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Operator Quantization and Deformation on the Heisenberg Modules

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Contents

Abstract

This dissertation explores the connection between Heisenberg modules over noncommutative tori–a well-studied object in noncommutative geometry–and the theories of operator-quantization along with Banach bundles associated to the deformation of C^*-algebras. This investigation is motivated by deep results in Gabor analysis.

The well-known fact that Heisenberg modules over higher dimensional noncommutative tori may be viewed terms Gabor frames allows us to establish the deformation of Banach bundles of Heisenberg modules over noncommutative tori. Furthermore, we show that the generated Banach space of this bundle of Heisenberg modules implements the Morita equivalence between the generated C^*-algebras of bundles of twisted group C^*-algebras coming from lattices in the time-frequency plane. Furthermore, building on the deformation results of Feichtinger and Kaiblinger on Gabor frames, we show that the constructed bundle of Heisenberg modules generalizes the Gabor frame stability results of Feichtinger's algebra to Heisenberg modules.

Recall that to each lattice in the time-frequency plane there is associated a Heisenberg module. We provide a detailed description of the adjointable maps of these Heisenberg module. Building on the observation that these adjointable maps are those that commute with all time-frequency shifts of the given lattice, i.e. operator-translation invariant in the language of quantum harmonic analysis. Using techniques from C^*-algebras and von Neumann algebras associated to Heisenberg modules, we show that every lattice operator-translation invariant map may be characterized as operator-periodization of finite rank operators with generators coming from the Heisenberg modules. We also introduce the related notion of operator-modulation invariant operators of a lattice, and show that these operators can also be characterized as Fourier-Wigner operator-periodizations of finite-rank operators by generators of the Heisenberg module.

Zusammenfassung

In dieser Dissertation untersuchen wir die Verbindung zwischen Heisenbergmodulen über nicht-kommutativen Tori–einem wohl-studierten Objekt der Nicht-kommutativen Geometrie–und den Theorien der Operator-Quantisierung sowie Banach-Bündeln, die mit der Deformation von C^*-algebren in Verbindung stehen. Diese Untersuchung wird durch tiefgreifende Ergebnisse in der Gabor-Analyse motiviert.

Der wohlbekannte Fakt, daßman Heisenbergmodule über mehr-dimensionalen nicht-kommutativen Tori auch als Gabor Frames interpretieren kann, erlaubt es uns die Deformierungen von Heisenbergmodulen über deformierten nicht-kommutativen Tori zu etablieren. Weiters zeigen wir, daßder konstruierte Banachraum dieses Bündels von Heisenbergmodulen implementiert die Morita-Äquivalenz zwischen der erzeugten C^*-Algebra des Bündels von verschränkten Gruppenalgebren assoziert zu den Gittern in der Zeit-Frequency Ebene. Überdies zeigen wir, daßdieses Bündel von Heisenbergmodulen die Stabilitätsresultate für Gabor Frames von Feichtinger und Kaiblinger für die Feichtinger Algebra zu den Heisenbergmodulen verallgemeinert.

Bekannterweise gehört zu jedem Gittern in der Zeit-Frequenz Ebene ein Heisenbergmodul. Wir geben eine detailierte Beschreibung der adjungierten Abbildungen dieser Heisenbergmodule. Basierend auf der Einsicht, dass diese adjungierten Abbildungen genau diese sind, die mit allen Zeit-Frequenz Verschiebungen des Gitters vertauschen, d.h. sie sind im Sinne der Quantenharmonischen Analyse invariant unter Operatortranslaten vom Gitter sind. Mit der Hilfe von Methoden der Theorie der C^*-Algebren und von Neumann Algebren beweisen wir, dass jede mit Operatortransaten-vertauschende Abbildung eine Operator-Periodisierung von endlichen Rang-Operatoren von Generatoren des Heisenbergmoduls ist. Wir führen weiters den Begriff der operator-modulation-invarianten Operatoren eines Gitters ein und karaktererisen sie als die Fourier-Wigner Periodisierung von endlichen Rang Operatoren der Generatoren des Heisenbergmoduls.

Chapter 1

Introduction

This dissertation contains contributions on the link between Gabor analysis and noncommutative geometry. To set the stage, let us highlight two key motivating results, from [Gel41, Neg71] and [Ser55, Swa62] respectively:

> **(Gelfand's Duality Theorem).** *There is a duality of categories between locally compact Hausdorff spaces and commutative C^*-algebras.*
>
> **(Serre-Swan Theorem).** *For a fixed compact Hausdorff space M, there is an equivalence of categories between complex vector bundles over M and finitely generated projective modules over $C(M)$.*

The general principle of noncommutative geometry is to extend such equivalences and dualities from commutative C^*-algebras to noncommutative ones. To this end, we firstly see that general C^*-algebras correspond to formal 'noncommutative topological spaces'. Consequently, finitely generated projective modules over C^*-algebras must correspond to complex vector bundles over noncommutative topological spaces.

Perhaps the simplest noncommutative topological spaces are the higher-dimensional noncommutative n-tori, which are C^*-algebras that can be obtained by a straightforward deformation of the relations between the generators of $C(\mathbb{T}^n)$. Studying the vector bundles over these noncommutative spaces lead us to the *Heisenberg modules* [Rie88], which are finitely generated projective modules over noncommutative n-tori.

On the one hand, Gabor analysis is a branch of harmonic analysis that studies representations of functions via time-frequency shifts [Gab47]. On the other hand, the construction method for

Heisenberg modules involves techniques from harmonic analysis on phase-space. This led to the establishment of connections between noncommutative geometry and Gabor analysis [Lue07]. The intimate relationship between the two subjects, through Heisenberg modules, will be our central focus.

A secondary goal of this thesis is to survey the literature surrounding the Heisenberg modules from the perspective of harmonic analysis. I had found this to be a worthwhile endeavor, especially because some of the material did not seem quite related at first glance. The next five chapters will be dedicated to a detailed exposition of the background material. I made the decision to start studying the foundations from a sufficiently abstract setting, opting to treat the setting for our results and applications as particular cases of earlier discussions. In this work, we generally assume basic knowledge of operator algebras and some rudimentary measure space theory.

The dissertation is structured as follows:

Chapter 2. We introduce Hilbert C^*-modules, the formal complex vector bundles over C^*-algebras. Imprimivity bimodules will also be studied; they are a special kind of Hilbert C^*-module that implements Morita equivalences for C^*-algebras. Heisenberg modules are examples of imprimitivity bimodules. In this chapter we shall also introduce frame theory for Hilbert C^*-modules, which is general enough to encompass frame theory on the Heisenberg modules, and frame theory on the Hilbert space of square integrable functions.

Chapter 3. We give a detailed review of abstract analysis on locally compact groups. Our primary goals in this chapter include: a derivation of Pontryagin duality, to set up notation for the Fourier transform and its inversion, and Weil's formula. Since we will eventually be working with integrated forms of group representations, we shall end with a quick section on vector-valued integration on locally compact groups.

Chapter 4. We introduce the time-frequency shifts and the time-frequency plane. We also define the *Heisenberg 2-cocycle* here. Building up from the language established from the previous chapter, we look at Fourier analysis on the time-frequency plane, deriving symplectic versions of some of the relevant results from Chapter 3. Next, we discuss Gabor analysis and Feichtinger's algebra, which motivate much of this dissertation. Finally, we touch upon operator quantization based on Feichtinger's algebra, where we close with a proof of Janssen's representation, setting us up perfectly for the construction of the Heisenberg modules.

Chapter 5. We investigate the theory of twisted C^*-dynamical systems from which we can construct twisted crossed-products. We show that twisted group C^*-algebras are twisted crossed-products. Using Feichtinger's algebra along with Janssen's representation, we construct Heisenberg modules as imprimitivity bimodules between twisted group C^*-algebras. We also discuss the localization scheme, allowing us to treat Heisenberg modules as square integrable functions.

Chapter 6. In the sixth chapter, we delve into the theory of Banach bundles, which are bundles whose fibers are generally infinite-dimensional Banach spaces. Banach bundles provide the mathematical foundation that would allow us to rigorously define what it means for a family of Heisenberg modules, indexed by a topological space to 'continuously vary'. We will also discuss the special case of Banach bundles whose fibers are all C^*-algebras, called C^*-bundles. One of the most relevant results in this chapter is an existence theorem ensuring the existence of a Banach bundle with plenty of continuous sections. All subsequent bundles in the next chapter, for our results and applications, will be constructed based on this theorem.

Chapter 7. This chapter presents my main contributions to the field. It is split into two sections (I) "Linear Deformations of Heisenberg modules and Gabor Frames" and (II) "On Modulation and Translation Invariant Operators and the Heisenberg Modules".

Appendix. I have included a brief appendix with a detailed discussion on universal C^*-algebras for quick reference. Another entry in the appendix is dedicated to reviewing reformulations of topological properties in terms of nets.

In Chapter 7, results without citations are original work. While Chapters 2-6 cover mostly established material, I chose to include proofs for the majority of the results in the discussions. Most of these will be cited, taking on the form of a direct citation in the theorem header, or somewhere in the preceding paragraph. However, some of these results will be claimed as original work. These include modified versions of known results, and independent proofs of established results.

Chapter 2

Hilbert C^*-Modules

Here we shall give a detailed discussion of Hilbert C^*-modules, which one can think of as a generalization of Hilbert spaces C^*-algebras are used in place of complex numbers for the coefficients. We are primarily motivated by the fact that the Heisenberg module is a particular instance of a Hilbert C^*-module, and so it is imperative that we familiarize ourselves with the general structure that underpins our object of study. We begin with a general introduction to the basic theory of Hilbert C^*-modules, briefly examining its parallels with, and divergences from, Hilbert space theory. Some space will be dedicated to a discussion of the so-called "multiplier algebras", which we can think of as a unitization scheme for C^*-algebras. These algebras are classically defined in terms of double centralizers, but we shall exploit the fact that they can equivalently be constructed from the adjointable maps of Hilbert C^*-modules. We then further examine the case of Hilbert C^*-modules equipped with symmetry conditions– called imprimitivity bimodules. The chapter concludes with a section on module frames, where we discuss a version of frame theory on Hilbert C^*-modules. All our frame-theory related needs will be outsourced from this section given that module frame encompass classical frame theory.

We shall follow both Lance's [Lan95] and Williams's [Wil07] expositions on the basic theory on Hilbert C^*-modules up to the discussion of imprimitivity bimodules. On the other hand, we will use the recent article by [AB17] as our main resource for module frames. There, the authors provided a careful treatment of modules frames for which the coefficient C^*-algebras are not necessarily assumed to be unital.

2.1 Basic Theory

Definition 2.1. Let A be a C^*-algebra and Z a complex vector space. Z is said to be an **inner product left A-module** if it is a left A-module compatible with complex scalar multiplication in the sense that:

$$\lambda(a \cdot z) = (\lambda a) \cdot z = a \cdot (\lambda z), \qquad \forall (\lambda, a, z) \in \mathbb{C} \times A \times Z.$$

Furthermore, there exists ${}_A\langle \cdot, \cdot \rangle : Z \times Z \to A$ such that for all $z, w, y \in Z$, $a \in A$, and scalars $\lambda, \mu \in \mathbb{C}$:

1. ${}_A\langle \lambda z + \mu y, w \rangle = \lambda \, {}_A\langle z, w \rangle + \mu \, {}_A\langle y, w \rangle$

2. ${}_A\langle a \cdot z, w \rangle = a \, {}_A\langle z, w \rangle$

3. ${}_A\langle z, w \rangle^* = {}_A\langle w, z \rangle$

4. ${}_A\langle z, z \rangle \geqslant 0$

5. **(Positive-Definiteness)**. ${}_A\langle z, z \rangle = 0 \implies z = 0$

An **A-submodule** of Z is a subvector space of Z that is closed with respect to the A-action.

Remark 2.1.

1. We will usually use the juxtaposition $a \cdot z := az$ as a shorthand for the module action.

2. It follows from Items 2 and 3 that ${}_A\langle z, w \rangle a = {}_A\langle z, a^* w \rangle$ for all $a \in A$ and $z, w \in Z$.

3. One can see from axioms (2) and (3), that the linear span ${}_A\langle Z, Z \rangle := \operatorname{span}\{{}_A\langle z, w \rangle : z, w \in Z\}$ is an ideal in A.

4. Note that axioms (1) and (3) above imply that the module inner-product is module-linear in the first component, and module-antilinear in the second one. If we omit axiom 5 above, we call Z a **semi inner-product A-module**. It is not too difficult to show that there is a canonical way of constructing an inner-product A-module from a semi inner-product A-module by quotienting out from Z the A-submodule $N = \{z \in Z : {}_A\langle z, z \rangle = 0\}$, and endowing Z/N with obvious operations.

5. For the rest of this section, we will use the symbol $\|\cdot\|$ to denote the norm on the C^*-algebra A. Shortly, we will show that inner-product A-modules can also be endowed with a norm. We again use the same symbol $\|\cdot\|$ to denote this norm, leaving it to the reader to figure out which norm we mean by context.

Given a C^*-algebra A, then from a combination of [Dix82, 1.6.8] and [Dix82, 1.6.9], we obtain that for any $a, b \in A$:

$$(ab)^*(ab) \leqslant \|a\|^2 b^* b. \tag{2.1}$$

Proposition 2.1 ([Lan95, Proposition 1.1])**.** *If Z is a left semi-inner-product A-module and $z, y \in Z$, then*

$$_A\langle y, z\rangle \, _A\langle z, y\rangle \leqslant \| \, _A\langle z, z\rangle \| \, _A\langle y, y\rangle .$$

Proof. For any positive $c \in A^+$, and $b \in A$, Equation (2.1) implies that

$$b^*(c^{1/2}c^{1/2})b \leqslant \|c^{1/2}\|^2 b^* b \implies b^* c b \leqslant \|c\| b^* b \implies b c b^* \leqslant \|c\| b b^*. \tag{2.2}$$

Let $z, y \in Z$ and $a \in A$. Suppose WLOG that $\| \, _A\langle z, z\rangle \| = 1$. If we take $c = \, _A\langle z, z\rangle$ in the inequality above, we have:

$$\begin{aligned}
0 &\leqslant \, _A\langle az - y, az - y\rangle \\
&= \, _A\langle az, az - y\rangle - \, _A\langle y, az - y\rangle \\
&= \, _A\langle az, az\rangle - \, _A\langle az, y\rangle - \, _A\langle y, az\rangle + \, _A\langle y, y\rangle \\
&= a \, _A\langle z, z\rangle a^* - a \, _A\langle z, y\rangle - \, _A\langle y, z\rangle a^* + \, _A\langle y, y\rangle \\
&\leqslant aa^* - a \, _A\langle z, y\rangle - \, _A\langle y, z\rangle a^* + \, _A\langle y, y\rangle .
\end{aligned}$$

The inequality above is true for all $a \in A$, so take $a = \, _A\langle y, z\rangle$ in particular. From this, we find that $0 \leqslant - \, _A\langle y, z\rangle \, _A\langle z, y\rangle + \, _A\langle y, y\rangle \iff \, _A\langle y, z\rangle \, _A\langle z, y\rangle \leqslant \| \, _A\langle z, z\rangle \| \, _A\langle z, z\rangle$ as required. ☐

As expected, we can endow an inner-product A module with a norm, similar to how we define one for a Hilbert space. The following follows easily from definition.

Proposition 2.2. *Let A be a C^*-algebra and Z be an inner-product A-module, consider*

$$\|z\| := \| {}_A\langle z, z\rangle \|^{1/2}. \tag{2.3}$$

Then $\| \cdot \|$ is a norm for Z. If Z is only a semi-inner-product A-module, then $\| \cdot \|$ is only a seminorm for Z.

From Proposition 2.1, we have the following estimates.

Proposition 2.3 ([Lan95])**.** *Let A be a C^*-algebra and Z an inner-product A-module.*

1. $\| {}_A\langle z, y\rangle \| \leqslant \|z\|\|y\| \qquad \forall z, y \in Z.$

2. *We can define an A-valued analogue of a norm in Z via: $|z| := {}_A\langle z, z\rangle^{1/2}$. We have the following estimate*

$$| {}_A\langle z, y\rangle | \leqslant \|z\||y| \qquad \forall z, y \in E,$$

where $|a| = (a^*a)^{1/2}$ for $a \in A$.

3. *The norm on Z also makes Z a normed A-module because*

$$\|az\| \leqslant \|a\|\|z\| \qquad \forall (z, a) \in Z \times A.$$

Proof.

1. Since $\| {}_A\langle z, z\rangle \| = \|z\|^2$, then exactly from Proposition 2.1

$${}_A\langle y, z\rangle \, {}_A\langle z, y\rangle \leqslant \|z\|^2 \, {}_A\langle y, y\rangle .$$

So we have, by the fact that the norm in a C^*-algebra preserves the poset of positive elements [Dix82, 1.6.9] that:

$$\| {}_A\langle z, y\rangle \|^2 \leqslant \|x\|^2\|y\|^2 \iff \| {}_A\langle x, y\rangle \| \leqslant \|x\|\|y\|.$$

2. Recall from [Mur90, 2.2.6] that square roots preserve the poset of positivity. That is, for

$0 \leqslant a \leqslant b$ in A, we have $a^{1/2} \leqslant b^{1/2}$. It now follows from Proposition 2.1, that we have

$$\left({}_A\langle y, z \rangle \, {}_A\langle z, y \rangle \right)^{1/2} \leqslant \left(\| \, {}_A\langle z, z \rangle \, \| \, {}_A\langle z, z \rangle \right)^{1/2}$$

which is equivalent to what we want to prove.

3. Note that $\|az\|^2 = \| \, {}_A\langle az, az \rangle \, \| = \|a \, {}_A\langle z, z \rangle \, a^*\| \leqslant \| \, {}_A\langle z, z \rangle \, \| \|aa^*\| = \|z\|^2 \|a\|^2$. Taking square roots, we arrive at our desired conclusion.

□

Notice that the estimates above immediately demonstrate that the definition of the norm (2.3) on Z gives us continuity of the module inner-product and module action for free.

Definition 2.2. Let A be a C^*-algebra, and Z an inner-product A-module. Z is said to be a Hilbert **A-module** or a **Hilbert C*-module over the C*-algebra A** if it is complete with respect to its norm.

In the sequel, we shall study an important 'completion scheme' for a special kind of Hilbert C^*-module. We shall need a technical lemma which gives a 'canonical' way of constructing a Hilbert A-module from dense subspaces. The next result shall give the details regarding the construction discussed in Item 4 of Remark 2.1.

Definition 2.3. Suppose that A_0 is a dense $*$-subalgebra of A, and Z_0 is a left A_0-module with a form ${}_{A_0}\langle \cdot, \cdot \rangle : Z_0 \times Z_0 \to A_0$ satisfying (an obvious version[1]) of Items 1 to 4 of Definition 2.1. Then Z_0 is said to be a **pre-Hilbert left $\mathbf{A_0}$-module**.

Lemma 2.1 ([RW98, Lemma 2.16]). *Let A_0 be a dense $*$-subalgebra of a C^*-algebra A, and Z_0 a pre-Hilbert left A_0-module, then there exists a Hilbert A-module Z and a linear map $q : Z_0 \to Z$ such that $q(Z_0)$ is dense in Z such that:*

1. $q(a_0 \cdot z_0) = a_0 \cdot q(z_0), \qquad \forall a_0 \in A_0, \; z_0 \in Z_0;$

2. ${}_A\langle q(z_0), q(w_0) \rangle = {}_{A_0}\langle z_0, w_0 \rangle, \qquad \forall z_0, w_0 \in Z_0.$

We call Z the ***completion*** *of Z_0.*

[1] Perhaps it might not be so obvious that the version of Item 4 of Definition 2.1 that we want is that: ${}_{A_0}\langle z, z \rangle \geqslant 0$ with respect to the completion A.

Proof. Let $N := \{z_0 \in Z : {}_{A_0}\langle z_0, z_0\rangle = 0\}$. We may only have a semi inner-product A_0 structure on Z_0, but Proposition 2.1 still applies in this case. In particular, it implies that for all $n \in N$ and $z_0 \in Z_0$, we have that

$$ {}_{A_0}\langle z_0, n\rangle = 0 = {}_{A_0}\langle n, z_0\rangle \tag{2.4}$$

which further implies that N is a vector subspace and A_0-submodule of Z_0. Now consider the canonical vector space quotient map $q : Z_0 \to Z_0/N$. Let us define on the quotient an A_0-valued inner-product and A_0-module action via:

$$ {}_{A_0}\langle q(z_0), q(w_0)\rangle := {}_{A_0}\langle z_0, w_0\rangle,$$
$$ a_0 \cdot q(z_0) := a_0 \cdot z_0$$

for all $z_0, w_0 \in Z_0$ and $a_0 \in A_0$. We verify that these are well-defined, suppose that $z_0, w_0 \in Z_0$ such that $q(z_0) = q(w_0)$, then it follows from Equation (2.4) for all $a_0 \in A_0$:

$$ {}_{A_0}\langle a_0 \cdot (z_0 - w_0), a_0 \cdot (z_0 - w_0)\rangle = a_0\, {}_{A_0}\langle z_0 - w_0, z_0 - w_0\rangle\, a_0^* = 0.$$

Therefore $a_0 \cdot (z_0 - w_0) \in N$, equivalently $a \cdot q(z_0 - w_0) = 0 \iff a_0 \cdot q(z_0) = a_0 \cdot q(w_0)$. On the other hand, if $z_1, w_1, z_2, w_2 \in Z_0$ such that $q(z_1) = q(z_2)$ and $q(w_1) = q(w_2)$ then we have:

$$ {}_{A_0}\langle z_1, w_1\rangle - {}_{A_0}\langle z_2, w_2\rangle = {}_{A_0}\langle z_1, w_1 - w_2\rangle + {}_{A_0}\langle z_1 - z_2, w_2\rangle = 0 + 0 = 0.$$

Therefore ${}_{A_0}\langle q(z_1), q(w_1)\rangle = {}_{A_0}\langle q(z_2), q(w_2)\rangle$.

Now, note that for any $z_0 \in Z_0$, we have

$$\|q(z_0)\| := \|\, {}_{A_0}\langle z_0, z_0\rangle \,\|^{1/2}.$$

If $\|q(z_0)\| = 0 \iff \|\, {}_{A_0}\langle z_0, z_0\rangle \,\|^{1/2}$, then ${}_{A_0}\langle z_0, z_0\rangle = 0$, therefore $z_0 \in N$, hence $q(z_0) = 0$. This shows that the map $\|\cdot\| : Z_0/N \to \mathbb{R}$ defined above is a norm on Z_0/N. Denote by Z the completion of Z_0/N with respect to the norm just defined.

We can again use Estimate (2.1) to show that for all $z_0 \in Z_0$ and $a_0 \in A_0$:

$$\|a_0 \cdot q(z_0)\|^2 = \|a_0\, {}_{A_0}\langle z_0, z_0\rangle\, a_0^*\|$$

$$= \|(a_0\, {}_{A_0}\langle q(z_0), q(z_0)\rangle^{1/2})(a_0\, {}_{A_0}\langle q(z_0), q(z_0)\rangle^{1/2})^*\|$$
$$\leqslant \|a_0\|^2 \|q(z_0)\|^2.$$

This implies $\|a \cdot q(z_0)\| \leqslant \|a_0\|\|q(z_0)\|$. Therefore the left A_0-module action is a linear and bounded map on the dense subspace Z_0/N of Z, therefore the left A_0-module action extends to all of Z satifying the following:

$$\|a_0 \cdot z\| \leqslant \|a_0\|\|z\|. \tag{2.5}$$

However, Estimate (2.5) can be used to show that the left A_0-module action can be extended to all of A via $a \cdot z = \lim_i a_i \cdot z$ for any net $\{a_i\}_i \subseteq A_0$ converging to $a \in A$.

