect on long run economic outcomes whenever stationary sunspot equilibria necessarily exist and whenever agents condition their subjective expectations on a two - state sunspot process. After that, I will discuss the strong E - stability of stationary sunspot equilibria using the recursive least squares formulation of agents' learning process in an example given by Evans (1989).

4.2. Weak Stability of Rational Expectations Equilibria in an OLG Model

This section will analyse the circumstances under which rational expectations equilibria of an overlapping generations model with extrinsic uncertainty can be asymptotically attained through some adaptive learning algorithm if it is assumed that agents can correctly specify the number of states which are associated with different behaviour in these equilibria.

As pointed out above, it is convenient to model agents' learning process through the Robbins - Monro algorithm discussed in section 4.1.1 which can be easily analysed

using stochastic approximation techniques. However, note that the analysis of the recursive least squares algorithm discussed above would lead to the same conclusions as will also be seen in a later section. Moreover, it is only necessary to show that stationary sunspot equilibria can be attained by some plausible adaptive learning process in order to demonstrate that they are, at least under certain circumstances, realistic economic outcomes. Hence, it is for now indeed sufficient to concentrate on the Robbins - Monro algorithm which has already been found to be a plausible description of agents' learning behaviour.

Therefore, I will first discuss the associated differential equation of this particular algorithm, before analysing the stability of both the monetary steady state and the class of stationary two - state sunspot equilibria if agents initially believe that the extrinsic uncertainty has no effect on the equilibrium outcomes and if they initially believe that economic outcomes are perfectly correlated with a two - state sunspot process.

4.2.1. Stochastic Approximation of the Robbins - Monro Algorithm

In order to analyse the Robbins - Monro algorithm discussed in section 4.1.1 using the techniques introduced in section 3.3, it is necessary to verify that the algorithm is indeed expressed in the general form for a stochastic recursive algorithm and that moreover all assumptions stated in section 3.3.2 are satisfied.

In order to see that the learning algorithm given by the recursive equations (33) - (34) is already in the general form (24), note that these equations can also be expressed as:

$$\begin{aligned}\hat{n}_{it} &= \hat{n}_{it-1} + \frac{h}{t} q_{it-1}^{-1} \mathbb{1}_{\{S_{t-1}=i\}} [R_{t-1} u'(R_{t-1} \hat{n}_{it-1}) - v'(\hat{n}_{it-1}) + \nu_{t-1}] \\ q_{it} &= q_{it-1} + \frac{h}{t} h^{-1} (\mathbb{1}_{\{S_t=i\}} - q_{it-1})\end{aligned}$$

From this formulation it is obvious that this recursive algorithm is in the general form (24), where the vector of estimates θ_t is given by the $2k \times 1$ vector $(\hat{n}_{1t} \dots \hat{n}_{kt} \ q_{1t} \dots q_{kt})'$, where the sequence of gain parameters is given by $\gamma_t = ht^{-1}$, and where the state variables are given by R_{t-1} , $\mathbb{1}_{\{S_t=i\}}$ for $i = 1, \dots, k$, $\mathbb{1}_{\{S_{t-1}=i\}}$ for $i = 1, \dots, k$, and by ν_{t-1} . Moreover, it can be seen that the function $H(\theta_{t-1}, X_t)$ which is used to update the estimates is given by the following vector

valued function:

$$\begin{pmatrix} H_{\hat{n}_1}(\theta_{t-1}, X_t) \\ \vdots \\ H_{\hat{n}_k}(\theta_{t-1}, X_t) \\ H_{q_1}(\theta_{t-1}, X_t) \\ \vdots \\ H_{q_k}(\theta_{t-1}, X_t) \end{pmatrix} = \begin{pmatrix} q_{1t}^{-1} \mathbb{1}_{\{S_{t-1}=1\}} [R_{t-1} u' (R_{t-1} \hat{n}_{1t-1}) - v' (\hat{n}_{1t-1}) + \nu_{t-1}] \\ \vdots \\ q_{kt}^{-1} \mathbb{1}_{\{S_{t-1}=k\}} [R_{t-1} u' (R_{t-1} \hat{n}_{kt-1}) - v' (\hat{n}_{kt-1}) + \nu_{t-1}] \\ h^{-1} (\mathbb{1}_{\{S_t=1\}} - q_{1t-1}) \\ \vdots \\ h^{-1} (\mathbb{1}_{\{S_t=k\}} - q_{kt-1}) \end{pmatrix}$$

where the subscript $\hat{n}_i$ refers to the expression which is used to update the labour supply associated with state i and correspondingly the subscript q_i refers to the expression which is used to update the relative number of times state i has been observed.

However, in order to apply the results stated in section 3.3.3 to analyse the stability of rational expectations equilibria under the dynamics of the system of stochastic difference equations given by the postulated learning algorithm, it is still necessary to verify the assumptions made in section 3.3.2.

Therefore, first note that it has already been argued above that the gain sequence $\gamma_t = t^{-1}$ satisfies assumption A.1. Hence, it must still hold that $\sum_{t=1}^{\infty} \gamma_t = \infty$ and that $\sum_{t=1}^{\infty} \gamma_t^2 < \infty$ if this gain sequence is multiplied by a positive (and finite) constant h , meaning that assumption A.1 is also satisfied for the modified gain sequence used here.

Moreover, since the utility functions $u(\cdot)$ and $v(\cdot)$ have been assumed to be twice continuously differentiable and since the labour supply estimates calculated according to the postulated stochastic recursive algorithm are bounded on $[\underline{n}, \bar{n}]$, it follows immediately that $H(\theta, x)$ is continuously differentiable with bounded derivatives and thus Lipschitz continuous, meaning that assumption A.3 is satisfied.

Next, it has to be verified that the state variables are indeed generated by a process of the form postulated in assumption B.1. In order to do this, Woodford (1990) decomposes the state variable R_{t-1} , which has been defined by (32) as $\frac{\sum_{j=1}^k \mathbb{1}_{\{S_t=h\}} \hat{n}_{jt-1}}{\sum_{j=1}^k \mathbb{1}_{\{S_{t-1}=j\}} \hat{n}_{jt-2}}$, into the two components $\tilde{n}_t := \sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_{jt-1}$ and $\tilde{n}_{t-1} := \sum_{j=1}^k \mathbb{1}_{\{S_{t-1}=j\}} \hat{n}_{jt-2}$ which capture the estimates for labour supply associated with the state of nature observed in periods t and $t-1$ made one period earlier. Thus, denoting the vector of state variables as $X_t = (\tilde{n}_t \ \tilde{n}_{t-1} \ \nu_{t-1} \ \mathbb{1}_{\{S_t=1\}} \ \dots \ \mathbb{1}_{\{S_t=k\}} \ \mathbb{1}_{\{S_{t-1}=1\}} \ \dots \ \mathbb{1}_{\{S_{t-1}=k\}})'$ it can be seen that they follow the following process of the form postulated in as-

sumption B.1:

$$X_t = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{=A(\theta_{t-1})} X_{t-1} + \underbrace{\begin{pmatrix} 0 & 0 & \hat{\mathbf{n}}_{t-1} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & I_k \end{pmatrix}}_{=B(\theta_{t-1})} e_t$$

where e_t must be chosen as the $(2k+1) \times 1$ vector $\left(\nu_{t-1} \quad (\mathbf{1}_{\{S_t=i\}})_{i=1}^k \quad (\mathbf{1}_{\{S_{t-1}=i\}})_{i=1}^k \right)'$, and where accordingly the coefficient matrix $B(\theta_{t-1})$ is a $(2k+3) \times (2k+1)$ matrix whose entry I_k denotes the $k \times k$ identity matrix and whose entry $\hat{\mathbf{n}}_{t-1}$ denotes a $1 \times k$ vector whose entries are given by $\hat{n}_{1t-1}, \dots, \hat{n}_{kt-1}$.

Although this assumption could be verified, the particular form of e_t used in this argument results in the problem that assumption B.2 is not satisfied. In order to see this, note that the assumption of the e_t 's being realization of random vectors which are independently distributed over time is clearly violated since by the assumption of the extrinsic uncertainty being captured by a Markov process the states of nature in two consecutive periods are correlated implying that also the indicator functions signalling the state of nature in period $t-1$ and period t must be correlated. Nevertheless, Woodford (1990) argues that all results given in section 3.3.3 hold since other assumptions can be strengthened in the present application.

Moreover, as Woodford argues also the last assumption postulated in section 3.3.2 is satisfied, meaning that it is indeed possible to approximate the stochastic recursive algorithm (33) - (34) by the following associated differential equation⁵⁶:

$$\frac{d\theta}{d\tau} = \lim_{t \rightarrow \infty} \mathbb{E} [H(\theta, \bar{X}_t)] =: h(\theta)$$

where, following the heuristic argument presented in section 3.3.1, it is necessary to fix certain estimates $\theta = (\hat{n}_1 \dots \hat{n}_k \ q_1 \dots q_k)'$ and to calculate $\bar{X}_t$, the state variables which would be generated by these constant estimates through the law of motion postulated in assumption B.1, in order to derive this associated differential equation.⁵⁷ As could be seen above the only state variables depending on estimates

⁵⁶Note that assumption A.2 which has not been discussed is only necessary to guarantee the existence of the right hand side of this differential equation. However, as will be seen immediately, this expression is well defined.

⁵⁷Recall that in section 3.3.1 it has been argued that for t sufficiently large the estimates calculated according to the stochastic recursive algorithm are approximately constant, meaning that the

are $\tilde{n}_t$ and thus also $\tilde{n}_{t-1}$, or in other words the real wage R_{t-1} , which is then given by:

$$\bar{R}_{t-1} = \frac{\sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_j}{\sum_{j=1}^k \mathbb{1}_{\{S_{t-1}=j\}} \hat{n}_j}$$

Using this notation, the right hand side of the system of associated ordinary differential equations is given by:

$$\begin{aligned} h_{\hat{n}_i}(\theta) &= \lim_{t \rightarrow \infty} \mathbb{E} \left[q_i^{-1} \mathbb{1}_{\{S_{t-1}=i\}} \left[\frac{\sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_j}{\hat{n}_i} u' \left(\frac{\sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_j}{\hat{n}_i} \hat{n}_i \right) - v'(\hat{n}_i) + \nu_{t-1} \right] \right] \\ h_{q_i}(\theta) &= \lim_{t \rightarrow \infty} \mathbb{E} [h^{-1}(\mathbb{1}_{\{S_t=i\}} - q_i)] \end{aligned}$$

where the subscripts of $h(\theta)$ have the same interpretation as above, and where it has been used in the definition of the real wage that in period $t-1$ state i is observed. Since $\sum_{j=1}^k \mathbb{1}_{\{S_t=j\}} \hat{n}_j$ is just the labour supply estimate for the particular state of nature which is observed in period t , it follows that its expected value is given by $\sum_{j=1}^k \pi_{ij} \hat{n}_j$.⁵⁸ Moreover, by the definition of the indicator function, $\mathbb{E}[\mathbb{1}_{\{S_{t-1}=i\}}]$ is given by $Prob(S_{t-1} = i)$, i.e. by the probability of observing state i in period $t-1$. Letting time go to infinity, this probability is given by the fraction of times state i has been observed. This long run proportion will be denoted as q_i^* . Furthermore, it is assumed that $\mathbb{E}[\nu_{t-1}]$ is zero.

