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The Quantisation of the Electromagnetic Field in the Coulomb Gauge and its Physical Implications

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Abstract

The aim of this work is to shine light on the precise processes by which locality is preserved in (semi-)quantum regimes under the imposition of the Coulomb gauge, by example of a simple double particle system. One of these two, acting as the *sender*, is conceptualised as a classical particle, while the other, acting as a *probe* and the surrounding field are treated as quantum objects. A seemingly novel approach of transforming the Dyson series is employed to garner a representation of the probe particle's density matrix as an infinite polynomial in time. The first three degrees, acting as three increasing levels of approximation, are fully laid out and examined for dependencies on the source particle that would imply faster than light signalling. Finally, a position measurement is performed, to see whether any of the terms in question might extend their effect to measurement results.

Kurzfassung

Das Ziel dieser Arbeit ist es, anhand des Beispiels eines simplen Zweipartikelsystems die genauen Prozesse zu beleuchten, wonach die Lokalität in (Semi-)Quantensystemen, welche der Coulomb-Eichung unterstehen, erhalten bleibt. Eines dieser Teilchen, der *Sender*, wird dabei als klassiches Partikel behandelt, das zweite *Empfänger*-Teilchen und das umliegende Feld hingegen als Quantenobjekte. Eine scheinbar neuartige Transformation zur Dyson-Serie erlaubt es, eine Dastellung der Dichtematrix des Empfängerteilchens als unendliches Polynom in der Zeitdomäne zu erlangen. Die ersten drei Grade dieses Polynoms, entsprechend dreier steigender Stufen der Näherung, werden ermittelt und dabei auf Einflusse des Senderpartikels untersucht, welche eine superluminale Informationsübertragung implizieren könnten. Letztlich wird eine Messung in der Ortsdomäne durchgeführt, um zu sehen ob sich diese etwaigen Terme auch in den Messergebnissen niederschlagen.

Contents

1. Introduction

1.1. The Coulomb Gauge

For any physical system, the associated Lagrangian will come with some number of redundant degrees of freedom. These can be regulated via the *choice* of a gauge. One area in which this strikes particular relevance is electrodynamics, where the *gauge field* takes the form of the electromagnetic four-potential and multiple options for gauges arise, each pair of which can be transitioned between via a group of *gauge transformations*. Even though the various candidates are, in principle, all equal in viability, some have established themselves as particularly helpful in many aspects and thus the most popular. Foremost of all stand the *Lorentz* and *Coulomb gauges*, the latter of which will be the primary object of interest in this work, as its manifestation in a quantum regime raises some questions with regard to causality.[3] The key issue of concern rests on the following fundamentals [6][1]:
As every four potential A_μ consists of a scalar component A_0 and a vector potential $\vec{A}$, to enforce the Coulomb gauge embodies nothing more than the imposition that the divergence of the latter be zero:

$$\nabla \cdot \vec{A} = 0 \tag{1.1}$$

Consider the relations between electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the four potential:

$$\vec{E} = -\nabla A_0 - \frac{\partial \vec{A}}{\partial t} \tag{1.2}$$

$$\vec{B} = \nabla \times \vec{A} \tag{1.3}$$

Consider also the Maxwell equations, with ρ and $\vec{j}$ denoting the electric charge and current density, respectively:

$$\nabla \cdot \vec{E} = \rho \tag{1.4}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.5}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.6}$$

$$\nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} \tag{1.7}$$

The constraints of keeping the above equations satisfied compounded with the gauge will force the four potential into the following shape [6][7]:

$$A_0(\vec{x},t) = \frac{1}{4\pi}\int_{\mathbb{R}} d\vec{x}' \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}',t) \tag{1.8}$$

$$\vec{A}(\vec{x},t) = \nabla \times \int_{\mathbb{R}} d\vec{x}' \frac{1}{|\vec{x}-\vec{x}'|}\vec{B}(\vec{x}',t) \tag{1.9}$$

While easily glanced over, these integrals reveal something very peculiar: At each point in time are both aspects of the four potential dependent on *every point in space*, implying an instantaneous transfer of information over great distances and thereby a breach of the principle of locality. This is commonly placated by the argument that the potential itself does not directly affect the actors of any given system (e.g. the motions of particles), only the *EM-field* given via (1.2) and (1.3) does, serving as an intermediary to relay $\vec{A}'s$ influence to matter. In a classical context, this field can be shown to be local after all, via one of two ways: One would be to invoke the gauge equivalence principle and defer to the Lorentz gauge, in which the preservation of locality is more readily apparent. The other, more involved option would be to simply show it by direct calculation.[1][2]
Another interesting point arises when considering the electric field's longitudinal component $\vec{E}_{||}$ and transversal component $\vec{E}_{\perp}$ individually:

$$\vec{E}_{||} = -\nabla A_0 \tag{1.10}$$

$$\vec{E}_{\perp} = -\frac{\partial}{\partial t}\vec{A} \tag{1.11}$$

Both are, once again, non-local! The fact that the total field $\vec{E}$ is provably local then necessitates that these non-local aspects somehow exactly cancel each other as the two parts combine into the whole.[1]

$$\vec{E} = \vec{E}_{||} + \vec{E}_{\perp} = \vec{E}_{||,local} + \cancel{\vec{E}_{||,nonlocal}} + \vec{E}_{\perp,local} + \cancel{\vec{E}_{\perp,nonlocal}} \tag{1.12}$$

In the quantum regime, a number of problems arise. For one thing, it is not clear how the classical gauge equivalence principle is to be translated in the quantum case, making any proof by reference to the Lorentz gauge unworkable. Further, the electric field's longitudinal and transversal aspects will fall into completely different Hilbert spaces - the former turning into an operator on the space of particles and and the latter into a quantum field operator - and thus there can be no argument of shared terms cancelling as they combine along eq. (1.12) above.[1]
It is widely assumed that locality still holds in the quantum realm, and that this should be independent of the choice of gauge. For the Coulomb gauge however, the *process by which it is preserved* and the *point* in any procedure beginning with the definition of a system at its most basic and ending in the obtainment of the relevant measurement results, at which 'non-locality collapses', are far from clear. This is precisely what this work aims to shine more light on.

In the following chapter, a simple double particle system will be introduced. A method will be developed to gain an expression for the state of one of the particles as it develops in time, and thereby also the expectation value of its position in space. It will arise that both will have to take the form of infinite polynomials in time, rising degrees of which can be investigated individually as approximations of similarly rising detail. Should any constant of the polynomial then show any influence of the second particles movement, this would imply an instantaneous transfer of information and violate locality. In subsequent chapters 3-5, this investigation is carried out in the first three degrees of approach. This leads to a couple of general statements that can be made in chapter 6, and in chapter 7, the findings are discussed, as is the prospect of approaching higher or making modifications to the system.

2. Preliminaries

2.1. Establishing the System

The system to be examined in the coming pages is, at its heart, very simple. Consider nothing but two particles, both embedded in the same *field* F. The *probe particle* P and *source particle* S will act as sender and receiver of one bit of information at some space-like separation. Let F and P be modeled as quantum objects and S as a classical particle with associated current[1] j and position $\vec{y}$.

$$j_\mu(\vec{x}, t) = e\delta(\vec{x} - \vec{y}(t)) \begin{pmatrix} 1 \\ \dot{\vec{y}}(t) \end{pmatrix} \tag{2.1}$$

A transfer of information between the particles could then take the form of S being sent on one of two trajectories with corresponding currents j_A and j_B, imparting a different influence on P, whereupon the signal could be appropriately decoded.
As shown in the coming sections, it is possible to derive an expression for the probe particle's state as it evolves over an infinitesimal amount of time past $t = 0$. More precisely, it will take the shape of a polynomial in time, with constants that may or may not reference the source's movement - and thereby a correspondence to either j_A or j_B. Should any such dependencies be found, it would mean that a possible change in currents and trajectories would spread between the particles *instantaneously*. This would prove that the state behaves non-locally, and if the effect continues downstream to the level of measurement results, it would constitute superluminal signalling.

2.2. The Hamiltonian

All considerations will occur in the Schrödinger picture and thereby with time-independent operators. The Hamiltonian of the full system will be the following [1]:

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}(\vec{q})\right)^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e}{4\pi}\int_{\mathbb{R}} d\vec{x}\, \frac{j^0(\vec{x}, t)}{|\vec{x} - \vec{q}|} - \int_{\mathbb{R}} d\vec{x}\, \vec{j}(\vec{x}, t) \cdot \vec{A}(\vec{x}) \tag{2.2}$$

[1] The four-current as defined here is itself not relativistic. However, over the following arguments and calculations it will be quite apparent that the lack of a Lorentz factor makes no difference for the issues at hand and is therefore omitted in an effort towards simplicity.

Herein, $\vec{q}$ and $\vec{p}$ denote P's associated position and momentum operators, respectively. The constants m and e represent the probe's mass and the elementary charge, while $\vec{A}$ will be the potential, defined as follows[1]:

$$\vec{A}_{\vec{x}} := \vec{A}(\vec{x}) = (2\pi)^{-3/2} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2|\vec{k}|}} \left(e^{i\vec{k}\vec{x}} a_{\vec{k},\lambda} + e^{-i\vec{k}\vec{x}} a^{\dagger}_{\vec{k},\lambda} \right) \vec{\epsilon}_{\lambda} \tag{2.3}$$

Insert this into equation (2.2) to arrive at the Hamiltonian's definitive form:

$$H = \frac{1}{2m} \left(\vec{p} - e\vec{A}_{\vec{q}} \right)^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi} \frac{1}{|\vec{y}(t) - \vec{q}|} - e\, \dot{\vec{y}}(t) \cdot \vec{A}_{\vec{y}(t)} \tag{2.4}$$

These terms can be referred to, from left to right, as the *momentum term(s)*, the *free field term*, the *Coulomb term* and the *interaction term*.

2.3. The Initial State

The field is simply taken to be the vacuum field $|0\rangle$. Some notes on other choices of field can be found in section 7. The initial state of the probe particle $|\phi\rangle$ can be taken in its most general form. The initial state of the full system is then the following:

$$|\psi_0\rangle_{FP} = |0\rangle_F \otimes |\phi\rangle_P \tag{2.5}$$

This can also be expressed as a density matrix:

$$\rho_{0FP} = |0\rangle \langle 0|_F \otimes |\phi\rangle \langle \phi|_P \tag{2.6}$$

The density matrix corresponding to the probe's initial state will be denoted by Φ:

$$\Phi_P = tr_F(\rho_{0FP}) = |\phi\rangle \langle \phi|_P \tag{2.7}$$

For the sake of brevity, the indices F and P will be omitted going forward.

2.4. Time Evolution

2.4.1. The Dyson Series

Note that the Hamiltonian, as defined in eq. (2.4), is not constant in time, as the Coulomb and interaction terms introduce an indirect dependence via the source's position and velocity. This makes it somewhat unwieldy, as the time evolution of any state can no longer be calculated directly, but only via the *Dyson series*, an infinite sum of progressively more intricately nested integrals [4]:

$$|\psi(T)\rangle = \sum_{g=0}^{\infty} (-i)^g \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{g-1}} dt_g\, H(t_1)H(t_2)\cdots H(t_g)\, |\psi_0\rangle \tag{2.8}$$

By implementing a number of substitutions, this can be transformed into a polynomial in time with constant coefficients, yielding a more manageable form of time evolution.

2.4.2. Polynomialisation

Begin by substituting $t_m = \Pi_{j=1}^{m} \alpha_j T$:

$$|\psi(T)\rangle = \sum_{g=0}^{\infty} (-i)^g T^g \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_g \prod_{j=1}^{g} \alpha_j^{g-j} H(\Pi_{s=1}^{j} \alpha_s T) |\psi_0\rangle \tag{2.9}$$

Each Hamiltonian can also be expanded around the origin via the Taylor series:

$$H(\tau) = \sum_{n=1}^{\infty} \frac{\tau^k}{n!} H_0^{[n]} \qquad \textit{with } H_0^{[n]} = \frac{d^n}{dt^n} H(t)|_{t=0} \tag{2.10}$$

Insert this appropiately into eq. (2.9), reorder the terms and integrate over the now isolated variables α_j:

$$\begin{aligned}
|\psi(T)\rangle = & \sum_{g=0}^{\infty} (-i)^g T^g \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_g \prod_{j=1}^{g} \alpha_j^{g-j} \\
& \sum_{n_1, n_2, \ldots, n_g = 0}^{\infty} \prod_{k=1}^{g} \alpha_k^{(\sum_{z=k}^{g} n_z)} T^{(\sum_{z=1}^{g} n_z)} \frac{1}{n_k!} H_0^{[n_k]} |\psi_0\rangle \\
= & \sum_{g=0}^{\infty} (-i)^g \sum_{n_1, n_2, \ldots, n_g = 1}^{\infty} T^{(g + \sum_{z=1}^{g} n_z)} \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_g \\
& \prod_{j=1}^{g} \alpha_j^{\left(g - j + \sum_{z=j}^{g} n_z\right)} \frac{1}{n_j!} H^{[n_j]} |\psi_0\rangle \\
= & \sum_{g=0}^{\infty} (-i)^g \sum_{n_1, n_2, \ldots, n_g = 0}^{\infty} T^{(g + \sum_{z=1}^{g} n_z)} \prod_{j=1}^{g} \frac{1}{(g - j + 1 + \sum_{z=j}^{g} n_z)} \frac{1}{n_j!} H_0^{[n_j]} |\psi_0\rangle
\end{aligned} \tag{2.11}$$

This might seem imposing at first glance. A look at section A.4, where select terms are calculated in detail, might help in gaining a more practical understanding.