We can essentially apply the same approach to the inner-products. Since $\|\, {}_{A_0}\langle q(z_0), q(w_0)\rangle \,\| \leqslant \|\, {}_{A_0}\langle q(z_0), q(z_0)\rangle \,\|\|\, {}_{A_0}\langle q(w_0), q(w_0)\rangle \,\|$ for all $z_0, w_0 \in Z_0$ (via Proposition 2.1) we can define for all $z, w \in Z$

$${}_A\langle z, w\rangle := \lim_i {}_{A_0}\langle q(z_i), q(w_i)\rangle$$

for any nets $q(z_i) \to z$ and $q(w_i) \to w$. By arguing using continuity and working on the necessary dense subspaces, it is routine to check that Axioms 1 to 3 of Definition 2.1 hold for the left A-module action and the A-valued inner-product on Z. Axiom 4 of Definition 2.1 follows from the fact that for any C^*-algebra A, the set of positive elements A^+ is a closed set (which easily follows from [Mur90, Lemma 2.2.2] and setting $t = 1$). Lastly, if $z \in Z$ with $q(z_i) \to z$ such that ${}_A\langle z, z\rangle = 0$, then we have $\|0 - q(z_i)\| = \|q(z_i)\| \to \|z\| = 0$. Therefore $q(z_i) \to 0$, and by uniqueness of limits, $z = \lim_i q(z_i) = 0$ as required. □

Example 2.1. Let A be a C^*-algebra and I be a countable index set. Let us define the space

$$\ell^2_A(I) := \left\{ \mathbf{a} = \{a_i\}_{i \in I} \subseteq A : \sum_{i \in I} a_i a_i^* \text{ converges in the norm of } A \right\}$$

Equipping $\ell^2_A(I)$ with the obvious left A action and inner-product:

1. $a \cdot \mathbf{a} := (aa_i)_{i \in I}$;

2. ${}_A\langle \mathbf{a}, \mathbf{a}'\rangle = \sum_{i \in I} a_i (a_i')^*$

makes $\ell^2_A(I)$ a Hilbert A-module. Note an important special case of this construction is when $|I| = 1$, in which case we see that A itself is a Hilbert A-module.

Proposition 2.4 ([Lan95]). *Let $\{e_i\}_i$ be an approximate identity in a C^*-algebra A, and Z a Hilbert A-module. We have $z = \lim_i e_i z$, therefore $AZ := \{az : a \in A,\ z \in Z\}$ is a dense submodule of Z.*

Proof. Note that we can write, for each $z \in Z$:

$$e_i\, {}_A\langle z, z\rangle\, e_i = ({}_A\langle z, z\rangle^{1/2}\, e_i)^*({}_A\langle z, z\rangle^{1/2}\, e_i).$$

Therefore, $\lim_i e_i\, {}_A\langle z, z\rangle\, e_i = {}_A\langle z, z\rangle$. Now we have

$${}_A\langle z - e_i z, z - e_i z\rangle = {}_A\langle z, z\rangle - e_i\, {}_A\langle z, z\rangle - {}_A\langle z, z\rangle\, e_i + e_i\, {}_A\langle z, z\rangle\, e_i,$$

hence it follows from what we have derived above that $\lim_i {}_A\langle z - e_i z, z - e_i z\rangle = 0$, which also implies $\lim_i \|z - e_i z\| = 0$. Therefore $z = \lim_i e_i z$ as required. □

Corollary 2.1 ([Lan95]). *Let A be a unital C^*-algebra with unit 1_A, and Z a Hilbert A-module. Then $1_A z = z$.*

Proof. Let $\{e_i\}_i$ be an approximate identity in A, then it follows from the definition of an approximate identity and unitality of A that $\lim_i e_i = 1_A$. Furthermore, from Proposition 2.4, it follows that $1_A z = \lim_i e_i z = z$ as required. □

Proposition 2.5 ([Lan95]). *Let Z be a Hilbert A- module. Consider ${}_A\langle Z, Z\rangle = \operatorname{span}\{{}_A\langle z, w\rangle : z, w \in Z\}$, then $\overline{{}_A\langle Z, Z\rangle Z} = Z$.*

Proof. It is not hard to see that $I := \overline{{}_A\langle Z, Z\rangle}$ is a closed two-sided ideal of A, and therefore contains its own approximate identity. Let $\{f_i\}_i$ be an approximate identity of I, and $z \in Z$. Because ${}_A\langle z, z\rangle \in I$, then we can run exactly the same computation as in Proposition 2.4 to conclude that $\lim_i f_i z = z$. Therefore $Z = \overline{{}_A\langle Z, Z\rangle Z} = Z$. □

One of the things that carry over from the theory of Hilbert spaces are the following norm characterizations.

Theorem 2.1 ([RW98, Lemma 2.16]). *Let Z be a Hilbert A-module, then for any $z \in Z$,*

1. $\|z\| = \sup\{\|\, {}_A\langle z, y\rangle \| : y \in Z,\ \|y\| \leqslant 1\}$;

2. $\|z\| = \sup\{\| {}_A\langle z, y\rangle \| : y \in E,\ \ \|y\| = 1\}$;

3. $\|z\| = \sup\left\{\frac{\| {}_A\langle z,y\rangle\|}{\|y\|} : y \in Z,\ \ y \neq 0\right\}$.

Proof. We will only prove item 1, the proof for the rest of the charaterizations are similar. Let $z, y \in Z$ where $\|y\| \leqslant 1$, then we know from Proposition 2.1 Item 1 that:

$$\| {}_A\langle z, y\rangle \| \leqslant \|x\|\|y\| = \|x\|,$$

so $\sup\{\| {}_A\langle z, y\rangle \| : y \in E,\ \ \|y\| \leqslant 1\} \leqslant \|x\|$. On the other hand, suppose $z \neq 0$, then

$$\begin{aligned}
\|z\| &= \frac{1}{\|z\|}\|z\|^2 \\
&= \frac{1}{\|x\|}\| {}_A\langle z, z\rangle \| \\
&= \left\| {}_A\left\langle z, \frac{z}{\|z\|}\right\rangle\right\| \\
&\leqslant \sup\{\| {}_A\langle z, y\rangle \| : y \in Z,\ \ \|y\| \leqslant 1\}.
\end{aligned}$$

This proves the theorem. □

So far, it seems that the theory of Hilbert C^*-modules parallels the theory of Hilbert spaces. In the sequel, we shall show where the two theories digress. We start by defining orthogonal complements for closed submodules.

Definition 2.4. Given a Hilbert A-module Z, we define the **orthogonal complement** of a closed A-submodule $W \subseteq Z$ to be

$$W^{\perp} := \{z \in Z : {}_A\langle z, w\rangle = 0,\ \ \forall w \in W\}.$$

Some of the known properties of the orthogonal complement carries over to Hilbert C^*-modules, like the proposition below.

Proposition 2.6. *$W^{\perp}$ is a closed A-submodule of Z.*

Because the module inner-product is also continuous for Hilbert C^*-modules, the proof for the above proposition is essentially the same as in the Hilbert space case. We shall show below that the orthogonal complement decomposition for Hilbert spaces no longer hold for Hilbert C^*-modules in general.

Example 2.2. Take the Cantor set $\mathcal{C}$, it is a closed subset of $[0,1]$ which is nowhere dense in it [SSS78]. This implies that if $X = [0,1]$ then $X\backslash\overline{\mathcal{C}} = X\backslash\mathcal{C}$ is dense in X. Consider the unital C^*-algebra $A = C(X)$. By Example 2.1, we know that we can also take $Z = A$, so that Z is a Hilbert A-module. One can see that $W := C(X\backslash\mathcal{C})$ is a *proper closed A-submodule* of Z if we identify all the functions W to be those in Z that is trivial in $\mathcal{C}$. If we define:

$$\begin{aligned} W^\perp &:= \{w \in W : {}_A\langle w, z\rangle = 0 \text{ for all } z \in Z\} \\ &= \{w \in W : w(t)\overline{z(t)} = 0 \text{ for all } t \in [0,1] \text{ and } z \in Z\}. \end{aligned}$$

we see that any $w \in W^\perp$ must be $\{0\}$. We can define the direct sum of Hilbert C^*-modules in a completely analogous way as in Hilbert spaces, but we see from our particular example that $W \oplus W^\perp \cong W \ncong Z$.

The same example also shows another failure of the Hilbert space theory to our setting. One is not assured the existence of adjoints for bounded linear maps between Hilbert A-modules. Take $Z = A = C(X)$, $W = C(X\backslash\mathcal{C})$ as in above. Consider the inclusion map $\iota : W \hookrightarrow Z$, which is a bounded, A-linear (and complex-linear) map. Suppose that there exists an adjoint $\iota^* : Z \to W$ satisfying:

$$ {}_A\langle \iota w, f\rangle = {}_A\langle w, \iota^* f\rangle, \qquad \forall (f,w) \in Z \times W.$$

If we take $f = 1$ (identity in $Z = C(X)$) in particular, then we find that $\iota^*(1)(t) = 1$ for all $t \in X\backslash\mathcal{C}$ and $\iota^*(1)(t) = 0$ for all $t \in \mathcal{C}$. But this function is not continuous, and therefore is not contained in W.

Since we cannot take the existence of adjoints for granted, we define the proper maps between Hilbert C^*-modules to come with an adjoint a priori. We give a formal definition below.

Definition 2.5. Let Z and W be Hilbert A-modules, the map $T : Z \to W$ is called an **adjointable** or **A-adjointable** map from Z to W if there exists another map $T^* : W \to Z$ such that for all $z \in Z$ and $w \in W$:

$$ {}_A\langle Tz, w\rangle = {}_A\langle z, T^* w\rangle .$$

We denote the set of all A-adjointable maps via $\mathcal{L}_A(Z,W)$.

The Hilbert C^*-modules Z, W are Banach spaces, it is then natural to ask if the adjointable maps $\mathcal{L}_A(Z, W)$ has any relation to the space of bounded linear operators $\mathcal{L}(Z, W)$. It turns out that requiring the existence of an adjoint is strong enough to make the adjointables continuous, and even respect the A-module structure of the involved Hilbert C^*-modules.

Proposition 2.7 ([Lan95]). *For any $T \in \mathcal{L}_A(Z, W)$, T is complex-linear, A-linear, and bounded. Also, $\|T^*\| = \|T\|$ (w.r.t. operator norm) for all $T \in \mathcal{L}_A(Z, W)$, and that the adjoint T^* is unique.*

Proof. Note that for any $z \in Z$, $w \in W$, $a \in A$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} {}_A\langle T(\alpha z), w\rangle &= {}_A\langle \alpha z, T^* w\rangle \\ &= \alpha\ {}_A\langle Tz, w\rangle \\ &= {}_A\langle \alpha Tz, w\rangle . \end{aligned}$$

We can similarly show that

$${}_A\langle T(az), w\rangle = {}_A\langle a(Tz), w\rangle$$

Therefore ${}_A\langle T(\alpha z) - \alpha Tz, w\rangle = 0$ and ${}_A\langle T(az) - aTz, w\rangle = 0$, so we know from Proposition 2.1 that $\|t(\alpha x) - \alpha(tx)\| = 0$ and $\|t(xa) - (tx)a\| = 0$. This proves complex-linearity and A-linearity of T.

Next, let Z_1 be the closed unit ball in Z, consider the collection $\{t_z : W \to A \mid z \in Z_1\}$ where

$$t_z(w) := {}_A\langle Tz, w\rangle \qquad \forall w \in W.$$

We have that for any $w \in W$,

$$\sup_{z \in Z_1} \|t_z(w)\| \leqslant \|T^* w\| < \infty.$$

Therefore by Banach-Steinhaus and Proposition 2.1

$$\sup_{z \in Z_1} \|T_z\|_{W \to A} = \sup_{z \in Z_1} \sup_{\|w\| \leqslant 1} \|\ {}_A\langle Tz, w\rangle\ \| = \sup_{z \in Z_1} \|Tz\| = \|T\| < \infty.$$

Therefore $t : Z \to W$ is a bounded operator. We also have

$$\|T^*\| = \sup_{\|w\| \leqslant 1} \|T^* w\| = \sup_{\|w\| \leqslant 1} \sup_{\|z\| \leqslant 1} \|\ {}_A\langle T^* w, z\rangle\ \| = \sup_{\|z\| \leqslant 1} \sup_{\|w\| \leqslant 1} \|\ {}_A\langle Tz, w\rangle\ \| = \|T\|.$$

Note that we can exchange the order of supremums because $\{\| {}_A\langle Tz, w\rangle \| : \|z\|, \|w\| \leqslant 1\}$ is a bounded set.

To show the uniqueness of adjoints, let $T_1^*, T_2^* \in \mathcal{L}_A(Z, W)$ be adjoints of T, then a simple computation shows that $\|T_1^* w - T_2^* w\| = 0$ for all $w \in W$, this proves uniqueness. $\square$

We now know that $\mathcal{L}_A(Z, W) \subseteq \mathcal{L}(Z, W)$, but it turns out that this inclusion is proper, meaning one can construct a non-adjointable (in a Hilbert left A-module sense) map in $\mathcal{L}(Z, W)$ for certain Z, W. For a counter example, see [Pas73].

Let Z, W, Y be Hilbert A-modules, it is easy to see that for $T \in \mathcal{L}_A(Z, W)$ and $S \in \mathcal{L}_A(W, Y)$,

1. $T^* \in \mathcal{L}_A(W, Z)$, and
2. $ST \in \mathcal{L}_A(Z, Y)$.

In particular, $\mathcal{L}(Z) := \mathcal{L}_A(Z, Z)$ is a C^*-algebra because

$$\|T\|^2 \geqslant \|T^*T\| \geqslant \sup_{\|z\| \leqslant 1} \| {}_A\langle T^*Tz, z\rangle \| = \sup_{\|z\| \leqslant 1} \|Tz\|^2 = \|T\|^2$$

Proposition 2.8 ([Lan95]). *If $T \in \mathcal{L}_A(Z, W)$ and $z \in Z$, then $|Tz|^2 \leqslant \|T\|^2 |z|^2$, hence $|Tz| \leqslant \|T\| |z|$.*

Proof. Let $q : A \to \mathbb{C}$ be a state (positive linear functional of norm 1) in A, then $p(\cdot, \cdot) := q({}_A\langle \cdot, \cdot \rangle) : Z \times Z \to \mathbb{C}$ is a positive sesquilinear-form on Z. By repeated application of Proposition 2.3 Item 1, we have for $z \in Z$,

$$\begin{aligned}
p(T^*Tz, z) &\leqslant p(T^*Tz, T^*Tz)^{1/2} p(z, z)^{1/2} \\
&= p((T^*T)^2 z, z)^{1/2} p(z, z)^{1/2} \\
&\leqslant \left[p((T^*T)^2 z, (T^*T)^2 z)^{1/2} p(z, z)^{1/2} \right]^{1/2} p(z, z)^{1/2} \\
&= p((T^*T)^4 z, z)^{\frac{1}{4}} p(z, z)^{\frac{1}{4} + \frac{1}{2}} \\
&\vdots \\
&\leqslant p((T^*T)^{2^n} z, z)^{\frac{1}{2^n}} p(z, z)^{\sum_{i=1}^n \frac{1}{2^n}} \\
&\leqslant \|T^*T\| (\|z\|^2)^{\frac{1}{2^n}} p(z, z)^{1 - \frac{1}{2^{n-1}}}
\end{aligned}$$

As $n \to \infty$, we have $q({}_A\langle T^*Tz, z\rangle) \leqslant \|T\|^2 q({}_A\langle z, z\rangle)$, and because this is true for all states $q : A \to \mathbb{C}$ then by the basic theory of C^*-algebras (GNS theorem), ${}_A\langle Tz, Tz\rangle \leqslant \|T\|^2 \, {}_A\langle z, z\rangle$

which is the same as $|Tz|^2 \leqslant \|T\|^2 |z|^2$ which implies $|Tz| \leqslant \|T\||z|$. ☐

Lemma 2.2 ([Lan95, Lemma 3.1]). *Suppose that Z is a Hilbert A-module, and let $T^* = T \in \mathcal{L}_A(Z)$. If there exists some $k > 0$ where*

$$\|Tz\| \geqslant k\|z\| \qquad \forall z \in Z. \tag{2.6}$$

Then T is invertible in $\mathcal{L}_A(Z)$.

Proof. Our strategy is to prove that $0 \notin \sigma_{\mathcal{L}_A(Z)}(T)$, i.e. 0 is not in the spectrum of T, equivalently T is invertible in $\mathcal{L}_A(Z)$. Suppose then, for contradiction that $0 \in \sigma_{\mathcal{L}_A(Z)}$. Let Id_Z be the identity in $\mathcal{L}_A(Z)$, and consider the C^*-subalgebra of $\mathcal{L}_A(Z)$ generated by Id_Z and T, denoted by $C^*(\mathrm{Id}_Z, T)$. By the continuous functional calculus, there is an isometric $*$-isomorphism $\Psi_T : C(\sigma(T)) \to C^*(\mathrm{Id}, T)$ such that $\Psi_T(\mathrm{id}) = T$ for the identity $\mathrm{id} \in C(\sigma_{\mathcal{L}_A(Z)}(T))$. Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function with the following properties:

1. $f(0) = 1 = \|f\|$;
2. $f(\lambda) = 0$ for all $|\lambda| \geqslant \frac{1}{2}k$.

One can use Urysohn's lemma to construct such a function. Since $0 \in \sigma_{\mathcal{L}_A(Z)}(T)$, then we can consider the restriction $f \in C(\sigma_{\mathcal{L}_A(Z)}(T))$, so that we can write $\Psi_T(f) := f(T) \in C^*(\mathrm{Id}_Z, T)$. Consider the function $g = \mathrm{id} \cdot f \in C(\sigma_{\mathcal{L}_A(Z)}(T))$, and it is not hard to see that $\|g\| \leqslant \frac{1}{2}k$. It follows from the homomorphic property of Ψ_T that $\Psi_T(g) = T\Psi_T(f) = Tf(T)$, and so $\|Tf(T)\| = \|\Psi_T(g)\| = \|g\| \leqslant \frac{1}{2}k$. Choose $z_0 \in Z$ with $\|z_0\| = 1$ and $\frac{1}{2} < \|f(T)z_0\|$. It follows that $\|Tf(T)z_0\| \leqslant \frac{1}{2}k < k\|f(T)z_0\|$, but this contradicts (2.6). Therefore $0 \notin \sigma_{\mathcal{L}_A(Z)}(T)$ as required. ☐

Definition 2.6. Let Z, W be Hilbert A-modules. We say that $U \in \mathcal{L}_A(Z, W)$ is **unitary** if $UU^* = \mathrm{Id}_W$ and $U^*U = \mathrm{Id}_Z$.

Remark 2.2. If $U \in \mathcal{L}_A(Z, W)$ then of course $U : Z \to W$ is a A-linear, surjective, and also isometric. The former two properties are immediate from definition. To show that U is an isometry, note that for $z \in Z$,

$$\|Uz\|^2 = \|{}_A\langle Uz, Uz\rangle\| = \|{}_A\langle z, U^*Uz\rangle\| = \|{}_A\langle z, z\rangle\| = \|z\|^2.$$

What deserves to be called a theorem is the converse, we shall prove that if $T : Z \to A$ is A-linear, surjective, and isometric, then T is unitary. Interestingly, we do not need to assume that $T \in \mathcal{L}_A(Z, W)$. Before we proceed, we note that the A-valued norm for a Hilbert A-module Z satisfies the familiar polarization identity, with similar proof.

Proposition 2.9 (Polarization Identity). *For any $z, w \in Z$,*

$$\begin{aligned} {}_A\langle z, w\rangle &= \frac{1}{4}\left[\left(|z+w|^2 - |z-w|\right) - i\left(|z+iw|^2 - |z-iw|^2\right)\right] \\ &= \frac{1}{4}\sum_{n=0}^{3} |z + i^n w|^2. \end{aligned}$$

As a corollary, for any adjointable map $T \in \mathcal{L}_A(Z)$,

$${}_A\langle z, Tw\rangle = \frac{1}{4}\sum_{n=0}^{3} {}_A\langle z + i^n w, T(z + i^n w)\rangle .$$

Since we know that $\mathcal{L}_A(Z)$ is a C^*-algebra, it would be useful to have a criterion that we can check for positive elements. The result below shows that using the Polarization identity, the criterion for positive operators in Hilbert spaces also hold for adjointable maps on Hilbert C^*-modules.

Proposition 2.10 ([RW98, Lemma 2.28]). *Let T be a complex linear map in Z. Then $T \geqslant 0$ in $\mathcal{L}_A(Z)$ if and only if for all $z \in Z$, ${}_A\langle Tz, z\rangle \geqslant 0$.*

Proof. If $T \geqslant 0$, then there exists an $S \in \mathcal{L}_A(Z)$ such that $T = S^*S$, then for $z \in Z$: ${}_A\langle Tz, z\rangle = {}_A\langle S^*Sz, z\rangle = {}_A\langle Sz, Sz\rangle \geqslant 0$.

Conversely, suppose ${}_A\langle Tz, z\rangle \geqslant 0$ for all $z \in Z$. Then by definition of positivity, we immediately have ${}_A\langle Tz, z\rangle = {}_A\langle z, Tz\rangle \geqslant 0$. It also follows from the Polarization Identity that for all $z, w \in Z$:

$$\begin{aligned} {}_A\langle z, Tw\rangle &= \frac{1}{4}\sum_{n=0}^{3} i^n \; {}_A\langle z + i^n w, T(z + i^n w)\rangle \\ &= \frac{1}{4}\sum_{n=0}^{3} i^n \; {}_A\langle T(z + i^n w), z + i^n w\rangle \\ &= {}_A\langle Tz, w\rangle . \end{aligned}$$

Therefore $T \in \mathcal{L}_A(Z)$ with $T = T^*$. Denote the spectrum of T on $\mathcal{L}_A(Z)$ by, $\sigma_{\mathcal{L}_A(Z)}(T)$, and the C^*-subalgebra generated by the identity Id_Z and T by $C^*(\mathrm{Id}_Z, T)$. Then by the continuous functional calculus, there is an isometric $*$-isomorphism $\Psi_T : C(\sigma(T)) \to C^*(\mathrm{Id}, T)$ such that $\Psi_T(\mathrm{id}) = T$ for the identity $\mathrm{id} \in C(\sigma_{\mathcal{L}_A(Z)}(T))$. Now, we have $f = \frac{\mathrm{id} + |\,\mathrm{id}\,|}{2}$ and $g = \frac{|\mathrm{id}| - \mathrm{id}}{2}$ are both in $C(\sigma_{\mathcal{L}_A}(T))$, with $\mathrm{id} = f - g$ such that $fg = gf = 0$, and $f, g \geqslant 0$. It follows that $T = \Psi_T(f - g)$, and that for all $z \in Z$

$$0 \leqslant {}_A\langle T(\Psi_T(g)z), \Psi_T(g)z\rangle = {}_A\big\langle \Psi_T(f-g)\Psi_T(g)z, \Psi_T(g)z\big\rangle = -\,{}_A\big\langle \Psi_T(g)^3 z, z\big\rangle.$$

Because $\Psi_T(g)^3 \geqslant 0$, it follows from the equation above that ${}_A\big\langle \Psi_T(g)^3 z, z\big\rangle = {}_A\big\langle z, \Psi_T(g)^3 z\big\rangle = 0$ for all z. It follows from the Polarization identity again that for all $z \in Z$:

$$\|\Psi_T(g)^3 z\| = \frac{1}{4}\sum_{n=0}^{3} i^n\, {}_A\big\langle \Psi_T(g)^3 z + i^n z, \Psi_T(g)^3(\Psi_T(g)^3 z + i^n z)\big\rangle = 0.$$

So $\Psi_T(g)^3 = 0$ hence $\Psi_T(g) = 0$, but then $T = \Psi_T(f) \geqslant 0$. □

Going back to unitary maps, the following lemma will be the result that will bridge us from our desired converse.

