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Popescu's nested approximation theorem

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Zusammenfassung

Diese Arbeit behandelt Popescus Satz über genestete Approximation. Dieser besagt, dass wenn ein System von algebraischen Potenzreihen in zwei Mengen von Variablen x, y eine formale Lösung $\widehat{y}(x)$ besitzt, die in den x-Variablen genestet ist, es auch eine algebraische Lösung gibt, die mit der formalen Lösung bis zu einem beliebig hohen Grad übereinstimmt und ebenfalls genestet ist. Dies ist eine Verallgemeinerung des Artinschen Approximationssatzes, der keine genesteten Lösungen erfordert.

Das erste Kapitel wird eine Wiederholung der grundlegenden Begriffe von formalen, konvergenten und algebraischen Potenzreihen sein. Darüber hinaus werden der Weierstraß'sche Vorbereitungssatz und der Weierstraß'sche Divisionssatz dieser Ringe eingeführt, die in späteren Beweisen eine zentrale Rolle spielen werden. Am Ende des Kapitels wird eine Wiederholung der Henselschen Ringe und der Kodierung von algebraischen Potenzreihen erfolgen.

Da die genestete Approximation ein Verallgemeinerung der Artinschen Approximation ist, wird in Kapitel 3 der Beweis des zweiteren Satzes sowohl im algebraischen als auch im konvergenten Fall präsentiert. Dazu gibt es einige Vorbereitungen in Kapitel 2. Kapitel 3 beginnt mit einem einfachen Fall der Artinschen Approximation, um ein Verständnis dafür zu entwickeln, wie der Beweis funktioniert, gefolgt von dem allgemeinen Beweis.

Eines der Hauptprobleme bei der genesteten Approximation ist, dass sie im Allgemeinen nur für den algebraischen Fall gilt. Ein Gegenbeispiel für den konvergenten Fall wird in Kapitel 5 präsentiert. Es gibt jedoch einen Spezialfall, der auf Denef und Lipshitz zurückgeht: Wenn es nur zwei Nester gibt und das erste Nest nur aus einer einzigen Variablen besteht, gilt dieser Beweis auch im konvergenten Fall. Der Beweis findet sich in Kapitel 4. Er hat den Vorteil, dem in Kapitel 3 gegebenen Beweis sehr ähnlich zu sein.

Der allgemeine Beweis von Popescus Satz über genestete Approximation ist weitaus schwieriger. In Kapitel 6 wir eine Rückführung des Problems auf Popescus Satz über das Glätten von Ringhomomorphismen gezeigt, dessen Beweis in dieser Arbeit nicht gegeben wird.

Kapitel 7 führt den Begriff der starken Artinschen Approximation ein und gibt einen Beweis zu einer schwächeren aber allgemeiner gültigen Version der starken Artinschen Approximation.

Abschließend werden in Kapitel 8 zwei Anwendungen von Popescus Satz über genestete Approximation vorgestellt.

Abstract

This thesis is about Popescu's nested approximation theorem. It states that if a system of algebraic power series in two sets of variables x, y has a formal solution $\widehat{y}(x)$ that is nested in the x variables, it also has an algebraic solution that agrees with the formal solution up to an arbitrarily high degree and which is also nested. It is a generalization of the Artin approximation theorem, which does not require nestedness.

The first chapter will be a revision of the basic notions of formal, convergent and algebraic power series. Further, the Weierstrass preparation theorem and the Weierstrass division theorem of these rings will be introduced, as they will play central roles in some of the proofs given in the later chapters. At the end of the chapter, there will be a revision of Henselian rings and encoding of algebraic power series.

As nested approximation is a generalization of Artin approximation, chapter 3 will present the proof of Artin approximation in both the algebraic and convergent case. To this end, there will be some preparation in chapter 2. Chapter 3 will start with an easy case of Artin approximation to get an idea how the proof works, followed by the general proof.

One of the key problems with nested approximation is that it in general only holds in the algebraic setting. A counterexample for the convergent setting is presented in chapter 5. However, there is a special case going back to Denef and Lipshitz: If there are only two nests and the first nest only consists of a single variable, the nested approximation also holds in the convergent case. The proof will be presented in chapter 4. It has the advantage of being very similar to the proof given in chapter 3.

The general proof of Popescu's nested approximation theorem is highly non-trivial. Chapter 6 will present the proof idea as a consequence of Popescu's theorem on smoothing ring homomorphisms, which will not be proven in this paper.

Chapter 7 will introduce the notion of strong Artin approximation and give a proof to a weaker but more general version of Artin's strong approximation theorem.

Finally, chapter 8 will give two applications of Popescu's nested approximation theorem.

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Contents

1 Introduction and basic notions

Let $\mathbb{K}$ denote a field and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote multivariables. Then let $\mathbb{K}[[x]]$, $\mathbb{K}\langle x\rangle$ and (if $\mathbb{K}$ is valued) $\mathbb{K}\{x\}$ denote the rings of formal, algebraic and respectively convergent power series. The two central theorems of this paper will be the following:

Theorem 1 (Artin approximation [Art68])**.** *Let $f(x,y) \in \mathbb{K}\{x,y\}^r$ (or $\mathbb{K}\langle x,y\rangle^r$) be a vector of convergent (or respectively algebraic) power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ of f vanishing at 0,*

$$f(x, \widehat{y}(x)) = 0.$$

Then there exists, for any $c \in \mathbb{N}$, a convergent (or resp. algebraic) power series solution $y(x) \in \mathbb{K}\{x\}^m$ (or resp. $\mathbb{K}\langle x\rangle^m$),

$$f(x, y(x)) = 0,$$

which coincides with $\widehat{y}(x)$ up to degree c,

$$y(x) \equiv \widehat{y}(x) \mod (x)^{c+1}.$$

In other words, the set of convergent (or resp. algebraic) solutions $y(x) \in \mathbb{K}\{x\}$ of $f(x, y(x)) = 0$ is dense in the set of formal solutions of f with respect to the Krull topology.

Theorem 2 (Popescu's nested approximation theorem [Pop86])**.** *Let $f(x,y) \in \mathbb{K}\{x,y\}^r$ be a vector of polynomials or algebraic power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ vanishing at 0,*

$$f(x, \widehat{y}(x)) = 0.$$

Furthermore, assume that each $\widehat{y}_i(x) \in \mathbb{K}[[x_1, \ldots, x_{n_i}]]$ for some $1 \leq n_i \leq n$. Then there exists, for any $c \in \mathbb{N}$, an algebraic power series solution $y(x) \in \mathbb{K}\langle x\rangle^m$,

$$f(x, y(x)) = 0,$$

which coincides with $\widehat{y}(x)$ up to degree c,

$$y(x) \equiv \widehat{y}(x) \mod (x)^{c+1}$$

and for which $y_i(x) \in \mathbb{K}\langle x_1, ..., x_{n_i}\rangle$.

A proof for Theorem 1 will be given in chapter 3, Theorem 2 will be proven in the simple case that $x = (x_1, x_2)$ in chapter 4 and the general proof will be outlined in chapter 7. Also notice that Theorem 2 only holds for the algebraic case and not for the convergent one. A famous counterexample by Gabrielov ([Gab71]) will be presented in chapter 5. In order to prove these theorems, we will need to take a closer look at the rings $\mathbb{K}[[x]], \mathbb{K}\{x\}$ and $\mathbb{K}\langle x\rangle$.

1.1 Basic notions of the power series rings

Let $x = (x_1, \ldots, x_n)$ be a multivariable. The ring of formal power series $\mathbb{K}[[x]]$ is the x-adic completion of the localization $\mathbb{K}[x]_{(x)}$ of the polynomial ring at the ideal $(x_1, \ldots, x_n)$. Its elements

can be written in the form $f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ with $c_\alpha \in \mathbb{K}$. We call $\text{ord}(f) := \min\{\alpha \mid c_\alpha \neq 0\}$ the order of f and define $\text{ord}(0) = \infty$.
If $\mathbb{K}$ is a valued and complete field with absolute value $|.|$, one can introduce for every $s \in \mathbb{K}$ the following pseudo norm on $\mathbb{K}[[x]]$:

$$|f|_s = \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| s^{|\alpha|}.$$

A formal power series f is called convergent if there exists an $s \neq 0 \in \mathbb{K}$ such that $|f|_s < \infty$. If we denote by $\mathbb{K}\{x\}_s$ the Banach-$\mathbb{K}$-algebra of elements f (one can show that this is indeed complete), such that $|f|_s < \infty$, then the ring of convergent power series $\mathbb{K}\{x\}$ can be interpreted as the union $\bigcup_{s \neq 0 \in \mathbb{K}} \mathbb{K}\{x\}_s$. A useful fact about the ring of convergent power series needed later on is the following: (cf. [GR71]):

Lemma 1.1. *If $f \in \mathbb{K}\{x\}_s$ then for $0 < r < s$ the partial derivatives $\partial_{x_i} f$ belongs to $\mathbb{K}\{x\}_r$.*

Proof. So let $0 < s < r$. Then for $k \in \mathbb{N}$ the term $k \cdot s^{-1} \cdot (\frac{s}{r})^k$ is bounded as k goes to ∞. So let $b \in \mathbb{K}$ such a bound. Then

$$|\partial_{x_i} f|_r = \sum_{\alpha \in \mathbb{N}^n} |\alpha_i| \cdot |c_\alpha| r^{|\alpha - e_i|} = \sum_{\alpha \in \mathbb{N}^n} |\alpha_i| \cdot |c_\alpha| r^{|\alpha|} \cdot r^{-1} \leq \sum_{\alpha \in \mathbb{N}^n} |\alpha_i| \cdot |c_\alpha| (\frac{s}{r})^{|\alpha|} \cdot s^{|\alpha|} \cdot r^{-1} \leq b \cdot |f|_s$$

and hence, $\partial_{x_i} f$ is convergent. □

A power series is called algebraic if it satisfies a polynomial equation, i.e. if there exists a $P(x,t) \neq 0 \in \mathbb{K}[x,t]$ such that $P(x, f(x)) = 0$. We call such an annihilating polynomial $P(x,t)$ minimal polynomial of f if it is a polynomial of minimal degree with respect to t annihilating f. As the set of algebraic power series can therefore also be interpreted as the intersection of $\mathbb{K}[[x]]$ and the algebraic closure of the quotient field of $K[x]$ (i.e. $\mathbb{K}(x)$) they are closed under addition and multiplication and a ring. We will once again need that the ring $\mathbb{K}\langle x \rangle$ is closed under derivation.

Lemma 1.2. *If $f \in \mathbb{K}\langle x \rangle$ then $\frac{\partial f}{\partial x_i} \in \mathbb{K}\langle x \rangle$ for $i = 1, \ldots n$.*

Proof. Let f be an algebraic power series and let $P(x,t) \in \mathbb{K}[x,t]$ be a minimal polynomial of f. We now take the derivative $\frac{\partial}{\partial x_i}$ on both sides of the equation $0 = P(x, f(x))$ to get that

$$0 = \frac{\partial P(t,x)}{\partial x_i}(x, f(x)) + \frac{\partial f}{\partial x_i} \cdot \frac{\partial P(t,x)}{\partial t}(x, f(x)).$$

If $\frac{\partial P(t,x)}{\partial t}(x, f(x))$ is non-zero we can write

$$\frac{\partial f}{\partial x_i} = \frac{-\frac{\partial P(t,x)}{\partial x_i}(x, f(x))}{\frac{\partial P(t,x)}{\partial t}(x, f(x))}$$

and hence $\frac{\partial f}{\partial x_i} \in \mathbb{K}(x)(f) = \mathbb{K}(x)[f]$ (as f is algebraic over the field $\mathbb{K}(x)$) and hence $\frac{\partial f}{\partial x_i}$ is algebraic over $\mathbb{K}(x)$. The assertion now follows from the fact that $\mathbb{K}\langle x \rangle = \mathbb{K}[[x]] \cap \overline{\mathbb{K}(x)}$ as $\frac{\partial f}{\partial x_i}$ is clearly an element of $\mathbb{K}[[x]]$. It only remains to show that $\frac{\partial P(t,x)}{\partial t}(x, f(x)) \neq 0$.
In characteristic 0 the polynomial $\frac{\partial P(t,x)}{\partial t}$ can not vanish at $f(x)$ as it has smaller degree in t than P, which is a minimal polynomial. In positive characteristic p the situation is a bit more delicate. Assume $\frac{\partial P(t,x)}{\partial t}$ vanishes at $f(x)$. Then $\frac{\partial P(t,x)}{\partial t} = 0$ as P is chosen to be minimal and hence P only contains $p-th$ powers of t. It therefore is not irreducible by using the Frobenius-homomorphism, contradicting P being a minimal polynomial of f. □

For the sake of completion, we want to mention that every algebraic power series is also convergent. For a proof see [Hau20]. The converse does not hold (cf. [Yur20]): Take the convergent power series $e^x = \sum_{n\geq 0} \frac{x^n}{n!} \in \mathbb{Q}\{x\}$. Assume that it is algebraic, i.e. $\exists P_1, \ldots, P_d \in \mathbb{Q}[x]$ such that $P(t,x) := \sum_{i=0}^{d} P_i(x)t^i$ annihilates e^x for some non-negative integer d. We may assume that $P(t,x)$ is irreducible. By Setting $x = 1$ in both sides of $0 = P(e^x, x)$ we get that

$$0 = P_0(1) + \ldots P_d(1)(e^x)^d.$$

If all $P_i(1)$ are zero, P is divided by $x - 1$ contradicting irreducibility of P. However, this would give a polynomial over $\mathbb{Q}$ annihilating e, which contradicts the transcendence of e over $\mathbb{Q}$.

1.2 The Weierstrass division theorem and its implications

This subsection will introduce the equivalent Weierstrass division theorem and Weierstrass preparation theorem, which are very powerful tools. They imply many other important properties of power series rings such as inter alia noetherianess, the inverse and implicit function theorem. The proofs in this section will, if not stated otherwise, be in line with [Hau20].
For the following, let A_n denote any of the rings $\mathbb{K}[[x_1, \ldots, x_n]], \mathbb{K}\{x_1, \ldots, x_n\}$ and $\mathbb{K}\langle x_1, \ldots, x_n\rangle$. An element $f(x)$ of A_n is said to be x_n-regular of order d if $f(0, \ldots, 0, x_n) = x_n^d u(x_n)$, where $u(x) \in \mathbb{K}[[x_n]]^*$ is a unit. In other words, the monomial x_n^d appears in the series expansion of $f(x)$ and d is the smallest such degree.

Theorem 3 (Weierstrass division Theorem (WDT))**.** *Let $f \in A_n$ be x_n-regular of order d. Then for any $g \in A_n$ there exist unique power series $h \in A_n$ and $r_0, \ldots, r_{d-1} \in A_{n-1}$ such that*

$$g(x) = f(x) \cdot h(x) + \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j,$$

where $x' = (x_1, \ldots, x_{n-1})$.

Equivalent to this is the following theorem

Theorem 4 (Weierstrass Preparation Theorem (WPT))**.** *Let $f \in A_n$ be x_n-regular of order d. Then there exist unique power series $h \in A_n^*$ and $r_0, \ldots, r_{d-1} \in A_{n-1}$ such that*

$$h(x) \cdot f(x) = x_n^d + \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j,$$

where $x' = (x_1, \ldots, x_{n-1})$. The right-hand side is then referred to as Weierstrassnormalform of f.

Corollary 1.1. *Theorem WDT and WPT are equivalent.*

Proof. WDT⇒WPT: Let $f(x) \in A_n$ be x_n-regular of order d. Now divide, x_n^d by f:

$$x_n^d = f(x) \cdot h(x) + \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j \Leftrightarrow f(x) \cdot h(x) = x_n^d - \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j.$$

There holds $h(0) \neq 0$ as otherwise the expression $f(x) \cdot h(x)$ does not contain a monomial of the form x_n^d because f is x_n-regular of order d. Thus, $h(x)$ is invertible.

WPT⇒WDT: W.L.O.G f is already in the form described, in Theorem 4. By linearity it suffices to show that we can divide a monomial x^c by f. This can be done by applying the Euclidean division of $A_{n-1}[x_n]$. □

We will first prove the formal WDT theorem. The proof given is according to [Lan05] and [Yur20].

Proof of formal WDT. We consider the ring $\mathbb{K}[[x]]$ as the ring of formal power series in x_n with coefficients in $\mathbb{K}[[x']]$. We then introduce for a power series $g = \sum_{i\geq 0} g_i(x')x_n^i \in \mathbb{K}[[x']][[x_n]]$ the maps, $\tau(g)$ and $\alpha(g)$: We define α to take the power series g and cut it off at degree d, whereas τ takes the remaining part of g and divides it by x_n^d i.e.

$$\tau(g) = \tau(\sum_{i\geq 0} g_i(x')x_n^i) = \sum_{i\geq d} g_i(x')x_n^{i-d}$$

$$\alpha(g) = \alpha(\sum_{i\geq 0} g_i(x')x_n^i) = \sum_{i=0}^{d-1} g_i(x')x_n^i.$$

By construction there holds that $g = \alpha(g) + \tau(g)\cdot x_n^d$ and furthermore $\tau(g) = 0$ if and only if g is a polynomial in x_n of degree less than d.
If there exists an $h \in \mathbb{K}[[x]]$ such that $\tau(g) = \tau(h\cdot f)$ there are unique $r_0, \dots r_{d-1} \in \mathbb{K}[[x']]$ such that $g = f\cdot h + \sum_{j=0}^{d-1} r_j(x')\cdot x_n^j$ by setting r_i the coefficient of x_n^i of $\alpha(g) - \alpha(h\cdot f)$. Therefore, it suffices to show that there is a unique h such that

$$\tau(g) = \tau(h\cdot f).$$

We now rewrite $f = \alpha(f) + \tau(f)\cdot x_n^d$ and use linearity of τ to get equivalently

$$\tau(g) = \tau(h\cdot(\alpha(f) + \tau(f)\cdot x_n^d)) = \tau(h\cdot\alpha(f)) + h\cdot\tau(f),$$

as $\tau(h\cdot\tau(f)\cdot x_n^d) = h\cdot\tau(f)$ by construction. Then $\tau(f)$ is invertible, as f is x_n-regular of order d. We can therefore replace h with $\widetilde{h} := h\cdot\tau(f)$ and rewrite the equation once again:

$$\tau(g) = \tau(\frac{\widetilde{h}\cdot\alpha(f)}{\tau(f)}) + \widetilde{h} = (\tau\circ(\frac{\alpha(f)}{\tau(f)}) + \mathrm{Id}_{\mathbb{K}[[x]]})\circ\widetilde{h},$$

where $\mathrm{Id}_{\mathbb{K}[[x]]}$ is the identity map. We now define the map $\Phi : \mathbb{K}[[x]] \to \mathbb{K}[[x]], l(x) \mapsto \tau(l\cdot\frac{\alpha(f)}{\tau(f)})$. If the map $\mathrm{Id}_{\mathbb{K}[[x]]} + \Phi$ has an inverse, then

$$\widetilde{h} = (\mathrm{Id}_{\mathbb{K}[[x]]} + \Phi)^{-1}\circ\tau(f)$$

is given uniquely and hence there are unique h and $r_0, \dots r_{d_1}$ satisfying the statement.
The formal inverse of $\mathrm{Id}_{\mathbb{K}[[x]]} + \Phi$ is given by $\sum_{i\geq 0}(-1)^i\Phi^i$. So the only thing left to show is that this formal inverse is well-defined as a power series. In order to see that, note that as f is x_n-regular of order d, it follows that $\alpha(f)(0,\dots,0,x_n) = 0$ and hence $\frac{\alpha(f)}{\tau(f)}$ is in the ideal generated by $x_1,\dots,x_{n-1}$. Thus Φ maps $(x_1,\dots,x_{n-1})^k$ to $(x_1,\dots,x_n)^{k+1}$ for all k. Hence, by induction for every $l \in \mathbb{K}[[x]]$ there holds

$$\Phi^k(l) \in (x_1,\dots,x_{n-1})^k\mathbb{K}[[x]].$$

As $\mathbb{K}[[x]]$ is complete in the x-adic topology, the formal inverse $\sum_{i\geq 0}(-1)^i\Phi^i$ is well-defined as a power series. □

For the algebraic version, we will prove WPT for characteristic 0, for a more general proof see [JP65]. The proof given here will be in line with [LT70] and [Yur20].