2.5. Decomposing the Density Matrix

Terms sharing the same exponent $G = g + \sum_{z=1}^{g} n_z$ on T may be grouped together, allowing for a decomposition of the wave function into a polynomial over T with constants $|\psi_G\rangle$:

$$|\psi(T)\rangle = \sum_{G=0}^{\infty} T^G |\psi_G\rangle \tag{2.12}$$

A corresponding density matrix ρ can also be constructed and similarly decomposed:

$$\rho(T) = \sum_{G, G' = 0}^{\infty} T^{G+G'} |\psi_G\rangle \langle\psi_{G'}| = \sum_{N=0}^{\infty} T^N \rho_N \tag{2.13}$$

A constant ρ_N of this polynomial is then defined as follows:

$$\rho_N = \sum_{G+G'=N} |\psi_G\rangle \langle\psi_{G'}| \tag{2.14}$$

The precise object of interest for this work is found in the reduced density matrix of the probe particle, which can be uncovered by taking the partial trace on F over the equations above:

$$\rho_P = tr_F(\rho(T)) = \sum_{N=0}^{\infty} T^N tr_F[\rho_N] \tag{2.15}$$

This implies that even for the smallest increment of time $T = dt$, all constants of the polynomial immediately lay their effects onto ρ, meaning that if any one of them shows any dependence on the source's movements[2], this equates to immediate signalling between particles. These constants $tr_F[\rho_N]$ can now be investigated, individually and for rising N corresponding to increased amount of detail, for any such dependencies. This makes it possible to structure the search for the mechanisms of locality in a concise manner of variable precision.

2.6. Measurement

The result ρ_P denotes *only the state* of the probe particle, a somewhat intangible object with limited practical applications. To provide a more workable end result, a measurement on this state will be performed. More specifically, the position measurement will be produced, at some location $\vec{r}$, with special attention payed to what might happen to any eventual non-local terms in the density matrix.

$$\langle\rho_P\rangle = \langle\vec{r}|\, tr_F(\rho(T))\, |\vec{r}\rangle \tag{2.16}$$

The position domain is chosen as it provides local measurement results with a relatively clear ontology as opposed to e.g. momentum measurements (and by extension also the clearest applicability to practical settings). Alternatives are discussed in section 7.4.

[2]That is, its initial velocity $\dot{\vec{y}}_0$, acceleration $\ddot{\vec{y}}_0$, and so on (compare again eq. (2.11)). The source's initial position $\vec{y}_0$ carries no information about its movement and is not relevant to this point.

3. Linear Order

Consider the density matrix constant relevant to a linear order expansion of ρ:

$$\rho_1 = |\psi_0\rangle \langle\psi_1| + |\psi_1\rangle \langle\psi_0| \tag{3.1}$$

Reference section A.4 for the structure of $|\psi_1\rangle$:

$$|\psi_1\rangle = -iH_0 |\psi_0\rangle \tag{3.2}$$

From this, we can derive the explicit form of ρ_1:

$$\rho_1 = i\rho_0 H_0 + h.c. \tag{3.3}$$

Take the partial trace on F to isolate the state of the probe particle.

$$tr_F(\rho_1) = i\, tr_F(\rho_0 H_0) + h.c. = i\Phi \langle 0| H_0 |0\rangle + h.c. \tag{3.4}$$

Reference the Hamiltonian's full expression in eq. 2.4 to see how it interacts with the vaccum states:

$$\langle 0| H_0 |0\rangle = \langle 0| \frac{1}{2m} \left(\vec{p} - e\vec{A}(\vec{q})\right)^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}|\, a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|} - e\, \dot{\vec{y}}_0 \cdot \vec{A}(\vec{y}_0) |0\rangle \tag{3.5}$$

As established in section A.1, uneven numbers of $\vec{A}$-instances will disappear between vacuum states. This immediately eliminates the interaction term, and with it the only instance of the source's velocity, barring any possibility of signalling at this order of approximation. For the sake of completeness and to illustrate the points made in section 6.2, however, should the following consderations also be noted.

The 'mixed' terms of the momentum set will vanish for the same reason stated above. The free field term will vanish as well, leaving only the Coulomb and remaining momentum terms. Reference eq. (A.7.1) for the relevant vacuum interaction to arrive at the following result:

$$\langle 0| H_0 |0\rangle = \frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\frac{1}{(2\pi)^3}\int_{\mathbb{R}} d\vec{k}\frac{1}{|\vec{k}|} \tag{3.6}$$

As this is plugged into eq. (3.4), the last term, being scalar, will vanish in the commutator, leading to the following result:

$$tr_F(\rho_1) = i\frac{1}{2m}\Phi\vec{p}^2 + i\frac{e^2}{4\pi}\Phi\frac{1}{|\vec{y}_0 - \vec{q}|} + h.c. \tag{3.7}$$

Moving into the position measurement, it is clear that the latter term will promptly vanish. Reference eq. (A.26) for the measurement result pertaining to the momentum term.

$$\langle\vec{r}| tr_F(\rho_1) |\vec{r}\rangle = i\frac{1}{2m}\langle\vec{r}| \Phi\vec{p}^2 |\vec{r}\rangle + h.c. = \frac{i}{2m}\left(\Delta\phi(\vec{r})\overline{\phi(\vec{r})} - \phi(\vec{r})\overline{\Delta\phi(\vec{r})}\right) \tag{3.8}$$

4. Second Order

Consider the constant relevant to ρ's second-order expansion:

$$\rho_2 = |\psi_0\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_0| \tag{4.1}$$

Reference section A.4 for the relevant wave function constants:

$$\begin{aligned}|\psi_1\rangle &= -iH_0|\psi_0\rangle \\ |\psi_2\rangle &= -\frac{1}{2}(H_0^2 + i\dot{H}_0)|\psi_0\rangle\end{aligned}$$

4.1. Double Hamiltonian, Single-Sided

Construct $|\psi_2\rangle\langle\psi_0| + h.c.$ to calculate the first and last part of eq. (4.1):

$$|\psi_2\rangle\langle\psi_0| + h.c. = -\frac{1}{2}H_0^2\rho_0 - i\frac{1}{2}\dot{H}_0\rho_0 + h.c. \tag{4.2}$$

The partial trace over the these terms will be the following:

$$tr_F[|\psi_2\rangle\langle\psi_0|] + h.c. = -\frac{1}{2}\langle 0|H_0^2|0\rangle\Phi - i\frac{1}{2}\langle 0|\dot{H}_0|0\rangle\Phi + h.c. \tag{4.3}$$

It should be quite apparent that H_0^2 will introduce a great number of terms. It is also self-evident from first principles that most of these carry no dependence on the source velocity and are thus ineligible to introduce signalling. In an effort to keep calculations as streamlined as possible, these terms will therefore be disregarded and only $\dot{\vec{y}}_0$-dependent ones taken for closer evaluation. Others, that is terms only depending on the position of the source or completely independent of it, will be grouped under the label $\mathcal{O}(\vec{y})$. The vacuum interaction $\langle 0|H_0^2|0\rangle$ can then be written out the following way:

$$\begin{aligned}&\langle 0|H_0^2|0\rangle \\ &= \langle 0|\left(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|}\right)\left(-e\,\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0}\right)|0\rangle \\ &+ \langle 0|\left(-e\,\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0}\right)\left(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|}\right)|0\rangle \\ &+ e^2\langle 0|\left(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0}\right)^2|0\rangle + \mathcal{O}(\vec{y}_0)\end{aligned} \tag{4.4}$$

As linear terms in $\vec{A}$ fall away[1], the remainders unify via eq. (A.20):

$$\langle 0| H_0^2 |0\rangle = 2\frac{e^2}{2m} \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\} (\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}) |0\rangle + e^2 \langle 0| (\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0})^2 |0\rangle + \mathcal{O}(\vec{y}_0) \tag{4.5}$$

4.2. Hamiltonian's Derivative

Reference eq. (2.4) to establish the Hamiltonian's time derivative. Recall that in the Schrödinger picture, no operators in H will exhibit any direct dependence on time. Only terms relating to the source current will thereby be subject to an indirect dependence, giving the derivative at $t = 0$ the following form:

$$\dot{H}_0 = -\frac{e^2}{4\pi} \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} - e\, \ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} - e\, \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0} \tag{4.6}$$

Notice the derivative $\vec{A}'_{\vec{y}_0} = \frac{d}{dt}\vec{A}(\vec{y}(t))|_{t=0}$. As demonstrated in section A.1, this will vanish between vacuum states just as well as a non-derived instance of the potential, leaving only one term in the vacuum interaction:

$$\langle 0| \dot{H}_0 |0\rangle = -\frac{e^2}{4\pi} \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \tag{4.7}$$

4.3. Double Hamiltonian, Double-Sided

The middle term in eq. (4.1) places operators on both sides of Φ:

$$|\psi_1\rangle \langle\psi_1| = H_0 \rho H_0 \tag{4.8}$$

Taking the partial trace of this necessitates the introduction of a *partial transpose on the field space*[2], which will be denoted on the respective operators by the superscript ^{T_F}. Once again, focus only on the terms relevant to signalling:

$$\begin{aligned}
tr_F[|\psi_1\rangle \langle\psi_1|] &= \sum_j \langle j| H_0 |0\rangle \Phi \langle 0| H_0 |j\rangle = \sum_j \langle 0| H_0^{T_F} |j\rangle \Phi \langle j| H_0^{T_F} |0\rangle \\
&= \langle 0| H_0^{T_F} \Phi H_0^{T_F} |0\rangle = \langle 0| (H_0 \Phi H_0)^{T_F} |0\rangle = \langle 0| H_0 \Phi H_0 |0\rangle \\
&= \langle 0| \Big(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|}\Big) \Phi \big(-e\, \dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle \\
&+ \langle 0| \big(-e\, \dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) \Phi \Big(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|}\Big) |0\rangle \\
&+ e^2 \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) \Phi \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle + \mathcal{O}(\vec{y}_0)
\end{aligned} \tag{4.9}$$

[1] ...according to section A.1.

[2] See section A.3 for a clarification.

As $(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0})$ is freely mobile around Φ, a familiar result emerges:

$$\begin{aligned}
&tr_F[|\psi_1\rangle \langle\psi_1|] \\
&= \langle 0| \big(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|}\big)\big(-e\,\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle\, \Phi \\
&+ \Phi \langle 0| \big(-e\,\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\frac{1}{2m}(\vec{p} - e\vec{A}_{\vec{q}})^2 + \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|}\big) |0\rangle \\
&+ e^2 \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2 |0\rangle\, \Phi + \mathcal{O}(\vec{y}_0)
\end{aligned} \tag{4.10}$$

By the same argument as in eq. (4.5), this reduces to the following result:

$$\begin{aligned}
tr_F(|\psi_1\rangle \langle\psi_1|) = &\frac{e^2}{2m} \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle\, \Phi \\
&+ \frac{e^2}{2m} \Phi \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle + e^2 \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2 |0\rangle\, \Phi + \mathcal{O}(\vec{y}_0)
\end{aligned} \tag{4.11}$$

4.4. Assembly and Measurement

Inserting the results above into the full expansion of ρ_2's partial trace will show that the results in eq. (4.5) and (4.11) exactly cancel each other, leaving only a commutator related to eq. (4.7):

$$\begin{aligned}
tr_F(\rho_2) = &-\frac{1}{2} \langle 0| H_0^2 |0\rangle\, \Phi - \frac{1}{2} \Phi \langle 0| H_0^2 |0\rangle + \langle 0| H_0^{T_F} \Phi H_0^{T_F} |0\rangle \\
&+ i\frac{1}{2} \Phi \langle 0| \dot{H}_0 |0\rangle - i\frac{1}{2} \langle 0| \dot{H}_0 |0\rangle\, \Phi \\
&= i\frac{e^2}{8\pi} \big[\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}, \Phi\big] + \mathcal{O}(\vec{y}_0)
\end{aligned} \tag{4.12}$$

The probe particle's state is therefore proven to depend on the source particle's acceleration.

$$tr_F(\rho_2(dt)) = i\, dt^2 \frac{e^2}{8\pi} \big[\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}, \Phi\big] + \mathcal{O}(\vec{y}_0) \tag{4.13}$$

However, as the Coulomb-derived operator in the commutator is only a construction of position operators, it is clear that it, and therefore the source-dependence, can only vanish under a position measurement.

5. Third order

The third order aspect of the total state's density matrix is the following:

$$\rho_3 = |\psi_3\rangle \langle\psi_0| + |\psi_2\rangle \langle\psi_1| + h.c. \tag{5.1}$$

Reference section $\boxed{\text{A.4}}$ to garner $|\psi_3\rangle$:

$$|\psi_3\rangle = \frac{1}{6}H_0^3 - \frac{1}{3}\dot{H}_0 H_0 - \frac{1}{6}H_0\dot{H}_0 - i\frac{1}{6}\ddot{H}_0 \tag{5.2}$$

Use this to expand ρ_3 into its full form:

$$\begin{aligned}
\rho_3 =& i\frac{1}{6}\left[H_0^3, \rho_0\right] \\
&+ i\frac{1}{2}\left(H_0\rho_0 H_0^2 - H_0^2\rho_0 H_0\right) \\
&+ \frac{1}{2}\dot{H}_0\rho_0 H_0 + \frac{1}{2}H_0\rho_0\dot{H}_0 - \frac{1}{3}\dot{H}_0 H_0\rho_0 - \frac{1}{3}\rho_0 H_0\dot{H}_0 - \frac{1}{6}\rho_0\dot{H}_0 H_0 - \frac{1}{6}H_0\dot{H}_0\rho_0 \\
&- i\frac{1}{6}\left[\ddot{H}_0, \rho_0\right]
\end{aligned} \tag{5.3}$$

As we take the partial trace over the field aspect and examine the individual terms more closely, we will have to tackle the whole construction in separate sections.

5.1. Triple Hamiltonian, Single-Sided

Start with the first line in eq. (5.3). The full extension of these terms would be quite extensive, yet the effort needed to arrive at the relevant results can be limited somewhat by a number of considerations. Firstly, focus again only on the terms containing at least one instance of the source velocity. Due to the nature of the Hamiltonian, such a dependence can only be introduced via its last term $(-e\,\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0})$. Recall that taking the partial trace on the field aspect will cause all terms to be in some way bracketed by vacuum state vectors and that, as was previously established, only terms featuring an even number of $\vec{A}$-instances[1] can prevail under such conditions. Thereby, between the three 'slots' given by the three instances of the Hamiltonian, only terms that use at least one for the source term and the other two to raise the number of potential instances to either two or four need to be considered.

$$
\begin{aligned}
&tr_F(H_0^3\rho_0) = \sum_j \langle j|\, H_0^3\, |0\rangle\, \Phi\, \langle 0|j\rangle = \langle 0|\, H_0^3\, |0\rangle\, \Phi \\
&= \langle 0|\, \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\}\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\}\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\}\big)\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)\big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-\frac{e}{2m}\{\vec{p},\vec{A}_{\vec{q}}\}\big)\big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \langle 0|\, \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)\big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\, |0\rangle\, \Phi \\
&+ \mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.4}
$$

[1] ...or no instances thereof.

