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On local and global aspects of the Langlands program

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Abstract

In this thesis we treat three topics situated in the Langlands program. Firstly, we prove rationality results for critical values of L-functions attached to representations in the residual spectrum of $\mathrm{GL}_4(\mathbb{A})$. We use the Jacquet-Langlands correspondence to describe their partial L-functions via cuspidal automorphic representations of the group $\mathrm{GL}_2'(\mathbb{A})$ over a quaternion algebra. Using methods inspired by existing results of Grobner and Raghuram we are then able to compute the critical values as a Shalika period up to a rational multiple. For the second topic, let $\mathbb{K}_v$ be a local non-archimedean field of residue characteristic p and $\overline{\mathbb{F}}_\ell$ an algebraic closure of a finite field of characteristic $\ell \neq p$. We extend the results of [82] concerning $\square$-irreducible representations of inner forms of $\mathrm{GL}_n(\mathbb{K}_v)$ to representations over $\overline{\mathbb{F}}_\ell$. As applications, we compute the Godement-Jacquet L-factor for any smooth irreducible representation over $\overline{\mathbb{F}}_\ell$ and show that the local factors of a representation agree with the ones of its C-parameter defined in [74]. Moreover, we reprove that the classification of irreducible representations via multisegments due to Vignéras and Mínguez-Sécherre is indeed exhaustive without using the results of [2]. We also characterize the irreducible constituents of certain parabolically induced representations, as was already done by Zelevinsky over $\mathbb{C}$. Finally, as the last result of the thesis, we prove a conjecture of Kudla and Rallis, see [72, Conjecture V.3.2]. Let χ be a unitary character, $s \in \mathbb{C}$ and W a symplectic vector space over a non-archimedean field with symmetry group $G(W)$. Denote by $I(\chi, s)$ the degenerate principal series representation of $G(W \oplus W)$. Pulling back $I(\chi, s)$ along the natural embedding $G(W) \times G(W) \hookrightarrow G(W \oplus W)$ gives a representation $I_{W,W}(\chi, s)$ of $G(W) \times G(W)$. Let π be an irreducible smooth complex representation of $G(W)$. We then prove

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \otimes \pi^\vee) = 1.$$

We moreover give analogous statements for W orthogonal or unitary. This gives in particular a new proof of the conservation relation of the local theta correspondence for symplectic-orthogonal and unitary dual pairs.

Zusammenfassung

In dieser Arbeit behandeln wir drei Themen, die im Langlands-Programm angesiedelt sind. Erstens beweisen wir neue Rationalitätresultate über kritischen Werte für L-Funktionen von Darstellungen von $\mathrm{GL}_4(\mathbb{A})$ im diskreten Spektrum. Wir verwenden die Jacquet-Langlands-Korrespondenz, um ihre partiellen L-Funktionen als L-Funktionen von automorphen Spitzenformen der Gruppe $\mathrm{GL}_2'(\mathbb{A})$ über einer Quaternionenalgebra zu beschreiben. Inspiriert von Resultaten von Grobner und Raghuram, können wir dann die kritischen Werte als Shalika-Periode bis auf ein rationalen Vielfaches berechnen. Für das zweite Thema sei $\mathbb{K}_v$ ein lokaler nicht-archimedischer Körper, dessen Restklassenkörper Charakteristik p hat und $\overline{\mathbb{F}}_\ell$ ein algebraischer Abschluss eines endlichen Feldes mit Charakteristik $\ell \neq p$. Wir verallgemeinern die Resultate von [82] bezüglich $\square$-irreduzibler Darstellungen von inneren Formen von $\mathrm{GL}_n(\mathbb{K}_v)$ zu Darstellungen über $\overline{\mathbb{F}}_\ell$. Wir wenden dies an um die Godement-Jacquet L-Faktoren für jede glatte irreduzible Darstellung über $\overline{\mathbb{F}}_\ell$ zu berechnen und zeigen, dass die lokalen Faktoren einer Darstellung mit denen ihres in [74] definierten C-Parameters übereinstimmen. Außerdem beweisen wir, dass die Klassifikation irreduzibler Darstellungen durch Multisegmente nach Vignéras und Mínguez-Sécherre tatsächlich surjektiv ist, ohne die Ergebnisse von [2] zu verwenden. Schließlich charakterisieren wir die irreduziblen Unterquotienten bestimmter parabolisch induzierter Darstellungen, wie dies bereits von Zelevinsky über $\mathbb{C}$ getan wurde. Schlussendlich beweisen wir eine Vermutung von Kudla und Rallis, siehe [72, Conjecture V.3.2]. Sei χ ein unitärer Charakter, $s \in \mathbb{C}$ und W ein symplektischer Vektorraum über einem nicht-archimedischen Körper mit Symmetriegruppe $G(W)$. Bezeichne mit $I(\chi, s)$ die degenerierte Hauptreihen-Darstellung von $G(W \oplus W)$. Schrenkt man $I(\chi, s)$ entlang der natürlichen Einbettung $G(W) \times G(W) \hookrightarrow G(W \oplus W)$ ein, so erhält man eine Darstellung $I_{W,W}(\chi, s)$ von $G(W) \times G(W)$. Sei π eine irreduzible glatte komplexe Darstellung von $G(W)$. Wir beweisen dann

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \times \pi^\vee) = 1.$$

Wir beweisen auch analoge Aussagen für Vektorräume W, die orthogonal oder unitär sind. Dies gibt insbesondere einen neuen Beweis der Konservierungsrelation der lokalen Thetakorrespondenz für symplektisch-orthogonale und unitäre duale Paare.

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Contents

Chapter 1

Introduction

What do three *Äpfel* and three *Birnen* have in common? Even without knowing any German, most people would agree that there is an equal amount of them, both sharing some incarnation of the number 3. A person familiar with the german variant of the idiom *apples and oranges* might even say that this is all they share and hence could, given some courage, define the number 3 precisely as the thing three *Äpfel* and three *Birnen* share. This observation elevates numbers from mere adjectives of objects to objects themselves. When and how humans made this discovery is unknown, however depending on the interpretation of archeological artifacts such as the *Ishango bone*, it would have been well more than 20000 years ago. As with everything else, humans were not satisfied by the pure existence of numbers, so they were swiftly given work. Through multiplication and addition we put them to use, to predict the stars, to tell our neighbour how many *Äpfel* he owes us, and plenty more useful things. But slowly the realization set in that numbers are not only useful but also surprisingly interesting in themselves, and thus some further investigation was considered necessary, an investigation which is still going on and whose end is not in sight.

1.1 Historical Overview

It was Euclid, who over 2300 years ago gave, among many other things, the study of numbers its logical framework, which nowadays is called the axiomatic method and is still in use to this day. By doing so, he was able to prove in Books VII-X of his *Elements* that the prime numbers hold the key to understanding many of mathematics's greatest secrets. Recall that a natural number is called *prime* if it is greater than 1 and only divisible by itself and 1. One of the most famous, and by many regarded as one of the most beautiful works of mathematics, can be found in Book IX of the *Elements*. In it, Euclid shows via a stunning argument that there have to exist infinitely many primes. A new age of mathematics has started, which promised deeper and deeper insights into number theory. However, over the next 2000 years, only incremental progress in the study of primes and numbers was made. Moreover, number theoretical problems like Fermat's last theorem seemed vehemently out of reach. Famously, Fermat claimed in 1637 to have a proof of his last theorem, *i.e.* showing that there exist no positive integers a, b, c and natural number $n > 2$ such that

$$a^n + b^n = c^n,$$

but did not publish his supposed proof. Although this statement by itself would be nothing more than a mathematical curiosity, it reminded us for many centuries how little we know about such seemingly easy problems. In 1994 the theorem was finally established as an almost inconsequential byproduct of the deep conjectures proven in [131]. Fermat's last theorem is maybe the most prominent example of a *diophantine equation*, a polynomial equation in several unknowns with integer coefficients. Whether or not a diophantine equation admits a solution is generally expected to be an unsolvable problem.

However, this does not mean that all hope is lost. Indeed, in the eighteenth century, a revolutionary idea emerged. Euler was the first to realize that certain problems about primes could be translated into an analytical property of a complex function. He achieves this by introducing the now-called Riemann-zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is a real number greater than 1. In 1737 he noticed in his work *Variae Observationes Circa Series Infinitas* that $\zeta(s)$ could be written as a product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

A new way of studying prime numbers was found, namely the above product formula shows that the infinitude of primes is implied by the existence of a pole of $\zeta(s)$ at $s = 1$, *i.e.* the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. As we will see this is no coincidence but rather the first instance of one of the great wonders of modern mathematics. It was slowly discovered that this principle extends far beyond the study of primes and today plays a crucial role in studying diophantine equations like the one in Fermat's last theorem.

This has already started to manifest itself in Euler's time. One of the first and deceptively simple-looking diophantine problems is the following: For which natural numbers m does the equation

$$x^2 = m$$

have an integer solution? Although the question seems rather trivial with the answer given tautologically by the square numbers, a lot of complexity arises at a more careful examination of the problem. Indeed, if the number m is truly enormous, like $m = 10^{100} + 1$, it is in practice not feasible to check whether m is a square or not by pure brute force computations. It thus may seem sensible to look at the solutions of $x^2 = m$ modulo some natural number q. If one can find a q such that

$$x^2 = m \mod q$$

admits no integer solution, one knows immediately that m is not a square number. For example, if we take as above $m = 10^{100} + 1$, one could take $q = 3$ and note that then $m = 2 \mod 3$. It is then not hard to check that 2 is not a square modulo 3, thus there exists no integer x with $x^2 = 10^{100} + 1$. We are thus led to the following definition. If p, q are natural numbers, we define the *Legendre symbol*

$$\left(\frac{q}{p}\right) = \begin{cases} 0 & \text{if } q = 0 \mod p, \\ 1 & \text{if there exists an integer } n \text{ with } n^2 = q \neq 0 \mod p, \\ -1 & \text{otherwise.} \end{cases}$$

Euler and Legendre were the first to conjecture the following.
Theorem 1 (Quadratic reciprocity law)**.** *Let p, q be odd prime numbers. Then*

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

At first sight, this is a striking statement since it is *a prior* not clear that $\left(\frac{q}{p}\right)$ and $\left(\frac{p}{q}\right)$ should be related at all. Moreover, quadratic reciprocity, together with some easier-to-prove properties of the Legendre symbol, allows one to give an algorithm to decide whether p is a square $\mod q$ for arbitrary natural numbers p, q. In 1801 Gauss gave the first and second proof in his *Disquisitiones Arithmeticae.* Throughout his life, he published another four, and two more were discovered after his death in his notes. Unfortunately, none of his proofs easily generalize to higher reciprocity laws, *i.e.* different equations than $x^2 = m$, and only 200 years later *Artin reciprocity*, the first instance of the *Langlands program*, was able to give a satisfactory explanation of quadratic reciprocity.

We now return to the Riemann-zeta function, as its story is far from over. In 1859 Riemann published his paper *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, where he shows how one could obtain many more interesting properties of the prime numbers by carefully studying the analytic properties $\zeta(s)$. In the same year, Hadamard and Poussin used this philosophy to prove the prime number theorem, stating that the number of primes lesser or equal to a real number x, denoted by $\pi(x)$, grows as $\frac{x}{\ln(x)}$, *i.e.*

$$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{\ln(x)}\right)} = 1.$$

Moreover, Riemann showed that $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$ and famously conjectured that the only zeros of $\zeta(s)$ have, apart from the trivial zeros

$$\{-2, -4, -6, \ldots\},$$

real part $\frac{1}{2}$. Finally, he proved the functional equation, for which it is helpful to introduce the completed version

$$\hat{\zeta}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

which satisfies

$$\hat{\zeta}(s) = \hat{\zeta}(1-s).$$

Here we denote as usual by $\Gamma(s)$ the Gamma-function. The factor $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$ might look artificial, however we will see later that it naturally emerges as the *archimedean contribution* to the Riemann-zeta function. Many mathematicians hoped now that attaching functions similar to the Riemann-zeta function to the many open problems of number theory would allow them to replicate its success story. This hope was realized gradually over the coming years by many of the best mathematicians, all of whom to list here would be impossible, so we will only mention those most important to the topics presented here.

At the beginning of the twentieth century, Artin established the aforementioned Artin reciprocity. It paved the way for several mathematicians, most notable Langlands, to initiate the Langlands program in the second half of the century, a far-reaching web of conjectures and theorems. The Langlands program has proven itself to be one of the most useful guiding principles in modern number theory. For an example of its success, one does not need to look further than one of the biggest achievements of modern mathematics, the proof of Fermat's last theorem by Wiles and Taylor. At the heart of the Langlands program lie so-called *L-functions*, which are the previously mentioned generalizations of the Riemann-zeta function. As the Riemann-zeta function, they are meromorphic functions, which admit a representation as a product and satisfy a functional equation. One can associate L-functions to many objects, not only of arithmetic nature. For example another source of L-functions comes from *automorphic representations*, objects closely tied to the subject of Harmonic analysis. The power of the Langlands program lies then in the fact that objects from different universes may have the same L-function. Thus one is equipped with the following powerful technique: Given a problem about, for example, a diophantine equation, one could hope to translate the problem into a problem of analytic nature about some L-function. Then one passes to Harmonic Analysis and picks the automorphic representation, whose L-function agrees with the first one. Reversing the process now allows us to transform the original problem into a problem on the automorphic side, where it will be hopefully easier to solve. Today, the Langlands program has grown much beyond what Langlands and his predecessors initially had in mind. Following the philosophy of Weil and inspired by developments in physics, the mathematical community found analogous conjectures and theorems in the world of geometry, which in turn fed back into the arithmetic, original Langlands program. Whether any of this answers the questions raised by our ancestors about their *Äpfel* and *Birnen* is not clear.

The goal of this thesis is not to build up to one final theorem but rather to offer the reader a selection of several phenomena occurring on the automorphic side of the Langlands program. We outline in Section 1.6, Section 1.7 and Section 1.8 our novel contributions to this topic. In Chapter 2, we recall the necessary notations, definitions, and preliminary results to finally present our results rigorously in Chapter 3, Chapter 4 and Chapter 5, and offer the respective proofs. These three chapters are based on preprints of the author of the thesis, namely [34], [35], and [36].

1.2 Artin and Dirichlet L-functions

We will now delve more deeply into the Langlands program and its conjectures. For $\mathbb{K}$ a field we denote by $\overline{\mathbb{K}}$ a separable closure of $\mathbb{K}$. At the starting point of our journey lies the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, or more generally $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ for a finite Galois extension $\mathbb{K}$ of $\mathbb{Q}$. Its importance lies in the fact that in many cases the study of arithmetical interesting problems leads naturally to a class of well-behaved representations of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ with coefficients over $\mathbb{C}$. The study of rational solutions of multivariate polynomials with coefficients in $\mathbb{K}$ is a prominent source of such representations, as the cohomology groups of the corresponding variety carry a natural $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$-action. Thus, understanding such representations becomes a central topic in modern number theory. As alluded to in the first part of this introduction, a natural approach is to associate to every well-behaved representation ρ of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ an L-function

$$L(s,\rho),$$

which was carried out by Artin in his three papers [4], [6], [7], where he already studied the first properties of these L-functions. The representations we will be interested in are the ones that are continuous with respect to the profinite topology, *i.e.* those, which can be obtained by pulling back a finite-dimensional $\mathbb{C}$-representation of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$, $\mathbb{L}/\mathbb{K}$ a finite Galois extension, to $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ via the canonical surjection

$$\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \twoheadrightarrow \mathrm{Gal}(\mathbb{L}/\mathbb{K}).$$

Motivated by the Riemann-zeta function, one would expect from such a L-function that firstly it is a meromorphic function on $\mathbb{C}$, secondly, $L(s,\rho)$ admits a representation as product of factors index by some set analogous to the primes for $\mathrm{Re}(s) >> 0$ and finally, it satisfies a functional equation. We will now briefly define Artin L-functions. Let $\mathbb{L}/\mathbb{K}$ be a finite Galois extension and $\rho\colon \mathrm{Gal}(\mathbb{L}/\mathbb{K}) \to \mathrm{GL}(V)$ a representation, where V is some finite dimensional $\mathbb{C}$-vector space. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$, let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_{\mathbb{L}}$ lying over a prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{K}}$ and set $k_{\mathfrak{p}} := \mathcal{O}_{\mathbb{K}}/\mathfrak{p}$. If $D_{\mathfrak{q}}$ is the group of elements in $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ preserving $\mathfrak{q}$, we obtain a surjective morphism

$$D_{\mathfrak{q}} \to \mathrm{Gal}(k_{\mathfrak{q}}/k_{\mathfrak{p}})$$

whose kernel is called the inertia subgroup $I_{\mathfrak{q}}$. Since $k_{\mathfrak{q}}$ is a finite field, one has a Frobenius morphism $\mathrm{Frob}_{\mathfrak{q}}$ in $\mathrm{Gal}(k_{\mathfrak{q}}/k_{\mathfrak{p}})$, whose inverse image in $D_{\mathfrak{q}}$ is denoted by $\sigma_{\mathfrak{q}}$. It follows that $\sigma_{\mathfrak{q}}$ defines a conjugacy class in $\mathrm{GL}(V^{I_{\mathfrak{q}}})$, where $V^{I_{\mathfrak{q}}}$ are the fixed vectors in V under the action of $I_{\mathfrak{q}}$. Thus the characteristic polynomial of $\rho(\sigma_{\mathfrak{q}})$, denoted by $P(\rho,t,\mathfrak{q})$ over $V^{I_{\mathfrak{p}}}$ is well defined. Artin then defines the local L-factor

$$L_{\mathfrak{q}}(s,\rho) := P(\rho, q_{\mathfrak{q}}^{-s}, \mathfrak{q})^{-1},\; q_{\mathfrak{q}} := \#k_{\mathfrak{q}}$$

and the L-function

$$L(s,\rho) := \prod_{\mathfrak{q}\,prime} L_{\mathfrak{q}}(s,\rho).$$

Since the product converges for s with real part large enough, this gives a good candidate for the desired L-function. If $\mathbb{L} = \mathbb{K} = \mathbb{Q}$ and ρ is the trivial representation, one obtains precisely the Riemann-zeta function. In order to be able to write the functional equation in a slightly more elegant fashion, one usually adds several factors consisting of Gamma-functions to $L(s,\rho)$ and thus obtaining a completed L-function $\Lambda(s,\rho)$. Now what about analytic continuation and the functional equation? To answer those questions, we need to go back a bit further in time. Dirichlet introduced in 1837 in his paper *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält* a certain class of meromorphic functions now called *Dirichlet L-functions*, which generalize the Riemann-zeta function *a priori* in a completely different direction than the Artin L-functions. Recall that a Dirichlet character is a multiplicative character $\chi\colon (\mathbb{Z}/m\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ for some $m \in \mathbb{N}$. Such a character can be extended by 0 to $\mathbb{Z}/m\mathbb{Z}$ and then pulled back to $\mathbb{Z}$, thus yielding a character $\chi\colon\mathbb{Z}\to\mathbb{C}^\times$, which we denote by the same symbol. To such χ one can associate the L-function

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \prod_{p \text{ prime}}\frac{1}{1-\chi(p)p^{-s}},\ s\in\mathbb{C}.$$

As the Riemman-zeta function, the series only converges for $\mathrm{Re}(s)>1$, however Dirichlet showed that $L(s,\chi)$ admits an analytic continuation and a functional equation, something which we still miss for Artin L-functions. If one could find for a Galois representation a Dirichlet character such that the respective L-functions agree, one would obtain those properties for free. Such a bold strategy was first carried out by Artin through his reciprocity law for one-dimensional representation. This relationship can be best formulated in the language of *adeles*, following the beautiful work of Tate [121].

1.2.1 Adeles Let $\mathbb{K}$ be a global field, *i.e.* a number field or the function field of a curve over a finite field. We will mostly focus on the case where $\mathbb{K}$ is a number field, however, most of what is said here can be straightforwardly extended to the function field case. We denote by $\mathcal{V}$ the set of places of $\mathbb{K}$, *i.e.* the set of equivalence classes of valuations, or equivalently of absolute values, of $\mathbb{K}$. We recall that an absolute value is a multiplicative function $|-|\colon\mathbb{K}\to\mathbb{R}_{\geq 0}$ such that

1. $|x|=0$ if and only if $x=0$,
2. $|x+y|\leq|x|+|y|$ for all $x,y\in\mathbb{K}$.

If the absolute value satisfies instead of (2) the stronger inequality

$$|x+y|\leq\max(|x|,|y|)$$

it is called *non-archimedean*. We call the non-archimedean places also finite places, denoted by $\mathcal{V}_f\subseteq\mathcal{V}$, and the infinite places we will refer to as the *archimedean places*, denoted by $\mathcal{V}_\infty$. There are always only a finite number of infinite places. If $\mathbb{K}=\mathbb{Q}$, $\mathcal{V}_f$ is the set of primes and $\mathcal{V}_\infty=\{\infty\}$. Here ∞ correspondence to the usual absolute value and a prime p to the valuation given by the order of divisibility by p. Completing $\mathbb{K}$ topologically with respect to the metric associated to the absolute value $v\in\mathcal{V}$ gives rise to fields $\mathbb{K}_v$. If $v\in\mathcal{V}_f$, we let $\mathcal{O}_v\subseteq\mathbb{K}_v$ be the ring of integers of $\mathbb{K}_v$, *i.e.* the set of elements of absolute value at most 1, $\mathfrak{p}_v$ the ideal in $\mathcal{O}_v$ of elements with absolute value strictly smaller than 1 and $k_v=\mathcal{O}_v/\mathfrak{p}_v$ the residue field, a finite field whose cardinality we denote by q_v. If $\mathbb{K}=\mathbb{Q}$, for a prime p, $\mathbb{K}_p=\mathbb{Q}_p$, $\mathcal{O}_p=\mathbb{Z}_p$, the p-adic integers and rationals, $\mathbb{K}_\infty=\mathbb{R}$ and $k_p=\mathbb{F}_p$. Moreover, if $v\in\mathcal{V}_\infty$, $\mathbb{K}_v\cong\mathbb{R}$ or $\mathbb{K}_v\cong\mathbb{C}$ for any global field $\mathbb{K}$. The respective places are called *real* and *imaginary* and if all infinite places of $\mathbb{K}$ are real, the number field is said to be *totally real*. From now on we mean by *almost all* all but finitely many. Then the ring of adeles is defined as

$$\mathbb{A}_\mathbb{K}=\{(a_v)\in\prod_{v\in\mathcal{V}}\mathbb{K}_v : a_v\in\mathcal{O}_v \text{ for almost all } v\in\mathcal{V}_f\}$$

and the ring of ideles $\mathbb{I}_\mathbb{K}=\mathbb{A}_\mathbb{K}^\times$ is the group of units in the adeles. Both $\mathbb{A}_\mathbb{K}$ and $\mathbb{I}_\mathbb{K}$ can be equipped with natural topologies giving them the structure of a locally compact topological ring respectively group. There are then natural inclusions

$$\mathbb{K}_v\hookrightarrow\mathbb{A}_\mathbb{K},\ \mathbb{K}\hookrightarrow\mathbb{A}_\mathbb{K},$$

all of which are continuous, $\mathbb{K}$ being equipped with the discrete topology, and the last one is given by diagonal inclusion. Moreover, the trace and norm map of a finite field extension $\mathbb{L}/\mathbb{K}$ can be extended to continuous maps

$$\mathrm{Tr}_{\mathbb{L}_w/\mathbb{K}_v}\colon \mathbb{L}_w \to \mathbb{K}_v,\ \mathrm{Nr}_{\mathbb{L}_w/\mathbb{K}_v}\colon \mathbb{L}_w \to \mathbb{K}_v,\ \mathrm{Tr}_{\mathbb{L}/\mathbb{K}}\colon \mathbb{A}_\mathbb{L} \to \mathbb{A}_\mathbb{K},\ \mathrm{Nr}_{\mathbb{L}/\mathbb{K}}\colon \mathbb{A}_\mathbb{L} \to \mathbb{A}_\mathbb{K}.$$

Here w is any place of $\mathbb{L}$ lying over v, *i.e.* the restriction of w to $\mathbb{K}$ equals v. Finally, the absolute values $|-|_v\colon \mathbb{K}_v \to \mathbb{R}_{\geq 0}$ for $v \in \mathcal{V}$ combine into an absolute value

$$|-|\colon \mathbb{A}_\mathbb{K} \to \mathbb{R}_{\geq 0},$$

with $|\mathbb{K}^\times| = 1$. For a group G we write G^{ab} for its abelinization of G.

Theorem 2 (Artin's reciprocity law, [5]). *Let $\mathbb{L}/\mathbb{K}$ be a Galois extension of two global fields. There exists a canonical isomorphism, called the* global symbol map*,*

$$\theta_{\mathbb{L}/\mathbb{K}}\colon \mathbb{K}^\times\backslash\mathbb{I}_\mathbb{K}/\mathrm{Nr}_{\mathbb{L}/\mathbb{K}}(\mathbb{I}_\mathbb{L}) \to \mathrm{Gal}(\mathbb{L}/\mathbb{K})^{ab}.$$

The Artin reciprocity law also has a "reverse"-direction.

Theorem 3 (Existence theorem). *If H is an open subgroup of $\mathbb{K}^\times\backslash\mathbb{I}_\mathbb{K}$ there exists a finite Galois extension $\mathbb{L}/\mathbb{K}$ such that $\mathrm{Nr}_{\mathbb{L}/\mathbb{K}}(\mathbb{I}_\mathbb{L}) = H$.*

Since the symbol maps for different Galois extensions behave nicely with each other one can summarize them as a continuous morphism

$$\theta_\mathbb{K}\colon \mathbb{K}^\times\backslash\mathbb{I}_\mathbb{K} \to \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})^{ab}.$$

Moreover, for v a place, the composition of $\theta_\mathbb{K}$ with the natural map $\mathbb{K}_v \to \mathbb{I}_\mathbb{K}/(\mathrm{Nr}_{\mathbb{L}/\mathbb{K}}(\mathbb{I}_\mathbb{K}))$ gives rise to an injective map, called the *local symbol map*,

$$\theta_v\colon \mathbb{K}_v^\times \hookrightarrow \mathrm{Gal}(\overline{\mathbb{K}_v}/\mathbb{K}_v).$$

Now this map is not an isomorphism but its image is a dense subgroup called the Weil group $\mathrm{W}_{\mathbb{K}_v}$, see [129]. It can be explicitly described as follows. For $v \in \mathcal{V}_f$ recall the surjective map $\mathrm{Gal}(\overline{\mathbb{K}_v}/\mathbb{K}_v) \twoheadrightarrow \mathrm{Gal}(\overline{k_v}/k_v) \cong \hat{\mathbb{Z}}$ whose kernel is called the inertia subgroup $I_{\mathbb{K}_v}$. A topological generator of $\mathrm{Gal}(\overline{k_v}/k_v)$ is given by the Frobenius and $\mathrm{W}_{\mathbb{K}_v}$ is then the inverse image of $\mathbb{Z} \subseteq \hat{\mathbb{Z}}$ in $\mathrm{Gal}(\overline{\mathbb{K}_v}/\mathbb{K}_v)$. If $v \in \mathcal{V}_\infty$, we define

$$\mathrm{W}_{\mathbb{K}_v} := \begin{cases} \mathbb{C}^\times & \text{if } \mathbb{K}_v \cong \mathbb{C}, \\ \mathbb{C}^\times \cup j\mathbb{C}^\times & \text{if } \mathbb{K}_v \cong \mathbb{R} \text{ and } j^2 = 1,\ jzj^{-1} = \overline{z}. \end{cases}$$

Now similarly to the local Weil group, there also exists a global Weil group $\mathrm{W}_\mathbb{K}$, which admits a continous map $\mathrm{W}_\mathbb{K} \to \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ with a dense image. We will not define it here, for a definition see for example [129], and just remark that it is uniquely determined by certain natural properties, for example it admits for each $v \in \mathcal{V}$ inclusions $\mathrm{W}_{\mathbb{K}_v} \hookrightarrow \mathrm{W}_\mathbb{K}$. Finally, Artin's reciprocity law gives an isomorphism

$$\mathbb{K}^\times\backslash\mathbb{I}_\mathbb{K} \to \mathrm{W}_\mathbb{K}^{ab}.$$

This is one of the reasons why it is more natural to work with Weil groups instead of the absolute Galois groups. They have several other advantages, but we will mention only one more. Namely, Weil groups are less sensitive to changes of the coefficient field of the representations we are interested in, *e.g.* from $\mathbb{C}$ to $\overline{\mathbb{Q}_\ell}$ for some prime ℓ. This is an important feature, especially if one is interested in Galois representations of geometric origin.

The second piece of the bridge between Dirichlet characters and certain Artin L-functions was best formalized by Tate in his thesis [121]. In it, he considers a Hecke-character, *i.e.* a continuous group homomorphism

$$\eta \colon \mathbb{I}_{\mathbb{K}} \to \mathbb{C}^\times$$

such that $\eta(\mathbb{K}^\times) = 1$, and associates to it its L-function $L(s, \eta)$. These characters and their L-functions were initially defined by [51]. As a special case, if $\mathbb{K} = \mathbb{Q}$ and χ is a Dirichlet-character, one can lift χ to a Hecke-character $\widetilde{\chi} \colon \mathbb{I}_{\mathbb{Q}} \to \mathbb{C}^\times$. On the other hand, every Hecke-character of finite order comes from a Dirichlet-character. The definition of $L(s, \eta)$ is as follows. Define for $v \in \mathcal{V}$ the space of Schwartz-functions

$$S(\mathbb{K}_v) := \begin{cases} \text{locally constant, compactly supported functions } \phi \colon \mathbb{K}_v \to \mathbb{C} & v \in \mathcal{V}_f, \\ \text{smooth, rapidly decreasing functions } \phi \colon \mathbb{K}_v \to \mathbb{C} & v \in \mathcal{V}_\infty. \end{cases}$$

We write η_v for the restriction of η to $\mathbb{K}_v^\times$. Next we fix for each $v \in \mathcal{V}$ a certain Haar-measure $\mathrm{d}x$ on $\mathbb{K}_v^\times$ and define for $\phi \in S(\mathbb{K}_v)$ and $s \in \mathbb{C}$ the integral

$$Z(\phi, \eta_v, s) := \int_{\mathbb{K}_v^\times} \eta_v(x) \phi(x) |x|^s \, \mathrm{d}x.$$

The main properties of these Zeta-factors are summarized as follows.

1. $Z(\phi, (\eta|\det|^t)_v, s) = Z(\phi, \eta_v, s + t)$ for $t \in \mathbb{R}$.
2. If η is unitary, $Z(\phi, \eta_v, s)$ converges absolutely for $\mathrm{Re}(s) > 0$.
3. If $v \in \mathcal{V}_f$, there exists a non-zero $P(\eta_v, s) \in \mathbb{C}[q_v^s, q_v^{-s}]$ such that

   $$P(\eta_v, s) Z(\phi, \eta_v, s) \in \mathbb{C}[q_v^s, q_v^{-s}]$$

   for all $\phi \in S(\mathbb{K}_v)$.
4. If $v \in \mathcal{V}_\infty$, there exists a non-zero meromorphic function $P(\eta_v, s)$ such that

   $$\frac{Z(\phi, \eta_v, s)}{P(\eta_v, s)} \in \mathbb{C}[s]$$

   for all $\phi \in S(\mathbb{K}_v)$.
5. In both cases one can choose $P(\eta_v, s)$ such that it is a $\mathbb{C}$-linear combination of $Z(\phi, \eta_v, s)'s$ and this fixes $P(\eta_v, s)$ up to a scalar.

We then normalize the $P(\eta_v, s)$ by demanding $P(\eta_v, 0) = 1$ and set

$$L(s, \eta_v) := P(\eta_v, s)^{-1}.$$

Finally, Tate defines

$$L_f(s, \eta) := \prod_{v \in \mathcal{V}_f} L(s, \eta_v), \; L(s, \eta) := \prod_{v \in \mathcal{V}} L(s, \eta_v)$$

and shows that these products converge for $\mathbb{R}(s) >> 0$, admit a meromorphic continuation and satisfy a functional equation. The advantages of going through all this work to define the L-factors are manifold, however, we will highlight at the moment only two. Firstly, the functional equation is essentially a consequence of replacing the Schwartz-function ϕ by its Fourier-transform $\hat{\phi}$ with respect to a suitable additive character and seeing how the integrals $Z(\phi, \eta, s)$ and $Z(\hat{\phi}, \eta^{-1}, 1-s)$ are related. Secondly, to prove analytic continuation, one can replace the local L-factors by a product of local Zeta-integrals of the form

$$Z(\phi, \eta, s) := \prod_{v \in \mathcal{V}} Z(\phi_v, \eta_v, s),$$

where for almost all v, ϕ_v is the characteristic function of $\mathcal{O}_v^\times$. Now this product can be written as an integral of a suitable function over $\mathbb{K}^\times \backslash \mathbb{A}_\mathbb{K}$ and a clever use of the Poisson formula shows that $Z(\phi, \eta, s)$ admits a natural analytic continuation. If χ is a Dirichlet-character with Hecke-lift $\widetilde{\chi}$, we also have

$$L(s, \chi) = L_f(s, \widetilde{\chi}).$$

Finally, with some extra work, one can show the following. Let $\mathfrak{D}_{\mathbb{K}/\mathbb{Q}}$ be the different of $\mathbb{K}$ and $C_\mathbb{K}^1 \subseteq \mathbb{K}^\times \backslash \mathbb{I}_\mathbb{K}$ the elements of absolute value 1.

Theorem 4 ([121])**.** *Let χ be a Hecke character. Then $L(s, \chi)$ is an entire function unless $\chi \cong |\det|^t$ for some $t \in \mathbb{C}$. In this case, the two poles are at $s = -t$ and $s = 1 - t$ with residues* $\mathrm{vol}(C_\mathbb{K}^1)$ *and* $\mathrm{vol}(C_\mathbb{K}^1)|\mathfrak{D}_{\mathbb{K}/\mathbb{Q}}|^{-\frac{1}{2}}$.

We will now combine this theory of L-functions of Hecke characters with the Artin reciprocity law and the Existence theorem. Note that the one-dimensional representations of a group G over $\mathbb{C}$ are in one-to-one correspondence with one-dimensional representations of G^{ab} and that in the case $\mathbb{K} = \mathbb{Q}$ the following goes back to the *Kronecker-Weber theorem*. We call a Hecke-character η algebraic if its restriction to $\mathbb{K}_v^\times$, $v \in \mathcal{V}_\infty$ is of the form $|\det|^a$, $a \in \mathbb{Z}$. Thus we obtained a canonical bijection

$$\begin{tikzcd}
\{\text{Hecke-characters of } \mathbb{A}_\mathbb{Q}\} \arrow[r, leftrightarrow, "\cong"] & \{\text{characters of } \mathrm{W}_\mathbb{Q}\} \\
\{\text{algebraic Hecke-characters}\} \arrow[u, hook] \arrow[r, leftrightarrow, "\cong"] & \{\text{characters of } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\} \arrow[u, hook] \\
\{\text{Dirichlet characters}\} \arrow[u, hook] \arrow[r, leftrightarrow, "\cong"] & \{\text{characters of finite order of } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\} \arrow[u, hook]
\end{tikzcd}$$

and similarly for other global fields $\mathbb{K}$. By carefully checking the construction of the local symbol maps, one even obtains that the above bijection respects the local L-factors of Hecke characters and the local Artin L-factors. We thus can find for every one-dimensional continuous representation ρ of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ a Hecke-character η of $\mathbb{A}_\mathbb{K}$ such that

$$L(s, \rho) = L_f(s, \eta),\ \Lambda(s, \rho) = L(s, \eta)$$

and hence we obtain for those representations the analytic continuation and functional equation of $L(s, \rho)$ for free from the respective properties of $L_f(s, \eta)$! Moreover, by Theorem 4 one also showed the one-dimensional case of the Artin conjecture, due to its name-sake.

Conjecture 1 (Artin conjecture)**.** *Let ρ be an irreducible, unitary, continuous representation of* $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. *Then $L(s, \rho)$ is entire if and only if $\rho \ncong 1$.*

If ρ is not a character but a more general representation, the conjecture and many other questions about $L(s, \rho)$ remain wide-open. It is therefore natural to look for the right generalizations of Hecke-characters. And indeed, a good candidate, namely *automorphic forms*, has been found. The first appearance of automorphic forms can be traced back to the famous theory of modular forms, which, as we will see, correspond to automorphic forms of GL_2.

1.3 Modular forms

The simplest incarnation of modular forms can be described as follows. Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half plane, which admits an action by $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

A modular form of weight $k \in \mathbb{Z}$ is then a holomorphic function

$$f : \mathfrak{h} \to \mathbb{C},$$

which firstly transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$ as follows

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = (cz + d)^k f(z).$$

Any such f can be expanded into a Fourier-series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, \; q = e^{2\pi i z},$$

since taking $a = b = d = 1$, $c = 0$ yields $f(z + 1) = f(z)$. The second condition a modular form has to satisfy is that

$$a_n = 0 \text{ if } n < 0.$$

It is called *cuspidal* if $a_n = 0$ for all $n \leq 0$, *i.e.* f is bounded at $i\infty$. To a cuspidal modular form, Hecke associates in [52] its L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which converges for $\mathrm{Re}(s) >> 0$, and its completed version

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f).$$

Crucially, the Mellin inversion formula allows Hecke to give an integral representation

$$\Lambda(s, f) = \int_0^\infty f(ix) x^s dx.$$

This in turn makes it possible to show that firstly, $\Lambda(s, f)$ admits an analytic continuation to an entire function on $\mathbb{C}$ and it satisfies a functional equation

$$\Lambda(s, f) = i^k \Lambda(k - s, f).$$

Hecke manages to prove even more. Let $\overline{a} = (a_1, a_2, \ldots)$ be a series of complex numbers growing not faster than n^k for some $k \in \mathbb{N}$ and set for all Dirichlet-characters $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$

$$L_\chi(s, \overline{a}) = \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^s}, \; \Lambda_\chi(s, \overline{a}) = \left(\frac{m}{2\pi}\right)^s \Gamma(s) L_\chi(s, \overline{a}).$$

Then $L_\chi(s, \overline{a})$ converges for $\mathrm{Re}(s) >> 0$ and if for all such χ, $\Lambda_\chi(s, \overline{a})$ admits an analytic continuation to an entire function and admits a functional equation,

$$f(z) := \sum_{n=1}^{\infty} a_n q^n, \; q = e^{i\pi z}$$

is a cuspidal modular form. This is the famous Converse Theorem of Hecke, see [52].
Remark. At this point, one cannot but mention the connection between modular forms and elliptic curves. Recall that an elliptic curve E over $\mathbb{Q}$ is a smooth projective curve of genus 1. By counting the number of solutions over a finite field

$$a_p := p + 1 - \#E(\mathbb{F}_p)$$

one can define an L-function $L(s, E)$ via an Euler product. It is the famous conjecture of Taniyama and Shimura, proved by Wiles in order to settle Fermat's last theorem, which states that $L(s, E)$ and suitable twists of it by Dirichlet characters χ admit an analytic continuation to an entire function and satisfy a functional equation as those of the L-functions of modular forms. But as we just saw, this implies that $L(s, E) = L(s, f)$ for some modular form f. The implications of such a result cannot be overstated. Furthermore, it suggests a natural extension of Artin L-functions to *motivic* L-functions, associated with Grothendieck's motifs, of which Galois representations are a special case. The L-functions of elliptic curves still hold many secrets. The most famous question of this sort is the Birch and Swinnerton-Dyer conjecture, which postulates that the order of vanishing of $L(s, E)$ at $s = 0$ is equal to the rank of the abelian group $E(\mathbb{Q})$. Moreover, the residue is conjectured to be the product of a rational number, which contains valuable arithmetic information about E, and a period, which supposedly stores transcendental information. We will at a later point offer a viewpoint on this period from the automorphic side.

1.4 Automorphic forms

We now make the conceptual leap from modular forms to automorphic representations, as was promised earlier. Standard references are [20] and [21]. For the moment we will just focus on the theory of GL_n. Let $\mathbb{K}$ be a number field and $v \in \mathcal{V}$. We set

$$K_v := \begin{cases} \mathrm{GL}_n(\mathcal{O}_v) & \text{if } v \in \mathcal{V}_f, \\ \mathrm{SO}_n(\mathbb{R}) & \text{if } \mathbb{K}_v \cong \mathbb{R}, \\ \mathrm{U}_n(\mathbb{R}) & \text{if } \mathbb{K}_v \cong \mathbb{C}. \end{cases}$$

We set $K_\infty := \prod_{v \in \mathcal{V}_\infty} K_v$. As for the adeles, we define

$$\mathrm{GL}_n(\mathbb{A}_\mathbb{K}) := \{(a_v) \in \prod_{v \in \mathcal{V}} \mathrm{GL}_n(\mathbb{K}_v) : a_v \in K_v \text{ for almost all } v \in \mathcal{V}_f\},$$

$$\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) := \{(a_v) \in \prod_{v\in\mathcal{V}_f} \mathrm{GL}_n(\mathbb{K}_v) : a_v \in K_v \text{ for almost all } v \in \mathcal{V}_f\},$$

$$\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},\infty}) := \prod_{v\in\mathcal{V}_\infty} \mathrm{GL}_n(\mathbb{K}_v)$$

and we can equip it with a natural topology making it a locally compact, topological group. We have natural inclusions

$$\mathrm{GL}_n(\mathbb{K}_v) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}}),\ \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}}),\ \mathrm{GL}_n(\mathbb{K}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}}).$$

For $g \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})$, we define

$$||g||:= \prod_{v\in\mathcal{V}} \max(|(g_v)_{i,j}|_v, |\det(g_v)|_v^{-1}|).$$

Let ω be a unitary Hecke character, $\mathfrak{g}$ the complexified Lie algebra of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},\infty})$ and $Z_n \subseteq \mathrm{GL}_n$ the center consisting of diagonal matrices of the form $\mathrm{diag}(z,\dots,z)$. An automorphic form with central character ω is a continuous function

$$\varphi\colon \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}}) \to \mathbb{C}$$

satisfying the following.

1. φ is $\mathrm{GL}_n(\mathbb{K})$-invariant.
2. $\varphi(zg) = \omega(z)f(g)$ for $\mathrm{diag}(z,\dots,z) \in Z_n(\mathbb{A}_{\mathbb{K}})$.
3. For each $g_f \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})$ the function
   $$g_\infty \mapsto \varphi(g_\infty g_f),\ g_\infty \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},\infty})$$
   is smooth.
4. There exists an open compact subgroup $K \subseteq \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})$ fixing φ under right translation.
5. The space spanned by K_∞-right translates of φ is finite dimensional.
6. There exists a constant C and $N \in \mathbb{Z}$ such that for all $g \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})$, $|\varphi(g)|\le C||g||^N$.
7. Let $Z(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$. Then there exists a natural action of $U(\mathfrak{g})$, obtained by deriving the action by right-translation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},\infty})$. We ask of φ that the space generated by $Z(\mathfrak{g})\varphi$ is finite-dimensional.

At first sight, this definition may seem like a mouthful, however these properties have direct analoga in modular forms. In particular, each modular form can be lifted to an automorphic form of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. If ϕ is such an automorphic form coming from a modular form, property (5) encodes then the weight of the modular form, (6) that the Fourier coefficients a_n vanish for $n < 0$ and property (7) is a consequence of the holomorphicity of the modular form. The first four properties determine then more or less how a modular form is supposed to be lifted to an automorphic form. The space of such automorphic forms is denoted by $\mathcal{A}(\mathrm{GL}_n(\mathbb{A}_{\mathbb{K}}),\omega)$. If $\alpha = (\alpha_1,\dots,\alpha_k)$, $\alpha_i \in \mathbb{Z}_{>0}$ is a composition of n, *i.e.* $\sum_{i=1}^k \alpha_i = n$, we let U_α be the subgroup of GL_n consisting of matrices of the form

$$\begin{pmatrix} 1_{\alpha_1} & * & * & \dots & * \\ 0 & 1_{\alpha_2} & * & \dots & * \\ 0 & 0 & \ddots & \dots & * \\ 0 & 0 & 0 & 1_{\alpha_{k-1}} & * \\ 0 & 0 & 0 & 0 & 1_{\alpha_k} \end{pmatrix},$$

where 1_m denotes the identity matrix in GL_m. We then say an automorphic form φ is cuspidal if

$$\int_{U_\alpha(\mathbb{K})\backslash U_\alpha(\mathbb{A}_\mathbb{K})} \varphi(n)\,\mathrm{d}n = 0$$

for all non-trivial compositions of n and some non-trivial Haar-measure $\mathrm{d}n$ on $U_\alpha(\mathbb{K})\backslash U_\alpha(\mathbb{A}_\mathbb{K})$. As the name suggests, the automorphic form associated to a modular form is cuspidal if and only if the modular form is cuspidal. The subspace of cuspidal automorphic forms is then denoted by

$$\mathcal{A}_0(\mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega) \subseteq \mathcal{A}(\mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega)$$

and the automorphic form corresponding to a cuspidal modular form is cuspidal. Finally, if ϕ is a automorphic form and

$$\int_{Z_n(\mathbb{A}_\mathbb{K})\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_\mathbb{K})} |\phi(n)|^2\,\mathrm{d}n < \infty,$$

we call it square-integrable. The completion of the space of square-integrable automorphic forms with respect to the L^2-norm is denoted

$$L^2(\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega)$$

and every cuspidal automorphic form is square-integrable. The space

$$L^2(\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega)$$

is preserved under right translation by $\mathrm{GL}_n(\mathbb{A}_\mathbb{K})$ and an irreducible subrepresentation of it is called a *discrete series representation*. The same cannot be said of the space $\mathcal{A}(\mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega)$, it is only preserved under right translation by K_∞, $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})$ and the induced action of $\mathfrak{g}$. We thus define an (irreducible) *automorphic representation* as an irreducible $(\mathfrak{g}, K_\infty, \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}))$-subquotient of $\mathcal{A}(\mathrm{GL}_n(\mathbb{A}_\mathbb{K}),\omega)$. For $v \in \mathcal{V}_f$ we call a representation (π_v, V) of $\mathrm{GL}_n(\mathbb{K}_v)$ *smooth* if every vector $\xi \in V$ is fixed by an open, compact subgroup and if moreover for every open compact subgroup $K \subseteq \mathrm{GL}_n(\mathbb{K}_v)$ the space of K-fixed vectors V^K is finite-dimensional, we call it *admissible*. This is equivalent to the statement that for any irreducible representation ρ_v of K_v, the ρ_v-isotypic part of π is finite-dimensional. In particular, one can show that every irreducible smooth representation is admissible. One can define admissibility similarly for $(\mathfrak{g}, K_\infty)$-modules and $(\mathfrak{g}, K_\infty, \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}))$-modules. If $v \in \mathcal{V}_f$, we call an admissible representation π_v of $\mathrm{GL}_n(\mathbb{K}_v)$ *unramified* if the space of K_v-fixed vectors is one-dimensional.

Let $\{\pi_v, v \in \mathcal{V}_f\}$ be a set of irreducible admissible representations of which almost all are unramified. If π_v is unramified, we fix a K_v-fixed vector ξ_v^0 and recall the restricted tensor-product

$$\bigotimes_{v\in\mathcal{V}_f}{}' \pi_v,$$

i.e. the subspace of the usual tensor product spanned by

$$\{\bigotimes_{v\in\mathcal{V}_f} \xi_v : \xi_v \in \mathcal{V}_v \text{ and for almost all } v,\ \xi_v = \xi_v^0\}.$$

Theorem 5 (Flath's theorem, [38]). *For any admissible, irreducible* $(\mathfrak{g}, K_\infty, \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}))$*-module* Π *there exists an irreducible* $(\mathfrak{g}, K_\infty)$*-module* π_∞ *and a family of representations* $\{\pi_v, v \in \mathcal{V}_f\}$ *as above such that*

$$\pi_\infty \otimes {\bigotimes_{v \in \mathcal{V}_f}}' \pi_v \cong \Pi$$

are isomorphic as $(\mathfrak{g}, K_\infty, \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}))$*-modules.*

Theorem 6 ([49]). *If* Π *is an irreducible automorphic representation, it is an admissible* $(\mathfrak{g}, K_\infty, \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}))$*-module.*

Going back to our prime example of automorphic forms, modular forms, the admissibility of an automorphic representation associated to a modular form is connected to the fact that the vector space of modular forms of a given weight is finite-dimensional.

This allows one to reduce many problems of automorphic representations from the global to the local level. For example, we have the following multiplicity one statement.

Theorem 7 (Strong multiplicity one, [104],[62],[61]). *Let* Π *and* Π' *be automorphic representations of* $\mathrm{GL}_n(\mathbb{A}_\mathbb{K})$ *and denote the corresponding local representations by* π_v, π'_v, $v \in \mathcal{V}$. *If* Π *and* Π' *are discrete series representation and for almost all* $v \in \mathcal{V}_f$, $\pi_v \cong \pi'_v$, *then* $\Pi \cong \Pi'$.

Crucially, the proof of this theorem given in [62] required a good notion of automorphic (Rankin-Selberg) L-functions.

We are now in a position to define local and global L-functions of $\mathrm{GL}_n(\mathbb{A}_\mathbb{K})$. For $n = 1$ this was first done in Tate's thesis and, for $n \geq 2$, Tate's approach was generalized in [44]. Indeed, for a smooth irreducible representation π_v of $\mathrm{GL}_n(\mathbb{K}_v)$ one can define as Tate in his thesis local L-factors by defining $L(s, \pi_v)$ to be the generator of a fractional ideal $\mathcal{L}(\pi_v) \subseteq \mathbb{C}(q^{s+\frac{1-n}{2}})$ normalized by the condition $L(0, \pi_v) = 1$. The ideal $\mathcal{L}(\pi_v)$ is in turn spanned by Zeta-integrals very similar to the ones we already saw in the discussions about Tate's thesis. Analogously to the situation there, it is not much more work to show that the local L-factors admit a functional equation. This matter will be discussed in much more detail later in this thesis. That being said, we can now define for an automorphic representation

$$L_f(s, \Pi) = \prod_{v \in \mathcal{V}_f} L(s, \pi_v).$$

As it turns out, this product only converges in general if $\mathrm{Re}(s) >> 0$ and Π is a discrete series and in particular, if Π is cuspidal. This is done by finding a representation of $L_f(s, \Pi)$ as a certain integral. Then one can again analogously to the case of Hecke-characters show that $L_f(s, \Pi)$ can be completed to an L-function $L(s, \Pi) = L(s, \pi_\infty) L_f(s, \Pi)$ by also accounting for the archimedean part and that $L(s, \Pi)$ admits an meromorphic continuation and admits a functional equation

$$L(s, \Pi) = \epsilon(s, \Pi, \psi) L(1 - s, \Pi^\vee),$$

where $\Pi^\vee$ is the dual representation of Π, ϵ is an entire function, depending on a chosen continuous, nontrivial additive character $\psi: \mathbb{K} \backslash \mathbb{A}_\mathbb{K} \to \mathbb{C}$. Moreover, one can show the following.

Theorem 8 ([44]). *Let* Π *be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_\mathbb{K})$. *Then* $L(s, \Pi)$ *is entire if and only if* $\Pi \ncong |-|^{is}$ *for some* $s \in \mathbb{R}$.

Of course, it is no coincidence that this is reminiscent of Artin's conjecture. This brings us to some of the cornerstones of the Langlands program.

1.5 Langlands correspondence

Since Artin's reciprocity law gave such an elegant way of proving Artin's conjecture and understanding Artin L-functions of one-dimensional Galois representations, we will now briefly describe its conjectural extension to arbitrary irreducible Galois representations. We will restrict ourselves to the case where $\mathbb{K}$ is a number field and start with the local side of the story. We will not be fully precise in this section, as stating all conjectures and definitions would go beyond this introduction. In particular, we will not discuss functoriality.

1.5.1 Local Langlands correspondence Let $v \in \mathcal{V}$, which will assume for simplicity to be non-archimedean. Recall that there exists a continuous map $\mathrm{W}_{\mathbb{K}_v} \hookrightarrow \mathrm{Gal}(\overline{\mathbb{K}_v}/\mathbb{K}_v)$ with dense image. We let ν_v be the unique character of $\mathrm{W}_{\mathbb{K}_v}$ trivial on $I_{\mathbb{K}_v}$ sending a lift of the Frobenius to q_v^{-1}. A Deligne representation of $\mathrm{W}_{\mathbb{K}_v}$ is a pair (Φ, U), where Φ is a finite-dimensional representation of $\mathrm{W}_{\mathbb{K}_v}$ such that an open subgroup acts trivially and U an element in $\mathrm{Ho}_{\mathrm{W}_{\mathbb{K}_v}}(\nu_v\Phi, \Phi)$, [31], [111], [120]. We call (Φ, U) semi-simple if Φ is semi-simple and let the length of (Φ, U) be the dimension of Φ. As for usual Galois representations, one can define the local L-factor of a Deligne representation by

$$P(T, \Phi, U) \coloneqq \det((1 - T\Phi(\sigma_v)\big|_{\mathrm{Ker}(U)^{I_{\mathbb{K}_v}}}), \; L(s, \Phi, U) \coloneqq P(q_v^{-s}, \Phi, U),$$

where σ_v is an element in the inverse image of the Frobenius in $\mathrm{W}_{\mathbb{K}_v}$. The following correspondence was first conjectured in [76] and formulated in its modern form in [19].

Theorem 9 (Local Langlands correspondence)**.** *There exists a bijection*

$$\begin{array}{c} \{\textit{semi-simple Deligne representations of length } n\} \\ \updownarrow \\ \{\textit{irreducible smooth representations of } \mathrm{GL}_n(\mathbb{K}_v)\} \end{array},$$

which respects the local L-factors on both sides. Moreover, if $n = 1$ *this is nothing more than the local symbol map.*

For a proof see [53], [50] and [113] if the characteristic of $\mathbb{K}_v$ is 0, and [86] for the non-zero case. If $\mathbb{K}_v$ is archimedean the proof is due to Langlands in [80]. Note that a Deligne representation is nothing but a representation of the *Weil-Deligne group* $\mathrm{W}'_{\mathbb{K}_v} \coloneqq \mathrm{W}_{\mathbb{K}_v} \ltimes \mathbb{K}_v$, where $\mathrm{W}_{\mathbb{K}_v}$ acts on $\mathbb{K}_v$ by

$$wxw^{-1} = \nu_v(w)x, \; x \in \mathbb{K}_v.$$

1.5.2 Global Langlands correspondence In the global setting, a precise conjectural statement as in the local case does not yet exist. The main problem is that there is currently no global analogon of the Weil-Deligne group known, there exist only conjectural definitions of such a group, which is usually denoted by $L_{\mathbb{K}}$, see [77], [110]. One expects that it is an extension of $\mathrm{W}_{\mathbb{K}}$ by a compact group and a possible construction has been proposed in [3]. We assume for a moment the existence of $L_{\mathbb{K}}$.

Conjecture 2 (Langlands reciprocity)**.** *There exists a bijection*

$$\begin{array}{c} \{\textit{irreducible, } n\textit{-dimensional representations of } L_{\mathbb{K}}\} \\ \updownarrow \\ \{\textit{cuspidal automorphic representations of } \mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})\}. \end{array}$$

This bijection, among other things, respects the L-factors, behaves well with respect to passing to representations of $\mathrm{W}'_{\mathbb{K}_v}$ *upstairs and to the local components* π_v *downstairs, and if* $n = 1$ *it specializes to Artin reciprocity.*

Moreover, one can define a certain condition on the infinite part π_∞ of an automorphic representation called *algebraicity.* Then such algebraic automorphic representations should correspond to the motivic Galois group, whose existence is also conjectural. If $n = 2$ the automorphic forms corresponding to cuspidal modular forms are the algebraic cuspidal representations.

Some progress has been made on the above conjecture. As we saw before, the case $n = 1$ is completely settled by Artin reciprocity. Moreover, for certain automorphic representations one can already construct a corresponding representation upstairs, see for example [30] or [27]. On the other hand, the passage from downstairs to upstairs has been achieved see for example [79] in the special cases if $n = 2$, [122], or if $\mathbb{K} = \mathbb{Q}$ and $n = 2$ [131] and [23].

We finish this section with a small remark on the case where $\mathbb{K}$ is not a number field but rather the function field of a curve over a finite field. In this case, much more progress was made. In [33], for the case $n = 2$, and in [75], for the case $n \geq 3$ the authors managed to show that there exists a bijection between cuspidal automorphic forms of $\mathrm{GL}_n(\mathbb{A}_\mathbb{K})$ and n-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ which are unramified outside a finite set of places. Moreover, said bijection plays well with the passing to the local representations and L-factors. What differentiates the function field case from the number field case is that one can reinterpret both sides of the bijection as certain constructions in the world of algebraic geometry. To be more precise, the field $\mathbb{K}$ corresponds to a curve X and by considering certain sheaves on the stack of vector bundles over X, one could use geometric methods to prove the conjecture. This in turn has lead to several attempts of repeating the same success story for number fields. However, it is not clear *a priori* what the right replacement of the curve X in the arithmetic context should be. In recent years, several people, see for example [37], offered a candidate for such a theory and managed to *geometrize* the local Langlands correspondence. This is surprising in the sense that the function field case and the number field case are usually expected to behave according to similar philosophies, however in this case, it is the function field case and the case of local, non-archimedean fields of characteristic 0 that have similar features.

Having now presented the Langlands program in very rough strokes, we will in the next three sections describe the problems treated in this thesis.

1.6 Critical values of L-functions

One of the expected properties of L-functions that we have not yet mentioned is that they have *critical values*, which store interesting information about the underlying object. The first instance of this was the Basel problem, which asked what the precise value of the series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is. This was answered by Euler in 1735 in his work *De Summis Serierum Reciprocarum*, where he showed that it equals to $\frac{\pi^2}{6}$. More generally, one can show that for k a positive integer

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!}B_{2k},$$

where B_n is the n-th (positive) Bernoulli-number, *i.e.*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

The functional equation then gives an explicit description of the Riemann-zeta function at negative odd integers. More generally, if Π is a discrete series representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})$, we recall that $L(s,\Pi)$ satisfies a functional equation

$$L(s,\Pi) = \epsilon(s,\Pi)L(1-s,\Pi^\vee).$$

We then call $s_0 \in \frac{n-1}{2} + \mathbb{Z}$ a critical value if both $L(s,\pi_\infty)$ and $L(1-s,\pi_\infty^\vee)$ do not have a pole at $s = s_0$. The expectation is that if s_0 is a critical value, then $L(s_0,\Pi)$ is of the form $\alpha \cdot \omega$, where α is a rational number carrying some form of *arithmetical* information about Π and ω is a period, which carries *transcendental* information about Π. For a more detailed survey see for example [70]. Recall that a period is an integral of an algebraic function over an algebraic domain (over $\mathbb{K}$). For the Riemann-zeta function, we recall that its completed version is of the form

$$\hat{\zeta}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and has the functional equation $\hat{\zeta}(s) = \hat{\zeta}(1-s)$. Since $\zeta(s)$ has no pole except at 1 and the poles of Γ are at the non-positive integers, it follows that the critical values of $\hat{\zeta}$ are precisely the even non-negative and odd negative integers. The above conjecture has its root in the theory of L-functions attached to motives, where it is now known as Beilinson's conjectures, see [15] and builds on conjectures of Deligne, see [32]. We will not state them here as this would go far beyond the scope of this introduction. Let us only remark on the following. Firstly, Beilinson's conjectures not only give a conjectural value for critical values but for all $s_0 \in \frac{n-1}{2} + \mathbb{Z}$. Moreover, if the L-function vanishes at s_0 one can also describe the order of vanishing of L and its residue in terms of the underlying motif. As was already mentioned, the most famous case of these conjectures is the BSD-conjecture for elliptic curves. The period of the corresponding L-function is given by an elliptic integral multiplied by a regulator, an invariant defined using the rational points of the elliptic curve.

By the philosophy of Langlands, we are thus naturally led to look for similar phenomena in the world of automorphic forms. Here one uses an explicit description of an automorphic representation, for example, the *Whittaker* or the *Shalika model* and we will focus on the latter. Moreover, we will only be able to express the critical value of the L-function up to a multiple in a number field, hence we can only pin down the period-part of it. Let us first recall the already made progress on problems of this kind before we describe in more detail what the new contribution of the thesis will be.

The authors of [90] and [114] were the first to study the properties of critical values of L-functions associated to modular forms, which was generalized to cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{K}})$ in [48] and [55]. The critical restriction, which also will be present in this thesis, is that one only can consider automorphic representations which are *cohomological*, which, loosely speaking, means that the automorphic representation can be detected in the cohomology of a certain locally symmetric space. More concretely, this boils down to the following definition. If Π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})$, we call it cohomological if the relative Lie-algebra cohomology

$$H^*(\mathfrak{g}, K_\infty, \pi_\infty \otimes E_\mu)$$

does not vanish for some highest weight representation E_μ. The specific model of the representation enters in the explicit description of the period. Namely, the period arises by comparing two different *rational structures* of the representations, one that can be associated to the model and one that comes from its cohomological interpretation. This has been pursued by [68], [89], [108], [109] in the case of Whittaker models and by [47] in the case of Shalika models, which will also be the model considered in this thesis. More specifically, the author considers automorphic cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}})$, where $\mathbb{K}$ is a totally real number field. Let

$$\mathcal{S} \coloneqq \Delta\mathrm{GL}_n \rtimes U_{(n,n)} = \left\{ s = \begin{pmatrix} h & X \\ 0 & h \end{pmatrix} : \, h \in \mathrm{GL}_n, \, X \in M_{n,n} \right\}$$

be the Shalika subgroup of GL_{2n}. We say that an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}})$ with central character ω admits a Shalika model with respect to a Hecke-character η and additive character ψ, if $\eta^n = \omega$ and if the Shalika period

$$g \mapsto \mathcal{S}_\psi^\eta(\phi)(g) \coloneqq \int_{Z_{2n}(\mathbb{A}_{\mathbb{K}})\mathcal{S}(\mathbb{K})\backslash\mathcal{S}(\mathbb{A}_{\mathbb{K}})} \phi(sg)\,\psi(\mathrm{tr}(X))^{-1}\,\eta(\det(h))^{-1}\,\mathrm{d}s,$$

for some $g \in \mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}})$ and a cusp form ϕ of Π, does not vanish. This gives an inclusion into the space of Shalika-functionals $\Pi \hookrightarrow \mathrm{ind}_{\mathcal{S}(\mathbb{A}_{\mathbb{K}}}^{\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}})}(\psi \otimes \eta)$, *i.e.* the space of functions $f\colon \mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}}) \to \mathbb{C}$ satisfying

$$f(sg) = \psi(\mathrm{tr}(X))\,\eta(\det(h))\,f(g) \text{ for all } g \in \mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}}),\ s \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})$$

and some smoothness condition, on which $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{K}})$ acts by right translation. The image of this inclusion is denoted by $\mathcal{S}_\psi^\eta(\Pi)$. The reason why one can then compare the so obtained Shalika-period with critical values of L-functions is the Shalika-zeta integral, see [39],

$$\zeta(s,\phi) \coloneqq \int_{\mathrm{GL}_n(\mathbb{A}_{\mathbb{K}})} \mathcal{S}_\psi^\eta(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det(g_1)|^{s-\frac{1}{2}}\,\mathrm{d}g_1.$$

In the Shalika-model there exists a special vector in $\mathcal{S}_\psi^\eta(\Pi)$, such that the Shalika-zeta integral computes the partial L-function $L^S(s,\Pi) = \prod_{v\in\mathcal{V}\smallsetminus S} L(s,\pi_v)$ for a finite set $S \subseteq \mathcal{V}$. Now one can reinterpret the Shalika-zeta integral as an instance of Poincaré duality and hence the claim that the critical values are the Shalika-period up to some rational multiple follows from the algebraic nature of Poincaré duality.

In the first chapter of this thesis we combine the above ideas together with the Jacquet-Langlands correspondence, see [11], [13], to obtain new results on critical values of discrete series. To do so we fix again a totally real field $\mathbb{K}$ and a central division algebra of degree 2 over $\mathbb{K}$. We then replace the group GL_n by GL'_n, the invertible matrices over $\mathbb{D}$. Moreover, one can extend the definition of a Shalika model analogously to cuspidal automorphic representations of $\mathrm{GL}'_{2n}(\mathbb{A}_\mathbb{K})$. Due to some technical restrictions we are at the moment only able to treat the case of cuspidal automorphic representations of $\mathrm{GL}'_2(\mathbb{A}_\mathbb{K})$. However there we can show via similar arguments to [47] and using the results of [46] that the critical values of a cuspidal automorphic representations of $\mathrm{GL}'_2(\mathbb{A}_\mathbb{K})$ are, up to a rational multiple, the Shalika-period. Finally, the Jacquet-Langlands correspondence gives a natural correspondence between discrete series representation of $\mathrm{GL}'_2(\mathbb{A}_\mathbb{K})$ and discrete series representations of $\mathrm{GL}_4(\mathbb{A}_\mathbb{K})$, which respects the L-functions. This in turn gives the desired result.

1.7 Modular representations

In the second section of the thesis we focus on the local representation theory of GL_n. We thus fix a local non-archimedean field $\mathbb{K}_v$ with residue characteristic p.

The classification of irreducible, smooth admissible representations of $\mathrm{GL}_n(\mathbb{K}_v)$ over an algebraically closed field R is of great importance in the Langlands program. This has been achieved by Bernstein and Zelevinsky in [132], [16], [17] for the case $R = \mathbb{C}$ and was later extended by Tadić in [118] to the case $G_n \coloneqq \mathrm{GL}_n(\mathbb{D}_v)$, where $\mathbb{D}_v$ is a central division algebra over $\mathbb{K}_v$ of degree d_v and char $\mathbb{K}_v = 0$. For char $\mathbb{K}_v \neq 0$, the classification was completed in [12]. The main focus of the fourth chapter of the thesis will lie on the case R of characteristic $\ell \coloneqq \operatorname{char} R$ possibly non-zero. Moreover, we restrict ourselves to the case $\ell \neq p$ to ensure the existence of Haar measures on reductive groups over $\mathbb{K}_v$. The theory of such representations has been explored and developed in [123], [124]. Passing from the $\ell = 0$ to the $\ell \neq 0$ case, many results carry over, however, also several new phenomena appear. Vignéras [124] and Mínguez-Sécherre [101] generalized the classification of Bernstein and Zelevinsky and Tadic to the case of $\ell \neq p$. The irreducible representations are classified by purely combinatorial objects called *multisegments*, formal finite sums of segments. To each multisegment $\mathfrak{m}$ an irreducible representation $\mathrm{Z}(\mathfrak{m})$ of G_n, for a suitable n, can be assigned and every irreducible representation of G_n is isomorphic to a representation coming from such a multisegment. However, in the case $\ell \neq 0$ it might happen that two multisegments give rise to isomorphic representations. Because of this one has to restrict the map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ to so-called *aperiodic* multisegments to obtain a bijection. The three main results of this section are the following. Firstly, we extend the theory of ρ-*derivatives*, see [83], [64], [66], [67], to R of non-zero characteristic. We quickly recall the necessary notions to define these derivatives. If $(\pi_1, V_1), (\pi_2, V_2)$ are admissible representations of G_{n_1} and G_{n_2}, we let $P_{(n_1,n_2)}(\mathbb{D}_v)$ be the parabolic subgroup of $G_{n_1+n_2}$ containing the upper triangular matrices with Levi-component the block diagonal matrices $G_{n_1} \times G_{n_2}$ and unipotent component $U_{(n_1,n_2)}(\mathbb{D}_v)$. We then denote by $\pi_1 \times \pi_2$ the normalized parabolically induced representation of $\pi_1 \otimes \pi_2$, *i.e.* the space of locally constant functions $f\colon G_n \to V_1 \otimes V_2$ such that for $g \in G_n$, $m = (m_1, m_2) \in G_{n_1} \times G_{n_2}$, $u \in U_{(n_1,n_2)}(\mathbb{K}_v)$

$$f(mng) = |\det(m_1)|_v^{\frac{d_v n_2}{2}} |\det(m_2)|_v^{-\frac{d_v n_1}{2}} (\pi_1(m_1) \otimes \pi_2(m_2)) f(g).$$

The group G_n acts on this space by right-translation. We call an admissible representation ρ *square-irreducible* if $\rho \times \rho$ is irreducible. If ρ is a square-irreducible representation of G_m and π, σ are irreducible representations of G_n and G_{n-m} respectively, $m \geq n$, we call σ a *left*- respectively *right-derivative* of π with respect to ρ if we have non-zero inclusions $\pi \hookrightarrow \rho \times \sigma$

respectively $\pi \hookrightarrow \sigma \times \rho$. Then σ is, up to isomorphism, unique if it exists, and π is the unique subrepresentation of $\rho \times \sigma$ respectively $\sigma \times \rho$.

This in turn allows us to achieve two things. Firstly, we are able to compute the L-factors of irreducible representations defined in [94] explicitly. Let us recall here their definition. Let $M_{n,n}(\mathbb{D}_v)$ be the space of $n \times n$ matrices over $\mathbb{K}_v$ and denoted by $S(M_{n,n}(\mathbb{D}_v))$ the space of Schwartz-functions with entries in R. Let ϕ be such a Schwartz-function and f be a matrix coefficient of an irreducible representation π of G_n. One can then define a Laurent-series $Z(\phi, f, T)$ whose k-th coefficient is given by

$$\int_{g \in G_n,\, |\det(g)| = q_v^{-k}} f(g)\phi(g)\, \mathrm{d}g.$$

It turns out that then $\mathrm{Z}(\phi, f, T) \in R(T)$ and that one can then show that by varying $Z(\phi, f, Tq^{\frac{1-d_v n}{2}})$ over all Schwartz-functions and matrix coefficients of π, one obtains a fractional ideal of $\mathcal{L}(\pi) \subseteq R[T, T^{-1}]$. We then define $L(\pi, T)$ as the generator of $\mathcal{L}(\pi)$ such that $L(\pi, 0) = 1$. In particular, if $R = \mathbb{C}$ and one sets $T = q^{-s}$ one obtains the Godement-Jacquet standard L-factors, which we discussed in Section 1.4. In the second part we the give an explicit description of $L(\pi, T)$ in terms of a multisegment parametrizing π. Moreover, we check that they behave well with the modular Local Langlands correspondence and the L-factor of the corresponding C-parameters, see [74]. The crucial fact that allows this approach is that a *cuspidal* representation has trivial L-factor if it is not square-irreducible and hence we are able to proceed by an inductive argument. Recall that a cuspidal representation of G_n is an irreducible, smooth representations such that for $n_1 + n_2 = n$, $n_i \in \mathbb{Z}_{>0}$ there exists no representations π_1, π_2 of $G_{n_1} \times G_{n_2}$ such that $\pi_1 \times \pi_2$ contains π as a subrepresentation. Secondly, we give a new proof that the map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ is surjective, *i.e.* every irreducible representation is isomorphic to $\mathrm{Z}(\mathfrak{m})$ for a suitable multisegment $\mathfrak{m}$. Such a new proof is of interest since the existing on heavily relied on the powerful, but abstract, machinery developed in [2], [25] of geometric representation theory. Our proof is entirely contained in the much more elementary setting of the representation theory of G_n.

1.8 Theta correspondence

The principle of the Langlands functoriality suggests that comparing automorphic representations or their local counterparts of different groups offers a promising way of understanding these representations. One of the most powerful tools of this kind is the so-called *theta correspondence*, which had its root in the foundations of quantum mechanics. The theta correspondence was first formally introduced in [56] and was for example used to show that the (naive) generalized Ramanujan conjecture is false in non-trivial examples, see [58]. The standard references for this topic are [57] and [97]. For the longest time the theta correspondence seemed to exist outside of the Langlands program and although useful, did not immediately connect to the principle of functoriality. However in recent years several attempts were made to remedy this apparent problem, which, among other things, led to the development of the *relative Langlands program*. In this thesis, we will however focus on more concrete, representation-theoretic question which arise from the theta correspondence. More precisely, we will study the local theta correspondence in type I.

The basic idea of the theta correspondence is the following. Assume we have two groups $\mathcal{G}$ and $\mathcal{H}$, a representation π of $\mathcal{G}$, and would like to find a natural recipe of how to construct from π a representation of $\mathcal{H}$. One possibility is to find a large group $\mathcal{M}$ containing $\mathcal{G}\times\mathcal{H}$ and a natural representation ω of $\mathcal{M}$ such that for all π we are interested in, there exists a non-zero $\mathcal{G}$-invariant map $\pi\to\omega$. We can then consider the maximal $(\pi,\mathcal{G})$-isotypic quotient of ω, *i.e.* the maximal representation Π of $\mathcal{H}$ such that there exists a surjective, non-zero $\mathcal{G}\times\mathcal{H}$-equivariant map

$$\omega\to\Pi\otimes\pi.$$

In such a situation Π is usually denoted by $\Theta(\pi)$. We now could ask several things from the map $\pi\mapsto\Theta(\pi)$, for example, if π is irreducible, can the same be said about $\Theta(\pi)$? In most cases this will not be true, but since we are lucky, in what will follow, $\Theta(\pi)$ will either be 0 or of finite length and will have a unique irreducible quotient, denoted by $\theta(\pi)$. The next question is then how much information about π is stored in $\theta(\pi)$, *i.e.* if $\theta(\pi)\cong\theta(\pi')\neq 0$, is then $\pi\cong\pi'$?

For this introduction, we will now focus on a particular instance of the theta correspondence and will show how one can implement the above ideas. Let $\mathbb{K}_v$ be a non-archimedean local field of characteristic different from 2, residue characteristic p, and $\psi_v\colon\mathbb{K}_v\to\mathbb{C}^\times$ a non-trivial additive character. The group $\mathcal{H}$ in the above motivation will be of the form $\mathrm{Sp}_m(\mathbb{K}_v)$. The group $G(V)$, where V is a quadratic space over $\mathbb{K}_v$ of even dimension n and $G(V)$ is the group of invertible linear transformations preserving the quadratic form, will play the role of $\mathcal{G}$ in the above motivation. However, as we will see later, if n is small compared to m it might happen that for π an irreducible representation of $\mathrm{Sp}_m(\mathbb{K}_v)$, $\Omega(\pi)=0$. It will thus be useful to consider a tower of quadratic spaces $\{V_r,\, r\in\mathbb{N}\}$ with $V_0=V$ and $\dim_{\mathbb{K}_v}V_r=n+2r$ and replace V by a suitable V_r. This is done as follows.

Recall that to a quadratic space V over $\mathbb{K}_v$, one can associate three invariants. The first being its dimension, the second its discriminant-character $\Delta_{\mathrm{disc}}\colon\mathbb{K}_v^\times\to\mathbb{C}^\times$ and the third its Hasse-invariant $\epsilon(V)\in\{\pm1\}$. Recall that the discriminant-character and the Hasse invariant are defined as follows. For $a,b\in\mathbb{K}_v^\times$ define the Hilbert-symbol

$$(a,b)_{\mathbb{K}_v}:=\begin{cases}1 & \text{if } z^2=ax^2+by^2 \text{ has a solution in } (\mathbb{K}_v^\times)^3,\\ -1 & \text{otherwise.}\end{cases}$$

After choosing a suitable basis, the quadratic form of V is represented by a diagonal matrix $\mathrm{diag}(a_1,\dots,a_n)$. Then one defines

$$\epsilon(V):=\prod_{1\leq i<j\leq n}(a_i,a_j)_{\mathbb{K}_v}$$

and

$$\det(V):=\prod_{i=1}^n a_i.$$

Finally define

$$\Delta_{\mathrm{disc}}(x):=(x,(-1)^{\frac{n(n-1)}{2}}\det(V))_{\mathbb{K}_v}.$$

Then these three invariants are already enough to determine the quadratic space V uniquely. Furthermore, denote the hyperbolic space

$$\mathbb{H}=\left(\mathbb{K}_v^2,\begin{pmatrix}0&1\\1&0\end{pmatrix}\right).$$

For a given dimension n and discriminant character Δ_{disc} there exist thus two quadratic spaces with Hasse-invariant ± 1, denoted by $V^{\pm}$, and Witt towers

$$\{V_r^{\pm} := V^{\pm} \oplus \mathbb{H}^r,\, r \in \mathbb{N}\}.$$

It is now easy to check that the discriminant character of each $V_r^{\pm}$ is still Δ_{disc} and the dimension of $V_r^{\pm}$ is even, as is n. Denote now by $(-,-)$ the quadratic form on V and by $\langle -,- \rangle$ the natural symplectic form on $\mathbb{K}_v^m$. Finally, define $\mathbb{W} := V \otimes \mathbb{K}_v^m$ together with its naturally induced symplectic form

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle := (x_1, x_2)\langle y_1, y_2 \rangle.$$

One thus has an embedding $G(V) \times \mathrm{Sp}_m(\mathbb{K}_v) \hookrightarrow \mathrm{Sp}(\mathbb{W})$ and thus a natural candidate for the group $\mathcal{M}$ could be $\mathrm{Sp}(\mathbb{W})$. It turns out that one cannot find a representation ω of $\mathrm{Sp}(\mathbb{W})$ such that $\Theta(\pi)$ is most of the time non-zero and if it is non-zero it is well-behaved. However, nature offers a way out of this dilemma and we only have to pay a small prize. It comes in the form of the metaplectic group $\mathrm{Mp}(\mathbb{W})$ and the Weil representation ω_{ψ_v} of $\mathrm{Mp}(\mathbb{W})$, which depends on the fixed additive character ψ_v. We will not recall the definition of the metaplectic group here and refer to [72]. At the moment we just note that it is a non-trivial extension

$$1 \to \{\pm 1\} \to \mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W}) \to 0$$

and it admits a natural representation called the Weil representation ω_{ψ_v}. Moreover, since we chose n to be even, the inclusion $G(V) \times \mathrm{Sp}_m(\mathbb{K}_v) \hookrightarrow \mathrm{Sp}(\mathbb{W})$ factors through $\mathrm{Mp}(\mathbb{W})$ and hence we can take in this case $\mathcal{M} = \mathrm{Mp}(\mathbb{W})$ and ω the Weil representation. Thus if π is an irreducible smooth representation of $\mathrm{Sp}_m(\mathbb{K}_v)$ we denote by $\Theta_{V,m,\psi_v}(\pi) = \Theta(\pi)$ its lift to $G(V)$. It was shown in [71] that $\Theta_{V,m,\psi_v}(\pi)$ is either 0 or a smooth representation of finite length of $G(V)$. The following was then conjectured in [56], [57] and proven by [128] in the case $p \neq 2$ and in [57] for archimedean fields. In [42] a new proof without the assumption on p was given and in [40] the remaining case of quaternionic dual pairs was covered.
Theorem 10 (Howe Duality Conjecture for Type I, [128], [42], [40])**.** *Let π, π' be irreducible smooth representations of* $\mathrm{Sp}_m(\mathbb{K}_v)$*. Then the following holds.*

1. *The representation $\theta(\pi)$ is either irreducible or* 0*.*
2. *If $\theta(\pi) \cong \theta(\pi') \neq 0$, then $\pi \cong \pi'$.*

It thus remains to determine for which quadratic spaces V the theta lift of a given π is non-zero. This is answered by the following theorem. We denote by $m^{\pm}_{\Delta_{\mathrm{disc}}}(\pi)$ the first occurrence of an irreducible smooth representation π of $\mathrm{Sp}_m(\mathbb{K}_v)$ in the theta correspondence in the Witt tower $\{V_r^{\pm}\}$, *i.e.* the smallest $\dim_{\mathbb{K}_v} V_r^{\pm}$ such that $\Theta_{V_r^{\pm},m,\psi_v}(\pi) \neq 0$. The following conservation relation was conjectured in [73] and proven in [117].
Theorem 11 (Conservation Relation, [117])**.** *For any irreducible smooth representation π of* $\mathrm{Sp}_m(\mathbb{K}_v)$

$$m^{+}_{\Delta_{\mathrm{disc}}}(\pi) + m^{-}_{\Delta_{\mathrm{disc}}}(\pi) = 2m + 4.$$

To prove this theorem, Kudla and Rallis proposed in [73] the following strategy, which was in the end however not the approach used by Sun in [117], who managed to achieve the result via different methods. Let χ be a unitary character of $\mathbb{K}_v^{\times}$ and $s \in \mathbb{C}$ and equip $\mathbb{K}_v^m \oplus \mathbb{K}_v^m$ with the standard symplectic form $\langle .,. \rangle$ on the first copy of $\mathbb{K}_v^m$ and $-\langle .,. \rangle$ on the second copy of

$\mathbb{K}_v^m$. Let $I(\chi,s)$ be the degenerate principal series of $\mathrm{Sp}_{2m}(\mathbb{K}_v)$. This representation is defined as follows. Let X be the isotropic subspace of $\mathbb{K}_v^{2m}$ equipped with the standard symplectic structure given by the first copy of $\mathbb{K}_v^m$ Set $P(X)$ to the stabilizer of X under the action of $\mathrm{Sp}_{2m}(\mathbb{K}_v)$. Then $P(X)$ has a Levi-component isomorphic to $\mathrm{GL}_m(\mathbb{K}_v)$ and we denote its unipotent part by $N(X)$. Define $I(\chi,s)$ as the set of locally constant functions

$$\{f\colon \mathrm{Sp}_{2m}(\mathbb{K}_v)\to\mathbb{C}:\ f(mng)=|\det(h)|^{s+\frac{2m+1}{2}}\chi(\det(h))f(g),$$

$$n\in N(X),\, h\in \mathrm{GL}_m(\mathbb{K}_v),\, g\in \mathrm{Sp}_{2m}(\mathbb{K}_v)\}.$$

Consider the natural embedding

$$\iota\colon \mathrm{Sp}_m(\mathbb{K}_v)\times \mathrm{Sp}_m(\mathbb{K}_v)\to \mathrm{Sp}_{2m}(\mathbb{K}_v)$$

and define the restriction of $I(\chi,s)$ to $\mathrm{Sp}_m(\mathbb{K}_v)\times\mathrm{Sp}_m(\mathbb{K}_v)$ as

$$I_{m,m}(\chi,s):=\iota^*(I(\chi,s)).$$

In [73] the authors construct a non-zero morphism $I_{m,m}(\chi,s)\to\pi\otimes\pi^\vee$ for all irreducible smooth representations π of $\mathrm{Sp}_m(\mathbb{K}_v)$ and conjectured that this morphism is up to a scalar unique. The main result of the third section is the following theorem, which positively answers [72, Conjecture V.3.2], *cf.* also [73, Conjecture 1.2].
Theorem 12. *For all irreducible smooth representations π of $G(W)$,*

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Sp}_m(\mathbb{K}_v)\times\mathrm{Sp}_m(\mathbb{K}_v)}(I_{m,m}(\chi,s),\pi\otimes\pi^\vee)=1.$$

In [73, §4] it was proven that Theorem 11 holds under the assumption of Theorem 12. In the same paper, see [73, Theorem 1.1, Lemma 1.4], Theorem 12 was verified for $\pi=1$ the trivial representation. In the last part of the thesis we will give a proof of this conjecture.

Chapter 2

Notations and preliminaries

In the course of this chapter we will fix the necessary notation and recall the preliminary results on which we will build later on. As the topics discussed can be roughly divided into local and global phenomena, we will keep the same separation here. However, unlike in the introduction and the rest of the thesis, we will start with the local side of the story and then take care of the global side. If a symbol can both appear in the local and global context, like a division algebra over a field, we will use a subscript v to distinguish the former from the later, *e.g.* a division algebra $\mathbb{D}_v$ over a local non-archimedean field $\mathbb{K}_v$ of degree d_v versus a division algebra $\mathbb{D}$ over a number field $\mathbb{K}$ of degree d. The subscript v will denote the valuation of the underlying local field. Moreover, we reserve capital letters like Π for global representations and lower-case letters like π for local representations whenever possible. Thus we will denote the local representations corresponding to a global representation Π usually by π_v.

2.1 Miscellaneous

For $n \in \mathbb{N}$ we let ζ_n be the group of n-th roots of unity in $\mathbb{C}^\times$ and if S is a set we let $\mathbb{N}(S)$ be the commutative monoid consisting of functions $S \to \mathbb{N}$ with finite support. Let $n \in \mathbb{Z}_{>0}$ and denote by S_n the n-th symmetric group. A tuple

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t),\ \alpha_i \in \mathbb{Z}_{>0}$$

will be called a *composition* of n if $\sum_{i=1}^t \alpha_i = n$. Note that throughout this text we assume compositions to be tuples, whereas we use the phrase *partition* for unordered tuples. The reverse composition $\overline{\alpha}$ of α is denoted by $\overline{\alpha} := (\alpha_t, \dots, \alpha_1)$ and we call α ordered if

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t.$$

For $j \in \{1, \dots, t\}$ let $(\beta_{1,j}, \beta_{2,j}, \dots, \beta_{i_j,j})$ be a composition of α_j. We then call

$$(\beta_{1,1}, \dots, \beta_{i_1,1}, \beta_{1,2}, \dots, \beta_{i_2,2}, \dots, \beta_{1,t}, \dots, \beta_{i_t,t})$$

a subcomposition of α. If $\alpha' = (\alpha'_1, \dots, \alpha'_{t'})$ is a second composition of n we define the intersection $\alpha \cap \alpha'$ recursively as follows.

$$\alpha \cap \alpha' := \begin{cases} (\alpha_1, (\alpha'_2, \dots, \alpha'_{t'}) \cap (\alpha_2, \dots, \alpha_t)) & \text{if } \alpha_1 = \alpha'_1, \\ (\alpha_1, (\alpha'_1 - \alpha_1, \dots, \alpha'_{t'}) \cap (\alpha_2, \dots, \alpha_t)) & \text{if } \alpha_1 < \alpha'_1, \\ (\alpha'_1, (\alpha'_2, \dots, \alpha'_{t'}) \cap (\alpha_1 - \alpha'_1, \dots, \alpha_t)) & \text{if } \alpha_1 > \alpha'_1. \end{cases}$$

If α and α' are compositions of n we say $\alpha \leq \alpha'$ if

$$\sum_{i=1}^{j} \alpha_j \leq \sum_{i=1}^{j} \alpha'_j$$

for all $j \in \{1, \ldots, \min(t, t')\}$.

2.1.1 The symmetric group In this section we quickly review the representation theory of S_n over an algebraically closed field R, not necessarily of characteristic 0. For a more thorough introduction see for example [63]. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of n and denote by $\mathbf{1}_n$ the trivial representation of S_n. We set

$$\mathrm{S}_\lambda := \mathrm{S}_{\lambda_1} \times \ldots \times \mathrm{S}_{\lambda_t},$$

$$\mathbf{1}_\lambda := \mathbf{1}_{\lambda_1} \otimes \ldots \otimes \mathbf{1}_{\lambda_t}$$

and

$$M^\lambda := \mathrm{Ind}_{\mathrm{S}_\lambda}^{\mathrm{S}_n}(\mathbf{1}_\lambda) = \mathrm{Hom}_{\mathrm{S}_\lambda}(\mathrm{S}_n, \mathbf{1}_\lambda),$$

called the *Young permutation module*. The representation M^λ contains the *Specht-module* S^λ with multiplicity 1, *cf.* [63, §4]. If $R = \overline{\mathbb{Q}}_\ell$, S^λ is irreducible for all λ and the Specht-modules parameterize all irreducible representations of S_n, however if $R = \overline{\mathbb{F}}_\ell$, S^λ might be reducible. To bypass this issue in the case $R = \overline{\mathbb{F}}_\ell$, one equips M^λ with a natural S_n-invariant and symmetric bilinear pairing $\langle \cdot, \cdot \rangle$. Under this pairing, let $(S^\lambda)^\perp$ be the orthogonal complement of S^λ in M^λ and set

$$D_\lambda := S^\lambda / (S^\lambda \cap (S^\lambda)^\perp).$$

We recall that λ is ℓ-regular if for all

$$a_i := \#\{j \in \{1, \ldots, t\} : \lambda_j = i\},$$

$a_i < \ell$. Now D_λ is irreducible if it is non-zero and it is non-zero if and only if λ is ℓ-regular. Moreover, if μ is a second partition of n, D_λ appears in M^μ if and only if $\lambda \leq \mu$. Finally M^μ is contained in M^λ in the Grothendieck group of R-representations of S_n if and only if $\mu \leq \lambda$. More precisely, the multiplicity of S^λ in M^μ is the Kostka number $K_{\lambda,\mu}$.

2.1.2 Locally compact groups We will fix our conventions regarding locally compact groups and their Haar measures. Let G be a locally compact group, *i.e.* a Hausdorff topological group with locally compact topology. Let $R \in \{\mathbb{C}, \overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$. We denote a left respectively right Haar-measure on G with coefficients in R by $\mathrm{d}_l g$ respectively $\mathrm{d}_r g$. The modular character of G is denoted by δ_G and if G is unimodular we denote the left and right Haar-measure by d. Since these Haar-measure are only well defined up to a scalar multiple, we will fix, when necessary, a choice by fixing the volumes of certain compact subgroups of G. If G is unimodular and H is a unimodular subgroup of G, the quotient $H \backslash G$ can be equipped with a natural choice of a quotient measure, depending on a choice of Haar-measure on both G and H.

2.1.3 Adelic notation We expand marginally on the notation already introduced in the introduction.

Local notation Let $\mathbb{K}_v$ be a local non-archimedean field and $\mathbb{D}_v$ a central division algebra over $\mathbb{K}_v$ of degree d_v. We let $\mathcal{O}_v$ be the ring of integers of $\mathbb{K}_v$ and fix a uniformizer ϖ_v, *i.e.* a generator of the maximal ideal of $\mathcal{O}_v$. We extend the valuation v of $\mathbb{K}_v$ to a valuation v' of $\mathbb{D}_v$ by

$$v'(x) := \frac{1}{d_v} v\left(\mathrm{Nr}_{\mathbb{D}_v/\mathbb{K}_v}(x)\right).$$

Define the ring of integers of $\mathbb{D}_v$ as

$$\mathcal{O}'_v := \{x \in \mathbb{D}_v : v'(x) \geq 0\}.$$

Global notation Let $\mathbb{K}$ be a number field with $\dim_{\mathbb{Q}} \mathbb{K} = r$, $\mathcal{O}$ its ring of integers, and let $\mathbb{D}$ be a central division algebra of degree d over $\mathbb{K}$. Recall the set of places $\mathcal{V}$, which decomposes into the finite places $\mathcal{V}_f$ and the infinite places $\mathcal{V}_\infty$. For $v \in \mathcal{V}$, one has $\mathbb{D} \otimes \mathbb{K}_v \cong M_{r_v,r_v}(\mathbb{D}_v)$, where $\mathbb{D}_v$ is a central division algebra of dimension d_v^2 over $\mathbb{K}_v$ and $d_v r_v = d$. If $\mathbb{D}_v = \mathbb{K}_v$ we call v a split place of $\mathbb{D}$. From now on we assume that $\mathbb{D}$ is non-split at all infinite places. Equivalently, for $v \in \mathcal{V}_\infty$, $\mathbb{D}_v = M_{\frac{d}{2},\frac{d}{2}}(\mathbb{H})$, where $\mathbb{H}$ denotes the Hamilton quaternions. We denote the places where $\mathbb{D}$ is non-split by $\mathcal{V}_{\mathbb{D}}$. Finally, let $\mathfrak{D}$ be the absolute different of $\mathbb{K}$, *i.e.* $\mathfrak{D}^{-1} := \{x \in \mathbb{K} : \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(x\mathcal{O}) \subseteq \mathbb{Z}\}$.