Proof of algebraic WPT. So let $f \in \mathbb{K}\langle x\rangle$ be x_n-regular of order d. As the statement holds for $\mathbb{K}[[x]]$, if there are $h \in \mathbb{K}\langle x\rangle$ and $r_0, \dots, r_{d-1} \in \mathbb{K}\langle x'\rangle$ such that $f \cdot h = x_n + \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j$, then they are already unique. For reasons of readability, we will write $r(x)$ for $x_n + \sum_{j=0}^{d-1} r_j(x') \cdot x_n^j$ for the rest of the proof.
Furthermore, if f_1 and f_2 are algebraic power series that are x_n-regular of order d_1 and d_2 and can be written as $h_1 \cdot f_1 = R_1(x)$ and $h_2 \cdot f_2 = R_2(x)$ respectively, where $h_i \in \mathbb{K}\langle x\rangle^*$ and $R_i \in \mathbb{K}\langle x'\rangle[x_n]_{\leq d_i}$ for $i = 1, 2$, then

$$h_1 \cdot h_2 \cdot f_1 \cdot f_2 = R_1 \cdot R_2.$$

Then $h_1 \cdot h_2$ is once again algebraic and a unit, $f_1 \cdot f_2$ is x_n-regular of order $d_1 + d_2$ and $R_1 \cdot R_2$ is a polynomial in x_n of degree $d_1 + d_2$ whose coefficients are algebraic power series in x'. Hence, we may assume that f is irreducible.
We now apply the formal WPT theorem to f to write

$$h \cdot f = r,$$

where once again $h \in \mathbb{K}[[x]]^*$ and $r \in \mathbb{K}[[x']][x_n]_{\leq d}$. We now want to show that both h and r are algebraic. As f is assumed to be irreducible and h is a unit, r is an irreducible element of $r \in \mathbb{K}[[x']][x_n]_{\leq d}$ and $r(x)$ is separable as we assumed characteristic 0, i.e. r has d different roots β_j in an algebraic closure of the Quotient field of $\mathbb{K}[[x']]$. Now rewrite

$$f(x) = h \cdot \prod_{j=0}^{d}(x_n - \beta_j).$$

As f is algebraic, there is a polynomial $P = \sum_{i=0}^{k} P_i(x)t^i \in \mathbb{K}[x, t]$ such that $P(x, f(x)) = 0$. We may assume w.l.o.g. that P is a minimal polynomial of f. By construction

$$0 = P(x', \beta_j, f(x', \beta_j)) = P_0(x\prime, \beta_j),$$

as $f(x', \beta_j) = 0$ for $j = 1, \dots, d$. Furthermore, as P is the minimal polynomial of f its constant term $P_0 \not\equiv 0$ as otherwise $\frac{P}{t}$ would be an annihilating polynomial of lower degree. Thus, β_j is a root of $P_0(x', t) \in \mathbb{K}[x', t]$ and hence, algebraic over $\mathbb{K}[x]$, and therefore $x_n - \beta_j$ is algebraic over $[K][x]$ for $j = 1, \dots, d$. As a consequence, $r = \prod_{j=0}^{d}(x_n - \beta_j)$ is algebraic.
It remains to show that h is algebraic. We will show the more general statement: if $F, R \neq 0$ are algebraic power series and $H := F/R$ is a formal power series, then H is algebraic. Recall that $\mathbb{K}\langle x\rangle = \mathbb{K}[[x]] \cap \overline{\mathbb{K}(x)}$. As F, R are algebraic over $\mathbb{K}[x]$, they are algebraic over $\mathbb{K}(x)$. Thus, $\mathbb{K}(x)[F, R]$ is an algebraic extension and consequently $\mathbb{K}(x)[F, R] = \mathbb{K}(x)(F, R)$ implying that H is algebraic over $\mathbb{K}(x)$. As $H \in \mathbb{K}[[x]]$ by assumption, $H \in \mathbb{K}\langle x\rangle$ and the proof is complete. □

Proof of convergent WDT. Let $f \in \mathbb{K}\{x\}$ be x_n-regular of order d and let $\mathbb{K}\{x'\}[x_n]_{<d}$ denote the $\mathbb{K}\{x'\}$ module of polynomials in x_n with degree less than d, and let $(f) = f \cdot \mathbb{K}\{x\}$ be the ideal

generated by f. The statement is then equivalent to the direct sum composition

$$\mathbb{K}\{x\} = f \cdot \mathbb{K}\{x\} \bigoplus \mathbb{K}\{x'\}[x_n]_{<d}.$$

As f is x_n-regular of order d and convergent, it can be written uniquely as $f = h(x) \cdot x_n^d + \sum_{i=0}^{d-1} a_i(x')x_n^i$, where $h \in \mathbb{K}\{x\}^*$ and $a_i \in \mathbb{K}\{x'\}$ with $a_i(0) = 0$. This can be done by expanding it in f and cutting it into an initial- and a tail-part at degree d. We may assume w.l.o.g. that $h(0) = 1$. Now let $s \in \mathbb{K}$ be small with respect to the valuation on $\mathbb{K}$ and $k \in \mathbb{N}$. We define for any $\alpha = (\alpha', \alpha_n) \in \mathbb{N}^n$ the map $|\alpha|_k := k \cdot |\alpha'| + \alpha_n$ and introduce the following pseudo-norm on $\mathbb{K}\{x\}$. For $g = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ define

$$|g|_{s,k} = \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| s^{|\alpha|_k}.$$

Let $\mathbb{K}\{x\}_{(s,k)}$ denote the Banach algebra of power series with finite norm $|.|_{s,k}$. The goal is now to show the existence of $s_0 \in \mathbb{K}$ and $k_0 \in \mathbb{N}$ such that for all $0 < s < s_0$ (with respect to the valuation on $\mathbb{K}$) and $k > k_0$ there holds

$$\mathbb{K}\{x\}_{(s,k)} = f \cdot \mathbb{K}\{x\}_{(s,k)} \bigoplus \mathbb{K}\{x'\}[x_n]^{<d}_{(s,k)},$$

where $\mathbb{K}\{x'\}[x_n]^{<d}_{(s,k)} := \mathbb{K}\{x'\}[x_n]_{<d} \cap \mathbb{K}\{x\}_{(s,k)}$. We now rewrite $f = x_n^d + f'$, where $f' = (h - 1) \cdot x_n^d + \sum_{i=0}^{d-1} a_i(x')x_n^i$. Notice that for s_0 small enough and k_0 large enough the following holds for all $0 < s < s_0$ and $k < k_0$: as both $(1 - h)$ and a_i are both convergent power series with no constant term, both $|(1 - h)|_{s,k}$ and $|a_i|_{s,k}$ for all $i = 1, \ldots, d - 1$ will tend to zero for s_0 small enough. For k large enough they will be dominated by $|x_n^d|_{s,k}$ by definition of $|.|_{s,k}$, hence there exists a constant $0 < c < 1$ such that

$$|f'|_{s,k} \leq c \cdot |x_n^d|_{s,k}.$$

Now consider the linear map

$$u : f \cdot \mathbb{K}\{x\}_{(s,k)} \times \mathbb{K}\{x'\}[x_n]^{<d}_{(s,k)} \to \mathbb{K}\{x\}_{(s,k)}$$

$$u(a, b) := a \cdot g + b.$$

We now want to show that u is an isomorphism of Banach spaces. We therefore write $u = v + w$, where $v(a, b) = a \cdot x_n^d + b$ and hence $w(a, b) = f' \cdot x_n^d$. It is clear that v is an isomorphism as the inverse function is given by $v^{-1} : g \to (\tau(g), \alpha(g))$, with $\tau(g), \alpha(g)$ given like in the proof of the formal Weierstrass division theorem. Therefore, it suffices to show that $u \circ v^{-1} = \mathrm{Id}_{\mathbb{K}}\{x\}_{(s,k)} + w \circ v^{-1}$ is an isomorphism. Its formal inverse is given by $\sum_{j \geq 0} (w \circ v^{-1})^j$. There holds for any $g \in \mathbb{K}\{x\}_{(s,k)}$

$$|w \circ v^{-1}(g)|_{s,k} = |w(\tau(g), \alpha(g))|_{s,k} = |f' \cdot \tau(g)|_{s,k} \leq |f'|_{s,k} \cdot |\tau(g)|_{s,k} \leq c \cdot |x_n^d|_{s,k} \cdot |\tau(g)|_{s,k}.$$

Thus, $w \circ v^{-1}$ has operator norm smaller than one and hence $\sum_{j \geq 0} (w \circ v^{-1})^j$ converges to a continous linear operator on $\mathbb{K}\{x\}_{(s,k)}$, proving the statement. □

We will now show that theses two theorems actually imply the properties of the ring A_n mentioned above:

Corollary 1.2. *The ring A_n is noetherian.*

Proof. Let I be an ideal of A_n. Then, after applying a linear change of coordinates, we may assume that there is an element f in I that is x_n-regular of order d. Now consider the ideal $J = (f) \subseteq I$. Then I/J is an A_n-submodule of A_n/J, which by induction on n and WDT is a finitely generated A_{n-1}-module and hence a finitely generated A_n-module. Therefore, I is finitely generated. □

Corollary 1.3. *The ring A_n is a UFD.*

Proof. Let $f \in A_n$. If f factors into $f = g \cdot h$, where g, h are not units, then ord(g),ord(h) < ord(f) and thus there is a factorization into finitely many irreducible parts.
Let $f = u \cdot g_1 \cdot \ldots \cdot g_r$ be such a factorization, where $u \in A_n^*$ and g_i are irreducible. We may assume that all the g_i are x_n regular by applying a linear automorphism of A_n of the form $x_i \to x_i + x_n^{k_i}$ for $i = 1, \ldots, n-1$. After multiplying with suitable units and their inverse, we may assume that all the g_i are in Weierstrass normal form, i.e. they are polynomial in x_n and can thus be interpreted as elements of $A_{n-1}[x_n]$, which is a UFD if A_{n-1} is UFD. By induction, one gets that A_n is a UFD as the base case $n = 1$ is trivial. □

The following two theorems will play an important role in the proof of the Artin approximation theorem.

Theorem 5 (Inverse function theorem (INFT))**.** *Let $f = (f_1, \ldots, f_n) \in A_n^n$ such that $f(0) = 0$ and the Jacobian matrix $\partial_x f$ of f is invertible in 0. Then there exists a unique $h \in A_n^n$ vanishing at 0 such that $h \circ f = f \circ h = \mathrm{Id}$.*

Equivalent to this is

Theorem 6 (Implicit function theorem (IMFT))**.** *Let $g = (g_1, \ldots, g_m) \in A_{n+m}^m$ be a vector of power series in $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ with $g(0,0) = 0$ such that $\partial_y g$ is invertible at 0. Then there exists a unique $h \in A_n^m$ vanishing at 0, such that $g(x, h(x)) = 0$.*

Equivalence of INFT and IMFT. First assume INFT and let $g = (g_1, \ldots, g_m) \in A_{n+m}^m$, $g(0) = 0$ such that $\partial_y g$ of g is invertible in 0. Now consider the map $f(x, y) = (x, g(x, y)) \in A_{n+m}^{n+m}$. Clearly, $f(0,0) = 0$ and the Jacobian matrix J_f is of the form

$$J_f = \begin{pmatrix} \mathbb{1}_n & 0 \\ \partial_x g & \partial_y g \end{pmatrix}.$$

Hence, the determinant of J_f is the same as the determinant of $\partial_y g$ and J_f is invertible in 0. By INFT there exist an element $\widetilde{h} \in A_{n+m}^{n+m}$ vanishing at 0 that is inverse to f, i.e. $f \circ h = \mathrm{Id}$. It will be of the form $\widetilde{h} = (x, h(x, y))$ for some $h \in A_{n+m}^m$. By construction $g_i(x, h(x, y)) = y_i$ for $i = 1, \ldots, m$. Therefore, $h(x, 0)$ will satisfy $g(x, h(x)) = 0$.
Conversely, assume IMFT and let $f = (f_1, \ldots, f_n) \in A_n^n$, $f(0) = 0$ such that $\partial_x f$ is invertible in 0. Let $g(x, y) = f(y) - x \in A_{n+n}^n$. Then $\partial_y g = \partial_x f$ is invertible in 0 and thus by IMFT there exists an $h \in A_n^n$ such that $g(x, h(x)) = 0$. By construction $f(h(x)) = x$, meaning that $f \circ h = \mathrm{Id}$. Furthermore,

$$h(x) = h \circ (f \circ h)(x) = (h \circ f) \circ h(x)$$

implies that $h \circ f = \mathrm{Id}$. □

While both of these theorems can be proven directly via a Newton algorithm, we will show here that they are implied by the Weierstrass theorems. This enables us – later on in chapter 4 – to

show for a ring only the analogue of WPT/WDT, getting INFT/IMFT as a consequence. While the proof below is in line with [Hau20], the idea goes back to [vdE90].

WPT/WDT implies INFT/IMFT. The general idea is to prove INFT by applying induction over n. However, for the base case, we will show that WPT implies IMFT with only one y variable. Even though this is more general than the base step, it also gives the base case via the equivalence of INFT and IMFT.

So let $g \in A_{n+1}$ such that $g(0,0) = 0$ and $\partial_y g(0,0) \neq 0$. We can therefore write $g(x,y) = y + \widetilde{g}(x,y)$, where $\widetilde{g} \in \langle y^2 \rangle A_{n+1}$. Thus g is y-regular of order 1. Therefore, by WPT, there exist $h(x,y) \in A_{n+1}^*$ and $r(x) \in A_n$ such that

$$h(x,y) \cdot g(x,y) = y - r(x)$$

and thus

$$g(x, r(x)) = h(x.r(x))^{-1}(r(x) - r(x)) = 0.$$

This substitution is well-defined as $r(0) = 0$ because otherwise $h(x,y) \cdot g(x,y)$ would have a constant term, contradicting $g(0,0) = 0$. Thus, r fulfills all the conditions of an implicit solution of g.

We will now show the more general case: let $f = (f_1, \ldots, f_n) \in A_n^n$ be a vector of power series, such that $f(0) = 0$ and the Jacobian $\partial_x f$ is invertible. In this case, we may assume that after applying a linear automorphism that the linear term of g_i is x_i. As this is the induction step, let $n \geq 1$ be arbitrary. By assumption, f_n is x_n-regular of order 1. If we now introduce a new set of variables $y = (y_1, \ldots, y_n)$, then $f_n - y_n$ will once again be x_n-regular of order 1. The Weierstrass normal form is given by $x_n - k(x', y_n)$, where $x' = (x_1, \ldots, x_{n-1})$ and $k \in A_n$. Now consider the ideal

$$I = \langle y_1 - f_1, \ldots, y_n - f_n \rangle = \langle y_1 - f_1, \ldots, y_{n-1} - f_{n-1}, x_n - k(x', y_n) \rangle$$

of A_{2n}. We now divide all generators of I by $x_n - k(x', y_n)\rangle$ using WDT, i.e. we write $f_i(x) = h_i(x, y_n) \cdot (x_n - k(x', y_n)) + \widetilde{f_i}(x', y_n)$. Therefore, the generators can be reduced to

$$I = \langle y_1 - \widetilde{f_1}(x', y_n), \ldots, y_{n-1} - \widetilde{f}_{n-1}(x', y_n), x_n - k(x', y_n) \rangle.$$

Then by induction, there exist power series $h_i(y))$ for $i = 1, \ldots, n$ such that

$$I = \langle x_1 - h_1(y), \ldots, x_{n-1} - h_{n-1}(y), x_n - k(x', y_n) \rangle.$$

Now dividing $k(x', y_n)$ successively by the x_i regular series $x_i - h_i(y)$ of order 1 will yield a power series $h_n(y)$ which is independent of x such that

$$I = \langle x_1 - h_1(y), \ldots, x_{n-1} - h_{n-1}(y), x_n - k(x', y_n) \rangle.$$

One can show that this is equivalent to $(h_1, \ldots, h_n)$ being inverse to $(f_1, \ldots, f_n)$. For details, see [vdE90]. □

1.3 Henselian rings and henselization

A local ring $R = (R, \mathfrak{m}, K = R/\mathfrak{m})$ over K is called *Henselian* if it satisfies the univariate polynomial implicit function theorem in the following sense: for every univariate polynomial $P(y) \in R[y]$ whose reduction $\overline{P}$ in $K[y]$ has a root $\alpha \in K$ with $\overline{P}'(\alpha) \neq 0$ (that is, α is a simple root of $\overline{P}$) there exists a root $a \in R$ of P with reduction $\overline{a} = \alpha$ in K. Here, P' denotes the derivative $\partial_y P$ of P with respect to y and $\overline{P}'$ its residue class in $K[y]$.
An étale extension of a ring R is of the form $S = R[x_1, \ldots, x_n]_g/(f_1, \ldots, f_n)$, where $g \in R[x_1, \ldots, x_n]$ and $f_i \in R[x_1, \ldots, x_n]_g$ such that the image of the Jacobian determinant $\det(\frac{\partial f_j}{\partial x_i})$ is invertible in S.
S is called standard étale if it is of the form $S \cong R[t]_g/f$ for $f, g \in R[x]$ monic with f' invertible in S.

Theorem 7 (Henselian rings cf. [Hoc17])**.** *For a local ring $R = (R, \mathfrak{m}, K)$, the following conditions are equivalent:*

1. *Every monic polynomial $P \in R[y]$ which factors modulo $\mathfrak{m}$, say, $\overline{P} = p_1 \cdot p_2$ in $K[y]$ with relatively prime $p_i \in K[y]$, already factors over R, say, $P = P_1 \cdot P_2$ with $P_i \in R[y]$ of reduction $\overline{P_i} = p_i$ in $K[y]$.*
2. *R is Henselian*
3. *A localization S of an étale extensions of R at a prime Q lying over $\mathfrak{m}$ such that $S/Q \cong R/\mathfrak{m}$ is a trivial extension of R. This type of extension is called pointed étale extension.*
4. *For every collection of polynomials $P = (P_1, \ldots, P_m)$ in $R[y_1, \ldots, y_m]$ whose reductions $\overline{P_i}$ in $K[y_1, \ldots, y_m]$ have a common root $\alpha \in K^m$ with invertible $\partial_y \overline{P}(\alpha)$ (that is, α is a simple root of $\overline{P} = (\overline{P_1}, \ldots, \overline{P_m})$) there exists a root $a \in R^m$ of P with reduction $\overline{a} = \alpha$ in K.*

For the proof of this theorem, we will need the following famous theorem of Chevalley:

Theorem 8 (Chevalley)**.** *Let $R = (R, \mathfrak{m}, K)$ be a local ring and let S be a finitely generated R-algebra with a prime ideal $\mathfrak{q}$ lying over $\mathfrak{m}$. Then f is étale over R if and only if there exist points $a \in R \setminus \mathfrak{m}$ and $b \in S \setminus \mathfrak{q}$ such that there is a standard étale Algebra E over R_a, such that S_b is an isomorphic image of E.*

The proof of this theorem, which requires Zariski's Main theorem among others, goes beyond the scope of this thesis. We therefor will not prove it here. If interested, one can find proofs in [Hoc17] pp. 27, [Sta18] Proposition 10.144.4 or [Mil80] pp.25. We can now use it to give a proof of theorem 7:

Proof of theorem 7. "(1)⇒(2)": Let P be a polynomial in $R[y]$. Let $\beta \in R$ (with reduction $\overline{\beta} \bmod \mathfrak{m}$) be the leading coefficient of P and let $\alpha \in K$ be a simple root of $\overline{P} \in K[y]$ (i.e. $\overline{P}(\alpha) = 0$ and $\overline{P}'(\alpha) \neq 0$). Therefore, $\overline{P} = \overline{\beta}(y - \alpha)q(y)$ where $q(y) \in K[y]$ monic. Since α is a simple root, $y - \alpha$ and $q(y)$ are relatively prime. By (1) this lifts to a factorization $P = \beta(y - a)Q(y)$ for some $a \in R$ and some $Q \in R[y]$ such that $\overline{a} = \alpha$ and $\overline{Q} = q$. Therefore, a is a root of P with reduction α as required.