Utilising some of the identities in section A.7 with special consideration of the mobility of $(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0})$ and recognising that some terms are self-adjoint leads to the following result of the partial trace minus its hermitian conjugate.

$$
\begin{aligned}
&tr_F(H_0^3\rho_0) - h.c.\\
&= 3\frac{e^2}{2m}\langle 0|\left(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\right)^2|0\rangle\, p^2\Phi\\
&+ \frac{3e^4}{4\pi}\langle 0|\left(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\right)^2|0\rangle\, \frac{1}{|\vec{y}_0 - \vec{q}|}\Phi\\
&+ 3\frac{e^4}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2|0\rangle\,\Phi\\
&+ 3\frac{e^2}{4m^2}\vec{p}^2\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\Phi\\
&+ 3\frac{e^2}{4m^2}\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\vec{p}^2\Phi\\
&+ 3\frac{e^4}{8\pi m}\frac{1}{|\vec{y}_0 - \vec{q}|}\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\Phi\\
&+ 3\frac{e^4}{8\pi m}\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\frac{1}{|\vec{y}_0 - \vec{q}|}\Phi\\
&+ 3\frac{e^4}{4m^2}\langle 0|\,\vec{A}_{\vec{q}}^2\{\vec{p}, \vec{A}_{\vec{q}}\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\Phi\\
&+ 3\frac{e^4}{4m^2}\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\vec{A}_{\vec{q}}^2\,|0\rangle\,\Phi\\
&+ 2\frac{e^2}{2m}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\,\Phi\\
&- 2\frac{e^2}{2m}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\Phi\,\langle 0|\left\{\vec{p}, \vec{A}_{\vec{q}}\right\}a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)|0\rangle\\
&+ \mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.5}
$$

5.2. Triple hamiltonian, Double-Sided

Taking the partial trace of the second line in eq. (5.3) will induce a operational structure that is significantly different to that of the first, but roughly equivalent concerning the ability of limiting the computational effort. The *partial transpose in F* makes another brief appearance and the ordering places Φ so that operators act on it from both sides, yet this does nothing to change the fact that all relevant terms must feature at least one source term and in total either two or four instances of the potential - transposed or not.

$$
\begin{aligned}
&tr_F(H_0\rho_0H_0^2) = \sum_j \langle j| H_0 |0\rangle \, \Phi \, \langle 0| H_0^2 |j\rangle = \sum_i \langle 0| H_0^{T_F} |j\rangle \, \Phi \, \langle j| H_0^{2T_F} |0\rangle \\
&= \langle 0| H_0^{T_F} \Phi H_0^{2T_F} |0\rangle \\
&= \langle 0| \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \Phi \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \Phi \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} \Phi \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} - e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\}\big)^{T_F} \Phi \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} \Phi \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\}\big)^{T_F} \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\}\big)^{T_F} \Phi \big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\vec{A}_{\vec{q}}^2\big)^{T_F} \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \Phi \big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)^{T_F} \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-\frac{e}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\}\big)^{T_F} \Phi \big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)^{T_F} \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \\
&+ \langle 0| \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} \Phi \big(\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda}\big)^{T_F} \big(-e\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^{T_F} |0\rangle \, \Phi \\
&+ \mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.6}
$$

Once again, apply the identities in section A.7 and identify self-adjoint terms, destined for cancellation, to arrive at the following result:

$$
\begin{aligned}
&tr_F(H\rho_0 H^2) - h.c. \\
&= \frac{e^2}{2m}\Phi \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2 |0\rangle\, p^2 \\
&+ \frac{e^4}{4\pi}\Phi \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2 |0\rangle\, \frac{1}{|\vec{y}_0 - \vec{q}|} \\
&+ \frac{e^4}{2m}\Phi \langle 0| \vec{A}_{\vec{q}}^2 \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)^2 |0\rangle \\
&+ \frac{e^2}{4m^2}\Phi \vec{p}^2 \langle 0| \big\{\vec{p}, \vec{A}_{\vec{q}}\big\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle \\
&+ \frac{e^2}{4m^2}\Phi \langle 0| \big\{\vec{p}, \vec{A}_{\vec{q}}\big\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle\, \vec{p}^2 \\
&+ \frac{e^4}{8\pi m}\Phi \frac{1}{|\vec{y}_0 - \vec{q}|} \langle 0| \big\{\vec{p}, \vec{A}_{\vec{q}}\big\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle \\
&+ \frac{e^4}{8\pi m}\Phi \langle 0| \big\{\vec{p}, \vec{A}_{\vec{q}}\big\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle\, \frac{1}{|\vec{y}_0 - \vec{q}|} \\
&+ \frac{e^4}{4m^2}\Phi \langle 0| \vec{A}_{\vec{q}}^2 \{\vec{p}, \vec{A}_{\vec{q}}\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle \\
&+ \frac{e^4}{4m^2}\Phi \langle 0| \big\{\vec{p}, \vec{A}_{\vec{q}}\big\}\big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\vec{A}_{\vec{q}}^2 |0\rangle \\
&+ \mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.7}
$$

Note this this result forms an almost complete complex conjugate to the final expression of the first part, only divided by 3 and leaving the latter's last two terms unaccounted for.

5.3. Mixed Terms

Next, turn your attention to the third line in eq. (5.3):

$$R := \frac{1}{2}\dot{H}_0\rho_0 H_0 - \frac{1}{3}\dot{H}_0 H_0\rho_0 - \frac{1}{6}\rho_0\dot{H}_0 H_0 + h.c. \tag{5.8}$$

$$tr_F(R) = \frac{1}{2}tr_F(\dot{H}_0\rho_0 H_0) - \frac{1}{3}\langle 0|\,\dot{H}_0 H_0\,|0\rangle\,\Phi - \frac{1}{6}\Phi\,\langle 0|\,\dot{H}_0 H_0\,|0\rangle + h.c. \tag{5.9}$$

To tackle the first trace term, reference eq.(4.6) for the terms in $\dot{H}_0$ and consider them individually.

$$tr_F(\dot{H}_0\rho_0 H_0) = \sum_j \langle j|\, -\frac{e^2}{4\pi}\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3} - e\,\ddot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0} - e\,\dot{\vec{y}}_0\cdot\vec{A}'_{\vec{y}_0}\,|0\rangle\,\Phi\,\langle 0|\,H_0\,|j\rangle \tag{5.10}$$

Note that the first term of $\dot{H}_0$ does not act on the field, letting $\langle j|$ slide right through to combine with the vacuum and form δ_{j0}. Meanwhile, the following two terms *only* act on the field, therefore collapsing into a scalar when brought between $\langle j|$ and $|0\rangle$ and so, in a manner much alike that of eq. (4.10), able to be brought to the rightmost side of Φ, where applying $\sum_j |j\rangle\langle j| = \mathbf{I}$ will significantly simplify the expression. This leads us to the following partial result:

$$\begin{aligned} tr_F(\dot{H}_0\rho_0 H_0) = &-\frac{e^2}{4\pi}\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Phi\big(\frac{1}{2m}\vec{p}^2 + e^2\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\,|0\rangle\big) \\ &+ e^2\Phi\,\langle 0|\,\big(\frac{1}{2m}\{\vec{p},\vec{A}_{\vec{q}}\} + \dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0}\big)\big(\ddot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0\cdot\vec{A}'_{\vec{y}_0}\big)\,|0\rangle \\ &+ \mathcal{O}(\vec{y}_0) \end{aligned} \tag{5.11}$$

To process the other terms in eq. (5.9), reference eq. (2.4) and (4.6) to see which terms of their product will both feature some dependence on the source's movement and not vanish when brought between vacuum states:

$$\begin{aligned} \langle 0|\,\dot{H}_0 H_0\,|0\rangle = &-\frac{e^2}{4\pi}\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Big(\frac{1}{2m}\vec{p}^2 + e^2\frac{1}{|\vec{y}_0-\vec{q}|} + \frac{e^2}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\,|0\rangle\Big) \\ &+ e^2\,\langle 0|\,\big(\ddot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0\cdot\vec{A}'_{\vec{y}_0}\big)\big(\frac{1}{2m}\{\vec{p},\vec{A}_{\vec{q}}\} + \dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0}\big)\,|0\rangle \\ &+ \mathcal{O}(\vec{y}_0) \end{aligned} \tag{5.12}$$

As some of their latter terms cancel[2], the results above combine to form the following expansion of eq. (5.9):

$$
\begin{aligned}
tr_F(R) = &- \frac{1}{2}\frac{e^2}{4\pi}\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}\Phi\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\,|0\rangle\big) \\
&+ \frac{1}{3}\frac{e^2}{4\pi}\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\,|0\rangle\big)\Phi \\
&+ \frac{1}{6}\frac{e^2}{4\pi}\Phi\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}\big(\frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\langle 0|\,\vec{A}_{\vec{q}}^2\,|0\rangle\big) \\
&+ h.c. + \mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.13}
$$

5.4. Commutator

Starting in the bottom line, with the commutator on the Hamiltonian's second time derivative. Reference eq. (2.4) or (4.6) to obtain the terms therein:

$$
\begin{aligned}
\ddot{H}_0 = &- \frac{e^2}{4\pi}\frac{\dot{\vec{y}}_0^2 + \vec{y}_0 \cdot \ddot{\vec{y}}_0 - \vec{q} \cdot \ddot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \\
&+ 3\frac{e^2}{4\pi}\frac{(\vec{y} \cdot \dot{\vec{y}})^2 - 2(\vec{q} \cdot \dot{\vec{y}})(\vec{y} \cdot \dot{\vec{y}}) + (\vec{q} \cdot \dot{\vec{y}})^2}{|\vec{y} - \vec{q}|^5} \\
&- e\left(\dddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\right) - 2e\left(\ddot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\right) - e\left(\dot{\vec{y}}_0 \cdot \vec{A}''_{\vec{y}_0}\right)
\end{aligned}
\tag{5.14}
$$

Taking the partial trace over the commutator will lead all the terms linear in $\vec{A}$ to vanish and only the two Coulomb-derived terms to remain:

$$
\begin{aligned}
tr_F(\ddot{H}\rho_0) - h.c. = &- \frac{e^2}{4\pi}\Big[\frac{\dot{\vec{y}}_0^2 + \vec{y}_0 \cdot \ddot{\vec{y}}_0 - \vec{q} \cdot \ddot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}, \Phi\Big] \\
&+ 3\frac{e^2}{4\pi}\Big[\frac{(\vec{y}_0 \cdot \dot{\vec{y}}_0)^2 - 2(\vec{q} \cdot \dot{\vec{y}}_0)(\vec{y}_0 \cdot \dot{\vec{y}}_0) + (\vec{q} \cdot \dot{\vec{y}}_0)^2}{|\vec{y}_0 - \vec{q}|^5}, \Phi\Big]
\end{aligned}
\tag{5.15}
$$

[2] reference section A.2

5.5. Assembly

When inserting the results of the previous four sections into eq. (5.3), most terms of the first and second parts will cancel, leaving a comparatively manageable final result for the state:

$$
\begin{aligned}
&tr_F(\rho_3) \\
&= i\frac{1}{3}\frac{e^2}{2m}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0|\,\{\vec{p},\vec{A}_{\vec{q}}\}a^{\dagger}_{\vec{k},\lambda}a_{\vec{k},\lambda}(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0})\,|0\rangle\,\Phi \\
&-\frac{1}{2}\frac{e^2}{4\pi}\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Phi\Big(\frac{1}{2m}\vec{p}^2+\frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|}+\frac{e^2}{2m}\langle 0|\,\vec{A}^2_{\vec{q}}\,|0\rangle\Big) \\
&+\frac{1}{3}\frac{e^2}{4\pi}\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Big(\frac{1}{2m}\vec{p}^2+\frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|}+\frac{e^2}{2m}\langle 0|\,\vec{A}^2_{\vec{q}}\,|0\rangle\Big)\Phi \\
&+\frac{1}{6}\frac{e^2}{4\pi}\Phi\frac{(\vec{y}_0-\vec{q})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Big(\frac{1}{2m}\vec{p}^2+\frac{e^2}{4\pi}\frac{1}{|\vec{y}_0-\vec{q}|}+\frac{e^2}{2m}\langle 0|\,\vec{A}^2_{\vec{q}}\,|0\rangle\Big) \\
&+i\frac{1}{6}\frac{e^2}{4\pi}\frac{\dot{\vec{y}}_0^{\,2}+\vec{y}_0\cdot\ddot{\vec{y}}_0-\vec{q}\cdot\ddot{\vec{y}}_0}{|\vec{y}_0-\vec{q}|^3}\Phi \\
&-i\frac{1}{2}\frac{e^2}{4\pi}\frac{(\vec{y}_0\cdot\dot{\vec{y}}_0)^2-2(\vec{q}\cdot\dot{\vec{y}}_0)(\vec{y}_0\cdot\dot{\vec{y}}_0)+(\vec{q}\cdot\dot{\vec{y}}_0)^2}{|\vec{y}_0-\vec{q}|^5}\Phi \\
&+h.c.+\mathcal{O}(\vec{y}_0)
\end{aligned}
\tag{5.16}
$$

5.6. Position Measurement

Note that all the terms stemming from the fourth part, and many of the third part feature no momentum operators and will thus, under a position measurement, cancel with their own conjugates. The remainders will compile into the following form:

$$
\begin{aligned}
&\langle\vec{r}|\,tr_F(\rho_3)\,|\vec{r}\rangle \\
&= i\frac{1}{3}\frac{e^2}{2m}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0,\vec{r}|\,\{\vec{p},\vec{A}_{\vec{q}}\}a^{\dagger}_{\vec{k},\lambda}a_{\vec{k},\lambda}(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0})\Phi\,|0,\vec{r}\rangle \\
&- i\frac{1}{3}\frac{e^2}{2m}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0,\vec{r}|\,\Phi\{\vec{p},\vec{A}_{\vec{q}}\}a^{\dagger}_{\vec{k},\lambda}a_{\vec{k},\lambda}(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0})\,|0,\vec{r}\rangle \\
&+\frac{1}{6}\frac{e^2}{8\pi m}\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\Delta\phi_{\vec{r}}\overline{\phi_{\vec{r}}}+\frac{1}{6}\frac{e^2}{8\pi m}\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\phi_{\vec{r}}\Delta\overline{\phi_{\vec{r}}} \\
&-\frac{1}{6}\frac{e^2}{8\pi m}\Delta\Big(\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\phi_{\vec{r}}\Big)\overline{\phi_{\vec{r}}}-\frac{1}{6}\frac{e^2}{8\pi m}\phi_{\vec{r}}\Delta\Big(\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\overline{\phi_{\vec{r}}}\Big)
\end{aligned}
\tag{5.17}
$$