We will also fix a non-trivial additive character

$$\psi\colon \mathbb{K}\backslash\mathbb{A}_{\mathbb{K}} \to \mathbb{C}^\times$$

as follows. First define

$$\psi_{\mathbb{Q}} := \psi_{\mathbb{R}} \otimes \bigotimes_{p \text{ prime}} \psi_p\colon \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^\times$$

by

$$\psi_{\mathbb{R}}(t) := e^{-2\pi i t},\ \psi_p(t) := e^{2\pi i \lambda_p(t)},\ \lambda\left(\sum_{k\in\mathbb{Z}} a_k p^k\right) = \sum_{k<0} a_k p^k.$$

By composing $\psi_{\mathbb{Q}}$ with the trace map $\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}\colon \mathbb{A}_{\mathbb{K}} \to \mathbb{A}_{\mathbb{Q}}$, we obtain the desired character $\psi\colon \mathbb{K}\backslash\mathbb{A}_{\mathbb{K}} \to \mathbb{C}^\times$. Note that the finite places where ψ ramifies correspond precisely to the prime ideals $\mathfrak{p}$ not dividing $\mathfrak{D}$. We will write from now on $\mathbb{A}$ for the adeles of $\mathbb{K}$.

2.1.4 Deligne representations Let $\mathbb{K}_v$ be as in Section 2.1.3. We recall $\mathrm{W}_{\mathbb{K}_v}$, the Weil group of $\mathbb{K}_v$, the inertia subgroup $I_{\mathbb{K}_v}$ and ν_v, the unique unramified character of $\mathrm{W}_{\mathbb{K}_v}$ acting on the Frobenius by $\nu_v(\mathrm{Frob}) = q_v^{-1}$. Let $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ be a field of coefficients, where ℓ is prime different from the residue characteristic of $\mathbb{K}_v$. A Deligne R-representation of $\mathrm{W}_{\mathbb{K}_v}$ is a pair (Φ, U), where Φ is a finite-dimensional, smooth representation of $\mathrm{W}_{\mathbb{K}_v}$ over R and U an element in $\mathrm{Hom}_{\mathrm{W}_{\mathbb{K}_v}}(\nu_v\Phi, \Phi)$. We quickly discuss several properties of such representations. For a more complete discussion, including proofs, see [74]. We call (Φ, U) semi-simple if Φ is semi-simple and if $R = \overline{\mathbb{F}}_\ell$, let $o(\Phi)$ be the smallest natural number k such that

$$\Phi \cong \nu_v^k \Phi.$$

Let

$$\mathbb{Z}[\Phi] := \begin{cases} \{\Phi, \ldots, \nu_v^{o(\Phi)-1}\Phi\} & R = \overline{\mathbb{F}}_\ell, \\ \{\nu_v^k \Phi : k \in \mathbb{Z}\} & R = \overline{\mathbb{Q}}_\ell. \end{cases}$$

We say a semi-simple R-Deligne representation is supported on $\mathbb{Z}[\Phi]$ if the underlying $\mathrm{W}_{\mathbb{K}_v}$-representation is the direct sum of elements in $\mathbb{Z}[\Phi]$. We say two indecomposable representations (Φ, U) and (Φ', U') are equivalent if $(\Phi', U') \cong (\Phi, \lambda U)$ for some non-zero scalar $\lambda \in R$ and two general semi-simple R-Deligne representations are equivalent if their respective indecomposable parts are equivalent. We denote by $\mathrm{Rep}_{ss}(\mathrm{W}_{\mathbb{K}_v}, R)$ the set of equivalence classes of semi-simple R-Deligne representations and by $\mathrm{Rep}_{ss,n}(\mathrm{W}_{\mathbb{K}_v}, R)$ the subset of representations of length n, where the length is the dimension of the underlying $\mathrm{W}_{\mathbb{K}_v}$-representation. We call an R-Deligne representation (Φ, U) nilpotent if U is nilpotent and recall that over $R = \overline{\mathbb{Q}}_\ell$, the equivalence classes of semi-simple representations are in bijections with equivalence classes of nilpotent semi-simple representations. We denote the latter set by $\mathrm{Nil}_{ss}(\mathrm{W}_{\mathbb{K}_v}, R)$ and by $\mathrm{Nil}_{ss,n}(\mathrm{W}_{\mathbb{K}_v}, R)$ its subset consisting of representations of length n.

For $r \in \mathbb{Z}_{\geq 1}$, we let $[0, r-1]$ be the R-Deligne representation whose underlying $\mathrm{W}_{\mathbb{K}_v}$-representation is

$$\bigoplus_{i=0}^{r-1} \nu_v^k$$

and U is defined by

$$U(x_0, \dots, x_{r-1}) = (0, x_0, \dots, x_{r-2}).$$

For Φ a general irreducible representation of $\mathrm{W}_{\mathbb{K}_v}$ we can then define

$$[0, r-1] \otimes \Phi$$

and these parametrize all nilpotent semi-simple R-Deligne representations up to equivalence. Over $\overline{\mathbb{F}}_\ell$ there exists a second class of indecomposable semi-simple representations of the following form. Let Φ be an irreducible representation of $\mathrm{W}_{\mathbb{K}_v}$, denote

$$\Psi(\Phi) := \bigoplus_{k=0}^{o(\Phi)-1} \nu_v^k \Phi$$

and pick an isomorphism $I: \nu_v^{o(\Phi)} \Phi \xrightarrow{\cong} \Phi$. We then define $C(I, \Phi) = C(\Phi)$ as the $\overline{\mathbb{F}}_\ell$-Deligne representation with underlying $\mathrm{W}_{\mathbb{K}_v}$-representation $\Psi(\Phi)$ and

$$U(x_0, \dots x_{o(\Phi)-1}) = (I(x_{o(\Phi)-1}), x_0, \dots, x_{o(\Phi)-2}).$$

Then $C(\Phi)$ depends up to equivalence only on $\mathbb{Z}[\Phi]$ and the underlying morphism U of $C(\phi)$ is a bijection. The indecomposable semi-simple $\overline{\mathbb{F}}_\ell$-Deligne representations are then classified up to equivalence by

$$[0, r-1] \otimes \Phi \text{ and } [0, r-1] \otimes C(\Phi),$$

see [74, Main Theorem 1].

We call a semi-simple $\overline{\mathbb{Q}}_\ell$-representation Φ of $\mathrm{W}_{\mathbb{K}_v}$ *integral* if it admits a $\mathrm{W}_{\mathbb{K}_v}$-stable $\overline{\mathbb{Z}}_\ell$-lattice L, which generates Φ. One can then define its reduction mod ℓ $\mathrm{r}_\ell(\Phi)$ by taking the semi-simplification of $L \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$, which is independent of the chosen lattice. We let $\mathrm{Nil}^e_{ss}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{Q}}_\ell)$ be the semi-simple nilpotent representations whose underlying $\mathrm{W}_{\mathbb{K}_v}$-representation is integral. One obtains hence a map

$$\mathrm{r}_\ell: \mathrm{Nil}^e_{ss}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Nil}_{ss}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{F}}_\ell)$$

sending

$$[0, r-1] \otimes \Phi \mapsto [0, r-1] \otimes \mathrm{r}_\ell(\Phi).$$

2.2 Reductive groups

We will quickly introduce the reductive groups relevant to us and fix our notation regarding tori and parabolic subgroups. If P is a parabolic subgroup of G, we denote the opposite parabolic subgroup by $\overline{P}$.

2.2.1 The general linear group Let K be a field and V a n-dimensional K-vector space. Then $\mathrm{GL}(V)$ is a linear algebraic group over K. Choosing a basis of V gives an identification of $\mathrm{GL}(V)$ with GL_n, the n-th general linear group over K. The standard choice of a torus T_n in GL_n is then given by the diagonal matrices with character group $X^*(T_n) \cong \mathbb{Z}^n$, where $\mu = (\mu_1, \ldots, \mu_n)$ corresponds to

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^{n} t_i^{\mu_i}.$$

The roots of GL_n with respect to T_n are then given by

$$\{e_i - e_j : 1 \leq i, j \leq n,\ i \neq j\},$$

where $e_i := (0, \ldots, \overbrace{1}^{i\text{-th}}, \ldots, 0)$. A weight $\mu \in X^*(T_n)$ is called *dominant* if

$$\mu_1 \geq \ldots \geq \mu_n$$

and *regular* if

$$\mu_1 > \ldots > \mu_n.$$

To each dominant weight μ one can associate a highest weight representation E_μ of $\mathrm{GL}_n(\mathbb{C})$ as follows. The weight μ induces a line bundle L_μ on $B_n(\mathbb{C})\backslash\mathrm{GL}_n(\mathbb{C})$ and the space of global sections

$$\Gamma(B_n(\mathbb{C})\backslash\mathrm{GL}_n(\mathbb{C}), L_\mu)$$

carries a natural action of $\mathrm{GL}_n(\mathbb{C})$. If μ is dominant, the so obtained finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ is irreducible and denoted by E_μ and is called the *highest weigh representation* corresponding to μ. Throughout the rest of this thesis we fix the choice of positive roots to $\{e_i - e_j : 1 \leq i < j \leq n\}$, corresponding to the standard Borel subgroup B_n given by the upper triangular matrices. Recall that the parabolic subgroups over $\mathbb{K}_v$ containing B_n are then parameterized by compositions of n. In other words, to $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of n we associate the parabolic subgroup P_α of GL_n containing the upper triangular matrices and having as a Levi-component the block-diagonal matrices $M_\alpha = \mathrm{GL}_{\alpha_1} \times \ldots \times \mathrm{GL}_{\alpha_k}$ and unipotent component U_α. Then $\overline{P_\alpha}$ is conjugated to $P_{\overline{\alpha}}$. If V is a finite-dimensional K-vector space and

$$\mathcal{F} = \{0 = F_0 \subseteq \ldots F_k = V\}$$

is a flag in V, the stabilizer $P(\mathcal{F})$ in $\mathrm{GL}(V)$ is a parabolic subgroup conjugated to the standard parabolic subgroup corresponding to the composition

$$(\dim_{\mathrm{K}} F_1/F_0, \ldots, \dim_{\mathrm{K}} F_k/F_{k-1}).$$

2.2.2 The general linear group over a division algebra Let K be a field and D a central division algebra over K of degree d. Let $M_{n,n}$ be the variety whose K points are the $n \times n$ matrices with entries in K and let $M'_{n,n}$ be the variety whose K points are the $n \times n$ matrices with entries in D. For $A \in M'_{n,n}(\mathrm{K})$ define $\det'(A)$ as follows. Choose an isomorphism $\phi : M'_{n,n}(\mathrm{K}) \otimes \overline{\mathrm{K}} \to M_{nd,nd}(\overline{\mathrm{K}})$ and set $\det'(A) \coloneqq \det(\phi(A \otimes 1))$. This map is independent of the chosen isomorphism and defined over K. Thus, we can extend it to a map $\det' : M'_{n,n} \to M_{1,1}$. Similarly we define a trace map $\mathrm{Tr}(A) \coloneqq \mathrm{Tr}(\phi(A \otimes 1))$ and extended it to a map $\mathrm{Tr} : M'_{n,n} \to M_{1,1}$. Having fixed all of these conventions, let us define the reductive K-group

$$\mathrm{GL}'_n \coloneqq \{A \in M'_{n,n} : \det'(A) \neq 0\}.$$

and denote the center of GL'_n by $Z'_n \cong \mathrm{GL}_1$. Again we can assign to each composition α of n a standard parabolic subgroup P_α of GL'_n defined over $\mathbb{K}_v$, containing the upper triangular matrices and having as a Levi-component the block-diagonal matrices $M_\alpha = \mathrm{GL}'_{\alpha_1} \times \ldots \times \mathrm{GL}'_{\alpha_k}$ and unipotent component denoted by U'_α. Then $\overline{P'_\alpha}$ is again conjugated to $P'_{\overline{\alpha}}$. We extend the notions of highest weight representations of GL_n to GL'_n as follows. If $\mathrm{K} = \mathbb{R}$ and $\mathrm{D} = \mathbb{H}$, a representation E_μ of $\mathrm{GL}'_n(\mathbb{R})$ is called a highest weight representation with dominant weight $\mu \in \mathbb{Z}^{2n}$, if the corresponding complexified representation of $\mathrm{GL}_{2n}(\mathbb{C})$ is a highest weight representation with weight μ. Define finally $H_n = \mathrm{GL}_n \times \mathrm{GL}_n$, $H'_n = \mathrm{GL}'_n \times \mathrm{GL}'_n$.

2.2.3 Classical groups Let $\mathbb{K}_v$ be a field as in Section 2.1.3 of characteristic different from 2 and let $\mathbb{L}_v$ be either $\mathbb{L}_v = \mathbb{K}_v$ or a quadratic extension of $\mathbb{L}_v$. Let $\mathfrak{c} \in \mathrm{Gal}(\mathbb{L}_v/\mathbb{K}_v)$ be the generator of $\mathrm{Gal}(\mathbb{L}_v/\mathbb{K}_v)$. If $[\mathbb{L}_v : \mathbb{K}_v] = 2$, we let $\mathbb{L}_v^1$ be the elements of norm 1 in $\mathbb{L}_v$.

Let $\epsilon \in \{\pm 1\}$ and W a finite dimensional vector space over $\mathbb{L}_v$ together with a non-degenerate ϵ-hermitian and $\mathfrak{c}$-sesquilinear form

$$\langle .,. \rangle : W \times W \to \mathbb{L}_v,$$

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle,\ \langle x, y \rangle = \epsilon \mathfrak{c}(\langle y, x \rangle).$$

Let $G(W)$ be the symmetry group of $\langle .,. \rangle$, *i.e.*

$$G(W) \coloneqq \{g \in \mathrm{GL}(W) : \langle gx, gy \rangle = \langle x, y \rangle \text{ for all } x, y \in W\}.$$

Then

$$G(W) = \begin{cases} \text{symplectic group } \mathrm{Sp}(W) & \mathbb{L}_v = \mathbb{K}_v, \epsilon = -1. \\ \text{orthogonal group } \mathrm{O}(W) & \mathbb{L}_v = \mathbb{K}_v, \epsilon = 1. \\ \text{unitary group } \mathrm{U}(W) & [\mathbb{L}_v : \mathbb{K}_v] = 2. \end{cases}$$

We call a subspace X of W *isotropic* if the ϵ-hermitian form vanishes on X. Let q_W be Witt index of W, *i.e.* the maximal dimension of an isotropic subspace of W. If X is an isotropic subspace of W, write $W = X \oplus W' \oplus Y$, such that $X \oplus Y$ and W' are non-degenerate and Y is isotropic. Let $P(X)$ be the stabilizer of X in $G(W)$. Then $P(X)$ is a maximal parabolic subgroup of $G(W)$ defined over $\mathbb{K}_v$ and every maximal parabolic subgroup is of this form. The Levi decomposition of $P(X) = M(X) \ltimes N(X)$ has Levi-component $M(X) = \mathrm{GL}(X) \times G(W')$, the stabilizer of Y in $P(X)$. More generally, if

$$\mathcal{F} = \{0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_r \subseteq X_{r+1} = V\}$$

is a flag of isotropic subspaces of W, its stabilizer is a parabolic subgroup $P(\mathcal{F}) = M(\mathcal{F}) \ltimes N(\mathcal{F})$ with Levi-component

$$M(\mathcal{F}) = \mathrm{GL}(X_1) \times \ldots \times \mathrm{GL}(X_r) \times G(W'),$$

where $W = X_r \oplus W' \oplus Y_r$ as above. We observe here that the parabolic subgroup $P(X)$ is conjugated to its opposite parabolic subgroup $\overline{P(X)}$ by an element which acts on $\mathrm{GL}(X_i)$ as $g \mapsto \mathfrak{c}({}^t g^{-1})$ and on $G(W')$ trivially. From now on we will write by abuse of notation $G(W) = G(W)(\mathbb{K}_v)$.

2.2.4 Metaplectic groups Let $\mathbb{K}_v$ be a local non-archimedean field of characteristic different than 2 and fix an additive, continous character $\psi_v : \mathbb{K}_v \to \mathbb{C}^\times$. For a more precise treatment of the contents of this section see [72]. Let W be a finite-dimensional symplectic space over $\mathbb{K}_v$. Choose a Witt decomposition W and recall the Heisenberg group $H(W) = W \oplus \mathbb{K}_v$, equipped with multiplication

$$(v_1, x_1) \cdot (v_2, x_2) = (v_1 + v_2, x_1 + x_2 + \frac{1}{2}\langle v_1, v_2\rangle)$$

and center $\mathbb{K}_v$. The Stone van-Neumann theorem implies that $H(W)$ has a unique irreducible $\mathbb{C}$-representation ω_{ψ_v} for each choice of central character ψ_v, which can be explicitly described as a compactly induced representation. Since $G(W)$ acts on $H(W)$ by acting on the W-component, one obtains for each $g \in G(W)$ an irreducible representation $\omega_{\psi_v} \circ g$. From the explicit description of ω_{ψ_v} it can be deduced that the central character of $\omega_{\psi_v} \circ g$ is still ψ_v, thus it follows $\omega_{\psi_v} \circ g \cong \omega_{\psi_v}$. Denote by X the underlying vector space of ω_{ψ_v}. Hence there exists $A_{\psi_v}(g) \in \mathrm{GL}(X)$ such that

$$A_{\psi_v}(g) \circ \omega_{\psi_v} \circ g \cong \omega_{\psi_v} \circ A_{\psi_v}(g)$$

and by Schur's lemma this is up to a scalar unique. Moreover, since one can show that ω_{ψ_v} is unitary, we constructed a map

$$A_{\psi_v} : \mathrm{Sp}(W) \to \mathrm{GL}(X)/S^1.$$

We then set

$$\mathrm{Mp}^{S^1}(W) := \mathrm{Sp}(W) \times_{\mathrm{GL}(X)/S^1} \mathrm{GL}(X),$$

a S^1-fibration of $\mathrm{Sp}(W)$. It turns out that this S^1-fibration factors through a ζ_2 vibration, which we from now on will denote by $\mathrm{Mp}(W)$, and is independent of ψ_v. The representation $\omega_{\psi_v} : \mathrm{Mp}(W) \to \mathrm{GL}(X)$ is called the Weil-representation.

2.3 Representation theory of local groups

Let $\mathbb{K}_v$ be as in Section 2.1.3 and with residue characteristic p and absolute value $|-|_v$. Let $\ell \neq p$ be a prime and G a connected reductive group over $\mathbb{K}_v$ with center $Z(G)$. Moreover, we fix an algebraically closed field $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \mathbb{C}\}$, fix a square root of q_v in R, and we let $\mathfrak{Rep}_G = \mathfrak{Rep}_G(R)$ be the category of smooth, finite length representations of $G(\mathbb{K}_v)$ over R with morphisms the $G(\mathbb{K}_v)$-equivariant R-linear morphisms. From now on we assume all representations to be smooth. For $\pi \in \mathfrak{Rep}_G$ we denote by $\pi^\vee$ the contragradient representation of π, given by the $G(\mathbb{K}_v)$-equivariant morphisms from π to the trivial representations, which admit an open compact subgroup fixing it. Then the contravariant functor

$$(-)^{\vee}: \mathfrak{Rep}_G(R) \to \mathfrak{Rep}_G(R)$$

is well-defined, an involution and exact. The set of isomorphism classes of irreducible representations in $\mathfrak{Rep}_G(R)$ is denoted by $\mathfrak{Irr}_G(R)$. If $\pi, \pi' \in \mathfrak{Rep}_G$, we write $\pi \hookrightarrow \pi'$ if π is a subrepresentation of π' and $\pi \twoheadrightarrow \pi'$ if π' is quotient of π. The *socle* respectively *cosocle* of π, *i.e.* the maximal semi-simple subrepresentation respectively quotient, are denoted by $\mathrm{soc}(\pi)$ respectively $\cos(\pi)$.

If $\iota: H \to G$ is a morphism of reducitve groups over $\mathbb{K}_v$ and $\pi \in \mathfrak{Rep}_G(R)$ we write $\iota^*(\pi)$ for the pullback of π to $H(\mathbb{K}_v)$. We let $\mathfrak{R}_G(R)$ be the Grothendieck group of $\mathfrak{Rep}_G(R)$ and denote the natural map

$$[-]: \mathfrak{Rep}_G(R) \to \mathfrak{R}_G(R).$$

For an element $[\mathrm{X}] \in \mathfrak{R}_G(R)$, we write $[\mathrm{X}] \geq 0$ if the multiplicity of all isomorphism classes of irreducible representations in $[\mathrm{X}]$ is greater or equal than 0 and we write $[\mathrm{X}] \geq [\mathrm{Y}]$ if $[\mathrm{X}] - [\mathrm{Y}] \geq 0$.

For $P = M \rtimes U$ a closed subgroup of $G(\mathbb{K}_v)$ such that $M \cap U = \{1\}$ and M normalizes U, and (τ, V) a smooth representation of M we let $\# - \mathrm{Ind}_P^{G(\mathbb{K}_v)}(\tau)$ be the compactly supported induction of the inflation of τ to P to $G(\mathbb{K}_v)$. The underlying vector space is the set of all functions $f: G(\mathbb{K}_v) \to V$ satisfying

1. $f(mug) = \tau(m) f(g)$ for all $m \in M$, $u \in U$ and $g \in G(\mathbb{K}_v)$,
2. there exists an open compact subgroup $K_f \subseteq G(\mathbb{K}_v)$ such that for all $k \in K_f$ and $g \in G(\mathbb{K}_v)$ $f(gk) = f(g)$,
3. f is compactly supported modulo P.

The group G acts then on $\# - \mathrm{Ind}_P^{G(\mathbb{K}_v)}(\tau)$ by right translations. We denote by $\mathrm{Ind}_P^{G(\mathbb{K}_v)}(\tau)$ the normalized compactly-supported induction of τ, *i.e.*

$$\mathrm{Ind}_P^{G(\mathbb{K}_v)}(\tau) \coloneqq \# - \mathrm{Ind}_P^{G(\mathbb{K}_v)}(\delta_P^{\frac{1}{2}} \tau),$$

where δ_P is the modular character of P. For (π, V) a representation of G, we denote by $\# - r_P(\pi)$ the reduction of π to M. To be more precise, let $V^U \subseteq V$ be the subspace of V spanned by the vectors of the form

$$\{\pi(u)v - v, \text{ for } u \in U, v \in V\}$$

and let $V_U \coloneqq V/V^U$. Then π restricted to M gives a well-defined action of M on V_U, which is by definition the reduction of V. We denote by $r_P(\pi)$ normalized reduction, *i.e.*

$$r_P(\pi) \coloneqq \delta_P^{-\frac{1}{2}} (\# - r_P(\pi)).$$

If $P = M \rtimes U$ is a parabolic subgroup of G defined over $\mathbb{K}_v$ with respective Levi-decomposition, $P(\mathbb{K}_v) \backslash G(\mathbb{K}_v)$ is compact and therefore the third condition on the functions $f \in \mathrm{Ind}_{P(\mathbb{K}_v)}^{G(\mathbb{K}_v)}(\tau)$ is superfluous. In this case we call this induction *parabolic induction* and denote it by abuse of notation $\mathrm{Ind}_P^G(\tau)$. Moreover, we call normalized reduction with respect to $U(\mathbb{K}_v)$ the *Jacquet-functor*, the image of a representation under the Jacquet-functor $r_P(\pi) = r_{P(\mathbb{K}_v)}(\pi)$ its *Jacquet module*, and we thus obtain functors

$$\mathrm{Ind}_P^G\colon \mathfrak{Rep}_R(M) \to \mathfrak{Rep}_R(G),\ r_P\colon \mathfrak{Rep}_R(G) \to \mathfrak{Rep}_R(M),$$

which are exact, send representations of finite length to representations of finite length and satisfy $\mathrm{Ind}_P^G(\tau)^\vee \cong \mathrm{Ind}_P^G(\tau^\vee)$ and $r_{\overline{P}}(\tau^\vee) \cong r_P(\tau)^\vee$. Moreover, r_P is left adjoint to Ind_P^G and Ind_P^G is left adjoint to $r_{\overline{P}}$, *i.e.*

$$\mathrm{Hom}_M(r_P(\pi),\tau) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\tau),\ \text{(Frobenius reciprocity)}$$

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\tau),\pi) = \mathrm{Hom}_M(\tau, r_{\overline{P}}(\pi)),\ \text{(Bernstein reciprocity).} \tag{1}$$

for all $\pi \in \mathfrak{Rep}_G$ and $\tau \in \mathfrak{Rep}_M$. Let P be a parabolic subgroup of G with Levi-component $M = M_1 \times M_2$ and ρ a representation of M_1. If π is a representation of G, denote by $\mathrm{Jac}_\rho(\pi)$ the (M_1,ρ)-invariant vectors of $r_{\overline{P}}(\pi)$, *i.e.* the maximal representation σ such that $\rho \otimes \sigma \hookrightarrow r_{\overline{P}}(\pi)$. An irreducible representation ρ of $G(\mathbb{K}_v)$ is called *cuspidal* if $r_P(\rho) = 0$ for all nontrivial parabolic subgroups P of G and *supercuspidal* if ρ is not the subquotient of a nontrivially parabolically induced representation. We denote the corresponding subsets of $\mathfrak{Irr}_G$ as $\mathfrak{C}_G$ resp. $\mathfrak{S}_G$. From Frobenius reciprocity follows immediately that if ρ is supercuspidal it is cuspidal. If R is of characteristic 0, also the other implication, *i.e.* cuspidal implying supercuspidal, is true. However, if $R = \overline{\mathbb{F}}_\ell$, there exist cuspidal non-supercuspidal representations, as we will discuss in more detail later on.

2.3.1 Geometric Lemma We will now quickly recap the theory of ℓ-spaces and their sheaves in order to state the Geometric Lemma and give a short sketch of its proof. For a precise treatment see [16]. A topological space X is called an ℓ-space if it is Hausdorff and the open compact sets form a base of the topology. The ring of smooth, *i.e.* locally constant functions on X is denoted by $C^\infty(X)$ and the ring of compactly supported smooth functions, also called Schwartz-functions, is denoted by $S(X)$. An ℓ-sheaf on X is a sheaf $\mathscr{F}$ which is a module over the sheaf of smooth functions. The category of ℓ-sheaves on X is denoted by $\mathrm{Sh}(X)$. For $\mathscr{F} \in \mathrm{Sh}(X)$ we let $\mathscr{F}(X)$ be its global sections and $\mathscr{F}_c(X)$ its compactly supported global sections.

Proposition 2.3.1 ([16] Proposition 1.14)**.** *The functor $\mathscr{F} \mapsto \mathscr{F}_c(X)$ induces an equivalence of categories from $\mathrm{Sh}(X)$ to $S(X)$-modules M such that*

$$S(X) \cdot M = M.$$

Proposition 2.3.2 ([16] 1.16)**.** *If U is an open subspace of X, $Z := X \setminus U$ and $\mathscr{F} \in \mathrm{Sh}(X)$, there is a short exact sequence*

$$0 \to \mathscr{F}_c(U) \to \mathscr{F}_c(X) \to \mathscr{F}_c(Z) \to 0$$

An ℓ-group H is a topological group H, which is an ℓ-space. For example, if G is a reductive group over a $\mathbb{K}_v$, then its $\mathbb{K}_v$-points are an ℓ-group. An action of an ℓ-group H on an ℓ-sheaf $\mathscr{F} \in \mathrm{Sh}(X)$ is a continous action of H on X and a morphism

$$\gamma\colon H \to \mathrm{Aut}(X, \mathscr{F}),$$

such that H acts on $\mathscr{F}_c(X)$ smoothly. For a fixed action γ_0 of H on X, we let $\mathrm{Sh}(X,H)$ be the category of H-sheaves, where the action of H restricted to X is γ_0. We denote the functor $\mathscr{F} \mapsto \mathscr{F}_c(X)$ by

$$\mathrm{Sec}\colon \mathrm{Sh}(X,H) \to \mathfrak{Rep}_H.$$

Moreover, if Q is closed subgroup of H and Z a locally closed and Q-invariant subspace of X there exists a restriction functor

$$\mathrm{Res} = \mathrm{Res}_{Z,Q}\colon \mathrm{Sh}(X,H) \to \mathrm{Sh}(Z,Q).$$

Proposition 2.3.3 ([16] Proposition 2.23)**.** *Let P be a closed subgroup of H and set $X = P\backslash H$. Then the functor*

$$\mathrm{Res}\colon \mathrm{Sh}(X,H) \to \mathrm{Sh}(*,P) = \mathfrak{Rep}_P$$

has an inverse, denoted by

$$\mathrm{Ind}\colon \mathrm{Sh}(*,P) \to \mathrm{Sh}(X,H).$$

Moreover, $\mathrm{Sec} \circ \mathrm{Ind}\colon \mathfrak{Rep}_P \to \mathfrak{Rep}_H$ *is* $\# - \mathrm{Ind}_P^H$.

Let $P = M_P \rtimes N_P$, $Q = M_Q \rtimes N_Q$ be closed subgroups of H with $M_P \cap N_P = \{1\}$, M_P normalizes N_P and similarly for Q such that there only finitely many Q-orbits in $P\backslash H$. Define for $\mathfrak{w} \in P\backslash H/Q$ represented by an element w the groups

$$M_P' \coloneqq M_P \cap w^{-1}M_Q w,\ M_Q' \coloneqq wM_P'w^{-1},\ N_Q' \coloneqq M_P \cap w^{-1}N_Q w,$$

$$N_P' \coloneqq M_Q \cap wN_P w^{-1}.$$

Let

$$\delta_1 \coloneqq \delta_{N_P}^{\frac{1}{2}} \cdot \delta_{N_P \cap w^{-1}Qw}^{-\frac{1}{2}},\ \delta_2 \coloneqq \delta_{N_Q}^{\frac{1}{2}} \cdot \delta_{N_Q \cap wPw^{-1}}^{-\frac{1}{2}}$$

be characters of M_P' respectively M_Q'. Finally, let $w\colon \mathfrak{Rep}_{M_P'} \to \mathfrak{Rep}_{M_Q'}$ be the pullback by conjugation by w and set $\delta \coloneqq \delta_1 \circ w^{-1} \circ \delta_2$. Order the Q-orbits of $P\backslash H$ as $\mathfrak{w}_1, \dots, \mathfrak{w}_l$ such that $\overline{O_{\mathfrak{w}_i}} = \bigcup_{i \le j} O_{\mathfrak{w}_j}$, where $O_{\mathfrak{w}}$ is the Q-orbit in $P\backslash H$ associated to $\mathfrak{w}$.

Define for $\mathfrak{w} \in P\backslash H/Q$ the functor

$$F(w) \coloneqq \mathrm{Ind}_{M_P'N_P'}^{M_Q} \circ w \circ \delta \circ r_{M_Q'N_Q'}\colon \mathfrak{Rep}_M \to \mathfrak{Rep}_N.$$

Then the following holds.

Lemma 2.3.4 (Geometric Lemma, [17] Lemma 2.11)**.** *The functor*

$$F \coloneqq r_Q \circ \mathrm{Ind}_P^H\colon \mathfrak{Rep}_M \to \mathfrak{Rep}_N$$

has a filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l = F$ *with subquotients* $F_{i-1}\backslash F_i \cong F(w_i)$.

Let us comment on its proof, as the details will be important later on. An induced representation $\mathrm{Ind}_P^H(\tau)$ corresponds to H-sheaf $\mathscr{F}$ on $P\backslash H$. Note that we have a filtration of $P\backslash H$ by

$$\varnothing \subseteq O_{\mathfrak{w}_1} \subseteq \ldots \subseteq \bigcup_{j \le i} O_{\mathfrak{w}_j} \subseteq \ldots \subseteq \bigcup_{j \le l} O_{\mathfrak{w}_j} = P\backslash H$$

inducing a filtration of Q-representations

$$0 \subseteq \mathscr{F}_c(O_{\mathfrak{w}_1}) \subseteq \ldots \subseteq \mathscr{F}_c(\bigcup_{j \le i} O_{\mathfrak{w}_j}) \subseteq \ldots \subseteq \mathrm{Ind}_P^H(\tau).$$

We then set $\tau_{\mathfrak{w}} := r_Q(\mathscr{F}_c(O_{\mathfrak{w}}))$, which are the subquotients of the above filtration of $r_Q(\mathrm{Ind}_P^H(\tau))$ by Proposition 2.3.2. One can then define the isomorphism

$$A_{\mathfrak{w}}\colon \tau_{\mathfrak{w}} \to F(w)(\tau)$$

by

$$A_{\mathfrak{w}}([f])(m) = \int_{N'_P\backslash N_Q} p(f(wmn))\,\mathrm{d}n,$$

where $m \in M_Q$, $\mathrm{d}n$ is a Haar-measure on N_Q, $[f]$ is the equivalence class of a section in $\tau_{\mathfrak{w}}$ and p is the projection $p\colon \tau \twoheadrightarrow r_{N'_Q}(\tau)$.

2.3.2 Integral structures Let $\overline{\mathbb{Z}}_\ell$ be the ring of integers of $\overline{\mathbb{Q}}_\ell$ with residue field $\overline{\mathbb{F}}_\ell$. We choose the square roots in R in a compatible way, *i.e.* if $R = \overline{\mathbb{Q}}_\ell$, we let $\sqrt{q_v} \in \overline{\mathbb{Z}}_\ell$, such that its reduction modulo the maximal ideal in $\overline{\mathbb{Z}}_\ell$ gives the chosen square root in $\overline{\mathbb{F}}_\ell$.

A $\overline{\mathbb{Q}}_\ell$-representation (π, V) of $G(\mathbb{K}_v)$ is called *integral* if it is admissible and admits an integral structure, *i.e.* a $G(\mathbb{K}_v)$-stable $\overline{\mathbb{Z}}_\ell$-submodule $\mathfrak{o}$ of V such that the natural map $\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell \to V$ is an isomorphism. If $P = M \rtimes U$ is a parabolic subgroup of $G(\mathbb{K}_v)$ and σ a representation of M with integral structure $\mathfrak{o}$, then $\mathrm{Ind}_P^G(\mathfrak{o})$ is an integral structure of $\mathrm{Ind}_P^G(\sigma)$, see [123, II.4.14]. For a representation π with integral structure $\mathfrak{o}$ one can consider the reduction mod ℓ of π, which is defined as

$$\mathrm{r}_\ell(\pi) := \mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell.$$

This is not an invariant of π, it depends on the chosen integral structure. However, its image in the Grothendieck group is. This leads to the following theorem.

Theorem 2.3.5 ([123, II.4.12],[102, Theorem A], [123, II.4.14, II.5.11])**.** *Let $\mathfrak{R}_G^{en}(\overline{\mathbb{Q}}_\ell)$ be the subgroup of $\mathfrak{R}_G(\overline{\mathbb{Q}}_\ell)$ generated by irreducible integral representations. The morphism* reduction mod ℓ *between the groups*

$$\mathrm{r}_\ell : \mathfrak{R}_G^{en}(\overline{\mathbb{Q}}_\ell) \to \mathfrak{R}_G(\overline{\mathbb{F}}_\ell)$$

is well defined. Moreover, r_ℓ satisfies the following properties.

1. *If $\widetilde{\rho}$ is a cuspidal representation over $\overline{\mathbb{Q}}_\ell$, it admits an integral structure if and only if its central character is integral.*
2. *If ρ is a supercuspidal representation over $\overline{\mathbb{F}}_\ell$ then there exists an integral cuspidal representation $\tilde{\rho}$ of $G(\mathbb{K}_v)$ over $\overline{\mathbb{Q}}_\ell$ such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$. We refer to such $\widetilde{\rho}$ as a lift of ρ.*
3. *The map r_ℓ commutes with parabolic induction,* i.e. *the natural map*

$$\mathrm{Ind}_P^G(\mathfrak{o}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \to \mathrm{Ind}_P^G(\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_l)$$

is an isomorphism.

2.3.3 Representation theory of the general linear group Let $\mathbb{D}_v$ be a central division algebra of degree d_v over $\mathbb{K}_v$, GL'_n the general linear group of $\mathbb{D}_v$ and denote $G_n := \mathrm{GL}'_n(\mathbb{K}_v)$. We write

$$\mathfrak{Rep}_n = \mathfrak{Rep}_n(R) = \mathfrak{Rep}_{G_n}(R),\ \mathfrak{R}_n = \mathfrak{R}_n(R) = \mathfrak{R}_{G_n}(R),$$

$$\mathfrak{Irr}_n = \mathfrak{Irr}_n(R) = \mathfrak{Irr}_{G_n}(R).$$

and

$$\mathfrak{Rep} = \mathfrak{Rep}_R = \bigcup_{n\in\mathbb{N}} \mathfrak{Rep}_{G_n}(R),\ \mathfrak{R} = \mathfrak{R}(R) = \bigcup_{n\in\mathbb{N}} \mathfrak{R}_n(R),$$

$$\mathfrak{Irr} = \mathfrak{Irr}(R) = \bigcup_{n\in\mathbb{N}} \mathfrak{Irr}_{G_n}(R).$$

Similarly, we define

$$\mathfrak{C} = \mathfrak{C}(R) = \bigcup_{n\in\mathbb{N}} \mathfrak{C}_{G_n}(R),\ \mathfrak{S} = \mathfrak{S}(R) = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_{G_n}(R).$$

We let

$$G_\alpha \coloneqq G_{\alpha_1} \times \ldots \times G_{\alpha_t}$$

and every irreducible representation π of G_α is then of the form

$$\pi \cong \pi_1 \otimes \ldots \otimes \pi_t,$$

for $[\pi_i] \in \mathfrak{Irr}_{\alpha_i}$ and $i \in \{1,\ldots,t\}$. As above let $\mathfrak{Rep}_\alpha$ be the category of smooth, finite length representations of G_α. The Grothendieck group of representations in $\mathfrak{Rep}_\alpha$ of finite length is denoted by $\mathfrak{R}_\alpha$, and we write $\mathfrak{Irr}_\alpha$ for the set of isomorphism classes of irreducible representations of G_α. If $\alpha = (n)$ we write $r_\beta \coloneqq r_{P_\beta}$, $\mathrm{Ind}_\beta \coloneqq \mathrm{Ind}_{P_\beta}^{G_n}$ and if

$$(\beta_{1,1},\ldots,\beta_{i_1,1},\beta_{1,2},\ldots,\beta_{i_2,2},\ldots,\beta_{1,t},\ldots,\beta_{i_t,t})$$

is a subcomposition of α and

$$\pi = \pi_{1,1} \otimes \ldots \otimes \pi_{i_1,1} \otimes \pi_{1,2} \otimes \ldots \otimes \pi_{i_2,2} \otimes \ldots \otimes \pi_{1,t} \otimes \ldots \otimes \pi_{i_t,t}$$

is an object of $\mathfrak{Rep}_\beta$, we write

$$\pi_{1,1} \times \ldots \times \pi_{i_1,1} \otimes \pi_{1,2} \times \ldots \times \pi_{i_2,2} \otimes \ldots \otimes \pi_{1,t} \times \ldots \times \pi_{i_t,t} \coloneqq \mathrm{Ind}_{P_\beta}^{G_\alpha}(\pi).$$

In particular, parabolic induction induces a product

$$\times : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}, ([\pi_1],[\pi_2]) \mapsto [\pi_1 \times \pi_2],$$

which equips $\mathfrak{R}$ with the structure of an commutative algebra, see [123, 1.16] for $\ell > 2$, [16, Theorem 1.9] for the case $R = \mathbb{C}$ and [101, Proposition 2.6] in general. We also use this opportunity to state that the modular character of the $\mathbb{K}_v$-points of the parabolic subgroup corresponding to a composition $\alpha = (\alpha_1,\ldots,\alpha_k)$ is given for $m = (m_1,\ldots,m_k) \in_\alpha$ by

$$\delta_{P_\alpha}(m) = \prod_{i=1}^{k} |\det{}'(m_i)|^{d_v\delta_i},\ \delta_i = -\sum_{j=1}^{i-1} \alpha_j + \sum_{j=i+1}^{k} \alpha_j.$$

If π is a representation we will write $\pi^k \coloneqq \overbrace{\pi \times \ldots \times \pi}^{k}$.

Combinatorial Geometric Lemma Finally, we recall the following combinatorial description of the Geometric Lemma in the case of G_n, for example [17]. Let $\alpha = (n)$, $\alpha_1 = (m_1, \dots, m_t)$, $\alpha_2 = (n_1, \dots, n_{t'})$, $\pi_1 \otimes \dots \otimes \pi_t \in \mathfrak{Rep}_{\alpha_1}$ and $\mathrm{Mat}^{\alpha_1,\alpha_2}$ the set of $t \times t'$ matrices $B = (b_{i,j})$ with non-negative integer coefficients such that for all $(i,j) \in \{1, \dots, t\} \times \{1, \dots, t'\}$

$$\sum_{j=1}^{t'} b_{i,j} = m_i, \sum_{i=1}^{t} b_{i,j} = n_j.$$

For $B \in \mathrm{Mat}^{\alpha_1,\alpha_2}$ and $i \in \{1, \dots, t\}$, $\beta_{B,i} \coloneqq (b_{1,i}, \dots, b_{i,t'})$ is a composition of m_i. Let l_i be the length of $r_{\beta_{B,i}}(\pi_i)$ and write the irreducible decomposition factors of $r_{\beta_{B,i}}(\pi_i)$ as

$$\sigma_i^{(k)} = \sigma_{i,1}^{(k)} \otimes \dots \otimes \sigma_{i,t'}^{(k)}, \ k \in \{1, \dots, l_i\}$$

with $[\sigma_{i,j}^{(k)}] \in \mathfrak{Irr}_{b_{i,j}}$. For $j \in \{1, \dots, t'\}$ and a sequence $\underline{k} = (k_1, \dots, k_t)$ with $k_i \in \{1, \dots, l_i\}$ set

$$\tau_j^{B,\underline{k}} \coloneqq \sigma_{1,j}^{(k_1)} \otimes \dots \otimes \sigma_{t,j}^{(k_t)}.$$

Then

$$[r_{\alpha_2}(\pi_1 \times \dots \times \pi_t)] = \sum_{B \in \mathrm{Mat}^{\alpha_1,\alpha_2}, \underline{k}} [\tau_1^{B,\underline{k}} \otimes \dots \otimes \tau_{t'}^{B,\underline{k}}].$$

We will also use this opportunity to state two lemmas, which were proven in [82] over $\overline{\mathbb{Q}}_\ell$, but the presented proofs work as well *muta mutandis* over $\overline{\mathbb{F}}_\ell$.

Lemma 2.3.6 ([82, Lemma 2.1]). *Let $P = M \rtimes U$ be a parabolic subgroup of $G = G_n$ and $Q_1 = N_1 \rtimes V_1$, $Q_2 = N_2 \rtimes V_2$ parabolic subgroups of G containing P. Let $\pi \in \mathfrak{Rep}_M$, $\tau_1 \in \mathfrak{Rep}_{N_1}$, $\tau_2 \in \mathfrak{Rep}_{N_2}$ such that there exist injective morphisms*

$$\iota_1 \colon \tau_1 \to \mathrm{Ind}_{N_1 \cap P}^{N_1}(\pi), \ \iota_2 \colon \tau_2 \to \mathrm{Ind}_{N_2 \cap P}^{N_2}(\pi)$$

and there exists

$$\iota_3 \colon \mathrm{Ind}_{Q_1}^G(\tau_1) \to \mathrm{Ind}_{Q_2}^G(\tau_2)$$

such that

$$\mathrm{Ind}_{Q_1}^G(\iota_1) = \mathrm{Ind}_{Q_2}^G(\iota_2) \circ \iota_3.$$

Then there exists a representation τ of M and an injective map $\iota \colon \tau \to \pi$ such that ι_1 factors through $\mathrm{Ind}_{N_1 \cap P}^{N_1}(\iota)$ and $\mathrm{Ind}_{N_2 \cap P}^{N_2}(\iota)$ factors through ι_2.

Corollary 2.3.1 ([82, Corollary 2.2], [67, Lemma 3.1]). *Let $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ and*

$$\pi_i \in \mathfrak{Rep}_{n_i}, \ i \in \{1, 2, 3\}.$$

Let σ be a subrepresentation of $\pi_1 \times \pi_2$ and τ a subrepresentation of $\pi_2 \times \pi_3$ such that as subrepresentations of $\pi_1 \times \pi_2 \times \pi_3$

$$\sigma \times \pi_3 \hookrightarrow \pi_1 \times \tau.$$

Then there exists a subrepresentation ω of π_2 such that

$$\sigma \hookrightarrow \pi_1 \times \omega, \ \omega \times \pi_3 \hookrightarrow \tau.$$

In particular, if π_2 is irreducible and $\sigma \neq 0$, then $\sigma = \pi_2$

Multisegments We will now fix our notations regarding segments, the central combinatorial objects in the classification of [16] and [101]. Let ρ, ρ' be cuspidal representations of G_m and $G_{m'}$. Then by [102, § 4.5] there exists an unramified character v_ρ of G_m such that $\rho \times \rho'$ is irreducible if and only if $\rho' \cong \rho v_\rho$ or $\rho' \cong \rho v_\rho^{-1}$. We set

$$\mathbb{Z}[\rho] := \{[\rho v_\rho^k], k \in \mathbb{Z}\}$$

and let $o(\rho)$ be the cardinality of $\mathbb{Z}[\rho]$. One can associate to ρ a finite extension $\mathrm{k}(\rho)$ of k_v of cardinality $q(\rho)$, see [102, Section 4], and define a number $e(\rho)$ as the smallest integer k such that

$$1 + q(\rho) + \ldots + q(\rho)^{k-1} = 0$$

as elements in R. If the characteristic of R is 0, we set $e(\rho) = \infty$. Then $e(\rho)$ satisfies

$$e(\rho) = \begin{cases} o(\rho) & \text{if } o(\rho) > 1, \\ \ell & \text{if } o(\rho) = 1, \end{cases}$$

by [102, Lemma 4.41].

There exist surjective maps with finite fibers

$$\mathrm{cusp} : \mathfrak{Irr} \to \mathbb{N}(\mathfrak{C}) \text{ and } \mathrm{scusp} : \mathfrak{Irr} \to \mathbb{N}(\mathfrak{S})$$

called *cuspidal support* and *supercuspidal support*. They are defined by

$$\mathrm{cusp}^{-1}([\rho_1] + \ldots + [\rho_k]) = \{[\pi] \in \mathfrak{Irr} : \pi \hookrightarrow \rho_{\sigma(1)} \times \ldots \times \rho_{\sigma(k)} \text{ for some } \sigma \in S_k\},$$

$$\mathrm{scusp}^{-1}([\rho_1] + \ldots + [\rho_k]) = \{[\pi] \in \mathfrak{Irr} : [\pi] \leq [\rho_1 \times \ldots \times \rho_k]\}.$$

We call k the length of the cuspidal support $[\rho_1] + \ldots + [\rho_k]$. A cuspidal representation ρ is called $\square$-*irreducible* if $o(\rho) > 1$ and we say an irreducible representation has $\square$-irreducible cuspidal support if its cuspidal support consists of $\square$-irreducible representations.

Lemma 2.3.7. *Let π be an irreducible representation and $\mathfrak{s} = \mathrm{cusp}(\pi)$. We write*

$$\mathfrak{s} = \mathfrak{s}_1 + \ldots + \mathfrak{s}_k$$

with $\mathfrak{s}_i \in \mathbb{N}(\mathbb{Z}[\rho_i])$ and $\mathbb{Z}[\rho_i] \neq \mathbb{Z}[\rho_j]$ for $i \neq j$. Then there exist irreducible representations $\pi_1, \ldots, \pi_k$ with $\mathrm{cusp}(\pi_i) = \mathfrak{s}_i$ and

$$\pi \cong \pi_1 \times \ldots \times \pi_k$$

Proof. Since parabolic induction is exact, we can find irreducible representations $\pi_1, \ldots, \pi_k$ such that $\mathrm{cusp}(\pi_i) = \mathfrak{s}_i$ and $\pi \hookrightarrow \pi_1 \times \ldots \times \pi_k$. In [101, Proposition 5.9] it was proven that if $k = 2$, $\pi_1 \times \ldots \times \pi_k$ is irreducible and their method extends easily to the general case $k > 2$. Thus the claim follows. $\square$

For ρ a cuspidal representation of G_m and integers $a \leq b$ we define a *segment* as the sequence

$$[a, b]_\rho := (\rho v_\rho^a, \rho v_\rho^{a+1}, \ldots, \rho v_\rho^b).$$

The length of such a segment is

$$l([a, b]_\rho) := b - a + 1$$

and its degree is

$$\deg([a,b]_\rho) := m(b-a+1).$$

We will also sometimes consider a formal, empty segment [] of length and degree 0. Two segments $[a,b]_\rho$ and $[a',b']_{\rho'}$ are said to be equivalent if they have the same length and $[\rho v_\rho^{a+i}] = [\rho' v_{\rho'}^{a'+i}]$ for all $i \in \mathbb{Z}$. The set of equivalence classes of segments will be denoted by $\mathcal{S}eg$ and for a fixed ρ the set of segments of the form $[a,b]_\rho$ will be denoted as $\mathcal{S}eg(\rho)$ and we call them ρ-segments.

If $\Delta = [a,b]_\rho$ is a segment, we let $a_\rho(\Delta) = a$ and $b_\rho(\Delta) = b$ seen as elements in $\mathbb{Z}/(o(\rho)\mathbb{Z})$. We write

$$^-\Delta := [a+1,b]_\rho,\ \Delta^- := [a,b-1]_\rho,\ \Delta^+ := [a,b+1]_\rho,\ ^+\Delta := [a-1,b]_\rho,\ [a]_\rho := [a,a]_\rho$$

and $\Delta^\vee := [-b,-a]_{\rho^\vee}$. The operation $\Delta \mapsto {}^-\Delta$ will sometimes be called *shortening the segment* Δ *by* 1 *on the left* and similarly for the others. If $a+1 > b$, $^-\Delta$ is the empty segment and similarly for the other operations. A segment $\Delta = [a,b]_\rho$ *precedes* $\Delta' = [a',b']_{\rho'}$ if the sequence

$$(\rho v_\rho^a, \ldots, \rho v_\rho^b, \rho' v_{\rho'}^{a'}, \ldots, \rho' v_{\rho'}^{b'})$$

contains a subsequence which is, up to isomorphism, a segment of length greater than $l(\Delta)$ and $l(\Delta')$. The segments Δ and Δ' are called *unlinked* if Δ does not precede Δ' and Δ' does not precede Δ.

A *multisegment* $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ is a formal finite sum of equivalence classes of segments, *i.e.* an element in $\mathbb{N}[\mathcal{S}eg]$. We extend the notion of length and degree linearly to multisegments as well as the dual operation $(-)^\vee$. The set of multisegments is denoted by $\mathcal{MS}$ and $\mathcal{MS}(\rho)$ is the set of multisegments $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ such that the equivalence class of Δ_i is contained in $\mathcal{S}eg(\rho)$ for all $i \in \{1, \ldots, n\}$. A multisegment $\mathfrak{m}$ contains a multisegment $\mathfrak{m}'$ if there exists a multisegment $\mathfrak{m}''$ such that $\mathfrak{m}$ is equivalent to $\mathfrak{m}' + \mathfrak{m}''$. Moreover, $\mathfrak{m}$ is called *aperiodic* if it does not contain a multisegment of the form

$$[a,b]_\rho + [a+1,b+1]_\rho + \ldots + [a+e(\rho)-1, b+e(\rho)-1]_\rho.$$

We let $\mathcal{M}_{ap}$ be the set of aperiodic multisegments and $\mathcal{M}(\rho)_{ap} := \mathcal{M}_{ap} \cap \mathcal{MS}(\rho)$. Let $\mathfrak{m} = [a_1,b_1]_{\rho_1} + \ldots + [a_k,b_k]_{\rho_k}$ be a multisegment. We set

$$\mathfrak{m}^1 := [b_1]_{\rho_1} + \ldots + [b_k]_{\rho_k},$$

$$\mathfrak{m}^- := [a_1, b_1 - 1]_{\rho_1} + \ldots + [a_k, b_k - 1]_{\rho_k}$$

and define recursively $\mathfrak{m}^{-(s+1)} := (\mathfrak{m}^{-s})^-$ and $\mathfrak{m}^{s+1} := (\mathfrak{m}^{-s})^1$. Let l be the largest natural number such that $\mathfrak{m}^l \neq 0$ and define the composition

$$\mu_\mathfrak{m} := (\deg(\mathfrak{m}^1), \ldots, \deg(\mathfrak{m}^l))$$

of $\deg(\mathfrak{m})$.

We call $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ unlinked if Δ_i and Δ_j are pairwise unlinked for all $i \neq j$, $i,j \in \{1, \ldots, k\}$. As for representations, there exists a surjective map

$$\mathrm{cusp}_\mathcal{M} : \mathcal{M}_{ap} \to \mathbb{N}(\mathfrak{C}),$$

which maps $[a,b]_\rho \mapsto [\rho v_\rho^a] + \ldots + [\rho v_\rho^b]$ and is extended linearly to multisegments. We call a multisegment *banal* if its cuspidal support does not contain the cuspidal line $\mathbb{Z}[\rho]$.

Classification We are now going to recall the constructions of [101] and [123]. Let ρ be a supercuspidal representation of G_m and $n = e(\rho)\ell^r$ for some $r \in \mathbb{Z}_{\geq 0}$. Then the representation

$$\rho \times \rho v_\rho \times \ldots \times \rho v_\rho^{n-1}$$

contains a unique cuspidal subquotient, which is denoted by $\mathrm{St}(\rho, n)$ and $o(\mathrm{St}(\rho, n)) = 1$. Moreover, every cuspidal non-supercuspidal representation is of the above form and if $\mathrm{St}(\rho, n) \cong \mathrm{St}(\rho', n')$ then $n = n'$ and $\mathbb{Z}[\rho] = \mathbb{Z}[\rho']$, see [101, Section 6]. In particular, every $\square$-irreducible cuspidal representation is supercuspidal. We let $\mathfrak{C}_m^\square$ be the set of isomorphism classes of $\square$-irreducible cuspidal representations of G_m and $\mathfrak{C}^\square := \bigcup_{m\geq 0} \mathfrak{C}_m^\square$. We write

$$\mathrm{scusp}_{\mathcal{M}} \colon \mathcal{M} \to \mathbb{N}(\mathfrak{S})$$

for the linear map sending a segment

$$[a, b]_{\mathrm{St}(\rho,n)} \mapsto \sum_{i=0}^{n-1} ([\rho v_\rho^{a+i}] + \ldots + [\rho v_\rho^{b+i}]).$$

Fix a cuspidal representation ρ of G_m. To each segment $[a, b]_\rho$ we are going to associate two irreducible representations $\mathrm{Z}([a, b]_\rho)$ and $\mathrm{L}([a, b]_\rho)$. This requires us to first consider $\mathscr{H}_R(n, q(\rho))$, the R-Hecke-algebra generated by $S_1, \ldots, S_{n-1}, X_1, \ldots, X_n$ satisfying the relations

1. $(S_i + 1)(S_i - q(\rho)) = 0$, $1 \leq i \leq n - 1$
2. $S_i S_j = S_j S_i$, $|i - j| \geq 2$
3. $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$, $1 \leq i \leq n - 2$
4. $X_i X_j = X_j X_i$, $1 \leq i, j \leq n$
5. $X_j S_i = S_i X_j$, $i \notin \{j, j - 1\}$
6. $S_i X_i S_i = q(\rho) X_{i+1}$, $1 \leq i \leq n - 1$

We let $\mathfrak{Irr}(\Omega_{\rho,n})^*$ be the set of all isomorphism classes of irreducible representations π of G_{nm} whose cuspidal support is up to inertial equivalence contained in $\mathbb{N}(\mathbb{Z}[\rho])$. By [102, §4.4] there exists a bijection

$$\xi_{\rho,n} \colon \mathfrak{Irr}(\Omega_{\rho,n})^* \xrightarrow{\cong} \{\text{isomorphism classes of simple } \mathscr{H}(n, q(\rho))\text{-modules}\}.$$

Let $a, b \in \mathbb{Z}$ such that $b - a + 1 = n$. Then $\mathscr{H}_R(n, q(\rho))$ has two 1-dimensional modules, $\mathscr{Z}(a, b)$, defined by

$$S_i \mapsto q(\rho),\ X_j \mapsto q(\rho)^{a+j-1}$$

and $\mathcal{L}(a, b)$, defined by

$$S_i \mapsto -1,\ X_j \mapsto q(\rho)^{b-j+1}.$$

To a segment $[a, b]_\rho$ we can associate now an irreducible subrepresentation $\mathrm{Z}([a, b]_\rho)$ resp. irreducible quotient $\mathrm{L}([a, b]_\rho)$ of

$$\rho v_\rho^a \times \ldots \times \rho v_\rho^b$$

by demanding

$$\xi_{\rho,n}(\mathrm{Z}([a, b]_\rho) = \mathscr{Z}(a, b) \text{ and } \xi_{\rho,n}(\mathrm{L}([a, b]_\rho) = \mathcal{L}(a, b).$$

For the empty segment, we let $Z([])$ be the trivial representation of the trivial group. Note that equivalent segments give rise to isomorphic representations. The representations $\mathrm{Z}([a,b]_\rho)$ and $\mathrm{L}([a,b]_\rho)$ behave very well under parabolic restriction.

Lemma 2.3.8 ([101, Lemma 7.16])**.** *Let $k \in \{a,\dots,b\}$. Then*

$$r_{((k-a)m,(b-k+1)m)}(\mathrm{Z}([a,b]_\rho)) = \mathrm{Z}([a,k-1]_\rho) \otimes \mathrm{Z}([k,b]_\rho),$$
$$r_{((b-k+1)m,(k-a)m)}(\mathrm{L}([a,b]_\rho)) = \mathrm{L}([k,b]_\rho) \otimes \mathrm{L}([a,k-1]_\rho).$$

The structure of representations induced from representations of the form $\mathrm{Z}([a,b]_\rho)$ or $\mathrm{L}([a,b]_\rho)$ is more complex and will occupy us in later chapters. A starting point is the following theorem.

Theorem 2.3.9 ([101, Theorem 7.26])**.** *Let $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$. Then the following are equivalent.*

1. $\mathrm{Z}(\Delta_1) \times \dots \times \mathrm{Z}(\Delta_k)$ *is irreducible.*
2. $\mathrm{L}(\Delta_1) \times \dots \times \mathrm{L}(\Delta_k)$ *is irreducible.*
3. $\mathfrak{m}$ *is unlinked.*

If some of the segments Δ_i are linked, the induced representation is no longer irreducible. The following two lemmata already hint how these induced representations might look like.

Lemma 2.3.10 ([101, Proposition 7.17])**.** *If $\rho \in \mathfrak{C}^\square$ then $\mathrm{Z}([a,b]_\rho)$ is the socle of*

$$\mathrm{Z}([a,b-1]) \times \rho v_\rho^b \text{ and } \rho v_\rho^a \times \mathrm{Z}([a+1,b])$$

and the cosocle of

$$\mathrm{Z}([a+1,b]) \times \rho v_\rho^a \text{ and } \rho v_\rho^b \times \mathrm{Z}([a,b-1]).$$

Next, we recall the definition of a residually non-degenerate representation. We will use an alternative definition from the one in [101], however it was proven in Theorem 9.10 of said paper that the following definition is equivalent to theirs. An irreducible representation π is called *residually non-degenerate* if there exists a multisegment $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$, $\Delta_i = [a_i,b_i]_{\rho_i}$ for $i \in \{1,\dots,k\}$ such that

$$[\pi] = [\mathrm{L}(\Delta_1) \times \dots \times \mathrm{L}(\Delta_k)]$$

and $l(\Delta_i) < e(\rho_i)$ for $i \in \{1,\dots,k\}$. More generally, for $\alpha = (\alpha_1,\dots,\alpha_t)$ a composition, we call a representation of the form $\pi = \pi_1 \otimes \dots \otimes \pi_t$ of G_α residually non-degenerate if each of the π_i's is residually non-degenerate.

Lemma 2.3.11 ([101, Corollary 8.5])**.** *If π is residually non-degenerate and*

$$[\pi] \leq [\mathrm{Z}(\Delta_1) \times \dots \times \mathrm{Z}(\Delta_k)]$$

then $l(\Delta_i) = 1$ for all $i \in \{1,\dots,k\}$.

Let ρ be a cuspidal representation of G_m, α a composition of nm and π an irreducible representation of G_{nm}. We say π is *α-degenerate* if $r_\alpha(\pi)$ contains a residually non-degenerate representation. It follows from the definition given in [101, §8] that if α' is a permutation of α then π is α-degenerate if and only if it is α'-degenerate. For $\mathfrak{m} = \Delta_1 + \dots + \Delta_n$ we define

$$\mathrm{I}(\mathfrak{m}) = [\mathrm{Z}(\Delta_1) \times \dots \times \mathrm{Z}(\Delta_k)],$$

which is well-defined since $\times$ is commutative in the Grothendieck group.

Theorem 2.3.12 ([101, Section 8]). *Let* $\mathfrak{m}$ *be a multisegment of the form* $\mathfrak{m} = [a_1]_{\rho_1} + \ldots + [a_k]_{\rho_k}$. *Then* $\mathrm{I}(\mathfrak{m})$ *contains a unique residually non-degenerate representation denoted* $\mathrm{Z}(\mathfrak{m})$ *and any cuspidal representation is residually non-degenerate.*

For $\mathfrak{m}$ a multisegment and l maximal such that $\mathfrak{m}^l \neq 0$ we set

$$\mathrm{St}(\mathfrak{m}) := \mathrm{Z}(\mathfrak{m}^l) \otimes \ldots \otimes \mathrm{Z}(\mathfrak{m}^1).$$

We can now define an irreducible representation $\mathrm{Z}(\mathfrak{m})$ for a general multisegment $\mathfrak{m}$.
Theorem 2.3.13 ([101, Theorem 9.19]). *Let* $\mathfrak{m}$ *be a multisegment. Then* $\mathrm{I}(\mathfrak{m})$ *contains a unique irreducible subquotient denoted by* $\mathrm{Z}(\mathfrak{m})$ *which is* $\mu_{\mathfrak{m}}$*-degenerate. The multiplicity of* $\mathrm{Z}(\mathfrak{m})$ *in* $\mathrm{I}(\mathfrak{m})$ *is* 1 *and the unique irreducible constituent of* $r_{\overline{\mu_{\mathfrak{m}}}}(\mathrm{Z}(\mathfrak{m}))$ *which is residually non-degenerate is* $\mathrm{St}(\mathfrak{m})$.

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ *is an ordered composition of* $\deg(\mathfrak{m})$ *and* π *is an irreducible* α*-degenerate subquotient of* $\mathrm{I}(\mathfrak{m})$ *then* $\alpha \leq \mu_{\mathfrak{m}}$ *with equality if and only if* $\pi = \mathrm{Z}(\mathfrak{m})$.

The theorem gives rise to a map

$$\mathrm{Z} : \mathcal{MS} \to \mathfrak{Irr}.$$

Proposition 2.3.14 ([101, Proposition 9.28]). *Let* $\mathfrak{m}$ *be a multisegment and write* $\mathfrak{m} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_k$ *as a sum of multisegments. Then*

$$[\mathrm{Z}(\mathfrak{m})] \leq [\mathrm{Z}(\mathfrak{m}_1) \times \ldots \times \mathrm{Z}(\mathfrak{m}_k)]$$

and $\mathrm{Z}(\mathfrak{m})$ *appears with multiplicity* 1.
Theorem 2.3.15 ([101, Theorem 9.36]). *The map* $\mathrm{Z} : \mathcal{M} \to \mathfrak{Irr}$ *restricted to aperiodic multisegments is injective and* $\mathrm{Z}(\mathfrak{m})^\vee \cong \mathrm{Z}(\mathfrak{m}^\vee)$. *Moreover, if* $\mathfrak{m}$ *is an multisegment*

$$\mathrm{scusp}_{\mathcal{M}}(\mathfrak{m}) = \mathrm{scusp}(\mathrm{Z}(\mathfrak{m}))$$

and if $\mathfrak{m}$ *is moreover aperiodic, then*

$$\mathrm{cusp}_{\mathcal{M}}(\mathfrak{m}) = \mathrm{cusp}(\mathrm{Z}(\mathfrak{m})).$$

Lemma 2.3.16. *Let* $\mathfrak{m}$ *be a multisegment. Then*

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{Z}(\mathfrak{m}^-) \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

Proof. By Theorem 2.3.13

$$[r_{\overline{\mu_{\mathfrak{m}}}}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathfrak{m})].$$

Let $\pi \in \mathfrak{Irr}_{\deg(\mathfrak{m}^-)}$ such that

$$[r_{\overline{\mu_{\mathfrak{m}^-}}}(\pi)] \geq [\mathrm{St}(\mathfrak{m}^-)],$$

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1)}(\mathrm{I}(\mathfrak{m}))] \geq [r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\pi \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

The first inequality implies that π is $\mu_{\mathfrak{m}^-}$-degenerate. On the other hand, the second inequality implies, using Lemma 2.3.11 and the Geometric Lemma, that $[\pi] \leq \mathrm{I}(\mathfrak{m}^-)$. Theorem 2.3.13 forces therefore $\pi \cong \mathrm{Z}(\mathfrak{m}^-)$. □

In the same article it was proven that Z is also surjective, using by passing to the Hecke-algebra and invoking a counting argument. We will reprove in later sections this surjectivity for all representations with □-irreducible cuspidal support.

Let us finally remark on how reduction $\mod \ell$ and the above classification interact. The first thing to note is the following, see [101, Theorem 9.39]. If $\tilde{\rho}$ is cuspidal integral representations of G_n over $\overline{\mathbb{Q}}_\ell$ and ρ is a cuspidal representation of G_n over $\overline{\mathbb{F}}_\ell$ such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$ then for integers $a \leq b$, $\mathrm{Z}([a,b]_{\tilde{\rho}})$ admits an integral structure and

$$\mathrm{r}_\ell(\mathrm{Z}([a,b]_{\tilde{\rho}}) = \mathrm{Z}([a,b]_\rho).$$

A lift of a segment $[a,b]_\rho$ over $\overline{\mathbb{F}}_\ell$ to $\overline{\mathbb{Q}}_\ell$ is a segment $[a',b']_{\widetilde{\rho}}$ such that $\widetilde{\rho}$ is a lift of ρ and $a' \leq b' \in \mathbb{Z}$ with $a-b = a'-b'$ and $a = a' \mod o(\rho)$. A multisegment $\widetilde{\mathfrak{m}} = \widetilde{\Delta}_1 + \ldots + \widetilde{\Delta}_k$ over $\overline{\mathbb{Q}}_\ell$ is a lift of a multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ over $\overline{\mathbb{F}}_\ell$ if $\widetilde{\Delta}_i$ is a lift of Δ_i for $i \in \{1,\ldots,k\}$. In this case we write

$$\mathrm{r}_\ell(\widetilde{\mathfrak{m}}) = \mathfrak{m}.$$

Theorem 2.3.17 ([101, Theorem 9.39])**.** *Let* $\mathfrak{m}$ *be a multisegment and* $\widetilde{\mathfrak{m}}$ *a lift of* $\mathfrak{m}$*. Then* $\mathrm{Z}(\widetilde{\mathfrak{m}})$ *admits an integral structure and* $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{m}}))$ *contains* $\mathrm{Z}(\mathfrak{m})$ *with multiplicity* 1*.*

Haar measures revisited If one tries to translate the computation of local L-factors from $\overline{\mathbb{Q}}_\ell$ to $\overline{\mathbb{F}}_\ell$, one quickly runs into the problem that, contrary to the situation in $\overline{\mathbb{Q}}_\ell$, for a compactly supported smooth function $f\colon G_n \to R$ the following identity does not necessarily have to hold:

$$\int_{G_n} f(g)\,\mathrm{d}g = \int_P \int_{K_n} f(pk)\,\mathrm{d}k\,\mathrm{d}_l p,$$

where P is a parabolic subgroup of G, K_n is the open compact subgroup $K_n \coloneqq \mathrm{GL}_n(\mathcal{O}_v')$ and $\mathrm{d}g$, $\mathrm{d}k$, $\mathrm{d}_l p$ are left Haar-measures on the respective groups, see for example in [94]. However one can save the situation via the following method, *cf.* [74, §2.2]. For H a closed subgroup of G_n let $S(H,R)$ be the set of smooth, compactly supported functions on H taking values in R. If $H' \subseteq H$ is a closed subgroup, we let $S(H'\backslash H) = S(H'\backslash H, \delta, R)$ be the space of smooth functions on H which transform by $\delta \coloneqq \delta_H^{-1}\delta_{H'}$ under left translation by H'. We let $S(H'\backslash H, \delta, \overline{\mathbb{Z}}_\ell)$ be the functions with values in $\overline{\mathbb{Z}}_\ell$ and we denote reduction mod ℓ by

$$\mathrm{r}_\ell\colon S((H'\backslash H, \delta, \overline{\mathbb{Z}}_\ell) \to S(H'\backslash H, \delta, \overline{\mathbb{F}}_\ell).$$

We will denote from now on by $\mathrm{d}_l h$ a left Haar-measure on H and by $\mathrm{d}_r h$ a right Haar-measure on H, and we choose measures compatible with r_ℓ, i.e. for $f \in S(H'\backslash H, \delta, \overline{\mathbb{Z}}_\ell)$,

$$\int_{H'\backslash H} f(h)\,\mathrm{d}_r h \in \overline{\mathbb{Z}}_\ell$$

and

$$\mathrm{r}_\ell\left(\int_{H'\backslash H} f(h)\,\mathrm{d}_r h\right) = \int_{H'\backslash H} \mathrm{r}_\ell(f(h))\,\mathrm{d}_r h$$

and similarly for $\mathrm{d}_l h$. In [74, §2.2] the authors show that if one replaces the left Haar-measure $\mathrm{d}_r p$ with a right Haar-measure $\mathrm{d}_r p$ on P and $(K_n \cap P)\backslash K_n$, above identity holds also in

$\overline{\mathbb{F}}_\ell$:

$$\int_{G_n} f(g)\,\mathrm{d}g = \int_P \int_{(K_n\cap P)\backslash K_n} f(pk)\delta_P^{-1}(p)\,\mathrm{d}k\,\mathrm{d}_r p. \tag{2}$$

Moreover for H a closed subgroup of G_n and $f \in S(H\backslash G_n)$ we can find $F \in S(G_n)$ such that

$$f(g) = \int_H F(gh)\delta^{-1}(h)\,\mathrm{d}_r h$$

and write in this situation $f = F^H$. As a consequence we have for $f \in S(P\backslash G_n)$ and P a parabolic subgroup of G_n

$$\int_{P\backslash G_n} F^P(g)\,\mathrm{d}g = \int_{(K_n\cap P)\backslash K_n} \int_P F(pk)\delta_P^{-1}(p)\,\mathrm{d}_r p\,\mathrm{d}k = \int_{(K_n\cap P)\backslash K_n} F^P(k)\,\mathrm{d}k. \tag{3}$$

Finally, we recall the following facts about smooth functions on G_n. Let $C^\infty(G_n)$ denote the representation of smooth R-valued functions on which $G_n \times G_n$ acts by left-right translation. Its dual, $S(G_n)$ is given by the compactly supported smooth functions often referred to as *Schwartz-functions* and such a Schwartz-function ϕ acts on $f \in C^\infty(G_n)$ by

$$\phi(f) := \int_{G_n} f(g)\phi(g)\,\mathrm{d}g,$$

where $\phi \in S(G_n)$.

Godement-Jacquet local L-factors We section we will recall the Godement-Jacquet L-functions for general irreducible representations. Originally due to [44] in the case of $R = \mathbb{C}$, we will recall the definitions and results as presented in [94] for general R. To formulate the functional equation we need to fix moreover an additive character $\psi_v\colon \mathbb{K}_v \to \mathbb{C}^\times$.

Let (π, V) be a representation over R of G_n and let for $v \in V, v^\vee \in V^\vee$

$$f := g \mapsto v^\vee(\pi(g)v)$$

be a matrix coefficient of π. In general we say that a smooth function f is a matrix coefficient of π if it is the finite sum of functions of the above form. Let $S_R(M_{n,n}(\mathbb{D}_v))$ be the space of Schwartz-functions on $M_{n,n}(\mathbb{D}_v)$. For $H \subseteq G_n$, we write for $N \in \mathbb{Z}$,

$$H(N) := \{h \in H : |\det{}'(h)| = q_v^{-N}\}.$$

For a matrix coefficient f we denote by

$$f^\vee := g \mapsto f(g^{-1})$$

and for $\phi \in S_R(M_{n,n}(\mathbb{D}_v))$ we let $\widehat{\phi}$ be its Fourier transform with respect to our fixed additive character ψ_v, see [94, Section 1.3]. For f a matrix coefficient of π and $\phi \in S_R(M_{n,n}(\mathbb{D}_v))$ one can then construct a formal Laurent series $Z(\phi, T, f) \in R((T))$, which is linear in f and ϕ. The coefficient of T^N of $Z(\phi, T, f)$ is given by the integral

$$\int_{G_n(N)} \phi(g)f(g)\,\mathrm{d}g.$$

Theorem 2.3.18 ([94, Theorem 2.3]). *Let (π, V) be a representation of G_n which is a subquotient of a representation induced from irreducible representations.*

1. *There exists a non-zero $P'(\pi, T) \in R[T]$ such that for each matrix coefficient f of π and $\phi \in S_R(M_{n,n}(\mathbb{D}_v))$*
$$P(\pi, T)Z(\phi, T, f) \in R[T, T^{-1}].$$

2. *There exists a non-zero $\gamma(T, \psi_v, f) \in R(T)$ such that for each matrix coefficient f and $\phi \in S_R(M_{n,n}(\mathbb{D}_v))$*
$$Z(\widehat{\phi}, T^{-1}q^{-\frac{d_v n+1}{2}}, f^\vee) = \gamma(T, \pi, \psi_v)Z(\phi, Tq^{-\frac{d_v n-1}{2}}, f).$$

Note that the statement of this theorem in [94] is only stated for irreducible representations, however the proof that is given there, together with [94, Proposition 2.5], shows that it is true for all the above representations.

Corollary 2.3.2 ([94, Corollary 2.4]). *Let $\mathcal{L}(\pi)$ be the R-subspace of $R((T))$ generated by*
$$Z(\phi, q^{-\frac{d_v n-1}{2}}T, f)$$
where f ranges over all matrix coefficients of π and ϕ over all Schwartz-functions in $S_R(M_{n,n}(\mathbb{D}_v))$. Then $\mathcal{L}(\pi)$ is a fractional ideal of $R[T, T^{-1}]$ containing the constant functions. It admits a generator
$$\frac{1}{P(\pi, T)}, \; P(\pi, T) \in R[T]$$
which can be normalized by the condition $P(\pi, 0) = 1$.

One then defines
$$L(\pi, T) := \frac{1}{P(\pi, T)}$$
and $\epsilon(T, \pi, \psi_v)$ by
$$\gamma(T, \pi, \psi_v) = \epsilon(T, \pi, \psi_v)\frac{L(\pi^\vee, q_v^{-1}T^{-1})}{L(\pi, T)}.$$
Then
$$\epsilon(T, \pi, \psi_v)\epsilon(q_v^{-1}T^{-1}, \pi^\vee, \psi_v) = \omega(-1),$$
where ω is the central character of π and $\epsilon(T, \pi, \psi_v) = C_{\pi,\psi_v}T^{k(\pi,\psi_v)}$ for some constant $C_{\pi,\psi_v} \in R$ and $k(\pi, \psi_v) \in \mathbb{Z}$. If $R = \overline{\mathbb{Q}}_\ell$ and π is an entire representation then $P(\pi, T)$ has integer coefficients and hence so do the rational functions $\gamma(T, \pi, \psi_v)$ and $\epsilon(T, \pi, \psi_v)$. We write for a rational function $F \in \overline{\mathbb{Z}}_\ell[T, T^{-1}]$, $\mathrm{r}_\ell(F) \in \overline{\mathbb{F}}_\ell[T, T^{-1}]$ for the coefficient-wise reduction $\bmod \ell$ of F and $\mathrm{r}_\ell\left(\frac{1}{F}\right) := \frac{1}{\mathrm{r}_\ell(F)}$ if $\mathrm{r}_\ell(F) \neq 0$. If $R = \mathbb{C}$, setting $T = q_v^{-s}$ gives the usual Godement-Jacquet L-factors of [44].

Lemma 2.3.19 ([94, Corollary 4.2]). *Let $\widetilde{\pi}$ be an entire admissible representation over $\overline{\mathbb{Q}}_\ell$ and $\mathfrak{o}$ an integral structure of $\widetilde{\pi}$. Assume that both $\widetilde{\pi}$ and $\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ satisfy the condition of Theorem 2.3.18. Then $\mathrm{r}_\ell(\gamma(T, \widetilde{\pi}, \widetilde{\psi_v}))$ is well defined, the polynomial $P(\mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell, T)$ divides $\mathrm{r}_\ell(P(\widetilde{\pi}, T))$ in $\overline{\mathbb{F}}_\ell[T]$ and*
$$\gamma(T, \mathfrak{o} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell, \psi_v) = \mathrm{r}_\ell(\gamma(T, \widetilde{\pi}, \widetilde{\psi_v})) = \mathrm{r}_\ell(\epsilon(T, \widetilde{\pi}, \widetilde{\psi_v}))\frac{\mathrm{r}_\ell(L(\widetilde{\pi}^\vee, q_v^{-1}T^{-1}))}{\mathrm{r}_\ell((L(\widetilde{\pi}, T))}.$$

Another useful lemma is the following:
Lemma 2.3.20 ([94, Proposition 2.5]). *Let π be a representation of G_n satisfying the conditions of Theorem 2.3.18. Moreover, let τ be a subquotient of π. Then $P(\tau, T)$ divides $P(\pi, T)$ and*

$$\gamma(T, \pi, \psi_v) = \gamma(T, \tau, \psi_v).$$

2.3.4 Representation theory of classical and metaplectic groups Unlike for the general linear group, we will say much less about the representation theory of classical groups. In this section we assume the field of coefficients $R = \mathbb{C}$ and use the notation of Section 2.2.3. Before we start with the classical groups, we need to introduce the following twist of representations of $\mathrm{GL}(V)$, where V is a finite-dimensional $\mathbb{L}_v$-vector space. Choose an isomorphism $\alpha\colon V \xrightarrow{\sim} \mathbb{L}_v^n$ and set $\mathfrak{c}(g) := \alpha^{-1}(\mathfrak{c}(\alpha(g)))$, where $\mathfrak{c}(\alpha(g))$ is the natural action of $\mathfrak{c}$ on $\mathrm{GL}_n(\mathbb{L}_v)$. We then set for $\pi \in \mathfrak{Rep}_{\mathrm{GL}(V)}$, ${}^{\mathfrak{c}}\pi(g) := \pi(\mathfrak{c}(g))$. The isomorphism class of ${}^{\mathfrak{c}}\pi$ is independent of the chosen isomorphism α, since all of those differ by an inner automorphism of V.

We put ourselves in the setup of Section 2.2.3. Fix X a maximal isotropic subspace of W, $W = X \oplus W' \oplus Y$ as above and assume we can choose a basis of $\{e_1, \ldots, e_{q_W}\}$ of X and a basis $\{f_1, \ldots, f_{q_W}\}$ of Y such that

$$\langle e_i, f_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Extend then $e_1, \ldots, e_{q_W}, f_1, \ldots, f_{q_W}$ to a basis of W and consider $G(W)$ in this basis. For $k \in \{1, \ldots, q_W\}$ let

$$X_k := \langle e_1, \ldots, e_k \rangle_{\mathbb{L}_v},\ Y_k := \langle f_1, \ldots, f_k \rangle_{\mathbb{L}_v}$$

and write $W_k := W'$ for the non-degenerate part in the decomposition

$$W = X_k \oplus W' \oplus Y_k.$$

More generally, for $a \leq b$, $a, b \in \{0, \ldots, q_W\}$ we write $X_{a,b}$ for the subspace of X spanned by $\{e_{a+1}, \ldots, e_b\}$ and similarly for $Y_{a,b}$. For a composition α of $k \in \{1, \ldots, q_W\}$ let P_α be the stabilizer of the flag

$$0 \subseteq X_{\alpha_1} \subseteq X_{\alpha_1+\alpha_2} \subseteq \ldots \subseteq X_k$$

with Levi-component M_α. For $\tau = \tau_1 \otimes \ldots \otimes \tau_t \otimes \sigma \in \mathfrak{Rep}_{M_\alpha}$ we denote

$$\tau_1 \times \ldots \times \tau_t \rtimes \sigma := \mathrm{Ind}_{P_\alpha}^{G(W)} \tau.$$

We take the opportunity to note at this point that

$$\delta_{P_\alpha} = \underbrace{|\det|}_{G_{\alpha_1}}^{p(\alpha)_1} \otimes \ldots \otimes \underbrace{|\det|}_{G_{\alpha_t}}^{p(\alpha)_t},$$

where $p(\alpha)_i := n - (2\sum_{j=1}^{i-1} \alpha_j) - \alpha_i - \eta$ and

$$\eta := \begin{cases} \epsilon & \text{if } \mathbb{L}_v = \mathbb{K}_v, \\ 0 & \text{if } [\mathbb{K}_v : \mathbb{L}_v] = 2. \end{cases}$$

If we deal at the same time with classical and general groups, we denote parabolic subgroups of classical groups by P_α and those of general linear groups by Q_α.

Let $k \in \{1, \dots, q_W\}$. An element $(m, g) \in G_k \times G(W_k)$ of the Levi-component of of $P(X_k)$ corresponds to an element of the form

$$\begin{pmatrix} m & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \mathfrak{c}({}^t m^{-1}) \end{pmatrix} \in G(W)$$

and an element $n \in N_k$ in the unipotent part of $P(X_k)$ is of the form

$$\begin{pmatrix} 1 & x & y - \frac{1}{2} x \mathfrak{c}({}^t x) \\ 0 & 1 & -\mathfrak{c}({}^t x) \\ 0 & 0 & 1 \end{pmatrix} \in G(W)$$

for $x \in \mathrm{Hom}(W_k, X_k)$ and $y \in \mathrm{Hom}(Y_k, X_k)$ with $\mathfrak{c}({}^t y) = y$. Finally, we denote by $P'_{(k)}$ the stabilizer of the flag $X_{q_W - k, q_W}$.

Mœglin-Vignéras-Waldspurger involution Recall the Mœglin-Vignéras-Waldspurger involution MVW: $\mathfrak{Rep}_{G(W)} \to \mathfrak{Rep}_{G(W)}$, see [97, p.91], which is covariant, exact and satisfies

1. $\mathrm{MVW} \circ \mathrm{MVW} = \mathrm{id}$,
2. $\pi^{\mathrm{MVW}} \cong \pi^\vee$ if π is irreducible,
3. For $r \in \{1, \dots, q_W\}$, $\alpha = (\alpha_1, \dots, \alpha_k)$ a composition of r, $\tau_i \in \mathfrak{Irr}_{\alpha_i}$, $\sigma \in \mathfrak{Irr}_{G(W_r)}$,

$$(\tau_1 \times \dots \times \tau_k \rtimes \sigma)^{\mathrm{MVW}} \cong {}^{\mathfrak{c}}\tau_1 \times \dots \times {}^{\mathfrak{c}}\tau_t \rtimes \sigma^{\mathrm{MVW}}.$$

As a consequence, if $t \in \{1, \dots, q_W\}$, $\pi \in \mathfrak{Irr}_{G(W)}$, $\rho \in \mathfrak{Rep}_t$, $\delta \in \mathfrak{Irr}_{G(W_t)}$ the following are equivalent.

$$\pi \hookrightarrow \rho \rtimes \delta \Leftrightarrow \pi^\vee \hookrightarrow {}^{\mathfrak{c}}\rho \rtimes \delta^\vee \Leftrightarrow {}^{\mathfrak{c}}\rho^\vee \rtimes \delta \twoheadrightarrow \pi \Leftrightarrow \rho^\vee \rtimes \delta^\vee \twoheadrightarrow \pi^\vee.$$

Indeed, the first equivalence follows from the covariance of MVW and properties 2 and 3. The second equivalence follows from duality and the third follows again from the covariance of MVW and properties 2 and 3.

Combinatorial Geometric Lemma Next, we give a combinatorial description of the Geometric Lemma for classical groups, see [119, Lemma 5.1], [14]. Let $t, r \in \{1, \dots, q_W\}$, $\pi \in \mathfrak{Rep}_t$ and $\sigma \in \mathfrak{Rep}_{G(W_t)}$. We will give a more combinatorial description of the Geometric Lemma for $r_{P_{(r)}}(\pi \rtimes \sigma)$. Let $k_1, k_2, k_3 \in \mathbb{N}$ such that $k_1 + k_2 + k_3 = t$ and $k_1 + k_3 \leq r$, $r + k_2 \leq q_W$. Then the $P_{(r)}$-orbits in $P_{(t)} \backslash G(W)$ are indexed by above triples and a t-dimensional isotropic subspace $U \in P_{(t)} \backslash G(W)$ is contained in the orbit corresponding to k_1, k_2, k_3 if and only if $\dim_{\mathbb{L}_v}(U \cap X_r) = k_1$ and $\dim_{\mathbb{L}_v}(p_{Y_r}(U)) = k_3$, where p_{Y_r} is the projection to Y_r. We then get a description of the semisimplified version of the Geometric Lemma as follows. Write the semisimplification of $r_{(k_1, k_2, k_3)}(\pi)$ as the sum of irreducible representations of the form

$\pi_1 \otimes \pi_2 \otimes \pi_3$ and the semisimplification of $r_{P_{(r-k_1-k_2)}}(\sigma)$ as the sum of irreducible representations of the form $\pi_4 \otimes \sigma'$. Then

$$[r_{P_{(r)}}(\pi \rtimes \sigma)] = \sum [\pi_1 \times \pi_4 \times {}^{\mathfrak{c}}\pi_3^\vee \otimes \pi_2 \rtimes \sigma'],$$

where the sum is over all k_1, k_2, k_3 and $\pi_1, \pi_2, \pi_3, \pi_4, \sigma'$ as above.

We will now give an explicit representative w_{k_1,k_2,k_3} of the $P_{(r)}$-orbit in the Grassmanian $P_{(t)} \backslash G(W)$ corresponding to the triple k_1, k_2, k_3. For $k \in \{1, \dots, q_W\}$ we set

$$\alpha_k := \begin{pmatrix} 0 & 1_k & 0 & 0 \\ 1_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_k \\ 0 & 0 & 1_k & 0 \end{pmatrix} \in G(X_k \oplus X_k \oplus Y_k \oplus Y_k)$$

and

$$\beta_k := \begin{cases} \begin{pmatrix} 0 & 1_k \\ \epsilon 1_k & 0 \end{pmatrix} & G(X_k \oplus Y_k) \text{ symplectic or orthogonal}, \\ \left(\begin{pmatrix} 0 & 1_k \\ -1_k & 0 \end{pmatrix}, 1 \right) & G(X_k \oplus Y_k) \text{ metaplectic}, \\ \begin{pmatrix} 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & -1_k \\ \epsilon 1_k & 0 & 0 & 0 \\ 0 & -\epsilon 1_k & 0 & 0 \end{pmatrix} & G(X_k \oplus Y_k) \text{ unitary}, \end{cases}$$

where in the last case we fix a basis $\langle x'_1, \dots, x'_k, \gamma x'_1, \dots, \gamma x'_k \rangle_E$ of X_k with $\gamma \notin F$ and similarly for Y_k. Set $a := \min(t, r)$. For a triple k_1, k_2, k_3 decompose W as

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_6 \oplus W_7,$$

$$W_1 := X_{k_1} \oplus Y_{k_1},\ W_2 := X_{k_1, k_1+k_3} \oplus Y_{k_1, k_1+k_3}$$

$$W_3 := X_{k_1+k_3, a} \oplus X_{q_W - a + k_1 + k_3, q_W},\ W_4 := Y_{k_1+k_3, a} \oplus Y_{q_W - a + k_1 + k_3, q_W},$$

$$W_5 := X_{a,t} \oplus X_{q_W - k_2, q_W - a + k_1 + k_3},\ W_6 := Y_{a,t} \oplus Y_{q_W - k_2, q_W - a + k_1 + k_3},$$

$$W_7 := X_{t, q_W - k_2} \oplus Y_{t, q_W - k_2} \oplus W_{q_W}.$$

We have a natural embedding

$$\iota_{k_1,k_2,k_3} \colon \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3) \times \mathrm{GL}(W_4) \times \mathrm{GL}(W_5) \times \mathrm{GL}(W_6) \times \mathrm{GL}(W_7) \to$$
$$\to \mathrm{GL}(W).$$

We then set

$$w_{k_1,k_2,k_3} =$$
$$= w_{k_1,k_2,k_3,r,G(W)} := \iota_{k_1,k_2,k_3}(1_{W_1}, \beta_{k_3}, \alpha_{a-k_1-k_3}, \alpha_{a-k_1-k_3}, \alpha_{t-a}, \alpha_{t-a}, 1_{W_7}).$$

We will also define a second representative of the orbit parametrized by k_1, k_2, k_3 denoted by v_{k_1,k_2,k_3} as follows. Let $b = \max(t, r)$ and decompose

$$W = W_1 \oplus W'_2 \oplus W'_3 \oplus W'_4 \oplus W'_5,\ W_1 = X_{k_1} \oplus Y_{k_1},$$

$$W_2' := X_{a-k_3,b} \oplus Y_{a-k_3,b},$$
$$W_3' := X_{k_1,a-k_3} \oplus X_{b,b+a-k_3-k_1},\ W_4' := Y_{k_1,a-k_3} \oplus Y_{b,b+a-k_3-k_1},$$
$$W_5' := W_{b+a-k_3-k_1}.$$

As above we have an embedding

$$\iota'_{k_1,k_2,k_3} : \mathrm{GL}(W_1) \times \mathrm{GL}(W_2') \times \mathrm{GL}(W_3') \times \mathrm{GL}(W_4') \times \mathrm{GL}(W_5') \to \mathrm{GL}(W).$$

If $G(W)$ is not metaplectic we set

$$v_{k_1,k_2,k_3} = v_{k_1,k_2,k_3,r,G(W)} := \iota'_{k_1,k_2,k_3}(1_{W_1}, \gamma, \alpha_{a-k_1-k_3}, \alpha_{a-k_1-k_3}, 1_{W_5}) \in G(W), \tag{4}$$

where γ is the matrix $\beta_{k_3} \in \mathrm{Hom}(X_{r-k_3,r} \oplus Y_{t-k_3,t}, X_{r-k_3,r} \oplus Y_{t-k_3,t})$ plus the matrix

$$\begin{cases} \alpha_{t-r} \in \mathrm{Hom}(X_{r,t} \oplus Y_{r-k_3,t-k_3}, X_{r,t} \oplus Y_{r-k_3,t-k_3}) & \text{if } r \le t, \\ \alpha_{r-t} \in \mathrm{Hom}(X_{t-k_3,r-k_3} \oplus Y_{t,r}, X_{t-k_3,r-k_3} \oplus Y_{t,r}) & \text{if } r \ge t. \end{cases}$$

As a finally remark we note that

$$\det(Z(G(W))) = \begin{cases} 1 & W \text{ symplectic}, \\ \{\pm 1\} & W \text{ orthogonal and } n \text{ odd}, \\ 1 & W \text{ orthogonal and } n \text{ even}, \\ \mathbb{L}_v^1 & W \text{ unitary}. \end{cases}$$

2.4 Automorphic representations

Let $\mathbb{K}$ be a totally real number field with $\dim_{\mathbb{Q}} \mathbb{K} = r$, ring of integers $\mathcal{O}$, and $\mathbb{D}$ a central division algebra over $\mathbb{K}$ of degree d. We will highlight the basic properties and constructions regarding automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_n'(\mathbb{A})$.