"(2)⇒(3)": Let S be a pointed étale extension of R. We may assume by theorem 8 that S is a localization of a standard étale extension of R near Q lying over $\mathfrak{m}$ and therefore of the form

$S = (R[x]_g/F)_Q$ where $F, g \in R[x]$, F is monic, F' is invertible in S, and Q lies over $\mathfrak{m}$. Let $kan : S \to S/Q = K$ and let $\lambda :=\text{kan}(x) \in K$. Then for $f = \overline{F} \in K[x]$, we get $f(\lambda) = 0$, and since F' is invertible in S, we know that $f' \notin Q$, and $f'(\lambda \neq 0)$. This means λ is a simple root of f in K and therefore by (2) $\exists r \in R$ such that $\overline{r} = \lambda$ and $F = (x - r)G(x)$ for some $G \in R[x]$. Since λ is a simple root of f, the image of $G(x)$ in S has to be invertible in S. Hence,

$$S \cong (R[x]_g/F)_Q \cong (R[x]_g/(x - r)G(x))_Q \cong (R[x]_g/(x - r))_Q.$$

Therefore, S is a localization of $R[x]/(x - r) \cong R$ and, since R is local, $S \cong R$.

"(3)⇒(4)": Consider a collection of polynomials $P = (P_1, \ldots, P_m)$ in $R[y_1, \ldots, y_m]$, whose reductions $\overline{P_i}$ in $K[y_1, \ldots, y_m]$ have a common root $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^m$ with $\partial_y \overline{P}(\alpha)$ invertible in K like in (4), and let Q be the kernel of $\phi : R[y_1, \ldots, y_m] \to K$ with $\phi_{|R} = R \twoheadrightarrow K$ and $\phi(y_i) = \alpha_i$. Since $\partial_y \overline{P}(\alpha)$ is invertible, $S := R[y_1, \ldots, y_m]_Q/(P_1, \ldots, P_n)$ is a pointed étale extension of R and therefore $S \cong R$. Lifting the solution α is equivalent to giving an R-algebra mapping $R[y_1, \ldots, y_m]/(P_1, \cdots, P_n) \to R$ such that under the composite $R[y_1, \ldots, y_m]/(P_1, \ldots, P_n) \to R \twoheadrightarrow K$ the elements y_j map to the α_j. This is again equivalent to mapping Q into $\mathfrak{m}$ and, therefore, to giving a local algebra map $S \to R$. However, since $S \cong R$ there exists a unique such map.

"(4)⇒(1)": Now let $P = y^n + c_{n-1}y^{n-1} + \ldots + c_0$ for some $c_i \in R$ be a monic polynomial in $R[y]$ with factorization $\overline{P} = p_1 \cdot p_2$ for some monic coprime p_1, p_2 in $K[y]$ of degrees d and e. Let $p_1 = \sum_{i=0}^{d} g_i y^i$ and $p_2 = \sum_{i=0}^{e} h_i y^i$ for some $g_i, h_i \in K$ with $g_d = h_e = 1$. We want a factorization $P = P_1 \cdot P_2$ in $R[y]$ such that the residues of P_1, P_2 in $K[y]$ are p_1, p_2. We assume $G_i, H_i \in R$ are the coefficients of P_1 and P_2, respectively. This leads to

$$y^n + c_{n-1}y^{n-1} + \ldots + c_0 = (y^d + G_{d-1}y^{d-1} + \ldots + G_0)(y^e + H_{e-1}y^{e-1} + \ldots + H_0).$$

Comparing the coefficients leads to a system of $n = d + e$ equations in n indeterminates:

$$\begin{aligned} c_0 &= G_0 H_0 \\ c_1 &= G_0 H_1 + G_1 H_0 \\ &\vdots \\ c_{n-1} &= G_{d-1} + H_{e-1}. \end{aligned}$$

However, since $\overline{P} = p_1 \cdot p_2$, we know that this system of equations has a solution in $K[y]$ given by $(g, h) = (g_0, \ldots, g_d, h_0, \ldots, h_e)$. In order to lift this solution using (3), we have to show that the

Jacobian determinant does not vanish at (g, h). The Jacobian determinant here is:

$$J = \begin{pmatrix} H_0 & H_1 & H_2 & \cdots & H_{e-1} & 1 & 0 & \cdots & 0 \\ 0 & H_0 & H_1 & \cdots & H_{e-2} & H_{e-1} & 1 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots & \\ 0 & \cdots & 0 & H_0 & H_1 & \cdots & H_{e-2} & H_{e-1} & 1 \\ G_0 & G_1 & G_2 & \cdots & G_{d-1} & 1 & 0 & \cdots & 0 \\ 0 & G_0 & G_1 & \cdots & G_{d-2} & G_{d-1} & 1 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots & \\ 0 & \cdots & 0 & G_0 & G_1 & \cdots & G_{d-2} & G_{d-1} & 1 \end{pmatrix}.$$

Then

$$J(g, h) = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{e-1} & 1 & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_{e-2} & h_{e-1} & 1 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots & \\ 0 & \cdots & 0 & h_0 & h_1 & \cdots & h_{e-2} & h_{e-1} & 1 \\ g_0 & g_1 & g_2 & \cdots & g_{d-1} & 1 & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{d-2} & g_{d-1} & 1 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots & \\ 0 & \cdots & 0 & g_0 & g_1 & \cdots & g_{d-2} & g_{d-1} & 1 \end{pmatrix}.$$

is the transpose of the Sylvester matrix of p_2 and p_1. Since p_1, p_2 are relatively prime, their resultant $\operatorname{res}(p_2, p_1) \neq 0$. But $\operatorname{res}(p_2, p_1) = \det(\operatorname{Syl}(p_2, p_1))$ (for details see [vdW13] pp. 102) and therefore $J(g, h)$ is invertible. Finally, (4) gives a solution $(G_1, \ldots, G_d, H_1, \ldots, H_e)$ such that $P = P_1 \cdot P_2$. □

It is very easy to show with what we have established so far that the ring A_n satisfies condition (2) of theorem 7:

Lemma 1.3. *The ring A_n is Henselian.*

Proof. Let $P(y) \in A_n[y]$ whose reduction $\overline{P}$ in $A_n/(x)[y] \cong \mathbb{K}$ has a root $\alpha \in \mathbb{K}$ with $\overline{P}'(\alpha) \neq 0$. After applying the automorphism $y \mapsto y + \alpha$ of A_n, we may assume that $\alpha = 0$. As P can be interpreted as an element of A_{n+1}, there exists an element $a \in A_n$ with $P(a) = 0$. By the uniqueness of IMFT, there also holds $a(0) = \overline{a} = \alpha$. □

Another important construction is the Henselization R^h of a local ring $R = (R, \mathfrak{m}, K)$. It is the smallest extension of R such that one of the conditions of theorem 7 holds. It can be proven that every local ring admits a Henselization that is unique up to isomorphism (cf.[Hoc17] pp.46). Condition 3 of theorem 7 suggests constructing the Henselization of R as the direct Limit $R^h = \varinjlim_{S \in \mathcal{R}} S$, where $\mathcal{R}$ is the directed set containing one representative of each isomorphism class of pointed étale extension of R. One can show that this definition of R^h actually leads to the smallest Henselian ring containing R (for details, see [Hoc17] pp.46).
One can show the following property of $\mathbb{K}\langle x \rangle$

Theorem 9. *The ring $\mathbb{K}\langle x \rangle$ is the Henselization of the ring $\mathbb{K}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$.*

For a proof, see [Yur20].

1.4 Encoding algebraic power series

Theorem 9 will enable us to find a useful way to encode algebraic power series, which can be used to prove that a system of algebraic power series like in theorem 1 can be replaced with a system of polynomial equations (see lemma 2.2 in chapter 2). Even though we will only need theorem 10, we will also prove theorem 11 for the sake of completeness. The problem one encounters when trying to encode algebraic power series f is that the minimal polynomial P_f is not a good way to do so, as it could also encode all the conjugated roots of P_f. The following idea goes back to [AM65] and is often referred to as Artin-Mazur theorem (for example cf. [AMR92]). The idea is to show for any $f \in \mathbb{K}\langle x\rangle$ with $f(0) = 0$ the existence of a natural number k and a vector of polynomials called mother code $P(x, y_1, \dots, y_k) \in \mathbb{K}[x, y]^k$ that satisfies the assumption of IMFT (i.e. the Jacobian matrix of P with respect to y is invertible at 0) such that there exist some algebraic power series $h_2, \dots, h_k \in \mathbb{K}\langle x\rangle$ with $h_i(0) = 0$ such that $P(x, f(x), h_2(x), \dots, h_k(x)) = 0$. The vector $(f, h_2, \dots, h_k)$ is called the baby series vector of the mother code P. Notice that it makes sense to call it the baby series verctor, as it is unique by IMFT. While there are constructive proofs of the mother code of an algebraic power series f, if given a Taylor expansion of f to an arbitrarily high degree (cf. [AMR92] p.36 or [ACJH14]p.11), we will give a non-constructive proof. We will need the following theorem from [DL80] p.60. For that, we call an algebraic power series $e(x) \in \mathbb{K}\langle x\rangle$ étale algebraic if $e(0) = 0$ and for its minimal polynomial $P_e(x, t)$ there holds $\partial_t P_e(0, 0) \neq 0$ (i.e. it is the unique solution of $P(x, t) = 0$ by IMFT). The proofs given are in line with [Yur20] pp.42.

Theorem 10 ([DL87])**.** *Let $f \in \mathbb{K}\langle x\rangle$ be an algebraic power series. Then there exist polynomials $C(x, t), D(x, t) \in \mathbb{K}[x, t]$ with $D(0, 0) \neq 0$ and an étale algebraic power series $e(x)$ such that*

$$f = \frac{C(x, e(x))}{D(x, e(x))}.$$

Proof. Theorem 9 states that $\mathbb{K}\langle x\rangle$ is the Henselization of $R := \mathbb{K}[x]_{(x)}$, i.e. $\mathbb{K}\langle x\rangle = \varinjlim_{S \in \mathcal{R}} S$, where $\mathcal{R}$ is a set of pointed étale extensions of R. Thus, there is a pointed étale extension $S_{\mathfrak{q}}$ of R containing f, i.e. there is an étale R-algebra S with a prime $\mathfrak{q}$ lying over $x_1, \dots, x_n$ such that $f \in S_q$. By applying theorem 8, there exist elements $a \in R \setminus (x)$ and $b \in S \setminus \mathfrak{q}$ such that S_b is isomorphic to a standard étale algebra over R_a. As R is a local ring, $R_a \cong R$. Hence, there exist monic $g, l \in R[t]$ with l' invertible in $R[t]_g$ and an isomorphism φ such that

$$\varphi : S_b \xrightarrow{\sim} R[t]_g/(l).$$

As localization commutes with taking quotient rings, we can rewrite this to

$$\widetilde{\varphi} : S_b \xrightarrow{\sim} (R[t]/(L))_G,$$

where $L \in R[t]$ and $G \in R[t]/(L)$ such that neither $\partial_t L$ nor G is in the image of $\mathfrak{q}$ under $\widetilde{\varphi}$.
Thus, $S_q \cong (R[t]/(L))_{\widetilde{\varphi}(\mathfrak{q})}$. Then $R[t]/(L) \cong R[e]$ for some algebraic power series e with minimal polynomial L. Hence, the theorem is proven if we can show that e is étale algebraic. As $\partial_t L$ is not in $\widetilde{\varphi}(\mathfrak{q})$ and $\mathfrak{q}$ lies over (x), there holds $\partial_t L(0) \neq 0$ (and also the denominator of f does not vanish at 0). If $e(0) \neq 0$, one can replace L with $\widetilde{e}(x) = h(e) - e(0)$. One can easily see that this $\widetilde{e}$ will verify all the conditions needed. □

This theorem now enables us to give a non-constructive proof of the Artin Mazur theorem. It will even enable us to show that $k = 2$ always suffices:

Theorem 11 (Artin-Mazur)**.** *Let $f \in \mathbb{K}\langle x\rangle$ be an algebraic power series with $f(0) = 0$. Then there exists a mother code $P = (P_1, P_2) \in \mathbb{K}[x][y_1, y_2]$ and by IMFT an algebraic power series $h \in \mathbb{K}\langle x\rangle$ with $h(0) = 0$ such that (f, h) is the baby vector of P.*

Proof. Let $Q_f \in \mathbb{K}[x, y_1]$ be the minimal polynomial of f. If $\partial_{y_1} Q_f(0, 0) \neq 0$, a mother code for f is trivially given by $P = (Q_f, y_2)$. As the Jacobian of P with respect to $y = (y_1, y_2)$ is given by

$$J_P = \begin{pmatrix} \partial_{y_1} Q_f & 0 \\ 0 & 1 \end{pmatrix},$$

and P is a mother code as the determinant of this Jacobian matrix does not vanish at 0.
So the interesting case is if $\partial_{y_1} Q_f(0, 0) = 0$. In this case, we can use theorem 10 to find an étale algebraic power series $e(x)$ and polynomials $C(x, y_2), D(x, y_2) \in \mathbb{K}[x, y_2],\ D(0, 0) \neq 0$ with

$$D(x, h(x)) \cdot f(x) = C(x, h(x)).$$

Now let Q_e be the minimal polynomial of e. We claim that $P = (y_1 \cdot D(x, y_2) - C(x, y_2), Q_e(x, y_2)$ is a mother code with baby vector (f, h) as $P(x, f(x), e(x)) = 0$ and the Jacobian of P with respect to y is

$$J_P = \begin{pmatrix} D(x, y_2) & y_1 \cdot \partial_{y_2} D(x, y_2) + \partial_{y_2} C(x, y_2) \\ 0 & \partial_{y_2} Q_e(x, y_2). \end{pmatrix}.$$

By assumption, $\det(J_P)(0, 0) = D(x, y_2)(0, 0) \cdot \partial_{y_2} Q_e(0, 0) \neq 0$ and thus P is indeed a mother code for f. □

2 Basic techniques for working with systems of power series

In this section we will show some basic techniques for working with systems of equations like in theorem 1. We will need the existence of a large enough minor of the Jacobian matrix of the systems of equations that does not vanish at the formal solution $\widehat{y}$. After applying a suitable automorphism to make this minor x_n regular, one can then use Weierstrass division to prove theorem 1 in chapter 3. We will therefore show that there also exists a system of equations for our $\widehat{y}$ with such a minor. The statement below is only stated and proven for algebraic power series, however, the convergent case is completely analogous.

Lemma 2.1. *Let $\mathbb{K}$ be an arbitrary field, and let $f(x,y) \in \mathbb{K}\langle x,y\rangle^r$ be a vector of algebraic power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ vanishing at 0,*

$$f(x,\widehat{y}(x)) = 0.$$

Then there exists a system of algebraic equations $F = (F_1(x,y),\ldots,F_s(x,y)) \in \mathbb{K}\langle x,y\rangle$ such that $F(x,\widehat{y}(x)) = 0$ with an $s\times s$-minor g of $\partial_{\mathrm{y}}F$ that does not vanish on $\widehat{y}(x)$ (i.e. $g(x,\widehat{y}(x)) \neq 0$) and such that if $y(x)$ is an algebraic solution of F approximating $\widehat{y}(x)$ up to an arbitrarily high degree c, then $y(x)$ is already a solution of f.

Proof of Lemma. Let I be the ideal generated by $(f_1,\ldots,f_r)$ in $\mathbb{K}\langle x,y\rangle$. We may replace I by the kernel of the substitution homomorphism $K\langle x,y\rangle \to K[[x]]$, where $l(x,y) \mapsto l(x,\widehat{y}(x))$. Therefore, we may assume that I is prime. Let s be the height of I. In other words, if Y denotes the analytic germ $(\mathbb{K}^{n+m},0)$ and $X = V(I)$ then $s = \operatorname{codim}_Y(X)$, as I being prime implies that all the components of $V(I)$ have the same dimension. We now use the following proposition:

Proposition 1 ([Mum99] p.44 Cor.4)**.** *Let U be an affine Variety and U a closed irreducible subset and let $s = codim_U(Z)$. Then there exists a regular sequence $F_1,\ldots,F_s$ such that Z is a component, of the complete intersection $V(F_1,\ldots,F_s)$.*

We can therefore assume that X is an irreducible component of a complete intersection variety X^* and there exists an ideal $I^* \subseteq I$ generated by a regular sequence $F = (F_1,\ldots,F_s)$ of length s of elements of I such that I is a prime component of I^*, and let J be the intersection of the other prime components of I^*. By the Jacobian Criterion ([Mum99] p.168 Cor.1 to Prop 2) there exists an $(s\times s)$-minor g of the Jacobian DF such that g is not in I^* meaning $g(x,\widehat{y}(x)) \neq 0$. The partial derivatives of $0 = F(x,\widehat{y}(x))$ with respect to the x_i are

$$0 = \partial_{x_i}F(x,\widehat{y}(x)) + \partial_{x_i}\widehat{y}\cdot\partial_{\mathrm{y}}F(x,\widehat{y}(x))$$

and equivalently

$$\partial_{x_i}F(x,\widehat{y}(x)) = -\partial_{x_i}\widehat{y}\cdot\partial_{\mathrm{y}}F(x,\widehat{y}(x)).$$

We may therefore assume that g can be chosen within the relative Jacobian $\partial_{\mathrm{y}}F$ with respect to the y variables. If $X = X^*$, we have already found the minor g we were looking for.
Now assume that $X \neq X^*$ and $\widetilde{y}(x)$ is an algebraic solution of F that approximates $\widehat{y}(x)$ up to a chosen degree. Our goal is to show that $\widetilde{y}(x)$ is also a solution of f for all $f \in I$. As $X \neq X^*$, there is $h(x,y) \in K\langle x,y\rangle$ that does not vanish on $V(I)$ but on $V(J)$. Therefore, $h(x,\widehat{y}(x)) \neq 0$ and as $\widetilde{y}(x)$ approximates $\widehat{y}(x)$, to a high degree also $h(x,\widetilde{y}(x)) \neq 0$.

Now consider the map $\Pi : K\langle x, y\rangle \to K\langle x\rangle$, where $l(x, y) \mapsto l(x, \widetilde{y}(x))$ and let $P = ker(\Pi)$. Then P is a prime ideal, as $K\langle x\rangle$ is an integral domain. However, as $\widetilde{y}(x)$ is a solution of $F_1, \ldots, F_s$ we get

$$I * J \subseteq I \cap J = I^* \subseteq P.$$

As $h(x, \widetilde{y}(x)) \neq 0$ we have $J \nsubseteq P$ and hence, as P is Prime, $I \subseteq P$. (This can be interpreted geometrically as $\widetilde{y}(x)$ having to ultimately end up in one of the components of the complete intersection.) Therefore, $f(x, \widetilde{y}(x)) = 0 \; \forall f \in I$. □

The second basic technique we want to show is that if we have a system of algebraic equations like in theorems 1 and 2, it can be replaced with a system of polynomials.

Lemma 2.2 (Reduction to the polynomial case)**.** *Let $f(x, y) \in \mathbb{K}\langle x, y\rangle^r$ be a vector of algebraic power series in two sets of variables x and y. Assume given a formal power series solution $\widehat{y}(x)$ of f vanishing at 0,*

$$f(x, \widehat{y}(x)) = 0.$$

Then f can be replaced by a vector of polynomials, in the sense that if there is an approximating solution of the polynomial system, it will yield an approximating solution of the algebraic system.