Consider the distribution of the Laplace operator as it appears in eq. (5.17)'s final line:

$$\Delta\left(\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\phi_{\vec{r}}\right)=\Delta\left(\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\right)\phi_{\vec{r}}+\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\Delta\phi_{\vec{r}}+2\vec{\partial}_j\left(\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}\right)\vec{\partial}_j\phi_{\vec{r}} \tag{5.18}$$

The gradient and Laplace operator act on the Coulomb term in the following way:

$$\partial_j\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}=3\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^5}(\vec{y}_0-\vec{r})_j-\frac{(\dot{\vec{y}}_0)_j}{|\vec{y}_0-\vec{r}|^3} \tag{5.19}$$

$$\begin{aligned}\partial_j^2\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}&=3\partial_j\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^5}(\vec{y}_0-\vec{r})_j-\partial_j\frac{(\dot{\vec{y}}_0)_j}{|\vec{y}_0-\vec{r}|^3}\\&=15\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^7}(\vec{y}_0-\vec{r})_j^2-6\frac{(\vec{y}_0-\vec{r})(\dot{\vec{y}}_0)_j}{|\vec{y}-\vec{r}|^5}-3\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^5}\\\rightarrow\Delta\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}&=\sum_{j=1}^{3}\partial_j^2\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}=0\end{aligned} \tag{5.20}$$

Inserting these combined results into eq. (5.17) leads to its following reformulation:

$$\begin{aligned}\langle\vec{r}|\,tr_F(\rho_3)\,|\vec{r}\rangle=\;&i\frac{1}{3}\frac{e^2}{2m}\int_{\mathbb{R}}d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0,\vec{r}|\,\{\vec{p},\vec{A}_{\vec{q}}\}a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0})\Phi\,|0,\vec{r}\rangle\\&-i\frac{1}{3}\frac{e^2}{2m}\int_{\mathbb{R}}d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}|\,\langle 0,\vec{r}|\,\Phi\{\vec{p},\vec{A}_{\vec{q}}\}a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}(\dot{\vec{y}}_0\cdot\vec{A}_{\vec{y}_0})\,|0,\vec{r}\rangle\\&+\frac{1}{3}\frac{e^2}{8\pi m}\left(\frac{\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3}-3\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^5}(\vec{y}_0-\vec{r})\right)\vec{\partial}_{\vec{r}}\Phi_{\vec{r}}\end{aligned} \tag{5.21}$$

At this point it is not at all obvious how these terms should be able to cancel, yet a last transformation ramains to be done.

5.7. Final Transformation

Reference eq. (A.25) to perform the first transformation, then apply the measurement identities in section 5.17, expand and simplify:

$$\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| \langle 0, \vec{r}| \{\vec{p}, \vec{A}_{\vec{q}}\} a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda} (\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}) \Phi |0, \vec{r}\rangle - h.c.$$

$$\begin{aligned}
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}\vec{y}_0} \langle \vec{r}| \{ (\vec{p} \cdot \dot{\vec{y}}_0), e^{-i\vec{k}\vec{q}} \} \Phi |\vec{r}\rangle \\
&- \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{(\dot{\vec{y}}_0 \cdot \vec{k})}{|\vec{k}|^2} e^{i\vec{k}\vec{y}_0} \langle \vec{r}| \{ (\vec{p} \cdot \vec{k}), e^{-i\vec{k}\vec{q}} \} \Phi |\vec{r}\rangle \\
&- h.c.
\end{aligned}$$

$$\begin{aligned}
&= -i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\dot{\vec{y}}_0 \cdot \vec{\partial}\phi_{\vec{r}}) \overline{\phi_{\vec{r}}} \\
&- i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}\vec{y}_0} (\dot{\vec{y}}_0 \cdot \vec{\partial}(e^{-i\vec{k}\vec{r}} \phi_{\vec{r}})) \overline{\phi_{\vec{r}}} \\
&+ i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{(\dot{\vec{y}}_0 \cdot \vec{k})}{|\vec{k}|^2} e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\vec{k} \cdot \vec{\partial}\phi_{\vec{r}}) \overline{\phi_{\vec{r}}} \\
&+ i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{(\dot{\vec{y}}_0 \cdot \vec{k})}{|\vec{k}|^2} e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\vec{k} \cdot \vec{\partial}(e^{-i\vec{k}\vec{r}} \phi_{\vec{r}})) \overline{\phi_{\vec{r}}} \\
&- h.c.
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&= -2i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\dot{\vec{y}}_0 \cdot \vec{\partial}\phi_{\vec{r}}) \overline{\phi_{\vec{r}}} \\
&- \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\dot{\vec{y}}_0 \cdot \vec{k}) \phi_{\vec{r}} \overline{\phi_{\vec{r}}} \\
&+ 2i \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{(\dot{\vec{y}}_0 \cdot \vec{k})}{|\vec{k}|^2} e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\vec{k} \cdot \vec{\partial}\phi_{\vec{r}}) \overline{\phi_{\vec{r}}} \\
&+ \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{\vec{k}^2}{|\vec{k}|^2} e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\dot{\vec{y}}_0 \cdot \vec{k}) \phi_{\vec{r}} \overline{\phi_{\vec{r}}} \\
&- h.c.
\end{aligned}$$

$$\begin{aligned}
&= -i \frac{1}{(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\dot{\vec{y}}_0 \cdot \vec{\partial}_{\vec{r}}) \Phi_{\vec{r}} \\
&+ i \frac{1}{(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{(\dot{\vec{y}}_0 \cdot \vec{k})}{|\vec{k}|^2} e^{i\vec{k}(\vec{y}_0 - \vec{r})} (\vec{k} \cdot \vec{\partial}_{\vec{r}}) \Phi_{\vec{r}}
\end{aligned}$$

The integrals over $\vec{k}$ act as Fourier Transforms, whose identities are given in section A.9. Their application reduces the result to the following:

$$
\begin{aligned}
&\int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| \langle 0,\vec{r}| \{\vec{p},\vec{A}_{\vec{q}}\} a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda} (\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}) \Phi |0,\vec{r}\rangle - h.c. \\
&= -i\,\delta(\vec{y}_0 - \vec{r})(\dot{\vec{y}}_0 \cdot \vec{\partial}_{\vec{r}})\Phi_{\vec{r}} \\
&\quad + i\frac{1}{4\pi}\left(\frac{\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^3} - 3\frac{(\vec{y}_0-\vec{r})\cdot\dot{\vec{y}}_0}{|\vec{y}_0-\vec{r}|^5}(\vec{y}_0-\vec{r})\right)\partial_{\vec{r}}\Phi_{\vec{r}}
\end{aligned}
\tag{5.23}
$$

When inserted into eq. (5.21), the latter term of the above equation will exactly cancel out the other remainder, leaving only one term behind:

$$
\langle\vec{r}|\, tr_F(\rho_3)\, |\vec{r}\rangle = \frac{1}{3}\frac{e^2}{2m}\delta(\vec{y}_0-\vec{r})(\dot{\vec{y}}_0\cdot\vec{\partial}_{\vec{r}})\Phi_{\vec{r}} + \mathcal{O}(\vec{y}_0)
\tag{5.24}
$$

6. Summary and Final Considerations

6.1. Conclusion of First Three Orders

Gather all previous results, specifically equations (4.12) and (5.16) to compile the full density matrix of the probe particle up to the third order of approximation:

$$\begin{aligned}
& tr_F(\rho) \\
& = i\,dt^2 \frac{e^2}{8\pi} \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \Phi \\
& + i\,dt^3 \frac{1}{3} \frac{e^2}{2m} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}| \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\} a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle \, \Phi \\
& - dt^3 \frac{1}{2} \frac{e^2}{4\pi} \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \Phi \Big(\frac{1}{2m} \vec{p}^2 + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m} \langle 0| \vec{A}^2_{\vec{q}} |0\rangle \Big) \\
& + dt^3 \frac{1}{3} \frac{e^2}{4\pi} \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \Big(\frac{1}{2m} \vec{p}^2 + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m} \langle 0| \vec{A}^2_{\vec{q}} |0\rangle \Big) \Phi \\
& + dt^3 \frac{1}{6} \frac{e^2}{4\pi} \Phi \frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \Big(\frac{1}{2m} \vec{p}^2 + \frac{e^2}{4\pi} \frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m} \langle 0| \vec{A}^2_{\vec{q}} |0\rangle \Big) \\
& + i\,dt^3 \frac{1}{6} \frac{e^2}{4\pi} \frac{\dot{\vec{y}}_0^2 + \vec{y}_0 \cdot \ddot{\vec{y}}_0 - \vec{q} \cdot \ddot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3} \Phi \\
& - i\,dt^3 \frac{1}{2} \frac{e^2}{4\pi} \frac{(\vec{y}_0 \cdot \dot{\vec{y}}_0)^2 - 2(\vec{q} \cdot \dot{\vec{y}}_0)(\vec{y}_0 \cdot \dot{\vec{y}}_0) + (\vec{q} \cdot \dot{\vec{y}}_0)^2}{|\vec{y}_0 - \vec{q}|^5} \Phi \\
& + h.c. + \cdots
\end{aligned} \tag{6.1}$$

As demonstrated in sections 5.6 and 5.7, this will, under a position measurement, almost completely collapse and leave only the following term:

$$\langle \vec{r}|\, tr_F(\rho)\, |\vec{r}\rangle = dt^3 \frac{1}{3} \frac{e^2}{2m} \delta(\vec{y}_0 - \vec{r}) \big(\dot{\vec{y}}_0 \cdot \vec{\partial}_{\vec{r}}\big) \Phi_{\vec{r}} + \cdots \tag{6.2}$$

Evidently, there would be some superluminal influence of $\dot{\vec{y}}_0$, were in not for the delta function $\delta(\vec{y}_0 - \vec{r})$. This limits the result cases where $\vec{y}_0 = \vec{r}$, i.e. where the source and probe particles occupy the same point in space. Instant signalling through a spatial distance of 0 does not violate locality, and so it remains intact.

6.2. Higher Orders

The derivation of ρ's first three orders of approximation clearly shows that the effort required in doing so rises quasi-exponentially with each order. From this position, examining the forth or any further orders in a similarly detailed fashion seems inexpedient and will be omitted. However, some pattern can be discerned, in which each order of approximation will invariably contribute one term of a certain structure to the density matrix. An analytical expression for these terms can be found via some select considerations.
Recall, once again, the full extension of the Dyson series via eq. (2.11) :

$$|\psi(T)\rangle = \sum_{g=0}^{\infty}(-i)^g \sum_{n_1,n_2,\ldots,n_g=0}^{\infty} T^{(g+\sum_{z=1}^{g} n_z)} \prod_{j=1}^{g} \frac{1}{(g-j+1+\sum_{z=j}^{g} n_z)} \frac{1}{n_j!} H_0^{[n_j]} |\psi_0\rangle$$

Consider the full derivation in sections 2.4 and 2.5 to see that the exponent $G = g + \sum_{z=1}^{g} n_z$ in $|\psi(t)\rangle = \sum_{g=0}^{\infty} T^G |\psi_G\rangle$ is determined by the contributions of two sources: The initial integral reformulation produces g and the products of Hamiltonian Taylor expansions add the sum $\sum_{z=1}^{g} n_z$. Note that in the index summation process[1] of determining any wave function constant $|\psi_G\rangle$ for $G \geq 1$, there is invariably a step, the natural starting point even, of setting $g = 1$. The constraint of G will then 'spawn' only a single index n_1 that is bound to the value $n_1 = G - 1$. This, in turn, will induce only a single instance of the Hamiltonian operator, derived $G - 1$ times in the time domain and evaluated at $t = 0$ to act on the ground state. The fixedness of indices will also bring the respective prefactor into a controlled form. It can therefore be said that each wave function constant must contain the following term:

$$|\psi_G\rangle = -i\frac{1}{G!} H_0^{[G-1]} |\psi_0\rangle + \cdots \tag{6.3}$$

Reference eq. (2.4) for the full expression of the Hamiltonian and recall that in the Schrödinger picture, it is only the last two terms which through their dependence on the source also exhibit one on time. Any time derivative of the Hamiltionian will then take the following form:

$$\frac{d^{G-1}}{dt^{G-1}} H\Big|_{t=0} = \frac{e^2}{4\pi} \frac{d^{G-1}}{dt^{G-1}} \Big(\frac{1}{|\vec{y}(t) - \vec{q}|}\Big)\Big|_{t=0} - e\frac{d^{G-1}}{dt^{G-1}} \big(\dot{\vec{y}}(t) \cdot \vec{A}_{\vec{y}(t)}\big)\Big|_{t=0} \tag{6.4}$$

As the last term is linear in $\vec{A}$, it will vanish between vacuum states, making the result for the $G-1$-nth derivative of the Hamiltonian equivalent to that of the same derivative on its coulomb term.

$$\langle 0| \frac{d^{G-1}}{dt^{G-1}} H |0\rangle\big|_{t=0} = \frac{e^2}{4\pi} \langle 0| \frac{d^{G-1}}{dt^{G-1}} \Big(\frac{1}{|\vec{y}(t) - \vec{q}|}\Big) |0\rangle\big|_{t=0} = \frac{e^2}{4\pi} \frac{d^{G-1}}{dt^{G-1}} \Big(\frac{1}{|\vec{y}(t) - \vec{q}|}\Big)\Big|_{t=0} \tag{6.5}$$

[1]Again, reference section A.4 to see this illustrated.