For $v \in \mathcal{V}_\infty$, let $Z'_{n,v}$ be the center of $\mathrm{GL}'_n(\mathbb{K}_v)$ and let K'_v be the product of the maximal compact subgroup of $\mathrm{GL}'_n(\mathbb{K}_v)$ and connected component of $Z'_{n,v}$, *i.e.*

$$K'_v := \mathrm{Sp}\left(\frac{nd}{2}\right)\mathbb{R}_{>0},\ K'_\infty := \prod_{v \in \mathcal{V}_\infty} K'_v.$$

Note that $\mathrm{Sp}(n)$ does not denote the standard algebraic symplectic group, we denote it by Sp_n, rather it denotes the *compact symplectic group*

$$\mathrm{Sp}(n) := \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}_n(\mathbb{R})$$

or, alternatively the quaternionic unitary group. The group $\mathrm{Sp}(n)$ is a real Lie group of dimension $\dim_{\mathbb{R}} \mathrm{Sp}(n) = n(2n+1)$, it is compact and simply connected. Similarly, we fix for $v \in \mathcal{V}_\infty$

$$K_v := \mathrm{SO}_n(\mathbb{R})\mathbb{R}_{>0},\ K_\infty := \prod_{v \in \mathcal{V}_\infty} K_v.$$

Moreover, we fix also open compact subgroups K'_v of $\mathrm{GL}'_n(\mathbb{K}_v)$ for $v \in \mathcal{V}_f$ as follows. Note that $\mathrm{GL}'_n(\mathbb{K}_v)$ consists of invertible $nd_v \times nd_v$ matrices with entries in $\mathbb{D}_v$. We then let K'_v be those matrices in $\mathrm{GL}'_n(\mathbb{K}_v) = \mathrm{GL}_{nd_v}(\mathbb{D}_v)$ which have entries in $\mathcal{O}'_v$. Denote for $v \in \mathcal{V}_\infty$ by $\mathfrak{g}'_v$ the Lie algebra of $\mathrm{GL}'_n(\mathbb{K}_v)$ and by $\mathfrak{g}'_\infty$, the Lie algebra of $\mathrm{GL}'_{n,\infty} := \prod_{v\in\mathcal{V}_\infty} \mathrm{GL}'_n(\mathbb{K}_v)$.

To ensure that the periods we will consider in later sections are well defined, we will also have to fix a Haar measure on $H'_n(\mathbb{K}_v)$ for all $v \in \mathcal{V}_f$. We do this by setting the volumes of the two copies of K'_v in $H'_n(\mathbb{K}_v)$ with respect to the measures to 1. Taking the product of those measures over all $v \in \mathcal{V}_f$, we obtain a Haar measure $\mathrm{d}_f g_1 \times \mathrm{d}_f g_2$ on $H'_n(\mathbb{A}_f)$. This in turn determines the volume

$$c := \mathrm{vol}\left(Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})/\mathbb{R}^r_{>0}\right) = 2^r \cdot \mathrm{vol}\left(\overbrace{\mathbb{K}^\times\backslash\mathbb{A}_f^\times \times \ldots \times \mathbb{K}^\times\backslash\mathbb{A}_f^\times}^{r}\right).$$

Now for $v \in \mathcal{V}_\infty$ let $\mathrm{d}_v g_1$ and $\mathrm{d}_v g_2$ be the Haar measures on the two copies of $\mathrm{GL}'_n(\mathbb{K}_v)$ such that $\mathrm{Sp}\left(\frac{nd}{2}\right) \subseteq \mathrm{GL}_{\frac{nd}{2}}(\mathbb{H})$ has volume 1. Set

$$\mathrm{d}_\infty g_1 := c \prod_{v\in\mathcal{V}_\infty} \mathrm{d}_v g_1,\ \mathrm{d}_\infty g_2 := \prod_{v\in\mathcal{V}_\infty} \mathrm{d}_v g_2,$$

which then gives a Haar measure $\mathrm{d}g_1 \times \mathrm{d}g_2$ on $H'_n(\mathbb{A})$ where

$$\mathrm{d}g_i := \mathrm{d}_f g_i\, \mathrm{d}_\infty g_i,\ i = 1, 2.$$

2.4.1 Discrete series Let us now fix our notations regarding automorphic representations, automorphic forms, and in particular, discrete series representations. We call an irreducible $(\mathfrak{g}'_\infty, K'_\infty, \mathrm{GL}'_n(\mathbb{A}))$-subquotient Π' of the space of automorphic forms $\mathcal{A}(\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'_n(\mathbb{A}))$ on $\mathrm{GL}'_n(\mathbb{A})$ an (irreducible) automorphic representation of $\mathrm{GL}'_n(\mathbb{A})$. We call Π' *cuspidal* if it is generated by a cusp form ϕ, *i.e.* an automorphic form ϕ such that

$$\int_{U(\mathbb{K})\backslash U(\mathbb{A})} \phi(ug)\, \mathrm{d}u = 0,$$

for all for all $g \in \mathrm{GL}'_n(\mathbb{A})$ and all non-trivial parabolic subgroups P of GL'_n with Levi-decomposition $P = MU$.

Let

$$\omega\colon Z'_n(\mathbb{K})\backslash Z'_n(\mathbb{A}) \to \mathbb{C}$$

be a continous, unitary character. As in the introduction we denote the L^2-completion of the square-integrable functions on $\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'_n(\mathbb{A})$ with central character ω by

$$L^2\left(\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'_n(\mathbb{A}), \omega\right).$$

This is a representation of $\mathrm{GL}'_n(\mathbb{A})$ via the right regular action. If $\widetilde{\Pi}$ is an irreducible subrepresentation of $L^2(\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'_n(\mathbb{A}), \omega)$, we will denote by $\widetilde{\Pi}^\infty$ the smooth vectors in $\widetilde{\Pi}$, *cf.* [45, Chapter 11]. Moreover, the subspace of smooth, K'_∞-finite vectors in $\widetilde{\Pi}$ carries the structure of a $(\mathfrak{g}'_\infty, K'_\infty, \mathrm{GL}'_n(\mathbb{A}))$-module. The automorphic representations which can be obtained in this way will be called *discrete series representations* and every cuspidal representation is a discrete series representation. If it is clear from context, we will implicitly use the representation Π' if we talk about $(\mathfrak{g}'_\infty, K'_\infty, \mathrm{GL}'_n(\mathbb{A}))$-modules and the corresponding representation $\widetilde{\Pi}$ if we talk about $\mathrm{GL}'_n(\mathbb{A})$-representations.

Coming with those two ways of looking at a discrete series representation Π', we have two ways of writing it as a restricted tensor product, *cf.* [45, Chapter 14]. We again denote by $\widetilde{\Pi}$ the corresponding subrepresentation of $L^2\left(\mathrm{GL}'_n\left(\mathbb{K}\right)\backslash\mathrm{GL}'_n\left(\mathbb{A}\right),\omega\right)$. Then the smooth vectors $\widetilde{\Pi}^{\infty}$ admit a decomposition

$$\widetilde{\Pi}^{\infty} \cong \overline{\bigotimes}_{\mathrm{pr}\, v\in\mathcal{V}_\infty} \widetilde{\pi}_v^{\infty} \otimes_{\mathrm{in}} \sideset{}{'}\bigotimes_{v\in\mathcal{V}_f} \widetilde{\pi}_v^{\infty},$$

where $\overline{\otimes}_{\mathrm{pr}}$ denotes taking the completed projective tensor product, $\otimes_{\mathrm{in}}$ denotes the inductive tensor product and $\widetilde{\pi}_v^{\infty}$ are $\mathrm{GL}'_n(\mathbb{K}_v)$-representations. For $v\in\mathcal{V}_\infty$, taking K'_v-finite vectors gives a $(\mathfrak{g}'_v, K'_v)$-module π'_v. This gives us a second decomposition $\Pi'\cong\otimes'_{v\in\mathcal{V}}\pi'_v$, which now is a restricted tensor product of $(\mathfrak{g}'_\infty, K'_\infty)$- respectively $\mathrm{GL}'_n(\mathbb{K}_v)$-modules. Throughout the paper we will therefore mean $(\widetilde{\pi}_v)^{\infty}$ if we treat π'_v as a $\mathrm{GL}'_n\left(\mathbb{K}_v\right)$-representation. We denote by $S_{\Pi'}\subseteq\mathcal{V}_f$ the finite set of places where Π' ramifies. The central character of Π' will be denoted by $\omega=\omega_{\Pi'}$.

For $v\in\mathcal{V}_\infty$ and $\left(\rho,W\right)$, $\rho=\rho_1\otimes\ldots\otimes\rho_k$ an irreducible representation of $M'_\alpha\left(\mathbb{K}_v\right)$ we set

$$\begin{aligned} \# - \mathrm{Ind}_{P'_\alpha}^{G'_n}\left(\rho\right) := \{ f\in C^{\infty}\left(G,W\right) : f\left(mng\right)=\rho\left(m\right)f\left(g\right),\\ m\in M'_\alpha\left(\mathbb{K}_v\right), n\in U'_\alpha\left(\mathbb{K}_v\right), g\in\mathrm{GL}'_n\left(\mathbb{K}_v\right)\}, \end{aligned} \tag{5}$$

on which $\mathrm{GL}'_n\left(\mathbb{K}_v\right)$ acts by right translation. We equip $\#-\mathrm{Ind}_{P'_\alpha}^{G'_n}$ with the subspace topology induced from the Fréchet space $C^{\infty}(\mathrm{GL}'_n(\mathbb{K}_v),W)$. The space

$$\mathrm{Ind}_{P'_\alpha}^{G'_n}\left(\rho\right) := \# - \mathrm{Ind}_{P'_\alpha}^{G'_n}\left(\rho\otimes\delta_{P'_\alpha}^{\frac{1}{2}}\right)$$

is called the normalized parabolically induced representation.

If (Σ,W) is a discrete series representation of $M'_\alpha(\mathbb{A})$, with the corresponding $M'_\alpha(\mathbb{A})$-representation $(\widetilde{\Sigma},\widetilde{W})$ and μ a character of $P'_\alpha(\mathbb{A})$ we define $\mathrm{Ind}_{P'_\alpha(\mathbb{A})}^{\mathrm{GL}'_n(\mathbb{A})}\left(\Sigma\otimes\mu\right)$ to be the space of smooth functions $f\colon\mathrm{GL}'_n(\mathbb{A})\to\widetilde{W}^{\infty}$ satisfying the normalized global analogon of the equivariance condition (5). The so-obtained space admits a natural topology with which the $\mathrm{GL}'_n(\mathbb{A})$-action by right translations is continuous. It admits a decomposition

$$\mathrm{Ind}_{P'_\alpha(\mathbb{A})}^{\mathrm{GL}'_n(\mathbb{A})}\left(\Sigma\otimes\mu\right) \cong \overline{\bigotimes}_{\mathrm{pr}\, v\in\mathcal{V}_\infty}\mathrm{Ind}_{P'_\alpha}^{G'_n}\left(\left(\tilde{\sigma}_v\otimes\mu_v\right)^{\infty}\right) \otimes_{\mathrm{in}} \sideset{}{'}\bigotimes_{v\in\mathcal{V}_f}\mathrm{Ind}_{P'_\alpha}^{G'_n}\left(\sigma_v\otimes\mu_v\right).$$

Similarly, we define for GL_n parabolic and normalized parabolic induction.

Mœglin-Waldspurger classification We also recall the following well-known description of discrete series representations known as the Mœglin-Waldspurger classification.

Theorem 2.4.1 ([11],[99])**.** *Let $k,l\in\mathbb{Z}_{\geq1}$, $n=lk$ and Σ' be a cuspidal unitary automorphic representation of $\mathrm{GL}'_l\left(\mathbb{A}\right)$. Then the parabolically induced $\mathrm{GL}'_n(\mathbb{A})$-representation*

$$\Sigma'|\det{}'|^{\frac{k-1}{2}}\times\ldots\times\Sigma'|\det{}'|^{\frac{1-k}{2}}$$

admits a unique irreducible quotient, denoted by $\mathrm{MW}\left(\Sigma',k\right)$. It is a discrete series representation of $\mathrm{GL}'_n\left(\mathbb{A}\right)$ and moreover for every discrete series representation Π' of $\mathrm{GL}'_n\left(\mathbb{A}\right)$, there exists l,k and Σ' as above such that $\Pi'\cong\mathrm{MW}\left(\Sigma',k\right)$. The analogous statement for GL_n instead of GL'_n holds also true.

Godement-Jacquet global L-functions For Π' a discrete series representation of $\mathrm{GL}'_n(\mathbb{A})$, we define

$$L(s,\Pi') := \prod_{v\in\mathcal{V}} L(s,\pi'_v),$$

which is well-defined for $\mathrm{Re}(s) >> 0$ and admits an analytic continuation to a meromorphic function, *cf.* [44, Theorem 13.8]. For $v \in \mathcal{V}_f$ we recalled the construction of the local L-factors in Section 2.3.3, for $v \in \mathcal{V}_\infty$ we refer to [44, §8]. The local ϵ-factors $\epsilon(s,\pi'_v,\psi_v)$ are equal to 1 for almost all $v \in \mathcal{V}$, hence

$$\epsilon(s,\Pi',\psi) := \prod_{v\in\mathcal{V}} \epsilon(s,\pi'_v,\psi_v)$$

is well-defined and

$$L(s,\Pi') = \epsilon(s,\Pi',\psi)L(1-s,\Pi'^{\vee}).$$

Moreover, if Π' is cuspidal and not a unitary character of the form $|\det|^{it}$, the L-function $L(s,\Pi')$ is entire. For S a finite subset of V, we write $L^S(s,\Pi') := \prod_{v\notin S} L(s,\pi'_v)$ respectively for its analytic continuation. In particular, we set $L(s,\Pi'_f) := L^{\mathcal{V}_\infty}(s,\Pi')$.

2.4.2 Jacquet-Langlands correspondence We will now quickly recall the basic notions of the Jacquet-Langlands correspondence. For a complete discussion see [11], [13]. Let $v \in \mathcal{V}$ be a place and recall that to each irreducible unitary representation of π_v of $\mathrm{GL}_{dn}(\mathbb{K}_v)$ respectively π'_v of $\mathrm{GL}'_n(\mathbb{K}_v)$ we can associate a trace character χ_{π_v} respectively $\chi_{\pi'_v}$. Moreover, for $g \in \mathrm{GL}_{dn}(\mathbb{K}_v)$ and $g' \in \mathrm{GL}'_n(\mathbb{K}_v)$ semisimple, we write $g \leftrightarrow g'$ if they have the same characteristic polynomial. Following [29] we call π_v d_v-compatible if there exists a unitary irreducible representation π'_v of $\mathrm{GL}'_n(\mathbb{K}_v)$ such that

$$\chi_{\pi_v}(g) = \epsilon(\pi_v)\chi_{\pi'_v}(g) \text{ if } g \leftrightarrow g',$$

where $\epsilon(\pi_v) \in \{-1,1\}$.

Let $U'_{cp}(\mathrm{GL}_{dn}(\mathbb{K}_v))$ be the set of unitary d_v-compatible irreducible representations of $\mathrm{GL}_{dn}(\mathbb{K}_v)$ and let $U'(\mathrm{GL}'_n(\mathbb{K}_v))$ be the set of unitary irreducible representations of $\mathrm{GL}'_n(\mathbb{K}_v)$. Moreover, let $U_{cp}(\mathrm{GL}_{dn}(\mathbb{K}_v))$ respectively $U(\mathrm{GL}'_n(\mathbb{K}_v))$ be the set of representations of the form $\Pi\otimes|\det|^s$ respectively $\Pi'\otimes|\det'|^s$ for $\Pi \in U'_{cp}(\mathrm{GL}_{dn}(\mathbb{K}_v))$ respectively $\Pi' \in U'(\mathrm{GL}'_n(\mathbb{K}_v))$. Then there exists a map called the *local Jacquet-Langlands correspondence*

$$\mathrm{LJ}_v : U_{cp}(\mathrm{GL}_{dn}(\mathbb{K}_v)) \to U\left(\mathrm{GL}'_n(\mathbb{K}_v)\right)$$

with the following properties, see [29]:

1. If $\pi_v = \widetilde{\pi}_v \otimes |\det'|^s$ with $\widetilde{\pi}_v$ a unitary d_v-compatible irreducible representations of $\mathrm{GL}_{dn}(\mathbb{K}_v)$,
$$\mathrm{LJ}_v(\pi_v) = \mathrm{LJ}_v(\widetilde{\pi}_v) \otimes |\det'|^s.$$
2. If v is a split place of $\mathbb{D}$, LJ_v is the identity.
3. LJ_v restricted to square integrable representations is a bijection onto the square integrable representations of $\mathrm{GL}'_n(\mathbb{K}_v)$.
4. LJ_v commutes with parabolic induction.

Similarly, there is a global correspondence going from the unitary discrete series representations of $\mathrm{GL}'_n(\mathbb{A})$ into the set of unitary discrete series representations of $\mathrm{GL}_{nd}(\mathbb{A})$ which is denoted by JL and called the *global Jacquet-Langlands correspondence*. It satisfies the following properties:

1. $\mathrm{LJ}_v\left(\left(\mathrm{JL}(\Pi')\right)_v\right) \cong \pi'_v$ for all $v \in \mathcal{V}$.
2. JL is injective.
3. If $\mathrm{JL}(\Pi')$ is cuspidal, then Π' is cuspidal.

Crucially, if Π' is cuspidal $\mathrm{JL}(\Pi')$ does not have to be cuspidal.

2.4.3 Cohomological automorphic representation Let us fix our notations regarding relative Lie algebra cohomology and cohomological automorphic representation. For each irreducible $(\mathfrak{g}'_\infty, K'_\infty)$-module

$$\pi'_\infty \cong \bigotimes_{v \in \mathcal{V}_\infty} \pi'_v$$

we denote by $H^r(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty)$ the $(\mathfrak{g}'_\infty, K'_\infty)$-cohomology of degree r of π'_∞. By the Künneth formula

$$H^r\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty\right) \cong \bigoplus_{\sum_{v \in \mathcal{V}_\infty} r_v = r} \ \bigotimes_{v \in \mathcal{V}_\infty} H^{r_v}\left(\mathfrak{g}'_v, K'_v, \pi'_v\right)$$

A $(\mathfrak{g}', K'_\infty)$-module π'_∞ is called cohomological if there exists a highest weight representation E_μ of $\mathrm{GL}'_n(\mathbb{R})$ such that $H^r(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu)$ is nonzero for some r. We call an automorphic representation $\Pi' \cong \pi'_\infty \otimes \Pi'_f$ of $\mathrm{GL}'_n(\mathbb{A})$ cohomological if its archimedian component π'_∞ is cohomological. The analogous definition can be made for GL_n.

Chapter 3

Critical values of discrete series representations

Let us concretize the goal of this chapter and give an outlook on the techniques used to achieve it. All definitions and results will be recalled with more rigour and in greater detail throughout this chapter. This chapter is based on [34]

3.1 Summary

Let

$$\mathcal{S} := \Delta \mathrm{GL}'_n \rtimes U'_{(n,n)} = \left\{ \begin{pmatrix} h & X \\ 0 & h \end{pmatrix} : h \in \mathrm{GL}'_n,\, X \in M'_{n,n} \right\}$$

be the Shalika subgroup of GL'_{2n}. We say that an irreducible cuspidal automorphic representation Π' of $\mathrm{GL}'_{2n}(\mathbb{A})$ with central character ω admits a Shalika model with respect to a character η, if $\eta^n = \omega$ and if the Shalika period

$$\mathcal{S}^{\eta}_{\psi}(\phi)(g) := \int_{Z'_{2n}(\mathbb{A})\mathcal{S}(\mathbb{K})\backslash \mathcal{S}(\mathbb{A})} \phi(sg)\, \psi(\mathrm{Tr}(X))^{-1}\, \eta\left(\det{}'(h)\right)^{-1} \mathrm{d}s \neq 0$$

does not vanish for some $\phi \in \Pi'$ and $g \in \mathrm{GL}'_{2n}(\mathbb{A})$.

In the split case, *i.e.* $\mathbb{D} = \mathbb{K}$, it is well know that Π' admits a Shalika model with respect to η if and only if the twisted partial exterior square L-function $L^S\left(s, \Pi', \bigwedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$. In the non-split case there is currently no analogous theorem known, however, in the special case $n = 1$ and $\mathbb{D}$ a quaternion division algebra the following was proved in [41].

Theorem 3.1.1 ([41, Theorem 1.3])**.** *Assume $\mathbb{D}$ is a quaternion division algebra and Π' a cuspidal irreducible automorphic representation of $\mathrm{GL}'_2(\mathbb{A})$. If $\mathrm{JL}(\Pi')$ is cuspidal and irreducible, the following assertions are equivalent.*

1. *Π' admits a Shalika model with respect to η.*
2. *The twisted partial exterior square L-function $L^S\left(s, \Pi', \bigwedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$ and for all $v \in \mathcal{V}_{\mathbb{D}}$, π'_v is not isomorphic to a parabolically induced representation*

$$|\det{}'|_v^{\frac{1}{2}} \tau_1 \times |\det{}'|_v^{-\frac{1}{2}} \tau_2,$$

where τ'_i are representations of $\mathrm{GL}'_1(\mathbb{K}_v)$ with central character η_v.

If $\mathrm{JL}(\Pi')$ *is not cuspidal,* $\mathrm{JL}(\Pi') = \mathrm{MW}(\Sigma, 2)$ *for some cuspidal irreducible representation* Σ *of* $\mathrm{GL}_2(\mathbb{A})$*. Then the following assertions are equivalent.*

1. Π' *admits a Shalika model with respect to* η.
2. *The central character* ω_Σ *of* Σ *equals* η.
3. *The twisted partial exterior square L-function* $L^S\left(s, \Pi', \wedge^2 \otimes \eta^{-1}\right)$ *has a pole at* $s = 2$.

For the rest of the introduction assume $\mathbb{D}$ is a quaternion algebra and Π' be an irreducible cuspidal cohomological automorphic representation of $\mathrm{GL}'_2(\mathbb{A})$ with respect to a coefficient system $E^\vee_\mu$. Note that for a cuspidal Π' and $\sigma \in \mathrm{Aut}(\mathbb{C})$ one can define the σ-twist ${}^\sigma\Pi'_f$ of the finite part of Π'_f. Following [46] we extend this to a σ-twist ${}^\sigma\Pi'$ of Π', which is a discrete series representation of $\mathrm{GL}'_2(\mathbb{A})$. In [46] it was shown that if moreover $\mathrm{JL}(\Pi')$ is cuspidal, ${}^\sigma\Pi'$ is again cuspidal. We prove that the assumption of $\mathrm{JL}(\Pi')$ being cuspidal is not necessary and extend their argument using the Mœglin-Waldspurger classification to the case when $\mathrm{JL}(\Pi')$ is residual. Using the above criterion for admitting a Shalika model, we see that if Π' admits a Shalika model then so does ${}^\sigma\Pi'$. Let $\mathbb{Q}\left(\Pi'_f\right)$ be the field fixed by the automorphisms fixing Π'_f. In [46] it was shown that $\mathbb{Q}\left(\Pi'_f\right)$ is a number field and that $\mathbb{Q}\left(\Pi'_f\right) = \mathbb{Q}\left(\mathrm{JL}(\Pi')_f\right)$. Following [65], [47] we define a finite extension $\mathbb{Q}(\Pi', \eta)$ of $\mathbb{Q}\left(\Pi'_f\right)$ and a $\mathbb{Q}(\Pi', \eta)$-structure on the Shalika model $\mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right)$ of Π'_f.

As in [47] we will make use of a *numerical coincidence*, which is together with Theorem 3.1.1, Theorem 3.6.3 and Theorem 3.4.6 the reason why we must limit ourselves to the case $\mathbb{D}$ being quaternion and $n = 1$. Let q_0 be the lowest degree in which the $(\mathfrak{g}', K'_\infty)$-cohomology of $\pi'_\infty \otimes E^\vee_\mu$ does not vanish. Then we have $q_0 = \dim_\mathbb{Q} \mathbb{K}$ and

$$\dim_\mathbb{C} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \Pi'_\infty \otimes E^\vee_\mu\right) = 1.$$

By fixing a basis vector of this one-dimensional vector space, we can define an isomorphism

$$\Theta_{\Pi'}\colon \mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right) \to H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \Pi' \otimes E^\vee_\mu\right),$$

where the right hand side inherits a $\mathbb{Q}(\Pi', \eta)$-structure from its geometric realization as automorphic cohomology. Thus we can normalize the above isomorphism by a factor $\omega\left(\Pi'_f\right)$, the so-called Shalika period, such that it respects the $\mathbb{Q}(\Pi', \eta)$-structures of both sides. Analogously to [47] we compute how $\omega\left(\Pi'_f\right)$ behaves under twisting with a Hecke character χ of $\mathrm{GL}_1(\mathbb{A})$ lifted to $\mathrm{GL}'_2(\mathbb{A})$ via the determinant map. Let $\mathcal{G}\left(\chi_f\right)$ be the Gauss sum of χ_f. Then

$$\sigma\left(\frac{\omega\left(\Pi'_f \otimes \chi_f\right)}{\mathcal{G}\left(\chi_f\right)^4 \omega\left(\Pi'_f\right)}\right) = \frac{\omega\left({}^\sigma\Pi'_f \otimes {}^\sigma\chi_f\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)^4 \omega\left({}^\sigma\Pi'_f\right)}$$

for $\sigma \in \mathrm{Aut}(\mathbb{C})$.

The next ingredient is the Shalika zeta-integral, first introduced in [39], and extended to $\mathrm{GL}'_2(\mathbb{A})$,

$$\zeta(s, \phi) := \int_{\mathrm{GL}'_1(\mathbb{A})} S^\eta_\psi(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}g_1$$

and its local analogs. As in [47] we fix a special vector $\xi^0_{\Pi'_f} \in \mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right)$ such that

$$\zeta_v\left(\frac{1}{2}, \xi^0_{\Pi'_f}\right) = L\left(\frac{1}{2}, \pi_v\right)$$

if v is a finite place at which ψ and Π' are unramified. By [39] the period integral over $H'_1 = \mathrm{GL}'_1 \times \mathrm{GL}'_1$ of a cusp form is precisely the Shalika zeta integral. To show the invariance of this period integral under the action of a Galois group, we first interpret it as an instance of Poincaré duality of the top cohomology group of the space

$$\mathbf{S}^{H'_1}_{K'_f} = H'_1(\mathbb{K})\backslash H'_1(\mathbb{A}) / \left(K'_\infty \cap H'_{1\infty}\right)\iota^{-1}\left(K'_f\right),$$

where $\iota\colon H'_1 \hookrightarrow \mathrm{GL}'_2$ is the block-diagonal embedding and K'_f a small enough open compact subgroup of $\mathrm{GL}'_2\left(\mathbb{A}_f\right)$. To make the whole story work it is crucial that $\dim_{\mathbb{R}} \mathbf{S}^{H'_1}_{K'_f} = q_0$, which only works if we restrict ourselves to the case $n = 1$ and $\mathbb{D}$ being a quaternion algebra, the aforementioned numerical coincidence. Since we assume that $\mathrm{JL}(\Pi')$ is residual, we then compute that the critical values of $L(s, \Pi')$ are all half-integers $s = \frac{1}{2} + m$, $m \in \mathbb{Z}$. We must moreover assume that for the weight μ there exists an integer p such that for all infinite places v of $\mathbb{K}$ the v-th component $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,4})$ of μ satisfies $-\mu_{v,2} \le p \le -\mu_{v,3}$. In this case we call μ *admissible* and we say $\frac{1}{2}$ is compatible with μ if we can choose $p = 0$.

Since we assume $\mathbb{K}$ to be totally real, we show as in [47] that a certain representation $E_{(0,-w)}$ of H'_1 appears in the coefficient system $E^\vee_\mu$ of π'_∞ if $\frac{1}{2}$ is compatible with μ, which in turn lets us map the fixed special vector $\xi^0_{\Pi'_f}$ first to

$$H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^\eta_\psi(\Pi') \otimes E^\vee_\mu\right)$$

and then interpret it as an element of

$$H^{q_0}_c\left(\mathbf{S}^{H'_1}_{K'_f}, \mathcal{E}^\vee_\mu\right),$$

which we than map to

$$H^{q_0}_c\left(\mathbf{S}^{H'_1}_{K'_f}, \mathcal{E}_{(0,-w)}\right)$$

using the map from above, where $\mathcal{E}^\vee_\mu$ and $\mathcal{E}_{(0,-w)}$ are the sheaves on $\mathbf{S}^{H'_1}_{K'_f}$ associated to $E^\vee_\mu$ and $E_{(0,-w)}$. Finally, applying Poincaré duality to this last space, we show that the resulting number is essentially the value of the L-function $L(s, \Pi')$ at $s = \frac{1}{2}$. Now the final result of [47] for critical values of the L-function follows analogously in our case, namely if $s = \frac{1}{2} + m$ and μ is admissible, there exist periods $\omega\left(\Pi'_f\right)$ and $\omega(\pi'_\infty, m)$ such that

$$\sigma\left(\frac{L\left(\frac{1}{2} + m, \Pi'_f \otimes \chi_f\right)}{\omega\left(\Pi'_f\right)\mathcal{G}\left(\chi_f\right)^4 \omega(\pi'_\infty, m)}\right) = \frac{L\left(\frac{1}{2} + m, {}^\sigma\Pi'_f \otimes {}^\sigma\chi_f\right)}{\omega\left({}^\sigma\Pi'_f\right)\mathcal{G}\left({}^\sigma\chi_f\right)^4 \omega(\pi'_\infty, m)}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi', \eta))$. Let $\mathbb{Q}(\Pi', \eta, \chi)$ be the compositum of $\mathbb{Q}(\Pi', \eta)$ and $\mathbb{Q}(\chi)$. This implies that

$$\frac{L\left(\frac{1}{2} + m, \Pi'_f \otimes \chi_f\right)}{\omega\left(\Pi'_f\right)\mathcal{G}\left(\chi_f\right)^4 \omega(\pi'_\infty, m)} \in \mathbb{Q}\left(\Pi', \eta, \chi\right)$$

and hence, proves the main result.

Theorem 3.1.2. *Let* Π *be a non-cuspidal discrete series representation of* $\mathrm{GL}_4(\mathbb{A})$ *written as* $\Pi \cong \mathrm{MW}(\Sigma|\det'|^{\frac{1}{2}} \times \Sigma|\det'|^{-\frac{1}{2}})$ *via the Mœglin-Waldspurger classification, where* Σ *is a cuspidal irreducible representation of* $\mathrm{GL}_2(\mathbb{A})$. *Assume moreover that there exists an irreducible cuspidal cohomological representation* Π' *of* $\mathrm{GL}_2'(\mathbb{A})$ *with* $\mathrm{JL}(\Pi') = \Pi$ *which is cohomological with respect to coefficient system* $E_\mu^\vee$ *and* μ *is admissible. Let* χ *be a finite order Hecke-character of* $\mathrm{GL}_1(\mathbb{A})$ *and* $s = \frac{1}{2} + m$ *a critical value of* $L(s, \Pi')$. *Then*

$$\frac{L\left(\frac{1}{2}+m, \Pi_f \otimes \chi_f\right)}{\omega\left(\Pi_f'\right)\mathcal{G}\left(\chi_f\right)^4 \omega\left(\pi_\infty', m\right)} \in \mathbb{Q}\left(\Pi', \omega_\Sigma, \chi\right).$$

3.2 Cohomological unitary dual

We start by recalling the classification of the cohomological irreducible unitary dual of $\mathrm{GL}_n(\mathbb{H})$ due to [126] and explicitly described in [46]. Let $\mathfrak{g}'$ be the Lie algebra of $\mathrm{GL}_n(\mathbb{H})$ and let $\mathfrak{k}'$ be the Lie algebra of $^{(}n)$, which determines a Cartan involution $\theta'(X) = -\bar{X}^T$ of $\mathfrak{g}'$. Moreover, let $\mathfrak{h}'$ be a maximal compact, θ'-stable Cartan-algebra $\mathfrak{h}' = \mathfrak{a}' \oplus \mathfrak{t}'$, with

$$\mathfrak{t}' = \left\{\begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix} : x_j \in \mathbb{R}\right\} \text{ and } \mathfrak{a}' = \left\{\begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} : y_j \in \mathbb{R}\right\}.$$

Furthermore, let E_λ be a highest weight representation of $\mathrm{GL}_n(\mathbb{H})$, where λ is a highest weight with respect to the subalgebra $\mathfrak{h}'_{\mathbb{C}}$. To each composition $n = \sum_{i=0}^r n_i$ written as

$$\underline{n} = [n_0, \ldots, n_r]$$

with $n_0 \geq 0$ and $n_i > 0$ we can associate a θ'-stable, parabolic subalgebra $\mathfrak{q}'_{\underline{n}}$ of $\mathfrak{g}'_{\mathbb{C}}$ whose Levi-decomposition we will denote as $\mathfrak{q}'_{\underline{n}} = \mathfrak{l}'_{\underline{n}} + \mathfrak{u}'_{\underline{n}}$, *cf.* [46, Section 4] for more details. We further assume that $\lambda\big|_{\mathfrak{a}'} = 0$ and that λ can be extended to an admissible character of $\mathfrak{l}'_{\underline{n}} \supseteq \mathfrak{h}'_{\mathbb{C}}$.

Theorem 3.2.1 ([46, Theorem 4.9]). *Let* E_λ *be a self-dual highest weight representation of* $\mathrm{GL}_n(\mathbb{H})$.

1. *To each ordered composition* $\underline{n} = [n_0, \ldots, n_r]$ *of* n *with* $n_0 \geq 0, n_i > 0$ *one can assign an irreducible unitary representation* $A_{\underline{n}}(\lambda)$ *of* $\mathrm{GL}_n(\mathbb{H})$.
2. *All such representations are cohomological with respect to* E_λ *and every cohomological representation is of this form.*
3. *The Poincaré polynomial of* $H^*\left(\mathfrak{g}', ^{(}n)\mathbb{R}_{\geq 0}, E_\lambda \otimes A_{\underline{n}}(\lambda)\right)$ *is*

$$P(\underline{n}, X) = \frac{X^{\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^- \cap \mathfrak{u}'_{\underline{n}}\right)}}{1+X} \prod_{i=1}^{r}\prod_{j=1}^{n_i}\left(1 + X^{2j-1}\right)\prod_{j=1}^{n_0}\left(1+X^{4j-3}\right).$$

Here $\mathfrak{g}_{\mathbb{C}}^-$ *is the* -1*-eigenspace of* θ' *acting on* $\mathfrak{g}'_{\mathbb{C}}$.

For later use, we compute the following.

Lemma 3.2.2. *Let* $\underline{n} = [n_0, n_1, \dots, n_r]$ *be a composition of* n*. Then*

$$\dim_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}^{-} \cap \mathfrak{u}'_{\underline{n}}) = \sum_{i=1}^{r} \binom{n_i}{2} + 2 \sum_{0 \le i < j \le r} n_i n_j.$$

Proof. We first recall the definition of $\mathfrak{u}'_{\underline{n}}$, *cf.* [46, §4.2]. Let

$$x = \operatorname{diag}(\underbrace{0, \dots, 0}_{n_0}, \underbrace{1, \dots, 1}_{n_1}, \dots, \underbrace{r \dots, r}_{n_r}) \in i\mathfrak{t}'$$

and let $\Delta(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ respectively $\Delta(\mathfrak{g}^{-}_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ be the set of roots coming from $\mathfrak{t}'_{\mathbb{C}}$. We have the explicit description

$$\Delta(\mathfrak{g}^{-}_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) = \{\pm e_i \pm e_j, 1 \le i < j \le n\},$$

where $e_j(H) = ix_j$ for $H = \operatorname{diag}(ix_1 + y_1, \dots, ix_n + y_n)) \in \mathfrak{h}'$. Moreover,

$$\mathfrak{u}'_{\underline{n}} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) \\ \alpha(x) > 0}} (\mathfrak{g}'_{\mathbb{C}})_{\alpha}$$

and therefore

$$\mathfrak{u}'_{\underline{n}} \cap \mathfrak{g}^{-}_{\mathbb{C}} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}^{-}_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}) \\ \alpha(x) > 0}} (\mathfrak{g}'_{\mathbb{C}})_{\alpha}.$$

Hence

$$\dim_{\mathbb{C}}(\mathfrak{g}^{-}_{\mathbb{C}} \cap \mathfrak{u}'_{\underline{n}}) = \#\{\alpha \in \Delta(\mathfrak{g}^{-}_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}}),\ \alpha(x) > 0\},$$

which is easily seen to be equal to the above explicit formula. □

Our next step is to showing that if Σ is a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and $k \in \mathbb{N}$, then Σ is cohomological if $\mathrm{MW}(\Sigma, k)$ is. The author would like to thank Harald Grobner for pointing out the argument presented here. Before we start, we need to recall the following theorem.

Theorem 3.2.3 ([112, Theorem 1.8])**.** *Let* G *be a connected, semisimple real Lie group with finite center and Lie algebra* $\mathfrak{g}$*. Fix a maximal connected subgroup* K *of* G *with Lie algebra* $\mathfrak{k}$ *and moreover, let* π *be an irreducible unitary smooth representation of* G *with central character* χ_π*. Finally, let* U *be a finite-dimensional* $(\mathfrak{g}, K)$*-module admitting an infinitesimal character* $\chi_U = \chi_{\pi^\vee}$*. Then*

$$H^*(\mathfrak{g}, \mathfrak{k}, \pi \otimes U) \neq 0.$$

We denote for a real Lie group G by Z_G its center and by Z_G^0 the connected component of the latter.

Lemma 3.2.4. *Let* $\underline{G}$ *be a connected reductive group over* $\mathbb{K}$*,* $v \in \mathcal{V}_\infty$ *and* $G \coloneqq \underline{G}(\mathbb{K}_v)$*. Let* π *be an irreducible unitary representation of* G *and* E_λ *a finite dimensional highest weight representation of* G *over* $\mathbb{C}$ *such that* Z_G^0*, acts trivially on* $E_\lambda \otimes \pi$ *and* $\chi_{E_\lambda} = \chi_{\pi^\vee}$*. Then*

$$H^*(\mathfrak{g}, (Z_G \cdot K)^0, \pi \otimes E_\lambda) \neq 0.$$

Proof. Note that E_λ always admits a central character. Recall that

$$H^*(\mathfrak{g}, (Z_G \cdot K)^0, \pi \otimes E_\lambda) =$$

$$= H^*(\mathfrak{g}, \mathfrak{z}_G \oplus \mathfrak{k}, \pi \otimes E_\lambda),$$

where $\mathfrak{z}_G$ is the Lie algebra of Z_G and we use that K has finite center. Since $\mathfrak{z}_G$ acts trivially on $\pi \otimes E_\lambda$, the Künneth formula gives a decomposition

$$H^*(\mathfrak{g}, \mathfrak{z}_G \oplus \mathfrak{k}, \pi \otimes E_\lambda) \cong \bigotimes_{a+b=*} H^a(\mathfrak{z}_G, \mathfrak{z}_G, \mathbb{C}) \otimes H^b(\mathfrak{g}/\mathfrak{z}_G, \mathfrak{k}, \pi \otimes E_\lambda) =$$

$$= H^*(\mathfrak{g}/\mathfrak{z}_G, \mathfrak{k}, \pi \otimes E_\lambda).$$

The latter does not vanish by Theorem 3.2.3, since both π and E_λ admit the right central characters and the image of $\mathfrak{g}/\mathfrak{z}_G$ under the exponential map generates the connected, semisimple real Lie group G/Z_G . □

Let $\underline{G}$ be either GL_n or GL'_n, $v \in \mathcal{V}_\infty$, $G = \underline{G}(\mathbb{K}_v)$ and $K = K_v$ or K'_v. Moreover, let $\underline{P}$ be a standard parabolic subgroup of $\underline{G}$ and set $P = \underline{P}(\mathbb{K}_v) = L \rtimes U$. Write $L = M \times A^0$, where $A^0 = Z_L^0$. Next, let (π, V) be an irreducible, unitary representation of L. Denote now by $\mathfrak{b}_\mathbb{C}$ the complexified Cartan subalgebra of the Lie algebra of M coming from our fixed choice of Cartan subalgebra of L, *i.e.* the diagonal matrices if $\underline{G} = \mathrm{GL}_n$ or $\mathfrak{h}'$ if $\underline{G} = \mathrm{GL}'_n$. Let $\mathfrak{a}^\vee_{P,\mathbb{C}}$ be the complexified dual of the Lie-algebra of A^0 and fix $\mu \in \mathfrak{a}^\vee_{P,\mathbb{C}}$. We let $\mathfrak{p}_\mathbb{C}$ be the complexified Lie algebra of P and let ρ be the half-sum of all positive roots of $\mathfrak{p}_\mathbb{C}$ with respect to our fixed Cartan subalgebra. Denote by Δ_M the simple roots of M, W the Weyl-group of G and

$$W^P = \{w \in W : w^{-1}(\alpha) > 0 \text{ for all } \alpha \in \Delta_M\}.$$

We write

$$\mathrm{Ind}_P^G(\pi, \mu) = \{f : G \to V \text{ smooth} : f(maug) = a^{\rho+\mu}\pi(m)f(g),$$

$$a \in A^0,\ m \in M,\ u \in U,\ g \in G\}$$

and use the standard parametrization of infinitesimal characters, i.e. for a highest weight representation E_λ, $\chi_{\lambda+\rho} = \chi_{E_\lambda}$.

Proposition 3.2.5. *If τ is a non-zero $(\mathfrak{g}, (Z_G \cdot K)^0)$-module, which appears as a quotient of $\mathrm{Ind}_P^G(\pi, \mu)$ and is cohomolocigal with respect to some highest weight representation $E_\lambda^\vee$, then π is cohomological as a $(\mathfrak{l}, (Z_L \cdot (L \cap K)^0)$-module with respect to $E^\vee_{w(\lambda+\rho)-\rho}$, where w is some element of W^P.*

Proof. We notice that without loss of generality A^0 acts trivially on π. Moreover, if χ_π denotes the infinitesimal character of π, $\mathrm{Ind}_P^G(\pi, \mu)$ and hence also τ have infinitesimal character $\chi_{\pi+\mu}$. On the other hand, τ is by assumption cohomological with respect to $E_\lambda^\vee$ and hence it has to have infinitesimal character $\chi_{\lambda+\rho}$ by [22, Theorem I.5.3]. Therefore the infinitesimal character of $\pi\big|_M$ is equal to

$$\chi_{\lambda+\rho-\mu}\big|_{\mathfrak{b}_\mathbb{C}}$$

and hence $\pi\big|_M$ has non-vanishing cohomology with respect to $E^\vee_{w(\lambda+\rho)-\rho\big|_{\mathfrak{b}_\mathbb{C}}}$ by Lemma 3.2.4. Now for any Konstant-representative $w \in W^P$

$$\chi_{\lambda+\rho-\mu\big|_{\mathfrak{b}_\mathbb{C}}} = \chi_{w(\lambda+\rho)\big|_{\mathfrak{b}_\mathbb{C}}}.$$

Note now that $w(\lambda+\rho)-\rho\big|_{\mathfrak{b}_\mathbb{C}}$ is a dominant weight, see [22, III.3.2], and hence the last character is equal to

$$\chi_{E^\vee_{w(\lambda+\rho)-\rho\big|_{\mathfrak{b}_\mathbb{C}}}}.$$

We can choose now w as in [22, III. Theorem 3.3] such that $Z^0_L = A^0$ acts trivially on

$$\pi \otimes E^\vee_{w(\lambda+\rho)-\rho}.$$

Hence π is by Lemma 3.2.4 cohomological with respect to $E^\vee_{w(\lambda+\rho)-\rho}$. ☐

Corollary 3.2.1. *Let Σ be a cuspidal irreducible representation of $\mathrm{GL}_n(\mathbb{A})$ or $\mathrm{GL}'_n(\mathbb{A})$ and $k \in \mathbb{N}$. Then Σ is cohomological if $\mathrm{MW}(\Sigma,k)$ is cohomological.*

Proof. Since $\mathrm{MW}(\Sigma,k)^\infty_v$ is the quotient of

$$(\sigma^\infty_v|\det'|^{\frac{k-1}{2}} \times \ldots \times \sigma^\infty_v|\det'|^{\frac{1-k}{2}})_v$$

for all $v \in \mathcal{V}_\infty$, the claim follows from Proposition 3.2.5, because

$$K'_v = \mathrm{Sp}(\frac{nd}{2})\mathbb{R}_{>0} = (\mathrm{Sp}(\frac{nd}{2})\mathbb{R})^0,\ K_v = \mathrm{SO}(n)\mathbb{R}_{>0} = (\mathrm{O}(n)\mathbb{R})^0.$$

☐

Lemma 3.2.6. *Assume Π' is a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ such that $\mathrm{JL}(\Pi')$ is not a cuspidal representation of $\mathrm{GL}_{2dn}(\mathbb{A})$. Let $2dn = kl$ and let Σ be a unitary cuspidal irreducible representation of $\mathrm{GL}_l(\mathbb{A})$ such that $\mathrm{JL}(\Pi') = \mathrm{MW}(\Sigma,k)$.*

Then l is even and σ_v is cohomological with respect to some highest weight representation E_{λ_v}, $\lambda_v = \left(\lambda_{v,1},\ldots,\lambda_{v,\frac{l}{2}}\right)$. For each $v \in \mathcal{V}_\infty$, π'_v is of the form $\pi'_v = A_{\underline{nd}}(\lambda'_v)$ for

$$\underline{nd} = [0,\overbrace{k,\ldots,k}^{\frac{l}{2}}]$$

and $\lambda_{v,\frac{l}{2}} = \lambda'_{v,nd}$. In particular, the lowest respectively highest degree in which the cohomology group

$$H^q(\mathfrak{g}'_v, \mathrm{Sp}(nd)\mathbb{R}_{\geq 0}, \pi'_v \otimes E_{\lambda'_v})$$

does not vanish is

$$q = nd\,(nd-1) - \frac{nd}{2}\,(k-1) \text{ respectively } q = nd\,(nd-1) + \frac{nd}{2}\,(k+1) - 1.$$

Proof. Fix an infinite place $v \in \mathcal{V}_\infty$. By [46, Theorem 5.2] $\mathrm{MW}(\Sigma, k)$ is cohomological and thus by Corollary 3.2.1 so is Σ. By [11, Theorem 18.2], $k|d$ and hence, l has to be even. Since the archimedean component of a cohomological cuspidal irreducible unitary representation of $\mathrm{GL}_l(\mathbb{A})$ must be tempered we may write

$$\sigma_v = A_{[0,2,\ldots,2]}(\lambda_v)$$

and let

$$\pi'_v = A_{\underline{nd}}\left(\lambda'_v\right)$$

for suitable $\underline{nd}$ and λ_v, λ'_v, with $\underline{nd} = [n_0, n_1, \ldots, n_{l'}]$, see [46, Section 5.5]. Furthermore, σ_v is fully induced from representations of $\mathrm{GL}_2(\mathbb{R})$. To proceed with the proof, we are quickly going to recap the construction of $A_{\underline{nd}}(\lambda'_v)$ in the proof of [46, Theorem 5.2]. Let

$$\rho_{\mathfrak{gl}_m(\mathbb{H})} = \left(\frac{2m-1}{2}, \ldots, -\frac{2m-1}{2}\right)$$

respectively

$$\rho_{\mathfrak{gl}_m(\mathbb{C})} = \left(\left(\frac{m-1}{2}, \ldots, -\frac{m-1}{2}\right), \left(\frac{m-1}{2}, \ldots, -\frac{m-1}{2}\right)\right)$$

the smallest algebraically integral element in the interior of the dominant Weyl chamber of $\mathrm{GL}_m(\mathbb{H})$ respectively $\mathrm{GL}_m(\mathbb{C})$. Define now

$$\mu := \left(\rho_{\mathfrak{gl}_{n_0}(\mathbb{H})}, \rho_{\mathfrak{gl}_{n_1}(\mathbb{C})}, \ldots, \rho_{\mathfrak{gl}_{n_{l'}}(\mathbb{C})}\right)$$

and let P' be a certain complex parabolic subgroup of $\mathrm{GL}_{dn}(\mathbb{H})$, which we will specify in a moment, and having Levi-factor $\prod_{i=1}^{nd} \mathrm{GL}_1(\mathbb{H})$. For any integer $s > 0$ and $u \in \mathbb{C}$ we set

$$D(u, s) := D(s) \otimes |\det|^{-\frac{u}{2}},$$

where $D(s)$ is the unique irreducible discrete series representation of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ of lowest $\mathrm{O}(2)$-type $s+1$. We also set

$$F(u, s) := F(s) \otimes |\det'|^{-\frac{u}{2}},$$

where $F(s)$ is the unique irreducible representation of $\mathrm{SL}_1(\mathbb{H})$ of dimension s. Moreover, recall the Levi decomposition $\mathfrak{q}'_{\underline{nd}} = \mathfrak{l}'_{\underline{nd}} + \mathfrak{u}'_{\underline{nd}}$ and let $\rho(\underline{nd}) = (\rho(\underline{nd})_1, \ldots, \rho(\underline{nd})_{nd})$ be the half-sum of all roots appearing in $\mathfrak{u}'_{\underline{nd}}$. Let $k_i = \lambda'_i + \rho(\underline{nd})_i$. We set

$$\sigma = \bigotimes_{i=1}^{nd-n_0} F(0, k_i).$$

Then P' can be chosen such that (P', σ, μ) is a Langlands-datum and $A_{\underline{nd}}(\lambda'_v)$ is the unique irreducible quotient of the induced representation $\mathrm{Ind}_{P'}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\sigma, \mu)$. Since $\sigma_v |\det'|^{\frac{k+1}{2}-j}$ is essentially tempered for every $v \in \mathcal{V}_\infty$ and $j \in \{1, \ldots, k\}$ and σ_v is cohomological,

$$\sigma_v |\det'|^{\frac{k+1}{2}-j} \cong \mathrm{Ind}_{P_j}^{\mathrm{GL}_l(\mathbb{K}_v)}(\sigma_j)$$

by [46, Section 5.5], where

$$\sigma_j = \bigotimes_{i=1}^{\frac{l}{2}} D(2j-k-1, k_{i,j})$$

for certain $k_{i,j} \in \mathbb{Z}_{>0}$ and P_j is the standard parabolic subgroup of upper triangular matrices with block size $\overbrace{(2,\ldots,2)}^{\frac{l}{2}}$. Recall furthermore

$$A_{\underline{nd}}\left(\lambda_v'\right) = \pi_v' = \mathrm{LJ}_v\left(\left(\mathrm{JL}\left(\Pi'\right)\right)_v\right) = \mathrm{LJ}_v\left(\mathrm{MW}\left(\sigma_v, k\right)\right).$$

By [46, Theorem 5.2] and its proof the last term is equal to the Langlands quotient of

$$\mathrm{Ind}_P^{\mathrm{GL}_{nd}(\mathbb{H})}\left(\bigotimes_{j=1}^{k}\bigotimes_{i=1}^{\frac{l}{2}} F\left(0, k_{i,j}\right), \mu'\right),$$

where now P is the standard parabolic subgroup of type $\overbrace{(1,\ldots,1)}^{nd}$ of $\mathrm{GL}_{nd}(\mathbb{H})$ and

$$\mu' = \left(\overbrace{\frac{k-1}{2},\ldots,\frac{k-1}{2}}^{\frac{l}{2}},\ldots,\overbrace{-\frac{k-1}{2},\ldots,-\frac{k-1}{2}}^{\frac{l}{2}}\right).$$

Comparing μ and μ' and using the uniqueness of the Langlands quotient implies then that $n_0 = 0$ and $n_1 = \ldots = n_{l'} = k$. Moreover, by we have $\lambda_{v,\frac{l}{2}} + 1 = k_{\frac{l}{2},k} = k_{nd} = \lambda'_{v,nd} + 1$

From Lemma 3.2.2 we obtain that

$$\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^{-} \cap \mathfrak{u}'_{\underline{nd}}\right) = 2\binom{nd}{2} - \frac{l}{2}\binom{k}{2} = nd\,(nd-1) - \frac{nd}{2}\,(k-1).$$

Therefore, by Theorem 3.2.1 the lowest degree of non-vanishing cohomology is $nd\,(nd-1) - \frac{nd}{2}\,(k-1)$ and the highest degree of non-vanishing cohomology is

$$\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}^{-} \cap \mathfrak{u}'_{\underline{nd}}\right) - 1 + \sum_{j=1}^{\frac{l}{2}}\sum_{i=1}^{k}(2i-1) = nd\,(nd-1) + \frac{nd}{2}\,(k+1) - 1.$$

□

Remark. To extend the ideas of [47] to the case $\mathrm{GL}'_{2n}(\mathbb{A})$ we need the following numerical coincidence. Namely, it will be necessary that either the lowest or highest degree in which the cohomology group

$$H^*\left(\mathfrak{g}'_v, K'_v, \pi_v \otimes E^{\vee}_{\lambda_v}\right)$$

does not vanish is

$$q_0 = (nd)^2 - (nd) - 1 = \dim_{\mathbb{R}}\left(H'_{\frac{nd}{2}}(\mathbb{R}) / \left(\mathrm{Sp}\left(\frac{nd}{2}\right) \times \mathrm{Sp}\left(\frac{nd}{2}\right) \times \mathbb{R}_{>0}\right)\right).$$

By Lemma 3.2.6 the only possible value for n is therefore $n = 1$, $d = 2$ and the composition $\underline{2n}$ of $2n = 2$ has to be $\underline{2n} = [0, 2]$.

3.3 Rational structures

Next we will recall the action of $\mathrm{Aut}(\mathbb{C})$ on representations. Let Π'_f be a representation of $\mathrm{GL}'_n(\mathbb{A}_f)$ on some complex vector space W and $\sigma \in \mathrm{Aut}(\mathbb{C})$. We define the σ-twist ${}^\sigma\Pi'_f$ as follows, *cf.* [127]. Let W' be a complex vector space which allows a σ-linear isomorphism $t: W' \to W$. We then set

$$ {}^\sigma\Pi'_f := t^{-1} \circ \Pi'_f \circ t. $$

An explicit example of such a space W' is the space

$$ W' = W \otimes_{\mathbb{C}} {}_\sigma\mathbb{C}, $$

where ${}_\sigma\mathbb{C}$ is $\mathbb{C}$ as a field but $\mathbb{C}$ acts on ${}_\sigma\mathbb{C}$ via σ^{-1}. Then W' is a $\mathbb{C}$ vector space via the right action of $\mathbb{C}$ on $\mathbb{C}$ and the map $t: W \to W \otimes_{\mathbb{C}} {}_\sigma\mathbb{C}$ is given by $w \mapsto w \otimes 1$. Similarly, we define the σ-twist ${}^\sigma\pi'_v$ of a local representation π'_v with $v \in \mathcal{V}_f$. For a highest weight representation E_μ of $\mathrm{GL}'_{n,\infty}$, we define

$$ \left({}^\sigma E_\mu\right)_v := \left(E_\mu\right)_{\sigma^{-1}\circ v}, $$

where v is seen as an embedding $\mathbb{K} \hookrightarrow \mathbb{C}$ and hence, $\sigma^{-1} \circ v$ defines an infinite place of $\mathbb{K}$. For Π'_f as above let

$$ \mathfrak{S}\left(\Pi'_f\right) := \{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^\sigma\Pi'_f \cong \Pi'_f\} $$

and let

$$ \mathbb{Q}\left(\Pi'_f\right) := \{z \in \mathbb{C} : \sigma(z) = z, \text{ for all } \sigma \in \mathfrak{S}\left(\Pi'_f\right)\} $$

be the rationality field of Π'_f. Analogously we define for a highest weight representation E_μ and a local representation π'_v the fields $\mathbb{Q}(E_\mu)$ and $\mathbb{Q}(\pi'_v)$. Moreover, if $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n and $\rho = \rho_1 \otimes \ldots \otimes \rho_k$ an irreducible representation of $M'_\alpha(\mathbb{K}_v)$, $v \in \mathcal{V}_f$,

$$ {}^\sigma\mathrm{ind}_{P'_\alpha(\mathbb{K}_v)}^{\mathrm{GL}_n(\mathbb{K}_v)}(\rho) = \mathrm{ind}_{P'_\alpha(\mathbb{K}_v)}^{\mathrm{GL}_n(\mathbb{K}_v)}({}^\sigma\rho), \; \sigma \in \mathrm{Aut}(\mathbb{C}) $$

and therefore

$$ {}^\sigma(\rho_1 \times \ldots \times \rho_k) = ({}^\sigma\rho_1 \times \ldots \times {}^\sigma\rho_k)\, \epsilon_\sigma^{dn-1}, \; \epsilon_\sigma := \frac{|\det'|^{\frac{1}{2}}}{\sigma\left(|\det'|^{\frac{1}{2}}\right)} \tag{6} $$

and similarly for the split case GL_n.

Finally, we say that the representation Π'_f, π'_v or E_μ with underlying vector space W is defined over some field $\mathbb{L} \subseteq \mathbb{C}$ if there exists an $\mathbb{L}$-vector space $W_\mathbb{L} \subseteq W$, stable under the group action of $\mathrm{GL}'_n(\mathbb{A}_f)$, $\mathrm{GL}'_n(\mathbb{K}_v)$ respectively $\prod_{v\in\mathcal{V}_\infty} \mathrm{GL}'_n(\mathbb{K})$, such that the natural map $W_\mathbb{L} \otimes_\mathbb{L} \mathbb{C} \to W$ is an isomorphism. In this case we say W admits an $\mathbb{L}$-structure. Let E_μ be a highest weight representation and let $\mathbb{L}$ be a minimal field extension of $\mathbb{K}$ such that $\mathbb{D}$ splits over $\mathbb{L}$. Then E_μ is defined over $\mathbb{L}$, see [46, Lemma 7.1] and we set

$$ \mathbb{Q}(\mu) := \mathbb{L} \cdot \mathbb{Q}(E_\mu). $$

Lemma 3.3.1 ([26, Proposition 3.2]). *Let $v \in \mathcal{V}_f$ and π'_v an irreducible representation of $\mathrm{GL}'_n(\mathbb{K}_v)$. Then π'_v admits an $\mathbb{Q}(\pi'_v)$-structure.*

Note that in the reference the lemma is only proven in the case GL_n. However, the proof carries over analogously, since the Langlands classification via multisegments used in it is also valid for GL'_n.

Theorem 3.3.2 ([46, Theorem 8.1, Proposition 8.2, Theorem 8.6])**.** *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and let μ be a highest weight such that Π' is cohomological with respect to E_μ. Then Π'_f is defined over the number field*

$$\mathbb{Q}(\Pi') \coloneqq \mathbb{Q}(\mu)\,\mathbb{Q}(\Pi'_f).$$

Moreover, let $S \subseteq \mathcal{V}$ be a finite set containing all places where Π'_f ramifies. Then $\mathbb{Q}(\Pi'_f)$ is the compositum of the number fields $\mathbb{Q}(\pi'_v), v \in \mathcal{V}_f - S$.

We also have the following theorem by the same authors.

Theorem 3.3.3 ([46, Proposition 7.21])**.** *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_n(\mathbb{A})$ and let μ be a highest weight such that Π' is cohomological with respect to E_μ. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ the representation ${}^\sigma\Pi'_f$ is the finite part of a discrete series representation ${}^\sigma\Pi'$ of $\mathrm{GL}'_n(\mathbb{A})$ which is cohomological with respect to ${}^\sigma E_\mu$. Moreover, if E_μ is regular, ${}^\sigma\Pi$ is cuspidal.*

Definition. We say the $\mathrm{Aut}(\mathbb{C})$-orbit of an cuspidal irreducible representation Π' of either $\mathrm{GL}'_n(\mathbb{A})$ or $\mathrm{GL}_n(\mathbb{A})$ is cuspidal cohomological if ${}^\sigma\Pi'$ is cuspidal and cohomological for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

We will now show that the regularity condition on E_μ is not needed.

Proposition 3.3.4. *Let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_n(\mathbb{A})$. Then ${}^\sigma\Pi'$ is cuspidal for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. Moreover,*

$${}^\sigma\mathrm{JL}(\Pi') = \mathrm{JL}({}^\sigma\Pi')$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Proof. If $\Pi \coloneqq \mathrm{JL}(\Pi')$ is cuspidal, the two claims are proven in [46, Theorem 7.30]. More precisely, they are proven under the assumption that Π is so-called *regular algebraic*, which by [26, Lemma 3.14] is equivalent to Π being cohomological. Since JL sends cohomological representations to cohomological representations, this shows the first claims.

If Π is not cuspidal, it is still a discrete series and we can write $\Pi = \mathrm{MW}(\Sigma, k)$ for some $k \geq 1$ by Theorem 2.4.1. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. We will proceed by showing that ${}^\sigma\Pi'$ is cuspidal by induction on the $\mathbb{K}$-rank of GL'_n. If $n = 1$, we already know that ${}^\sigma\Pi'$ is cuspidal. For $n > 1$, let Θ, Σ', s and t be such that

$$\mathrm{MW}(\Theta, s) = {}^\sigma\Pi',\ \mathrm{MW}(\Sigma', t) = \mathrm{JL}(\Theta)$$

and hence, by [13, Theorem 18.2]

$$\mathrm{JL}({}^\sigma\Pi') = \mathrm{MW}(\Sigma', st).$$

Note that

$$\begin{aligned}{}^\sigma\mathrm{MW}(\Sigma,k)_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}} &= {}^\sigma\mathrm{JL}(\Pi')_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}} \overset{(2.4.2.(2))}{=} {}^\sigma\left(\Pi'_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}}\right) \overset{(2.4.2)}{=} \\ &= ({}^\sigma\Pi')_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}} \overset{(2.4.2.(2))}{=} \mathrm{JL}({}^\sigma\Pi')_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}} = \mathrm{MW}(\Sigma', st)_{\mathcal{V}\smallsetminus\mathcal{V}_\mathbb{D}}.\end{aligned} \tag{7}$$

We will need the following intermediate lemma.

Lemma 3.3.5. *We have*

$$^{\sigma}\mathrm{MW}\left(\Sigma,k\right)=\mathrm{MW}\left({}^{\sigma}\Sigma\chi_{\sigma},k\right)$$

for some quadratic character χ_σ *with* ${}^{\sigma^{-1}}\chi_\sigma=\chi_{\sigma^{-1}}^{-1}$.

Proof. Let $v\in\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}$ be a place where $\mathrm{MW}(\Sigma,k)$ and Σ are unramified. We can write $\sigma_v=\mathrm{L}(\mathfrak{m})$ and $\mathrm{MW}(\Sigma,k)_v=\mathrm{L}(\mathfrak{m}')$ for some multisegments $\mathfrak{m}$ and $\mathfrak{m}'$ both consisting only of segments of length 1 and with unramified cuspidal support. Moreover, $\mathfrak{m}$ and $\mathfrak{m}'$ determine each other and

$$\cos(\mathrm{L}(\mathfrak{m})|\det|^{\frac{k-1}{2}}\times\ldots\times\mathrm{L}(\mathfrak{m})|\det|^{\frac{1-k}{2}})=\mathrm{L}(\mathfrak{m}').$$

by Theorem 2.4.1. Applying [26, Lemma 3.5(ii)] both to $\mathrm{L}(\mathfrak{m})$ and $\mathrm{L}(\mathfrak{m}')$ yields

$$\cos({}^{\sigma}\mathrm{L}(\mathfrak{m})|\det|^{\frac{k-1}{2}}\chi_\sigma\times\ldots\times{}^{\sigma}\mathrm{L}(\mathfrak{m})|\det|^{\frac{1-k}{2}}\chi_\sigma)={}^{\sigma}\mathrm{L}(\mathfrak{m}'),$$

where χ_σ is ϵ_σ if both k and dn are odd and the trivial character otherwise. By [78, Lemma 1] $\mathrm{MW}\left({}^{\sigma}\Sigma\epsilon_\sigma,k\right)_v$ has to be the unique constituent of

$$\left(|\det|^{\frac{k-1}{2}}\Sigma\chi_\sigma\times\ldots\times|\det|^{\frac{1-k}{2}}\Sigma\chi_\sigma\right)_v$$

with a K_v'-fixed vector for almost all places $v\in\mathcal{V}_f$. Similarly, ${}^{\sigma}\mathrm{MW}\left(\Sigma,k\right)_v$ has to be the unique constituent of

$$^{\sigma}\left(|\det|^{\frac{k-1}{2}}\Sigma\times\ldots\times|\det|^{\frac{1-k}{2}}\Sigma\right)_v$$

with a K_v'-fixed vector for almost all places $v\in\mathcal{V}_f$. Thus, ${}^{\sigma}\mathrm{MW}\left(\Sigma,k\right)$ and $\mathrm{MW}\left({}^{\sigma}\Sigma\chi_\sigma,k\right)$ have to agree at almost all places and the claim follows then from Strong Multiplicity One, *cf.* [11, §4.4]. □

Hence, it follows from (7) that $st=k$ and ${}^{\sigma}\Sigma\chi_\sigma=\Sigma'$ by Strong Multiplicity One. Assume now that $s>1$. By Corollary 3.2.1, we know that Θ is cohomological and since $s>1$ the induction hypothesis implies that ${}^{\sigma^{-1}}\Theta$ is cuspidal. We thus can consider the discrete series representation $\mathrm{MW}\left({}^{\sigma^{-1}}\Theta,t\right)$. Finally,

$$\mathrm{JL}\left({}^{\sigma^{-1}}\Theta\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}={}^{\sigma^{-1}}\mathrm{JL}\left(\Theta\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}={}^{\sigma^{-1}}\mathrm{MW}\left({}^{\sigma}\Sigma\chi_\sigma,t\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}\overset{(3.3.5)}{=}$$

$$=\mathrm{MW}\left(\Sigma^{\sigma^{-1}}\chi_\sigma\chi_{\sigma^{-1}},t\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}=\mathrm{MW}\left(\Sigma,t\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}.$$

Therefore,

$$\mathrm{JL}\left(\mathrm{MW}\left({}^{\sigma^{-1}}\Theta,s\right)\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}=\mathrm{MW}\left(\Sigma,k\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}=|JL|\left(\Pi'\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}},$$

which implies $\mathrm{MW}\left({}^{\sigma^{-1}}\Theta,s\right)=\Pi'$ by Strong Multiplicity One and the injectivity of JL, a contradiction by Theorem 2.4.1. Thus, $s=1$ and hence, ${}^{\sigma}\Pi'$ is cuspidal. Moreover,

$$^{\sigma}\mathrm{JL}\left(\Pi'\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}={}^{\sigma}\mathrm{MW}\left(\Sigma,k\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}=\mathrm{MW}\left({}^{\sigma}\Sigma\epsilon_\sigma,k\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}=\mathrm{JL}\left({}^{\sigma}\Pi\right)_{\mathcal{V}\smallsetminus\mathcal{V}_{\mathbb{D}}}$$

and the second claim follows again from Strong Multiplicity One. □

3.4 Shalika models

Let $U'_{(n,n)}$ and $\mathcal{S}$ be the following two subgroups of GL'_{2n}. We recall

$$U'_{(n,n)} = \left\{ \begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix} : X \in M'_{n,n} \right\}$$

and define the Shalika subgroup

$$\mathcal{S} := \Delta \mathrm{GL}'_n \rtimes U'_{(n,n)} = \left\{ \begin{pmatrix} h & X \\ 0 & h \end{pmatrix} : h \in \mathrm{GL}'_n, X \in M'_n \right\}.$$

Let ψ be the additive character fixed in Section 2.1.3. We extend this character to $\mathcal{S}(\mathbb{A})$ by setting

$$\psi(s) := \psi(\mathrm{Tr}(X)), \, \eta(s) := \eta\left(\det'(h)\right),$$

for $s = \begin{pmatrix} h & X \\ 0 & h \end{pmatrix}$. Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and we assume there exists a Hecke character η of $\mathrm{GL}_1(\mathbb{A})$ such that for all $a \in \mathrm{GL}_1(\mathbb{A})$

$$\eta \circ \det' \overbrace{\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}}^{n} = \omega \overbrace{\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}}^{2n}.$$

Let S_η be the set of places where η ramifies. For $\phi \in \Pi'$ a cusp form and $g \in \mathrm{GL}'_{2n}(\mathbb{A})$ we define the Shalika period integral by

$$\mathcal{S}^\eta_\psi(\phi)(g) := \int_{Z'_{2n}(\mathbb{A})\mathcal{S}(\mathbb{K})\backslash\mathcal{S}(\mathbb{A})} \phi(sg)\,\psi(s)^{-1}\,\eta(s)^{-1}\,\mathrm{d}s.$$

Note that this is well defined since

$$Z'_{2n}(\mathbb{A})\,\Delta\mathrm{GL}'_n(\mathbb{K})\backslash\Delta\mathrm{GL}'_n(\mathbb{A})$$

has finite measure and $U'_{(n,n)}(\mathbb{K})\backslash U'_{(n,n)}(\mathbb{A})$ is compact. If there exists a ϕ such that $\mathcal{S}^\eta_\psi(\phi)$ does not vanish for some $g \in \mathrm{GL}'_{2n}(\mathbb{A})$, this gives a nonzero intertwining operator of $\mathrm{GL}'_{2n}(\mathbb{A})$-representations

$$\mathcal{S}^\eta_\psi \colon \Pi' \to \mathrm{Ind}^{\mathrm{GL}'_{2n}(\mathbb{A})}_{\mathcal{S}(\mathbb{A})}(\eta \otimes \psi),$$

where the second space is the vector-space consisting of smooth functions

$$\mathrm{Ind}^{\mathrm{GL}'_{2n}(\mathbb{A})}_{\mathcal{S}(\mathbb{A})}(\eta \otimes \psi) := \{ f \in C^\infty\left(\mathrm{GL}'_{2n}(\mathbb{A})\right), f(sg) = \eta(s)\,\psi(s)\,f(g),$$

$$s \in \mathcal{S}(\mathbb{A}) \}$$

and $\mathrm{GL}'_{2n}(\mathbb{A})$ acts by right-translation. In this case we say that Π' admits a Shalika model with respect to η. For $v \in \mathcal{V}$ we define local Shalika models of $\Pi' \cong \otimes'_{v\in\mathcal{V}} \pi'_v$ as follows. Set

$$\mathrm{Ind}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}_{\mathcal{S}(\mathbb{K}_v)}(\eta_v \otimes \psi_v) := \{ f \in C^\infty\left(\mathrm{GL}'_{2n}(\mathbb{K}_v)\right), f(sg) = \eta_v(s)\,\psi_v(s)\,f(g),$$

$$s \in \mathcal{S}(\mathbb{K}_v)\}.$$

If v is a finite place in V, we say π'_v admits a local Shalika model if there exists a non-zero intertwiner

$$\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v)$$

of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$-representations. For $v \in \mathcal{V}_\infty$ a priori, π'_v is by our conventions not an honest $\mathrm{GL}(\mathbb{K}_v)$-representation and therefore we have to consider $(\pi'_v)^\infty$, the sub-space of smooth vectors in π'_v. Then $\mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v)$ and $(\pi'_v)^\infty$ are both Fréchet spaces and admit a natural, smooth $\mathrm{GL}'_{2n}(\mathbb{K}_v)$-action. We say π'_v admits a local Shalika model with respect to η_v if there exists a non-zero, continuous intertwining operator of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$-representations

$$\left(\pi'_v\right)^\infty \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_n(\mathbb{K}_v)}(\eta_v \otimes \psi_v).$$

If Π' admits a global Shalika model with respect to η, then so does π'_v with respect to η_v for all $v \in \mathcal{V}$. Note that the reverse direction, *i.e.* the existence of a local Shalika model for each π'_v, $v \in \mathcal{V}$ implying the existence of a Shalika model for Π', is not true in general, see [41, Theorem 1.4].

3.4.1 Existence In the case of GL_n we have the following characterization of Shalika models.

Theorem 3.4.1 ([60, Theorem 1])**.** *Let Π be a cuspidal irreducible representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the following assertions are equivalent.*

1. *There exists $\phi \in \Pi$ and $g \in \mathrm{GL}_{2n}(\mathbb{A})$ such that $\mathcal{S}_\psi^\eta(\phi)(g) \neq 0$.*
2. *Let $S \subset V$ be a finite subset of places containing $\mathcal{V}_\infty$ and the finite places where Π and η ramify. Then the twisted partial exterior square L-function $L^S\left(s, \Pi, \bigwedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$.*

If $\mathbb{D}$ does not split over $\mathbb{K}$ there is no longer such a nice criterion. In the case $n = 2$ and $\mathbb{D}$ a quaternion algebra we have the following criterion by [41], which is a consequence of the global theta correspondence.

Theorem 3.4.2 ([41, Theorem 1.3])**.** *Assume $\mathbb{D}$ is a quaternion algebra, Π' a cuspidal irreducible representation of $\mathrm{GL}'_2(\mathbb{A})$ and η the Hecke character we fixed above. Let $S \subset V$ be a finite subset of places containing $\mathcal{V}_\infty$ and the finite places where Π and η ramify. If $\mathrm{JL}(\Pi')$ is cuspidal the following assertions are equivalent.*

1. *Π' admits a Shalika model with respect to η.*
2. *The twisted partial exterior square L-function $L^S\left(s, \Pi', \bigwedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 1$ and for all $v \in \mathcal{V}_\mathbb{D}$ the representation π'_v is not of the form $|\det'|_v^{\frac{1}{2}}\tau_1 \times |\det'|_v^{-\frac{1}{2}}\tau_2$, where τ_1 and τ_2 are representations of $\mathrm{GL}'_1(\mathbb{K}_v)$ with central character η_v.*

If $\mathrm{JL}(\Pi')$ is not cuspidal it is of the form $\mathrm{MW}(\Sigma, 2)$ for some cuspidal irreducible representation Σ of $\mathrm{GL}_2(\mathbb{A})$. Then the following assertions are equivalent.

1. *Π' admits a Shalika model with respect to η.*
2. *The central character ω_Σ of Σ equals η.*
3. *The twisted partial exterior square L-function $L^S\left(s, \Pi', \bigwedge^2 \otimes \eta^{-1}\right)$ has a pole at $s = 2$.*

Thus, the situation is much more delicate in the case where $\mathrm{JL}(\Pi')$ is cuspidal because of the second, local condition. On the other hand, if $\mathrm{JL}(\Pi')$ is not cuspidal we have a priori $\eta^2 = \omega_\Pi = \omega_\Sigma^2$, hence, η and ω_Σ only differ by a quadratic character at most.

3.4.2 Shalika zeta-integrals The connection between L-functions and Shalika models can first be seen from the next two theorems, which are extensions of [39, Proposition 2.3, Proposition 3.1, Proposition 3.3].

Theorem 3.4.3. *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$. Assume Π' admits a Shalika model with respect to η and let $\phi \in \Pi'$ be a cusp form. Consider the integrals*

$$\Psi(s,\phi) := \int_{Z'_{2n}(\mathbb{A})H'_n(\mathbb{K})\backslash H'_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right) \left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta(h_2)^{-1}\, \mathrm{d}h_1\, \mathrm{d}h_2,$$

$$\zeta(s,\phi) := \int_{\mathrm{GL}'_n(\mathbb{A})} S_\psi^\eta(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det'(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}g_1.$$

Then $\Psi(s,\phi)$ converges absolutely for all s and $\zeta(s,\phi)$ converges absolutely if $\mathrm{Re}(s) >> 0$. Moreover, if $\zeta(s,\phi)$ converges absolutely, $\Psi(s,\phi) = \zeta(s,\phi)$.

In [39] this statement was proven for $\mathbb{D} = \mathbb{K}$, and we will show in Section 3.7 that their proof extends with some small adjustments to the case of $\mathbb{D}$ being a division algebra. Let $\xi_\phi \in \mathcal{S}_\psi^\eta(\Pi')$ and choose an isomorphism $\mathcal{S}_\psi^\eta(\Pi') \xrightarrow{\cong} \otimes_{v\in\mathcal{V}} \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$. Assume the image of ξ_ϕ can be written as a pure tensor

$$\xi_\phi \mapsto \bigotimes_{v\in\mathcal{V}} \xi_{\phi,v} \in \bigotimes_{v\in\mathcal{V}} \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v).$$

We can now consider the local version of the above integral

$$\zeta_v(s,\xi_{\phi,v}) := \int_{\mathrm{GL}'_n(\mathbb{K}_v)} \xi_{\phi,v}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det'_v(g_1)|^{s-\frac{1}{2}}\, \mathrm{d}_v g_1,$$

where $\xi_{\phi,v} \in \mathcal{S}_{\psi_v}^\eta(\pi'_v)$. The local Shalika integrals are then connected to the local L-factors by the following theorem.

Theorem 3.4.4. *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and assume Π' admits a Shalika model with respect to η. Then for each place $v \in \mathcal{V}$ and $\xi_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$ there exists an entire function $P(s,\xi_v)$, with $P(s,\xi_v) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$ if $v \in \mathcal{V}_f$, such that*

$$\zeta_v(s,\xi_v) = P(s,\xi_v)\, L(s,\pi_v)$$

and hence, $\zeta_v(s,\xi_v)$ can be analytically continued to $\mathbb{C}$. Moreover, for each place v there exists a vector ξ_v such that $P(s,\xi_v) = 1$. If v is a place where neither Π' nor ψ ramify this vector can be taken as the spherical vector $\xi_{\pi'_v}$ normalized by $\xi_{\pi'_v}(\mathrm{id}) = 1$.

In the case $\mathbb{K} = \mathbb{D}$ this the existence of such a holomorphic P was proven in [39] and in [85, Corollary 5.2] it was shown that P is actually a polynomial in $\mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$. Theorem 3.4.3 and Theorem 3.4.4 imply for $\xi_{\phi,f} \cong \otimes'_{v\in\mathcal{V}_f} \xi_{\phi,f}$ and $\mathrm{Re}(s) >> 0$

$$\zeta_f(s,\xi_{\phi,f}) := \int_{\mathrm{GL}'_n(\mathbb{A}_f)} \xi_{\phi,f}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right) |\det'(g_1)|_f^{s-\frac{1}{2}}\, \mathrm{d}_f g_1 =$$

$$= \prod_{v\in\mathcal{V}_f} P(s,\xi_{\phi,v})\, L(s,\pi_v).$$

3.4.3 Aut($\mathbb{C}$)**-action** Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and assume Π' admits a Shalika model with respect to η. It is natural to ask whether ${}^{\sigma}\Pi'$ admits a Shalika model with respect to ${}^{\sigma}\eta$ assuming that ${}^{\sigma}\Pi'$ is cuspidal. In the split case it was proven in the appendix of [47] that if Π' admits a Shalika model with respect to η, then ${}^{\sigma}\Pi'$ admits one with respect to ${}^{\sigma}\eta$.

Definition. We say the $\mathrm{Aut}(\mathbb{C})$-orbit of Π' admits a Shalika model with respect to η if ${}^{\sigma}\Pi'$ is cuspidal and admits a Shalika model with respect to ${}^{\sigma}\eta$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In the case of $n = 1$ and $\mathbb{D}$ a quaternion algebra Theorem 3.4.2 allows us to prove the following.

Lemma 3.4.5. *Let $\mathbb{D}$ be a quaternion algebra and Π' a cuspidal irreducible cohomological representation of $\mathrm{GL}'_2(\mathbb{A})$. If Π' admits a Shalika model with respect to η then ${}^{\sigma}\Pi'$ admits one with respect to ${}^{\sigma}\eta$.*

Proof. Note first that by Proposition 3.3.4 ${}^{\sigma}\Pi'$ is cuspidal. Assume first that $\mathrm{JL}(\Pi')$ is not cuspidal, *i.e.*

$$\mathrm{JL}\left(\Pi'\right) = \mathrm{MW}(\Sigma, 2)$$

for some cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. From Corollary 3.2.1, Proposition 3.3.4 and Lemma 3.3.5 it follows that

$$\mathrm{JL}\left({}^{\sigma}\Pi'\right) = {}^{\sigma}\mathrm{JL}\left(\Pi'\right) = \mathrm{MW}\left({}^{\sigma}\Sigma, 2\right).$$

Since the central character ω_{Σ} of Σ equals by assumption η, the central character of ${}^{\sigma}\Sigma$ equals ${}^{\sigma}\eta$. Thus we are done by Theorem 3.4.2.

Next assume $\mathrm{JL}(\Pi')$ is cuspidal and hence, $\mathrm{JL}(\Pi')$ admits a Shalika model with respect to η by Theorem 3.4.1 and Theorem 3.4.2. Thus, ${}^{\sigma}\mathrm{JL}(\Pi') = \mathrm{JL}({}^{\sigma}\Pi')$ admits also a Shalika model with respect to ${}^{\sigma}\eta$ by [47, Theorem 3.6.2] and hence,

$$L^S\left(s, {}^{\sigma}\Pi, \bigwedge^2 \otimes {}^{\sigma}\eta^{-1}\right)$$

has a pole at $s = 1$. Moreover, if v is a non-split place of $\mathbb{D}$ and ${}^{\sigma}\pi_v$ were of the form

$${}^{\sigma}\pi_v \cong |\det{}'|_v^{\frac{1}{2}}\tau_1 \times |\det{}'|_v^{-\frac{1}{2}}\tau_2,$$

where τ_i are representations of $\mathrm{GL}'_1(\mathbb{K}_v)$ with central character ${}^{\sigma}\eta$. This would lead to the contradiction, since by (6)

$$\pi_v = {}^{\sigma^{-1}}\left(|\det{}'|_v^{\frac{1}{2}}\tau_1 \times |\det{}'|_v^{-\frac{1}{2}}\tau_2\right) =$$

$$= |\det{}'|_v^{\frac{1}{2}\sigma^{-1}}\tau_1 \times |\det{}'|_v^{-\frac{1}{2}\sigma^{-1}}\tau_2.$$

□

Remark. We have currently no proof in the general case $n > 2$ and unfortunately the methods of [41] do not generalize well beyond the quaternion case. Hence, we can only conjecture the following.

Conjecture 3.4.1. *Let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_n(\mathbb{A})$ such that $\mathrm{JL}(\Pi')$ is residual. If Π' admits a Shalika model with respect to η, then so does the $\mathrm{Aut}(\mathbb{C})$-orbit of Π'.*

In [47] the authors define an action of $\mathrm{Aut}(\mathbb{C})$ on a given Shalika model and we will generalize this now to our setting. Let ψ_f be the finite part of the additive character ψ, which takes values in $\mu_\infty \subseteq \mathbb{C}^\times$, the subgroup of all roots of unity of $\mathbb{Q}^*$. We will associate to an element $\sigma \in \mathrm{Aut}(\mathbb{C})$ an element $t_\sigma \in \mathbb{A}^*$ such that for all $x \in \mathbb{A}$

$$\sigma(\psi(x)) = \psi(t_\sigma x).$$

More explicitly, we construct t_σ by first restricting σ to $\mathbb{Q}(\mu_\infty)$ and sending it to $\prod_p \mathbb{Z}_p^*$ via the global symbol map of Artin reciprocity

$$\mathrm{Aut}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \xrightarrow{\cong} \widehat{\mathbb{Z}}^* = \prod_{p \text{ prime}} \mathbb{Z}_p^*,$$

then embed the so obtained element into $\mathbb{A}$ via the diagonal embedding $\mathbb{Z}_p \hookrightarrow \prod_{v|p} \mathcal{O}_v$. Next we define the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on the finite part $\mathcal{S}_{\psi_f}^{\eta_f}(\Pi'_f)$ by sending ξ_f to

$$g_f \mapsto {}^\sigma\xi_f(g_f) \coloneqq \sigma\left(\xi_f\left(\mathbf{t}_\sigma^{-1} g_f\right)\right), \; g_f \in \mathrm{GL}'_{2n}(\mathbb{A}_f),$$

where

$$\mathbf{t}_\sigma = \mathrm{diag}\left(\overbrace{t_\sigma, \dots, t_\sigma}^{n}, \overbrace{1, \dots, 1}^{n}\right).$$

This gives a σ-linear intertwining operator

$$\sigma^* \colon \mathrm{Ind}_{\mathcal{S}(\mathbb{A}_f)}^{\mathrm{GL}'_{2n}(\mathbb{A}_f)}(\eta_f \otimes \psi_f) \to \mathrm{Ind}_{\mathcal{S}(\mathbb{A}_f)}^{\mathrm{GL}'_{2n}(\mathbb{A}_f)}({}^\sigma\eta_f \otimes \psi_f), \; \xi_f \mapsto {}^\sigma\xi_f. \tag{8}$$

Completely analogously we define a σ-linear intertwining operator

$$\sigma^* \colon \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v) \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}({}^\sigma\eta_v \otimes \psi_v)$$

for every finite place v, where we use $t_{\sigma,v}$ and $\mathbf{t}_{\sigma,v}$ instead of t_σ and $\mathbf{t}_\sigma$.

3.4.4 Uniqueness Let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ which admits a Shalika model with respect to η. Let $v \in \mathcal{V}_f$ be a finite place. In order to proceed we need the local uniqueness of the Shalika model, *i.e.* for every irreducible representation π'_v of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$ the claim that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\left(\pi'_v, \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v)\right) \le 1.$$

By Frobenius reciprocity every such map corresponds uniquely to a Shalika functional

$$\lambda \in \mathrm{Hom}_{\mathcal{S}(\mathbb{K}_v)}\left(\pi'_v, \eta_v \otimes \psi_v\right),$$

i.e. a map

$$\lambda \colon \pi'_v \to \mathbb{C}$$

such that $\lambda(\pi'_v(s)\phi) = \eta_v(s)\psi_v(s)\lambda(\phi)$ for $s \in \mathcal{S}(\mathbb{K}_v)$ and $\phi \in \pi'_v$ and vice versa.

Definition. We say that the $\mathrm{Aut}(\mathbb{C})$-orbit of Π' has a unique local Shalika model if ${}^{\sigma}\pi'_v$ has a unique Shalika model for all $v \in \mathcal{V}_f$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In the split case or when $\mathbb{D}$ is a quaternion algebra the following was proven in [103].
Theorem 3.4.6 ([103, Theorem 3.4])**.** *Let $\mathbb{D}$ be a field or a quaternion algebra. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\left(\pi'_v, \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v)\right) \le 1.$$

This is yet another reason why we will have to restrict ourselves to the case $\mathbb{D}$ being quaternion in the end. Combining Proposition 3.3.4, Lemma 3.4.5, and Theorem 3.4.6 we have proved the following.
Theorem 3.4.7. *Let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_2(\mathbb{A})$ which admits a Shalika model with respect to η and assume that $\mathbb{D}$ is a quaternion algebra. Then the $\mathrm{Aut}(\mathbb{C})$-orbit of Π' is cuspidal cohomological, admits a Shalika model with respect to η and has a unique local Shalika model.*

For the rest of the chapter let us collect the following decorations of a cuspidal irreducible representation Π' of $\mathrm{GL}(\mathbb{A})$:

1. Π' is cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$.
2. The $\mathrm{Aut}(\mathbb{C})$-orbit of Π' is cuspidal cohomological and admits a Shalika model with respect to η.
3. The $\mathrm{Aut}(\mathbb{C})$-orbit of Π' has a local unique Shalika model.

Moreover, we also fix a splitting ${}^{\sigma}\Pi' \cong {}^{\sigma}\pi'_\infty \otimes {}^{\sigma}\Pi'_f$, ${}^{\sigma}\Pi'_f \xrightarrow{\cong} \otimes'_{v \in \mathcal{V}_f} {}^{\sigma}\pi'_v$ and a Shalika model of ${}^{\sigma}\pi_v$

$$\mathcal{S}_{\psi_v}^{{}^{\sigma}\eta_v} : {}^{\sigma}\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}({}^{\sigma}\eta_v \otimes \psi_v)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, $v \in \mathcal{V}_f$.
Lemma 3.4.8. *For Π' as in 3.4.4, $v \in \mathcal{V}_f$ and the action of (8) we have*

$$\sigma^*\left(\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)\right) = \mathcal{S}_{\psi_v}^{{}^{\sigma}\eta_v}({}^{\sigma}\pi'_v)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. For any finite extension $\mathbb{K}$ of $\mathbb{Q}(\pi'_v, \eta_v)$ we have a $\mathbb{K}$-structure

$$\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)_{\mathbb{K}} := \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)^{\mathrm{Aut}(\mathbb{C}/\mathbb{K})}$$

on $\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$.