Proof. Using theorem 10 we get that $f \in (\mathbb{K}[x, y, z]_g/h)^r$, where h is irreducible, monic with respect to z, $\partial_z h(0, 0, 0) \neq 0$ and $g \in 1 + (x, y, z)\mathbb{K}[x, y, z]^r$. Now let a be the highest power in the denominators of the f_i and let $f^* := g^a f \in \mathbb{K}[x, y, z]$.
Now consider the system of equations

$$f^*(x, y, z) = 0 \tag{1}$$

$$h(x, y, z) = 0, \tag{2}$$

with formal solution $(\widehat{y}(x), \widehat{z}(x, \widehat{y}(x)))$. Assume $(y(x), z(x))$ is an algebraic solution approximating the formal solution to a given degree. By construction, we have both $h(x, y(x), \widehat{z}(x, y(x))) = 0$ and $h(x, y(x), z(x)) = 0$. As h satisfies the conditions for IMT, we get that $z(x) = \widehat{z}(x, y(x))$. Thus,

$$0 = f^*(x, y(x) m \widehat{z}(x, y(x))) = g^a(x, y(x), \widehat{z}(x, y(x))) \cdot f(x, y(x)). \tag{3}$$

As g is a unit in $\mathbb{K}\langle x\rangle^r$ there holds $f(x, y(x)) = 0$. □

Thirdly, we want to show that in theorem 1, it would be enough to show that every algebraic (or convergent) system f with a formal solution $\widehat{y}(x)$ also has an algebraic (or respectively convergent) solution $y(x)$, meaning that one can show that the approximation $\mathrm{mod}\langle x\rangle^c$ can be expressed in terms of another formal or respectively algebraic equation. The proof and definitions below are in line with 2, and we will only show the case that f is polynomial (and thus this includes the case that f is algebraic), however, the case for f convergent is analogous. For that, let R be a commutative noetherian ring with identity, $\mathfrak{m} \subseteq R$ be a maximal ideal, and $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then we say $(R, \mathfrak{m})$ has the

- weak approximation property if for every finite system of polynomials $f \in R[y]^r$ with a solution $\widehat{y}$ in $\widehat{R}^m$, there exists a solution $y \in R^m$.
- approximation property if for every finite system of polynomials $f \in R[y]^r$ with a solution $\widehat{y}$ in $\widehat{R}^m$ and for every $c \in \mathbb{N}$, there exists a solution $y \in R^m$ such that $y \equiv \widehat{y} \bmod \mathfrak{m}^c$.

Furthermore, there is a notion of a strong approximation property, for details see chapter 6.
It is obvious that the approximation property implies the weak approximation property, however as mentioned above, we will show that these two notions are equivalent.

Lemma 2.3. *If $(R, \mathfrak{m})$ has the weak approximation property, it also has the approximation property.*

Proof. Let $\mathfrak{m}$ generated by $x_1, \dots, x_n$ and let $f \in R[Y]$ be a finite system of equations with a formal solution $\widehat{y} \in \widehat{R}^m$. Furthermore, let $c \in \mathbb{N}$. Then choose any elements $\widetilde{y}_1, \dots, \widetilde{y}_m$ such that $\widetilde{y}_i \equiv \widehat{y}_i \bmod \mathfrak{m}^c\widehat{R}$. In other words, each $\widetilde{y}_i$ has the form

$$\widetilde{y}_i = \widehat{y}_i + \sum_{\alpha \in \mathbb{N}^n, |\alpha| = c} \widehat{z}_{i,\alpha} \cdot x^\alpha.$$

Thus, we can look at the polynomial equations $g_i = Y_i - \sum_{\alpha \in \mathbb{N}^n, |\alpha| = c} Z_{i,\alpha} \cdot x^\alpha - \widetilde{y}_i \in R[Y, Z]$. Then $f, g \in R[Y, Z]^{r+m}$ is a finite system of polynomial equations with formal solutions $(\widehat{y}, \widehat{z})$, where $\widehat{y}_i$ and $\widehat{z}_i \in \widehat{R}$, and thus by the weak approximation property, there exist $y_i, z_i \in R$ giving a solution to f, g. By construction of the g_i, there holds $y_i \equiv \widetilde{y}_i \equiv \widehat{y}_i \bmod \mathfrak{m}^c$ and hence R also has the approximation property. □

3 Proof of Artin's approximation theorem

This section will present the proof of theorem 1. We will first give the proof for the simplest case, where there is only one equation f and two binary variables $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We do this because the general case (which is proven below) is more tedious, and it is easier to lose sight of the key ideas of the proof because of the many variables and indices.

3.1 Proof for the simplest case

Proof of Artin approximation in the case $r = 1, m = n = 2$. The proof consists of two main steps:
Step 1 We will first show that whether a formal, convergent or algebraic power series $y(x)$ is a solution of f (i.e. $f(x, y(x)) = 0$) can be reduced to a system in just x_1. So let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $f \in \mathbb{K}\langle x, y\rangle$ (or $\mathbb{K}\{x, y\}$ respectively), and let $y(x) \in K[[x]]$ (or $\mathbb{K}\langle x\rangle$ and $\mathbb{K}\{x\}$ respectively) with $y(0) = 0$ be a solution $f(x, y(x)) = 0$. We may assume using lemma 2.1 that $\partial_{\mathtt{y}_1} f(x, y(x)) \neq 0$ and let $g = \partial_{\mathtt{y}_1}$. For readability reasons, we will leave out the x arguments of y for the rest of the proof.
After applying a linear change of coordinates $x_1 \mapsto x_1 + lx_2, x_2 \mapsto x_2$ for a suitable $l \in \mathbb{K}$ we may assume that $g(x, y)$ is x_2-regular of order d. We now apply formal (or respectively convergent/algebraic) Weierstrass division by dividing y_1 by $x_2 \cdot g(x, y)$ and y_2 by $x_2 \cdot g(x, y)^2$ to get

$$\begin{aligned} y_1(x) &= v_1(x) \cdot g(x, y) + z_1(x) \\ y_2(x) &= v_2(x) \cdot g(x, y)^2 + z_2(x), \end{aligned} \tag{4}$$

where $z_1(x) \in \mathbb{K}[[x_1]][x_2]_{\leq d}$, $z_2(x) \in \mathbb{K}[[x_1]][x_2]_{\leq 2d}$, and $v_i(x) \in x \cdot \mathbb{K}[[x]]$. There will hold $z_1(x) \in \mathbb{K}\langle x_1\rangle[x_2]_{\leq d}$, $z_2(x) \in \mathbb{K}\langle x_1\rangle[x_2]_{\leq 2d}$, and $v_i(x) \in x \cdot \mathbb{K}\langle x\rangle$ in the algebraic case and $z_1(x) \in \mathbb{K}\{x_1\}[x_2]_{\leq d}$, $z_2(x) \in \mathbb{K}\{x_1\}[x_2]_{\leq 2d}$, and $v_i(x) \in x \cdot \mathbb{K}\{x\}$ in the convergent case. Once again, for readability reasons, we will leave out the x arguments of z and v. These z_i are only polynomial in x_2, so we now show that we can express everything in terms of $z = (z_1, z_2)$:

Lemma 3.1. *The ideals generated by $g(x, y)$ and $g(x, z)$ are the same in $\mathbb{K}[[x]]$ (or $\mathbb{K}\langle x\rangle$ and $\mathbb{K}\{x\}$ respectively).*

Proof. Let $v = (v_1, g(x, y)v_2)$ and consider the Taylor expansion of $g(x, z) = g(x, y - g(x, y) \cdot v)$ in the point $(x, g(x, y))$:

$$g(x, z) = g(x, y) - g(x, y) \cdot v \cdot \partial_{\mathtt{y}} g(x, y) + g(x, y)^2 \cdot h(x) \tag{5}$$

where $h(x) \in x \cdot \mathbb{K}[[x]]^2$ (or $x \cdot \mathbb{K}\langle x\rangle$ and $x \cdot \mathbb{K}\{x\}$ respectively). We can now factor out $g(x, y)$ to get

$$g(x, z) = g(x, y)(1 + v \cdot \partial_{\mathtt{y}} g(x, y) + h(x)) \tag{6}$$

and $(1 + v \cdot \partial_{\mathtt{y}} g(x, y) + h(x))$ is invertible. Hence, the two ideals are the same. □

We can therefore rewrite (4) as

$$\begin{aligned} y_1 &= a_1(x) \cdot g(x, z) + z_1 \\ y_2 &= a_2(x) \cdot g(x, z)^2 + z_2, \end{aligned} \tag{7}$$

where the $a_i(x)$ are once again elements of $x \cdot \mathbb{K}[[x]]$ (or $x \cdot \mathbb{K}\langle x\rangle$ and $x \cdot \mathbb{K}\{x\}$ respectively). We will also leave out the x arguments of the a_i for the rest of the proof.

Let $a = (a_1, g(x,z)a_2)$ and consider the Taylor expansion of $f(x,y) = f(x, a \cdot g(x,z) + z)$ in the point (x,z):

$$f(x,z) + a \cdot g(x,z) \cdot \partial_{\mathtt{y}} f(x,z) + g(x,z) \cdot q(x,a,g(x,z)) = 0 \tag{8}$$

where $q(x,w,t)$ is an algebraic respectively convergent power series in $x = (x_1, x_2)$, $w = (w_1, w_2)$ and t, and where q is at least quadratic in w. Now use the fact that $g(x,z) = \partial_{\mathtt{y}_1} f$ to get

$$\begin{gathered} g(x,z)^2 \cdot q(x,a,g(x,z)) = \\ g(x,z) \cdot \partial_{\mathtt{y}_1} f(x,z) \cdot q(x,a,g(x,z)) \end{gathered} \tag{9}$$

and

$$\begin{gathered} a \cdot g(x,z) \cdot \partial_{\mathtt{y}} f(x,z) = \\ g(x,z) \cdot [\partial_{\mathtt{y}_1} f(x,z) \cdot a_1 + \partial_{\mathtt{y}_2} f(x,z) \cdot a_2 \cdot g(x,z)] = \\ g(x,z) \cdot \partial_{\mathtt{y}_1} f(x,z) \cdot [a_1 + \partial_{\mathtt{y}_2} f(x,z) \cdot a_2]. \end{gathered} \tag{10}$$

Combine this with (8) to get

$$f(x,z) + g(x,z) \cdot \partial_{\mathtt{y}_1} f(x,z) \cdot \Phi(x, a_1, a_2, z) = 0, \tag{11}$$

where

$$\Phi(x, w_1, w_2, \mathtt{z}) = (w_1 + \partial_{y_2} f(x, \mathtt{z}) \cdot w_2 + q(x, w_1, g(x,\mathtt{z}) \cdot w_2, g(x,\mathtt{z}))). \tag{12}$$

Then Φ is algebraic, vanishing in zero, and $\partial_{w_1}\Phi(0,0,0,0)$ does not vanish and is therefore invertible. Setting $b(x,\mathtt{z}) = \Phi(x, a_1, a_2, \mathtt{z})$ we get, that

$$f(x,z) + g(x,z) \cdot \partial_{\mathtt{y}_1} f(x,z) \cdot b(x,z) = 0, \tag{13}$$

and equivalently

$$f(x,z) + g(x,z)^2 \cdot b(x,z) = 0. \tag{14}$$

Thus $f(x,z)$ is in the ideal generated by $g(x,z)^2$. This ideal membership can be reformulated to a system of equations by dividing $f(x,\mathtt{z})$ by $g(x,\mathtt{z})^2$ (with respect to x_2) and setting the remainder, which is polynomial in x_2, to zero. By comparing coefficients in x_2 this leads to a finite system of equations in x_1. Now assume we are given an algebraic or formal solution of this system of equations. It will give rise to a power series $z = (z_1, z_2)$ that is polynomial in x_2 and a power series $b(x,\mathtt{z}) \in \mathbb{K}\langle x, \mathtt{z}\rangle$ (or $\mathbb{K}\{x,\mathtt{z}\}$ resp.) such that

$$f(x,z) + g(x,z)^2 \cdot b(x,z) = 0 \tag{15}$$

and equivalently

$$f(x,z) + g(x,z) \cdot \partial_{\mathtt{y}_1} f(x,z) \cdot b(x,z) = 0. \tag{16}$$

Now consider the algebraic respectively convergent equation

$$\Psi(x, w_1, w_2) = \Phi(x, w_1, w_2, z) - b(x,z) = 0. \tag{17}$$

As mentioned above $\partial_{w_1}\Phi$ is invertible, hence $\partial_{w_1}\Psi$ is as well. If we now choose any algebraic respectively convergent power series a_2 we can apply the implicit function theorem to $\Psi(x, w_1, a_2, z)$

to get a unique algebraic respectively convergent power series $a_1 \in \mathbb{K}\langle x\rangle$ (or $\mathbb{K}\{x\}$ resp.) such that

$$\Phi(x, a_1, a_2, z) = b(x, z). \tag{18}$$

Setting $a = (a_1, g(x,z) \cdot a_2)$ we can repeat (9) and (10) in reverse order to get

$$f(x,z) + a \cdot g(x,z) \cdot \partial_{\mathtt{y}} f(x,z) + g(x,z)^2 \cdot q(x, a, g(x,z)) = 0. \tag{19}$$

Finally setting $y_i = a_i \cdot g(x,z) + z_i$ and repeating the proof of the lemma 3.1, we get that the left-hand side of (48) is the Taylor expansion of $f(x,y)$. Therefore, our newly constructed y is an algebraic or respectively convergent solution of f.

Step 2 Now let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $f \in \mathbb{K}\langle x, y\rangle$, and let $\widehat{y}(x) \in K[[x]]$ with $\widehat{y}(0) = 0$ be a formal solution $f(x, \widehat{y}(x)) = 0$. As seen in Step 1, this leads to a system of equations in x_1, to which the coefficients $\widehat{k}_{ij}(x_1)$ of

$$\widehat{z} = (\sum_{l=0}^{d} \widehat{k}_{1l}(x_1) x_2^l, \sum_{l=0}^{2d} \widehat{k}_{2l}(x_1) x_2^l)$$

give a formal solution. For the following, let $\widehat{z}(x)$ and $\widehat{a}''(x)$ be derived from $\widehat{y}(x)$ like in step 1. We can now apply induction to get an algebraic respectively convergent solution given by $k_{ij}(x_1)$ where the k_{ij} approximate the $\widehat{k}_{ij}$ up to degree c. Now, Setting $z_i(x) = \sum_{l=0}^{2d} \widehat{k}_{il}(x_1) x_2^l$ and choosing $a_2(x)$ to be the truncation of $\widehat{a}_2(x)$ at degree c, we will get an algebraic respectively convergent solution $y(x)$ of $f(x,y)$ that approximates $\widehat{y}(x)$ up to degree c by construction. □

This completes the proof of the simplest case, where we just divide by a $\partial_{\mathtt{y}_1} f(x, \widehat{y})$ that does not vanish on $\widehat{y}(x)$. However, in a more general setting this will have to be replaced by a maximal minor, which exists by lemma 2.1. This makes all the calculations longer, however, the proof will essentially be the same.

3.2 The general proof

Proof of algebraic Artin approximation theorem (cf. .[Hau17]p. 603-607). Like above, the proof consists of two steps:

Step 1 We will once again first show that whether a formal, convergent or algebraic power series $y(x)$ is a solution of f (i.e. $f(x, y(x)) = 0$) can be reduced to a system in just x'. So let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, $f \in \mathbb{K}\langle x, y\rangle^r$ (or $\mathbb{K}\{x, y\}^r$ respectively), and let $y(x) \in K[[x]]^m$ (or $\mathbb{K}\langle x\rangle^m$ and $\mathbb{K}\{x\}^m$ respectively) with $y(0) = 0$ be a solution $f(x, y(x)) = 0$. Using Lemma 2.1, we may assume that there is a minor g of $\partial_{\mathtt{y}} f$ such that $g(x, y(x)) \neq 0$. Let $\mathtt{y}'$ denote the $\mathtt{y}_i$ used for g and let $\mathtt{y}''$ denote the others. For readability reasons, we will leave out the x arguments of y for the rest of the proof.

We may once again assume that $g(x,y)$ is x_n-regular of order d by applying a suitable linear change of coordinates. We now apply formal (or respectively convergent/algebraic) Weierstrass division by dividing y_i by $x_n \cdot g(x,y)$ if $\mathtt{y}_i \in \mathtt{y}'$ and by $x_n \cdot g(x,y)^2$ if $\mathtt{y}_i \in \mathtt{y}''$ to get

$$\begin{aligned} y_i(x) &= v_i(x) \cdot g(x,y) + z_i(x) \quad \text{for } \mathtt{y}_i \in \mathtt{y}' \\ y_i(x) &= v_i(x) \cdot g(x,y)^2 + z_i(x) \quad \text{for } \mathtt{y}_i \in \mathtt{y}'', \end{aligned} \tag{20}$$

where $z_i(x) \in \mathbb{K}[[x']][x_n]_{\leq 2d}$ and $v_i(x) \in x \cdot \mathbb{K}[[x]]$. There will hold $z_i(x) \in \mathbb{K}\langle x'\rangle[x_n]_{\leq 2d}$ and $v_i(x) \in x \cdot \mathbb{K}\langle x\rangle$ in the algebraic case and $z_i(x) \in \mathbb{K}\{x'\}[x_n]_{\leq 2d}$ and $v_i(x) \in x \cdot \mathbb{K}\{x\}$ in the convergent case. Once again, for readability reasons, we will leave out the x arguments of z and v. These z_i are only polynomial in x_n, so we now show that we can express everything in terms of $z = (z_1, \ldots, z_m)$:

Lemma 3.2. *The ideals generated by $g(x, y)$ and $g(x, z)$ are the same in $\mathbb{K}[[x]]$ (or $\mathbb{K}\langle x\rangle$ and $\mathbb{K}\{x\}$ respectively).*

The proof of this lemma is completely analogous to the proof of lemma 3.1. We can therefore rewrite (34) as

$$\begin{aligned} y_i &= a_i(x) \cdot g(x, z) + z_i \quad \text{for } \mathtt{y}_i \in \mathtt{y}' \\ y_i &= a_i(x) \cdot g(x, z)^2 + z_i \quad \text{for } \mathtt{y}_i \in \mathtt{y}'', \end{aligned} \tag{21}$$

where the $a_i(x)$ are once again elements of $x \cdot \mathbb{K}[[x]]$ (or $x \cdot \mathbb{K}\langle x\rangle$ and $x \cdot \mathbb{K}\{x\}$ respectively). We will also leave out the x arguments of the a_i for the rest of the proof.
Let a' be the vector of all a_i for which the corresponding $\mathtt{y}_i \in \mathtt{y}'$ and a'' denote the vector of those for which the corresponding $\mathtt{y}_i \in \mathtt{y}''$. Furthermore, let $a = (a', g(x, z)a'')$. Now consider the Taylor expansion of $f(x, y) = f(x, a \cdot g(x, z) + z)$ in the point (x, z):

$$f(x, z) + a \cdot g(x, z) \cdot \partial_{\mathtt{y}} f(x, z) + g(x, z) \cdot q(x, a, g(x, z)) = 0, \tag{22}$$

where $q(x, w, t)$ is an algebraic power series in x, $w = (w', w'')$ and t and q is at least quadratic in w. As $g = \partial_{\mathtt{y}'} f$ is a maximal minor of $\partial_{\mathtt{y}} f$, we can write the diagonal matrix $g \cdot \mathbb{1}_r$ as the product of $\partial_{\mathtt{y}'} f$ and its adjoint matrix $\partial^*_{\mathtt{y}'} f$. We can use this fact to get

$$\begin{gathered} g(x, z)^2 \cdot q(x, a, g(x, z)) = \\ g(x, z) \cdot \partial_{\mathtt{y}'} f(x, z) \cdot \partial^*_{\mathtt{y}'} f(x, z) \cdot q(x, a, g(x, z)) \end{gathered} \tag{23}$$

and

$$\begin{gathered} a \cdot g(x, z) \cdot \partial_{\mathtt{y}} f(x, z) = \\ g(x, z) \cdot [\sum_{\mathtt{y}_i \in \mathtt{y}'} \partial_{\mathtt{y}_i} f(x, z) \cdot a_i + \sum_{\mathtt{y}_i \in \mathtt{y}''} \partial_{\mathtt{y}_i} f(x, z) \cdot a_i \cdot g(x, z)] = \\ g(x, z) \cdot \partial_{\mathtt{y}'} f(x, z) \cdot [a' + \partial_{\mathtt{y}''} f(x, z) \cdot a'' \cdot \partial^*_{\mathtt{y}'} f(x, z)]. \end{gathered} \tag{24}$$