Lemma 2.3 ([Lan95]). *Suppose $U : Z \to W$ is an A-linear, surjective map that satisfies the pseudo-isometry condition:*

$$|Uz| = |z| \qquad \forall z \in Z$$

then $U \in \mathcal{L}_A(Z, W)$ and is unitary.

Proof. The pseudo-isometry condition along with the polarization identity implies that for any $z, y \in Z$,

$$\begin{aligned} {}_A\langle Uz, Uy\rangle &= \frac{1}{4}\left[\left(|U(z+y)|^2 - |U(z-y)|\right) - i\left(|U(z+iy)|^2 - |U(z-iy)|^2\right)\right] \\ &= \frac{1}{4}\left[\left(|z+y|^2 - |z-y|\right) - i\left(|z+iy|^2 - |z-iy|^2\right)\right] \\ &= {}_A\langle z, y\rangle. \end{aligned}$$

Then, we see that ${}_A\langle Uz, Uy\rangle = {}_A\langle z, y\rangle = {}_A\big\langle z, U^{-1}(Uy)\big\rangle$, since U is surjective, it follows from the uniqueness of the adjoint that $U^* = U^{-1}$. □

Lemma 2.4 ([Lan95, Lemma 3.4]). *Suppose that a, b are positive elements of a C^*-algebra A, and that $\|ac\| = \|bc\|$ for all $c \in A$, then $a = b$.*

Proof. Let us suppose WLOG[2] that a, b are in the closed unit ball of A (and are both nonzero so we exclude the trivial case). Our strategy is to prove that $a^2 = b^2$, since the uniqueness of square roots will imply that $a = b$. Suppose for contradiction that that WLOG $a > b$. Let us further assume for *eventual contradiction*, that $a^2 - b^2 \neq 0$ and WLOG that $a^2 - b^2 > 0$ so that that $\delta = \frac{1}{2}\sup\{\lambda : \lambda \in \sigma_A(a^2 - b^2)\} > 0$.

Consider the (possibly non-unital) functional calculus map wrt $a^2 - b^2$, $\Psi : C_0(\sigma(a^2 - b^2)\backslash\{0\}) \to C^*(\{a^2 - b^2\})$. Let $f \in C_0(\sigma(a^2 - b^2))$ with the following properties:

1. $0 \leqslant f(\lambda) \leqslant 1, \qquad \forall \lambda \in \sigma_A(a^2 - b^2)$
2. $\lambda \leqslant \delta \implies f(\lambda) = 0$
3. $f(2\delta) = 1$.

It is then immediate that $\|f\| = |f(\lambda)| \sup_{\lambda \in \sigma_A(a^2 - b^2)} = 1$. Let $c := \Psi(f) = f(a^2 - b^2) \in A$, then

$$
\begin{aligned}
\|c(a^2 - b^2)c\| &= \|\Psi(f)\Psi(\mathrm{id})\Psi(f)\| \\
&= \|\Psi(f \cdot \mathrm{Id} \cdot f)\| \\
&= \|\lambda \mapsto \lambda f^2(\lambda)\|_{C_0(\sigma(a^2 - b^2)\backslash\{0\})} \qquad \text{(since } \Psi \text{ is an isometry)} \\
&= \sup |\lambda f^2(\lambda)|_{\lambda \in \sigma_A(a^2 - b^2)} \\
&= 2\delta f(2\delta) = 2\delta.
\end{aligned}
$$

Furthermore, by the properties of f, we know that $\lambda f^2(\lambda) > \delta f^2(\lambda)$ for all $\lambda \in \sigma_A(a^2 - b^2)$. Therefore $\mathrm{Id} \cdot f^2 > \delta f^2$ in $C_0(\sigma(a^2 - b^2)\backslash\{0\})$, and because Ψ is an injective $*$-homomorphism, we have that (injective $*$-homomorphisms on C^*-algebras preserve positivity)

$$
\begin{aligned}
\Psi(\mathrm{Id} \cdot f^2) = \Psi(f \cdot \mathrm{Id} \cdot f) \geqslant \Psi(\delta f^2) &\iff \Psi(f)\Psi(\mathrm{Id})\Psi(f) > \delta \Psi(f)^2 \\
&\iff c(a^2 - b^2)c > \delta c^2.
\end{aligned}
$$

Let κ be a state in A such that $\kappa(cb^2c) = \|cb^2c\|$ (existence is guaranteed by considering pure states). Because $\|b\| \leqslant 1$, then it follows from (2.1) that $cb^2c \leqslant \|b\|^2 c^2 \leqslant c^2$, which implies that

$$
\begin{aligned}
&\kappa(cb^2c) \leqslant \kappa(c^2) \\
\implies &\kappa(ca^2c) \geqslant \kappa(cb^2c + \delta c^2) \geqslant (1 + \delta)\kappa(cb^2c).
\end{aligned}
$$

[2] Otherwise we can divide a and b by their norms.

It follows that $\|ca^2c\| \geqslant (1+\delta)\|cb^2c\|$. Therefore, if $cb^2c \neq 0$, then $\|ca^2c\| > \|cb^2c\|$. On the other hand, if $cb^2c = 0$, then $2\delta = \|c(a^2-b^2)c\| = \|ca^2c\|$. In either case, $\|ca^2c\| > \|cb^2c\| \iff \|(ac)^*(ac)\| > \|(bc)^*(bc)\| \iff \|ac\|^2 > \|bc\|^2 \implies \|ac\| > \|bc\|$, which is a contradiction. Therefore $a^2 = b^2$ as required. $\square$

Theorem 2.2 ([Lan95, Theorem 3.5])**.** *Let A be a C^*-algebra, let Z, W be Hilbert A-modules and let $U : Z \to W$ be a linear map, then TFAE:*

1. *U is A-linear, surjective, and isometric;*
2. *U is a unitary element of $\mathcal{L}_A(Z, W)$.*

Proof. That 2 implies 1 follows from the remark below Definition 2.6. For the converse, it suffices to prove that the isometry property follows from the pseudo-isometry property (Lemma 2.3). To do this, let $c \in A$, $z \in Z$, and recall that $|Uz| = {}_A\langle Uz, Uz\rangle^{1/2} \in A$. We have the following computation:

$$
\begin{aligned}
\|a|Uz|\| &= \|(a|Uz|)(a|Uz|)^*\|^{1/2} \\
&= \|a\, {}_A\langle Uz, Uz\rangle\, a^*\|^{1/2} \\
&= \|\, {}_A\langle U(az), U(az)\rangle\, \|^{1/2}, \qquad (A\text{-linearity}) \\
&= \|U(az)\|^{1/2} \\
&= \|az\|^{1/2}, \qquad (\text{Isometry}) \\
&= \|a|z|\|.
\end{aligned}
$$

Because the equation above is true for all $a \in A$ and, $|Uz|$ and $|z|$ are both positive in A, then by Lemma 2.4, $|Uz| = |z|$ as required. $\square$

Let Z, W be Hilbert A-modules and $z \in Z$, $w \in W$, then we define an analogous **rank-one operator** for $\mathcal{L}_A(W, Z)$:

$$\theta_{w,z}(y) = {}_A\langle y, w\rangle z \qquad \forall y \in W. \tag{2.7}$$

Now if $u \in Z$, then

$${}_A\langle \theta_{w,z}(y), u\rangle = {}_A\langle {}_A\langle y, w\rangle z, u\rangle$$

$$
\begin{aligned}
&= {}_A\langle y, w\rangle\, {}_A\langle z, u\rangle \\
&= {}_A\langle y, {}_A\langle u, z\rangle\, w\rangle \\
&= {}_A\langle y, \theta_{z,w}(u)\rangle .
\end{aligned}
$$

Therefore $\theta_{w,z} \in \mathcal{L}_A(W, Z)$ indeed, with adjoint $\theta^*_{w,z} = \theta_{z,w}$. We have the following formulae for rank-one adjointables, they all follow from straightforward computations as in above.

Proposition 2.11. *Let Z, W, Y be Hilbert A-modules, $z \in Z$, and $w \in W$, then*

$$
\begin{aligned}
T\theta_{w,z} &= \theta_{w,Tz}, \qquad \forall T \in \mathcal{L}_A(Z, Y); \\
\theta_{w,z}S &= \theta_{S^*w,z}, \qquad \forall S \in \mathcal{L}_A(Y, W).
\end{aligned} \tag{2.8}
$$

Definition 2.7. Let Z, W be Hilbert A-modules, we denote by $\mathcal{K}_A(W, Z)$ the closed linear span of the subset $\{\theta_{w,z} : z \in Z, w \in W\}$ in $\mathcal{K}_A(W, Z)$. We also define $\mathcal{K}(Z) := \mathcal{K}_A(Z, Z)$. When we are talking about adjointable operators with respect to Hilbert A-modules, we call $\mathcal{K}_A(W, Z)$ the **compact operators** from W to Z.

Corollary 2.2. *$\mathcal{K}_A(Z)$ is a closed two-sided ideal of the C^*-algebra $\mathcal{L}_A(Z)$.*

Proof. It follows directly from 2.11. □

Obviously, the model for the definition above are the actual compact operators in a Hilbert space, which can be characterized as the limit of finite-rank operators. The characterization, unfortunately, does not carry over perfectly. There are operators $T \in \mathcal{K}_A(W, Z)$ that are not compact operators from W to Z when they are seen as Banach spaces (recall that compact operators can be defined for Banach spaces as well, not just Hilbert space).

Recall from Example 2.1 that by taking $|I| = 1$, any C^*-algebra A is also a Hilbert C^*-module over itself. The module A-action is just given by the usual multiplication, while the module inner-product is given by ${}_A\langle a, b\rangle = ab^*$.

Proposition 2.12 ([RW98, Example 2.26])**.** $A \cong \mathcal{K}_A(A)$.

Proof. For each $a \in A$, define the operator $R_a : A \to A$, via $R_a(b) = ba$ for all $b \in B$. We see that $R_a \in \mathcal{L}_A(A)$ when we identify A as a *left* Hilbert A-module, with adjoint $R^*_a = R_{a^*}$. Note that $\left\| R_a\left(\frac{a^*}{\|a\|}\right)\right\| = \|a\| \leqslant \|R_a\| \leqslant \|a\|$, therefore $\|R_a\| = \|a\|$, therefore the map

$$
R : a \mapsto R_a \tag{2.9}
$$

is an isometry from A to $\mathcal{L}_A(A)$. It follows by elementary computations that $a \mapsto R_a$ is a $*$-homomorphism, and so all that remains is to show that the image of the mapping is actually $\mathcal{K}_A(A)$. Note that if we take the product $ab \in A$, then for all $c \in A$

$$R_{ab}(c) = cab = {}_A\langle c, a^* \rangle b = \theta_{a^*,b}(c) \in \mathcal{K}_A(A).$$

However, one can always write any element of a C^*-algebra as a closed linear span of products like a^*b by the existence of approximate identities. Therefore the image of $a \mapsto R_a$ is indeeed $\mathcal{K}_A(A)$. ☐

Proposition 2.13. *If A is unital, then $\mathcal{K}_A(A) \cong \mathcal{L}_A(A)$.*

Proof. This follows from the proof Proposition 2.12. The $*$-isometric homomorphism $a \mapsto R_a$ becomes an isomorphism to $\mathcal{L}_A(A)$ because for all $a, c \in A$

$$R_a(c) = R_{a \cdot 1_A}(c) = \theta_{a^*,1_A}(c).$$

☐

Let A be a unital C^*-algebra, and if we again consider A as a Hilbert left A-module, then we see that the identity can be written as a "compact operator" $\mathrm{Id}_A = \theta_{1_A,1_A} \in \mathcal{K}_A(A)$, but it is not an actual compact operator in $\mathcal{L}(A)$ unless A is finite-dimensional. This example can also be seen in [Lan95], which shows that compact operators in $\mathcal{K}_A(W,Z)$ are far from compact operators in $\mathcal{L}(W,Z)$.

Let us describe a different kind of topology for the adjointables $\mathcal{L}_A(Z,W)$, similar to the strong operator topology on bounded linear operators on Hilbert spaces. Let us first recall the notion seminorm topology.

Definition 2.8. Let V be vector space of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\mathscr{P}$ a family of semi-norms on V. Consider the family of subsets $S_{\mathscr{P}}$ in V where:

$$S_{\mathscr{P}} := \bigcup_{(p,x,\varepsilon) \in \mathscr{P} \times V \times \mathbb{R}_{>0}} \{v \in V : p(v - x) < \varepsilon\}.$$

Then $S_{\mathscr{P}}$ is a subbase of V, and the topology generated by $S_{\mathscr{P}}$ is called the **semi-norm topology corresponding to** $\mathscr{P}$. We also call any topological vector spaces whose topology is generated by a family of seminorms **locally convex spaces**.

Remark 2.3. Note that a net $\{x_i\}_i$ in V converges in the seminorm topology with respect to the seminorms $\mathscr{P}$ to x if and only if for all $p \in \mathscr{P}$: $p(x_i - x) \to 0$.

Definition 2.9. Let Z, W be Hilbert A-modules, the **$*$-strong topology** on $\mathcal{L}_A(Z, W)$ is the seminorm topology generated by the following family of seminorms:

$$\{T \mapsto \|Tz\| : z \in Z\} \cup \{T \mapsto \|T^* w\| : w \in W\}.$$

Remark 2.4. It follows that a net $\{T_i\}_i$ in $\mathcal{L}_A(Z, W)$ **converges $*$-strongly** to T if and only if for all $(z, w) \in Z \times W$: $T_i z \to Tz$ and $T_i^* w \to T^* w$.

Proposition 2.14 ([Lan95, Proposition 1.3])**.** *Let Z, W be Hilbert A-modules, then $\mathcal{K}_A(Z, W)$ is $*$-strongly dense in $\mathcal{L}_A(Z, W)$.*

Proof. To start, let $\{E_i\}_i$ be an approximate identity for the ideal $\mathcal{K}_A(Z, W)$, we shall first prove that for any $z \in Z$

$$\lim_i E_i z = z.$$

We do this by showing that it's true for the dense subset ${}_A\langle Z, Z\rangle Z$ of Z. Suppose $z = {}_A\langle u, v\rangle w \in {}_A\langle Z, Z\rangle Z$, then

$$E_i z = E_\lambda({}_A\langle u, v\rangle w) = (E_i \circ \theta_{v,w})(u) \to \theta_{v,w}(u) = z$$

as required. Now, let $T \in \mathcal{L}_A(Z, W)$, then

$$\|(TE_i - T)z\| \leqslant \|T\| \|E_i z - z\| \to 0, \tag{2.10}$$

which is true for all $z \in Z$, this shows that $\{TE_i\}_i$ converges to T in the seminorms $\{T \mapsto \|Tz\| : z \in Z\}$. For the other half of the family of the seminorms, we can deduce this from (2.10). Let $w \in W$, take $z' = (E_i T^* - T^*)w \in Z$

$$\|(E_i T^* - T^*)w\|^2 = {}_A\langle (TE_i - T)(E_i T^* - T^*)w, w\rangle = {}_A\langle (TE_i - T)z', w\rangle \to 0$$

Therefore $\|(E_i T^* - T^*)w\| \to 0$ for all $w \in W$. Hence for any $T \in \mathcal{L}_A(Z, W)$, a net $\{TE_i\}_i \subseteq \mathcal{K}_A(Z, W)$ exists that converges to T. Therefore by net density criterion B.6, the $*$-strong closure of $\mathcal{K}_A(Z, W)$ is $\mathcal{L}_A(Z, W)$. $\square$

We emphasize now that Hilbert C^*-modules are also bonafide modules, and so we can borrow from the basic theory of modules the following notions.

Definition 2.10. We say that a Hilbert A-module Z is **finitely generated** if there exist finite elements, say $\{z_i\}_{i=1}^n \subseteq Z$, such that for all $z \in Z$, there exist $\{a_i\}_{i=1}^n \subseteq A$ where

$$z = a_1 z_1 + a_2 z_2 + ... + a_n z_n.$$

Definition 2.11. Let Z be a Hilbert A-module. Denote by $\mathrm{Hom}_A(Z, A)$ the class of all continuous (i.e. bounded) A-linear maps from Z to A (not necessarily adjointable), then Z is said to be **projective** if there exist an index set I, families $\{w_i\}_{i \in I} \subseteq Z$, and $\{\varphi\}_{i \in I} \subseteq \mathrm{Hom}_A(Z, A)$ such that for all $z \in Z$, $\varphi_i(z) = 0$ for all but finitely many $i \in I$ and

$$z = \sum_{i \in I} \varphi_i(z) w_i.$$

Remark 2.5. Note that when $|I| < \infty$ in the definition above, then Z is both finitely generated and projective, i.e. **finitely generated projective**.

The space $\mathrm{Hom}_A(Z, A)$ takes on the role of continuous linear functionals for Hilbert C^*-modules. In light of this we have the following definition.

Definition 2.12. A Hilbert A-module Z is said to be **self-dual** if for each $\varphi \in \mathrm{Hom}_A(Z, A)$, there exists a $w \in Z$ such that $\varphi(z) = {}_A\langle z, w \rangle$ for all $z \in Z$.

2.2 The Multiplier Algebra

As one studies the theory of C^*-algebras (or even just Banach $*$-algebras), one quickly finds that the notion of adjoining an identity to a nonunital algebra is an important idea (for example, we need unitality to define the spectrum of an element). The usual way of adjoining an identity to a C^*-algebra A is by defining $A^1 := A \oplus \mathbb{C}$ with the multiplication $(a_1, \lambda_1)(a_2, \lambda_2) = (a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2)$, and with involution $(a, \lambda)^* = (a^*, \overline{\lambda})$. We see that $(0, 1)$ is the identity in A^1. We can define the C^*-norm for A^1 using the language of Hilbert C^*-modules: we can extend the $R : A \to \mathcal{K}_A(A)$ map, to $R^1 : A^1 \to \mathcal{L}_A(A)$. R^1 will be a $*$-isometric homomorphism on the unitization A^1 by sending $(a, \lambda) \mapsto R_a + \lambda \, \mathrm{Id} \in \mathcal{L}_A(A)$. The range of this extended map R^1 is

$\mathbb{C}\,\mathrm{Id} + \mathcal{K}_A(A) \subseteq \mathcal{L}_A(A)$, and of course it remains to show that this is closed (hence a C^*-algebra). A proof is supplied in [RW98, Example 2.41]. Once we have done all the necessary verification, we have A^1 with the norm $\|(a,\lambda)\| := \|R^1(a,\lambda)\|$ is a unital C^*-algebra with $\|a\| = \|(a,0)\| = \|R^1(a,0)\| = \|R_a\|$, so we can also see that $A \cong \mathcal{K}_A(A) \hookrightarrow \mathbb{C}\,\mathrm{Id} + \mathcal{K}_A(A) \cong A^1$. Therefore A embeds as an ideal in A^1 (recall that $\mathcal{K}_A(A)$ is an ideal in $\mathcal{L}_A(A)$). This embedding turns out to be a *minimal* because of the following result.

Theorem 2.3 ([SP21, 2.2.4])**.** *Let A be a non-unital C^*-algebra with unitization A^1, and B another unital C^*-algebra. Every $*$-homomorphism $\varphi : A \to B$ extends to a unique unital $*$-homomorphism $\varphi^1 : A^1 \to B$ via $\varphi^1((a,\lambda)) := \varphi(a) + \lambda 1_B$. If φ is injective, then so is φ^1.*

A corollary of the result above is that: if there is any embedding of A to some unital C^*-algebra B, say via $\iota : A \to B$, then it is always that case that A^1 embeds in B as well (via ι^1). So in this sense A^1 is the *minimal* unitization for A because it is always embedded in every unital C^*-algebra that A embeds to.

We have recollected the minimal unitization result for A since it motivates the notion of *maximal* unitization for A called the *multiplier algebra*, which is also an embedding of A as an ideal to a unital C^*-algebra. As noted by Lance [Lan95], one has to be very careful with how we define our poset, i.e. requiring *essentiality* of A as an embedded ideal to some unital C^*-algebra, so that we indeed have an actual maximal unitization. Note that throughout this document, when we say that I is some ideal of a C^*-algebra, what we mean is a closed two-sided ideal. We shall keep this standing assumption unless otherwise stated.

Definition 2.13. An ideal I in a C^*-algebra A is *essential* if I has nonzero intersection with every other nonzero ideals of A.

Let X, Y be subsets of a C^*-algebra A, then we define XY to be the linear span of elements of the form xy, where $(x,y) \in X \times Y$. For the closed linear span, we have $[XY] := \overline{XY}$.

Proposition 2.15. *Let I, J be closed ideals of a C^*-algebra A, then $[IJ] = I \cap J$.*

Proof. The fact that $[IJ] \subseteq I \cap J$ is immediate. It is sufficient to show that if $a \in I \cap J$, then $a \in \overline{IJ}$ for the specific case when a is positive, since every element can be written as a linear combination of positive elements (follows from [Mur90, 2.2.2] and from the fact that all elements in a C^*-algebra can be written is in the linear span of self-adjoint elements). Since the intersection of ideals is itself an ideal, and ideals are C^*-algebras themselves, then $a \in (I \cap J)^+$ implies that

$a^{1/2} \in (I \cap J)$ as well. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity for I, then $a = \lim_\lambda (a^{1/2}e_\lambda)a^{1/2}$, since $a^{1/2}e_\lambda \in I$ for all $\lambda \in \Lambda$ and $a^{1/2} \in J$, then $a \in [IJ]$. $\square$

Proposition 2.16 ([RW98, Lemma 2.36])**.** *Let I be an ideal of A. The following are equivalent:*

1. *I is an essential ideal of A;*

2. *For all $a \in A$, $aI = \{ab : b \in I\} = \{0\} \implies a = 0$.*

Proof. ($\implies$). Suppose I is an essential ideal of A. Let $a \in A$ such that $aI = \{0\}$, and consider the ideal generated by a, $J_a := \overline{span} AaA = \overline{span}\{bac : b, c \in A\}$. Observe that $aI = \{0\}$ if and only if $I \cap J_a = [IJ_a] = \{0\}$. Therefore by hypothesis, $I \cap J_a = \{0\}$, hence by essentiality of I, it must be that $J_a = 0$, but this implies $a = 0$, as required.
($\impliedby$). Conversely, assume $aI = \{ab : b \in I\} = \{0\} \implies a = 0$ for all $a \in A$. Let J be an ideal of A such that $J \cap I = [IJ] = \{0\}$, then for all $j \in J$, $jI = \{0\}$. By hypothesis, $j = 0$, therefore $J = \{0\}$, whence I is essential. $\square$

Definition 2.14. Let A be a C^*-algebra, a unital C^*-algebra B is said to be a **unitization** of A if there exists an injective $*$-homomorphism $\iota_B : A \to B$ (i.e. an embedding) such that $\iota_B(A)$ is an essential ideal in B.

Remark 2.6. This definition captures the idea that there are no nontrivial unitizations for a C^*-algebra that is already unital. Suppose A is a unital C^*-algebra, then the identity map is one unitization, and we cannot find anything better than this. Suppose there exists a unital C^*-algebra B containing A as an ideal, with $B\backslash A \neq \varnothing$. Choose $b \in B\backslash A$, then $b1_A \in A$ and $b - b1_A \neq 0$ (otherwise $b \in A$), but then we have $(b - b1_A)A = 0$, so A cannot be an essential ideal in B.