Via equation (2.14), it is clear that that any density matrix constant ρ_G will contain the term $|\psi_G\rangle \langle\psi_0|$ and its conjugate:

$$\rho_G = |\psi_G\rangle \langle\psi_0| + |\psi_0\rangle \langle\psi_G| + \cdots = -i\frac{1}{G!}H_0^{[G-1]}\rho_0 + i\frac{1}{G!}\rho_0 H_0^{[G-1]} + \cdots = i\frac{1}{G!}\left[\rho_0, H_0^{[G-1]}\right] \tag{6.6}$$

Taking the partial trace on F will then amount to bracketing all terms in vacuum states. Inserting eq. (6.4) finally leads to the following result:

$$tr_F(\rho_G) = i\frac{1}{G!}\langle 0| \left[\rho_0, H_0^{[G-1]}\right] |0\rangle + \cdots = i\frac{e^2}{4\pi}\frac{1}{G!}\frac{d^{G-1}}{dt^{G-1}}\Big(\big[\Phi, \frac{1}{|\vec{y}(t) - \vec{q}|}\big]\Big)\Big|_{t=0} + \cdots \tag{6.7}$$

This is precisely the term that appeares in equations (3.7), (4.13) and (5.16) as the respective pre-measurement result for the reduced density matrix in all three orders of approach. Via eq. (2.13), it will appear in any and all layers of approximation:

$$tr_F(\rho(T)) = \sum_{G=0}^{\infty} T^G tr_F(\rho_G) = i\frac{e^2}{4\pi}\sum_{G=0}^{\infty} T^G \frac{1}{G!}\frac{d^{G-1}}{dt^{G-1}}\Big(\big[\Phi, \frac{1}{|\vec{y}(t) - \vec{q}|}\big]\Big)\Big|_{t=0} + \cdots \tag{6.8}$$

However, as it contains no instance of momentum operators, the commutator and thereby the whole terms will collapse under position measurements.

7. Discussion

7.1. General Conclusion

Via the process shown in section 6.2, the reduced density matrix denoting P's state exhibits an instantaneous dependency on S's movement, arising as early as the second order of approximation. All terms of this sort do, however, cancel under a position measurement. In fact, all non-local contributions to the measurement result have been shown to cancel up to the third order. In the case of the latter, the non-local terms of the density matrix are quite extensive, and their cancellation is, from a purely mathematical standpoint, not obvious and even somewhat surprising relative to the initial impression. These findings all point towards the conjecture that while P's quantum state is non-local, the cancellations in the measurement process happen at all orders and locality is preserved, just as in the classical case. A look into the fourth order approximation might be interesting in this regard, yet given the effort required in processing its projected scale and the weight of all previous findings, its pursuit seems neither worthwhile nor advisable. The final conclusion will thus have to land on a tentative conjecture that locality is preserved, with a more systematic investigation left for future work.

7.2. Modified Hamiltonians

The system as it was introduced in section 2.1 supposes free particles, without any additional potential acting upon them. In any realistic application, such a potential $\vec{V}$ would have to be accounted for, and implemented into the Hamiltonian as follows:

$$\tilde{H} = H + V(\vec{q}) \tag{7.1}$$

For example, the probe could take the form of the nucleus of an atom resting at the coordinate origin, with an electron circling around it, giving a potential $V(\vec{q}) = \frac{e^2}{4\pi}\frac{1}{|\vec{q}|}$. Any such scenario would increase the computational effort even further, yet as V can only ever depend on particle *locations*, its behaviour with regard to cancellations during the measurement process can be expected to resemble that of the Coulomb term in H. That is, the two of them could be combined into a single term that is marked by only featuring constructions of $\vec{q}$ on the probe particle space, and any terms stemming from it can be assumed to be more or less equivalent in conduct under a position measurement to those previously introduced by the Coulomb field, only with the latter switched out for the new sum. This *equivalent behaviour* would be, as previously demonstrated, to cancel, and thus there is at least a strong indication that the stated conclusion regarding signalling is invariant under V.

Another alternative would be to use, as the probe, not a particle but a complex scalar field or a Dirac field. This would complicate the structure of the system significantly, but would bring the benefit of presenting fully relativistic theory. The Lagrangian of the probe as a complex scalar field would be the following [5]:

$$L_P = \int d^3x \left(-\vec{A}^*(\Box + m^2)\vec{A} - ie\vec{A}_\mu\big(\vec{A}^*(\partial_\mu \vec{A}) - (\partial_\mu \vec{A}^*)\vec{A}\big) + e^2|\vec{A}|^2 A_\mu A^\mu \right) \tag{7.2}$$

As the system's formation would be so fundamentally different, it is not possible to draw any strict conclusions from the results gathered here. Should another non-relativistic setup lead to evidence of non-local behaviour, though, one would have to check whether this follows into a fully relativistic frame via a system such as this.

7.3. Alternative Field States

The main argument of this work unfolds under the imposition that F's initial state be the vacuum $|0\rangle$. To somewhat generalise the results, the same procedure of calculations cam be repeated with other choices of field states. This raises the computational effort by a considerable margin, as the field-potential interactions can no longer depend on uneven numbers of $\vec{A}$-instances going to zero, and the sums of resulting ladder operator permutations grow even larger. To illustrate, consider the following:

$$\vec{A} \sim (a + a^\dagger) \tag{7.3}$$

$$|\alpha\rangle = D(\alpha)\,|0\rangle\,,\;\; D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{7.4}$$

Therefore, while $\langle 0|\,\vec{A}^m\,|0\rangle \sim \langle 0|\,(a + a^\dagger)^m\,|0\rangle$ remains comparatively controlled at 2^m terms and only needs further examination for even m as per the argument in section A.1, the switch to coherent states introduces the constant α and its conjugate as objects for possible permutations with the ladder operators, leading to many more terms needing to be considered:

$$\begin{aligned} &\langle\alpha|\,\vec{A}^m\,|\alpha\rangle \sim \langle\alpha|\,\vec{A}^{m-1} D(\alpha)D^\dagger(\alpha)\vec{A}D(\alpha)\,|0\rangle \\ = &\langle\alpha|\,\vec{A}^{m-1} D(\alpha)(a + a^\dagger + \alpha + \overline{\alpha})\,|0\rangle = \langle 0|\,(a + a^\dagger + \alpha + \overline{\alpha})^m\,|0\rangle \end{aligned} \tag{7.5}$$

Of the resulting 4^m states, many will vanish as ladder operators and vacuum states combine, yet the added terms' computational impact remains significant, especially in the more complex arrangements that appear in the advanced stages of the calculation. Note that no aspect of this argument can be accused of being based in mere speculation, as the author has worked though this entire scenario in all three orders of approximation. In special consideration of the reader's sanity and, with regard to the printing effort, the integrity of global forest reserves[1], such a detailed treatment will have to be omitted at this point and the reader is asked to lend their trust towards the conclusion that the final

[1] The needed additions of auxiliary calculations to the (already extensive) appendix alone would be an unreasonably excessive undertaking.

results for coherent states are, in all relevant aspects, equivalent to those for the vacuum state, as presented throughout this work. At various points, some terms might be added to the probe particle's density matrix and some of the respective constants might be shifted around, yet for all source-dependent instances, these variations will either vanish or cancel each other out during the measurement process, leading to an end result that is qualitatively identical. For a simple example, consider the linear order as per section 3. The respective approximation of the system's density matrix is the following:

$$\rho_1 = i\big[\rho_0, H_0\big] \tag{7.6}$$

The reduced density matrix for the probe is once again given by the partial trace of the above, which will, now that the initial state of the entire system is defined as $\rho_0 = |\alpha\rangle\langle\alpha| \otimes \Phi$, take the following form:

$$tr_F(\rho_1) = i\Phi\langle\alpha| H_0 |\alpha\rangle + h.c. \tag{7.7}$$

Recall H_0's full expression in equations (2.4) or (3.5) to see that $\langle\alpha| H_0 |\alpha\rangle$ will take the terms found in $\langle 0| H_0 |0\rangle$, as shown in section 3, and *add* a number of others:

$$\begin{aligned}
\langle\alpha| H_0 |\alpha\rangle =& \frac{1}{2m}\vec{p}^2 + \frac{e^2}{4\pi}\frac{1}{|\vec{y}_0 - \vec{q}|} + \frac{e^2}{2m}\frac{1}{(2\pi)^3}\int_{\mathbb{R}} d\vec{k}\frac{1}{|\vec{k}|} \\
& - \frac{e}{2m}(2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}\big\{(e^{i\vec{k}\vec{q}}\alpha_{\vec{k},\lambda} + e^{-i\vec{k}\vec{q}}\overline{\alpha}_{\vec{k},\lambda}), (\vec{p}\cdot\vec{\epsilon}_\lambda)\big\} \\
& + \int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}|\vec{k}||\alpha_{\vec{k},\lambda}|^2 \\
& - e\,(2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}(e^{i\vec{k}\vec{y}}\alpha_{\vec{k},\lambda} + e^{-i\vec{k}\vec{y}}\overline{\alpha}_{\vec{k},\lambda}), (\dot{\vec{y}}\cdot\vec{\epsilon}_\lambda)
\end{aligned} \tag{7.8}$$

Note that the last term even contains the source velocity and is therefore relevant with regard to signalling. However, it is also completely scalar and will therefore drop out in eq. (7.7). The same goes for the second-to-last row and, as shown before for the vacuum-case, the third term of the first row. The second line will prevail both in the density matrix and the position measurement result, yet has no effect towards signalling.
For another alternative, substituting a general number state $|n\rangle$ for the vacuum in the relevant equations of chapters 3-5 reveals a result completely identical to the one at hand.

7.4. Alternative Measurement Domains

While it has been established that there is no chance of superluminal signalling in a first order approximation of the probe's density matrix, the second order approximation, as seen in section 4, will yield the following commutator:

$$tr_F(\rho_2(dt)) = i\, dt^2 \frac{e^2}{8\pi} \big[\frac{(\vec{y}_0 - \vec{q}) \cdot \dot{\vec{y}}_0}{|\vec{y}_0 - \vec{q}|^3}, \Phi \big] + \mathcal{O}(\vec{y}_0) \tag{7.9}$$

Only under the location measurement is the dependency on $\dot{\vec{y}}_0$ lost. Under a momentum measurement however, the commutator would not vanish but turn into some construction of a derivative on Φ, and the source velocity would survive! Unfortunately though, momentum measurements on non-relativistic particles are not localised in general. This result is therefore not exceptional under the stated construction and does not contradict the previous findings. It might, however, present a worthwhile avenue for further research in a modified (fully relativistic) system!

Bibliography

[1] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and atoms: Introduction to quantum electrodynamics.* Wiley-VCH, 2004.

[2] C. W. Gardiner and P. Drummond. Causality in the coulomb gauge: A direct proof. *Physical review A*, 38(9):4897, 1988.

[3] M. Henneaux and C. Teitelboim. *Quantization of gauge systems.* Princeton university press, 1992.

[4] J. J. Sakurai and J. Napolitano. *Modern quantum mechanics.* Cambridge University Press, 2020.

[5] M. D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2013.

[6] B.-S. K. Skagerstam, K.-E. Eriksson, and P. K. Rekdal. Causality in quantum field theory with classical sources. *Journal of Physics Communications*, 3(8):082001, 2019.

[7] A. M. Stewart. Vector potential of the coulomb gauge. *European journal of physics*, 24(5):519, 2003.

A. Appendix - Clarifications and Auxillary Calculations

This section serves to establish some general points and prove certain identities where they might have distracted from the main line of argument in the main text.

A.1. Vacuum Interaction of the Potential

Reference eq.(2.3) to recall the definition of the potential:

$$\vec{A}_{\vec{x}} := \vec{A}(\vec{x}) = (2\pi)^{-3/2} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2|\vec{k}|}} \left(e^{i\vec{k}\vec{x}} a_{\vec{k},\lambda} + e^{-i\vec{k}\vec{x}} a^{\dagger}_{\vec{k},\lambda} \right) \vec{\epsilon}_{\lambda}$$

Over the course of many equations throughout this work, one is frequently confronted with products of local evaluations of the potential, referring to either the source or probe particle's position, all 'caught' between vacuum states. As shown above, each instance of $\vec{A}_{\vec{x}}$ features one of each ladder operator, in addition, as the only field operators, and so a larger number m of instances will, with regard to field space, form a sum of every possible permutation of m ladder operators. Bracketing these permutations with vacuum states will cause all uneven-numbered (and even most even-numbered) combinations to cancel. By extension, this means that any even-numbered amount m of local potentials will vanish under the states vacuum interaction.
At other points, the same scenario arises, only with some instances of $\vec{A}_{\vec{x}}$ switched for $\vec{A}'_{\vec{x}}$, the time derivative *of the potential's evaluation at* $\vec{x}$. Of course, $\vec{A}_{\vec{x}}$ exhibits no direct dependence on time, but $\vec{x}(t)$ may. The seond time derivative also appears briefly in section 5.4. See the full expressions below and note that taking this derivative does not change the potentials structure with regard to operators.