Combine this with (38) to get

$$f(x, z) + g(x, z) \cdot \partial_{\mathtt{y}'} f(x, z) \cdot \Phi(x, a', a'', z) = 0, \tag{25}$$

where

$$\Phi(x, w', w'', \mathtt{z}) = (w' + \partial^*_{\mathtt{y}'} f(x, \mathtt{z}) \cdot q(x, w', g(x, \mathtt{z}) \cdot w'', g(x, \mathtt{z})) + \partial^*_{\mathtt{y}'} f(x, \mathtt{z}) \cdot \partial_{\mathtt{y}''} f(x, \mathtt{z}) \cdot w''. \tag{26}$$

Then Φ is algebraic, vanishing in zero, and $\partial_{w'} \Phi(0, 0, 0, 0)$ does not vanish and is therefore invertible. Setting $b(x, \mathtt{z}) = \Phi(x, a', a'', \mathtt{z})$ we get, that

$$f(x, z) + g(x, z) \cdot \partial_{\mathtt{y}'} f(x, z) \cdot b(x, z) = 0, \tag{27}$$

and equivalently

$$\partial^*_{\mathtt{y}'} f(x,z) \cdot f(x,z) + g(x,z)^2 \cdot b(x,z) = 0. \tag{28}$$

Thus, $\partial^*_{\mathtt{y}'} f(x,z) \cdot f(x,z)$ is in the ideal generated by $g(x,z)^2$. This ideal membership can be reformulated to a system of equations by dividing $f(x,\mathtt{z})$ by $g(x,\mathtt{z})^2$ (with respect to x_n) and setting the remainder, which is polynomial in x_n, to zero. By comparing coefficients in x_n this leads to a finite system of equations in x'. Now assume we are given an algebraic or formal solution of this system of equations. It will give us a power series $z = (z_1, \ldots, z_m)$ that is polynomial in x_n and a power series $b(x,\mathtt{z}) \in \mathbb{K}\langle x,\mathtt{z}\rangle$ (or $\mathbb{K}\{x,\mathtt{z}\}$ resp.) such that

$$f(x,z) + g(x,z)^2 \cdot b(x,z) = 0. \tag{29}$$

Now consider the algebraic respectively convergent equation

$$\Psi(x,w',w'') = \Phi(x,w',w'',z) - b(x,z) = 0. \tag{30}$$

As mentioned above, $\partial_{w'}\Phi$ is invertible, hence $\partial_{w'}\Psi$ is as well. If we now choose any algebraic respectively convergent power series a'', we can apply the implicit function theorem to $\Psi(x,w',a'',z)$ to get a unique algebraic respectively convergent power series $a' \in \mathbb{K}\langle x\rangle^m$ (or $\mathbb{K}\{x\}^r$ resp.) such that

$$\Phi(x,a',a'',z) = b(x,z). \tag{31}$$

Setting $a = (a', g(x,z)\cdot a'')$, we can repeat (39) and (40) in reverse order to get

$$f(x,z) + a \cdot g(x,z) \cdot \partial_{\mathtt{y}} f(x,z) + g(x,z)^2 \cdot q(x,a,g(x,z)) = 0. \tag{32}$$

Finally, setting $y_i = a_i \cdot g(x,z) + z_i$ and repeating the proof of the lemma 3.2, we get that the left-hand side of (48) is the Taylor expansion of $f(x,y)$. Therefore, our newly constructed y is an algebraic or respectively convergent solution of f.

Step 2 We can now use this reduction to use induction on the number n of x variables to prove the general statement. The case $n = 0$ is trivial, as in this case the formal solution $\widehat{y}(x)$ is constant and therefore already algebraic. Now assume the statement proven for all natural numbers smaller than $n-1$ and let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ and $f \in \mathbb{K}\langle x,y\rangle^r$. Assume we are given a formal solution $\widehat{y}(x) \in K[[x]]^m$ with $\widehat{y}(0) = 0$. For the following, let $\widehat{z}(x)$ and $\widehat{a}''(x)$ be derived from $\widehat{y}(x)$ like in step 1. This once again leads to a system of equations in x' to which the vector of coefficients $\widehat{k}_{ij}(x')$ of

$$\widehat{z} = (\sum_{l=0}^{2d} \widehat{k}_{1l}(x') x_n^l, \ldots, \sum_{l=0}^{2d} \widehat{k}_{ml}(x') x_n^l)$$

is a formal solution. We can now apply induction to get an algebraic respectively convergent solution given by $k_{ij}(x')$, where the k_{ij} approximate the $\widehat{k}_{ij}$ up to degree c. Setting $z_i(x) = \sum_{l=0}^{2d} k_{il}(x') x_n^l$ and choosing $a''(x)$ to be the truncation of $\widehat{a}''(x)$ at degree c, we will get an algebraic respectively convergent solution $y(x)$ of $f(x,y)$ that approximates $\widehat{y}(x)$ up to degree c by construction.

□

4 Popescu's nested approximation theorem in the simplest case

This chapter will be about theorem 2 in the case where there are only two components of x variables. For clarity, we will call them x and t and the different components of the solutions will be called y and z, where y will only depend on t and z will depend on t and x. In this setting $x = (x_1, \dots, x_n), y = (y_1, \dots, y_m), z = (z_1, \dots, z_k)$ may be multivariables but t has to be a single variable. The upside of this simple case is that it is also true for convergent power series, hence we will prove this case. The algebraic case works analogously. The following proof goes back to ideas of [DL80]. The exact statement we want to prove is the following:

Theorem 12 (Popescu's nested approximation theorem [Pop86])**.** *Let $f(t, x, y, z) \in \mathbb{C}\{x, y\}^r$ be a vector of convergent power series in a single t variable and some multivariables $x = (x_1, \dots, x_n), y = (y_1, \dots, y_m)$, and $z = (z_1, \dots, z_k)$. Assume given a formal power series solution $\widehat{y}(t), \widehat{z}(t, x)$ vanishing at 0 (i.e. $\widehat{y} \in (t)\mathbb{C}[[t]]^m, \widehat{z} \in (t, x)\mathbb{C}[[t, x]]^k$),*

$$f(t, x, \widehat{y}(t), \widehat{z}(t, x)) = 0.$$

Then there exists, for any $c \in \mathbb{N}$, a convergent power series solution $y(t) \in \mathbb{C}\{t\}^m, z(t, x) \in \mathbb{C}\{t, x\}^k$,

$$f(t, x, y(t), z(t, x)) = 0,$$

which coincides with $\widehat{y}(t), \widehat{z}(t, x)$ up to degree c,

$$y(t) \equiv \widehat{y}(t) \mod (t)^{c+1}$$

$$z(t, x) \equiv \widehat{z}(t, x) \mod (t, x)^{c+1}.$$

The general idea is to use repeated approximation in suitable rings that "separate" our variables and try to repeat an analogon of the proof in the previous section. Hence, the rings used should have a Weierstrass division. We therefore look at the following:

$$\mathbb{C}[[t]]\text{“}x\text{”} := \{q(h_1, \dots, h_s, x) \in \mathbb{C}[[t, x]], q(H_1, \dots, H_s, x) \in \mathbb{C}\{H_1, \dots, H_s, x\}, h_1, \dots, h_s \in (t)\mathbb{C}[[t]]$$

$$\text{for some } s \in \mathbb{N}\}.$$

Lemma 4.1. *Let $R = \mathbb{C}[[t]]\text{“}x\text{”}$. Then*

- *R is a ring,*
- *R is closed under partial derivatives with respect to the x_i,*
- *R has a Weierstrass division algorithm.*

Proof. For the rest of the proof let $q(h_1(t), \dots, h_s(t), x)$ and $p(i_1(t), \dots, i_r(t), x)$ be two elements of R, i.e. $h_j, i_j \in (t)\mathbb{C}[[t]]$, $q \in \mathbb{C}\{H_1, \dots, H_s, x\}$, and $p \in \mathbb{C}\{I_1, \dots, I_r, x\}$.
(1): Both p, q can also be seen as elements of $\mathbb{C}\{H_1, \dots, H_s, I_1, \dots, I_r, x\}$ and thus $p + q$ and $p \cdot q$ are elements of $\mathbb{C}\{H_1, \dots, H_s, I_1, \dots, I_r, x\}$ as well. Hence, $(p + q)(h_1, \dots, h_s, i_1, \dots, i_r, x)$ and $(p \cdot q)(h_1, \dots, h_s, i_1, \dots, i_r, x)$ are elements of R.
(2): As $\partial_{x_i} p \in \mathbb{C}\{H_1, \dots, H_s, x\}$ by lemma 1.1, $\partial_{x_i} p \in \mathbb{C}\{h_1, \dots, h_s, x\} \in R$.

(3) Now let q be x_n-regular of order d. We once again interpret p and q as elements of $\mathbb{C}\{H_1,\dots,H_s,I_1,\dots,I_r,x\}$ and can use the Weierstrass division of this ring to write

$$p(H_1,\dots,H_s,I_1,\dots,I_R,x) = A(H_1,\dots,H_s,I_1,\dots,I_R,x)\cdot q(H_1,\dots,H_s,I_1,\dots,I_R,x)$$

$$+\sum_{ij=0}^{d-1} B_j(H_1,\dots,H_s,I_1,\dots,I_R,x')\cdot x_n^d,$$

where $A\in\mathbb{C}\{H_1,\dots,H_s,I_1,\dots,I_r,x\}$ and $B_j\in\mathbb{C}\{H_1,\dots,H_s,I_1,\dots,I_r,x'\}$. As this equality will hold after substituting the h_j and i_j for the H_j and I_j respectively, the claim holds. □

Note that (3) also implies that INFT and IMFT hold in the ring $\mathbb{C}[[t]]\text{“}x\text{”}$. The problem with replicating the proof of section 3 is that the existence of a minor that does not vanish at the formal solution will not be sufficient. We can not just use a linear automorphism to assume this minor to be x_n-regular at the formal solution, as it might temper with t as well. To illustrate, consider the following two simple examples:

$$f_1 = t + x_1\cdot x_2), f_2 = t\cdot x_1 + t\cdot x_2.$$

f_1 can be made x_2 regular by applying the automorphism $x_1\to x_1+x_2$. However, for f_2 no such automorphism exists that fixes t. This could create problems if we look at the variables seperately. To circumvent this, we will be able to show that if t is a single variable, one can use blow ups (in this case so-called Neron blow ups) to show the existence of a minor of $\partial_z f$ that is not congruent zero modulo (t) at the formal solution $\widehat{z}$. We can thus apply a linear automorphism of the form $t\mapsto t,\ x_i\mapsto x_i+x_n^{l_i}$ for $i=1,\dots,n-1$ and $x_n\mapsto x_n$ to assume this minor to be x_n regular. Before doing so, we note that after substituting y for $\widehat{y}(t)$ in the defining equation, the proof of lemma 2.1 also works in this case, so we assume that we have a minor of the Jacobian matrix of our system which does not vanish at the formal solution.

4.1 Neron-desingularisation

Let $t, x=(x_1,\dots,x_n), z=(z_1,\dots,z_m)$ be variables and let $\widehat{z}\in\mathbb{C}[[t]][[x]]$. Furthermore, let

$$\varphi:\mathbb{C}[[t]]\text{“}x,Z\text{”}\to\mathbb{C}[[t]][[X]]$$

be given by $z\mapsto\widehat{z}$. Assume for $I=\ker\varphi=\langle q_1,\dots,q_r\rangle$ there exists an $r\times r$-minor of $\frac{\partial q_i}{\partial z_j}$ that does not vanish at $\widehat{z}$.
We now want to show that there even exists an $r\times r$-minor M of $\frac{\partial q_i}{\partial z_j}$ such that $M(t,x,\widehat{z}(t,x))\not\equiv 0 \bmod (t)$.
Assume the opposite: I.e. for all $r\times r$-minors M of $\frac{\partial q_i}{\partial z_j}$ there holds $M(t,x,\widehat{z}(t,x))\in(t)\mathbb{C}[[t]][[x]]$. Consider the map

$$\psi:\mathbb{C}[[t]]\text{“}x,Z\text{”}\to\mathbb{C}[[x]]$$

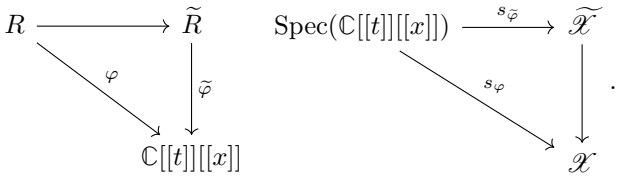
$$\psi=\pi\circ\varphi,$$

where $\pi:\mathbb{C}[[t]][[x]]\to\mathbb{C}[[x]],\ t\mapsto 0$.
Let $J=\ker(\psi)$ and let $R=\mathbb{C}[[t]]\text{“}X,Z\text{”}/I$ and $\mathscr{X}=\mathrm{Spec}(R)$.

Claim $J \subseteq \mathrm{Sing}(\mathscr{X})$: By assumption, for all $r \times r$-minors M of $\frac{\partial q_i}{\partial z_j}$ $\varphi(M) \in (t)\mathbb{C}[[t]][[x]$ and hence $\psi(M) = 0$. Therefore, if $\widetilde{M}$ is the ideal generated by these minors, then $\widetilde{M} \subseteq J$ and thus $V(J) \subseteq V(\widetilde{M})$. The claim holds as $\mathrm{Sing}(\mathscr{X}) = V(\widetilde{M})$.
Let $t, b_1, \ldots, b_j$ be the generators of J. The idea is now to resolve the singularity at J using blowups. So let $\widetilde{\mathscr{X}}$ be the t-chart of the blow up of $\mathscr{X}$ at J, i.e. $\widetilde{\mathscr{X}} = \mathrm{Spec}\widetilde{R}$, where

$$\widetilde{R} := \mathbb{C}[[t]]\text{“}X, Z\text{”}/(q_1, \ldots, q_r)[\frac{b_1}{t}, \ldots, \frac{b_j}{t}] \cong \mathbb{C}[[t]]\text{“}X, Z\text{”}[W]/(q_1, \ldots, q_r, tW_1 - b_1, \ldots, tW_j - b_j).$$

Note that $\widetilde{\mathscr{X}} \to \mathscr{X}$ is an isomorphism for $t \neq 0$ using the relation $t \cdot W_i - b_i = 0$. Using the definition of J, we get that $b_i(t, x, \widehat{Z}(t, x)) = t \cdot w_i(t, x)$ for some $w_i \in \mathbb{C}[[t]][[x]]$. Hence, the morphism s_φ :Spec$(\mathbb{C}[[t]][[x]]) \to \mathscr{X}$ can be extended to a morphism $s_{\widetilde{\varphi}}$:Spec$(\mathbb{C}[[t]][[x]]) \to \widetilde{\mathscr{X}}$ with commutating diagrams



We now introduce a measure of singularity on φ:

$$l(\varphi) = \min_{M \in \widetilde{M}} \mathrm{ord}_t\, M(t, x, \widehat{z}(t, x)).$$

It follows from above that if $l(\varphi) > 0$ then $\mathscr{X}$ is singular at φ. Equivalently, if $\mathscr{X}$ is regular at φ, then $l(\varphi) = 0$. On the contrary, if $l(s) = 0$, then $\mathscr{X}$ is regular at φ.

Theorem 13 (Neron-desingularisation)**.** *Let φ and $\widetilde{\varphi}$ be given like above. Then*

$$l(\widetilde{\varphi}) \leq l(\varphi)$$

with equality if, and only if, $l(\varphi) = 0$.

Proof. Using algebraic geometry, we can assume that $J = (t, Z_1, \ldots, Z_m)$ (for details see [Art69]). So let $\mathscr{X} = V(q_1, \ldots q_r)$, $q_i \in \mathbb{C}[[t]]\text{“}x, Z\text{”}$. As the blow up is given by the equations

$$t \cdot W_i = Z_i,$$

one can eliminate the Z variable and interpret $\widetilde{\mathscr{X}}$ as Zero set of polynomials in W over $\mathbb{C}[[t]][[x]]$. Now let

$$q_i(Z) = t \cdot a_{i0} + \sum_{j=0}^{m} a_{ij} Z_j + (\text{degree} \geq 2)$$

and substitute Z_i with $t \cdot W_i$:

$$q_i(t \cdot W) = t \cdot a_{i0} + \sum_{j=0}^{m} t \cdot a_{ij} W_j + t^2(\text{degree} \geq 2).$$

Note that the constant term with respect to W is in (t) as $q_i(t \cdot W) = 0$ and thus it is sensible to

write $t \cdot a_{i0}$. We can therefor rewrite $q_i(t \cdot W) = t \cdot v_i(W)$, where

$$v_i(W) = a_{i0} + \sum_{j=0}^{m} a_{ij} W_j + t(\text{degree} \geq 2). \tag{33}$$

Hence, $\widetilde{\mathscr{X}}$ is the vanishing set of the v_i and some additional relations to eliminate t-torsion. Deriving the equation $q_i(t \cdot W) = t \cdot v_i(W)$ leads to

$$t \cdot \frac{\partial v_i}{W_j}(W) = \frac{\partial}{W_j} q(t \cdot W)_i = t \cdot \frac{\partial q_i}{Z_j}(t \cdot W) = t \cdot \frac{\partial q_i}{Z_j}(Z).$$

Therefore, the Jacobian of $\mathscr{X}$ is a submatrix of the Jacobian of $\widetilde{\mathscr{X}}$ if we use generators q_i and v_i respectively and hence $l(\widetilde{\varphi}) \leq l(\varphi)$. Let $.^\circ$ denote reduction modulo t. Then by (33) the

$$v_i^\circ = a_{i0}^\circ + \sum_{j=0}^{m} a_{ij}^\circ W_j$$

and thus for the Jacobian matrix $J = \frac{\partial v_i}{W_j}(W) = \frac{\partial q_i}{Z_j}(Z)$ there holds that J° is the matrix a_{ij}° (note that here the a_{ij}° are power series in x). Hence, there also holds that $J^\circ = (a_{ij}^\circ)$ at φ. If we assume $\mathscr{X}$ to be not smooth at φ (i.e. $l(\varphi) \neq 0$), this matrix can not be of full rank ($= r$) as otherwise there would be a minor with $\operatorname{ord}_t = 0$ at φ. Now let M be a $r \times r$-minor of $\frac{\partial q_i}{Z_j}(Z)$ such that $l(\varphi) = \operatorname{ord}_t M(t, x, \widehat{z}(t, x))$. And without loss of generality, let it be the minor $\frac{\partial q_i}{Z_j}(Z)$ for $1 \leq i, j \leq r$. As the matrix (a_{ij}°) does not have full rank, we can perform an invertible linear transformation of $q_1, \ldots, q_r$ such that $a_{1j}^\circ = 0$ (i.e. $a_{1j} \cong 0 \bmod t$). With this new system one gets

$$q_1(Z) = t \cdot a_{10} + \sum_{j=0}^{m} t \cdot \alpha_{1j} Z_j + t^2(\text{degree} \geq 2)$$

$$q_1(t \cdot W) = t \cdot a_{10} + t^2(\sum_{j=0}^{m} \alpha_{1j} Z_j + (\text{degree} \geq 2)).$$

As $q_1 \in I$, there holds $q_1(t, x, \widehat{z}(t, x)) = 0$ and thus $a_{10}(t, x) \equiv 0 \bmod t$. This implies that $v_1(W) \cong 0 \bmod t$. One can therefore replace v_1 with $\frac{v_1}{t}$ and this new polynomial will still vanish on $\widetilde{\mathscr{X}}$. This replacement will also give for all minors using $\frac{v_1}{W_j}$ that $l(\widetilde{\varphi}) \leq l(\varphi) - 1$. □

4.2 The approximation Property of $\mathbb{C}[[t]]\text{“}x\text{”}$

We can now prove the following theorem:

Theorem 14. *[[DL80]] The ring $\mathbb{C}[[t]]\text{“}x\text{”}$ has the approximation property, i.e. if $q \in \mathbb{C}[[t]]\text{“}x, z\text{”}^r$ with formal solution $\widehat{z}(x) \in (t, x)\mathbb{C}[[t]][[x]]^k$, then there exists for every $c \in \mathbb{N}$ a solution $z(x) \in \mathbb{C}[[t]]\text{“}x\text{”}^k$ agreeing with $\widehat{z}$ up to degree c.*

The proof is essentially the same as the one given in chapter 3 using the fact that the ring $\mathbb{C}[[t]]\text{“}x\text{”}$ has a Weierstrass division:

Proof. Step 1 We will first show that whether an element $z(t, x)$ of $\mathbb{C}[[t, x]]^k$ or $\mathbb{C}[[t]]\text{“}x\text{”}^k$ is a solution of q (i.e. $q(t, x, z(t, x)) = 0$) can be reduced to a system in just $x' = (x_1, \ldots, x_{n-1})$. So let $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_k)$, $q \in \mathbb{C}[[t]]\text{“}x, z\text{”}^r$, and let $z(t, x) \in (t, x)\mathbb{C}[[t, x]]^k$ be a solution

$q(t, x, z(t, x)) = 0$.
Using the lemma above, we may assume that there is a minor g of $\partial_{\mathbf{z}} q$ such that $g(t, x, z(t, x)) \not\equiv 0 \bmod (t)$. This means that in the series expansion of $g(t, x, z(t, x))$ there appears a term of the form $\lambda \cdot x^d, \lambda \in \mathbb{C}$. Therefore, we can apply a linear change in the x-variables and assume $g(t, x, z(t, x))$ to be x_n-regular of order d. Let $\mathbf{z}'$ denote the $\mathbf{z}_i$ used for g and let $\mathbf{z}''$ denote the others. For readability reasons, we will leave out the x and t arguments of z for the rest of the proof.
We now apply formal Weierstrass division of $\mathbb{C}[[t, x]]$ by dividing z_i by $x_n \cdot g(t, x, z)$ if $\mathbf{z}_i \in \mathbf{z}'$ and by $x_n \cdot g(t, x, z)^2$ if $\mathbf{z}_i \in \mathbf{z}''$ to get

$$\begin{aligned} z_i(t, x) &= v_i(t, x) \cdot g(t, x, z) + w_i(t, x) \quad \text{for } \mathbf{z}_i \in \mathbf{z}' \\ z_i(t, x) &= v_i(t, x) \cdot g(t, x, z)^2 + w_i(t, x) \quad \text{for } \mathbf{z}_i \in \mathbf{z}'', \end{aligned} \tag{34}$$

where $w_i(t, x) \in \mathbb{C}[[t, x']][x_n]_{\leq 2d}$ and $v_i(x) \in x \cdot \mathbb{C}[[t, x]]$. Once again, for readability reasons, we will leave out the t and x arguments of w and v. These w_i are only polynomial in x_n, so we now show that we can express everything in terms of $w = (w_1, \ldots, w_k)$:

Lemma 4.2. *The ideals generated by $g(t, x, z)$ and $g(t, x, w)$ are the same in $\mathbb{K}[[t, x]]$.*

Proof. Let $v = (v_1, g(t, x, z)v_2)$ and consider the Taylor expansion of $g(t, x, w) = g(t, x, z - g(t, x, z) \cdot v)$ in the point $(t, x, g(x, z))$:

$$g(t, x, w) = g(t, x, z) - g(t, x, z) \cdot v \cdot \partial_{\mathbf{z}} g(t, x, z) + g(t, x, z)^2 \cdot h(t, x), \tag{35}$$

where $h(t, x) \in x \cdot \mathbb{C}[[t, x]]^k$. We can now factor out $g(t, x, z)$ to get

$$g(t, x, w) = g(t, x, z)(1 + v \cdot \partial_{\mathbf{z}} g(t, x, z) + h(t, x)), \tag{36}$$

and $(1 + v \cdot \partial_{\mathbf{z}} g(t, x, z) + h(t, x))$ is invertible. Hence, the two ideals are the same. □

Note that this also shows that $g(t, x, w)$ is x_n-regular of order d. We can therefore rewrite (34) as

$$\begin{aligned} z_i &= a_i(t, x) \cdot g(t, x, w) + w_i \quad \text{for } \mathbf{z}_i \in \mathbf{z}' \\ z_i &= a_i(t, x) \cdot g(t, x, w)^2 + w_i \quad \text{for } \mathbf{z}_i \in \mathbf{z}'', \end{aligned} \tag{37}$$

where the $a_i(t, x)$ are once again elements of $x \cdot \mathbb{K}[[t, x]]$. We will also leave out the t and x arguments of the a_i for the rest of the proof.
Let a' be the vector of all a_i for which the corresponding $\mathbf{z}_i \in \mathbf{z}'$, and a'' denote the vector of those for which the corresponding $\mathbf{z}_i \in \mathbf{z}''$. Furthermore, let $a = (a', g(t, x, w)a'')$. Now consider the Taylor expansion of $q(t, x, z) = q(t, x, a \cdot g(t, x, w) + w)$ in the point (t, x, w):

$$q(t, x, w) + a \cdot g(t, x, w) \cdot \partial_{\mathbf{z}} q(t, x, w) + g(t, x, w) \cdot u(t, x, a, g(x, w)) = 0 \tag{38}$$

where $u(t, x, A, G)$ is a convergent power series in t, x, $A = (A', A'')$, and G. Furthermore, u is at least quadratic in A. As $g = \partial_{\mathbf{z}'} q$ is a maximal minor of $\partial_{\mathbf{z}} q$, we can write the diagonal matrix

$g \cdot \mathbb{1}_r$ as the product of $\partial_{\mathtt{z}'} q$ and its adjoint matrix $\partial^*_{\mathtt{z}'} q$. We can use this fact to get

$$\begin{gathered} g(t,x,w)^2 \cdot u(t,x,a,g(t,x,w)) = \\ g(t,x,w) \cdot \partial_{\mathtt{z}'} q(t,x,w) \cdot \partial^*_{\mathtt{z}'} q(t,x,w) \cdot u(t,x,a,g(t,x,w)) \end{gathered} \tag{39}$$

and

$$\begin{gathered} a \cdot g(t,x,w) \cdot \partial_{\mathtt{z}} q(t,x,w) = \\ g(t,x,w) \cdot [\sum_{\mathtt{z}_i \in \mathtt{z}'} \partial_{\mathtt{z}_i} q(t,x,w) \cdot a_i + \sum_{\mathtt{z}_i \in \mathtt{z}''} \partial_{\mathtt{z}_i} q(t,x,w) \cdot a_i \cdot g(t,x,w)] = \\ g(t,x,w) \cdot \partial_{\mathtt{z}_1} q(t,x,w) \cdot [a' + \partial_{\mathtt{z}''} q(t,x,w) \cdot a'' \cdot \partial^*_{\mathtt{z}'} q(t,x,w)] \end{gathered} \tag{40}$$

Combine this with (38) to get

$$q(t,x,w) + g(t,x,w) \cdot \partial_{\mathtt{z}'} q(x,w) \cdot \Phi(t,x,a',a'',w) = 0, \tag{41}$$

where

$$\Phi(t,x,A',A'',\mathtt{w}) = (\mathtt{A}' + \partial^*_{\mathtt{z}'} \mathtt{q}(\mathtt{t},\mathtt{x},\mathtt{w}) \cdot \mathtt{u}(\mathtt{t},\mathtt{x},\mathtt{w}', \mathtt{g}(\mathtt{t},\mathtt{x},\mathtt{w}) \cdot \mathtt{A}'', \mathtt{g}(\mathtt{t},\mathtt{x},\mathtt{w})) + \partial^*_{\mathtt{z}'} \mathtt{q}(\mathtt{t},\mathtt{x},\mathtt{w}) \cdot \partial_{\mathtt{z}''} \mathtt{q}(\mathtt{t},\mathtt{x},\mathtt{w}) \cdot \mathtt{A}'' \tag{42}$$

Then Φ is convergent, vanishing in zero, and $\partial_{A'} \Phi(0,0,0,0,0)$ does not vanish and is therefore invertible. Setting $b(t,x,\mathtt{w}) = \mathtt{\Phi}(\mathtt{t},\mathtt{x},\mathtt{a}',\mathtt{a}'',\mathtt{w})$ we get, that

$$q(t,x,w) + g(t,x,w) \cdot \partial_{\mathtt{z}'} q(t,x,w) \cdot b(t,x,w) = 0, \tag{43}$$

and equivalently

$$\partial^*_{\mathtt{z}'} q(t,x,w) \cdot q(t,x,w) + g(t,x,w)^2 \cdot b(t,x,w) = 0. \tag{44}$$

Thus, $\partial^*_{\mathtt{z}'} q(t,x,w) \cdot q(t,x,w)$ is in the ideal generated by $g(t,x,w)^2$. This ideal membership can be reformulated to a system of equations by using the Weierstrass division in the ring $\mathbb{C}[[t]]\text{“}x\text{”}$ to divide $q(t,x,\mathtt{w})$ by $g(t,x,\mathtt{w})^2$ (with respect to x_n) and setting the remainder, which is polynomial in x_n, to zero. By comparing coefficients in x_n this leads to a finite system of equations in x'. Now assume we are given a solution in the ring $\mathbb{C}[[t]]\text{“}x'\text{”}$ of this system of equations. It will give us a power series $w = (w_1, \ldots, w_k)$ that is polynomial in x_n and a power series $b(t,x,\mathtt{w}) \in \mathbb{C}[[\mathtt{t}]]\text{“}\mathtt{x},\mathtt{w}\text{”}$ such that

$$q(t,x,w) + g(t,x,w)^2 \cdot b(t,x,w) = 0. \tag{45}$$

Now consider the algebraic respectively convergent equation

$$\Psi(t,x,A',A'') = \Phi(t,x,A',A'',w) - b(t,x,w) = 0. \tag{46}$$

As mentioned above $\partial_{A'} \Phi$ is invertible, hence $\partial_{A'} \Psi$ is as well. If we now choose any power series $a'' \in \mathbb{C}[[t]]\text{“}x\text{”}$, we can apply the implicit function theorem to $\Psi(t,x,A',a'',w)$ to get a unique algebraic respectively convergent power series $a' \in \mathbb{C}[[t]]\text{“}x\text{”}^k$ such that

$$\Phi(t,x,a',a'',w) = b(t,x,w). \tag{47}$$

Setting $a = (a', g(x,w) \cdot a'')$, we can repeat (39) and (40) in reverse order to get

$$q(t,x,w) + a \cdot g(t,x,w) \cdot \partial_{\mathtt{z}} q(t,x,w) + g(t,x,w)^2 \cdot u(t,x,a,g(t,x,w)) = 0. \tag{48}$$

Finally setting $z_i = a_i \cdot g(t,x,w) + w_i$ and repeating the proof of the lemma 4.2, we get that the left-hand side of (48) is the Taylor expansion of $q(t,x,z)$. Therefore, our newly constructed z is a solution of q.

Step 2 We can now use this reduction to use induction on the number n of x variables to prove the general statement. The case $n = 0$ is trivial, as in this case the formal solution $\widehat{z}(t,x)$ is only dependent on t and therefore this case is covered by Artin's approximation theorem (cf. [Art68]).

Now assume the statement proven for all natural numbers smaller than $n-1$ and let $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_k)$ and $q \in \mathbb{C}[[t]]\text{“}x, y\text{”}^r$. Assume we are given a formal solution $\widehat{z}(x) \in (t,x)K[[x,t]]^k$. For the following, let $\widehat{w}(x)$ and $\widehat{a}''(x)$ be derived from $\widehat{z}(x)$ like in step 1. This leads to a system of equations in x', to which the vector of coefficients $\widehat{e}_{ij}(x')$ of

$$\widehat{w} = (\sum_{l=0}^{2d} \widehat{e}_{1l}(x')x_n^l, \ldots, \sum_{l=0}^{2d} \widehat{e}_{ml}(x')x_n^l)$$

is a formal solution. We can now apply induction to get a solution given by $e_{ij}(x') \in \mathbb{C}[[t]]\text{“}x'\text{”}$ where the e_{ij} approximate the $\widehat{e}_{ij}$ up to degree c. Setting $w_i(x) = \sum_{l=0}^{2d} e_{il}(x')x_n^l$ and choosing $a''(x)$ to be the truncation of $\widehat{a}''(x)$ at degree c, we will get a solution $z(x) \in \mathbb{C}[[t]]\text{“}x\text{”}$ of $q(x,y)$ that approximates $\widehat{z}(x)$ up to degree c by construction. □

4.3 The proof of theorem 12

We are now finally prepared to prove theorem 12:

Proof. So let $f(t,x,y,z) \in \mathbb{C}^r$ be a system of equations with a nested solution $\widehat{y}(t), \widehat{z}(t,x)$ and let c be a natural number. f can be interpreted as an element of $\mathbb{C}[[t]]\text{“}x,z\text{”}$ by fixing the $\widehat{y}(t)$ and therefore theorem 14 can be applied to approximate $\widehat{z}(t,x)$ within $\mathbb{C}[[t]]\text{“}x\text{”}$. This leads to a solution $\widetilde{z}(t,x) \in \mathbb{C}[[t]]\text{“}x\text{”}^k$, where

$$\widetilde{z}_i(t,x) \equiv \widehat{z}_i(t,x) \mod (t,x)^{c+1}$$

for $i = 1, \ldots, k$. These $\widetilde{z}_i(t,x)$ can be written as $\sum_{i\geq 0} g_{ji}(\widehat{h})x^i$, where $\sum_{i\geq 0} g_{ji}(Z)x^i \in \mathbb{C}\{H,x\}$ and $\widehat{h} = (\widehat{h}_1(t), \ldots, \widehat{h}_s(t)) \in (t)\mathbb{C}[[t]]$. We can rewrite these as $\sum_{i\geq 0} p_i(\widehat{h}(t))x^i$, where $\widehat{h}(t) = (\widehat{h}_1(t), \ldots, \widehat{h}_s(t)) \in \mathbb{C}[[t]]^s$.

We can now insert $\widetilde{z}$ into $f(t,x,y,z)$ to get $f(t,x,y,\sum_{i\geq 0} p_i(h(t))x^i)$ and expand this power series in x to get $\sum_{i\geq 0} P_i(t,y,h)x^i$, where the P_i are elements of $\mathbb{C}\{t,y,h\}$ as f is convergent. As the ring $\mathbb{C}\{t,y,h\}$ is noetherian, there is a finite $J \subseteq \mathbb{N}$ such that P_i can be reduced to a finite system $\{P_j \mid j \in J\}$ for which $(\widehat{y}(t), \widehat{t})$ is a formal solution. We now apply theorem 1 to get a convergent $(y(t), h(t))$ such that

$$y_i(t) \equiv \widehat{y}_i(t) \mod (t)^{c+1} \text{ for } i = 1, \ldots, m$$

$$h_j(t) \equiv \widehat{h}_j(t) \mod (t)^{c+1} \text{ for } j = 1, \ldots, s$$

and such that $P_j(t, y(t), h(t)) = 0 \forall j \in J$ and therefore $P_i(t, y(t), h(t)) = 0 \forall i$. Set $z_i(t,x) = \sum_{i\geq 0} g_{ji}(h)x^i$ for $i = 1, \ldots, k$. Then by construction $z(t,x) = (z_1(t,x), \ldots, z_k(t,x))$ is a convergent power series vector with $f(t,x,y(t),z(t,x)) = 0$ and

$$z_i(t,x) \equiv \widetilde{z}_i(t,x) \equiv \widehat{z}_i(t,x) \mod (t,x)^{c+1} \text{ for } i = 1, \ldots, k.$$

Hence, $z(t,x)$ and $y(t,x)$ are as required. $\square$

5 Gabrielov's counterexample for convergent nested approximation

As mentioned above, the convergent case of theorem 2 does not hold in general. Thus, this section will present Gabrielov's (cf. [Gab71]) counterexample. It is given by looking at relations between convergent power series:
Let $\mathbb{K}$ be a valued field and let $f_1(x), \ldots, f_m(x) \in \mathbb{K}\{x\}$ be algebraic equations. We denote by

$$\mathrm{Rel}(f_1, \ldots, f_m) := \{r(y) \in \mathbb{K}\{y_1, \ldots, y_m\} \mid r(f_1, \ldots, f_m) = 0)\}$$

and

$$\widehat{\mathrm{Rel}}(f_1, \ldots, f_m) := \{\widehat{r}(y) \in \mathbb{K}[[y_1, \ldots, y_m]] \mid \widehat{r}(f_1, \ldots, f_m) = 0)\}$$

the module of analytic and respectively formal relations of $(f_1, \ldots, f_m)$.
By looking at the homomorphism

$$\alpha : \mathbb{K}\{x, y\} \to \mathbb{K}\{x\},\ h(x, y_1, \ldots, y_m) \mapsto h(x, f_1(x), \ldots, f_m(x)),$$

one can see that $\mathrm{Rel}(f_1, \ldots, f_m) \subseteq \ker \alpha$, i.e. $r \in \mathrm{Rel}(f_1, \ldots, f_m)$ is an element of the ideal I generated by $\langle y_1 - f_1(x), \ldots, y_m - f_m(x)\rangle$ of $\mathbb{K}\{x, y\}$. One can show in the same way that $\widehat{r} \in \widehat{\mathrm{Rel}}(f_1, \ldots, f_m)$ is in the ideal $\widehat{I}$ of $\mathbb{K}[[x, y]]$ generated once again by $\langle y_1 - f_1(x), \ldots, y_m - f_m(x)\rangle$. Therefore for $r(y) \in \mathrm{Rel}(f_1, \ldots, f_m)$ and $\widehat{r}(y) \in \widehat{\mathrm{Rel}}(f_1, \ldots, f_m)$ there holds

$$r(y) = \sum_{i=1}^{m} a_i(x, y) \cdot (y_i - f_i(x))$$

$$\widehat{r}(y) = \sum_{i=1}^{m} \widehat{a_i}(x, y) \cdot (y_i - f_i(x)),$$

for some convergent power series $a_i \in \mathbb{K}\{x, y\}$ and formal power series $\widehat{a_i} \in \mathbb{K}[[x, y]]$. Now assume there is a convergent power series $r(y)$ and some formal power series $\widehat{a_i}(x, y)$ such that $r(y) = \sum_{i=1}^{m} \widehat{a_i}(x, y) \cdot (y_i - f_i(x))$. If we can find an example where there are no convergent $a_1(x, y), \ldots, a_m(x, y)$ such that $\sum_{i=1}^{m} a_i(x, y) \cdot (y_i - f_i(x))$, then this leads to a counterexample of theorem 2 with the convergent equation $E(x, y, R, A) = R - \sum_{i=1}^{m} A_i \cdot (y_i - f_i(x))$, which has the formal nested solution $r(y)$ and $\widehat{a_1}(x, y), \ldots \widehat{a}_m(x, y)$.
Gabrielov found an explicit example. Let $x = (x_1, x_2)$ and let f_1, f_2, f_3 be the following convergent functions:

$$f_1(x) = x_1, \quad f_2(x) = x_1 \cdot x_2, \quad f_3(x) = x_1 \cdot x_2 \cdot e^{x_2}.$$

Then there are no non trivial relations between f_1, f_2 and f_3 (cf. [É65]) p.115). We now define the power series

$$\Phi_k(y_1, y_2, y_3) = y_1^{k-1} \cdot y_3 - \sum_{i=1}^{k} \frac{y_1^{k-i} \cdot y_2^i}{(i-1)!}.$$

We now note that

$$\Phi_k(f_1(x), f_2(x), f_3(x)) = x_1^k \cdot x_2 e^{x_2} - \sum_{i=1}^{k} \frac{x_1^k \cdot x_2^i}{(i-1)!} = x_1 \cdot x_2 \cdot (\sum_{i \geq 0} \frac{x_2^i}{i!} - \sum_{i=0}^{k-1} \frac{x_2^i}{i!}) = \sum_{i \geq 0} \frac{x_1^k \cdot x_2^{i+k+1}}{(i+k)!}.$$

Thus setting $P(y_1, y_2, y_3) = \sum_{k\geq 1} k! \cdot \Phi_k(y_1, y_2, y_3) \in \mathbb{C}[[y_1, y_2, y_3]]$, we can define

$$f_4(x) = P(f_1(x), f_2(x), f_3(x)) = \sum_{k\geq 1} \sum_{i\geq 0} \frac{k!}{(i+k)!} x^k \cdot x_2^{k+j+1}.$$

Then $f_4(x)$ is a convergent power series, however, $P(y_1, y_2, y_3)$ is only formal and not covergent. Thus

$$\widehat{r}(y_1, y_2, y_3, y_4) = y_4 - P(y_1, y_2, y_3)$$

is a formal realtion between the convergent power series $f_1(x), f_2(x), f_3(x)$ and $f_4(x)$. The claim is, that there exists no non-trivial convergent solution. Therefore, we assume the existence of such a convergent relation $r(y_1, y_2, y_3, y_4)$, which without loss of generality is irreducible (otherwise if $0 = r(f_1(x), f_2(x), f_3(x), f_4(x)) = r_1(f_1(x), f_2(x), f_3(x), f_4(x)) \cdot r_2(f_1(x), f_2(x), f_3(x), f_4(x))$ and thus, as rings of power series are an integral domain, either $r_1(f_1(x), f_2(x), f_3(x), f_4(x)) = 0$ or $r_2(f_1(x), f_2(x), f_3(x), f_4(x)) = 0$). As $\widehat{r}$ is y_4-regular of order 1, we can use Weierstrass division to write $r(y)$ uniquely as

$$r(y_1, \ldots, y_4) = a(y_1, \ldots, y_4) \cdot \widehat{r}(y_1, \ldots, y_4) + b(y_1, y_2, y_3),$$

for some formal power series $a \in \mathbb{C}[[y_1, \ldots, y_4]]$ and $b \in \mathbb{C}[[y_1, \ldots, y_3]]$. Substituting y_i for $f_i(x)$ leads to $0 = b(f_1(x), f_2(x), f_3(x))$, as both r and $\widehat{r}$ are relations of the $f_i(x)$. However, as mentioned before, there are no relations between the f_i, and thus $b = 0$ and $\widehat{r}$ divides r. Consequently, $a(y)$ is a unit, as r was assumed to be irreducible. This implies furthermore that $r(y)$ is y_4-regular of order 1 as well and hence by WPT there is a unit $c(y)$ such that r can be uniquely written as

$$c(y) \cdot r(y) = y_4 - \widetilde{P}(y_1, y_2, y_3).$$

The uniqeness implies that $P = \widetilde{P}$ and thus $r(y)$ is not convergent, contradicting the assumption. While this counterexample may seem random, there is actually a way to construct such counter examples. For details see [ACJHK18].