With the language of unitization in place, we can finally discuss what we mean by **maximal** unitization. Let B_1, B_2 be unital C^*-algebras which are unitizations of A, we say that $B_1 \leqslant B_2$ if and only if there exists an injective $*$-homomorphism $\phi : B_1 \to B_2$ such that the following diagram commutes

$$\begin{tikzcd} & B_1 \arrow[d, "\phi"] \\ A \arrow[ur, "\iota_{B_1}"] \arrow[r, "\iota_{B_2}"] & B_2 \end{tikzcd}$$

This definition is consistent with what we already know about A^1 for a nonunital A.

Proposition 2.17. *If A is a nonunital C^*-algebra, then $\iota : A \hookrightarrow A^1$ via $\iota(a) = (a, 0)$ is a minimal unitization of A.*

Proof. We only really need to check that $\iota(A)$ is an essential ideal in A^1. We can show this by using Item 2 of Proposition 2.16. Let $(b, \lambda) \in A^1$ such that $(b, \lambda)\iota(A) = \{0\}$, equivalently, $(b, \lambda)(a, 0) = 0$ for all $a \in A$. In the case, $\lambda = 0$, we obtain $ba = 0$ for all $a \in A$, and therefore $bb^* = 0$ in particular, which means $\|bb^*\| = \|b\|^2 = 0$, hence $b = 0$. Otherwise, if $\lambda \neq 0$, then we obtain that $(b, \lambda)(a, 0) = 0 \implies -\frac{1}{\lambda}ba = a$ for all $a \in A$. By involutions in the last equality, $a(-\frac{1}{\lambda}b)^* = a$ for all $a \in A$ as well. However, the two last equalities show that (take $a = -\frac{1}{\lambda}b$ in the last equality, and take $a = (-\frac{1}{\lambda}b)^*$ in the preceding one)

$$-\frac{1}{\lambda}b = \left(-\frac{1}{\lambda}b\right)\left(-\frac{1}{\lambda}b\right)^* = \left(-\frac{1}{\lambda}b\right)^*.$$

This means for all $a \in A$, $(-\frac{1}{\lambda}b)a = a(-\frac{1}{\lambda}b)$, so $\frac{1}{\lambda}b$ is an identity in A, which is a contradiction. Therefore $\lambda = 0$, and we are back in the first case where we have shown that $b = 0$, i.e. it must be that $(b, \lambda) = 0$ as required. The minimality of the unitization follows from the paragraph after Theorem 2.3. □

Proposition 2.18 ([RW98, Example 2.43])**.** *Let A be a nonunital C^*-algebra, the map $R : A \to \mathcal{L}_A(A)$ defines a unitization of A.*

Proof. We already know that $R(A) = \mathcal{K}_A(A)$, so all that remains is to show that the ideal of compact operators $\mathcal{K}_A(A)$ is essential in $\mathcal{L}_A(A)$. Let $T \in \mathcal{L}_A(A)$ such that $T\theta_{a,b} = 0$ for all $a, b \in A$ (equivalently $TK = 0$ for all $K \in \mathcal{K}_A(A)$). This means that for all $a, b, c \in A$:

$$(T \circ \theta_{a,b})(c) = {}_A\langle c, a\rangle Tb = 0. \tag{2.11}$$

Let $\{e_i\}_i$ be an approximate identity in the closed two-sided ideal $\overline{{}_A\langle A, A\rangle}$, then the condition (2.11) implies that for any $c \in A$

$$Tc = \lim_i e_i Tc = 0,$$

therefore $T = 0$ as required. □

Of course we do not know yet where the unitization $R : A \hookrightarrow \mathcal{L}_A(A)$ lands in the poset of all unitizations for A. Our big claim here is that it is the *unique maximal* unitization for A.

Definition 2.15. Let A be a C^*-algebra, the multiplier algebra of A, denoted $M(A)$, is defined to be $M(A) := \mathcal{L}_A(A)$.

Remark 2.7. In most references (e.g. [RW98],[Lan95],[SP21]), there is talk of *minimal* and *maximal* unitizations, but interestingly, what seems to get proven is that in the poset that we have just defined, A^1 is not just a minimal unitization, but a least element in the poset. Meaning, it is comparable to all other unitization, and is less than all of them (recall there is a difference between least/greatest element and minimal/maximal element, but the former implies the latter). The proof for the multiplier algebra, i.e. the "maximal" unitization, is actually a proof of existence of the greatest element in our poset.

Note that the crucial part of the proof for minimality of A^1 is Theorem 2.3, our goal is to give an analogue of this for the multiplier algebra $M(A)$, but for this we need the following definition.

Definition 2.16. Let Z be a Hilbert A-module, and B a C^*-algebra, a $*$-homomorphism $\alpha : B \to \mathcal{L}_A(Z)$ is called **nondegenerate** if

$$[\alpha(B)Z] = \overline{\operatorname{span}}\{\alpha(b)z : b \in B, z \in Z\} = Z.$$

That is, $\operatorname{span}\{\alpha(b)z : b \in B, z \in Z\}$ is dense in Z.

Note that this definition of nondegeneracy is consistent with the definition for representations on C^*-algebras. Indeed, this is because if H is a Hilbert space, then it is a Hilbert C^*-module over the C^*-algebra of complex number $\mathbb{C}$, and $\mathcal{L}_{\mathbb{C}}(H) = \mathcal{L}(H)$.

Proposition 2.19. *Let A be a C^*-algebra, then the embedding $R : A \to \mathcal{K}_A(A) \subseteq \mathcal{L}_A(A)$ is a nondegenerate $*$-homomorphism.*

Proof. Writing out the $*$-strong density of $\mathcal{K}_A(A)$ in $\mathcal{L}_A(A)$ (Proposition 2.14) gives us our desired result. □

Proposition 2.20. *Let A, J, C be C^*-algebras, such that J is an ideal of A, and let Z be a Hilbert C-module. Suppose $\alpha : J \to \mathcal{L}_C(Z)$ is a nondegenerate $*$-homomorphism. Then α extends uniquely to a $*$-homomorphism $\overline{\alpha} : A \to \mathcal{L}_C(Z)$. If α is injective and J is essential in A then $\overline{\alpha}$ is injective.*

Proof. The extension statement above is a rehash of the extension statement for nondegenerate representations of C^*-algebra, and in fact can be proven almost verbatim. We will only prove the statement for injectivity of the extension whenever J is as essential ideal of A.

Suppose that J is essential in A, and that $\alpha : J \to \mathcal{L}_C(Z)$ is injective. We shall show that $\mathrm{Ker}(\overline{\alpha})$ is the trivial ideal of A. Recall that the extension of α is defined using the nondegeneracy of it on J : we have on the dense subset $\alpha(J)Z$, $\overline{\alpha}(a)(\alpha(j)z) = \alpha(aj)z$ for all $a \in A$, $j \in J$ and $z \in Z$. Therefore if $a \in \mathrm{Ker}(\overline{\alpha})$, then $\overline{\alpha}(a)[\alpha(J)Z] = \{0\}$. This implies that for all $j \in J$, $\alpha(aj) = 0$, and by injectivity of α, we have $aj = 0$. Therefore $[\mathrm{Ker}(\overline{\alpha})J] = \{0\}$, so by essentiality of J, $\mathrm{Ker}(\overline{\alpha}) = \{0\}$, hence $\overline{\alpha}$ is injective. □

We have the following main result. Note that in keeping with the literature, by 'maximal' what we really mean is 'greatest element'.

Corollary 2.3. *Let A be a C^*-algebra, then*

1. *The embedding $R : A \to \mathcal{L}_A(A)$ is a maximal unitization;*
2. *$\mathcal{L}_A(A)$ is the unique maximal unitization of A, in the sense that if B is another maximal unitization, then $B \cong \mathcal{L}_A(A)$.*

Proof.

1. We already know that $R : A \to \mathcal{L}_A(A)$ is a unitization, what remains is to prove that it is maximal. We also know from Proposition 2.19 that R is a nondegenerate $*$-homomorphism. Now, let $\alpha : A \to \mathscr{A}$ be another unitization of A, so that by definition α is injective with $\alpha(A)$ an essential ideal in $\mathscr{A}$. We can then define the map $R \circ \alpha^{-1} : \alpha(A) \to \mathcal{L}_A(A)$, but then $R \circ \alpha^{-1}$ must be a nondegenerate injective $*$-homomorphism. Furthermore, $\alpha(A)$ is an essential ideal in $\mathscr{A}$, and so by Proposition 2.20, we have an injective extension $\overline{R \circ \alpha^{-1}} : \mathscr{A} \to \mathcal{L}_A(A)$. Hence we have constructed a $*$-homomorphism satisfying the following commutative diagram:

$$\begin{tikzcd} & \mathscr{A} \arrow[d, "\overline{R\circ\alpha^{-1}}"] \\ A \arrow[ur, "\alpha"] \arrow[r, "R"] & \mathcal{L}_A(A). \end{tikzcd}$$

2. What we have shown is that $\mathcal{L}_A(A)$ is *the* greatest element in the poset of unitizations of A, and so it must be unique. However, to see the explicit isomorphism, let $\alpha : A \to B$ be another maximal (greatest) unitization of A, then it must be that there exists an injective $*$-homomorphism such that $\rho : \mathcal{L}_A(A) \to B$ with its own commutative diagram showing

$\mathcal{L}_A(A) \leqslant B$. However we can repeat what we have done in Item 1 above, and conclude that there is an injective $*$-homomorphism such that $\overline{R \circ \alpha^{-1}} : B \to \mathcal{L}_A(A)$ satisfying the commutative diagram above (replace $\mathscr{A}$ with B). A rudimentary computation shows that $\overline{R \circ \alpha^{-1}} \circ \rho$ is the identity map, and therefore $B \cong \mathcal{L}_A(A)$.

□

This is our main theorem for this section, justifying why we can call $M(A) := \mathcal{L}_A(A)$ *the* multiplier algebra, making it the unique maximal unitization for A.

Corollary 2.4. *Let A, B be C^*-algebras. Let $\phi : B \to M(A) := \mathcal{L}_A(A)$ be a nondegenerate $*$-homomorphism, then it extends uniquely to $\overline{\phi} : M(B) \to M(A)$.*

Proof. Consider the unitization $R : B \to \mathcal{L}_B(B) = M(B)$, so that B is embedded as a dense ideal to $M(B)$. So that we can identify the mapping ϕ to a mapping $\phi : R(B) \to M(A)$, we apply the extension result of Proposition 2.20 to this map. □

Corollary 2.5. *If A is unital then $A \cong M(A)$.*

Proof. This follows from the result $A \cong \mathcal{L}_A(A)$ when A is unital (Proposition 2.13). □

The next result will allow us to get a more concrete realization of the multiplier algebra. We first need the notion of an idealiser.

Definition 2.17. Let A be a C^*-algebra, with subalgebras $C \subseteq B \subseteq A$. We say that B_C is the **idealiser** of C in B, defined

$$B_C := \{b \in B : bC \subseteq C \text{ and } Cb \subseteq C\}.$$

Note that the idealiser of C in B is the largest ideal of B containing C.

Theorem 2.4. *Let A, C be C^*-algebras and Z be a Hilbert C-module. Suppose that $\alpha : A \to \mathcal{L}_C(Z)$ is a nondegenerate injective $*$-homomorphism and let B be the idealiser of $\alpha(A)$ in $\mathcal{L}_C(Z)$, that is:*

$$B := \{T \in \mathcal{L}_C(Z) : T\alpha(A) \subseteq \alpha(A) \textit{ and } \alpha(A)T \subseteq \alpha(A)\}.$$

Then $B \cong M(A)$.

Proof. Note that $\alpha(A) \subseteq B$, and in-fact, $\alpha(A)$ is an ideal of B, this is pretty much all what is needed (we used the definition of B here). Our strategy is to first show that A embeds as an essential ideal in B via α. To do this, we only need to show that $\alpha(A)$ is an essential ideal of B. Let $T \in B$, and suppose $T\alpha(A) = \{0\}$, this means that $T\alpha(A)Z = \{0\}$, hence by nondegeneracy of α, $T[\alpha(A)Z] = T(Z) = \{0\}$, so $T = 0$. Therefore $\alpha(A)$ is indeed an essential ideal of B. What we have shown is that $\alpha : A \to \alpha(A) \subseteq B$ is a unitization of A, we are done if we can show that this is maximal.

Let $\alpha_D : A \to D$ be another unitization of A. We will pretty much use the same technique as in Corollary 2.3. Via Proposition 2.20, we have a nondegenerate injective $*$-homorphism $\alpha \circ \alpha_D^{-1} : \alpha(A) \to B$, since $\alpha_D(A)$ is an essential ideal in D, then it extends to an injective $*$-homomorphism $\overline{\alpha \circ \alpha_D^{-1}} : D \to B$, where $(\overline{\alpha \circ \alpha_D^{-1}})(\alpha_D(a)) = \alpha(a)$, so $D \leqslant B$. Therefore B is maximal, as required. □

The theorem says that simply by existence of a nondegenerate injective $*$-homomorphism $\alpha : A \to \mathcal{L}_C(Z)$, we can compute the multiplier algebra of A.

Theorem 2.5. *If Z is a Hilbert A-module, then $\mathcal{L}_A(Z) \cong M(\mathcal{K}_A(Z))$.*

Proof. We know that $\mathcal{K}_A(Z)$ is a (closed two-sided) ideal of the C^*-algebra $\mathcal{L}_A(Z)$ and that the inclusion map $\iota : \mathcal{K}_A(Z) \to \mathcal{L}_A(Z)$ is a nondegenerate injective $*$-homomorphism by Proposition 2.14. Therefore by Theorem 2.4, if

$$B := \{T \in \mathcal{L}_A(Z) : T\iota(\mathcal{K}_A(Z)) \subseteq \iota(\mathcal{K}_A(Z)) \text{ and } \iota(\mathcal{K}_A(Z))T \subseteq \iota\mathcal{K}_A(Z)\},$$

then $B \cong M(\mathcal{K}_A(Z))$. However, $B = \mathcal{L}_A(Z)$ so the result follows. □

Example 2.3. If we take the Hilbert space case, $Z = H$, then we see that $M(\mathcal{K}(H)) \cong \mathcal{L}(H)$.

The notion of multiplier algebras can be used to generalize the notion of 2-cocycles for C^*-algebras, which in turn allows us to define twisted C^*-dynamical systems. See Section 5.1 of Chapter 5.

2.3 Imprimitivity Bimodules

We may have failed to recognize the elephant in the room. So far we have been discussing Hilbert *left* C^*-modules, but there must be a notion of a Hilbert *right* C^*-module, no? Indeed it is straightforward to define one, but we need to be careful and precisely identify the differences in these two related structures. We give an explicit definition below.

Definition 2.18. Let B be a C^*-algebra and Z a complex vector space. Z is said to be an **inner-product right B-module** if it is a right B-module compatible with complex scalar multiplication in the sense that:

$$\lambda(z \cdot b) = z \cdot (\lambda b) = (\lambda z) \cdot b.$$

Furthermore, there exists $\langle \cdot, \cdot \rangle_B : Z \times Z \to B$ such that for all $z, w, y \in Z$, $b \in B$, and scalars $\lambda, \mu \in \mathbb{C}$:

1. $\langle w, \lambda z + \mu y \rangle_B = \lambda \langle w, z \rangle_B + \mu \langle w, y \rangle_B$

2. $\langle w, z \cdot b \rangle_B = \langle w, z \rangle_B \cdot b$

3. $\langle w, z \rangle_B^* = \langle z, w \rangle_B$

4. $\langle z, z \rangle_B \geqslant 0$ and $\langle z, z \rangle_B = 0 \implies z = 0$

A **B-submodule** of Z is a subvector space of Z that is closed with respect to the B-action.

An important distinction of inner-product left modules from the inner-product right modules is the fact that the module inner-product for the latter are linear in the second component, while antilinear in the first component. Furthermore, all of the results that we have from the previous two sections can be modified *mutatis mutandis* to give a Hilbert right C^*-module versions of themselves. We can also induce a norm on inner-product right B-modules, and we can also define Hilbert right B-modules as those where the induced norm is complete. Adjointable maps can also be defined similarly, and for $b_1, b_2 \in B$, one can also define the basic rank-one adjointable operators $\theta_{b_1,b_2} : Z \to Z$ via $\theta_{b_2,b_1}(z) = b_2 \langle b_1, z \rangle_B$. In the literature, the notation ${}_AZ$ and Z_B are sometimes used to denote that Z is a inner-product A module or an inner-product B module respectively. We will adapt this convention only when it is needed to be emphasized. Otherwise we will strive to keep the presentation of the modules as uncluttered as possible. For a fixed C^*-algebra A, it is also a right Hilbert A-module over itself by proper modifications of Example 2.1.

Example 2.4. Define

$$\ell^2_A(I) := \left\{ \mathbf{a} = (a_i)_{i\in I} \subseteq A : \sum_{i\in I} a_i a_i^* \text{ converges in the norm of } A \right\}$$

Equipping $\ell^2_A(I)$ with the obvious right A action and inner-product:

1. $\mathbf{a} \cdot a := (a_i a)_{i\in I}$;

2. $\langle \mathbf{a}, \mathbf{a}' \rangle_A = \sum_{i\in I} (a_i)^* a_i'$.

As usual, we get A as a Hilbert right A-module in the case $|I| = 1$.

We can then see that the mapping $L : A \to \mathcal{L}_A(A_A)$, with $L : a \mapsto L_a$ and $L_a(c) = ac$ is the same map that shows A is isomorphic to the compact adjointables $\mathcal{K}_A(A_A)$. The upshot of all this is that we could have also defined the multiplier algebra of A as $\mathcal{L}_A(A_A)$.

Having ourselves to the notion of a Hilbert right C^*-module, we are quickly led to the idea that there must be some useful notion of a bimodule between two C^*-algebras that respect their own Hilbert left-right C^*-module structures. This leads to imprimitivity bimodules between C^*-algebras, but we first need to go through the following intermediate definition before we can give a full definition of these bimodules. We shall closely follow the exposition in [RW98].

Definition 2.19. Let A be a C^*-algebra and Z a Hilbert left A-module, then Z is said to be **full** if the ideal ${}_A\langle Z, Z\rangle = \operatorname{span}\{{}_A\langle z, w\rangle : z, w \in Z\}$ is dense in A. We have an analogous definition for Hilbert right C^*-modules.

Definition 2.20. Let A, B be C^*-algebras, if Z be a complex vector space satisfying the following:

1. Z is a full Hilbert left A-module and a full Hilbert right B-module;

2. For all $a \in A$, $b \in B$, $z, w \in Z$:

$$\langle az, w\rangle_B = \langle z, a^* w\rangle_B\,, \text{ and } {}_A\langle zb, w\rangle = {}_A\langle z, wb^*\rangle;$$

3. For all $z, y, w \in Z$:

$$ {}_A\langle z, y\rangle w = z \langle y, w\rangle_B\,, \tag{2.12}$$

then Z is called an **$\mathbf{A}-\mathbf{B}$-imprimitivity bimodule**.

Proposition 2.21. *Suppose Z is an $A-B$ imprimitivity bimodule, then Z is a legitimate $A-B$ bimodule in the sense that for all $a \in A$, $z \in Z$, $b \in B$, and $\lambda \in \mathbb{C}$:*

1. $a(zb) = (az)b$;

2. $(\lambda a)(zb) = a(z(\lambda b))$.

Proof. Take an arbitrary $w \in Z$, we have

$$\begin{aligned}\langle a(zb) - (az)b, w\rangle_B &= \langle (zb), a^* w\rangle_B - \langle az, w\rangle_B\, b^* \\ &= \langle z, a^* w\rangle_B\, b^* - \langle z, a^* w\rangle_B\, b^* = 0.\end{aligned}$$

Since the equation above is true for all $w \in Z$, and Z is a Hilbert right B-module, then it follows from (an analogue) of Theorem 2.1 that $a(zb) = (az)b$. The second formula can be proven similarly. $\square$

For the next few pages, we shall occupy ourselves with various results that would help us construct an $A-B$-imprimitivity bimodule.

Lemma 2.5 ([RW98, Lemma 3.7])**.** *Let Z be an $A-B$-bimodule satisfying Axioms* 1 *and* 3 *of Definition 2.20. Z is an $A-B$-imprimitivity bimodule if and only if for all $a \in A$, $b \in B$, and $z \in Z$*

2'. $\langle az, az\rangle_B \leqslant \|a\|^2 \langle z, z\rangle_B$ *and* ${}_A\langle zb, zb\rangle \leqslant \|b\|^2\, {}_A\langle z, z\rangle$.

Proof. Suppose Z is an $A-B$-imprimitivity bimodule as in Definition 2.20, then Item 2 says that the left action of A are adjointable maps in $\mathcal{L}_B(Z_B)$, and the right action of B are adjointable maps in $\mathcal{L}_A({}_AZ)$. That is, for each $(a, b) \in A \times B$, if $L_a(z) := az$ and $R_b := zb$, then $L_a \in \mathcal{L}_B(Z_B)$ and $R_b \in \mathcal{L}_A({}_AZ)$. It follows from Proposition 2.8 that for all $z \in Z$

$$\begin{aligned}\langle az, az\rangle_B &\leqslant \|L_a\|^2 \langle z, z\rangle_B \leqslant \|a\|^2 \langle z, z\rangle_B \\ {}_A\langle zb, zb\rangle &\leqslant \|R_b\|^2\, {}_A\langle z, z\rangle \leqslant \|b\|^2\, {}_A\langle z, z\rangle.\end{aligned}$$

Conversely, if we suppose $2'$, then the maps $z \mapsto az$ and $z \mapsto zb$ are all linear and bounded maps on Z such that (via Proposition 2.3 Item 1) for all $z, w \in Z$

$$\begin{aligned} \| \langle az, aw \rangle_B \| &\leqslant \|az\| \|w\| \leqslant \|a\| \|z\| \|w\|, \qquad \text{(wrt norm in } B) \\ \| {}_A\langle zb, wb \rangle \| &\leqslant \|zb\| \|w\| \leqslant \|b\| \|z\| \|w\|, \qquad \text{(wrt norm in } A). \end{aligned} \tag{2.13}$$

It now follows from 3 of Definition 2.20 that for all $z, w, z_1, z_2 \in Z$

$$\begin{aligned} \langle {}_A\langle z_1, z_2 \rangle z, w \rangle_B &= \langle z_1 \langle z_2, z \rangle_B , w \rangle_B = \langle z, z_2 \rangle_B \langle z_1, w \rangle_B \\ &= \langle z, z_2 \langle z_1, w \rangle_B \rangle_B = \langle z, {}_A\langle z_2, z_1 \rangle w \rangle_B \\ &= \langle z, ({}_A\langle z_1, z_2 \rangle)^* w \rangle_B \end{aligned}$$

Therefore $\langle az, w \rangle_B = {}_B\langle z, a^* z \rangle$ for all in $a \in {}_A\langle Z, Z \rangle$, but we know from axiom 1 of Definition 2.20 that ${}_A\langle Z, Z \rangle$ is dense in Z, and since we have Estimates 2.13, we know that the formula must extend to all $a \in A$. We can prove ${}_A\langle zb, w \rangle = {}_A\langle z, wb^* \rangle$ similarly. $\square$

Lemma 2.6. *If Z is a full Hilbert left A-module, then Z is a full Hilbert right $\mathcal{K}_A({}_AZ)$-module, when equipped with the following structure:*

1. $z \cdot T = Tz, \qquad \forall T \in \mathcal{K}_A({}_AZ)$*;*

2. $\langle w, z \rangle_{\mathcal{K}_A({}_AZ)} = \theta_{w,z}$*.*

Furthermore, the induced norms coincide, i.e. for all $z \in Z$:

$$\| {}_A\langle z, z \rangle \|^{1/2} = \| \langle z, z \rangle_{\mathcal{K}_A({}_AZ)} \|^{1/2}.$$

Proof. Let us denote $B = \mathcal{K}_A({}_AZ)$ to make the computations slicker. Now, first we need to check the axioms of Definition 2.18 one-by-one. Let $z, w, y \in Z$, and $\lambda, \mu \in \mathbb{C}$.