$$\begin{aligned} \vec{A}'_{\vec{x}(t)} &= \frac{d}{d\tau} \vec{A}(\vec{x}(\tau))|_{\tau=t} = \nabla \vec{A}(\vec{x}(\tau)) \cdot \dot{\vec{x}}(\tau)|_{\tau=t} \\ &= (2\pi)^{-3/2} \frac{d}{d\tau} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2|\vec{k}|}} \nabla \left(a_{\vec{k},\lambda} e^{i\vec{k}\vec{x}(\tau)} + a^{\dagger}_{\vec{k},\lambda} e^{-i\vec{k}\vec{x}(\tau)} \right) \cdot \dot{\vec{x}}(\tau) \vec{\epsilon}_{\lambda}|_{\tau=t} \\ &= i(2\pi)^{-3/2} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2|\vec{k}|}} \left(a_{\vec{k},\lambda} e^{i\vec{k}\vec{x}(t)} - a^{\dagger}_{\vec{k},\lambda} e^{-i\vec{k}\vec{x}(t)} \right) \left(\dot{\vec{x}}(t) \cdot \vec{k} \right) \vec{\epsilon}_{\lambda} \end{aligned} \tag{A.1}$$

$$\begin{aligned}
\vec{A}''_{\vec{x}(t)} &= \frac{d}{d\tau}\vec{A}'(\vec{x}(\tau))|_{\tau=t} \\
&= i(2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}\Big(a_{\vec{k},\lambda}e^{i\vec{k}\vec{x}(\tau)} - a^{\dagger}_{\vec{k},\lambda}e^{-i\vec{k}\vec{x}(\tau)}\Big)\big(\ddot{\vec{x}}(\tau)\cdot\vec{k}\big)\vec{\epsilon}_{\lambda}|_{\tau=t} \\
&- (2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}\Big(a_{\vec{k},\lambda}e^{i\vec{k}\vec{x}(\tau)} + a^{\dagger}_{\vec{k},\lambda}e^{-i\vec{k}\vec{x}(\tau)}\Big)\big(\dot{\vec{x}}(\tau)\cdot\vec{k}\big)^{2}\vec{\epsilon}_{\lambda}|_{\tau=t} \\
&= (2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}a_{\vec{k},\lambda}e^{i\vec{k}\vec{x}(t)}\Big(i\big(\ddot{\vec{x}}(t)\cdot\vec{k}\big) - \big(\dot{\vec{x}}(t)\cdot\vec{k}\big)^{2}\Big)\vec{\epsilon}_{\lambda} \\
&- (2\pi)^{-3/2}\int_{\mathbb{R}} d\vec{k}\sum_{\lambda=1}^{2}\frac{1}{\sqrt{2|\vec{k}|}}a^{\dagger}_{\vec{k},\lambda}e^{-i\vec{k}\vec{x}(t)}\Big(i\big(\ddot{\vec{x}}(t)\cdot\vec{k}\big) + \big(\dot{\vec{x}}(t)\cdot\vec{k}\big)^{2}\Big)\vec{\epsilon}_{\lambda}
\end{aligned}$$

Appropriately, the argument laid out regarding odd numbers of potential instances vanishing between vacuum states extends fully to these derivatives. That is, any assortment of $\vec{A}$, $\vec{A}'$ and $\vec{A}''$ whose total number is not amultiple of two vanished when bracketed by vacuum state vectors. Note that even most even-numbered permutations of ladder operators vanish between vacuum states. Reference section A.5 for the explicit calculation of interactions between the vacuum and relevant ladder operator combinations and section A.7 to see them applied in practice application.

A.2. Third Order Cancellations

The calculation below proves the mutual cancellation of some terms as they occur in section 5.3, more explicitly in the assembly of eq. (5.13).

$$\begin{aligned}
&\frac{e^2}{2}\Phi \langle 0| \big(\frac{1}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} + \dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\big) |0\rangle \\
&-\frac{e^2}{3} \langle 0| \big(\frac{1}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} + \dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\big) |0\rangle \Phi \\
&-\frac{e^2}{6}\Phi \langle 0| \big(\frac{1}{2m}\{\vec{p}, \vec{A}_{\vec{q}}\} + \dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big)\big(\ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\big) |0\rangle \\
&+h.c. \\
&=\frac{1}{3}\frac{e^2}{2m}\Phi \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\}\big(\ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\big) |0\rangle \\
&-\frac{1}{3}\frac{e^2}{2m} \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\}\big(\ddot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0} + \dot{\vec{y}}_0 \cdot \vec{A}'_{\vec{y}_0}\big) |0\rangle \Phi \\
&+h.c. \\
&=0
\end{aligned} \tag{A.2}$$

A.3. The Partial Transpose

Take any arbitrary operator O, defined the following way, by some appropriate choice of bases states across field- and probe particle space:

$$O = \sum_{a,b,c,d} |a\rangle \langle b|_F \otimes |c\rangle \langle d|_P \, O_{abcd} \tag{A.3}$$

Its partial transpose with regard to the field space will then be denoted O^{T_F}, and defined the following way:

$$O^{T_F} = \sum_{a,b,c,d} |b\rangle \langle a|_F \otimes |c\rangle \langle d|_P \, O_{abcd} \tag{A.4}$$

Also note the following:

$$\langle v_1| O^{T_F} |v_2\rangle = \langle v_2| O |v_1\rangle \tag{A.5}$$

This entire concept makes only a rather brief appearance in sections 4.3 and 5.2, in the process of examining the partial trace of any construction of the form $O_1\rho_0 O_2$:

$$tr_F(O_1\rho_0 O_2) = \sum_j \langle j| O_1 |0\rangle \Phi \langle 0| O_2 |j\rangle = \sum_j \langle 0| O_1^{T_F} |j\rangle \Phi \langle j| O_2^{T_F} |0\rangle = \quad \langle 0| O_1^{T_F} \Phi O_2^{T_F} |0\rangle \tag{A.6}$$

If O_1 and O_2 would only act on the field space, the above would simplify quite neatly to $\langle 0| O_2 \Phi O_1 |0\rangle$, but their probe-particle related aspects make the construction slightly more convoluted. However, in all relevant applications of this work, these aspects turn out to be very neatly defined and separable, making it possible to isolate the field aspects and apply eq. (A.5) only to them. To illustrate, consider a separation $O = O_F \otimes O_P$:

$$
\begin{aligned}
&tr_F(O_1\rho_0 O_2) = \langle 0| \left(O_{1,F} \otimes O_{1,P}\right)^{T_F} \Phi\left(O_{2,F} \otimes O_{2,P}\right) 2^{T_F} |0\rangle \\
&= \langle 0| O_{1,F}^T O_{2,F}^T |0\rangle \otimes O_{1,P}\Phi O_{2,P} = \langle 0| \left(O_{2,F} O_{1,F}\right)^T |0\rangle \otimes O_{1,P}\Phi O_{2,P} \\
&= \langle 0| O_{2,F} O_{1,F} |0\rangle \otimes O_{1,P}\Phi O_{2,P}
\end{aligned}
\tag{A.7}
$$

As demonstrated in section A.7, all relevant choices of $O_{1,F}$ and $O_{2,F}$ turn out to commute between vacuum states, allowing for the simplifications seen in section 5.2. In section 4.3, the entire set of states and operators is palindromic, allowing for a simplification via eq. (A.5) that circumvents any need for a more detailed treatment altogether.

A.4. Derivation of Wave Function Constants

Recall the equation for the Polynomialised Dyson Series from eq. (2.11):

$$|\psi(T)\rangle = \sum_{g=1}^{\infty}(-i)^g \sum_{n_1,n_2,\ldots,n_g=0}^{\infty} T^{(g+\sum_{z=1}^{g} n_z)} \Pi_{j=1}^{g} \frac{1}{g-j+1+\sum_{z=j}^{g} n_z} \frac{1}{n_j!} H_0^{[n_j]} |\psi_0\rangle$$

As discussed, this sections the wave function into sections $|\psi(T)\rangle = \sum_{d=1}^{\infty} T^d |\psi_d\rangle$. The individual wave function constants $|\psi_d\rangle$ can be calculated by fixing $d = g + \sum_{z=1}^{g} n_z$ in the equation above, then summing over individual indices g, and the sub-indices n that this constraint induces.

A.4.1. Derivation of $|\psi_1\rangle$

Set $g + \sum_{z=1}^{g} n_z = 1$. This limits the range of indices for g to $g = 1$ and thereby 'spawns' only one n-index $n_1 = 0$. Inserting this into the Dyson Polynomial gives the following result:

$$\begin{aligned} |\psi_1\rangle &= \big(-i\big)^1 \Big(\frac{1}{1-1+1+n_1} \frac{1}{n_1!} H_0^{[n_j]}\Big)\Big|_{n_1=0} |\psi_0\rangle \\ &= -i \frac{1}{1-1+1+0} \frac{1}{0!} H_0^{[0]} |\psi_0\rangle \\ &= -iH_0 |\psi_0\rangle \end{aligned} \tag{A.8}$$

A.4.2. Derivation of $|\psi_2\rangle$

Set $g + \sum_{z=1}^{g} n_z = 2$. For $g = 1$, this will spawn one sub-index $n_1 = 1$. For $g = 2$, two sub-indices with constraint $n_1 + n_2 = 0$ are created, translating to $n_1, n_2 = 0$. Inserting this into the Dyson Polynomial gives the following result:

$$\begin{aligned} |\psi_2\rangle &= \big(-i\big)^1 \Big(\frac{1}{1-1+1+n_1} \frac{1}{n_1!} H_0^{[n_j]}\Big)\Big|_{n_1=1} |\psi_0\rangle \\ &+ \big(-i\big)^2 \Bigg(\frac{1}{2-1+1+n_1+n_2} \frac{1}{2-2+1+n_2} \frac{1}{n_1!} \frac{1}{n_2!} H_0^{[n_1]} H_0^{[n_2]}\Bigg)\Bigg|_{n_1,n_2=0} |\psi_0\rangle \\ &= -i \frac{1}{1-1+1+1} \frac{1}{1!} H_0^{[1]} |\psi_0\rangle \\ &- \big(\frac{1}{2-1+1+0+0} \frac{1}{2-2+1+0} \frac{1}{0!} \frac{1}{0!} H_0^{[0]} H_0^{[0]} |\psi_0\rangle \\ &= -\frac{1}{2}\big(i\dot{H}_0 + H_0^2\big) \end{aligned} \tag{A.9}$$

A.4.3. Derivation of $|\psi_3\rangle$

Set $g + \sum_{z=1}^{g} n_z = 3$. For $g = 1$, this will spawn one sub-index $n_1 = 2$. For $g = 2$, two sub-indices with constraint $n_1 + n_2 = 1$ are created, giving two possible combinations $n_1 = 0, n_2 = 1$ and $n_1 = 1, n_2 = 0$. For $g = 3$, three sub-indices with constraint $n_1 + n_2 + n_3 = 0$ appear, translating to $n_1, n_2, n_3 = 0$. Inserting this into the Dyson Polynomial gives the following result:

$$
\begin{aligned}
&|\psi_3\rangle \\
&= (-i)^1 \left(\frac{1}{1-1+1+n_1} \frac{1}{n_1!} H_0^{[n_j]} \right) \Big|_{n_1=2} |\psi_0\rangle \\
&+ (-i)^2 \left(\frac{1}{2-1+1+n_1+n_2} \frac{1}{2-2+1+n_2} \frac{1}{n_1!} \frac{1}{n_2!} H_0^{[n_1]} H_0^{[n_2]} \right) \Big|_{n_1=1,n_2=0} |\psi_0\rangle \\
&+ (-i)^2 \left(\frac{1}{2-1+1+n_1+n_2} \frac{1}{2-2+1+n_2} \frac{1}{n_1!} \frac{1}{n_2!} H_0^{[n_1]} H_0^{[n_2]} \right) \Big|_{n_1=0,n_2=1} |\psi_0\rangle \\
&+ (-i)^3 \left(\frac{1}{3-1+1+n_1+n_2+n_3} \right. \\
&\quad \left. \frac{1}{3-2+1+n_2+n_3} \frac{1}{3-3+1+n_3} \frac{1}{n_1!} \frac{1}{n_2!} \frac{1}{n_2!} \frac{1}{n_3!} H_0^{[n_1]} H_0^{[n_2]} H_0^{[n_3]} \right) \Big|_{n_1,n_2,n_3=0} |\psi_0\rangle \\
&= -i \frac{1}{1-1+1+2} \frac{1}{2!} H_0^{[2]} |\psi_0\rangle \\
&- \frac{1}{2-1+1+1+0} \frac{1}{2-2+1+0} \frac{1}{1!} \frac{1}{0!} H_0^{[1]} H_0^{[0]} |\psi_0\rangle \\
&- \frac{1}{2-1+1+0+1} \frac{1}{2-2+1+1} \frac{1}{0!} \frac{1}{1!} H_0^{[0]} H_0^{[1]} |\psi_0\rangle \\
&+ i \frac{1}{3-1+1+0+0+0} \frac{1}{3-2+1+0+0} \frac{1}{3-3+1+0} \frac{1}{0!} \frac{1}{0!} \frac{1}{0!} \frac{1}{0!} H_0^{[0]} H_0^{[0]} H_0^{[0]} |\psi_0\rangle \\
&= \left(-i \frac{1}{6} \ddot{H}_0 - \frac{1}{3} \dot{H}_0 H_0 - \frac{1}{6} H_0 \dot{H}_0 + i \frac{1}{6} H_0^3 \right) |\psi_0\rangle
\end{aligned}
\tag{A.10}
$$

A.5. Interactions of Vacuum states and Ladder Operators

It is self-evident that every permutation of an odd-number of ladder operators will cancel when acted upon from both sides by the vacuum state.
All permutations of two or four ladder operators will similarly go to zero, except for the following exceptions. Note the identity $[a_1, a_2^\dagger] = a_1 a_2^\dagger - a_2^\dagger a_1 = \delta_{12}$, which will be applied liberally.

$$\langle 0| a_1 a_2^\dagger |0\rangle = \langle 0| a_2^\dagger a_1 + \delta_{12} |0\rangle = \delta_{12} \tag{A.11}$$

$$\langle 0| a_1 a_2^\dagger a_3 a_4^\dagger |0\rangle = \langle 0| (a_2^\dagger a_1 + \delta_{12})(a_4^\dagger a_3 + \delta_{34}) |0\rangle = \delta_{12}\delta_{34} \tag{A.12}$$

$$\begin{aligned} \langle 0| a_1 a_2 a_3^\dagger a_4^\dagger |0\rangle &= \langle 0| a_1 (a_3^\dagger a_2 + \delta_{23}) a_4^\dagger |0\rangle = \langle 0| a_1 a_3^\dagger a_2 a_4^\dagger |0\rangle + \delta_{23} \langle 0| a_1 a_4^\dagger |0\rangle \\ &= \delta_{13}\delta_{24} + \delta_{14}\delta_{23} \end{aligned} \tag{A.13}$$

