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Solutions to the static, spherical symmetric Einstein-Vlasov-Maxwell system with nonvanishing cosmological constant

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Abstract

The self-consistent motion of an ensemble of collisionless, charged particles under the effect of its own gravitational and electromagnetic (EM) field as well as the corresponding metric and EM field can be described by the Einstein-Vlasov-Maxwell system (EVMS), a system of coupled differential equations consisting of the Einstein equations, the Maxwell equations and the Vlasov equation. Since this system is rather complicated, we limit ourselves to the static, spherical symmetric case, including however the cosmological constant. To prove the existence of solutions to this system, we combine the work of [1], where the proof is done for a vanishing cosmological constant, with the work of [2], in which the existence of solutions is proven for the Einstein-Vlasov system, so the electrical neutral case, but with a nonvanishing cosmological constant. Both of the sources consider the static, spherical symmetric system. In the present thesis we prove the existence of solutions to the regular system as well as the existence of solutions in presence of a charged black hole at the center of the spherical symmetry. The proofs in the present work require weak particle charges, a weak black hole charge and a small absolute value for the cosmological constant.

Kurzfassung

Die selbstkonsistente Bewegung eines Ensembles von kollisionslosen, geladenen Teilchen unter Einfluss seines eigenen Gravitationsfeldes und seines eigenen elektromagnetischen (EM) Feldes sowie das zugehörige Metrikfeld und EM-Feld können durch das Einstein-Vlasov-Maxwell System (EVMS) beschrieben werden. Dies ist ein System aus gekoppelten Differentialgleichungen, bestehend aus den Einstein Gleichungen, den Maxwell Gleichungen und der Vlasov Gleichung. Aufgrund der Komplexität dieses Gleichungssystems beschränken wir uns hier auf den statischen, sphärisch symmetrischen Fall, wobei wir die kosmologische Konstante aber inkludieren. Um die Existenz von Lösungen von diesem System zu beweisen, kombinieren wir die Arbeit von [1], in der der Beweis für verschwindende kosmologische Konstante durchgeführt wurde, mit der Arbeit von [2], in der die Existenz von Lösungen für das Einstein-Vlasov System, also den elektrisch neutralen Fall, aber mit nichtverschwindender kosmologischer Konstante bewiesen ist. In beiden Arbeiten wird jeweils das statische, sphärisch symmetrische System betrachtet. In der vorliegenden Masterarbeit wird die Existenz von Lösungen für das reguläre System bewiesen, sowie die Existenz von Lösungen in Anwesenheit eines geladenen, schwarzen Lochs im Zentrum der Kugelsymmetrie. Die Beweise, wie sie hier gemacht werden, erfordern schwache Teilchenladungen, eine schwache Ladung des schwarzen Lochs und einen kleinen Absolutbetrag der kosmologischen Konstante.

Contents

1. Introduction

The study of the dynamics of charged particles within the framework of general relativity is a fundamental problem in gravitational physics. A prominent model that describes the self-consistent motion of such a particle ensemble under the influence of its own gravitational and electromagnetic field is the Einstein-Vlasov-Maxwell system (EVMS), which neglects collisions between the particles, and models the ensemble as a collisionless gas, describing the evolution of its particle distribution function $f(t, \vec{x}, \vec{v})$ by the Vlasov equation. The Vlasov equation, being dependent on both the metric and the charge distribution, is coupled to the Einstein and Maxwell equations, which, in turn, depend on the particle distribution function. Thus, the EVMS forms a complex system of coupled differential equations that describes the motion of collisionless, charged particles in general relativity.

Since in many cases collisions between the components of an astrophysical system happen rarely, it often makes sense to describe the evolution of the particle distribution function of such systems by the Vlasov equation. While the uncharged Einstein-Vlasov system can be useful to model objects like globular clusters, galaxies or galaxy clusters, which are assumed to have zero net charge, the Einstein-Vlasov-Maxwell system can be a good attempt for the description of objects where charges do play a role for the dynamics. Such objects are for example charged galactic nebulae which are clouds of ionized matter. These include planetary nebulae, which are ejected from stars of intermediate mass (about $1 - 8$ solar masses) at a late stage of their lives. However an issue here is that it is often not reasonable to assume static conditions, which makes it hard to study the EVMS. (see [3])

The complexity of the EVMS has prompted to investigate specific cases, where the system of equations is simplified far enough such that it can be analysed rigorously. One possibility is to consider spherical symmetry and static conditions of the ensemble as a whole. These assumptions reduce the dimensionality of the problem significantly which facilitates the study of the system. Besides the physical applications it is also useful to investigate this system, because its solutions share some properties with the axially symmetric, rotating solutions of the uncharged Einstein-Vlasov system. Therefore, we can possibly learn something about those uncharged solutions with a different symmetry by finding out about the charged solutions in spherical symmetry. (see [3]) Previous works on the spherical symmetric, static system have established the existence of solutions in certain cases. For instance, in 2015 Håkan Andreasson, David Fajman and Maximilian Thaller studied the electrically uncharged system with a cosmological constant of small absolute value for which they showed the existence of local solutions, global regular solutions as well as global solutions with a Schwarzschild singularity, i.e. a black hole, at the symmetry center (see [2]). In 2019 Maximilian Thaller proved the existence of local and global regular solutions for small particle charges, but in absence of a cosmological constant (see [1]). The objective of the present thesis is to combine the two just mentioned approaches and establish the existence of local solutions, global regular solutions and global solutions with a charged black hole at the center, each for the charged EVMS and in presence of a nonvanishing cosmological constant, where all the charges as well as the absolute value of the cosmological constant are small.

The thesis is based entirely on theoretical investigations and the proofs will be mostly constructed by combining methods from the two mentioned works of Thaller and Andréasson, Fajman, and Thaller. It is divided in two main parts. The first part treats the regular case, where all the particle properties are non-divergent everywhere, including at the center. We will start off with the explanation of the setup, where we will use an appropriate Ansatz for the static, spherical symmetric system and based on this write down the equations of the EVMS. By choosing suitable boundary conditions, we impose a regular center. After the setup, we approach step by step the proof for the existence of global solutions, for the time being with a positive cosmological constant $\Lambda > 0$. At first we proof the local existence on a radial interval $[\mathring{r}, \mathring{r} + \delta]$ with an arbitrary $\mathring{r} \geq 0$ and some small $\delta > 0$. In the next chapter we establish a continuation criterion, which guarantees the existence of solutions as long as the denominator of a certain one of the reduced Einstein equations is positive. Using this and a background solution with vanishing cosmological constant and a spatial support bounded above by some radius $R_0 > 0$, we show that there exist solutions to our system of interest at least until $R_0 + \Delta R$ with some positive number $\Delta R > 0$ and that these solutions also have a bounded spatial support. To do so it is necessary to assume that Λ is sufficiently small. As a last step, we show that beyond the support of the matter region we can extend the solutions by a vacuum solution in a smooth way, which finally gives us the existence of global regular solutions. In the last chapter of part I we briefly discuss the case $\Lambda < 0$ and reconsider the parts in the proof that are affected by the different sign. The second part of the thesis considers the system with a singularity, specifically with a charged black hole, at the center. In this case we still use the same setup as in the first part with different boundary conditions and an effective electric energy as an auxiliary quantity to avoid troubles inside the black hole event horizon. We initially assume the black hole charge Q_0 and the particle charges q_0 to have the same sign to simplify calculations. In the first chapter of part two we study the scenario where we have a weakly charged black hole surrounded by weakly charged matter shells and vanishing cosmological constant. We use this result further as a background solution for the next chapter, where we reintroduce a nonvanishing, positive cosmological constant and show that staying close to the background solution, i.e. taking a small value of the cosmological constant, there exist global solutions to this non-regular system. Then, again we discuss the case for a negative cosmological constant and show the existence of global solutions with a charged black hole at the center and $\Lambda < 0$. In the last chapter we treat the case where the black hole charge and the charge of the particles of the matter region do not have the same sign, that is to say where $sgn(Q_0) \neq sgn(q_0)$, where sgn is the signum function. In this scenario the contribution from the effective electromagnetic energy will for small enough radii be positive, so there it acts as a binding energy and we have to reconsider some arguments in the proofs. However, in the end it is still possible to show the existence of solutions also for the different signs of the charges.

Part I.

Regular solutions

2. Setup

We consider the static, spherical symmetric Einstein-Vlasov-Maxwell system (EVMS) with positive cosmological constant $\Lambda > 0$. Therefore, we choose the Ansatz

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\Theta^2 + \sin^2\Theta d\varphi^2\right) \tag{2.1}$$

for the metric, where $\mu(r)$, $\lambda(r) \in \mathbb{R}$, t is the time coordinate and (r, Θ, φ) are the standard coordinates of a spherical coordinate system. Furthermore, we consider all particles to be of the same mass and charge (given by q_0) and their distribution function f to be a function only of the variables r, ω and L, where r is the radial coordinate,

$$\omega = e^{\lambda(r)}p^{(r)} \tag{2.2}$$

is the radial momentum and

$$L = r^4\left(\left(p^{(\Theta)}\right)^2 + \sin^2\Theta\left(p^{(\varphi)}\right)^2\right) \tag{2.3}$$

is the square of the angular momentum. $\left(p^{(r)}, p^{(\Theta)}, p^{(\varphi)}\right)$ denote the momentum components in spherical coordinates. We can write the relevant matter quantities, i.e. the energy density $\rho(r)$, the radial pressure $p(r)$, the transversal pressure $p_T(r)$ and the charge density $\rho_q(r)$ in terms of integrals over the particle distribution function:

$$\rho(r) = \frac{\pi}{r^2}\int_0^\infty\int_{-\infty}^\infty f(r,\omega,L)\sqrt{1+\omega^2+\frac{L}{r^2}}d\omega dL \tag{2.4}$$

$$p(r) = \frac{\pi}{r^2}\int_0^\infty\int_{-\infty}^\infty f(r,\omega,L)\frac{\omega^2}{\sqrt{1+\omega^2+\frac{L}{r^2}}}d\omega dL \tag{2.5}$$

$$p_T(r) = \frac{\pi}{2r^4}\int_0^\infty\int_{-\infty}^\infty f(r,\omega,L)\frac{L}{\sqrt{1+\omega^2+\frac{L}{r^2}}}d\omega dL \tag{2.6}$$

$$\rho_q(r) = q_0 e^{\lambda(r)}\frac{\pi}{r^2}\int_0^\infty\int_{-\infty}^\infty f(r,\omega,L)d\omega dL \tag{2.7}$$

With our symmetry considerations the Vlasov equation looks like this:

$$\omega\frac{\partial f}{\partial r} + \left(\frac{L}{r^3} + q_0\frac{q(r)}{r^2}e^{\mu(r)+\lambda(r)} - \mu'(r)\left(1 + w^2 + \frac{L}{r^2}\right)\right)\frac{\partial f}{\partial\omega} = 0 \tag{2.8}$$

where $q(r)$ is the charge contained in a sphere of radius r around the center, determined by the only non-trivial Maxwell equation of the static, spherical symmetric system

$$q'(r) = 4\pi r^2 \rho_q(r) \tag{2.9}$$

There are two independent, conserved quantities along the characteristics of the Vlasov equation, namely L and the total energy

$$E = e^{\mu(r)}\sqrt{1+\omega^2+\frac{L}{r^2}} - I_q(r) \tag{2.10}$$

where

$$I_q(r) = q_0 \int_0^r e^{\mu(s)+\lambda(s)} \frac{q(s)}{s^2} ds \tag{2.11}$$

gives the electric contribution to the particle energy. We note that $I_q(r)$ is always positive, even for negative charges q_0, because $q(r) \propto q_0$, as can be seen in the definition of $q(r)$ in the Maxwell equation (2.9) combined with the definition of the charge distribution function $\rho_q(r)$ in (2.7), while the other factor is positive, so we always have $I_q(r) = q_0^2 C(r)$ with some positive function $C(r)$ that is independent of q_0. Therefore, any Ansatz $f(r,\omega,L) = \Phi(E,L)$ solves the Vlasov equation. For convenience we define

$$\epsilon := \sqrt{1+\omega^2+\frac{L}{r^2}} \tag{2.12}$$

.

and use in the following the polytropic Ansatz

$$f(r,\omega,L) = \Phi(E,L) = \alpha\left[1-\frac{E}{E_0}\right]_+^k [L-L_0]_+^l = \alpha\left[1-\frac{\epsilon e^{\mu}-I_q}{E_0}\right]_+^k [L-L_0]_+^l \tag{2.13}$$

where $\alpha, k \geq 0,\ l > -\frac{1}{2},\ k < 3l + \frac{7}{2},\ E_0 > 0$ and $L_0 \geq 0$ and

$$[x]_+ = \begin{cases} x & , \text{for } x \geq 0 \\ 0 & , \text{for } x < 0 \end{cases}$$

So, E_0 defines a cutoff energy in the sense that for $E > E_0$ we have $f = 0$ which means that for total energies higher than E_0 the particle distribution function and with it all the matter quantities in (2.4)-(2.7) vanish. As argued in [1], we can set $E_0 = 1$ by rescaling the time coordinate. L_0 is a cutoff radial momentum squared, in the sense that for $L < L_0$ we have $f = 0$, so for too small total angular momenta the particle distribution function and the matter quantities vanish again. With this Ansatz in (2.13) we can write the matter quantities (2.4), (2.5) and (2.7) as

$$\rho(r) = g(r,\mu(r),I_q(r)) \tag{2.14}$$

$$p(r) = h(r,\mu(r),I_q(r)) \tag{2.15}$$

$$\rho_q(r) = q_0 e^{\lambda(r)} k(r,\mu(r),I_q(r)) \tag{2.16}$$

with functions

$$g(r,\mu,I_q) = \alpha c_l r^{2l} \int_{\sqrt{1+\frac{L_0}{r^2}}}^{e^{-\mu}(E_0+I_q)} \left(1 - \frac{\epsilon e^{\mu} - I_q}{E_0}\right)^k \epsilon^2 \left(\epsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{l+\frac{1}{2}} d\epsilon \tag{2.17}$$

$$h(r,\mu,I_q) = \frac{\alpha c_l}{2l+3} r^{2l} \int_{\sqrt{1+\frac{L_0}{r^2}}}^{e^{-\mu}(E_0+I_q)} \left(1 - \frac{\epsilon e^{\mu} - I_q}{E_0}\right)^k \left(\epsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{l+\frac{3}{2}} d\epsilon \tag{2.18}$$

$$k(r,\mu,I_q) = \alpha c_l r^{2l} \int_{\sqrt{1+\frac{L_0}{r^2}}}^{e^{-\mu}(E_0+I_q)} \left(1 - \frac{\epsilon e^{\mu} - I_q}{E_0}\right)^k \epsilon \left(\epsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{l+\frac{1}{2}} d\epsilon \tag{2.19}$$

with

$$c_l = 2\pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds \tag{2.20}$$

.

Here we understand $g = h = k = 0$ if in the integrals in (2.17)-(2.19) the upper integral limits are smaller or equal the lower integral limits, that is, for

$$e^{-\mu} (E_0 + I_q) \leq \sqrt{1 + \frac{L_0}{r^2}} \tag{2.21}$$

or equivalently

$$\frac{\sqrt{1 + \frac{L_0}{r^2}} e^{\mu} - I_q}{E_0} \geq 1 \tag{2.22}$$

We can express this condition also as

$$0 \geq \ln(E_0 + I_q) - \frac{1}{2} \ln \left(1 + \frac{L_0}{r^2}\right) - \mu =: \gamma(r) \tag{2.23}$$

where we defined the new quantity $\gamma(r)$. If on the other hand, the upper integral limits are greater than the lower ones, the integrals in (2.17)-(2.19) and therefore also the matter quantities $\rho(r)$, $p(r)$ and $\rho_q(r)$ will be positive. This means that we have vacuum, in the sense that all the matter quantities are zero, at a radius r if and only if $\gamma(r) \leq 0$. We note further some properties of the three functions g, h and k in (2.17)-(2.19). At first, since for positive g and h we have in their definition in (2.17) and (2.18) that $\epsilon \geq \sqrt{1 + \frac{L_0}{r^2}} \geq 1$, we can conclude that

$$h \leq \frac{g}{2l+3} \tag{2.24}$$

Furthermore, we state the following lemma

Lemma 2.1. *The functions $g(r,\mu,I_q)$, $h(r,\mu,I_q)$ and $k(r,\mu,I_q)$ defined in (2.17)-(2.19) have the following properties:*

i. They are continuously differentiable

ii. They and all their partial derivatives are increasing in the first and third argument

iii. They and their partial derivatives with respect to the first and third argument are non-increasing in the second argument

iv. Their partial derivatives with respect to the second argument are non-decreasing in the second argument.

Lemma 2.1 is proven in [1] and we will need it later, in the course of the proof of the existence of solutions. In the considered system we get two independent, non-trivial Einstein equations, which read

$$e^{-2\lambda(r)}\left(2r\lambda'(r)-1\right)+1-r^2\Lambda=8\pi r^2\rho(r)+\frac{q^2(r)}{r^2} \tag{2.25}$$

$$e^{-2\lambda(r)}\left(2r\mu'(r)+1\right)-1+r^2\Lambda=8\pi r^2 p(r)-\frac{q^2(r)}{r^2} \tag{2.26}$$

We can solve the differential equation (2.25) for $u(r):=e^{-2\lambda(r)}$. It takes the form

$$-ru'(r)-u(r)=-1+8\pi r^2\rho(r)+\frac{q^2(r)}{r^2}+r^2\Lambda \tag{2.27}$$

This is an ordinary, inhomogeneous, linear differential equation, which we can solve by conventional methods and we find the general solution

$$u(r)=e^{-2\lambda(r)}=1-\frac{8\pi}{r}\int_0^r s^2\rho(s)ds-\frac{1}{r}\int_0^r\frac{q^2(s)}{s^2}ds-\frac{r^2\Lambda}{3}+\frac{C}{r} \tag{2.28}$$

with an integration constant C. Imposing a regular center with $\lambda(0)=0$ and $\frac{8\pi}{r}\int_0^r s^2\rho(s)ds=\frac{1}{r}\int_0^r\frac{q^2(s)}{s^2}ds=0$ we find

$$e^{-2\lambda}=1-\frac{8\pi}{r}\int_0^r s^2\rho(s)ds-\frac{1}{r}\int_0^r\frac{q^2(s)}{s^2}ds-\frac{r^2\Lambda}{3} \tag{2.29}$$

Now we can plug this in to the second Einstein equation (2.26) and, using the expressions (2.14) and (2.15) for the matter quantities in terms of g and h, we are left with the reduced Einstein equation

$$\mu'=\frac{4\pi\left(rh(r,\mu,I_q)+\frac{1}{r^2}\int_0^r s^2g(s,\mu,I_q)ds-\frac{q^2(r)}{8\pi r^3}+\frac{1}{8\pi r^2}\int_0^r\frac{q^2(s)}{s^2}ds-\frac{r\Lambda}{12\pi}\right)}{1-\frac{8\pi}{r}\int_0^r s^2g(s,\mu,I_q)ds-\frac{1}{r}\int_0^r\frac{q^2}{s^2}ds-\frac{r^2\Lambda}{3}} \tag{2.30}$$

where in $g(s,\mu,I_q)$ we understand that $\mu=\mu(s)$ and $I_q=I_q(s)$, so the second and third argument always depend on the first argument of g and the same holds for the functions h and k, which we will use later. All in all we have found now, that using the Ansatz (2.13) for the particle distribution function, equation (2.29) for $e^{-2\lambda}$ and solving equations (2.30) and (2.9) solves the whole EVMS. At the origin we impose the boundary conditions

$$q(0) = 0 \tag{2.31}$$

$$\mu(0) = \mu_C \tag{2.32}$$

for the system.

3. Local existence

In this chapter we will show that locally there exists a solution to the EVMS with positive cosmological constant. We consider the following theorem:

Theorem 3.1 (local existence)**.** *Let* $\mathring{r} \geq 0$ *and* $0 \leq I_q(\mathring{r}) < \infty$, $\infty > \mu(\mathring{r}) > -\infty$, $\infty > \lambda(\mathring{r}) \geq 0$ *and* $e^{-2\lambda(\mathring{r})} \geq a$ *with an arbitrary constant* $0 < a \leq 1$*. In case that* $\mathring{r} = 0$*, let* $\lambda(\mathring{r}) = I_q(\mathring{r}) = 0$*. Let* $\Lambda > 0$*. Then there exists a* $\delta > 0$ *such that a solution* (μ, λ, q) *to the system of Einstein* (2.30), (2.25) *and Maxwell equations* (2.9) *exists on the interval* $[\mathring{r}, \mathring{r} + \delta]$*. The solution is unique for the chosen polytropic Ansatz* (2.13) *of the particle distribution function* f*.*

Proof. The proof of the theorem can be done by a contraction argument similar to the one in [1], where it is done for the case $\Lambda = 0$. In this proof we will consider the two not integrated Einstein equations (2.26) and (2.29), instead of the solution (2.29) for $e^{-2\lambda(r)}$ and the resulting equation (2.30) for $\mu(r)$, to stay close to the proof in [1]. We consider the set

$$\mathcal{C} = \left\{(u, v, w) \in \left(C^0([\mathring{r}, \mathring{r} + \delta], \mathbb{R})\right)^2 \times C^1([\mathring{r}, \mathring{r} + \delta], \mathbb{R}) \,|\, \forall r \in [\mathring{r}, \mathring{r} + \delta] : (3.1) - (3.5)\right\}$$

with conditions:

$$(u, v, w)(\mathring{r}) = \left(\mathring{\mu}, e^{-2\mathring{\lambda}}, \mathring{I}\right) \tag{3.1}$$

$$0 \leq w'(\mathring{r}) \leq |q_0| \frac{e^{\mathring{\mu}-1}}{\sqrt{2}} \tag{3.2}$$

$$(u, v, w)(r) \in [\mathring{\mu} - 1, \mathring{\mu} + 1] \times \left[\frac{a}{2}, 2\right] \times \left[\mathring{I}, \mathring{I} + 1\right] \tag{3.3}$$

$$\frac{8\pi}{r} \int_{\mathring{r}}^{r} s^2 g(s, u, w) ds + \frac{1}{q_0^2 r} \int_{\mathring{r}}^{r} s^2 \left(w'(s)\right)^2 v(s) e^{-2u(s)} ds + \frac{r^2 \Lambda}{3} \leq \frac{2 - a}{4} \tag{3.4}$$

for all $r \in [\mathring{r}, \mathring{r} + \delta]$ and if $\mathring{r} = 0$ also

$$1 - v(r) \leq C r^2 \tag{3.5}$$

with a constant C which does not depend on δ. We also define a norm on this set:

$$\|(u, v, w)\|_{\mathcal{C}} = \sup_{r \in [\mathring{r}, \mathring{r}+\delta]} |u(r)| + \sup_{r \in [\mathring{r}, \mathring{r}+\delta]} |v(r)| + \sup_{r \in [\mathring{r}, \mathring{r}+\delta]} |w(r)| + \sup_{r \in [\mathring{r}, \mathring{r}+\delta]} \left|w'(r)\right| \tag{3.6}$$

Now we define the operator

$$\begin{array}{cccc} \mathcal{T}: & \mathcal{C} & \to & \left(C^0([\mathring{r},\mathring{r}+\delta],\mathbb{R})\right)^2 \times C^1([\mathring{r},\mathring{r}+\delta],\mathbb{R}) \\ & (u,v,w) & \mapsto & (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)[u,v,w] \end{array} \tag{3.7}$$

with components

$$\begin{aligned} \mathcal{T}_1[u,v,w](r) = \mathring{\mu} + \int_{\mathring{r}}^{r} \frac{1}{v(s)} \Bigg(& 4\pi s h(s,u,w) + \frac{1-v(s)}{2s} \\ & - \frac{\left(w'(s)\right)^2 s}{2q_0^2} v(s) e^{-2u(s)} - \frac{s\Lambda}{2} \Bigg) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \mathcal{T}_2[u,v,w](r) = e^{-2\mathring{\lambda}} & - \frac{8\pi}{r} \int_{\mathring{r}}^{r} s^2 g(s,u,w) ds \\ & - \frac{1}{q_0^2 r} \int_{\mathring{r}}^{r} s^2 \left(w'(s)\right)^2 v(s) e^{-2u(s)} ds - \frac{r^2\Lambda}{3} \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathcal{T}_3[u,v,w](r) = \mathring{I} + q_0 \int_{\mathring{r}}^{r} \frac{e^{u(s)}}{s^2\sqrt{v(s)}} \Bigg(& \mathring{q} \\ & + 4\pi q_0 \int_{\mathring{r}}^{s} \frac{\sigma^2}{\sqrt{v(s)}} k(\sigma,u,w) d\sigma \Bigg) ds \end{aligned} \tag{3.10}$$

where

$$\mathring{q} = \frac{w'(\mathring{r})}{q_0} \mathring{r}^2 e^{-(\mathring{\lambda}+\mathring{\mu})} \tag{3.11}$$

and we introduced the notation $\mathring{x} = x(\mathring{r})$. If we set $\Lambda = 0$ this brings us exactly to the operator defined in [1], which is proven to be a contraction. Furthermore, the set $\mathcal{C}$ is the same, with the only difference, that our condition (3.4) is a bit more strict, because we added a Λ-term. Since the definition of the third component $\mathcal{T}_3$ stays exactly the same, but we have the same (or one even more strict) conditions, $\mathcal{T}_3$ still fulfills

$$\begin{aligned} \sup_{r\in[\mathring{r},\mathring{r}+\delta]} \left|\mathcal{T}_3\left[u_1,v_1,w_1\right] - \mathcal{T}_3\left[u_2,v_2,w_2\right]\right| + \sup_{r\in[\mathring{r},\mathring{r}+\delta]} \left| \frac{d\mathcal{T}_3\left[u_1,v_1,w_1\right]}{dr} - \frac{d\mathcal{T}_3\left[u_2,v_2,w_2\right]}{dr} \right| \\ \leq C\delta \|(u_1-u_2),(v_1-v_2),(w_1-w_2)\|_{\mathcal{C}} \end{aligned} \tag{3.12}$$