Proof. For the first assertion, note that the representation ${}^{\sigma}\pi'_v$ has on the on hand the unique Shalika model $\mathcal{S}_{\psi_v}^{{}^{\sigma}\eta_v}({}^{\sigma}\pi'_v)$ with respect to ${}^{\sigma}\eta_v$, but on the other hand, the σ-linear map

$$\pi'_v \xrightarrow{\cong} \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v) \xrightarrow{\sigma^*} \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}({}^{\sigma}\eta_v \otimes \psi_v)$$

gives rise to a linear map

$${}^{\sigma}\pi'_v \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}({}^{\sigma}\eta_v \otimes \psi_v).$$

Therefore, the assumed local uniqueness of the Shalika model implies that up to a scalar those two maps have to agree and hence, their image is identical.

For the second assertion, we follow the line of reasoning of [65, Theorem 3.1]. We fix the following Shalika functional

$$\phi_v \mapsto \lambda(\phi_v) := \mathcal{S}_{\psi_v}^{\eta_v}(\phi_v)(1).$$

First we show that there exists a $\mathbb{Q}(\pi'_v)$-rational structure $\pi'_{v,\mathbb{Q}(\pi'_v)}$ of π'_v such that

$$\lambda\left(\pi'_{v,\mathbb{Q}(\pi'_v)}\right) \subseteq \overline{\mathbb{Q}}.$$

Let $\widetilde{\pi'_v}$ be a $\mathbb{Q}(\pi'_v)$-rational structure of π'_v, see Lemma 3.3.1. Therefore, $\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}$ is a $\overline{\mathbb{Q}}$-rational structure of π'_v. Since both η_v and ψ_v take values in $\overline{\mathbb{Q}}^*$,

$$\left(\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}\right) \otimes \left(\eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{K}_v)} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \left(\pi'_v \otimes \eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{K}_v)},$$

where the subscripts indicate that we take the $\mathcal{S}(\mathbb{K}_v)$-invariant vectors. Frobenius reciprocity ensures that the right-hand side is one dimensional, hence, so is the left-hand side. Thus,

$$\left(\widetilde{\pi'_v} \otimes_{\mathbb{Q}(\pi'_v)} \overline{\mathbb{Q}}\right) \otimes \left(\eta_v^{-1} \otimes \psi_v^{-1}\right)_{\mathcal{S}(\mathbb{K}_v)}$$

is non-zero and we can apply Frobenius reciprocity again to obtain a Shalika functional λ' on π'_v which by the local uniqueness differs by some constant c from λ, *i.e.* $\lambda' = c\lambda$. Setting

$$\pi'_{v,\mathbb{Q}(\pi'_v)} := c\widetilde{\pi'_v}$$

we obtain the desired rational structure. We define an action of $\mathrm{Aut}(\mathbb{C})$ on the space of η_v-Shalika functionals of π'_v by setting

$${}^{\sigma}\lambda(u) = \sigma\left(\lambda\left(\beta_\sigma^{-1}\left(\pi'_v(\mathbf{t}_{\sigma,v}^{-1})u\right)\right)\right),$$

where $\beta_\sigma\colon \pi'_v \to \pi'_v$ is defined by

$$\beta_\sigma(zu) = \sigma(z)\,u,\ u \in \pi'_{v,\mathbb{Q}(\pi'_v)},\ z \in \mathbb{C}.$$

It is easy to see that for σ fixing $\mathbb{Q}(\pi'_v, \eta_v)$, ${}^{\sigma}\lambda$ is again a η_v-Shalika functional of π'_v.

Since $\lambda \neq 0$, we can fix a vector $u_0 \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ such that $\lambda(u_0) \neq 0$. Then there exists an open subgroup $\Gamma \subseteq \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$, in the usual profinite topology on $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$, such that

$$\mathbf{t}_\sigma u_0 = u_0,\ \sigma(\lambda(u_0)) = \lambda(v_0)$$

for all $\sigma \in \Gamma$, since λ takes values in $\overline{\mathbb{Q}}$. By the uniqueness of the local Shalika model we have constants c_σ such that ${}^{\sigma}\lambda = c_\sigma \lambda$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v, \eta_v))$. Then one easily computes

$$c_\sigma \lambda(u_0) = {}^{\sigma}\lambda(u_0) = \sigma\left(\lambda\left(\beta_\sigma^{-1}\left(\pi'_v(\mathbf{t}_{\sigma^{-1},v}^{-1})u_0\right)\right)\right) = \lambda(u_0)$$

and therefore $c_\sigma = 1$, which shows that Γ fixes λ. Recall the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$ given by ${}^{\sigma}\xi(u) = \sigma\left(\xi\left(\pi'_v(\mathbf{t}_{\sigma^{-1},v}^{-1})u\right)\right)$. Using the embedding of

$$\mathcal{S}_{\psi_v}^{\eta_v}\colon \pi'_v \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}(\eta_v \otimes \psi_v),$$

we can pull this action back to π'_v and denote it by $u \mapsto \sigma(u)$. It is clear from how λ was defined that for $u \in \pi'_v$, $g \in \mathrm{GL}'_{2n}(\mathbb{K}_v)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$\lambda\left(\pi'_v(g)\sigma(u)\right) = \sigma\left(\lambda\left(\pi'_v\left(\mathbf{t}_{\sigma^{-1},v}g\right)u\right)\right).$$

Thus, for $\sigma \in \Gamma$, $u \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ and $g \in \mathrm{GL}'_{2n}(\mathbb{K}_v)$ we deduce

$$\lambda\left(\pi'_v(g)\left(\sigma(u)\right)\right) = \lambda\left(\pi'_v(g)u\right).$$

But the irreducibly of π'_v now implies that $\sigma(u) - u = 0$, since otherwise π'_v would be spanned by $\mathrm{GL}'_{2n}(\mathbb{K}_v)\cdot(\sigma(u) - u)$ and therefore we arrive at the contradiction $\lambda = 0$. Hence, $\sigma(u) = u$ for all $u \in \pi'_{v,\mathbb{Q}(\pi'_v)}$ and $\sigma \in \Gamma$. This implies that $\pi'_{v,\mathbb{Q}(\pi'_v)}$ is contained in $\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}$ and therefore

$$\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})} = \pi'_{v,\mathbb{Q}(\pi'_v)} \otimes_{\mathbb{Q}(\pi'_v,\eta_v)} \overline{\mathbb{Q}}.$$

But now every vector in $\pi'_{v,\mathbb{Q}(\pi'_v)} \otimes \overline{\mathbb{Q}}$ is fixed by an open subgroup of

$$\mathrm{Aut}\left(\overline{\mathbb{Q}}/\mathbb{Q}\left(\pi'_v, \eta_v\right)\right)$$

and therefore by [115, Proposition 11.1.6]

$$\pi_v'^{\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v))} = \pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})^{\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v))}}$$

is a $\mathbb{Q}(\pi'_v, \eta_v)$-rational structure of $\pi_v'^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}$ and hence, of π'_v. We set

$$\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)_{\mathbb{Q}(\pi'_v,\eta_v)} := \left(\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)\right)^{\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v))},$$

which is the image of $\pi_v'^{\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v))}$ under the map

$$\mathcal{S}^{\eta_v}_{\psi_v} \colon \pi'_v \xrightarrow{\cong} \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right).$$

Since this map respects the actions of $\mathrm{Aut}(\mathbb{C})$ and $\pi_v'^{\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi'_v,\eta_v))}$ is defined over $\mathbb{Q}(\pi'_v, \eta_v)$, so is $\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\pi'_v,\eta_v)}$. Moreover, it follows that the canonical map

$$\mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)_{\mathbb{Q}(\pi'_v,\eta_v)} \otimes_{\mathbb{Q}(\pi'_v,\eta_v)} \mathbb{C} \to \mathcal{S}^{\eta_v}_{\psi_v}\left(\pi'_v\right)$$

is an isomorphism and hence, the claim is proven. □

We introduce the following notation. Let $v \in \mathcal{V}_f$, $\sigma \in \mathrm{Aut}(\mathbb{C})$ and

$$f \in \mathbb{C}(q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}).$$

We denote by f^σ the rational function obtained by applying σ to all coefficients of f for some $\sigma \in \mathrm{Aut}(\mathbb{C})$, which is the same as applying σ to the coefficients of f considered as a Laurent-series. Moreover,

$$\sigma\left(f\left(\frac{1}{2}\right)\right) = f^\sigma\left(\frac{1}{2}\right).$$

Lemma 3.4.9. *Let Π' be a cuspidal irreducible automorphic representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ with local representations π_v of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$ for $v \in \mathcal{V}$. Then for every finite place v*

$$L^\sigma\left(s, \pi'_v\right) = L\left(s, {}^\sigma\pi'_v\right),$$

and hence, if $L(s, \pi'_v)$ has no pole at $s = \frac{1}{2}$, $L\left(\frac{1}{2}, \pi'_v\right) \in \mathbb{Q}\left(\pi'_v\right)$.

Proof. For the first claim, note that by [11, Theorem 6.18], we have an explicit description of the local L-factors and that for π'_v a representation of $\mathrm{GL}'_m(\mathbb{K}_v)$, $L(s + \frac{md-1}{2}, \pi'_v) \in \mathbb{C}(q_v^s, q_v^s)$. We denote then for $f \in \mathbb{C}(q_v^s, q_v^s)$ by ${}^\sigma f$ the coefficient-wise application of σ. Note that for m even we thus have that ${}^\sigma L(s + \frac{md-1}{2}, \pi'_v) = L^\sigma(s + \frac{md}{2}, \pi'_v)$. One can then carry over the proof of [26, Lemma 4.6] *muta mutandis* from the case GL_m to GL'_m to obtain that

$${}^\sigma L\left(s + \frac{md-1}{2}, \pi'_v\right) = L\left(s + \frac{md-1}{2}, {}^\sigma\pi'_v\right).$$

Thus, for $m = 2n$,

$$L^\sigma(s + nd, \pi'_v) = L\left(s + nd, {}^\sigma\pi'_v\right)$$

and since $q_v^{nd} \in \mathbb{Q}$, the first claim follows. For the second claim, it is enough to observe that in this case

$$\sigma\left(L\left(\frac{1}{2}, \pi'_v\right)\right) = L^\sigma\left(\frac{1}{2}, \pi'_v\right) = L\left(\frac{1}{2}, {}^\sigma\pi'_v\right) = L\left(\frac{1}{2}, \pi'_v\right)$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi'_v))$. □

Lemma 3.4.10. *For Π' as in 3.4.4 and $v \in \mathcal{V}_f$ there exists a vector $\xi^0_{\Pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\Pi',\eta)}$ such that*

$$\zeta_v\left(s, \xi^0_{\Pi',v}\right) = L\left(s, \pi'_v\right)$$

if $v \notin S_{\Pi'_f,\psi}$ and

$$P^\sigma(s, \xi^0_{\Pi',v}) = P(s, {}^\sigma\xi^0_{\Pi',v})$$

for all $\sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\Pi', \eta_v)\right)$ if $v \in S_{\Pi'_f,\psi}$.

Proof. We again follow the proof of [65, Theorem 3.1]. For $v \notin S_{\Pi'_f,\psi}$ we choose $\xi^0_{\Pi',v}$ to be the normalized spherical vector of Theorem 3.4.4. Note that $\xi^0_{\Pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\Pi',\eta_v)}$, since for $v \notin S_{\Pi'_f,\psi}$ the normalization $\xi^0_{\Pi',v}(1) = 1$ implies that ${}^\sigma\xi^0_{\Pi',v}(1) = 1$ and hence, because $\sigma^*(\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)) = \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, ${}^\sigma\xi^0_{\Pi',v} = \xi^0_{\pi'_v}$. Thus, $P(s, \xi^0_{\Pi'_f,v}) = 1$ for all $v \notin S_{\Pi'_f,\psi}$ and therefore $\zeta_v\left(s, \xi^0_{\Pi',v}\right) = L\left(s, \pi'_v\right)$.

For $v \in S_{\Pi'_f,\psi}$, pick any non-zero $\xi^0_{\Pi',v} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)_{\mathbb{Q}(\Pi',\eta)}$ and recall that

$$P\left(s, \xi^0_{\Pi',v}\right) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}],$$

see Theorem 3.4.4, and the L-function $L\left(s, \pi'_v\right)$ does not vanish at $s = \frac{1}{2}$, as it is the reciprocal of a polynomial. Since

$$\frac{\zeta_v\left(s, \xi^0_{\Pi',v}\right)}{L\left(s, \pi'_v\right)} = P\left(s, \xi^0_{\Pi',v}\right), \ \frac{1}{L\left(s, \pi'_v\right)} \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}],$$

we have

$$\zeta_v\left(s,\xi^0_{\Pi',v}\right)\in\mathbb{C}\left(q_v^{s-\frac{1}{2}},q_v^{\frac{1}{2}-s}\right).$$

From the definition of $\zeta_v\left(s,\xi^0_{\Pi',v}\right)$ it follows that the k-th coefficient of $q_v^{s-\frac{1}{2}}$ in $\zeta_v\left(s,\xi^0_{\Pi',v}\right)$ is

$$c_k\left(\xi^0_{\Pi',v}\right):=\int_{\substack{\mathrm{GL}'_n(\mathbb{K}_v),\\ |\det'(g_1)|=q^{-k}}}\xi^0_{\Pi',v}\left(\begin{pmatrix}g_1&0\\0&1\end{pmatrix}\right)\mathrm{d}_v g_1,$$

which vanishes for $k<<0$ and is a finite sum, see the proof of Theorem 3.4.4. Hence, by a change of variables,

$$c_k\left({}^{\sigma}\xi^0_{\Pi',v}\right)=\sigma\left(c_k\left(\xi^0_{\Pi',v}\right)\right)$$

for all $\sigma\in\mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}\left(\eta_v\right)\right)$. It follows that for all $s\in\mathbb{C}$, $\sigma\in\mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}\left(\eta_v\right)\right)$ $\zeta_v^{\sigma}\left(s,\xi^0_{\Pi',v}\right)=\zeta_v\left(s,{}^{\sigma}\xi^0_{\Pi',v}\right)$ by analytic continuation. Lemma 3.4.9 shows then that

$$P^{\sigma}\left(s,\xi^0_{\Pi',v}\right)L^{\sigma}\left(s,\pi'_v\right)=\zeta_v^{\sigma}\left(s,\xi^0_{\Pi',v}\right)=\zeta_v\left(s,{}^{\sigma}\xi^0_{\Pi',v}\right)=$$
$$=P\left(s,{}^{\sigma}\xi^0_{\Pi',v}\right)L\left(s,{}^{\sigma}\pi'_v\right)=P\left(s,{}^{\sigma}\xi^0_{\Pi',v}\right)L^{\sigma}\left(s,\pi'_v\right)$$

for all $\sigma\in\mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\Pi',\eta_v)\right)$ and hence,

$$P^{\sigma}(s,\xi^0_{\Pi',v})=P(s,{}^{\sigma}\xi^0_{\Pi',v}).$$

□

We let $\xi_{\Pi'_f,0}\in\mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right)$ be the image of $\otimes_{v\in\mathcal{V}_f}\xi^0_{\Pi',v}$ under the fixed isomorphisms of Section 3.4.4.

3.5 Periods

In this section we will closely follow the strategy of [47]. Throughout the rest of the section let Π' be an automorphic representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in 3.4.4. Let μ be the highest weight such that Π' is cohomological with respect to $E^{\vee}_{\mu}$ and assume that $\mathrm{JL}(\Pi')=\mathrm{MW}(\Sigma,k)$ for some $k>1$ and Σ a cuspidal irreducible representation of $\mathrm{GL}_l(\mathbb{A})$ with $lk=2nd$. Note that we also fixed a splitting isomorphism

$$\Pi'\xrightarrow{\cong}\pi'_{\infty}\otimes\Pi'_f\xrightarrow{\cong}\bigotimes_{v\in\mathcal{V}_{\infty}}\pi'_v\otimes\Pi'_f.$$

and $\mathrm{JL}\left({}^{\sigma}\Pi'\right)={}^{\sigma}\mathrm{MW}(\Sigma,k)=\mathrm{MW}\left({}^{\sigma}\Sigma,k\right)$ by Lemma 3.3.5 and Proposition 3.3.4. We also have that

$$\left({}^{\sigma}\Pi'\right)_{\infty}\xrightarrow{\cong}\bigotimes_{v\in\mathcal{V}_{\infty}}\pi'_{\sigma^{-1}\circ v}.$$

Indeed, by Theorem 3.3.3 ${}^{\sigma}\Pi'$ is cohomological with respect to ${}^{\sigma}E^{\vee}_{\mu}$ and therefore Theorem 3.2.1 and Lemma 3.2.6 show that for $v\in\mathcal{V}_{\infty}$ ${}^{\sigma}\pi'_v\cong A_{\underline{n'}}(\lambda_v)$, where λ_v is determined by $\mu_{\sigma^{-1}\circ v}$ and $\underline{n'}$ is determined by k and l. For $\sigma\in\mathrm{Aut}(\mathbb{C})$ we thus can fix a splitting isomorphism

$$\left({}^{\sigma}\Pi'\right)_{\infty}\otimes{}^{\sigma}\Pi'_f\xrightarrow{\cong}\bigotimes_{v\in\mathcal{V}_{\infty}}\pi'_{\sigma^{-1}\circ v}\otimes{}^{\sigma}\Pi'_f.$$

Let us give an example that satisfies all of those properties.

Example. Let for a moment $\mathbb{K} = \mathbb{Q}$. Then in [46, § 6.11] the following representation was constructed. Set π_∞ to the Langlands quotient of $F(1, s+2) \times F(-1, s+2)$ for s a positive integer. This representation is cohomological with the coefficient system given by the weight vector $\left(\frac{s}{2}, \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}\right)$. Moreover, π_∞ can be extended to a cuspidal irreducible automorphic representation of $\mathrm{GL}_2'(\mathbb{A})$ with $\mathbb{D}$ a quaternion algebra and is regular algebraic if k is even.

3.5.1 Orbifolds Let K_f' be a compact open subgroup of $\mathrm{GL}_{2n}'\left(\mathbb{A}_f\right)$ and denote the block diagonal embedding by $\iota\colon H_n' \hookrightarrow \mathrm{GL}_{2n}'$. We set

$$\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'} := \mathrm{GL}_{2n}'(\mathbb{K}) \backslash \mathrm{GL}_{2n}'(\mathbb{A}) / K_\infty' K_f'$$

and

$$\mathbf{S}_{K_f'}^{H_n'} := H_n'(\mathbb{K}) \backslash H_n'(\mathbb{A}) / \left(K_\infty' \cap H_{n,\infty}'\right) \iota^{-1}\left(K_f'\right).$$

Let $r = \dim_{\mathbb{Q}} \mathbb{K}$ and note that if we consider $\mathbf{S}_{K_f'}^{H_n'}$ as an orbifold, its real dimension is

$$\dim_{\mathbb{R}} \mathbf{S}_{K_f'}^{H_n'} = r\left((nd)^2 - nd - 1\right).$$

Lemma 3.5.1. *The embedding ι induces a proper map*

$$\iota\colon \boldsymbol{S}_{K_f'}^{H_n'} \to \boldsymbol{S}_{K_f'}^{\mathrm{GL}_{2n}'}.$$

Proof. It follows from [8, Lemma 2.7] that

$$H_n'(\mathbb{K}) \backslash H_n'(\mathbb{A}) / \iota^{-1}\left(K_f'\right) \to \mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}$$

is proper. But this map factors as

$$H_n'(\mathbb{K}) \backslash H_n'(\mathbb{A}) / \iota^{-1}\left(K_f'\right) \to \mathbf{S}_{K_f'}^{H_n'} \to \mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}.$$

Since the first map is surjective and the composition is proper, the second map is proper. □

Next let $E_\mu^\vee$ be a highest weight representation of $\mathrm{GL}_{2n,\infty}'$ and consider the locally constant sheaf $\mathcal{E}_\mu^\vee$ on $\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}$, whose espace étalé is

$$\mathrm{GL}_{2n}'(\mathbb{A})/K_\infty' K_f' \times_{\mathrm{GL}_{2n}'(\mathbb{K})} E_\mu^\vee,$$

We consider its cohomology groups of compact support

$$H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right),\ H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, \mathcal{E}_\mu^\vee\right).$$

Both carry a natural structure of a module of the Hecke algebra

$$\mathcal{H}_{K_f'}^{\mathrm{GL}_{2n}'} = S\left(K_f' \backslash \mathrm{GL}_{2n}'\left(\mathbb{A}_f\right) / K_f'\right)$$

respectively

$$\mathcal{H}_{K_f'}^{H_n'} = S\left(\iota^{-1}(K_f') \backslash H_n'\left(\mathbb{A}_f\right) / \iota^{-1}(K_f')\right),$$

where the product is as usual given by convolution. Now since ι is proper, it defines a map between compactly supported cohomology groups

$$\iota^*: H_c^*(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee) \to H_c^*(\mathbf{S}_{K_f'}^{H_n'}, \mathcal{E}_\mu^\vee).$$

Recall that for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ there exists then a σ-linear isomorphism $\sigma: E_\mu^\vee \to {}^\sigma E_\mu^\vee$ of $\mathrm{GL}_{2n}'(\mathbb{K})$-representations. Thus, there exist natural σ-linear isomorphisms of Hecke algebra-modules

$$\sigma_{\mathrm{GL}_{2n}'}^*: H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, {}^\sigma\mathcal{E}_\mu^\vee\right)$$

and

$$\sigma_{H_n'}^*: H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, \mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, {}^\sigma\mathcal{E}_\mu^\vee\right),$$

as well as a morphism

$${}^\sigma\iota^*: H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, {}^\sigma\mathcal{E}_\mu^\vee\right) \to H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, {}^\sigma\mathcal{E}_\mu^\vee\right).$$

Then the following diagram commutes.

$$\begin{tikzcd}
H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, {}^\sigma\mathcal{E}_\mu^\vee\right) \arrow[r, "\iota^*"] \arrow[d, "\sigma_{\mathrm{GL}_{2n}'}^*"] & H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, \mathcal{E}_\mu^\vee\right) \arrow[d, "\sigma_{H_n'}^*"] \\
H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, {}^\sigma\mathcal{E}_\mu^\vee\right) \arrow[r, "{}^\sigma\iota^*"] & H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, {}^\sigma\mathcal{E}_\mu^\vee\right)
\end{tikzcd} \tag{9}$$

Lemma 3.5.2 ([46, Lemma 7.3])**.** *The* $\mathcal{H}_{K_f'}^{\mathrm{GL}_{2n}'}$*-module* $H_c^*\left(\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right)$ *and the* $\mathcal{H}_{K_f'}^{H_n'}$*-module* $H_c^*\left(\mathbf{S}_{K_f'}^{H_n'}, \mathcal{E}_\mu^\vee\right)$ *are defined over* $\mathbb{Q}(\mu)$ *by taking* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$*-invariant vectors under the action given by above* $\sigma_{\mathrm{GL}_{2n}'}^*$ *respectively* $\sigma_{H_n'}^*$.

If $K_f' \subseteq K_f''$ consider the canonical map $\mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'} \to \mathbf{S}_{K_f''}^{\mathrm{GL}_{2n}'}$. This allows us to define the space

$$\mathbf{S}^{\mathrm{GL}_{2n}'} := \varprojlim_{K_f'} \mathbf{S}_{K_f'}^{\mathrm{GL}_{2n}'}$$

as a projective limit. Note that $\mathcal{E}_\mu^\vee$ naturally extends to $\mathbf{S}^{\mathrm{GL}_{2n}'}$ and hence, the cohomology $H_c^*\left(\mathbf{S}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right)$ is a $\mathrm{GL}(\mathbb{A}_f)$-module.

Proposition 3.5.3 ([46, Proposition 7.16, Theorem 7.23])**.** *There exists an inclusion of the space*

$$H_{cusp}^*\left(\mathrm{GL}_{2n}', E_\mu^\vee\right) := \bigoplus_{\Pi' \ cuspidal} H^*\left(\mathfrak{g}_\infty', K_\infty', \pi_\infty' \otimes E_\mu^\vee\right) \otimes \Pi_f'$$

into $H_c^*\left(\mathbf{S}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right)$ *respecting the* $\mathrm{GL}_{2n}'\left(\mathbb{A}_f\right)$*-action. We denote by*

$$H_c^*\left(\mathbf{S}^{\mathrm{GL}_{2n}'}, \mathcal{E}_\mu^\vee\right)\left(\Pi_f'\right)$$

the image of

$$H^*\left(\mathfrak{g}_\infty', K_\infty', \pi_\infty' \otimes E_\mu^\vee\right) \otimes \Pi_f'$$

under this inclusion.

If K'_f fixes Π'_f, we obtain an inclusion $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right) \hookrightarrow H_c^*\left(\boldsymbol{S}_{K'_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ and we denote its image again by $H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right)$. Moreover, the isomorphism $\sigma^*_{\mathrm{GL}'_{2n}}$ respects the decomposition, i.e. if Π' and ${}^\sigma\Pi'$ are both cuspidal then

$$\sigma^*_{\mathrm{GL}'_{2n}}\left(H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right)\right) = H_c^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left({}^\sigma\Pi'_f\right)$$

for $\sigma \in \mathrm{Aut}(\mathbb{C})$ and thus if the $\mathrm{Aut}(\mathbb{C})$-orbit of Π' is cuspidal, the cohomology group

$$H_c^*\left(\boldsymbol{S}_{K'_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right)$$

is defined over $\mathbb{Q}(\Pi')$.

Let q_0 be the lowest degree in which the cohomology

$$H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E_\mu^\vee\right)$$

does not vanish. Thus, by Theorem 3.2.1

$$\mathbb{C} \cong H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) \otimes E_\mu^\vee\right) = \left(\bigwedge^{q_0}\left(\mathfrak{g}'_\infty, /\mathfrak{k}'_\infty\right)^* \otimes \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) \otimes E_\mu^\vee\right)^{K'_\infty}.$$

We fix once and for all a generator of

$$H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) \otimes E_\mu^\vee\right)$$

as follows. First fix an Künneth-isomorphism

$$\mathfrak{K}: H^*\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) \otimes E_\mu^\vee\right) \xrightarrow{\cong} \bigotimes_{v \in \mathcal{V}_\infty} H^*\left(\mathfrak{g}'_v, K'_v, \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right) \otimes E_{\mu_v}^\vee\right),$$

which is determined by the already fixed isomorphism

$$\mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) \cong \bigotimes_{v \in \mathcal{V}_\infty} \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right),$$

and let $q_{0,v}$ be the lowest degree in which the cohomology

$$H^{q_{0,v}}\left(\mathfrak{g}'_v, K'_v, \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right) \otimes E_{\mu_v}^\vee\right) = \left(\bigwedge^{q_{0,v}}\left(\mathfrak{g}'_v/\mathfrak{k}'_v\right)^* \otimes \mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right) \otimes E_{\mu_v}^\vee\right)^{K'_v}$$

does not vanish and similarly we fix Künneth-isomorphisms $\mathfrak{K}_\sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. For $v \in \mathcal{V}_\infty$ we then choose a generator of this space of the form

$$[\pi'_v] := \sum_{\underline{i}=\left(i_1,\dots,i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E_{\mu_v}^\vee} X_{\underline{i}}^* \otimes \xi_{v,\alpha,\underline{i}} \otimes e_\alpha^\vee, \tag{10}$$

where

1. Pick a $\mathbb{L}$-basis $\{Y_i\}$ of $\mathfrak{h}'_v/(\mathfrak{h}'_v \cap \mathfrak{k}'_v)$.
2. Extend $\{Y_i\}$ to a $\mathbb{L}$-basis $\{X_i\}$ of $\mathfrak{g}'_v/\mathfrak{k}'_v$, set $\{X_i^*\}$ to the corresponding dual basis of $(\mathfrak{g}'_v/\mathfrak{k}'_v)^*$ and $X_{\underline{i}}^* := \bigwedge_{i\in\underline{i}} X_i^*$.
3. A $\mathbb{Q}(\mu)$-basis $e_\alpha^\vee$ of $E_{\mu_v}^\vee$.
4. For each α and $\underline{i}$ a vector $\xi_{v,\alpha,\underline{i}} \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$.

We then set $[\pi'_\infty] := \mathfrak{K}^{-1}(\otimes_{v\in\mathcal{V}_\infty}[\pi'_v])$. We further assume that the X_i's are a extension of a basis of $\mathfrak{h}'_\infty/(\mathfrak{h}'_\infty \cap \mathfrak{k}'_\infty)$, where $\mathfrak{h}'_\infty$ is the Lie algebra at infinity of $H'_n(\mathbb{A})$. Finally for $\sigma \in \mathrm{Aut}(\mathbb{C})$ we set

$$\sigma([\pi'_\infty]) := [({}^\sigma\Pi')_\infty] := \mathfrak{K}_\sigma^{-1}\left(\bigotimes_{v\in\mathcal{V}_\infty}[\pi'_{\sigma^{-1}\circ v}]\right).$$

Let K'_f be an open compact subgroup of $\mathrm{GL}'_{2n}(\mathbb{A}_f)$ which fixes Π'. A choice of such a generator $[\pi'_\infty]$ fixes an isomorphism of $\mathcal{H}_{K'_f}^{\mathrm{GL}'_{2n}}$-module

$$\Theta_{\Pi'}: \mathcal{S}_{\psi_f}^{\eta_f}(\Pi'_f) \xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\Pi'_f)$$

defined by

$$\mathcal{S}_{\psi_f}^{\eta_f}(\Pi') \xrightarrow{\cong} \mathcal{S}_{\psi_f}^{\eta_f}(\Pi') \otimes H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\pi'_\infty) \otimes E_\mu^\vee\right) \xrightarrow{\cong}$$
$$\xrightarrow{\cong} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_\psi^\eta(\Pi') \otimes E_\mu^\vee\right) \xrightarrow{\cong} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \Pi' \otimes E_\mu^\vee\right) \xrightarrow{\cong}$$
$$\xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\Pi'_f),$$

where the first isomorphism is the one induced by $[\pi_\infty]$ and the third isomorphism is the one induced by the inverse of $\Pi' \xrightarrow{\cong} \mathcal{S}_\psi^\eta(\Pi')$.

Theorem 3.5.4. *For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{L})$ there exists a complex number*

$$\omega({}^\sigma\Pi'_f) = \omega({}^\sigma\Pi'_f, [{}^\sigma\pi'_\infty]) \in \mathbb{C}^\times$$

such that

$$\Theta_{{}^\sigma\Pi',0} := \omega({}^\sigma\Pi'_f)^{-1}\,\Theta_{{}^\sigma\Pi'}$$

is $\mathrm{Aut}(\mathbb{C})$ *invariant,* i.e.

$$\begin{array}{ccc}
\mathcal{S}_{\psi_f}^{\eta_f}(\Pi'_f) & \xrightarrow{\Theta_{\Pi',0}} & H_c^{q_0}(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee)(\Pi'_f) \\
\downarrow{\scriptstyle\sigma^*} & & \downarrow{\scriptstyle\sigma^*_{\mathrm{GL}'_{2n}}} \\
\mathcal{S}_{\psi_f}^{{}^\sigma\eta_f}({}^\sigma\Pi'_f) & \xrightarrow{\Theta_{{}^\sigma\Pi',0}} & H_c^{q_0}(\mathbf{S}^{\mathrm{GL}'_{2n}}, {}^\sigma\mathcal{E}_\mu^\vee)({}^\sigma\Pi'_f)
\end{array}$$

commutes. Hence, $\Theta_{\Pi',0}$ maps the $\mathbb{Q}(\Pi',\eta)$-structure of $\mathcal{S}_{\psi_f}^{\eta_f}(\Pi')$ to the $\mathbb{Q}(\Pi',\eta)$-structure of

$$H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\Pi'_f)$$

and $\omega(\Pi'_f)$ is well defined up to multiplication by an element of $\mathbb{Q}(\Pi',\eta)$.

Proof. Since

$$\Theta_{\Pi'}: \mathcal{S}_{\psi_f}^{\eta_f}\left(\Pi'_f\right) \xrightarrow{\cong} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right)$$

is a morphism of irreducible $\mathrm{GL}'_{2n}\left(\mathbb{A}_f\right)$-modules it follows from Schur's Lemma that there exists a complex number $\omega\left(\Pi'_f\right)$ such that $\Theta_{\Pi',0} = \omega\left(\Pi_f\right)^{-1}\Theta_{\Pi'}$ maps one $\mathbb{Q}\left(\Pi', \eta\right)$-structure onto the other, since rational structures are unique up to homothethies, see [26, Proposition 3.1]. Consider now the vector $\xi^0_{\Pi'_f}$ from Lemma 3.4.10 which generates $\mathcal{S}_{\psi_f}^{\sigma\eta_f}(\Pi'_f)$ as a $\mathrm{GL}'_{2n}\left(\mathbb{A}_f\right)$-module. After rescaling $\omega\left({}^\sigma\Pi'_f\right)$ by an element of $\mathbb{Q}\left(\Pi', \eta\right)$ we have the equality

$$\sigma^*_{\mathrm{GL}'_{2n}}\left(\Theta_{\Pi',0}\left(\xi^0_{\Pi'_f}\right)\right) = \Theta_{{}^\sigma\Pi',0}\left(\xi^0_{{}^\sigma\Pi'_f}\right),$$

since both sides of the equation lie in the same $\mathbb{Q}(\Pi', \eta)$-structure. Thus we proved the assertion. □

3.5.2 Behavior under twisting As in [47] we discuss now how the above introduced periods behave under twisting with an algebraic character $\chi = (\widetilde{\chi} \circ \det') \cdot |\det'|^b$, where $\widetilde{\chi}$ is a Hecke character of $\mathrm{GL}_1(\mathbb{A})$ of finite order and $b \in \mathbb{Z}$. In particular, for $v \in \mathcal{V}_\infty$ the character $\widetilde{\chi}(\det')_v$ is the trivial one. For the rest of this section fix such a character χ. The following is an easy consequence of the respective definitions.

Lemma 3.5.5. *The representation $\Pi' \otimes \chi$ is cohomological with respect to $(E_{\mu+b})^\vee = E_\mu^\vee \otimes \bigotimes_{v \in S_\infty} |\det'_v|^{-b}$. If Π' admits a Shalika model with respect to η then $\Pi \otimes \chi$ admits one with respect to $\chi^2\eta$ and hence, $\omega\left(\Pi'_f \otimes \chi_f\right)$ is well defined up to a multiple of $\mathbb{Q}\left(\Pi', \chi, \eta\right)^*$.*

We fix a splitting isomorphism

$$\chi \xrightarrow{\cong} \chi_\infty \otimes \chi_f \xrightarrow{\cong} \bigotimes_{v \in \mathcal{V}_\infty} |\det'_v|^b \otimes \chi_f,$$

which extends to a splitting isomorphism

$$\Pi' \otimes \chi \xrightarrow{\cong} \pi'_\infty \otimes \chi_\infty \otimes \Pi'_f \otimes \chi_f \xrightarrow{\cong} \bigotimes_{v \in \mathcal{V}_\infty} \pi'_v \otimes |\det'_v|^b \otimes \Pi'_f \otimes \chi_f.$$

Having already fixed the generator $[\pi'_v]$ we set

$$[\pi'_v \otimes \chi_v] := \sum_{\underline{i} = \left(i_1, \dots, i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}} \otimes |\det'_v|^{-b} \xi_{v,\alpha,\underline{i}} \otimes e^\vee_\alpha$$

and

$$[\pi'_\infty \otimes \chi_\infty] := \mathfrak{K}_\chi^{-1}\left(\bigotimes_{v \in \mathcal{V}_\infty} [\pi'_v \otimes \chi_v]\right),$$

where $\mathfrak{K}_\chi$ is defined as the map

$$\mathfrak{K}_\chi: H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_\infty}^{\chi^2_\infty\eta_\infty}\left(\pi'_\infty \otimes \chi_\infty\right) \otimes (E_{\mu+b})^\vee\right) \to$$

$$\to \bigotimes_{v \in \mathcal{V}_\infty} H^{q_0}(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_{\psi_v}^{\chi^2_v\eta_v}\left(\pi'_v \otimes \chi_v\right) \otimes (E_{\mu_v+b})^\vee),$$

corresponding to the splitting isomorphism

$$\pi'_\infty \otimes \chi_\infty \xrightarrow{\cong} \bigotimes_{v \in \mathcal{V}_\infty} \pi'_v \otimes |\det'_v|^{-b}.$$

Note that for $[\pi'_v]$ as in (10) we have then

$$[\pi'_v \otimes \chi_v] := \sum_{\underline{i}=\left(i_1,\ldots,i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}} \otimes \xi_{v,\alpha,\underline{i}} |\det'_v|^b \otimes e^\vee_\alpha.$$

We quickly recall the definition of the Gauss sum of χ_f, see [130, VII, §7]. For $v \in \mathcal{V}_f$ let $\mathfrak{c}_v$ be the conductor of χ_v and choose $c \in \mathrm{GL}_1\left(\mathbb{A}_f\right)$ such that for all $v \in \mathcal{V}_f$

$$\mathrm{ord}_v(c_v) = -\mathrm{ord}_v(\mathfrak{c}) - \mathrm{ord}_v(\mathfrak{D}).$$

Set

$$\mathcal{G}\left(\chi_v, \psi_v, c_v\right) := \int_{\mathcal{O}_v^\times} \chi_v\left(u_v\right)^{-1} \psi\left(c_v^{-1} u_v\right) \mathrm{d}_v,$$

where d_v is a Haar measure on $\mathcal{O}_v^\times$ normalized such that $\mathcal{O}_v^\times$ has volume 1. Then $\mathcal{G}\left(\chi_v, \psi_v, c_v\right)$ is nonzero for all finite places and 1 at the places where χ and ψ is unramified, see [43, Equation 1.22]. Hence, the global Gauss sum

$$\mathcal{G}\left(\chi_f, c\right) := \prod_{v \in \mathcal{V}_f} \mathcal{G}\left(\chi_v, \psi_v, c_v\right)$$

is well-defined. From now on we fix one such c and write $\mathcal{G}\left(\chi_f\right) := \mathcal{G}\left(\chi_f, c\right)$. If $\chi = (\widetilde{\chi} \circ \det') \cdot |\det'|^b$ is a character of GL'_{2n} as above, we set

$$\mathcal{G}\left(\chi_f\right) := \mathcal{G}\left(\widetilde{\chi}_f |\cdot|_f^b\right).$$

The periods we defined in Theorem 3.5.4 behave under twisting with such a character as follows.

Theorem 3.5.6. *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in Section 3.5. Let $\chi = (\widetilde{\chi} \circ \det') \cdot |\det'|^b$ with $\widetilde{\chi}$ a Hecke character of $\mathrm{GL}_1(\mathbb{A})$ of finite order and $b \in \mathbb{Z}$. For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ we have*

$$\sigma\left(\frac{\omega\left(\Pi'_f \otimes \chi_f\right)}{\mathcal{G}\left(\chi_f\right)^{nd} \omega\left(\Pi'_f\right)}\right) = \frac{\omega\left({}^\sigma\Pi'_f \otimes {}^\sigma\chi_f\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)^{nd} \omega\left({}^\sigma\Pi'_f\right)}.$$

In order to prove this we need three lemmata about the following maps. The first map is

$$\begin{aligned} S_\chi \colon \mathcal{S}^\eta_\psi\left(\Pi'\right) &\to \mathcal{S}^{\eta\chi^2}_\psi\left(\Pi' \otimes \chi\right) \\ \xi &\longmapsto \left(g \mapsto \chi\left(\det'(g)\right) \xi(g)\right), \end{aligned}$$

which splits under our fixed splitting isomorphisms into the two maps

$$\begin{aligned} S_{\chi_f} \colon \mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right) &\to \mathcal{S}^{\eta_f \chi_f^2}_{\psi_f}\left(\Pi'_f \otimes \chi_f\right) \\ \xi_f &\longmapsto \left(g_f \mapsto \chi_f\left(\det'(g_f)\right) \xi_f(g_f)\right) \end{aligned}$$

and

$$\begin{aligned} S_{\chi_\infty}\colon \mathcal{S}_{\psi_\infty}^{\eta_\infty}\left(\pi'_\infty\right) &\to \mathcal{S}_{\psi_\infty}^{\eta_\infty\chi_\infty^2}\left(\pi'_\infty \otimes \chi_f\right) \\ \xi_\infty &\longmapsto \left(g_\infty \mapsto \chi_\infty\left(\det'\left(g_\infty\right)\right)\xi_\infty\left(g_\infty\right)\right). \end{aligned}$$

Moreover, we define

$$\begin{aligned} A_\chi\colon \Pi' &\to \Pi' \otimes \chi \\ \phi &\longmapsto \left(g \mapsto \chi\left(\det'(g)\right)\phi(g)\right), \end{aligned}$$

where we consider $\phi \in \Pi'$ as a cusp form.

Lemma 3.5.7. *With S_{χ_f} as above,*

$$\sigma^* \circ S_{\chi_f} = \sigma\left(\left(\chi_f\left(t_\sigma\right)\right)^{-nd}\right) S_{{}^\sigma\chi_f} \circ \sigma^* = \left(\frac{\sigma\left(\mathcal{G}\left(\chi_f\right)\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)}\right)^{-nd} S_{{}^\sigma\chi_f} \circ \sigma^*.$$

Let $1_{E_\mu^\vee}$ be the identity map on $E_\mu^\vee$ and let $\left(A_\chi \otimes 1_{E_\mu^\vee}\right)^*$ be the induced map on cohomology

$$\left(A_\chi \otimes 1_{E_\mu^\vee}\right)^* : H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f\right) \to H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)\left(\Pi'_f \otimes \chi_f\right).$$

Lemma 3.5.8. *With A_χ as above,*

$$\left(A_\chi \otimes 1_{E_\mu^\vee}\right)^* \circ \Theta_{\Pi'} = \Theta_{\Pi'\otimes\chi} \circ \mathcal{S}_{\chi_f}.$$

Lemma 3.5.9. *For any $\sigma \in \mathrm{Aut}\left(\mathbb{C}/\mathbb{Q}(\mu)\right)$ we have*

$$\sigma^*_{\mathrm{GL}'_{2n}} \circ \left(A_\chi \otimes 1_{E_\mu^\vee}\right)^* = \left(A_{{}^\sigma\chi} \otimes 1_{{}^\sigma E_\mu^\vee}\right)^* \circ \sigma^*_{\mathrm{GL}'_{2n}}$$

We will first show how Theorem 3.5.6 follows from the above lemmata.

Proof of Theorem 3.5.6. We compute

$$\left(A_{{}^\sigma\chi} \otimes 1_{{}^\sigma E_\mu^\vee}\right)^* \circ \sigma^*_{\mathrm{GL}'_{2n}} \circ \Theta_\Pi$$

in two different ways. On the one hand

$$\left(A_{{}^\sigma\chi} \otimes 1_{{}^\sigma E_\mu^\vee}\right)^* \circ \sigma^*_{\mathrm{GL}'_{2n}} \circ \Theta_{\Pi'} \overset{(3.5.4)}{=} \left(\frac{\sigma\left(\omega\left(\Pi'_f\right)\right)}{\omega\left({}^\sigma\Pi'_f\right)}\right)\left(A_{{}^\sigma\chi} \otimes 1_{{}^\sigma E_\mu^\vee}\right)^* \circ \Theta_{{}^\sigma\Pi'} \circ \sigma^* \overset{(3.5.8)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\Pi'_f\right)\right)}{\omega\left({}^\sigma\Pi'_f\right)}\right)\Theta_{{}^\sigma\Pi'\otimes{}^\sigma\chi} \circ \mathcal{S}_{{}^\sigma\chi_f} \circ \sigma^*.$$

But on the other hand, we see

$$\left(A_{{}^\sigma\chi} \otimes 1_{{}^\sigma E_\mu^\vee}\right)^* \circ \sigma^*_{\mathrm{GL}'_{2n}} \circ \Theta_{\Pi'} \overset{(3.5.9)}{=} \sigma^*_{\mathrm{GL}'_{2n}} \circ \left(A_\chi \otimes 1_{E_\mu^\vee}\right)^* \circ \Theta_{\Pi'} \overset{(3.5.8)}{=}$$

$$= \sigma^*_{\mathrm{GL}'_{2n}} \circ \Theta_{\Pi'\otimes\chi} \circ S_{\chi_f} \overset{(3.5.4)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\Pi'_f \otimes \chi_f\right)\right)}{\omega\left({}^\sigma\Pi'_f \otimes {}^\sigma\chi_f\right)} \right) \Theta_{{}^\sigma\Pi' \otimes {}^\sigma\chi} \circ \sigma^* \circ S_{\chi_f} \overset{(3.5.7)}{=}$$

$$= \left(\frac{\sigma\left(\omega\left(\Pi'_f \otimes \chi_f\right)\right)}{\omega\left({}^\sigma\Pi'_f \otimes {}^\sigma\chi_f\right)} \right) \left(\frac{\sigma\left(\mathcal{G}\left(\chi_f\right)\right)}{\mathcal{G}\left({}^\sigma\chi_f\right)} \right)^{-nd} \Theta_{{}^\sigma\Pi' \otimes {}^\sigma\chi} \circ S_{{}^\sigma\chi_f} \circ \sigma^*.$$

Hence, the desired equality follows. $\square$

Proof of Lemma 3.5.7. The first equality follows by inserting the definition of σ^* and noticing that the determinant of $\det'(\mathbf{t}_\sigma) = t_\sigma^{nd}$. The second equality follows from [24, Theorem 2.4.3], which states

$$\sigma\left(\mathcal{G}\left(\chi_f\right)\right) = \sigma\left(\chi_f\left(t_\sigma\right)\right) \mathcal{G}\left({}^\sigma\chi_f\right).$$

$\square$

Proof of Lemma 3.5.8. We proceed as in the proof of [109, Proposition 2.3.7]. It is enough to show that the following three diagrams commute. The first one is

$$\begin{tikzcd}
\mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi'_f\right) \arrow[r] \arrow[d] & \otimes_{\mathcal{S}^{\eta_f}_{\psi_f}(\Pi'_f)} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty}_{\psi_\infty}\left(\pi'_\infty\right) \otimes E^\vee_\mu\right) \arrow[d] \\
\mathcal{S}^{\eta_f\chi_f^2}_{\psi_f}\left(\Pi'_f\right) \arrow[r] & \otimes_{\mathcal{S}^{\eta_f\chi_f^2}_{\psi_f}(\Pi'_f\otimes\chi_f)} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty\chi_\infty^2}_{\psi_\infty}\left(\pi'_\infty \otimes \chi_\infty\right) \otimes \left(\left(E_{\mu+b}\right)^\vee\right)\right)
\end{tikzcd}$$

Here the map on the right is

$$S_{\chi_f} \otimes \left(S_{\chi_\infty} \otimes 1_{E^\vee_\mu}\right)^*,$$

where $\left(S_{\chi_\infty} \otimes 1_{E^\vee_\mu}\right)^*$ is the map on cohomology induced by the isomorphism

$$S_{\chi_\infty} \otimes 1_{E^\vee_\mu} \colon \mathcal{S}^{\eta_\infty}_{\psi_\infty}\left(\pi'_\infty\right) \otimes E^\vee_\mu \to \mathcal{S}^{\eta_\infty\chi_\infty^2}_{\psi_\infty}\left(\pi'_\infty \otimes \chi_\infty\right) \otimes \left(E_{\mu+b}\right)^\vee.$$

From this, it follows that by our choice of generators, $[\pi_\infty]$ is mapped to $[\pi_\infty \otimes \chi_\infty]$, and hence, this diagram commutes. The second diagram is

$$\begin{tikzcd}
\otimes_{\mathcal{S}^{\eta_f}_{\psi_f}(\Pi'_f)} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty}_{\psi_\infty}\left(\pi'_\infty\right) \otimes E^\vee_\mu\right) \arrow[r] \arrow[d] & H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta}_{\psi}\left(\Pi'\right) \otimes E^\vee_\mu\right) \arrow[d] \\
\otimes_{\mathcal{S}^{\eta_f\chi_f^2}_{\psi_f}(\Pi'_f\otimes\chi_f)} H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta_\infty\chi_\infty^2}_{\psi_\infty}\left(\pi'_\infty \otimes \chi_\infty\right) \otimes E^\vee_\mu\right) \arrow[r] & H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}^{\eta\chi^2}_{\psi}\left(\Pi' \otimes \chi\right) \otimes \left(E_{\mu+b}\right)^\vee\right)
\end{tikzcd}$$

It is induced from the following commutative diagram, and hence, commutative itself.

$$\begin{tikzcd}
\mathcal{S}^{\eta_f}_{\Pi_f}\left(\Pi_f\right) \otimes \mathcal{S}^{\eta_\infty}_{\pi_\infty}\left(\pi_\infty\right) \otimes E^\vee_\mu \arrow[r] \arrow[d] & \mathcal{S}^{\eta}_{\Pi}\left(\Pi\right) \otimes E^\vee_\mu \arrow[d] \\
\mathcal{S}^{\eta_f\chi_f^2}_{\Pi_f}\left(\Pi_f \otimes \chi_f\right) \otimes \mathcal{S}^{\eta_\infty\chi_\infty^2}_{\pi_\infty}\left(\pi_\infty \otimes \chi_\infty\right) \otimes E^\vee_\mu \arrow[r] & \mathcal{S}^{\eta\chi^2}_{\Pi}\left(\Pi \otimes \chi\right) \otimes \left(E_{\mu+b}\right)^\vee
\end{tikzcd}$$

Here the left respectively right map going from top to bottom is

$$S_{\chi_f} \otimes S_{\chi_\infty} \otimes 1_{E_\mu^\vee} \text{ respectively } S_\chi \otimes 1_{E_\mu^\vee}.$$

And the third diagram is

$$\begin{tikzcd}
H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_\psi^\eta(\Pi') \otimes E_\mu^\vee\right) \arrow[r] \arrow[d] & H_c^{q_0}\left(\mathbf{S}_{K'_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)(\Pi'_f) \arrow[d] \\
H^{q_0}\left(\mathfrak{g}'_\infty, K'_\infty, \mathcal{S}_\psi^{\eta\chi^2}(\Pi' \otimes \chi) \otimes (E_{\mu+b})^\vee\right) \arrow[r] & H_c^{q_0}\left(\mathbf{S}_{K'_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_{\mu+b}^\vee\right)(\Pi'_f \otimes \chi_f),
\end{tikzcd}$$

where the left respectively right map going from top to bottom is

$$\left(S_\chi \otimes 1_{E_\mu^\vee}\right)^* \text{ respectively } \left(A_\chi \otimes 1_{E_\mu^\vee}\right)^*,$$

which therefore clearly commutes. Thus, all three diagrams commute, and the lemma is proven. □

Proof of Lemma 3.5.9. We will proceed as in the proof [109, Proposition 2.3.6]. Recall that the cuspidal cohomology $H_c^*(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee)(\Pi'_f)$ embeds injectively into the cohomology

$$H_c^q\left(\mathbf{S}_{K'_f}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$$

for a suitable small open compact subgroup K'_f. If moreover K'_f fixes χ, we can extend the map A_χ to

$$A_{\chi,K_\chi}: H_c^q\left(\mathbf{S}_{K_\chi}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right) \to H_c^q\left(\mathbf{S}_{K_\chi}^{\mathrm{GL}'_{2n}}, \mathcal{E}_{\mu+b}^\vee\right),$$

coming from the map $\mathcal{E}_\mu^\vee \to \mathcal{E}_{\mu+b}^\vee$ given simply by twisting with χ_∞. After passing to the projective limit, we thus obtain a map

$$A'_\chi: H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right) \to H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_{\mu+b}^\vee\right).$$

Recall for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ the σ-linear morphism

$$\sigma^*_{\mathrm{GL}'_{2n}}: H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right) \to H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_{\sigma\mu}^\vee\right)$$

which is induced by the σ-linear morphism $l_\sigma: E_\mu^\vee \to {}^\sigma E_\mu^\vee$ which permutes the places according to σ. We now identify $H_c^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ with the de Rahm cohomology $H_{dR}^q\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$. The maps A_χ and $\sigma^*_{\mathrm{GL}'_{2n}}$ have then analogous incarnations $A_{\chi,dR}$ and $\sigma^*_{\mathrm{GL}'_{2n},dR}$ for the de Rahm cohomology. It therefore suffices to prove that $A_{\chi,dR}$ is $\sigma^*_{\mathrm{GL}'_{2n},dR}$ invariant. Now for $\omega \in H_{dR}^*\left(\mathbf{S}^{\mathrm{GL}'_{2n}}, \mathcal{E}_\mu^\vee\right)$ represented by a section one has

$$\sigma^*_{\mathrm{GL}'_{2n},dR}(A_{\chi,dR}(\omega))(g) = l_\sigma \circ A_{\chi,dR}(\sigma^*_{\mathrm{GL}'_{2n},dR}(\omega))(g),\ g \in \mathbf{S}^{\mathrm{GL}'_{2n}}.$$

By fixing again a small enough compact open subgroup K'_f as above, we get the explicit description

$$l_\sigma \circ A_{\chi,K'_f,dR}(\omega)(g) = \sigma(\chi(g)) \cdot l_\sigma \circ \omega(g) = A_{\sigma(\chi),dR,K'_f}(\sigma^*_{\mathrm{GL}'_{2n},dR}(\omega))(g)$$

for $g \in \mathbf{S}^{\mathrm{GL}'_{2n}}_{K'_f}$. Passing again to the inductive limit, we therefore obtain that

$$\sigma^*_{\mathrm{GL}'_{2n},dR}(A_{\chi,dR}(\omega)) = A_{\sigma(\chi),dR}(\omega).$$

Finally,

$$\sigma(\chi) = {}^\sigma\chi \tag{11}$$

since both ${}^\sigma(\widetilde{\chi} \circ \mathrm{Nrd})$ and $\widetilde{\chi} \circ \mathrm{Nrd}$ are trivial at infinity by Theorem 3.2.1 and

$$|\det'|^b = {}^\sigma|\det'|^b = \sigma(|\det'|^b)$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. □

3.6 Critical values of of L-functions and their cohomological interpretation

Let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ as in Section 3.5 and consider the standard global L-function $L(s,\Pi')$ of Π'. Recall that a critical point of $L(s,\Pi')$ is in our case a point $s_0 \in \frac{1}{2} + \mathbb{Z}$ such that both $L(s,\pi'_\infty)$ and $L(1-s,\pi'^\vee_\infty)$ are holomorphic at s_0. Write for $v \in \mathcal{V}_\infty$

$$\mu_v = \left(\mu_{v,1},\ldots,\mu_{v,2nd}\right).$$

We further assume that $\mathrm{JL}(\Pi')$ is residual, *i.e.* $\mathrm{JL}(\Pi') = \mathrm{MW}(\Sigma,k)$ for some cuspidal irreducible representation Σ of GL_l with $lk = 2nd$, $k > 1$. Recall that l is even by Lemma 3.2.6. To calculate the critical values of the L-function it suffices to consider the meromorphic contribution of the local L-factors from the infinite places. Let μ' be the highest weight such that Σ is cohomological with respect to $E^\vee_{\mu'}$. Therefore, the L-factor of $L(s,\pi'_\infty)$ is by Lemma 3.2.6, [46, Theorem 5.2] and [13, Theorem 19.1.(b)] up to a non-vanishing holomorphic function of the form

$$\prod_{v\in\mathcal{V}_\infty}\prod_{i=1}^{k}\prod_{j=1}^{\frac{l}{2}}\Gamma\left(s+\mu'_{v,j}+\frac{l}{2}-j+\frac{k}{2}-i+1\right)$$

and the one of $L(1-s,\pi'^\vee_\infty)$ is of the form

$$\prod_{v\in\mathcal{V}_\infty}\prod_{i=1}^{k}\prod_{j=1}^{\frac{l}{2}}\Gamma\left(-s-\mu'_{v,\frac{l}{2}-j+1}+\frac{l}{2}-j+\frac{k}{2}-i+2\right).$$

Recall that the poles of the Gamma function lie precisely on the non-positive integers and it is non-vanishing everywhere else. We say $s = \frac{1}{2}$ is *compatible with* μ if for all $v \in \mathcal{V}_\infty$

$$-\mu'_{v,\frac{l}{2}} \le 0 \le -\mu'_{v,\frac{l}{2}+1}$$

and we say μ is admissible if there exists $p \in \mathbb{Z}$ such that

$$-\mu'_{v,\frac{l}{2}} \le p \le -\mu'_{v,\frac{l}{2}+1}$$

for all $v \in \mathcal{V}_\infty$. If k is even, all half integers are critical values

$$\mathrm{Crit}\left(\Pi'\right) := \{s = \frac{1}{2} + m : m \in \mathbb{Z}\}.$$

If k is odd, a critical value $s_0 = \frac{1}{2} + m$ of $L(s, \Pi')$ has to satisfy

$$-\mu'_{v,j} - \frac{l}{2} + j + i - \frac{k+1}{2} \leq m \leq -\mu'_{v,l-j+1} + \frac{l}{2} - j - i + \frac{k+1}{2}$$

for all $v \in \mathcal{V}_\infty$, $j \in \{1, \ldots, \frac{l}{2}\}$ and $i \in \{1, \ldots, k\}$. Since $\mu'_{v,1} \geq \ldots \geq \mu'_{v,l}$, the critical values of Π' are in this case

$$\mathrm{Crit}\left(\Pi'\right) \coloneqq \{s = \frac{1}{2} + m : -\mu'_{v,\frac{l}{2}} + \frac{k-1}{2} \leq m \leq -\mu'_{v,\frac{l}{2}+1} + \frac{1-k}{2},\ m \in \mathbb{Z},\ v \in \mathcal{V}_\infty\}.$$

Note that if k is odd and there exists a critical value, μ is admissible and that by Lemma 3.2.6, $\mu'_{v,\frac{l}{2}} = \mu_{v,nd}$ and $\mu'_{v,\frac{l}{2}} = \mu_{v,nd+1}$.

Following [47] we define a map $\mathcal{T}^*$.

Proposition 3.6.1 ([47, Proposition 6.3.1]). *Let Π' be an irreducible representation as in 3.5 and assume moreover that $n = 1$. Assume $\frac{1}{2}$ is a critical value of $L(s, \Pi')$ and compatible with μ. Denote by w_v the weight such that $E_{\mu_v} \cong E^\vee_{\mu_v} \otimes \det'^{w_v}$ for each $v \in S_\infty$. Let $E_{(0,-w_v)}$ be the representation $\mathbf{1} \otimes \det^{-w_v}$ of $H'(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H'(\mathbb{C})}\left(E^\vee_{\mu_v}, E_{(0,-w_v)}\right) = 1$$

for all $v \in S_\infty$.

We then let $E_{0,-w} \coloneqq \bigotimes_{v \in \mathcal{V}_\infty} E_{(0,-w_v)}$ and write $\mathcal{E}_{(0,-w)}$ for the corresponding locally constant sheaf of $\mathbf{S}^{H'_n}_{K'_f}$. Since $\frac{1}{2}$ is assumed to be compatible with μ and since $\mathbb{Q}(\mu)$ contains the splitting field of $\mathbb{D}$, [65, Lemma 4.8] shows that there exists a map $\mathcal{T} = \otimes_{v \in S_\infty} \mathcal{T}_{\circ v}$ in above space which is defined over $\mathbb{Q}(\mu)$ and we fix a choice of such a map. Lifting this map to cohomology we obtain a morphism

$$\mathcal{T}^* : H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, \mathcal{E}^\vee_\mu\right) \to H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, \mathcal{E}_{(0,-w)}\right).$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$ we define the twist of $\mathcal{T}$ as

$${}^\sigma\mathcal{T} = \bigotimes_{v \in S_\infty} \mathcal{T}_{\sigma^{-1} \circ v}$$

and denote the corresponding morphism on the cohomology by ${}^\sigma\mathcal{T}^*$. Since $\mathcal{T}$ is defined over $\mathbb{Q}(\mu)$, we therefore obtain for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$ that ${}^\sigma\mathcal{T}^* = \mathcal{T}^*$. We then have the following commutative diagram for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu))$.

$$\begin{tikzcd}
H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, \mathcal{E}^\vee_\mu\right) \arrow[r, "\mathcal{T}^*"] \arrow[d, "\sigma^*_{H'_n}"] & H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, \mathcal{E}_{(0,-w)}\right) \arrow[d, "\sigma^*_{H'_n}"] \\
H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, {}^\sigma\mathcal{E}^\vee_\mu\right) \arrow[r, "{}^\sigma\mathcal{T}^*"] & H^*_c\left(\mathbf{S}^{H'_n}_{K'_f}, {}^\sigma\mathcal{E}_{(0,-w)}\right)
\end{tikzcd} \tag{12}$$

The next step consists of translating the computation of a critical value of $L(s, \Pi')$ into an instance of Poincaré duality. But in order to apply Poincaré duality, the highest or lowest degree in which $H^*\left(\mathfrak{g}'_\infty, K'_\infty, \Pi' \otimes E^\vee_\mu\right)$ vanishes has to equal the dimension $\dim_{\mathbb{R}} \mathbf{S}^{H'_n}_{K'_f}$, which implies $n = 1, d = 2$ and $k = 2$. Indeed, Lemma 3.2.6 implies that

$$r\left((nd)^2 - nd - 1\right) = r\left((nd)^2 - nd - \frac{nd}{2}(k-1)\right) \text{ or}$$

$$r\left((nd)^2 - nd - 1\right) = r\left((nd)^2 - nd + \frac{nd}{2}(k-1) + 1\right)$$

and only the first equation can be satisfied, which leads to the above restriction.

3.6.1 Poincaré duality We therefore let Π' be a representation as in 3.5 and assume moreover that $\mathbb{D}$ is quaternion and $n = 1$ and μ is admissible. By Theorem 3.4.7 the respective conditions on the $\mathrm{Aut}(\mathbb{C})$-orbit on Π' hold unconditionally in this case and we set $q_0 := r$, which is the real dimension of $\mathbf{S}^{H'_1}_{K'_f}$ and the lowest degree in which $H^*(\mathfrak{g}'_\infty, K'_\infty, \Pi' \otimes E^\vee_\mu)$ does not vanish.

If Π' is as above and $\frac{1}{2}$ is a critical value of $L(s, \Pi')$ which is compatible with μ, we choose K'_f small enough so that $\iota^{-1}\left(K'_f\right)$ can be written as a product $K_{f,1} \times K_{f,2}$, η_f is trivial on $\det'\left(K_{f,2}\right)$ and K'_f fixes Π'_f. Let $\mathcal{C}$ be the set of connected components of $\mathbf{S}^{H'_1}_{K'_f}$. Using [18, Theorem 5.1] we see that $\mathcal{C}$ is finite and each $C \in \mathcal{C}$ is a quotient of

$$H'_{1,\infty} / \left(K'_\infty \cap H'_{1,\infty}\right)$$

by a discrete subgroup of $H'_1(\mathbb{K})$. Recall that the Y_i's from in Section 3.5.1 give a basis of $\mathfrak{h}'_\infty / (\mathfrak{h}'_\infty \cap \mathfrak{k}'_\infty)$. Thus, they give an orientation on $H'_{1,\infty} / \left(K'_\infty \cap H'_{1,\infty}\right)$, since $\mathfrak{h}'_\infty / (\mathfrak{h}'_\infty \cap \mathfrak{k}'_\infty)$ is parallelizable and therefore on each $C \in \mathcal{C}$ we can now consider $\mathbf{1} \times \eta^{-1}$ as a global section of $\mathcal{E}_{(0,-w)}$ and denote the corresponding cohomology class as

$$[\eta] \in H^{q_0}_c\left(\mathbf{S}^{H'_1}_{K'_f}, \mathcal{E}_{(0,-w)}\right).$$

Poincaré duality on each connected component of $\mathbf{S}^{H'_1}_{K'_f}$ gives rise to a surjective map

$$H^{q_0}_c\left(\mathbf{S}^{H'_1}_{K'_f}, \mathcal{E}_{(0,-w)}\right) \to \mathbb{C}$$

defined by

$$\theta \mapsto \int_{\mathbf{S}^{H'_1}_{K'_f}} \theta \wedge [\eta] := \sum_{C \in \mathcal{C}} \int_C \theta \wedge [\eta]$$

Lemma 3.6.2. *This map commutes with twisting by an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$,* i.e.

$$\sigma\left(\int_{S^{H'_1}_{K'_f}} \theta \wedge [\eta]\right) = \int_{S^{H'_1}_{K'_f}} \sigma^*_{H'_1}(\theta) \wedge [{}^\sigma\eta].$$

Proof. Indeed,

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f'}} \theta \wedge [\eta]\right) = \int_{\mathbf{S}^{H_1'}_{K_f'}} \sigma^*_{H_1'}(\theta \wedge [\eta]) = \int_{\mathbf{S}^{H_1'}_{K_f'}} \sigma^*_{H_1'}(\theta) \wedge \sigma^*_{H_1'}([\eta]).$$

Finally, by (11) we have that $\sigma^*_{H_1'}([\eta]) = [{}^\sigma\eta]$. □

The goal now is to compute the composition of maps

$$\begin{array}{ccccc} H_c^{q_0}\left(\mathbf{S}^{\mathrm{GL}_2'}_{K_f'}, \mathcal{E}_\mu^\vee\right) & \xrightarrow{\iota^*} & H_c^{q_0}\left(\mathbf{S}^{H_1'}_{K_f'}, \mathcal{E}_\mu^\vee\right) & \xrightarrow{\mathcal{T}^*} & H_c^{q}\left(\mathbf{S}^{H_1'}_{K_f'}, \mathcal{E}_{(0,-w)}\right) \\ {\scriptstyle\Theta_{\Pi',0}}\big\uparrow & & & & \big\downarrow{\scriptstyle\int_{\mathbf{S}^{H_1'}_{K_f'}}} \\ \mathcal{S}^{\eta_f}_{\psi_f}\left(\Pi_f'\right) & & & & \mathbb{C}, \end{array}$$

i.e.

$$\xi_{\phi_f} \mapsto \int_{\mathbf{S}^{H_1'}_{K_f'}} \mathcal{T}^* \iota^* \Theta_{\Pi,0}\left(\xi_{\phi_f}\right) \wedge [\eta]$$

and relate this expression to the critical values of the L-function. To proceed we need the following non-vanishing result. Recall for $v \in \mathcal{V}_\infty$

$$[\pi_v'] = \sum_{\underline{i}=\left(i_1,\dots,i_{q_{0,v}}\right)} \sum_{\alpha=1}^{\dim E^\vee_{\mu_v}} X^*_{\underline{i}} \otimes \xi_{v,\alpha,\underline{i}} \otimes e^\vee_\alpha.$$

For each $\underline{i}$ write

$$\iota^*\left(X_{\underline{i}}\right) = s\left(\underline{i}\right) Y_1^* \wedge \ldots \wedge Y_{q_0}^*,$$

where $s\left(\underline{i}\right)$ is some complex number. If $\frac{1}{2} \in \mathrm{Crit}(\Pi')$ is compatible with μ, we know that $\mathcal{T}$ exists and $\zeta_v(s,\cdot) = P(s,\cdot)L(s,\pi_v')$ for $v \in \mathcal{V}_\infty$ by Theorem 3.4.4. Since $\frac{1}{2}$ is critical, we know that $\zeta_v(\frac{1}{2},\cdot)$ is well defined for all vectors in the Shalika model. We thus can set

$$c\left(\pi_v'\right) := \sum_{\underline{i}} \sum_{\alpha=1}^{\dim E_{\mu_v}} s\left(\underline{i}\right) \mathcal{T}\left(e^\vee_\alpha\right) \zeta_v\left(\frac{1}{2}, \xi_{v,\alpha,\underline{i}}\right)$$

and $c\left(\pi_\infty'\right) := \prod_{v\in\mathcal{V}_\infty} c\left(\pi_v'\right)$.

For $s = \frac{1}{2} + m \in \mathrm{Crit}\left(\Pi'\right)$, the assumption on μ being admissible gives an integer p such that $\Pi' \otimes |\det'|^p$ has critical value $\frac{1}{2}$ and $\frac{1}{2}$ is compatible with $\mu - p$. We fix a smallest such p, which exists, since $-\mu_{v,2} \le p$ for all $v \in \mathcal{V}_\infty$, and set

$$\Pi'\left(m\right) = \Pi' \otimes |\det|^p \tag{13}$$

and

$$c\left(\pi_\infty', m\right) := c\left(\Pi'\left(m\right)_\infty\right).$$

Theorem 3.6.3 ([116, Theorem A.3])**.** *For* Π' *as in 3.6.1 and* $s = \frac{1}{2} + m \in \mathrm{Crit}\left(\Pi'\right)$*, the expression* $c\left(\pi_\infty', m\right)$ *does not vanish.*

Proof. We will assume without loss of generality that $s = \frac{1}{2}$ is critical. Since $c(\pi'_\infty) = \prod_{v\in\mathcal{V}_\infty} c(\pi'_v)$ we fix a place $v \in \mathcal{V}_\infty$. We set $H := \mathrm{GL}_1(\mathbb{H}) \times \mathrm{GL}_1(\mathbb{H})$ and $G := \mathrm{GL}_2(\mathbb{H})$ with maximal compact subgroup K'_H respectively K'. Since $\frac{1}{2}$ is critical, it follows from Theorem 3.4.3 that the local zeta integral at v gives a functional

$$\zeta_v\left(\frac{1}{2}, \cdot\right) \in \mathrm{Hom}_H\left(\pi'_v, \chi\right),$$

where $\chi = 1_{\mathrm{GL}_1(\mathbb{H})} \otimes \det'^{w_v}$. It is nonzero, since $\zeta_v(s, \cdot) = P(s, \cdot)L(s, \pi'_v)$ and there exists by Theorem 3.4.4 $\xi_{\Pi',v}$ such that $P(s, \xi_{\Pi',v}) = 1$. Since the L-factors at infinity are products of Gamma-functions and non-vanishing holomorphic functions, $\zeta_v(s, \xi_{\Pi',v})$ also never vanishes. Thus $\zeta_v(s, \cdot)$ never vanishes and hence $\zeta_v(\frac{1}{2}, \cdot)$ is non-zero. Let j_2 be the inclusion

$$j_2 \colon \mathfrak{h}'/\mathfrak{k}'_H \hookrightarrow \mathfrak{g}'/\mathfrak{k}'$$

and consider now the map

$$\begin{aligned} \mathrm{Hom}\left(\mathfrak{g}'/\mathfrak{k}', \pi'_v \otimes E^\vee_{\mu_v}\right) &\to \mathrm{Hom}\left(\mathfrak{h}'/\mathfrak{k}'_H, \chi \otimes E_{(0,-w_v)}\right) \\ f &\longmapsto \left(\zeta_v\left(\frac{1}{2}, \cdot\right) \otimes \mathcal{T}_v\right) \circ f \circ j_2 \end{aligned}$$

By [116, Theorem A.3] the induced map

$$c \colon H^1\left(\mathfrak{g}'_\infty, K', \pi'_v \otimes E^\vee_{\mu_v}\right) \to H^1\left(\mathfrak{h}', K'_H, \chi \otimes E_{(0,-w_v)}\right)$$

does not vanish on the one dimensional space $H^1\left(\mathfrak{g}'_\infty, K', \pi'_v \otimes E^\vee_{\mu_v}\right)$. Since it is generated by $[\pi'_v]$ we conclude that $c(\pi'_v) \neq 0$. □

We set

$$c(\pi_\infty, m)^{-1} := \omega(\pi_\infty, m).$$

Remark. The proof of [116, Theorem A.3] relies crucially on the numerical coincidence, *i.e.* that either the lowest or highest nonvanishing degree of the $(\mathfrak{g}'_\infty, K'_\infty)$-cohomology

$$H^*\left(\mathfrak{g}'_\infty, K'_\infty, \pi'_\infty \otimes E^\vee_\mu\right)$$

is $\dim_\mathbb{R} \mathbf{S}^{H'_1}_{K'_f}$.

Theorem 3.6.4. *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_2(\mathbb{A})$ as in 3.6.1. Assume $s = \frac{1}{2} \in \mathrm{Crit}(\Pi')$ is compatible with μ and let $\xi^0_{\Pi'_f}$ be the vector of Lemma 3.4.10. Then*

$$\int_{\mathbf{S}^{H'_1}_{K'_f}} \mathcal{T}^*\iota^*\Theta_{\Pi',0}\left(\xi^0_{\Pi'_f}\right) \wedge [\eta] = \frac{L\left(\frac{1}{2}, \Pi'_f\right)\prod_{v\in S_{\Pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\Pi',v}\right)}{\omega\left(\Pi'_f\right)\omega\left(\pi'_\infty\right)\mathrm{vol}\left(\iota^{-1}\left(K'_f\right)\right)}$$

for every small enough open compact subgroup K'_f of $\mathrm{GL}'_2\left(\mathbb{A}_f\right)$.

Proof. The proof of this theorem can be carried out in the same way as the proof of [47, Theorem 6.7.1]. We only include it for completeness. Recall from Section 2.4 that

$$c = \mathrm{vol}\left(\mathbb{K}^\times \backslash \mathbb{A}^* / \mathbb{R}^r_{>0}\right).$$

We choose K'_f such that it fixes $\xi^0_{\Pi'_f}$.

Plugging $\xi^0_{\Pi'_f}$ in the definition of the terms of the integral and using the K'_f-invariance of $\xi^0_{\Pi'_f}$ we obtain the following identity.

$$\int_{\mathbf{S}^{H'_1}_{K'_f}} \mathcal{T}^* \iota^* \Theta_{\Pi',0}\left(\xi^0_{\Pi'_f}\right) \wedge [\eta] =$$

$$= \mathrm{vol}\left(\iota^{-1}\left(K'_f\right)\right)^{-1} c^{-1} \omega\left(\Pi'_f\right)^{-1} \sum_{\underline{i},\alpha} s\left(\underline{i}\right) \mathcal{T}\left(e_\alpha\right) \int_{H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})/\mathbb{R}^d_+} \eta \phi^0_{\underline{i},\alpha}\big|_{H'_1(\mathbb{A})} \,\mathrm{d}h,$$

where

$$\phi^0_{\underline{i},\alpha} := \left(\mathcal{S}^\eta_\psi\right)^{-1}\left(\bigotimes_{v\in\mathcal{V}_\infty} \xi_{v,\underline{i},\alpha} \otimes \xi^0_{\Pi'_f}\right).$$

We compute now the latter integral over $H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})/\mathbb{R}^d_+$ for fixed $\underline{i}$ and α. Again plugging in the definitions yields

$$\int_{H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})/\mathbb{R}^d_+} [\eta] \phi^0_{\underline{i},\alpha}\big|_{H'_1(\mathbb{A})} \,\mathrm{d}h =$$

$$= \int_{Z'(\mathbb{A})H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})} \int_{Z'(\mathbb{K})\backslash Z'(\mathbb{A})/\mathbb{R}^d_+}$$

$$\left(\phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \eta^{-1}\left(\det{}'\left(h_2 z\right)\right) dz\right) \mathrm{d}h_1 \,\mathrm{d}h_2.$$

We can now pull the $z = \mathrm{diag}\left(a,a\right)$-contribution out of $\phi^0_{\underline{i},\alpha}$ and $\eta^{-1}\left(\det'\right)$, which yields a factor of $\omega\left(z\right)\eta\left(\det'(a)\right)^{-1} = 1$ and hence, the integral simplifies to

$$c \int_{Z'(\mathbb{A})H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \eta^{-1}\left(\det{}'(h_2)\right) \mathrm{d}h_1 \,\mathrm{d}h_2.$$

Recall the equality of Theorem 3.4.3 and the properties of the special vector $\xi^0_{\Pi'_f}$

$$\int_{Z'(\mathbb{A})H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta^{-1}\left(\det{}'(h_2)\right) \mathrm{d}h_1 \,\mathrm{d}h_2 =$$

$$= \zeta_\infty\left(s, \xi^0_{\infty,\underline{i},\alpha}\right) \zeta_f\left(s, \xi^0_{\Pi'_f}\right) = \zeta_\infty\left(s, \xi^0_{\infty,\underline{i},\alpha}\right) L(s, \Pi'_f) \prod_{v\in S_{\Pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\Pi',v}\right)$$

for $\mathrm{Re}(s) >> 0$. But the integral converges absolutely for all s hence, we obtain the equality for all s. Recall that $L(s,\Pi)$ is an entire function and hence, $L\left(\frac{1}{2}, \Pi'_f\right) \in \mathbb{C}$ since $s = \frac{1}{2}$ is critical. Therefore,

$$\int_{Z'(\mathbb{A})H'_1(\mathbb{K})\backslash H'_1(\mathbb{A})} \phi^0_{\underline{i},\alpha}\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} z\right) \eta^{-1}\left(\det{}'(h_2)\right) \mathrm{d}h_1 \,\mathrm{d}h_2 =$$

$$= \zeta_\infty\left(\frac{1}{2}, \xi^0_{\infty,\underline{i},\alpha}\right) L(\frac{1}{2}, \Pi'_f) \prod_{v\in S_{\Pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\Pi',v}\right).$$

Plugging this in the above sum over $\underline{i}$ and α, we obtain the desired identity. $\square$

We are now ready to prove our analog of [47, Theorem 7.1.2].

Theorem 3.6.5. *Let $\mathbb{D}$ be a quaternion algebra and let Π' be a cuspidal irreducible cohomological representation of $\mathrm{GL}_2'(\mathbb{A})$ which admits a Shalika model with respect to η. Let μ be the highest weight such that Π' is cohomological with respect to $E_\mu^\vee$ and assume μ is admissible. We further assume that* $\mathrm{JL}(\Pi')$ *is residual,* i.e. $\mathrm{JL}(\Pi') = \mathrm{MW}(\Sigma, 2)$ *for Σ a cuspidal irreducible cohomological representation of* $\mathrm{GL}_2(\mathbb{A})$*. Let moreover $\chi = \widetilde{\chi} \circ \det'$, where $\widetilde{\chi}$ is a Hecke-character of $\mathrm{GL}_1(\mathbb{A})$ of finite order. Then, for $\frac{1}{2} + m$ a critical value of Π',*

$$\frac{L\left(\frac{1}{2}+m, \Pi_f' \otimes \chi_f\right)}{\omega\left(\Pi_f'\right) \mathcal{G}\left(\chi_f\right)^4 \omega\left(\pi_\infty', m\right)} \in \mathbb{Q}\left(\Pi', \chi, \eta\right).$$

Proof. Again the proof can be adapted from [47] to the situation at hand. To show the claim it is enough to show that the ratio stays invariant under all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi', \chi, \eta))$. First assume that $m = 0$, $\frac{1}{2}$ is compatible with μ and $\chi = 1$. We are going to compute

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f'}} \mathcal{T}^* \iota^* \Theta_{\Pi',0}\left(\xi^0_{\Pi_f'}\right) \wedge [\eta]\right) \tag{14}$$

for some $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi', \chi, \eta))$ in two different ways, where K_f' is a sufficiently small open compact subgroup of $\mathrm{GL}(\mathbb{A}_f)$. On the one hand, (14) equals by Theorem 3.6.4 and Lemma 3.4.10 to

$$\sigma\left(\frac{L\left(\frac{1}{2}, \Pi_f'\right) \prod_{v \in S_{\Pi_f', \psi}} P\left(\frac{1}{2}, \xi^0_{\Pi', v}\right)}{\omega\left(\Pi_f'\right) \omega\left(\pi_\infty'\right) \mathrm{vol}\left(\iota^{-1}\left(K_f'\right)\right)}\right) =$$

$$= \sigma\left(\frac{L\left(\frac{1}{2}, \Pi_f'\right)}{\omega\left(\Pi_f'\right) \omega\left(\pi_\infty'\right)}\right) \cdot \frac{\prod_{v \in S_{\Pi_f', \psi}} P\left(\frac{1}{2}, \xi^0_{\Pi', v}\right)}{\mathrm{vol}\left(\iota^{-1}\left(K_f'\right)\right)},$$

where we used that $\mathrm{vol}(\iota^{-1}(K_f')) \in \mathbb{Q}^*$. On the other hand , by pulling σ into the integral (14), we compute

$$\sigma\left(\int_{\mathbf{S}^{H_1'}_{K_f'}} \mathcal{T}^* \iota^* \Theta_{\Pi',0}\left(\xi^0_{\Pi_f'}\right) \wedge [\eta]\right) \overset{(3.6.2)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f'}} \sigma^*_{H_1'}(\mathcal{T}^* \iota^* \Theta_{\Pi',0}\left(\xi^0_{\Pi_f'}\right)) \wedge [{}^\sigma\eta] \overset{(12)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f'}} {}^\sigma\mathcal{T}^* \sigma^*_{H_1'}(\iota^* \Theta_{\Pi',0}\left(\xi^0_{\Pi_f'}\right)) \wedge [{}^\sigma\eta] \overset{(9)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f'}} {}^\sigma\mathcal{T}^* \iota^* \sigma^*_{\mathbf{S}^{\mathrm{GL}_2'}_{K_f'}}(\Theta_{\Pi',0}\left(\xi^0_{\Pi_f'}\right)) \wedge [{}^\sigma\eta] \overset{(3.5.4)}{=}$$

$$= \int_{\mathbf{S}^{H_1'}_{K_f'}} {}^\sigma\mathcal{T}^* \iota^* \Theta_{{}^\sigma\Pi',0}\left(\xi^0_{{}^\sigma\Pi_f'}\right) \wedge [{}^\sigma\eta].$$

By Theorem 3.4.7 ${}^{\sigma}\Pi'$ admits a Shalika model with respect to ${}^{\sigma}\eta$ and is cohomological and therefore the last integral equals by Theorem 3.6.4

$$\frac{L\left(\frac{1}{2}, {}^{\sigma}\Pi'_f\right)}{\omega\left({}^{\sigma}\Pi'_f\right)\omega\left({}^{\sigma}\pi'_\infty\right)} \cdot \frac{\prod_{v\in S_{\Pi'_f,\psi}} P\left(\frac{1}{2}, \xi^0_{\Pi',v}\right)}{\operatorname{vol}\left(\iota^{-1}\left(K'_f\right)\right)},$$

which proves the assertion.

If $\frac{1}{2}+m$ is an arbitrary critical value and $\chi = 1$, consider $\Pi'(m) = \Pi' \otimes |\det'|^p$, where p is as in (13) and hence, $\frac{1}{2}$ is compatible with $\mu - p$. Recall also that $\mathcal{G}\left(|\det'|^n_f\right) = 1$, thus Theorem 3.5.6 proves the claim. Finally, to obtain the result for $\frac{1}{2}+m$ is an arbitrary critical value and $\chi \neq 1$, we apply Theorem 3.5.6 again and note that $\pi'_\infty = \pi'_\infty \otimes \chi_\infty$, since χ is of finite order. □

Recall that since $\mathrm{JL}(\Pi') = \mathrm{MW}(\Sigma, 2)$, the partial L-functions of the discrete series representations Π' and $\mathrm{MW}(\Sigma, 2)$ coincide. We therefore obtain a new result on critical values for residual representations of GL_4. Note that for any place $v \in \mathcal{V}_f$,

$$L\left(\frac{1}{2}+m, \pi'_v \otimes \chi_v\right) \in \mathbb{Q}\left(\Pi', \chi\right)$$

by Lemma 3.4.9 and by Theorem 3.4.2 Π' admits a Shalika model with respect to ω_Σ. The following is therefore an easy consequence of Theorem 3.6.5.

Theorem 3.6.6. *Let* $\Pi = \mathrm{MW}(\Sigma, 2)$ *be a discrete series representation of* $\mathrm{GL}_4(\mathbb{A})$ *such that there exists a cuspidal irreducible representation* Π' *of* $\mathrm{GL}'_2(\mathbb{A})$ *with* $\mathrm{JL}(\Pi') = \Pi$*, which is cohomological with respect to the coefficient system* $E^\vee_\mu$ *and* μ *admissible. Let* χ *be a finite order Hecke-character of* $\mathrm{GL}_1(\mathbb{A})$ *and let* $s = \frac{1}{2}+m$ *be a critical value of* Π'*. Then*

$$\frac{L\left(\frac{1}{2}+m, \Pi_f \otimes \chi_f\right)}{\omega\left(\Pi'_f\right)\mathcal{G}\left(\chi_f\right)^4\omega\left(\pi'_\infty, m\right)} \in \mathbb{Q}\left(\Pi', \omega_\Sigma, \chi\right).$$

3.7 Proof of Theorem 3.4.3 & Theorem 3.4.4

We will now show how to adapt the proof given in [39] to the situation at hand. Almost all of the arguments remain unchanged and we only include them for completeness. Throughout this section Π' will be a cuspidal irreducible representation of $\mathrm{GL}(\mathbb{A})$ which admits a Shalika model with respect to η and $\phi \in \Pi$ will be a cusp form.

If H is an algebraic subgroup of GL'_{2n} containing Z'_{2n} we denote by

$$H^0(\mathbb{A}) = \{h \in H(\mathbb{A}) : |\det'(h)| = 1\}.$$

Given a Haar-measure $\mathrm{d}h$ on $H(\mathbb{A})$, there exists a Haar measure $\mathrm{d}z$ on the center

$$Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})$$

such that for all $s \in \mathbb{C}$ and f a smooth function on $H(\mathbb{A})$

$$\begin{gathered}\int_{Z_{2n}(\mathbb{A})H(\mathbb{K})\backslash H(\mathbb{A})} f(h)|\det'(h)|^s \,\mathrm{d}h = \\ \int_{Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})} |\det'(z)|^s \int_{H(\mathbb{K})\backslash H^0(\mathbb{A})} f(hz)\,\mathrm{d}h\,\mathrm{d}z,\end{gathered} \tag{15}$$

assuming the first integral converges. Indeed, this follows from the fact that $H^0(\mathbb{A})\backslash H(\mathbb{A})$ can be identified with $Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A}) \times Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A})$ and that the integral of $|\det'|^s$ over $Z'^0_{2n}(\mathbb{A})\backslash Z'_{2n}(\mathbb{A})$ is the same as the integral of $|\det'|^s$ over $Z'_{2n}(\mathbb{K})\backslash Z_{2n}(\mathbb{A})$. We will denote by 1_n the n-dimensional identity matrix. Let $q, p \in \mathbb{Z}^+$ be such that $p + q = 2n$, let $U'_{(q,p)} \subseteq P'_{(q,p)}$ be the corresponding unipotent subgroup of GL'_{2n} and let A be the group of diagonal matrices of GL_{2n} embedded into GL'_{2n}. We identify $U'_{(q,p)}$ from time to time with the linear space of $p \times q$ matrices $M'_{q,p}$. To each $\beta \in M'_{q,p}(\mathbb{K})$ we associate the character θ_β of $U'_{(q,p)}(\mathbb{A})$

$$u = \begin{pmatrix} 1_p & v \\ 0 & 1_q \end{pmatrix} \mapsto \psi\left(\mathrm{Tr}'\left(v\beta\right)\right).$$

Moreover, let $H = \mathrm{GL}'_p \times \mathrm{GL}'_q$ be the Levi-component of $P_{(q,p)}$. Then for

$$\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in H\left(\mathbb{K}\right)$$

it is straightforward to see that

$$\theta_\beta\left(\gamma^{-1}u\gamma\right) = \theta_{\gamma_2^{-1}\beta\gamma_1}\left(u\right).$$

The additive group $M'_{q,p}\left(\mathbb{A}\right)$ is isomorphic to $M_{dq,dp}\left(\mathbb{A}\right)$. It is well known that the additive characters of the latter group are parametrized by the space of linear functionals $\mathrm{Hom}_{\mathbb{K}}\left(M_{dq,dp}\left(\mathbb{K}\right), \mathbb{K}\right)$ by identifying $X \in \mathrm{Hom}_{\mathbb{K}}\left(M_{dq,dp}\left(\mathbb{K}\right), \mathbb{K}\right)$ with the additive character

$$v \mapsto \psi(X(v))).$$

Identifying $M_{dq,dp}\left(\mathbb{K}\right)$ with $M'_{q,p}\left(\mathbb{K}\right)$ again, we obtain that all characters of $U'_{(q,p)}\left(\mathbb{A}\right)$ are of the form θ_β and $\theta_\beta = \theta_{\beta'}$ if and only if β'. This allows us to consider for a cuspform $\phi \in \Pi'$ its Fourier expansion

$$\phi\left(g\right) = \sum_{M'_{q,p}(\mathbb{K})} \phi_\beta\left(g\right),$$

where

$$\phi_\beta\left(g\right) := \int_{U'_{(q,p)}(\mathbb{K})\backslash U'_{(q,p)}(\mathbb{A})} \phi\left(gu\right)\theta_\beta\left(u\right)du.$$

It is again easy to see that

$$\phi_\beta\left(\gamma g\right) = \phi_{\gamma_2^{-1}\beta\gamma_1}\left(g\right)$$

and $\phi_0 = 0$, since ϕ is cuspidal.

Lemma 3.7.1.

$$\int_{H(\mathbb{K})\backslash H^0(\mathbb{A})} \sum_{\beta\in M'_{q,p}(\mathbb{K})} |\phi_\beta\left(h\right)| \mathrm{d}h < \infty$$

Proof. It suffices to show that the integral is finite over a standard Siegel set of H^0, *i.e.* let H_{2n} be the Cartan subgroup of GL'_{2n} consisting of the diagonal matrices with entries in a fixed maximal subfield $\mathbb{E} \subseteq \mathbb{D}$, let Ω be a compact subset of $\mathrm{GL}'_{2n}\left(\mathbb{A}\right)$, let C be a positive constant and let $S\left(C\right)$ be the connected component of 1_{2n} of the diagonal matrices

$$a = \mathrm{diag}\left(a_1, \ldots, a_p, a_{p+1}, \ldots, a_{2n}\right)$$

with $a \in H_{2n}$ satisfying $\left|\frac{a_i}{a_{i+1}}\right| \geq C$ for $i \neq p, 2n$ and $\prod_{i=1}^{p} a_i = \prod_{i=p+1}^{2n} a_i = 1$, see [105, Theorem 4.8]. Hence, we have to show that there exists a constant D such that

$$\sum_{\beta \in M'_{q,p}} |\phi_\beta (a\omega)| < D$$

for all $a \in S(C)$ and $\omega \in \Omega$. We consider the function $u \mapsto \phi(ua\omega)$, $u \in U'_{(q,p)}(\mathbb{A})$ as a smooth, periodic function in u for fixed a and ω. Then its Fourier series is also smooth and converges absolutely. To prove that this convergence is uniform, *i.e.* independent of a and ω, it suffices to show like in the proof of [39, Lemma 2.1] that firstly, there exists a compact open subgroup $U_f \subseteq U'_{(q,p)}(\mathbb{A}_f)$ such that

$$\phi\left(uu'a\omega\right) = \phi(ua\omega)$$

for all $u' \in U_f$ and secondly, there exists a constant D' independent of a and ω such that for any X of the enveloping universal algebra of $\mathfrak{u}_{(q,p),\infty}$, the Lie algebra of $\prod_{v \in \mathcal{V}_\infty} U_{(q,p)}(\mathbb{K}_v)$,

$$|\lambda(X)\phi(ua\omega)| < D'.$$

Here we denote by λ the left action of U_∞ and ρ its right action. The existence of U_f as above follows immediately from the smoothness of ϕ, since Ω is compact and $S(C)$ normalizes $U'_{(q,p)}(\mathbb{A}_f)$.