6 A proof-sketch for Popescu's nested approximation theorem

The goal of this chapter is to give a sketch of how theorem 2 is proven in general. The proofs in this chapter are in line with [Spi99]. The proof relies on Popescu's theorem on smoothing ring homomorphisms, which will not be proven here but rather taken as a blackbox. Proofs can be found in [Pop86] and [Spi99]. In order to state said theorem, we need the following definitions of smooth and regular ring extensions:

Definition 6.1 ([Mat70] p.34)**.** *Let A be a commutative ring and B an A-module. Furthermore, let*

$$S : \ldots \to N \to N' \to N'' \to \ldots$$

be a sequence of A-modules. B is called flat A-module if the sequence

$$S \otimes_A M : \ldots \to N \otimes_A M \to N' \otimes_A M \to N'' \otimes_A M \to \ldots$$

is exact whenever S is exact.
Let $\phi : A \to C$ be a homomorphism of commutative rings. ϕ is called flat if it makes C a flat A-module.

Definition 6.2 ([Mat70] p.89)**.** *Let $(A, \mathfrak{m})$ be a noetherian local ring. A is called regular if the minimal number of generators of $\mathfrak{m}$ is the same as the Krull dimension of A.*
A noetherian ring B is called regular at every of its prime ideals is regular.

Definition 6.3 ([Mat70] p.248)**.** *Let A be a commutative ring containing a field $\mathbb{K}$. A is called geometrically regular over $\mathbb{K}$ if for any finite extension $\mathbb{K}'$ the ring $A \otimes_{\mathbb{K}} \mathbb{K}'$ is a regular ring.*

Definition 6.4 ([Mat70] p.249)**.** *A homomorphism $\phi : A \to B$ between commutative rings is called regular if*

1. *ϕ is flat*
2. *and for every $\mathfrak{p}$ primeideal of A the fibre $B \otimes_A \mathbb{K}(\mathfrak{p})$ is geometrically regular over $\mathbb{K}(\mathfrak{p})$, where $\mathbb{K}(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$.*

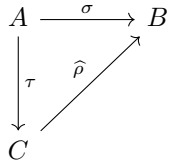
In this case we also say that B is regular over A.
A noetherian ring A is called a G-ring if for all $\mathfrak{p}$ primeideals of A the canonical map

$$\sigma : A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$$

is regular, where $\widehat{A_{\mathfrak{p}}}$ denotes the $\mathfrak{p}$-adic completion of $A_{\mathfrak{p}}$.

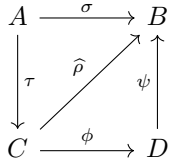
With these definitions we can now state Popescu's theorem:

Theorem 15 (Popescu's Theorem [Pop86])**.** *Consider the following commutative diagram of ring homomorphisms:*



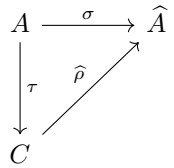
(49)

Assume that C is a finitely generated A-algebra and σ is regular. Then there exists D a smooth finitely generated A-algebra and ring homomorphisms $\phi : C \to D$ and $\psi : D \to B$ such that the commutative diagram (49) can be extended to:



(50)

We will first prove an analogon of theorem 1 using Popescu's theorem, which will later on be used to prove theorem 2. For that, let $(A, \mathfrak{m})$ be a Henselian G-ring and let $f = (f_1, \ldots, f_r) \in A[Y_1, \ldots, Y_m]$ be a system of polynomial equations with "formal" solutions $\widehat{y}_1, \ldots, \widehat{y}_m \in \widehat{A}$, where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A. Let J be the ideal defined by $f_1, \ldots, f_r \in A[Y_1, \ldots, Y_m]$ and let C denote the ring $A[Y_1, \ldots, Y_m]/J$. Notice that the solutions $\widehat{y}_1, \ldots, \widehat{y}_m$ define a homomorphism $\widehat{\rho} : C \to \widehat{A}$, i.e. the setting is the following commutative diagram:



(51)

The ring A has the approximation property if there exists a homomorphism $\rho : C \to A$ agreeing with $\widehat{\rho}$ modulo $\mathfrak{m}^c$ for every $c \in \mathbb{N}$. Notice that ρ is of the form $Y_i \to y_i$ for some $y_i \in A$. If such a ρ exists that means that $f(y_1, \ldots, y_m) = 0$ for all $f \in J$. Hence, ρ gives a solution to the approximation problem.
As we have assumed A to be a G-ring we can – after applying theorem 15 – replace C with a smooth ring D. In the following, we will once again assume that D is a complete intersection – for details see [Spi99] p.407. The algebraic concept of smooth is related to that of submersions in geometry. The idea is now to replace the smooth ring D with an étale E which corresponds to what is the local normal form of a submersion in geometry.

Proposition 2 ([Spi99] p.438)**.** *Let $\sigma : A \to B$ be a homomorphism of commutative rings and D be a smooth A algebra of finite type, which is a complete intersection. Furthermore, let $\psi : D \to B$ be an A-algebra homomorphism and let J be an ideal of A such that $\psi(D) \subseteq \sigma(A) + JB$ and such that elements of $\psi(D) \cap (1 + JB)$ are units in B. Let $A[Y_1, \ldots, Y_m]/I$ be a representation of D*

and let r be the height of I.
Then there exists a smooth finite Type A Algebra that is étale over the polynomial ring $A[Y_{r+1}, \dots, Y_m]$.

Proof. Let D be given as above,$I = (f_1, \dots, f_r)$, and $\frac{\partial f}{\partial u}$ denote the matrix $(\frac{\partial f_i}{\partial u_j})$. As D is smooth, there holds $\Delta_f D = D$ and hence there is a matrix $G = (g_{ij})_{1 \leq i \leq n, 1 \leq j \leq r}$ such that $\frac{\partial f}{\partial u} G = \mathrm{Id}_r$ is the identity matrix.
Now introduce new variables $t = (t_1, \dots, t_r)$ and consider the ring $D[t]$. Then ψ can be extended to $D[t]$ via $t_i \mapsto 0$.
By assumption, $\psi(D) \subseteq \sigma(A) + JB$ and thus there are elements $a_{ij} \in A$ such that $\psi(g_{ij}) \equiv \sigma(a_{ij}) \bmod JB$. Let G_0 be the matrix with entries a_{ij}, then by construction there holds $\psi(\frac{\partial f}{\partial u}) G_0 \equiv Id_r \bmod JB$. Hence, the determinant of $\psi(\frac{\partial f}{\partial u}) G_0 \equiv 1 \bmod JB$ and is thus by assumption a unit in B. Now let $x = \det \frac{\partial f}{\partial u} G_0$. As $\psi(x)$ is invertible, ψ can be extended to $D[t]_x$. One can then easily verify that $D[t]_x$ has the required properties after applying a linear change of variables $u_i \mapsto u_i - \sum_{j=1}^{r} a_{ij} t_j$. □

Note that this step is explicitly done using Neron-desingularization in [Art69]. We can now use this to show theorem 1 in the following case:

Theorem 16 (Approximation property of Henselian rings)**.** *Let $(A, \mathfrak{m})$ be a Henselian G-ring and let $C = A[Y_1, \dots, Y_m]/(f_1, \dots, f_r)$ correspond to a system of equation with formal solution $\widehat{\rho} : C \to \widehat{A}$. Then for every $c \in \mathbb{N}$ there exists a solution $\rho : C \to A$ that agrees with $\widehat{\rho}$ modulo $\mathfrak{m}^c$.*

Proof. Using theorem 15 we can first assume C to be smooth over A as A is a G-ring and C is of finite representation. As said above, we can assume $(f_1, \dots, f_r)$ to be a complete intersection. The maximal ideal $\widehat{\mathfrak{m}}$ of $\widehat{A}$ satisfies the required properties of J in the assumptions of proposition 2. Hence, we can assume C to be étale over $A[Y_{r+1}, \dots, Y_m]$ (i.e. C is of the form $A[Y_{r+1}, \dots, Y_m][t_1, \dots, t_k]_h/(g_1, \dots, g_s]$, where $h \in A[Y_{r+1} \dots, Y_m, t_1, \dots, t_k]$ and the determinant of the Jacobian matrix $(\frac{\partial g_i}{\partial t_j})$ is invertible in C).
Now fix $c \in \mathbb{N}$ and choose elements $x_{r+1}, \dots, x_m \in A$ such that $x_i - \widehat{\rho}(Y_i) \in \mathfrak{m}^c \widehat{A}$. We then consider $C' = C/(Y_{r+1} - x_{r+1}, \dots, Y_m - x_m)$. It is of finite representation over A and its Jacobian matrix is of the form

$$\begin{pmatrix} Id_{m-r+1} & 0 \\ M & \frac{\partial g_i}{\partial t_j} \end{pmatrix}.$$

Thus, its determinant is invertible in C' and C' is an étale algebra over A. We now define ρ via $\rho(Y_i) = x_i$ for $r+1 \leq i \leq m$. Note, that as C is étale over $A[Y_{r+1} \dots, Y_m]$, the values of $\rho(x_j)$ for $1 \leq j \leq r$ are already determined by the values of $\rho(x_{r+1}), \dots \rho(x_m)$. As $\widehat{\rho}$ is a solution and $\rho(x_i) \cong \widehat{\rho}(x_i) \bmod \mathfrak{m}$, the map ρ defines a section $C' \to A/\mathfrak{m}^c$. It therefore especially defines a section $C' \to A/\mathfrak{m}$ which lifts to A by the Henselian property of A. □

The goal is now to use this method of proof to show theorem 2. This is done by first generalizing theorem 15 and proposition 2 to a "nested" system. For that we consider diagrams of the following form:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \dots & \xrightarrow{\alpha_{k-1}} & A_k = A \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow & & \downarrow{\scriptstyle \tau_k} \\
C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & \dots & \xrightarrow{\gamma_{k-1}} & C_k = C \\
\downarrow{\scriptstyle \widehat{\rho}_1} & & \downarrow{\scriptstyle \widehat{\rho}_2} & & \downarrow & & \downarrow{\scriptstyle \widehat{\rho}} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \dots & \xrightarrow{\beta_{k-1}} & B_k = B
\end{array} \tag{52}$$

In the notation of theorem 2, the k will correspond to the number of nests, the A_i will corresponds to the $\mathbb{K}\langle x_1, \dots, x_{n_i}\rangle$ and the B_i will correspond to the according rings of formal power series. Finally, the $\widehat{y}_i$ will induce the $\widehat{\rho}_i$ and the C_i will be $A_{i-1}[y_i]/J$, where J is the ideal generated by the kernel of the substitution homomorphism $y_i \to \widehat{y}_i$. As all the proofs and arguments below are induction arguments, we will consider the case $k = 2$. We first generalize Popescu's theorem and proposition 2 in the following way:

Theorem 17 (Teissier cf.[Spi99] p.439-441)**.** *Consider the following diagram of noetherian rings*

$$\begin{array}{ccc}
A_1 & \xrightarrow{\alpha_1} & A_2 = A \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\
C_1 & \xrightarrow{\gamma_1} & C_2 = C \\
\downarrow{\scriptstyle \widehat{\rho}_1} & & \downarrow{\scriptstyle \widehat{\rho}} \\
B_1 & \xrightarrow{\beta_1} & B_2 = B
\end{array} \tag{53}$$

Assume that the C_i are of finite type over the A_i. Further, let $A'_1 = A_1$ and $A'_2 = A_2 \otimes_{A_1} B_1$ and assume that A'_i is noetherian and the induced $\sigma_i : A'_i \to B_i$ are regular. Then there exist D_1 smooth of finite type over A_1 and D_2 smooth of finite type over $D_1 \otimes_{A_1} A_2$ such that the diagram can be extended to

$$\begin{array}{ccc}
A_1 & \xrightarrow{\alpha_1} & A_2 = A \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\
C_1 & \xrightarrow{\gamma_1} & C_2 = C \\
\downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} \\
D_1 & \xrightarrow{\delta_1} & D_2 = D \\
\downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\
B_1 & \xrightarrow{\beta_1} & B_2 = B
\end{array} \tag{54}$$

If there once again exists an ideal $J \subseteq A$ such that for $J_i = A_i \cap J$: Elements of $1 + J_i B_i$ are units of B_i and $\frac{A_i}{J_i} \cong \frac{B_i}{J_i B_i}$ then D_1, D_2 can once again be replaced by D_1', D_2' such that D_1 is étale over $A_1[y_{r_1+1}, \ldots, y_{m_1}]$ and D_2 is étale over $D_1 \otimes_{A_1} A_2[y_{r_2+1}, \ldots, y_{m_2}]$, where r_i is the height of the ideal defining D_i.

For $k \geq 2$ one generalizes to $A_i' = A_i \otimes_{A_{i-1}} B_{i-1}$, D_i smooth of finite type over $D_{i-1} \otimes_{A_{i-1}} A_i$ and D_i' étale over $D_{i-1} \otimes_{A_{i-1}} A_i[y_{r_i+1}, \ldots, y_{m_i}]$.
As mentioned above, the general proof is via induction on k, where the base case $k = 1$ is exactly theorem 15 and proposition 2. We therefore only prove the induction step for $k = 2$.

Proof. Consider the diagram

$$\begin{array}{ccc} A_2' & \xrightarrow{\sigma_2'} & B_2 \\ \downarrow{\scriptstyle \tau_1'} & \nearrow{\scriptstyle \widehat{\rho}_2'} & \\ C_2 \otimes_{A_1} B_1 & & \end{array} \tag{55}$$

where $\tau_1', \widehat{\rho}_2'$ are the maps induced by $\tau_1, \widehat{\rho}_1$ and $\sigma_2' = \widehat{\rho}_2' \circ \tau_1'$. Then by assumption the map σ_2' is regular. As C_2 is of finite type over A_2, the ring $C_2 \otimes_{A_1} B_1$ is of finite type over A_2'. By the base case, there exists a D' smooth over A_2' (or respectivly étale over $A_i[y_{\widehat{r}+1}, \ldots, y_{\widehat{m}}]$) such that $\widehat{\rho}_2'$ factors through D':

$$\begin{array}{ccccc} A_2' & \xrightarrow{\sigma_2'} & B_2 & & \\ \downarrow{\scriptstyle \tau_1'} & \nearrow{\scriptstyle \widehat{\rho}_2'} & \uparrow & & \\ C_2 \otimes_{A_1} B_1 & & \xrightarrow{\Phi'} & & D' \end{array} \tag{56}$$

The idea is to replace B_1 with a finite type $\widehat{\rho}_1(C_1)$ subalgebra C' of B, such that both the maps τ_1' and Φ' are defined over C', i.e. we consider the diagram

$$\begin{array}{ccc} A_2 \otimes_{A_1} C' & \longrightarrow & B_2 \\ \downarrow & \nearrow & \uparrow \\ C_2 \otimes_{A_1} C' & \xrightarrow{\widehat{\phi}} & \widehat{D} \end{array} \tag{57}$$

where $\widehat{D}$ is smooth over $A_2 \otimes_{A_1} C'$ (resp. étale over $A_2 \otimes_{A_1} C'[y]$ where y is like above). We can finally consider the diagram

$$\begin{array}{c} A_1 \\ \downarrow \\ C' \\ \downarrow \\ B_1 \end{array} \tag{58}$$

and use induction to get a D_1 of the wanted form. We can now set $D_2 = \widehat{D} \otimes_{C'} D_1$. As D_1 is smooth over C' (or respectively étale over a suited polynomial ring $C'[y]$) and $\widehat{D}$ is smooth over $A_2 \otimes_{A_1} C'$ (resp. étale over a suited $A_2 \otimes_{A_1} C'[y]$), the ring D is smooth over $A_2 \otimes_{A_1} D_1$ (or respectively étale over $A_2 \otimes_{A_1} D_1[y_{r_2}, \ldots, y_{m_2}]$).
Hence, the statement is proven if we find maps $C_2 \to D_2$ and $D_2 \to B_2$ making the diagram 54 commute. For the first map consider the composition of the maps $C_2 \to C_2 \otimes_{A_1} C', x \mapsto x \otimes 1$, $\widehat{\Phi} : C_2 \otimes_{A_1} C' \widehat{D}$ and $\widehat{D} \to D_2, x \mapsto x \otimes 1$. The map $D_2 \to B_2$ is obtained from the maps $D_1 \to B_1$ and $\widehat{D} \to B_2$. □

As said above, the general case is proven via induction on k. However, one always just uses the base step using the obvious replacements except in the step 58 where one would consider the diagram

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & \ldots & \longrightarrow & A_{k-2} & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_1 & \longrightarrow & C_2 & \longrightarrow & \ldots & \longrightarrow & C_{k-2} & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & \ldots & \longrightarrow & B_{k-2} & \longrightarrow & B_{k-1} \end{array} \tag{59}$$

Under the right assumptions, this nested "smoothing" now gives us a way to finally prove theorem 2:

Theorem 18 (Nested approximation for Henselian systems, cf. [Spi99] p. 441)**.** *Consider a diagram like in 53 with all the assumptions made in theorem 17. Furthermore, assume that there is an ideal $\mathfrak{m} \subseteq A$ such that for $\mathfrak{m}_i = \mathfrak{m} \cap A_i$ the $\mathfrak{m}_i$-adic completion $\widehat{A}_i$ of A_i and the $\mathfrak{m}_i B_i$-adic completion $\widehat{B}_i$ of B_i satisfy $\widehat{A}_i = \widehat{B}_i$ and such that $(A_i, \mathfrak{m}_i)$ is a Henselian G-ring. Then for any $c \in \mathbb{N}$ there exists a $\rho : C \to A$ such that*

1. *ρ agrees with $\widehat{\rho} \bmod \mathfrak{m}^c$ and*
2. *$\rho(C_i) \subseteq (A_i)$.*

Once again the theorem has been stated for the case $k = 2$ but the proof works recursively over k and we will hence only indicate the case $k = 2$.