In the definition of $\mathcal{T}_2$ we just added a Λ-term, which is however independent of the variables (u,v,w) and therefore just cancels out in the difference $\mathcal{T}_2(u_1,v_1,w_1) - \mathcal{T}_2(u_2,v_2,w_2)$, so we also still have

$$\sup_{r\in[\mathring{r},\mathring{r}+\delta]} |\mathcal{T}_2[u_1,v_1,w_1]-\mathcal{T}_2[u_2,v_2,w_2]| \leq C\delta\|(u_1-u_2),(v_1-v_2),(w_1-w_2)\|_{\mathcal{C}} \quad (3.13)$$

Only in $\mathcal{T}_1$ we have to work a bit, because the additional Λ-term here depends on v. We can do the estimate

$$\begin{aligned}
&\sup_{r\in[\mathring{r},\mathring{r}+\delta]} |\mathcal{T}_1[u_1,v_1,w_1]-\mathcal{T}_1[u_2,v_2,w_2]| \\
&\leq \\
&\sup_{r\in[\mathring{r},\mathring{r}+\delta]} \left| \int_{\mathring{r}}^{r} \left(\frac{1}{v_1(s)} \left(4\pi s h(s,u_1,w_1) + \frac{1-v_1(s)}{2s} - \frac{(w_1'(s))^2 s}{2q_0^2} v_1(s) e^{-2u_1(s)} \right) \right.\right. \\
&\left.\left. - \frac{1}{v_2(s)} \left(4\pi s h(s,u_2,w_2) + \frac{1-v_2(s)}{2s} - \frac{(w_2'(s))^2 s}{2q_0^2} v_2 e^{-2u_2(s)} \right) \right) ds \right| \\
&+ \sup_{r\in[\mathring{r},\mathring{r}+\delta]} \left| \int_{\mathring{r}}^{r} \left(\frac{1}{v_1(s)} - \frac{1}{v_2(s)} \right) \left(-\frac{s\Lambda}{2} \right) ds \right|
\end{aligned} \quad (3.14)$$

The first part corresponds to $\sup_{r\in[\mathring{r},\mathring{r}+\delta]} |\mathcal{T}_1[u_1,v_1,w_1]-\mathcal{T}_1[u_2,v_2,w_2]|$ in the case $\Lambda = 0$, which we already know is $\leq C\delta\|(u_1-u_2),(v_1-v_2),(w_1-w_2)\|_{\mathcal{C}}$. Thus, it remains to estimate the second part further:

$$\begin{aligned}
\sup_{r\in[\mathring{r},\mathring{r}+\delta]} \left| \int_{\mathring{r}}^{r} \left(\frac{1}{v_1(s)} - \frac{1}{v_2(s)} \right) \left(-\frac{s\Lambda}{2} \right) ds \right| &\leq \frac{(\mathring{r}+\delta)\Lambda}{2} \sup_{r\in[\mathring{r},\mathring{r}+\delta]} \int_{\mathring{r}}^{r} \left| \frac{1}{v_1(s)} - \frac{1}{v_2(s)} \right| ds \\
&= \frac{(\mathring{r}+\delta)\Lambda}{2} \int_{\mathring{r}}^{\mathring{r}+\delta} \frac{2}{a} \left| \frac{a}{2v_1(s)} - \frac{a}{2v_2(s)} \right| ds \\
&\leq \frac{(\mathring{r}+\delta)\Lambda}{2} \int_{\mathring{r}}^{\mathring{r}+\delta} \frac{2}{a} \left| \frac{2}{a} v_1(s) - \frac{2}{a} v_2(s) \right| ds \\
&\leq \frac{(\mathring{r}+\delta)\Lambda}{2} \frac{4}{a^2} \sup_{r\in[\mathring{r},\mathring{r}+\delta]} |v_1(r)-v_2(r)| \int_{\mathring{r}}^{\mathring{r}+\delta} ds \\
&= \frac{2}{a^2} \Lambda (\mathring{r}+\delta)\delta \sup_{r\in[\mathring{r},\mathring{r}+\delta]} |v_1(r)-v_2(r)| \\
&= (C\delta + C\delta^2) \sup_{r\in[\mathring{r},\mathring{r}+\delta]} |v_1(r)-v_2(r)| \\
&\leq C\left(\delta+\delta^2\right) \|(u_1-u_2),(v_1-v_2),(w_1-w_2)\|_{\mathcal{C}}
\end{aligned} \quad (3.15)$$

with a finite constant C independent of δ. In the second line of the latter calculation we used that the supremum of the integral is of course where the integration interval is maximal, since the integrand is strictly positive for all s. While in the third line, we used that v_1 and v_2 are bigger equal $\frac{a}{2}$ according to condition (3.3) and that for $x, y \geq 1$ always holds

$$\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} \leq y - x \tag{3.16}$$

With this estimate we finally obtain

$$\|\mathcal{T}\left[u_1, v_1, w_1\right] - \mathcal{T}\left[u_2, v_2, w_2\right]\|_{\mathcal{C}} \leq C\left(\delta + \delta^2\right) \|(u_1 - u_2), (v_1 - v_2), (w_1 - w_2)\|_{\mathcal{C}} \tag{3.17}$$

which makes $\mathcal{T}$ a contraction, since we can choose δ sufficiently small such that $C\left(\delta + \delta^2\right) < 1$. Then we use Banach's fixed point theorem which guarantees the existence of a unique fixed point for $\mathcal{T}$ and since $\mathcal{T}[\mu, e^{-2\lambda}, I_q] = (\mu, e^{-2\lambda}, I_q)$, this fixed point is precisely $(\mu, e^{-2\lambda}, I_q)$ solving the two not-yet combined Einstein equations (2.26) and (2.29), as well as the Maxwell equation (2.9), in the sense that I_q solves (2.11) with $q(r)$ solving (2.9). Therefore, the EVMS has a local solution on $[\mathring{r}, \mathring{r} + \delta]$.

□

4. Continuation criterion

Having shown that locally there exists a solution, we proceed now to show that the solution exists at least as long as the denominator of the Einstein equation (2.30) is positive. We state the following lemma:

Lemma 4.1 (Continuation criterion)**.** *Let R_C be the largest radius such that a, for the polytropic Ansatz* (2.13) *of f unique, local solution (μ, λ, q) to the EVMS exists on $[0, R_C)$, in the sense that μ solves the Einstein equation* (2.30)*, λ is given by* (2.29) *and q solves the Maxwell equation* (2.9) *with initial values $\mu(0) = \mu_C$, $q(0) = 0$. Let $\Lambda > 0$. Then there exists a radius $R_D \leq R_C$ where the denominator of equation* (2.30) *converges towards zero, meaning that*

$$\lim_{r \to R_D} \inf \left(1 - \frac{8\pi}{r} \int_0^r s^2 g(s, \mu, I_q) ds - \frac{1}{r} \int_0^r \frac{q^2(s)}{s^2} ds - \frac{r^2 \Lambda}{3} \right) = 0 \tag{4.1}$$

Proof. For simplicity we call this denominator in the following $\tilde{d}\,(r, \mu, I_q)$ or simply $d(r)$ because $\mu = \mu(r)$ and $I_q = I_q(r)$. Since we impose a regular center, per definition we have

$$d(0) = 1 > 0 \tag{4.2}$$

Furthermore, $d(r)$ is a continuous function, therefore, considering the case where $d(r) > 0$ does not hold on the whole interval $[0, R_C)$ it has to become zero at some point on the interval. So, what is left to proof, is that Lemma 4.1 also holds for $d(r) > 0$ on the whole interval. We do this by assuming the Lemma to be wrong and show that this leads to a contradiction. So, we assume

$$d(r) \geq a \tag{4.3}$$

for all $r \in [0, R_C)$, with an arbitrary positive constant $0 < a \leq 1$, prohibiting d to converge to zero. We show now that in this case the requirements of the local existence Lemma 3.1 are met for all $\mathring{r} \in [0, R_C)$ and that therefore, by choosing $\mathring{r}$ sufficiently close to R_C, we can continue the solution beyond R_C. From (4.3) we immediately see that $e^{-2\lambda} \geq a$ on the whole interval $[0, R_C)$, since $e^{-2\lambda(r)}$ corresponds precisely to the denominator $d(r)$. Per definition (2.29) we always have $e^{-2\lambda} \leq 1$, thus λ is positive and bounded for all $r \in [0, R_C)$ and it is left to show that also μ and I_q stay bounded on the whole interval. Let us start with μ. We can estimate its absolute value to

$$\begin{aligned}
|\mu(r)| &= \left|\mu_C + \int_0^r \mu'(s)ds\right| \\
&\leq |\mu_C| + \frac{4\pi}{d}\left(\int_0^r sh(s,\mu,I_q)ds + \int_0^r \frac{1}{s^2}\int_0^s \sigma^2 g(\sigma,\mu,I_q)d\sigma ds + \frac{r^2\Lambda}{24\pi}\right. \\
&\quad \left. + \frac{1}{8\pi}\left|\int_0^r \left(\frac{1}{s^2}\int_0^s \frac{q^2(\sigma)}{\sigma^2}d\sigma - \frac{q^2(s)}{s^3}\right)ds\right|\right) \\
&\leq |\mu_C| + \frac{4\pi}{a}\left(r\int_0^r h(s,\mu,I_q)ds + \int_0^r \frac{1}{s^2}\int_0^s \sigma^2 g(\sigma,\mu,I_q)d\sigma ds + \frac{r^2\Lambda}{24\pi}\right. \\
&\quad \left. + \frac{1}{8\pi}\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds\right)
\end{aligned} \tag{4.4}$$

Where we used in the first step the Einstein equation (2.30) along with the assumption $d \geq a$ while in the second step we estimated $s \leq r$ in the first integral and for the charge terms used the fact that

$$\frac{q^2(r)}{r^3} - \frac{1}{r^2}\int_0^r \frac{q^2(s)}{s^2}ds = \frac{d}{dr}\left(\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds\right) \tag{4.5}$$

Since $d \geq a$, neither $\frac{8\pi}{r}\int_0^r s^2 g(s,\mu,I_q)ds$ nor $\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds$ can diverge, because it would make d negative. Consequently, $g(r,\mu,I_q)$ has to stay finite on the whole interval $r \in [0, R_C)$ and due to inequality (2.24), also $h(r,\mu,I_q)$ stays finite. Because of the finite integral limits also the integral of h is finite so, keeping in mind that we consider a finite radius, all the terms in (4.4) are bounded and we obtain

$$|\mu(r)| \leq C \tag{4.6}$$

with a positive, finite constant C. For $I_q(r)$ we know that it is per definition positive, so bounded from below and since we assume $d \geq a$, it follows that $\frac{q^2(r)}{r^2} \leq C$. As argued by [1] this implies $I_q \leq C$. Consequently the requirements of the local existence Lemma 3.1 are met for all $\mathring{r} \in [0, R_C)$ and there exists a solution on $[\mathring{r}, \mathring{r}+\delta]$. Therefore, we can choose $\mathring{r}$ so close to R_C such that $\mathring{r}+\delta > R_C$ and thus extend the solution beyond R_C. However, this fails to coincide with the definition of R_C as maximal radius of existence and we conclude that the assumption (4.3) for the denominator of the Einstein equation was wrong and that instead $d(r)$ does converge to 0 at some point in $[0, R_C)$ and the Lemma is proven.

□

5. Existence beyond the non-vacuum region

In [1] it has been proven that for a sufficiently small charge parameter q_0 there exists a, for the specific Ansatz (2.13) of f, unique global solution (μ, λ, q) to the EVMS in the case of a vanishing cosmological constant with initial data $\mu(0) = \mu_C$ with a negative central value μ_C and $q(0) = 0$ and f is of bounded spatial support with boundary R_0. In the following we let for each quantity x of the system with $\Lambda = 0$ denote x_Λ the corresponding quantity for the EVMS in the case $\Lambda > 0$. In this chapter we use the solution established in [1] as a background solution and show that if we start with the same initial conditions, and stay close to this background solution, that is, for a sufficiently small cosmological constant Λ, also a solution to the EVMS with nonvanishing cosmological constant exists at least up to a radius $R_0 + \Delta R$, with some finite $\Delta R > 0$ and the corresponding particle density function f_Λ is again of bounded spatial support. We state this in the following proposition:

Proposition 5.1 (Existence beyond the non-vacuum region). *Let (μ, λ, q) be the unique global solution to the EVMS for $\Lambda = 0$, $\mu(0) = \mu_C$ with $-\infty < \mu_C < 0$ and $q(0) = 0$ where f is of bounded spatial support $[0, R_0)$ with $\gamma(R_0) = 0$. Let $(\mu_\Lambda, \lambda_\Lambda, q_\Lambda)$ be the unique local solution on $[0, \delta]$ to the EVMS for $\Lambda > 0$ and the same initial values $\mu_\Lambda(0) = \mu_C$ and $q_\Lambda(0) = 0$ as in the $\Lambda = 0$ case. Then the solution $(\mu_\Lambda, \lambda_\Lambda, q_\Lambda)$ exists at least on the interval $[0, R_0 + \Delta R]$, with some finite $\Delta R > 0$, and f_Λ is of bounded spatial support with boundary $R_{0\Lambda} < R_0 + \Delta R$ if*

$$\Lambda < \min\left\{\frac{1}{C_d(R_0+\Delta R)}\frac{1}{18} - \frac{q_0^2 C_d(R_0+\Delta R)}{C_d(0)}, \frac{|\gamma(R_0+\Delta R)|}{C_\gamma(R_0+\Delta R)} - \frac{q_0^2 C_\gamma(R_0+\Delta R)}{C_\gamma(0)}\right\} \tag{5.1}$$

where there exist functions $C_d(r)$ and $C_\gamma(r)$, which are positive, growing in r and independent of q_0 and Λ, such that

$$|d(r) - d_\Lambda(r)| \leq C_d(r)\left(\Lambda + q_0^2\right) \tag{5.2}$$

and

$$|\gamma_\Lambda(r) - \gamma(r)| \leq C_\gamma(r)\left(\Lambda + q_0^2\right) \tag{5.3}$$

Furthermore, we choose the absolute value of q_0 small enough such that

$$\frac{q_0^2 C_d(R_0+\Delta R)}{C_d(0)} < \frac{1}{18}\frac{1}{C_d(R_0+\Delta R)} \tag{5.4}$$

and

$$\frac{q_0^2 C_\gamma(R_0+\Delta R)}{C_\gamma(0)} < \frac{|\gamma(R_0+\Delta R)|}{C_\gamma(R_0+\Delta R)} \tag{5.5}$$

to ensure that the upper bound of Λ *is strictly positive.*

Proof. To proof proposition 5.1, we firstly proof the following Lemma:

Lemma 5.1. *Let* $R > 0$ *be such that* $|\gamma(R)| \neq 0$ *and let* $(\mu_\Lambda, \lambda_\Lambda, q_\Lambda)$, (μ, λ, q) *be solutions to the EVMS on* $[0, R]$ *with* $\Lambda > 0$ *and* $\Lambda = 0$ *respectively, with common initial values* $\mu(0) = \mu_\Lambda(0) = \mu_C$ *with* $0 > \mu_C > -\infty$ *and* $q(0) = q_\Lambda(0) = 0$*. Furthermore, let the denominator in the differential equation* (2.30) *for* μ *in the case of vanishing cosmological constant be* $d(r) \geq a > \frac{1}{18}$ *and in the case* $\Lambda > 0$ *let* $d_\Lambda(r) \geq a_\Lambda := a - \frac{1}{18} > 0$ *and let* $|\gamma_\Lambda(r) - \gamma(r)| \leq |\gamma(R)|$ *for all* $r \in [0, R]$*. Then*

$$|g(r,\mu_\Lambda,I_\Lambda) - g(r,\mu,I)| + |h(r,\mu_\Lambda,I_\Lambda) - h(r,\mu,I)| \leq \left(q_0^2 + \Lambda\right) C_{gh}(r) \tag{5.6}$$

where $C_{gh}(r)$ *is increasing in* r *and independent of* q_0 *and* Λ.

Proof. For convenience we introduce the following notation $g(r, \mu_\Lambda, I_\Lambda) =: g_\Lambda$ and $g(r, \mu, I) =: g$ and estimate

$$|g_\Lambda - g| \leq |\underbrace{g(r,\mu_\Lambda,I_\Lambda) - g(r,\mu,I_\Lambda)}_{=:\Delta_\mu g}| + |\underbrace{g(r,\mu,I_\Lambda) - g(r.\mu,I)}_{\Delta_I g}| \tag{5.7}$$

$\Delta_I g$ can be estimated further by

$$\begin{aligned} |\Delta_I g| &\leq \underbrace{\sup_{J\in[I,I_\Lambda]} \left|\frac{\partial g(r,\mu,J)}{\partial J}\right|}_{=:C(r)} \underbrace{|I_\Lambda(r) - I(r)|}_{\leq q_0^2 C(r)} \\ &\leq q_0^2 C(r) \end{aligned} \tag{5.8}$$

with some positive function $C(r)$ which is growing in r. Here we used the fact that the supremum can be written as such a function $C(r)$, since $g \in \mathcal{C}^1$, as stated under the first point in lemma 2.1, and under the assumptions of Lemma 5.1, more precisely for positive denominators of the Einstein equation, both I and I_Λ are bounded by some $q_0^2 C(r)$ with a positive, continuous function $C(r)$ which is growing in r and independent of q_0, which also makes $[I, I_\Lambda]$ a finite interval. This bound has been shown in [1] for I, just that in the present thesis we also pulled out the second q_0-factor to make $C(r)$ always positive and independent of q_0. Even if this bound has only been shown for $I(r)$, the same argumentation still works for I_Λ, because it is defined in the same way as I, just with the charge distribution corresponding to the solution with $\Lambda > 0$. The first $C(r)$, so the one with which we defined the supremum, is also growing in r because of the second property in lemma 2.1, which implies that the partial derivative of $g(r,\mu,J)$ with respect to J is an increasing function of r. Consequently also the final $C(r)$ in the second line is increasing and positive. For the first term in (5.7) we find

$$\begin{aligned}|\Delta_\mu g| &\leq \sup_{u\in[\mu,\mu_\Lambda]}\left|\frac{\partial g(r,u,I_\Lambda)}{\partial u}\right||\mu_\Lambda-\mu| \\ &=: C(r)|\mu_\Lambda-\mu|\end{aligned} \tag{5.9}$$

where due to the condition $|\gamma_\Lambda-\gamma|\leq|\gamma(R)|$ and since both, I_Λ and I are bounded also $|\mu_\Lambda-\mu|$ is bounded, so $[\mu,\mu_\Lambda]$ is a finite interval and due to the first and the last property of g in lemma 2.1, we can write the supremum again as some continuous function $C(r)$, growing in r. For the absolute value of the difference between μ_Λ and μ we know that it is bounded, but we want to understand better the dependence on q_0, Λ and r, so we define the charge terms in the reduced Einstein equation (2.30) as a new function

$$u(r) := \frac{1}{d(r)}\left(-\frac{q^2(r)}{2r^3}+\frac{1}{2r^2}\int_0^r\frac{q^2(s)}{s^2}ds\right) \tag{5.10}$$

and equivalently $u_\Lambda(r)$ with $q_\Lambda(r)$ instead of $q(r)$ and $d_\Lambda(r)$ instead of $d(r)$. With this we do the following estimate, using the Einstein equation (2.30) for $\Lambda=0$ and $\Lambda>0$ respectively

$$\begin{aligned}|\mu_\Lambda-\mu| &= \left|\int_0^r \mu_\Lambda'(s)-\mu'(s)ds\right| \\ &= \left|\int_0^r \mu_\Lambda'(s)-u_\Lambda(s)+u_\Lambda(s)-\left(\mu'(s)-u(s)+u(s)\right)ds\right| \\ &\leq \int_0^r \left|\mu_\Lambda'(s)-u_\Lambda(s)-\left(\mu'(s)-u(s)\right)\right|ds+\left|\int_0^r u_\Lambda(s)ds\right|+\left|\int_0^r u(s)ds\right| \\ &\leq \int_0^r\frac{4\pi}{d_\Lambda(s)}\left(\left|\frac{s\Lambda}{12\pi}\right|+s|h(s,\mu_\Lambda,I_\Lambda)-h(s,\mu,I)|\right. \\ &\quad\left.+\frac{1}{s^2}\int_0^s\sigma^2|g(s,\mu_\Lambda,I_\Lambda)-g(s,\mu,I)|d\sigma\right)ds \\ &\quad+\int_0^r\left|\frac{4\pi}{d_\Lambda(s)}-\frac{4\pi}{d(s)}\right|\left(sh(s,\mu,I)+\frac{1}{s^2}\int_0^s\sigma^2 g(\sigma,\mu,I)d\sigma\right)ds \\ &\quad+\left|\int_0^r\frac{1}{2d_\Lambda(s)}\left(-\frac{q_\Lambda^2}{s^3}+\frac{1}{s^2}\int_0^s\frac{q_\Lambda^2(\sigma)}{\sigma^2}d\sigma\right)ds\right| \\ &\quad+\left|\int_0^r\frac{1}{2d(s)}\left(-\frac{q^2}{s^3}+\frac{1}{s^2}\int_0^s\frac{q^2(\sigma)}{\sigma^2}d\sigma\right)ds\right|\end{aligned} \tag{5.11}$$

where we subtracted and added the terms $\frac{4\pi}{d_\Lambda(s)}\left(sh(s,\mu,I)+\frac{1}{s^2}\int_0^s\sigma^2g(\sigma,\mu,I)d\sigma\right)$ inside of the absolute value under the first integral in the third line. We do further estimates for the terms of the estimation above. It is assumed that $d \geq a$ and, as we noted before, in equation (4.5), the charge terms without the $\frac{1}{2d(s)}$-factor that appear inside of the integral of the latter estimate are the derivative of $\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds$. Thus, we can estimate

$$\begin{aligned} \left|\int_0^r \frac{1}{2d(s)}\left(-\frac{q^2}{s^3}+\frac{1}{s^2}\int_0^s \frac{q^2(\sigma)}{\sigma^2}d\sigma\right)ds\right| &\leq \frac{1}{2a}\left|\int_0^r \frac{d}{ds}\left(\frac{1}{s}\int_0^s \frac{q^2(\sigma)}{\sigma^2}d\sigma\right)ds\right| \\ &= \frac{1}{2a}\left|\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds\right| \\ &= \frac{1}{2a}\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds \end{aligned} \tag{5.12}$$

where in the last step we dropped the absolute value, because the term is anyway positive. As argued before, during the proof of the continuation criterion, for $d \geq a > 0$ we have

$$\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds \leq C \tag{5.13}$$

we can also pull out the q_0-factor because $q(r) \propto q_0$ as we recognized earlier, and write

$$\frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds \leq q_0^2 C \tag{5.14}$$

with a finite positive constant C that is independent of q_0 now and it follows that

$$\left|\int_0^r \frac{1}{2d(s)}\left(-\frac{q^2}{s^3}+\frac{1}{s^2}\int_0^s \frac{q^2(\sigma)}{\sigma^2}d\sigma\right)ds\right| \leq \frac{q_0^2 C}{2a} = q_0^2 C \tag{5.15}$$

Since we also have $d_\Lambda \geq a_\Lambda > 0$, we can do the same estimate for the charge terms of the solution with $\Lambda > 0$ and also obtain

$$\left|\int_0^r \frac{1}{2d_\Lambda(s)}\left(-\frac{q_\Lambda^2}{s^3}+\frac{1}{s^2}\int_0^s \frac{q_\Lambda^2(\sigma)}{\sigma^2}d\sigma\right)ds\right| \leq q_0^2 C \tag{5.16}$$

Furthermore, we observe

$$\begin{aligned} \int_0^r \frac{1}{s^2}\int_0^s \sigma^2\left|g_\Lambda(\sigma)-g(\sigma)\right|d\sigma ds &\leq \int_0^r \frac{s^2}{s^2}\int_0^s \left|g_\Lambda(\sigma)-g(\sigma)\right|d\sigma ds \\ &= \int_0^r\int_0^r \left|g_\Lambda(\sigma)-g(\sigma)\right|d\sigma ds \\ &= r\int_0^r \left|g_\Lambda(\sigma)-g(\sigma)\right|d\sigma \end{aligned} \tag{5.17}$$