A.6. Calculation of Vacuum-Potential Interactions

A.6.1. Type I

$$
\begin{aligned}
&\langle 0|\, \vec{A}(\vec{x}_1) \cdot \vec{A}(\vec{x}_2)\, |0\rangle \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \sum_{\lambda_1,\lambda_2=1}^{2} \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}} \langle 0| \cdots + a_{\vec{k}_1,\lambda_1} a^{\dagger}_{\vec{k}_2,\lambda_2} e^{i(\vec{k}_1\vec{x}_1 - \vec{k}_2\vec{x}_2)}\, |0\rangle \left(\vec{\epsilon}_{1_{\lambda_1}} \cdot \vec{\epsilon}_{2_{\lambda_2}}\right) \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \sum_{\lambda_1,\lambda_2=1}^{2} \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}} \delta(\vec{k}_1 - \vec{k}_2)\delta_{\lambda_1\lambda_2} e^{i(\vec{k}_1\vec{x}_1 - \vec{k}_2\vec{x}_2)} \vec{\epsilon}_{1_{\lambda_1}} \cdot \vec{\epsilon}_{2_{\lambda_2}} \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{1}{|\vec{k}|} e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}
\end{aligned}
\tag{A.14}
$$

A.6.2. Type II

$$
\begin{aligned}
&\langle 0| \big(\vec{v}_1 \cdot \vec{A}(\vec{x}_1)\big)\big(\vec{v}_2 \cdot \vec{A}(\vec{x}_2)\big)\, |0\rangle \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \sum_{\lambda_1,\lambda_2=1}^{2} \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}} \langle 0| \cdots + a_{\vec{k}_1,\lambda_1} a^{\dagger}_{\vec{k}_2,\lambda_2} (\vec{v}_1 \cdot \vec{\epsilon}_{1_{\lambda_1}}) e^{i\vec{k}_1\vec{x}_1} (\vec{v}_2 \cdot \vec{\epsilon}_{2_{\lambda_2}}) e^{-i\vec{k}_2\vec{x}_2)}\, |0\rangle \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \sum_{\lambda_1,\lambda_2=1}^{2} \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}} \delta(\vec{k}_1 - \vec{k}_2)\delta_{\lambda_1\lambda_2} (\vec{v}_1 \cdot \vec{\epsilon}_{1_{\lambda_1}}) e^{i\vec{k}_1\vec{x}_1} (\vec{v}_2 \cdot \vec{\epsilon}_{2_{\lambda_2}}) e^{-i\vec{k}_2\vec{x}_2} \\
&= \frac{1}{2(2\pi)^3} \sum_{\lambda=1}^{2} \int_{\mathbb{R}} d\vec{k} \frac{1}{|\vec{k}|} (\vec{v}_1 \cdot \vec{\epsilon}_{\lambda}) e^{i\vec{k}\vec{x}_1} (\vec{v}_1 \cdot \vec{\epsilon}_{\lambda}) e^{-i\vec{k}\vec{x}_2}
\end{aligned}
\tag{A.15}
$$

A.6.3. Type III

$$
\begin{aligned}
&\langle 0|\left(\vec{v}_1\cdot\vec{A}(\vec{x}_1)\right)\left(\vec{v}_2\cdot\vec{A}(\vec{x}_2)\right)\left(\vec{A}(\vec{x}_3)\cdot\vec{A}(\vec{x}_4)\right)|0\rangle \\
&=\frac{1}{4(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3\,d\vec{k}_4 \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^{2}\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2||\vec{k}_3||\vec{k}_4|}} \\
&\qquad \langle 0|\big(\cdots + a_{\vec{k}_1,\lambda_2}a^\dagger_{\vec{k}_2,\lambda_2}a_{\vec{k}_3,\lambda_3}a^\dagger_{\vec{k}_4,\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_2}e^{i(\vec{k}_3\vec{x}_3-\vec{k}_4\vec{x}_4)} \\
&\qquad\qquad + a_{\vec{k}_1,\lambda_1}a_{\vec{k}_2,\lambda_2}a^\dagger_{\vec{k}_3,\lambda_3}a^\dagger_{\vec{k}_4,\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{i\vec{k}_2\vec{x}_2}e^{-i(\vec{k}_3\vec{x}_3+\vec{k}_4\vec{x}_4)}\big)|0\rangle\left(\vec{\epsilon}_{3_{\lambda_3}}\cdot\vec{\epsilon}_{4_{\lambda_4}}\right) \\
&=\frac{1}{4(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3\,d\vec{k}_4 \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^{2}\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2||\vec{k}_3||\vec{k}_4|}} \\
&\quad \big(\delta(\vec{k}_1-\vec{k}_2)\delta(\vec{k}_3-\vec{k}_4)\delta_{\lambda_1\lambda_2}\delta_{\lambda_3\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_2}e^{i(\vec{k}_3\vec{x}_3-\vec{k}_4\vec{x}_4)} \\
&\qquad + \delta(\vec{k}_1-\vec{k}_3)\delta(\vec{k}_2-\vec{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{i\vec{k}_2\vec{x}_2}e^{-i(\vec{k}_3\vec{x}_3+\vec{k}_4\vec{x}_4)} \\
&\qquad\quad + \delta(\vec{k}_1-\vec{k}_4)\delta(\vec{k}_2-\vec{k}_3)\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{i\vec{k}_2\vec{x}_2}e^{-i(\vec{k}_3\vec{x}_3+\vec{k}_4\vec{x}_4)}\big)\left(\vec{\epsilon}_{3_{\lambda_3}}\cdot\vec{\epsilon}_{4_{\lambda_4}}\right) \\
&=\frac{1}{2(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2 \sum_{\lambda=1}^{2}\frac{1}{|\vec{k}_1||\vec{k}_2|}(\vec{v}_1\cdot\vec{\epsilon}_{1_\lambda})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{1_\lambda})e^{-i\vec{k}_1\vec{x}_2}e^{i\vec{k}_2(\vec{x}_3-\vec{x}_4)} \\
&+\frac{1}{4(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2 \sum_{\lambda_1,\lambda_2=1}^{2}\frac{1}{|\vec{k}_1||\vec{k}_2|}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{i\vec{k}_2\vec{x}_2} \\
&\quad \left(e^{-i(\vec{k}_1\vec{x}_3+\vec{k}_2\vec{x}_4)}+e^{-i(\vec{k}_2\vec{x}_3+\vec{k}_1\vec{x}_4)}\right)\left(\vec{\epsilon}_{1_{\lambda_1}}\cdot\vec{\epsilon}_{2_{\lambda_2}}\right)
\end{aligned}
\tag{A.16}
$$

A.6.4. Type IV

$$
\begin{aligned}
&\langle 0|\left(\vec{v}_1\cdot\vec{A}(\vec{x}_1)\right)\left(\vec{A}(\vec{x}_2)\cdot\vec{A}(\vec{x}_3)\right)\left(\vec{v}_4\cdot\vec{A}(\vec{x}_4)\right)|0\rangle\\
&=\frac{1}{4(2\pi)^6}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3\,d\vec{k}_4\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^{2}\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2||\vec{k}_3||\vec{k}_4|}}\\
&\qquad\langle 0|\left(\cdots+a_{\vec{k}_1,\lambda_2}a^{\dagger}_{\vec{k}_2,\lambda_2}a_{\vec{k}_3,\lambda_3}a^{\dagger}_{\vec{k}_4,\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(-\vec{k}_2\vec{x}_2+\vec{k}_3\vec{x}_3)}(\vec{v}_4\cdot\vec{\epsilon}_{4_{\lambda_4}})e^{-i\vec{k}_4\vec{x}_4}\right.\\
&\qquad\qquad\left.+a_{\vec{k}_1,\lambda_1}a_{\vec{k}_2,\lambda_2}a^{\dagger}_{\vec{k}_3,\lambda_3}a^{\dagger}_{\vec{k}_4,\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(\vec{k}_2\vec{x}_2-\vec{k}_3\vec{x}_3)}(\vec{v}_4\cdot\vec{\epsilon}_{4_{\lambda_4}})e^{-i\vec{k}_4\vec{x}_4}\right)|0\rangle\left(\vec{\epsilon}_{2_{\lambda_2}}\cdot\vec{\epsilon}_{3_{\lambda_3}}\right)\\
&=\frac{1}{4(2\pi)^6}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\,d\vec{k}_3\,d\vec{k}_4\sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^{2}\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2||\vec{k}_3||\vec{k}_4|}}\\
&\quad\left(\delta(\vec{k}_1-\vec{k}_2)\delta(\vec{k}_3-\vec{k}_4)\delta_{\lambda_1\lambda_2}\delta_{\lambda_3\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(-\vec{k}_2\vec{x}_2+\vec{k}_3\vec{x}_3)}(\vec{v}_4\cdot\vec{\epsilon}_{4_{\lambda_4}})e^{-i\vec{k}_4\vec{x}_4}\right.\\
&\qquad+\delta(\vec{k}_1-\vec{k}_3)\delta(\vec{k}_2-\vec{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(\vec{k}_2\vec{x}_2-\vec{k}_3\vec{x}_3)}(\vec{v}_4\cdot\vec{\epsilon}_{4_{\lambda_4}})e^{-i\vec{k}_4\vec{x}_4}\\
&\qquad\quad\left.+\delta(\vec{k}_1-\vec{k}_4)\delta(\vec{k}_2-\vec{k}_3)\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(\vec{k}_2\vec{x}_2-\vec{k}_3\vec{x}_3)}(\vec{v}_4\cdot\vec{\epsilon}_{4_{\lambda_4}})e^{-i\vec{k}_4\vec{x}_4}\right)\left(\vec{\epsilon}_{2_{\lambda_2}}\cdot\vec{\epsilon}_{3_{\lambda_3}}\right)\\
&=\frac{1}{2(2\pi)^6}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\sum_{\lambda=1}^{2}\frac{1}{|\vec{k}_1||\vec{k}_2|}(\vec{v}_1\cdot\vec{\epsilon}_{1_\lambda})e^{i\vec{k}_1\vec{x}_1}e^{i\vec{k}_2(\vec{x}_2-\vec{x}_3)}(\vec{v}_2\cdot\vec{\epsilon}_{1_\lambda})e^{-i\vec{k}_1\vec{x}_4}\\
&+\frac{1}{2(2\pi)^6}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\sum_{\lambda_1,\lambda_2=1}^{2}\frac{1}{|\vec{k}_1||\vec{k}_2|}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}e^{i(-\vec{k}_1\vec{x}_2+\vec{k}_2\vec{x}_3)}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_4}\left(\vec{\epsilon}_{1_{\lambda_1}}\cdot\vec{\epsilon}_{2_{\lambda_2}}\right)
\end{aligned}
\tag{A.17}
$$

A.6.5. Type V

$$
\begin{aligned}
&\langle 0|\,(\vec{v}_1\cdot\vec{A}(\vec{x}_1))\sum_{\lambda=1}\int_{\mathbb{R}}d\vec{k}\,|\vec{k}|\,a^{\dagger}_{\vec{k},\lambda}a_{\vec{k},\lambda}(\vec{v}_2\cdot\vec{A}(\vec{x}_2))\,|0\rangle\\
&=\frac{1}{2(2\pi)^3}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\,d\vec{K}\sum_{\lambda_1,\lambda_2,\Lambda=1}^{2}\frac{|\vec{K}|}{\sqrt{|\vec{k}_1||\vec{k}_2|}}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_2}\\
&\qquad\langle 0|\left(\cdots+a_{\vec{k}_1,\lambda_1}a^{\dagger}_{\vec{K},\Lambda}a_{\vec{K},\Lambda}a^{\dagger}_{\vec{k}_2,\lambda_2}\right)|0\rangle\\
&=\frac{1}{2(2\pi)^3}\int_{\mathbb{R}}d\vec{k}_1\,d\vec{k}_2\,d\vec{K}\sum_{\lambda_1,\lambda_2,\Lambda=1}^{2}\frac{|\vec{K}|}{\sqrt{|\vec{k}_1||\vec{k}_2|}}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_2}\\
&\qquad\delta(\vec{k}_1-\vec{K})\delta(\vec{K}-\vec{k}_2)\delta_{\lambda_1\Lambda}\delta_{\Lambda\lambda_2}\\
&=\frac{1}{2(2\pi)^3}\sum_{\lambda=1}^{2}\int_{\mathbb{R}}d\vec{k}(\vec{v}_1\cdot\vec{\epsilon}_{1_{\lambda_1}})e^{i\vec{k}_1\vec{x}_1}(\vec{v}_2\cdot\vec{\epsilon}_{2_{\lambda_2}})e^{-i\vec{k}_2\vec{x}_2}
\end{aligned}
\tag{A.18}
$$

A.7. Identities of Vacuum-Potential Interactions

A.7.1. Type I

$$eq.(\boxed{A.14}) \rightarrow \langle 0|\,\vec{A}^2(\vec{q})\,|0\rangle = \frac{1}{2(2\pi)^3}\sum_{\lambda=1}^{2}\int_{\mathbb{R}} d\vec{k}\frac{1}{|\vec{k}|} \tag{A.19}$$

A.7.2. Type II

$$\begin{aligned}
&eq.(\boxed{A.15}) \rightarrow \langle 0|\,\{\vec{p},\vec{A}(\vec{q})\}\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)\,|0\rangle\\
&= \frac{1}{2(2\pi)^3}\sum_{\lambda=1}^{2}\int_{\mathbb{R}} d\vec{k}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_\lambda)}{|\vec{k}|}e^{i\vec{k}\vec{y}_0}\{(\vec{p}\cdot\vec{\epsilon}_\lambda), e^{-i\vec{k}\vec{q}}\}\\
&= \langle 0|\,\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)\{\vec{p},\vec{A}(\vec{q})\}\,|0\rangle
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
&eq.(\boxed{A.15}) \rightarrow \langle 0|\,\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)^2\,|0\rangle\\
&= \frac{1}{2(2\pi)^3}\sum_{\lambda=1}^{2}\int_{\mathbb{R}} d\vec{k}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_\lambda)^2}{|\vec{k}|}
\end{aligned} \tag{A.21}$$