Therefore, it follows that $h_{\hat{n}_i}(\theta)$ can be written as:

$$h_{\hat{n}_i}(\hat{n}_1, \dots, \hat{n}_k, q_1, \dots, q_k) = \frac{q_i^*}{q_i} \left[\sum_{j=1}^k \pi_{ij} \frac{\hat{n}_j}{\hat{n}_i} u'(\hat{n}_j) - v'(\hat{n}_i) \right]$$

Correspondingly, by using again that $\lim_{t \rightarrow \infty} \mathbb{E}[\mathbb{1}_{\{S_t=i\}}]$ is given by q_i^* , the component of $h(\theta)$ associated with q_i can be written as:

$$h_{q_i}(\hat{n}_1, \dots, \hat{n}_k, q_1, \dots, q_k) = h^{-1}(q_i^* - q_i)$$

correct estimates influencing the law of motion of the state variables can be replaced by a constant value.

⁵⁸More formally, this can also be seen by noting that the expected value of the indicator function $\mathbb{1}_{\{S_t=j\}}$ conditional on the state of nature in period $t-1$ being i is given by π_{ij} , i.e. the probability of reaching state j from state i . Since the expression is multiplied by the indicator function for the state of nature in period $t-1$ being i , the expectation can indeed be conditioned on this event (in the case any other state of nature is observed in period $t-1$ the expression is multiplied by zero).

meaning that the qualitative behaviour of agents' labour supply choice can be analysed through the following system of differential equations:

$$\begin{aligned}
\frac{d\hat{n}_1}{d\tau} &= \frac{q_1^*}{q_1} \left[\sum_{j=1}^k \pi_{1j} \frac{\hat{n}_j}{\hat{n}_1} u'(\hat{n}_j) - v'(\hat{n}_1) \right] \\
&\vdots \\
\frac{d\hat{n}_k}{d\tau} &= \frac{q_k^*}{q_k} \left[\sum_{j=1}^k \pi_{kj} \frac{\hat{n}_j}{\hat{n}_k} u'(\hat{n}_j) - v'(\hat{n}_k) \right] \\
\frac{dq_1}{d\tau} &= h^{-1} (q_1^* - q_1) \\
&\vdots \\
\frac{dq_k}{d\tau} &= h^{-1} (q_k^* - q_k)
\end{aligned}$$

From this formulation of the system of associated differential equations, it is obvious that each of the last k differential equations of this system can be analysed separately and that moreover the relative number of times any state i has been observed up to a certain period converges globally to its long run equivalent⁵⁹

Next consider the first k differential equations of this system: Comparing these associated differential equations with equation (16) shows that the subsystem consisting of these differential equations can also be written as:

$$\frac{d}{d\tau} \begin{pmatrix} \hat{n}_1 \\ \vdots \\ \hat{n}_k \end{pmatrix} = \text{diag} \left(\frac{q_1^*}{q_1} \quad \dots \quad \frac{q_k^*}{q_k} \right) F(\hat{n}_1, \dots, \hat{n}_k, \Pi) \quad (38)$$

where, as above, $F(\hat{n}_1, \dots, \hat{n}_k, \Pi)$ denotes the $k \times 1$ vector whose i 's component is given by (16), and where $\text{diag} \left(\frac{q_1^*}{q_1} \quad \dots \quad \frac{q_k^*}{q_k} \right)$ is a $k \times k$ diagonal matrix with diagonal elements $\frac{q_1^*}{q_1}, \dots, \frac{q_k^*}{q_k}$. Since however a qualitative analysis of a system of differential equations is only concerned with the asymptotic behaviour of the trajectories for $\hat{n}_1, \dots, \hat{n}_k$ and since it has already been shown that $\frac{q_i^*}{q_i}$ converges globally to one, it is sufficient to analyse only the system of associated differential equation (38), where the vector F is premultiplied by the $k \times k$ identity matrix.

Note that through the formulation of the associated differential equations for $\hat{n}_1, \dots, \hat{n}_k$

⁵⁹This can be seen by noting that the derivative of $h^{-1}(q_i^* - q_i)$ with respect to q_i is equal to $-h^{-1}$ which is by assumption smaller than zero regardless of the value of q_i .

given above, it also becomes apparent that the monetary steady state and all stationary sunspot equilibria found under the hypothesis of rational expectations constitute steady states of this system of associated differential equations.

Moreover, by the contraposition of Theorem 6 it can be concluded that only locally asymptotically stable steady states of this system of associated differential equations are possible outcomes of the adjustment process (33) - (34) used by agents to determine their labour supply. As usually, a given steady state is locally asymptotically stable under the dynamics of a certain differential equation if all eigenvalues of the Jacobi matrix of the right hand side of this differential equation evaluated at this particular steady state have negative real part.

Therefore, Woodford (1990) arrives at the conclusion that the labour supply calculated according to the learning algorithm (33) - (34) converges to some arbitrarily small neighbourhood of a certain rational expectations equilibrium $\mathbf{n} = (n_1, \dots, n_k)'$ associated with the transition probability matrix Π with probability zero, unless all eigenvalues of $DF(\mathbf{n}, \Pi)$ have negative real part, where, as above, $DF(\mathbf{n}, \Pi)$ denotes the Jacobi matrix of F evaluated at the rational expectations equilibrium $\mathbf{n}$. Since the determinant of a matrix is given by the product of its eigenvalues, it must hold that for a rational expectations equilibrium satisfying this condition, $(-1)^k \det DF(\mathbf{n}, \Pi)$, which has been defined as $\Delta(\mathbf{n}, \Pi)$ in Theorem 5, is positive. By contraposition it follows that if for some rational expectations equilibrium $\mathbf{n}$ $\Delta(\mathbf{n}, \Pi)$ is negative, this equilibrium cannot constitute an outcome of the adjustment process for labour supply. In particular, it follows that if the sufficient condition for the existence of stationary sunspot equilibria given in Theorem 5 is fulfilled, labour supply cannot be at the monetary steady state level in the long run if in the short run agents adjust their labour supply according to the algorithm (33) - (34).

However, even if the monetary steady state is not stable under the adjustment process for labour supply when agents believe in the influence of an extrinsic sunspot process⁶⁰, it could be the case that it is stable when agents do not believe in this influence, i.e. when they, due to the lack of knowledge on how to form rational expectations, simply try to find the labour supply maximizing their expected lifetime utility (given their subjective perceptions about the structure of the economy) numerically by observing the "errors" made by their predecessors. Formally, such a situation can be analysed by setting the number of possible states agents believe

⁶⁰Note that it has so far been assumed that agents believe in the dependence of economic outcomes on an extrinsic k - state Markov process.

to be associated with different prices to one. After discussing the implications of these perceptions, I will increase the number of states to two and will discuss in greater detail how the results on the stability of the monetary steady state obtained from a correct specification of this equilibrium are affected by this change in agents' perceptions about the structure of the economy they live in.

4.2.2. Weak Stability of the Monetary Steady State

In order to analyse the weak stability of any rational expectations equilibrium under the dynamics of an adaptive learning algorithm, it must, as already pointed out above, be assumed that agents' subjective perceptions about the data generating process coincide with the functional form of the process actually generating the data in this equilibrium. Since in the monetary steady state each observation on prices is drawn from the same distribution (or more precisely, since prices are constant over time), it must be assumed that agents do not believe that the distribution of the real wage varies with the realization of an extrinsic stochastic process, or in other words that only one possible state of nature influences this distribution which is such that the price level is constant over time, in order to investigate the weak stability of this rational expectations equilibrium under the postulated adjustment process for labour supply.

However, for $k = 1$ the variable q_i clearly has no meaning since in this case only one possible state of nature exists, meaning that the fraction of times this unique state is observed must be one. Moreover, since states of nature cannot change and since as argued above agents are assumed to believe in a constant price level, the real wage calculated for a fixed estimate $\hat{n}$ for labour supply associated with the unique state of nature is always one.

Therefore, the system of associated differential equations (38) reduces to:

$$\frac{d\hat{n}}{d\tau} = u'(\hat{n}) - v'(\hat{n})$$

Hence, the monetary steady state n^* is locally asymptotically stable under the dynamics of this associated differential equation if $u''(n^*) - v''(n^*)$ is negative.

Moreover, since it has been assumed that $u(\cdot)$ is strictly concave and $v(\cdot)$ is strictly convex on their entire domain, it is even the case that $u'(\hat{n}) - v'(\hat{n})$ is decreasing for all $\hat{n} \in [\underline{n}, \bar{n}]$, meaning that the monetary steady state is even globally stable under the dynamics of the associated differential equation. In order to see that any trajectory of the learning algorithm must then converge to this rational expectations

equilibrium, note that $[\underline{n}, \bar{n}]$ is a compact set⁶¹ which is infinitely often visited by the trajectories of the postulated stochastic recursive algorithm describing agents' learning behaviour⁶² and which is contained in the domain of attraction of the monetary steady state under the dynamics of the associated differential equation. Moreover, note that by assumption the random shock to preferences has bounded support implying that e_t is bounded with probability one for all t . Therefore, all assumptions stated in Theorem 7 are satisfied provided that the invariant set I is chosen to consist just of the monetary steady state. Hence, it can indeed be concluded by this theorem that the labour supply calculated according to the adjustment process (33) - (34) converges to its monetary steady state level. In other words, this means that the monetary steady state will always be attained asymptotically if agents do not believe in the influence of an extrinsic sunspot process and gradually adjust the labour supply of their predecessors in order to find the labour supply maximizing their subjectively expected lifetime utility.

What might at first glance seem to weaken this result is that the possibility of the autarchy equilibrium which would also be consistent with the belief that the sunspot process did not matter was a priori ruled out as an outcome of the adjustment process by assuming that agents constrain their labour supply to lie within a compact set which is bounded above zero. However, the autarchy equilibrium does not constitute a fixed point of the associated differential equation analysed above since by assumption it holds that $\lim_{\hat{n} \rightarrow 0} u'(\hat{n}) = \infty$ and that $v'(0)$ is finite. This is due to the fact that the belief of prices or labour supply being equal in all periods automatically results in the belief that the real wage is equal to one, which is however no longer consistent with the autarchy equilibrium since the real wage in this particular rational expectations equilibrium is equal to $\frac{0}{0}$ which is not defined, but certainly different from one. Therefore, the monetary steady state would be weakly stable even if the constraint that the labour supply is bounded above zero is not taken into account. In other word, this means that the monetary steady state is the unique outcome which can be expected to be observed in the long run when agents are not able to form rational expectations in the short run, but are adjusting their labour supply adaptively using the postulated Robbins - Monro algorithm based on the perception that prices are constant over time. This result has also been demonstrated for other learning rules by Lucas (1986) and Evans (1989).

⁶¹Note that in a metric space a bounded and closed set is also compact.

⁶²Recall that the labour supply calculated according to agents learning rule is by construction constrained to lie within this set.

However, as indicated above, the monetary steady state loses its stability if agents believe based on observations for the random disturbances to preferences that prices vary with the state of nature captured by some extrinsic sunspot process and if the sufficient condition for the existence of stationary sunspot equilibria is satisfied. It is however not clear per se whether in such a case the adjustment algorithm (33) - (34) will instead converge to one of the stationary sunspot equilibria, or whether it will show a completely different qualitative behaviour. This issue will be discussed in the next section for a case in which agents believe in the influence of a two - state sunspot process.

4.2.3. Weak Stability of Stationary Two - State Sunspot Equilibria

In order to discuss the weak stability of stationary two - state sunspot equilibria under the postulated Robbins - Monro learning algorithm, consider a situation in which agents believe that the state of nature determining the perceived distribution function of the real wage can be described by the realization of a two - state Markov process. More precisely, it will be assumed in this section that agents believe that prices are perfectly correlated with an observable two - state Markov process, meaning that their perceptions coincide with the class of stationary two - state sunspot equilibria, but that they need to find the concrete values for prices (i.e. the unknown parameter in the distribution function G_i) numerically through the Robbins - Monro algorithm discussed in section 4.1.1.