Due to $d \geq a$ and $d_\Lambda \geq a_\Lambda$ we find

$$
\begin{aligned}
\left|\frac{1}{d_\Lambda}-\frac{1}{d}\right| &= \frac{|d-d_\Lambda|}{d_\Lambda d} \le \frac{|d-d_\Lambda|}{a_\Lambda a} \\
&\le \frac{1}{a_\Lambda a}\left(\left|\frac{r^2\Lambda}{3}\right| + \frac{8\pi}{s}\int_0^s \sigma^2\,|g_\Lambda(\sigma)-g(\sigma)|\,d\sigma + \underbrace{\left|\frac{1}{s}\int_0^s \frac{q^2(\sigma)}{\sigma^2}d\sigma\right|}_{\le q_0^2 C}\right. \\
&\quad \left. + \underbrace{\left|\frac{1}{s}\int_0^s \frac{q_\Lambda^2(\sigma)}{\sigma^2}d\sigma\right|}_{\le q_0^2 C}\right)
\end{aligned}
\tag{5.18}
$$

where in the last step we plugged in the explicit expressions for the denominators of the Einstein equation (2.30) for $\Lambda = 0$ and $\Lambda > 0$ respectively. Since we know how the functions g and h behave with respect to each argument (see Lemma 2.1), especially that they are growing in r and I_q and that they are non-increasing in μ, we can also deduce

$$
\begin{aligned}
g(s,\mu,I) &\le g\left(\max_{s\in[0,r]} s,\ \min_{s\in[0,r]} \mu(s),\ \max_{s\in[0,r]} I_q(s)\right) \\
&= g(r,\mu(0),I_q(r)) =: C(r)
\end{aligned}
\tag{5.19}
$$

with an increasing function $C(r)$. In the second line we used the fact that $\mu(s)$ and $I_q(s)$ are growing functions of s. The same estimate holds for $h(s,\mu,I_q)$. Plugging in those estimates in (5.11), we obtain

$$
\begin{aligned}
|\mu_\Lambda-\mu| &\le \Lambda C(r) + q_0^2 C(r) + r\int_0^r \left(|g_\Lambda-g|+|h_\Lambda-h|\right)ds + C(r)\int_0^r |g_\Lambda-g|\,d\sigma \\
&\le \left(\Lambda+q_0^2\right)C(r) + C(r)\int_0^r \left(|g_\Lambda-g|+|h_\Lambda-h|\right)ds
\end{aligned}
\tag{5.20}
$$

where all functions $C(r)$ are positive and increasing in r. With this we also have an estimate of $\Delta_\mu g$ and, using this along with the result for the estimate of $\Delta_I g$, we end up with the estimate

$$
|g_\Lambda-g| \le \left(q_0^2+\Lambda\right)C(r) + C(r)\int_0^r \left(|g_\Lambda-g|+|h_\Lambda-h|\right)ds \tag{5.21}
$$

where the first function $C(r)$ is positive and growing in r and the second function $C(r)$ is besides positive and growing in r also continuous. We can do the exact same steps for $|h_\Lambda-h|$ and therefore conclude

$$|g_\Lambda - g| + |h_\Lambda - h| \leq \left(q_0^2 + \Lambda\right) C(r) + C(r) \int_0^r \left(|g_\Lambda - g| + |h_\Lambda - h|\right) ds \tag{5.22}$$

Now we can use a form of Grönwall's inequality which leads to the inequality

$$|g_\Lambda - g| + |h_\Lambda - h| \leq \left(q_0^2 + \Lambda\right) C(r) \exp \int_0^r C(r) ds = \left(q_0^2 + \Lambda\right) C_{gh}(r) \tag{5.23}$$

with $C_{gh}(r) > 0$, increasing in r and independent of q_0 and Λ and Lemma 5.1 is proven. □

With Lemma 5.1 in mind, we can proceed to proof Proposition 5.1. We do this by noting that up to a certain radius, the conditions of the Lemma are fulfilled and then we show that for sufficiently small $|q_0|$ and Λ this radius is bigger equal $R_0 + \Delta R$. We can plug the last estimate which we had for $|g_\Lambda - g| + |h_\Lambda - h|$, so equation (5.23), back in the estimate for $|\mu_\Lambda - \mu|$ in equation (5.20) and find that under the conditions of Lemma 5.1

$$|\mu_\Lambda - \mu| \leq \left(\Lambda + q_0^2\right) C_\mu(r) \tag{5.24}$$

with a positive, increasing function $C_\mu(r)$ that is independent of q_0 and Λ. Furthermore we note that for any $x \geq 0$

$$e^x = 1 + x + \frac{x^2}{2} + ... \geq 1 + x \tag{5.25}$$

Applying an ln to this, we have $x \geq \ln(1 + x)$ and it follows for the absolute value of the difference between the gamma functions (where $\gamma(r)$ is defined in (2.23) and $\gamma_\Lambda(r)$ is the equivalent quantity in the system with nonvanishing $\Lambda > 0$)

$$\begin{aligned} |\gamma_\Lambda - \gamma| &\leq |\mu_\Lambda - \mu| + \ln(I_\Lambda + 1) + \ln(I + 1) \\ &\leq \left(\Lambda + q_0^2\right) C_\mu(r) + I_\Lambda + I \\ &\leq \left(\Lambda + q_0^2\right) C_\mu(r) + q_0^2 C(r) \\ &\leq \left(\Lambda + q_0^2\right) C_\gamma(r) \end{aligned} \tag{5.26}$$

where we defined a new function $C_\gamma(r) := C_\mu(r) + C(r)$ that has the same properties as $C_\mu(r)$ and $C(r)$ meaning that it is positive, growing in r and independent of Λ and q_0^2. Now we define two radii, namely once

$$r^* := \inf\{r \in [0, R) | d_\Lambda(r) = a_\Lambda\} \tag{5.27}$$

which is the smallest radius where $d_\Lambda(r) = a_\Lambda$, where a_Λ is defined as in Lemma 5.1 as a constant $a_\Lambda = a - \frac{1}{18}$ and a is again chosen to be bigger than $\frac{1}{18}$ such that $a_\Lambda > 0$. This radius r^* is for sure bigger than 0 since we are starting with $d_\Lambda(0) = 1$ and $d(r)$ is a continuous function. Secondly, we define the radius

$$\tilde{r} := \inf\{r \in [0,R) | \, |\gamma_\Lambda(r) - \gamma(r)| > |\gamma(R_0 + \Delta R)|\} \tag{5.28}$$

which is the smallest radius where $|\gamma_\Lambda(r) - \gamma(r)| > |\gamma(R_0 + \Delta R)|$. This radius is also for sure bigger than zero since we are starting with $I_\Lambda(0) = I(0) = 0$ and $\mu_\Lambda(0) = \mu(0) = \mu_C < 0$. Thus, we also get $\gamma_\Lambda(0) = \gamma(0)$ and therefore, their difference is zero at the beginning. We further define

$$\tilde{r}^* := \min\{r^*, \tilde{r}\} \tag{5.29}$$

such that for $r \leq \tilde{r}^*$ we have $d_\Lambda > a_\Lambda$ and $|\gamma_\Lambda(r) - \gamma(r)| \leq |\gamma(R_0 + \Delta R)|$. These inequalities along with $d \geq a > 0$ are precisely the conditions for Lemma 5.1. Due to how the background solution is constructed, namely using a nontrivial Ansatz for the particle distribution function up to the boundary of the support of the matter functions, which is at a radius up to which the denominator of the Einstein equation is for sure positive, and then continuing the particle distribution function with the constant zero function, also $d \geq a > 0$ is guaranteed.

Thus, the requirements for the Lemma are met for all $r \leq \tilde{r}^*$. Therefore within this interval $[0, \tilde{r}^*]$ we can estimate

$$\begin{aligned} |d_\Lambda(r) - d(r)| &\leq \frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2|g_\Lambda - g|ds + \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds + \frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}ds \\ &\leq \left(q_0^2 + \Lambda\right) C_d(r) \end{aligned} \tag{5.30}$$

with the positive, increasing function $C_d(r)$ that is independent of q_0 and Λ.

In the next step we show that $\tilde{r}^* > R_0 + \Delta R$, by assuming the opposite and coming across a contradiction. So, considering $\tilde{r}^* \leq R_0 + \Delta R$, for all $r \in [0, \tilde{r}^*]$ we have $r \leq R_0 + \Delta R$ and thus

$$\begin{aligned} d_\Lambda(r) &\geq d(r) - q_0^2 C_d(r) - \Lambda C_d(r) \\ &> d(r) - q_0^2 C_d(r) - \frac{1}{18}\frac{C_d(r)}{C_d(R_0 + \Delta R)} + \frac{C_d(r)}{C_d(0)} q_0^2 C_d(R_0 + \Delta R) \\ &\geq a - q_0^2 C_d(R_0 + \Delta R) - \frac{1}{18} + q_0^2 C_d(R_0 + \Delta R) \\ &= a - \frac{1}{18} = a_\Lambda \end{aligned} \tag{5.31}$$

where in the second line we used one of the first upper bound for Λ in (5.1) and in the third line the fact that $C_d(r)$ is growing in r and consequently $\frac{C_d(r)}{C_d(R_0+\Delta R)} \leq 1$ and $\frac{C_d(r)}{C_d(0)} \geq 1$. Due to this estimate, we know that $d_\Lambda(r)$ is strictly bigger than a_Λ for all $\tilde{r}^*$ which means that in the interval $[0, \tilde{r}^*]$ the $d_\Lambda(r)$ never falls down to the value a_Λ and consequently, recalling the definition (5.27) of r^*, it follows $r^* > \tilde{r}^*$. Looking at the absolute value of the difference between γ_Λ and γ we found that under the conditions of

Lemma 5.1 and thus for all $r \in [0, \tilde{r}^*]$ the estimate (5.26) holds. With the second upper bound for Λ in (5.1) and the fact that $C_\gamma(r)$ is a growing function of r, we obtain for all $r \leq \tilde{r}^*$

$$\begin{aligned} |\gamma_\Lambda(r) - \gamma(r)| &\leq q_0^2 C_\gamma(r) + \Lambda C_\gamma(r) \\ &< q_0^2 C_\gamma(r) + |\gamma(R_0 + \Delta R)| \underbrace{\frac{C_\gamma(r)}{C_\gamma(R_0 + \Delta R)}}_{\leq 1} - q_0^2 C_\gamma(R_0 + \Delta R) \underbrace{\frac{C_\gamma(r)}{C_\gamma(0)}}_{\geq 1} \\ &\leq q_0^2 C_\gamma(R_0 + \Delta R) + |\gamma(R_0 + \Delta R)| - q_0^2 C_\gamma(R_0 + \Delta R) \\ &= |\gamma(R_0 + \Delta R)| \end{aligned} \tag{5.32}$$

This inequality tells us that for all $r \in [0, \tilde{r}^*]$ we have $|\gamma_\Lambda - \gamma|$ strictly lower than $|\gamma(R_0 + \Delta R)|$ and therefore $\tilde{r} > \tilde{r}^*$. Thus, we have both r^* and $\tilde{r}$ strictly bigger than $\tilde{r}^*$ which contradicts the definition of $\tilde{r}^*$ and we can conclude that the assumption $\tilde{r}^* \leq R_0 + \Delta R$ was wrong. Knowing that a solution exists at least as long as $d_\Lambda(r) > 0$, which we showed in the continuation criterion 4.1, we thus showed that there exists a solution at least up to $R_0 + \Delta R$. We also showed that at least up to that radius we have (5.32) fulfilled. This also implies that

$$|\gamma_\Lambda(R_0 + \Delta R) - \gamma(R_0 + \Delta R)| \leq |\gamma(R_0 + \Delta R)| \tag{5.33}$$

which means that $\gamma_\Lambda(R_0 + \Delta R)$ has the same sign as $\gamma(R_0 + \Delta R)$. We know that the support of the background solution is bounded by R_0, therefore, at $R_0 + \Delta R$ we have vacuum, i.e. $\gamma(R_0 + \Delta R) \leq 0$ and consequently also

$$\gamma_\Lambda(R_0 + \Delta R) \leq 0 \tag{5.34}$$

and we can conclude that at $R_0 + \Delta R$ there is vacuum also for the solution with positive cosmological constant and that thus the support of the matter quantities of these solutions is bounded by some $R_{0\Lambda} \leq R_0 + \Delta R$.

□

6. Global regular solutions with charge and $\Lambda > 0$

In the previous chapter we saw, that under suitable conditions there exists a solution to the EVMS with positive cosmological constant on $[0, R_0 + \Delta R]$ and the particle distribution function is of bounded support. Beyond the support we can glue a correctly shifted Reissner-Nordström-de-Sitter metric to this solution, which allows us to construct a global static solution to the EVMS. The Reissner-Nordström-de-Sitter metric

$$-\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2\Lambda}{3}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2\Lambda}{3}\right)^{-1} dr^2 + r^2 \left(d\Theta^2 + \sin^2 \Theta d\varphi^2\right) \tag{6.1}$$

is a solution to the vacuum Einstein equations. We formulate the above statement in the following theorem

Theorem 6.1 (Global existence). *Let f_Λ be of the form* (2.13) *and $\rho_\Lambda(r)$, $p_\Lambda(r)$ and $\rho_{q,\Lambda}(r)$ of* (2.4), (2.5) *and* (2.7), *respectively. Then for every central value - $\infty < \mu_C < 0$ there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every $0 < |q_0| \leq C_1$ and every $0 < \Lambda \leq C_2$ there exists a unique global solution $(\mu_\Lambda(r), \lambda_\Lambda(r), q_\Lambda(r))$ to the static, spherical symmetric EVMS, which is defined by equations* (2.8), (2.9), (2.25) *and* (2.26), *with initial data $\mu_\Lambda(0) = \mu_C$ and $\lambda_\Lambda(0) = q_\Lambda(0) = 0$. The solution coincides with a shifted Reissner-Nordström-de-Sitter metric in the vacuum region.*

Proof. According to Proposition 5.1 for any $-\infty < \mu_C < 0$ there exists a solution at least up to $R_0 + \Delta R$ for any $0 < \Delta R < \infty$ where the matter quantities are bounded on $[0, R_{0\Lambda}]$ if Λ and $|q_0|$ are sufficiently small and where the background solution is supported on R_0. Since the particle distribution function vanishes in the vacuum region we drop the original Ansatz (2.13) at $R_{0\Lambda}$ and instead continue with the constant zero function $f_\Lambda = 0$. This can be done in a continuous way and such that the new function

$$f_\Lambda^{global} = \begin{cases} \alpha \left[1 - \frac{\epsilon e^{\mu} - I_q}{E_0}\right]_+^k [L - L_0]_+^l, & \text{for } r \in [0, R_{0\Lambda}] \\ 0 & \text{for } r \in [0, r_C] \end{cases} \tag{6.2}$$

has the same regularity as the original Ansatz (2.13). The radius r_C is the cosmological horizon until which we construct the solution. We call the solution then global in the sense that it exists until r_C. With this new function, from $R_{0\Lambda}$ on, the matter quantities are constantly zero, while the charge remains constantly equal the total charge Q of the whole particle ensemble. The Einstein equation (2.30) in this vacuum region takes the form

$$\begin{aligned}\mu'_\Lambda(r) = 4\pi \left(1 - \frac{8\pi}{r}\int_0^{R_{0\Lambda}} s^2 g_\Lambda(s)ds - \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3}\right)^{-1} \\ \left(\frac{1}{R_{0\Lambda}^2}\int_0^{R_{0\Lambda}} s^2 g_\Lambda(s)ds - \frac{Q^2}{8\pi r^3} + \frac{1}{8\pi r^2}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r\Lambda}{12\pi}\right)\end{aligned} \tag{6.3}$$

We note that

$$4\pi \int_0^{R_{0\Lambda}} s^2 g_\Lambda(s)ds = M \tag{6.4}$$

is the total mass and that the second line in the vacuum Einstein equation (6.3) corresponds to the derivative of the inverse of the first line without the 4π factor and divided by 8π. Thus, we can write

$$\begin{aligned}\mu'_\Lambda(r) &= 4\pi \left(1 - \frac{2M}{r} - \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3}\right)^{-1} \cdot \frac{1}{8\pi}\frac{d}{dr}\left(1 - \frac{2M}{r} - \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3}\right) \\ &= \frac{1}{2}\frac{d}{dr}\ln\left(1 - \frac{2M}{r} - \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3}\right) \\ &= -\frac{d}{dr}\lambda_\Lambda(r)\end{aligned} \tag{6.5}$$

where we noted the relation to λ_Λ which we obtained by inspecting equation (2.29) for $e^{-2\lambda_\Lambda}$ in the vacuum. This means that in the vacuum region the general solution for $\mu_\Lambda(r)$ corresponds to the general solution for $\lambda_\Lambda(r)$, given in (2.28), with a global negative sign in front. So, we have

$$e^{2\mu_\Lambda} = e^{-2\lambda_\Lambda} = 1 - \frac{8\pi}{r}\int_0^r s^2\rho_\Lambda(s)ds - \frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3} + \frac{K}{r} \tag{6.6}$$

We note that, since this is valid only in the vacuum region, we can write

$$\frac{8\pi}{r}\int_0^r s^2\rho_\Lambda(s)ds = \frac{8\pi}{r}\int_0^{R_{0\Lambda}} s^2\rho_\Lambda(s)ds = \frac{2M}{r} \tag{6.7}$$

and

$$
\begin{aligned}
\frac{1}{r}\int_0^r \frac{q_\Lambda^2(s)}{s^2}ds &= \frac{1}{r}\left(\underbrace{\int_0^{R_{0\Lambda}} \frac{q_\Lambda^2(s)}{s^2}ds}_{=const.} + \int_{R_{0\Lambda}}^r \frac{q_\Lambda^2(s)}{s^2}ds\right) \\
&= \frac{1}{r}\left(C + Q^2\int_{R_{0\Lambda}}^r \frac{1}{s^2}ds\right) \\
&= \frac{1}{r}\left(C + Q^2\left(-\frac{1}{r}+\frac{1}{R_{0\Lambda}}\right)\right) \\
&= \frac{C}{r} - \frac{Q^2}{r^2}
\end{aligned}
\tag{6.8}
$$

Since C is a constant, we can just absorb it in the integration constant K and end up with

$$
e^{2\mu_\Lambda} = e^{-2\lambda_\Lambda} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2\Lambda}{3} + \frac{K}{r} \tag{6.9}
$$

which corresponds precisely to the components of a shifted Nordström-de-Sitter metric. We can determine the integration constant C in such a way to match this vacuum solution to our non-vacuum solution at the boundary of the support $R_{0\Lambda}$, meaning that we demand that as the non-vacuum solution approaches $R_{0\Lambda}$ from below has to give the same value than if the vacuum solution approaches $R_{0\Lambda}$ from above. This limit for the vacuum solution is

$$
\lim_{r\searrow R_{0\Lambda}}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2\Lambda}{3} + \frac{K}{r}\right) = 1 - \frac{R_{0\Lambda}^2\Lambda}{3} - \frac{2M}{R_{0\Lambda}} + \frac{Q^2}{R_{0\Lambda}^2} + \frac{K}{R_{0\Lambda}} \tag{6.10}
$$

So, the required boundary conditions read

$$
\lim_{r\nearrow R_{0\Lambda}} e^{2\mu_\Lambda(r)} = 1 - \frac{R_{0\Lambda}^2\Lambda}{3} - \frac{2M}{R_{0\Lambda}} + \frac{Q^2}{R_{0\Lambda}^2} + \frac{K}{R_{0\Lambda}} \tag{6.11}
$$

and

$$
\lim_{r\nearrow R_{0\Lambda}} e^{-2\lambda_\Lambda(r)} = 1 - \frac{R_{0\Lambda}^2\Lambda}{3} - \frac{2M}{R_{0\Lambda}} + \frac{Q^2}{R_{0\Lambda}^2} + \frac{K}{R_{0\Lambda}} \tag{6.12}
$$

Since we know that the non-vacuum solution for $e^{-2\lambda_\Lambda(r)}$ and initial condition $\lambda_\Lambda(0) = 0$ is given by (2.29), we find

$$
1 - \underbrace{\frac{8\pi}{R_{0\Lambda}}\int_0^{R_{0\Lambda}} s^2\rho_\Lambda(s)ds}_{=\frac{2M}{R_{0\Lambda}}} - \frac{1}{R_{0\Lambda}}\int_0^{R_{0\Lambda}} \frac{q_\Lambda^2(s)}{s^2}ds - \frac{R_{0\Lambda}^2\Lambda}{3} = 1 - \frac{R_{0\Lambda}^2\Lambda}{3} - \frac{2M}{R_{0\Lambda}} + \frac{Q^2}{R_{0\Lambda}^2} + \frac{K}{R_{0\Lambda}} \tag{6.13}
$$

Solving for K gives

$$K = -\int_0^{R_{0\Lambda}} \frac{q_\Lambda^2(s)}{s^2} ds - \frac{Q^2}{R_{0\Lambda}} \tag{6.14}$$

So, at $R_{0\Lambda}$ we can continue our non-vacuum solution in a continuous way by

$$e^{2\mu_\Lambda} = e^{-2\lambda_\Lambda} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2\Lambda}{3} - \frac{1}{r}\int_0^{R_{0\Lambda}} \frac{q_\Lambda^2(s)}{s^2} ds - \frac{Q^2}{rR_{0\Lambda}} \tag{6.15}$$

□

7. Negative cosmological constant

In case of a negative cosmological constant $\Lambda < 0$ all the above results are still true as long as the absolute value of Λ remains sufficiently small. The proofs for $\Lambda > 0$ can still be applied with some slight changes, mostly just taking the absolute value of Λ instead of Λ. In the following paragraphs we will discuss in detail what will be different when changing the sign of the cosmological constant.

In the local existence theorem $\lambda(\mathring{r}) \geq 0$ is now not required anymore, because due to the negative sign of the cosmological constant $e^{-2\lambda(r)} > 1$ is possible, as can be seen by inspecting the formula (2.29) for $e^{-2\lambda}$. Thus, it is also allowed to start from a negative $\lambda(\mathring{r}) > -\infty$. In the proof of local existence we then change the fourth condition (3.4) to

$$\frac{8\pi}{r}\int_{\mathring{r}}^{r} s^2 g_\Lambda(s,u,w)ds + \frac{1}{q_0^2 r}\int_{\mathring{r}}^{r} s^2 \left(w'(s)\right)^2 v(s)e^{-2u(s)}ds \leq \frac{2-a}{4} \tag{7.1}$$

i.e. we drop the Λ-term of the condition, which would give a negative contribution for $\Lambda < 0$ and would therefore relax the condition. However, in the paper [1] where the proof is done for vanishing cosmological constant, the fourth condition also reads as in (7.1) and since we strongly rely on this proof, also in case $\Lambda < 0$ condition (7.1) is sufficient to show the local existence. The definition of the operator $\mathcal{T}$ (see equations (3.7) to (3.11)) remains the same and showing that it is a contraction with respect to the defined norm (3.6) still works in the same way as we did for positive Λ, because in the norm only absolute values matter. So, we do not care about the sign of Λ, we just need to replace Λ with $|\Lambda|$ in inequality (3.15).

The only thing that changes in the continuation criterion is that instead of showing $\lambda \geq 0$ on the whole interval $[0, R_C)$, we need to show $\lambda > -\infty$ according to how we changed the requirement for the boundary condition $\lambda(\mathring{r})$ in the local existence theorem. Since we have a finite radius $r \in [0, R_C)]$, we can estimate

$$e^{-2\lambda} \leq 1 - \frac{r^2\Lambda}{3} \leq 1 + \frac{R_C^2|\Lambda|}{3} < \infty \tag{7.2}$$

which means for λ

$$\lambda > -\infty \tag{7.3}$$

When estimating the absolute value of μ, again we need to keep the absolute value for Λ when doing the calculation in (4.4).