1. It follows from the definition of rank-one operators in $\mathcal{K}_A(A_Z) = B$ that: $\langle w, \lambda z + \mu y \rangle_B = \lambda \theta_{w,z} + \mu \theta_{w,y}$;

2. Let $T \in B$, then it follows from Equation 2.8 that $\langle w, z \cdot T \rangle_B = \langle w, Tz \rangle_B = \theta_{w,Tz} = T\theta_{w,z} = \langle w, z \rangle_B \cdot T$;

3. $\langle w, z \rangle_B^* = \theta_{w,z}^* = \theta_{z,w} = \langle z, w \rangle_B$;

4. We can use Proposition 2.10 to prove positivity of $\langle w, w\rangle_B \in \mathcal{K}_A({}_AZ) \subseteq \mathcal{L}_A({}_AZ)$. We compute ${}_A\langle z\langle w, w\rangle_B, z\rangle = {}_A\langle \theta_{w,w}z, z\rangle = {}_A\langle {}_A\langle z, w\rangle w, z\rangle = ({}_A\langle z, w\rangle)({}_A\langle z, w\rangle)^* \geqslant 0$. Therefore $\theta_{w,w} \geqslant 0$.

5. Lastly, we want to show positive definiteness. Suppose $\langle w, w\rangle_B = \theta_{w,w} = 0$, then we can use the computations from the previous item (take $z = w$) so that: $0 = \| {}_A\langle w\langle w, w\rangle_B, w\rangle \| = \| {}_A\langle w, w\rangle \|^2$, therefore ${}_A\langle w, w\rangle = 0$, which implies $w = 0$.

We have already shown that Z is a inner-product right $B = \mathcal{K}_A({}_AZ)$-module. We still need to show fullness, but it follows from the fact that $\mathcal{K}_A({}_AZ)$ is the norm completion of the span of rank-one operators $\theta_{w,z} = \langle w, z\rangle_B$, so indeed $\mathcal{K}_A({}_AZ) = \overline{\langle Z, Z\rangle_B}$. Lastly, we want to show that the induced norm from B is the same as the induced norm from A. It follows from Proposition 2.1 that:

$$
{}_A\left\langle z\langle w, w\rangle_B^{1/2}, z\langle w, w\rangle_B^{1/2}\right\rangle = {}_A\langle z\langle w, w\rangle_B, z\rangle = ({}_A\langle z, w\rangle)({}_A\langle w, z\rangle) = \| {}_A\langle w, w\rangle \| \, {}_A\langle z, z\rangle .
$$

Taking the A-norm and supremum over all $\|z\| = 1$, this implies $\| \langle w, w\rangle_B^{1/2} \|^2 = \| \langle w, w\rangle_B \| \leqslant \| {}_A\langle w, w\rangle \|$. On the other hand, if we take $z = w$ above, we find that $\| {}_A\langle\langle w, w\rangle_B w, w\rangle \| = \| {}_A\langle w, w\rangle \|^2$, and then it follows from Proposition 2.3 Item 1 that $\| \langle w, w\rangle_B \| \geqslant \| {}_A\langle w, w\rangle \|$, therefore $\| \langle w, w\rangle_B \| = \| {}_A\langle w, w\rangle \|$ as required. □

Proposition 2.22 ([RW98, Proposition 3.8]). *Every full Hilbert left A-module ${}_AZ$ is a $A - \mathcal{K}_A({}_AZ)$-imprimitivity bimodule. Conversely, if Z is an $A - B$-imprimitivity bimodule, then $B \cong \mathcal{K}_A({}_AZ)$, where the isomorphism $\phi : B \to \mathcal{K}_A({}_AZ)$ is fully determined by $\phi(\langle w, z\rangle_B) = \theta_{w,z}$ for each $w, z \in Z$.*

Proof. Suppose Z is an full Hilbert left A-module, then we already know from the previous Lemma that Z is also a full right Hilbert $\mathcal{K}_A({}_AZ)$-module. We only need to check axioms 2 and 3 of Definition 2.20. For axiom 2, it is trivial that for any $T \in \mathcal{K}_A({}_AZ)$, $z, w \in Z$

$$
{}_A\langle z \cdot T, w\rangle = {}_A\langle Tz, w\rangle = {}_A\langle z, T^*w\rangle = {}_A\langle z, w \cdot T\rangle .
$$

On the other hand, let $a \in A$, we have, from the definition of rank-one operators that

$$
\langle a \cdot z, w\rangle_{\mathcal{K}_A(Z)} = \theta_{az,w} = \theta_{z,a^*w} = \langle z, a^* \cdot w\rangle_{\mathcal{K}_A(Z)}
$$

as required. Next we check axiom 3, let $y \in Z$, so that

$$z \cdot \langle w, y \rangle_{\mathcal{K}_A(Z)} := z \cdot \theta_{w,y} := \theta_{w,y}(z) = {}_A\langle z, w \rangle \cdot y,$$

as required.

For the converse, assume that Z is an $A-B$-imprimitivity bimodule, consider the map $\phi : B \to \mathcal{L}_A({}_AZ)$ via $\phi(b)(z) = z \cdot b$, which is well-defined by the fact that Z is an $A-B$-imprimitivity bimodule (i.e. $\phi(b) \in \mathcal{L}_A({}_AZ)$ truly). This map is a $*$-homomorphism between C^*-algebras, and therefore it follows from the basic theory of C^*-algebras that ϕ automatically is continuous (i.e bounded) and has a closed image. All that remains is to prove that ϕ is a C^*-isomorphism between B and $\mathcal{K}_A({}_AZ)$. To prove surjectivity, we have that for all $w, z, y \in Z$: $\phi(\langle w, z \rangle_B)(y) = y \cdot \langle w, z \rangle_B = {}_A\langle y, w \rangle \cdot z = \theta_{w,z}(y) = y \cdot \langle w, z \rangle_{\mathcal{K}_A(Z)}$. Therefore $\phi(\langle w, z \rangle_B) = \langle w, z \rangle_{\mathcal{K}_A(Z)}$, and therefore it follows from the fact that ϕ has a closed image that $\phi(B) = \overline{\phi(\overline{\langle Z, Z \rangle_B})} = \overline{\phi(\langle Z, Z \rangle_B)} = \overline{\langle Z, Z \rangle_{\mathcal{K}_A(Z)}} = \mathcal{K}_A(Z)$. To show injectivity, suppose $b \in B$ such that $\phi(b) = 0$, and $\varepsilon > 0$. Since $\langle Z, Z \rangle_B$ is a dense ideal in B, we can find an approximate identity, $\{b_i\}_{i \in I}$ in B such that for each $i \in I$, $b_i \in \langle Z, Z \rangle_B$. Therefore, we have a $j \in J$ such that: $\|b - b_j b\| < \varepsilon$. But then, if we write explicitly $b_j = \sum_{k=1}^{n} \langle w_k, z_k \rangle_B$, we have

$$\begin{aligned} \|b\| &\leqslant \left\| b - \sum_{k=1}^{n} \langle w_k, z_k \rangle_B \, b \right\| + \left\| \sum_{k=1}^{n} \langle w_k, z_k \rangle_B \, b \right\| \\ &< \varepsilon + \left\| \sum_{k=1}^{n} \langle w_k, \phi(b)(z_k) \rangle_B \right\| = \varepsilon. \end{aligned}$$

Therefore $\|b\| = 0$, equivalently, $b = 0$. This proves the isomorphism. $\square$

As is usual with analysis, the spaces that we encounter in the wild are usually defined via completion of dense subspaces. Hence it would be useful to have a completion criterion wherein we only check for certain axioms on the dense *-subalgebras A_0 and B_0 of A and B respectively, against a dense subspace Z_0 of Z from which we can conclude that Z is an actual $A-B$-imprimitivity bimodule. The dense subspaces involved are captured in the following definition.

Definition 2.21. Let A and B be C^*-algebras with dense $*$-subalgebras A_0 and B_0 respectively. An $\mathbf{A_0 - B_0}$**-pre-imprimitivity bimodule** is a vector space Z_0 which is an A_0-B_0-bimodule such that

1. Z_0 is a left semi-inner product A_0-module via ${}_{A_0}\langle \cdot, \cdot \rangle : Z_0 \times Z_0 \to A_0$ and a right semi-inner

product B_0-module via $\langle\cdot,\cdot\rangle_{B_0} : Z_0 \times Z_0 \to B_0$ (i.e. they satisfy all inner-product module axioms except positive-definiteness);

2. ${}_{A_0}\langle Z_0, Z_0\rangle$ and $\langle Z_0, Z_0\rangle_{B_0}$ are dense ideals of A_0 and B_0 (hence of A and B) respectively;

3. For all $a \in A_0$, $b \in B_0$, and $z \in Z_0$,

$$\langle a \cdot z, a \cdot z\rangle_{B_0} \leqslant \|a\|^2 \langle z, z\rangle_{B_0},$$
$${}_{A_0}\langle z \cdot b, z \cdot b\rangle \leqslant \|b\|^2 \, {}_{A_0}\langle z, z\rangle$$

in the completions B and A respectively;

4. For all $z, y, w \in Z_0$

$${}_{A_0}\langle z, y\rangle \cdot w = z \cdot \langle y, w\rangle_{B_0}.$$

If Z is a $A - B$-imprimitivity bimodule, we could have asked earlier if the induced norm on Z as a Hilbert left A-module coincides with the induced norm on Z as a Hilbert right B-module, this actually turns out to be the case, but we can prove a stronger result that also includes this.

Proposition 2.23 ([RW98, Proposition 3.11]). *Let A, B be C^*-algebras with $A_0 \subseteq A$ and $B_0 \subseteq B$ as dense $*$-subalgebras. If Z is an $A_0 - B_0$-pre-imprimitivity bimodule, then the following seminorms coincide:*

$${}_{A_0}\|z\| := \|\, {}_{A_0}\langle z, z\rangle \,\|^{1/2} = \|\, \langle z, z\rangle_{B_0} \,\|^{1/2} = \|z\|_{B_0}.$$

Proof. Let $z \in Z$, then we have the following computation:

$$\begin{aligned}
\|\, {}_{A_0}\langle z, z\rangle \,\|^2 &= \|\, {}_{A_0}\langle z, z\rangle \, {}_{A_0}\langle z, z\rangle \,\| \\
&= \|\, {}_{A_0}\Big\langle z \langle z, z\rangle_{B_0}, z\Big\rangle \,\| \\
&= \|\, {}_{A_0}\Big\langle z, z \cdot \langle z, z\rangle_{B_0}\Big\rangle \,\| \\
&\leqslant {}_{A_0}\|z\| \cdot {}_{A_0}\|z \cdot \langle z, z\rangle_{B_0} \,\| \\
&= \|\, {}_{A_0}\langle z, z\rangle \,\|^{1/2} \cdot \|\, {}_{A_0}\langle z, z\rangle \,\|^{1/2} \|\, \langle z, z\rangle_{B_0} \,\|, \qquad \text{(Item 3, Definition 2.21)}
\end{aligned}$$

This gives us, $\|\, {}_{A_0}\langle z, z\rangle \,\| \leqslant \|\, \langle z, z\rangle_{B_0} \,\|$. We can obtain the reverse inequality by running a similar computation starting at $\| \langle z, z \rangle_{B_0} \|^2$. Therefore ${}_{A_0}\|z\| = \| {}_{A_0}\langle z, z \rangle \|^{1/2} = \| \langle z, z \rangle_{B_0} \|^{1/2} = \|z\|_{B_0}$, as required. □

Remark 2.8. In the case $A_0 = A$, and $B_0 = B$, and $Z_0 = Z$, then we obtain the coincidence of induced *norms* result for $A - B$-imprimitivity bimodules since they are also $A - B$-pre-imprimitivity bimodules.

Finally we give the main result of this section, that $A_0 - B_0$-pre-imprimitivity bimodules 'complete' into actual $A - B$-imprimitivity bimodules.

Corollary 2.6 ([RW98, Proposition 3.12]). *Let A and B be C^*-algebras with dense $*$-subalgebras A_0 and B_0 respectively. If Z_0 is an A_0-B_0-preimprimitivity bimodule, then there is an A-B-imprimitivity bimodule Z and an A_0-B_0-bimodule homomorphism (meaning it respects the bimodule structure) $q : Z_0 \to Z$ such that $q(Z_0)$ is dense and for $z, w \in Z_0$*

$$\begin{aligned} \langle q(z), q(w) \rangle_B &= \langle z, w \rangle_{B_0}, \\ {}_A\langle q(z), q(w) \rangle &= {}_{A_0}\langle z, w \rangle. \end{aligned} \tag{2.14}$$

Proof. It follows from Proposition 2.23 that $N = \{z_0 \in Z_0 : {}_{A_0}\langle z_0, z_0 \rangle = 0\} = \{z_0 \in Z_0 : \langle z_0, z_0 \rangle_{B_0} = 0\}$. Therefore the exact vector space canonical quotient map $q : Z_0 \to Z_0/N$ in Lemma 2.1 allows us to construct a Hilbert left$A$ and Hilbert right B module Z as the norm-completion of $q(Z_0)$, satisfying Equations (2.14). The left A-module action, the right B-module action, the inner-products ${}_A\langle \cdot, \cdot \rangle$, and $\langle \cdot, \cdot \rangle_B$ are all continuous, and so by density arguments, the associativity condition Item 3 of Definition 2.20 is satisfied by Z. From the same density argument, we also have that

$$\begin{aligned} \langle az, az \rangle_B &\leqslant \|a\|^2 \langle z, z \rangle_B, \\ {}_A\langle zb, zb \rangle &\leqslant \|b\|^2 {}_A\langle z, z \rangle \end{aligned}$$

for all $a \in A$, $b \in B$, and $z \in Z$. Fullness follows from Item 2 of Definition 2.21. Therefore it follows from Lemma 2.5 that Z is an $A - B$-imprimitivity bimodule. □

Definition 2.22. Let A, B be C^*-algebras such that there exists Z, an $A - B$-imprimitivity bimodule, then one says that A and B are **Morita equivalent.**

We shall not get into the details of why Morita equivalence is an actual equivalence relation on C^*-algebras. However, this notion borrowed from the theory of modules over rings [Mor58], turned out to be quite useful when studying C^*-algebras. In particular, one can use the nice symmetric structure of an imprimitivity bimodule, say between A and B to construct a correspondence between representations (modulo unitary equivalence) and ideals between A and B. We do not find much use of these facts in this thesis. What will be useful, however, is the picture that we get from noncommutative geometry. Recall Gelfand's, and Serre-Swan's theorems from Chapter 1. We shall obtain a particular 'noncommutative' incarnation of a 'vector bundle' over C^*-algebras in the unital case. We first need the following lemma.

Lemma 2.7. *Let I be a two-sided (not necessarily closed) ideal in a unital C^*-algebra B, then $\overline{I} = B$ implies $I = B$.*

Proof. Since $1_B \in \overline{I}$, and $1_B \in \text{Inv}(B)$, where $\text{Inv}(B)$ is the open set of invertible elements of B, then we know that $I \cap \text{Inv}(B) \neq \varnothing$, therefore there exists an invertible element in I, say $x \in I$. However, this means $xx^{-1} = 1_B \in I$, but $1_B \in I$ implies $I = B$. □

Corollary 2.7. *Suppose Z is an $A-B$-imprimitivity bimodule. B is unital if and only if Z is a finitely generated projective and self-dual Hilbert left A-module.*

Proof. Suppose that B is unital. By the fact that Z is full, it follows from Lemma 2.7 that we have $\langle Z, Z\rangle_B = B$. In particular, there exists $z_1, .., z_n, w_1, ..., w_n \in Z$ such that $\sum_{k=1}^n \langle z_k, w_k\rangle_B = 1_B$ for the identity $1_B \in B$. Therefore for each $z \in Z$:

$$z = z \cdot 1_B = z \cdot \sum_{k=1}^{n} \langle z_k, w_k\rangle_B = \sum_{k=1}^{n} {}_A\langle z, z_k\rangle w_k.$$

Since for each $k = 1, ..., n$, the map $z \mapsto {}_A\langle z, z_k\rangle$ is in $\text{Hom}_A(Z, A)$, then it follows from Definition 2.11, that Z is finitely generated and projective as a Hilbert left A-module. Now, $\varphi \in \text{Hom}_A(Z, A)$, it follows from the equation above that for all $z \in Z$:

$$\begin{aligned}\varphi(z) &= \varphi\left(\sum_{k=1}^{n} {}_A\langle z, z_k\rangle w_k\right)\\ &= \sum_{k=1}^{n} {}_A\langle z, z_k\rangle \varphi(w_k) = {}_A\left\langle z, \sum_{k=1}^{n} \varphi(w_k)^* z_k\right\rangle\end{aligned}$$

Taking $w := \sum_{k=1}^n \varphi(w_k)^* z_k$, we find that $\varphi(z) = {}_A\langle z, w\rangle$, so Z is self-dual.

For the converse, assume that Z is finitely generated projecive, and self-dual as a Hilbert left A-module. This means there exist a finite family $\{\varphi_k\}_{k=1}^n \subseteq \text{Hom}_A(Z, A)$ and elements

$\{w_k\}_{k=1}^n \subseteq Z$ such that

$$z = \sum_{k=1}^{n} \varphi_k(z) w_k.$$

By self-duality, we have $\{z_k\}_{k=1}^n \subseteq Z$ such that $\varphi_k(z) = {}_A\langle z, z_k\rangle$ for all $z \in Z$ and $k = 1, ..., n$. Therefore we have that:

$$\mathrm{Id}(z) = z = \sum_{k=1}^{n} \varphi_k(z) w_k = \sum_{k=1}^{n} \theta_{z_k, w_k}(z),$$

given the identity operator $\mathrm{Id} : Z \to Z$. The equation above implies that $\mathrm{Id} \in \mathcal{K}_A({}_A A)$, so from Proposition 2.22, $\mathcal{K}_A({}_A A) \cong B$ is unital, as required. □

We end with an important observation of what happens when one of the C^*-algebras involved is unital.

Corollary 2.8. *Let Z be an $A-B$-imprimitivity bimodule with B unital. Then $\mathcal{L}_A({}_A Z)$ consists of finite-rank operators.*

We proceed as in the previous Corollary. If B is unital, then there exists $\{z_k\}_{k=1}^n$ and $\{w_k\}_{k=1}^n$ in Z such that:

$$z = z \cdot 1_B = z \cdot \sum_{k=1}^{n} \langle z_k, w_k\rangle_B = \sum_{k=1}^{n} {}_A\langle z, z_k\rangle w_k.$$

If $T \in \mathcal{L}_A({}_A Z)$, then:

$$T(z) = \sum_{k=1}^{n} {}_A\langle z, z_k\rangle T w_k = \sum_{k=1}^{n} \theta_{z_k, T w_k}(z).$$

So $T = \sum_{k=1}^n \theta_{z_k, T w_k}$ is of finite-rank as required.

Remark 2.9. Note that most of the results that we have shown here are 'symmetric', for example, in the above Proposition: A is unital if and only if Z is a finitely generated and self-dual Hilbert right B-module.

Proposition 2.24. *Let Z be an $A-B$-imprimitivity bimodule with B unital. Let $T = \sum_{i=1}^n \theta_{h_i, g_i}$ be an operator in $\mathcal{L}_A({}_A Z)$. Then T is unitary in $\mathcal{L}_A({}_A Z)$ if and only if*

$$\sum_{i,j} \langle h_i, g_i\rangle_B \langle g_j, h_j\rangle_B = 1_B = \sum_{i,j} \langle g_i, h_i\rangle_B \langle h_j, g_j\rangle_B . \tag{2.15}$$

Proof. Let us suppose that Equation (2.6) is true, then

$$T(T^*(z)) = T\left(\sum_{j=1}^{n} \theta_{g_j,h_i}(z)\right) = T\left(\sum_{j=1}^{n} {}_A\langle z, g_j\rangle \cdot h_j\right) = \sum_{j=1}^{n} {}_A\langle z, g_j\rangle \cdot T(h_j)$$
$$= \sum_{i,j} {}_A\langle z, g_j\rangle \, {}_A\langle h_j, h_i\rangle \cdot g_i = \sum_{i,j} z \cdot \langle g_j, h_j\rangle_B \, \langle h_i, g_i\rangle_B = z \cdot 1_B = z.$$

We can make a similar computation to show that $T^*(T(z)) = z$ for all $z \in Z$. Hence T is unitary.

On the other hand, suppose that T is unitary, then we still have for all $z \in Z$, $z = T(T^*(z)) = z \cdot \sum_{i,j} \langle g_i, h_i\rangle_B \, \langle h_j, g_j\rangle_B$. If we let $b = \sum_{i=1}^{n} \langle h_i, g_i\rangle_B$, then $z = T(T^*(z)) = z \cdot b^*b$. Therefore, for all $z, w \in Z$:

$$\langle z, w\rangle_B = \langle z, T(T^*(w))\rangle_B = \langle z, w \cdot b^*b\rangle_B \,.$$

This implies that for all $z, w \in Z$, $\| \langle z, w\rangle_B \cdot 1_B\| = \| \langle z, w\rangle_B \cdot b^*b\|$, but by fullness of Z as a Hilbert right B-module, it must follow from Lemma 2.4 that $1_B = b^*b = \sum_{i,j} \langle g_i, h_i\rangle_B \, \langle h_j, g_j\rangle_B$. We obtain $1_B = bb^*$ by considering $z = T^*(T(z))$ instead. □

In light of Corollary 2.8, the result above is actually a full-characterization of unitary adjointable maps in an imprimitivity bimodule when at least one of the coefficient C^*-algebras is unital.

Remark 2.10. Note that we can rewrite Equation (2.6) as

$$\sum_{i=1}^{n} \langle h_i, T^* g_i\rangle_B = 1_B = \sum_{i=1}^{n} \langle g_i, T h_i\rangle_B \,.$$

The above equation is equivalent to saying that $\{h_i\}_{i=1}^n$ and $\{T^* g_i\}_{i=1}^n$ as well as $\{g_i\}_{i=1}^n$ and $\{T h_i\}_{i=1}^n$ are **dual pair of generators** for Z in the sense that for any $z \in Z$:

$$z = \sum_{i=1}^{n} {}_A\langle z, g_i\rangle T h_i = \sum_{i=1}^{n} {}_A\langle z, h_i\rangle T^* g_i.$$

In the case $n = 1$ and $T = \mathrm{Id}_Z$, we recover the so-called **Wexler-Raz identity for modules** such as Z, i.e. $\theta_{h,g} = \mathrm{Id}_Z$ if and only if

$$\langle h, g\rangle_B = 1_B. \tag{2.16}$$

2.4 Module Frames

A description of frames was first given by Duffin and Schaeffer in [DS52], and was then popularized by Debauchies, Grossman, and Meyer in [DGM86]. Frame theory has now become a household tool for signal processing (harmonic analysis), providing a coherent theory of non-orthogonal expansions. To expound, frames can be thought of as a (countable) family of $\{h_i\}_{i\in I}$ of elements in a Hilbert space H from which all elements $h \in H$ can be reconstructed. That is, $h = \sum_{i\in I} c_i h_i$ for some $\{c_i\}_{i\in I} \subseteq \mathbb{C}$, but for which this family of coefficients $\{c_i\}_{i\in I}$ need not be unique, in particular $\{h_i\}_{i\in I}$ need not be a basis. This non-uniqueness of the representation-coefficients may sound unconventionally unnecessary as it generally introduces redundancy in the representation of elements in H, that is, we are in effect using more elements than absolutely necessary for such representations. There are practical motivations for this redundancy, the canonical example comes from signal processing, where one usually samples and analyzes (coefficients) a signal for perfect reconstruction. Using a redundant sample (a frame) gives robustness in our reconstruction since any corruption or deletion in our samples do not necessarily mean that we can no longer reconstruct the signal from our possibly corrupted set.