As argued above, this learning algorithm can be analysed by stochastic approximation through the associated differential equation given by (38), where since q_i converges globally to q_i^* , it is only necessary to consider the following system of differential equations for a qualitative analysis:

$$\frac{d}{d\tau} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix} = F(\hat{n}_1, \hat{n}_2, \Pi)$$

In order to apply the results introduced in section 3.3.3, Woodford (1990) assumes that for any fixed point $\mathbf{n} = (n_1 \ n_2)'$ of this associated differential equation the eigenvalues of $DF(\mathbf{n}, \Pi)$, i.e. of the Jacobi matrix of F evaluated at this steady state, are different from zero, meaning that the fixed point is referred to as hyperbolic. As Woodford argues, this assumption guarantees that the steady states of the associated differential equation are isolated which also implies that there only exist finitely many of these fixed points. As Woodford furthermore notes, it follows then that

any trajectory of the differential equation must converge to a fixed point, a periodic orbit, or to a set consisting of finitely many steady states and finitely or infinitely many trajectories which converge to a fixed point both as time goes to minus and to plus infinity, but which do not pass through one of the fixed points.

In order to see that in the present example periodic orbits can however be ruled out as a possible qualitative behaviour of a trajectory of the associated differential equation, note that it follows from (16) that the derivative of $F_i(\hat{n}_1, \hat{n}_2, \Pi)$ with respect to $\hat{n}_i$, i.e. the i 'th diagonal element of the Jacobi matrix of F , can be written as:

$$[DF(\hat{n}_1, \hat{n}_2, \Pi)]_{ii} = \pi_{ii} u''(\hat{n}_i) - \pi_{ij} \frac{\hat{n}_j}{\hat{n}_i^2} u'(\hat{n}_j) - v''(\hat{n}_i) \quad \text{for } i, j \in \{1, 2\} \text{ and } i \neq j$$

which is strictly negative since $u''(\cdot)$ is assumed to be strictly negative, $u'(\cdot)$ and $v''(\cdot)$ are assumed to be positive and since π_{ij} , n_j and n_i are strictly positive constants. Hence, Woodford (1990) argues that in particular $\frac{\partial F_1(\hat{n}_1, \hat{n}_2, \Pi)}{\partial \hat{n}_1} + \frac{\partial F_2(\hat{n}_1, \hat{n}_2, \Pi)}{\partial \hat{n}_2}$, evaluated at any point within the set $[\underline{n}, \bar{n}]^2$, must be negative. Since this also implies that this term cannot change its sign on $[\underline{n}, \bar{n}]^2$ and that it is not identical to zero, Woodford concludes by applying Bendixson's criterion that the associated differential equation cannot have a periodic orbit.⁶³

Moreover, Woodford also notes that it can be ruled out by the same technique that a trajectory of the associated differential equation converges to a set containing a so called Jordan curve⁶⁴ which is made up of trajectories and fixed points. Under this condition and the assumption that all fixed points of the associated differential equation are hyperbolic, Woodford can show that also the third possibility for the asymptotic behaviour of a trajectory of the associated differential equation can be ruled out. In order to obtain this result, he demonstrates that if there existed only finitely many fixed points of the associated differential equation and if the trajectories of labour supply were inward pointing on the boundary of $[\underline{n}, \bar{n}]^2$, this

⁶³Bendixson's criterion states that if a two dimensional differential equation given by $\frac{dx}{dt} = H(x, y)$ and $\frac{dy}{dt} = G(x, y)$ is such that $\frac{\partial H}{\partial x} + \frac{\partial G}{\partial y}$ is not identical to zero and does not change its sign on a certain domain, it cannot have periodic orbits which are entirely contained in this domain.

In the present application, (x, y) is given by $(\hat{n}_1, \hat{n}_2)$, H corresponds to F_1 and G corresponds to F_2 . Furthermore, the set under consideration is given by $[\underline{n}, \bar{n}]^2$. Since it has been shown in Lemma 4 and Lemma 5 that all trajectories of F are inward pointing at the boundary of this set, it can be indeed concluded that no trajectory in this set can converge to a periodic orbit.

⁶⁴A Jordan curve is defined as a bijective mapping from the unit circle, meaning that each point on the Jordan curve is the image of a point on the unit circle and distinct points on the unit circle are mapped into distinct points on the Jordan curve. Moreover, both the mapping from the unit circle and the inverse of this mapping are continuous, meaning that the Jordan curve is homeomorphic to the unit circle.

asymptotic behaviour would eventually result in the existence of a Jordan curve made up of some of the fixed points and the trajectories which converge to one of these fixed points as time goes to minus infinity and to another one as time goes to plus infinity which would, by the result stated above, yield a contradiction.

Therefore, Woodford (1990) concludes that if the associated differential equation is two dimensional, meaning that the argument given above can be applied, any trajectory in $[\underline{n}, \bar{n}]^2$ must indeed converge to a fixed point.

Since the set of all steady states of the associated differential equation clearly constitutes an invariant set and since, by the result of Woodford just indicated, the domain of attraction of this set includes all of the compact set $[\underline{n}, \bar{n}]^2$ in which the trajectories of the stochastic recursive algorithm must lie for all t (and thus of course infinitely often), all requirements stated in Theorem 7 are fulfilled. Hence, this theorem can be used to conclude that the trajectories of the learning algorithm (33) - (34) must converge with probability one to the set $I \times \{(q_1^* \quad q_2^*)'\}$, where I denotes the invariant set just described, and where the symbol $\times$ denotes the cartesian product of two sets. Furthermore, as argued above, it must be the case that if the trajectories converge to a certain steady state or fixed point $\mathbf{n}$, $\Delta(\mathbf{n}, \Pi)$ is positive. Therefore, Woodford (1990) arrives at the following result:

Theorem 10 *Suppose $k = 2$. Then the learning algorithm (33) - (34) converges with probability one to a rational expectations equilibrium $\mathbf{n}$ for which labour supply in both states of nature is strictly positive and for which $\Delta(\mathbf{n}, \Pi) > 0$.*

In other words, this means that if agents condition their labour supply on an extrinsic two - state Markov process and the sufficient condition for the existence of stationary sunspot equilibria is satisfied, the adjustment process for labour supply must converge with probability one to one of the stationary two - state sunspot equilibria. In particular, it must converge to a sunspot equilibrium for which $\Delta(\mathbf{n}, \Pi) > 0$, i.e. which is associated with an index of +1. As outlined in section 2.5, it follows from the Poincaré - Hopf Index Theorem that in this case at least two such stationary two - state sunspot equilibria exist. This also implies that in this case the monetary steady state cannot be reached by the postulated learning algorithm, meaning that it can only be weakly stable, that is stable under a correctly specified perceived law of motion for prices. Therefore, it can only be observed as an outcome of the adjustment process for labour supply if agents do not believe that the distribution of the preference shocks and the real wage depends on some extrinsic variable although past observations might indicate this.

If however the sufficient condition for the existence of stationary sunspot equilibria is not satisfied, i.e. if $\Delta(\mathbf{n}^*, \Pi) > 0$, the monetary steady state is again the unique possible outcome of the adjustment process for labour supply, provided that there indeed do not exist any stationary sunspot equilibria.

As Woodford (1990) points out, this analysis shows however that extrinsic uncertainty can even matter for the economic allocation if agents only condition their labour supply on the realization of an extrinsic stochastic process since they believe, based on past observations, that by doing so they can approximate the labour supply maximizing their subjectively expected lifetime utility better than by not taking this process into account.

However, it is not possible to discuss the plausibility of any particular sunspot equilibrium by means of this theorem since it only refers to the whole set of rational expectations equilibria for which $\Delta(\mathbf{n}, \Pi) > 0$ holds.

In order to address this issue nevertheless, it is necessary to use a local approach as done in Theorem 8 introduced in section 3.3.3. This theorem however only refers to locally asymptotically stable fixed points of the associated differential equation. Therefore, it is necessary to show that stationary two - state sunspot equilibria can indeed be locally asymptotically stable under the dynamics of the associated differential equation.

In order to see that if the sufficient condition for the existence of stationary sunspot equilibria is satisfied, there must also exist stationary two - state sunspot equilibria at which Theorem 8 can be applied, note that, as argued above, there must in this case necessarily exist two sunspot equilibria with $\Delta(\mathbf{n}, \Pi) > 0$. Unfortunately, it has so far only been argued that $\Delta(\mathbf{n}, \Pi)$ must be positive for locally asymptotically stable fixed points of the associated differential equation, but not that conversely any rational expectations equilibrium with $\Delta(\mathbf{n}, \Pi) > 0$ is indeed locally asymptotically stable. As demonstrated by Woodford (1990) this converse result holds however as well provided the extrinsic uncertainty is described by a two - state Markov process as assumed here.

In order to see this, consider a sunspot equilibrium $\mathbf{n} = (n_1, n_2)'$ with $\Delta(\mathbf{n}, \Pi) > 0$ and recall that all diagonal elements of the Jacobi matrix of F evaluated at any rational expectations equilibrium $\mathbf{n} = (n_1, n_2)'$ must be negative. Therefore, also the trace of $DF(n_1, n_2, \Pi)$ must be negative, and since the eigenvalues of a 2×2 matrix A are calculated as $\lambda_{1,2} = \frac{tr[A]}{2} \pm \sqrt{\frac{tr[A]^2}{4} - det[A]}$, where $tr[A]$ denotes the trace of the matrix A , it follows that at least one eigenvalue of $DF(n_1, n_2, \Pi)$ is

negative. Knowing this, $\Delta(n_1, n_2, \Pi) := (-1)^2 \lambda_1 \lambda_2 > 0$ ⁶⁵ implies however that also the second eigenvalue of $DF(n_1, n_2, \Pi)$ must be negative, meaning that the sunspot equilibrium under consideration is indeed a locally asymptotically steady state of the associated differential equation.

Hence, if the sufficient condition for the existence of stationary sunspot equilibria stated in Theorem 3 is satisfied, there must exist at least two stationary two - state sunspot equilibria at which Theorem 8 can be applied and for which it must thus hold that if a trajectory of labour supply enters a compact neighbourhood of such an equilibrium at a point in time sufficiently large, the adjustment process converges to this particular sunspot equilibrium with probability arbitrarily close to one. Moreover, it follows from what has been noted in the discussion of this theorem that if the rate of adaption to newly observed data, which is for the learning algorithm under consideration controlled by the positive parameter h , were sufficiently small, a trajectory of the adjustment process would converge to any of these two - state sunspot equilibria with probability arbitrarily close to one even if just the initial conditions for labour supply lied within a compact neighbourhood of this particular steady state.

Therefore, contrasting the conjecture of Woodford (1990) that it is often not possible to single out a unique rational expectations equilibrium which can be attained by the postulated adaptive learning algorithm, it is possible to do so, at least if a two - state sunspot process is used in agents' learning algorithm and if initial conditions for $\hat{n}_{1t}, \dots, \hat{n}_{kt}$ are also taken into account.

Clearly, it is however also of interest to analyse the qualitative behaviour of the postulated learning algorithm if agents conditioned their subjective expectations on a sunspot process with an arbitrary, but finite number of states. The results obtained through such an analysis could also be used as conditions for strong stability of the two - state sunspot equilibria considered so far since, as will be explained further below, increasing the number of states is basically equivalent to assuming that agents included additional extrinsic stochastic processes which are independent of each other in their learning algorithm.