In Proposition 5.1 about the existence beyond the non-vacuum region it is necessary to adapt the requirements a bit. Namely, instead of requiring the upper bound (5.1) for

Λ we require the same upper bound for the absolute value of Λ and with functions $C_d(r)$ and $C_\gamma(r)$, which are growing in r and independent of q_0 and Λ, such that

$$|d(r) - d_\Lambda(r)| \leq C_d(r) \left(|\Lambda| + q_0^2\right) \tag{7.4}$$

and

$$|\gamma_\Lambda(r) - \gamma(r)| \leq C_\gamma(r) \left(|\Lambda| + q_0^2\right) \tag{7.5}$$

The requirements for Lemma 5.1 do not have to be changed (apart from $\Lambda > 0$), but the statement is then that the distance between the g- and h-functions is bounded above by

$$|g(r, \mu_\Lambda, I_\Lambda) - g(r, \mu, I)| + |h(r, \mu_\Lambda, I_\Lambda) - h(r, \mu, I)| \leq \left(q_0^2 + |\Lambda|\right) C_{gh}(r) \tag{7.6}$$

So, compared to the case with positive cosmological constant we changed now Λ to $|\Lambda|$. The first part in the proof of this Lemma, where we estimate $|\Delta_I g|$, can be adopted as it is for the case $\Lambda < 0$, because it is only about how g changes with the electromagnetic energy, so the cosmological constant plays no role. In the second part, where we estimate $|\Delta_\mu|$, all the Λ-dependent terms only depend on the absolute value of Λ, as we can see in equations (5.11) and (5.18). Hence, we can also leave this part unchanged, just when collecting the results for the distance between μ_Λ and μ in (5.20), we can not drop the absolute value of Λ, instead we have to write

$$|\mu_\Lambda - \mu| \leq \left(|\Lambda| + q_0^2\right) C(r) + C(r) \int_0^r \left(|g_\Lambda - g| + |h_\Lambda - h|\right) ds \tag{7.7}$$

and as a consequence we also get the absolute value of Λ instead of just Λ in all the subsequent inequalities that we derived from the distance between μ_Λ and μ. This concerns inequality (5.21), (5.22) and (5.23) for the distances between g_Λ and g and between h_Λ and h, inequality (5.24) for another estimate for the distance between the μ-functions, inequality (5.26) and (5.32) for the distance $|\gamma_\Lambda - \gamma|$ and inequality (5.30) and (5.31) for the denominators d_Λ and d. Thanks to the fact that we changed the condition for Λ in Proposition 5.1 to a condition for $|\Lambda|$ we can still plug in these upper bounds in inequality (5.31) for d_Λ and in (5.32) for the distance between the γ-functions and finally obtain the same result as we had in the case for positive Λ.

The global existence theorem also still holds for a negative cosmological constant, requiring now that $|\Lambda| \leq C_2$ with some positive constant C_2 and with the difference that in this case we can continue the solution beyond the non-vacuum region by a Reissner-Nordström-Anti-de-Sitter solution. This has the same metric components (6.1) as the Reissner-Nordström-de-Sitter metric, but given that Λ is negative now, it does not possess a cosmological horizon any more, but does exist until $r \to \infty$. The gluing process works exactly the same, because the sign of Λ is not relevant at any point there.

Part II.

Solutions with central black hole

In the second part of this work, we want to prove the existence of static, spherical symmetric solutions to the system with a black hole singularity at the center. The metric of a vacuum region surrounding a charged black hole with a nonvanishing, positive cosmological constant is determined by the Reissner-Nordström-de-Sitter solution

$$e^{2\mu_\Lambda(r)} = 1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} - \frac{r^2\Lambda}{3} \tag{7.8}$$

$$e^{2\lambda_\Lambda(r)} = \left(1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} - \frac{r^2\Lambda}{3}\right)^{-1} \tag{7.9}$$

and is valid in a vacuum region outside the black hole region, where the black hole region is defined as the region inside the black hole event horizon $r_{B\Lambda}$, which is the second largest zero of the Reissner-Nordström-de-Sitter metric component $e^{2\mu_\Lambda(r)}$ defined in (7.8). Here M_0 and Q_0 are the total mass and the total charge of the black hole, respectively. Choosing these black hole parameters as well as the cutoff angular momentum L_0 appropriately, Λ positive and the absolute values of the charges q_0 and Q_0 sufficiently small, we can construct solutions with the following features: For small $r > r_{B\Lambda}$, we set $f_\Lambda = 0$, such that there is a vacuum region and the metric is given by the Reissner-Nordström-de-Sitter metric and the charge is constantly equal the total black hole charge Q_0. Moving further out, one comes across a region $[r_{-\Lambda}, r_{+\Lambda}]$, where a nontrivial Ansatz for f_Λ yields vacuum, so we can continue the zero function continuously by this nontrivial Ansatz and the vacuum solution continuously by a nonvacuum solution which in a region outside of $r_{+\Lambda}$ gives us a matter region of bounded support. Beyond the support we can glue again the constant zero function to the nontrivial Ansatz of f_Λ in a continuous way, leading to another, now shifted, Reissner-Nordström-de-Sitter solution where the mass parameter equals the total mass of the system, so the black hole mass M_0 plus the total mass of the matter region and the charge parameter equals the total charge of the system, so the black hole charge Q_0 plus the total charge of the matter region.

To investigate the system with a charged black hole in the center we can mostly take the same setup as for the regular case (see chapter 2) with the only difference that now we do not implement a regular center, so we do not set $\lambda(0) = q(0) = 0$, but instead we require $q(r_{B\Lambda}) = Q_0$ and $\lambda(r_{B\Lambda}) = \lambda_0$ such that $e^{2\lambda_0}$ corresponds to the component (7.9) of the non-shifted Reissner-Nordström-de-Sitter solution at radius $r_{B\Lambda}$. Furthermore we adapt definition (2.11) of the electromagnetic energy contribution $I_q(r)$. Since the inside of the black hole plays no role, we can choose some radius r_v which lies in the vacuum region between the black hole and the matter region as a lower integral limit and end up with the new definition

$$\tilde{I}_\Lambda(r) := q_0 \int_{r_v}^{r} \frac{q_\Lambda(s)}{s^2} e^{\mu_\Lambda(s)+\lambda_\Lambda(s)} ds \tag{7.10}$$

of course this does not change the r derivative compared to the regular case and we have $\tilde{I}'_\Lambda(r) = I'_\Lambda(r)$. Consequently the adapted energy

$$\tilde{E} := e^{\mu_\Lambda}(r)\sqrt{1+\omega^2+\frac{L}{r^2}} - \tilde{I}_\Lambda(r) \tag{7.11}$$

defined with $\tilde{I}_\Lambda$ is conserved along the characteristics of the Vlasov equation, because

$$\frac{d\tilde{E}}{ds} = \frac{dE}{ds} = 0 \tag{7.12}$$

and we can choose any Ansatz $f_\Lambda(r,\omega,L) = \Phi(\tilde{E},L)$ to solve the Vlasov equation. As for the regular solutions we choose the polytropic Ansatz

$$f_\Lambda(r,\omega,L) = \alpha\left[1-\frac{\tilde{E}}{E_0}\right]_+^k [L-L_0]_+^l \tag{7.13}$$

To prove the existence of solutions to this system with a singularity at the center we want to use the charged solution with central singularity and with $\Lambda = 0$ as a background solution, so first we need to prove that this background solution exists.

8. Charged black holes surrounded by charged shells for $\Lambda = 0$

As proven in [4] there exist global solutions $(\mu(r), \lambda(r))$ to the uncharged EVS with $\Lambda = 0$ with a black hole singularity at the center and the matter quantities are supported on a shell $\{r_+ < r < R_0\}$. We can make use of this result to show that, introducing sufficiently weak charges to the system, so staying sufficiently close to the uncharged solutions, there still exist global solutions with matter quantities of finite spatial support. In the following let for each quantity x in the uncharged case denote x_q the corresponding quantity for the charged solution to the EVMS. We state the existence of charged solutions in the following theorem.

Theorem 8.1 (Existence of background solution). *Let $E_0 = 1$ and $q_0, L_0, M_0 > 0$ as well as $M_0^2 > Q_0^2$ and $L_0 > 16M_0^2$. Then there exists a solution $(\mu_q(r), \lambda_q(r), q(r))$ on (r_{BQ_0}, ∞) to the charged EVMS with $\Lambda = 0$ and where q_0 and Q_0 have the same sign and the absolute value for both of them is sufficiently small, where r_{BQ_0} is the event horizon of the charged black hole. The spatial support of f_q is contained in a shell $\{r_{+q} < r < R_{0q}\}$. In the complement of this shell the solution is respectively given by a correctly shifted Reissner-Nordström metric. The solution is unique if we choose the polytropic Ansatz* (7.13) *in an interval $[r_0, R_{0q}]$, where r_0 is defined as in* (8.50).

Proof. At first we want to show that under the conditions of Theorem 8.1 there exists a region $[r_{-q}, r_{+q}]$ outside the black hole horizon where a nontrivial Ansatz (7.13) gives us a vacuum solution, more precisely an unshifted Reissner-Nordström solution

$$e^{2\mu_q(r)} = 1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} \tag{8.1}$$

$$e^{2\lambda_q(r)} = \left(1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2}\right)^{-1} \tag{8.2}$$

so that we can start with a unshifted Reissner-Nordström metric and $f_\Lambda = 0$ at r_{BQ_0} and then glue this to the nontrivial Ansatz inside the interval $[r_{-q}, r_{+q}]$. Mathematically this means that we need to check that there exists a region outside of the black hole horizon where the vacuum condition (2.22) holds with the adapted electromagnetic energy $\tilde{I}_{qv}$ in a vacuum region that reads

$$\tilde{I}_{qv}(r) = q_0 \int_{r_v}^{r} \frac{q(s)}{s^2} ds = q_0 q(r_v)\left(\frac{1}{r_v} - \frac{1}{r}\right) = q_0 Q_0 \left(\frac{1}{r_v} - \frac{1}{r}\right) \tag{8.3}$$

and where $e^{2\mu}$ corresponds to the first metric component (8.1) of the Reissner-Nordström metric such that the condition then reads

$$\sqrt{1+\frac{L_0}{r^2}}\sqrt{1-\frac{2M_0}{r}+\frac{Q_0^2}{r^2}} - q_0 Q_0 \left(\frac{1}{r_v}-\frac{1}{r}\right) \geq 1 \tag{8.4}$$

We note that $\tilde{I}_{qv}(r)$ is bigger than zero, because q_0 and Q_0 have the same sign and r is bigger equal r_v and it is bounded above with boundary

$$\tilde{I}_{qv}(r) \leq \frac{q_0 Q_0}{r_v} \tag{8.5}$$

which we will use later. Now we start looking only at the first part of the latter inequality (8.4), so the part with the square roots on the left-hand side and analyze it in terms of zeros and ones. We define this part as a function

$$a_{Q_0}(r) := \sqrt{1-\frac{2M_0}{r}+\frac{Q_0^2}{r^2}} \cdot \sqrt{1+\frac{L_0}{r^2}} \tag{8.6}$$

and define another function

$$a(r) := \sqrt{1-\frac{2M_0}{r}} \cdot \sqrt{1+\frac{L_0}{r^2}} \tag{8.7}$$

which serves as a background function to which we stay close by choosing a black hole charge of small absolute value. Basically, we want to find the following configuration which is exemplarily depicted in figure 8.1

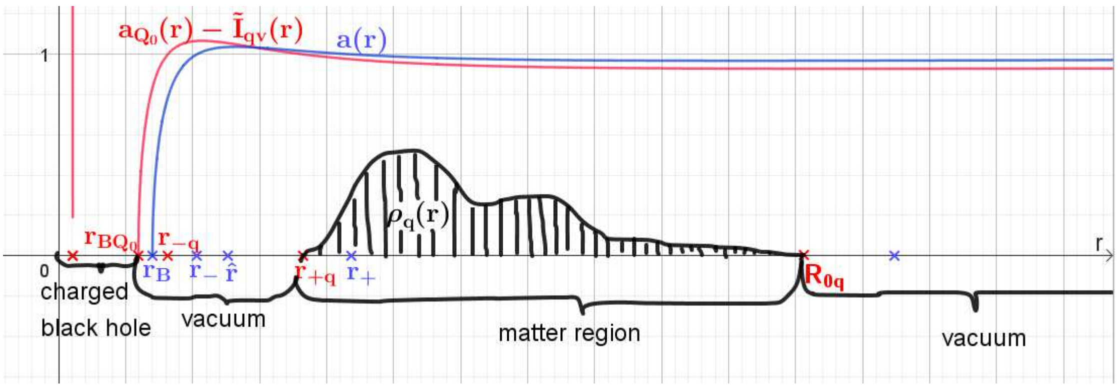

Figure 8.1.: Sketch of an exemplary configuration of a weakly charged black hole surrounded by a weakly charged matter shell for $\Lambda = 0$

It is only a qualitative graphic to help imagine the scenario. On the x-axis the radius is plotted, while the y-axis is dimensionless. We see in blue the function $a(r)$ for the background solution and in red the function $a_{Q_0}(r) - \tilde{I}_{qv}(r)$. In the region where the respective function is bigger than one, there is a vacuum region for the corresponding solution. r_{BQ_0} and r_B are the black hole horizons for the respective solution, r_- and

r_{-q} the first radius outside of the black hole, where the function takes the value one, r_+ and r_{+q} are the second ones outside of the black hole, $\hat{r}$ is the radius, where $a(r)$ attains its maximum and R_{0q} is the boundary of the support of the matter quantities of the charged solution. In the figure $\rho_q(r)$ is depicted to exemplify the matter quantities, as well corresponding to the charged solution. To show the existence of this configuration we define further

$$v(r) := 1 - \frac{2M_0}{r} \tag{8.8}$$

$$v_{Q_0}(r) := 1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} \tag{8.9}$$

and notice that the zeros of $v(r)$ and $v_{Q_0}(r)$ coincide with the zeros of $a(r)$ and $a_{Q_0}(r)$ respectively and since $v(r)$ and $v_{Q_0}(r)$ correspond to the first metric components of the Schwarzschild and the Reissner-Nordström metric respectively, their zeros give rise to the horizons of the corresponding system. Like that, the only zero of $a(r)$ is the Schwarzschild black hole horizon

$$r_B = 2M_0 \tag{8.10}$$

Under the condition $L_0 > 16M_0^2$ the equation $a(r) = 1$ becomes the quadratic equation

$$-2M_0 r^2 + L_0 r - 2M_0 L_0 = 0 \tag{8.11}$$

which yields the two positive, finite solutions

$$r_\pm = \frac{L_0}{4M_0} \pm \sqrt{\frac{L_0^2}{16M_0^2} - L_0} \tag{8.12}$$

We observe that $r_B < r_-$ by completing the square inside the square root:

$$\begin{aligned} r_- &= \frac{L_0}{4M_0} - \sqrt{\frac{L_0^2}{16M_0^2} - L_0 + 4M_0^2 - 4M_0^2} = \frac{L_0}{4M_0} - \sqrt{\left(\frac{L_0}{4M_0} - 2M_0\right)^2 - 4M_0^2} \\ &> \frac{L_0}{4M_0} - \sqrt{\left(\frac{L_0}{4M_0} - 2M_0\right)^2} = \frac{L_0}{4M_0} - \left(\frac{L_0}{4M_0} - 2M_0\right) = 2M_0 = r_B \end{aligned} \tag{8.13}$$

The condition $M_0^2 > Q_0^2$ makes sure that a_{Q_0} has two positive, real zeros, namely at $r = M_0 \pm \sqrt{M_0^2 - Q_0^2}$. The black hole horizon for the solution to the charged EVMS corresponds to the bigger zero and is thus given by

$$r_{BQ_0} = M_0 + \sqrt{M_0^2 - Q_0^2} \tag{8.14}$$

From there we see immediately that $r_{BQ_0} < r_B$. The equation $a_{Q_0}(r) = 1$ for finite r becomes a cubic equation. It is known that for a generic cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{8.15}$$

the discriminant is given by

$$\frac{4\left(b^2 - 3ac\right)^3 - \left(2b^3 - 9abc + 27a^2 d\right)^2}{27a^2} \tag{8.16}$$

whose sign determines the number of real solutions to the cubic equation. Especially, if the sign of the discriminant is positive, the cubic equation has three distinct real solutions which are given by

$$\begin{aligned} x_1 &= -\frac{b}{3a} + \sqrt{-\frac{4p}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right)\right) \\ x_{2,3} &= -\frac{b}{3a} - \sqrt{-\frac{4p}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right) \pm \frac{\pi}{3}\right) \end{aligned} \tag{8.17}$$

with

$$p = \frac{3ac - b^2}{3a} \tag{8.18}$$

and

$$q = \frac{-9abc + 27a^2 d + 2b^3}{27a^3} \tag{8.19}$$

In our case the cubic equation $a_{Q_0}(r) = 1$ for finite r reads

$$-2M_0 r^3 + \left(Q_0^2 + L_0\right) r^2 - 2M_0 L_0 r + Q_0^2 L_0 = 0 \tag{8.20}$$

Looking at its discriminant

$$D = \frac{72(Q_0^2 + L_0)Q_0^2 M_0^2 L_0^2 - 64M_0^4 L_0^3 - 108Q_0^4 M_0^2 L_0^2 + 4(Q_0^2 + L_0)^2 M_0^2 L_0^2 - 4(Q_0^2 + L_0)^3 Q_0^2 L_0}{1728M_0^4} \tag{8.21}$$

we can investigate how many real solutions there are to this cubic equation. Since we only need to find out the sign of the discriminant, we can break it down to the following expression

$$\tilde{D} := 4M_0^2 L_0 \left(-2Q_0^4 + 5Q_0^2 L_0 - 4M_0^2 L_0\right) + M_0^2 L_0^3 - Q_0^2 \left(Q_0^2 + L_0\right)^3 \tag{8.22}$$

by dropping a factor $\frac{4L_0}{1728M_0^4}$ which has a positive sign, so D and $\tilde{D}$ have the same sign. Imposing the conditions $M_0^2 > Q_0^2$ and $L_0 > 16M_0^2$ one can show that the expression in (8.22) is strictly bigger than zero (you can find the calculation in section A.2 of the appendix) which also means for the discriminant $D > 0$ and therefore, there exist three distinct real solutions to the cubic equation. So, $a_{Q_0}(r)$ has three finite, real values r with $a_{Q_0}(r) = 1$ given by

$$\begin{aligned} r_1 &= \frac{Q_0^2 + L_0}{6M_0} + \sqrt{-\frac{4p}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right)\right) \\ r_{2,3} &= \frac{Q_0^2 + L_0}{6M_0} - \sqrt{-\frac{4p}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}}\right) \pm \frac{\pi}{3}\right) \end{aligned} \tag{8.23}$$

with

$$p = L_0 - \frac{\left(Q_0^2 + L_0\right)^2}{12M_0^2} \tag{8.24}$$

and

$$q = \frac{L_0(Q_0^2 + L_0)}{6M_0} - \frac{Q_0^2 L_0}{2M_0} - \frac{Q_0^6 + 3Q_0^4 L_0 + 3Q_0^2 L_0^2 + L_0^3}{108M_0^2} \tag{8.25}$$

Since

$$\lim_{r \searrow 0} a_{Q_0}(r) = \infty \tag{8.26}$$

and $a_{Q_0}(r)$ is a continuous function as long as $v_{Q_0}(r) > 0$, the smallest of the three solutions (8.23) is smaller than the first zero of $a_{Q_0}(r)$ and therefore also smaller than the event horizon r_{BQ_0}, so it will not be relevant for our solution. Because of this, we focus on the two larger radii of the set $\{r_1, r_2, r_3\}$. We call the second largest one r_{-Q_0} and the largest one r_{+Q_0}. It can be proved that of the following three expressions

$$\cos x \tag{8.27}$$

$$-\cos\left(x - \frac{\pi}{3}\right) \tag{8.28}$$

$$-\cos\left(x + \frac{\pi}{3}\right) \tag{8.29}$$

at any x the second lowest expression is always bigger equal $-\frac{1}{2}$. This allows us to do the estimate

$$r_{-Q_0} \geq \frac{Q_0^2 + L_0}{6M_0} - \frac{1}{2}\sqrt{-\frac{4p}{3}} \tag{8.30}$$

plugging in the expression (8.24) for p we get

$$r_{-Q_0} \geq \frac{Q_0^2 + L_0 - \sqrt{\left(Q_0^2 + L_0\right)^2 - 12L_0 M_0^2}}{6M_0} \tag{8.31}$$

it can be shown (see appendix section A.3) that

$$\frac{Q_0^2 + L_0 - \sqrt{\left(Q_0^2 + L_0\right)^2 - 12L_0M_0^2}}{6M_0} > M_0 - \sqrt{M_0^2 - Q_0^2} \tag{8.32}$$

so we have

$$r_{-Q_0} > M_0 - \sqrt{M_0^2 - Q_0^2} \tag{8.33}$$

meaning that r_{-Q_0} is bigger than the first zero of $a_{Q_0}(r)$. Furthermore, since $v_{Q_0}(r)$ is continuous, between its two zeros it is either strictly positive or strictly negative, so we only have to check one point between the two zeros which we choose to be the average of the two zeros $\frac{M_0+\sqrt{M_0^2-Q_0^2}+M_0-\sqrt{M_0^2-Q_0^2}}{2} = M_0$ and we obtain

$$v(M_0) = 1 - \frac{2M_0}{M_0} + \frac{Q_0^2}{M_0^2} = -1 + \frac{Q_0^2}{M_0^2} < 0 \tag{8.34}$$

where we used the condition $M_0^2 > Q_0^2$. So, $v_{Q_0}(r)$ is strictly negative between its two zeros and consequently, $a_{Q_0}(r)$ is not defined in $\mathbb{R}$ for $M_0 - \sqrt{M_0^2 - Q_0^2} < r < r_{BQ_0}$. Therefore, we can conclude that

$$r_{-Q_0} > r_{BQ_0} \tag{8.35}$$

There is also a lower bound on the largest of the three cosines in (8.27) - (8.29) which is given by $+\frac{1}{2}$. From that, we can derive a lower bound on r_{+Q_0} and find

$$r_{+Q_0} \geq r_- \tag{8.36}$$

(see appendix section A.4 for the calculation). In the interval (r_-, r_+), we have $a(r) > 1$ as can be easily verified for example by calculating

$$\begin{aligned} a\left(\frac{r_- + r_+}{2}\right) = a\left(\frac{L_0}{4M_0}\right) &= \sqrt{\left(1 - \frac{8M_0^2}{L_0}\right)\left(1 + \frac{16M_0^2L_0}{L_0^2}\right)} \\ &= \sqrt{1 + \frac{16M_0^2}{L_0} - \frac{8M_0^2}{L_0}\left(1 + \frac{16M_0^2}{L_0}\right)} \\ &= \sqrt{1 + \frac{16M_0^2}{L_0}\left(1 - \frac{1}{2} - \underbrace{\frac{8M_0^2}{L_0}}_{\leq\frac{1}{2}}\right)} \geq 1 \end{aligned} \tag{8.37}$$

One can conclude that due to this result along with continuity reasons $a(r) > 1$ in the whole interval (r_-, r_+). Since $a_{Q_0}(r) \geq a(r)$ at any r, also $a_{Q_0}(r) > 1$ in this region. Furthermore, due to continuity of $a_{Q_0}(r)$ in the positive plane, we get

$$r_{-Q_0} < r_- < r_+ < r_{+Q_0} \tag{8.38}$$

In the next step, we establish an upper bound on r_B. To this end we note that $v'(r) = \frac{2M_0}{r^2}$ and keep in mind that $r_{BQ_0} < r_B$. We estimate

$$\begin{aligned} -v(r_{BQ_0}) &= \int_{r_{BQ_0}}^{r_B} v'(s)ds - \underbrace{v(r_B)}_{=0} \\ &\geq \int_{r_{BQ_0}}^{r_B} \inf_{s\in[r_{BQ_0},r_B]} v'(s)ds \\ &= \frac{2M_0}{r_B^2}(r_B - r_{BQ_0}) \end{aligned} \tag{8.39}$$

Since

$$-v(r_{BQ_0}) = -\left(\underbrace{v_{Q_0}(r_{BQ_0})}_{=0} - \frac{Q_0^2}{r_{BQ_0}^2}\right) = \frac{Q_0^2}{r_{BQ_0}^2} \tag{8.40}$$

we find

$$r_B - r_{BQ_0} \leq \frac{Q_0^2}{r_{BQ_0}^2}\frac{r_B^2}{2M_0} = \frac{Q_0^2}{\left(M_0 + \sqrt{M_0^2 - Q_0^2}\right)^2}\frac{r_B^2}{2M_0} \tag{8.41}$$

where we plugged in the explicit expression (8.14) for r_{BQ_0}. So, decreasing Q_0^2, or equivalently the absolute value of Q_0, decreases the numerator on the right hand side of (8.41), while it increases the denominator and especially we find

$$\lim_{|Q_0|\to 0} \frac{Q_0^2}{\left(M_0 + \sqrt{M_0^2 - Q_0^2}\right)^2}\frac{r_B^2}{2M_0} = 0 \tag{8.42}$$

This allows us to make the distance between r_B and r_{BQ_0} by choosing $|Q_0|$ sufficiently small, namely so small that we obtain $r_B < r_{-Q_0}$. All together we now have the configuration

$$r_{BQ_0} < r_B < r_{-Q_0} < r_- < r_+ < r_{+Q_0} \tag{8.43}$$

In the end we want to find a region, where we can continue the initial Ansatz $f_q = 0$ by a nontrivial Ansatz (7.13), i.e. a region where also the nontrivial Ansatz yields vacuum. To do so, we still need to take the second term in condition (8.4), i.e. the $\tilde{I}_{qv}(r)$-term, into account. We remember that inside of a vacuum region $[r_{-q}, r_{+q}]$ we have the estimate (8.5) for the electromagnetic energy contribution. Furthermore, we note that $a(r)$ has a unique maximum

$$\hat{r} = \frac{L_0}{2M_0} - \sqrt{\frac{L_0^2}{4M_0^2} - 3L_0} \tag{8.44}$$

between r_- and r_+ and estimate the difference between $a(\hat{r})$ and $a(\hat{r}) - \tilde{I}_{qv}(\hat{r})$ to

$$a(\hat{r}) - \left(a(\hat{r}) - \tilde{I}_{qv}(\hat{r})\right) = \tilde{I}_{qv}(\hat{r}) < \frac{q_0 Q_0}{r_v} \tag{8.45}$$

Thus, in this step it is sufficient to choose either the absolute value of q_0 or of Q_0 small enough, such that $a(\hat{r}) - (a(\hat{r}) - \tilde{I}_{qv}(\hat{r})) \leq a(\hat{r}) - 1$, which means that $a(\hat{r}) - \tilde{I}_{qv}(\hat{r}) > 1$ and guarantees the existence of exactly two ones $\{r_a, r_b\}$ of the function $a(r) - \tilde{I}_{qv}(r)$ between r_- and r_+. Since $a(r) - \tilde{I}_{qv}(r) \leq a_{Q_0}(r) - \tilde{I}_{qv}(r) \leq a_{Q_0}(r)$ for all r, $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ has exactly two ones r_{-q} and r_{+q} in the interval $[r_{-Q_0}, r_{+Q_0}]$ such that

$$r_{-Q_0} \leq r_{-q} \leq r_a \leq r_b \leq r_{+q} \leq r_{+Q_0} \tag{8.46}$$

and $a_{Q_0}(r) - \tilde{I}_{qv}(r) > 1$ between r_{-q} and r_{+q}. In this region $[r_{-q}, r_{+q}]$ we have

$$a_{Q_0}(r) - \tilde{I}_{qv}(r) = e^{\mu_q(r)}\sqrt{1 + \frac{L_0}{r^2}} - \tilde{I}_{qv}(r) > 1 \tag{8.47}$$

with $e^{\mu_q(r)} = \sqrt{1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2}}$ which is precisely the condition (8.4) for having a vacuum and thus, also an Ansatz $f_q(r, \omega, L) = \alpha \left[1 - \frac{\tilde{E}}{E_0}\right]_+^k [L - L_0]_+^l$ yields vacuum in the interval $[r_{-q}, r_{+q}]$. This allows us to continue f_q by such a nontrivial Ansatz at any radius inside this interval.