We shall deal with frames in their various incarnations in this work. In keeping with the theme of incorporating operator algebras with harmonic analysis, we shall give a brief introduction to module frame theory in particular. The reader interested in the classical case in infinite dimensions should consult [Chr03].

Module frames can be thought of as the generalization of the notion of frames as we have described above, where one replaces the Hilbert space with a Hilbert C^*-module instead. Naturally, we also replace the scalars with C^*-algebras. Originally introduced and developed by Frank and Larson in [FL99, FL02]. It has since found itself useful in both the theory of operator algebras e.g. [RT03] where it has shown connections with Kasparov's stabilization theorem, and harmonic analysis e.g. [AE20, AJL20] where module frames provided a natural intermediary language for the results connecting operator algebras and time-frequency analysis via Heisenberg modules. We will eventually see the results of this section applied in the latter example with our dicussion of the Heisenberg modules (c.f. Chapter 5). Throughout the rest of this document, we will simply refer to module frames as frames, unless we need to emphasize the module structure of the ambient space.

Definition 2.23. Let Z be a Hilbert left A-module. A countable family $\mathbf{z} = \{z_i\}_{i\in I} \subseteq Z$ is said to be a **(module) frame** for Z if there exists two constants $C, D > 0$ such that

$$C \; {}_A\langle z, z\rangle \leqslant \sum_{i\in I} {}_A\langle z, z_n\rangle \; {}_A\langle z, z_n\rangle^* \leqslant D \; {}_A\langle z, z\rangle, \qquad \forall z \in Z,$$

where the middle sum converges in the norm of A. C and D are called the **lower frame** or **upper constants** respectively. If only the upper frame constant exists for $\mathbf{z}$, then it is called a **Bessel family** of Z. If $C = D$, then $\mathbf{z}$ is called a **tight frame**. If $A = B = 1$, then $\mathbf{z}$ is called a **Parseval frame**.

Remark 2.11. We will generally follow [AB17], defining module frames over not necessarily unital C^*-algebras, in contrast to the original paper [FL99].

Remark 2.12. If $\mathbf{z}$ forms a Parseval frame, we see that $\sum_{i=1}^{n} {}_A\langle z, z_n\rangle \; {}_A\langle z, z_n\rangle = {}_A\langle z, z\rangle$ for all $z \in Z$. This motivates the terminology since it is a generalization of Parseval's identity in functional analysis.

Example 2.5. Every Hilbert space H is a Hilbert left $\mathbb{C}$-module. Therefore a family $\{h_i\}_{i\in I}$ is a frame if and only if there exists $C, D > 0$ such that

$$C\|h\|_2^2 \leqslant \sum_{i\in I} |\langle h, h_i\rangle_2|^2 \leqslant D\|h\|_2^2, \qquad \forall h \in H.$$

As expected, we recover the definition of frame theory in Hilbert spaces. We shall also take this opportunity to emphasize that there is a perfect correspondence of morphisms between H as a Hilbert $\mathbb{C}$-module and as a Hilbert space. This comes from the fact that every bounded linear operator on H is automatically adjointable: $\mathcal{L}_{\mathbb{C}}(H) = \mathcal{L}(H)$. In particular: compact, positive, invertible, and unitary operators in $\mathcal{L}_{\mathbb{C}}(H)$ are exactly the same compact, positive, invertible, and unitary operators in $\mathcal{L}(H)$.

It will do us well to keep this example in mind, especially when we get into the business of doing Gabor analysis, which is primarily written in terms of frame theory on the Hilbert space of square-integrable functions (see Chapter 4).

Recall the Hilbert left A-module $\ell^2_A(I)$ given in Example 2.1. We can now define the two fundamental operators in frame theory.

Definition 2.24. Let Z be a Hilbert left A-module. For a Bessel family $\mathbf{z} = \{z_i\}_{i\in I} \subseteq Z$. The **analysis** and **synthesis** operators with respect to $\mathbf{z}$ are denoted by $\Phi_{\mathbf{z}} : Z \to \ell^2_A(I)$, and

$\Psi_{\mathbf{z}} : \ell^2_A(I) \to Z$ respectively. For all $z \in Z$ and $\mathbf{a} = \{a_i\}_{i\in I} \in \ell^2_A(I)$, we have:

$$\begin{aligned} \Phi_{\mathbf{z}}(z) &= \{{}_A\langle z, z_i\rangle\}_{i\in i} \\ \Psi_{\mathbf{z}}(\mathbf{a}) &= \sum_{i\in I} a_i z_i \end{aligned} \tag{2.17}$$

We still have to justify the well-definedness of both the analysis and synthesis operators. Our first hint for well-definedness is the requirement that $\mathbf{z}$ must be a Bessel family, this means that for all $z \in Z$, the sum $\sum_{i\in I} {}_A\langle z, z_i\rangle\, {}_A\langle z, z_i\rangle^*$ converges in the norm of A, which is exactly what it means for $\{{}_A\langle z, z_i\rangle\}_{i\in I}$ to be a member in $\ell^2_A(I)$. Hence $\Phi_{\mathbf{z}}$ is well-defined. Next we shall show that $\Psi_{\mathbf{z}}$ is also well-defined. We utilize the module norm characterization from Item 1 of Theorem 2.1 along with the Cauchy-Schwarz inequality from Item 1 of Proposition 2.3. Let J be any subfamily of I, in the following computation (partially from [AB17, Theorem 2.1]), we put explicit subscripts in the module norms involved for easier parsing:

$$\begin{aligned} \left\| \sum_{j\in J} a_j z_j \right\|_Z^2 &= \sup\left\{ \left\| {}_A\left\langle \sum_{j\in J} a_j z_j, w \right\rangle \right\|_A^2 : \|w\|_Z \leqslant 1 \right\} \\ &= \sup\left\{ \left\| \sum_{j\in J} a_j\, {}_A\langle z_j, w\rangle \right\|_A^2 : \|w\|_Z \leqslant 1 \right\} \\ &= \sup\left\{ \left\| {}_A\langle \{a_j\}_{j\in J}, \{{}_A\langle w, z_j\rangle\}_{j\in J}\rangle \right\|_{\ell^2_A(I)} : \|w\|_Z \leqslant 1 \right\} \\ &\leqslant \sup\left\{ \left\| \sum_{j\in J} a_j a_j^* \right\|_A \cdot \left\| \sum_{j\in J} {}_A\langle w, z_i\rangle\, {}_A\langle w, z_i\rangle^* \right\|_A : \|w\|_Z \leqslant 1 \right\} \\ &\leqslant D \left\| \sum_{j\in J} a_j a_j^* \right\|. \end{aligned}$$

Since the computation is valid for all J, we can replace J with $I\backslash F$ for F any finite subfamily of I. This implies that the estimate above approaches 0 for monotone increasing values of F. This means $\Psi_{\mathbf{z}}(\mathbf{a}) = \sum_{i\in I} a_i z_i \in Z$ as required.

Proposition 2.25. *The analysis and synthesis operators with respect to a Bessel sequence are both adjointable maps, and are in fact adjoints of each other.*

Proof. Let $\mathbf{z} = \{z_i\}_{i\in I}$ be a Bessel family, it follows from the continuity of the inner-products that for all $z \in Z$ and $\mathbf{a} \in \ell^2_A(I)$:

$$ {}_A\langle \Phi_{\mathbf{z}} z, \mathbf{a} \rangle = \sum_{i \in I} {}_A\langle z, z_i \rangle a_i^* = \sum_{i \in I} {}_A\langle z, a_i z_i \rangle = {}_A\left\langle z, \sum_{i \in I} a_i z_i \right\rangle = {}_A\langle z, \Psi_{\mathbf{z}}(\mathbf{a}) \rangle . $$

□

Definition 2.25. Let Z be a Hilbert left A-module, with $\mathbf{z} = \{z_i\}_{i \in I} \subseteq Z$ a Bessel sequence. We define the **frame** operator $\theta_{\mathbf{z}} \in \mathcal{L}_A(Z)$ associated with $\mathbf{z}$ to be the composition of the associated analysis, then the associated synthesis operator:

$$ \theta_{\mathbf{z}} = \Psi_{\mathbf{z}} \circ \Phi_{\mathbf{z}} = \Phi^*_{\mathbf{z}} \circ \Phi_{\mathbf{z}} = \Psi_{\mathbf{z}} \circ \Psi^*_{\mathbf{z}}. \tag{2.18} $$

One can also generalize this a bit further by considering different Bessel families for the analysis and synthesis operators. Suppose $\mathbf{w} = \{w_i\}_{i \in I}$ is another Bessel family, then the **mixed frame** operator $\theta_{\mathbf{z},\mathbf{w}} \in \mathcal{L}_A(Z)$ is given by:

$$ \theta_{\mathbf{z},\mathbf{w}} = \Psi_{\mathbf{w}} \circ \Phi_{\mathbf{z}} = \Phi^*_{\mathbf{w}} \circ \Phi_{\mathbf{z}} = \Psi_{\mathbf{w}} \circ \Psi^*_{\mathbf{z}}. \tag{2.19} $$

Remark 2.13. We shall first agree that, for the rest of the manuscript, we will abbreviate and refer to mixed frame operators simply as frame operators unless we need to explicitly differentiate between the two.

If $\mathbf{z} = \{z_i\}_{i=1}^N, \mathbf{w_i}_{i=1}^N$ are finite families in Z, it is easy to see that we can write the associated frame operators in terms of rank-one operators:

$$ \theta_{\mathbf{z},\mathbf{w}} = \sum_{i=1}^N \theta_{z_i, w_i}. $$

Therefore in $\theta_{\mathbf{z},\mathbf{w}} \in \mathcal{K}_A(Z)$. However, if $\mathbf{z}$ or $\mathbf{w}$ are countably infinite, we may be tempted[3] to conclude that every frame operator is compact. This is the perfect time to take a pause and examine how we interpret $\theta_{\mathbf{z},\mathbf{w}}$ as an infinite series of operators.

Proposition 2.26. *Let $\mathbf{z} = \{z_i\}_{i \in I}, \mathbf{w} = \{w_i\}_{i \in I}$ be Bessel families of a Hilbert left A-module Z. Let F be a finite subfamily of I, define $\mathbf{z}^F = \{z_i\}_{i \in F}, \mathbf{w}^F = \{w_i\}_{i \in F}$. Then $\theta^*_{\mathbf{z},\mathbf{w}} = \theta_{\mathbf{w},\mathbf{z}}$ and $\theta_{\mathbf{z}^F,\mathbf{w}^F} \to \theta_{\mathbf{z},\mathbf{w}}$ in $*$-strong topology of $\mathcal{L}_A(Z)$ as $F \to I$ (for monotone increasing F)*

[3] I was.

Proof. We find that $\theta^*_{\mathbf{z}^F,\mathbf{w}^F} = \theta_{\mathbf{w}^F,\mathbf{z}^F}$ for any finite subfamily $F \subseteq I$. By proceeding to do an estimate similar to our proof of well-defined of the synthesis operator, we find that both $\theta_{\mathbf{z}^F,\mathbf{w}^F} z \to \theta_{\mathbf{z},\mathbf{w}} z$ and $\theta_{\mathbf{w}^F,\mathbf{z}^F} z \to \theta_{\mathbf{w}^F,\mathbf{z}^F} z$ in Z, for all $z \in Z$. Equivalently $\theta_{\mathbf{z}^F,\mathbf{w}^F} \to \theta_{\mathbf{z},\mathbf{w}}$ in $*$-strong topology. Furthermore, it follows from the definition of $*$-strong topology that the adjoint operator on $\mathcal{L}_A(Z)$ is a continuous map with respect to the $*$-strong topology. Hence it follows from what we have just shown that $\theta_{\mathbf{z},\mathbf{w}} = \theta_{\mathbf{w},\mathbf{z}}$. □

Proposition 2.27. *Let Z be a Hilbert left A-module, and $\mathbf{z} = \{z_i\}_{i\in I}$ be a Bessel family. Then every frame operator $\theta_{\mathbf{z}}$ is positive.*

Proof. We have $\theta_{\mathbf{z},\mathbf{z}} = \theta_{\mathbf{z}}$, so it follows from Proposition 2.26 that $\theta^*_{\mathbf{z}} = \theta^*_{\mathbf{z},\mathbf{z}} = \theta_{\mathbf{z},\mathbf{z}} = \mathbf{z}$. Therefore $\theta_{\mathbf{z}}$ is self-adjoint. To prove positivity of $\theta_{\mathbf{z}}$, we have that ${}_A\langle \theta_{\mathbf{z}} z, z\rangle = \sum_{i\in I} {}_A\langle z, z_i\rangle ({}_A\langle z, z_i\rangle)^* \geqslant 0$, so it follows from Proposition 2.10 that $\theta_{\mathbf{z}}$ is positive. □

So far we have not really used the the frame condition in any of the operators of interest that we have introduced. The lower frame bound now makes an appearance in the following important result regarding the frame operator. To the best of the author's knowledge, the proof presented below, especially the backwards direction, is new (although this is a commonly cited result).

Theorem 2.6. *Let Z be a Hilbert left A-module and $\mathbf{z} = \{z_i\}_{i\in I}$ a Bessel family. We have the following: $\mathbf{z} = \{z_i\}_{i\in I}$ is a frame if and only if the frame operator $\theta_{\mathbf{z}}$ is invertible in $\mathcal{L}_A(Z)$.*

Proof. Suppose $\mathbf{z}$ is a frame in Z with lower frame bound $C > 0$. It follows from Item 2 Theorem 2.1 that (again with explicit subscripts in the norms for easier parsing):

$$\begin{aligned}
\|\theta_{\mathbf{z}} z\|_Z &= \sup\left\{\left\| \left\langle \sum_{i\in I} {}_A\langle z, z_i\rangle z_i, w\right\rangle\right\|_A : \|w\|_Z = 1\right\} \\
&\geqslant \left\| \left\langle \sum_{i\in I} {}_A\langle z, z_i\rangle z_i, \frac{z}{\|z\|_Z}\right\rangle\right\|_A \\
&= \left\| \sum_{i\in I} {}_A\langle z, z_i\rangle ({}_A\langle z, z_i\rangle)^* \frac{1}{\|z\|_Z}\right\|_A \\
&\geqslant C\| {}_A\langle z, z\rangle \|_A \frac{1}{\|z\|_Z} \\
&= C\|z\|_Z.
\end{aligned}$$

It follows from self-adjointness of $\theta_{\mathbf{z}}$ and Lemma 2.2 that $\theta_{\mathbf{z}}$ is invertible in $\mathcal{L}_A(Z)$.

Conversely, suppose that $\theta_{\mathbf{z}}$ is invertible in $\mathcal{L}_A(Z)$. By virtue of $\theta_{\mathbf{z}}$ being adjointable we already know that for all $z \in Z$, $\sum_{i\in I} {}_A\langle z, z_i\rangle \left({}_A\langle z, z_i\rangle\right)^* = {}_A\langle \theta_{\mathbf{z}} z, z\rangle$ converges in the norm of A. Furthermore, θ_z is positive (so it is also self-adjoint), and therefore $\theta_{\mathbf{z}}^{1/2}$ and $\theta_{\mathbf{z}}^{-1/2}$ exists in $\mathcal{L}_A(Z)$. Using Proposition 2.8 we find that for all $z \in Z$:

$$\begin{aligned} \left|\theta_{\mathbf{z}}^{1/2} z\right|^2 \leqslant \left\|\theta_{\mathbf{z}}^{1/2}\right\|^2 |z|^2 &\iff \\ {}_A\langle \theta_{\mathbf{z}} z, z\rangle \leqslant \left\|\theta_{\mathbf{z}}^{1/2}\right\|^2 |z|^2 &\iff \\ \sum_{i\in I} {}_A\langle z, z_i\rangle \left({}_A\langle z, z_i\rangle\right)^* \leqslant \left\|\theta_{\mathbf{z}}^{1/2}\right\|^2 {}_A\langle z, z\rangle &. \end{aligned}$$

Therefore $\mathbf{z}$ is a Bessel family with upper frame bound $D := \left\|\theta_{\mathbf{z}}^{1/2}\right\|^2 = \|\theta_{\mathbf{z}}\|$. To prove the existence of lower frame bound for $\mathbf{z}$, we again use Proposition 2.8, so that for all $z \in Z$:

$$\begin{aligned} \left|\theta_{\mathbf{z}}^{-1/2}(\theta_{\mathbf{z}}^{1/2} z)\right|^2 \leqslant \left\|\theta_{\mathbf{z}}^{-1/2}\right\|^2 \left|\theta_{\mathbf{z}}^{1/2} z\right|^2 &\iff \\ {}_A\langle z, z\rangle \leqslant \left\|\theta_{\mathbf{z}}^{-1/2}\right\|^2 {}_A\langle \theta_{\mathbf{z}} z, z\rangle &\iff \\ \frac{1}{\left\|\theta_{\mathbf{z}}^{-1/2}\right\|^2} {}_A\langle z, z\rangle \leqslant \sum_{i\in I} {}_A\langle z, z_i\rangle \left({}_A\langle z, z_i\rangle\right)^* &. \end{aligned}$$

Hence $\mathbf{z}$ possesses a lower frame bound $C := \frac{1}{\left\|\theta_{\mathbf{z}}^{-1/2}\right\|^2} = \frac{1}{\left\|\theta_{\mathbf{z}}^{-1}\right\|}$. Therefore $\mathbf{z}$ is a frame for Z. □

Proposition 2.28. *Let $\mathbf{z} = \{z_i\}_{i\in I}$ a Bessel family for a Hilbert A-module Z. Then*

$$\|\theta_{\mathbf{z}}\| = \inf\left\{D^{1/2} : D \textit{ is a Bessel bound for } \mathbf{z}\right\}.$$

Proof. We use the fact that $\theta_{\mathbf{z}}$ is positive so that for all $w \in Z$:

$${}_A\langle \theta_{\mathbf{z}} w, w\rangle = {}_A\left\langle \theta_{\mathbf{z}}^{1/2} w, \theta_{\mathbf{z}}^{1/2} w\right\rangle \leqslant (\|\theta_{\mathbf{z}}\|^{1/2}))^2 \, {}_A\langle w, w\rangle .$$

Therefore $\|\theta_{\mathbf{z}}\|$ is a Bessel bound for z, which implies that

$$\|\theta_{\mathbf{z}}\|^{1/2} \geqslant \inf\left\{D^{1/2} : D \text{ a Bessel bound for } \mathbf{z}\right\}. \tag{2.20}$$

On the other hand, if D is a Bessel bound for z, we find that

$${}_A\langle \theta_{\mathbf{z}} w, w\rangle^{1/2} \leqslant D^{1/2} \, {}_A\langle w, w\rangle^{1/2} , \qquad \forall w \in Z$$

$$\implies \|\theta_{\mathbf{z}}^{1/2} w\| \leqslant D^{1/2}\|w\|, \qquad \forall w \in Z$$
$$\implies \|\theta_{\mathbf{z}}^{1/2}\| \leqslant D^{1/2}.$$

The last inequality, paired with (2.20), finally gives us $\|\theta_{\mathbf{z}}\|^{1/2} = \{D^{1/2} : D \text{ is a Bessel bound for } z\}$, from which the result follows. ☐

Proposition 2.29. *Let Z be a Hilbert left A-module, and $z \in Z$. Consider the frame operator asssociated with a single element z, θ_z. We have*

$$\|z\| = \|\theta_z\|^{1/2} \tag{2.21}$$

Proof. The required equation follows exactly from Lemma 2.6. ☐

An important corollary of invertibility of the frame operator associated to a frame is that we have a reconstruction formula for Z.

Corollary 2.9 (Reconstruction Formula)**.** *Let $\mathbf{z} = \{z_i\}_{i\in I}$ be a frame for a Hilbert left A-module Z, then for all $z \in Z$:*

$$\begin{aligned} z &= \theta_{\mathbf{z}}^{-1}\theta_{\mathbf{z}} z = \sum_{i\in I} {}_A\langle z, z_i\rangle\, \theta_{\mathbf{z}}^{-1} z \\ &= \theta_{\mathbf{z}}\theta_{\mathbf{z}}^{-1} z = \sum_{i\in I} {}_A\left\langle \theta_{\mathbf{z}}^{-1} z, z_i\right\rangle z_i = \sum_{i\in I} {}_A\left\langle z, \theta_{\mathbf{z}}^{-1} z_i\right\rangle z_i. \end{aligned} \tag{2.22}$$

An interesting connection between the algebraic structure of Z, and frames in Z, is summarized in the following result, which can be found in [AE20, Proposition 2.6].

Corollary 2.10. *Let Z be a Hilbert left A-module. Consider the frame operator associated with the fmaily $\mathbf{z} = \{z_i\}_{i\in I}$. We have that $\mathbf{z}$ is a frame if and only if $\{z_i\}_{i\in I}$ generates Z as a module.*

We can now go back to our original motivation. In the Hilbert space case, $Z = H$, we find that a $\mathbf{h} = \{h_i\}_{i\in I} \subseteq H$ is a frame for H if and only if for all $h \in H$:

$$h = \theta_{\mathbf{h}}\theta_{\mathbf{h}}^{-1} h = \sum_{i\in I} \left\langle h, \theta_{\mathbf{h}}^{-1} h_i\right\rangle_2 h_i.$$

Therefore, we can reconstruct any vector h using the not-necessarily orthogonal family $\{h_i\}_{i\in I}$. Furthermore, we also have a way of computing the non-unique coefficients: $c_i = \left\langle h, \theta_{\mathbf{h}}^{-1} h_i\right\rangle_2$.

In the reconstruction formula (2.22), notice that if we define $\mathbf{w} = \{w_i\}_{i \in I}$ via $w_i = \theta_{\mathbf{z}}^{-1} z_i$ then we can rewrite (2.22) to

$$\theta_{\mathbf{z},\mathbf{w}} = \theta_{\mathbf{w},\mathbf{z}} = \mathrm{Id}_Z \,.$$

The equation above motivates one of the key concepts in frame theory.

Definition 2.26. Let $\mathbf{z} = \{z_i\}_{i \in I}$ be a frame for a Hilbert left A-module Z, we say that the family $\mathbf{w} = \{w_i\}_{i \in I}$ is a **dual frame** of $\mathbf{z}$ if:

$$\theta_{\mathbf{z},\mathbf{w}} = \mathrm{Id}_Z \,. \tag{2.23}$$

Dual frames always exist for $\mathbf{z}$, and in particular if we choose $\mathbf{w} = \{\theta_{\mathbf{z}}^{-1} z_i\}_{i \in I}$, then $\mathbf{w}$ is called the **canonical dual frame** of $\mathbf{z}$.

Before we proceed, we need to justify the terminology dual *frame*. It only makes sense to show that dual frames are indeed frames.