Unfortunately, as already indicated above, it is not possible to make precise statements for this more general case by using the present specification of agents' learning behaviour. As Woodford (1990) argues, this is due to the fact that for $k > 2$ it is not possible to employ Bendixson's criterion to show that indeed all trajectories in

⁶⁵Note that the determinant of a matrix is equal to the product of its eigenvalues.

$[\underline{n}, \bar{n}]^k$ will converge to a fixed point, i.e. to the invariant set I introduced above. Therefore, it can no longer be verified that all trajectories of the stochastic recursive algorithm visit a compact subset of the domain of attraction of I under the dynamics of the associated differential equation infinitely often which is however required for the application of Theorem 7.

Moreover, although Theorem 8 could in principal still be used to show that locally asymptotically stable steady states of the associated differential equation can be attained through the postulated learning process, it is not any longer possible to show that there exist stationary sunspot equilibria which are indeed locally asymptotically stable under the dynamics of the associated differential equation even if the sufficient condition for the existence of sunspot equilibria is met. In order to see this, note that the only property known of the stationary sunspot equilibria, which must necessarily exist in this case is that at least for two of them $\Delta(\mathbf{n}, \Pi)$ is positive. However, if $\Delta(\mathbf{n}, \Pi) := (-1)^k \prod_{j=1}^k \lambda_j > 0$ and k were even, it could be the case that the number of negative eigenvalues is even, but that there also exist positive eigenvalues. Conversely, if k were odd, it could be the case that the number of negative eigenvalues is odd, but that there also exist positive eigenvalues. In other words, for $k > 2$ $\Delta(\mathbf{n}, \Pi) > 0$ does not any longer imply that all eigenvalues of $DF(\mathbf{n}, \Pi)$ are negative and that thus the rational expectations equilibrium under consideration is locally asymptotically stable under the dynamics of the associated differential equation. Therefore, it is in the more general case not clear whether there exist stationary sunspot equilibria at which Theorem 8 can be applied and which must thus be considered as possible outcomes of the learning process (33) - (34) at least for appropriate initial conditions.

Hence, the only possibility to demonstrate convergence to a sunspot equilibrium in this more general case is to simulate the model and thus explicitly determine the rational expectations equilibria and their properties under the associated differential equation. However, results obtained from this analysis would, as the analysis of the recursive least squares algorithm, depend on the postulated functional form of agents' utility.

Unlike the stability results, instability results for rational expectations equilibria under the dynamics of adaptive learning algorithms can however, as already noted in section 4.2.1, be extended to this more general case. In other words, this means that also if agents condition their expectations on a sunspot process with arbitrarily, but finitely many states, it must still hold that the monetary steady state cannot be

reached through the learning algorithm considered here provided that the sufficient condition for the existence of stationary sunspot equilibria is violated. This means that the instability of the monetary steady state found for $k = 2$ in this situation cannot be overturned by the perceptions that even more states of nature are associated with different prices. The next section will demonstrate more generally that an instability result for a certain rational expectations equilibrium can never be altered by overparametrized beliefs of agents.

Moreover, it will be demonstrated that under special circumstances it is possible to show that the monetary steady state is strongly stable, i.e. that it is also stable under the dynamics of the considered learning algorithm if agents believe that the distribution of the preference shocks and the real wage depends on the realization of an extrinsic sunspot process with arbitrarily many states. After that, I will give an example by Evans (1989) for an economy in which agents use recursive least squares estimation to forecast the real wage, which will in particular allow conclusions about the strong stability of two - state sunspot equilibria.

4.3. Strong Stability of Rational Expectations Equilibria in an OLG Model

4.3.1. Instability Results

In the previous section it could already be seen that if stationary sunspot equilibria necessarily exist, the monetary steady state cannot be reached asymptotically by the learning algorithm (33) - (34) regardless of the number of states agents use to sort their observations. Also intuitively it seems plausible that an equilibrium which was found to be unstable under a certain set of explanatory variables, which is here given by the states of nature agents believe to be associated with different prices or alternatively by the different extrinsic processes agents use in their learning algorithm, should also be unstable when a larger set of explanatory variables is used. This result is formally demonstrated by Woodford (1990):

Therefore, he assumes that a certain rational expectations equilibrium $\mathbf{n} = (n_1 \dots n_k)'$ is unstable under the dynamics of the associated differential equation if agents condition their expectations only on the sunspot process (S_t) which can take on values in the state space $\{1, \dots, k\}$ and which, as usually, is associated with the transitions probability matrix $\Pi = (\pi_{ij})_{1 \leq i \leq k; 1 \leq j \leq k}$, meaning that by Theorem 7 this equilibrium cannot be attained by the considered learning algorithm under the postulated law of motion.

If agents only used this sunspot process in order to adjust the labour supply of their predecessors, it follows from rewriting (16) as

$$F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi) = \frac{1}{\hat{n}_i} \left[\sum_{j=1}^k \pi_{ij} \hat{n}_j u'(\hat{n}_j) - \hat{n}_i v'(\hat{n}_i) \right] \quad (39)$$

that the derivative of $F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)$ with respect to $\hat{n}_j$ for $j \neq i$ is given by:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_j} = \frac{1}{\hat{n}_i} \pi_{ij} U'(\hat{n}_j)$$

where, as in section 2, $U'(\hat{n}_j)$ is given by $\hat{n}_j u''(\hat{n}_j) + u'(\hat{n}_j)$.

Furthermore, it follows from (39) that the derivative of $F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)$ with respect to $\hat{n}_i$ is given by:

$$\begin{aligned} \frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_i} &= -\frac{1}{\hat{n}_i^2} \left[\sum_{j=1}^k \pi_{ij} \hat{n}_j u'(\hat{n}_j) - \hat{n}_i v'(\hat{n}_i) \right] + \frac{\pi_{ii}}{\hat{n}_i} [\hat{n}_i u''(\hat{n}_i) + u'(\hat{n}_i)] - \\ &\quad - \frac{1}{\hat{n}_i} [\hat{n}_i v''(\hat{n}_i) + v'(\hat{n}_i)] \end{aligned}$$

Since the rational expectations equilibrium under consideration must be a root of F , $\sum_{j=1}^k \pi_{ij} n_j u'(n_j) - n_i v'(n_i)$ must be zero. Evaluating the derivative stated above at the given rational expectations equilibrium therefore yields the following expression:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_i} \Big|_{(\hat{n}_1, \dots, \hat{n}_k)' = \mathbf{n}} = \pi_{ii} \frac{1}{n_i} U'(n_i) - \frac{1}{n_i} V'(n_i)$$

where $U'(\cdot)$ is defined as above and where, as in section 2, $V'(n_i)$ is defined as $n_i v''(n_i) + v'(n_i)$.

Furthermore, for an eigenvector $e = (e_1, \dots, e_k)' \in \mathbb{R}^k$ corresponding to any eigenvalue λ of the matrix $DF(n_1, \dots, n_k, \Pi)$, it must hold that $DF(n_1, \dots, n_k, \Pi)e = \lambda e$. In particular, this equality must of course also hold for the i 'th component of these vectors. Since for the left hand side of this equation the i 'th component is obtained by multiplying the i 'th row of the matrix DF , i.e. $\left(\frac{\partial F_i}{\partial \hat{n}_1} \dots \frac{\partial F_i}{\partial \hat{n}_k} \right)$, by the vector e , it must therefore hold that:

$$\left[\frac{1}{n_i} \pi_{ii} U'(n_i) - \frac{1}{n_i} V'(n_i) \right] e_i + \sum_{j \neq i} \frac{1}{n_i} \pi_{ij} U'(n_j) e_j = \lambda e_i$$

Multiplying this equation by n_i and rearranging then yields:

$$\sum_{j=1}^k \pi_{ij} U'(n_j) e_j = [V'(n_i) + \lambda n_i] e_i$$

which must of course hold for all $i = 1, \dots, k$.

Woodford (1990) then assumes that agents can also observe a second sunspot process (W_t) which is independent of (S_t) and which can take on values in the state space $\{1, \dots, p\}$.⁶⁶ The transitions probability matrix of this second sunspot process is assumed to be given by $\Psi = (\psi_{ab})_{1 \leq a \leq p; 1 \leq b \leq p}$. If agents now used both extrinsic processes as an indicator for the distribution of the preference shocks and the real wage, there would be $k \cdot p$ possible states of nature which can be depicted as ai for $0 \leq i \leq k$ and $0 \leq a \leq p$.⁶⁷ Since the two sunspot processes are assumed to be independent, the transition probability from state ai to any other state bj is given by the product $\pi_{ij} \cdot \psi_{ab}$.

Furthermore, the sunspot equilibrium $(n_1, \dots, n_k)'$ considered above can now be expressed as a degenerate sunspot equilibrium of cardinality $p \cdot k$ and order k for which n_{ai} , i.e. the labour supply associated with state ai , is given by n_i for all $a = 1, \dots, p$ and all $i = 1, \dots, k$.

Hence, the marginal utility resulting from an increase of the labour supply estimate for state ai is now given by:

$$\tilde{F}_{ai}(\hat{N}, \Pi, \Psi) = \frac{1}{\hat{n}_{ai}} \left[\sum_{b=1}^p \sum_{j=1}^k \psi_{ab} \pi_{ij} \hat{n}_{bj} u'(\hat{n}_{bj}) - \hat{n}_{ai} v'(\hat{n}_{ai}) \right]$$

where the $p \times k$ matrix $\hat{N}$ is such that the element in its a 'th row and its i 'th column is given by $\hat{n}_{ai}$, i.e. by the labour supply estimate for state ai .

Calculating as above the partial derivatives of $\tilde{F}_{ai}$ with respect to $\hat{n}_{ai}$ and with respect to the labour supply estimate for any other state of nature, $\hat{n}_{bj}$, where either $b \neq a$ or $j \neq i$, and evaluating these derivatives at the degenerate sunspot

⁶⁶In the terminology of Azariadis and Guesnerie (1982) such a second extrinsic variable is sometimes referred to as a "moonspot" process.

⁶⁷Therefore, as indicated above, introducing an additional sunspot process, which is independent of the original process, is equivalent to increasing the possible number of states.

equilibrium described above, gives:

$$\begin{aligned}\frac{\partial \tilde{F}_{ai}(\hat{N}, \Pi, \Psi)}{\partial \hat{n}_{ai}} \Big|_{\hat{N}=N} &= \frac{1}{n_i} \psi_{aa} \pi_{ii} U'(n_i) - \frac{1}{n_i} V'(n_i) \\ \frac{\partial \tilde{F}_{ai}(\hat{N}, \Pi, \Psi)}{\partial \hat{n}_{bj}} \Big|_{\hat{N}=N} &= \frac{1}{n_i} \psi_{ab} \pi_{ij} U'(n_j)\end{aligned}$$

where N is the $p \times k$ matrix for which for all $i = 1, \dots, k$ all entries in the i 'th column are equal to n_i , i.e. the labour supply associated with state i in the rational expectations equilibrium under consideration.