To prove the second claim, we fix $v \in \mathcal{V}_\infty$, a root α of H_{2n} in $U'_{(q,p)}$ and a root vector X_α of α in the Lie algebra of $\mathfrak{u}_{(q,p),\infty}$. Recall that since H_{2n} is a Cartan subgroup, such root vectors span $\mathfrak{u}_{(q,p),\infty}$. Then

$$\lambda(-X_\alpha)\phi(ua\omega) = \alpha(a_v)^{-1}\rho\left(\mathrm{ad}(\omega^{-1})X_\alpha\right)\phi(ua\omega).$$

Now $\mathrm{ad}(\omega^{-1})X_\alpha$ is a linear combination of basis elements of $\mathfrak{g}'_v$, whose coefficients are bounded, because Ω is compact. Since $a \in S(C)$, $\alpha(a_v)^{-1}$ is bounded by a constant multiple of $|a_p|^{-M}|a_{2n}|^M$ for some $M \geq 0$.

Therefore, $\lambda(-X_\alpha)\phi(ua\omega)$ is bounded above by

$$\sum_j |a_p|^{-M_j}|a_{2n}|^{M_j}|\phi_j(ua\omega)|,$$

for $\phi_j \in \Pi$. The following lemma will be useful in this and the following proof.

Lemma 3.7.2 ([39, Lemma 2.2])**.** *Let ϕ be a cusp form of a reductive group G, which is invariant under the split component of the center of G. Let R be a maximal proper parabolic subgroup of G, let δ_R be the module of the group $R(\mathbb{A})$ and let Ω be a compact subset of $G(\mathbb{A})$. Then for every $M \geq 0$ there exists a constant D such that*

$$\delta_R(r)^M |\phi(r\omega)| \leq D$$

for all $r \in R(\mathbb{A})$ and $\omega \in \Omega$.

Since ua is contained in $P'_{(q-1,p+1)}(\mathbb{A})$ and $P'_{(q+1,p-1)}$ with respective modules

$$\delta_{P'_{(p-1,q+1)}}(ua) = |a_p|^{-2nd},\ \delta_{P'_{(p+1,q-1)}}(ua) = |a_{p+1}|^{2nd},$$

we deduce that $|a_p|^{-M_j}|a_{2n}|^{M_j}|\phi_j(ua\omega)|$ is bounded above. This finishes the proof. □

The next step is to observe that even though we are dealing with matrices over a division algebra, Gauss elimination still holds true in $M'_{q,p}(\mathbb{K})$. Therefore, the $H(\mathbb{K}) = \mathrm{GL}'_p(\mathbb{K}) \times \mathrm{GL}'_q(\mathbb{K})$-orbits on $M'_{q,p}(\mathbb{K})$ under the action $\gamma \cdot \beta = \gamma_2^{-1}\beta\gamma_1$ are precisely given by the possible ranks of the matrices. To be more precise, we say a matrix β has rank r if it is in the orbit of

$$\beta_r := \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}.$$

The stabilizer $H_{1_r}(\mathbb{K})$ of the matrix β_r is the subgroup of matrices of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \text{ with } g_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix},\ g_2 = \begin{pmatrix} a_2 & 0 \\ c_1 & d_1 \end{pmatrix}, \tag{16}$$

where d_1 is a square matrix of dimension r, a_1 is a square matrix of dimension $q-r$ and a_2 is a square matrix of dimension $p-r$. Now we can write

$$\phi(g) = \sum_{r=1}^{\min(q,p)} \sum_{\gamma \in H_{1_r}(\mathbb{K}) \backslash H(\mathbb{K})} \phi_{\beta_r}(\gamma g).$$

Next comes the proof of the generalization of [39, Proposition 2.1].

Proposition 3.7.3. *Let $q > p$. Then for any cusp form $\phi \in \Pi'$,*

$$\int_{G'_q(\mathbb{K}) \backslash G'^0_q(\mathbb{A})} \phi\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 = 0.$$

Proof. By Lemma 3.7.1 we can exchange the integral and sum in the Fourier series of ϕ, *i.e.*

$$\int_{G'_q(\mathbb{K}) \backslash G'^0_q(\mathbb{A})} \phi\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 =$$

$$= \int_{G'_q(\mathbb{K}) \backslash G'^0_q(\mathbb{A})} \sum_{r=1}^{p} \sum_{\gamma \in H_{1_r}(\mathbb{K}) \backslash H(\mathbb{K})} \phi_{\beta_r}\left(\gamma \begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1 =$$

$$= \sum_{r=1}^{q} \sum_{\gamma \in H_{1_r}(\mathbb{K}) \backslash H(\mathbb{K})} \int_{G'_q(\mathbb{K}) \backslash G'^0_q(\mathbb{A})} \phi_{\beta_r}\left(\gamma \begin{pmatrix} g_1 & 0 \\ 0 & 1_p \end{pmatrix}\right) \mathrm{d}g_1.$$

This equals by (16)

$$\sum_{r=1}^{q} \sum_{\gamma_2} \int_{P_{(q-r,r)}(\mathbb{K}) \backslash G^0_q{}'(\mathbb{A})} \phi_{\beta_r}\left(\begin{pmatrix} g_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\right) \mathrm{d}g_1,$$

where γ_2 is a matrix of $G'_q(\mathbb{K})$ modulo matrices of the form $\begin{pmatrix} a_2 & 0 \\ c_1 & 1_r \end{pmatrix}$. Taking the integral first over

$$U'_{(q-r,r)}(\mathbb{K}) \backslash U'_{(q-r,r)}(\mathbb{A})$$

and then over

$$P_{(q-r,r)}(\mathbb{K})\, U'_{(q-r,r)}(\mathbb{A}) \backslash R(\mathbb{A})$$

shows that the claim follows if

$$\int_{U'_{(q-r,r)}(\mathbb{K}) \backslash U'_{(q-r,r)}(\mathbb{A})} \phi_{\beta_r}(vg)\, \mathrm{d}v \tag{17}$$

vanishes for all $g \in \mathrm{GL}(\mathbb{A})$. To see this, observe first that the character θ_{β_r} can be extended to $U'_{(q,p)}(\mathbb{A})U'_{(q-r,r)}(\mathbb{A})$, trivially on $U'_{(q-r,r)}$, since $U'_{(q-r,r)}$ normalizes $U'_{(q,p)}$. Thus, the integral (17) can be written as

$$\int_{U'_{(q,p)}U'_{(q-r,r)}(\mathbb{K})\backslash U'_{(q,p)}U'_{(q-r,r)}(\mathbb{A})} \phi\left(u'g\right)\theta_{\beta_r}\left(u'\right)du'.$$

This integral however factors through an integral over

$$U'_{(p-r,p-r)}(\mathbb{K})\backslash U'_{(p-r,q-r)}(\mathbb{A}),$$

on which θ_{β_r} is trivial. Since ϕ is cuspidal, the assertion follows. □

3.7.1 Proof of Theorem 3.4.3 Before we begin the proof, let us remark the following.

Remark. If Π' admits a Shalika model with respect to η, π'_v admits a Shalika model with respect to η_v, *i.e.* a continous intertwining map

$$\left(\pi'_v\right)^\infty \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\left(\psi_v \otimes \eta_v\right)$$

if $v \in \mathcal{V}_\infty$ and a morphism of $\mathrm{GL}(\mathbb{K}_v)$-representations

$$\pi'_v \to \mathrm{Ind}_{\mathcal{S}(\mathbb{K}_v)}^{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\left(\psi_v \otimes \eta_v\right)$$

if $v \in \mathcal{V}_f$.

In both cases Frobenius reciprocity gives us a continuous morphism

$$\lambda_v \in \mathrm{Hom}_{\mathcal{S}(\mathbb{K}_v)}\left(\left(\pi'_v\right)^\infty, \psi_v \otimes \eta_v\right) \text{ respectively } \lambda_v \in \mathrm{Hom}_{\mathcal{S}(\mathbb{K}_v)}\left(\pi'_v, \psi_v \otimes \eta_v\right).$$

If $v \in \mathcal{V}_\infty$ the so obtained map is a priori just an intertwiner of group actions, but not necessarily continuous. However, the space of smooth vectors satisfies the Heine-Borel property, *i.e.* a subset of $(\pi'_v)^\infty$ is compact if and only if it is bounded on bounded sets and closed. Since a linear map of Fréchet spaces is continuous if and only if it is bounded, the claim follows. If $v \in \mathcal{V}_f$ we obtain λ_v without any extra steps.

For a cuspform $\phi = \otimes_{v\in\mathcal{V}}\phi_v \in \Pi'$ we have that $|\phi(g)| \leq \beta(\phi)$, where β is a semi-norm on $(\pi'_\infty)^\infty \otimes \Pi_f^{K'_f}$ for any open compact subgroup K'_f, since cusp forms are of rapid decay. Letting λ be the Shalika functional associated via Frobenius reciprocity to our Shalika model, we obtain that $g \mapsto \lambda(\Pi'(g)\phi)$ is bounded. Thus, if restrict λ to smooth vectors obtain so the local Shalika functionals λ_v, $v \in \mathcal{V}$, we also have that $g_v \mapsto \lambda_v(\pi'_v(g_v)\phi_v)$ is bounded for all $v \in \mathcal{V}$.

Theorem 3.7.4. *Let* Π' *be a cuspidal irreducible representation of* $\mathrm{GL}'_{2n}(\mathbb{A})$. *Assume* Π' *admits a Shalika model with respect to* η *and let* $\phi \in \Pi'$ *be a cusp form. Consider the integrals*

$$\Psi(s,\phi) := \int_{Z'_{2n}(\mathbb{A})H'_n(\mathbb{K})\backslash H'_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right)\left|\frac{\det'(h_1)}{\det'(h_2)}\right|^{s-\frac{1}{2}} \eta(h_2)^{-1}\,\mathrm{d}h_1\,\mathrm{d}h_2,$$

$$\zeta(s,\phi) := \int_{\mathrm{GL}'_n(\mathbb{A})} \mathcal{S}_\psi^\eta(\phi)\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}\right)|\det'(g_1)|^{s-\frac{1}{2}}\,\mathrm{d}g_1.$$

Then $\Psi(s,\phi)$ *converges absolutely for all* s *and* $\zeta(s,\phi)$ *converges absolutely if* $\mathrm{Re}(s) >> 0$. *Moreover, if* $\zeta(s,\phi)$ *converges absolutely,* $\Psi(s,\phi) = \zeta(s,\phi)$.

Proof. We apply Lemma 3.7.2 to the case $R' = P'_{(n,n)} \subseteq \mathrm{GL}'_{2n}$ to see that

$$\phi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right)\left|\frac{\det'(h_1)}{\det'(h_2)}\right|^M$$

is bounded above for any M, hence, $\Psi(s,\phi)$ converges absolutely. Indeed, recall that

$$Z'_n(\mathbb{A})\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'_n(\mathbb{A})$$

has finite volume and hence

$$Z'_{2n}(\mathbb{A})H'_n(\mathbb{K})\backslash H'_n(\mathbb{A}) = (1_n \times Z'_n(\mathbb{A}))\Omega,$$

where Ω has finite volume. Since above M can be chosen arbitrarily small, the claim follows. For a suitable measure $\mathrm{d}z$ on $Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})$ we have by (15)

$$\Psi(s,\phi) = \int_{Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})} |\det'(z)|^{s-\frac{1}{2}}$$

$$\int_{\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})}\int_{\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 z & 0 \\ 0 & h_2 \end{pmatrix}\right)\eta(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}z.$$

Inserting the Fourier series we see that the contribution of the matrices with rank $r < n$ is 0 by Proposition 3.7.3 and hence,

$$\begin{aligned}\int_{\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})}\int_{\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})} \phi\left(\begin{pmatrix} h_1 z & 0 \\ 0 & h_2 \end{pmatrix}\right)\eta(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2 = \\ = \int_{\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})\times\mathrm{GL}'_n(\mathbb{K})\backslash\mathrm{GL}'^0_n(\mathbb{A})} \sum_{\gamma_1\in\mathrm{GL}'_n(\mathbb{K})} \\ \phi_{\beta_n}\left(\begin{pmatrix} \gamma_1 h_1 z & 0 \\ 0 & h_2 \end{pmatrix}\right)\eta(h_2)\,\mathrm{d}h_1\,\mathrm{d}h_2.\end{aligned} \tag{18}$$

Contracting the sum and the integral and performing a change of variables, it follows that (18) is equal to

$$\int_{\mathrm{GL}'^0_n(\mathbb{A})} \mathcal{S}^\eta_\psi(\phi)\left(\begin{pmatrix} gz & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}\right)\eta(g)\,\mathrm{d}g.$$

Thus,

$$\begin{aligned}\Psi(s,\phi) = \\ = \int_{Z'_{2n}(\mathbb{K})\backslash Z'_{2n}(\mathbb{A})} |\det'(z)|^{s-\frac{1}{2}} \int_{\mathrm{GL}'^0_n(\mathbb{A})} \mathcal{S}^\eta_\psi(\phi)\left(\begin{pmatrix} gx & 0 \\ 0 & 1_n \end{pmatrix}\right)\eta(g)\,\mathrm{d}g\,\mathrm{d}z = \\ = \zeta(s,\phi),\end{aligned} \tag{19}$$

where the last equation is valid only once we show that $\zeta(s,\phi)$ converges absolutely for $\mathrm{Re}(s) >> 0$. To show the convergence we use the Dixmier-Malliavin theorem.

Theorem 3.7.5 (Dixmier-Malliavin theorem)**.** *Suppose G to be a Lie group and π a continuous representation of G on a Fréchet space V. Then every smooth vector $v \in V^\infty$ can be represented as a finite sum*

$$v = \sum_k \pi(f_k)\,v_k,$$

with $v_k \in V$, f_k a smooth, compactly supported function on G and

$$\pi(f)\,w := \int_G f(x)\,\pi(x)\,w\,\mathrm{d}x$$

for some fixed Haar measure on G.

Remark. Note that if G is a reductive group over $\mathbb{K}$ and (Π'_f, V_f) is a smooth representation of $G(\mathbb{A}_f)$, we can write $v = \Pi'_f(v) := \int_{G(\mathbb{A}_f)} \phi(x)\,\Pi'(x)\,v\,\mathrm{d}x$ for some smooth, *i.e.* locally constant, function ϕ as every vector in V_f is fixed by some open compact subgroup.

We consider the action of $\mathrm{GL}'_{2n}(\mathbb{A})$ on $\mathcal{S}^\eta_\psi(\Pi')$. The cusp form ϕ is a smooth vector in Π', where we consider Π' as a proper $\mathrm{GL}'_{2n}(\mathbb{A})$-subrepresentation of the corresponding L^2-space. Applied to our case this yields that $\mathcal{S}^\eta_\psi(\phi)(g)$ can be written as a finite sum

$$\sum_k \int_{U'_{(n,n)}(\mathbb{A})} \xi_k\left(g\begin{pmatrix}1_n & u\\ 0 & 1_n\end{pmatrix}\right)\phi_k(u)\,\mathrm{d}u,$$

where the ϕ_k are compactly supported, smooth functions on $U'_{(n,n)}(\mathbb{A})$ and $\xi_k \in \mathcal{S}^\eta_\psi(\Pi)$. Moreover, all ξ_k satisfy the equivariance property under η and ψ and are therefore bounded by the remark of Section 3.7.1. Applying this, we deduce

$$\mathcal{S}^\eta_\psi\left(\begin{pmatrix}g_1 & 0\\ 0 & 1_n\end{pmatrix}\right) = \sum_k \xi_k\left(\begin{pmatrix}g_1 & 0\\ 0 & 1_n\end{pmatrix}\right)\widehat{\phi_k}(g_1),$$

where $\widehat{\phi_k}$ is the Fourier transform of ϕ_k. Recalling the definition of $\zeta(s,\phi)$, we obtain

$$\zeta(s,\phi) = \int_{\mathrm{GL}'_n(\mathbb{A})} \sum_k \xi_k\left(\begin{pmatrix}g_1 & 0\\ 0 & 1_n\end{pmatrix}\right)\widehat{\phi_k}(g_1)\,|\mathrm{det}'(g_1)|^{s-\frac{1}{2}}\,\mathrm{d}g_1$$

Since the ξ_k are bounded, $\zeta(s,\phi)$ is thus bounded by a multiple of

$$\sum_k \int_{\mathrm{GL}'_n(\mathbb{A})} \widehat{\phi_k}(g_1)\,|\det g_1|^{s-\frac{1}{2}}\,dg_1,$$

which converges absolutely for s with real part sufficiently large by [44, Theorem 13.8] and thus $\zeta(s,\phi)$ converges for $\mathrm{Re}(s) >> 0$. □

3.7.2 Proof of Theorem 3.4.4

Theorem 3.7.6. *Let Π' be a cuspidal irreducible representation of $\mathrm{GL}'_{2n}(\mathbb{A})$ and assume Π' admits a Shalika model with respect to η. Then for each place $v \in \mathcal{V}$ and $\xi_v \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$ there exists an entire function $P(s,\xi_v)$, with $P(s,\xi_v) \in \mathbb{C}[q_v^{s-\frac{1}{2}}, q_v^{\frac{1}{2}-s}]$ if $v \in \mathcal{V}_f$, such that*

$$\zeta_v(s,\xi_v) = P(s,\xi_v)\,L(s,\pi'_v)$$

and hence, $\zeta_v(s,\xi_v)$ can be analytically continued to $\mathbb{C}$. Moreover, for each place v there exists a vector ξ_v such that $P(s,\xi_v) = 1$. If v is a place where neither Π' nor ψ ramify this vector can be taken as the spherical vector $\xi_{\pi'_v}$ normalized by $\xi_{\pi'_v}(\mathrm{id}) = 1$.

We follow closely the proofs of [39, Proposition 3.1, Proposition 3.2]. We denote by $S(M_{s,t})$ respectively $S(M'_{s,t})$ the space of Schwartz-functions on $M_{s,t}(\mathbb{K}_v)$ respectively $M'_{s,t}(\mathbb{K}_v)$.

Proof. The first step is to prove the following lemma

Lemma 3.7.7. *There exists, depending on ξ_v, a positive Schwartz-function $\Theta \in S(M_{n,n})$, such that*

$$\left|\xi_v\left(\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix} g\right)\right)\right| \leq \Theta\left(b^{-1} g_1 a\right)$$

for the Iwasawa decomposition

$$g = u\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k,\ u \in U'_{(n,n)}(\mathbb{K}_v),\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H'_n(\mathbb{K}_v),\ k \in K'_v.$$

Proof. We first assume that we are in the archimedean case. Using the Dixmier-Malliavin theorem, it is enough to prove the claim in the case ξ_v being of the form

$$\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)} \xi_{v,1}(gh)\,\Psi(h)\,\mathrm{d}_v h,$$

where Ψ is smooth function of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$ with compact support. Write g as

$$h = \begin{pmatrix} 1_n & u_1 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} k_1,\ g = \begin{pmatrix} 1_n & u_2 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} k_2.$$

We compute

$$\begin{aligned}\xi_v\left(\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix} g\right)\right) &= \\ = \psi(\mathrm{Tr}(g_1 u_2)) \int_{\mathrm{GL}'_n(\mathbb{K}_v)\times\mathrm{GL}'_n(\mathbb{K}_v)\times K'_v} &\xi_{v,1}\left(\begin{pmatrix} g_1 a_2 a_1 & 0 \\ 0 & b_2 b_1 \end{pmatrix} k_1\right) \\ \Xi\left(b_2^{-1} g_1 a_2; k, k_2, a_1, b_1\right) &|\mathrm{det}'_v\left(a_1 b_1^{-1}\right)|^{-nd}\,\mathrm{d}_v a_1\,\mathrm{d}_v b_1\,\mathrm{d}_v k_1,\end{aligned}$$

where $\Xi(v; k_1, k_2, a_1, b_2)$ is the Fourier transform of

$$u_1 \mapsto \Psi\left(k_2^{-1}\begin{pmatrix} 1_n & u_1 \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} k_2\right)$$

This function and its derivatives have compact support, which is independent of the parameters k_2, a_1, b_1, k_1. Thus, the respective Fourier transform are contained in a bounded set in the space of Schwartz-functions on $U'_{(n,n)}(\mathbb{K}_v)$. Hence, there exists a positive Schwartz-function Θ_1 and a function Θ_2 with compact support such that

$$|\Xi(v; k_1, k_2, a_1, b_1)| \leq \Theta_1(v)\,\Theta_2(a_1, b_1).$$

This is enough to show the majorization, since $\xi_{v,1}$ is bounded by the remark of Section 3.7.1.

In the case where $\mathbb{K}_v$ is non-archimedean we do not need the Dixmier-Malliavin lemma, since we automatically can write ξ_v in integral form by the remark after Theorem 3.7.5. □

In the next step we let $v \in \mathcal{V}$ be a place and consider integrals of the form

$$Z(\xi_v, \Psi, s) := \int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)} \xi_v(g)\,\Psi(g)\,|\mathrm{det}'_v(g)|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_v g$$

for $\xi_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$ and $\Psi \in S\left(M'_{2n,2n}\right)$. Since ξ_v is bounded, this integral converges for $\mathrm{Re}(s) >> 0$, see for example the proof of [44, Theorem, 3.3].

Lemma 3.7.8. *The function*

$$\frac{Z(\xi_v,\Psi,s)}{L(s,\pi'_v)}$$

is meromorphic and if $v\in\mathcal{V}_f$

$$\frac{Z(\xi_v,\Psi,s)}{L(s,\pi'_v)}\in\mathbb{C}[q_v^{s-\frac{1}{2}},q_v^{\frac{1}{2}-s}].$$

Moreover, there exists $\xi_{v,j}\in\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$, $\Psi_j\in S\left(M'_{2n,2n}\right)$ such that we can write the local L-factor as a finite sum of the form

$$L\left(s,\pi'_v\right)=\sum_j Z(\xi_{v,j},\Psi_j,s).$$

Proof. We first assume that $\mathbb{K}_v$ is non-archimedean. Let $I(\pi'_v)$ be the $\mathbb{C}$ vector-space spanned by the integrals of the form

$$Z(f,\Psi,s):=\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\Psi(g)\,f(g)\,|\mathrm{det}'_v(g)|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_vg,$$

where f is a smooth matrix coefficient of π'_v and $\Psi\in S\left(M'_{2n,2n}\right)$. To be more precise, the integrals converge for $\mathrm{Re}(s)>>0$ and admit a meromorphic continuation. By [44, Theorem 3.3] $I(\pi'_v)$ is a $\mathbb{C}[q_v^{s-\frac{1}{2}},q_v^{\frac{1}{2}-s}]$-ideal in $\mathbb{C}(q_v^{s-\frac{1}{2}})$ generated by $L(s,\pi'_v)$.

We will now show that the $\mathbb{C}$-vector space spanned by the $Z(\xi_v,\Psi,s)$ is $I(\pi'_v)$, which consequently will show the claims of the lemma. To do so we introduce the space $\mathscr{U}$ consisting of smooth matrix coefficients of the form

$$g\mapsto\int_{K'_v}\xi_v\left(k^{-1}g\right)e(x)\,\mathrm{d}_vk,\ g\in\mathrm{GL}(\mathbb{K}_v),\ \xi_v\in\mathcal{S}_{\psi_v}^{\eta_v}\left(\pi'_v\right),$$

where e is an idempotent under the usual convolution product on the functions supported on K'_v.

Given ξ_v and Ψ, we define

$$g\mapsto f(g):=\int_{K'_v}\xi_v\left(x^{-1}g\right)e(k)\,\mathrm{d}_vk,\ g\in\mathrm{GL}(\mathbb{K}_v),$$

which is a smooth matrix coefficient of π'_v and hence, for such f

$$Z(\xi_v,\Psi,s)=\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}f(g)\,\Psi(g)\,|\mathrm{det}'_v(g)|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_vg\in\mathscr{U}.$$

On the other hand, for every $f\in\mathscr{U}$ and Schwartz-function Ψ there exists $\xi_v\in\mathcal{S}_{\psi_v}^{\eta_v}(\pi'_v)$, $\Psi'\in S\left(M'_{2n,2n}\right)$ such that $Z(f,\Psi,s)=Z(\xi_v,\Psi',s)$. Indeed,

$$Z(f,\Psi,s)=\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\int_{K'_v}\xi_v\left(k^{-1}g\right)e(k)\,\Psi(g)\,|\mathrm{det}'_v(g)|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_vk\,\mathrm{d}_vg=$$

$$=\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\xi_v(g)\int_{K'_v}(e(k)\,\Psi(kg)|\,\mathrm{det}'_v(g)|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_vk\,\mathrm{d}_vg=Z(\xi_v,\Psi',s),$$

where $\Psi'(g) := \int_{K'_v} e(k)\,\Psi(kg)\,\mathrm{d}_v k$. This shows that the space spanned by the $Z(\xi_v, \Psi, s)$ is the space spanned by $\mathscr{U}$. It therefore suffices to show that the span of $\mathscr{U}$ is $I(\pi'_v)$. But since $\mathscr{U}$ is closed under right translations under $\mathrm{GL}(\mathbb{K}_v)$ and π'_v is irreducible, any smooth matrix coefficient f of π'_v can be written as a finite sum

$$f(g) = \sum_i f_i(g_i g)$$

for some suitable $g_i \in \mathrm{GL}(\mathbb{K}_v)$ and $f_i \in \mathscr{U}$. Therefore, the final claim follows because then

$$Z(f, \Psi, s) = \int_{\mathrm{GL}(\mathbb{K}_v)} \sum_i f_i(g_i g)\,\Psi(g)\,|\det g|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_v g =$$

$$= \int_{\mathrm{GL}(\mathbb{K}_v)} \sum_i f_i(g g_i)\,\Psi\left(g_i^{-1} g g_i\right)|\det g|^{s-\frac{2nd-1}{2}}\,\mathrm{d}_v g,$$

where the last expression is of the desired form. In the case where $\mathbb{K}_v$ is archimedean, we argue as above, replacing the action of $\mathrm{GL}'_{2n}(\mathbb{K}_v)$ by the action of the Lie algebra and K'_v and using the Dixmier-Malliavin lemma. □

We return now to the proof of Theorem 3.7.6 and assume from now on that v is archimedean, since the non-archimedean case can be dealt with analogously. We start with the second assertion. We will only prove the archimedean case, since the non-archimedean case follows analogously. Let $\mathrm{SL}'_{2n} := \{g \in \mathrm{GL}'_{2n} : \det'_v(g) = 1\}$ and $K_0 := \mathrm{SL}'_{2n}(\mathbb{K}_v) \cap K'_v$. Using the Iwasawa decomposition we can write

$$Z(\xi_v, \Psi, s) = \int_{H'_n(\mathbb{K}_v)\times U'_{(n,n)}(\mathbb{K}_v)\times K_0} \xi_v\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix} k\right)\Psi\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix} k\right)$$

$$|\det'_v(a)|^{s-\frac{1}{2}}|\det'_v(b)|^{s-\frac{nd-1}{2}}\,\mathrm{d}_v a\,\mathrm{d}_v b\,\mathrm{d}_v x\,\mathrm{d}_v k.$$

We introduce the function

$$\Xi(u, t, w; k) := \int_{H'_n(\mathbb{K}_v)} \Psi\left(\begin{pmatrix} x & y \\ 0 & w\end{pmatrix} k\right)\left(\mathrm{Tr}'(yt) - \mathrm{Tr}'(xu)\right)\mathrm{d}_v x\,\mathrm{d}_v y. \tag{20}$$

If we put issues of convergence aside for a moment, the Fourier inversion formula and a change of variables imply that

$$Z(\xi_v, f, s) = \int_{\mathrm{SL}'_{2n}(\mathbb{K}_v)\times \mathrm{GL}'_n(\mathbb{K}_v)} \xi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1_n\end{pmatrix} x\right)|\det'_v(a)|^{s-\frac{1}{2}}\,\mathrm{d}\mu_\Psi(x)\,\mathrm{d}_v a, \tag{21}$$

where we define the measure μ_Ψ on $\mathrm{SL}'_{2n}(\mathbb{K}_v)$ by

$$\int_{\mathrm{SL}'_{2n}(\mathbb{K}_v)} \Psi(x)\,\mathrm{d}\mu_\Psi(x) :=$$

$$= \int_{\mathrm{GL}'_n(\mathbb{K}_v)\times U'_{(n,n)}(\mathbb{K}_v)\times K'_v} f\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & 1_n\end{pmatrix}\begin{pmatrix} 1_n & u \\ 0 & 1_n\end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & b\end{pmatrix} k\right)$$

$$\Xi\left(u, b, b^{-1}; k\right)|\det'_v(b)|^{nd}\,\mathrm{d}_v b\,\mathrm{d}_v u\,\mathrm{d}_v k.$$

Let us now argue how to put the issues of convergence to rest in the integral of (21). Following [39] we consider the unimodular subgroup Q of GL'_{2n} consisting of matrices of the form

$$Q = \left\{ \begin{pmatrix} b^{-1} & u \\ 0 & b \end{pmatrix} : b \in \mathrm{GL}'_n,\ u \in M'_{n,n} \right\}.$$

Thus, $\mathrm{d}\mu_\Psi = \mu_\Psi(q,k)\ \mathrm{d}_v q\ \mathrm{d}_v k$, where μ_Ψ is a smooth function on $Q(\mathbb{K}_v) \times K'_v$ and it and its derivatives are rapidly decreasing, *i.e.*

$$\|q\|^N |\mu_\Psi(q,k)|$$

is bounded for all natural numbers N, where $\|q\|$ denotes the usual height of q. Recall the majorization α of the beginning of the proof and the remark before Theorem 3.7.4 and that we obtained from Lemma 3.7.7 that

$$\int_{\mathrm{SL}'_{2n}(\mathbb{K}_v)\times\mathrm{GL}'_n(\mathbb{K}_v)} \left| \xi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} x \right) |\det'_v(a)|^{s-\frac{1}{2}} \right| \mathrm{d}\mu_\Psi(x)\ \mathrm{d}_v a \leq$$

$$\leq \int_{\mathrm{GL}'_n(\mathbb{K}_v)\times Q(\mathbb{K}_v)} \Theta(b^{-1}ab^{-1}) |\det'_v(a)|^{\mathrm{Re}(s)-\frac{1}{2}} |\mu_\Psi(q,k)|\, \mathrm{d}_v q\, \mathrm{d}_v k\, \mathrm{d}_v a$$

for a suitable Schwartz-function Θ. After changing $a \mapsto bab$, we can bound this integral for $\mathrm{Re}(s) >> 0$ by a multiple of

$$\int_{Q(\mathbb{K}_v)} |\det'_v(b)|^{2\mathrm{Re}(s)-1} |\mu_\Psi(q,k)|\, \mathrm{d}_v q\, \mathrm{d}_v k\, \mathrm{d}_v a,$$

which converges since μ_Ψ is rapidly decreasing. Thus, we justified the rewriting of the integral (21) and showed that the operator

$$\int_{Q\times K'_v} \pi'_v(qk)\, \mu_\varphi(q,k)\ \mathrm{d}_v q\, \mathrm{d}_v k \tag{22}$$

preserves the local Shalika model, since a priori the operator does not preserve smoothness.

By collecting the results so far, we can prove the following. By Lemma 3.7.8 we find $\xi_{v,j} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, $\Psi_j \in S\left(M'_{2n,2n}\right)$ such that

$$L\left(s, \pi'_v\right) \overset{(3.7.8)}{=} \sum_j \int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)} \xi_{v,j}(g)\, \Psi_j(g)\, |\det'_v(g)|^{s-nd-\frac{1}{2}}\, \mathrm{d}_v g \overset{(21)}{=}$$

$$= \sum_j \int_{\mathrm{GL}'_n(\mathbb{K}_v)} \xi'_{v,j}\left(\begin{pmatrix} g_1 & 0 \\ 0 & 1_n \end{pmatrix} \right) |\det'_v(g)|^{s-\frac{1}{2}}\, \mathrm{d}_v g_1,$$

where

$$\xi'_{v,j}(g) = \int_{\mathrm{SL}'_{2n}(\mathbb{K}_v)} \xi_{v,j}(gx)\, \mu_{\Psi_j}(x)\ \mathrm{d}_v x.$$

Since we showed that (22) preserves $\mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$, we have $\xi'_{v,j} \in \mathcal{S}^{\eta_v}_{\psi_v}(\pi'_v)$ and therefore we proved the second claim of Theorem 3.7.6.

Next, we show the first claim of Theorem 3.7.6. We apply the Dixmier-Malliavin lemma to $Q \times K'_v$ and write

$$\xi_v(g) = \sum_j \int_{\mathrm{GL}'_n(\mathbb{K}_v)\times U'_{(n,n)}(\mathbb{K}_v)\times K'_v} \xi_{v,j}\left(g \begin{pmatrix} b^{-1} & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & b \end{pmatrix} k \right)$$

$$\Gamma_j(u,b,k)\,|\det_v'(b)|^{nd}\,\mathrm{d}_v b\,\mathrm{d}_v u\,\mathrm{d}_v k,$$

where Γ_j are smooth functions with compact support on $\mathrm{GL}_n'(\mathbb{K}_v)\times U_{(n,n)}'(\mathbb{K}_v)\times K_v'$. Let Λ_j be the projection of the support of Γ_j to $U_{(n,n)}'(\mathbb{K}_v)$ and let $\Psi\in S\left(M_{n,n}'\right)$ be such that $\Psi\left(b^{-1}\right)=1$ for $b\in\bigcup_j\Lambda_j$, where we identify $U_{(n,n)}'$ with $M_{n,n}'$. Define

$$\Gamma_j'\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix};k\right):=$$

$$=\int_{U_{(n,n)}'(\mathbb{K}_v)\times U_{(n,n)}'(\mathbb{K}_v)}\Gamma_j(u,b,k)\,\Psi(v)\,\psi(\mathrm{Tr}(xu-yv))\,\mathrm{d}_v u\,\mathrm{d}_v v.$$

Then $\zeta_v(s,\xi_v)$ can be written as

$$\zeta_v(s,\xi_v)=$$

$$=\sum_j\int_{H_n'(\mathbb{K}_v)\times U_{(n,n)}'(\mathbb{K}_v)\times K_v'}\xi_{v,j}\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}k\right)\Gamma_j'\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix};k\right) \tag{23}$$

$$|\det_v'(a)|^{s+nd-\frac{1}{2}}||\det_v'(b)|^{s+nd-\frac{1}{2}}\,\mathrm{d}_v a\,\mathrm{d}_v b\,\mathrm{d}_v x\,\mathrm{d}_v k.$$

Let

$$\Omega_1:=\{(a,b)\in M_{n,2n}':(a,b):\mathbb{D}^{2n}\to\mathbb{D}^n\text{ surjective}\}.$$

The group $\mathrm{GL}_n'(\mathbb{K}_v)$ acts from the left and the group K_v' from the right on $\Omega_1(\mathbb{K}_v)$. The resulting action of $\mathrm{GL}_n'(\mathbb{K}_v)\times K_v'$ is transitive. The stabilizer of $(0,1_n)$ is the group $\left(k_2^{-1},k\right)$, where

$$k=\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}\in K_v'\cap P_{(n,n)}'(\mathbb{K}_v)$$

for some k_1. Let

$$\Omega:=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_{2n,2n}':(c,d)\in\Omega_1\right\}.$$

Let $\mathscr{S}(\Omega)$ be the space of smooth functions $\varphi\colon\Omega(\mathbb{K}_v)\to\mathbb{C}$ such that

1. $|a|^{2n}|b|^{2n}\varphi(g)$, $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is bounded for all $n\in\mathbb{Z}\mathbb{Z}$,
2. The projection of the support of ϕ to $\Omega_1(\mathbb{K}_v)$ is compact,
3. If D is a differential operator which commutes with additive changes in (a,b) then $D\varphi\in\mathscr{S}(\Omega)$.

Analogously we define the space

$$\Omega_0:=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in M_{2n,2n}':\det_v'(d)\neq 0\right\}$$

and $\mathscr{S}(\Omega_0\times K_v')$. The natural map

$$r\colon\Omega_0(\mathbb{K}_v)\times K_v'\to\Omega(\mathbb{K}_v),\ (p,k)\mapsto pk$$

is surjective, proper, and a submersion and the inverse image of pk is

$$r^{-1}(pk)=\{\left(pk'^{-1},k'k\right):k'\in K_v'\cap P_{(n,n)}'(\mathbb{K}_v)\}.$$

Lemma 3.7.9. *Let $\varphi \in \mathscr{S}(\Omega_0 \times K')$. Then*

$$\varphi_*(pk) = \int_{K'_v \cap P_{(n,n)}(\mathbb{K}_v)} \varphi\left(pk'^{-1}, k'k\right) \mathrm{d}_v k'$$

belongs to $\mathscr{S}(\Omega)$.

We postpone the proof of this lemma to see how it finishes the proof of the first claim of Theorem 3.7.6. Recall

$$\zeta\left(s, \pi'_v\right) \overset{(23)}{=} \sum_j \int_{H'_n(\mathbb{K}_v) \times U'_{(n,n)}(\mathbb{K}_v) \times K'_v} \xi_{v,j}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right)$$

$$\Gamma'_j\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; k\right) |\det'_v(a)|^{s+nd-\frac{1}{2}} ||\det'_v(b)|^{s+nd-\frac{1}{2}} \,\mathrm{d}_v a \,\mathrm{d}_v b \,\mathrm{d}_v x \,\mathrm{d}_v k.$$

Observe that changing Γ'_j by

$$(p,k) \mapsto \Gamma'_j\left(pk'^{-1}, k'k\right)$$

for some fixed $k' \in K'_v \cap P'_{(n,n)}(\mathbb{K}_v)$ does not change the integral. Thus,

$$\int_{H'_n(\mathbb{K}_v) \times U'_{(n,n)}(\mathbb{K}_v) \times K'_v} \xi_{v,j}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right)$$

$$\Gamma'_j\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; k\right) |\det'_v(a)|^{s-\frac{1}{2}} ||\det'_v(b)|^{s+nd-\frac{1}{2}} \,\mathrm{d}_v a \,\mathrm{d}_v b \,\mathrm{d}_v x \,\mathrm{d}_v k =$$

$$= \int_{\Omega_0(\mathbb{K}_v) \times K'_v} \xi_{v,j}(r(p,k)) |\det'_v(p)|^{s-nd-\frac{1}{2}} \int_{K'_v \cap P'_{(n,n)}(\mathbb{K}_v)} \Gamma'_j(pk'^{-1}; k'k) \,\mathrm{d}k' \,\mathrm{d}p \,\mathrm{d}k,$$

where we extend $\xi_{v,j}$ trivially to $\Omega(\mathbb{K}_v)$ and $\mathrm{d}p$ is a right $P'_{(n,n)}(\mathbb{K}_v)$-invariant measure on $\Omega_0(\mathbb{K}_v)$. Thus,

$$\zeta_v\left(s, \pi'_v\right) = \sum_j \int_{\Omega(\mathbb{K}_v)} \xi_{v,j}(g) \Gamma'_{j*}(g) |\det'_v(g)|^{s-nd-\frac{1}{2}} \,\mathrm{d}_v g =$$

$$= \sum_j \int_{\mathrm{GL}(\mathbb{K}_v)} \xi_{v,j}(g) \Gamma'_{j*}(g) |\det'_v(g)|^{s-nd-\frac{1}{2}} \,\mathrm{d}_v g,$$

for a suitable $\mathrm{GL}(\mathbb{K}_v)$-invariant measure on $\Omega(\mathbb{K}_v)$. Extending Γ'_{j*} to a Schwartz-function of $M'_{2n}(\mathbb{K}_v)$ proves the first claim of Theorem 3.7.6 by Lemma 3.7.8.

Proof of Lemma 3.7.9. It is easy to check that φ_* is well defined and that the first two properties are satisfied, so it remains to check the third. Let D be a differential operator of order 1 on Ω, which is independent of (a,b). Since r is submersive, there exists a pullback differential operator D^* on $\Omega_0(\mathbb{K}_v) \times K'_v$ such that $(D^*\phi)_* = D\phi_*$, hence, it is enough to show that D^* leaves $\mathscr{S}(\Omega_0 \times K')$ invariant. Assume that D is an operator in (a,b), hence, without loss of generality it acts on a function φ' by

$$\frac{d}{dt}\varphi'\left(\begin{pmatrix} a+tX & b+tX \\ c & d \end{pmatrix}\right)\bigg|_{t=0}$$

at a matrix X. Then we can choose D^* such that it acts on a function φ by

$$\frac{d}{dt}\varphi\left(\begin{pmatrix} x+tXk^{-1} & y+tYk^{-1} \\ 0 & m \end{pmatrix};k\right)\Bigg|_{t=0},$$

which is a differential operator in the variables x, y, whose coefficients depend only on k. Therefore, the obtained function stays in $\mathscr{S}\left(\Omega_0 \times K'_v\right)$.

The second possibility is that D is a differential operator on Ω_1. Since any such operator is the linear combination of operators defined by invariant vector fields on $\mathrm{GL}'_n(\mathbb{K}_v)$ and K'_v. First assume that D acts on φ' by

$$\frac{d}{dt}\varphi'\begin{pmatrix} a & b \\ \exp(tX)\,c & \exp(tX)\,d \end{pmatrix}\Bigg|_{t=0}$$

where X is an element of the Lie algebra of $\mathrm{GL}'_n(\mathbb{K}_v)$. Then we can choose again D^* such that it acts on φ by

$$\frac{d}{dt}\varphi\left(\begin{pmatrix} x & y \\ 0 & \exp(tX)\,m \end{pmatrix};k\right)\Bigg|_{t=0},$$

which clearly leaves $\mathscr{S}\left(\Omega_0 \times K'\right)$ invariant.

Finally, for an element $Y \in \mathfrak{k}'_v$, the value of D on φ' is the difference, of the two operators D_1 and D_2 applied to φ', given by

$$\frac{d}{dt}\varphi'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\exp(tY)\right)\Bigg|_{t=0}$$

and

$$\frac{d}{dt}\varphi'\begin{pmatrix} a\exp(tY) & b\exp(tY) \\ c & d \end{pmatrix}\Bigg|_{t=0}.$$

We can then choose D_1^* to act as

$$\frac{d}{dt}\phi\left(p, k\exp(tY)\right)\big|_{t=0},$$

which preserves $\mathscr{S}\left(\Omega_0 \times K'\right)$. By the first case we considered, D_2* does so too, since it is a differential operator in (a, b) with polynomial coefficients. □

The last step in the proof of Theorem 3.7.6 concerns the special case of π'_v and ψ_v being unramified. Thus, assume π'_v is unramified and let $\xi_{v,0}$ be the corresponding vector in the Shalika model, *i.e.* the one fixed by $\mathrm{GL}_{2d_vn}\left(\mathcal{O}'_v\right)$. Then, using [44, Lemma 6.10], we know that

$$L\left(s, \pi'_v\right) = \int_{\mathrm{GL}_{2d_vn}(\mathcal{O}'_v)} f(g)\left|\det'_v(g)\right|^{s-nd-\frac{1}{2}} \mathrm{d}_v g,$$

where f is a spherical function attached to π'_v, *i.e.* the matrix coefficient of π'_v is of the form $g \mapsto v_0^\vee(\pi'_v(g)v_0)$, where v_0 and $v_0^\vee$ are non-zero vectors of π'_v and $\pi_v'^\vee$ fixed by by the maximal open compact subgroup $\mathrm{GL}_{2d_vn}\left(\mathcal{O}'_v\right)$. Let Ψ_v be the characteristic function of $\mathrm{GL}_{2d_vn}\left(\mathcal{O}'_v\right)$. Then following the proof of Lemma 3.7.8 shows that

$$L\left(s, \pi'_v\right) = \int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)} \xi_{v,0}(g)\,\Psi_v(g)\left|\det'_v(g)\right|^{s-nd-\frac{1}{2}} \mathrm{d}_v g$$

Recall, that $\zeta\left(s,\xi_{v,0}\right)$ can by (21) also be written as

$$\zeta_v\left(s,\xi_{v,0}\right)=$$

$$=\int_{H'_n(\mathbb{K}_v)\times U'_{(n,n)}(\mathbb{K}_v)\times K'_v}\xi_{v,0}\left(\begin{pmatrix}a & 0\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}b^{-1} & 0\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}1_n & u\\ 0 & 1_n\end{pmatrix}\begin{pmatrix}1_n & 0\\ 0 & b\end{pmatrix}k\right)$$

$$\Xi\left(u,b,b^{-1};k\right)\left|\det'_v\left(a\right)\right|^{s-\frac{1}{2}}\left|\det'_v\left(b\right)\right|^{nd}\mathrm{d}_v a\,\mathrm{d}_v b\,\mathrm{d}_v u\,\mathrm{d}_v k, \tag{24}$$

where we plugged in the definition of μ_{Ψ_v} and Ξ is defined by (20). It is easy to see that Ξ vanishes unless u, b and b^{-1} have entries in $\mathcal{O}'_v$, since the conductor of ψ_v is $\mathcal{O}_v$. Therefore, (24) equals to

$$\int_{\mathrm{GL}'_{2n}(\mathbb{K}_v)}\xi_{v,0}\left(g\right)\Psi_v\left(g\right)\left|\det'_v\left(g\right)\right|^{s-nd-\frac{1}{2}}\mathrm{d}_v g,$$

which proves the claim. $\square$

Chapter 4

Modular representations of the general linear group

Again we will start by giving an outline of what awaits us in this chapter and sketch the main ideas and techniques used to achieve our results. We use the conventions of Section 2.3.3. This chapter is based on [35].

4.1 Summary

Following [83], we call an irreducible representation π of G_n $\square$- *irreducible* if $\pi \times \pi$ is irreducible. We study these representations in Section 4.2.2 using the theory of intertwining operators introduced in [28], which we recall in Section 4.2.1. The main proposition of this section is then the following.

Proposition 4.1.1 (*cf.* [83], [66], [67])**.** *Let $\pi \in \mathfrak{Irr}$. The following are equivalent.*

1. *π is $\square$-irreducible.*
2. *For all $\sigma \in \mathfrak{Irr}$, $\mathrm{soc}(\pi \times \sigma)$ is irreducible and appears with multiplicity 1 in $\pi \times \sigma$.*
3. *For all $\sigma \in \mathfrak{Irr}$, $\mathrm{soc}(\sigma \times \pi)$ is irreducible and appears with multiplicity 1 in $\sigma \times \pi$.*

If any of the above statements hold true, the maps

$$\sigma \mapsto \mathrm{soc}(\sigma \times \pi), \sigma \mapsto \mathrm{soc}(\pi \times \sigma)$$

are injective.

Note that over $\overline{\mathbb{Q}}_\ell$, this was already proven in [83]. We adapt their argument, which is originally due to [66], [67]. The proposition allows us to give the following definition. Let $\pi' \in \mathfrak{Irr}$ and π as in the proposition. If $\pi' \cong \mathrm{soc}(\sigma \times \pi)$ respectively $\pi' \cong \mathrm{soc}(\pi \times \sigma)$, we define the right-$\pi$-derivative $\mathcal{D}_{r,\pi}(\pi') := \sigma$ respectively the left-π-derivative $\mathcal{D}_{l,\pi}(\pi') := \sigma$. If π' does not lie in the image of these maps, we set the derivatives to 0.

One of the main differences between the $\overline{\mathbb{F}}_\ell$-case and the complex case is that not all cuspidal representations are $\square$-irreducible. Recall that if ρ is a cuspidal representation, then ρ is $\square$-irreducible if and only if $o(\rho) > 1$. The behavior of $\square$-irreducible and $\square$-reducible cuspidal representations differs quite drastically. For example, if ρ is a $\square$-reducible cuspidal representation, its L-factor will always be trivial, and $\rho \times \rho$ it is of length 2 and semisimple if $\ell > 2$ and of length 3 if $\ell = 2$. We say an irreducible representation has $\square$-irreducible cuspidal support if its cuspidal support consists of $\square$-irreducible cuspidal representations.

In Section 4.3 we study the specific situation where ρ is a $\square$-irreducible cuspidal representation. The notion of ρ-derivatives was first introduced independently in [93] and [64] and further developed in [81] in the case G_n and $R = \mathbb{C}$. Moreover, the theory was extended to classical groups in [9]. Here we extend the notion of derivatives to the case $R = \overline{\mathbb{F}}_\ell$ and observe that most of the results carry over.

Corollary 4.1.1. *If $\mathcal{D}_{r,\rho}(\pi) \cong \mathcal{D}_{r,\rho}(\pi') \neq 0$ or $\mathcal{D}_{l,\rho}(\pi) \cong \mathcal{D}_{l,\rho}(\pi') \neq 0$, then $\pi \cong \pi'$. Furthermore,*

$$\mathcal{D}_{r,\rho}(\pi)^\vee \cong \mathcal{D}_{l,\rho^\vee}(\pi^\vee).$$

For ρ a $\square$-irreducible cuspidal representation, we moreover define four maps

$$\mathrm{soc}(\rho, \cdot\,), \mathrm{soc}(\,\cdot\,, \rho), \mathcal{D}_{r,\rho}, \mathcal{D}_{l,\rho} \colon \mathcal{M}_{ap}(\rho) \to \mathcal{M}_{ap}(\rho)$$

using a combinatorial description such that

$$\mathcal{D}_{l,\rho}(\mathrm{soc}(\rho, \mathfrak{m})) = \mathfrak{m} = \mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m}, \rho)).$$

We then can compute the ρ-derivatives of the corresponding representation $\mathrm{Z}(\mathfrak{m})$ simply as follows.

Proposition 4.1.2. *Let $\mathfrak{m}$ be an aperiodic multisegment and ρ a $\square$-irreducible cuspidal representation.*

$$\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho}(\mathfrak{m})), \mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{l,\rho}(\mathfrak{m})).$$

As a corollary we obtain that $\mathrm{Z}(\mathrm{soc}(\rho, \mathfrak{m})) = \mathrm{soc}(\rho \times \mathrm{Z}(\mathfrak{m}))$. In particular, this is used to give a new proof of the following theorem.

Theorem 4.1.3. *Let π be an irreducible smooth representation of G_n whose cuspidal support is $\square$-irreducible cuspidal. Then there exists a multisegment $\mathfrak{m}$ such that $\mathrm{Z}(\mathfrak{m}) \cong \pi$.*

Remark. There are two main reasons why such a new proof could be of interest. The first is that the original proof is quite involved and relies heavily on the techniques from geometric representation theory. Secondly, the existing proof does not give much information on the inverse Z^{-1} of Z, which is something that we will need in the computations of the L-factors. Let us now expand on both of these points. We start by recalling the existing proof of the surjectivity of Z in [101]. One first reduces the claim to the situation where all representations and multisegments involved have cuspidal support $\mathfrak{s}$ contained in the cuspidal line spanned by ρ. Let $\mathcal{M}(\mathfrak{s})$ be the set of aperiodic multisegments with cuspidal support $\mathfrak{s}$ and $\mathfrak{Irr}(\mathfrak{s})$ the set of isomorphism classes of irreducible representations with cuspidal support $\mathfrak{s}$. It is then possible to define a central character $\chi_\mathfrak{s}$ of the affine Hecke-algebra $\mathscr{H}_R(n, q(\rho))$ with the property that there exists a bijection between $\mathfrak{Irr}(\mathfrak{s})$ and

$$\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s}),$$

the set of isomorphism classes modules of $\mathscr{H}_R(n, q(\rho))$ with central character $\chi_\mathfrak{s}$. The proof of the classification of simple modules of the Hecke algebra has two very different flavors, depending on whether $o(\rho) > 1$ or $o(\rho) = 1$. If $o(\rho) = 1$, the central character of $\mathscr{H}_R(n, 1)$ is always trivial and hence the representation theory of the Hecke-algebra reduces in this case to the representation theory of the group algebra $R[\mathrm{S}_n]$ or said differently, to the modular representation theory of the symmetric group. Although some questions on modular representations of the symmetric groups remain to this day unsolved, the classification of irreducible representations falls not among them. It is a classical result that the irreducible representations are classified by ℓ-regular compositions and there is an explicit construction of these, see Section 2.1.1. Thus one has an explicit description of the inverse map Z^{-1} in this case.

On the other hand, if we assume that $o(\rho) > 1$, the situation is much more involved. Here one uses [2, Theorem B] to show that $\mathcal{M}(\mathfrak{s})$ and $\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s})$ have the same cardinality and hence obtains as a consequence that the irreducible representations are classified by aperiodic multisegments. In the proof of Theorem B of [2] the author first uses the geometric representation theory of [25] to show that in the case $R = \overline{\mathbb{F}}_\ell$, $\mathfrak{Irr}(\mathscr{H}_R(n, q(\rho)), \mathfrak{s})$ is parametrized by simple modules with a particular central character $\tilde{\chi}_{\mathfrak{s}}$ of an affine Hecke-algebra $\mathscr{H}_{\mathbb{C}}(n, \xi)$ over $\mathbb{C}$ at a root of unity ξ. Using geometric representation theory again, the author then argues that those are classified by the aperiodic multisegments. Thus we proved that $\mathfrak{Irr}(\mathfrak{s})$ is finite and of the same cardinality as $\mathcal{M}(\mathfrak{s})$. Since the map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ induces an injective map between these two sets, it has to be bijective. This gives unfortunately not the kind of information on the inverse map Z^{-1} we need later on. However, since a cuspidal representation is $\square$-irreducible if and only if $o(\rho) > 1$, this second, harder, case is covered by our new surjectivity proof, and thus gives us a more representation-theoretic description of Z^{-1}.

Let us now return to the remaining contents of this chapter. We continue by discussing how the Aubert-Zelevinsky dual of an irreducible representation interacts with taking derivatives. Recall that the Aubert-Zelevinsky dual, see for example [132], [10], [98], [96], is a map

$$(-)^*\colon \mathfrak{Irr} \to \mathfrak{Irr},\ \pi \mapsto \pi^*$$

preserving the cuspidal support. For $\mathfrak{m}$ an aperiodic multisegment, we write $\langle \mathfrak{m} \rangle \coloneqq \mathrm{Z}(\mathfrak{m})^*$. Using derivatives we then compute in Section 4.4.3 the Godement-Jacquet local L-factors mod ℓ introduced in [94].

Theorem 4.1.4. *Fix an additive character ψ_v of $\mathbb{K}_v$ and let $\Delta = [a, b]_\rho$ be a segment over R. If $\rho \cong \chi$ for an unramified character χ of $\mathbb{K}_v$ and $q^d \neq 1$, then*

$$L(\langle \Delta \rangle, T) = \frac{1}{1 - \chi(\varpi_v) q_v^{-d_v b + \frac{1-d}{2}} T}.$$

Otherwise

$$L(\langle \Delta \rangle, T) = 1.$$

More generally, if $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ is an aperiodic multisegment then

$$L(\langle \mathfrak{m} \rangle, T) = \prod_{i=1}^{k} L(\langle \Delta_i \rangle, T),\ \epsilon(T, \langle \mathfrak{m} \rangle, \psi_v) = \prod_{i=1}^{k} \epsilon(T, \langle \Delta_i \rangle, \psi_v)$$

and

$$\gamma(T, \langle \mathfrak{m} \rangle, \psi_v) = \prod_{i=1}^{k} \gamma(T, \langle \Delta_i \rangle, \psi_v).$$

In [74] a map

$$\mathrm{C}\colon \mathfrak{Irr}_n \xrightarrow{\cong} \{\mathrm{C}-\text{parameters of length } n\}$$

was introduced, where the right-hand-side consists of certain semi-simple R-Deligne representations. For a precise definition see Section 4.4.3. In [74] to each such C-parameter (Φ, U) an L-factor $L((\Phi,U),T)$, an ϵ-factor $\epsilon(T,(\Phi,U),\psi_v)$ and a γ-factor $\gamma(T,(\Phi,U),\psi_v)$ is then associated.

Corollary 4.1.2. *The map* C *respects the local factors,* i.e. *for* $\pi \in \mathfrak{Irr}$

$$L(\pi,T) = L(\mathrm{C}(\pi),T),\ \epsilon(T,\pi,\psi_v) = \epsilon(T,\mathrm{C}(\pi),\psi_v),$$

$$\gamma(T,\pi,\psi_v) = \gamma(T,\mathrm{C}(\pi),\psi_v).$$

In Section 4.5 we describe the structure of the representations of the form $\mathrm{I}(\mathfrak{m})$, which is heavily motivated by the analysis done in [132] in the case $R = \overline{\mathbb{Q}}_\ell$. Fix a cuspidal representation ρ and a cuspidal support $\mathfrak{s}$ of the form

$$\mathfrak{s} = d_0[\rho] + \ldots + d_{o(\rho)-1}[\rho v_\rho^{o(\rho)-1}],$$

where d_i be the multiplicity of ρv_ρ^i, $i \in \{0,\ldots,o(\rho)-1\}$ in $\mathfrak{s}$. Let $Q = A^+_{o(\rho)-1}$ be the affine Dynkin quiver with $o(\rho)$ vertices numbered $\{0,\ldots,o(\rho)-1\}$ and an arrow from i to j if $j = i+1 \mod o(\rho)$. Hence $\mathfrak{s}$ gives rise to a graded vector space

$$V(\mathfrak{s}) \coloneqq \bigoplus_{i=1}^{o(\rho)-1} \mathbb{C}^{d_i}.$$

We let $N(V(\mathfrak{s}))$ be the $\mathbb{C}$-variety of nilpotent representations of Q with underlying vector space $V(\mathfrak{s})$. Then $N(V(\mathfrak{s}))$ admits an action by

$$\mathrm{GL}_{d_0}(\mathbb{C}) \times \ldots \times \mathrm{GL}_{d_{o(\rho)-1}}(\mathbb{C}),$$

whose orbits are in bijection with irreducible representations $\mathrm{Z}(\mathfrak{m})$ whose cuspidal support is $\mathfrak{s}$. We write $X_\mathfrak{m}$ for the orbit corresponding to $\mathrm{Z}(\mathfrak{m})$ and write $\mathfrak{n} \le \mathfrak{m}$ if $X_\mathfrak{m} \subseteq \overline{X_\mathfrak{n}}$

Theorem 4.1.5. *Let* $\mathfrak{m},\mathfrak{n} \in \mathcal{M}(\rho)$ *not necessarily aperiodic. Then* $\mathrm{Z}(\mathfrak{n})$ *appears as a subquotient of* $\mathrm{I}(\mathfrak{m})$ *if and only if the cuspidal supports of* $\mathfrak{n}$ *and* $\mathfrak{m}$ *agree and* $\mathfrak{n} \le \mathfrak{m}$.

It is thus expected that the singularities of the space $N(V(\mathfrak{s}))$ allow one to obtain a deeper understanding of the irreducible representations with cuspidal support $\mathfrak{s}$. In this spirit, we extended [82, Conjecture 4.2] to representations over $\overline{\mathbb{F}}_\ell$.

4.2 Intertwining operators & square-irreducible representations

We will quickly discuss the intertwining operators of [28] and □-irreducible representations, *cf.* [82].

4.2.1 Intertwining operators We start with the intertwining operators in arbitrary characteristic introduced in [28, §7] and state some lemmas, which are probably already well-known to the experts. Fix for the rest of the section a parabolic subgroup $P = P_{(n_1,n_2)} = M \rtimes U$ with Levi-component $M = G_{n_1} \times G_{n_2}$ of $G = G_n$ and σ a smooth representation of M of finite length. We write $R(T)$ and denote by $\sigma_{R(T)} = \sigma \otimes_R R(T)$ the basechange of σ to $R(T)$. Moreover, we equip $R(T)$ with a valuation $v\colon R(T) \to \mathbb{R}$ which restricts to 0 on R and satisfies $v(T) < 0$. Finally, we denote by $\psi_{un} : M \to R(T)$ the generic unramified character sending

$$(m_1, m_2) \mapsto T^{-\log_{q_v}(|\det'(m_1)|_v)}.$$

We write $P' = P_{(n_2,n_1)} = M' \rtimes U'$ with Levi-component $M' = G_{n_2} \times G_{n_1}$ and we write $\overline{\sigma}$ for the representation obtained by twisting σ with the inner automorphism given by the element

$$w := \begin{pmatrix} 0 & 1_{n_1} \\ 1_{n_2} & 0 \end{pmatrix}. \tag{25}$$

For example, if $\sigma = \sigma_1 \otimes \sigma_2$, we have $\overline{\sigma_1 \otimes \sigma_2} = \sigma_2 \otimes \sigma_1$. We define the $R(T)$-representation

$$\sigma_{un} := \sigma_{R(T)} \otimes \psi_{un}.$$

The representation σ_{un} is admissible. Indeed, for any open compact subgroup $K \subseteq G$, we have $|\det'(K)|_v = 1$, and hence the character ψ_{un} does not play a role when considering K-fixed vectors. The claim follows then from [54, Lemma III.1(ii)]. By [28, Lemma 3.7]

$$\mathrm{Hom}_{G_n}(\mathrm{Ind}_P^G(\sigma_{un}), \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}))$$

is non-zero and contains a certain class of non-zero morphisms $J_P(\sigma)$'s in it, which are up to a choice of a Haar-measure uniquely determined by the following condition. If $f \in \mathrm{Ind}_P^G(\sigma_{un})$ is compactly supported in PwP', *cf.* (25), then

$$J_P(\sigma)(f)(1) = \int_{U'} f(wu'1)\,\mathrm{d}u'.$$

For $k \in \mathbb{Z}$ we denote by $\mathfrak{l}_k := \sigma \otimes_R (T-1)^k R[T]_{(T-1)} \otimes_{R[T]_{(T-1)}} \psi_{un}$.
Lemma 4.2.1. *Let $\sigma \in \mathfrak{Rep}_M$. Then there exists a morphism*

$$J_P(\sigma)\colon \mathrm{Ind}_P^G(\sigma_{un}) \to \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}),$$

which sends $\mathrm{Ind}_P^G(\mathfrak{l}_0)$ to $\mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0})$ and defines a non-zero morphism R_σ

$$\begin{tikzcd} \mathrm{Ind}_P^G(\mathfrak{l}_0) \arrow[r, "J_P(\sigma)"] \arrow[d, two heads, "T\mapsto 1"] & \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0}) \arrow[d, two heads, "T\mapsto 1"] \\ \mathrm{Ind}_P^G(\sigma) \arrow[r, "R_\sigma"] & \mathrm{Ind}_{P'}^G(\overline{\sigma}) \end{tikzcd}$$

Such R_σ is called an intertwining operator*.*

Proof. Recall that the Geometric Lemma gives inclusions

$$\begin{tikzcd} \overline{\sigma_{un}} \arrow[r, hook] & r_{P'}(\mathrm{Ind}_P^G(\sigma_{un})) \\ \overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[u, hook] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[u, hook] \end{tikzcd}$$

which are determined by a suitable choice of Haar-measures, *i.e.* up to a scalar in $R[T]_{(T-1)}$. We will use the notation of [28, §2] and note that by [28, Lemma 3.7] σ_{un} is $(\overline{P}, P)$-regular. Following [28, Lemma 2.10], we can therefore define a retraction r of the inclusion

$$\begin{tikzcd} \overline{\sigma_{un}} \arrow[r, hook] \arrow[rr, bend right, "1"'] & r_{P'}(\mathrm{Ind}_P^G(\sigma_{un})) \arrow[r, "r"] & \overline{\sigma_{un}}, \end{tikzcd}$$

Moreover, this retraction defines by Frobenius reciprocity the intertwiner $J_P(\sigma)$ up to scalar multiplication in $R(T)$.

The lattice $r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0))$ is an integral structure of $r_{P'}(\mathrm{Ind}_P^G(\sigma_{un}))$ in the sense of [123, I.9] by [28, Proposition 6.7]. The image of $r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0))$ under r is an $R[T]_{(T-1)}$-integral structure $\mathfrak{o}$ of $\overline{\sigma_{un}}$ by [123, I.9.3]. We can now fix the maximal $k \in \mathbb{Z}$ such that $\mathfrak{o} \subseteq \mathfrak{l}_k$. The existence of such k follows from [123, Remark p.80], in which it is shown that any two integral structures of an admissible representation are commensurable. We thus obtain the following diagram.

$$\begin{tikzcd} \overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[rr, bend right, "\iota"'] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[r, "r"] & \overline{\mathfrak{l}_k}, \end{tikzcd}$$

where $\iota\colon \mathfrak{l}_0 \hookrightarrow \mathfrak{l}_k$ is the natural inclusion and in particular $k \leq 0$. We now multiply r by $(T-1)^{-k}$ and hence the image of $r' := r(T-1)^{-k}$ is contained in $\overline{\mathfrak{l}_0}$ but not in $\overline{\mathfrak{l}_1}$. We thus have a diagram

$$\begin{tikzcd} \overline{\mathfrak{l}_0} \arrow[r, hook] \arrow[rr, bend right, "(T-1)^{-k}\cdot"'] & r_{P'}(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \arrow[r, "r'"] & \overline{\mathfrak{l}_0} \end{tikzcd}$$

and we fix $J_P(\sigma)$ to the intertwiner coming from Frobenius reciprocity applied to r'. Moreover, we set $r(\sigma) := -k \geq 0$, which is well defined. Indeed, the only possible choice we made is choosing the inclusion

$$\overline{\sigma_{un}} \hookrightarrow r_{P'}(\mathrm{Ind}_P^G(\sigma_{un}))$$

up to a scalar in $R[T]_{(T-1)}$, which does not affect k. Next observe that the sequence

$$0 \longrightarrow \mathrm{Ind}_P^G(\mathfrak{l}_1) \longrightarrow \mathrm{Ind}_P^G(\mathfrak{l}_0) \xrightarrow{T\mapsto 1} \mathrm{Ind}_P^G(\sigma) \longrightarrow 0$$

is exact since the representation in the sequence are admissible. We thus have the following commutative diagram.

$$\begin{tikzcd} \mathrm{Ind}_P^G(\sigma_{un}) \arrow[r, "J_P(\sigma)"] & \mathrm{Ind}_{P'}^G(\overline{\sigma}_{un}) \\ \mathrm{Ind}_P^G(\mathfrak{l}_0) \arrow[u, hook] \arrow[r, "J_P(\sigma)"] \arrow[d, two heads, "T\mapsto 1"] & \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_0}) \arrow[u, hook] \arrow[d, two heads, "T\mapsto 1"] \\ \mathrm{Ind}_P^G(\sigma) \arrow[r, "R_\sigma"] & \mathrm{Ind}_{P'}^G(\overline{\sigma}) \end{tikzcd}$$

Now if R_σ vanishes, $J_P(\sigma)(\mathrm{Ind}_P^G(\mathfrak{l}_0)) \subseteq \mathrm{Ind}_{P'}^G(\overline{\mathfrak{l}_1})$, however applying Frobenius reciprocity would give a contradiction to the construction of $J_P(\sigma)$. $\square$

Note that $J_P(\sigma)$ in the above lemma is unique up to a unit in $R[T]_{(T-1)}$ and R_σ is unique up to a unit in R. We call $r(\sigma)$ the order of the pole of the intertwining operator at σ. Note that $r(\sigma) = 0$ if and only if for all $f \in \operatorname{Ind}_P^G(\sigma)$ with support contained in PwP',

$$R_\sigma(f)(g) = \int_{U'} f(wu')\,\mathrm{d}u'.$$

If $\sigma = \pi_1 \otimes \pi_2$, we will also denote by

$$R_{\pi_1,\pi_2}\colon \pi_1 \times \pi_2 \to \pi_2 \times \pi_1$$

the so obtained morphism and r_{π_1,π_2} the order of vanishing. The next lemma is the generalization of the list of properties needed to deal with square irreducible representations and collected for $R = \mathbb{C}$ in [82, Lemma 2.3].

Lemma 4.2.2. *Let $0 \to \sigma_1 \hookrightarrow \sigma \twoheadrightarrow \sigma_2 \to 0$ be a short exact sequence of representations of M and π_1, π_2, π_3 representations of G_{k_i}, $i \in \{1,2,3\}$.*

1. *If $M = G$, R_σ is a scalar.*
2. *The order of the poles satisfies $r(\sigma) \geq r(\sigma_1)$ and*
$$J_P(\sigma)\big|_{\sigma_1} = (T-1)^{r(\sigma)-r(\sigma_1)} J_P(\sigma_1).$$
3. *R_σ restricts either to an intertwining operator or to 0,* i.e. *up to a scalar*
$$R_\sigma\big|_{\sigma_1} = \begin{cases} R_{\sigma_1} & \textit{if } r(\sigma) = r(\sigma_1), \\ 0 & \textit{otherwise.} \end{cases}$$
4. *If $\sigma = \pi_1 \times \pi_2 \otimes \pi_3$, $r_{\pi_1\times\pi_2,\pi_3} \leq r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$ and*
$$(T-1)^{r_{\pi_1,\pi_2}+r_{\pi_2,\pi_3}-r_{\pi_1\times\pi_2,\pi_3}} J_P(\sigma) = (J_P(\pi_1 \otimes \pi_3) \times 1_{\pi_2}) \circ (1_{\pi_1} \times J_P(\pi_2 \otimes \pi_3)).$$
Therefore,
$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) = \begin{cases} R_{\pi_1\times\pi_2,\pi_3} & \textit{if } r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}, \\ 0 & \textit{otherwise,} \end{cases}$$
up to a scalar.
5. *Moreover, $r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$ if at least one of the R_{π_i,π_3}, $i \in \{1,2\}$ is an isomorphism or if π_3 is irreducible.*
6. *If R_σ is an isomorphism, then $R_{\overline{\sigma}} \circ R_\sigma$ is a scalar.*

Proof. The first point is trivially true. Since the following diagram commutes up to a non-zero scalar in $R(T)$, (2) and (3) also follow immediately.

$$\begin{tikzcd}
\operatorname{Ind}_P^G(\sigma_{2un}) \arrow[r, "J_P(\sigma_2)"] & \operatorname{Ind}_{P'}^G(\overline{\sigma_2}_{un}) \\
\operatorname{Ind}_P^G(\sigma_{un}) \arrow[r, "J_P(\sigma)"] \arrow[u, two heads] & \operatorname{Ind}_{P'}^G(\overline{\sigma}_{un}) \arrow[u, two heads] \\
\operatorname{Ind}_P^G(\sigma_{1un}) \arrow[r, "J_P(\sigma_1)"] \arrow[u, hook] & \operatorname{Ind}_{P'}^G(\overline{\sigma_1}_{un}) \arrow[u, hook]
\end{tikzcd}$$

The first part of (4) follows from the following, up to multiplication by some non-zero element of $R(T)$, commutative diagram.

$$\begin{tikzcd}
\mathrm{Ind}_P^G((\pi_1 \times \pi_2 \otimes \pi_3)_{un}) \arrow[r, "J_P(\pi_1\times\pi_2\otimes\pi_3)"] \arrow[d, "1_{\pi_1}\times J_P(\pi_2\otimes\pi_3)"'] & \mathrm{Ind}_{P'}^G((\pi_3 \otimes \pi_1 \times \pi_2)_{un}) \\
\mathrm{Ind}_{P_{k_1,k_3,k_2}}^G((\pi_1 \otimes \pi_3 \otimes \pi_2)_{un}) \arrow[ur, bend right, "J_P(\pi_1\otimes\pi_3)\times 1_{\pi_2}"'] &
\end{tikzcd}$$

To see that it commutes, apply [28, Proposition 7.8(ii)] to the bottom two arrows and [28, Proposition 7.8(iii)] to the top arrow. Then the commutativity follows from [28, Proposition 7.8(i)]. For (6), [28, Proposition 7.8(i)] tells us that $J_P(\sigma) \circ J_{P'}(\overline{\sigma})$ is a scalar in $R(T)$. Thus, after multiplying by the suitable normalization factors and setting $T = 1$, the claim follows. Moreover, it also is easy to check that this diagram implies the claims regarding the poles.

It remains to prove (5) and here we argue as in [82, Theorem 2.3]. If one of the R_{π_i,π_3} is an isomorphism, the composition

$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) \neq 0$$

and hence $r_{\pi_1\times\pi_2,\pi_3} = r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$. If π_3 is irreducible and $r_{\pi_1\times\pi_2,\pi_3} < r_{\pi_1,\pi_2} + r_{\pi_2,\pi_3}$, then

$$(R_{\pi_1,\pi_3} \times 1_{\pi_2}) \circ (1_{\pi_1} \times R_{\pi_2,\pi_3}) = 0$$

and hence

$$\pi_1 \times \mathrm{Im}(R_{\pi_2\times\pi_3}) \hookrightarrow \mathrm{Ker}(R_{\pi_1,\pi_3}) \times \pi_2 \hookrightarrow \pi_1 \times \pi_3 \times \pi_2.$$

Twisting with the inner automorphism given by

$$w = \begin{pmatrix} 0 & 0 & 1_{k_2} \\ 0 & 1_{k_3} & 0 \\ 1_{k_1} & 0 & 0 \end{pmatrix},$$

we obtain that

$$\mathrm{Im}(R_{\pi_2\times\pi_3})^{w_1} \times \pi_1 \hookrightarrow \pi_2 \times \mathrm{Ker}(R_{\pi_1,\pi_3})^{w_2} \hookrightarrow \pi_2 \times \pi_3 \times \pi_1,$$

where $*^{w'}$ denotes the twist by the inner automorphism$g \mapsto w'gw'^{-1}$, and

$$w_1 = \begin{pmatrix} 0 & 1_{k_2} \\ 1_{k_3} & 0 \end{pmatrix}, \, w_2 = \begin{pmatrix} 0 & 1_{k_3} \\ 1_{k_1} & 0 \end{pmatrix}.$$

By Corollary 2.3.1, the irreducibility of π_3 implies that $\mathrm{Ker}(R_{\pi_1,\pi_3}) = \pi_3 \times \pi_1$, a contradiction. □

4.2.2 Square irreducible representations Using the above mod ℓ-analogous of the results of [82] , one can arrive via the same arguments to the following results. We say $\Pi \in \mathfrak{Rep}_n$ is *SI* if Π is socle-irreducible, and its socle appears with multiplicity one in its composition series. Similarly, we say Π is *CSI*, if is cosocle-irreducible, and its cosocle appears with multiplicity one in its composition series. A representation $\pi \in \mathfrak{Rep}_n$ is called square-irreducible or $\square$-irreducible if $\pi \times \pi \in \mathfrak{Irr}_{2n}$. The following three theorems follow *muta mutandis* as in [82], thanks to Lemma 4.2.2 and Corollary 2.3.1.

Lemma 4.2.3 ([82, Corollary 2.5], [67, Corollary 3.3]). *A representation of $\pi \in \mathfrak{Rep}_n$ is $\square$-irreducible if and only if one of the following equivalent conditions holds.*

1. $\pi \times \pi$ *is irreducible.*
2. $\pi \times \pi$ *is SI.*
3. $\dim_R \operatorname{End}_{G_{2n}}(\pi \times \pi) = 1$.
4. $R_{\pi,\pi}$ *is a scalar.*

Note that a cuspidal representation is square-irreducible if and only if it is $\square$-irreducible cuspidal and we denote the subset of square-irreducible representations of $\mathfrak{Irr}_n$ by $\mathfrak{Irr}_n^{\square}$. More generally, for $\Delta = [a,b]_\rho$ a segment, $\mathrm{Z}(\Delta)$ is $\square$-irreducible if and only if $b-a+1 < o(\rho)$.

Lemma 4.2.4 ([82, Lemma 2.8], [66, Theorem 3.1]). *Let $\pi \in \mathfrak{Irr}_n^{\square}$ and $\tau \in \mathfrak{Irr}_m$. Then both $\pi \times \tau$ and $\tau \times \pi$ are SI and CSI.*

Lemma 4.2.5 ([84, Theorem 4.1.D]). *Let $\pi \in \mathfrak{Irr}_n^{\square}$. The maps*

$$\operatorname{soc}(\pi \times \cdot\,) = \cos(\,\cdot \times \pi)\colon \mathfrak{Irr} \to \mathfrak{Irr},\ \operatorname{soc}(\,\cdot \times \pi) = \cos(\pi \times \cdot\,)\colon \mathfrak{Irr} \to \mathfrak{Irr}$$

are injective.

Lemma 4.2.6. *If $\pi \in \mathfrak{Irr}_n^{\square}$ then $\pi^k \in \mathfrak{Irr}_{nk}^{\square}$ for all $k \in \mathbb{Z}_{>0}$.*

Proof. By Lemma 4.2.3 it is enough to show that R_{π^k,π^k} is a scalar. We start by showing that $R_{\pi^k,\pi}$ is a scalar for all k, which we do by induction on k. By assumption, we know that $R_{\pi,\pi}$ is a scalar. By Lemma 4.2.2(4) and (5)

$$R_{\pi^k,\pi} = (R_{\pi^{k-1},\pi} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi})$$

is a scalar by the induction hypothesis on k. By Lemma 4.2.2(6) also R_{π,π^k} is a scalar. Next we prove by induction on k that R_{π^k,π^k} is a scalar. Again by Lemma 4.2.2(4) and (5), we know that

$$R_{\pi^k,\pi^{k-1}} = (R_{\pi^{k-1},\pi^{k-1}} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi^{k-1}})$$

is a scalar. By Lemma 4.2.2(6) also R_{π^{k-1},π^k} is a scalar. Applying a third time Lemma 4.2.2(4) and (5) we see that

$$R_{\pi^k,\pi^k} = (R_{\pi^{k-1},\pi^k} \times 1_\pi) \circ (1_{\pi^{k-1}} \times R_{\pi,\pi^k}),$$

which therefore is a scalar. □

Lemma 4.2.7. *Let ρ be a cuspidal representation such that $o(\rho) = 1$ and $\mathfrak{m} \in \mathcal{M}(\rho)_{ap}$. Then $\mathrm{Z}(\mathfrak{m})$ is not square-irreducible.*

Proof. Assume otherwise that $\mathrm{Z}(\mathfrak{m})$ is square-irreducible. Then by Lemma 4.2.6 for all $k \in \mathbb{Z}_{>0}$ we have that $\mathrm{Z}(\mathfrak{m})^k$ is square-irreducible and in particular irreducible. Hence $\mathrm{Z}(\mathfrak{m})^k \cong \mathrm{Z}(k\mathfrak{m})$ by Proposition 2.3.14. But if $k > \ell = e(\rho)$, the multisegment $k\mathfrak{m}$ is no longer aperiodic, and hence its cuspidal support is no longer contained in $\mathbb{N}(\mathbb{Z}[\rho])$. But on the other hand, the cuspidal support of $\mathrm{Z}(\mathfrak{m})^k$ must be contained in $\mathcal{M}(\rho)$, a contradiction. □

4.2.3 P-regular representations Motivated by the notion of (P,Q) -regularity introduced in [28, §2], we will introduce a similar condition called P-regularity. Note that being (P,Q)-regular is a slightly more fine-tuned notion, but for our purposes the following more blunt definition is sufficient.

Let P be a parabolic subgroup of $G = G_n$ with Levi-decomposition $P = M \rtimes U$ and opposite parabolic subgroup $\overline{P}$. A representation σ of M of finite length is called *P-regular* if the natural restriction map

$$\mathrm{Hom}_{G\times G}(S(G), \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma^\vee)) \to$$

$$\to \mathrm{Hom}_{\overline{P}\times P}(S(\overline{P}P), \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma^\vee))$$

is an isomorphism of R-vector spaces.

Lemma 4.2.8. *Let $\sigma \in \mathfrak{Rep}_M$.*

1. *If σ is irreducible and P-regular, then*

$$\dim_R \mathrm{Hom}_{G\times G}(S(G), \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma^\vee)) = 1.$$

2. *If σ is irreducible and it appears in $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^{G}(\sigma))$ with multiplicity 1, it is P-regular.*

Proof. For the first claim, note that by Frobenius reciprocity, we have

$$\dim_R \mathrm{Hom}_{\overline{P}\times P}(S(\overline{P}P), \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma^\vee)) =$$

$$= \dim_R \mathrm{Hom}_M(r_{\overline{P}\times P}(S(\overline{P}P)), \sigma \otimes \sigma^\vee) =$$

$$= \dim_R \mathrm{Hom}_M(S(M), \sigma \otimes \sigma^\vee) = \dim_R \mathrm{Hom}_M(\sigma, \sigma) = 1.$$

For the second claim, note that by the first part and the existence of intertwining operators, it follows that it is enough to show that a non-zero map $f\colon S(G) \mapsto \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \mathrm{Ind}_{P}^{G}(\sigma^\vee)$ restricts to a non-zero map on $S(\overline{P}P)$. We now apply Frobenius reciprocity with respect to r_P to obtain a non-zero map

$$f'\colon r_{\{1\}\times P}(S(G))) \cong \mathrm{Ind}_{P\times M}^{G\times M}(S(M)) \to \mathrm{Ind}_{\overline{P}}^{G}(\sigma) \otimes \sigma^\vee$$

and Frobenius reciprocity with respect to $r_{\overline{P}}$ to obtain a non-zero map

$$f''\colon r_{\overline{P}\times\{1\}}(\mathrm{Ind}_{P\times M}^{G\times M}(S(M))) \to \sigma \otimes \sigma^\vee.$$

After applying the Geometric Lemma to the left side, we will show that f'' restricts to a non-zero map on

$$F(1)(S(M)) = S(M) \hookrightarrow r_{\overline{P}\times\{1\}}(\mathrm{Ind}_{P\times M}^{G\times M}(S(M))) \to \sigma \otimes \sigma^\vee.$$

Indeed, let $w_i \in G$ be a permutation matrix and assume that $Pw_i\overline{P}$ is the smallest element in $P\backslash G/\overline{P}$ such that f' does not vanish on $F(w_i)(S(M))$, *i.e.* we obtain a non-zero map

$$f''\colon F(w_i)(S(M)) \to \sigma \otimes \sigma^\vee.$$

Using the specific form of $F(w_i)$, *cf.* [17, Geometric Lemma], and applying first Bernstein reciprocity and then Frobenius reciprocity we obtain a non-zero morphism

$$S(M) \to F(w_i^{-1})(\sigma) \otimes \sigma^\vee.$$

where here $F(w_i^{-1})(\sigma)$ is now the subquotient coming from the Geometric Lemma applied to $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$. This in turn gives a non-zero morphism

$$\sigma \to F(w_i^{-1})(\sigma).$$

By assumption σ appears only with multiplicity 1 in $r_{\overline{P}}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$ and $F(1)(\sigma) \cong \sigma$. Thus $w_i^{-1} \in \overline{P}$ and hence $Pw_i\overline{P} = P\overline{P}$. We therefore showed that f'' does not vanish on $F(1)(S(M))$. But now $F(1)(S(M)) = r_{\overline{P}\times P}(S(\overline{P}P))$ and thus by Frobenius reciprocity f does not vanish on on $S(\overline{P}P)$. □

Assume now $P = P_{(n_1,n_2)}$, $\sigma = \sigma_1 \otimes \sigma_2$ is P-regular, irreducible and either σ_1 or σ_2 is □-irreducible. In this case we call σ *strongly* P-regular. By Lemma 4.2.4 $\pi := \cos(\mathrm{Ind}_P^G(\sigma)) \cong \mathrm{soc}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$ is irreducible. Moreover, by Lemma 4.2.8(1), the only morphism up to a scalar in

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_{\overline{P}}^G(\sigma))$$

is given by

$$\mathrm{Ind}_P^G(\sigma) \twoheadrightarrow \pi \hookrightarrow \mathrm{Ind}_{\overline{P}}^G(\sigma),$$

since

$$\dim_R \mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_{\overline{P}}^G(\sigma)) =$$

$$= \dim_R \mathrm{Hom}_{G\times G}(S(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)) = 1.$$

For the next corollary, we identify for every admissible representation σ of M

$$\mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee) \cong \mathrm{Ind}_{\overline{P}}^G(\sigma^\vee)^\vee \otimes \mathrm{Ind}_P^G(\sigma)^\vee$$

Corollary 4.2.1. *Let $\sigma \in \mathfrak{Irr}_M$ be a strongly P-regular representation and define* $\pi = \mathrm{soc}(\mathrm{Ind}_{\overline{P}}^G(\sigma))$. *Let*

$$T \in \mathrm{Hom}_{\overline{P}\times P}(S(\overline{P}P), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee)),$$

written as

$$\phi \mapsto (f_1 \otimes f_2 \mapsto T(\phi)(f_1 \otimes f_2)),\ f_1 \otimes f_2 \in \mathrm{Ind}_{\overline{P}}^G(\sigma^\vee) \otimes \mathrm{Ind}_P^G(\sigma).$$

Then for all $f_1 \otimes f_2 \in \mathrm{Ind}_{\overline{P}}^G(\sigma^\vee) \otimes \mathrm{Ind}_P^G(\sigma)$ there exists a matrix coefficient f of π such that for all $\phi \in S(\overline{P}P)$

$$T(\phi)(f_1 \otimes f_2) = \int_{\overline{P}P} \phi(p) f(p)\,\mathrm{d}p.$$

Proof. Since σ is irreducible, Lemma 4.2.8(1) tells us that the natural inclusion

$$\mathrm{Hom}_{G\times G}(S(G), \pi \otimes \pi^\vee) \hookrightarrow \mathrm{Hom}_{G\times G}(S(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma)^\vee)$$

is an isomorphism. Note that every morphism in the former space is given by sending a matrix coefficient f and a Schwartz-function $\phi \in S(G)$ to

$$\int_G f(g)\phi(g)\,\mathrm{d}g.$$

Now by P-regularity every map in

$$\mathrm{Hom}_{\overline{P}\times P}(S(\overline{P}P), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma^\vee))$$

is just the restriction of a map in

$$\mathrm{Hom}_{G\times G}(S(G), \mathrm{Ind}_{\overline{P}}^G(\sigma) \otimes \mathrm{Ind}_P^G(\sigma)^\vee),$$

hence it has to be of the form

$$\phi \in S(\overline{P}P) \mapsto \int_{\overline{P}P} \phi(p)f(p)\,\mathrm{d}p.$$

□

Next, we see that strongly P-regular representation will interact nicely with L-factors. Recall that for π an irreducible representation, we had

$$P(\pi,T) = \frac{1}{L(\pi,T)}.$$

Proposition 4.2.9. *Let $P = P_{(n_1,n_2)} = M \rtimes U$ be a parabolic subgroup of G_n and $\sigma = \sigma_1 \otimes \sigma_2$ a strongly P-regular representation. Set $\pi = \cos(\mathrm{Ind}_P^G(\sigma))$. Then $P(\sigma_2,T)$ divides $P(\pi,T)$.*

Before we come to the proof, let us observe some parallels between this statement and a special case of it, namely [59, Theorem 2.7]. In there Jacquet uses the following fact, which only holds true over $\overline{\mathbb{Q}}_\ell$ in full generality. For a multisegment $\mathfrak{m}$ over $\overline{\mathbb{Q}}_\ell$, let $\langle\mathfrak{m}\rangle$ be the representation associated to $\mathfrak{m}$ via the Langlands classification. If $\rho\nu_\rho^b$ a cuspidal representation in the cuspidal support of $\mathfrak{m}$ with b maximal, let $\Delta_1 + \ldots + \Delta_k$ be the sub-multisegment of $\mathfrak{m}$ consisting of all segments of the form $[a,b]_\rho$. Then

$$\bigtimes_{i=1}^k \langle\Delta_i\rangle$$

is square-irreducible and

$$\sigma = \bigtimes_{i=1}^k \langle\Delta_i\rangle \otimes \langle\mathfrak{m} - \sum_{i=1}^k \Delta_i\rangle$$

is strongly-P regular with

$$\cos(\mathrm{Ind}_P^{G_n}(\sigma)) = \langle\mathfrak{m}\rangle.$$

Thus Proposition 4.2.9 can be seen as a generalization of this fact over $\overline{\mathbb{Q}}_\ell$ and as we will see, also its proof follows essentially the same idea.