Proposition 2.30. *Let $\mathbf{z} = \{z_i\}_{i \in I}$ be a frame for a Hilbert left A-module Z with upper frame bound D. Suppose $\mathbf{w} = \{w_i\}_{i \in I}$ is a dual frame for $\mathbf{z}$, then $\mathbf{w}$ is a frame.*

Proof. We need to prove that $\mathbf{w}$ is a Bessel family, and subsequently, that $\theta_{\mathbf{w}}$ is invertible. However, this all follows from the simple computation:

$$\begin{aligned} \theta_{\mathbf{w}} \circ \theta_{\mathbf{z}} &= \Psi_{\mathbf{w}} \circ \Phi_{\mathbf{w}} \circ \Psi_{\mathbf{z}} \circ \Phi_{\mathbf{z}} \\ &= \Psi_{\mathbf{w}} \circ \theta_{\mathbf{z},\mathbf{w}} \circ \Phi_{\mathbf{z}} \\ &= \Psi_{\mathbf{w}} \circ \Phi_{\mathbf{z}} \\ &= \theta_{\mathbf{w},\mathbf{z}} = \mathrm{Id}_Z \,. \end{aligned}$$

Therefore if we can show that $\theta_{\mathbf{w}}$ is a Bessel family, then it follows immediately it is invertible with inverse $\theta_{\mathbf{z}}$ from the computation above. However, the Bessel property follows from the exact same computation. Fix $z \in Z$, find $z' \in Z$ such that $\theta_z z' = z$. Then

$$\begin{aligned} {}_A\langle \theta_{\mathbf{w}} z, z \rangle &= {}_A\langle \theta_{\mathbf{w}}(\theta_{\mathbf{z}} z'), \theta_{\mathbf{z}} z' \rangle \\ &= {}_A\langle z', \theta_{\mathbf{z}} z' \rangle \\ &= {}_A\langle \theta_{\mathbf{z}} z', z' \rangle \end{aligned}$$

$$
\begin{aligned}
&\leqslant D\ {}_A\langle z', z'\rangle \\
&= D\ {}_A\langle \theta_{\mathbf{z}}^{-1} z, \theta_{\mathbf{z}}^{-1} z\rangle \\
&\leqslant D \left\|\theta_{\mathbf{z}}^{-1}\right\|^2\ {}_A\langle z, z\rangle .
\end{aligned}
$$

We have shown that $\theta_{\mathbf{w}}$ is a Bessel family and invertible, as required. □

Remark 2.14. Every frame generates a Parseval frame. To see this, let $\mathbf{z}$ be a frame so that $\theta_{\mathbf{z}}$ is invertible. Define $w_i = \theta_{\mathbf{z}}^{-1/2} z_i$, and $\mathbf{w} = \{w_i\}_{i\in I}$. The following computation shows everything that we need to know: $z = (\theta_{\mathbf{z}}^{-1/2} \circ \theta_{\mathbf{z}} \circ \theta_{\mathbf{z}}^{-1/2} z) z = \sum_{i\in I}\ {}_A\left\langle z, \theta_{\mathbf{z}}^{-1/2} z_i\right\rangle \theta_{\mathbf{z}}^{-1/2} z_i = \sum_{i\in I}\ {}_A\langle z, w_i\rangle w_i = \theta_{\mathbf{w}} z$ for all $z \in Z$. Therefore $\theta_{\mathbf{w}} = \mathrm{Id}_Z$, henceforth ${}_A\langle \theta_{\mathbf{w}} z, z\rangle = {}_A\langle z, z\rangle$ for all $z \in Z$, as required. From our computation, it also follows that the canonical dual frame of the generated Parseval frame is itself.

Dual frames are not unique, and one can consult [Chr03] for examples. There is quite a large amount of literature pertaining to dual frames, and their computation (for the canonical dual frame case, this is quite tricky as it involves inverting the frame operator) [DHM23, CC00, Sto15]. Another point to note, which ties together some of the concepts discussed in the previous sections, is that if $\mathbf{z}$ is a frame, with dual frame $\mathbf{w}$, then $\mathbf{z}$ and $\mathbf{w}$ are also dual pair of generators (see Remark 2.10) for the Hilbert left A-module Z.

As a final remark, note that there is an obvious way of reconstructing all that we've done here to Hilbert right C^*-modules. However, the theory of module frames for Hilbert right C^*-modules will not play much role in this study.

Chapter 3

Harmonic Analysis

One of the goals of this monograph is to present results in time-frequency analysis in terms of abstract harmonic analysis. We shall give an introduction to topological groups, developing the necessary language to discuss and prove the basic results of harmonic analysis on locally compact abelian (LCA) groups. We close the chapter with a short discussion of vector-valued integration on LCA groups.

We shall mostly follow the rapid introduction developed in the first chapter of Williams' monograph [Wil07] for the introductory material on general topological groups and locally compact spaces. Once we have established a certain sophistication of our tools, we will follow Folland's text [Fol15] as we delve into the abelian case in particular. We invite the reader to consult Appendix B for a concise overview of necessary net theoretic results. We will use net-convergence characterizations to capture topological results. We also assume some familiarity with measure and integration theory.

3.1 Topological Groups

Definition 3.1. A topologial group is a group G together with a topology τ such that

1. Points (i.e. singletons) are closed in (G, τ);
2. The map mult : $(s, r) \mapsto sr$ is continuous from $G \times G$ to G;
3. The map inv : $s \mapsto s^{-1}$ is continuous from G to G.

Regarding Item 1 above, we see that it is satisfied when the topology τ for G is assumed to be Hausdorff. As it turns out, it is an equivalent condition for Hausdorfness (see Lemma 3.3) below.

Proposition 3.1. *Finite products of topological groups is a topological group.*

Proof. We do the proof for two products, the proof can then be extended inductively. Let G_1, G_2 be topological groups. The product $G_1 \times G_2$ will then be equipped with the product topology, i.e. the initial topology[1] with respect to the projection maps $\text{proj}_1 : G_1 \times G_2 \to G_1$ and $\text{proj}_2 : G_1 \times G_2 \to G_2$. Suppose mult_i, inv_i are the multiplication and inversion maps for G_i $(i = 1, 2)$. We can define[2] the group multiplication $\text{mult} : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ and inversion $\text{inv} : G_1 \times G_2 \to G_1 \times G_2$ via following:

$$\text{mult}((s_1, s_2), (r_1, r_2)) := (s_1 r_1, s_2 r_2) = \big(f_1((s_1, s_2), (r_1, r_2)), f_2((s_1, s_2), (r_1, r_2))\big)$$
$$\text{inv}(s_1, s_2) := (s_1^{-1}, s_2^{-1}) = \big(g_1(s_1, s_2), g_2(s_1, s_2)\big)$$

Note that for $i = 1, 2$:

$$f_i((s_1, s_2), (r_1, r_2)) := \big((\text{mult}_i)(\text{proj}_1(s_1, s_2), \text{proj}_2(r_1, r_2)), (\text{mult}_i(\text{proj}_2(s_1, s_2), \text{proj}_2(r_1, r_2))\big)$$
$$g_i((s_1, s_2), (r_1, r_2)) := \big(\text{inv}_1(\text{proj}_1(s_1, s_2), \text{inv}_2(\text{proj}_2(r_1, r_2))\big).$$

Since each f_i and g_i can be written via continuous functions as above, it follows that they are also continuous. Therefore mult and inv are also continuous. □

The following gives us basic examples of topological groups.

Example 3.1. Let n, m, and d be positive integers:

1. Any group with discrete[3] topology is a topological group;
2. The groups $\mathbb{R}^n$, $\mathbb{T}^m$, and $\mathbb{Z}^d$ are all topological groups.

Lemma 3.1. *If G is a topological group, then inversion, $s \mapsto s^{-1}$ is a homeomorphism from G to G, while for any fixed $r \in G$ the maps induced by multiplication: $s \mapsto sr$ and $s \mapsto rs$ are also homeomorphisms from G to G.*

Proof. The inversion map $\text{inv} : s \mapsto s^{-1}$ is a continuous map whose inverse is itself, so it is a homeomorphism in G. Let $r \in G$. For multiplication, we shall only prove the homeomorphic

[1] meaning the *weakest topology* that one can endow on $G_1 \times G_2$ that makes each projection map continuous
[2] Indeed, for the rest of this document, we will always assume this definition.
[3] By 'discrete' we mean a space whose topology consists of all its subsets i.e. all the open (hence closed) sets are just its subsets.

property for the map $l_r : G \to G$ via $s \mapsto rs$. We already know l_r is continuous because it lifts through the continuous map $\mathrm{const}_r \times \mathrm{id} : G \to G \times G$ following the commutative diagram:

$$\begin{tikzcd}
 & G \times G \arrow[d, "\mathrm{mult}"] \\
G \arrow[ur, "\mathrm{const}_r \times \mathrm{id}"] \arrow[r, "l_r"] & G
\end{tikzcd}$$

where $\mathrm{const}_r \times \mathrm{id} : s \mapsto (r, s)$. It can easily be seen that the inverse of l_r is $l_{r^{-1}}$, we can apply the continuity result to $r^{-1} \in G$ to conclude that $l_{r^{-1}}$ is also continuous, which implies that l_r is a homeomorphism. We have a similar proof for the right multiplication $s \mapsto sr$. $\square$

Definition 3.2. Let X be a topological space. For $x \in X$, a family $\mathcal{N}_x$ of neighborhoods[4] of x is called a **neighborhood basis of x** if for every neighborhood W of x, there exists $N \in \mathcal{N}_x$ such that $N \subseteq W$.

For a topological group G, we shall use the following notations: for a fixed $r \in G$, $Vr := \{vr \in G : v \in V\}$, $rV = \{rv \in G : v \in V\}$, $V^{-1} = \{v^{-1} \in G : v \in V\}$, and $V^2 = \{v^2 \in G : v \in V\}$, for any subset $V \subseteq G$. Now with the language of neighborhood basis, we have the following corollary directly following Lemma 3.1.

Corollary 3.1. *Let G be a topological group. Let $\mathcal{N}_e$ be a neighborhood basis of the neutral element $e \in G$. Then $\{Nr : N \in \mathcal{N}_e\}$ and $\{rN : N \in \mathcal{N}_e\}$ are both neighborhood bases of r. In particular, if $\mathcal{N}$ consists of open neighborhoods, then $\beta = \{Vr : V \in \mathcal{N},\ r \in G\}$ is a basis for the topology of G.*

Lemma 3.2 ([Wil07, Lemma 1.12])**.** *Let V be a neighborhood of e in G. Then we have the following inclusion $V \subseteq \overline{V} \subseteq V^2$.*

Proof. We only need to prove $\overline{V} \subseteq V^2$. Let $s \in \overline{V}$, so every neighborhood of s intersects V. However, we know that sV^{-1} is a neighborhood of s, so that $sV^{-1} \cap V \neq \varnothing$, hence there exist $t, v \in V$ such that $st^{-1} = v \iff s = vt$, from which it follows that $s \in V^2$ as required. $\square$

Recall that X is a **regular space** if and only if every closed subset C and a point $p \in X \backslash C$ admit non-intersecting neighborhoods.

Lemma 3.3 ([Wil07, Lemma 1.13])**.** *If G is a topological group, then G is Hausdorff and regular.*

[4] Here we use the definition that W is a neighborhood of x iff x is an interior point of W. This means that W is not necessarily open.

Proof. We start with regularity. Let $C \subseteq G$ be closed and $s \notin C$. Without loss of generality we can assume that $e = s$ (c.f. Corollary 3.1). The set $W := G \backslash C$ is open, then it follows from the continuity of $\text{mult} : G \times G \to G$ that there exists an open neighborhood $V \ni e$ such that

$$\text{mult}(V \times V) = V^2 \subseteq W.$$

It follows from Lemma 3.2 that $\overline{V} \subseteq V^2 \subseteq W = G \backslash C$. Therefore $U := G \backslash \overline{V}$ is an open neighborhood of C while V is an open neighborhood of e such that $U \cap V = \varnothing$. This shows us that G is regular. Now by assumption, each singleton is closed in G, so it follows from regularity that G is also Hausdorff. □

3.2 Locally Compact Spaces

Definition 3.3. A Hausdorff topological space X is **locally compact** if each point $x \in X$ has a compact neighborhood.

Since we always assume Hausdorffness, it is not hard to prove the following equivalent definition.

Proposition 3.2. *A Hausdorff topological space X is locally compact if and only if for any $x \in X$, every open neighborhood of x has a compact closure.*

Proof. Let us first do the forward direction, and suppose X is locally compact. Take $x \in X$, and let $K_x \ni x$ be a compact neighborhood. Let $\operatorname{int} K_x$ be the interior of the compact set K_x, then $x \in \operatorname{int} K_x \subseteq K_x$. Since in a Hausdorff space: closed subsets of a compact set are compact, and compact sets are closed; it follows that $x \in \overline{\operatorname{int} K_x} \subseteq K_x$, therefore $\operatorname{int} K_x$ is an open neighborhood of x with compact closure.

Conversely, let $x \in X$ and let us suppose that every open neighborhood of x has a compact closure. Take $U_x \ni x$ an open neighborhood of x, then $x \in U_x \subseteq \overline{U_x}$, and $\overline{U_x}$ is compact by assumption. Therefore we have constructed a compact neighborhood of x via $\overline{U_x}$. □

We will use either of these equivalent notions for the rest of the manuscript.

Note that it follows from Proposition 3.2, and the fact that the closure of product of open sets is the product of closure of open sets that we have the following basic result.

Proposition 3.3. *Finite products of locally compact spaces are locally compact.*

We also have the following result which usually runs in the background of proofs regarding locally compact spaces.

Proposition 3.4. *Let X be a locally compact space. Then for every point $x \in X$ and every open neighborhood $U_x \ni x$ of x, there exists another open neighborhood $V_x \ni x$ such that $V_x \subseteq U_x$ where $\overline{V_x}$ is compact and $x \in V_x \subseteq \overline{V_x} \subseteq U_x$.*

Proof. Let $x \in X$, and let U_x be an open neighborhood of x, since X is locally compact then there exists a compact neighborhood of x, given by K_x such that $K_x \subseteq U_x$ (by possibly intersecting any compact neighborhood of x with U_x[5]) Since K_x is a neighborhood, there exists another open set $V_x \ni x$ such that $x \in V_x \subseteq K_x$. Because X is Hausdorff, we have the following inclusions:

$$x \in V_x \subseteq \overline{V_x} \subseteq \overline{K_x} = K_x \subseteq U_x.$$

We are done since $\overline{V_x}$ must be compact as a closed subset of a compact set. □

The hypothesis of Hausdorffness is assumed so that we can define local compactness in terms of neighborhoods instead of having to work with neighborhood bases. We do not lose much with this assumption since we will mostly be working with topological groups anyway.

Example 3.2. Speaking of topogical groups, the product $G = \mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^d \times F$ for a finite group F is an example of a locally compact *abelian* group. In fact such groups are called **elementary locally compact abelian groups**.

The next following results will be useful in generating examples of locally compact spaces coming from subsets.

Definition 3.4. A subset Y of a topological space X is said to be **locally closed** if each point in Y has an open neighborhood U in X such that $Y \cap U$ is closed in U with respect to the relative topology.

Lemma 3.4 ([Wil07, Lemma 1.25])**.** *If X is a topological space and $Y \subseteq X$, then the following are equivalent.*

1. *Y is locally closed in X;*

2. *Y is open in $\overline{Y}$;*

[5]It is not hard to show that the intersection of a compact set and an open set is compact.

3. *$Y = C \cap O$ for some closed set C, and open set O in Y.*

Proof. ($1 \implies 2$). Let $y \in Y$, we know by local closedness that there exists an open neighborhood P of y such that $P \cap Y$ is closed in P. Since $y \in P \cap \overline{Y}$ as well, it suffices to show that $P \cap \overline{Y} \subseteq Y$ (i.e. we would then have shown that every point in Y is interior to it with respect to the relative topology induced by $\overline{Y}$). If $x \in P \cap \overline{Y}$, so there exists a net $\{x_i\}_i$ in Y such that $x_i \to x$, since P is an open neighborhood of x, $\{x_i\}_i$ will eventually be in P, and so $\{x_i\}_i$ is eventually in $P \cap Y$. We know that $P \cap Y$ is closed in P, and therefore the limit $x \in P$ of the net $\{x_i\}_i$ which is eventually in $P \cap Y$ must be in it, i.e. $x \in P \cap Y \subseteq Y$. Therefore we have shown that $P \cap \overline{Y} \subseteq Y$.

($2 \implies 3$). If Y is open in $\overline{Y}$, then there exists an open set $O \subseteq X$ such that $Y = \overline{Y} \cap O$, we are done.

($3 \implies 1$). Suppose $Y = C \cap O$ as in the hypothesis. Let $y \in Y$, and choose $U = O$, an open neighborhood of y. We want to show that $U \cap Y$ is open in U, so let $\{x_i\}_i$ be a net in $U \cap Y$ that converges to x in U. We are done if we can show that $x \in Y$ due to Proposition B.1. Now, because $\{x_i\}_i \subseteq Y = C \cap O$, and C is closed, it follows that $x \in C$. Furthermore, by our choice of U, $x \in O = U$. Therefore $x \in C \cap O = U \cap Y$, as required. □

Lemma 3.5. *Let X be a locally compact space. $Y \subseteq X$ is locally compact if and only if Y is locally closed.*

Proof. Assume that Y is locally closed so that there exists, from Lemma 3.4, a closed set $C \subseteq X$ and open set $O \subseteq X$ such that $Y = C \cap O$. Fix $y \in Y$ with $U \ni y$ an open neighborhood in X such that $\overline{U}$ is compact. It now follows from the fact that the intersection of compact and open set is a compact set, that $y \in \overline{U} \cap Y = \overline{Y} \cap C \cap O \subseteq Y$ is compact in Y. That is, $y \in U \cap Y$ is an open neighborhood in the relative topology of Y, whose relative closure $\overline{U} \cap Y$ is compact. Therefore it follows from Proposition 3.2 Y is locally compact.

Conversely, suppose Y is locally compact so that for any fixed $y \in Y$, there exists an open neighborhood $U \ni y$ of X such that $U \cap Y$ is has a compact closure in Y. That is: $C_Y := \overline{U} \cap \overline{Y} \cap Y \subseteq Y$ is closed in Y. By Hausdorffness, C_Y is compact in Y, therefore closed in Y, hence also closed in X. To show local closedness in Y, we shall show that $U \cap Y$ is closed in U. Suppose $\{x_i\}_i$ is a net in $U \cap Y$ that converges in U. Since $U \cap Y \subseteq C_Y$ and C_Y is closed, then $x \in C_Y = \overline{U} \cap \overline{Y} \cap Y$, therefore $x \in Y$. We have shown local closedness of Y. □

Remark 3.1. Note in particular that any closed or open subsets S of a locally compact space X are locally compact since they can be trivially written as $S = S \cap X$, so the assertion follows from Lemma 3.4 and 3.5.

Example 3.3. The general linear group $\mathrm{GL}_n(\mathbb{R})$ of $n \times n$ matrices with entries in $\mathbb{R}$ is a locally compact space because it can be seen as an open subset of $\mathbb{R}^{n^2}$. This follows from the fact that the determinant map $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous, and therefore $\det^{-1}(\mathbb{R}\backslash\{0\}) = \mathrm{GL}_n(\mathbb{R})$ is open.

In the following, we endow a topology for continuous functions between two topological spaces. It will be useful in our discussion of the so-called Pontryagin dual (see Definition 3.9) of an LCA group.

Definition 3.5. Suppose that X and Y are topological space and that $C(X,Y)$ is the collection of continuous functions from X to Y. The **compact-open topology** on $C(X,Y)$ is the topology generated by (i.e. via subbasis) the sets

$$U(K,V) := \{f \in C(X,Y) : f(K) \subseteq V\}$$

for each compact $K \subseteq X$ and open $V \subseteq Y$.

In the locally compact space case, we have quite an easy net convergence criterion.

Proposition 3.5 ([Wil07, Lemma 1.30])**.** *Let X, Y be locally compact spaces, endow $C(X,Y)$ with the compact-open topology. We have the net convergence $f_i \to f$ in $C(X,Y)$ if and only if $x_i \to x$ in X implies $f_i(x_i) \to f(x)$ in Y.*

Proof. ($\Longrightarrow$). Suppose $f_i \to f$ in $C(X,Y)$ and $x_i \to x$ in X. Take any open neighborhood $V \ni f(x)$, and from this construct a compact subset K in X such that $x \in K$, and $f \in U(K,V)$ (these all exist from the fact that f and Proposition 3.4). Then $\{f_i\}_i$ is eventually in $U(K,V)$ and $\{x_i\}_i$ is eventually in K, together they imply that $\{f_i(x_i)\}_i$ is eventually in V. Since V is an arbitrary neighborhood of $f(x)$, we have $f_i(x_i) \to f(x)$ as required.
($\Longleftarrow$). Suppose $x_i \to x$ in X implies $f_i(x_i) \to f(x)$ in Y. We want to show that $f_n \to f$ in $C(X,Y)$, that is, $\{f_i\}_i$ is eventually in $U(K,V)$ for some compact K and open V. Suppose for contradiction that $f_i \nrightarrow f$, then we can find some (after careful negation of net-convergence definition) $U(K,V) \ni f$ and subnet $\{f_i\}_i$ (after passing to a subnet and relabeling) such that $f_i \notin U(K,V)$ for all i. By compactness of K (recall that K is compact iff every net has a convergent subnet from Proposition B.7), we can find an $\{x_i\}_i$ (again, after passing to a subnet and relabeling) such that $x_i \to x$. By hypothesis $f_i(x_i) \to f(x)$, since $\{f_i(x_i)\} \subseteq X\backslash V$ and $X\backslash V$ is closed, then $f(x) \in X\backslash V$, i.e. $f(x) \notin V$, which is a contradiction. $\square$

An important difference between locally compact Hausdorff spaces and compact Hausdorff spaces is that in the locally compact space case, we no longer have normality and paracompactness of the space for free, and therefore we do not exactly have: Urysohn's Lemma, Tietze extension Theorem, and existence of Partitions of Unity. However there exist separate versions of these results for the locally compact group case. We shall give their full statement for posterity. We prove only the version of Urysohn's Lemma since the proof for the other two are similar (this assumes that we are familiar with the proofs of the original versions of these results). The following comes from [Wil07, Lemma 1.41-1.43], and will be relevant in some of our constructions in Chapter 6.

Lemma 3.6 (Urysohn). *Suppose that X is a locally compact space and that V is an open neighborhood of a compact set K in X. Then there is an $f \in C_c(X)$ such that $\operatorname{Im} f \subseteq [0,1]$ and that $f(x) = 1$ when $x \in K$ and $f(x) = 0$ when $x \notin V$.*

Proof. Due to Proposition 3.4, each point $x \in K$ contains an open neighborhood $U_x \ni x$ whose closures are compact, and such that $U_x \subseteq \overline{U_x} \subseteq V$. Now by compactness of K, there exists $x_1, ..., x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} U_{x_i} \subseteq \bigcup_{i=1}^{n} \overline{U_{x_i}} \subseteq V.$$

Now, if we take $W = \bigcup_{i=1}^{n} U_{x_i}$, then it is a an open neighborhood of K whose closure is compact, such that $K \subseteq W \subseteq \overline{W} \subseteq V$. Because $\overline{W}$ is a compact Hausdorff space, then it is normal, and therefore we can use Urysohn's Lemma for normal spaces on $\overline{W}$: there exists an $h \in C(\overline{W})$ such that $0 \leqslant h \leqslant 1$ on $\overline{W}$ with $h \equiv 1$ on K, and $h \equiv 0$ on $\overline{W} \backslash W$. We define

$$f(x) = \begin{cases} h(x), & \text{if } x \in \overline{W}, \\ 0, & \text{if } x \notin \overline{W}. \end{cases}$$

By Pasting Lemma $f \in C_c(X)$, with all the required properties. □

The proofs for Tietze Extension and existence of partitions of unity follow the same principle of constructing a compact set out of X wherein we can use the known compact version of the result as in the one given for Urysohn's Lemma above.