Proof. As the $(A_i, \mathfrak{m}_i)$ are Henselian G-rings, we may assume using theorem 17 that C_1 is étale over $A_1[y_{r_1+1}, \ldots, y_{m_1}]$ and C_2 is étale over $C_1 \otimes_{A_1} A_2[y_{r_2+1}, \ldots, y_{m_2}]$. We can now – after fixing $c \in \mathbb{N}$ – construct $\rho_1 : C_1 \to A_1$ agreeing with $\widehat{\rho}_1$ mod $\mathfrak{m}_1^c$ the same way as in the proof of Artin Approximation for Henselian rings. The map $\alpha_1 \circ \rho_1 : C_1 \to A_2$ induces a map $\widetilde{\rho_2} : C_1 \otimes_{A_1} A_2 \to A_2$. As C_2 is étale over $C_1 \otimes_{A_1} A_2[y_{r_2+1}, \ldots, y_{m_2}]$, one can extend $\widetilde{\rho}_2$ to $\rho_2 : C_2 \to A_2$ by prescribing values for y_j for $r_2 + 1 \leq j \leq m_2$. This is once again done in the same way it was done for the case $k = 1$: Take elements $x_i \in A_2$ such that $x_i \cong \widehat{\rho}(y_i)$ mod $\mathfrak{m}_2^c$ and use the Henselian property of A_2 to lift the section $C_2/(y_{r_2+1} - x_{r_2+1}, \ldots, y_{m_2} - x_{m_2}) \to A_2/\mathfrak{m}_2^c$ to $\rho : C_2 \to A_2$. □

It only remains to show that this actually proves theorem 2: Consider a system of equations with formal solutions like in the assumptions of said theorem. After applying lemma 2.2, we may assume that f is a polynomial system. Let $y = (y_1, \ldots, y_{m_1}, \ldots, y_{m_k})$ be a numbering such that $\widehat{y}_1, \ldots, \widehat{y}_{m_i}$ only depend on $x_1, \ldots, x_{n_i}$ (i.e. $\widehat{y}_1, \ldots, \widehat{y}_{m_i} \in \mathbb{K}[[x_1, \ldots, x_{n_i}]] =: B_i$). We can then consider A_i to be the rings $\mathbb{K}\langle x_1, \ldots, x_{n_i}\rangle$ with the maximal ideals $\mathfrak{m}_i = (x_1, \ldots, x_{n_i})$. Then there holds that $(A_i, \mathfrak{m}_i)$ are Henselian G-rings and $\widehat{A}_i = \widehat{B}_i = B_i$, where $\widehat{A}_i$ is the $\mathfrak{m}_i$-adic completion of A_i and $\widehat{B}_i$ is analogous. Furthermore, $(x_1, \ldots, x_{n_i})$ satisfies the assumption of theorem 17 that the units of B_i are the elements of the form $1 + \mathfrak{m}_i B_i$ as $(\mathbb{K}[[x_1, \ldots, x_{n_i}]], (x_1, \ldots, x_{n_i}))$ is a local ring and

$$\frac{A_i}{\mathfrak{m}_i} = \frac{\mathbb{K}\langle x_1, \ldots, x_{n_i}\rangle}{(x_1, \ldots, x_{n_i})} \cong \mathbb{K} \cong \frac{\mathbb{K}[[x_1, \ldots, x_{n_i}]]}{(x_1, \ldots, x_{n_i})} = \frac{B_i}{\mathfrak{m}_i B_i}.$$

Now consider I_i to be the ideal generating the kernel of $\mathbb{K}\langle x_1, \ldots, x_{n_i}, y_1, \ldots, y_{m_i}\rangle \to \mathbb{K}[[x_1, \ldots, x_{n_i}]]$, $y_i \mapsto \widehat{y}_i(x)$. After applying theorem 10 we can assume the I_i to be generated by Polynomials. Now let $C_i := A_i[y_1, \ldots, y_{m_i}]/I_i$. Then the C_i are of finite representation over A_i and have a morphism $\widehat{\rho}_i : C_i \to B_i$ defined once again via $y_i \mapsto \widehat{y}_i(x)$.
Finally, let $A_1' = A_1$ and $A_i\prime = A_i \otimes_{A_{i-1}} B_{i-1}$ for $2 \leq i \leq k$. I.e.

$$A_i\prime = \mathbb{K}\langle x_1, \ldots, x_{n_i}\rangle \otimes_{\mathbb{K}\langle x_1, \ldots, x_{n_{i-1}}\rangle} \mathbb{K}[[x_1, \ldots, x_{n_{i-1}}]].$$

Thus, the maps $\sigma : A_i' \to B_i$ are regular (cf. [Sta24]). Hence, the theorem above holds and there is a solution morphism $\rho : \mathbb{K}\langle x\rangle[y]/f \Rightarrow \mathbb{K}\langle x\rangle$ agreeing with $\widehat{\rho}$ up to an arbitrarily high degree such that $\rho(C_i) \subseteq A_i$, which means the solution $y(x)$ induced by the images of the y_i under ρ is nested. Note that in the convergent case the rings which would correspond to the C_i above must not be of finite representation and thus the proof does not work in this case.

7 Strong Artin approximation

When talking about the approximation property of rings, one should mention also the notion of the strong approximation property.

Theorem 19 (Strong Artin approximation [Art69])**.** *Let $f(x,y) \in \mathbb{K}\{x,y\}^r$ be a vector of convergent (or resp. algebraic) power series in two sets of variables x and y. Then there exists a function $\beta : \mathbb{N} \to \mathbb{N}$ such that for any $c \in \mathbb{N}$ and any solution $\widehat{y}$ of f at order $\beta(c)$, i.e.*

$$f(x, \widehat{y}(x) = 0) \mod (x)^{\beta},$$

there exists a convergent power series solution $y(x) \in \mathbb{K}\{x\}^m$ (or resp. $\mathbb{K}\langle x\rangle^m$),

$$f(x, y(x)) = 0,$$

which coincides with $\widehat{y}(x)$ up to degree c,

$$y(x) \equiv \widehat{y}(x) \mod (x)^{c+1}.$$

This theorem holds also in a much more general setting than just in power series rings over fields. [Gre66] showed that for any Henselian discrete valuation ring V such that $\mathrm{Quot}(V) \to \mathrm{Quot}(\widehat{V})$ is separable, the strong approximation property holds in $V[y]$. [Pop86] and [PP75] proved that also polynomial rings over complete local rings have the strong approximation property, while [Wav75] showed that the same holds for systems of analytic equations. [DL80] proved that Weierstrass systems also have the strong approximation property. For a proof of theorem 19 see [Art69] or [Ron18] pp.34.
We will only give a proof for the following theorem, telling that if we have a solution to an arbitrarily high degree, we have a solution in general. But first we need some new notions: Let $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_m)$ be two sets of multivariables, and let $G : \mathbb{C}[[x]]^p \to \mathbb{C}[[z]]^q$ be a map. Then G is called polyfinite if

1. G is continuous with respect to the Krull-topology and
2. for every $l \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that the restriction

$$G_{l,k} : \mathbb{C}[[x]]^p/(x)^{k+1} \to \mathbb{C}[[z]]^q/(z)^{l+1}$$

is a polynomial map.

Let $y = (y_1, \ldots, y_p) \in \mathbb{C}[[x]]^p$, where $y_i = \sum_{\alpha \in \mathbb{N}^n} y_{i,\alpha} x^i$. One can expand $G(y)$ as power series in z to get $G_j(y) = \sum_{\beta \geq 0} Gj, \beta(y) z^{\beta}$ for $j = 1, \ldots, q$. Then condition 2) implies that $G_{j,\beta}(y)$ is a polynomial in only finite many y_α.
For example, if $z = x$ and $n = m = 1$ the map $y(x) \mapsto g(x, y(x), y'(x), \ldots, y^{(d)}(x))$ with $g \in \mathbb{C}[x, Y_1, \ldots, Y_{d+1}]$ is a polyfinite map. This means the following theorem holds for more general system of equations then theorem 19, however it just tells us the existence of a solution, that does not have to approximate the solutions to a given degree.

Theorem 20. *Let $G : \mathbb{C}[[x]]^p \to \mathbb{C}[[z]]^q$ be a polyfinite map. Then there exists $h \in \mathbb{N}$ depending on G such that:*

if there is a solution $\widehat{y} \in \mathbb{C}[[x]]^p$ *of* $G(y) = 0$ *up to degree* $h+1$*, i.e.*

$$G(\widehat{y}) = 0 \mod (z)^{h+1},$$

then there already exists a solution $y \in \mathbb{C}[[x]]^p$ *with* $G(y) = 0$*.*

While there is a shorter proof of this theorem using non standarad algebra (cf. [AvdDvdH23]) we want to give a proof going back to [Mor01]. It requires the following theorems:

Lemma 7.1 (Chevalley)**.** *Let* $\alpha : X \to Y$ *be a dominant morphism of finite type of affine schemes* X *and* Y *over a field* $\mathbb{K}$*. Then* $\alpha(X)$ *contains a non empty open set of* Y*.*

A proof for this Lemma can be found in [Mat70].

Proof of theorem 20. Let $R = \mathbb{C}[[x]]^p$, $R_k = R/(x^{k+1})$, and $\pi_k : R \to R_k$ be the projection map. As G is continuous, there holds

$$y' - y \to 0 \Rightarrow G(y') - G(y) \to 0,$$

i.e. there exists a sequence $(m_k)_{k\in\mathbb{N}} \to \infty$ such that

$$y' = y \mod (x)^{k+1} \Rightarrow G(y') = G(y) \mod (z)^{m_k+1}.$$

Now let

$$V_k = \pi_k(\{y \in R \mid \operatorname{ord} G(y) \geq (m_k)\}).$$

Then $V_k \subseteq R_k$ is a Zariski-closed set as G is a polyfinite map.
Let $l \geq k$ be a natural number and $y \in R_l$ and let $\pi_{l,k} : R_l \to R_k$ be the projection map. Then $\pi_{l,k}(y) = \widetilde{y} \cong y \bmod (x^{k+1})$ and therefore $G(\widetilde{y}) \cong G(y) \bmod (z^{m_k+1})$. Now if $y \in V_r$, then $\operatorname{ord} G(y) \geq m_l \geq m_k$. Hence, the restriction $\pi_{l,k} : V_l \to V_k$ is well-defined. Therefore,

$$V_k \supseteq \overline{\pi_{k+1,k}(V_{k+1})} \supseteq \overline{\pi_{k+2,k}(V_{k+2})} \supseteq \dots$$

is a descending chain of closed sets in V_k. As the Z-topology on V_k is noetherian, there exists $s_k \in \mathbb{N}$ such that $\forall l \geq s_k : W_k := \overline{\pi_{s_k,k}(V_{s_k})} = \overline{\pi_{l,k}(V_l)}$.
Now let $l \geq k$ and $W_l = \overline{\pi_{l',l}(V_{l'})}$, with $l' \geq s_l, s_k$. Then

$$\overline{\pi_{l,k}(W_l)} = \overline{\pi_{l,k}(\overline{\pi_{l',l}(V_{l'})})} = \overline{\pi_{l',k}(V_{l'})} = W_k.$$

Hence, the restriction $\pi_{l,k} : W_l \to W_k$ is well-defined and dominant (has dense image).
Now look at the case $k = 0$ and choose $h \geq s_0$. By assumption, $\exists \widehat{y} \in R$ such that $G(\widehat{y}) = 0 \bmod (z^{h+1})$. I.e. $\operatorname{ord} G(\widehat{y}) \geq h+1 \geq s_0$. Hence, $\pi_{s_0}(\widehat{y}) \in V_{s_0}$ and therefore, $\pi_0(\widehat{y}) = \pi_{so,o}(\pi_{s_0}(\widehat{y})) \in W_0$. This shows that W_0 is non-empty and consequently all W_k are non-empty, as they have dense image in W_0.
Using lemma 7.1, the image of $\pi_{l,0} : W_l \to W_0$ contains an open set as it is dominant. As W_0 with the Zariski topology is a Bair-space, $U_0 = \bigcup_{l\geq 0} \pi_{l,0}(W_l)$ is non-empty and dense. If k is once again arbitrary, similarly for $l \geq s_k$ the V_i are closed and have dense image in W_k and once again by Chevalley's theorem $U_k = \bigcup_{l\geq s_k} \pi_{l,k}(W_l) \subseteq W_k$ is dense. Now let $l' \geq k$. We can assume that $s_{l'} \geq s_k$, and as $\pi_{l,k}(V_l)$ is a descending chain in W_k, one can rewrite $U_k = \bigcup_{l\geq s_{l'}} \pi_{l,k}(W_l) =$

$\bigcup_{l \geq s_{l'}} \pi_{l,l'}(\pi_{l',k}(W_l))$. Thus, the restriction $\pi_{l,k} : U_l \to U_k$ is onto, and hence, any element of U_k successfully lifts to an element of $U_l \subseteq V_l$ for all $L \geq k$. By construction of V_l, this limit is a power series and a solution as $m_l \to \infty$.

□

8 Application

8.1 Polynomial systems with one catalytic variable

This chapter will be about 2 applications of Popescu's nested approximation theorem. The first of these will be about showing algebraicity of certain generating functions in combinatorics. The notions will be in line with [BMJ05]. We begin this chapter with an easy example for motivation: Generalized Dyck paths. We want to count the number of walks with n steps consisting of $(1,1)$ and (1,-1) in $\mathbb{N}^2$. Let $a_{n,k}$ be the number of such walks with n steps ending in (n,k) and let $F(t,x) := a_{n,k}t^n x^k$ be its generating function. One easily sees that the generating function satisfies the following equation

$$F(t,x) = 1 + txF(t,x) + \frac{t}{x}\left(F(t,x) - F(t,0)\right), \tag{60}$$

One arrives at this formula at considering the last step of a walk. The one corresponds to the empty path, the second term corresponds to the case where the last step taken was $(1,1)$ and the third term corresponds to $(1,-1)$ taking in account that at this point k can not be zero. Furthermore, the case $k = 0$ is well known as the Dyck paths, for which it is known that $a_{2n,0}$ is the n-th catalan number. With similar argumentation, one gets that

$$F(t,0) = \frac{1-\sqrt{1-4t^2}}{2t^2},$$

which is clearly an algebraic function, as it satisfies the equation

$$4t^4F(t,0)^2 - 4t^2F(t,0) + 4t^2.$$

We are interested whether $F(t,x)$ is an algebraic power series as well.
We now consider the equation 60 but replace $F(t,0)$ with an arbitrary $F_1(t)$ and end up with

$$F(t,x) = 1 + txF(t,x) + \frac{t}{x}\left(F(t,x) - F_1(t)\right).$$

We can see that this already determines $F(t,x)$ uniquely as multiplying both sides with x and setting $x = 0$ we get that $F_1(t) = F(t,0)$. However, this means we have the algebraic power series

$$P(t,x,F,F_1) = x + tx^2F + x(F - F_1) - xF$$

with formal nested solution $F(t,x)$ and $F_1(t)$. Using Popescu's nested approximation theorem, we get that for any $c \in \mathbb{N}$ the existence of algebraic solutions ${}_cF(t,x), {}_cF_1(t)$ agreeing with F and F_1 up to degree c. However, as P uniquely determines F and F_1, we get that ${}_cF(t,x) = F(t,x)$ and ${}_cF_1(t) = F_1(t)$ and thus $F(t,x)$ is algebraic as well.
We now generalize the motivational example: We consider m-tuples of power series $F(t,x), F_1(t), \ldots, F_m(t)$ determined uniquely by a polynomial equation P in x and t. We call such systems a polynomial system in one catalytic variable x. Then we have the following:

Lemma 8.1 (Bousquet-Mélou, Jehanne)**.** *Let $P(t,x,F,F_1,\ldots,F_m)$ be a polynomial system in one catalytic variable x. Then $F(t,x)$ and $F_1(t),\ldots,F_k(t)$ are algebraic power series.*

The proof is analogous to the case of the generalized Dyck paths.

This is indeed a useful theorem as these polynomial systems in one catalytic variable arise in several cases in combinatorics, e.g. planar maps and quadrangular dissections of the disk (for details see [BMJ05] chapter 3). The proof given by Bousquet-Mélou and Jehanne actually does not use popescu's nested approximation theorem, and is therefore longer. However, it has the advantage of giving an explicit minimal polynomial for $F(t,x), F_1(t), \dots, F_m(t)$. As we always only have a single variable t and two nests, here one can actually use the simpler version of nested approximation stated in chapter 4. Furthermore, by nature of these generating functions, one usually has an equation of the form $x^d F(t,x) = P(t,x,F(t,x),F_1(t),\dots,F_m(t))$. Thus, the first derivative with respect to F contains the term x^d and is hence x-regular. Therfore, one does not even need the Neron-desingularisation.

8.2 Decomposition of algebraic space germs

A second application of nested approximation comes in the field of decomposition of analytic germs. It is in line with [Hau17] p.623. We call an analytic germ $(X,0)$ over a valued field $\mathbb{K}$ decomposable if it can be written as a Cartesian product $(X,0) = (Z_1,0)\times(Z_2,0)$ for some $(Z_1,0),(Z_2,0)$ positive dimensional analytic germs. We call $(X,0)$ incomposable if no such $(Z_1,0),(Z_2,0)$ exist. One can show that such decompositions are unique up to isomorphism (for more details, see [Hau90]).
We are now interested in the following question: Let $(X,0)$ be an analytic germ defined by algebraic power series and assume that $(X,0) = (Z_1,0)\times(Z_2,0)$ where $(Z_i,0)$ are formal spaces. Does there also exist a factorization of $(X,0)$ into a product defined by algebraic equations. The answer to this question is affirmative, and the proof will use nested approximation.
Let $(X,0)$ be defined by the algebraic equations, $f_1,\dots,f_r$ and let $J \subseteq \mathbb{K}\langle x\rangle$ be the according ideal. Further, let I be the completion of J in $\mathbb{K}[[x]]$. By assumption, we can separate x into two sets of variables x_1, x_2 such that there is a formal automorphism ϕ of $\mathbb{K}[[x]]$ with $\phi(I)(x) = I_1(x_1)+I_2(x_2)$ for some ideals $I_j \subseteq \mathbb{K}[[x_j]]$. We encode the components of $\phi(x_i)$ in a vector $\varphi(x)$. Then the above decomposition is equivalent to the existence of a $2r\times r$ matrix $A = (A_1,A_2) \in M_{2r,r}(\mathbb{K}[[x]])$ such that

$$A_1 \cdot f(\varphi(x)) = f(\varphi(x_1,0)),$$

$$A_1 \cdot f(\varphi(x)) = f(\varphi(x_1,0)).$$

Both these equations are examples of nested systems of equations (the first one is nested with $(x_1)\subseteq(x_1,x_2)$ and the second with $(x_2)\subseteq(x_1,x_2)$). As the f_i are algebraic, we can use Popescu's nested approximation theorem to get two automorphisms ψ_1,ψ_2 of $\mathbb{K}\langle x\rangle$ with corresponding vectors σ_1,σ_2 and two matrices $B_1,B_2 \in M_{r,r}(\mathbb{K}[[x]]$ such that

$$B_j \cdot f(\sigma(x)) = f(\sigma(x_j)).$$

Taking the inverses Ψ_j of ψ_j and writing them as $\Psi_j = (\Psi_{1j},\Psi_{2j})$ one can show that $(\Psi_{11},\Psi 22)$ is once again an automorphism with inverse ρ that satisfies

$$B_j \circ \Psi_j \circ \rho \cdot (f\circ\rho) = f\circ\sigma_j$$

and thus the claim follows as the σ_j are algebraic.

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