Before we start with the proof, we define the following. Let $H\colon G_n \times G_n \to R$ be a smooth function such that

$$H(u_1mg_1, u_2mg_2) = H(g_1,g_2)$$

for all $u_1 \in \overline{U}$, $u_2 \in U$, $m \in M, g_1, g_2 \in G_n$ and

$$m \mapsto H(g_1, mg_2)$$

is a matrix coefficient of $\sigma \otimes \delta_P^{\frac{1}{2}}$. Then for $\phi \in S(\overline{P}P)$, define the integral

$$T_H(\phi) := \int\limits_{U \times M \times \overline{U}} H(1,m)\phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}m.$$

Lemma 4.2.10. *The integrals $T_H(\phi)$ are well defined for $\phi \in S(\overline{P}P)$ and define a non-zero $\overline{P} \times P$-equivariant morphism*

$$S(\overline{P}P) \to \mathrm{Ind}_{\overline{P}}^{G_n}(\sigma) \otimes \mathrm{Ind}_P^{G_n}(\sigma^\vee).$$

Proof. By the assumption on ϕ the integral converges. Note that above H's are precisely linear combinations of functions of the form

$$H(g_1, g_2) = f^\vee(g_1)(f(g_2))$$

for $f^\vee \in \mathrm{Ind}_{\overline{P}}^{G_n}(\sigma^\vee)$, $f \in \mathrm{Ind}_P^{G_n}(\sigma)$. □

Proof of Proposition 4.2.9. Having introduced the above machinery, we can now mimic the proof of [59, Theorem 2.7]. We recall

$$\delta_P(m_1, m_2) = |\mathrm{det}'(m_1)|_v^{dn_2}|\mathrm{det}'(m_2)|_v^{-dn_1}.$$

Note that the claim of the proposition is equivalent to

$$\mathcal{L}(\sigma_2) \subseteq \mathcal{L}(\pi),$$

cf. Corollary 2.3.2. To show this, we need to find for each $\phi_2 \in S_R(M'_{n_2,n_2}(\mathbb{K}_v))$ and matrix coefficient f_2 of σ_2, a $\phi \in S_R(M'_{n,n}(\mathbb{K}_v))$ and a matrix coefficient f of π such that

$$Z(\phi, Tq^{-\frac{d_v n-1}{2}}, f) = Z(\phi_2, Tq^{-\frac{d_v n_2 -1}{2}}, f_2).$$

Choose now a coefficient f_1 of σ_1 and $\phi_1 \in S_R(M'_{n_1,n_1}(\mathbb{K}_v))$ supported in a suitable neighbourhood of 1_n such that $Z(\phi_1, Tq^{-\frac{d_v n_1 -1}{2}}, f_1) = 1$ and choose

$$\phi_{1,2} \in S(\overline{U}),\ \phi_{2,1} \in S(U)$$

which are supported in a suitable neighbourhood of 1_n such that

$$\phi(\overline{u}um) = \phi_{1,2}(\overline{u})\phi_{2,1}(u)\phi_1(m_1)\phi_2(m_2),\ m = (m_1, m_2)$$

satisfies

$$\int\limits_{\overline{U} \times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u = \phi_1(m_1)\phi_2(m_2).$$

Note that in particular for all $N \in \mathbb{Z}$,

$$\phi\chi_{G_n(N)} \in S(\overline{P}P),$$

where $\chi_{G_n(N)}$ is the characteristic function of $G_n(N)$ and ϕ can be extended by 0 to a function on $S_R(M_{n,n}(\mathbb{D}_v))$. Moreover, the coefficient of T^N in

$$Z(\phi_2, Tq^{-\frac{d_v n_2 - 1}{2}}, f_2) = Z(\phi_1, Tq^{-\frac{d_v n_1-1}{2}}, f_1) Z(\phi_2, Tq^{-\frac{d_v n_2-1}{2}}, f_2)$$

is

$$\int_{M(N)} |\det{}'(m_1)|_v^{\frac{d_v n_1-1}{2}} |\det{}'(m_2)|_v^{\frac{d_v n_2-1}{2}} f_1(m_1) f_2(m_2)$$
$$\int_{\overline{U}\times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}(m_1,m_2),$$

which can be rewritten as

$$= \int_{M(N)} |\det{}'(m)|_v^{\frac{d_v n-1}{2}} \delta_P(m)^{\frac{1}{2}} f_1(m_1) f_2(m_2) \int_{\overline{U}\times U} \phi(\overline{u}^{-1}um)\,\mathrm{d}\overline{u}\,\mathrm{d}u\,\mathrm{d}m. \tag{26}$$

Note that here we used that $\phi_1(m_1)$ is supported in a sufficiently small neighborhood of 1 and hence in the support of ϕ_1, we have $|\det'(m_1)|_v = 1$. We can now find H as in Lemma 4.2.10 such that $H(1,m) = f(m_1)f(m_2)\delta_P(m_1,m_2)^{\frac{1}{2}}$ and hence we find by Corollary 4.2.1 a matrix coefficient f of π, which is independent of N, such that above integral (26) equals

$$\int_{G_n} |\det{}'(g)|_v^{\frac{d_v n-1}{2}} f(g)(\phi\chi_{G_n(N)})(g)\,\mathrm{d}ug = \int_{G_n(N)} |\det{}'(g)|_v^{\frac{d_v n-1}{2}} f(g)\phi(g)\,\mathrm{d}g.$$

This in turn is nothing else than the coefficient of T^N in

$$Z(\phi, Tq^{-\frac{d_v n-1}{2}}, f).$$

Thus $\mathcal{L}(\sigma_2) \subseteq \mathcal{L}(\pi)$. $\square$

4.3 Derivatives

In this section we define derivatives with respect to a $\square$-irreducible cuspidal representation, following the ideas of [93] and [64], who establish all results in this section in the case $R = \overline{\mathbb{Q}}_\ell$. Derivatives will also give rise to most of the P-regular representations we will encounter throughout the rest of the chapter.

4.3.1 Basic properties of derivatives Let $\rho, \pi \in \mathfrak{Irr}$. We call a representation $\pi' \in \mathfrak{Irr}$ a *right-ρ-derivative* respectively a *left-ρ-derivative* of π if

$$\pi \hookrightarrow \pi' \times \rho \text{ respectively } \pi \hookrightarrow \rho \times \pi'.$$

As a consequence of Lemma 4.2.4 and Lemma 4.2.5 we obtain the following lemma and its corollary.

Lemma 4.3.1. *Let $\rho \in \mathfrak{Irr}^\square$ and $\pi \in \mathfrak{Irr}$. The left- and right-ρ-derivative of π are up to isomorphism unique. We denote them by $\mathcal{D}_{l,\rho}(\pi)$ and $\mathcal{D}_{r,\rho}(\pi)$ respectively and set them to* 0 *if no such representation exists.*

Moreover, if $\mathcal{D}_{r,\rho}(\pi) \neq 0$, then $\pi = \mathrm{soc}(\mathcal{D}_{r,\rho}(\pi) \times \rho)$ and it appears in its decomposition series with multiplicity 1*. Analogously, if $\mathcal{D}_{l,\rho}(\pi) \neq 0$, then $\pi = \mathrm{soc}(\rho \times \mathcal{D}_{l,\rho}(\pi))$ and it appears in its decomposition series with multiplicity* 1*.*

Corollary 4.3.1. *Let* $\pi_1, \pi_2 \in \mathfrak{Irr}$ *and* $\rho \in \mathfrak{Irr}^{\square}$. *If* $\mathcal{D}_{l,\rho}(\pi_1) \cong \mathcal{D}_{l,\rho}(\pi_2) \neq 0$ *or* $\mathcal{D}_{r,\rho}(\pi_1) \cong \mathcal{D}_{r,\rho}(\pi_2) \neq 0$, *then* $\pi_1 \cong \pi_2$.

Lemma 4.3.2. *Let* $\rho \in \mathfrak{C}_m^{\square}$ *and* $\pi \in \mathfrak{Irr}_n$. *Then* $\mathcal{D}_{r,\rho}(\pi) \neq 0$ *if and only if there exists an irreducible representation* $\pi' \in \mathfrak{Irr}_{n-m}$ *such that*

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)].$$

Similarly, $\mathcal{D}_{l,\rho}(\pi) \neq 0$ *if and only if there exists an irreducible representation* $\pi' \in \mathfrak{Irr}_{n-m}$ *such that*

$$[\rho \otimes \pi'] \leq [r_{(m,n-m)}(\pi)].$$

Proof. We only prove the claim for right-derivatives, the one for left-derivatives follows analogously. Note that if π admits a right-derivative π', we obtain by Frobenius reciprocity $r_{(n-m,m)}(\pi) \twoheadrightarrow \pi' \otimes \rho$. On the other hand, let π' be such that

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)].$$

We now argue inductively on n that then π admits a right derivative. Choose ρ' a cuspidal representation of $G_{m'}$ and $\tau \in \mathfrak{Irr}_{n-m'}$ such that $\pi \hookrightarrow \tau \times \rho'$. If $\rho' \cong \rho$ we are done, thus we assume from now on otherwise. Choose k maximal such that there exists $\tau' \in \mathfrak{Irr}_{n-km-m'}$

$$\tau \hookrightarrow \tau' \times \rho^k.$$

We thus have $\pi \hookrightarrow \tau' \times \sigma$, where σ is an irreducible subquotient of $\rho^k \times \rho'$. By induction we note that the maximality of k implies that there exists no $\tau'' \in \mathfrak{Irr}_{n-(k+1)m-m'}$ such that

$$[\tau'' \otimes \rho] \leq [r_{(n-(k+1)m-m',m)}(\tau')]. \tag{27}$$

We will now show that σ admits a right-derivative with respect to ρ, which would show the claim. Firstly,

$$[\pi' \otimes \rho] \leq [r_{(n-m,m)}(\pi)] \leq [r_{(n-m,m)}(\tau' \times \sigma)].$$

The Geometric Lemma, the fact that ρ is supercuspidal and (27) imply therefore that $k > 0$ and there exists $\sigma' \in \mathfrak{Irr}$ such that

$$[\sigma' \otimes \rho] \leq [r_{((k-1)m+m',m)}(\sigma)].$$

We first note that if $\rho' \not\cong \rho v_\rho^{\pm 1}$, by Lemma 2.3.7 $\sigma \cong \rho^k \times \rho' \cong \rho' \times \rho^k$ in which the claim would follow immediately. Therefore we assume from now on $\rho' \in \{\rho v_\rho, \rho v_\rho^{-1}\}$ and we differentiate two cases, namely $o(\rho) > 2$ and $o(\rho) = 2$.

Case 1: $o(\rho) > 2$

Then $\rho \times \rho'$ has two subquotients σ_1 and σ_2 such that $\rho^{k-1} \times \sigma_1$ and $\rho^{k-1} \times \sigma_2$ are irreducible by Theorem 2.3.9. Thus if $k > 1$, the claim follows from the commutativity of parabolic induction. If $k = 1$,

$$\sigma' \otimes \rho \leq [r_{(m,m)}(\sigma)]$$

implies thus by Lemma 2.3.8 that $\sigma \hookrightarrow \rho' \times \rho$.

Case 1: $o(\rho) = 2$

Then $\rho \times \rho'$ has three subquotients $\sigma_1 = \mathrm{Z}([0,1]_\rho)$, $\sigma_2 = \mathrm{Z}([1,2]_\rho)$ and $\sigma_3 = St(\rho, 2)$ cuspidal. Note that $\rho^{k-1} \times \sigma_3$ is irreducible by Lemma 2.3.7 and hence if $\sigma \hookrightarrow \rho^{k-1} \times \sigma_3$, we have that $k > 1$ and the claim follows from the commutativity of parabolic induction. If $k = 1$, it follows again from Lemma 2.3.8 that $\sigma = \sigma_2 \hookrightarrow \rho v_\rho^{-1} \times \rho$. Thus we assume $k > 1$ and also note that if $\sigma \hookrightarrow \rho^{k-1} \times \sigma_2$, then we are done since, $\sigma_2 \hookrightarrow \rho v_\rho^{-1} \times \rho$. We are therefore in the situation where $\sigma \hookrightarrow \rho^{k-1} \times \sigma_1$. Next we show that

$$[\rho \times \sigma_1] = [\mathrm{Z}([0,2]_\rho)] + [\mathrm{Z}([0,1]_\rho + [0,0]_\rho)] \tag{28}$$

and

$$\begin{aligned} r_{(2m,m)}(\mathrm{Z}([0,2]_\rho)) &= \mathrm{Z}([0,1]_\rho) \otimes \rho, \\ r_{(2m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho)) &= \rho \times \rho \otimes \rho v_\rho. \end{aligned} \tag{29}$$

Indeed, note that by the Geometric Lemma

$$[r_{(2m,m)}(\mathrm{Z}([0,1]_\rho) \times \rho)] = [\mathrm{Z}([0,1]_\rho) \otimes \rho] + [\rho \times \rho \otimes \rho v_\rho].$$

Since furthermore $\mathrm{Z}([0,1]_\rho) \times \rho$ does not contain a cuspidal representation by Lemma 2.3.11, it is at most of length 2. On the other hand, by Lemma 2.3.10 and Theorem 2.3.13 it also contains the two representations of (28) as subquotients. Since by Lemma 2.3.8 the parabolic restriction of $\mathrm{Z}([0,2]_\rho)$ is of the above-mentioned form, the claim follows. In particular, if $\sigma \hookrightarrow \rho^{k-2} \times Z([0,2]_\rho)$ we are done by Lemma 2.3.10 and therefore we can assume that

$$\sigma \hookrightarrow \rho^{k-2} \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho).$$

From (29) we obtain that $k > 2$. It is now enough to show that

$$\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho)$$

is irreducible, since then

$$\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho) \cong \mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho.$$

To see this note that by Proposition 2.3.14

$$[\mathrm{Z}([0,1]_\rho + 2[0,0]_\rho] \le [\rho \times \mathrm{Z}([0,1]_\rho + [0,0]_\rho)]$$

and by Theorem 2.3.13

$$[r_{(m,3m)}(\mathrm{Z}([0,1]_\rho + 2[0,0]_\rho))] \ge [\rho \otimes \mathrm{Z}([1,1]_\rho + 2[0,0]_\rho)],$$

$$\mathrm{Z}([0,0]_\rho + [1,1]_\rho + [0,0]_\rho) \cong \mathrm{Z}([0,0]_\rho + [1,1]_\rho) \times \rho,$$

since $\mathrm{Z}([0,0]_\rho + [1,1]_\rho) \cong \mathrm{St}(\rho, 2)$ is cuspidal. By Frobenius reciprocity we thus have

$$[r_{(m,2m,m)}(\mathrm{Z}([0,0]_\rho + 2[0,0]_\rho))] \ge [\rho \otimes \mathrm{Z}([0,0]_\rho + [1,1]_\rho) \otimes \rho],$$

and hence there exists an irreducible π_3 such that

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + 2[0,0]_\rho))] \ge [\pi_3 \otimes \rho].$$

On the other hand the Geometric Lemma and (29) imply

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] \geq [\rho^3 \otimes \rho v_\rho].$$

Since $\rho^3 \otimes \rho v_\rho$ is residually non-degenerate, Theorem 2.3.13 implies that the irreducible subquotient of

$$\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho,$$

whose parabolic restriction contains $\rho^3 \otimes \rho v_\rho$ must be $\mathrm{Z}([0,1]_\rho + 2[0,0]_\rho)$. But

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] = [\rho^3 \otimes \rho v_\rho] + [\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \otimes \rho]$$

by the Geometric Lemma and (29) and hence $\pi_3 \cong \mathrm{Z}([0,1]_\rho + [0,0]_\rho)$ and therefore

$$[r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho)] = [r_{(3m,m)}(\mathrm{Z}([0,1]_\rho + 2[0,0]_\rho))].$$

This implies that any other possible subquotient π_4 of $\mathrm{Z}([0,1]_\rho + [0,0]_\rho) \times \rho$ must satisfy

$$r_{(3m,m)}(\pi_4) = 0.$$

But this yields a contradiction as follows. The cuspidal support of π_4 is either $3[\rho] + [\rho v_\rho]$ or $Z([0,0]_\rho + [1,1]_\rho) + 2[\rho]$. In both cases, it follows that either the right-derivative of π_4 with respect to ρ or ρv_ρ is non-zero. But then Frobenius reciprocity gives the desired contradiction. $\square$

Remark. Note that for ρ a cuspidal non supercuspidal representation of G_m, the *if*-direction of Lemma 4.3.2 does not hold in general. For example, let $\rho \in \mathfrak{C}^\square$ and

$$\pi = \mathrm{Z}([-1,0]_\rho + [1,1]_\rho + \ldots + [e(\rho)-1, e(\rho)-1]_\rho).$$

Then the cuspidal support of π is

$$[\rho] + \ldots + [\rho v_\rho^{e(\rho)-2}] + 2 \cdot [\rho v_\rho^{e(\rho)-1}]$$

and therefore does not contain the cuspidal representation $\mathrm{St}(\rho, e(\rho))$, but by Theorem 2.3.13, $r_{(m,e(\rho)m)}(\pi)$ contains $\rho v_\rho^{-1} \otimes \mathrm{St}(\rho, e(\rho))$.

Recall that ρ^l is irreducible if ρ is $\square$-irreducible for all $l \in \mathbb{Z}_{>0}$ by Theorem 2.3.9.

Lemma 4.3.3. *Let $\pi \in \mathfrak{Irr}_n$, $\rho \in \mathfrak{C}_m^\square$, $l \in \mathbb{Z}_{\geq 0}$ and $\pi' \in \mathfrak{Irr}_{n-lm}$ be such that*

$$\pi \hookrightarrow \pi' \times \rho^l$$

and π' admits no right derivative with respect to ρ. Then $\pi' \otimes \rho^l$ is strongly $P_{(lm,n-lm)}$-regular. Moreover, if π'' is an irreducible representation such that $\pi'' \otimes \rho^l$ appears in $r_{(n-lm,lm)}(\pi)$ or $r_{(n-lm,lm)}(\pi' \times \rho^l)$, then $\pi'' \cong \pi'$, and $\pi' \otimes \rho^l$ appears in $r_{(n-lm,lm)}(\pi' \times \rho^l)$ with multiplicity 1. Finally, π appears with multiplicity 1 in $\pi' \times \rho^l$.

Proof. To prove the first claim, we observe that for such maximal l,

$$r_{(n-lm,lm)}(\pi' \times \rho^l)$$

contains the representation $\pi' \otimes \rho^l$ with multiplicity one by Lemma 4.3.2 and the Geometric Lemma. Thus it is $P_{(lm,n-lm)}$-regular by Lemma 4.2.8(2), since $\overline{P_{(lm,n-lm)}}$ is conjugated to $P_{(n-lm,lm)}$. Moreover, it is also clear that if $\pi'' \otimes \rho^l$ appears in $r_{(n-lm,lm)}(\pi' \times \rho^l)$, $\pi'' \cong \pi'$. Since by Frobenius reciprocity $r_{(n-lm,lm)}(\pi)$ contains $\pi' \otimes \rho^l$, π appears with multiplicity 1 in $\pi' \times \rho^l$ and it is the only irreducible subrepresentation.

Finally, to see that $\pi' \otimes \rho^l$ is strongly $P_{(lm,n-lm)}$-regular, recall that

$$r_{(n-lm,lm)}(\pi)^\vee \cong r_{\overline{P_{(n-lm,lm)}}}(\pi^\vee)$$

by Bernstein reciprocity. Thus

$$[\pi' \otimes \rho^l] \leq [r_{(n-lm,lm)}(\pi)]$$

if and only if

$$[\pi'^\vee \otimes (\rho^\vee)^l] \leq [r_{\overline{P_{(n-lm,lm)}}}(\pi^\vee)].$$

Since $\overline{P_{(n-lm,lm)}}$ is conjugated to $P_{(lm,n-lm)}$ this is equivalent to

$$[(\rho^\vee)^l \otimes \pi'^\vee] \leq [r_{(lm,n-lm)}(\pi^\vee)].$$

Now one can prove completely analogously to above that this is equivalent to $\pi^\vee \hookrightarrow (\rho^\vee)^l \times \pi'^\vee$ and $\pi'^\vee$ does not admit a left derivative with respect to $\rho^\vee$. But this is implies that $\rho^l \times \pi' \twoheadrightarrow \pi$. □

Remark. Note that Lemma 4.3.3 allows one to give an alternative proof of Lemma 4.3.1 and Corollary 4.3.1 for □-irreducible cuspidal representations.

We now define

$$\mathcal{D}^k_{r,\rho}(\pi) := \overbrace{\mathcal{D}_{r,\rho} \circ \cdot \circ \mathcal{D}_{r,\rho}}^{k}(\pi),\ \mathcal{D}^k_{l,\rho}(\pi) := \overbrace{\mathcal{D}_{l,\rho} \circ \cdot \circ \mathcal{D}_{l,\rho}}^{k}(\pi),\ \mathcal{D}^0_{r,\rho} := \mathcal{D}^0_{l,\rho} := 1,$$

$d_{r,\rho}(\pi)$ the maximal k such that $\mathcal{D}^k_{r,\rho}(\pi) \neq 0$, $\mathcal{D}_{r,\rho,max}(\pi) := \mathcal{D}^{d_{l,\rho}(\pi)}_{l,\rho}(\pi)$, $d_{l,\rho}(\pi)$ the maximal k such that $\mathcal{D}^k_{l,\rho}(\pi) \neq 0$ and $\mathcal{D}_{l,\rho,max}(\pi) := \mathcal{D}^{d_{l,\rho}(\pi)}_{l,\rho}(\pi)$. As a corollary of Lemma 4.3.2 we obtain the following.

Corollary 4.3.2. *For $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}^\square_m$, $d_{r,\rho}(\pi)$ is the maximal k such that there exists $\pi' \in \mathfrak{Irr}_{n-mk}$ with*

$$[\pi' \otimes \rho^k] \leq [r_{(n-km,km)}(\pi)].$$

In this case $\pi' \cong \mathcal{D}_{r,\rho,max}(\pi)$ and $\pi' \otimes \rho^k$ appears with multiplicity 1 in the parabolic restriction. Analogously, $d_{l,\rho}(\pi)$ is the maximal k such that there exists $\pi' \in \mathfrak{Irr}_{n-mk}$ with

$$[\rho^k \otimes \pi'] \leq [r_{(km,n-km)}(\pi)].$$

In this case $\pi' \cong \mathcal{D}_{l,\rho,max}(\pi)$ and $\rho^k \otimes \pi'$ appears with multiplicity 1 in the parabolic restriction.

Finally,

$$\mathcal{D}_{r,\rho,max}(\pi) \otimes \rho^{d_{r,\rho}(\pi)} \text{ respectively } \rho^{d_{l,\rho}(\pi)} \otimes \mathcal{D}_{l,\rho,max}(\pi)$$

are strongly $P_{(d_{r,\rho}(\pi)m,n-d_{r,\rho}(\pi)m)}$- respectively $P_{(n-d_{l,\rho}(pi)m,d_{l,\rho}(\pi)m)}$-regular.

Lastly, dualizing behaves well with respect to taking derivatives.

Lemma 4.3.4. *Let $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}_m^{\square}$. Then*

$$\mathcal{D}_{r,\rho}(\pi)^{\vee} \cong \mathcal{D}_{l,\rho^{\vee}}(\pi^{\vee}),\ d_{r,\rho}(\pi) = d_{l,\rho^{\vee}}(\pi^{\vee}),\ \mathcal{D}_{r,\rho,max}(\pi)^{\vee} \cong \mathcal{D}_{l,\rho^{\vee},max}(\pi^{\vee}).$$

Proof. By the contravariance of dualizing we have

$$\pi^{\vee} \cong (\mathrm{soc}(\mathcal{D}_{r,\rho}(\pi) \times \rho))^{\vee} \cong \cos(\mathcal{D}_{r,\rho}(\pi)^{\vee} \times \rho^{\vee}) \overset{Lemma\ 4.2.5}{\cong} \mathrm{soc}(\rho^{\vee} \times \mathcal{D}_{r,\rho}(\pi)^{\vee}),$$

therefore $\mathcal{D}_{r,\rho}(\pi)^{\vee} \cong \mathcal{D}_{l,\rho^{\vee}}(\pi^{\vee})$ and from this, the other claims follow straightforwardly. □

4.3.2 Derivatives and multisegments In this section we will answer the following question: Given an aperiodic multisegment $\mathfrak{m}$ and $\rho \in \mathfrak{C}_m^{\square}$, what is $\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m}))$ and $\mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m}))$? To answer this we will define four maps

$$\mathcal{D}_{l,\rho}, \mathcal{D}_{r,\rho}, \mathrm{soc}(\,\cdot\,,\rho), \mathrm{soc}(\rho,\,\cdot\,)\colon \mathcal{M}_{ap} \to \mathcal{M}_{ap}.$$

In the case $R = \overline{\mathbb{Q}}_\ell$ this has already been done independently in [64, Theorem 2.2.1] and [93, Theorem 7.5].

Best matching functions Similarly to [81, §4.3], we make the following definition. Fix $\rho \in \mathfrak{C}^{\square}$ and let $\mathfrak{m} = \sum_{i=1}^{n} \Delta_i$ be a multisegment. A best matching function for $\mathfrak{m}$ with respect to ρ is a tuple (A, f), where $A \subseteq \{1, \dots, n\}$, possibly empty, and $f\colon A \to \{1, \dots, n\}$ an injective function satisfying the following properties.

1. If $a \in A$, then $\Delta_a \in \mathcal{S}eg(\rho)$ and $b_\rho(\Delta_a) = 0$.
2. If $a \in A$, then $\Delta_{f(a)} \in \mathcal{S}eg(\rho)$, $b_\rho(\Delta_{f(a)}) = -1$ and $l(\Delta_a) \le l(\Delta_{f(a)})$.
3. If $a \in A$, $b \in A^{\complement} = \{1, \dots, n\} \setminus A$ such that $\Delta_b \in \mathcal{S}eg(\rho)$, $b_\rho(\Delta_b) = 0$ and $l(\Delta_a) \le l(\Delta_b) \le l(\Delta_{f(a)})$, then $\Delta_b = \Delta_a$.
4. If $a \in A$, $c \in f(A)^{\complement} = \{1, \dots, n\} \setminus f(A)$ such that $\Delta_c \in \mathcal{S}eg(\rho)$, $b_\rho(\Delta_c) = -1$ and $l(\Delta_a) \le l(\Delta_c) \le l(\Delta_{f(a)})$, then $\Delta_c = \Delta_{f(a)}$.
5. For $b \in A^{\complement}$, $c \in f(A)^{\complement}$ such that $\Delta_b, \Delta_c \in \mathcal{S}eg(\rho), b_\rho(\Delta_b) = 0$, $b_\rho(\Delta_c) = -1$, then $l(\Delta_c) < l(\Delta_b)$.

Example. Let $\rho \in \mathfrak{C}^{\square}$. If

$$\mathfrak{m} = [-1,0]_\rho + [-1,0]_\rho + [-3,0]_\rho + [-2,-1]_\rho + [-3,-1]_\rho + [-3,-1]_\rho,$$

setting $A = \{1,2\}$, $f(1) = 4$, $f(2) = 5$ gives a best matching function with respect to ρ.

We will assosciate to a best matching function (A, f) of a multisegment $\mathfrak{m}$ the multisegments $\mathfrak{m}_f := \sum_{a \in A} \Delta_a$ and $f(\mathfrak{m}_f) := \sum_{a \in A} \Delta_{f(a)}$. If $\mathfrak{m}, \mathfrak{m}'$ are two multisegments and (A, f) and (A', f') are two best matching functions of $\mathfrak{m}$ and $\mathfrak{m}'$ with respect to ρ, we say f and f' are equivalent if $\mathfrak{m}_f = \mathfrak{m}'_{f'}$ and $f(\mathfrak{m}_f) = f'(\mathfrak{m}'_{f'})$.

Lemma 4.3.5. *Given $\mathfrak{m}$ a multisegment and $\rho \in \mathfrak{C}^{\square}$ there exists a best matching function (A, f) of $\mathfrak{m}$ with respect to ρ, which can be constructed explicitly. Moreover, all best matching functions of $\mathfrak{m}$ with respect to ρ are equivalent.*

Proof. Write $\mathfrak{m} = \sum_{i=1}^{n} \Delta_i$. First assume that either there exists no ρ-segment $\Delta \in \mathfrak{m}$ with $b_\rho(\Delta) = 0$ or for every ρ-segment $\Delta \in \mathfrak{m}$ with $b_\rho(\Delta) = 0$ there exists no ρ-segment $\Delta' \in \mathfrak{m}$ with $b_\rho(\Delta') = -1$ and $l(\Delta) \leq l(\Delta')$. Then properties (1) and (2) force A to be empty. Moreover, if we take A to be empty, the properties (1), (2), (3), and (4) are trivially true, and property (5) is true by assumption, thus we are done in this case.

If above assumption is not satisfied, we can fix a copy Δ of the longest ρ-segment in $\mathfrak{m}$ with $b_\rho(\Delta) = 0$ such that there exists a ρ-segment $\Delta' \in \mathfrak{m}$ with $b_\rho(\Delta') = -1$ and $l(\Delta) \leq l(\Delta')$. We also fix Γ, a copy of the ρ-segment of minimal length such that $b_\rho(\Gamma) = -1$ and $l(\Delta) \leq l(\Gamma)$.

Let us first construct a best matching function of $\mathfrak{m}$ via induction on the number of segments. Without loss of generality, we can assume $\Gamma = \Delta_{n-1}$ and $\Delta = \Delta_n$. Let $f' \colon A' \to \{1, \dots, n-2\}$ be a best matching function of $\mathfrak{m} - \Delta_{n-1} - \Delta_n$. We set $A = A' \cup \{n\}$ and extend f' to $f \colon A \to \{1, \dots, n\}$ by setting $f(n) = n-1$. We now show that f is a best matching function. Note that since f' is a best matching function, properties (1), (2), and (5) are trivially satisfied. For properties (3) and (4), the only non-trivial cases occur if $a = n$. If there would exist Δ_b or Δ_c as in property (3) and (4) such that

$$l(\Delta_n) < l(\Delta_b) \leq l(\Delta_{n-1}) \text{ respectively } l(\Delta_n) \leq l(\Delta_c) < l(\Delta_{n-1}),$$

we would obtain a contradiction to the maximality of Δ respectively the minimality of Γ.

Finally, to show that $\mathfrak{m}_f$ and $f(\mathfrak{m}_f)$ are uniquely determined by $\mathfrak{m}$, we will argue also inductively on the number of segments. We start by showing that if (A, f) is a best matching function then $\Delta \in \mathfrak{m}_f$ and $\Gamma \in f(\mathfrak{m}_f)$. Note that by property (5) at least one of $n-1, n$ has to be an element of A respectively $f(A)$. If $n \in A$, property (4) implies $f(\Delta_n) = \Gamma = \Delta_{n-1}$ by the minimality of Γ. If on the other hand $n-1 \in f(A)$, we let $a' \in A$ be such that $f(a') = n-1$ and assume $\Delta \notin \mathfrak{m}_f$. By the maximality of Δ, we have that $l(\Delta_{a'}) \leq l(\Delta) \leq l(\Gamma)$. Now property (3) implies $\Delta_{a'} = \Delta$, a contradiction. We thus can assume without loss of generality that $n \in A$ and $n-1 \in f(A)$. Next, we show that every best matching function (A, f) of $\mathfrak{m}$ is equivalent to a best matching function (A, g) of $\mathfrak{m}$ such that $g(n) = n-1$. We let $a' \in A$ be such that $n-1 = f(a')$. Define

$$g \colon A \to \{1, \dots, n\},\ g(a) = \begin{cases} n-1 & \text{if } a = n, \\ f(n) & \text{if } a = a', \\ f(a) & \text{otherwise.} \end{cases}$$

It is clear that f and g are equivalent, it thus remains to show that g is a best matching function. Property (1) is trivially satisfied. Property (2) is non-trivial only for a'. But since by the maximality of $\Delta = \Delta_n$, $l(\Delta_{a'}) \leq l(\Delta_n)$, it follows from property (2) of f. Property (3) and (4) are trivially true for all a except $a \in \{a', n\}$. Let Δ_b, Δ_c be like in properties (3) and (4). For $a = n$, if

$$l(\Delta_n) \leq l(\Delta_b) \leq l(\Delta_{g(n)}) = l(\Gamma),$$

the maximality of $\Delta = \Delta_n$ implies $\Delta_n = \Delta_b$. If

$$l(\Delta_n) \leq l(\Delta_c) \leq l(\Delta_{g(n)}) = l(\Gamma),$$

the minimality of Γ implies $\Gamma = \Delta_c$. For $a = a'$, if

$$l(\Delta_{a'}) \leq l(\Delta_b) \leq l(\Delta_{g(a')}) = l(\Delta_{f(n)}),$$

the maximality of $\Delta = \Delta_n$ implies

$$l(\Delta_b) \leq l(\Delta_n) \leq l(\Gamma) = l(\Delta_{f(a')})$$

and hence by property (3) of f, $\Delta_b = \Delta_{a'}$. Finally, if

$$l(\Delta_{a'}) \leq l(\Delta_c) \leq l(\Delta_{g(a')}) = l(\Delta_{f(n)}),$$

we first show $l(\Delta_c) \geq l(\Delta_n)$. Indeed, if $l(\Delta_c) < l(\Delta_n)$, we would obtain that $l(\Delta_{a'}) \leq l(\Delta_c) < l(\Gamma) = l(\Delta_{f(a')})$, contradicting property (3) of f. Thus $l(\Delta_n) \leq l(\Delta_c) \leq l(\Delta_{f(n)})$ and hence by property (3) of f, $\Delta_c = \Delta_{f(n)}$.

Thus we can replace f by an equivalent best matching function sending n to $n-1$ and hence we can restrict f to a best matching function

$$f' \colon A \setminus \{n\} \to \{1, \dots, n-2\}$$

of $\mathfrak{m} - \Delta_{n-1} - \Delta_n$. By the induction hypothesis, we know that all best matching functions of $\mathfrak{m} - \Delta_{n-1} - \Delta_n$ are equivalent and hence all best matching functions of $\mathfrak{m}$ are equivalent. □

Let $\mathfrak{m}$ be a multisegment and $\rho \in \mathfrak{C}^\square$ with a corresponding best matching function (A, f). The above lemma allows us to make the following definition. A ρ-segment $\Delta \in \mathfrak{m} - \mathfrak{m}_f$ with $b_\rho(\Delta) = 0$ is called *free* and a ρ-segment $\Delta \in \mathfrak{m} - f(\mathfrak{m}_f)$ with $b_\rho(\Delta) = -1$ is called *extendable*. We then set

1. $d_{r,\rho}(\mathfrak{m})$ to the number of free ρ-segments in $\mathfrak{m}$,
2. $$\mathcal{D}_{r,\rho}(\mathfrak{m}) := \begin{cases} 0 & \text{if there exists no free } \rho\text{-segment in } \mathfrak{m}, \\ \mathfrak{m} - \Delta + \Delta^- & \text{otherwise,} \end{cases}$$
   where Δ is the shortest free ρ-segment of $\mathfrak{m}$,
3. $\mathcal{D}_{r,\rho,max}(\mathfrak{m})$ to the multisegment obtained from $\mathfrak{m}$ by replacing all free ρ-segments Δ by Δ^-,
4. $$\operatorname{soc}(\mathfrak{m}, \rho) := \begin{cases} \mathfrak{m} + [0,0]_\rho & \text{if there exists no extendable } \rho\text{-segment in } \mathfrak{m}, \\ \mathfrak{m} + \Delta^+ - \Delta & \text{otherwise,} \end{cases}$$
   where Δ is the longest extendable ρ-segment,
5. $d_{l,\rho}(\mathfrak{m}) := d_{r,\rho^\vee}(\mathfrak{m}^\vee)$, $\mathcal{D}_{l,\rho}(\mathfrak{m}) := \mathcal{D}_{r,\rho^\vee}(\mathfrak{m}^\vee)^\vee$,
   $\mathcal{D}_{l,\rho,max}(\mathfrak{m}) := \mathcal{D}_{r,\rho^\vee,max}(\mathfrak{m}^\vee)^\vee$, $\operatorname{soc}(\rho, \mathfrak{m}) := \operatorname{soc}(\mathfrak{m}^\vee, \rho^\vee)^\vee$.

Lemma 4.3.6. *Let $\mathfrak{m}$ be a multisegment and $\rho \in \mathfrak{C}^\square$. Then any best matching functions with respect to ρ of $\mathfrak{m}$ and $\operatorname{soc}(\mathfrak{m}, \rho)$ are equivalent. If $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$, its best matching functions with respect to ρ are also equivalent to those of $\mathfrak{m}$.*

Proof. We only show the claim regarding $\operatorname{soc}(\mathfrak{m}, \rho)$, the claim regarding $\mathcal{D}_{r,\rho}(\mathfrak{m})$ follows analogously. Let $\mathfrak{m} = \sum_{i=1}^{n-1} \Delta_i + \Delta_n^-$ and $\operatorname{soc}(\mathfrak{m}, \rho) = \sum_{i=1}^{n} \Delta_i$. We fix a best matching function (A, f) of $\mathfrak{m}$.

Since by construction $\mathfrak{m}_f$ and $f(\mathfrak{m}_f)$ are contained in $\mathrm{soc}(\mathfrak{m}, \rho)$, it is enough by Lemma 4.3.5 to show that (A, f) satisfies the 5 properties of being a best matching function $\mathrm{soc}(\mathfrak{m}, \rho)$. The properties (1), (2), and (4) of 4.3.2 are trivially satisfied.

The properties (3) and (5) of 4.3.2 are satisfied for all b except for $b = n$. We start with property (3). Assume there exists $a \in A$ such that

$$l(\Delta_{f(a)}) \geq l(\Delta_n) > l(\Delta_a). \tag{30}$$

Firstly, this implies that Δ_n cannot be of length 1, hence we can assume Δ_n^- to be non-zero. Thus we have $l(\Delta_n^-) \geq l(\Delta_a)$. In this case, we can apply property (3) of f to a and n since f is a best matching function of $\mathfrak{m}$ and Δ_n^- is an extendable segment. Thus we obtain that $\Delta_{f(a)} = \Delta_n^-$, contradicting (30).

For property (5), if there exists an extendable segment Δ_c such that $l(\Delta_c) \geq l(\Delta_n)$, we would obtain that $l(\Delta_c) > l(\Delta_n^-)$, which would contradict the maximality of Δ_n^-. □

Lemma 4.3.7. *Let $\mathfrak{m}$ be a multisegment and $\rho \in \mathfrak{C}^\square$. Then*

$$\mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m}, \rho)) = \mathfrak{m}.$$

If moreover $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$,

$$\mathrm{soc}(\mathcal{D}_{r,\rho}(\mathfrak{m}), \rho) = \mathfrak{m}$$

and similarly for $\mathcal{D}_{l,\rho}$ and $\mathrm{soc}(\rho, \cdot)$*.*

Proof. We only prove,

$$\mathcal{D}_{r,\rho}(\mathrm{soc}(\mathfrak{m}, \rho)) = \mathfrak{m},$$

the other claims follow analogously. Then the claim is equivalent to the following claim. If Δ is the shortest free ρ-segment in $\mathrm{soc}(\mathfrak{m}, \rho)$, the segment Δ^- is the longest extendable ρ-segment in $\mathfrak{m}$.

Let Δ^- be the longest extendable ρ-segment in $\mathfrak{m}$. By Lemma 4.3.6 Δ is a free ρ-segment of $\mathrm{soc}(\mathfrak{m}, \rho)$. If Δ' would be a shorter free ρ-segment of $\mathrm{soc}(\mathfrak{m}, \rho)$, it would be a free ρ-segment of $\mathfrak{m}$ by Lemma 4.3.6 and $l(\Delta') \leq l(\Delta^-)$. This would contradict property (5) of a best matching function. □

Finally, we show that the four maps indeed take aperiodic multisegments to aperiodic multisegments.

Lemma 4.3.8. *If $\mathfrak{m}$ is aperiodic,*

$$\mathcal{D}_{r,\rho}(\mathfrak{m}), \mathcal{D}_{l,\rho}(\mathfrak{m}), \mathrm{soc}(\mathfrak{m}, \rho), \mathrm{soc}(\rho, \mathfrak{m})$$

are aperiodic.

Proof. Write $\mathfrak{m} = \sum_{i=1}^n \Delta_i$ and let (A, f) be a best matching function of $\mathfrak{m}$ with respect to ρ. We only show the claim for $\mathcal{D}_{r,\rho}(\mathfrak{m})$, the others follow completely analogously. We assume by contradiction otherwise, *i.e.* $\mathcal{D}_{r,\rho}(\mathfrak{m})$ contains a multisegment equivalent to

$$[a, 0]_\rho + \ldots + [a + e(\rho) - 1, e(\rho) - 1]_\rho,$$

for a suitable $a \in \mathbb{Z}_{<0}$. If $[a+e(\rho)-1, e(\rho)-1]_\rho$ appears in $\mathfrak{m}$, $\mathfrak{m}$ cannot be aperiodic, since by the construction of $\mathcal{D}_{r,\rho}(\mathfrak{m})$, $\mathfrak{m}$ also has to contain

$$[a,0]_\rho + \ldots + [a+e(\rho)-2, e(\rho)-2]_\rho.$$

Thus we assume now that $[a+e(\rho)-1, e(\rho)-1]_\rho$ does not appear in $\mathfrak{m}$ and will produce a contradiction.

Let Δ be a the shortest free ρ-segment in $\mathfrak{m}$, *i.e.* the segment such that $\mathcal{D}_{r,\rho}(\mathfrak{m}) = \mathfrak{m} + \Delta^- - \Delta$. Since we assumed that $[a+e(\rho)-1, e(\rho)-1]_\rho$ does not appear in $\mathfrak{m}$, we have $\Delta^- = [a+e(\rho)-1, e(\rho)-1]_\rho$ and hence $\Delta = [a-1,0]_\rho$. Since $\mathfrak{m}$ contains then both $[a,0]_\rho$ and $[a-1,0]_\rho$ and

$$l([a,0]_\rho) < l([a-1,0]_\rho),$$

the minimality of Δ implies that $[a,0]_\rho \in \mathfrak{m}_f$. Let $a' \in A$ be such that $\Delta_{a'} = [a,0]_\rho$. By property (2) and (3) of 4.3.2 we obtain that

$$2-a = l([a-1,0)]_\rho) > l(\Delta_{f(a')}) \geq l([a,0]_\rho) = 1-a$$

and hence $l(\Delta_{f(a')}) = -a+1$. This implies that

$$\Delta_{f(a')} = [a-1,-1]_\rho = [a+e(\rho)-1, e(\rho)-1]_\rho,$$

a contradiction. □

Lemma 4.3.9. *Let* $\mathfrak{m}$ *and* $k \in \{0,\ldots,d_{r,\rho}(\mathfrak{m})\}$. *Then* $\mathcal{D}^k_{r,\rho}(\mathfrak{m})$ *is the multisegment obtained from* $\mathfrak{m}$ *by shortening the shortest* k *free* ρ*-segments in* $\mathfrak{m}$ *by one on the right.*

Proof. Note that by Lemma 4.3.6 we have that a best matching function of $\mathfrak{m}$ is a best matching function of $\mathcal{D}^k_{r,\rho}(\mathfrak{m})$ and from this the claim follows straightforwardly. □

Corollary 4.3.3. *Let* $\mathfrak{m}$ *be an aperiodic multisegment and* $\rho \in \mathfrak{C}^\square$. *Then*

$$\mathcal{D}_{r,\rho,max}(\mathfrak{m})^- = \mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-).$$

Set $l = d_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-)$ *and* k *to the largest positive integer such that*

$$\mathfrak{m}^1 - (l+k)[0,0]_\rho + l[-1,-1]_\rho$$

contains at least as many copies of $[-1,-1]_\rho$ *as of* $[0,0]_\rho$. *Then*

$$d_{r,\rho}(\mathfrak{m}) = l+k,\ \mathcal{D}_{r,\rho,max}(\mathfrak{m})^1 = \mathfrak{m}^1 - (l+k)[0,0]_\rho + l[-1,-1]_\rho.$$

Proof. Write $\mathfrak{m} = \sum_{i=1}^n \Delta_i$ such that there exists $m \in \{1,\ldots,n\}$ with $l(\Delta_i) = 1$ if and only if $i > m$. Let (A,f) be a best matching function of $\mathfrak{m}$ with respect to ρ. Set $A^- := A \cap \{1,\ldots,m\}$. Then the restriction of f

$$f\colon A^- \to \{1,\ldots,m\}$$

is a best matching function of $\mathfrak{m}^-$ with respect to ρv_ρ^{-1}. Note that (A^-, f) is well defined since $l(\Delta_a) \leq l(\Delta_{f(a)})$ and hence if $l(\Delta_a) > 1$, then also $l(\Delta_{f(a)}) > 1$. Thus $a \in A, a \leq m$ implies $f(a) \leq m$. Moreover, by Lemma 4.3.5 it is enough to show that the 5 properties of 4.3.2 are satisfied, which is easy to check.

By Lemma 4.3.9, we can compute the above quantities in two steps. First dealing with the free segments of length 1 and then with the ones of length greater 1. Namely, let k be the number of free segments with respect to ρ of length 1 in $\mathfrak{m}$. Then

$$\mathcal{D}^k_{r,\rho}(\mathfrak{m})^- = \mathfrak{m}^-$$

and $\mathcal{D}^k_{r,\rho}(\mathfrak{m})$ has no free segments with respect to ρ of length 1. Denote the number of free segments in $\mathcal{D}^k_{r,\rho}(\mathfrak{m})$ with respect to ρ by l. Now $l = d_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-)$ since (f^-, A) is a best matching function of $\mathfrak{m}^-$. Moreover, we obtain by the same reason that

$$\mathcal{D}^{d_{r,\rho}(\mathfrak{m})}_{r,\rho}(\mathfrak{m})^- = \mathcal{D}^l_{r,\rho}(\mathfrak{m})^- = \mathcal{D}^l_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-)$$

and

$$\mathcal{D}_{r,\rho,max}(\mathfrak{m})^1 = \mathfrak{m}^1 - (k+l)[0,0]_\rho + l[-1,-1]_\rho.$$

Finally, if $\mathcal{D}_{r,\rho,max}(\mathfrak{m})^1$ would contain more copies of $[0,0]_\rho$ than of $[-1,-1]_\rho$, $\mathcal{D}_{r,\rho,max}(\mathfrak{m})$ would have to contain a free segment with respect to ρ, since every best matching function of $\mathcal{D}_{r,\rho,max}(\mathfrak{m})$ pairs up the segments ending in 0 and ending in -1. Thus if there are more copies of $[0,0]_\rho$ than of $[-1,-1]_\rho$ in $\mathcal{D}_{r,\rho,max}(\mathfrak{m})^1$, there needs to exist a free segment in $\mathcal{D}_{r,\rho,max}(\mathfrak{m})$. This however contradicts Lemma 4.3.9. . ☐

We will now make the relation between derivatives of multisegments and derivatives of representations explicit.

Theorem 4.3.10. *For $\mathfrak{m}$ an aperiodic multisegment and $\rho \in \mathfrak{C}^\square$, we have*

$$\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho}(\mathfrak{m})),\ \mathcal{D}_{l,\rho}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{l,\rho}(\mathfrak{m})),$$

$$d_{r,\rho}(\mathrm{Z}(\mathfrak{m})) = d_{r,\rho}(\mathfrak{m}),\ d_{l,\rho}(\mathrm{Z}(\mathfrak{m})) = d_{l,\rho}(\mathfrak{m})$$

and

$$\mathcal{D}_{r,\rho,max}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})),\ \mathcal{D}_{l,\rho,max}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{l,\rho,max}(\mathfrak{m})).$$

We will take great care in the proof not to use the surjectivity of Z for representations of $\square$-irreducible cuspidal support over $\overline{\mathbb{F}}_\ell$, since we want to eventually use the above theorem to give a new proof of it. We also remark that the theorem is already known over $\overline{\mathbb{Q}}_\ell$, *cf.* [64, Theorem 2.2.1] and [93, Theorem 7.5]. Let $\mathfrak{m} = \sum_{i=1}^n \Delta_i$ be an aperiodic multisegment over $\overline{\mathbb{F}}_\ell$ and ρ a supercuspidal representation. We call a lift $\widetilde{\mathfrak{m}} = \sum_{i=1}^n \widetilde{\Delta_i}$ of $\mathfrak{m}$, $\widetilde{\Delta_i}$ a lift of Δ_i for all $i \in \{1,\dots,n\}$, *ρ-right-derivative compatible* if there exists a lift $\widetilde{\rho}$ of ρ such that for all ρ-segments in $\mathfrak{m}$ ending in 0 respectively -1, their lift in $\widetilde{\mathfrak{m}}$ ends in 0 respectively -1 and is a segment over $\widetilde{\rho}$. Similarly, a lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$ is said to be *ρ-left-derivative compatible* if there exists a lift $\widetilde{\rho}$ of ρ such that for all ρ-segments in $\mathfrak{m}$ starting in 0 respectively 1, their lift in $\widetilde{\mathfrak{m}}$ starts in 0 respectively 1 and is a segment over $\widetilde{\rho}$. If $\rho \in \mathfrak{C}^\square$, there exists for all $\mathfrak{m} \in \mathcal{M}(\rho)$ such lifts since ρ is supercuspidal. The following is then easy to see.

Lemma 4.3.11. *Let $\widetilde{\mathfrak{m}}$ be a ρ-right-derivative compatible lift of an aperiodic multisegment $\mathfrak{m}$ and $\widetilde{\rho}$ the corresponding lift of ρ. Then $\mathrm{r}_\ell(\mathcal{D}_{r,\widetilde{\rho}}(\widetilde{\mathfrak{m}})) = \mathcal{D}_{r,\rho}(\mathfrak{m})$. More precisely, if (A,f) is a best matching function with respect to $\widetilde{\rho}$ of $\widetilde{\mathfrak{m}}$, (A,f) is a best matching function of $\mathfrak{m}$ with respect to ρ.*

A similar statement holds for ρ-left-compatible lifts.

Lemma 4.3.12. *Let $\rho \in \mathfrak{C}_m^{\square}$, $\widetilde{\mathfrak{m}}$ be a ρ-right-derivative compatible lift of an aperiodic multisegment $\mathfrak{m}$ and $\widetilde{\rho}$ the corresponding lift of ρ. Let moreover $\alpha = (\deg(\mathcal{D}_{r,\rho,max}(\mathfrak{m})), md_{r,\rho})$ and $\pi \in \mathfrak{Irr}$. Then the multiplicity of*

$$\pi \otimes \rho^{d_{r,\rho}}$$

in

$$\mathrm{r}_\ell(r_\alpha(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) \times \widetilde{\rho}^{d_{r,\rho}}))$$

is the multiplicity of π in $\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})))$. An analogous statement holds for ρ-left-derivative compatible lifts.

Proof. We will only deal with the case of the ρ-right-compatible lift $\widetilde{\mathfrak{m}}$. Recall that parabolic restriction commutes with reduction mod ℓ on the level of Grothendieck groups. Now since $\widetilde{\mathfrak{m}}$ is a ρ-right-compatible lift $\widetilde{\mathfrak{m}}$ and Theorem 4.3.10 is known over $\overline{\mathbb{Q}}_\ell$, there exists no lift $\widetilde{\rho}'$ of ρ such that

$$\mathcal{D}_{r,\widetilde{\rho}'}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}}))) \neq 0.$$

Thus Lemma 4.3.2 and the Geometric Lemma show that the $(G_{md_{r,\rho}}, \rho^{d_{r,\rho}})$-invariant part of

$$[\mathrm{r}_\ell(r_\alpha(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) \times \widetilde{\rho}^{d_{r,\rho}}))]$$

is

$$[\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}}))) \otimes \rho^{d_{r,\rho}}].$$

From this the claim follows immediately. □

Before we start with the proof of Theorem 4.3.10, we need the following lemmas.

Lemma 4.3.13. *Let $\mathrm{Z}(\mathfrak{n})$ be a residually-degenerate representation and $\rho \in \mathfrak{C}^{\square}$ of G_m. Let d_i be the multiplicity of $[i,i]_\rho$ in $\mathfrak{n}$, $i \in \{-1, 0\}$. Then $\mathcal{D}_{r,\rho}(\mathrm{Z}(\mathfrak{n})) \neq 0$ if and only if $d_{-1} < d_0$. Similarly, $\mathcal{D}_{l,\rho v_\rho^{-1}}(\mathrm{Z}(\mathfrak{n})) \neq 0$ if and only if $d_{-1} > d_0$.*

Proof. Via Lemma 2.3.7 we can reduce the lemma to the case where $\mathfrak{n}$ has cuspidal support in $\mathbb{N}(\mathbb{Z}[\rho])$. Thus $\mathfrak{n}$ is an aperiodic, banal multisegment consisting of segments of length 1. Let $\widetilde{\mathfrak{n}}$ be a ρ-right-derivative compatible lift of $\mathfrak{n}$. Since $\mathfrak{n}$ is banal, it follows that $\widetilde{\mathfrak{n}}$ can be chosen such that $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{n}})) = \mathrm{Z}(\mathfrak{n})$ by [100, §6]. Let $d = \min(d_{-1}, d_0)$ and write

$$\mathfrak{n} = \overbrace{[-1,-1]_\rho + \ldots + [-1,-1]_\rho}^{d_{-1}} + \overbrace{[0,0]_\rho + \ldots + [0,0]_\rho}^{d_0} + \ldots.$$

Setting $A = \{d_{-1}+1, \ldots, d_{-1}+d\}$ and $f(d_{-1}+i) = i$ for all $i \in \{1, \ldots, d\}$ gives a best matching function (A, f) of $\mathfrak{n}$ with respect to ρ. We let $\widetilde{\rho}$ be the lift of ρ in the definition of $\widetilde{\mathfrak{n}}$.

Thus $d_{-1} < d_0$ if and only if $\mathcal{D}_{r,\rho}(\mathfrak{n}) \neq 0$ which is equivalent to $\mathcal{D}_{r,\widetilde{\rho}}(\widetilde{\mathfrak{n}}) \neq 0$ by Lemma 4.3.11. Since the lemma holds over $\overline{\mathbb{Q}}_\ell$, we have by Lemma 4.3.2 that this is equivalent to the existence of $\widetilde{\tau} \in \mathfrak{Irr}$ such that $\widetilde{\tau} \otimes \widetilde{\rho}$ appears in $r_{(\deg \mathfrak{n} - m, m)}(\mathrm{Z}(\widetilde{\mathfrak{n}}))$. Because parabolic restriction commutes with reduction mod ℓ, we obtain from Lemma 4.3.2 that this is equivalent to $\mathrm{Z}(\mathfrak{n})$ admitting a right ρ-derivative. The second claim follows from an analogous argument. □

Lemma 4.3.14. *Let π be a non-residually degenerate representation of G_n such that there exists $\rho \in \mathfrak{C}_m^{\square}$ with*

$$[r_{(m,n-m)}(\pi)] \geq [\rho v_\rho^{-1} \otimes \mathrm{Z}(\mathfrak{n})]$$

and $\mathfrak{n}$ is residually degenerate. Then

$$[r_{(n-m,m)}(\pi)] \geq [\mathrm{Z}(\mathfrak{n} + [-1,-1]_\rho - [0,0]_\rho) \otimes \rho].$$

Proof. Using Lemma 2.3.7 it is easy to see that it is enough to show the claim for π having cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$. If $[\pi] \leq \mathrm{I}(\mathfrak{m})$ with $\mathfrak{m}$ a multisegment containing at least one segment not of length 1, Theorem 2.3.13 shows that above situation can happen if and only if $\mathfrak{m} = \mathfrak{n} + [-1,0]_\rho - [0,0]_\rho$ and $\pi \cong \mathrm{Z}(\mathfrak{m})$. In this case it is also easy to see that the only $(\deg(\mathfrak{n}), m)$-degenerate representation appearing in the parabolic restriction $r_{(\deg(\mathfrak{n}),m)}(\mathrm{I}(\mathfrak{m}))$ is $\mathrm{Z}(\mathfrak{n} + [-1,-1]_\rho - [0,0]_\rho) \otimes \rho$. Since the parabolic restriction of π must contain a $(\deg(\mathfrak{n}), m)$-degenerate representation, we are done in this case.

We thus assume that π does not appear as a subquotient of such a $\mathrm{I}(\mathfrak{m})$. We write $\mathfrak{n}' \coloneqq \mathfrak{n} + [-1,-1]_\rho$ and let d_i be the multiplicity of $[i,i]_\rho$ in $\mathfrak{n}'$, $i \in \{-1, \dots, o(\rho)-2\}$. Choose a lift $\widetilde{\rho}$ of ρ to $\overline{\mathbb{Q}}_\ell$ and set

$$\widetilde{\mathfrak{n}} \coloneqq d_{-1}[-1,-1]_{\widetilde{\rho}} + d_0[0,0]_{\widetilde{\rho}} + \dots + d_{o(\rho)-2}[o(\rho)-2, o(\rho)-2]_{\widetilde{\rho}}.$$

Then $\widetilde{\mathfrak{n}}$ is a ρv_ρ^{-1}-left-derivative compatible lift of $\mathfrak{n}'$ and $[\pi] \leq \mathrm{I}(\mathfrak{n}') = \mathrm{r}_\ell(\mathrm{I}(\widetilde{\mathfrak{n}}))$. Let $\pi' \in \mathfrak{Irr}$ be such that $[\widetilde{\pi}] \leq \mathrm{I}(\widetilde{\mathfrak{n}})$ and $[\pi] \leq \mathrm{r}_\ell(\widetilde{\pi})$. Since over $\overline{\mathbb{Q}}_\ell$ we already have the subjectivity of Z, we have $\widetilde{\pi} \cong \mathrm{Z}(\widetilde{\mathfrak{k}})$ for some multisegment $\widetilde{\mathfrak{k}}$. Since

$$[\pi] \leq \mathrm{r}_\ell(\widetilde{\pi}) \leq \mathrm{r}_\ell(\mathrm{I}(\widetilde{\mathfrak{k}})) = \mathrm{I}(\mathrm{r}_\ell(\widetilde{\mathfrak{k}})),$$

the observations of the previous paragraph imply that $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{n}}$. By Lemma 4.3.13 and Lemma 4.3.2 we have that $d_{-1} > d_0$ and

$$\begin{aligned}
&[\rho v_\rho^{-1} \otimes (\rho v_\rho^{-1})^{d_{-1}-d_0-1} \otimes \mathrm{Z}(\mathfrak{n}' - (d_{-1} - d_0)[-1,-1]_\rho)] \leq \\
&\leq [r_{((m,(d_{-1}-d_0-1)m, \deg(\mathfrak{m})-(d_{-1}-d_0)m)}(\rho v_\rho^{-1} \otimes \mathrm{Z}(\mathfrak{n}))] \leq \\
&\leq [r_{((m,(d_{-1}-d_0-1)m, \deg(\mathfrak{m})-(d_{-1}-d_0)m)}(\pi)].
\end{aligned}$$

Thus

$$[(\rho v_\rho^{-1})^{d_{-1}-d_0} \otimes \mathrm{Z}([\mathfrak{n}' - (d_{-1} - d_0)[-1,-1]_\rho)] \leq [r_{(d_{-1}-d_0)m, \deg(\mathfrak{m})-(d_{-1}-d_0)m)}(\pi)].$$

Since we already know Theorem 4.3.10 to be true over $\overline{\mathbb{Q}}_\ell$, we know by Corollary 4.3.2 that

$$(\widetilde{\rho} v_{\widetilde{\rho}}^{-1})^{d_{-1}-d_0} \otimes \mathrm{Z}([\widetilde{\mathfrak{n}} - (d_{-1} - d_0)[-1,-1]_{\widetilde{\rho}})$$

appears with multiplicity 1 in $r_{(d_{-1}-d_0)m, \deg(\mathfrak{m})-(d_{-1}-d_0)m)}(\mathrm{Z}(\widetilde{\mathfrak{n}}))$. Furthermore, since parabolic restriction commutes with reduction mod ℓ, Corollary 4.3.2 also shows that

$$(\rho v_\rho^{-1})^{d_{-1}-d_0} \otimes \mathrm{Z}(\mathfrak{n}' - (d_{-1} - d_0)[-1,-1]_\rho) \tag{31}$$

appears in

$$r_{(d_{-1}-d_0)m, \deg(\mathfrak{m})-(d_{-1}-d_0)m)}(\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{n}}))). \tag{32}$$

By Lemma 4.3.12 and Theorem 2.3.17 (31) appears with multiplicity 1 in (32). Moreover, by Lemma 4.3.13 (31) appears also in the parabolic restriction of $\mathrm{Z}(\mathfrak{n}')$, which appears in $\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{n}}))$ by Theorem 2.3.17. Thus $\pi \cong \mathrm{Z}(\mathfrak{n}')$, a contradiction to the fact that π is not residually degenerate. $\square$

Lemma 4.3.15. *Let $\mathfrak{m}$ be an aperiodic multisegment, $\rho \in \mathfrak{C}_m^{\square}$ and define the composition* $\alpha := \alpha_{\mathfrak{m},\rho} = (\mu_{\overline{\mathcal{D}_{r,\rho,max}(\mathfrak{m})}}, md_{r,\rho}(\mathfrak{m}))$. *Then*

$$[r_\alpha(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \otimes \rho^{d_{r,\rho}(\mathfrak{m})}].$$

Proof. We argue by induction on t, the maximal length of a segment in $\mathfrak{m}$. Note that we have by Lemma 2.3.16

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{Z}(\mathfrak{m}^-) \otimes \mathrm{Z}(\mathfrak{m}^1)]$$

and hence by the induction hypothesis, we know that

$$[r_{(\alpha_{\mathfrak{m}^-,\rho v_\rho^{-1}},\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes (\rho v_\rho^{-1})^{d_{r,\rho v_\rho^{-1}}} \otimes \mathrm{Z}(\mathfrak{m}^1)].$$

Let $\pi \in \mathfrak{Irr}_N$, $N := m \cdot d_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-) + \deg(\mathfrak{m}_1)$, $l := d_{r,\rho v_\rho^{-1}}(\mathfrak{m}^-)$ be such that

$$[r_{(\deg \mathfrak{m}-N,N)}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \pi], \tag{33}$$

$$[r_{(N-\deg(\mathfrak{m}^1),\deg(\mathfrak{m}_1))}(\pi)] \geq [(\rho v_\rho^{-1})^l \otimes \mathrm{Z}(\mathfrak{m}^1)]. \tag{34}$$

From Theorem 2.3.13 it follows that $ml \leq \deg \mathfrak{m}_1$ and the maximal composition α' such that π is α'-degenerate is

$$(\deg(\mathfrak{m}_1), ml).$$

Since we know from (34) that

$$[r_{(m,\dots,m,\deg(\mathfrak{m}_1))}(\pi)] \geq [\rho v_\rho^{-1} \otimes \dots \rho v_\rho^{-1} \otimes \mathrm{Z}(\mathfrak{m}^1)]$$

we can just apply Lemma 4.3.14 l-times and obtain

$$[r_{(\deg(\mathfrak{m}_1),m,\dots,m)}(\pi)] \geq [\mathrm{Z}(\mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho) \otimes \rho \otimes \dots \otimes \rho].$$

Now since $o(\rho) > 1$ and ρ is supercuspidal, this implies

$$[r_{(\deg(\mathfrak{m}_1),N-\deg(\mathfrak{m}^1))}(\pi)] \geq [\mathrm{Z}(\mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho) \otimes \rho^l]. \tag{35}$$

Write $\mathfrak{n} = \mathfrak{m}^1 + l[-1,-1]_\rho - l[0,0]_\rho$. Combining (33) and (35) shows that

$$[r_{(\deg \mathfrak{m}-N,\deg(\mathfrak{m}_1),N-\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}) \otimes \rho^l]. \tag{36}$$

Write now $\mathfrak{n} = \mathfrak{n}' + k[0,0]_\rho$ such that $\mathfrak{n}'$ contained at least as many copies of $[-1,-1]_\rho$ as of $[0,0]_\rho$. Applying Lemma 4.3.13 k-times in combination with Lemma 4.3.2, we see that

$$[r_{(\deg(\mathfrak{n}'),m,\dots,m)}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}') \otimes \rho \otimes \dots \otimes \rho]$$

or equivalently

$$[r_{(\deg(\mathfrak{n}'),km)}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}') \otimes \rho^k]. \tag{37}$$

Combining (37) and (36), we obtain that

$$[r_{(\deg \mathfrak{m}-N,\deg(\mathfrak{n}'),N-\deg(\mathfrak{n}'))}(\mathrm{Z}(\mathfrak{m}))] \geq [\mathrm{St}(\mathcal{D}_{r,\rho v_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}') \otimes \rho^{l+k}].$$

Now $l+k$ is the number of free ρ-segments in $\mathfrak{m}$ since l is by Corollary 4.3.3 the number of free ρ-segments of length greater than 1 and k is the number of free ρ-segments of length precisely 1. Moreover, Corollary 4.3.3 also shows that

$$\mathcal{D}_{r,\rho,max}(\mathfrak{m})^- = \mathcal{D}_{r,\rho\nu_\rho^{-1},max}(\mathfrak{m}^-),\ \mathcal{D}_{r,\rho,max}(\mathfrak{m})^1 = \mathfrak{m}^1 - (k+l)[0,0]_\rho + l[1,1]_\rho.$$

Thus

$$\mathrm{St}(\mathcal{D}_{r,\rho\nu_\rho^{-1},max}(\mathfrak{m}^-)) \otimes \mathrm{Z}(\mathfrak{n}') = \mathrm{St}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})),\ \rho^{l+k} = \rho^{d_{r,\rho}(\pi)},$$

which finishes the induction step. $\square$

Proof of Theorem 4.3.10. We will only proof $\mathcal{D}_{r,\rho,max}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m}))$, the other claims follow then quickly using Corollary 4.3.2 and Lemma 4.3.4.

We will show that $\mathrm{Z}(\mathfrak{m}) = \mathrm{soc}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \times \rho^{d_{r,\rho}(\mathfrak{m})})$. To do so we first can reduce the claim via Lemma 2.3.7 to the assumption that $\mathfrak{m}$ has cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$. We choose a ρ-right-derivative compatible lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$. Then $\mathrm{r}_\ell(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) = \mathcal{D}_{r,\rho,max}(\mathfrak{m})$ and from the $\overline{\mathbb{Q}}_\ell$-case we already know that

$$\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\widetilde{\mathfrak{m}})) = \mathcal{D}_{r,\rho,max}(\mathrm{Z}(\widetilde{\mathfrak{m}})).$$

We set $\pi := \mathrm{soc}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \times \rho^{d_{r,\rho}(\mathfrak{m})})$ and $\alpha := (\deg(\mathcal{D}_{r,\rho,max}(\mathfrak{m})), m \cdot d_{r,\rho}(\mathfrak{m}))$. By Theorem 2.3.17

$$[\pi] \le [\mathrm{soc}(\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \times \rho^{d_{r,\rho}(\mathfrak{m})})] \le [\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\widetilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \widetilde{\rho}^{d_{r,\rho}(\mathfrak{m})})]$$

and

$$[\mathrm{Z}(\mathfrak{m})] \le [\mathrm{r}_\ell(\mathrm{Z}(\widetilde{\mathfrak{m}}))] \le [\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\widetilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \widetilde{\rho}^{d_{r,\rho}(\mathfrak{m})})].$$

Moreover, Frobenius reciprocity implies $r_\alpha(\pi)$ contains

$$\mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})) \otimes \rho^{d_{r,\rho}(\mathfrak{m})}. \tag{38}$$

and Lemma 4.3.15 implies there exists $\pi' \in \mathfrak{Irr}$ which is $\mu_{\overline{\mathcal{D}_{r,\rho,max}(\mathfrak{m})}}$-degenerate such that

$$[r_\alpha(\mathrm{Z}(\mathfrak{m}))] \ge [\pi' \otimes \rho^{d_{r,\rho}(\mathfrak{m})}].$$

Lemma 4.3.12 shows that

$$[\pi'] \le \mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\widetilde{\rho},max}(\widetilde{\mathfrak{m}}))) \le \mathrm{r}_\ell(\mathrm{I}(\mathcal{D}_{r,\widetilde{\rho},max}(\widetilde{\mathfrak{m}}))) = \mathrm{I}(\mathcal{D}_{r,\rho,max}(\mathfrak{m})).$$

Hence Theorem 2.3.13 shows that $\pi \cong' \mathrm{Z}(\mathcal{D}_{r,\rho,max}(\mathfrak{m}))$. But Lemma 4.3.12 also shows that (38) appears with multiplicity 1 in

$$r_\alpha(\mathrm{r}_\ell(\mathrm{Z}(\mathcal{D}_{r,\widetilde{\rho},max}(\widetilde{\mathfrak{m}})) \times \widetilde{\rho}^{d_{r,\rho}(\mathfrak{m})})). \tag{39}$$

Therefore $\pi \cong \mathrm{Z}(\mathfrak{m})$. $\square$

4.4 Applications of ρ-derivatives

In this section we give several applications of the above developed machinery.

4.4.1 Surjectivity of Z The first consequence is the surjection of the map Z for representations with □-irreducible cuspidal support over $\overline{\mathbb{F}}_\ell$.

Theorem 4.4.1. *Let $\pi \in \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell)$ with □-irreducible cuspidal support. Then there exists an aperiodic multisegment $\mathfrak{m}$ such that $\pi \cong \mathrm{Z}(\mathfrak{m})$.*

Proof. We argue by induction on n. Let π be an irreducible representation of G_n with □-irreducible cuspidal support and let $\pi' \in \mathfrak{Irr}$ and ρ be a □-irreducible cuspidal representation such $\pi \hookrightarrow \pi' \times \rho$. By the induction hypothesis, we know that there exists an aperiodic multisegment $\mathfrak{m}'$ such that $\pi' \cong \mathrm{Z}(\mathfrak{m}')$. By Theorem 4.3.10 we have then that

$$\pi \cong \mathrm{soc}(\pi' \times \rho) = \mathrm{Z}(\mathrm{soc}(\mathfrak{m}', \rho)).$$

□

4.4.2 Aubert-Zelevinsky dual We will now give an explicit description of the Aubert-Zelevinsky dual π^* using the theory of derivatives. Recall the Grothendieck group $\mathfrak{R} = \bigcup_{n\in\mathbb{N}} \mathfrak{R}_n$. In [132] the Aubert-Zelevinsky involution

$$\mathrm{D}\colon \mathfrak{R} \to \mathfrak{R},$$

which sends $[\pi] \in \mathfrak{Irr}_n$ to

$$\mathrm{D}([\pi]) := \sum_{\alpha} (-1)^{r(\alpha)} [\mathrm{Ind}_\alpha \circ r_\alpha(\pi)]$$

was introduced, where the sum is over all compositions of n and $r(\alpha)$ is the number of elements of α. For τ a representation of finite length, we write

$$\mathrm{D}(\tau) = \sum_{\pi\in\mathfrak{Irr}} k_\pi[\pi],\ k_\pi \in \mathbb{Z},$$

where all but finitely many k_π are non-zero. We say $\pi \in \mathfrak{Irr}$ appears in $\mathrm{D}(\tau)$ if $k_\pi \neq 0$. The map D satisfies the following properties.

Theorem 4.4.2. *Let* D *be the Aubert-Zelevinsky involution.*

1. *If π_1, π_2 are irreducible representations then*

$$\mathrm{D}([\pi_1 \times \pi_2]) = \mathrm{D}([\pi_1]) \times \mathrm{D}([\pi_2]).$$

2. *If β is a composition of n and π an irreducible representation then*

$$\mathrm{D}([r_\beta(\pi)]) = \mathrm{Ad}(w_\beta) \circ r_\beta(\mathrm{D}([\pi])),$$

   where w_β is the longest element of the Weyl group $P_\beta \backslash G_n / P_{(1,\dots,1)}$.

3. *If π is irreducible, there exists a unique irreducible representation π^* with the same cuspidal support as π appearing in $\mathrm{D}([\pi])$ with sign $(-1)^{r(\pi)}$, where $r(\pi)$ is the length of the cuspidal support of π.*

Proof. For (1) and (2) see [10, Theorem 1.7] and [100, Proposition A.2]. For (3) see [98] for the case $R = \overline{\mathbb{Q}}_\ell$ and [96, Theorem 2.5] for the general case $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$. □

Theorem 4.4.3. *Let $\pi \in \mathfrak{Irr}_n$ and $\rho \in \mathfrak{C}^{\square}$. Then*

$$\mathcal{D}_{r,\rho}(\pi)^* \cong \mathcal{D}_{l,\rho}(\pi^*).$$

Proof. If $\pi = \rho^k$, the claim follows from Theorem 4.4.2(1). Thus we can assume that $\mathcal{D}_{r,\rho,max}(\pi) \neq 0$. By Frobenius reciprocity

$$[r_{(n-d_{r,\rho}(\pi)m,d_{r,\rho}(\pi)m)}(\pi)] \geq [\mathcal{D}_{r,\rho,max}(\pi) \otimes \rho^{d_{r,\rho}(\pi)}]$$

and hence by Theorem 4.4.2 (2)

$$r_{(d_{r,\rho}(\pi)m,n-d_{r,\rho}(\pi)m)}(\mathrm{D}([\pi]))]$$

contains the representation $\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{r,\rho,max}(\pi)^*$ with a non-zero coefficient

$$(-1)^{d_{r,\rho}(\pi)}(-1)^{r(\mathcal{D}_{r,\rho,max}(\pi))}.$$

Recall that $(-1)^{r(\pi)}[\pi^*]$ is the unique constituent of $\mathrm{D}([\pi])$ with cuspidal support the same as π, therefore it follows that

$$(-1)^{r(\pi)}[r_{(d_{r,\rho}(\pi)m,n-d_{r,\rho}(\pi)m)}(\pi^*)] \geq$$
$$\geq (-1)^{d_{r,\rho}(\pi)}(-1)^{r(\mathcal{D}_{r,\rho,max}(\pi))}[\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{r,\rho,max}(\pi)^*] =$$
$$= (-1)^{r(\pi)}[\rho^{d_{r,\rho}(\pi)} \otimes \mathcal{D}_{l,\rho,max}(\pi^*)],$$

where the last equality follows from the induction hypothesis. The claim then follows from Corollary 4.3.2. $\square$

If $\mathfrak{m}$ is an aperiodic multisegment $\mathfrak{m}$, we set $\mathfrak{m}^*$ to the aperiodic multisegment such that $\mathrm{Z}(\mathfrak{m})^* = \mathrm{Z}(\mathfrak{m}^*)$.

For $\mathfrak{m}$ an aperiodic multisegment with $\square$-irreducible cuspidal support we can thus offer the following algorithm to compute $\mathfrak{m}^*$, where $\mathfrak{m}^*$ is the aperiodic multisegment such that $\mathrm{Z}(\mathfrak{m})^* = \mathrm{Z}(\mathfrak{m}^*)$. It works recursively on $\deg(\mathfrak{m})$. Let $\rho \in \mathfrak{C}^{\square}$ such that $\mathcal{D}_{r,\rho}(\mathfrak{m}) \neq 0$. We then compute $\mathfrak{m}^*$ by first computing $\mathcal{D}_{r,\rho}(\mathfrak{m})$. From this we can already compute $\mathcal{D}_{r,\rho}(\mathfrak{m})^*$ recursively. But by Theorem 4.4.2 and Theorem 4.3.10 $\mathcal{D}_{r,\rho}(\mathfrak{m})^* = \mathcal{D}_{l,\rho}(\mathfrak{m}^*)$. By Lemma 4.3.1 and Theorem 4.3.10

$$\mathrm{soc}(\rho, \mathcal{D}_{r,\rho}(\mathfrak{m})^*) = \mathfrak{m}^*.$$

Note that we thus rediscover the algorithm of computing the Aubert-Zelevinsky dual of [87].

4.4.3 Godement-Jacquet local factors In this section we will compute the local L-factors associated to an irreducible smooth representation over $\overline{\mathbb{F}}_\ell$ and show that the map

$$\mathrm{C}\colon \mathfrak{Irr}_n \xrightarrow{\cong} \{\mathrm{C}-\text{parameters of length } n\}$$

defined in [74] respects these local L-factors. We start by proving the following lemma, which in the case $R = \overline{\mathbb{Q}}_\ell$ was proved in [59, Proposition 2.3].

Lemma 4.4.4. *Let $\sigma_1, \dots, \sigma_k$ be irreducible representations of $G_{n_1}, \dots, G_{n_k}$ over R. Then*

$$L(\sigma_1 \times \ldots \times \sigma_k, T) = \prod_{i=1}^{k} L(\sigma_i, T)$$

and

$$\gamma(T, \sigma_1 \times \ldots \times \sigma_k, \psi_v) = \prod_{i=1}^{k} \gamma(T, \sigma_i, \psi_v).$$

Proof. We can mimic the proof of [59, Proposition 2.3] more or less *muta mutandis*. We start by showing the inclusion

$$\mathcal{L}(\sigma_1 \times \ldots \times \sigma_k) \subseteq \prod_{i=1}^{k} \mathcal{L}(\sigma_i).$$

Set $\sigma \coloneqq \sigma_1 \times \ldots \times \sigma_k$, $\alpha = (n_1, \dots, n_k)$ and $P \coloneqq P_\alpha$ with Levi-component M and unipotent component U. Recall that

$$\delta_P(m) = \prod_{i=1}^{k} |\det{}'(m_i)|_v^{d_v \delta_i},\ \delta_i \coloneqq -n_1 - \ldots - n_{i-1} + n_{i+1} + \ldots + n_k.$$

Moreover, recall that as in [59, Proposition 2.3] the matrix coefficients of σ are exactly given by integrals of the form

$$f(g) = \int\limits_{P\backslash G_n} H(g'g, g')\, \mathrm{d}g', \tag{40}$$

$$H(g_1, g_2) = h^\vee(g_1)(h(g_2)),\ h \in \sigma_1 \times \ldots \times \sigma_k,\ h^\vee \in \sigma_1^\vee \times \ldots \times \sigma_k^\vee.$$

Note that H is smooth, it satisfies for all $p \in P$, $g_1, g_2 \in G_n$

$$H(pg_1, pg_2) = \delta_P(p) H(g_1, g_2)$$

and $m \mapsto H(mg_1, g_2)$ is a matrix coefficient of $\sigma_1 \otimes \ldots \otimes \sigma_k \otimes \delta_P^{\frac{1}{2}}$. Furthermore, given such H, the function

$$f(g) = \int\limits_{P\backslash G_n} H(g'g, g')\, \mathrm{d}g' = \int\limits_{(K_n \cap P)\backslash K_n} H(kg, k)\, \mathrm{d}k$$

is a matrix coefficient of σ. Fix now $\phi \in S_{\overline{\mathbb{F}}_\ell}(M_{n,n}(\mathbb{D}_v))$ and as in [59, Proposition 2.3] we see that the coefficient of T^N in $Z(\phi, Tq^{-\frac{d_v n - 1}{2}}, f)$ is

$$\int\limits_{(K_n \cap P)\backslash K_n} \int\limits_{(K_n \cap P)\backslash K_n} \int\limits_{P(N)} \phi(k^{-1} p k') \delta_P(p)^{-1} |\det{}'(p)|_v^{\frac{d_v n - 1}{2}} H(pk', k)\, \mathrm{d}_r p\, \mathrm{d}k\, \mathrm{d}k' \tag{41}$$

We write for

$$m = (m_1, \ldots m_k),\ u = \begin{pmatrix} 1_{n_1} & \dots & u_{i,j} \\ 0 & \ddots & \vdots \\ 0 & 0 & 1_{n_k} \end{pmatrix},\ p(m,u) \coloneqq \begin{pmatrix} m_1 & \dots & u_{i,j} \\ 0 & \ddots & \vdots \\ 0 & 0 & m_k \end{pmatrix}$$

and express $\mathrm{d}_r p$ as

$$\mathrm{d}_r p = \mathrm{d}u\, \mathrm{d}m \prod_{i=1}^{k} |\det{}'(m_i)|_v^{\frac{d(n - \delta_i - n_i)}{2}}.$$

Thus (41) equals to

$$\int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{M(N)}\ \int\limits_{U} \phi(k^{-1}p(m,u)k')\,\mathrm{d}u$$

$$\prod_{i=1}^{k}|\det{}'(m_i)|_v^{\frac{d_v(n_i-n-\delta_i)}{2}}|\det{}'(m)|_v^{\frac{d_vn-1}{2}}H(mk',k)\,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}k'.$$

For $m\in M$ and $k,k'\in(K_n\cap P)\backslash K_n$ we set

$$\phi(m;k,k'):=\int\limits_{U}\phi(k^{-1}p(m,u)k)\,\mathrm{d}u,\ h(m;k,k'):=H(mk',k)\delta_P^{-\frac{1}{2}}(m). \tag{42}$$

Thus the coefficient of T^N is

$$\int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{M(N)} \phi(m;k,k')h(m;k,k')\prod_{i=1}^{k}|\det{}'(m_i)|_v^{\frac{d_vn_i-1}{2}}\,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}k'. \tag{43}$$

As in [59, Proposition 2.3] one sees then that this is the coefficient of T^N of a finite sum of elements of the form

$$\prod_{i=1}^{k}Z(\phi_i,Tq^{-\frac{d_vn_i-1}{2}},f_i)$$

with f_i a matrix coefficient of σ_i and $\phi_i\in S_{\overline{\mathbb{F}}_\ell}(M'_{n_i,n_i}(\mathbb{K}_v))$ and hence

$$\mathcal{L}(\sigma_1\times\ldots\times\sigma_k)\subseteq\prod_{i=1}^{k}\mathcal{L}(\sigma_i).$$

Next we show that

$$\prod_{i=1}^{k}\mathcal{L}(\sigma_i)\subseteq\mathcal{L}(\sigma_1\times\ldots\times\sigma_k).$$

Given $\phi_i\in S_{\overline{\mathbb{F}}_\ell}(M'_{n_i,n_i}(\mathbb{K}_v))$, we can find $\phi\in S_{\overline{\mathbb{F}}_\ell}(M_{n,n}(\mathbb{D}_v))$ such that

$$\int\limits_{U}\phi(p(m,u))\,\mathrm{d}u=\prod_{i=1}^{k}\phi_i(m_i),\ m=(m_1,\ldots,m_k)\in M.$$

Moreover, we choose two smooth functions $\xi,\xi'\in S((K_n\cap P)\backslash K_n)$ such that for all $p\in P$ and $N\in\mathbb{Z}$

$$\int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{(K_n\cap P)\backslash K_n}\ \int\limits_{P}(\phi\chi_{G_n(N)})(k^{-1}pk')\xi(k)\xi'(k')\delta_P^{-1}(p)\,\mathrm{d}_rp\,\mathrm{d}k\,\mathrm{d}k'=$$

$$=\int\limits_{P(N)}\phi(p)\delta_P^{-1}(p)\,\mathrm{d}_rp,$$

where $\chi_{G_n(N)}$ denotes the characteristic function of $G_n(N)$. For example, one could take the characteristic functions of sufficiently small open compact subgroups in G_n whose pro-order is invertible in R and rescale them appropriately. Projecting them into $S((K_n \cap P)\backslash K_n)$ gives then the desired functions. Then the coefficient of T^N in

$$\prod_{i=1}^{k} Z(\phi_i, Tq^{-\frac{d_v n_i - 1}{2}}, f_i)$$

is

$$\int_{M(N)} \prod_{i=1}^{k} f_i(m_i)\phi_i(m_i)|\det{}'(m_i)|_v^{\frac{d_v n_i - 1}{2}} \,\mathrm{d}m =$$

$$= \int_{(K_n\cap P)\backslash K_n} \int_{(K_n\cap P)\backslash K_n} \int_{P(N)} |\det{}'(p)|_v^{\frac{d_v n-1}{2}} \delta_P^{\frac{1}{2}}(m)\phi(k^{-1}pk')\delta_P^{-1}(p) \tag{44}$$

$$\prod_{i=1}^{k} f_i(m_i)\xi(k)\xi'(k') \,\mathrm{d}k \,\mathrm{d}k' \,\mathrm{d}_r p.$$

Setting

$$H_1(p,k,k') \coloneqq \prod_{i=1}^{k} f_i(m_i)\xi(k)\xi'(k')$$

and choosing H as in (40) with $H(pk,k') = H_1(p,k,k')$ shows that (44) is the coefficient of T^N in $Z(\phi, Tq^{-\frac{d_v n-1}{2}}, f)$ as in (41).

Finally, to prove the equality of the γ-factors, we go back to (42) and define analogously for $\sigma_1^\vee \times \ldots \times \sigma_k^\vee$, $h^\vee(m;k,k')$ and $\phi^\vee(m;k,k')$. Observe that if $h^\vee$ is defined via $f^\vee$, the dual matrix coefficient of f, then

$$h^\vee(m;k,k') = h(m^{-1};k,k')$$

and if $\phi^\vee(m;k,k')$ is defined using the Fourier-transform of ϕ, then

$$\phi^\vee(m;k,k') = (\widehat{\phi})(m;k,k').$$

The claim follows then by writing $Z(\phi, Tq^{-\frac{d_v n-1}{2}}, f)$ and $Z(\widehat{\phi}, T^{-1}q^{-\frac{d_v n+1}{2}}, f^\vee)$ in the form of (43). $\square$

We write for $\mathfrak{m}$ an aperiodic multisegment $\langle \mathfrak{m}\rangle \coloneqq \mathrm{Z}(\mathfrak{m})^*$.

Theorem 4.4.5. *Let $\Delta = [a,b]_\rho$ be a segment over R. If $\rho \cong \chi$ for an unramified character χ of $\mathbb{K}_v$ and $q^d \neq 1$, then*

$$L(\langle\Delta\rangle, T) = \frac{1}{1 - \chi(\varpi_v) q_v^{-d_v b + \frac{1-d}{2}} T}.$$

Otherwise

$$L(\langle\Delta\rangle, T) = 1.$$

More generally, if $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ is an aperiodic multisegment then

$$\mathcal{L}(\langle\mathfrak{m}\rangle) = \mathcal{L}(\langle\Delta_1\rangle \times \ldots \times \langle\Delta_k\rangle)$$

and

$$L(\langle\mathfrak{m}\rangle, T) = L(\langle\Delta_1\rangle \times \ldots \times \langle\Delta_k\rangle, T) = \prod_{i=1}^{k} L(\langle\Delta_i\rangle, T).$$

If $R = \overline{\mathbb{Q}}_\ell$ this is [59, Theorem 2.7] and the case $\mathfrak{m}$ a banal multisegment was covered in [94, Theorem 3.1, Theorem 5.7]. From now on we thus assume that $R = \overline{\mathbb{F}}_\ell$.

Proof of Theorem 4.4.5. The idea of the proof is to apply Proposition 4.2.9 together with Corollary 4.3.2. Indeed, note that this shows that

$$P(\mathcal{D}_{r,\rho,max}(\pi),T)|P(\pi,T) \tag{45}$$

for all □-irreducible cuspidal unramified representations ρ. Furthermore, note that by Theorem 4.4.3 we have a precise description of $\mathcal{D}_{r,\rho,max}(\langle\mathfrak{m}\rangle)$ in terms of the multisegment $\mathfrak{m}$.

We start by proving the claim that

$$L(\langle[0,b]_\rho\rangle,T) = \frac{1}{1-\chi(\varpi_v)q_v^{-d_v b+\frac{1-d}{2}}T}$$

if ρ is unramified and □-irreducible and 1 otherwise. We proceed by induction on $l([0,b]_\rho)$. Note that the base-case is [94, Theorem 3.1] and if ρ is either ramified or not □-irreducible, Lemma 2.3.20 implies

$$P(\langle[0,b]_\rho\rangle,T)|_v P(\rho^{b+1},T) \overset{Lemma\ 4.4.4}{=} 1,$$

which proves the claim. Thus we assume ρ unramified and □-irreducible. Then the claim follows from the induction-hypothesis since

$$\mathcal{D}_{r,\rho,max}(\langle[0,b]_\rho\rangle) = \langle[1,b]_\rho\rangle$$

by Theorem 4.4.3 and Lemma 2.3.10 and hence by (45)

$$1-\rho(\varpi_v)q_v^{-d_v b+\frac{1-d}{2}}T|P(\langle[0,b]_\rho\rangle,T).$$

On the other hand, since ρ is supercuspidal, we can find a lift $[0,b]_{\widetilde{\rho}}$ with $\widetilde{\rho}$ unramified and hence $\mathrm{r}_\ell(\langle[0,b]_{\widetilde{\rho}}\rangle)$ contains $[0,b]_\rho$. By Lemma 2.3.20 and Lemma 2.3.19 also

$$P(\langle[0,b]_\rho\rangle,T)|1-\rho(\varpi_v)q_v^{-d_v b+\frac{1-d}{2}}T,$$

proving the claim for a segment.

Next we compute the L-function of $\langle\mathfrak{m}\rangle$. Let us argue by induction on $\deg(\mathfrak{m})$ and assume that we have proven the claim already for all multisegments of degree lesser than $\deg(\mathfrak{m})$, the base case being [94, Theorem 3.1]. We can write $\langle\mathfrak{m}\rangle = \pi_1\times\pi_2$, where π_1 has □-irreducible unramified cuspidal support and π_2 has □-reducible or ramified cuspidal support by Lemma 2.3.7. Since a representation induced from □-reducible cuspidal representations or ramified cuspidal support has trivial L-factor by Lemma 4.4.4, $L(\pi_2,T) = 1$ by Lemma 2.3.20. Again by Lemma 4.4.4

$$L(\langle\mathfrak{m}\rangle,T) = L(\pi_1,T)L(\pi_2,T) = L(\pi_1,T)$$

and hence we can assume that $\langle\mathfrak{m}\rangle$ has □-irreducible cuspidal unramified support.

From the $\overline{\mathbb{Q}}_\ell$-case we obtain that for any lift $\widetilde{\mathfrak{m}}$ of $\mathfrak{m}$

$$\prod_{i=1}^k P(\langle \Delta_i \rangle, T) = \mathrm{r}_\ell(P(\langle \widetilde{\mathfrak{m}} \rangle), T),$$

since we assumed $\langle \mathfrak{m} \rangle$ to have $\square$-irreducible unramified cuspidal support. From (45) and Theorem 4.3.10 we know that

$$P(\mathcal{D}_{r,\rho,max}(\langle \mathfrak{m} \rangle), T) | P(\langle \mathfrak{m} \rangle, T)$$

and from Lemma 2.3.20 and Lemma 4.4.4 it follows that

$$P(\langle \mathfrak{m} \rangle, T) | \prod_{i=1}^k P(\langle \Delta_i \rangle, T).$$

Fix now a $\square$-irreducible cuspidal unramified character ρ such that $\mathcal{D}_{r,\rho}(\pi) \neq 0$. Let moreover (A, f) be a best matching function of $\mathfrak{m}^\vee$ with respect to $\rho^\vee$. By Theorem 4.3.10, Theorem 4.4.2 and the induction-hypothesis we then have that

$$\prod_{i=1}^k P(\langle \Delta_i \rangle, T) P(\mathcal{D}_{r,\rho,max}(\langle \mathfrak{m} \rangle), T)^{-1} = P(\rho, T)^{-k},$$

where k is the multiplicity of $[0,0]_\rho$ in $\mathfrak{m} - (\mathfrak{m}_f^\vee)^\vee$. Similarly, for a fixed $a \in \mathbb{Z}$ such that $\mathcal{D}_{\rho^\vee v_\rho^{-a}, r}(\langle \mathfrak{m}^\vee \rangle) \neq 0$, we let (A', f') be a best matching function of $\mathfrak{m}$ with respect to ρv_ρ^a. Set k' to the multiplicity of $[-a,-a]_{\rho^\vee}$ in $\mathfrak{m}^\vee - (\mathfrak{m}_{f'})^\vee$. We thus obtain that

$$P(\langle \mathfrak{m} \rangle, T) P(\rho, T)^l = \prod_{i=1}^k P(\langle \Delta_i \rangle, T)$$

for some $l \leq k$ and

$$P(\langle \mathfrak{m}^\vee \rangle), T) P(\rho^\vee v_\rho^{-a}, T)^{l'} = \prod_{i=1}^k P(\langle \Delta_i^\vee \rangle, T)$$

for $l' \leq k'$. We will now show that $l = l' = 0$. By Lemma 2.3.19(2)

$$\frac{\prod_{i=1}^k P(\langle \Delta_i \rangle, T)}{\prod_{i=1}^k P(\langle \Delta_i^\vee \rangle, q_v^{-1} T^{-1})} \mathrm{r}_\ell(\epsilon(T, \langle \widetilde{\mathfrak{m}} \rangle, \widetilde{\psi_v})) =$$

$$= \frac{\mathrm{r}_\ell(L(\langle \widetilde{\mathfrak{m}}^\vee \rangle, q_v^{-1} T^{-1}))}{\mathrm{r}_\ell((L(\langle \widetilde{\mathfrak{m}} \rangle, T))} \mathrm{r}_\ell(\epsilon(T, \langle \widetilde{\mathfrak{m}} \rangle, \widetilde{\psi_v})) = \frac{L(\langle \mathfrak{m}^\vee \rangle, q_v^{-1} T^{-1})}{L(\langle \mathfrak{m} \rangle, T)} \epsilon(T, \langle \mathfrak{m} \rangle, \psi_v).$$

Rewriting, we obtain that

$$\frac{P(\rho, T)^l}{P(\rho^\vee v_\rho^{-a}, q_v^{-1} T^{-1})^{l'}} = \frac{\epsilon(T, \langle \mathfrak{m} \rangle, \psi_v)}{\mathrm{r}_\ell(\epsilon(T, \langle \widetilde{\mathfrak{m}} \rangle, \widetilde{\psi_v}))}.$$

Recall now that the right side is a unit in $\overline{\mathbb{F}}_\ell[T, T^{-1}]$ and hence so has to be left side. But

$$\frac{P(\rho, T)^l}{P(\rho^\vee v_\rho^{-a}, q_v^{-1} T^{-1})^{l'}} = \frac{(1 - \rho(\varpi_v) q_v^{\frac{1-d_v}{2}} T)^l}{(1 - \rho^\vee(\varpi_v^{-1}) q_v^{a d_v + \frac{1-d_v}{2} - 1} T^{-1})^{l'}}$$

and in order for this to be a unit either $l = l' = 0$, in which case we are done, or $l = l' > 0$ and $-a+1 = 0 \mod o(\rho)$. We let now d_0 be the multiplicity of $[0,0]_\rho$ and d_1 be the multiplicity $[1,1]_\rho$ in $\mathfrak{m}$. By property (5) of Section 4.3.2 and Theorem 4.4.2 we obtain that if $[0,0]_\rho$ appears with non-zero multiplicity in $\mathfrak{m} - (\mathfrak{m}_f^\vee)^\vee$, then there have to exist more copies of $[0,0]_\rho$ in $\mathfrak{m}$ than copies of $[1,1]_\rho$. Thus if $k \geq l > 0$, we have $d_1 < d_0$. But on the other hand, the same argument shows that if $[-1,-1]_{\rho^\vee} = [-a,-a]_{\rho^\vee}$ appears with non-zero multiplicity in $\mathfrak{m}^\vee - (\mathfrak{m}_{f'})^\vee$, then there have to exist more copies of $[-1,-1]_{\rho^\vee}$ in $\mathfrak{m}^\vee$ than copies of $[0,0]_{\rho^\vee}$. Thus if $k' \geq l' > 0$, we have $d_0 < d_1$. But since we already showed that $l = l'$, the only possibility is $l = l' = 0$. □

We obtain as a corollary the following.