In the second part of the proof we show now that for such a nontrivial Ansatz (7.13) there exists a unique solution to the system at least up to a radius $R_0 + \Delta R$, where R_0 is the boundary of the support of the uncharged background solution and ΔR is a constant bigger zero. The matter quantities of the solution have bounded support such that beyond the support we can glue another trivial Ansatz $f_q = 0$ to the particle distribution function and get again a (now shifted) Reissner-Nordström solution which exists until $r \to \infty$. Since

$$r_{-Q_0} \leq r_- \leq r_a \leq r_b \leq r_+ \leq r_{+Q_0} \tag{8.48}$$

we have

$$\max\{r_-, r_{-q}\} < \min\{r_+, r_{+q}\} \tag{8.49}$$

So we can use the smaller radius of r_+ and r_{+q} to perform the gluing process, because this will allow us to do the gluing at the same radius for the uncharged solution. For this purpose, we define

$$r_0 := \min\{r_+, r_{+q}\} \tag{8.50}$$

After integrating the Einstein equation (2.25) for $\lambda_q(r)$ from r_0 to r and setting $\Lambda = 0$ we get

$$e^{-2\lambda_q(r)} = 1 - \frac{8\pi}{r}\int_{r_0}^{r} s^2 g_q(s)ds - \frac{1}{r}\int_{r_0}^{r}\frac{q^2(s)}{s^2}ds + \frac{K}{r} \tag{8.51}$$

with an integration constant K which we fix by imposing instead of a regular center now the condition

$$e^{-2\lambda_q(r_0)} = 1 - \frac{2M_0}{r_0} + \frac{Q_0^2}{r_0^2} \tag{8.52}$$

to match the non-vacuum solution to the unshifted Reissner-Nordström solution at r_0. This leads to the constant

$$K = -2M_0 + \frac{Q_0^2}{r_0} \tag{8.53}$$

and we can write

$$\frac{K}{r} = \frac{r_0}{r}\left(\frac{-2M_0}{r_0} + \frac{Q_0^2}{r_0^2}\right) = \frac{r_0}{r}\left(e^{-2\lambda_q(r_0)} - 1\right) \tag{8.54}$$

and consequently

$$\begin{aligned} e^{-2\lambda_q(r)} &= 1 - \frac{8\pi}{r}\int_{r_0}^{r} s^2 g_q(s)ds - \frac{1}{r}\int_{r_0}^{r}\frac{q^2(s)}{s^2}ds + \frac{r_0}{r}\left(e^{-2\lambda_q(r_0)} - 1\right) \\ &= 1 - \frac{2}{r}\left(\frac{r_0}{2}\left(1 - e^{-2\lambda_q(r_0)}\right) + 4\pi\int_{r_0}^{r} s^2 g_q(s)ds + \frac{1}{2}\int_{r_0}^{r}\frac{q^2}{s^2}ds\right) \end{aligned} \tag{8.55}$$

plugging this in to the second nontrivial Einstein equation (2.26) for $\mu'(r)$ we end up with the single Einstein equation

$$\mu_q'(r) = \frac{4\pi r h_q(r) + \frac{r_0}{2r^2}\left(1 - e^{-2\lambda_q(r_0)}\right) + \frac{4\pi}{r^2}\int_{r_0}^{r} s^2 g_q(s)ds - \frac{q^2(r)}{2r^3} + \frac{1}{2r^2}\int_{r_0}^{r}\frac{q^2(s)}{s^2}ds}{1 - \frac{2}{r}\left(\frac{r_0}{2}\left(1 - e^{-2\lambda_q(r_0)}\right) + 4\pi\int_{r_0}^{r} s^2 g_q(s)ds + \frac{1}{2}\int_{r_0}^{r}\frac{q^2}{s^2}ds\right)} \tag{8.56}$$

for $r \geq r_0$ with boundary condition

$$e^{2\mu_{0q}} := e^{2\mu_q(r_0)} = 1 - \frac{2M_0}{r_0} + \frac{Q_0^2}{r_0^2} \tag{8.57}$$

Since the right-hand side of the Einstein equation (8.56) is regular for all $r \geq r_0$, a solution exists locally. Furthermore, the equation only differs from the Einstein equation (2.30) with regular center by some additional terms depending on r_0, M_0 and Q_0, which are however bounded and well behaved on any finite interval. Therefore, also the continuation criterion applies, which guarantees a solution at least as long as the denominator is positive. Now we show, similar to what we did in the proof for regular solutions, that the maximum radius of existence R_C is bigger equal $R_0 + \Delta R$, with some finite and positive ΔR, if we choose the absolute value of the charges q_0 and Q_0 small enough. We call the denominator in the uncharged Einstein equation

$$d_{M_0}(r) := 1 - \frac{2}{r}\left(M_0 + 4\pi \int_{r_0}^{r} s^2 g(s) ds\right) \tag{8.58}$$

and in the charged case

$$d_{M_0 Q_0}(r) := 1 - \frac{2}{r}\left(M_0 + 4\pi \int_{r_0}^{r} s^2 g_q(s) ds\right) + \frac{Q_0^2}{r_0 r} - \frac{1}{r}\int_{r_0}^{r} \frac{q^2(s)}{s^2} ds \tag{8.59}$$

and define

$$\Delta d_0 := \alpha \cdot d_{M_0 Q_0}(r_0) = \alpha\left(1 - \frac{2M_0}{r_0} + \frac{Q_0^2}{r_0^2}\right) \tag{8.60}$$

with a positive constant $\alpha < 1$. Similar to the proof for regular solutions we define the radii

$$r^* := \inf\{r \in (r_0, R_C) | d_{M_0 Q_0}(r) = \Delta d_0\} \tag{8.61}$$

and

$$\tilde{r} := \inf\{r \in (r_0, R_C) | |\gamma_q(r) - \gamma(r)| > |\gamma(R_0 + \Delta R)|\} \tag{8.62}$$

and their minimum

$$\tilde{r}^* := \min\{r^*, \tilde{r}\} \tag{8.63}$$

and note that there exists a solution for all $r \in (r_0, \tilde{r}^*)$, because the continuation criterion applies. So, we want to show that $\tilde{r}^* \geq R_0 + \Delta R$. To do so we estimate the distance between $\mu_q(r)$ and $\mu(r)$ for $r \leq \tilde{r}^*$. For the uncharged solution we also glue the trivial Ansatz for the particle distribution function to a nontrivial one at r_0 and impose the conditions

$$e^{-2\lambda(r_0)} = 1 - \frac{2M_0}{r_0} \tag{8.64}$$

and

$$e^{2\mu_0} := e^{2\mu(r_0)} = 1 - \frac{2M_0}{r_0} \tag{8.65}$$

to match the solution to the nonshifted Schwarzschild metric at r_0. Firstly, we calculate the difference between the initial values of μ and μ_q

$$
\begin{aligned}
|\mu_0 - \mu_{0q}| &= \frac{1}{2}\ln\left(\left(1-\frac{2M_0}{r_0}+\frac{Q_0^2}{r_0^2}\right)\left(1-\frac{2M_0}{r_0}\right)^{-1}\right) \\
&= \frac{1}{2}\ln\left(1+\frac{Q_0^2}{r_0^2}\left(1-\frac{2M_0}{r_0}\right)^{-1}\right) =: C_{0Q_0}
\end{aligned}
\tag{8.66}
$$

where C_{0Q_0} is a constant growing in Q_0^2 or equivalently in $|Q_0|$ and vanishing as $Q_0 \to 0$. For computational convenience we set g_q, g, h_q and h equal to 0 for $r \in (0, r_0)$. With this we estimate

$$
\begin{aligned}
|\mu(r) - \mu_q(r)| &\leq C_{0Q_0} + \int_{r_0}^{r} |\mu'(s) - \mu_q'(s)| ds \\
&\leq C_{0Q_0} + \int_{r_0}^{r} \frac{1}{d_{M_0Q_0}(s)} \Bigg(4\pi s |h_q(s) - h(s)| \\
&+ \frac{4\pi}{s^2}\int_0^s \sigma^2 \left|g_q(\sigma) - g(\sigma)\right| d\sigma + \left|\frac{Q_0^2}{2r_0 s^2}\right| \\
&+ \left| -\frac{q^2}{2s^3} + \frac{1}{2s^2}\int_{r_0}^{s} \frac{q^2(\sigma)}{\sigma^2} d\sigma \right| \Bigg) ds \\
&+ \int_{r_0}^{r} \left| \frac{4\pi}{d_{M_0Q_0}(s)} - \frac{4\pi}{d_{M_0}(s)} \right| \left(sh(s) + \frac{1}{s^2}\int_0^s \sigma^2 g(\sigma) d\sigma + \frac{M_0}{s^2} \right) ds
\end{aligned}
\tag{8.67}
$$

where we plugged in the Einstein equation (8.56) for the charged an uncharged case respectively and then subtracted and added a term $\frac{1}{d_{M_0Q_0}(s)}\left(sh(s) + \frac{1}{s^2}\int_0^s \sigma^2 g(\sigma)d\sigma + \frac{M_0}{s^2}\right)$ inside of the absolute value under the integral. We note that there is a part which is really similar as in (5.11) for the regular solutions and we can still make use of estimate (5.17) just with g_Λ replaced by g_q. Also estimate (5.19) still applies for h and g, however now without the third argument of the functions, because h and g belong to the uncharged Einstein-Vlasov system and thus do not depend on any electric energy. When estimating the charge terms we can also still make use of estimates (5.12) and (5.13). However, we have to be a bit careful when pulling out the q_0 factor and can not directly adopt estimate (5.14) as it was in the regular case, because there we had $q(r) \propto q_0$, while now we have $q(r) \propto (Q_0 + q_0 C(r))$ with some positive, continuous function $C(r)$. Instead we have now

$$
\frac{q^2(r)}{r^2} \leq (Q_0 + q_0)^2 C \tag{8.68}
$$

for $r \geq r_0$ with a finite positive constant C which is independent of the charge parameters Q_0 and q_0. So, we can estimate for $r \in (r_0, \tilde{r}^*)$

$$\begin{aligned}\int_{r_0}^{r} \frac{1}{d_{M_0Q_0}(s)} \left| -\frac{q^2}{2s^3} + \frac{1}{2s^2} \int_{r_0}^{s} \frac{q^2(\sigma)}{\sigma^2} d\sigma \right| ds &\leq \frac{1}{\Delta d_0} (Q_0 + q_0)^2 C(r) \\ &\leq (Q_0 + q_0)^2 C(r)\end{aligned} \tag{8.69}$$

with a positive, continuous function $C(r)$, growing in r and independent of Q_0 and q_0. Since we are considering $r \leq \tilde{r}^*$, we have $d_{M_0Q_0}(r) > \Delta d_0 > 0$. Furthermore, we have Vlasov matter, defined in (2.4)-(2.7), and therefore

$$\begin{aligned} p + 2p_T &= \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^{\infty} f(r, \omega, L) \frac{\omega^2}{\sqrt{1 + \omega^2 + \frac{L}{r^2}}} d\omega dL \\ &\quad + \frac{\pi}{r^4} \int_0^\infty \int_{-\infty}^{\infty} f(r, \omega, L) \frac{L}{\sqrt{1 + \omega^2 + \frac{L}{r^2}}} d\omega dL \\ &= \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^{\infty} f(r, \omega, L) \frac{\omega^2 + \frac{L}{r^2}}{\sqrt{1 + \omega^2 + \frac{L}{r^2}}} d\omega dL \\ &\leq \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^{\infty} f(r, \omega, L) \frac{1 + \omega^2 + \frac{L}{r^2}}{\sqrt{1 + \omega^2 + \frac{L}{r^2}}} d\omega dL \\ &= \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^{\infty} f(r, \omega, L) \sqrt{1 + \omega^2 + \frac{L}{r^2}} d\omega dL = \rho \end{aligned} \tag{8.70}$$

Consequently the generalized Buchdahl inequality with Schwarzschild singularity at the center holds for all $r \in [\frac{9M_0}{4}, \infty)$ (see [2]) which reads

$$\frac{2M_0 + 8\pi \int_{r_0}^{r} s^2 \rho ds}{r} \leq \frac{8}{9} \tag{8.71}$$

Since $r_0 \geq \hat{r}$ and using the definition (8.44) for $\hat{r}$ we find

$$\begin{aligned} r_0 \geq \hat{r} &= \frac{L_0}{2M_0} - \sqrt{\frac{L_0^2}{4M_0^2} - 3L_0} \\ &= \frac{L_0}{2M_0} - \sqrt{\left(\frac{L_0}{2M_0} - 3M_0 \right)^2 - 9M_0^2} \\ &\geq \frac{L_0}{2M_0} - \sqrt{\left(\frac{L_0}{2M_0} - 3M_0 \right)^2} \\ &= \frac{L_0}{2M_0} - \frac{L_0}{2M_0} + 3M_0 = 3M_0 \geq \frac{9M_0}{4} \end{aligned} \tag{8.72}$$

where in line two we completed the square by adding and subtracting a factor of $9M_0^2$ under the square root. This inequality means that $r_0 \geq \frac{9M_0}{4}$, so the generalized Buchdahl inequality (8.71) holds in the EVS with central black hole on all $r \in [r_0, \infty)$. Using this result we get for the denominator of the uncharged system

$$\begin{aligned} d_{M_0}(r) &= 1 - \frac{2}{r}\left(\frac{r_0}{2}\frac{2M_0}{r_0} + 4\pi \int_{r_0}^{r} s^2 g(s) ds\right) \\ &= 1 - \frac{2M_0 + 8\pi \int_{r_0}^{r} s^2 \rho ds}{r} \geq 1 - \frac{8}{9} = \frac{1}{9} \end{aligned} \tag{8.73}$$

Thus, we can estimate again similarly to what we did in the regular case,

$$\left|\frac{1}{d_{M_0Q_0}} - \frac{1}{d_{M_0}}\right| = \frac{|d_{M_0} - d_{M_0Q_0}|}{d_{M_0Q_0} d_{M_0}} \leq \frac{9\,|d_{M_0} - d_{M_0Q_0}|}{\Delta d_0} \tag{8.74}$$

In difference to the regular case, we have now in estimate (8.67) for the difference between μ and μ_q additional terms depending on Q_0 and M_0. We find

$$\left|\frac{Q_0^2}{2r_0 s^2}\right| \leq \frac{Q_0^2}{2r_0^3} =: C_{Q_0} \tag{8.75}$$

with a constant again growing in $|Q_0|$ and vanishing as $Q_0 \to 0$ and for the black hole mass term we get

$$\frac{M_0}{s^2} \leq \frac{M_0}{r_0^2} =: C \tag{8.76}$$

We still need an estimate for the difference between the two denominators, so we calculate

$$\begin{aligned} |d_{M_0}(r) - d_{M_0Q_0}(r)| &\leq \frac{8\pi}{r}\int_0^r s^2 |g_q(s) - g(s)| ds \\ &\quad + \frac{1}{r}\int_{r_0}^{r} \left|\frac{q^2(s)}{s^2}\right| ds + \frac{Q_0^2}{r_0 r} \\ &\leq 8\pi r \int_0^r |g_q(s) - g(s)| ds + (Q_0 + q_0)^2 C(r) + \frac{Q_0^2}{r_0^2} \\ &= 8\pi r \int_0^r |g_q(s) - g(s)| ds + (q_0^2 + q_0 Q_0) C(r) + C_{Q_0} \end{aligned} \tag{8.77}$$

with a positive, continuous function $C(r)$, growing in r and independent of q_0 and Q_0 and a positive constant C_{Q_0} that is independent of r and q_0, growing in $|Q_0|$ and converging to zero as Q_0 does. Collecting all the estimates, we obtain

$$|\mu(r) - \mu_q(r)| \leq r C_{Q_0} + (q_0^2 + q_0 Q_0) C(r) + C(r) \int_0^r (|g_q - g| + |h_q - h|)\, ds \tag{8.78}$$

where the constant C_{Q_0} is still positive, independent of q_0, growing in $|Q_0|$ and vanishes at $Q_0 = 0$ and both $C(r)$ are continuous, positive, independent of all charges and growing in r. As in equation (5.9) the difference between the μ-functions gives us an estimate for the absolute value of $\Delta_\mu g$. However, to get an upper bound for $|g_q - g|$ we still need to estimate $|\Delta_{\tilde{I}} g|$, which now changes a bit because $\tilde{I}$ is due to the presence of the charged black hole not proportional to q_0^2 anymore and can also not be estimated by $q_0^2 C(r)$ as an upper bound anymore. This is because the total charge $q(r)$ is now a sum of the black hole charge and the particle charge inside the radius r. However, we are considering all charges to be of the same sign, so $q(r)$ has still the same sign as q_0, especially we find

$$\begin{aligned}\tilde{I}_q(r) &= q_0 \int_{r_v}^{r} \frac{q(s)}{s^2} e^{\mu(s)+\lambda(s)} ds \\ &= q_0 \int_{r_v}^{r} \frac{Q_0 + 4\pi \int_{r_v}^{s} \sigma^2 \rho_q(\sigma) d\sigma}{s^2} e^{\mu(s)+\lambda(s)} ds \\ &\propto q_0 Q_0 C(r) + q_0^2 C(r)\end{aligned} \tag{8.79}$$

which is positive because of the same sign of Q_0 and q_0 with two positive functions $C(r)$ that are independent of q_0 and Q_0. Thus, we can estimate

$$|\Delta_I g| \leq (q_0 Q_0 + q_0^2) C(r) \tag{8.80}$$

and it results for the difference between the g- and h- functions that

$$|g_q - g| + |h_q - h| \leq \left(q_0 Q_0 + q_0^2\right) C(r) + r C_{Q_0}(r) + C(r) \int_0^r \left(|g_q - g| + |h_q - h|\right) ds \tag{8.81}$$

Again, a Grönwall argument leads to

$$|g_q - g| + |h_q - h| \leq (q_0 Q_0 + q_0^2) C(r) + r C_{Q_0} \tag{8.82}$$

Plugging this estimate in (8.78) gives

$$|\mu(r) - \mu_q(r)| \leq (q_0 Q_0 + q_0^2) C(r) + C_{Q_0}(r) \tag{8.83}$$

As in (5.26) this also means for the distance between the γ-functions

$$|\gamma_q(r) - \gamma(r)| \leq (q_0 Q_0 + q_0^2) C(r) + C_{Q_0}(r) \tag{8.84}$$

Thus, choosing the absolute value of the charges q_0 and Q_0 small enough we can assure

$$|\gamma_q(r) - \gamma(r)| \leq |\gamma(R_0 + \Delta R)| \tag{8.85}$$

for all $r \in (r_0, R_0 + \Delta R)$. Looking at the distance (8.77) of the denominators and keeping in mind estimate (8.82) for the difference between the g-functions, we obtain

$$|d_{M_0}(r) - d_{M_0Q_0}(r)| \leq (q_0Q_0 + q_0^2)C(r) + C_{Q_0}(r) \tag{8.86}$$

and see that by choosing $|q_0|$ and $|Q_0|$ small enough, we can reach on all $r \in (r_0, R_0 + \Delta R)$

$$|d_{M_0}(r) - d_{M_0Q_0}(r)| \leq \frac{1}{9} - \Delta d_0 \tag{8.87}$$

where we choose the constant α in (8.60) small enough, such that $\frac{1}{9} - \Delta d_0 \geq C > 0$. Due to the Buchdahl inequality we have $d_{M_0} \geq \frac{1}{9}$ and consequently we get

$$\underbrace{d_{M_0}(r) - \frac{1}{9}}_{\geq 0} + \Delta d_0 \leq d_{M_0Q_0}(r) \tag{8.88}$$

and thus

$$\Delta d_0 \leq d_{M_0Q_0}(r) \tag{8.89}$$

for all $r_0 \leq r \leq R_0 + \Delta R$ which, together with (8.85) means that

$$\tilde{r}^* \geq R_0 + \Delta R \tag{8.90}$$

Due to the same argumentation as in the case of regular solutions, using the properties of the solution to the uncharged system, we obtain the result, that a solution (μ_q, λ_q, q) exists at least until $R_0 + \Delta R$ and that the matter quantities are of bounded support with boundary $R_{0q} \leq R_0 + \Delta R$. Beyond the support there is again a vacuum region, where we can glue the nontrivial Ansatz (2.13) (with the modified energy) for the particle contribution function continuously to the constant zero function and thus continue the solution by a correctly shifted Reissner-Nordström solution with total mass parameter M which corresponds to the sum of the black hole mass M_0 and the mass $4\pi \int_{r_0}^{R_{0q}} s^2 g_q(s)ds$ of the matter shell and with total charge parameter Q which corresponds to the sum of the black hole charge Q_0 and the total charge $4\pi \int_{r_0}^{R_{0q}} s^2 \rho_q(s)ds$ of the matter region with boundary conditions

$$\begin{aligned} 1 - \frac{2M}{R_{0q}} + \frac{Q^2}{R_{0q}^2} + \frac{K}{R_{0q}} &= \lim_{r \nearrow R_{0q}} e^{-2\lambda_q(r)} \\ &= 1 - \frac{8\pi}{R_{0q}} \int_{r_0}^{R_{0q}} s^2 g_q(s)ds - \frac{1}{R_{0q}} \int_{r_0}^{R_{0q}} \frac{q^2(s)}{s^2}ds - \frac{2M_0}{R_{0q}} + \frac{Q_0^2}{r_0 R_{0q}} \end{aligned} \tag{8.91}$$

This fixes the constant K to

$$K = -8\pi \int_{r_0}^{R_{0q}} s^2 g_q(s)ds - \int_{r_0}^{R_{0q}} \frac{q^2(s)}{s^2}ds + \frac{Q_0^2}{r_0} - \frac{Q^2}{R_{0q}} \tag{8.92}$$

and the components of the shifted Reissner-Nordström solution beyond the support of the matter quantities read

$$e^{-2\lambda(r)} = e^{2\mu(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{r}\int_{r_0}^{R_{0q}} s^2 g_q(s) ds - \frac{1}{r}\int_{r_0}^{R_{0q}} \frac{q^2(s)}{s^2} ds + \frac{Q_0^2}{r r_0} - \frac{Q^2}{r R_{0q}} \tag{8.93}$$

We can use this metric until $r \to \infty$. $\square$

9. Charged black holes surrounded by charged shells for $\Lambda > 0$

Having proven the existence of the background solution of the system with a central black hole with $\Lambda = 0$, but nonzero charge parameters, we now use this result to proof the existence of solutions in the case $\Lambda > 0$. The procedure will be the same as in the previous chapter (chapter 8). We state the existence of solutions in the following theorem

Theorem 9.1 (Global existence with central black hole)**.** *Let $E_0 = 1$ and $L_0, M_0 > 0$ as well as $M_0^2 > Q_0^2$ and $L_0 > 16M_0^2$. Furthermore, let $(\mu_q(r), \lambda_q(r), q(r))$ denote the background solution, which exists under suitable conditions as we proved in Theorem 8.1, and has matter quantities supported on a shell $\{r_{+q} < r < R_{0q}\}$. Then there exists a unique solution $(\mu_\Lambda,\ \lambda_\Lambda,\ q_\Lambda)$ on $(r_{B\Lambda}, r_{C\Lambda})$ to the EVMS if $\Lambda > 0$ is small enough and the charges q_0 and Q_0 have the same sign and their absolute values are small enough, where $r_{B\Lambda}$ is the black hole horizon and $r_{C\Lambda}$ the cosmological horizon of the system. The spatial support of f_Λ is contained in a shell $\{r_{+\Lambda} < r < R_{0\Lambda}\}$. In the complement of this shell the solution is given by a (correctly shifted) Reissner-Nordström-de-Sitter metric.*

Proof. At first we proof again that we can construct the desired configuration under the conditions of theorem 9.1 to guarantee the existence of a region where we can continue $f_\Lambda = 0$ by a nontrivial Ansatz (7.13). To be more precise, we want to show that there exists an interval $[r_{-\Lambda}, r_{+\Lambda}]$ outside of the black hole horizon, where a nontrivial Ansatz (7.13) for the particle distribution function gives us a vacuum solution, namely an unshifted Reissner-Nordström-de-Sitter solution. As we did in the case with $\Lambda = 0$ we do this by showing the existence of a region outside the black hole horizon where $\gamma_\Lambda \leq 0$ where γ_Λ is the gamma function (2.23) with the adapted electromagnetic energy $\tilde{I}_\Lambda$, given in (7.10) and where e^μ corresponds to the first metric component of the unshifted Reissner-Nordström-de-Sitter solution (7.8). As we have seen at the very beginning, that is in the chapter Setup 2, the condition $\gamma_\Lambda \leq 0$ is equivalent to the condition (2.22), which in this case now reads

$$\sqrt{1+\frac{L_0}{r^2}}\sqrt{1-\frac{2M_0}{r}+\frac{Q_0^2}{r^2}-\frac{r^2\Lambda}{3}} - q_0Q_0\left(\frac{1}{r_v}-\frac{1}{r}\right) \geq 1 \tag{9.1}$$

where we used the to equation (8.3) equivalent definition

$$\tilde{I}_{\Lambda v}(r) = q_0\int_{r_v}^{r}\frac{q_\Lambda(s)}{s^2}ds = q_0 q_\Lambda(r_v)\left(\frac{1}{r_v}-\frac{1}{r}\right) = q_0Q_0\left(\frac{1}{r_v}-\frac{1}{r}\right) \tag{9.2}$$

of the electromagnetic energy in the vacuum region, which is still positive and bounded above by $\frac{q_0 Q_0}{r_v}$ as we found in the previous chapter for $\tilde{I}_{qv}(r)$. Another time we start off looking at the first part of the left-hand side and define it as an own function

$$a_\Lambda(r) := \sqrt{1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} - \frac{r^2\Lambda}{3}} \cdot \sqrt{1 + \frac{L_0}{r^2}} \tag{9.3}$$

In figure 9.1 there is again depicted an example of the desired configuration. Another time, it only serves as a relief to visualize the scenario.