Next, Woodford (1990) defines the vector $\tilde{e} \in \mathbb{R}^{kp}$ such that $\tilde{e}_{ai} = e_i$ for all $a = 1, \dots, p$ and all $i = 1, \dots, k$, where e_i is the i 'th component of the eigenvector e found above. Then, he considers the component of the product $D\tilde{F}(N, \Pi, \Psi) \tilde{e}$ associated with state ai :

$$\begin{aligned}\left(D\tilde{F}(N, \Pi, \Psi) \tilde{e}\right)_{ai} &= \sum_{b=1}^p \sum_{j=1}^k \frac{1}{n_i} \psi_{ab} \pi_{ij} U'(n_j) e_j - \frac{1}{n_i} V'(n_i) e_i \\ &= \underbrace{\left[\sum_{b=1}^p \psi_{ab}\right]}_{=1} \frac{1}{n_i} \left[\sum_{j=1}^k \pi_{ij} U'(n_j) e_j\right] - \frac{1}{n_i} V'(n_i) e_i\end{aligned}$$

Since it has been shown above that if e is an eigenvector of $DF(n_1, \dots, n_k, \Pi)$ corresponding to the eigenvalue λ , it must hold that $\sum_{j=1}^k \pi_{ij} U'(n_j) e_j = [V'(n_i) + \lambda n_i] e_i$, this expression can be rearranged further to yield:

$$\begin{aligned}\left(D\tilde{F}(N, \Pi, \Psi) \tilde{e}\right)_{ai} &= \frac{1}{n_i} V'(n_i) e_i + \lambda e_i - \frac{1}{n_i} V'(n_i) e_i \\ &= \lambda e_i \\ &= \lambda \tilde{e}_{aj}\end{aligned}$$

meaning that since ai was chosen arbitrarily, λ must also be an eigenvalue of this larger dimensional Jacobi matrix.

Since this relationship must moreover hold for all eigenvalues λ of $DF(n_1, \dots, n_k, \Pi)$, it follows that all eigenvalues of the Jacobi matrix of F evaluated at a certain (non - degenerate) rational expectations equilibrium must also be eigenvalues of $D\tilde{F}(N, \Pi, \Psi)$, i.e. of the Jacobi matrix determining the local stability of this rational expectations equilibrium under the dynamics of the associated differential equation if agents condition their expectations additionally on a second sunspot

process. Note that by induction it follows that this must hold for any arbitrary finite number of extrinsic stochastic processes used in the learning algorithm.

Since it has been assumed that the given rational expectations equilibrium $\mathbf{n} = (n_1, \dots, n_k)'$ is unstable when agents base their adjustment process for labour supply only on the original sunspot process, at least one λ must be positive. Therefore, it follows by the argument presented above that also at least one eigenvalue of $D\tilde{F}(N, \Pi, \Psi)$ must be positive, meaning that the rational expectations equilibrium under consideration is also unstable if agents use a more sophisticated way to sort the observations they consider relevant for their behaviour.

As already pointed out above, this result is consistent with the results obtained in the previous section about the monetary steady state if the sufficient condition for the existence of stationary sunspot equilibria is satisfied. However, it also follows from this result that in general rational expectations equilibria with $\Delta(\mathbf{n}, \Pi) < 0$ in the case the adjustment process for labour supply is based on a two - state sunspot process, i.e. rational expectations equilibria which cannot be stable in this context, can also not become stable when additional extrinsic variables are taken into account in the learning algorithm.

Since in general $D\tilde{F}$ has however also eigenvalues different from the k eigenvalues of the smaller dimensional matrix $DF(n_1, \dots, n_k, \Pi)$, it does not follow from this result that the weak stability of certain rational expectations equilibria also automatically translates into strong stability results for these equilibria. The next section will however discuss a special case for which this is true at the monetary steady state.

4.3.2. Strong Stability of the Monetary Steady State

In section 4.2 it has already been demonstrated that if the sufficient condition for the existence of stationary sunspot equilibria were not met, the monetary steady state would be stable under the dynamics of the considered learning algorithm provided that agents' perceptions were only consistent with prices being constant over time or with the class of two - state sunspot equilibria.

As already pointed out above, this sufficient condition for the existence of stationary sunspot equilibria is however always satisfied if the monetary steady state is indeterminate under perfect foresight dynamics, i.e. if the negative income effect of an increase in the real wage outweighs the positive substitution effect on labour supply by a sufficient margin.

In this section it will moreover be shown based on Theorem 8 that if the sufficient

condition for the existence of stationary sunspot equilibria is not satisfied and if instead the positive substitution effect dominates at the monetary steady state, it is locally stable under the dynamics of the Robbins - Monro learning algorithm discussed in section 4.1.1 even if agents condition their expectations on a Markov process with arbitrarily many states. In other words, this means that a trajectory of this stochastic recursive algorithm converges to the monetary steady state with probability arbitrarily close to one provided that it visits a compact neighbourhood of this equilibrium and provided that the parameter h controlling the rate of adaptation to newly observed data is small enough.

More precisely, Woodford (1990) states the following result:

Theorem 11 *If $u'(n^*) + n^*u''(n^*) > 0$ ⁶⁸, i.e. if at the monetary steady state an increase in the real wage results in a substitution away from leisure to old - age consumption, the monetary steady state is locally strongly stable under the dynamics of the learning algorithm (33) - (34).*

Proof: In order to show this, it is only necessary to show that all eigenvalues of the Jacobi matrix of the associated differential equation evaluated at the monetary steady state have negative real part regardless of the number of states of the sunspot process used in the learning algorithm. This is sufficient since then Theorem 8 can be applied to show that for the case of slow adaption, i.e. for h sufficiently small, a trajectory of the learning algorithm converges to the monetary steady state with probability arbitrarily close to one if it started in a compact neighbourhood of this equilibrium, meaning that it is locally stable in this situation. Moreover, it has been argued above that conditioning the adjustment process on more than one extrinsic stochastic process can also be modelled by increasing the number of possible states of the original process. Hence, if the monetary steady state is found to be locally stable under the postulated learning algorithm for any finite number of states, it must also be locally stable if agents condition their expectations on any finite number of independent sunspot processes and thus also locally strongly stable.

In order to show that the monetary steady state is indeed locally asymptotically stable under the dynamics of the associated differential equation, Woodford (1990) first shows that the Jacobi matrix $DF(\mathbf{n}^*, \Pi)$ is diagonally dominated, meaning that $\left| \frac{\partial F_i(\mathbf{n}^*, \Pi)}{\partial \hat{n}_i} \right| > \sum_{j \neq i} \left| \frac{\partial F_i(\mathbf{n}^*, \Pi)}{\partial \hat{n}_j} \right|$ for any $i = 1, \dots, k$:

⁶⁸Recall, that the elasticity of labour supply with respect to the real wage is positive at the monetary steady state if and only if this condition holds.

From (16) it follows that the partial derivative of F_i with respect to $\hat{n}_j$ is given by:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_j} = \pi_{ij} \frac{1}{\hat{n}_i} [u'(\hat{n}_j) + \hat{n}_j u''(\hat{n}_j)]$$

Evaluating this partial derivative at the monetary steady state, i.e. at $\mathbf{n}^* = (n^*, \dots, n^*)' \in \mathbb{R}^k$ yields:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_j} \Big|_{(\hat{n}_1, \dots, \hat{n}_k)' = \mathbf{n}^*} = \pi_{ij} \frac{1}{n^*} [u'(n^*) + n^* u''(n^*)]$$

from which it follows that

$$\sum_{\substack{j=1 \\ j \neq i}}^k \frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_j} \Big|_{(\hat{n}_1, \dots, \hat{n}_k)' = \mathbf{n}^*} = (1 - \pi_{ii}) \left[\frac{1}{n^*} u'(n^*) + u''(n^*) \right]$$

Moreover, as has already been shown above, the partial derivative of F_i with respect to $\hat{n}_i$ is given by:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_i} = \pi_{ii} u''(\hat{n}_i) - \frac{1}{\hat{n}_i^2} \sum_{j \neq i} \pi_{ij} \hat{n}_j u'(\hat{n}_j) - v''(\hat{n}_i)$$

Evaluating this partial derivative at the monetary steady state yields:

$$\frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_i} \Big|_{(\hat{n}_1, \dots, \hat{n}_k)' = \mathbf{n}^*} = \pi_{ii} u''(n^*) - (1 - \pi_{ii}) \frac{1}{n^*} u'(n^*) - v''(n^*)$$

Therefore, it follows that the sum over any row of the Jacobi matrix $DF(\mathbf{n}^*, \Pi)$ is given by:

$$\sum_{j=1}^k \frac{\partial F_i(\hat{n}_1, \dots, \hat{n}_k, \Pi)}{\partial \hat{n}_j} \Big|_{(\hat{n}_1, \dots, \hat{n}_k) = \mathbf{n}^*} = u''(n^*) - v''(n^*)$$

which, by the assumptions on preferences made in section 2.1, is strictly negative. Moreover, the partial derivative of F_i with respect to $\hat{n}_i$ is strictly negative, whereas all the partial derivatives of F_i with respect to the labour supply estimates associated with any other state of nature are strictly positive. Since this holds for any $i = 1, \dots, k$, it follows that the Jacobi matrix $DF(\mathbf{n}^*, \Pi)$ is indeed diagonally dominated. In order to see this, note that if this were not the case, meaning that a negative diagonal element were in absolute terms smaller than the positive sum over the other elements in its row, it could not hold that the sum over all elements in this row is negative, which must however hold as just demonstrated.

Since in the present case, in particular all diagonal elements of the matrix $DF(\mathbf{n}^*, \Pi)$ are negative, a theorem due to McKenzie (1960) applies, which basically states that if a matrix is diagonally dominated and all its diagonal elements are negative, all its eigenvalues have negative real part.

Therefore, Theorem 8 can indeed be applied at the monetary steady state, meaning that this rational expectations equilibrium is in the present context locally stable under the dynamics of the considered Robbins - Monro algorithm based on an extrinsic process with arbitrarily many states in the sense described above. Thus, by the argument given above, it must also be locally stable if agents used more than one sunspot process in order to sort their observations, meaning that the monetary steady state must be locally strongly stable under the learning algorithm (33) - (34).

However, it must be emphasized that this result on the strong stability of one particular rational expectations equilibrium under the dynamics of the Robbins - Monro algorithm could only be obtained since it has been assumed that leisure and consumption are at the monetary steady state gross substitutes. As already pointed out above, it is not in general possible to discuss strong stability results for rational expectations equilibria if agents' learning behaviour is characterized by the stochastic recursive algorithm (33) - (34). Therefore, I will next turn to an example of an economy for which Evans (1989) showed based on the analysis of E - stability that although two - state sunspot equilibria can be weakly E - stable, they can never be strongly E - stable.

4.3.3. Strong Stability of Two - State Sunspot Equilibria - An Example

Although in section 4.2.3 the set of two - state sunspot equilibria has been found to be weakly stable under the dynamics of a Robbins - Monro algorithm, it has not been possible to analyse whether this stability result would remain unchanged if agents slightly altered their perceptions. In order to address this remaining question, consider an economy in which agents do not recursively update the labour supply choice of their predecessors through the Robbins - Monro algorithm, but instead use the recursive least squares algorithm described in section 4.1.2 in order to obtain subjective expectations about the real wages they could possibly face and thus also about their lifetime utility.

One advantage of this specification of agents' learning behaviour is that, as has already been argued in section 3.4.1, it becomes possible to focus on the corresponding E - stability conditions which are determined through a mapping from agents' per-

ceived law of motion to the induced actual law of motion instead of conducting a direct analysis of the stochastic recursive algorithm using stochastic approximation techniques. Note that Evans (1989) uses this simpler approach without even referring to the real time learning process behind his analysis.

In the present application the perceived law of motion is clearly given by the belief that labour supply in any period t will be determined solely by the state of nature prevailing in this period. The mapping from these perceptions to the actual law of motion for labour supply is then given by the solution to the utility maximization problem given these perceptions, that is given the subjectively expected utility associated with old - age consumption expressed by (37). In order to determine this mapping it is thus necessary to explicitly solve agents' utility maximization problem. Therefore, it is also necessary to assume a specific form for the utility associated with old - age consumption, $u(c_{t+1})$, and the disutility associated with work, $v(n_t)$. This however also implies that, as pointed out by Woodford (1990), the following results cannot be extended to other specifications for preferences.