Corollary 4.4.1. *Let* $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ *be an aperiodic multisegment. Then*

$$\epsilon(T, \langle\mathfrak{m}\rangle, \psi_v) = \prod_{i=1}^{k} \epsilon(T, \langle\Delta_i\rangle, \psi_v)$$

and

$$\gamma(T, \langle\mathfrak{m}\rangle, \psi_v) = \prod_{i=1}^{k} \gamma(T, \langle\Delta_i\rangle, \psi_v).$$

Proof. Notice that if π is a subquotient of π', $\gamma(T,\pi,\psi_v) = \gamma(T,\pi',\psi_v)$ and hence the claim for the γ-factor follows from Lemma 4.4.4. By the functional equation, the claim also follows for the ϵ-factors. □

LLC and local factors In this subsection assume $\mathbb{D}_v = \mathbb{K}_v$. We recall the map

$$\mathrm{C}\colon \mathfrak{Irr}_n \xrightarrow{\cong} \{\mathrm{C-parameters\ of\ length\ } n\}$$

defined in [74]. We recall the local Langlands correspondence The local Langlands correspondence gives then a well-known canonical bijection

$$\mathrm{LLC}\colon \mathrm{Nil}_{ss,n}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\cong} \mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell)$$

and the following surjection introduced in [125, I.8.6]. We denote by $\mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell)_e$ the isomorphism classes of integral irreducible representations. For π in this set, choose α the maximal composition such that π is α-degenerate. Then map π to the unique irreducible constituent in $[\mathrm{r}_\ell(\pi)]$ which is α-degenerate. The so-constructed map is surjective and denoted by

$$\mathrm{J}_\ell\colon \mathfrak{Irr}_n(\overline{\mathbb{Q}}_\ell)_e \to \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell).$$

Theorem 4.4.6 ([125, Theorem 1.6, Theorem 1.8.5])**.** *There exists a bijection*

$$\mathrm{V}\colon \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell) \xrightarrow{\cong} \mathrm{Nil}_{ss,n}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{F}}_\ell),$$

which is uniquely characterized by the equality

$$\mathrm{V}(\mathrm{J}_\ell(\mathrm{LLC}(\Phi, U)^*)^*) = \mathrm{r}_\ell(\Phi, U).$$

We recall next the map

$$\mathrm{CV}\colon \mathrm{Nil}_{ss}(\mathrm{W}_{\mathbb{K}_v},\overline{\mathbb{F}}_\ell) \to \mathrm{Rep}_{ss}(\mathrm{W}_{\mathbb{K}_v},\overline{\mathbb{F}}_\ell),$$

cf. [74, §6.3]. A nilpotent $\overline{\mathbb{F}}_\ell$-Deligne representation supported in $\mathbb{Z}[\Phi]$ can always be written in the form

$$\Phi_{ap} \oplus \Phi_{cyc}$$

with

$$\Phi_{ap} = \bigoplus_{i\geq 1} \bigoplus_{k=1}^{o(\Phi)-1} c_{i,k}[0,i-1] \otimes \nu_v^k \Phi,$$

$c_{i,k} \in \mathbb{Z}_{\geq 0}$, $c_{i,k} = 0$ for $i >> 0$ and for each i there exists at least one k such that $c_{i,k} = 0$. Furthermore, Φ_{cyc} is of the form

$$\Phi_{cyc} = \bigoplus_{i\geq 1} a_i[0,i-1] \otimes \bigoplus_{k=1}^{o(\Phi)-1} \nu_v^k \Phi$$

with $a_i \in \mathbb{Z}_{\geq 0}$, $a_i = 0$ for $i >> 0$. Then

$$\mathrm{CV}(\Phi_{ap} \oplus \Phi_{cyc}) \coloneqq \Phi_{ap} \oplus \bigoplus_{i\geq 1} a_i[0,i-1] \otimes C(\Phi).$$

Finally, for a nilpotent $\overline{\mathbb{F}}_\ell$-Deligne representation $(\Phi_1,U_1)\oplus\cdots\oplus(\Phi_k,U_k)$ with each of the summands supported in another $\mathbb{Z}[\Phi']$,

$$\mathrm{CV}((\Phi_1,U_1)\oplus\cdots\oplus(\Phi_k,U_k)) \coloneqq \mathrm{CV}(\Phi_1,U_1)\oplus\cdots\oplus\mathrm{CV}(\Phi_k,U_k).$$

One can then finally define the map

$$\mathrm{C} \coloneqq \mathrm{CV}\circ\mathrm{V}\colon \mathfrak{Irr}_n(\overline{\mathbb{F}}_\ell) \to \mathrm{Rep}_{ss}(\mathrm{W}_{\mathbb{K}_v},\overline{\mathbb{F}}_\ell).$$

We note the following properties of C, see [74, Examples 6.5]. If ρ is supercuspidal, $\mathrm{C}(\rho)$ is banal if and only $\rho \in \mathfrak{C}^\square$ and in this case $\mathrm{C}(\rho)$ is unramified if and only if ρ is. On the other hand, if ρ is $\square$-reducible, $\mathrm{C}(\rho) = (\Phi, I_\Phi)$ for some bijective map I_Φ. More generally, if $\pi = \langle\mathfrak{m}\rangle \in \mathfrak{Irr}_n$ for an aperiodic multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ with $\Delta_i = [a_i,b_i]_{\rho_i}$ we have

$$\mathrm{C}(\pi) = \bigoplus_{i=1}^{k}[0,b_i-a_i]\otimes\nu_v^{a_i}\mathrm{C}(\rho_i).$$

For

$$(\Phi,U)\in\mathrm{Rep}_{ss}(\mathrm{W}_{\mathbb{K}_v},R),$$

one can associate to the Deligne-representation a local L-factor

$$L((\Phi,U),T) \coloneqq \det(1-T\Phi(\mathrm{Frob})|_{\mathrm{Ker}(U)^{I_{\mathbb{K}_v}}})^{-1} \in R(T),$$

an ϵ-factor

$$\epsilon(T,(\Phi,U),\psi_v)\in R[T,T^{-1}]^*$$

and γ-factor

$$\gamma(T,(\Phi,U),\psi_v)\in R(T)$$

to (Φ, U), *cf.* [74, §5]. Moreover, the γ-factor does only depend on Φ, all three factors are multiplicative with respect to $\oplus$ and for

$$(\widetilde{\Phi}, U) \in \mathrm{Nil}^e_{ss}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{Q}}_\ell),$$

one has

$$\mathrm{r}_\ell(\gamma(T, (\widetilde{\Phi}, U), \widetilde{\psi_v})) = \gamma(T, \mathrm{r}_\ell(\widetilde{\Phi}, U), \psi_v),$$

see [74, Proposition 5.6, Proposition 5.11].

Theorem 4.4.7. *The map* C *respects the local factors,* i.e. *for* $\pi \in \mathfrak{Irr}_n$

$$L(\pi, T) = L(\mathrm{C}(\pi), T),\ \epsilon(T, \pi, \psi_v) = \epsilon(T, \mathrm{C}(\pi), \psi_v),$$

$$\gamma(T, \pi, \psi_v) = \gamma(T, \mathrm{C}(\pi), \psi_v).$$

Proof. We start with the equality of the γ-factors. Since the γ-factor of π does only depend on the supercuspidal support of π by Lemma 2.3.20 and the γ factor of $\mathrm{C}(\pi)$ depends only on the underlying $\mathrm{W}_{\mathbb{K}_v}$-representation, the multiplicativity of the γ-factor of $\mathrm{Rep}_{ss}(\mathrm{W}_{\mathbb{K}_v}, \overline{\mathbb{F}}_\ell)$ and Lemma 4.4.4 show that it is enough to show the equality for a supercuspidal representations ρ. But by picking a lift $\widetilde{\rho}$ of ρ, we see that

$$\gamma(T, \rho, \psi_v) = \mathrm{r}_\ell(\gamma(T, \widetilde{\rho}, \widetilde{\psi_v})) = \mathrm{r}_\ell(\gamma(T, \mathrm{LLC}^{-1}(\widetilde{\rho}), \widetilde{\psi_v})) = \gamma(T, \mathrm{C}(\rho), \psi_v).$$

For the L-factor, note that by Theorem 4.4.5 both sides are multiplicative and hence it is enough to show the claim for a segment $\pi = \langle [a, b]_\rho \rangle$. It is now easy to check from the definition and the discussion above the theorem, that

$$L([0, b-a] \otimes \mathrm{C}(\rho v_\rho^a), T) = L(\mathrm{C}(\rho v_\rho^b), T).$$

Now if ρ is either non-banal, *i.e.* $\square$-reducible, or not an unramified character, it follows from the above discussion that either the underlying morphism in $\mathrm{C}(\rho v_\rho^b)$ is bijective or there exists no $I_{\mathbb{K}_v}$-fixed vector, hence the L-factor is trivial. On the other hand, if ρ is an unramified character,

$$\mathrm{C}(\rho v_\rho^b)(\mathrm{Frob}) = \rho(\varpi_v) q_v^{-b}.$$

We thus arrive at

$$L([0, b-a] \otimes \mathrm{C}(\rho v_\rho^a), T) = \begin{cases} 1 & \text{if } o(\rho) = 1 \text{ or } \rho \text{ is ramified,} \\ \frac{1}{1 - \rho(\varpi_v) q_v^{-b} T} & \text{if } \rho \in \mathfrak{C}^\square \text{ and } \rho \text{ is unramified.} \end{cases}$$

By Theorem 4.4.5 this concise with $L(\langle [a, b]_\rho \rangle, T)$. Finally, the equality of ϵ-factors follows from the functional equation [74, Definition 5.5]. $\square$

4.5 Quiver varieties

In this section we will describe certain quiver varieties and explain how they relate to the representation theory of G_n over $\overline{\mathbb{F}}_\ell$. We start by setting the following geometric stage.

4.5.1 Quiver varieties We fix a cuspidal representation ρ. Let $Q = A^+_{o(\rho)-1}$ be the affine Dynkin quiver with $o(\rho)$ vertices numbered $\{0, \dots, o(\rho)-1\}$ and an arrow from i to j if $j = i+1 \mod o(\rho)$, *i.e.* of Q is of the form

$$\begin{tikzcd} & \dots \arrow[dr, bend left] & \\ 0 \arrow[ur, bend left] & & o(\rho)-2 \arrow[dl, bend left] \\ & o(\rho)-1 \arrow[ul, bend left] & \end{tikzcd}$$

If $o(\rho) = \infty$, we let $A_\infty = A^+_\infty$

$$\dots \longrightarrow i-1 \longrightarrow i \longrightarrow i+1 \longrightarrow \dots$$

A finite dimensional representation of Q is a graded vector space

$$V = \bigoplus_{i=0}^{o(\rho)-1} V_i$$

over some algebraically closed and complete field K (e.g. $\mathbb{C}$) and a linear map $T_{i\to j} \in \mathrm{Hom}(V_i, V_j)$ for each edge $i \to j$. Fixing V we denote by $E(V)$ the affine space of representations of Q with underlying vector space V and equip it with the natural topology coming from K. The group

$$\mathrm{GL}(V) = \mathrm{GL}(V_0) \times \dots \times \mathrm{GL}(V_{o(\rho)-1})$$

acts on $E(V)$ by

$$(g_0, \dots, g_{o(\rho)-1}) \cdot (x_0, \dots, x_{o(\rho)-1}) \coloneqq (g_1 x_0 g_0^{-1}, g_2 x_1 g_1^{-1}, \dots, g_0 x_{o(\rho)-1} g_{o(\rho)-1}^{-1}).$$

We denote by $N(V) \subseteq E(V)$ the space of nilpotent representations, *i.e.* the collection of all $T \in E(V)$ such that for large enough $N \in \mathbb{N}$, $T^N = 0$, where T is seen as an element of $\mathrm{Hom}(V, V)$. Set

$$\mathfrak{s}_V \coloneqq (\dim_K V_0) \cdot [\rho] + \dots + (\dim_K V_{o(\rho)-1}) \cdot [\rho v^{o(\rho)-1}].$$

The $\mathrm{GL}(V)$-orbits of $N(V)$ are parametrized by multisegments with cuspidal support $\mathfrak{s}_V$ as follows, see [88, §15].

First assign to a segment $\Delta = [a, b]_\rho$ with cuspidal support

$$\mathrm{cusp}_{\mathcal{M}}(\Delta) = d_0 \cdot [\rho] + \dots + d_{o(\rho)-1} \cdot [\rho v^{o(\rho)-1}]$$

the vector space

$$V(\Delta) \coloneqq \bigoplus_{i=0}^{o(\rho)-1} K^{d_i}$$

together with a basis $e_a, e_{a+1}, \dots, e_b$, $e_i \in V_i$, where the index of e_i is seen modulo $o(\rho)$. The linear map $\lambda(\Delta)$ associated to Δ sends e_i to e_{i+1} for $a \le i < b$ and e_b to 0.

To a multisegment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_n$ we then associate the linear maps $\lambda(\mathfrak{m}) := \lambda(\Delta_1) \oplus \ldots \oplus \lambda(\Delta_n)$ of $V(\mathfrak{m}) := V(\Delta_1) \oplus \ldots \oplus V(\Delta_n)$. It is easy to see that if $\mathrm{cusp}(\mathfrak{m}) = \mathfrak{s}_V$ then $V(\mathfrak{m}) \cong V$ as graded vector spaces and that $\lambda(\mathfrak{m})$ and $\lambda(\mathfrak{m}')$ lie in the same orbit of G_V if and only if $\mathfrak{m}$ and $\mathfrak{m}'$ are equivalent. Denote the orbit of $\lambda(\mathfrak{m})$ in V as $X_{\mathfrak{m}}$. We define an order on the set of multisegments by setting $\mathfrak{n} \leq \mathfrak{m}$ if

$$X_{\mathfrak{m}} \subseteq \overline{X}_{\mathfrak{n}},$$

where $\overline{X}_{\mathfrak{n}}$ is the closure of $X_{\mathfrak{n}}$ in $N(V)$ with its topology being the one coming from $E(V)$.

To motivate the next definitions we recall first the case $R = \overline{\mathbb{Q}}_\ell$, which was treated in [132]. Assume for a moment that $R = \overline{\mathbb{Q}}_\ell$, let ρ be a cuspidal representation of G_m and let $\Delta = [a,b]_\rho$ and $\Delta' = [a',b']_\rho$ be two segments in $\mathcal{S}eg(\rho)$ such that Δ precedes Δ', *i.e.*

$$a + 1 \leq a' \leq b + 1 \leq b'.$$

Define the union and intersection of Δ and Δ' as

$$\Delta \cup \Delta' := [a,b']_\rho,\ \Delta \cap \Delta' := [a',b]_\rho$$

and observe that they are unlinked. In [132, §4] the following decomposition was proven.

$$[\mathrm{Z}(\Delta) \times \mathrm{Z}(\Delta')] = [\mathrm{Z}(\Delta \cup \Delta) \times \mathrm{Z}(\Delta \cap \Delta')] + [\mathrm{Z}(\Delta + \Delta')]. \tag{46}$$

Let $\mathfrak{m} \in \mathcal{M}(\rho)$ be a multisegment. Then in [133] and [132, §7] the authors proved that in this case $[\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$ if and only if $\mathfrak{n} \leq \mathfrak{m}$, see Section 4.5.1. In this section we are going to prove the analogous statement over $\overline{\mathbb{F}}_\ell$.

4.5.2 Elementary operations We recall the following combinatorial description of $X_{\mathfrak{m}} \subseteq \overline{X_{\mathfrak{n}}}$ in terms of the underlying multisegments. Fix a cuspidal ρ, Q and $V \cong V(\mathfrak{m}) \cong V(\mathfrak{n})$ as in Section 4.5.1. We define the notation of an elementary operation on a multisegment as follows. Pick two ρ-segments Δ and Δ' in $\mathfrak{m}$, which are by definition only defined up to equivalence. Next pick two representatives for Δ and Δ'

$$[a,b]_\rho = (\rho v_\rho^a, \ldots, \rho v_\rho^b),\ [a',b']_\rho = (\rho v_\rho^{a'}, \ldots, \rho v_\rho^{b'})$$

such that

$$a + 1 \leq a' \leq b + 1 \leq b' \tag{47}$$

and define the elementary operation with respect to (Δ, Δ')

$$\mathfrak{m} \mapsto \mathfrak{m} - \Delta - \Delta' + [a,b']_\rho + [a',b]_\rho.$$

If there do not exist representatives satisfying (47), we cannot perform an elementary operation with respect to (Δ, Δ'). If we can perform an elementary operation we call the segments linked.

Lemma 4.5.1 ([69], [106, Theorem 3.12])**.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two multisegments with the same support contained in* $\mathcal{M}(\rho)$*. Then* $\mathfrak{n} \leq \mathfrak{m}$ *if and only if* $\mathfrak{n}$ *can be obtained from* $\mathfrak{m}$ *via finitely many elementary operations.*

The next lemmas will allow us to have a better grip on elementary operations and their proofs are rather straightforward.

Lemma 4.5.2. *Let Δ_1, Δ_2, Γ_1, Γ_2 be segments in $\mathcal{S}eg(\rho)$, where Δ_1 or Δ_2 are also allowed be the empty segment and $\Delta_1 + \Delta_2 \preceq \Gamma_1 + \Gamma_2$. Then either*

$$\Delta_1 + \Delta_2^+ \preceq \Gamma_1 + \Gamma_2^+ \text{ and } (\Delta_1 + \Delta_2^+)^1 = (\Gamma_1 + \Gamma_2^+)^1 \text{ or} \tag{48}$$

$$\Delta_1^+ + \Delta_2 \preceq \Gamma_1 + \Gamma_2^+ \text{ and } (\Delta_1^+ + \Delta_2)^1 = (\Gamma_1 + \Gamma_2^+)^1. \tag{49}$$

If in the first case Δ_2 is the empty segment, we mean by Δ_2^+ the segment $[b+1, b+1]_\rho$, where b is such that $\Gamma_2 = [a, b]_\rho$ and analogously in the second case for Δ_1. Moreover,

$$\Delta_1^+ + \Delta_2^+ \preceq \Gamma_1^+ + \Gamma_2^+ \text{ and } (\Delta_1^+ + \Delta_2^+)^1 = (\Gamma_1^+ + \Gamma_2^+)^1.$$

Proof. We will prove the first claim by induction on the number of elementary operations necessary to obtain $\Delta_1 + \Delta_2$ from $\Gamma_1 + \Gamma_2$. We can moreover assume without loss of generality that $l(\Delta_1) > l(\Delta_2)$. Assume first that they differ by one and write the representatives of Γ_1 and Γ_2 as $[a, b]_\rho$ and $[a', b']_\rho$ such that the elementary operation is performed by

$$\Gamma_1 + \Gamma_2 \mapsto [a, b']_\rho + [a', b]_\rho$$

and the representatives satisfy (47). Now, since $l(\Delta_1) > l(\Delta_2)$,

$$\Delta_1 = [a, b']_\rho,\ \Delta_2 = [a', b]_\rho.$$

We have $a + 1 \leq a' \leq b + 1 \leq b' + 1$ and therefore

$$\Gamma_1 + \Gamma_2^+ \mapsto [a, b' + 1]_\rho + [a', b]_\rho$$

is an elementary operation, implying (49). Next we assume that again they differ by one elementary operation but this time the representative of Γ_1 is $[a', b']_\rho$ and the representative of Γ_2 is $[a, b]_\rho$, again satisfying (47). Again by $l(\Delta_1) > l(\Delta_2)$,

$$\Delta_1 = [a, b']_\rho,\ \Delta_2 = [a', b]_\rho.$$

Note that $a + 1 \leq a' \leq b + 2$. If $b + 2 \leq b'$, we are in the situation of (48), since

$$\Gamma_1 + \Gamma_2^+ \mapsto [a, b']_\rho + [a', b + 1]_\rho$$

is an elementary operation. On the other hand, if $b + 1 = b'$, then

$$\Gamma_1 + \Gamma_2^+ = \Delta_1 + \Delta_2^+$$

and we have (48).

If they differ by more than one elementary operation, choose $\mathfrak{n} = \Delta_1' + \Delta_2'$ such that $\mathfrak{n} \preceq \Gamma_1 + \Gamma_2$ differ by one elementary operation and $\Delta_1 + \Delta_2 \preceq \mathfrak{n}$. By the base case $\Delta_1' + \Delta_2'^+ \preceq \Gamma_1 + \Gamma_2^+$ or $\Delta_1'^+ + \Delta_2' \preceq \Gamma_1 + \Gamma_2^+$ and hence we obtain by the induction hypothesis $\Delta_1 + \Delta_2^+ \preceq \Gamma_1 + \Gamma_2^+$ or $\Delta_1^+ + \Delta_2 \preceq \Gamma_1 + \Gamma_2^+$. The claim regarding $(-)^1$ follows easily by the same induction argument.

For the second claim, we will also proceed on the number of elementary operations necessary to obtain $\Delta_1 + \Delta_2$ from $\Gamma_1 + \Gamma_2$. of Γ_1 and Γ_2 as $[a, b]_\rho$ and $[a', b']_\rho$ such that the elementary operation is performed by

$$\Gamma_1 + \Gamma_2 \mapsto [a, b']_\rho + [a', b]_\rho$$

and the representatives satisfy (47). Then $a+1 \le a' \le b+2 \le b'+1$ and hence

$$\Gamma_1^+ + \Gamma_2^+ \mapsto [a, b'+1]_\rho + [a', b+1]_\rho$$

is an elementary operation. The induction step follows now as in the second claim. $\square$

Next, we need the following lemma, which can already be found in [132] for complex representations. The argument presented there works analogously, however we will write it out for completeness.

Lemma 4.5.3. *Let* $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ *be multisegments with* $\mathfrak{n}_1^1 = \mathfrak{m}_1^1$ *and* $\mathfrak{n}_1^- + \mathfrak{n}_2 \le \mathfrak{m}_1^- + \mathfrak{m}_2$*. Then there exists* $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ *with* $\mathfrak{n}' \le \mathfrak{m}_1 + \mathfrak{m}_2$*,* $\mathfrak{n}_1'^1 = \mathfrak{n}_1^1$ *and* $\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{n}_1^- + \mathfrak{n}_2$*.*

Proof. Assume first that $\mathfrak{n}_1^- + \mathfrak{n}_2$ and $\mathfrak{m}_1^- + \mathfrak{m}_2$ differ by one elementary operation. Let $\Delta_1 + \Delta_2$ resp. $\Gamma_1 + \Gamma_2$ be the multisegments contained in $\mathfrak{m}_1^- + \mathfrak{m}_2$ resp. $\mathfrak{n}_1^- + \mathfrak{n}_2$ involved in the elementary operation. Applying Lemma 4.5.2 gives multisegments $\Delta_1^* + \Delta_2^*$ resp. $\Gamma_1^* + \Gamma_2^*$ such that $\Gamma_1^* + \Gamma_2^*$ is contained in $\mathfrak{m}_1 + \mathfrak{m}_2$, $\Delta_1^* + \Delta_2^* \le \Gamma_1^* + \Gamma_2^*$,

$$\Delta_1^* \in \{\Delta_1, \Delta_1^+\},\ \Delta_2^* \in \{\Delta_2, \Delta_2^+\},\ \Gamma_1^* \in \{\Gamma_1, \Gamma_1^+\},\ \Gamma_2^* \in \{\Gamma_2, \Gamma_2^+\}$$

and $(\epsilon_{\Gamma_1}\Gamma_1^* + \epsilon_{\Gamma_2}\Gamma_2^*)^1 = (\epsilon_{\Delta_1}\Delta_1^* + \epsilon_{\Delta_2}\Delta_2^*)^1$ with

$$\epsilon_\Lambda := \begin{cases} 1 & \text{if } \Lambda^* = \Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathfrak{n}' := \Delta_1^* + \Delta_2^* + \mathfrak{m}_1 + \mathfrak{m}_2 - \Gamma_1^* - \Gamma_2^*.$$

It is clear by construction that $\mathfrak{n}' \le \mathfrak{m}_1 + \mathfrak{m}_2$ via one elementary operation. Write $\Delta_1^* + \Delta_2^* - \Gamma_1^* - \Gamma_2^*$ as $\mathfrak{k}_1 + \mathfrak{k}_2$, where we allow that the coefficients of the segments in $\mathfrak{k}_i$'s are not necessarily positive integers and

$$\mathfrak{k}_1 = \epsilon_{\Delta_1}\Delta^* + \epsilon_{\Delta_2}\Delta_2^* - \epsilon_{\Gamma_1}\Gamma_1^* - \epsilon_{\Gamma_2}\Gamma_2^*.$$

Note that $\mathfrak{k}_1^- + \mathfrak{k}_2 = \Gamma_1 + \Gamma_2 - \Delta_1 - \Delta_2$ and $\mathfrak{k}_1^1 = 0$, where we extended the operations $(-)^-$ and $(-)^1$ in the straightforward way to multisegments with integer coefficients.

We now define $\mathfrak{n}_1' := \mathfrak{m}_1 + \mathfrak{k}_1$ and $\mathfrak{n}_2' := \mathfrak{m}_2 + \mathfrak{k}_2$. By construction, the coefficients of all segments in $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$ are positive integers because $\epsilon_{\Gamma_1} = 1$ if and only if Γ_1 was a segment of $\mathfrak{m}_1$ and similarly for Γ_2. Thus $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$ are well-defined multisegments. We therefore obtain

$$\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{m}_1^- + \mathfrak{m}_2 + \Delta_1 + \Delta_2 - \Gamma_1 - \Gamma_2 = \mathfrak{n}_1^- + \mathfrak{n}_2$$

since $\mathfrak{k}_1^- + \mathfrak{k}_2 = \Delta_1 + \Delta_2 - \Gamma_1 - \Gamma_2$ and $\mathfrak{n}_1'^1 = \mathfrak{m}_1^1 = \mathfrak{n}_1^1$, since $\mathfrak{k}_1^1 = 0$. Therefore we constructed $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ as desired.

If $\mathfrak{n}_1^- + \mathfrak{n}_2$ differs by more than one elementary operation from $\mathfrak{m}_1^- + \mathfrak{m}_2$ the claim follows by induction on the number of elementary operations. Indeed, pick $\mathfrak{k}$ such that $\mathfrak{m}_1^- + \mathfrak{m}_2$ is obtained from $\mathfrak{k}$ via one elementary operation and $\mathfrak{k} \le \mathfrak{m}_1^- + \mathfrak{m}_2$. By the induction hypothesis we know that there exists $\mathfrak{k}' = \mathfrak{k}_1' + \mathfrak{k}_2'$, $\mathfrak{k}_1'^- + \mathfrak{k}_2' = \mathfrak{k}'$, $\mathfrak{k}_1'^- = \mathfrak{m}_1^1$ with $\mathfrak{k}' \le \mathfrak{m}_1 + \mathfrak{m}_2$. Moreover, by applying the induction hypothesis a second time, we obtain $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$, $\mathfrak{n}_1'^- + \mathfrak{n}_2' = \mathfrak{n}_1^- + \mathfrak{n}_2$, $\mathfrak{n}' \le \mathfrak{k}' \le \mathfrak{m}_1 + \mathfrak{m}_2$ and $\mathfrak{n}_1'^1 = \mathfrak{k}_1'^1 = \mathfrak{m}_1^1 = \mathfrak{n}_1^1$. $\square$

We are now ready to prove the result announced at the beginning of the section.

Theorem 4.5.4. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two multisegments with the same cuspidal support contained in $\mathcal{M}(\rho)$. Then $\mathrm{Z}(\mathfrak{n})$ appears in $\mathrm{Z}(\mathfrak{m})$ if and only if $\mathfrak{n} \leq \mathfrak{m}$.*

Proof. **First direction: $\mathfrak{n} \leq \mathfrak{m} \Rightarrow [\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$**

By Lemma 4.5.1 and Theorem 2.3.13 it is enough to show that if $\mathfrak{n}$ is obtained from $\mathfrak{m}$ via one elementary operation, then $\mathrm{I}(\mathfrak{n}) \leq \mathrm{I}(\mathfrak{m})$. By the exactness of parabolic induction, it is thus enough that if $\Delta = [a,b]_\rho$ and $\Delta' = [a',b']_\rho$ satisfying (47), then

$$\mathrm{I}([a,b']_\rho + [a',b]_\rho) \leq \mathrm{I}(\Delta + \Delta').$$

If ρ is supercuspidal, we can use Theorem 2.3.5 to find $\tilde{\rho}$ such that $\mathrm{r}_\ell(\tilde{\rho}) = \rho$. By (46) we have that

$$\mathrm{I}([a,b']_{\widetilde{\rho}} + [a',b]_{\widetilde{\rho}}) \leq \mathrm{I}([a,b]_{\widetilde{\rho}} + [a',b']_{\widetilde{\rho}}).$$

Since parabolic restriction commutes with r_ℓ, we have thus that

$$\mathrm{I}([a,b']_\rho + [a',b]_\rho) = \mathrm{r}_\ell(\mathrm{I}([a,b']_{\widetilde{\rho}} + [a',b]_{\widetilde{\rho}})) \leq \mathrm{r}_\ell(\mathrm{I}([a,b]_{\widetilde{\rho}} + [a',b']_{\widetilde{\rho}})) = \mathrm{I}(\Delta + \Delta').$$

If ρ is not supercuspidal, $o(\rho) = 1$ and we argue as follows. Write $\mathfrak{n} = [a,b']_\rho + [a',b]_\rho$ and $\mathfrak{m} = \Delta + \Delta'$. Observe first that if $\mathrm{Z}(\mathfrak{n})$ does not have its cuspidal support contained in $\mathbb{N}(\mathbb{Z}[\rho])$ then it follows by [101, Proposition 9.32] that $\mathfrak{n}$ consists of two segments of equal length and $\ell = 2$. For such an $\mathfrak{n}$, $[\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{n}) \leq \mathrm{I}(\mathfrak{m})$ implies therefore $\mathfrak{n} = \Delta + \Delta'$ by Theorem 2.3.13. It thus follows that it is enough to show that the multiplicity of any irreducible representation with cuspidal support in $\mathbb{N}(\mathbb{Z}[\rho])$ in $\mathrm{I}(\mathfrak{n})$ is lesser or equal than its multiplicity in $\mathrm{I}(\mathfrak{m})$. Let n be the length of $\mathfrak{m}$. We can therefore utilize the map $\xi_{\rho,n}$ between irreducible representations with inertially equivalent to representations having cuspidal support $n \cdot [\rho]$ and irreducible modules of $\mathscr{H}(n,1)$. Note that this map can be upgraded in the following way, see [102, Proposition 4.19]. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of n and $\pi = \pi_1 \otimes \ldots \otimes \pi_k$ is an irreducible representation of M_α with cuspidal support in $n \cdot [\rho]$, then there exists a bijection between between the irreducible constituents of

$$\mathrm{Hom}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(\alpha_1,1)\otimes\ldots\otimes\mathscr{H}_{\overline{\mathbb{F}}_\ell}(\alpha_k,1)}(\mathscr{H}(n,1), \xi_{\rho,\alpha_1}(\pi_1) \otimes \ldots \otimes \xi_{\rho,\alpha_k}(\pi_k))$$

and the irreducible constituents of $\pi_1 \times \ldots \times \pi_k$ with cuspidal support $n \cdot [\rho]$. Moreover, the map respects multiplicities. Now for any irreducible representation with cuspidal support $n \cdot [\rho]$, the $\overline{\mathbb{F}}_\ell[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$-part of the Hecke-algebra acts trivially, see [91, Theorem 3.7]. Thus we can regard $\xi_{\rho,\alpha_i}(\pi_i)$ and $\xi_{\rho,n}(\pi)$ as $\overline{\mathbb{F}}_\ell[\mathrm{S}_n]$-modules and as well as the induced Hom-spaces. We are thus in the realm of the modular representation theory of S_n, which was already discussed in Section 2.1.1. The desired claim is then the last remark of this section.

Second direction: $\mathfrak{n} \leq \mathfrak{m} \Leftarrow [\mathrm{Z}(\mathfrak{n})] \leq \mathrm{I}(\mathfrak{m})$

We will proceed inductively and assume the statement to be true for all G_k with $k < \deg(\mathfrak{m})$. By Lemma 2.3.16

$$[r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{I}(\mathfrak{m}))] \geq [r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{Z}(\mathfrak{n}))] \geq [\mathrm{Z}(\mathfrak{n}^-) \otimes \mathrm{Z}(\mathfrak{n}^1)].$$

We apply the Geometric Lemma applied to $r_{(\deg(\mathfrak{m}^-),\deg(\mathfrak{m}^1))}(\mathrm{I}(\mathfrak{m}))$ and observe that then Lemma 2.3.8 implies that there exist $\mathfrak{m}_1, \mathfrak{m}_2$ with $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ such that

$$[\mathrm{Z}(\mathfrak{n}^-) \otimes \mathrm{Z}(\mathfrak{n}^1)] \leq \mathrm{I}(\mathfrak{m}_1^- + \mathfrak{m}_2) \otimes \mathrm{I}(\mathfrak{m}_1^1).$$

Thus $\mathfrak{m}_1^1 = \mathfrak{n}^1$ by Theorem 2.3.12 and by the induction hypothesis $\mathfrak{n}^- \leq \mathfrak{m}_1^- + \mathfrak{m}_2$. We apply Lemma 4.5.3 to the case $\mathfrak{n}_1 = \mathfrak{n}, \mathfrak{n}_2 = 0$ and obtain therefore the existence of $\mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2'$ such that $\mathfrak{n}' \leq \mathfrak{m}$, ${\mathfrak{n}_1'}^1 = \mathfrak{n}^1$ and $\mathfrak{n}^- = {\mathfrak{n}_1'}^- + \mathfrak{n}_2'$. For any multisegment $\mathfrak{k}$ we will write $\mathfrak{k}[1]$ for the multisegment obtain from $\mathfrak{k}$ by replacing every segment Δ in it by ${}^-(\Delta^+)$ and $\mathfrak{k}^+$ for the multisegment obtain from $\mathfrak{k}$ by replacing every segment Δ in it by Δ^+. Note that $(\mathfrak{k}^+)^1 = \mathfrak{k}^1[1]$.

Because $\mathfrak{n}^- = {\mathfrak{n}_1'}^- + \mathfrak{n}_2'$, it follows that $\mathfrak{n}$ is of the form

$$\mathfrak{n} = (\mathfrak{n}_1'^-)^+ + (\mathfrak{n}_2')^+ + \mathfrak{k}',$$

where $\mathfrak{k}' \in \mathcal{M}(\rho)$ is some multisegment consisting of segments of length 1. Thus $\mathfrak{n}^1$ is of the form

$$\mathfrak{n}^1 = \mathfrak{n}_1'^2[1] + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

But on the other hand $\mathfrak{n}^1 = \mathfrak{n}_1'^1$ and hence

$$\mathfrak{n}_1'^1 = \mathfrak{n}_1'^2[1] + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

Therefore

$$\mathfrak{n}_1' = (\mathfrak{n}_1'^-)^+ + \mathfrak{n}_2'^1[1] + \mathfrak{k}'.$$

Since $(\mathfrak{n}_2')^+ \leq \mathfrak{n}_2' + \mathfrak{n}_2'^1[1]$, it follows that

$$\mathfrak{n} = (\mathfrak{n}_1'^-)^+ + (\mathfrak{n}_2')^+ + \mathfrak{k}' \leq (\mathfrak{n}_1'^-)^+ + \mathfrak{n}_2' + \mathfrak{n}_2'^1[1] + \mathfrak{k}' = \mathfrak{n}'.$$

Thus $\mathfrak{n} \leq \mathfrak{n}' \leq \mathfrak{m}$. □

Corollary 4.5.1. *A multisegment $\mathfrak{m}$ in $\mathcal{M}(\rho)$ is unlinked if and only if $X_\mathfrak{m}$ is an open orbit of $N(V(\mathfrak{m}))$.*

Proof. Since there are only finitely many multisegments $\mathfrak{n} \in \mathcal{M}(\rho)$ with

$$\mathrm{cusp}_\mathcal{M}(\mathfrak{n}) = \mathrm{cusp}_\mathcal{M}(\mathfrak{m}),$$

we have

$$\overline{N(V)\backslash X_\mathfrak{m}} = \overline{\bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_\mathfrak{n}} = \bigcup_{\mathfrak{n}\neq\mathfrak{m}} \overline{X_\mathfrak{n}},$$
$$\bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_\mathfrak{n} = N(V)\backslash X_\mathfrak{m}.$$

Now $\bigcup_{\mathfrak{n}\neq\mathfrak{m}} \overline{X_\mathfrak{n}} = \bigcup_{\mathfrak{n}\neq\mathfrak{m}} X_\mathfrak{n}$ if and only if $X_\mathfrak{m}$ is not contained in $\overline{X_\mathfrak{n}}$ for any $\mathfrak{n} \neq \mathfrak{m}$. By Theorem 4.5.4 and Lemma 4.5.1 this is equivalent to $\mathfrak{m}$ being unlinked. □

Recall the following theorem of [95].

Theorem 4.5.5 ([95, Theorem 6.1])**.** *Let ρ be an unramified character of G_1 and $\mathfrak{m}$ an aperiodic multisegment in $\mathcal{M}(\rho)$. Then $\mathrm{Z}(\mathfrak{m})$ is unramified if and only if $\mathfrak{m}$ is unlinked. Moreover, if $\pi \in \mathfrak{Irr}(\Omega_{\rho',n})$ is an unramified representation for some cuspidal ρ', then ρ' is an unramified character and there exists and an unlinked multisegment $\mathfrak{m} \in \mathcal{M}(\rho')$ such that $\mathrm{Z}(\mathfrak{m}) \cong \pi$.*

Observe that an unlinked multisegment in $\mathcal{M}(\rho)$ is automatically aperiodic. Therefore, if $\mathfrak{m} \in \mathcal{M}(\rho)$ is unlinked, the cuspidal support of $\mathrm{Z}(\mathfrak{m})$ is contained in $\mathbb{N}(\mathbb{Z}[\rho])$ by [101, Proposition 9.34].

Corollary 4.5.2. *Let $\pi \cong \mathrm{Z}(\mathfrak{m}) \in \mathfrak{Irr}(\Omega_{\rho,n})$ be a representation. Then π is unramified if and only if the orbit $X_{\mathfrak{m}}$ is open and ρ is an unramified character.*

We will also use the opportunity to give a formula for the number of unramified representations with given cuspidal support. One has yet to find a use for it though.
Proposition 4.5.6. *Let ρ an unramified character,*

$$\mathfrak{s} = d_0[\rho] + \ldots + d_{o(\rho)-1}[\rho v^{o(\rho)-1}],\ d_i \in \mathbb{Z}_{\geq 0}$$

a fixed cuspidal support, $D := \min_i d_i$ and $m = \#\{i : d_i = D\}$. The number of unramified irreducible representations π with cuspidal support $\mathfrak{s}$, or equivalently, the number of irreducible components of $N(V(\mathfrak{s}))$, is m if $D > 0$ and 1 if $D = 0$.

Proof. If $D = 0$ the space $N(V) = E(V)$ has precisely one irreducible component and thus only one open orbit. Therefore there exists a unique unramified irreducible representation with cuspidal support $\mathfrak{s}$. We will now give a precise description of the corresponding multisegment $\mathfrak{m}_{\mathfrak{s}}$. Let $d_k = \max_i d_i$ and $j \leq k$ be such that

$$d_{j-1} < d_j = d_{j+1} = \ldots = d_k > d_{k+1}.$$

Set

$$\mathfrak{s}' := d_0[\rho] + \ldots + (d_j - 1)[\rho v^j] + \ldots + (d_k - 1)[\rho v^k] + \ldots + d_{o(\rho)-1}[\rho v^{o(\rho)-1}].$$

Then we can compute $\mathfrak{m}_{\mathfrak{s}}$ iteratively by noting that $\mathfrak{m}_{\mathfrak{s}} = \mathfrak{m}_{\mathfrak{s}'} + [j,k]$.

If $D > 0$ let $\mathfrak{m}_{\mathfrak{s}}$ be the set of irreducible unramified representations with cuspidal support $\mathfrak{s}$. Let

$$\mathfrak{s}^{+1} = (d_0 + 1) \cdot [\rho] + \ldots + (d_{o(\rho)-1} + 1) \cdot [\rho v^{o(\rho)-1}]$$

Then we construct an injective map $i\colon \mathfrak{m}_{\mathfrak{s}} \to \mathfrak{m}_{\mathfrak{s}^{+1}}$ by sending an unlinked segment $\mathfrak{m} = \Delta_1 + \ldots + \Delta_k$ with $l(\Delta_1) \geq l(\Delta_i)$ for $i \in \{1, \ldots, k\}$ and $\Delta_1 = [a,b]$ to $[a, b + o(\rho)] + \Delta_2 + \ldots + \Delta_k$, which is unlinked since $\mathfrak{m}$ is.

There exists also a map $p\colon \mathfrak{m}_{\mathfrak{s}^{+1}} \to \mathfrak{m}_{\mathfrak{s}}$ constructed as follows. Note that for $\Delta_1 + \ldots + \Delta_k = \mathfrak{m} \in \mathfrak{m}_{\mathfrak{s}^{+1}}$ there exists a unique Δ_i with $l(\Delta_i) \geq o(\rho)$. Indeed, $\rho v^{o(\rho)-1}$ and ρ must be contained in the same segment, which is then of length greater than $o(\rho)$, since otherwise the two segments would be linked. Since any two segments of length greater than $o(\rho) - 1$ are linked, uniqueness follows. Assume without loss of generality that $l(\Delta_1) \geq l(\Delta_i)$ for $i \in \{1, \ldots, k\}$ and write $\Delta_1 = [a,b]$. The map then sends $\mathfrak{m}$ to $[a, b - o(\rho)] + \Delta_2 + \ldots + \Delta_k$, which is again unlinked. Because $D > 0$ it follows that p is injective.

Since both i and p are injective if $D > 0$ and $p \circ i = \mathrm{id}$, $\#\mathfrak{m}_{\mathfrak{s}} = \#\mathfrak{m}_{\mathfrak{s}^{+1}}$ if $D > 0$. Thus it suffices to prove the claim for $D = 1$. Write

$$\mathfrak{s} = \mathfrak{s}' + [\rho] + \ldots + [\rho v^{o(\rho)-1}]$$

We know from the case $D = 0$ that we can write

$$\mathfrak{m}_{\mathfrak{s}'} = \mathfrak{m}_1 + \ldots + \mathfrak{m}_k \tag{50}$$

where $\mathfrak{m}_i = \Delta_i^1 + \ldots + \Delta_i^{l_i}$, $\Delta_i^j = [a_i^j, b_i^j]$ with $a_i^j \leq a_i^{j'} \leq b_i^{j'} \leq b_i^j$ for $j \leq j'$, $\Delta_i^j = \Delta_i^{j'}$ if and only if $j = j'$ and $b_i^1 + 1 < a_{i'}^1$ if $i < i'$.

We now construct m many unlinked segments with support $\mathfrak{s}$ as follows. For $c \in \{0, \dots, o(\rho)-1\}$ such that either $c \leq a_i^1 - 2$ or $c \geq b_i^1 + 2$ for all i it is easy to check that $\mathfrak{m}_{\mathfrak{s}} + [c, c + o(\rho) - 1]$ is unlinked. If $b_i + 1 = c$ or $c = a_i^1 - 1$ then the multisegment

$$\mathfrak{m}_1 + .. + \mathfrak{m}_{i-1} + [a_i^1, b_i^1 + o(\rho)] + [a_i^2, b_i^2] + \ldots + [a_i^{l_i}, b_i^{l_i}] + \mathfrak{m}_{i+1} + \ldots + \mathfrak{m}_k$$

is unlinked. Thus $\#\mathfrak{m}_{\mathfrak{s}} \geq m$. On the other hand if $\mathfrak{m} \in \mathfrak{m}_{\mathfrak{s}}$ then $p(\mathfrak{m}) = \mathfrak{m}_{\mathfrak{s}'}$ which shows that $\mathfrak{m}$ has to be of the form (50) and hence showing $\#\mathfrak{m}_{\mathfrak{s}} \leq m$. $\square$

4.5.3 $\square$-irreducible representations revisited We will use the opportunity to extend [82, Conjecture 4.2] to representations with $\square$-irreducible cuspidal support and give a conjectural classification of square-irreducible representations. Using Lemma 2.3.7 we can quickly reduce the classification to a classification for representations $\mathrm{Z}(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{M}(\rho)_{ap}$ for some cuspidal representation ρ. By Lemma 4.2.7 $o(\rho) > 1$ if $\mathrm{Z}(\mathfrak{m})$ has to have any chance of being square-irreducible. We fix for the rest of the section an aperiodic multisegment $\mathfrak{m} \in \mathcal{M}(\rho)_{ap}$ for some $\rho \in \mathfrak{C}^{\square}$. We set $Q^+ := A^+_{o(\rho)-1}$. Moreover, we recall the quiver representation $(V(\mathfrak{m}), \lambda(\mathfrak{m}))$ over $\mathbb{C}$ of Q^+ as in Section 4.5.1. Next, we denote by Q^- the quiver obtained from Q by inverting the arrows. For a fixed graded vector space $V = \oplus_{i=1}^{o(\rho)-1} V_i$ over $\mathbb{C}$ as in Section 4.5.1 we write $E^+(V)$ respectively $E^-(V)$ for the representations of Q^+ respectively Q^- with underlying vector space V and by $N^+(V)$ and $N^-(V)$ the subvariety of nilpotent representations. The third quiver of this story is denoted by $\overline{Q}$ and of the form

$$\begin{tikzcd}
 & \cdots \arrow[dr, leftrightarrow, bend left] & \\
0 \arrow[ur, leftrightarrow, bend left] & & o(\rho)-2 \\
 & o(\rho)-1 \arrow[ul, leftrightarrow, bend left] \arrow[ur, leftrightarrow, bend right] &
\end{tikzcd} .$$

We let $\overline{E}(V)$ be the affine variety of representations of Q with underlying vector space V. Again $\overline{E}(V)$ admits an action of

$$\mathrm{GL}(V) = \mathrm{GL}(V_0) \times \ldots \times \mathrm{GL}(V_{o(\rho)-1}).$$

Moreover, there exist two G_V-equivariant maps

$$E^+(V) \xleftarrow{p_+} \overline{E}(V) \xrightarrow{p_-} E^-(V).$$

The preimage of $N^+(V) \times N^-(V)$ is denoted by $\overline{N}(V)$. Finally, we recall the moment map

$$[\cdot, \cdot] \colon \overline{N}(V) \to \mathfrak{gl}(V),$$
$$(X_{i \mapsto i+1}, Y_{i+1 \mapsto i})_{i \in \{0, \dots, o(\rho)-1\}} \mapsto (X_{i \mapsto i+1} Y_{i+1 \mapsto i} - Y_{i \mapsto i-1} X_{i-1 \mapsto i})_{i \in \{0, \dots, o(\rho)-1\}}.$$

Let $\Lambda(V)$ be the kernel of $[\cdot, \cdot]$.

Recall now above fixed multisegment $\mathfrak{m}$ and $V := V(\mathfrak{m})$, $\lambda(\mathfrak{m})$ and $X_{\mathfrak{m}}$ from Section 4.5.1. To avoid confusion we denote the map $\lambda(\mathfrak{m})^+$ in $N^+(V(\mathfrak{m}))$ and $\lambda(\mathfrak{m})^-$ in $N^-(V(\mathfrak{m}))$ and similarly for $X^+_{\mathfrak{m}}$ and $X^-_{\mathfrak{m}}$. We now let $\mathcal{M}(V)_{ap}$ be the aperiodic segments $\mathfrak{n} \in \mathcal{M}(\rho)_{ap}$ with the same cuspidal support as $\mathfrak{m}$. For $\mathfrak{n} \in \mathcal{M}(V)_{ap}$ denote the closure of $p_+^{-1}(X^+_{\mathfrak{n}}) \cap \Lambda(V)$ in $\Lambda(V)$ by $\mathcal{C}^+_{\mathfrak{n}}$ and the closure of $p_-^{-1}(X^-_{\mathfrak{n}}) \cap \Lambda(V)$ in $\Lambda(V)$ by $\mathcal{C}^-_{\mathfrak{n}}$.

Lemma 4.5.7 ([88, Proposition 15.5]). *The maps* $\mathfrak{n} \mapsto \mathcal{C}^+_{\mathfrak{n}}$ *respectively* $\mathfrak{n} \mapsto \mathcal{C}^-_{\mathfrak{n}}$ *induce a bijection between* $\mathcal{M}(V)_{ap}$ *and the irreducible constituents of* $\Lambda(V)$.

Moreover, the interaction of $\mathcal{C}^+$ and $\mathcal{C}^-$ is governed by the Aubert-Zelevinsky duality.
Lemma 4.5.8 ([87, Propostion 5.2]). *For* $\mathfrak{n} \in \mathcal{M}(V)_{ap}$

$$\mathcal{C}^+_{\mathfrak{n}} = \mathcal{C}^-_{\mathfrak{n}^*}.$$

We thus can generalize [82, Conjecture 4.2] as follows.
Conjecture 4.5.1. *Let* $\mathfrak{m} \in \mathcal{M}(\rho)_{ap}$ *for some* $\rho \in \mathfrak{C}^\square$. *Then* $\mathrm{Z}(\mathfrak{m})$ *is square-irreducible if and only if* $\mathcal{C}^+_{\mathfrak{m}}$ *contains an open* $\mathrm{GL}(V)$*-orbit.*

At this point, some remarks are in order.
Remark. (1). If $\mathfrak{m}$ is banal, *i.e.* the cuspidal support of $\mathfrak{m}$ does not contain $[\rho]+\ldots+[\rho v_\rho^{o(\rho)-1}]$, then above conjecture reduces to [82, Conjecture 4.2].

(2). The conjecture is true for segments. First of all note that $\mathrm{Z}(\Delta)$, $\Delta = [a,b]_\rho$, is $\square$-irreducible if and only if $b-a+1 < o(\rho)$. Thus if $b-a+1 < o(\rho)$, we know from the banal case that $\mathcal{C}^+_\Delta$ admits an open orbit. On the other hand, if $b-a+1 \geq o(\rho)$, we can reduce the claim by [1, Corollay 8.7] to the case $b-a+1 = o(\rho)$. Indeed, if we assume $b-a+1 > o(\rho)$, let in the Corollary $C_1 = \mathcal{C}^+_{\Delta^-}$ and $C_2 = \mathcal{C}^+_{[b,b]_\rho}$. In the language of the above Corollary, it is then not hard to see that $C_1 * C_2 = \mathcal{C}^+_\Delta$. Thus we can assume without loss of generality that $a = 0$, $b = o(\rho)-1$. Then the claim that $\mathcal{C}^+_\Delta$ contains not an open orbit is equivalent to the claim that

$$T^*_{\lambda(\Delta)}X_\Delta = \{y \in N^-(V(\Delta)) : [\lambda(\Delta), y] = 0\}$$

does not contain an open orbit of the centralizer of $\lambda(\Delta)$. But in this case the centralizer is equal to $\{(a,\ldots,a) \in (C^*)^{o(\rho)}\}$, which acts trivially on $T^*_{\lambda(\Delta)}X_\Delta$. However, $T^*_{\lambda(\Delta)}X_\Delta$ is one-dimensional and hence $\mathcal{C}^+_\Delta$ does not contain an open orbit.
Example. Let us give the following family of examples of non-banal $\square$-irreducible representations. Let $\rho \in \mathfrak{C}^\square_m$ and consider $\mathfrak{m} = [0, o(\rho)-1]_\rho + [0,0]_\rho$. We claim that $\mathrm{Z}(\mathfrak{m})$ is $\square$-irreducible and $\mathfrak{C}^+_{\mathfrak{m}}$ admits an open orbit.

Let us first check that $\mathfrak{C}^+_{\mathfrak{m}}$ admits an open orbit. Note that

$$\mathfrak{m}^* = [0,0]_\rho + \ldots + [o(\rho)-2, o(\rho)-2]_\rho + [o(\rho)-1, o(\rho)]_\rho$$

We set

$$x := ((1,0), 1, \cdots, 1, 0) \in X^+_{\mathfrak{m}},\ x^* := ((0,1)^T, 0, \ldots, 0) \in X^-_{\mathfrak{m}^*}.$$

Here the i-th coordinate represents the map from $V_{i-1}(\mathfrak{m}) \to V_i(\mathfrak{m})$ respectively $V_i(\mathfrak{m}) \to V_{i-1}(\mathfrak{m})$. It suffices now to show that the stabilizer of x admits an open orbit in

$$T^*_x X_{\mathfrak{m}} = \{y \in N^-(V(\mathfrak{m})) : [x,y] = 0\}.$$

To see this, we observe that the stabilizer of x equals to

$$G_x := \{(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, a\ldots, a) \in \mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^\times)^{o(\rho)}\}.$$

Moreover, the set $\{(x,y)^T, 0, \ldots, 0\} : y \neq 0\}$ is an open subset of $T^*_x X_{\mathfrak{m}}$, which is equal to $G_x \cdot x^*$, finishing the argument.

We continue by showing that $\mathrm{Z}(\mathfrak{m})$ is $\square$-irreducible. To do so, we prove the following three lemmata.

Lemma 4.5.9. *Let $\rho \in \mathfrak{C}_m^{\square}$ and $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}(\rho)$ such that $\mathfrak{n}+\mathfrak{m}$ is aperiodic. Then $\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{n})$ is irreducible if and only if*

$$[r_{(n-m,m)}(\mathrm{Z}(\mathfrak{m}+\mathfrak{n}))] = [r_{(n-m,m)}(\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{n}))]$$

Proof. One direction is trivially true by Proposition 2.3.14. For the other one, assume that the above equality holds and $\mathrm{Z}(\mathfrak{n}) \times \mathrm{Z}(\mathfrak{n})$ is not irreducible. Then there exits either an irreducible quotient or subrepresentation π of $\mathrm{Z}(\mathfrak{n}) \times \mathrm{Z}(\mathfrak{m})$ different from $\mathrm{Z}(\mathfrak{n}+\mathfrak{m})$. Since π has the same cuspidal support as $\mathrm{Z}(\mathfrak{n}+\mathfrak{m})$, the claim follows from the exactness of parabolic restriction. □

Lemma 4.5.10. *Let $a \in \{0, \dots, o(\rho)-2\}$. Then*

$$\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho)$$

is irreducible.

Proof. We argue by induction on a. Theorem 4.3.10 and Lemma 4.3.2 allow us to compute

$$[r_{(o(\rho)m+(a+1)m,m)}(\mathrm{Z}([0,a]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho))] =$$

$$= [\mathrm{Z}([0,a-1]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v_\rho^a] +$$

$$+ [\mathrm{Z}([0,a]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}].$$

On the other hand, the Geometric Lemma gives that

$$[r_{(o(\rho)m+(a+1)m,m)}(\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho))] =$$

$$= [\mathrm{Z}([0,a-1]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v_\rho^a] +$$

$$+ [\mathrm{Z}([0,a]_\rho) \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}].$$

If $a = 0$, Theorem 2.3.9 implies

$$\rho \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \cong \rho^2 \times \mathrm{Z}([0,o(\rho)-2]_\rho) \cong$$

$$\cong \mathrm{Z}([0,0]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho)$$

and hence the parabolic restrictions of the two representations agree. If $a > 0$, the induction hypothesis and Theorem 2.3.9 imply that the parabolic restrictions of the two representations agree, hence the claim follows by Lemma 4.5.9. □

Lemma 4.5.11. *The representation*

$$\rho \times \rho \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho)$$

is irreducible.

Proof. We proceed analogously to the proof of the above lemma. Namely, the Geometric Lemma and Corollary 4.3.2 show that

$$[r_{(o(\rho)m+2m,m)}(\rho\times\rho\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho))]=$$

$$=2[\rho\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho)\otimes\rho]+[\rho\times\rho\times \mathrm{Z}([0,o(\rho)-2]_\rho+[0,0]_\rho)]\otimes\rho v_\rho^{-1}].$$

On the other hand Lemma 4.3.2 shows that

$$[r_{(o(\rho)m+2m,m)}(\mathrm{Z}([0,o(\rho)-1]_\rho+3[0,0]_\rho))]\geq$$

$$[\mathrm{Z}([0,o(\rho)-1]_\rho+2[0,0]_\rho))\otimes\rho]+[\mathrm{Z}([0,o(\rho)-2]_\rho+3[0,0]_\rho))\otimes\rho v_\rho^{-1}].$$

By Lemma 4.5.10

$$\mathrm{Z}([0,o(\rho)-1]_\rho+2[0,0]_\rho)\cong\rho\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho)$$

and by Theorem 2.3.9

$$\mathrm{Z}([0,o(\rho)-2]_\rho+3[0,0]_\rho))\cong \mathrm{Z}([0,o(\rho)-2]_\rho)\times\rho^3\cong\rho\times\rho\times \mathrm{Z}([0,o(\rho)-2]_\rho+[0,0]_\rho).$$

Thus if π is another subquotient than $\mathrm{Z}([0,o(\rho)-1]_\rho+3[0,0]_\rho)$ of

$$\rho\times\rho\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho),$$

it is either cuspidal, which cannot happen, or Frobenius reciprocity implies that π is a subrepresentation of

$$\mathrm{Z}([0,o(\rho)-1]_\rho+2[0,0]_\rho))\times\rho,$$

implying $\pi\cong \mathrm{Z}([0,o(\rho)-1]_\rho+3[0,0]_\rho)$ by the theory of derivatives. But

$$\mathrm{Z}([0,o(\rho)-1]_\rho+3[0,0]_\rho)$$

appears with multiplicity 1 in

$$\rho\times\rho\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho)$$

by Proposition 2.3.14, thus proving the claim. □

Lemma 4.5.12. *Let $b\in\{1,\dots,o(\rho)-2\}$. Then*

$$\rho\times \mathrm{Z}([0,b]_\rho)\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho)$$

is irreducible.

Proof. The proof proceeds analogously to the one of Lemma 4.5.10 by showing inductively on b that

$$[r_{(o(\rho)m+(b+2)m,m)}(\rho\times \mathrm{Z}([0,b]_\rho)\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho))]=$$

$$=[r_{(o(\rho)m+(b+2)m,m)}(\mathrm{Z}([0,0]_\rho+[0,b]_\rho+[0,o(\rho)-1]_\rho+[0,0]_\rho))].$$

The base case is Lemma 4.5.11. For $b>0$

$$[r_{(o(\rho)m+(b+2)m,m)}(\rho\times \mathrm{Z}([0,b]_\rho)\times \mathrm{Z}([0,o(\rho)-1]_\rho+[0,0]_\rho))]=$$

$$= [\mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho)) \otimes \rho] +$$
$$+ [\rho \times \mathrm{Z}([0,b-1]_\rho) \times \mathrm{Z}([0,o(\rho)-1]_\rho + [0,0]_\rho) \otimes \rho v^b] +$$
$$+ [\rho \times \mathrm{Z}([0,b]_\rho) \times \mathrm{Z}([0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}]$$

and

$$[r_{(o(\rho)m+(b+2)m,m)}(\mathrm{Z}([0,0]_\rho + [0,b]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho))] =$$
$$= [\mathrm{Z}([0,b]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho)) \otimes \rho] +$$
$$+ [\mathrm{Z}([0,b-1]_\rho + [0,o(\rho)-1]_\rho + [0,0]_\rho + [0,0]_\rho) \otimes \rho v^b] +$$
$$+ [\mathrm{Z}([0,b]_\rho + [0,0]_\rho + [0,o(\rho)-2]_\rho + [0,0]_\rho) \otimes \rho v_\rho^{-1}].$$

We use the induction hypothesis, Theorem 2.3.9 and Lemma 4.5.10 to show that the parabolic restrictions agree. We are now done by Lemma 4.5.9. ☐

We now come finally to the □-irreducibility of $\mathrm{Z}(\mathfrak{m})$, $\mathfrak{m} = [0,o(\rho)-1] + [0,0]_\rho$. Note that by Theorem 4.3.10 and Lemma 4.3.2

$$r_{(o(\rho)m,m)}(\mathrm{Z}(\mathfrak{m})) = \mathrm{Z}([0,o(\rho)-2] + [0,0]_\rho) \otimes \rho v_\rho^{-1}.$$

Thus the Geometric Lemma and Lemma 4.5.12 imply that

$$[r_{(2o(\rho)m+m,m)}(\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{m}))] = 2[\mathrm{Z}([0,o(\rho)-2] + [0,0]_\rho + \mathfrak{m}) \otimes \rho v_\rho^{-1}].$$

If $\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{m})$ is not irreducible, it has an irreducible subquotient π different from $\mathrm{Z}(\mathfrak{m}+\mathfrak{m})$ with the same cuspidal support. But by above computation and Frobenius reciprocity, we conclude that $\mathcal{D}_{r,\rho v_\rho^{-1}}(\pi) \cong \mathcal{D}_{r,\rho v_\rho^{-1}}(\mathrm{Z}(\mathfrak{m}+\mathfrak{m})) \neq 0$ and hence $\pi \cong \mathrm{Z}(\mathfrak{m}+\mathfrak{m})$. Since $\mathrm{Z}(\mathfrak{m}+\mathfrak{m})$ appears only with multiplicity 1 in $\mathrm{Z}(\mathfrak{m}) \times \mathrm{Z}(\mathfrak{m})$, the claim follows.

Chapter 5

On a conjecture of Kudla and Rallis

Again, we start by sketching the main results and their proof. For this section we return again to the classical case with coefficients $R = \mathbb{C}$ and use the conventions of Section 2.2.3 and Section 2.3.4. This chapter is based on [36].

5.1 Summary

We assume W in this introduction to be a symplectic vector space over $\mathbb{K}_v$ of dimension m. Thus $m = 2m'$ is even and we can decompose W as $X_{m'} \oplus Y_{m'}$, such that on $X_{m'}$ and $Y_{m'}$ the symplectic form vanishes. To be more precise, we pick a basis $\{x_1, \dots x_{m'}\}$ of $X_{m'}$ and $\{y_1, \dots y_{m'}\}$ of $Y_{m'}$ with

$$\langle x_i, y_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

and set for $k \in \{0, \dots, m'\}$,

$$X_k := \langle x_1, \dots x_k \rangle_{\mathbb{L}_v},\ Y_k := \langle y_1, \dots y_k \rangle_{\mathbb{L}_v},\ W_k := \langle x_{k+1}, \dots x_{m'}, y_{k+1}, \dots, y_{m'} \rangle_{\mathbb{L}_v}.$$

Here we use the notation $\langle S \rangle_{\mathbb{L}_v}$ for the $\mathbb{L}_v$-sub-vector space spanned by a set $S \subseteq W$.

The proof builds on three ingredients, the first being a well-known filtration, see [73, § 1],

$$0 = I_{-1} \subseteq I_0 \subseteq \dots \subseteq I_{m'} = I_{W,W}(\chi, s) \tag{51}$$

of $I_{W,W}(\chi, s)$ with subquotients

$$\sigma_t := I_{t-1} \backslash I_t \cong \operatorname{Ind}_{P_{(t)} \times P_{(t)}}^{G(W) \times G(W)} (\underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes \underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes S(G(W_t))).$$

We say that $\pi \in \mathfrak{Irr}_{G(W)}$ *does not appear on the boundary* if every non-zero morphism $I_{W,W} \to \pi \otimes \pi^\vee$ does not vanish on I_0.

The second ingredient is motivated by the filtration of the Jacquet module of the Weil representation in [71, Theorem 2.8] and [92, Proposition 3.2]. For $i, j \in \mathbb{N}$ let $\sigma_{i,j}$ be the space of locally constant, compactly supported functions $\mathrm{Hom}(F^j, F^i) \to \mathbb{C}$ on which $G_i \times G_j$ acts by

$$((g_1, g_2) \cdot f)(x) = f(g_1^{-1} x g_2).$$

Furthermore, for $r \in \mathbb{N}$, $k \in \{0, \dots, r\}$ recall that we denoted by $Q_{(k,r-k)}$ the standard parabolic subgroup of G_r corresponding to the composition $(k, r-k)$ of r and for $t \in \{1, \dots, m'\}$ let $P'_{(t)}$ be the parabolic subgroup fixing the flag $\langle x_{m'-t+1}, \dots, x_{m'} \rangle_{\mathbb{L}_v} \subseteq W$. We then prove the following.

Theorem 5.1.1. *Let $P = P_{(r)} \times G(W) \subseteq G(W) \times G(W)$, $r \in \{1, \dots, m'\}$ be a standard parabolic subgroup. The representation*

$$r_P(I_{W,W}(\chi, s))$$

admits a filtration whose subquotients $\tau_{k,j}$, $k \in \{1, \dots, r\}$, $j \in \{r, \dots, m'\}$ admit isomorphisms

$$A_{k,j} \colon \tau_{k,j} \xrightarrow{\cong} \mathrm{Ind}_{Q_{(k,r-k)} \times P'_{(j-r)} \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi |\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes$$

$$\otimes \sigma_{r-k,j} (\chi |\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi |\det|_v^{s+\frac{j}{2}}) \otimes \underbrace{\chi |\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j)))$$

of $G_r \times G(W_r) \times G(W)$-representations, where $P'_{(j-r)}$ is the parabolic subgroup of $G(W_r)$ defined above.

The main idea of the proof of this theorem is the following. The representation $I_{W,W}(\chi, s)$ corresponds to a sheaf $\mathscr{F}^{s,\chi}$ on the Lagrangian Grassmanian $\mathrm{L}_W := P(Y) \backslash G(W \oplus W)$, where $P(Y)$ is the Siegel parabolic subgroup in $G(W \oplus W)$. We find for each k, j a certain $P_{(r)} \times G(W)$-right-invariant, locally closed subset $\Gamma_{k,j}$ of L_W and show that for each point $x \in \Gamma_{k,j}$ the stabilizer of x under the action of the unipotent part $N_r \times 1_W$ of $P_{(r)} \times G(W)$ is, up to conjugation, independent of x. This allows us to write down an explicit isomorphism from $\tau_{k,j} := r_{P_{(r)} \times G(W)}(\mathscr{F}_c^{s,\chi}(\Gamma_{j,k}))$ to the representation

$$\mathrm{Ind}_{Q_{(k,r-k)} \times P'_{(j-r)} \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi |\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes$$

$$\otimes \sigma_{r-k,j} (\chi |\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi |\det|_v^{s+\frac{j}{2}}) \otimes \underbrace{\chi |\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j))).$$

Here $\mathscr{F}_c^{s,\chi}(\Gamma_{j,k})$ denotes the compactly supported sections on $\Gamma_{k,j}$. Moreover, since we also show that $\bigcup_{j,k} \Gamma_{k,j} = \mathrm{L}_W$, it follows straightforwardly from the definition of $\tau_{k,j}$ that they are the subquotients of a filtration of $r_{P_{(r)} \times G(W)}(I_{W,W}(\chi, s))$.

Finally, the third ingredient is the following theorem of [92].

Theorem 5.1.2 ([92, Theorem 1])**.** *Let $i, j \in \mathbb{N}$, $i \leq j$ and let $\pi \in \mathfrak{Irr}_i$. Then there exists $\pi' \in \mathfrak{Irr}_j$, unique up to isomorphism, such that*

$$\mathrm{Hom}_{G_i \times G_j}(\sigma_{i,j}, \pi \otimes \pi') \neq \{0\}.$$

Moreover, for such a π',

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_i \times G_j}(\sigma_{i,j}, \pi \otimes \pi') = 1.$$

Let now $\pi \in \mathfrak{Irr}_{G(W)}$. If $\pi \otimes \pi^\vee$ does not admit a morphism from any σ_t with $t > 0$, the claim follows straightforwardly. Otherwise we can find $r \in \{1, \ldots, m'\}$ and suitable irreducible smooth representations ρ of G_r and τ of $G(W_r)$ such that π is a quotient of $\operatorname{Ind}_{P_{(r)}}^{G(W)}(\rho \otimes \tau)$. The MVW-involution then allows us to realize π as a subrepresentation of $\operatorname{Ind}_{P_{(r)}}^{G(W)}(\rho^\vee \otimes \tau)$ and hence every morphism

$$f: I_{W,W}(\chi, s) \to \pi \otimes \pi^\vee$$

induces a morphism

$$f': I_{W,W}(\chi, s) \to \operatorname{Ind}_{P_{(r)}}^{G(W)}(\rho^\vee \otimes \tau) \otimes \pi^\vee.$$

Having done this, we can apply Frobenius reciprocity and use the filtration of

$$r_{P_{(r)} \times G(W)}(I_{W,W}(\chi, s))$$

to obtain some restrictions on f' and subsequently on f. Combining these with the first filtration (51) and the theorem of [92], allows us to reduce the claim of Theorem 12 to the following proposition.

Proposition 5.1.3. *Let σ_t be as above and let $\pi \in \mathfrak{Irr}_{G(W)}$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W) \times G(W)}(\sigma_t, \pi \otimes \pi^\vee) \leq 1.$$

This can be proven by induction on $\dim_{\mathbb{L}_v} W$ using a variant of a trick of [92]. Furthermore, we prove an analogous statement to Theorem 12 in the case $G(W)$ being unitary or orthogonal, see Theorem 5.5.3 for the precise statement.

Let us remark that to prove the Conservation relation in Type I in its full generality for above spaces V and W, one would have to extend Theorem 12 also to the case W symplectic and $G(W)$ replaced by $\mathrm{Mp}(W)$, the metaplectic cover of $G(W)$. Indeed, to handle the case where $\mathbb{L}_v = \mathbb{K}_v$, W is symplectic and $\dim_{\mathbb{K}_v} V = n$ is odd, the metaplectic group $\mathrm{Mp}(W)$ appears. In this case we are able to reduce the claim to an analogous statement of Theorem 5.1.2 for metaplectic covers of the general linear group, of which we however do not have a proof at the moment. Finally, the case W and V being right $\mathbb{D}_v$-vector spaces, *i.e.* the quaternionic case, has not been considered in this chapter. The main obstruction here is that the MVW-involution, *cf.* [97, p.91], does not extend easily to this setting.

5.2 Preliminary results

In this section we will collect several lemmas, which will be useful throughout the later parts of the proof. We start by recalling Howe-duality in type II as treated in [92].

5.2.1 Howe duality in type II Let $\sigma_{n,m}$ be the space of Schwartz-functions on

$$M_{n,m} := \operatorname{End}(E^m, E^n).$$

It carries a natural action of $G_n \times G_m$ by

$$(g_1, g_2) \cdot f := (x \mapsto f(g_1^{-1} x g_2)).$$

and admits a filtration $0 = S_{t+1} \subseteq \ldots \subseteq S_0 = \sigma_{n,m}$, $t = \min(n,m)$, where $S_i = S(M^i_{n,m})$ denotes the space of Schwartz-functions on $M^i_{n,m}$, the linear maps of rank greater than or equal to i. The subquotients are of the form

$$\omega_i := S_{i+1}\backslash S_i \cong S(M_{n,m,i}) \cong \# - \mathrm{Ind}_{Q_{(n-i,i)}\times Q_{(m-i,i)}}^{G_n\times G_m}(\underbrace{1}_{G_{n-i}} \otimes \underbrace{1}_{G_{m-i}} \otimes S(G_i)),$$

where $S(M_{n,m,i})$ denotes the space of Schwartz-functions on $M_{n,m,i}$, the linear maps of precisely rank i, see [97, 3.II.2]. The morphism

$$S(M_{n,m,i}) \xrightarrow{\cong} \# - \mathrm{Ind}_{Q_{(n-i,i)}\times Q_{(m-i,i)}}^{G_n\times G_m}(\underbrace{1}_{G_{n-i}} \otimes \underbrace{1}_{G_{m-i}} \otimes S(G_i))$$

sends f to $(g_1 \times g_2) \mapsto f(g_1^{-1}1_i' g_2)$, where

$$1_i' := \begin{pmatrix} 0 & 0 \\ 0 & 1_i \end{pmatrix}.$$

Theorem 5.2.1 ([92, Theorem 1])**.** *Let $n, m \in \mathbb{N}$, $n \le m$ and $\pi \in \mathfrak{Irr}_n$. Then there exists $\pi' \in \mathfrak{Irr}_m$, unique up to isomorphism, such that*

$$\mathrm{Hom}_{G_n\times G_m}(\sigma_{n,m}, \pi \otimes \pi') \neq \{0\}.$$

Moreover, $\dim_{\mathbb{C}} \mathrm{Hom}_{G_n\times G_m}(\sigma_{n,m}, \pi\otimes\pi') = 1$ *and π' is a quotient of*

$$|\det|_v^{\frac{m-2n-1}{2}} \times \ldots \times |\det|_v^{\frac{1-m}{2}} \times |\det|_v^{\frac{m-n}{2}} \pi^\vee.$$

Note that for α a composition of n, the representation

$$\underbrace{|\det|_v^{\frac{\alpha_1}{2}}}_{G_{\alpha_1}} \times \ldots \times \underbrace{|\det|_v^{\frac{\alpha_k}{2}}}_{G_{\alpha_k}} \cong \mathrm{Z}([1,\alpha_1]_{|\det|_v^{-\frac{1}{2}}}) \times \ldots \times \mathrm{Z}([1,\alpha_k]_{|\det|_v^{-\frac{1}{2}}}) \in \mathfrak{Irr}_n$$

is square-irreducible by Theorem 2.3.9.

Lemma 5.2.2. *Let $\alpha, k \in \mathbb{N}$, $s \in \mathbb{C}$, $n = k\alpha$ and $\rho := \underbrace{|\det|_v^{\frac{s}{2}}}_{G_\alpha}$, $\pi := \rho^k \in \mathfrak{Irr}_n$. If τ is a smooth representation, not necessarily irreducible, such that either $r_{(n-\alpha,\alpha)}(\pi) \twoheadrightarrow \tau\otimes\rho$ or $\rho\otimes\tau \hookrightarrow r_{\overline{P_{(n-\alpha,\alpha)}}}(\pi)$ then $\tau \cong \rho^{k-1}$.*

Proof. We will only show the first claim, the second follows by duality. First of all, it is easy to see from the Geometric Lemma that each irreducible subquotient of τ is isomorphic to ρ^{k-1} thus it suffices to show that τ is irreducible. It is enough to assume τ is of length 2. Moreover, since ρ is square-irreducible, it was shown in the proof of [83, Theorem 4.1.D] that the intertwining operator $R_{\rho,\tau}\colon \rho\times\tau \to \tau\times\rho$ has image isomorphic to π and the so obtained map $\pi \hookrightarrow \rho\times\tau$ is the map obtained by Frobenius-reciprocity from $r_{(n-\alpha,\alpha)}(\pi) \twoheadrightarrow \tau\otimes\rho$. Now if τ_1 is an irreducible subrepresentation of τ and $\tau_2 = \tau/\tau_1$ the corresponding irreducible quotient, we obtain a commutative diagram

$$\begin{tikzcd} \rho\times\tau_1 \arrow[r] \arrow[d, hook] & \tau_1\times\rho \arrow[d, hook] \\ \rho\times\tau \arrow[r, "R_{\rho,\tau}"] \arrow[d, two heads] & \tau\times\rho \arrow[d, two heads] \\ \rho\times\tau_2 \arrow[r] & \tau_2\times\rho \end{tikzcd}$$

where the top and bottom arrows are either 0 or the intertwining operator by Lemma 4.2.2(3). By above observation we know that the bottom arrow has to be non-zero and hence the order of the pole of $R_{\rho,\tau}$ and R_{ρ,τ_2} are equal, again by Lemma 4.2.2(3) This implies that also the pole of R_{ρ,τ_1} is equal to the pole of $R_{\rho,\tau}$, since $\tau_1\cong\rho^{k-1}\cong\tau_2$ and hence also the top arrow is non-zero by Lemma 4.2.2(3). This however contradicts the fact that $R_{\rho,\tau}$ has irreducible image π. □

Next, we prove the two following corollaries of Theorem 5.2.1, therefore we assume $G_n=\mathrm{GL}_n$ in this section.

Lemma 5.2.3. *Let $n\le m\in\mathbb{N}$, $\alpha=(\alpha_1,\dots,\alpha_k)$ a composition of n, $p\in\mathbb{Z}$ and π an irreducible representation of the form*

$$\pi:=\underbrace{|\det|_v^{\frac{n-\alpha_1}{2}+p}}_{G_{\alpha_1}}\times\dots\times\underbrace{|\det|_v^{\frac{n-\alpha_k}{2}+p}}_{G_{\alpha_k}}.$$

Let $a:=\max_i\alpha_i$, $b:=\min_i\alpha_i$ and assume $p\ge a$ or $p<b-n$. Let π' be the irreducible representation such that there exists a unique up to a scalar, non-zero morphism f

$$f\colon\sigma_{n,m}\to\pi\otimes\pi'.$$

Then f does not vanish on S_n and there exists no morphism $\omega_l\to\pi\otimes\pi'$ for $l<n$.

Proof. It is enough to show that for $l<n$, there exists no morphism $\omega_l\to\pi\otimes\pi'$. We note that

$$\omega_l\cong\mathrm{Ind}_{Q_{(n-l,l)}\times Q_{(m-l,l)}}^{G_n\times G_m}(|\det|_v^{\frac{l}{2}}\otimes|\det|_v^{-\frac{l}{2}}\otimes(|\det|_v^{\frac{l-n}{2}}\otimes|\det|_v^{\frac{m-l}{2}})S(G_l)).$$

and apply first (1) with respect to $\overline{Q_{n-l,n}}\times G_m$ and then Lemma 2.3.8 together with the Geometric Lemma to obtain a non-zero morphism

$$|\det|_v^{\frac{l}{2}}\to|\det|_v^{p}(\underbrace{|\det|_v^{-\frac{\alpha_1-l_1}{2}+\frac{n-\alpha_1}{2}}}_{G_{l_1}}\times\dots\times\underbrace{|\det|_v^{-\frac{\alpha_k-l_k}{2}+\frac{n-\alpha_k}{2}}}_{G_{l_k}})$$

for some composition $(l_1,\dots,l_k)$ of $n-l$ with $l_i\le\alpha_i$. By Theorem 2.3.9 all but one l_i are 0 and there is one i such that $l=n-l_i$. But this implies

$$\frac{l}{2}=-\frac{\alpha_i-l_i}{2}+\frac{n-\alpha_i}{2}+p$$

and hence

$$n>l=n-\alpha_i+p\ge n-a+p\text{ and }0\le l=n-\alpha_i+p\le n-b+p,$$

a contradiction. □

Lemma 5.2.4. *Let $n \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_k)$ a composition of n and π the irreducible representation*

$$\pi := \underbrace{|\det|_v^{\frac{n-\alpha_1}{2}}}_{G_{\alpha_1}} \times \dots \times \underbrace{|\det|_v^{\frac{n-\alpha_k}{2}}}_{G_{\alpha_k}}.$$

Let $a := \max_i \alpha_i$ and f the, unique up to a scalar, non-zero morphism

$$f\colon \sigma_{n,n} \to \pi \otimes \pi^\vee.$$

Then f vanishes on S_{n-a+1} and does not vanish on S_{n-a}.

Observe that by Theorem 2.3.9 the representation π of the corollary is indeed an irreducible representation.

Before we start with the proof, let us recall from [97, 3.III. 7], see also Section 2.3.3, how one constructs a morphism

$$P\colon \sigma_{n,n} \to \pi \otimes \pi^\vee \tag{52}$$

for general irreducible representations π. For $s \in \mathbb{C}$ let f_s be a matrix coefficient of $\pi|\det|_v^s$ and let ϕ be an element of $\sigma_{n,n}$. Let $\mathrm{d}g$ be a Haar measure on G_n and define for $\operatorname{Re} s >> 0$ the Godement-Jaqcuet zeta integral

$$P(s, \phi, f_s) := \frac{1}{L(s - \frac{n-1}{2}, \pi)} \int_{G_n} f_s(g)\phi(g)\,\mathrm{d}g,$$

where $L(s, \pi)$ is the standard L-function of π, see for example [44]. Then $P(s, \phi, f_s)$ is a polynomial in q^s and q^{-s} and can be analytically continued to $s = 0$. By specifying $s = 0$, one obtains a morphism $\sigma_{n,n} \to \pi \otimes \pi^\vee$.

Proof of Lemma 5.2.4. We first show that if there exists a non-zero morphism $\omega_l \to \pi \otimes \pi^\vee$ then $l = n$ or $l \in \{n - \alpha_1, \dots, n - \alpha_k\}$. As in Lemma 5.2.3 recall that if $l < n$

$$\omega_l \cong \operatorname{Ind}_{Q_{(n-l,l)} \times Q_{(n-l,l)}}^{G_n \times G_n} (|\det|_v^{\frac{l}{2}} \otimes |\det|_v^{-\frac{l}{2}} \otimes (|\det|_v^{\frac{l-n}{2}} \otimes |\det|_v^{\frac{n-l}{2}}) S(G_l)).$$

Applying first (1) with respect to $\overline{Q_{n-l,l}} \times G_n$ and then Lemma 2.3.8 together with the Geometric Lemma gives a non-zero morphism

$$|\det|_v^{\frac{l}{2}} \to \underbrace{|\det|_v^{-\frac{\alpha_1 - l_1}{2} + \frac{n-\alpha_1}{2}}}_{G_{l_1}} \times \dots \times \underbrace{|\det|_v^{-\frac{\alpha_k - l_k}{2} + \frac{n-\alpha_k}{2}}}_{G_{l_k}}$$

for some composition $(l_1, \dots, l_k)$ of $n - l$ with $l_i \leq \alpha_i$. By Lemma 2.3.8 it follows that all but one l_i are 0 and hence for this one non-zero $l_i = n - l$ we have

$$\frac{l}{2} = -\frac{\alpha_i - l_i}{2} + \frac{n - \alpha_i}{2}$$

which implies $l = n - \alpha_i$.

In order to show the lemma, it thus suffices by Theorem 5.2.1 to show that for the map P of (52), P vanishes on $S_{n-\alpha_i}$ for $n - \alpha_i > n - \alpha$ and S_n. Since π is an irreducible induced representation, we can assume without loss of generality that $\alpha_1 \leq \ldots \leq \alpha_k = a$. We start with the case $k = 1$ and hence π is the trivial representation. Then $L(s - \frac{n-1}{2}, \pi)$ has poles at $s = i$, $i \in \{0, \ldots, n-1\}$ and therefore $L(s - \frac{n-1}{2}, \pi)^{-1}$ vanishes at 0. Since for $f \in S_n \hookrightarrow \sigma_{n,n}$ the integral $\int_{G_n} f(g)\phi_0(g)\,\mathrm{d}g$ converges and $L(s - \frac{n-1}{2}, \pi)^{-1}$ vanishes at 0, $P(0, f, \phi)$ vanishes for all $f \in S_n$. This finishes the case $k = 1$.

If $k > 1$, we can use [59, Proposition 2.3], which shows that for fixed $\phi \in \sigma_{n,n}$ and matrix coefficient f of π we can write $P(0, \phi, f)$ as the finite linear combination of functionals of the form

$$\prod_{i=1}^{k} P(0, \phi_i, f_i),$$

for $\phi_i \in \sigma_{\alpha_i,\alpha_i}$ and f_i a matrix coefficient of the trivial character of G_{α_i}. Moreover, in the proof of [59, Proposition 2.3] the author shows that

$$\prod_{i=1}^{k} P(0, \phi_i, f_i) = P(0, \phi', f')$$

for f' a matrix coefficient of π and $\phi' \in \sigma_{n,n}$ satisfying

$$\int_{U_\alpha} \phi'(xmu)\,\mathrm{d}u = \phi_1(x_1m_1) \cdot \ldots \cdot \phi_k(x_km_k)$$

for all $m = (m_1, \ldots m_k)$ in the Levi-component of Q_α, $x_i \in M_{\alpha_i,\alpha_i}$,

$$x := \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_k \end{pmatrix}$$

and a suitable Haar-measure $\mathrm{d}u$ on the unipotent part U_α of Q_α. In particular, if we assume that $P(0, \phi', f')$ does not vanish for $\phi' \in S_k$ for $k > n - a$, we know from the case $k = 1$ that each of the ϕ_i does not vanish on 0. Therefore we obtain from the specific form of ϕ' that it does not vanish on some element of rank at most

$$\max_{u \in U_\alpha} \operatorname{rank}(u) = n - a.$$

This contradicts $\phi' \in S_k$. $\square$

5.2.2 Derivatives for classical groups

The following lemma will be useful later on.

Lemma 5.2.5 ([42, Lemma 5.2]). *Let ξ be a character of G_1 such that $\xi^\vee \not\cong \xi$, $r \in \{1, \ldots, q_W\}$ and $\delta \in \mathfrak{Irr}_{G(W_r)}$ such that $r_{\overline{P_{(1)}}}(\delta)$ does not contain an irreducible subquotient of the form $\xi \otimes \delta'$. Then $\xi^r \rtimes \delta$ has a unique irreducible quotient denoted by $\delta_{\xi,r}$, $\xi^r \otimes \delta$ appears in $r_{\overline{P_{(r)}}}(\xi^r \rtimes \delta)$ with multiplicity 1 and there exists no $\delta' \not\cong \delta$, $\delta' \in \mathfrak{Irr}_{G(W_r)}$ such that $\xi^r \otimes'$ appears in $r_{\overline{P_{(r)}}}(\xi^r \rtimes \delta)$.*

Moreover, if $\pi \in \mathfrak{Irr}_{G(W)}$ is such that $r_{\overline{P_{(1)}}}(\pi)$ does contain an irreducible subquotient of the form $\chi \otimes \delta^\vee$, then there exist r and $\delta \in \mathfrak{Irr}_{G(W_r)}$ such that $\pi \cong \delta_{\chi,r}$.

Let G' be now either G_n or $G(W)$. We denote by $S(G')$ the regular representation of $G' \times G'$, *i.e.* the space of Schwartz-functions on G' with action

$$(g_1, g_2) \cdot f \coloneqq (h \mapsto f(g_1^{-1} h g_2)).$$

If G' is metaplectic we consider ζ_2-equivariant Schwartz-functions: See also [97, 3.II.3] for the next lemma.

Lemma 5.2.6. *For $\pi, \pi' \in \mathfrak{Irr}_{G'}$, the space*

$$\mathrm{Hom}_{G' \times G'}(S(G'), \pi \otimes \pi') \neq 0$$

if and only if $\pi^\vee \cong \pi'$ and in this case it is 1 dimensional. More generally, if $\pi_1, \pi_2 \in \mathfrak{Rep}_{G'}$ of finite length, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G' \times G'}(S(G'), \pi_1 \otimes \pi_2) = \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi_1^\vee, \pi_2).$$

If χ is a character of G', then there exists an isomorphism

$$S(G') \xrightarrow{\cong} (\chi^\vee \otimes \chi) S(G').$$

Proof. Let $C^\infty(G') = S(G')^\vee$ be the vector space of smooth functions on G' on which $G' \times G'$ acts by left-right translation as on $S(G')$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G' \times G'}(S(G'), \pi_1^\vee \otimes \pi_2^\vee) = \dim_{\mathbb{C}} \mathrm{Hom}_{G' \times G'}(\pi_1 \otimes \pi_2, C^\infty(G')).$$

Note now that $C^\infty(G')$ is the trivial representation of $\Delta G'$ non-compactly induced to $G' \times G'$. The claim then follows from Frobenius-reciprocity. The last isomorphism is given by sending a function $f \mapsto \chi f$. □

Lemma 5.2.7. *Let G' be as above and P a parabolic subgroup of G' with Levi-decomposition $P = M \ltimes N$. Then*

$$r_{P \times G}(S(G')) = \mathrm{Ind}_{M \times P}^{M \times G'}(S(M))$$

Proof. Since $S(G') = \mathrm{Ind}_{\Delta G'}^{G' \times G'} 1$ we can apply the Geometric Lemma and note that $\Delta G' \backslash G' \times G'$ is a single $P \times G'$-orbit. □

If P is a maximal parabolic subgroup of G' with Levi-component $M = M_1 \times M_2$ and two representations of finite length of M of the form $\pi_1 \otimes \pi_2$, $\pi_1' \otimes \pi_2'$ we recall

$$\mathrm{Hom}_M(\pi_1 \otimes \pi_2, \pi_1' \otimes \pi_2') \cong \mathrm{Hom}_{M_1}(\pi_1 \otimes \pi_1') \otimes \mathrm{Hom}_{M_2}(\pi_2 \otimes \pi_2')$$

as $\mathbb{C}$-vector spaces, see for example [107, Theorem 1.1].