A.7.3. Type III

$$\begin{aligned}
&eq.(\boxed{A.16}) \rightarrow \langle 0|\,\{\vec{p},\vec{A}(\vec{q})\}\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)\vec{A}^2(\vec{q})\,|0\rangle\\
&= \frac{1}{2(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\sum_{\lambda=1}^{2}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_{1_\lambda})}{|\vec{k}_1||\vec{k}_2|}e^{i\vec{k}_1\vec{y}_0}\{(\vec{p}\cdot\vec{\epsilon}_{1_\lambda}), e^{-i\vec{k}_1\vec{q}}\}\\
&+ \frac{1}{2(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\sum_{\lambda_1,\lambda_2=1}^{2}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_{2_{\lambda_2}})}{|\vec{k}_1||\vec{k}_2|}e^{i\vec{k}_2\vec{y}_0}\{(\vec{p}\cdot\vec{\epsilon}_{1_\lambda}), e^{i\vec{k}_1\vec{q}}\}e^{-i(\vec{k}_1+\vec{k}_2)\vec{q}}\big(\vec{\epsilon}_{1_{\lambda_1}}\cdot\vec{\epsilon}_{2_{\lambda_2}}\big)\\
&= \langle 0|\,\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)\{\vec{p},\vec{A}(\vec{q})\}\vec{A}^2(\vec{q})\,|0\rangle\\
&= \langle 0|\,\{\vec{p},\vec{A}(\vec{q})\}\vec{A}^2(\vec{q})\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)\,|0\rangle \leftarrow eq.(\boxed{A.23})
\end{aligned}$$

$$\begin{aligned}
&eq.(\boxed{A.16}) \rightarrow \langle 0|\,\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)^2\vec{A}^2(\vec{q})\,|0\rangle\\
&= \frac{1}{2(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\sum_{\lambda=1}^{2}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_{1_\lambda})^2}{|\vec{k}_1||\vec{k}_2|}\\
&+ \frac{1}{2(2\pi)^6}\int_{\mathbb{R}} d\vec{k}_1\,d\vec{k}_2\sum_{\lambda_1,\lambda_2=1}^{2}\frac{(\dot{\vec{y}}_0\cdot\vec{\epsilon}_{1_{\lambda_1}})(\dot{\vec{y}}_0\cdot\vec{\epsilon}_{2_{\lambda_2}})}{|\vec{k}_1||\vec{k}_2|}e^{i(\vec{k}_1+\vec{k}_2)(\vec{y}_0-\vec{q})}\big(\vec{\epsilon}_{1_{\lambda_1}}\cdot\vec{\epsilon}_{2_{\lambda_2}}\big)\\
&= \langle 0|\,\vec{A}^2(\vec{q})\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)^2\,|0\rangle\\
&= \langle 0|\,\vec{A}^2(\vec{q})\big(\dot{\vec{y}}_0\cdot\vec{A}(\vec{y}_0)\big)^2\,|0\rangle \leftarrow eq.(\boxed{A.24})
\end{aligned} \tag{A.22}$$

A.7.4. Type IV

$$\begin{aligned}
&eq.(\boxed{A.17}) \rightarrow \langle 0| \left\{\vec{p}, \vec{A}(\vec{q})\right\} \vec{A}^2(\vec{q}) \big(\vec{y}_0 \cdot \vec{A}(\vec{y}_0)\big) |0\rangle \\
&= \frac{1}{2(2\pi)^6} \sum_{\lambda=1}^{2} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \frac{(\vec{y}_0 \cdot \vec{\epsilon}_{1_\lambda})}{|\vec{k}_1||\vec{k}_2|} e^{i\vec{k}_1 \vec{y}_0} \big\{ \big(\vec{p} \cdot \vec{\epsilon}_{1_\lambda}\big), e^{-i\vec{k}_1 \vec{q}} \big\} \\
&+ \frac{1}{2(2\pi)^6} \sum_{\lambda_1, \lambda_2=1}^{2} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \frac{(\vec{y}_0 \cdot \vec{\epsilon}_{1_{\lambda_1}})}{|\vec{k}_1||\vec{k}_2|} e^{i\vec{k}_1 \vec{y}_0} \big\{ \big(\vec{p} \cdot \vec{\epsilon}_{2_{\lambda_2}}\big), e^{-i\vec{k}_2 \vec{q}} \big\} e^{i(\vec{k}_2 - \vec{k}_1)\vec{q}} \big(\vec{\epsilon}_{1_{\lambda_1}} \cdot \vec{\epsilon}_{2_{\lambda_2}}\big) \\
&= \langle 0| \left\{\vec{p}, \vec{A}(\vec{q})\right\} \big(\dot{\vec{y}}_0 \cdot \vec{A}(\vec{y}_0)\big) \vec{A}^2(\vec{q}) |0\rangle \\
&= \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}(\vec{y}_0)\big) \left\{\vec{p}, \vec{A}(\vec{q})\right\} \vec{A}^2(\vec{q}) |0\rangle \leftarrow eq.(\boxed{A.22})
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
&eq.(\boxed{A.17}) \rightarrow \langle 0| \big(\vec{y}_0 \cdot \vec{A}(\vec{y}_0)\big) \vec{A}^2(\vec{q}) \big(\vec{y}_0 \cdot \vec{A}(\vec{y}_0)\big) |0\rangle \\
&= \frac{1}{2(2\pi)^6} \sum_{\lambda=1}^{2} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \frac{(\vec{y}_0 \cdot \vec{\epsilon}_{1_\lambda})^2}{|\vec{k}_1||\vec{k}_2|} \\
&+ \frac{1}{2(2\pi)^6} \sum_{\lambda_1, \lambda_2=1}^{2} \int_{\mathbb{R}} d\vec{k}_1\, d\vec{k}_2 \frac{(\vec{y}_0 \cdot \vec{\epsilon}_{1_{\lambda_1}})(\vec{y}_0 \cdot \vec{\epsilon}_{2_{\lambda_2}})}{|\vec{k}_1||\vec{k}_2|} e^{i(\vec{k}_1 - \vec{k}_2)(\vec{y}_0 - \vec{q})} \big(\vec{\epsilon}_{1_{\lambda_1}} \cdot \vec{\epsilon}_{2_{\lambda_2}}\big) \\
&= \langle 0| \big(\vec{y}_0 \cdot \vec{A}(\vec{y}_0)\big)^2 \vec{A}^2(\vec{q}) |0\rangle \leftarrow eq.(\boxed{A.22})
\end{aligned} \tag{A.24}$$

A.7.5. Type V

Note that $\sum_{\lambda=1}^{2} |\vec{\epsilon}_\lambda\rangle \langle\vec{\epsilon}_\lambda| = \mathbb{I} - \frac{1}{|\vec{k}^2|} \left|\vec{k}\right\rangle \left\langle\vec{k}\right|$.

$$\begin{aligned}
&eq.(\boxed{A.18}) \rightarrow \langle 0| \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}|\, a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} \{\vec{p}, \vec{A}_{\vec{q}}\} |0\rangle \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} \big(\dot{\vec{y}}_0 \cdot \vec{\epsilon}_\lambda\big) e^{i\vec{k}\vec{y}_0} \big\{ \big(\vec{p} \cdot \vec{\epsilon}_\lambda\big), e^{-i\vec{k}\vec{q}} \big\} \\
&= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k}\, e^{i\vec{k}\vec{y}_0} \big\{ \big(\vec{p} \cdot \dot{\vec{y}}_0\big), e^{-i\vec{k}\vec{q}} \big\} - \frac{1}{2(2\pi)^3} \int_{\mathbb{R}} d\vec{k} \frac{1}{|\vec{k}|^2} \big(\dot{\vec{y}}_0 \cdot \vec{k}\big) e^{i\vec{k}\vec{y}_0} \big\{ \big(\vec{p} \cdot \vec{k}\big), e^{-i\vec{k}\vec{q}} \big\} \\
&= \langle 0| \{\vec{p}, \vec{A}_{\vec{q}}\} \int_{\mathbb{R}} d\vec{k} \sum_{\lambda=1}^{2} |\vec{k}|\, a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda} \big(\dot{\vec{y}}_0 \cdot \vec{A}_{\vec{y}_0}\big) |0\rangle
\end{aligned} \tag{A.25}$$

A.8. Position Measurement

Let $f_{\vec{q}}$ be any construction of position operators.

A.8.1. Type I

$$
\begin{aligned}
\langle\vec{r}|\,\vec{p}^2\Phi\,|\vec{r}\rangle &= \int_{\mathbb{R}} d\vec{p}'\,\langle\vec{r}|\vec{p}'\rangle\,\langle\vec{p}'|\,\vec{p}^2\,|\phi\rangle\,\langle\phi|\vec{r}\rangle \\
&= \int_{\mathbb{R}} d\vec{p}'\vec{p}'^2\,\langle\vec{r}|\vec{p}'\rangle\,\langle\vec{p}'|\phi\rangle\,\langle\phi|\vec{r}\rangle \\
&= (2\pi)^{-\frac{3}{2}}\overline{\phi}_{\vec{r}}\int_{\mathbb{R}} d\vec{p}'\vec{p}'^2 e^{i\vec{r}\vec{p}'}\,\widetilde{\phi}_{\vec{p}'} \\
&= -(2\pi)^{-\frac{3}{2}}\overline{\phi}_{\vec{r}}\Delta_r\int_{\mathbb{R}} d\vec{p}' e^{i\vec{r}\vec{p}'}\,\widetilde{\phi}_{\vec{p}'} \\
&= -\overline{\phi}_{\vec{r}}\Delta_r\phi_{\vec{r}}
\end{aligned}
\tag{A.26}
$$

A.8.2. Type II

$$
\langle\vec{r}|\,\vec{p}^2\Phi f_{\vec{q}}\,|\vec{r}\rangle = f_{\vec{r}}\,\langle\vec{r}|\,\vec{p}^2\Phi\,|\vec{r}\rangle = -f_{\vec{r}}\overline{\phi}_{\vec{r}}\Delta_r\phi_{\vec{r}}
\tag{A.27}
$$

A.8.3. Type III

$$
\begin{aligned}
\langle\vec{r}|\,\vec{p}^2 f_{\vec{q}}\Phi\,|\vec{r}\rangle &= \int_{\mathbb{R}} d\vec{p}'d\vec{r}'\,\langle\vec{r}|\,\vec{p}^2\,|\vec{p}'\rangle\,\langle\vec{p}'|\vec{r}'\rangle\,\langle\vec{r}'|\,f_{\vec{q}}\,|\phi\rangle\,\langle\phi|\vec{r}\rangle \\
&= \int_{\mathbb{R}} d\vec{p}'d\vec{r}'\vec{p}'^2 f_{\vec{r}'}\,\langle\vec{r}|\vec{p}'\rangle\,\langle\vec{p}'|\vec{r}'\rangle\,\langle\vec{r}'|\phi\rangle\,\langle\phi|\vec{r}\rangle \\
&= (2\pi)^{-3}\overline{\phi}_{\vec{r}}\int_{\mathbb{R}} d\vec{r}'\phi_{\vec{r}'}f_{\vec{r}'}\int_{\mathbb{R}} d^3\vec{p}'\vec{p}'^2 e^{i\vec{r}\vec{p}'}e^{-i\vec{p}'\vec{r}'} \\
&= -(2\pi)^{-3}\overline{\phi}_{\vec{r}}\int_{\mathbb{R}} d\vec{r}' f_{\vec{r}'}\phi_{\vec{r}'}\Delta_r\int_{\mathbb{R}} d^3\vec{p}' e^{i\vec{r}\vec{p}'}e^{-i\vec{p}'\vec{r}'} \\
&= -\overline{\phi}_{\vec{r}}\int_{\mathbb{R}} d\vec{r}' f_{\vec{r}'}\phi_{\vec{r}'}\Delta_r\delta(\vec{r}-\vec{r}') \\
&= -\overline{\phi}_{\vec{r}}\Delta_r\int_{\mathbb{R}} d\vec{r}'\delta(\vec{r}-\vec{r}')f_{\vec{r}'}\phi_{\vec{r}'} \\
&= -\phi_{\vec{r}}\Delta_r\big(f_{\vec{r}}\phi_{\vec{r}}\big)
\end{aligned}
\tag{A.28}
$$

A.9. Fourier Transform

A.9.1. Type I

$$\int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}(\vec{a}-\vec{b})} = (2\pi)^3 \delta(\vec{a}-\vec{b}) \tag{A.29}$$

A.9.2. Type II

$$\begin{aligned}\int_{-\infty}^{\infty} d\vec{k} \frac{1}{|\vec{k}|} e^{i\vec{k}\vec{s}} &= 2\pi \int_0^{\infty} dr \int_0^{\pi} d\theta \frac{1}{r} e^{ir|s|cos(\theta)} r^2 sin(\theta) = 2\pi \int_0^{\infty} dr \int_{-1}^{1} dx\, r e^{ir|s|x} \\ &= 4\pi \frac{1}{|s|} \int_0^{\infty} dr\, sin(r|s|) = 4\pi \frac{1}{|s|^2} \left(1 - lim_{R\to\infty} cos(|s|R)\right) \simeq \frac{4\pi}{|s|^2}\end{aligned} \tag{A.30}$$

A.9.3. Type III

$$\begin{aligned}\int_{-\infty}^{\infty} d\vec{k} \frac{k_i k_j}{|k|^2} e^{i\vec{k}\vec{s}} =& - i\partial_{s_i} \int_{-\infty}^{\infty} d\vec{k} \frac{k_j}{|k|^2} e^{i\vec{k}\vec{s}} \\ =& - \partial_{s_i}\partial_{s_j} \int_{-\infty}^{\infty} d\vec{k} \frac{1}{|k|^2} e^{i\vec{k}\vec{s}} = -\partial_{s_i}\partial_{s_j} \int_0^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{1}{r^2} e^{ir|s|\,cos(\theta)} r^2 sin(\theta) \\ =& - 2\pi\partial_{s_i}\partial_{s_j} \int_0^{\infty} dr \int_0^{\pi} d\theta\, e^{ir|s|\,cos(\theta)} sin(\theta) = -2\pi\partial_{s_i}\partial_{s_j} \int_0^{\infty} dr \int_{-1}^{1} dx\, e^{ir|s|x} \\ =& - 2\pi\partial_{s_i}\partial_{s_j} \int_0^{\infty} dr \frac{1}{ir|s|} (e^{ir|s|} - e^{-ir|s|}) \\ &= -4\pi\partial_{s_i}\partial_{s_j} \frac{1}{|s|} \int_0^{\infty} dr \frac{sin(r|s|)}{r} = -2\pi^2 \partial_{s_i}\partial_{s_j} \frac{1}{|s|} \\ =& 2\pi^2 \partial_{s_i} \frac{s_j}{|s|^3} = 2\pi^2 \left(\frac{\delta_{ij}}{|s|^3} - 3\frac{s_i s_j}{|s|^5} \right)\end{aligned} \tag{A.31}$$