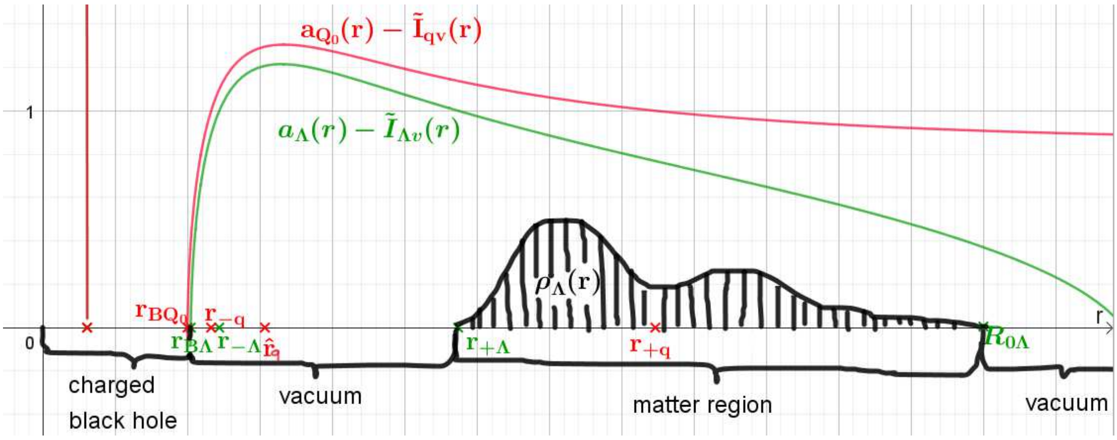

Figure 9.1.: Sketch of an exemplary configuration of a weakly charged black hole surrounded by a weakly charged matter shell for $\Lambda > 0$

As before, we see in red the function $a_{Q_0}(r) - \tilde{I}_{qv}(r)$, which plays here however the role of the background solution with the corresponding black hole horizon and the two ones $\{r_{-q}, r_{+q}\}$ of the function that are bigger than r_{BQ_0}. Moreover, the radius $\hat{r}$ indicates now where the maximum of $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ is. In green it is shown the graph for $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)$ with the black hole horizon $r_{B\Lambda}$ and the two ones $r_{-\Lambda}$ and $r_{+\Lambda}$ that are bigger than $r_{B\Lambda}$. Furthermore, we can see $\rho_\Lambda(r)$ as an example for one of the matter quantities with boundary of the support at $R_{0\Lambda}$. In the following we are going to show this configuration of the radii. We define

$$v_\Lambda(r) := 1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2} - \frac{r^2\Lambda}{3} \tag{9.4}$$

and notice again, that the zeros of $v_\Lambda(r)$ coincide with the zeros of $a_\Lambda(r)$ and because $v_\Lambda(r)$ is the first metric component of the Reissner-Nordström-de-Sitter metric its zeros give rise to the horizons of the system. The first component of the Schwarzschild-de-Sitter metric

$$v_{\Lambda, Q=0}(r) = 1 - \frac{2M_0}{r} - \frac{r^2\Lambda}{3} \tag{9.5}$$

possesses a black hole event horizon for Λ sufficiently small, which is given by the second largest zero of $v_{\Lambda,Q=0}(r)$. From the existence proof of the background solution, we also know that $v_{Q_0}(r)$, defined in (8.9), has a black hole horizon r_{BQ_0} (8.14). Since $v_{Q_0}(r) \geq v_\Lambda(r) \geq v_{\Lambda,Q=0}(r)$ for all r, and $v_\Lambda(r)$ is a continuous function, this implies that for small Λ also $v_\Lambda(r)$ has a finite black hole horizon, which we call $r_{B\Lambda}$. We want to find bounds on this radius and therefore calculate

$$v'_{Q_0}(r) = \frac{2M_0}{r^2} - \frac{2Q_0^2}{r^3} \tag{9.6}$$

and at r_{BQ_0} we find

$$\begin{aligned} v'_{Q_0}(r_{BQ_0}) &= \frac{1}{r_{BQ_0}^3}\left(2M_0 r_{BQ_0} - 2Q_0^2\right) \\ &= \frac{1}{r_{BQ_0}^3}\left(2M_0^2 + 2M_0\sqrt{M_0^2 - Q_0^2} - 2Q_0^2\right) > 0 \end{aligned} \tag{9.7}$$

where in the second line we plugged in the explicit expression (8.14) for r_{BQ_0}. So, in a neighbourhood of r_{BQ_0} the function $v_{Q_0}(r)$ comes from the negative plane, crosses zero at r_{BQ_0} and moves to the positive plane. Due to the fact that $v_{Q_0}(r) \geq v_\Lambda(r)$, this means that $r_{BQ_0} < r_{B\Lambda}$ and that also $v'_\Lambda(r_{B\Lambda}) > 0$. Together with the fact that $v_\Lambda(r)$ is decreasing in Λ this implies that

$$\frac{dr_{B\Lambda}}{d\Lambda} > 0 \tag{9.8}$$

which we will need later. In order to find an upper bound on $r_{B\Lambda}$, we do the following estimate

$$\begin{aligned} v_{Q_0}(r_{B\Lambda}) &= \int_{r_{BQ_0}}^{r_{B\Lambda}} v'_{Q_0}(s)ds + \underbrace{v_{Q_0}(r_{BQ_0})}_{=0} \\ &\geq \int_{r_{BQ_0}}^{r_{B\Lambda}} \inf_{s\in[r_{BQ_0}, r_{B\Lambda}]} v'_{Q_0}(s)ds \\ &= \int_{r_{BQ_0}}^{r_{B\Lambda}} \min\{v'_{Q_0}(r_{BQ_0}), v'_{Q_0}(r_{B\Lambda})\}ds \\ &= \min\{v'_{Q_0}(r_{BQ_0}), v'_{Q_0}(r_{B\Lambda})\}(r_{B\Lambda} - r_{BQ_0}) \end{aligned} \tag{9.9}$$

where $\inf_{s\in[r_{BQ_0}, r_{B\Lambda}]} v'_{Q_0}(s) = \min\{v'_{Q_0}(r_{BQ_0}), v'_{Q_0}(r_{B\Lambda})\}$, because $v'_{Q_0}(r)$ possesses only one extremum, which is however a maximum. To see this, we determine the finite radius where $v''_{Q_0} = 0$

$$
\begin{aligned}
0 = v''_{Q_0}(r) = -\frac{4M_0}{r^3} + \frac{6Q_0^2}{r^4} \\
\Rightarrow \quad r = \frac{3Q_0^2}{2M_0}
\end{aligned}
\tag{9.10}
$$

and check that the second derivative of $v'(r)$ is negative at this radius

$$
\begin{aligned}
v'''_{Q_0}(\frac{3Q_0^2}{2M_0}) &= \left(\frac{12M_0}{r^4} - \frac{24Q_0^2}{r^5}\right)_{|r=\frac{3Q_0^2}{2M_0}} = \frac{2^4 \cdot 12M_0^5}{3^4Q_0^8} - \frac{2^5 \cdot 24Q_0^2M_0^5}{3^5Q_0^{10}} \\
&= \frac{2^4 \cdot 12M_0^5\,(3-2)}{3^5Q_0^8} = -\frac{2^4 \cdot 12M_0^5}{3^5Q_0^8} < 0
\end{aligned}
\tag{9.11}
$$

For the difference between the black hole horizons we thus obtain

$$
r_{B\Lambda} - r_{BQ_0} \leq \frac{v_{Q_0}(r_{B\Lambda})}{\min\{v'_{Q_0}(r_{BQ_0}), v'_{Q_0}(r_{B\Lambda})\}}
\tag{9.12}
$$

Since $v_\Lambda(r_{B\Lambda}) = 0$ we find $v_{Q_0}(r_{B\Lambda}) = \frac{r_{B\Lambda}^2\Lambda}{3}$. Now we look at the cases $v'_{Q_0}(r_{BQ_0}) \leq v'_{Q_0}(r_{B\Lambda})$ and $v'_{Q_0}(r_{BQ_0}) \geq v'_{Q_0}(r_{B\Lambda})$ separately. In the first case inequality (9.12) becomes

$$
r_{B\Lambda} - r_{BQ_0} \leq \frac{r_{B\Lambda}^2\Lambda}{3v'_{Q_0}(r_{BQ_0})}
\tag{9.13}
$$

$v_{Q_0}(r)$ and therefore also $v'_{Q_0}(r)$ does not explicitly depend on Λ and also r_{BQ_0} is independent of Λ. Furthermore, according to (9.8), $r_{B\Lambda}$ is growing in Λ, so the right-hand-side of inequality (9.13) is growing in Λ. In the case that $v'_{Q_0}(r_{BQ_0}) \geq v'_{Q_0}(r_{B\Lambda})$, the inequality reads

$$
r_{B\Lambda} - r_{BQ_0} \leq \frac{r_{B\Lambda}^2\Lambda}{3} \cdot \frac{r_{B\Lambda}^3}{2M_0r_{B\Lambda} - 2Q_0^2} \leq \frac{r_{B\Lambda}^5\Lambda}{6(M_0r_{BQ_0} - Q_0^2)}
\tag{9.14}
$$

Now the denominator is again independent of Λ and the numerator is growing in Λ. So in both cases, we have the same behaviour and can reach a really small distance between $r_{B\Lambda}$ and r_{BQ_0} by choosing Λ really small. Doing so, we can reach

$$
r_{B\Lambda} < r_{-q}
\tag{9.15}
$$

where we defined r_{-q} as the lower for us relevant one of the function $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ in the existence proof of the background solution. In the next step we want to establish the existence of two radii $r_{-\Lambda}$ and $r_{+\Lambda}$ both bigger than the black hole horizon $r_{B\Lambda}$, at which $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r) = 1$ with definition . In the proof for the background solution we saw that $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ has two ones r_{-q} and r_{+q} between which the function is strictly bigger than one. Due to continuity reasons, the function has in this interval a unique maximum $\hat{r}_q$. The distance between $\left(a_{Q_0}(\hat{r}_q) - \tilde{I}_{qv}(\hat{r}_q)\right)^2$ and $\left(a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q)\right)^2$ is bounded above by

$$
\begin{aligned}
\left|\left(a_{Q_0}(\hat{r}_q) - \tilde{I}_{qv}(\hat{r}_q)\right)^2 - \left(a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q)\right)^2\right| &\leq \Lambda\frac{\hat{r}_q^2 + L_0}{3} + 2\,\underbrace{a_{Q_0}(\hat{r}_q)}_{=const.>1}\,\underbrace{\tilde{I}_{qv}(\hat{r}_q)}_{\leq\frac{q_0Q_0}{r_v}} \\
&\quad + 2\,\underbrace{|a_\Lambda(\hat{r}_q)|}_{=const.\geq 0}\,\underbrace{\tilde{I}_{\Lambda v}(\hat{r}_q)}_{\leq\frac{q_0Q_0}{r_v}} + \underbrace{\tilde{I}_{qv}^2(\hat{r}_q)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}} + \underbrace{\tilde{I}_{\Lambda v}^2(\hat{r}_q)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}} \\
&\leq \Lambda\frac{\hat{r}_q^2 + L_0}{3} + \left(q_0Q_0 + q_0^2Q_0^2\right)C
\end{aligned}
\tag{9.16}
$$

with a positive constant C. Choosing Λ and either the absolute value of q_0 or of Q_0 small enough we can reach

$$
\left|\left(a_{Q_0}(\hat{r}_q) - \tilde{I}_{qv}(\hat{r}_q)\right)^2 - \left(a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q)\right)^2\right| < \left(a_{Q_0}(\hat{r}_q) - \tilde{I}_{qv}(\hat{r}_q)\right)^2 - 1 \tag{9.17}
$$

and consequently

$$
a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q) > 1 \tag{9.18}
$$

where the possibility

$$
a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q) < -1 \tag{9.19}
$$

is excluded because choosing Λ and $|q_0|$ or $|Q_0|$ small makes the solution remain close to the background solution. Especially, since $a_{Q_0}(\hat{r}_q) > 1 + \tilde{I}_{qv}(\hat{r}_q) \geq 1$ and we choose Λ small we have at least $a_\Lambda(\hat{r}_q) > 1 - \epsilon$ with a really small number ϵ this means then that $a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q) > 1 - \epsilon - \frac{q_0Q_0}{r_v}$ which is by choosing also $|q_0|$ or $|Q_0|$ small for sure bigger than -1. Recording that $a_\Lambda(r) \leq a_{Q_0}(r)$ everywhere and therefore of course also $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r) \leq a_{Q_0}(r)$ and that $r_{-Q_0} < r_{-q} < \hat{r}_q < r_{+q} < r_{+Q_0}$ this implies that $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)$ has exactly two ones between r_{-Q_0} and r_{+Q_0} of which the first one is smaller than $\hat{r}_q$ and the second one is bigger than $\hat{r}_q$. We call the smaller one $r_{-\Lambda}$ and the bigger one $r_{+\Lambda}$. Thus, condition (9.1) is fulfilled in this region and we can at any radius inside this interval glue a nontrivial Ansatz (7.13) for the particle distribution function f_Λ to the initial constant zero function in a continuous way.

Now, we move on to prove that for the nontrivial Ansatz (7.13) under the conditions in Theorem 9.1 there exists a unique solution to the system at least until a radius $R_{0q}+\Delta R$, where R_{0q} is the boundary of the support of the background solution with $\Lambda = 0$, and ΔR is a positive constant. Furthermore we show that the matter quantities are of bounded support and that beyond the support we can again continue the particle distribution function continuously by a trivial Ansatz $f_\Lambda = 0$ and get again a (now shifted) Reissner-Nordström-de-Sitter solution which makes sense at least until the cosmological horizon $r_{C\Lambda}$. For the first gluing, which we have to do between $r_{-\Lambda}$ and $r_{+\Lambda}$ we want to choose a radius within this interval, where we can also perform the gluing process for the background solution. We have found

$$r_{-Q_0} < r_{-q} < \hat{r}_q < r_{+q} < r_{+Q_0} \tag{9.20}$$

as well as

$$r_{-Q_0} < r_{-\Lambda} < \hat{r}_q < r_{+\Lambda} < r_{+Q_0} \tag{9.21}$$

Due to this configuration of the radii, we know that

$$\max\{r_{-q}, r_{-\Lambda}\} < \min\{r_{+q}, r_{+\Lambda}\} \tag{9.22}$$

and define the minimum of r_{+q} and $r_{+\Lambda}$ as

$$r_{0\Lambda} := \min\{r_{+q}, r_{+\Lambda}\} \tag{9.23}$$

At this radius we can continue the trivial Ansatz $f_{q/\Lambda} = 0$ for both, the system with nonvanishing cosmological constant as well as for the background solution, by a nontrivial Ansatz of the form (7.13). Thus, $r_{0\Lambda}$ is our radius of choice to perform this first gluing processes. Integrating the Einstein equation (2.25) for $\lambda_\Lambda(r)$ from $r_{0\Lambda}$ to r leads to

$$e^{-2\lambda_\Lambda(r)} = 1 - \frac{8\pi}{r}\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds - \frac{1}{r}\int_{r_{0\Lambda}}^{r} \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3} + \frac{K}{r} \tag{9.24}$$

We impose the boundary condition

$$e^{-2\lambda_\Lambda(r_{0\Lambda})} = 1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2\Lambda}{3} \tag{9.25}$$

to match the solution resulting from the nontrivial Ansatz (7.13) for f_Λ to the unshifted Reissner-Nordström-de-Sitter solution at $r_{0\Lambda}$. This fixes the integration constant K to

$$\begin{aligned} K &= -2M_0 + \frac{Q_0^2}{r_{0\Lambda}} \\ &= r_{0\Lambda}\left(-\frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2\Lambda}{3}\right) + \frac{r_{0\Lambda}^3\Lambda}{3} \\ &= r_{0\Lambda}\left(e^{-2\lambda_\Lambda(r_{0\Lambda})} - 1\right) + \frac{r_{0\Lambda}^3\Lambda}{3} \end{aligned} \tag{9.26}$$

So, we have

$$e^{-2\lambda_\Lambda(r)} = 1 - \frac{8\pi}{r}\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds - \frac{1}{r}\int_{r_{0\Lambda}}^{r} \frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3} + \frac{r_{0\Lambda}}{r}\left(e^{-2\lambda_\Lambda(r_{0\Lambda})} - 1\right) + \frac{r_{0\Lambda}^3\Lambda}{3}$$
$$= 1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_{0\Lambda}^3}{r}\right) - \frac{2}{r}\left(\frac{r_{0\Lambda}}{2}\left(1 - e^{-2\lambda_\Lambda(r_{0\Lambda})}\right) + 4\pi\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds + \frac{1}{2}\int_{r_{0\Lambda}}^{r} \frac{q_\Lambda^2(s)}{s^2}ds\right) \tag{9.27}$$

Plugging this result in the second Einstein equation (2.26) for μ_Λ, we obtain

$$\mu_\Lambda'(r) = \frac{4\pi r h_\Lambda(r) - \Lambda\left(\frac{r}{3} + \frac{r_{0\Lambda}^3}{6r^2}\right) + \frac{r_{0\Lambda}}{2r^2}\left(1 - e^{-2\lambda_\Lambda(r_{0\Lambda})}\right) + \frac{4\pi}{r^2}\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds - \frac{q_\Lambda^2(r)}{2r^3} + \frac{1}{2r^2}\int_{r_{0\Lambda}}^{r}\frac{q_\Lambda^2(s)}{s^2}ds}{1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_{0\Lambda}^3}{r}\right) - \frac{2}{r}\left(\frac{r_{0\Lambda}}{2}\left(1 - e^{-2\lambda_\Lambda(r_{0\Lambda})}\right) + 4\pi\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds + \frac{1}{2}\int_{r_{0\Lambda}}^{r}\frac{q_\Lambda^2}{s^2}ds\right)} \tag{9.28}$$

for $r \geq r_{0\Lambda}$ with boundary condition

$$e^{2\mu_{0\Lambda}} := e^{2\mu_\Lambda(r_{0\Lambda})} = 1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2\Lambda}{3} \tag{9.29}$$

Like in the proof for the background solution, the right-hand side of the Einstein equation (9.28) is regular for all $r \geq r_{0\Lambda}$, so a solution exists locally and the additional terms including $r_{0\Lambda}$, M_0 and Q_0 are bounded and well behaved on any finite interval. Therefore, also the continuation criterion applies, which guarantees a solution at least as long as the denominator is positive. Now we show, also similar to what we did in the proof for regular solutions as well as for the background solution, that the maximum radius of existence R_C is bigger equal $R_{0q} + \Delta R$ if we choose Λ small enough. We call the denominators in the Einstein equation (9.28) for $\Lambda = 0$ and $\Lambda \neq 0$ respectively

$$d_{M_0}(r) := 1 - \frac{2}{r}\left(M_0 + 4\pi\int_{r_{0\Lambda}}^{r} s^2 g_q(s)ds\right) + \frac{Q_0^2}{r_{0\Lambda}r} - \frac{1}{r}\int_{r_{0\Lambda}}^{r}\frac{q^2(s)}{s^2}ds \tag{9.30}$$

and

$$d_{M_0\Lambda}(r) := 1 - \frac{2}{r}\left(M_0 + 4\pi\int_{r_{0\Lambda}}^{r} s^2 g_\Lambda(s)ds\right) + \frac{Q_0^2}{r_{0\Lambda}r} - \frac{1}{r}\int_{r_{0\Lambda}}^{r}\frac{q_\Lambda^2(s)}{s^2}ds - \frac{r^2\Lambda}{3} \tag{9.31}$$

and define

$$\Delta d_{0\Lambda} := \alpha_\Lambda \cdot d_{M_0\Lambda}(r_{0\Lambda}) = \alpha_\Lambda\left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2}\right) \tag{9.32}$$

with a positive constant $\alpha_\Lambda < 1$. Like in the proof for regular solutions and also for the background solution we define the radii

$$r_\Lambda^* := \inf\{r \in (r_{0\Lambda}, R_C) | d_{M_0\Lambda}(r) = \Delta d_{0\Lambda}\} \tag{9.33}$$

and

$$\tilde{r}_\Lambda := \inf\{r \in (r_{0\Lambda}, R_C) | \gamma_\Lambda(r) - \gamma(r)| > |\gamma(R_{0q} + \Delta R)|\} \tag{9.34}$$

and their minimum

$$\tilde{r}^*_\Lambda := \min\{r^*_\Lambda, \tilde{r}_\Lambda\} \tag{9.35}$$

For the background solution we also glue the trivial Ansatz to a nontrivial one at $r_{0\Lambda}$ and thus have the conditions

$$\lim_{r \searrow r_{0\Lambda}} e^{-2\lambda(r)} = 1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} \tag{9.36}$$

and

$$e^{2\mu_{0q}} := \lim_{r \searrow r_{0\Lambda}} e^{2\mu(r)} = 1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} \tag{9.37}$$

Again we estimate at first the distance between the initial values of the full solution and the background solution

$$\begin{aligned} |\mu_{0q} - \mu_{0\Lambda}| &= \frac{1}{2} \ln \left(\left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2}\right) \left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2 \Lambda}{3}\right)^{-1} \right) \\ &= \frac{1}{2} \ln \left(1 + \frac{r_{0\Lambda}^2 \Lambda}{3} \left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2 \Lambda}{3}\right)^{-1} \right) =: C_{0\Lambda} \end{aligned} \tag{9.38}$$

which we called $C_{0\Lambda}$, a positive constant growing in Λ and vanishing as $\Lambda \to 0$. We note that this constant also depends on $|Q_0|$. However, as long as the absolute value of Q_0 does not diverge, the constant $C_{0\Lambda}$ shows the desired behaviour with Λ, meaning that we can make it arbitrarily small by choosing Λ small. As we did in the proof for the background solution, we set g_Λ, g_q, h_Λ and h_q equal to 0 for $r \in (0, r_{0\Lambda})$. With this we estimate for $r_{0\Lambda} \leq r \leq \tilde{r}^*$

$$
\begin{aligned}
|\mu_q(r) - \mu_\Lambda(r)| \le{}& C_{0\Lambda} + \int_{r_{0\Lambda}}^{r} |\mu_q'(s) - \mu_\Lambda'(s)| ds \\
\le{}& C_{0\Lambda} + \int_{r_{0\Lambda}}^{r} \frac{1}{d_{M_0\Lambda}(s)} \Bigg(4\pi s |h_\Lambda(s) - h_q(s)| \\
&+ \frac{4\pi}{s^2} \int_0^s \sigma^2 \left| g_\Lambda(\sigma) - g_q(\sigma) \right| d\sigma + \left| \frac{s\Lambda}{3} \right| \\
&\left| - \frac{q_\Lambda^2}{2s^3} + \frac{1}{2s^2} \int_{r_{0\Lambda}}^{s} \frac{q_\Lambda^2(\sigma)}{\sigma^2} d\sigma \right| \Bigg) ds \\
&+ \int_{r_\Lambda}^{r} \left| \frac{4\pi}{d_{M_0\Lambda}(s)} - \frac{4\pi}{d_{M_0}(s)} \right| \left(s h_q(s) + \frac{1}{s^2} \int_0^s \sigma^2 g_q(\sigma) d\sigma + \left| \frac{M_0}{s^2} - \frac{Q_0^2}{2s^2 r_{0\Lambda}} \right| \right) ds \\
&+ \int_{r_{0\Lambda}}^{r} \frac{4\pi}{d_{M_0}(s)} \left| - \frac{q^2(s)}{8\pi s^3} + \frac{1}{8\pi s^3} \int_0^s \frac{q^2(\sigma)}{\sigma^2} d\sigma \right| ds
\end{aligned}
\tag{9.39}
$$