Lemma 5.2.8. *Let P be a parabolic subgroup of G with Levi-decomposition $P = M \ltimes N$, $\pi \in \mathfrak{Rep}_M$ not necessarily of finite length and $\tau \in \mathfrak{Rep}_G$ a representation of finite length. Let $f \colon \mathrm{Ind}_P^G(\pi) \to \tau$ be a non-zero morphism. Then there exists an irreducible subquotient σ of π and a non-zero morphism $\mathrm{Ind}_P^G(\sigma) \to \tau$.*

Proof. Assume first $G = P$ and take σ to be an irreducible subrepresentation of $\mathrm{Ker}(f)/\pi \hookrightarrow \tau$. Since f factors through $\pi \twoheadrightarrow \mathrm{Ker}(f)/\pi$ we obtain the desired morphism by restricting to σ. For general P, we obtain by Frobenius reciprocity a non-zero morphism $\pi \to r_P(\tau)$. Applying the first case to M and $P = M$ yields an irreducible subquotient σ of π together with a non-zero morphism $\sigma \to r_P(\tau)$. Again by Frobenius reciprocity, we obtain a non-zero morphism $\mathrm{Ind}_P^G(\sigma) \to \tau$. □

5.3 Filtrations

In this section, we study filtrations of $I_{W,W}(\chi, s)$ and its Jacquet module along a maximal parabolic subgroup. The first filtration was already introduced in [73].

Consider the space $W \oplus W$ where the second copy of W is equipped with the inner product $-\langle .,. \rangle$. This induces a natural morphism

$$\iota: G(W) \times G(W) \to G(W \oplus W),$$

which is an embedding except when $G(W) = \mathrm{Mp}(W)$. In this case it restricts to an embedding

$$\iota: \mathrm{Sp}(W) \times \mathrm{Sp}(W) \hookrightarrow \mathrm{Sp}(W \oplus W)$$

and we choose it so that $(\zeta, \zeta') \mapsto (\zeta\zeta')$ on the ζ_2-part. We write from now on W^i, X^i_a, $X^i_{a,b}, Y^i_a$, $Y^i_{a,b}$ for the i-th copy of the respective space in $W \oplus W$ for suitable a, b. Let χ be a unitary character of G_n, $s \in \mathbb{C}$ and $Y := X^1_{q_W} \oplus X^2_{q_W} \oplus \Delta W_{q_W} \subseteq W \oplus W$ and decompose the corresponding parabolic subgroup as $P(Y) = M(Y) \ltimes N(Y)$. We consider the representation, called the degenerate principal series representation of $G(W) \times G(W)$,

$$I_{W,W}(\chi, s) := \iota^*(\mathrm{Ind}_{P(Y)}^{G(W \oplus W)}(\chi|\det|_v^s)).$$

This representation will preoccupy us throughout the remaining chapter. Define the Grassmanian

$$\mathrm{L}_W := P(Y) \backslash G(W \oplus W)$$

as the space of n-dimensional isotropic subspaces of $W \oplus W$. The above embedding gives a $G(W) \times G(W)$-action on L_W with locally-closed orbits Ω_t for $t \in \{0, \dots, q_W\}$ given by

$$\Omega_t := \{U : \dim_E(U \cap W^1) = \dim_E(U \cap W^2) = t\}.$$

Set $\Omega^t := \bigcup_{t' \leq t} \Omega_{t'}$, which is an open subset of L_W. By Proposition 2.3.2 we have a filtration

$$0 = I_{-1} \subseteq I_0 \subseteq \dots \subseteq I_{q_W} = I_{W,W}(\chi, s),$$

where

$$I_t := \{\phi \in I_{W,W}(\chi, s) : \phi|_{\overline{\Omega_{t+1}}} = 0\}.$$

Define

$$\sigma_t := I_{t-1} \backslash I_t.$$

The following theorem was already proved in [73, §1]. To make later arguments clearer, we write it out.

Lemma 5.3.1 ([73, §1])**.** *The subquotients $\sigma_t := I_{t-1} \backslash I_t$ are of the form*

$$\sigma_t \cong (1 \otimes \chi|_{\mathrm{Z}(G(W))})\mathrm{Ind}_{P_{(t)} \times P_{(t)}}^{G(W) \times G(W)}(\underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes \underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes S(G(W_t))).$$

Proof. We introduce the following notation. If A, B, C, D are vector spaces over E,

$$a \in \mathrm{Hom}(A, C),\ b \in \mathrm{Hom}(B, C),\ c \in \mathrm{Hom}(A, D),\ d \in \mathrm{Hom}(B, D),$$

we denote by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Hom}(A \oplus B, C \oplus D)$$

the corresponding linear map. Pick $\delta_t \in G(W \oplus W)$ with $P(Y)\delta_t \in \Omega_t$ as follows. Recall $\alpha_{q_W - t}, \beta_t$ of Section 2.3.4 and for $V := (X^1_{t,q_W} \oplus Y^1_{t,q_W}) \oplus (X^2_{t,q_W} \oplus Y^2_{t,q_W})$ let

$$x_{0,V} := \begin{pmatrix} 0 & \alpha_{q_W-t} \\ \beta_t & 1_{2q_W-2t} \end{pmatrix} \in \mathrm{Hom}((X^1_{t,q_W} \oplus Y^2_{t,q_W}) \oplus (X^2_{t,q_W} \oplus Y^1_{t,q_W}), V).$$

We now set $V' := X^1_t \oplus Y_{t,1} \oplus X^2_t \oplus Y^2_t \oplus W^1_{q_W} \oplus W^2_{q_W}$ and define δ_t to be the image of $1_{V'} \times x_{0,V}$ (respectively $(1_{V'}, 1) \times x_{0,V}$ in the metaplectic case) in $G(W \oplus W)$ under the natural morphism

$$G(V') \times G(V) \to G(W \oplus W).$$

In the metaplectic case we again take the morphism respecting the multiplication. The stabilizer of $P(Y)\delta_t$ in L_W under the action of $G(W) \times G(W)$ is the subgroup

$$R_t \cong G_t \times G_t \times \Delta G(W_t) \ltimes (N_t \times N_t) \subseteq P_{(t)} \times P_{(t)}, \tag{53}$$

where N_t denotes the unipotent component of $P_{(t)}$. Next, let us recast the representation $I_{W,W}(\chi, s)$ in the language of ℓ-sheaves by sending the representation $\chi|\det|^s_v \delta^{\frac{1}{2}}_{P(Y)}$ through the following diagram.

$$\begin{tikzcd} \mathfrak{Rep}_{P(Y)} \arrow[r, "\mathrm{Ind}"] & \mathrm{Sh}(\mathrm{L}_W, G(W \oplus W)) \arrow[r, "\mathrm{Res}"] & \mathrm{Sh}(\mathrm{L}_W, G(W) \times G(W)) \arrow[d, "\mathrm{Sec}"] \\ \mathfrak{Rep}_{G_1} \arrow[u] & & \mathfrak{Rep}_{G(W) \times G(W)} \end{tikzcd}$$

By Proposition 2.3.3 this computes precisely $I_{W,W}(\chi, s)$. Using this, we denote by $\mathscr{F}^{s,\chi}$ the sheaf in $\mathrm{Sh}(\mathrm{L}_W, G(W) \times G(W))$ such that $\mathrm{Sec}(\mathscr{F}^{s,\chi}) = I_{W,W}(\chi, s)$. The filtration of L_W by Ω^t gives by Proposition 2.3.2 a filtration of $\mathrm{Sec}(\mathscr{F}^{s,\chi})$ via the short exact sequence

$$\begin{tikzcd} \mathrm{Sh}(\Omega^{t-1}, G(W) \times G(W)) \arrow[r] \arrow[d, mapsto] & \mathrm{Sh}(\Omega^t, G(W) \times G(W)) \arrow[r] \arrow[d, mapsto] & \mathrm{Sh}(\Omega_t, G(W) \times G(W)) \arrow[d, mapsto] \\ \mathscr{F}^{s,\chi}_c(\Omega^{t-1}) \arrow[r, hook] & \mathscr{F}^{s,\chi}_c(\Omega^t) \arrow[r, two heads] & \sigma_t \end{tikzcd}$$

But $\mathrm{Sh}(\Omega_t, G(W) \times G(W)) \xrightarrow{\mathrm{Sec}} \mathfrak{Rep}_{G(W) \times G(W)}$ is by Proposition 2.3.3 equivalent to the following composition.

$$\begin{tikzcd} \mathrm{Sh}(\Omega_t, G(W) \times G(W)) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{Sec}"] & \mathfrak{Rep}_{R_t} \arrow[d, "\mathrm{Ind}"] \\ \mathfrak{Rep}_{G(W) \times G(W)} & \mathrm{Sh}(\Omega_t, G(W) \times G(W)) \arrow[l, "\mathrm{Sec}"'] \end{tikzcd}$$

This gives the claimed induced representation. Namely, for

$$g = (m_1, m_2, g, g, n_1, n_2) \in R_t,$$

we have

$$\delta_t g \delta_t^{-1} = (\underbrace{m}_{H(Y)}, n) \in P(Y)$$

and by an easy calculation $|m| = |m_1 m_2|$. Then $G_t \times G_t$ acts by $\chi|\det|_v^s \delta_{P(Y)}^{\frac{1}{2}}$ on $\mathrm{Res}(\mathscr{F}^{s,\chi})$, $\Delta G(W_t)$ acts by $\chi\big|_{Z(G(W_t))}$ and $N_t \times N_t$ acts trivially. Moreover,

$$\mathfrak{Rep} R_t \xrightarrow{\mathrm{Ind}} \mathrm{Sh}(\Omega_t, G(W) \times G(W))$$

factors as

$$\mathfrak{Rep} R_t \xrightarrow{\mathrm{Ind}} \mathrm{Sh}(R_t \backslash P_{(t)} \times P_{(t)}) \xrightarrow{\mathrm{Ind}} \mathrm{Sh}(\Omega_t, G(W) \times G(W)).$$

Since the image of a character χ under

$$\mathfrak{Rep}_{\Delta G(W_t)} \xrightarrow{\mathrm{Ind}} \mathrm{Sh}(\Delta G(W_t) \backslash G(W_t) \times G(W_t), G(W_t) \times G(W_t)) \xrightarrow{\mathrm{Sec}}$$

$$\to \mathfrak{Rep}_{G(W_t) \times G(W_t)}$$

is the regular representation twisted by $(1 \otimes \chi^{-1})$, the final assertion follows by noting that $\mathrm{Sec} \circ \mathrm{Ind}$ gives $\# - \mathrm{Ind}$ and writing $1 = \delta_{P_{(t)} \times P_{(t)}}^{\frac{1}{2}} \delta_{P_{(t)} \times P_{(t)}}^{-\frac{1}{2}}$ to normalize the induction. $\square$

The isomorphism

$$\mathscr{F}_c^{s,\chi}(\Omega_t) \xrightarrow{\cong} (1 \otimes \chi\big|_{Z(G(W)})\mathrm{Ind}_{P_{(t)} \times P_{(t)}}^{G(W) \times G(W)} (\underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes \underbrace{\chi|\det|_v}_{G_t}{}^{s+\frac{t}{2}} \otimes S(G(W_t))).$$

can therefore be written out explicitly as

$$f \mapsto (\underbrace{(g_1 \times g_2)}_{G(W) \times G(W)} \mapsto (\underbrace{h}_{G(W_j)} \mapsto f(\delta_t \iota(g_1, h g_2)))).$$

An irreducible representation $\pi \in \mathfrak{Irr}_{G(W)}$ is said *to appear on the boundary component* if there exists a non-zero morphism $f\colon I_{W,W}(\chi, s) \to \pi \otimes \pi^\vee$ which vanishes on I_{t-1} and does not vanish on I_t for some $t > 0$. Note that in this case there exists then a non-zero morphism $\sigma_t \to \pi \otimes \pi^\vee$.

Remark. Note that one could a prioi hope that one gets a similar statement for $I_{W,W}(\chi, s)$ as for $\sigma_{n,n}$, in the sense that if there exists a non-zero morphism $I_{W,W}(\chi, s) \to \pi \otimes \pi'$, then this implies $\pi' \cong \chi\pi^\vee$. However, if for example W is symplectic, σ_{q_W} does not have to be cosocle irreducible, indeed for $s = -\frac{q_W}{2}$, σ_{q_W} is semisimple of length 4.

5.3.1 Filtration of the Jacquet functor Let $r \in \{1,\dots,q_W\}$, $P = P_{(r)} \times G(W) \subseteq G(W) \times G(W)$ be a standard parabolic subgroup and define the following subsets of L_W for $k \in \{1,\dots,r\}$, $j \in \{r,\dots,q_W\}$.

$$\Gamma^{k,j} := \{U : \dim_E(U \cap X_r^1) \le k,\, \dim_E(p_{Y_r^1}(U)) \ge r-k,$$
$$\dim_E(p_{X_r^1\backslash W^1}(U)) \ge 2q_W - j - k\},$$

where $p_{Y_r^1}$ respectively $p_{X_r^1\backslash W^1}$ is the projection to Y_r^1 respectively $X_r^1\backslash W^1$. This subspace is P-invariant and $\Gamma^{k,j}$ is open since $\dim_E(\cdot \cap X)$ is upper semicontinous and $\dim_E(p_X(-))$ is lower semicontinous for a suitable vector space X. We write $(k,j) \le (k',j')$ if $k \le k'$, $j \le j'$. Then $\Gamma^{k,j} \subseteq \Gamma^{k',j'}$ if and only if $(k,j) \le (k',j')$. Define

$$\Gamma_{k,j} := \Gamma^{k,j} \backslash \bigcup_{(k',j')<(k,j)} \Gamma^{k',j'} = \{U : \dim_E(U \cap X_r^1) = k,\, \dim_E(p_{Y_r^1}(U)) = r-k,$$
$$\dim_E(p_{X_r^1\backslash W^1}(U)) = 2q_W - j - k\}.$$

It is then clear that $\bigcup_{k,j}\Gamma_{k,j} = \mathrm{L}_W$ and the union is disjoint. Hence this gives a stratification of L_W by P-invariant locally closed subspaces of L_W. Recall the parabolic subgroup $P'_{(j-r)}$ defined in Section 2.3.4.

Theorem 5.3.2. *The representation $r_P(I_{W,W}(\chi,s))$ has a filtration with subquotients $\tau_{k,j}$, $k \in \{1,\dots,r\}$, $j \in \{r,\dots,q_W\}$ such that there exists isomorphisms*

$$A_{k,j}\colon \tau_{k,j} \xrightarrow{\cong} (1 \otimes \chi)\mathrm{Ind}_{Q_{(k,r-k)}\times P'_{(j-r)}\times P_{(j)}}^{G_r\times G(W_r)\times G(W)} (\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes$$
$$\otimes \sigma_{r-k,j}(\chi|\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \otimes \underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j))),$$

where $P_{(j-r)}$ is a parabolic subgroup of $G(W_r)$.

Proof. We write the Levi-decomposition of P as $P = M \ltimes N$. Let $\mathscr{F}^{s,\chi}$ be the sheaf in $\mathrm{Sh}(\mathrm{L}_W, G(W)\times G(W))$ such that $\mathscr{F}_c^{s,\chi}(\mathrm{L}_W) = I_{W,W}(\chi,s)$ as in the proof of Lemma 5.3.1. Observe that $\Gamma_{k,j}$ is covered by

$$\Gamma_{k,j,t} := \Omega_t \cap \Gamma_{k,j} = x_t w_{k,j-r,t+r-j-k}\iota(P),\ t \in \{j-r+k,\dots,j\}.$$

To see this it is enough to note that for $U = x_t w_{a,b,t-a-b}\iota(p)$, $a,b \in \mathbb{N}$, $a+b \le t$, $p \in P$

$$\dim_E(U \cap X_r^1) = a,\ \dim_E(p_{X_r^1\backslash W^1}(U)) = 2q_W - r - b - a,$$
$$\dim_E(p_{Y_r^1}(U)) = r - a.$$

We now set

$$\# - \tau_{k,j} := \mathscr{F}_c^{s,\chi}(\Gamma_{k,j})_N,\ \tau_{k,j} := \delta_P^{-\frac{1}{2}} \# - \tau_{k,j}$$

and since the $\Gamma_{k,j}$ cover L_W and are locally closed, we obtain a filtration of $r_P(I_{W,W}(\chi,s))$ with subquotients $\tau_{k,j}$'s by Proposition 2.3.2. Next, we will define a morphism

$$A'_{k,j}\colon \# - \tau_{k,j} \to (1 \otimes \chi)\# - \mathrm{Ind}_{Q_{k,r-k}\times P'_{(j-r)}\times P_{(j)}}^{G_r\times G(W_r)\times G(W)} (\underbrace{\chi|\det|_v^s \delta_{P(Y)}^{\frac{1}{2}}}_{G_k} \otimes$$

$$\otimes\sigma_{r-k,j}(\chi|\det|_v^{-s}\delta_{P(Y)}^{-\frac{1}{2}}|\det|_v^{n-j-k-\eta}\otimes\chi|\det|_v^{s}\delta_{P(Y)}^{\frac{1}{2}})\otimes\underbrace{\chi|^{s-r+k}\delta_{P(Y)}^{\frac{1}{2}}}_{G_{j-r}}\otimes S(G(W_j))).$$

To do so we first define a morphism $\phi : M_{r-k,j}\times G(W_j)\to G(W\oplus W)$. This is done as follows. Let $h=(x,g)$ be an element of the left side and let l be the rank of x and set $t=j-l$. We then write

$$x = m_2^{-1}1_l' t m_1 \tag{54}$$

with $m_1\in G_j$, $m_2\in G_{r-k}$ and

$$1_l' := \begin{pmatrix}0 & 0\\ 0 & 1_l\end{pmatrix}\in M_{r-k,j}.$$

Next let $\iota_1\colon G_j\to G(W)$ be the multiplicative morphism sending $m_1\in G_j$ to the element $(m_j,1_{W_j})$ in the Levi-subgroup of the standard parabolic $P_{(r)}$. Similarly, we define $\iota_2\colon G_{r-k}\to G(W)$ by sending first

$$m_2\mapsto\begin{pmatrix}1_k & 0\\ 0 & m_2\end{pmatrix}$$

and then embedding this element into the Levi subgroup of the standard parabolic $P_{(r)}$ of $G(W)$. Finally, let $\iota_3\colon G(W_j)\to G(W)$ by sending g to the element $(1_j,g)$ in the Levi subgroup of the standard parabolic subgroup $P_{(j)}$ of $G(W)$. We then set

$$\phi(h):=\iota(\iota_2(\mathfrak{c}({}^t m_2)),\iota_1(m_1^{-1}))\delta_t\iota(w_{k,j-r,t+r-j-k},1_W)\iota(\iota_2(m_2),\iota_1(m_1)\iota_3(g)),$$

where $w_{k,j-r,t+r-j-k}=w_{k,j-r,t+r-j-k,r}$ is the element we introduced in Section 2.3.4. We need to show that this is independent of the choice of m_1 and m_2. Recall that the stabilizer of $1_l'$ under the action of $G_{r-k}\times G_j$ in the sense of (54) is of the form

$$p'=\begin{pmatrix}p_3 & 0\\ p_2' & p_1'\end{pmatrix}\in G_{r-k},\ p=\begin{pmatrix}p_1 & p_2\\ 0 & p_3\end{pmatrix}\in G_j,$$

where $p_1\in G_t$, $p_2\in M_{t,l}$, $p_3\in G_l$, $p_1'\in G_{r-k-l}$, $p_2'\in M_{l,r-k-l}$. A straightforward calculation shows then that if $p_3=1_l$

$$\iota(1_W,\iota_1(p^{-1}))\delta_t\iota(w_{k,j-r,t+r-j+k},1_W)\iota(1_W,\iota_1(p))=\delta_t\iota(w_{k,j-r,t+r-j+k},1_W)$$

and

$$\iota(\iota_2(\mathfrak{c}({}^t p')),1_W)\delta_t\iota(w_{k,j-r,t+r-j+k},1_W)\iota(\iota_2(p'),1_W)=\delta_t\iota(w_{k,j-r,t+r-j+k},1_W).$$

Moreover, if $p_1=1_t$, $p_1'=1_{r-k-l}$, $p_2=0$, $p_2'=0$ one has the identity

$$\iota(\iota_2(\mathfrak{c}({}^t p')),\iota_1(p^{-1}))\delta_t\iota(w_{k,j-r,t+r-j+k},1_W)=$$
$$=\delta_t\iota(w_{k,j-r,t+r-j+k},1_W)\iota(\iota_2(p'^{-1}),\iota_1(p^{-1})).$$

Combining these three equalities gives $\phi(p'^{-1}1_l p,1_W)=\phi(1_l,1_W)$ and hence shows that ϕ is well defined. Note moreover, that if we restrict the morphisms to those elements of a fixed rank l, it is continuous and for $a=(m,g)\in G_{r-k}\times G(W_j)$, $b=(m',g')\in G_j\times G(W_j)$

$$\phi(a^{-1}hb)=\iota(\iota_2(\mathfrak{c}({}^t m)),\iota_1(m'^{-1})\iota_3(g^{-1}))\phi(h)\iota(\iota_2(m)\iota_3(g'),\iota_1(m')). \tag{55}$$

We then define $A'_{k,j}$ as the morphism sending a section

$$f \in \mathscr{F}_c^{s,\chi}(\Gamma_{k,j}),\ f\colon P(Y)\Gamma_{k,j} \to \mathbb{C}$$

on the left hand side to

$$\underbrace{g}_{G(W)} \times \underbrace{m}_{G_r \times G(W_r)} \mapsto \left(\underbrace{h}_{M_{r-k,j} \times G(W_j)} \mapsto \int_{N_{k,j}\backslash N} f(\phi(h)\iota(nm,g))\,\mathrm{d}n \right),$$

where we fix the choice of a Haar-measure on N and $N_{k,j} := N_{k,j,j}$ with

$$N_{k,j,t} := \iota^{-1}(\iota(N,1_W) \cap \iota(w_{k,j-r,t+r-j-k}^{-1}, 1_W)\delta_t^{-1}N(Y)\delta_t\iota(w_{k,j-r,t+r-j-k}, 1_W))$$

for $t \in \{j-r+k,\dots,j\}$. Putting now all issues of well-definedness aside for a moment, it follows from the properties of a Haar-measure that this morphism factors through $\mathscr{F}_c^{s,\chi}(\Gamma_{k,j})_N$.

Several things need now to be checked in order for this morphism to well defined. We start with the following lemma.

Lemma 5.3.3. *For all $t, t' \in \{j-r+k,\dots,j\}$*

$$N_{k,j,t} = N_{k,j,t'}.$$

Proof. Recall from the proof of Lemma 5.3.1 that the stabilizer of $x_t = P(Y)\delta_t$ under the action of $G(W) \times G(W)$ is equal to

$$R_t \cong G_t \times G_t \times \Delta G(W_t) \ltimes (N_t \times N_t).$$

Here we denote now by $N_t = N(X_t)$ again the unipotent part of the parabolic subgroup $P_{(t)} = P(X_t)$ in $G(W)$ and we write from now on $w(H) := w^{-1}Hw$ for any closed subgroup H of $G(W)$. Thus $N_{k,j,t}$ is

$$N_{k,j,t} = N_r \cap w_{k,j-r,t+r-j-k}(N_t).$$

Note the following equalities. Let $a, b \in \{0,\dots,q_W-1\}$. Firstly, $N(X_a) \cap N(X_{b,q_W})$ consists of those elements in $N(X_a)$ which are the identity except on Y_{0,q_W} and induce the 0-map in $\mathrm{Hom}(Y_b, X_b)$. Moreover,

$$N(X_{a,b}) \cap N(Y_{a,b}) = \{1\}$$

and if $a \le b$ $N(X_a) \cap N(X_b)$ consists of those elements in $N(X_a)$ which are the identity on $X_{a,b} \oplus Y_{a,b}$. Using this, the claim follows easily. Indeed, note that

$$w_{k,j-r,t+r-j-k}(N_t) = N(X_k \oplus Y_{k,t-j+r} \oplus X_{q_W-j+r,q_W}).$$

It follows that therefore $N_r \cap w_{k,j-r,t+r-j-k}(N_t)$ consists of the following elements of N_r. If $j-r \ne 0, k \ne r$ they are the identity except on $X_k \oplus Y_k \oplus Y_{q_W-j+r,q_W}$ and induce the 0-map in $\mathrm{Hom}(Y_{q_W-j+r}, X_{q_W-j+r})$. If $j = r, k \ne r$, it consists of those elements in N_r which are the identity everywhere except on $X_k \oplus Y_k$. Finally, if $k = r$ then $j = t$ and it consists of those elements in N_r which are the identity on $X_{r,j} \oplus Y_{r,j}$. This description is independent of t. $\square$

It implies that $n \mapsto f(\phi(h)\iota(nm,g))$ is invariant under $N_{k,j}$ and hence the integral makes sense. Let us check now that the integral converges. Indeed, let K be the compact support of f on $\mathrm{L}_W = P(Y)\backslash G(W \oplus W)$. From the definition of $N_{k,j}$ it follows that $x(N_{k,j}\backslash N)$ defines a closed subspace of L_W for all $x \in \Gamma_{k,j}$ and hence $xK \cap xN_{k,j}\backslash N$ is again compact and therefore we have convergence. Next, it is easy to see that $A'_{k,j}(f)$ is also compactly supported with respect to h and $A'_{k,j}(f)$ is locally constant with respect to g and m. A priori it is however only given that $A'_{k,j}(f)$ is locally constant with respect to h when we restrict it to elements h whose rank in $M_{r-k,j}$ is l for some fixed l. Indeed, for $a = (a', 1_{W_j}) \in G_{r-k} \times G(W_j)$, $b \in G_j \times G(W_j)$ both close to the identity, we have thanks to the $P(Y)$-invariance, f being locally constant, (55) and a change of variables

$$\int_{N_{k,j}\backslash N_r} f(\phi(h)\iota(nm,g))\,\mathrm{d}n = \int_{N_{k,j}\backslash N_r} f(\phi(b^{-1}ha)\iota(nm,g))\,\mathrm{d}n.$$

To show that $A'_{k,j}(f)$ is locally constant with respect to h without this restriction to a fixed rank, we argue as follows. Fix a rank l and let $i \in \{l+1, \dots, r-k\}$ and for $e \in F$

$$m_e := \begin{pmatrix} 0 & 0 & 0 \\ 0 & e1_{i-l} & 0 \\ 0 & 0 & 1_l \end{pmatrix} = 1'_i n'_e$$

with

$$n'_e := \begin{pmatrix} 1_{j-i} & 0 & 0 \\ 0 & e1_{i-l} & 0 \\ 0 & 0 & 1_l \end{pmatrix}.$$

Note then that the centralizer of $w_{k,j-r,r-k-l}^{-1}\delta_{j-l}^{-1}\delta_{j-i}w_{k,j-r,r-k-i}$ contains $\iota(G_r \times N_r, 1_W)$, as it is the identity plus a morphism in $\mathrm{Hom}(X,Y)$ represented by the matrix β_{i-l}, where

$$X := Y^1_{l+k,i+k} \oplus X^2_{j-i,j-l},\ Y := Y^2_{j-i,j-l} \oplus X^1_{l+k,i+k}.$$

Thus for all $p \in G_r \ltimes N_r$

$$\iota(p^{-1}, 1_W)\phi(1'_l, 1_{W_j})^{-1}\phi(m_e, 1_{W_j})\iota(p, 1_W) =$$

$$= \iota(1, \iota_1({n'}_e^{-1}))\phi(1'_l, 1_{W_j})^{-1}\phi(1'_i, 1_{W_j})\iota(1_W, \iota_1(n'_e)).$$

We can describe the last element explicitly as $1_{W\oplus W}$ plus a morphism in $\mathrm{Hom}(X,Y)$ represented by the matrix $e\beta_{i-l}$ and for any open neighborhood of the identity we can choose e such that the above element is contained in it. Using that f is locally constant and a change of variables we thus obtain that that

$$\int_{N_{k,j}\backslash N} f(\phi(1'_l, 1_{W_j})\iota(nm,g))\,\mathrm{d}n =$$

$$= \int_{N_{k,j}\backslash N} f(\phi(1'_l, 1_{W_j})\iota(nm,g)\phi(1'_l, 1_{W_j})^{-1}\iota(n^{-1}, 1_W)\phi(m_e, 1_{W_j})\iota(n, 1_W))\,\mathrm{d}n =$$

$$= \int_{N_{k,j}\backslash N} f(\phi(m_e, 1_{W_j})\iota(nm,g))\,\mathrm{d}n$$

and hence $A'_{k,j}(f)(g,m)(1_l, 1_{W_j}) = A'_{k,j}(f)(g,m)(m_e, 1_{W_j})$ for all g and m. If h is another element in $M_{r-k,j}$ of rank i different from $(m_e, 1_{W_j})$ such that h is close to $1'_l$ we can use that

f restricted to a fixed rank is locally constant together with the fact that ϕ is continuous on those elements to show that $A_{k,j}(f)$ is locally constant with respect to h.

Next, we need to check that $A'_{k,j}(f)$ behaves under left translation with respect to g and m as is required by parabolic induction. For the invariance regarding the unipotent part of $P_{(j)}$, $Q_{(k,r-k)}$ and $P'_{(j-r)}$ we argue as follows. Firstly, for n in the unipotent part of $Q_{(k,r-k)}$ or $P'_{(j-r)}$, we have that for $t \in \{j-r+k,\dots,j\}$, $w_{k,j-r,t-j-k+r} n w^{-1}_{k,j-r,t-j-k+r} \in \iota(N_t, 1_W)$ and hence the invariance follows from (53). For n in the unipotent part of $P_{(j)}$, then ${}_{k,j-r,t-j-k+r} n w^{-1}_{k,j-r,t-j-k+r} \in \iota(N_t, N_t)$ which again implies the invariance by (53).

Next, we discuss the required equivariance properties by the Levi-components. First, for $m_1 = (a_1, b_1) \in G_j \times G(W_j), m_2 = (a_2, b_2) \in G_{r-k} \times G(W_j)$ we observe that by (55) and the equivariance properties of f

$$\chi(m_1 m_2)|\det(m_2)|_v^s \delta_{P(Y)}^{\frac{1}{2}}(m_1)|\det(m_1)|_v^{-s}\delta_{P(Y)}^{-\frac{1}{2}}(m_1)\cdot$$

$$\cdot f(\phi(h)\iota(\iota_2(m_2)nm, \iota_1(m_1)g)) =$$

$$= f(\iota(\iota_2(\mathfrak{c}(a_2^t)\iota_3(b_2), \iota_1(a_1^{-1}))\phi(h)\iota(\iota_2(a_2), \iota_1(a_1)\iota_3(b_1)))\iota(nm, g)) =$$

$$= f(\iota(\iota_2(\mathfrak{c}(a_2^t), \iota_3(b_2^{-1})\iota_1(a_1^{-1})))\phi(h)\iota(\iota_2(a_2), \iota_1(a_1)\iota_3(b_1))\iota(nm, g)) =$$

$$= f(\phi(m_2^{-1} h m_1)\iota(nm, g)),$$

where we used for the second equality (53). Therefore

$$f(\phi(h)\iota(\iota_2(m_1)nm, \iota_1(m_1)g)) =$$

$$= \chi(m_1 m_2)|\det(m_2)|_v^s \delta_{P(Y)}^{\frac{1}{2}}(m_2)|\det(m_1)|_v^{-s}\delta_{P(Y)}^{-\frac{1}{2}}(m_2) f(\phi(m_1^{-1} h m_2)\iota(nm, g)).$$

Finally,

$$\int_{N_{k,j}\backslash N} f(\phi(h)\iota(n\iota_2(m_2)m, g))\,\mathrm{d}n =$$

$$= |m_2|^{n-j-k-\eta} \int_{N_{k,j}\backslash N} f(\phi(h)\iota(\iota_2(m_2)nm, g))\,\mathrm{d}n$$

by a change of variables and the explicit form of $N_{k,j}\backslash N$ we give in the proof of Lemma 5.3.3. The required invariance thus follows. Furthermore, for m' an element in $G_k \times G_{r-k} \times G_{j-r} \times G(W_j)$ of the form $m' = (m'_1, 1_{r-k}, m'_2, 1_{W_j}) \in G_k \times 1_{r-k} \times G_{j-r} \times 1_{W_j}$ and $t \in \{j-r+k,\dots,j\}$

$$w_{k,j-r,t-j-k+r} m' w^{-1}_{k,j-r,t-j-k+r} \in \iota(G_t, G_t)$$

and therefore we have by (53)

$$f(\phi(h)\iota(\iota_2(m')nm, g)) = \chi(m'_1 m'_2)|\det(m'_1 m'_2)|_v^s \delta_{P(Y)}^{\frac{1}{2}}(m'_1 m'_2) f(\phi(h)\iota(nm, g))$$

and hence a change of variables shows that

$$A_{k,j}(f)(g, m'm, h) =$$

$$= \chi(m'_1)|\det(m'_1)|^s \delta_{P(Y)}^{\frac{1}{2}}(m'_1)\chi(m'_2)|\det(m'_2)|^{s-r+k}\delta_{P(Y)}^{\frac{1}{2}}(m'_2) A_{k,j}(f)(g, m, h).$$

Finally, it is not hard to see that $A'_{k,j}$ is an $G_r \times G(W_r) \times G(W)$-intertwiner, since the morphism respects right translation.

Therefore $A'_{k,j}$ is a well-defined morphism. Note that after normalizing both the Jacquet module and induction we obtain a morphism

$$A_{k,j}\colon \tau_{k,j} \to (1 \otimes \chi)\mathrm{Ind}_{Q_{(k,r-k)} \times P'_{(j-r)} \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes$$

$$\otimes \sigma_{r-k,j}(\chi|\det|_v^{-s-\frac{r-k-j}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \otimes \underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j))).$$

Next we stratify $\Gamma_{k,j}$ by $\Gamma_{k,j,t}$, $t \in \{j-r+k, \ldots, j\}$, where

$$\Gamma_{k,j,t} \coloneqq \Omega_t \cap \Gamma_{k,j} = x_t w_{k,j-r,t+r-j-k} \iota(P)$$

and set $l \coloneqq j - t$. Recall from Section 5.2.1 that

$$(1 \otimes \chi)\mathrm{Ind}_{Q_{(k,r-k)} \times P'_{(j-r)} \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes \sigma_{r-k,j}(\chi|\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \otimes$$

$$\otimes \underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j)))$$

has a filtration by the rank of the linear maps in $\sigma_{r-k,j}$, *i.e.* by representations of the form

$$\tau'_{k,j,l} \coloneqq (1 \otimes \chi)\mathrm{Ind}_{Q_{(k,r-k)} \times P_{(j-r)} \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes \omega_l(\chi|\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \otimes$$

$$\otimes \underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j)))$$

for $l \in \{0, \ldots, r-k\}$. We will show that $A_{k,j}$ restricts to an isomorphism $r_P(\mathscr{F}^{s,\chi}(\Gamma_{k,j,t})) \xrightarrow{\cong} \tau'_{k,j,j-t}$. Note that by construction of $A_{k,j}$ and ϕ, the $A_{k,j}$ restricts a priori to a morphism $r_P(\mathscr{F}^{s,\chi}(\Gamma_{k,j,t})) \to \tau'_{k,j,j-t}$. To show that it is an isomorphism we use the Geometric Lemma. Namely, we can compute $r_P(\mathscr{F}^{s,\chi}(\Gamma_{k,j,t})))$ by applying the Geometric Lemma to

$$\sigma_t = \mathrm{Ind}_{G_t \ltimes N_t \times G_t \ltimes N_t \times \Delta G(W_t)}^{G(W) \times G(W)} (\chi|\det|_v^{s+\frac{t}{2}} \otimes \chi|\det|_v^{s+\frac{t}{2}} \otimes 1)$$

and obtain isomorphisms

$$A_{k,j,t}\colon r_P(\mathscr{F}^{s,\chi}(\Gamma_{k,j,t})) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} F(w_{k,j-r,t-j-k+r})(\mathrm{Ind}_{G_t \times G_t \times \Delta G(W_t)}^{G_t \times G(W_t) \times G(W)} (\chi|\det|_v^{s+\frac{t}{2}} \otimes \chi|\det|_v^{s+\frac{t}{2}} \otimes 1)) =$$

$$= \mathrm{Ind}_{P'}^{G_r \times G(W_r) \times G(W)} \circ \mathrm{Ind}_R^{R'} (w_{k,j-,t-j-k+r} \circ \delta \circ r_Q(|\det|_v^{s+\frac{t}{2}} \otimes |\det|_v^{s+\frac{t}{2}} \otimes 1)), \qquad (56)$$

where

$$P' = Q_{(k,l,r-k-l)} \times P'_{(j-r)} \times P_{(j-l,l)},\ R \coloneqq \Delta G_l \times \Delta G(W_j),$$

$$R' := G_l \times G_l \times G(W_j) \times G(W_j),\ Q := Q_{(k,j-r,t-j-k+r)} \times G_{j-l} \times \Delta G_l \times \Delta G(W_j).$$

Plugging in the definitions yields that above representation is

$$\pi_{k,j,t} := \mathrm{Ind}_{Q'_{(k,r-k-l,l)} \times P'_{(j-r)} \times P_{(t,l)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes \underbrace{\chi|\det|_v^{-s-j+\frac{l+r-k}{2}}}_{G_{r-k-l}} \otimes \underbrace{\chi|\det|_v^{s+\frac{t}{2}}}_{G_t} \otimes$$

$$\otimes S(G_l)(|\det|_v^{-s-j+\frac{l-k}{2}} \otimes |\det|_v^{s+j-\frac{l-k}{2}}) \otimes \underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}} \otimes S(G(W_j))).$$

Recall from our discussion of the Geometric Lemma we saw that $A_{k,j,t}$ is precisely

$$A_{k,j,t}(f) = \underbrace{g}_{G(W)} \times \underbrace{m}_{G_r \times G(W_r)} \mapsto \int_{N_{k,j,t} \backslash N} p(f(\delta_t \iota(w_{k,j-r,r-k-l}, 1_W)\iota(nm, g)))\,\mathrm{d}n$$

where

$$N_{k,j,t} = N_{k,j}$$

by Lemma 5.3.3 and p is the projection to

$$r_{H'}(\chi|\det|_v^{s+\frac{t}{2}} \otimes \mathrm{Ind}_{P_{(t)} \times \Delta G(W_t)}^{G(W_t) \times G(W)} (\chi|\det|_v^{s+\frac{t}{2}} \otimes 1))$$

with

$$H' = (N, 1_W) \cap (w_{k,j-r,t+r-j-k}^{-1}, 1_W) G_t \times G_t \times \Delta G(W_t)(w_{k,j-r,t+r-j-k}, 1_W)).$$

In this case, this means we just forget the H'-action. Thus

$$A_{k,j,t}(f)(g,m) =$$

$$= \int_{N_{k,j} \backslash N} f(\delta_t \iota(w_{k,j-r,r-k-l}, 1_W)\iota(nm, g))\,\mathrm{d}n = A_{k,j}(f)(g,m)(1'_l).$$

Comparing the two group-actions on each side of the equation, we obtain a commutative diagram of the following form

$$\begin{tikzcd} r_P(\mathscr{F}^{s,\chi}(\Gamma_{k,j,t})) \arrow[r, "A_{k,j,t}"] \arrow[d, "A_{k,j}"] & \pi_{k,j,t} \\ \tau'_{k,j,l} \arrow[ur, "B_l"] & \end{tikzcd}$$

where $B_l\colon \tau'_{k,j,l} \xrightarrow{\cong} \pi_{k,j,t}$ is the following parabolically induced isomorphism. Namely,

$$B_l := \mathrm{Ind}_{Q(k,r-k) \times G(W_r) \times P_{(j)}}^{G_r \times G(W_r) \times G(W)} (\underbrace{B}_{G_{r-k} \times G_j} \otimes \underbrace{1}_{G_k \times G(W_r) \times G(W_j)}),$$

and B is the composition

$$\omega_l(\chi|\det|_v^{-s-j+\frac{r-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \mathrm{Ind}_{Q_{r-k-l,l} \times Q_{j-l,l}}^{G_{r-k} \times G_j} (|\det|_v^{\frac{l}{2}} \otimes |\det|_v^{-\frac{l}{2}} \otimes$$

$$\otimes(|\det|_v^{\frac{l-r+k}{2}}\otimes|\det|_v^{\frac{j-l}{2}})S(G_l))(\chi|\det|_v^{-s-j+\frac{r-k}{2}}\otimes\chi|\det|_v^{s+\frac{j}{2}})\xrightarrow{\cong}$$
$$\xrightarrow{\cong}\mathrm{Ind}_{Q_{r-k-l,l}\times Q_{j-l,l}}^{G_{r-k}\times G_j}(|\det|_v^{-s-j+\frac{l+r-k}{2}}\otimes|\det|_v^{s+\frac{j-l}{2}}\otimes S(G_l)).$$

The last isomorphism in this composition is obtained by sending a function f' on $S(G_l)$ to $\chi|\det|_v^{-s-j+\frac{l}{2}}\cdot f'$ and then parabolically inducing this morphism to $G_{r-k}\times G_j$. Therefore $A_{k,j}(f)$ induces an isomorphism from to $r_P(\mathscr{F}_c^{s,\chi}(\Gamma_{k,j,t}))\to\tau'_{k,j,l}$ and the 5-Lemma shows then that $A_{k,j}$ is an isomorphism. □

As a corollary of the proof, we obtain the following. Let $\Omega^t=\bigcup_{i=0}^t\Omega_t$, $\Gamma_{k,j,t}:=\Omega_t\cap\Gamma_{k,j}$ and $\Gamma_{k,j}^t:=\Omega^t\cap\Gamma_{k,j}$. Moreover, set

$$\tau_{k,j,t}:=r_P(\mathscr{F}(\Gamma_{k,j,t})),\ \tau_{k,j}^t:=r_P(\mathscr{F}(\Gamma_{k,j}^t)).$$

Furthermore, $\tau_{k,j}$ has a filtration by

$$S_{k,j}^l:=(1\otimes\chi)\mathrm{Ind}_{Q_{(k,r-k)}\times P'_{(j-r)}\times P_{(j)}}^{G_r\times G(W_r)\times G(W)}(\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k}\otimes$$

$$\otimes S_l(\chi|\det|_v^{-s-j+\frac{r-k}{2}}\otimes\chi|\det|_v^{s+\frac{j}{2}})\otimes\underbrace{\chi|\det|_v^{s+k+\frac{j-r}{2}}}_{G_{j-r}}\otimes S(G(W_j)))$$

coming from the filtration of $\sigma_{r-k,j}$ by S_l, $l\in\{0,\dots,r-k\}$.

Corollary 5.3.1. *Then $A_{k,j}$ restricts to an isomorphism*

$$A_{k,j}\colon\tau_{k,j}^t\xrightarrow{\cong}S_{k,j}^{j-t}.$$

5.4 Behavior on a boundary component

Let ξ be a character of the form

$$\xi:=\chi|\det|_v^{s-\frac{1}{2}},\ s\in\mathbb{C},\ \chi^2=1$$

and set for $k\in\mathbb{N}$, $\rho_k:=\mathrm{Z}([1,k]_\xi)=\chi|\det|^{s+\frac{k}{2}}$. More generally, for $m\in\{0,\dots,q_W\}$, $\alpha=(\alpha_1,\dots,\alpha_k)$ a composition of m, define

$$\rho_\alpha:=\chi(\underbrace{|\det|_v^{s+\frac{\alpha_1}{2}}}_{G_{\alpha_1}}\times\dots\times\underbrace{|\det|_v^{s+\frac{\alpha_k}{2}}}_{G_{\alpha_k}})=\mathrm{Z}([1,\alpha_1]_\xi)\times\dots\times\mathrm{Z}([1,\alpha_k]_\xi),$$

which is irreducible by Theorem 2.3.9. Define

$$\sigma'_t:=\mathrm{Ind}_{P_{(t)}\times P_{(t)}}^{G(W)\times G(W)}(\rho_t\otimes\rho_t\otimes S(G(W_t)))$$

and write $\xi_a:=\xi|\det|_v^a$ for $a\in\mathbb{C}$.

In this subsection, we prove the following proposition.

Proposition 5.4.1. *Let $\pi \in \mathfrak{Irr}_{G(W)}$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \leq 1.$$

Proof. We prove the claim by induction on $\dim_E W$, the case $t = 0$ being Lemma 5.2.6. Thus assume that $t \geq 1$ and let $d := -2s$ be such that $\xi_d^\vee \cong \xi_1$. We also can assume without loss of generality that $\alpha_1 \geq \ldots \geq \alpha_k$ and set $t := \alpha_1$. Depending on d we will differentiate several cases.

Case 1: $d \in \{1, \ldots, t-1\}$

Note that in this case $\xi_t \not\cong \xi_t^\vee$. Indeed, if $\xi_t^\vee \cong \xi_t$ we would obtain that $2t = d + 1$ and therefore $2t < t + 1$, which gives a contradiction. Observe moreover, that if there exists a non-zero morphism

$$\sigma_t \to \pi \otimes \pi^\vee,$$

there exist by Lemma 5.2.8 representations $\tau \in \mathfrak{Irr}_{G(W_t)}$ such that $\rho_t \rtimes \tau \twoheadrightarrow \pi$. Since by Lemma 2.3.10 $\xi_t \times \rho_{t-1} \twoheadrightarrow \rho_t$ we can find by Lemma 2.3.10 $1 \leq r \in \mathbb{Z}_{>0}$ and $\delta \in \mathfrak{Irr}_{G(W_r)}$ such that

$$(\xi_t)^r \rtimes \delta \twoheadrightarrow \pi$$

and δ is not a quotient of a representation of the form $\xi_t \rtimes \delta'$ for $\delta' \in \mathfrak{Irr}_{G(W_{r+1})}$. Using the MVW-involution and the fact that $\xi_t \not\cong \xi_t^\vee$ we obtain that

$$\pi \cong \delta_{\xi_t,r} \hookrightarrow (\xi_t^\vee)^r \rtimes \delta,\ \pi^\vee \cong \delta_{\xi_t,r}^\vee \hookrightarrow (\xi_t^\vee)^r \rtimes \delta^\vee$$

by Lemma 5.2.5, where we also introduced this notation. We now have by Frobenius reciprocity that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \leq$$

$$\leq \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', (\xi_t^\vee)^r \rtimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee) =$$

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_r\times G(W_r)\times G(W)}(r_{P_{(r)}\times G(W)}(\sigma_t'), (\xi_t^\vee)^r \otimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee)$$

Next we apply the Geometric Lemma to $r_{P_{(r)}\times G(W)}(\sigma_t')$ and show that the only subquotient

$$F(v_{k_1,k_2,k_3})(\rho_t \otimes \operatorname{Ind}_{G(W_r)\times P_{(t)}}^{G(W_r)\times G(W)}(\rho_t \otimes S(G(W_t))))$$

admitting a morphism to $(\xi_t^\vee)^r \otimes \delta \otimes \pi^\vee$ corresponds to $k_1 = 0$, $k_2 = r-1$, $k_3 = 1$. Here we use the representative v_{k_1,k_2,k_3} chosen in (4). Indeed, if there would exist a non-zero morphism

$$F(v_{k_1,k_2,k_3})(\rho \otimes \operatorname{Ind}_{G(W_m)\times P_{(m)}}^{G(W_m)\times G(W)}(\rho \otimes S(G(W_m)))) \to (\xi_t^\vee)^r \otimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee),$$

we would obtain a morphism $\mathrm{Z}([1,k_1]_\xi) \times \rho' \times \mathrm{Z}([k_1+k_2+1,t]_\xi)^\vee \to (\xi_t^r)^\vee$ and a morphism $\mathrm{Z}([k_1+1,k_1+k_2]_\xi) \rtimes \rho'' \to \delta$ for suitable representations $\rho' \in \mathfrak{Irr}_{r-k_1-k_3}$ and $\rho'' \in \mathfrak{Irr}_{G(W_{k_2})}$. From the first morphism we obtain that $k_1 + k_2$ is either $t-1$ or t. From the second morphism we obtain that $k_1 + k_2$ cannot be t, since otherwise, we would have a surjective morphism

$$\xi_t \times \mathrm{Z}([k_1+1,t-1]_\xi) \rtimes \rho'' \twoheadrightarrow \mathrm{Z}([k_1+1,k_1+k_2]_\xi) \rtimes \rho'' \twoheadrightarrow \delta$$

and hence we obtain a contradiction to the assumption on δ by the MVW-involution. Moreover, $k_1 \leq 1$ with equality if and only if $d = t$, which we excluded. We thus showed that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', (\xi_t^r)^\vee \rtimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee) \leq$$

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_r\times G(W_r)\times G(W)}($$

$$\operatorname{Ind}_{Q_{(1,r-1)}\times P_{(t-1)}\times P_{(t,r-1)}}^{G_r\times G(W_r)\times G(W)}(S(G_{r-1}) \otimes \xi_t^\vee \otimes \rho_{t-1} \otimes \rho_t \otimes S(G(W_{t+r-1})),$$

$$(\xi_t^\vee)^r \otimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee).$$

Applying (1) with respect to $Q_{(1,r-1)}$ yields that the dimension of the last space is equal to the dimension of

$$\operatorname{Hom}_{G_1 l\times G_{r-1}\times G(W_r)\times G(W)}($$

$$\xi_t^\vee \otimes \operatorname{Ind}_{G_{r-1}\times P_{(t-1)}\times P_{(t,r-1)}}^{G_{r-1}\times G_1\times G(W_r)\times G(W)}(S(G_{r-1}) \otimes \rho_{t-1} \otimes \rho \otimes S(G(W_{t+r-1})),$$

$$r_{\overline{Q_{(1,r-1)}}}((\xi_t^\vee)^r) \otimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee).$$

We thus get a morphism $\xi_t^\vee \otimes S(G_{r-1}) \to r_{\overline{Q_{(1,r-1)}}}((\xi_t^\vee)^r)$, which by Lemma 5.2.2 is up to a scalar unique and factors through the inclusion

$$\xi_t^\vee \otimes S(G_{r-1}) \twoheadrightarrow \xi_t^\vee \otimes (\xi_t^\vee)^{r-1} \hookrightarrow r_{\overline{Q_{(1,r-1)}}}((\xi_t^\vee)^r).$$

Applying this to our Hom-space we obtain that the dimension is bounded by

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_{r-1}\times G(W_r)\times G(W)}($$

$$\operatorname{Ind}_{G_{r-1}\times P_{(t-1)}\times P_{(t,r-1)}}^{G_{r-1}\times G(W_r)\times G(W)}(S(G_{r-1}) \otimes \rho_{t-1} \otimes \rho \otimes S(G(W_{t+r-1}))),$$

$$(\xi_t^\vee)^{r-1} \otimes \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee).$$

Applying first (1) with respect to the parabolic subgroup $P(X_{t,t+r-1})$ contained in the second copy of $G(W)$, then Lemma 5.2.6 and then again (1) with respect to $P(X_{t,t+r-1})$, it follows that the dimension is equal to

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W_r)\times G(W)}(\operatorname{Ind}_{P_{(t-1)}\times P_{(t,r-1)}}^{G(W_r)\times G(W)}(\rho_{t-1} \otimes \rho_t \otimes \xi_t^{r-1} \otimes S(G(W_{t+r-1})),$$

$$\delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee).$$

Since $\xi_t^{r-1} \times \rho_t \cong \rho_t \times \xi_t^{r-1}$ by Theorem 2.3.9, the dimension of the last space is equal to

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W_r)\times G(W)}(\operatorname{Ind}_{P_{(t-1)}\times P_{(r-1,t)}}^{G(W_r)\times G(W)}(\rho_{t-1} \otimes \xi_t^{r-1} \otimes \rho_t \otimes S(G(W_{t+r-1}))$$

$$, \delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee) \leq$$

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W_r)\times G(W)}(\operatorname{Ind}_{P_{(t-1)}\times P_{(r,t-1)}}^{G(W_r)\times G(W)}(\rho_{t-1} \otimes \xi_t^r \otimes \rho_{t-1} \otimes S(G(W_{t+r-1})),$$

$$\delta \otimes (\xi_t^\vee)^r \rtimes \delta^\vee),$$

where we used for the second inequality again that $\xi_t \times \rho_{t-1} \twoheadrightarrow \rho_t$. Applying (1), we see that this is equal to

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_r\times G(W_r)\times G(W_r)}((\xi_t^\vee)^r \otimes \operatorname{Ind}_{P_{(t-1)}\times P_{(t-1)}}^{G(W_r)\times G(W_r)}(\rho_{t-1} \otimes \rho_{t-1} \otimes S(G(W_{t+r-1})),$$

$$\delta \otimes r_{\overline{P_{(r)}}}((\xi_t^\vee)^r \rtimes \delta^\vee)) =$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_r \times G(W_r) \times G(W_r)}((\xi_t^\vee)^r \otimes \sigma_{t-1}, \delta \otimes r_{\overline{P_{(r)}}}((\xi_t^\vee)^r \rtimes \delta^\vee)).$$

Since $\xi_t \not\cong \xi_t^\vee$ we can apply Lemma 5.2.5 to see that the dimension is equal to

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_r \times G(W_r) \times G(W_r)}((\xi_t^\vee)^r \otimes \sigma_{t-1}, (\xi_t^\vee)^r \otimes \delta \otimes \delta^\vee).$$

The induction hypothesis on $\dim_E W$ shows then that this space is 1-dimensional.

Case 2: $d \notin \{1, \ldots, t\}$ or $t = d$:

Note that if there exists a morphism

$$\sigma_t \to \pi \otimes \pi^\vee$$

then there exists by Lemma 5.2.8 $\tau' \in \mathfrak{Irr}_{G(W_t)}$ such that $\rho_t \times \tau' \twoheadrightarrow \pi$. Therefore there exists $\rho \in \mathfrak{Irr}_m$, $\rho := \rho_\alpha$ for $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\max_i \alpha_i = t$ and $\tau \in \mathfrak{Irr}_{G(W_m)}$ with $\rho \rtimes \tau \twoheadrightarrow \pi$ such that there exists no $b \in \{1, \ldots, t\}$ and $\tau' \in \mathfrak{Irr}_{G(W_{m+b})}$ with $\rho_b \rtimes \tau' \twoheadrightarrow \tau$. We write $\rho = \rho_t \times \rho'$. Using the MVW-involution, we obtain that $\pi \hookrightarrow \rho^\vee \rtimes \tau$.

We have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W) \times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \pi^\vee) =$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_t \times G(W_t) \times G(W)}(\rho_t \otimes \mathrm{Ind}_{G(W_t) \times P_{(t)}}^{G(W_t) \times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$r_{\overline{P_{(t)}}}(\rho^\vee \rtimes \tau) \otimes \pi^\vee) =$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_t \times G(W_t) \times G(W)}(\rho_t^\vee \otimes \mathrm{Ind}_{G(W_t) \times P_{(t)}}^{G(W_t) \times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$r_{P_{(t)}}(\rho^\vee \rtimes \tau) \otimes \pi^\vee).$$

We now apply the Geometric Lemma to $r_{P_{(t)}}(\rho \rtimes \tau)$ and see for which k_1, k_2, k_3 a morphism from the left side exists to the respective subquotient. We claim now that if $d \notin \{1, \ldots, t\}$ it is $k_1 = t$, $k_2 = m - t$, $k_3 = 0$ and if $t = d$ it is either $k_1 = t$, $k_2 = m - t$, $k_3 = 0$ or $k_1 = 0$, $k_2 = m - t$, $k_3 = t$. Assume

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_t \times G(W_t) \times G(W)}(\rho_t \otimes \mathrm{Ind}_{G(W_t) \times P_{(t)}}^{G(W_t) \times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$F(v_{k_1,k_2,k_3})(\rho^\vee \otimes \tau) \otimes \pi^\vee) \neq 0.$$

Plugging in the definition of

$$F(v_{k_1,k_2,k_3})(\rho^\vee \otimes \tau) = \mathrm{Ind}_{Q_{(k_1, t-k_1-k_2, k_3)} \times P_{(k_2)}}^{G_t \times G(W_t)} \circ v_{k_1,k_2,k_3} \circ$$

$$\circ r_{Q_{(k_1,k_2,k_3)} \times P_{(m-k_1-k_3)}}(\rho^\vee \otimes \tau)$$

and applying Frobenius reciprocity together with Lemma 2.3.8 we obtain $\rho_{k_3} \cong \rho_{k_3}^\vee$ and hence $k_3 = d$. Thus $k_3 = 0$ if $t \neq d$ and if $k_3 \neq 0$, $k_3 = d$ and hence $k_1 = 0$ and $k_2 = m - t$ in this case. Moreover, if $k_3 = 0$ then k_2 cannot be different from $m - t$ as it otherwise implies the existence of an irreducible representation $\tau' \in \mathfrak{Irr}_{G(W_{m+t-k_1})}$ and a non-zero morphism $\mathrm{Z}([k_1 - t, -1]_{\xi^\vee}) \otimes \tau' \hookrightarrow r_{P_{(t-k_1)}}(\tau)$ and hence a morphism $\mathrm{Z}([1, t - k_1]_\xi) \otimes \tau' \hookrightarrow r_{\overline{P_{(t-k_1)}}}(\tau)$.

By (1) this contradicts the assumption on τ. Thus, we have shown the claim that if $d \neq t$ $k_1 = t, k_2 = m-t, k_3 = 0$ and if $d = t$ $k_1 = t, k_2 = m-t, k_3 = 0$ or $k_1 = 0, k_2 = m-t, k_3 = t$.

Case 2.1. $d \notin \{1, \ldots, t\}$**:**

We just showed

$$\begin{gathered}\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \pi^\vee) \leq \\ \dim_{\mathbb{C}} \operatorname{Hom}_{G_t\times G(W_t)\times G(W)}(\rho_t^\vee \otimes \operatorname{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))), \\ r_{(t,m-t)}(\rho^\vee) \rtimes \tau \otimes \pi^\vee).\end{gathered} \tag{57}$$

We note that the $\rho^\vee$ is irreducible and that $\operatorname{Jac}_{\rho_t^\vee}(\rho^\vee)$, *cf.* end of Section 2.3.3, is irreducible. Indeed, since $\rho^\vee$ is irreducibly induced, we can write it as $\rho^\vee \cong (\rho_t^\vee)^k \times \rho_\beta^\vee$, where $\max_i \beta_i < t$. Then the Geometric Lemma gives that $\operatorname{Jac}_{\rho_t^\vee}(\rho^\vee) \hookrightarrow \operatorname{Jac}_{\rho_t^\vee}((\rho_t^\vee)^k) \times \rho_\beta^\vee$, which by Lemma 5.2.2 and Theorem 2.3.9 is irreducible and equal to $\rho' = (\rho_t^\vee)^{k-1} \times \rho_\beta^\vee$. Thus every morphism in (57) factors through a morphism in

$$\operatorname{Hom}_{G_t\times G(W_t)\times G(W)}(\rho_t^\vee \otimes \operatorname{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))), \rho_t^\vee \otimes \rho'^\vee \rtimes \tau \otimes \pi^\vee),$$

whose dimension is by (1) and Lemma 5.2.6 equal to

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W_t)\times G(W)}(\operatorname{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))), \rho'^\vee \rtimes \tau \otimes \pi^\vee) =$$

$$= \dim_{\mathbb{C}} \operatorname{Hom}_{G(W_t)\times G_t\times G(W_t)}(\rho_t \otimes S(G(W_t)), \rho'^\vee \rtimes \tau \otimes r_{\overline{P_{(t)}}}(\pi^\vee)) =$$

$$= \dim_{\mathbb{C}} \operatorname{Hom}_{G_t\times G(W_t)}(\rho_t \otimes \rho' \rtimes \tau^\vee, r_{\overline{P_{(t)}}}(\pi^\vee)) =$$

$$= \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\rho_t \times \rho' \rtimes \tau^\vee, \pi^\vee) = \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau).$$

But on the other hand, for each morphism $\sigma_t' \to \pi \otimes \pi^\vee$ and each embedding $\pi \hookrightarrow \rho^\vee \rtimes \tau$ we have a morphism

$$\sigma_t' \twoheadrightarrow \pi \otimes \pi^\vee \hookrightarrow \rho^\vee \rtimes \tau \otimes \pi^\vee,$$

and hence

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau) \geq \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \pi^\vee) \geq$$

$$\geq \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \cdot \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau).$$

In particular, we have $\operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \leq 1$.

Case 2.2: $t = d$**:**

Note that this is equivalent to $\rho_t \cong \rho_t^\vee$.

Since $t = d$

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_t\times G(W_t)\times G(W)}(\rho_t^\vee \otimes \operatorname{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$F(v_{0,m-d,d})(\rho^\vee \otimes \tau) \otimes \pi^\vee) =$$

$$= \dim_{\mathbb{C}} \operatorname{Hom}_{G_t\times G(W_t)\times G(W)}(\rho_t^\vee \otimes \operatorname{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$F(v_{d,m-d,0})(\rho^\vee \otimes \tau) \otimes \pi^\vee) =$$

$$= \dim_{\mathbb{C}} \mathrm{Hom}_{G_t \times G(W_t) \times G(W)}(\rho_t^\vee \otimes \mathrm{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$r_{(d,m-d)}(\rho^\vee) \rtimes \tau \otimes \pi^\vee).$$

As in Case 2.1 we see that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_t \times G(W_t) \times G(W)}(\rho_t^\vee \otimes \mathrm{Ind}_{G(W_t)\times P_{(t)}}^{G(W_t)\times G(W)}(\rho_t \otimes S(G(W_t))),$$

$$r_{(d,m-d)}(\rho^\vee) \rtimes \tau \otimes \pi^\vee) =$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \pi^\vee) = \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau).$$

Using the same arguments as in the beginning of Case 2 and Case 2.1 we obtain

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \pi^\vee) \leq 2 \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau). \tag{58}$$

In a completely analogous fashion, we can show that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \rho^\vee \rtimes \tau^\vee) \leq 2 \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \rho^\vee \rtimes \tau^\vee).$$

On the other hand, for each morphism $\pi \otimes \pi^\vee \hookrightarrow \rho^\vee \rtimes \tau \otimes \rho^\vee \rtimes \tau^\vee$ and morphism $\sigma_t' \twoheadrightarrow \pi \otimes \pi'$ we obtain a map in $\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(\sigma_t', \rho^\vee \rtimes \tau \otimes \rho^\vee \rtimes \tau^\vee)$ and hence

$$2 \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \rho^\vee \rtimes \tau^\vee) \geq$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau)^2.$$

Next we prove the following lemma.

Lemma 5.4.2.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \rho^\vee \rtimes \tau^\vee) \leq 2.$$

Proof. We write $\rho = \rho_t^k \times \tilde{\rho}$, where $\tilde{\rho} = \rho_\beta$ and $\max_i \beta_i < t = d$. Applying Frobenius reciprocity, and the Geometric Lemma, to

$$\mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \rho^\vee \rtimes \tau^\vee)$$

we obtain as above that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho \rtimes \tau^\vee, \rho^\vee \rtimes \tau^\vee) \leq 2 \dim_{\mathbb{C}} \mathrm{Hom}_{G_{kt} \times G(W_{kt})}(\rho_t^k \otimes \tilde{\rho} \rtimes \tau^\vee, \rho_t^k \otimes \tilde{\rho}^\vee \rtimes \tau^\vee).$$

Thus it is enough to show that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W_{kt})}(\tilde{\rho} \rtimes \tau^\vee, \tilde{\rho}^\vee \rtimes \tau^\vee) \leq 1.$$

But this follows by applying Frobenius reciprocity and using the Geometric Lemma completely analogously as in the beginning of Case 2 and 2.1. □

We thus proved that

$$4 \geq \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \cdot \dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau)^2$$

and hence either $\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \leq 1$ or

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\pi, \rho^\vee \rtimes \tau) = 1. \tag{59}$$

We assume from now on the second case and thus by (58) we obtain that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) \leq 2. \tag{60}$$

Fix an irreducible subrepresentation σ_1 of $\operatorname{Jac}_{\rho_t}(\pi)$. Firstly, by (1) and the MVW-involution, $\pi \hookrightarrow \rho_t \rtimes \sigma_1^\vee$. Applying Frobenius reciprocity to the map $\rho \rtimes \tau \twoheadrightarrow \pi \hookrightarrow \rho_t \rtimes \sigma_1$ and using the Geometric Lemma, gives with the usual argument $\rho' \rtimes \tau \twoheadrightarrow \sigma_1$. We thus obtain a map

$$\rho \rtimes \tau \twoheadrightarrow \rho_t \rtimes \sigma_1 \twoheadrightarrow \pi.$$

If $\operatorname{Jac}_{\rho_t}(\pi)$ admits a second subrepresentation σ_1', *i.e.* has a socle of length at least 2, the exact same argument would give a map

$$\rho \rtimes \tau \twoheadrightarrow \rho_t \rtimes \sigma_1' \twoheadrightarrow \pi$$

and hence $\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)}(\rho \rtimes \tau, \pi) > 1$, which contradicts our assumption of (59) by the MVW-involution.

Now we distinguish two cases, namely $\pi \cong \rho_t \rtimes \sigma_1$ or π is a proper quotient of $\rho_t \rtimes \sigma_1$. The latter is easy to deal with. By (58) it suffices to construct a non-zero map $\sigma_t' \to \rho^\vee \rtimes \tau \otimes \pi^\vee$ which has image not isomorphic to $\pi \otimes \pi^\vee$. This we can do as follows.

$$\sigma_t' \twoheadrightarrow \rho_t \rtimes \sigma_1 \otimes \rho_t \rtimes \sigma_1^\vee \twoheadrightarrow \rho_t \rtimes \sigma_1 \otimes \pi^\vee \hookrightarrow \rho^\vee \rtimes \tau \otimes \pi^\vee.$$

Thus we assume from now one that $\pi \cong \rho_t \rtimes \sigma_1$. We first assume that $\operatorname{Jac}_{\rho_t}(\pi)$ is not irreducible. It thus contains a non-semi-simple subrepresentation σ of length 2. Applying (1) we obtain a map $\rho_t \rtimes \sigma \twoheadrightarrow \pi$ such that the composition $\pi \cong \rho_t \rtimes \sigma_1 \hookrightarrow \rho_t \rtimes \sigma \twoheadrightarrow \pi$ is non-zero and hence a scalar since $\pi \cong \rho_t \rtimes \sigma_1$. Therefore

$$\rho_t \rtimes \sigma \cong \pi \oplus \rho_t \rtimes \sigma_2, \tag{61}$$

where σ_2 is the unique quotient of σ. Next we denote the image of the map

$$S(G(W_t)) \to (\sigma^{\mathrm{MVW}})^\vee \otimes \sigma^{\mathrm{MVW}}$$

corresponding to the identity map $\sigma^{\mathrm{MVW}} \to \sigma^{\mathrm{MVW}}$, *cf.* Lemma 5.2.6, by I.

Lemma 5.4.3. *The representation I is of length* 3 *and isomorphic to the kernel of the map* $(\sigma^{\mathrm{MVW}})^\vee \otimes \sigma^{\mathrm{MVW}} \twoheadrightarrow \sigma_1 \otimes \sigma_2^\vee$.

Proof. It is straightforward to see that I has to be contained in the kernel. Moreover I admits $\sigma_1 \otimes \sigma_1^\vee$ and $\sigma_2 \otimes \sigma_2^\vee$ as a quotient. Indeed, the composition

$$S(G(W_t)) \to (\sigma^{\mathrm{MVW}})^\vee \otimes \sigma^{\mathrm{MVW}} \twoheadrightarrow (\sigma^{\mathrm{MVW}})^\vee \otimes \sigma_2^\vee$$

has image $\sigma_2 \otimes \sigma_2^\vee$. We can argue similarly for $\sigma_1 \otimes \sigma_1^\vee$. Thus if I is not the kernel we obtain that it is isomorphic to $\sigma_1 \otimes \sigma_1^\vee \oplus \sigma_2 \otimes \sigma_2^\vee$, which cannot be a subrepresentation of $(\sigma^{\mathrm{MVW}})^\vee \otimes \sigma^{\mathrm{MVW}}$, since it contradicts the assumption that σ and hence σ^{MVW} are not semi-simple. $\square$

We thus have a surjective map

$$\sigma_t' \twoheadrightarrow \mathrm{Ind}_{P_{(t)}\times P_{(t)}}^{G(W)\times G(W)}(\rho_t \otimes \rho_t \otimes I) \cong \pi \otimes \pi^\vee \oplus \rho_t \rtimes \sigma_2 \otimes \rho_t \rtimes \sigma_2^\vee \oplus \rho_t \rtimes \sigma_2 \otimes \pi^\vee.$$

The last isomorphism stems from (61). In particular we have that $\pi \ncong \rho_t \rtimes \sigma_2$ by (60) and we constructed surjective maps

$$\sigma_t' \twoheadrightarrow \rho_t \rtimes \sigma_2 \otimes \pi^\vee,\ \sigma_t' \twoheadrightarrow \rho_t \rtimes (\sigma^{\mathrm{MVW}})^\vee \otimes \pi^\vee. \tag{62}$$

Note that by (61), $\rho_t \rtimes (\sigma^{\mathrm{MVW}})^\vee \cong \rho_t \rtimes \sigma$. Now applying the Geometric Lemma to the inclusion $\rho_t \otimes \sigma \hookrightarrow r_{\overline{P_{(t)}}}(\rho^\vee \rtimes \tau)$ implies that at least one of σ or σ_2 is a subrepresentation of $\rho'^\vee \rtimes \tau$. We pick one which is and denote it by σ'. We can now construct a non-zero map $\sigma_t' \to \rho^\vee \rtimes \tau \otimes \pi^\vee$ which has an image not isomorphic to $\pi \otimes \pi^\vee$. Namely, by (62) we have

$$\sigma_t' \twoheadrightarrow \rho_t \rtimes \sigma' \otimes \pi^\vee \hookrightarrow \rho^\vee \rtimes \tau \otimes \pi^\vee.$$

and thus we are done by (58).

Finally, assume $\mathrm{Jac}_{\rho_t}(\pi)$ is irreducible and hence isomorphic to σ_1. By (1) and Lemma 5.2.6 we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(\sigma_t', \pi \otimes \pi^\vee) = \dim_{\mathbb{C}} \mathrm{Hom}_{G(W)}(\rho_t \rtimes \mathrm{Jac}_{\rho_t}(\pi)^\vee, \pi^\vee) = 1.$$

□

5.5 Proof of Theorem 12

Let $\pi \in \mathrm{Irr}(G(W))$ be an irreducible representation, χ a unitary character and $s \in \mathbb{C}$. In this section, we study the space

$$\mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \otimes \chi\pi^\vee).$$

Proposition 5.5.1 ([73, §1])**.** *The above space is non-empty,* i.e.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \otimes \chi\pi^\vee) \geq 1.$$

We quickly recall the construction of a functional in this space. For $v \in \pi, v^\vee \in \pi^\vee$ write the matrix coefficient $\phi(g) \coloneqq v^\vee(\pi(g)v)$. For $\Psi \in I_{W,W}(\chi, s)$ we then define

$$\mathrm{Z}(s, \chi, \phi, \Psi) \coloneqq \int_{G(W)} \phi(g)\Psi(x_0\iota(g, 1))\,\mathrm{d}g.$$

The integral $\mathrm{Z}(s, \chi, \phi, \Psi)$ converges for $\mathfrak{Re}\, s >> 0$ and admits a meromorphic continuation to the whole complex plane. Moreover, it can be written as a rational function in q^{-s} and the leading term of the Laurent polynomial of $\mathrm{Z}(s, \chi, \phi, \Psi)$ at $s = s_0$ defines then an element in

$$\mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s_0), \pi \otimes \pi^\vee).$$

In the same paper the authors deal with the case $\pi = \pi^\vee = 1$ and π not appearing on the boundary if W is symplectic.

Theorem 5.5.2 ([73, Theorem 1.1, Lemma 1.4])**.** *Let W be a symplectic vector space and $\pi \in \mathfrak{Irr}_{G(W)}$ either trivial or not appearing on the boundary. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi,s), \pi \otimes \pi^\vee) = 1.$$

We will now generalize Theorem 5.5.2 to arbitrary representations.

Theorem 5.5.3. *Let W be now either a symplectic, orthogonal or unitary vector space over E and $G(W) \subseteq \mathrm{GL}(W)$ the corresponding symmetry group. Let π an irreducible representation of $G(W)$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi,s), \pi \otimes \chi\pi^\vee) = 1.$$

Proof. In the light of Proposition 5.4.1 it would be enough to show that there exists a unique $t \in \{0,\dots,q_W\}$, depending on π, such that every non-zero morphism in

$$\operatorname{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi,s), \pi \otimes \chi\pi^\vee)$$

does vanish on I_{t-1} and does not vanish on I_t. Indeed, assuming this and given two non-zero morphisms

$$f_1, f_2 \colon I_{W,W}(\chi,s) \to \pi \otimes \chi\pi^\vee,$$

both would induce a non-zero morphism on

$$f_1', f_2' \colon I_{t-1}\backslash I_t \cong \sigma_t \to \pi \otimes \chi\pi^\vee$$

and hence there would exists by Proposition 5.4.1 $\lambda \in \mathbb{C}$ such that $f_1' = \lambda f_2'$. The morphism $f_1 - \lambda f_2$ is then again an element of $\operatorname{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi,s), \pi \otimes \chi\pi^\vee)$, which vanishes on I_t and hence must be identically 0, proving that the Hom-space is 1-dimensional.

To show that every non-zero morphism vanishes on I_{t-1} and does not vanish on I_t for some t depending on π, we have to fix some notation. We set for $a \in \mathbb{C}$, $k \in \mathbb{N}$

$$\xi_a \coloneqq \chi|\det|_v^{s+a-\frac{1}{2}},\ \rho_k \coloneqq \mathrm{Z}([1,k]_{\xi_0}) = \chi|\det|^{s+\frac{k}{2}}.$$

Moreover, let $d \coloneqq -2s \in \mathbb{C}$ be such that $\xi_d \cong \xi_1^\vee$ as in the proof of Proposition 5.4.1. If π does not lie on the boundary of $I_{W,W}(\chi,s)$, the claim follows immediately. On the other hand, if it does, there exists $j \in \{1,\dots,q_W\}$ and some morphism $I_{W,W}(\chi,s) \to \pi \otimes \chi\pi^\vee$ which vanishes on I_{j-1} and not on I_j. It thus induces a morphism $\sigma_j \to \pi \otimes \chi\pi^\vee$ and hence there exists by Lemma 5.2.8 $\sigma \in \mathfrak{Irr}_{G(W_j)}$ such that $\rho_j \rtimes \sigma \twoheadrightarrow \pi$. In this case, let $\rho \in \mathfrak{Irr}_m$, $\tau \in \mathfrak{Irr}_{G(W_m)}$ be irreducible representations satisfying the following conditions.

1. There exists $1 \le k \in \mathbb{N}$, $t_1,\dots,t_k \in \{1,\dots,q_W\}$ such that

$$\rho \cong \rho_{t_1} \times \ldots \times \rho_{t_k}$$

   and we set $t \coloneqq \max_j t_j$.

2. We can realize π as a quotient $\rho \rtimes \tau \twoheadrightarrow \pi$.

3. There does not exist $t' \in \{1,\dots,q_W\}$ and $\tau' \in \mathfrak{Irr}_{G(W_{m+t'})}$ such that $\rho_{t'} \rtimes \tau' \twoheadrightarrow \tau$.

4. There does not exists $t' > t$ and a non-zero morphism $I_{W,W}(\chi,s) \to \pi \otimes \chi\pi^\vee$ vanishing on $I_{t'-1}$.

Note firstly that by Theorem 2.3.9 a representation ρ of this form is indeed irreducible and secondly such ρ and σ exist. Indeed, choose t_1 to be the maximal j such that there exists a morphism $f \in \mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \otimes \chi\pi^\vee)$ such that f vanishes on I_{j-1} and does not vanish on I_j. The so obtained morphism $\sigma_{t_1} \to \pi \otimes \chi\pi^\vee$ gives then $\sigma \in \mathfrak{Irr}_{G(W_{t_1})}$ such that $\rho_{t_1} \rtimes \sigma \twoheadrightarrow \pi$. We then can write σ as the quotient of $\rho_{t_2} \times \ldots \rho_{t_k} \rtimes \tau$ for a suitable τ which satisfies above requirement. Then π is the quotient of $\rho \rtimes \tau$ of the desired form. The maximality of t implies that it suffices to show that every morphism $I_{W,W}(\chi, s) \to \pi \otimes \chi\pi^\vee$ vanishes on I_{t-1}.

By the MVW-involution we can therefore realize π as a subrepresentation of $\rho^\vee \rtimes \tau$ and hence we obtain a non-zero morphism

$$f \colon I_{W,W}(s, \chi) \to \rho^\vee \rtimes \tau \otimes \chi\pi^\vee$$

for each morphism in $\mathrm{Hom}_{G(W)\times G(W)}(I_{W,W}(\chi, s), \pi \otimes \chi\pi^\vee)$. Applying Frobenius reciprocity to this morphism we get a morphism

$$f' \colon r_{P_{(m)}\times G(W)}(I_{W,W}(s, \chi)) \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee,$$

where

$$r_{P_{(m)}\times G(W)}(I_{W,W}(s, \chi))$$

admits a filtration with subquotients $\tau_{k,j}$, $k \in \{0, \ldots, m\}$, $j \in \{m, \ldots, q_W\}$ and $\tau_{k,j}$ has a filtration

$$\tau_{k,j}^{j-r+k} \cong S_{k,j}^{r-k} \subseteq \ldots \subseteq \tau_{k,j} = \tau_{k,j}^t \cong S_{k,j}^{j-t} \subseteq \ldots \subseteq \tau_{k,j}^j \cong S_{k,j}^0$$

with subquotients $\tau_{k,j,t} = \tau_{k,j}^{t-1} \backslash \tau_{k,j}^t$, $t \in \{j - r + k, \ldots, j\}$ by Theorem 5.3.2 and Corollary 5.3.1. Recall that

$$(1 \otimes \chi)\mathrm{Ind}_{Q_{(k,m-k)}\times P'_{(j-m)}\times P_{(j)}}^{G_m\times G(W_m)\times G(W)}(\chi|\det|_v^{s+\frac{k}{2}} \otimes$$

$$\otimes\sigma_{m-k,j}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes\chi|\det|_v^{s+\frac{j}{2}}) \otimes \chi|\det|_v^{s+k+\frac{j-m}{2}} \otimes S(G(W_j))) \cong \tau_{k,j}$$

and

$$(1 \otimes \chi)\mathrm{Ind}_{Q_{(k,m-k)}\times P'_{(j-m)}\times P_{(j)}}^{G_m\times G(W_m)\times G(W)}(\chi|\det|_v^{s+\frac{k}{2}} \otimes$$

$$\otimes\omega_{j-t}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes\chi|\det|_v^{s+\frac{j}{2}}) \otimes \chi|\det|_v^{s+k+\frac{j-m}{2}} \otimes S(G(W_j))) \cong \tau_{k,j,t}.$$

Assume now that f does not vanish on I_{t-1}. We have to differentiate three cases, depending on the value of d and t.

Case 1: $t \notin \{1, \ldots, d\}$**:**

We will now show that f' does not vanish on $\tau_{0,m}$ and there exists no non-zero morphism $\tau_{k,j} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$ for all other subquotients. Assume there exists a non-zero morphism $\tau_{k,j} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$. We apply first (1) with respect to $P_{(j)}$ to

$$(1 \otimes \chi)\mathrm{Ind}_{Q_{(k,m-k)}\times P'_{(j-m)}\times P_{(j)}}^{G_m\times G(W_m)\times G(W)}(\chi|\det|_v^{s+\frac{k}{2}} \otimes$$

$$\otimes\sigma_{m-k,j}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes\chi|\det|_v^{s+\frac{j}{2}}) \otimes \chi|\det|_v^{s+k+\frac{j-m}{2}} \otimes S(G(W_j))) \cong \tau_{k,j} \to$$

$$\to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$$

and obtain a non-zero morphism

$$\mathrm{Ind}_{Q_{(k,m-k)}\times G_j}^{G_m\times G_j}(\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes \sigma_{m-k,j}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}})) \to \rho^\vee \otimes \rho',$$

for some suitable irreducible representation ρ'. If $k > 0$, applying (1) with respect to $Q_{(k,m-k)}$, Lemma 2.3.8 and the Geometric Lemma show that $\rho_k \cong \rho_k^\vee$ and hence $k = d$, which contradicts the assumption on d. Thus $k = 0$. Moreover, if $j > m$, we obtain from Lemma 5.2.8 an irreducible representation τ' such that $\rho_{j-m} \rtimes \tau' \twoheadrightarrow \tau$ contradicting the assumption on τ. Indeed, the parabolic subgroup $P'_{(j-m)}$ is conjugated to the standard parabolic subgroup $P_{(j-m)}$ and twisting an irreducible representation by an inner automorphism does not change its isomorphism class. Note that the exact same proof shows that if there exists a non-zero morphism $\tau_{k,j,t'} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$ for some t' then $k = 0$ and $j = m$. Thus if f does not vanish on I_{t-1}, f' does not vanish on $\tau_{0,m}^{t-1}$.

Thus f' restricts to a non-zero morphism on $\tau_{0,m}$, which is a subrepresentation of

$$r_{P_{(m)}\times G(W)}(I_{W,W}(s,\chi))$$

since $\Gamma_{0,m} = \Gamma^{0,m}$ is an open subset of L_W, see the preamble of Theorem 5.3.2. However by Lemma 5.2.4 and Corollary 5.3.1, f' vanishes on $S_{0,m}^{m-t+1} \cong \tau_{0,m}^{t-1}$, a contradiction.

Case 2: $t = d$:

Recall that this is equivalent to $\rho_t^\vee \cong \rho_t$. We will now show that f' does not vanish on $\tau_{0,m}$ or $\tau_{d,m}$ and there exists no non-zero morphism $\tau_{k,j} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$ for all other subquotients. Assume there exists a non-zero morphism $\tau_{k,j} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$. We apply first (1) with respect to $P_{(j)}$ to

$$(1 \otimes \chi)\mathrm{Ind}_{Q_{(k,m-k)}\times P'_{(j-m)}\times P_{(j)}}^{G_m\times G(W_m)\times G(W)}(\chi|\det|_v^{s+\frac{k}{2}} \otimes$$

$$\otimes \sigma_{m-k,j}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}}) \otimes \chi|\det|_v^{s+k+\frac{j-m}{2}} \otimes S(G(W_j))) \cong \tau_{k,j} \to$$

$$\to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$$

and obtain a non-zero morphism

$$\mathrm{Ind}_{Q_{(k,m-k)}\times G_j}^{G_m\times G_j}(\underbrace{\chi|\det|_v^{s+\frac{k}{2}}}_{G_k} \otimes \sigma_{m-k,j}(\chi|\det|_v^{-s-j+\frac{m-k}{2}} \otimes \chi|\det|_v^{s+\frac{j}{2}})) \to \rho^\vee \otimes \rho',$$

for some suitable irreducible representation ρ'. Applying (1) with respect to $Q_{(k,m-k)}$, Lemma 2.3.8 and the Geometric Lemma show that $\rho_k \cong \rho_k^\vee$ and hence $k = d$ or $k = 0$. Moreover, if $j > m$, we obtain as in Case 1 an irreducible representation τ' such that $\rho_{j-m} \rtimes \tau' \twoheadrightarrow \tau$ contradicting the assumption on τ. Observe that the exact same proof shows that if there exists a non-zero morphism $\tau_{k,j,t'} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$ for some t' then $k = 0$ or $k = d$ and $j = m$. Note that by the preamble to Theorem 5.3.2 $\Gamma^{0,m} = \Gamma_{0,m}$ is an open subset of L_W and hence $\tau_{0,m}$ is a subrepresentation of $r_{P_{(m)}\times G(W)}(I_{W,W}(s,\chi))$. If f' does not vanish on I_{t-1} it therefore induces a non-zero morphism on $\tau_{0,m}^{t-1}$ or $\tau_{d,m}^{t-1}$. But $\Gamma_{d,m} \cap \Omega^{d-1} = \varnothing$ so

$\tau_{d,m}^{t-1} = 0$ and therefore f' induces a non-zero morphism $\tau_{0,m} \to \rho^\vee \otimes \tau \otimes \chi\pi^\vee$. However, by Corollary 5.3.1 and Lemma 5.2.4 this implies that f' vanishes on $S_{0,m}^{m-t+1} \cong \tau_{0,m}^{t-1}$. Thus we also arrive in this case at a contradiction.

Case 3: $d \in \{1, \dots, t-1\}$**:**

We will first show that then f does not vanish on I_{d-1}. Indeed, assume otherwise. We set in this case for $d < b \in \mathbb{N}$ $\rho'_b := \mathrm{Z}([d+1, b]_{\xi_0}) = \chi|\det|_v^{s+\frac{b+d}{2}}$. Since $\rho'_t \times \rho_d \twoheadrightarrow \rho_t$ by Lemma 2.3.10, we can define $\rho' \in \mathfrak{Irr}_{m'}$ and $\delta \in \mathfrak{Irr}_{G(W_{m'})}$ as follows.

1. There exists $1 \le l \in \mathbb{N}$, $d < b_1, \dots, b_l \in \{1, \dots, q_W\}$ such that
$$\rho' \cong \rho'_{b_1} \times \ldots \times \rho'_{b_l}.$$
2. We can realize π as a quotient $\rho' \rtimes \delta \twoheadrightarrow \pi$ and set $b := \max_i b_i$.
3. There does not exist $b' \in \{d, \dots, q_W\}$ and $\delta^\vee \in \mathfrak{Irr}_{G(W_{m+b'})}$ such that $\rho'_{b'} \rtimes \delta^\vee \twoheadrightarrow \delta$.

Note that we can assume that $b \ge t$. Indeed, by Theorem 2.3.9 we can assume without loss of generality that $t_1 = t = \max_i t_i$ and hence there exist by Lemma 5.2.8 $\tau' \in \mathfrak{Irr}_{G(W_t)}$ such that $\rho_t \rtimes \tau' \twoheadrightarrow \pi$. Since $\rho'_t \times \rho_d \times \tau' \twoheadrightarrow \rho_t \rtimes \tau' \twoheadrightarrow \pi$, we obtain $\tau'' \in \mathfrak{Irr}_{G(W_{t-d})}$ such that $\rho'_t \rtimes \tau'' \twoheadrightarrow \pi$. Writing τ'' as the quotient of $\rho'_{b_2} \times \ldots \rho'_{b_l} \rtimes \delta$ as desired shows that we can assume $b \ge t$.

By the MVW-involution we can therefore realize π as a subrepresentation of $\rho'^\vee \rtimes \delta$ and hence we obtain a non-zero morphism
$$f\colon I_{W,W}(s,\chi) \to \rho'^\vee \rtimes \delta \otimes \chi\pi^\vee.$$
Applying Frobenius reciprocity to this morphism we get a morphism
$$f''\colon r_{P_{(m')}\times G(W)}(I_{W,W}(s,\chi)) \to \rho'^\vee \otimes \delta \otimes \chi\pi^\vee,$$
where
$$r_{P_{(m')}\times G(W)}(I_{W,W}(s,\chi))$$
admits a filtration with subquotients $\tau_{k,j}$, $k \in \{0, \dots, m'\}$, $j \in \{m', \dots, q_W\}$ by Theorem 5.3.2. As in the previous cases one sees that the only $\tau_{k,j}$ admitting morphisms to $\rho'^\vee \otimes \tau \otimes \chi\pi^\vee$ have to satisfy $k = 0$. Moreover, if $j > m' + d$, we would obtain from Lemma 5.2.8 a morphism $\rho_{j-m'} \rtimes \delta^\vee \twoheadrightarrow \delta$ for a suitable $\delta^\vee$ and since $\rho'_{j-m'} \times \rho_d \twoheadrightarrow \rho_{j-m'}$ by Lemma 2.3.10, we would contradict the assumption on δ. Thus $j \le m'+d$. For $j < m'+d$, a morphism $\tau_{0,j} \to \rho'^\vee \otimes \tau \otimes \chi\pi^\vee$ does not vanish on $S_{0,j}^{m'} \cong \tau_{0,j}^{j-m'}$ by Lemma 5.2.3 and hence does not vanish on $\tau_{0,j}^{d-1}$. By exactly the same argument we obtain that f'' has to vanish on all $\tau_{k,j,t'}$ with $k \ne 0$ or $j \ne m'+d$.

This implies that if f vanishes on I_{d-1}, f'' restricts to a non-zero morphism on $\tau_{0,m'+d}$ and vanishes on all other $\tau_{k,j}$'s. But here we can again apply Corollary 5.3.1 and Lemma 5.2.4 to see that then f'' vanishes on $S_{0,m'+d}^{m'-b+1} \cong \tau_{0,m'+d}^{b-1}$ and hence f'' vanishes on $r_{P_{(m')}\times G(W)}(I_{b-1})$. Since we showed that $b \ge t$, f vanishes on I_{t-1}.

Therefore f does not vanish on I_{d-1}. We now apply this restriction to f'. As in Case 1 and 2, we see that the subquotients $\tau_{k,j}$ of $r_{P_{(m)}\times G(W)}(I_{W,W}(s,\chi))$ and the subquotients $\tau_{k,j,t'}$ of $\tau_{j,k}$ admit a morphism to $\rho^\vee \otimes \tau \otimes \chi\pi^\vee$ only if $k = 0$ and $j = m$ or $k = d$ and $j \ge m$.

Since for $k = d$, $\Gamma_{k,j} \cap \Omega^{d-1} = \varnothing$, we obtain that f' must restrict to a non-zero morphism on $\tau_{0,m}^{d-1} \cong S_{0,m}^{m-d+1}$ and in particular it does not vanish on $\tau_{0,m}$, which, as we observed before, is a subrepresentation of $r_{P_{(m)} \times G(W)}(I_{W,W}(s,\chi))$. But f' restricted to $\tau_{0,m}$ does vanish on $S_{0,m}^{m-t+1} \cong \tau_{0,m}^{t-1}$ by Corollary 5.3.1 and Lemma 5.2.4. Since $\tau_{0,m}^{t-1}$ contains $\tau_{0,m}^{d-1}$ we arrive at a contradiction. $\square$

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