Evans (1989) however considers the following specification of agents' utility function:

$$u(c_{t+1}) = \frac{1}{1-\sigma} c_{t+1}^{1-\sigma} \quad \sigma \geq 0, \sigma \neq 1$$

and

$$v(n_t) = \frac{1}{1+\kappa} n_t^{1+\kappa} \quad \kappa \geq 0$$

Clearly, the specification of the disutility associated with working does not impose a natural upper bound on labour supply, as assumed in section 2.1. However, it has already been shown in section 2.5 that the optimal labour supply will anyhow be bounded above zero and below the upper bound for the feasible labour supply. Therefore, it must still hold that agents will never want to choose an infinitely high labour supply, meaning that this modification to the assumptions made in section 2 is not crucial. This was also pointed out by Woodford (1990).

Using this utility functions, the labour supply at the monetary steady state can be explicitly calculated as:

$$n^* = \arg \max_n \frac{1}{1-\sigma} n^{1-\sigma} - \frac{1}{1+\kappa} n^{1+\kappa}$$

from which it follows that the monetary steady state is characterized by $n^* = 1$.

Moreover, it follows from (3) that the elasticity of labour supply with respect to the

real wage evaluated at the monetary steady is given by:

$$\begin{aligned}\varepsilon(1) &= \frac{(n^*)^{-\sigma} - \sigma (n^*)^{-1-\sigma}}{\sigma (n^*)^{-1-\sigma} + \kappa (n^*)^{\kappa-1}} \\ &= \frac{1 - \sigma}{\sigma + \kappa}\end{aligned}$$

Recall that in section 2.3 it has been shown that the condition $\varepsilon(1) < -\frac{1}{2}$ is sufficient for the existence of two - state sunspot equilibria. Evans (1989) however argues that in his example this condition is even necessary: More precisely, he shows that if $\sigma < 2 + \kappa$, i.e. if $\varepsilon(1)$ were greater than $-\frac{1}{2}$, each rational expectations equilibrium, i.e. each root of (12) - (13), would be associated with an index of +1. Since however, by the Poincaré - Hopf Index Theorem the sum of the indices at these roots must be +1, there can only exist one root of (12) - (13), meaning that in this case the monetary steady state must be the unique perfect foresight equilibrium provided that the state of nature can be described by a two - state Markov process.

With this background, it is now possible to show that the requirement for the weak stability of the set of two - state sunspot equilibria under the dynamics of the recursive least squares learning algorithm are in this example exactly the same as the conditions found in section 4.2.3. In order to see this, note that in any period in which state i is observed labour supply is chosen as:

$$n_i = \arg \max_{n_t} u^e \left(\frac{p_t}{p_{t+1}} n_t \right) - v(n_t)$$

where the subjective expectation about the utility obtained through old - age consumption is given by (37), and where it is additionally assumed that $k = 2$.

Using the market clearing condition and the perceived law of motion $n_t = \hat{n}_i$, the labour supply choice in period t can hence be written as:

$$n_i = \arg \max_{n_t} \sum_{j=1}^2 \pi_{ij} \frac{1}{1 - \sigma} \left(\frac{\hat{n}_j}{\hat{n}_i} n_t \right)^{1-\sigma} - \frac{1}{1 + \kappa} n_t^{1+\kappa}$$

meaning that the labour supply maximizing the subjectively expected utility in any period t in which state i is observed must satisfy the following first order condition:

$$\sum_{j=1}^2 \pi_{ij} \frac{\hat{n}_j}{\hat{n}_i} \left(\frac{\hat{n}_j}{\hat{n}_i} n_i \right)^{-\sigma} - n_i^{\kappa} = 0$$

which can be rearranged in order to obtain the labour supply maximizing the lifetime utility of an agent born in a period in which state i is observed implied by the perceived law of motion for prices and labour supply, as:

$$\begin{aligned} n_i^\kappa &= \sum_{j=1}^2 \pi_{ij} \frac{\hat{n}_j}{\hat{n}_i} \left(\frac{\hat{n}_j}{\hat{n}_i} n_i \right)^{-\sigma} \\ n_i^{\kappa+\sigma} &= \sum_{j=1}^2 \pi_{ij} \frac{\hat{n}_j}{\hat{n}_i} \left(\frac{\hat{n}_j}{\hat{n}_i} \right)^{-\sigma} \\ n_i &= \left[\sum_{j=1}^2 \pi_{ij} \left(\frac{\hat{n}_j}{\hat{n}_i} \right)^{1-\sigma} \right]^{\frac{1}{\kappa+\sigma}} \end{aligned}$$

Therefore, the non-linear mapping from agents' perceptions $(\hat{n}_1 \ \hat{n}_2)'$ to the actual labour supplies $(n_1 \ n_2)'$ is given by:

$$T_1(\hat{n}_1, \hat{n}_2) = \begin{pmatrix} \left[\pi_{11} + \pi_{12} \left(\frac{\hat{n}_2}{\hat{n}_1} \right)^{1-\sigma} \right]^{\frac{1}{\kappa+\sigma}} \\ \left[\pi_{22} + \pi_{21} \left(\frac{\hat{n}_1}{\hat{n}_2} \right)^{1-\sigma} \right]^{\frac{1}{\kappa+\sigma}} \end{pmatrix}$$

where the subscript of T indicates that only one extrinsic sunspot process has been used in the learning algorithm.

E - stability of any rational expectations equilibrium under the considered perceived law of motion is then determined through the following ordinary differential equation:

$$\frac{d}{d\tau} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix} = T_1(\hat{n}_1, \hat{n}_2) - \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix} \quad (40)$$

implying that any given stationary two - state sunspot equilibrium is weakly E - stable if all eigenvalues of the Jacobi matrix of the right hand side of this equation evaluated at this sunspot equilibrium have negative real part, or if equivalently all eigenvalues of DT_1 evaluated at the sunspot equilibrium under consideration have real part less than 1.

In order to see whether this condition can be satisfied, consider first the partial derivative of T_1 with respect to its first argument, which is given by the following

expression:

$$\frac{\partial T_1(\hat{n}_1, \hat{n}_2)}{\partial \hat{n}_1} = \begin{pmatrix} -\frac{1-\sigma}{\kappa+\sigma} \left[\pi_{11} + \pi_{12} \left(\frac{\hat{n}_2}{\hat{n}_1} \right)^{1-\sigma} \right]^{\frac{1-\kappa-\sigma}{\kappa+\sigma}} & \pi_{12} \left(\frac{\hat{n}_2}{\hat{n}_1} \right)^{1-\sigma} \frac{1}{\hat{n}_1} \\ \frac{1-\sigma}{\kappa+\sigma} \left[\pi_{22} + \pi_{21} \left(\frac{\hat{n}_1}{\hat{n}_2} \right)^{1-\sigma} \right]^{\frac{1-\kappa-\sigma}{\kappa+\sigma}} & \pi_{21} \left(\frac{\hat{n}_1}{\hat{n}_2} \right)^{-\sigma} \frac{1}{\hat{n}_2} \end{pmatrix}$$

Moreover, it follows from the first order condition for optimal labour supply that at a rational expectations equilibrium $\pi_{11} + \pi_{12} \left(\frac{\hat{n}_2}{\hat{n}_1} \right)^{1-\sigma}$ and $\pi_{22} + \pi_{21} \left(\frac{\hat{n}_1}{\hat{n}_2} \right)^{1-\sigma}$ are given by $n_1^{\kappa+\sigma}$ and by $n_2^{\kappa+\sigma}$, respectively. Since furthermore the partial derivative of T_1 with respect to its second argument is symmetric to the computation above, it follows that the Jacobi matrix DT_1 evaluated at any rational expectations equilibrium $(n_1 \ n_2)'$ is given by:

$$DT_1(n_1, n_2) = \begin{pmatrix} -\varepsilon(1) \pi_{12} n_1^{-(1+\kappa)} n_2^{1-\sigma} & \varepsilon(1) \pi_{12} n_1^{-\kappa} n_2^{-\sigma} \\ \varepsilon(1) \pi_{21} n_1^{-\sigma} n_2^{-\kappa} & -\varepsilon(1) \pi_{21} n_1^{1-\sigma} n_2^{-(1+\kappa)} \end{pmatrix}$$

From which it can easily be seen that the determinant of $DT_1(n_1, n_2)$ is equal to zero, implying that the eigenvalues of this Jacobi matrix are given by zero and the trace of the matrix. Since zero is clearly smaller than one, the only requirement for the weak E - stability of any stationary two - state sunspot equilibrium is that the trace of DT_1 evaluated at this particular equilibrium is smaller than one. Moreover, Evans (1989) shows that if the sufficient condition for the existence of sunspot equilibria is fulfilled, i.e. if $\varepsilon(1) < -\frac{1}{2}$, this requirement is satisfied if and only if the rational expectations equilibrium under consideration is associated with an index of +1. Since there must exist at least two stationary two - state sunspot equilibria associated with an index of +1 if the sufficient condition for the existence of stationary sunspot equilibria is satisfied, the results obtained by Evans (1989) concerning the weak stability of the set of two - state sunspot equilibria are consistent with those obtained by Woodford (1990) and discussed in section 4.2.3, i.e. the weak stability of the class of two - state sunspot equilibria under the dynamics of the Robbins - Monro algorithm.

Moreover, similar to the result obtained in section 4.2.3, the monetary steady state is also in this example E - stable if agents use a two - state Markov process, provided that $\varepsilon(1) > -\frac{1}{2}$ and that thus the sufficient condition for the existence of sunspot equilibria is violated. In order to see this, note that for $\varepsilon(1) > -\frac{1}{2}$, $tr[DT_1(n^*, n^*)] = -\varepsilon(1)(\pi_{12} + \pi_{21})$ is smaller than one for all transition probabilities which lie strictly between zero and one, meaning that the monetary steady

state indeed satisfies the condition for E - stability under the given perceived law of motion coinciding with the functional form of the class of stationary two - state sunspot equilibria.

However, as for the analysis of Woodford (1990), it is not possible to obtain any information on the location of the two - state sunspot equilibria which have been found to be (locally) E - stable. Therefore, Evans and Honkapohja (1994) and (2003) follow a different approach: They investigate the weak E - stability of certain classes of two - state sunspot equilibria, as for example the stationary two - state sunspot equilibria located in a neighbourhood of two distinct stationary perfect foresight equilibria. Evans and Honkapohja (1994) find that this class of sunspot equilibria is weakly E - stable if and only if both perfect foresight equilibria are weakly E - stable. Nevertheless, since in the model considered here the only weakly E - stable stationary perfect foresight equilibrium is the monetary steady state, this case is not relevant for the present application. In particular, it must be concluded that it is not possible for agents to adaptively learn a sunspot equilibrium for which one state of nature is associated with almost valueless money, whereas in the other state of nature prices are almost at their monetary steady state levels.