Similar to what we did in previous chapters, in the latter calculation we at first plugged in the Einstein equation (9.28) with $\Lambda = 0$ and $\Lambda > 0$ respectively and then subtracted and added a term $\frac{1}{d_{M_0\Lambda}(s)} \left(4\pi h_q(s) + \frac{4\pi}{s^2} \int_0^s g_q(\sigma) d\sigma + \frac{M_0}{s^2} - \frac{Q_0^2}{2s^2 r_{0\Lambda}} \right)$ inside of the absolute value under the integral. There is again a part similar to the regular case (5.11) and we can use some of our previous estimates. The estimate (8.69) for the charge terms can be adopted from the previous chapter. Also the M_0- and Q_0-terms we already estimated in the proof for the existence of the background solution. Now there is one additional Λ-term, which we estimate to

$$
\int_{r_{0\Lambda}}^{r} \frac{1}{d_{M_0\Lambda}(s)} \left| \frac{s\Lambda}{3} \right| ds \le \frac{\Lambda}{3\Delta d_{0\Lambda}} \left(\frac{r^2}{2} - \frac{r_{0\Lambda}^2}{2} \right) = \Lambda C(r) \tag{9.40}
$$

with $C(r) \ge 0$ continuous and growing in r. For the denominators we note that for $r \le \tilde{r}^*$ we have $d_{M_0\Lambda}(r) \ge \Delta d_{0\Lambda}$ and per construction of the background solution we always have $d_{M_0}(r) \ge b > 0$ with some positive constant b, because we used a nontrivial Ansatz (7.13) only in the interval $[r_0, R_{0q}]$ where the denominator is strictly positive. Thus, we obtain

$$
\left| \frac{1}{d_{M_0Q_0}} - \frac{1}{d_{M_0}} \right| = \frac{|d_{M_0} - d_{M_0Q_0}|}{d_{M_0Q_0} d_{M_0}} \le \frac{|d_{M_0} - d_{M_0Q_0}|}{b\Delta d_0} \tag{9.41}
$$

and it is left to estimate the difference between the denominators

$$
\begin{aligned}
|d_{M_0\Lambda}(r) - d_{M_0}(r)| \le{}& \left| \frac{r^2\Lambda}{3} \right| + 4\pi \int_0^r s^2 |g_\Lambda(s) - g_q(s)| ds \\
&+ \frac{1}{r} \int_{r_{0\Lambda}}^{r} \left| \frac{q^2(s)}{s^2} \right| ds + \frac{1}{r} \int_{r_{0\Lambda}}^{r} \left| \frac{q_\Lambda^2(s)}{s^2} \right| ds \\
\le{}& \Lambda C(r) + 4\pi r^2 \int_0^r |g_\Lambda(s) - g_q(s)| ds + (q_0 + Q_0)^2 C(r)
\end{aligned}
\tag{9.42}
$$

Collecting all the estimates, we obtain for all $r \in [r_{0\Lambda}, \tilde{r}^*]$ that

$$\begin{aligned} |\mu_q(r) - \mu_\Lambda(r)| &\leq C_{0\Lambda} + \left(\Lambda + (Q_0 + q_0)^2\right) C(r) + C(r) \int_0^r \left(|g_\Lambda - g_q| + |h_\Lambda - h_q|\right) ds \\ &= C_\Lambda(r) + (Q_0 + q_0)^2 C(r) + C(r) \int_0^r \left(|g_\Lambda - g_q| + |h_\Lambda - h_q|\right) ds \end{aligned} \tag{9.43}$$

with a positive function $C_\Lambda(r)$ that is growing in Λ and in r and vanishes as $\Lambda \to 0$ and this behaviour is not influenced as long as $|Q_0| < \infty$. Moreover, it is independent of q_0. The two positive functions $C(r)$ are growing in r and independent of Λ, Q_0 and q_0. Due to the same argument as in the proof of the previous chapter, it results

$$|g_\Lambda - g_q| + |h_\Lambda - h_q| \leq q_0 Q_0 C(r) + (Q_0 + q_0)^2 C(r) + C_\Lambda(r) + C(r) \int_0^r \left(|g_\Lambda - g| + |h_\Lambda - h|\right) ds \tag{9.44}$$

where we can combine

$$\begin{aligned} q_0 Q_0 C(r) + (Q_0 + q_0)^2 C(r) &= q_0 Q_0 C(r) + Q_0^2 C(r) + 2 q_0 Q_0 C(r) + q_0^2 C(r) \\ &\leq (Q_0 + q_0)^2 C(r) \end{aligned} \tag{9.45}$$

and finally

$$|g_\Lambda - g_q| + |h_\Lambda - h_q| \leq (Q_0 + q_0)^2 C(r) + C_\Lambda(r) + C(r) \int_0^r \left(|g_\Lambda - g| + |h_\Lambda - h|\right) ds \tag{9.46}$$

Again, a Grönwall argument leads to

$$|g_\Lambda - g_q| + |h_\Lambda - h_q| \leq (Q_0 + q_0)^2 C(r) + C_\Lambda(r) \tag{9.47}$$

Plugging this estimate in (10.17) gives

$$|\mu_q(r) - \mu_\Lambda(r)| \leq (Q_0 + q_0)^2 C(r) + C_\Lambda(r) \tag{9.48}$$

As in (5.26) this also means for the distance between the γ-functions

$$|\gamma_\Lambda(r) - \gamma_q(r)| \leq (Q_0 + q_0)^2 C(r) + C_\Lambda(r) \tag{9.49}$$

Thus, choosing $|Q_0|$, $|q_0|$ and Λ small enough we can assure

$$|\gamma_\Lambda(r) - \gamma_q(r)| \leq |\gamma(R_{0q} + \Delta R)| \tag{9.50}$$

for all $r \in (r_{0\Lambda}, R_{0q} + \Delta R)$. Next, we use estimate (9.47) for the difference of the g-functions to consider the distance (9.42) between the denominators. This results in

$$|d_{M_0}(r) - d_{M_0\Lambda}(r)| \leq (Q_0 + q_0)^2 C(r) + C_\Lambda(r) \tag{9.51}$$

So, choosing Λ, $|Q_0|$ and $|q_0|$ small enough, we can reach

$$|d_{M_0}(r) - d_{M_0\Lambda}(r)| \leq b - \Delta d_{0\Lambda} \tag{9.52}$$

for all $r \in [r_{0\Lambda}, R_{0q} + \Delta R]$, where we recall that b is a lower bound on $d_{M_0}(r)$ and $\Delta d_{0\Lambda}$ a lower bound on $d_{M_0\Lambda}(r)$ on the given interval and we choose the constant α_Λ in (8.60) small enough, such that $b - \Delta d_{0\Lambda} \geq C > 0$. Consequently we get

$$\underbrace{d_{M_0}(r) - b}_{\geq 0} + \Delta d_{0\Lambda} \leq d_{M_0\Lambda}(r) \tag{9.53}$$

and thus

$$\Delta d_{0\Lambda} \leq d_{M_0\Lambda}(r) \tag{9.54}$$

for all $r_{0\Lambda} \leq r \leq R_{0q} + \Delta R$ which, together with (9.50) means that

$$\tilde{r}^*_\Lambda \geq R_{0q} + \Delta R \tag{9.55}$$

Due to the same argumentation as in the case of regular solutions, using the properties of the background solution, we obtain the result, that a solution $(\mu_\Lambda, \lambda_\Lambda, q_\Lambda)$ exists at least until $R_{0q} + \Delta R$ and that the matter quantities are of bounded support with boundary $R_{0\Lambda} \leq R_{0q} + \Delta R$. Beyond the support there is again a vacuum region, where we can continue the nonvacuum solution by a correctly shifted Reissner-Nordström-de-Sitter solution up to the cosmological horizon. Here we have the boundary condition

$$1 - \frac{2M}{R_{0\Lambda}} + \frac{Q^2}{R_{0\Lambda}^2} - \frac{R_{0\Lambda}^2 \Lambda}{3} + \frac{K}{R_{0\Lambda}} = \lim_{r \nearrow R_{0\Lambda}} e^{-2\lambda_\Lambda(r)} \tag{9.56}$$

which again determines the constant K.

$\square$

10. Charged black holes surrounded by charged shells for $\Lambda < 0$

The global existence theorem with central black hole 9.1 can also be proven with a negative cosmological constant, if the absolute value of Λ is sufficiently small. Then, in the complement of the matter shell we continue the metric by a (correctly shifted) Reissner-Nordström-Anti-de-Sitter metric, which in its unshifted form is given by (6.1) and does, in contrast to the Reissner-Nordström-de-Sitter metric, not possess a cosmological horizon, so we can use it until $r \to \infty$. We need to adapt the proof of the global existence theorem for $\Lambda > 0$ a bit.

Especially in the first part where we show the existence of a vacuum region $[r_{-\Lambda}, r_{+\Lambda}]$ outside of the black hole horizon for a non-trivial Ansatz of form (7.13), we have to change the argumentation in some parts, because the functions $a_\Lambda(r)$, $v_\Lambda(r)$ and $v_{\Lambda,Q=0}$ behave quite differently if we change the sign of Λ. The scenario looks then something like in figure 10.1.

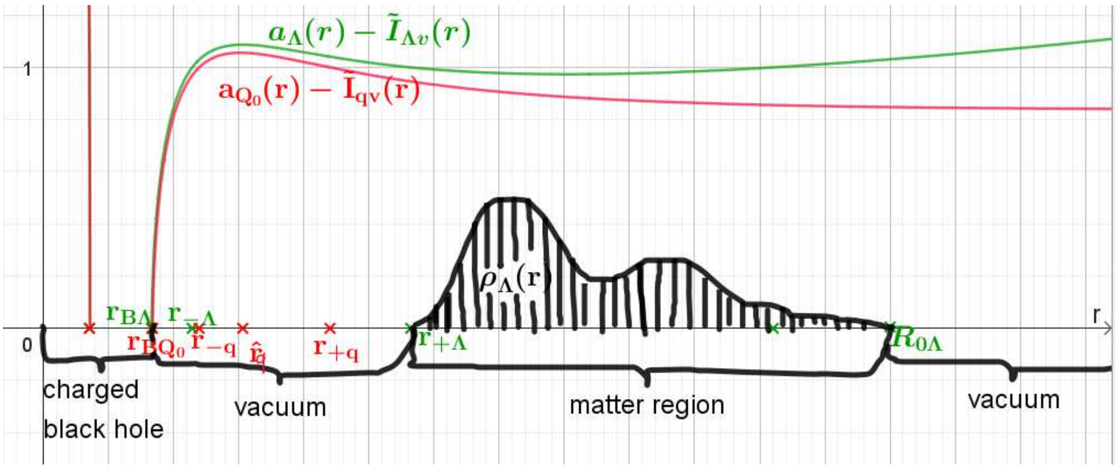

Figure 10.1.: Sketch of an exemplary configuration of a weakly charged black hole surrounded by a weakly matter shell for $\Lambda < 0$

We recognize that the main difference to the case for $\Lambda > 0$, which is illustrated in figure 9.1, is that there the background function $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ was bigger than $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)$, while now it is the other way round. This means that when estimating quantities for our function of interest (so the one with the Λ index) we have to take care when we use the background solution for the estimates and we can not just adopt the arguments we used for $\Lambda > 0$. Let us go through the configuration of the radii again. For positive Λ we argued that $v_{Q_0}(r) \geq v_\Lambda(r) \geq v_{\Lambda,Q=0}(r)$ and both, $v_{Q_0}(r)$ and $v_{\Lambda,Q=0}(r)$ possess a zero which corresponds to the black hole horizon. From this follows then that also $v_\Lambda(r)$ has to possess such a zero. However, for a negative sign of Λ we have the situation $v_{Q_0}(r) \leq v_\Lambda(r)$ and also $v_{\Lambda,Q=0}(r) \leq v_\Lambda(r)$, so the argument does not work anymore. So, we are going to argue that $v_{\Lambda,Q=0}(r)$ possesses a zero, functioning as a black hole horizon, and that we can then make the distance between $v_{\Lambda,Q=0}(r)$ and $v_\Lambda(r)$ sufficiently small by choosing a weak black hole charge Q_0, such that also $v_\Lambda(r)$ has a zero, corresponding to the black hole horizon $r_{B\Lambda}$ of the charged black hole of our solution of interest. The equation $v_{\Lambda,Q=0} = 0$ becomes for finite r a cubic equation

$$-2M_0 + r - \frac{r^3\Lambda}{3} = 0 \tag{10.1}$$

which for a negative cosmological constant we can write as

$$-2M_0 + r + \frac{r^3|\Lambda|}{3} = 0 \tag{10.2}$$

The discriminant Δ_v of this equation reads according to formula (8.22)

$$\Delta_v = \frac{-4|\Lambda|^3 - (-6M_0|\Lambda|)^2}{3|\Lambda|} \tag{10.3}$$

Given that this is strictly negative, there exists only one real solution to the cubic equation. Investigating the behaviour of the function $v_{\Lambda,Q=0}$ for $r \searrow 0$ and $r \to \infty$, we see that in the positive plane it comes from $-\infty$ and goes to $+\infty$ and because it is continuous in the positive plane, the only zero of the function has to be at some radius r_{B0} with $0 < r_{B0} < \infty$. This corresponds to the black hole horizon for the Schwarzschild-Anti-de-Sitter metric and is the only horizon that exists in this uncharged system. At any radius that lies between 0 and r_{B0} and that is located at a finite distance from 0 and r_{B0}, we have $v_{\Lambda,Q=0}(r) \leq a < 0$ with some finite negative constant a. Now, by choosing Q_0 sufficiently small, we can make the distance

$$v_\Lambda(r) - v_{\Lambda,Q=0}(r) = \frac{Q_0^2}{r^2} \tag{10.4}$$

at a radius $r < r_{B0}$ situated at a finite distance from 0 and r_{B0}, so small, such that also $v_\Lambda(r) < 0$ at this radius. Note however that $\lim_{r \searrow 0} v_\Lambda(r) = +\infty$. Moreover, we know that $v_\Lambda(r) > v_{\Lambda,Q=0}(r)$ at any r, so in the interval $[r_{B0}, \infty)$ we have for sure $v_\Lambda(r) > 0$. Due to continuity in the positive plane, it follows that $v_\Lambda(r)$ possesses two zeros in the positive plane, of which the bigger zero is the black hole horizon $r_{B\Lambda}$. We also note that there is no zero bigger than $r_{B\Lambda}$, so there is also no cosmological horizon for the Reissner-Nordström-Anti-de-Sitter metric. Due to the negative sign of Λ, we have for all r that $v_\Lambda(r) > v_{Q_0}(r)$ and consequently $r_{B\Lambda} < r_{BQ_0}$. Now let us look at the ones of the functions. We can estimate the distance between $a_\Lambda^2(r)$ and $a_{Q_0}^2(r)$ by

$$\left|a_\Lambda^2(r) - a_{Q_0}^2(r)\right| \le \left|-\frac{r^2\Lambda}{3}\left(1+\frac{L_0}{r^2}\right)\right| = \frac{r^2|\Lambda|}{3}\left(1+\frac{L_0}{r^2}\right) \tag{10.5}$$

after bringing $a_{Q_0}^2(r)$ to the other side and taking the square root, we obtain

$$|a_\Lambda(r)| \le \sqrt{\frac{r^2|\Lambda|}{3}\left(1+\frac{L_0}{r^2}\right) + a_{Q_0}^2(r)} \le r\sqrt{\frac{|\Lambda|}{3}\left(1+\frac{L_0}{r^2}\right)} + |a_{Q_0}(r)| \tag{10.6}$$

where in the last step we used the fact that for two positive numbers $a, b \ge 0$ it holds that

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b} \tag{10.7}$$

Then, for the distance between $\left(a_{Q_0}(r) - \tilde{I}_{qv}(r)\right)^2$ and $\left(a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)\right)^2$ in the intervals $[r_{BQ_0}, r_{-q})$ and (r_{+q}, ∞), where we already know that $0 \le a_{Q_0}(r) - \tilde{I}_{qv}(r) < 1$ we find

$$\begin{aligned}\left|\left(a_{Q_0}(r) - \tilde{I}_{qv}(r)\right)^2 - \left(a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)\right)^2\right| &\le |\Lambda|\frac{r^2+L_0}{3} + 2\underbrace{a_{Q_0}(r)}_{<1}\underbrace{\tilde{I}_{qv}(r)}_{\le\frac{q_0Q_0}{r_v}} \\ &\quad + 2\underbrace{|a_\Lambda(r)|}_{see(10.6)}\underbrace{\tilde{I}_{\Lambda v}(r)}_{\le\frac{q_0Q_0}{r_v}} + \underbrace{\tilde{I}_{qv}^2(r)}_{\le\frac{q_0^2Q_0^2}{r_v^2}} + \underbrace{\tilde{I}_{\Lambda v}^2(r)}_{\le\frac{q_0^2Q_0^2}{r_v^2}} \\ &\le |\Lambda|\frac{r^2+L_0}{3} + r\sqrt{\frac{|\Lambda|}{3}\left(1+\frac{L_0}{r^2}\right)}\frac{q_0Q_0}{r_v} \\ &\quad + \left(q_0Q_0 + q_0^2Q_0^2\right)C\end{aligned} \tag{10.8}$$

with a constant C independent of Λ, q_0 and Q_0. At a finite distance from the interval limits, we have $a_{Q_0}(r) - \tilde{I}_{qv}(r) \le a$ with a finite number $0 < a < 1$ and thus, by choosing a sufficiently small absolute value for Λ and a sufficiently small absolute value for q_0 or Q_0 we can reach

$$\left|\left(a_{Q_0}(r) - \tilde{I}_{qv}(r)\right)^2 - \left(a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)\right)^2\right| < 1 - a^2 \tag{10.9}$$

and it follows

$$\begin{aligned}\left(a_\Lambda(r) - \tilde{I}_{q\Lambda}(r)\right)^2 &< 1 - a^2 + \left(a_{Q_0}(r) - \tilde{I}_{qv}(r)\right)^2 \\ &\le 1 - a^2 + a^2 = 1\end{aligned} \tag{10.10}$$

Taking the square root gives then

$$-1 < a_\Lambda(r) - \tilde{I}_{q\Lambda}(r) < 1 \tag{10.11}$$

On the other hand, inside of the interval (r_{-q}, r_{+q}) we have $a_{Q_0}(r) - \tilde{I}_{qv}(r) > 1$ and the function has because of its continuity a unique maximum inside of this interval which we call $\hat{r}$ just as we did in the case for $\Lambda > 0$. Also the estimate is almost the same as in equation (9.16) for a positive cosmological constant, with the difference that now we keep the absolute value of Λ and finally end up with

$$\left|\left(a_{Q_0}(\hat{r}_q) - \tilde{I}_{qv}(\hat{r}_q)\right)^2 - \left(a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q)\right)^2\right| \leq |\Lambda|\frac{\hat{r}_q^2 + L_0}{3} + \left(q_0 Q_0 + q_0^2 Q_0^2\right) C \quad (10.12)$$

So, choosing $|\Lambda|$ and $|q_0|$ or $|Q_0|$ sufficiently small, we can, by performing the same steps as we did it for the positive cosmological constant reach

$$a_\Lambda(\hat{r}_q) - \tilde{I}_{\Lambda v}(\hat{r}_q) > 1 \quad (10.13)$$

Together with the previous result and due to continuity reasons we have now found that $a_\Lambda(r) - \tilde{I}_{q\Lambda}(r)$ possesses exactly two ones in the interval $[r_{BQ_0}, \infty)$, especially they are also bigger than the black hole horizon $r_{B\Lambda}$, because of $r_{B\Lambda} \leq r_{BQ_0}$. We name them again $r_{-\Lambda}$ and $r_{+\Lambda}$ like in the $\Lambda > 0$ case and note that the smaller one is situated below $\hat{r}_q$ and the bigger one above $\hat{r}_q$, so we can write

$$r_{B\Lambda} \leq r_{BQ_0} \leq r_{-\Lambda} \leq \hat{r}_q \leq r_{+\Lambda} \quad (10.14)$$

The interval $[r_{-\Lambda}, r_{+\Lambda}]$ is again a region, where a nontrivial Ansatz of the form (7.13) gives vacuum, and where consequently it is possible to continue the constant zero function $f_\Lambda = 0$ by such a nontrivial Ansatz in a continuous way.

The second part of the proof is really similar to the one with positive Λ, with the main difference that we need to work with the absolute value of Λ instead of just Λ. Due to the configuration (10.14) and

$$r_{BQ_0} \leq r_{-q} \leq \hat{r}_q \leq r_{+q} \quad (10.15)$$

we can again choose the radius $r_{0\Lambda} = \min\{r_{+q}, r_{+\Lambda}\}$ to perform the gluing process of the trivial Ansatz to the nontrivial Ansatz of the particle distribution function. The Einstein equations and their boundary conditions stay the same as for $\Lambda > 0$ and also the definitions of r^*, $\tilde{r}$ and $\tilde{r}^*$ remain unchanged. The first time where there is some modification required is when estimating the distance between the initial values for μ for the background solution and the full solution. For negative Λ we have

$$|\mu_{0q} - \mu_{0\Lambda}| = \frac{1}{2}\left|\ln\left(\left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2}\right)\left(1 - \frac{2M_0}{r_{0\Lambda}} + \frac{Q_0^2}{r_{0\Lambda}^2} - \frac{r_{0\Lambda}^2\Lambda}{3}\right)^{-1}\right)\right| =: C_{0\Lambda} \quad (10.16)$$

where the behaviour of $C_{0\Lambda}$ is now different, because as Λ has a negative sign, the denominator inside the logarithmic function is compared to the numerator bigger because it has an additional positive term $+\frac{r_{0\Lambda}^2|\Lambda|}{3}$. Therefore, the argument of the logarithm is smaller equal one which makes the logarithm itself negative. The lower the argument of the logarithm, the lower is the logarithm and thus the bigger its absolute value. So, the absolute value of the logarithm grows as we raise $|\Lambda|$. This means that $C_{0\Lambda}$ is a constant that grows with the absolute value of Λ and vanishes for $\Lambda = 0$. In all the further estimates ((9.39) - (9.42)) we just have to keep writing $|\Lambda|$ instead of just Λ and then everything works the same and leads to

$$\begin{aligned}|\mu_q(r) - \mu_\Lambda(r)| &\leq C_{0\Lambda} + \left(|\Lambda| + (Q_0 + q_0)^2\right) C(r) + C(r) \int_0^r \left(|g_\Lambda - g_q| + |h_\Lambda - h_q|\right) ds \\ &= C_\Lambda(r) + (Q_0 + q_0)^2 C(r) + C(r) \int_0^r \left(|g_\Lambda - g_q| + |h_\Lambda - h_q|\right) ds\end{aligned} \tag{10.17}$$

with a positive function $C_\Lambda(r)$ that is growing in $|\Lambda|$ and r, vanishing for $\Lambda = 0$ and shows this behaviour for any finite Q_0. Furthermore, it is independent of q_0. The two positive functions $C(r)$ are growing in r and independent of Λ, Q_0 and q_0. The next steps can be again performed as in the case for positive Λ and give us estimates for the difference between the g- and h-functions (see (9.47)), for the difference between the μ-functions (see(9.48)), for the difference between the γ-functions (see (9.49)) and for the difference between the denominators (see (9.51)). Finally, it is possible to choose $|Q_0|$, $|q_0|$ and $|\Lambda|$ sufficiently small to obtain

$$|\gamma_\Lambda(r) - \gamma_q(r)| \leq |\gamma(R_{0q} + \Delta R)| \tag{10.18}$$

as well as

$$|d_{M_0}(r) - d_{M_0\Lambda}(r)| \leq b - \Delta d_{0\Lambda} \tag{10.19}$$

on all $r \in [r_{0\Lambda}, R_{0q} + \Delta R]$. We recall that b is a lower bound on $d_{M_0}(r)$ and $\Delta d_{0\Lambda}$ a lower bound on $d_{M_0\Lambda}(r)$ on the given interval. Furthermore, we choose the constant α_Λ in (8.60) small enough, such that $b - \Delta d_{0\Lambda} \geq C > 0$ to find

$$\Delta d_{0\Lambda} \leq d_{M_0\Lambda}(r) \tag{10.20}$$

in the latter interval. With inequality (10.18) it follows again that

$$\tilde{r}_\Lambda^* \geq R_{0q} + \Delta R \tag{10.21}$$

Another time the argumentation in part I for the regular solutions applies and leads to the result that a solution $(\mu_\Lambda, \lambda_\Lambda, q_\Lambda)$ exists at least until $R_{0q} + \Delta R$, that the matter quantities are of bounded support with boundary $R_{0\Lambda} \leq R_{0q} + \Delta R$ and that beyond the support we can continue the solution by another Reissner-Nordström-Anti-de-Sitter solution, which is now shifted in such a way that we can perform the gluing process continuously at the radius $R_{0\Lambda}$. Now the Reissner-Nordström-Anti-de-Sitter solution can be used until $r \to \infty$, because there is no cosmological horizon.