However, Evans and Honkapohja (1994) were not able to show a similar result for the more interesting case in the model considered here, namely for the class of two - state sunspot equilibria located near a single steady state, as for example the monetary steady state. As has already been noted in section 2.4, stationary two - state sunspot equilibria exist arbitrarily close to the monetary steady state if and only if it is indeterminate under perfect foresight dynamics which, as has been shown in the proof of Theorem 1 in section 2.1, is the case if $\left| \frac{1+\varepsilon(1)}{\varepsilon(1)} \right| < 1$. Based on this result, Evans and Honkapohja (1994) only showed that the class of stationary two - state sunspot equilibria located in a neighbourhood around the monetary steady state is never weakly E - stable if $0 < \frac{1+\varepsilon(1)}{\varepsilon(1)} < 1$. Since in the present example $\varepsilon(1)$ is given by $\frac{1-\sigma}{\sigma+\kappa}$ this condition would however require that $\kappa < -\sigma$ which is a contradiction to the assumptions made on the utility function⁶⁹. Therefore, this case cannot arise in the present example and is thus also not relevant here. The relevant question whether the class of stationary two - state sunspot equilibria located near the monetary steady state can be weakly E - stable for $-1- < \frac{1+\varepsilon(1)}{\varepsilon(1)} < 0$ could however not be answered through the approach of Evans and Honkapohja (1994). Using a different technique, Evans and Honkapohja (2003) however finally show that

⁶⁹Recall that it has been assumed that both κ and σ are positive.

for $0 > \frac{1+\varepsilon(1)}{\varepsilon(1)} > -1$, i.e. for $\sigma > 2+\kappa$, weakly E - stable two - state sunspot equilibria exist in any neighbourhood around the monetary steady state. In order to establish this result Evans and Honkapohja rely on a bifurcation analysis: Therefore, they define the transition probability $\bar{\pi}_{11}$ such that $tr[DT_1(n^*, n^*)]$ is one, i.e. such that $\bar{\pi}_{11} = 2 + \frac{1}{\varepsilon(1)} - \pi_{22} = 1 + \frac{1+\kappa}{1-\sigma} - \pi_{22}$. Holding π_{22} fixed, Evans and Honkapohja then vary π_{11} about $\bar{\pi}_{11}$ and analyse the change in the qualitative behaviour of the dynamical system (40). Using the central manifold technique, they find that the qualitative change occurring at $\pi_{11} = \bar{\pi}_{11}$ is characterized by a so called transcritical bifurcation, meaning that for $\pi_{11} > \bar{\pi}_{11}$ the monetary steady state is stable under the considered dynamics, whereas for $\pi_{11} < \bar{\pi}_{11}$ the monetary steady state loses its stability to a stationary two - state sunspot equilibrium located close to the monetary steady state.

As already noted above, all the results just indicated only show that agents' learning process will necessarily converge to a two - state sunspot equilibrium if both the necessary and the sufficient condition for the existence of such an equilibrium are fulfilled and if agents can correctly specify the extrinsic stochastic process associated with this class of equilibria. It might however very well be the case that agents can also observe the evolution of a second extrinsic variable and believe that this variable also has an influence on the economy.

Let this second extrinsic variable be given by the two - state Markov process (W_t) which can take on values in the state space $\{1, 2\}$, and denote, as above, the probability of reaching state b from state a for this additional process as ψ_{ab} . As already pointed out above, agents who condition their expectations also on this process believe then that nature is in one of four possible states, denoted as ai , where i refers to the realization of the original sunspot process and a refers to the realization of the additional process which is assumed to be independent from the original process.

A two - state sunspot equilibrium which has been found to be weakly E - stable can then be written as a degenerate sunspot equilibrium of cardinality four and order two by setting $n_{ai} = n_i$ for all $a = 1, 2$ and all $i = 1, 2$. If this degenerate sunspot equilibrium is weakly E - stable, regardless of the additional process used in the expectations formation process, the original two - state sunspot equilibrium is said to be strongly E - stable, meaning that its stability under learning is robust with respect to changes in agents' perceptions about the number of relevant explanatory

variables.⁷⁰

Evans (1989) however argues that in his example a two - state sunspot equilibrium can never be strongly E - stable. In order to see this, note that, analogously to above, the mapping from agents perceptions $(\hat{n}_{11}, \hat{n}_{21}, \hat{n}_{12}, \hat{n}_{22})'$ to the actual law of motion for n_{ai} , the optimal labour supply associated with state of nature ai , is given by:

$$[T_2(\hat{n}_{11}, \hat{n}_{21}, \hat{n}_{12}, \hat{n}_{22})]_{ai} = \left[\sum_{b=1}^2 \sum_{j=1}^2 \pi_{ij} \psi_{ab} \left(\frac{\hat{n}_{bj}}{\hat{n}_{ai}} \right)^{1-\sigma} \right]^{\frac{1}{\kappa+\sigma}} \quad \forall a, i \in \{1, 2\}$$

As above, a rational expectations equilibrium $(n_{11}, n_{21}, n_{12}, n_{22})'$ is said to be E - stable under the considered perceived law of motion if all eigenvalues of the Jacobi matrix of T_2 evaluated at this equilibrium have real part less than one.

In order to show that there exist extrinsic processes for which this matrix has however at least one eigenvalue with real part greater than one, Evans (1989) first shows that this is the case if the transition probability matrix of the additional process were such that $\psi_{11} = \psi_{22} = 0$. He then argues by continuity of the eigenvalues that this result must also hold for transition probabilities which are sufficiently close to zero but which do not violate the requirement for the transition probabilities stated in the definition of stationary sunspot equilibria given in section 2.

Thus, consider the partial derivative of $[T_2]_{ai}$ with respect to $\hat{n}_{ai}$:

$$\frac{\partial [T_2]_{ai}}{\partial \hat{n}_{ai}} = -\frac{1-\sigma}{\kappa+\sigma} \hat{n}_{ai}^{-2+\sigma} \sum_{(j,b) \neq (i,a)} \pi_{ij} \psi_{ab} \hat{n}_{bj}^{1-\sigma} \left[\sum_{b=1}^2 \sum_{j=1}^2 \pi_{ij} \psi_{ab} \left(\frac{\hat{n}_{bj}}{\hat{n}_{ai}} \right)^{1-\sigma} \right]^{\frac{1-\kappa-\sigma}{\kappa+\sigma}}$$

Using the first order condition for n_{ai} , the optimal labour supply associated with state ai , in order to replace $\sum_{j=1}^2 \sum_{b=1}^2 \pi_{ij} \psi_{ab} \left(\frac{\hat{n}_{bj}}{\hat{n}_{ai}} \right)^{1-\sigma}$ by $n_{ai}^{\kappa+\sigma}$ and evaluating this partial derivative at the degenerate sunspot equilibrium under consideration yields that the diagonal elements of the Jacobi matrix of T_2 evaluated at this equilibrium

⁷⁰Note that, unlike in the previous sections, only a very specific overparametrization, namely the inclusion of one additional sunspot process with two (and not arbitrarily many) states is taken into account here. Therefore, the concept of strong E - stability used here would actually be weaker than the concept discussed in earlier sections. However, since it is the aim of this analysis to show that the sunspot equilibrium under consideration cannot be strongly E - stable it is enough to find just one overparametrized perceived law of motion under which it is not E - stable.

are given by:

$$\begin{aligned}
DT_2(n_1, n_2, n_1, n_2)_{ai, ai} &= -\frac{1-\sigma}{\kappa+\sigma} n_i^{-(1+\kappa)} \left[\pi_{ij} \underbrace{\psi_{ab}}_{=1} n_j^{1-\sigma} + \pi_{ij} \underbrace{\psi_{aa}}_{=0} n_j^{1-\sigma} + \pi_{ii} \underbrace{\psi_{ab}}_{=1} n_i^{1-\sigma} \right] \\
&= -\frac{1-\sigma}{\kappa+\sigma} n_i^{-(1+\kappa)} \underbrace{(\pi_{ij} n_j^{1-\sigma} + \pi_{ii} n_i^{1-\sigma})}_{=n_i^{1+\kappa}} \\
&= -\frac{1-\sigma}{\kappa+\sigma}
\end{aligned}$$

where the last equality holds since it must be the case that $n_i v'(n_i) = \pi_{ii} n_i u'(n_i) + \pi_{ij} n_j u'(n_j)$, where $j \neq i$ and $i \in \{1, 2\}$, if n_i is chosen optimally when only the original sunspot process is used.

Furthermore, the partial derivative of $[T_2]_{ai}$ with respect to any $\hat{n}_{bj}$, where either $j \neq i$ or $b \neq a$, is given by:

$$\frac{\partial [T_2(\hat{n}_{11}, \hat{n}_{21}, \hat{n}_{12}, \hat{n}_{22})]_{ai}}{\partial \hat{n}_{bj}} = \frac{1-\sigma}{\kappa+\sigma} \left[\sum_{m=1}^2 \sum_{l=1}^2 \pi_{il} \psi_{am} \left(\frac{\hat{n}_{ml}}{\hat{n}_{ai}} \right)^{1-\sigma} \right]^{\frac{1-\kappa-\sigma}{\kappa+\sigma}} \pi_{ij} \psi_{ab} \frac{\hat{n}_{bj}^{-\sigma}}{\hat{n}_{ai}^{1-\sigma}}$$

As above, the first order condition for optimal labour supply can be used to substitute $n_{ai}^{\kappa+\sigma}$ for $\sum_{l=1}^2 \sum_{m=1}^2 \pi_{il} \psi_{am} \left(\frac{\hat{n}_{ml}}{\hat{n}_{ai}} \right)^{1-\sigma}$. Evaluating this partial derivative at the degenerate sunspot equilibrium under consideration yields that the entry in the row corresponding the actual law of motion for n_{ai} and the column corresponding to the partial derivative with respect to $\hat{n}_{bj}$ of the Jacobi matrix DT_2 evaluated at this equilibrium, is given by:

$$DT_2(n_1, n_2, n_1, n_2)_{ai, bj} = \frac{1-\sigma}{\kappa+\sigma} n_i^{-\kappa} n_b^{-\sigma} \pi_{ij} \psi_{ab}$$

Since it is assumed that $\psi_{aa} = 0$ for $a = 1, 2$, it automatically follows from this that $[DT_2(n_1, n_2, n_1, n_2)]_{r, c} = 0$ for $(r, c) \in \{(1, 2), (2, 1), (3, 4), (4, 3)\}$, where r refers to the row and c refers to the column of the Jacobi matrix the given element is in, since for these entries it holds that $b = a$, meaning that the additional sunspot process does not change its state. Furthermore, it also follows for $a \neq b$ and $i, a \in \{1, 2\}$, that:

$$DT_2(n_1, n_2, n_1, n_2)_{ai, bi} = \frac{1-\sigma}{\kappa+\sigma} n_i^{-\kappa} n_i^{-\sigma} \pi_{ii} \underbrace{\psi_{ab}}_{=1}$$

Since n_i has been chosen optimally in the presence of the original two - state sunspot process, it must, as already noted above, hold that $n_i v'(n_i) = \pi_{ii} n_i u'(n_i) +$

$\pi_{ij}n_j u'(n_j)$. This can in the present example be rearranged to show that $n_i^{-\kappa}n_i^{-\sigma}\pi_{ii}$ can also be expressed as $1 - \pi_{ij}n_j^{1-\sigma}n_i^{-(1+\kappa)}$, implying that the expression above can be rewritten as:

$$DT_2(n_1, n_2, n_1, n_2)_{ai, bj} = -\frac{1-\sigma}{\kappa+\sigma}n_j^{1-\sigma}n_i^{-(1+\kappa)} + \frac{1-\sigma}{\kappa+\sigma}$$

Moreover, since $\psi_{ab} = 1$ for $a \neq b$, it immediately follows that the derivative of the mapping for optimal labour supply in state ai with respect to $\hat{n}_{bj}$ for $j \neq i$ and $b \neq a$ is given by:

$$DT_2(n_1, n_2, n_1, n_2)_{ai, bj} = \frac{1-\sigma}{\kappa+\sigma}n_i^{-\kappa}n_j^{-\sigma}\pi_{ij}$$

Collecting all these partial derivatives in the Jacobi matrix and comparing the last two results with the Jacobi matrix DT_1 evaluated at the stationary two - state sunspot equilibrium under consideration shows that the larger dimensional Jacobi matrix DT_2 evaluated at the corresponding degenerate sunspot equilibrium can be written as the following block matrix:

$$DT_2(n_1, n_2, n_1, n_2) = \begin{pmatrix} -\varepsilon(1)I_2 & DT_1(n_1, n_2) + \varepsilon(1)I_2 \\ DT_1(n_1, n_2) + \varepsilon(1)I_2 & -\varepsilon(1)I_2 \end{pmatrix}$$

where I_2 denotes the two dimensional identity matrix.

As usually, the eigenvalues of this matrix are given by the roots of the characteristic polynomial, i.e. by the roots of the determinant of $DT_2(n_1, n_2, n_1, n_2) - \lambda I_4$, which can also be written as:

$$\begin{vmatrix} -\varepsilon(1)I_2 - \lambda I_2 & DT_1(n_1, n_2) + \varepsilon(1)I_2 \\ DT_1(n_1, n_2) + \varepsilon(1)I_2 & -\varepsilon(1)I_2 - \lambda I_2 \end{vmatrix} = 0$$

Since the determinant of a matrix does not change when one row (resp. column) is added to another row (resp. column), it is possible to make the following transformations, without altering the roots of the characteristic polynomial:

$$\begin{vmatrix} -\varepsilon(1)I_2 - \lambda I_2 & DT_1(n_1, n_2) + \varepsilon(1)I_2 \\ DT_1(n_1, n_2) - \lambda I_2 & DT_1(n_1, n_2) - \lambda I_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} -DT_1(n_1, n_2) - 2\varepsilon(1)I_2 - \lambda I_2 & DT_1(n_1, n_2) + \varepsilon(1)I_2 \\ 0 & DT_1(n_1, n_2) - \lambda I_2 \end{vmatrix} = 0$$

Since this matrix is block - diagonal, the solutions for λ are given by the eigenvalues of $-DT_1(n_1, n_2) - 2\varepsilon(1)I_2$ and the eigenvalues of $DT_1(n_1, n_2)$, which, as noted

above, are given by 0 and $\text{tr}[DT_1(n_1, n_2)]$. Therefore, this example also illustrates a result obtained in section 4.3.1, namely that all eigenvalues of the Jacobi matrix evaluated at a certain rational expectations equilibrium must also be eigenvalues of the Jacobi matrix when agents use additional extrinsic sunspot processes in their learning rules.

In order to calculate the two remaining eigenvalues of $DT_2(n_1, n_2, n_1, n_2)$, note that the trace of $-DT_1(n_1, n_2) - 2\varepsilon(1)I_2$ is given by $-\text{tr}[DT_1(n_1, n_2)] - 4\varepsilon(1)$. Furthermore, since the determinant of $DT_1(n_1, n_2)$ is zero, it follows that the determinant of this matrix can be written as $2\varepsilon(1)\text{tr}[DT_1(n_1, n_2)] + 4\varepsilon(1)^2$. Since the eigenvalues of any 2×2 matrix A can be computed as the roots of $\lambda^2 - \text{tr}[A]\lambda + \det[A] = 0$, the missing two eigenvalues are given by:

$$\begin{aligned}\lambda_{1,2} &= \frac{-\text{tr}[DT_1(n_1, n_2)] - 4\varepsilon(1)}{2} \pm \\ &\quad \pm \sqrt{\frac{(-\text{tr}[DT_1(n_1, n_2)] - 4\varepsilon(1))^2}{4} - 2\varepsilon(1)\text{tr}[DT_1(n_1, n_2)] - 4\varepsilon(1)^2} \\ &= \frac{-\text{tr}[DT_1(n_1, n_2)] - 4\varepsilon(1)}{2} \pm \frac{\text{tr}[DT_1(n_1, n_2)]}{2}\end{aligned}$$

In particular, it can be seen from this that one eigenvalue of DT_2 evaluated at the degenerate sunspot equilibrium under consideration must be equal to $-2\varepsilon(1)$. If this eigenvalue were smaller than one so that the sunspot equilibrium under consideration could be strongly E - stable, it would be the case that $\varepsilon(1) > -\frac{1}{2}$. However, as argued above, it is in this case not possible that stationary sunspot equilibria exist since in the model discussed here $\varepsilon(1) < -\frac{1}{2}$ is not just sufficient, but indeed necessary for the existence of stationary sunspot equilibria.

Therefore, if $\psi_{11} = \psi_{22} = 0$, one eigenvalue of DT_2 must be greater than one, meaning that in this case the two - state sunspot equilibrium under consideration cannot be E - stable under the considered overparametrized perceived law of motion. As already indicated above, this must continue to hold if ψ_{11} and ψ_{22} are small, but strictly greater than zero.

Therefore, Evans (1989) concludes that although the class of stationary two - state sunspot equilibria is weakly stable if sunspot equilibria exist, there also exist sunspot processes which if included in the learning algorithm render this class of equilibria unstable. This has in particular been shown for two - state Markov processes for which the probability that states change from period to the next is high. Therefore, the class of two - state sunspot equilibria can never be strongly E - stable and thus it can be argued that although two - state sunspot equilibria might be observed

under certain circumstances, they will not be persistently observed, when agents use a recursive least squares learning rule as described here.

However, since it has been shown above that in the presence of sunspot equilibria the monetary steady state can also not be attained by the learning algorithm when any number of extrinsic processes are used, the question which equilibria are plausible outcomes of agents' learning behaviour remains. Evans (1989) tries to answer this question by simulating his model. In doing so, he can first confirm that a weakly E - stable two - state sunspot equilibrium is indeed not stable when another extrinsic stochastic process is used in the learning algorithm. Moreover, in his simulations, trajectories starting close to the given sunspot equilibrium, but allowing for a dependence on the realization of the additionally used process, converge to a degenerate sunspot equilibrium of higher order. Furthermore, Evans (1989) shows with his simulation that the distance between the smallest and the largest labour supply associated with a sunspot equilibrium which is attained by the learning process increases with the number of sunspot processes used in the learning algorithm. In other words, Evans (1989) demonstrates that if agents condition their expectations about prices on many different extrinsic variables, they are led to rather "extreme" actions.

However, what is more important about these results is that they indicate that the possibility of agents conditioning their expectations on the realization of certain extrinsic variables results in great difficulties for predicting on which, or even on which type of, rational expectations equilibrium an economy will asymptotically settle. In order to do this, it would be necessary to have precise information on agents' perceived law of motion. Moreover, slight changes in the parametrization of agents' perceived law of motion and small perturbations from a given class of rational expectations equilibria might lead the economy to a sunspot equilibrium of completely different order.

5. Conclusion

The previous analysis for the Samuelson overlapping generations model showed that if the monetary steady state is locally asymptotically stable under perfect foresight dynamics, i.e. if small deviations from the monetary steady state price level result in prices converging back to the monetary steady state, stationary sunspot equilibria

will necessarily exist under the hypothesis of rational expectations. Moreover, for two plausible learning rules, the results given in section 4 show that in this case the monetary steady state cannot be attained by agents who adjust their expectations adaptively and use the evolution of an extrinsic variable in doing so. Furthermore, it was demonstrated that at least for some special cases extrinsic uncertainty will even asymptotically have an effect on economic outcomes.

However, the analysis undertaken here is based on a very simple model which can only be regarded as a rather stylized description of real economies. For example, it has been assumed that all agents are identical and that thus all agents base their expectations on the same learning algorithm. It might however also be the case that only one part of the population believes in an influence of the extrinsic sunspot process on economic outcomes. Nevertheless, Woodford (1990) demonstrates that if the number of agents who use the extrinsic sunspot process in their learning algorithm is sufficiently large, the economy will still converge to a sunspot equilibrium, meaning that the results described above remain unchanged.

However, it seems in general justified to analyse the effect of extrinsic uncertainty in a simplified model such as the Samuelson overlapping generations model since modifications to the model which would allow for a better description of reality should make it even easier for stationary sunspot equilibria to exist: One possible modification discussed by Evans and Honkapohja (2001) is to assume that the production of one agent does not only depend on his specific labour input, as it has been assumed here, but also on the aggregate labour input in the economy. This concept of increasing social returns is based on the observation that a larger population can create more ideas implying that the individual worker can refer back to a greater range of solutions to potential problems occurring during his production process and can thus be more productive. As Evans and Honkapohja show, incorporating this modified production technology into the simple overlapping generations model discussed here can yield three stationary perfect foresight equilibria, in all of which money has a positive value and where two of them are (locally) stable under the dynamics of a learning algorithm provided that agents do not use any extrinsic process in their learning algorithm. Evans and Honkapohja (2001) then argue by a result due to Evans and Honkapohja (1994) that a sunspot equilibrium located in any neighbourhood of two perfect foresight equilibria is stable under learning if and only if the respective perfect foresight equilibria are both weakly E - stable. Therefore, they conclude that even if the simple model discussed here is augmented by a more realistic production technology the basic result that extrinsic uncertainty

can have an impact on economic outcomes remains unchanged.

A different restriction imposed by the Samuelson overlapping generations model is that agents can only consume in the second part of their lives. A different assumption, made for example by the Diamond overlapping generations model, namely that agents can also consume in the first part of their lives, would not change the results obtained through the present analysis, but would make it necessary to focus on the saving decision of young agents instead of the labour supply decision as done here. This is the case, since under the latter assumption the relevant economic decision is not how much time agents spend on production, but which part of their production they consume themselves when young and which part they save for their old age consumption, i.e. which part they sell on the goods market.

Another restriction which has been imposed by the present analysis is that agents who are not able to form rational expectations adjust their expectations adaptively. However, this is not the only possibility to model learning behaviour. Another way is to assume that agents base their expectations on so called eductive learning, i.e. that they solely adjust their expectations by a process of reasoning based on some initial common knowledge about the aggregate expectations in the economy, as described for example in Evans and Honkapohja (2001). The advantage of this approach is that coordination on a rational expectations equilibrium could occur instantaneously, while for adaptive learning this coordination can only be expected to occur asymptotically. However, the use of eductive learning rests on rather strong common knowledge assumptions and furthermore it also results in stricter stability conditions for some rational expectations equilibria.

Moreover, the aim of this thesis was to demonstrate that there exist plausible situations in which extrinsic uncertainty has an affect on economic outcomes. Since both learning algorithms introduced here seem to be a plausible description for the behaviour of agents, this could be accomplished.

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A. Appendix

A.1. Abstract (English)

This thesis aims at describing the circumstances under which economic outcomes can be influenced by the evolution of a purely extrinsic variable within the framework of a simple overlapping generations model. This question will be both addressed under the assumption of rational expectations and under the more realistic assumption of agents adjusting their expectations about economic variables adaptively from past observations by using econometric techniques. Therefore, this thesis will also provide a brief overview of the literature on learning in macroeconomic models and introduce the basic techniques which are employed to analyse different learning algorithms.

A.2. Abstract (German)

Das Ziel dieser Diplomarbeit ist es die Umstände zu beschreiben unter denen ökonomische Ereignisse in einem einfachen OLG Modell durch den Verlauf einer extrinsischen Variable beeinflusst werden können. Diese Frage wird sowohl unter der Annahme von rationalen Erwartungen als auch unter der realistischeren Annahme, dass Agenten ihre Erwartungen über die für sie relevanten ökonomischen Variablen adaptiv durch ökonometrische Methoden aus vergangenen Beobachtungen bilden, untersucht. Daher wird diese Diplomarbeit auch einen kurzen Überblick über die Literatur über Lernen in makroökonomischen Modellen geben und die grundlegenden Methoden die zu einer Analyse von unterschiedlichen Lernalgorithmen dienen vorstellen.

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