11. Black hole and matter region with opposite sign of charge

In part I for the regular solutions we did not make any assumption about the sign of q_0. The only restriction that we put on q_0 was that its absolute value should be sufficiently small, but whether the particles are positively or negatively charged did not matter for the existence of regular solutions. Also in part II of the present thesis, when investigating charged black holes of total charge Q_0 surrounded by charged matter shells, again of particle charge q_0, we did not limit ourselves to a fixed sign of the charges. However, we always let q_0 and Q_0 have the same sign, so we only considered the cases where the black hole and the matter region are either both positively or both negatively charged and where consequently there is a repulsive electric force between them. This assumption was just to keep the calculations simple, but in reality we can still proof the existence of solutions for two different signs of q_0 and Q_0 as long as both their absolute values stay low. Let us go through this case. The main thing that changes is that the sign of the electric energy contribution (7.10) can now also be negative, because the total charge $q_\Lambda(r)$ (or $q(r)$ for the background solution) has for a sufficiently small radius a different sign than q_0 due to the central black hole which has an opposite sign of the charge compared to the matter region. In the vacuum region $r \in [r_{B\Lambda}, r_{+\Lambda}]$ between the black hole and the matter region we especially have a strictly negative adapted electric energy

$$\tilde{I}_{\Lambda v}(r) = q_0 Q_0 \left(\frac{1}{r_v} - \frac{1}{r} \right) < 0 \tag{11.1}$$

However, the expression is still bounded where now we estimate its absolute value by

$$\left| \tilde{I}_{\Lambda v}(r) \right| \leq \left| \frac{q_0 Q_0}{r_v} \right| \tag{11.2}$$

The same holds for the background solution, so also there we have equation (11.1) and (11.2) where we replace $\tilde{I}_{\Lambda v}$ by $\tilde{I}_{qv}$. Due to the different sign of $\tilde{I}_{\Lambda v}(r)$ compared to the case where $sgn(q_0) = sgn(Q_0)$, also the graphs of $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ and $a_\Lambda(r) - \tilde{I}_{\Lambda v}(r)$ look a bit different. Namely all function values are slightly higher, and the effect gets stronger the higher r becomes, but is restricted by (11.2). So, since we still choose $|q_0 Q_0|$ small, the fundamental behaviour of the graphs do not change. However, we have to reconsider some arguments in the proofs taking into account the sign of the electric energy.

Let us start with chapter 8, where $\Lambda = 0$. The first time that $\tilde{I}_q(r)$ plays a role there is in equation (8.45) where we estimated the distance between $a(\hat{r})$ and $a(\hat{r}) - \tilde{I}_{qv}(\hat{r})$ to show that we can make this distance so small, such that also $a(\hat{r}) - \tilde{I}_{qv}(\hat{r}) > 1$. Then from $a(r) - \tilde{I}_{qv}(r) \leq a_{Q_0}(r) - \tilde{I}_{qv}(r) \leq a_{Q_0}(r)$ we deduced that $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ possesses exactly two ones in the interval $[r_{-Q_0}, r_{+Q_0}]$. Now we can not argue like that any more because $\tilde{I}_{qv}(r) < 0$ and consequently $a(r) - \tilde{I}_{qv}(r) > a(r)$ and also $a_{Q_0}(r) - \tilde{I}_{qv}(r) > a_{Q_0}(r)$. However from the last inequality it follows that between the two ones r_{-Q_0} and r_{+Q_0} of $a_{Q_0}(r)$ also $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ is for sure bigger than one. In this case we want to show that $a_{Q_0}(r) - \tilde{I}_{qv}(r) < 1$ at some point in the interval (r_{BQ_0}, r_{-Q_0}) and at another point in the interval (r_{+Q_0}, ∞), because then due to continuity the existence of two ones follow. So, we estimate

$$\left| a_{Q_0}(r) - \left(a_{Q_0}(r) - \tilde{I}_{qv}(r) \right) \right| < \left| \frac{q_0 Q_0}{r_v} \right| \tag{11.3}$$

This enables us to make the distance between $a_{Q_0}(r)$ and $a_{Q_0}(r) - \tilde{I}_{qv}(r)$ so small, such that inside of the above mentioned intervals, at a finite distance from the interval limits, where $a_{Q_0}(r) \leq a$ with some finite number $a < 1$, we can reach

$$a_{Q_0}(r) - \tilde{I}_{qv}(r) < 1 \tag{11.4}$$

and the existence of two ones r_{-q} and r_{+q} follows with

$$r_{-q} < r_{-Q_0} < r_- < r_+ < r_{+Q_0} < r_{+q} \tag{11.5}$$

So, the first part of the proof is complete. In the second part not a lot is changing. We can estimate the charges (cf. equation (8.68)) now by

$$\frac{q^2(r)}{r^2} \leq (Q_0 + q_0)^2 C \leq (Q_0^2 + q_0^2) C \tag{11.6}$$

and consequently also get the upper bound $(Q_0^2 + q_0^2)C(r)$ for the charge terms (8.69) which occur in the distance between the μ-functions. Furthermore, also for the distance between the denominators (8.77) we obtain

$$|d_{M_0}(r) - d_{M_0 Q_0}(r)| \leq 8\pi r \int_0^r |g_q(s) - g(s)| ds + q_0^2 C(r) + C_{Q_0} \tag{11.7}$$

where we absorbed the Q_0^2 term in the constant C_{Q_0}. Then for the distance between the μ-functions (8.78) we also find

$$|\mu(r) - \mu_q(r)| \leq r C_{Q_0} + q_0^2 C(r) + C(r) \int_0^r (|g_q - g| + |h_q - h|)\, ds \tag{11.8}$$

When estimating $|\Delta_I g|$, we can still use equation (8.79) which gives a proportionality of $\tilde{I}_q(r)$ to $q_0 Q_0 C(r) + q_0^2 C(r)$. With that we can estimate

$$|\Delta_I g| \leq \left(|q_0 Q_0| + q_0^2 \right) C(r) \tag{11.9}$$

which is the same as we had for the case that $sgn(q_0) = sgn(Q_0)$, just with the absolute value of q_0Q_0. The rest of the proof works analogously to chapter 8, just that we always take $|q_0Q_0|$ instead of q_0Q_0. This concerns estimate (8.81) - (8.84) and (8.86).

In the proof of the global existence theorem with black hole 9.1 in chapter 9 with positive cosmological constant we have to do similar changes. Another time we need to adapt the argument when showing the existence of two ones outside of the black hole radius. Similar to estimate (9.16) we find

$$\begin{aligned}\left|\left(a_{Q_0}(r)-\tilde{I}_{qv}(r)\right)^2-\left(a_\Lambda(r)-\tilde{I}_{\Lambda v}(r)\right)^2\right| &\leq \Lambda\frac{r^2+L_0}{3}+2\underbrace{a_{Q_0}(r)}_{=const.>1}\underbrace{\left|\tilde{I}_{qv}(r)\right|}_{\leq\left|\frac{q_0Q_0}{r_v}\right|}\\ &\quad+2\underbrace{|a_\Lambda(r)|}_{=const.\geq 0}\underbrace{\left|\tilde{I}_{\Lambda v}(r)\right|}_{\leq\left|\frac{q_0Q_0}{r_v}\right|}+\underbrace{\tilde{I}^2_{qv}(r)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}}+\underbrace{\tilde{I}^2_{\Lambda v}(r)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}}\\ &\leq\Lambda\frac{r^2+L_0}{3}+\left(|q_0Q_0|+q_0^2Q_0^2\right)C\end{aligned} \tag{11.10}$$

Again at the unique maximum $\hat{r}_q$ of $a_{Q_0}(r)-\tilde{I}_{qv}(r)$ we can take Λ and either $|q_0|$ or $|Q_0|$ sufficiently small to reach

$$a_\Lambda(\hat{r}_q)-\tilde{I}_{\Lambda v}(\hat{r}_q)>1 \tag{11.11}$$

Moreover, at a finite distance from the interval limits inside the intervals $(r_{B\Lambda}, r_{-q})$ and (r_{+q}, r_C), where $a_{Q_0}(r)-\tilde{I}_{qv}(r)\leq a$ with a finite number $a<1$, we can also reach

$$a_\Lambda(r)-\tilde{I}_{\Lambda v}(r)<1 \tag{11.12}$$

Then again, due to continuity reasons the existence of two ones $r_{-\Lambda}$ and $r_{+\Lambda}$ is guaranteed where

$$r_{-\Lambda}<\hat{r}_q<r_{+\Lambda} \tag{11.13}$$

For the second part of the proof we estimate this time

$$|\tilde{I}_\Lambda(r)|\leq|q_0Q_0C(r)+q_0^2C(r)|\leq q_0^2C(r) \tag{11.14}$$

and the same holds for $\tilde{I}_q(r)$ which then leads to

$$|\Delta_I g|\leq q_0^2C(r) \tag{11.15}$$

and we can proceed with the proof analogously to the case with the same sign of the charges in chapter 9.

For chapter 9 for the negative cosmological constant, estimate (11.16) becomes now

$$\begin{aligned}
\left|\left(a_{Q_0}(r)-\tilde{I}_{qv}(r)\right)^2-\left(a_\Lambda(r)-\tilde{I}_{\Lambda v}(r)\right)^2\right| &\leq |\Lambda|\frac{r^2+L_0}{3}+2\underbrace{a_{Q_0}(r)}_{<1}\underbrace{\left|\tilde{I}_{qv}(r)\right|}_{\leq\left|\frac{q_0Q_0}{r_v}\right|} \\
&\quad+2\underbrace{|a_\Lambda(r)|}_{see(10.6)}\underbrace{\left|\tilde{I}_{\Lambda v}(r)\right|}_{\leq\left|\frac{q_0Q_0}{r_v}\right|}+\underbrace{\tilde{I}^2_{qv}(r)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}}+\underbrace{\tilde{I}^2_{\Lambda v}(r)}_{\leq\frac{q_0^2Q_0^2}{r_v^2}} \\
&\leq |\Lambda|\frac{r^2+L_0}{3}+r\sqrt{\frac{|\Lambda|}{3}\left(1+\frac{L_0}{r^2}\right)}\left|\frac{q_0Q_0}{r_v}\right| \\
&\quad+\left(|q_0Q_0|+q_0^2Q_0^2\right)C
\end{aligned} \tag{11.16}$$

and we can draw the same conclusions as for the case where q_0 and Q_0 had the same sign.

12. Conclusions and outlook

In the present thesis we have studied the Einstein-Vlasov-Maxwell system under static and spherical symmetric conditions. We have investigated it in terms of the existence of solutions under certain conditions. In the first part of the thesis we treated the regular case, where we imposed a regular center with the initial conditions $\lambda(0) = q(0) = 0$ and $\mu(0) = \mu_C$ with some finite constant μ_C. It could be proved that under these assumptions there exist local solutions and that if we additionally require a negative central value μ_C, a weak particle charge q_0 and a small absolute value of the cosmological constant, there exist also global solutions. For the global solutions we have seen that the matter quantities are of bounded support and that beyond the support we can continue the solution by a vacuum solution. Depending on the sign of Λ this is either a Reissner-Nordström-de-Sitter metric (for $\Lambda > 0$), which can be used until the cosmological horizon r_C beyond which we have no access to any information, or a Reissner-Nordström-Anti-de-Sitter metric (for $\Lambda < 0$), which can be used until $r \to \infty$ because for a negative sign of Λ there is no cosmological horizon and we can in theory get information from every point in spacetime. The second part of the thesis concerned the system with a black hole singularity at the center. Also here we needed to assume a small absolute value for the cosmological constant, a weak particle charge q_0 and a weak black hole charge Q_0. It has been shown that then outside of the black hole event horizon there exists a radial interval $[r_{-\Lambda}, r_{+\Lambda}]$ where a nontrivial Ansatz for the particle distribution function that solves the Vlasov equation leads to a vacuum region. Therefore, it was possible to start with a trivial Ansatz $f_\Lambda = 0$ and a Reissner-Nordström(-Anti)-de-Sitter solution outside of the black hole horizon and then continue f_Λ inside of the vacuum region $[r_{-\Lambda}, r_{+\Lambda}]$ by a nontrivial Ansatz, which eventually gives a matter region starting from $r_{+\Lambda}$. Again it could be proved that the matter region has a bounded support and beyond the outer boundary the particle distribution function can be continued again by the constant zero function and the solution can be continued by another Reissner-Nordström(-Anti)-de-Sitter metric.

Further research of the spherical symmetric EVMS with nonvanishing cosmological constant could treat the massless case $m = 0$, as has been for example done by [1] for the EVMS without cosmological constant. Another idea could be to investigate different global topologies on which solutions to the system exist, such as they have been studied in paper [2] for the uncharged EVS, but with nonvanishing cosmological constant. In the present thesis we have made a few assumptions on the system, especially restricting ourselves to spherical symmetry and static conditions. Future research projects could also try to relax these conditions. For example one could move to a stationary, axial symmetric system. There have already been some investigations with these symmetry assumptions of the uncharged Einstein-Vlasov system with vanishing cosmological constant. Paper [5] provides a summary of the present situation and open problems for this case. According to the authors the stationary, axisymmetric system has been studied numerically, where many different morphologies could be generated depending on the specific form of the Ansatz for the particle distribution function. Especially toroidal solutions have been studied further numerically where a different behaviour has been discovered depending on whether the total angular momentum is greater or smaller than the total mass squared. Analytically it has been proved that there exist solutions to the static axial symmetric EVS where the matter region is compactly supported in the radial direction. Moreover, the existence of solutions to the stationary, axial symmetric EVMS, so where the particles are also considered to be charged, has been proved for the case that the solutions only slightly differ from spherical symmetric solutions. Additionally, another analytic approach has shown that stationary solutions can be constructed as bifurcations from the Kerr spacetime, representing a more general case of a black hole surrounded by Vlasov matter. Some open problems concerning the stationary, axial symmetric EVS are the existence of stationary, axial symmetric solutions that are far from spherical symmetry, highly relativistic solutions, the massless case, and solutions with a central black hole. All these results and open questions can also be investigated in the charged case, so extending the Einstein-Vlasov system to the Einstein-Vlasov-Maxwell system. Another attempt could be to include the cosmological constant in the equations and study its effects in the stationary axial symmetric scenario.

To conclude, there remains significant potential for a further exploration of the Einstein-Vlasov-Maxwell system beyond the conditions studied in this thesis. Particularly in the stationary, axial symmetric case there are a lot of open problems. These concern also still the uncharged case, for which the results found in this thesis could provide some assistance, due to the parallels between the charged, static system and the uncharged rotating system as we noted in the introduction.

Bibliography

[1] Maximilian Thaller. Existence of static solutions of the einstein-vlasov-maxwell system and the thin shell limit. *Society for Industrial and Applied Mathematics*, 51(3):2231–2260, 2019.

[2] Håkan Andreasson, David Fajman, and Maximilian Thaller. Static solutions to the einstein-vlasov system with a nonvanishing cosmological constant. *Society for Industrial and Applied Mathematics*, 47(4):2657–2688, 2015.

[3] Maximilian Thaller. *Compact Relativistic Matter Shells with Massless and Charged Particles*. Thesis for the degree of licentiate of philosophy, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden, 2017.

[4] Gerhard Rein. Static solutions of the spherically symmetric vlasov-einstein system. *Mathematical Proceedings of the Cambridge Philosophical Society*, 115(3):559–570, 1994.

[5] Ellery Ames and Håkan Andreasson. Stationary solutions of the axially symmetric einstein–vlasov system: present status and open problems. *Classical and Quantum Gravity*, 41, 2024.

A. Appendix

Here you can find some of the calculations and arguments whose results we used in the thesis, but which have not been carried out in the main body to avoid disrupting the flow of reading. At the corresponding points in the text a reference is made to the appendix.

A.1. Grönwall's inequality

During the proof of Lemma 5.1 in chapter 5 about the existence beyond the non-vacuum region, we made use of a form of Grönwall's inequality whose statement is the following: For any real-valued and continuous function $u(x)$ with

$$u(x) \leq \alpha(x) + \int_a^x \beta(s)u(s)ds \tag{A.1}$$

with a constant lower integral limit $a \leq x$, a real, continuous function $\beta(x)$ and a real, non-decreasing function $\alpha(x)$ whose negative part is integrable on $[a, x]$, it holds that

$$u(x) \leq \alpha(x) \exp \int_a^x \beta(s)ds \tag{A.2}$$

A.2. Proof that $\tilde{D} > 0$

To show that the equation $a_{Q_0}(r) = 1$ has three distinct, real solutions we need to show that $\tilde{D}$, defined in (8.22) is strictly positive. For this aim, at first we expand the brackets and find

$$\begin{aligned} \tilde{D} &= 4M_0^2 L_0 \left(-2Q_0^4 + 5Q_0^2 L_0 - 4M_0^2 L_0\right) + M_0^2 L_0^3 - Q_0^2 \left(Q_0^2 + L_0\right)^3 \\ &= -8L_0 M_0^2 Q_0^4 - 16L_0^2 M_0^4 - Q_0^8 - L_0^3 Q_0^2 - 3L_0 Q_0^6 - 3L_0^2 Q_0^4 + 20L_0^2 M_0^2 Q_0^2 + L_0^3 M_0^2 \end{aligned} \tag{A.3}$$

then we split up the second last term to remove step by step the negative terms. We write

$$20L_0^2 M_0^2 Q_0^2 = 3L_0^2 M_0^2 Q_0^2 + 1L_0^2 M_0^2 Q_0^2 + 16L_0^2 M_0^2 Q_0^2 \tag{A.4}$$

and do the following estimates

$$
\begin{aligned}
L_0^2 M_0^2 Q_0^2 &> 16 L_0 M_0^4 Q_0^2 > 16 L_0 M_0^2 Q_0^4 > 8 L_0 M_0^2 Q_0^4 + 8 L_0 Q_0^6 \\
&> 8 L_0 M_0^2 Q_0 4 + 3 L_0 Q_0^6 + 5 \cdot 16 M_0^2 Q_0^6 > 8 L_0 M_0^2 Q_0 4 + 3 L_0 Q_0^6 + 5 \cdot 16 Q_0^8 \\
&> 8 L_0 M_0^2 Q_0 4 + 3 L_0 Q_0^6 + Q_0^8
\end{aligned}
\tag{A.5}
$$

and

$$3 L_0^2 M_0^2 Q_0^2 > 3 L_0^2 Q_0^4 \tag{A.6}$$

This means for $\tilde{D}$ that

$$
\begin{aligned}
\tilde{D} &> -16 L_0^2 M_0^4 - L_0^3 Q_0^2 + 16 L_0^2 M_0^2 Q_0^2 + L_0^3 M_0^2 \\
&= -16 L_0^2 M_0^2 \left(M_0^2 - Q_0^2\right) + L_0^3 \left(M_0^2 - Q_0^2\right) \\
&= L_0^2 \underbrace{\left(M_0^2 - Q_0^2\right)}_{>0} \underbrace{\left(L_0 - 16 M_0^2\right)}_{>0} > 0
\end{aligned}
\tag{A.7}
$$

and the proof is complete.

A.3. Proof that $r_{-Q_0} > M_0 - \sqrt{M_0^2 - Q_0^2}$

To prove that $r_{-Q_0} > M_0 - \sqrt{M_0^2 - Q_0^2}$, it is sufficient to proof the inequality in (8.32), because we already realized in (8.31) that $r_{-Q_0} \geq \frac{Q_0^2 + L_0 - \sqrt{\left(Q_0^2 + L_0\right)^2 - 12 L_0 M_0^2}}{6 M_0}$. To proof that this latter expression is bigger than $M_0 - \sqrt{M_0^2 - Q_0^2}$, we firstly realize that

$$12 M_0^2 \left(3 M_0^2 - Q_0^2\right) > 0 \tag{A.8}$$

and that

$$12 M_0^2 \left(3 M_0^2 - Q_0^2\right) = 12 L_0 M_0^2 - \left(L_0 + Q_0^2\right)^2 + \left(L_0 + Q_0^2\right)^2 - 12 M_0^2 \left(L_0 + Q_0^2\right) + 36 M_0^4 \tag{A.9}$$

plugging this equality into inequality (A.8) and bringing the first two terms to the left-hand side, we obtain

$$\left(L_0 + Q_0^2\right)^2 - 12 M_0^2 \left(L_0 + Q_0^2\right) + 36 M_0^4 > \left(L_0 + Q_0^2\right)^2 - 12 L_0 M_0^2 \tag{A.10}$$

Since

$$\left(L_0 + Q_0^2\right)^2 - 12 L_0 M_0^2 > L_0^2 - 12 L_0 M_0^2 > 16 L_0 M_0^2 - 12 L_0 M_0^2 > 0 \tag{A.11}$$

we can take the square root of inequality (A.10). For this purpose we note that

$$\left(L_0 + Q_0^2\right)^2 - 12M_0^2\left(L_0 + Q_0^2\right) + 36M_0^4 = \left(L_0 + Q_0^2 - 6M_0^2\right)^2 \tag{A.12}$$

and find

$$\begin{aligned} L_0 + Q_0^2 - 6M_0^2 &> \sqrt{\left(L_0 + Q_0^2\right)^2 - 12L_0M_0^2} \\ &> \sqrt{\left(L_0 + Q_0^2\right)^2 - 12L_0M_0^2} - 6M_0\sqrt{M_0^2 - Q_0^2} \end{aligned} \tag{A.13}$$

We rearrange this to

$$L_0 + Q_0^2 - \sqrt{\left(L_0 + Q_0^2\right)^2 - 12L_0M_0^2} > 6M_0^2 - 6M_0\sqrt{M_0^2 - Q_0^2} \tag{A.14}$$

and divide as a last step by $6M_0$ which results in

$$\frac{L_0 + Q_0^2 - \sqrt{\left(L_0 + Q_0^2\right)^2 - 12L_0M_0^2}}{6M_0} > M_0 - \sqrt{M_0^2 - Q_0^2} \tag{A.15}$$

and proves (8.32) and thus $r_{-Q_0} > M_0 - \sqrt{M_0^2 - Q_0^2}$.

A.4. Proof that $r_{+Q_0} \geq r_-$

In order to show that $r_{+Q_0} \geq r_-$, we estimate

$$\begin{aligned} r_{+Q_0} - r_- &\geq \frac{Q_0^2 + L_0}{6M_0} + \frac{1}{2}\sqrt{\frac{4}{3}\left(\frac{\left(Q_0^2 + L_0\right)^2}{12M_0^2} - L_0\right)} - \frac{L_0}{4M_0} + \sqrt{\frac{L_0^2}{16M_0^2} - L_0} \\ &\geq \frac{L_0}{6M_0} + \frac{1}{2}\sqrt{\frac{L_0^2}{9M_0^2} - \frac{4L_0}{3}} - \frac{L_0}{4M_0} + \sqrt{\frac{L_0^2}{16M_0^2} - L_0} \\ &\geq \frac{1}{2}\sqrt{\frac{L_0^2}{9M_0^2} - \frac{4L_0}{3}} - \frac{L_0}{12M_0} \end{aligned} \tag{A.16}$$

Since

$$\begin{aligned} \frac{1}{4}\left(\frac{L_0^2}{9M_0^2} - \frac{4L_0}{3}\right) - \frac{L_0^2}{144M_0^2} &= \frac{L_0^2}{36M_0^2} - \frac{L_0^2}{144M_0^2} - \frac{L_0}{3} = \frac{3L_0^2}{144M_0^2} - \frac{L_0}{3} \\ &> \frac{48M_0^2L_0}{144M_0^2} - \frac{L_0}{3} = \frac{L_0}{3} - \frac{L_0}{3} = 0 \end{aligned} \tag{A.17}$$

we have

$$\frac{1}{4}\left(\frac{L_0^2}{9M_0^2} - \frac{4L_0}{3}\right) > \frac{L_0^2}{144M_0^2} \tag{A.18}$$

Due to positivity we can take the square root which leads to

$$\frac{1}{2}\sqrt{\frac{L_0^2}{9M_0^2} - \frac{4L_0}{3}} > \frac{L_0}{12M_0} \tag{A.19}$$

Using this in inequality (A.16) yields

$$r_{+Q_0} - r_- > 0 \tag{A.20}$$

and we are done.