Theorem 3.1 (Tietze Extension). *Suppose X is a locally compact space and that $K \subseteq X$ is compact. If $g \in C(K)$, then there is an $f \in C_c(X)$ such that $f_{|K} = g$.*

Lemma 3.7 (Existence of Partitions of Unity). *Suppose X is a locally compact space and that $\{U_i\}_{i=1}^n$ is a cover of a compact set $K \subseteq X$ by open sets with compact closures. Then for $i = 1, ..., n$ there exist $\varphi_i \in C_c(X)$ such that*

1. $0 \leqslant \varphi_i \leqslant 1$ *for all* $x \in X$*;*
2. $\operatorname{supp} \varphi \subseteq U_i$*;*
3. $\sum_{i=1}^n \varphi_i = 1$ *on* K *and* $\sum_{i=1}^n \varphi_i \leqslant 1$ *outside* K*.*

3.3 Locally Compact Groups and Their Subgroups

In this section we consider the case when the underlying topology on a topological group G is locally compact.

Lemma 3.8 ([Wil07, Lemma 1.34]). *A locally closed subgroup of a topological group is closed.*

Proof. Let H be a locally closed subgroup of a topological group G. Since local closedness for topological groups must be equivalent to local closedness using neighborhoods of the neutral element $e \in G$ (c.f. Corollary 3.1). Therefore there exists a neighborhood $W \ni e$ such that $W \cap H$ is closed in W. Next, by continuity of multiplication, we can find let U, V be neighborhoods of e in G such that

$$V^2 \subseteq U \subseteq U^2 \subseteq W.$$

Let $x \in \overline{H}$. Consider a net $\{x_i\}_i \subseteq H$ such that $x_i \to x$. By continuity of inversion, and since $x_i^{-1} \in H$, then $x_i^{-1} \to x^{-1} \in \overline{H}$. We know that $x^{-1}V$ is a neighborhood of x^{-1}, therefore $x^{-1}V \cap H \neq \varnothing$, so we can take $y \in x^{-1}V \cap H$. We now that $\{x_i\}_i$ is eventually in Vx, while by continuity of multiplication, $\{x_i y\}_i$ is eventually in

$$Vx(x^{-1}V \cap H) \subseteq V^2 \cap H \subseteq U \cap H \subseteq W \cap H.$$

In particular, $\{x_i y\}_i \subseteq U$ eventually, and therefore $xy \in \overline{U}$. Recall from Lemma 3.2 that $xy \in \overline{U} \subseteq U^2 \subseteq W$. So the net $\{x_i y\}_i \subseteq W \cap H$ has limit $xy \in W$, but $W \cap H$ is closed in $\overline{W}$, so $xy \in W \cap H$, but then

$$x = (xy)y^{-1} \in (W \cap H)(Vx \cap H) \subseteq H.$$

so $x \in H$. Therefore $H = \overline{H}$, hence H is closed. $\square$

Corollary 3.2. *If H is a subgroup of a locally compact group G such that H is either open, discrete, or locally compact, then H is closed.*

Proof. It suffices for all cases that we prove that H is a locally closed group due to Lemma 3.8. H being locally compact is equivalent to being locally closed (c.f. Lemma 3.5), every open set is also locally closed (c.f. Lemma 3.4). Finally, every discrete subset is also locally closed because all of its subsets are closed in the relative topology, therefore H is also closed in the discrete case. $\square$

Let H be a subgroup of a topological group G, the group of (left) cosets G/H can also be endowed with a topological group structure via the topology that makes the map

$$q : G \to G/H, \ g \mapsto gH.$$

continuous (aka the final topology on G/H with respect to q). Explicitly, $U \subseteq G/H$ is open if and only if

$$q^{-1}(U) \subseteq G$$

is open in G.

Lemma 3.9 ([Wil07, Lemma 1.44])**.** *If H is a subgroup of a topological group G, then the quotient map $q : G \to G/H$ is open and continuous.*

Proof. By our definition of the topology on G/H, q is automatically continuous. We only need to show open-ness. If V is open in G, we want to show that $q(V)$ is open in G/H. We have

$$\begin{aligned} q^{-1}(q(V)) &= \{g \in G : gH \in q(V)\} \\ &= \{g \in G : gH = vH, \ \text{for some } v \in V\} \\ &= \{g \in G : g \in vH, \text{for some } v \in V\} \\ &= \bigcup_{h \in H} Vh. \end{aligned}$$

Since each $Vh \cong V$ is open, then $q(V)$ is open in G/H. $\square$

At this point, it will be very useful to review the net-criterion for open surjective maps given by Proposition B.9 in the appendix. A straightforward application of this proposition on $q : G \to G/H$ gives us the following lemma.

Lemma 3.10 ([Wil07, Lemma 1.46]). *Suppose that H is a subgroup of a topological group G and that $\{s_i H\}_{i\in I}$ converging to sH. Then there is a subnet $\{s'_j\}_{j\in J}$ of $\{s_i\}_{i\in I}$ and another net $\{h_j\}_{j\in J}$ such that $s'_j h_j \to s$.*

Corollary 3.3 ([Wil07, 1.48]). *Let H be a subgroup of a topological group G. The left coset space G/H with the quotient topology is Hausdorff iff H is closed in H. If G is locally compact, then G/H is locally compact. If G is second countable, then G/H is second countable.*

Proof. We first prove that H closed $\implies$ G/H Hausdorff. Recall that a space is Hausdorff if and only if every convergent net in it has unique limit (c.f. Proposition B.2). Suppose $\{s_i H\}_i \subseteq G/H$ converges to both sH and rH in G/H. By Lemma 3.10, there exists (after passing to a subnet and relabeling) a net $\{h_i^s\}_i$ and $\{h_i^r\}_i$ in G such that

$$s_i h_i^s \to s, \text{ and } s_i h_i^r \to r.$$

Define $\{h_i\}_i = \{(h_i^s)^{-1} h_i^r\}$. We have

$$h_i = (h_i^s)^{-1} h_i^r = (h_i^s)^{-1} \cdot s_i^{-1} \cdot s_i \cdot h_i^r \to r^{-1} s$$

by continuity of inversion and multiplication. Since H is closed, then $r^{-1}s \in H$, which implies that $sH = rH$, so indeed the limit is unique, therefore G/H is Hausdorff.

Conversely, assume H is not closed, we want to show G/H is not Hausdorff. Then we negate the net-criterion for being a closed set (c.f. Corollary B.1), and find some net $\{h_i\}_i$ in H such that $h_i \to s$ with $s \in G\backslash H$. Now q is continuous, therefore

$$eH = q(h_i) \to q(s) \in \overline{\{eH\}}.$$

We have found a net, the constant net $\{eH\}_i$, to converge to two limits eH and sH, but $H \neq sH$ since $s \in G\backslash H$.

Next we want to prove that if G is second countable, then so is G/H. Let $\{V_n\}_n$ be a countable basis for G. Define $U_n := q(V_n)$ in G/H for each n. Since $\{V_n\}_n$ is a basis, and q is continuous, for each open $W \subseteq G/H$, and $s \in G$ such that $s \in q^{-1}(W)$, there is some V_n such that

$$s \in V_n \subseteq q^{-1}(W),$$

which implies that $q(s) \in q(V_n) = U_n \subseteq W$. By arbitrariness of s, we see that $\{U_n\}_n$ is also a basis for W, which is countable, so G/H is second countable.

Lastly, suppose G is locally compact, then we can find a basis in G (c.f Proposition 3.4) whose closures are compact. Now we can run the same argument as in the proof for second countability of G/H above and find a countable basis of G/H whose closures are all compact (since continuous functions send compact sets to compact sets). The resulting basis whose closures are all compact shows that G/H is locally compact. $\square$

We can use the same techniques as in the proof above to show that G/H is also a locally compact group.

Corollary 3.4. *Let G be a locally compact group, and H a closed subgroup, then G/H is also a locally compact group with respect to the usual coset operations.*

Proof. We only give proof for continuity of inversion. The proof for multiplication is similar. We want to show that if we have net-convergence $s_iH \to sH$ in G/H, then $s_i^{-1}H \to s^{-1}H$ as well. We use the priciple of convergence presented in Proposition B.3 Now we can take any subnet of $\{s_iH\}_{i\in I}$ relabel it and call it $\{s_iH\}_{i\in I}$ still. By Lemma 3.10, there is a subnet $\{s'_j\}_{j\in J}$ of $\{s_i\}_{i\in I}$ and another net $\{h_j\}_{j\in J}$ such that $s'_jh'_j \to s$. But inversion is continuous so that $h_j^{-1}(s'_j)^{-1} \to s^{-1}$. But q is continuous, so

$$q(h_j^{-1}(s'_j)^{-1}) = (s'_jh_j)^{-1} \to s^{-1}H.$$

This implies

$$(s'_j)^{-1}H = h_j(s'_jh_j)^{-1}H = (s'_jh_j)^{-1}H \to s^{-1}H.$$

What we have essentially shown is that every subnet of $\{s_i^{-1}\}_{i\in I}$ has a subnet that converges to s^{-1} as required. $\square$

3.4 The Haar Measure

We are now ready to discuss integration on locally compact space.

Definition 3.6. Let X be a locally compact space and $\mathrm{Bor}(X)$ be its Borel space (i.e. the σ-algebra generated by open sets of X). A (Borel) measure $\mu : \mathrm{Bor}(X) \to \mathbb{R}_{\geqslant 0}$ on X is called a **Radon measure** if μ satisfies the following:

1. (Inner-regularity). For each open $V \subseteq X$, we have

$$\mu(V) = \sup\{\mu(C) : C \subseteq V,\ C \text{ compact}\};$$

2. (Outer-regularity). For each $E \in \mathrm{Bor}(X)$,

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}.$$

In the special case that $X = G$ is a locally compact group, we call μ a **left Haar measure** on G if we further add

3. (Left-Translation Invariance). For all $s \in G$ and $E \in \mathrm{Bor}(X) : \mu(sE) = \mu(E)$.

Remark 3.2. Note that inner-regularity above makes sense because we defined locally compact spaces to be automatically Hausdorff, therefore all compact sets of X are closed, hence are inside $\mathrm{Bor}(X)$.

We also have an obvious analogous definition for right Haar measure by adding in right-translation invariance. We shall abbreviate and use the term "Haar measure" to mean left Haar measure; when we mean right Haar measure, we shall say so explicitly.

Proposition 3.6. *Let G be a locally compact group with nontrivial Haar measure μ. Then*

1. $\mu(U) > 0$ *for all nonempty open set* U;
2. $\mu(C) < \infty$ *for all compact set* C;
3. $\displaystyle\int_G f(s)d\mu(s) > 0$ *for all* $f \in C_c^+(G)\backslash\{0\}$.

Proof.

1. Suppose for contradiction that there exists a nonempty open set U such that $\mu(U) = 0$. Let K be any compact set in G, then we can cover K by finite translations of U, i.e. there exists $x_1, ..., x_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} x_i U.$$

This implies $\mu(K) \leqslant \sum_{i=1}^{n} \mu(x_i U) = \sum_{i=1}^{n} \mu(U) = 0$. Since K is arbitrary and μ is inner-regular, we find that

$$\mu(G) = \sup\{\mu(K) : K \subseteq G,\ K \text{ compact}\} = 0.$$

Therefore $\mu \equiv 0$, which is a contradiction to nontriviality.

2. Let $C \subseteq G$ be compact. Take $E \in \mathrm{Bor}(G)$ such that $\mu(E) < \infty$ (exists by nontriviality of μ). By outer regularity, there exists some open set $U \subseteq G$ with

$$\mu(U) < \mu(E) + 1,$$

and $E \subseteq U$. By Item 1 above,

$$0 < \mu(U) \leqslant \mu(E) + 1.$$

As in the preceeding argument, we can find some $x_1, ..., x_n \in G$ such that

$$C \subseteq \bigcup_{i=1}^{n} x_i U.$$

Therefore $\mu(C) \leqslant \sum_{i=1}^{n} \mu(x_i U) \leqslant n\mu(U) < \infty$.

3. Let $f \in C_c^+(G)\backslash\{0\}$, define the open set

$$U := f^{-1}\left(\left(\frac{1}{2}\|f\|_\infty, +\infty\right)\right) = \left\{x \in G : f(x) > \frac{1}{2}\|f\|_\infty\right\} \neq \varnothing.$$

We then have that

$$f(x) \geqslant \frac{1}{2}\|f\|_\infty \chi_U(x), \qquad \forall x \in G.$$

By monotonicity of the integral

$$\int_G f(s)d\mu(s) \geqslant \frac{1}{2}\|f\|_\infty \int_G \chi_U(s)d\mu(s) = \frac{1}{2}\|f\|_\infty \mu(U) > 0. \tag{3.1}$$

□

Remark 3.3. Note that the estimate in (3.1) shows us that if $\int_G f(s)d\mu(s) = 0$, then we see that $\|f\|_\infty = 0$, which implies $f = 0$ since f is continuous. We can use this to show that

$$\|f\|_1 := \int_G |f(s)|d\mu(s)$$

is positive-definite on $C_c(G)$, and in fact defines a norm. The notion of integrable space $L^1(G)$ can be extended to $1 \leqslant p < \infty$, and we find that

$$\|f\|_p := \left(\int_G |f(s)|^p d\mu(s)\right)^{1/p}$$

is also a norm on $C_c(G)$. We can define the $L^p(G)$ space to be the norm-completion of $C_c(G)$ with respect to $\|\cdot\|_p$. Unfortunately this does not extend to $p = \infty$, since the completion of $C_c(G)$ with respect to the sup-norm is $C_0(G)$. The advantage of this definition is that we skip a lot of measure-theoretic considerations when we just work on the seemingly simpler dense subspace $C_c(G)$. Of course we are working backwards from the classical results here, one can define each $L^p(G)$ space as an actual function space (modulo null-equivalence) which has $C_c(G)$ as a dense subspace, for this point of view we refer to [Rud86].

Remark 3.4. Functions in $L^1(G)$ are called, **integrable**, given a Radon measure ν (which may be complex or just positive), we can induce another Radon measure by integrating against some integrable function, say $f \in L^1(G)$. For each measurable $E \in \mathrm{Bor}(G)$, we define the new measure denoted by $fd\nu$ via

$$(fd\nu)(E) := \int_E f(s)d\nu(s). \tag{3.2}$$

Locally compact groups are well-behaved enough that we basically have uniqueness of Haar measures on G. Here we shall cite two of the most important results on locally compact spaces, whose proofs are very technical (the results themselves are enough for us), hence we refer to the following sources: [Fol15, Theorem 2.10] and [Rud86, Theorem 2.14] respectively.

Theorem 3.2 (Existence and Uniqueness of Haar Measures on Locally Compact Groups)**.** *Every locally compact group G has a Haar measure that is unique in the following sense: if μ and ν are two Haar measures on G, then there exists a unique positive real number $\lambda > 0$ such that*

$$\mu = \lambda\nu.$$

This existence and uniqueness theorem is usually paired with the following duality result.

Theorem 3.3 (Riesz-Markov-Kakutani Representation Theorem)**.** *Let X be a locally compact space. Then for any continuous linear functional on $C_c(X)$, i.e. $\psi : C_c(X) \to \mathbb{C}$, there exists a unique Radon measure $\mu_\psi : \mathrm{Bor}(X) \to \mathbb{C}$ such that for all $f \in C_c(X)$:*

$$\psi(f) = \int_X f(x) d\mu_\psi(x).$$

Before we derive some corollaries using the previous two major results, let us first set some notation. Let G be a locally compact group, for each $y \in G$, define for each function $f : G \to \mathbb{C}$ and $s \in G$

1. $L_y f(s) = f(y^{-1}s)$
2. $R_y f(s) = f(sy)$.

For the rest of the manuscript, we shall also reserve the greek letter μ or μ_G for Haar measures, and ν for general Radon measures.

Proposition 3.7 ([Fol15, Proposition 2.9])**.** *Let μ be a Radon-measure on a locally compact group G, and let $\widetilde{\mu}(E) := \mu(E^{-1})$ for each $E \in \mathrm{Bor}(G)$. Then:*

1. *μ is a Haar measure if and only if $\widetilde{\mu}$ is a right Haar masure;*
2. *μ is a Haar measure if and only if*

$$\int_G (L_y f)(s) d\mu(s) = \int_G f(s) d\mu(s) \tag{3.3}$$

for all $f \in C_c^+(G)$, and $y \in G$.

Proof.

1. μ is a Haar measure if and only if for all $s \in G$ and $E \in \mathrm{Bor}(G)$, $\mu(sE) = \mu(E) = \mu((E^{-1}s^{-1})^{-1}) = \widetilde{\mu}(E^{-1}s^{-1}) = \widetilde{\mu}(E^{-1})$. This is sufficient.

2. We only need to prove this for simple functions. Suppose that μ is a Haar measure. Then for any simple function $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, for $E_i \in \mathrm{Bor}(G)$ in each $i = 1, ..., n$, we have

$$\begin{aligned}\int_G (L_y\varphi)(s)d\mu(s) &= \sum_{i=1}^n a_i\mu(yE_i)\\ &= \sum_{i=1}^n a_i\mu(E_i)\\ &= \int_G \varphi(s)d\mu(s).\end{aligned}$$

Since any $C_c^+(G)$ can be approximated by simple functions[6], we know that the formula extends to $C_c^+(G)$.

Conversely, suppose the Equation (3.3) is true for all $C_c^+(G)$, then by the fact that $\operatorname{span} C_c^+(G) = C_c(G)$, and that $C_c(G)$, then Equation (3.3) extends to all of $C_c(G)$. Now if we define for each $y \in G$, the Borel measure $\mu_y(E) := \mu(yE)$ for each $E \in \mathrm{Bor}(G)$. We find that from the previous computation

$$\int_G (L_y f)(s)d\mu(s) = \int_G f(s)d\mu_y(s) = \int_G f(s)d\mu(s)$$

for all $f \in C_c(G)$. Therefore by Riesz-Markov-Kakutani 3.3, we have that $\mu(yE) = \mu(E)$ for all $E \in \mathrm{Bor}(G)$. Since $y \in G$ is arbitrary, then μ is left-invariant, hence it is a Haar measure.

□

Remark 3.5. We can have a linear functional version of Item 2 in the proposition above. If I is a positive linear functional on $C_c(G)$, then its corresponding Radon measure is a Haar measure if and only if $I(L_y\psi) = I(\psi)$ for all $y \in G$.

It will also be eventually useful for us to formally define here what we mean when we apply the translation transforms L_y and R_y for $y \in G$ to a Radon measure. For a Radon measure ν, $L_y\nu$ and $R_y\nu$ respectively are the Radon measures that are completely characterized by the following: for all $\psi \in C_c(G)$,

$$\int_G \psi(s)d(L_y\nu)(s) = \int_G (L_y\psi)(s)d\nu(s)$$

[6] Recall that the approximating sequence in simple approximation theorem in particular is an increasing sequence, so we can switch limits and integrals by monotone convergence theorem.

$$\int_G \psi(s)d(R_y\nu)(s) = \int_G (R_y\psi)(s)d\nu(s)$$

Studying the scaling factors that appear due to the Haar existence and uniqueness theorem lead us to the so called **modular function** $\Delta : G \to \mathbb{R}_{\geqslant 0}$, where $\Delta(r)$ is defined to be the unique scalar such that $\Delta(r)\mu_G = (R_r\mu_G)$ for each $r \in G$. We note that Δ is well-defined and does not depend on the initial choice of the Haar measure. Let us suppose that μ_1 and μ_2 are Haar measures for G, then there exists $c > 0$ such that $c\mu_2 = \mu_1$. Let Δ_1 and Δ_2 be the associated modular functions for μ_1 and μ_2 respectively. It follows that $c\Delta_2(g)\mu_2 = \Delta_1(g)\mu_1$ for all $g \in G$. However by uniqueness of Haar measures:

$$\frac{c\Delta_2(g)}{\Delta_1(g)} = c,$$

which implies that $\Delta_1(g) = \Delta_2(g)$ for all $g \in G$.

The next result is needed to prove continuity of the modular function. We refer the reader to [Wil07, Lemma 1.62]. We shall focus on its implications.

Lemma 3.11. *Suppose that $f \in C_c(G)$ and that $\varepsilon > 0$. Then there is a neighborhood V of the neutral element e in G such that $s^{-1}r \in V$ or $sr^{-1} \in V$ implies that*

$$|f(s) - f(r)| < \varepsilon.$$

The result above is nothing more than the usual uniform continuity that we are familiar with, one can see this quite clearly in the case $G = \mathbb{R}$.

Corollary 3.5. *Let G be a locally compact group. If $f \in C_c(G)$, then for any net $\{y_i\}_i \subseteq G$ such that $y_i \to y \in G$ we have*

$$\|R_{y_i}f - R_yf\|_\infty \to 0,$$
$$\|L_{y_i}f - L_yf\|_\infty \to 0.$$

Lemma 3.12. *Let μ be a Haar measure on a locally compact group G. The modular function $\Delta : G \to \mathbb{R}_{\geqslant 0}$ is well-defined (i.e. independent of the choice of Haar-measure), is a continuous (multiplicative) group homomorphism, and satisfies:*

$$\Delta(r)\int_G (R_rf)(s)d\mu(s) = \int_G f(s)d\mu(s) \tag{3.4}$$

for all $r \in G$ and $f \in C_c^+(G)$.

Proof. We have already shown well-definedness of $\Delta : G \to \mathbb{R}_{>0}$. The Equation (3.4) also follows quite easily by considering integrals of simple functions under the Haar measures $R_r\mu_G$ and μ_G (and extending the result via simple approximation). The homomorphism property also follows quite easily, let $s, t \in G$ and $E \in \mathrm{Bor}(G)$:

$$\Delta(st)\mu_G(E) = R_{st}(\mu_G)(E) = \mu_G(E(st)) = \Delta(t)\mu_G(Es) = \Delta(s)\Delta(t)\mu_G(E).$$

Hence $\Delta(st) = \Delta(s)\Delta(t)$. Since $\Delta > 0$, this is enough to prove that $\Delta : G \to \mathbb{R}^*$ is a homomorphism[7].

Finally, to show continuity, let $f \in C_c^+(G)$, and let $y_i \to y$ in G, it follows from Corollary 3.5 that $R_{y_i}f \to R_y f$ in where the support of $R_{y_i}f$ is eventually in the support of $R_y f$. Therefore $\int_G f(sy_i^{-1})d\mu_G(s) \to \int_G f(sy^{-1})d\mu_G(s)$. It follows from Equation (3.4) that:

$$\Delta(y_i)\int_G f(s)d\mu_G(s) \to \Delta(y)\int_G f(s)d\mu_G(s).$$

Therefore $\Delta(y_i) \to \Delta(y)$, as required. □

Definition 3.7. We say that a locally compact group G is **unimodular** if its modular function satisfies $\Delta \equiv 1$.

The following result is immediate from the definition of the modular function. This also shows us why it is usually treated informally as a gauge for measuring how close to being right-invariant a Haar measure is on a given locally compact group .

Lemma 3.13. *Let G be a locally compact group, then any Haar measure μ in G is simultaneously left and right invariant if and only if G is unimodular.*

As a quick corollary, we see that an abelian group is always unimodular.

Lemma 3.14. *If G is compact, then G is unimodular.*

Proof. If G is compact, then by continuity of the modular function, $\Delta(G)$ is compact in $\mathbb{R}_{\geqslant 0}$, hence by Heine-Borel, it must be closed and in particular, bounded. Suppose for contradiction that $\Delta \not\equiv 1$, then there exits $s \in G$ such that (WLOG, otherwise take s^{-1}) $\Delta(s) > 1$, then from

[7] It is important to recall that we treat $\mathbb{R}_{>0}$ as a subgroup under the multiplicative structure of $\mathbb{R}^*$

the homomorphic-property of the modular function, $\Delta(s^n) = \Delta(s)^n \to +\infty$. We have found a contradiction to boundedness of $\Delta(G)$. $\square$

3.5 The Pontryagin Dual

For the rest of the section, we now assume that our group G is locally compact, abelian (and so we shall use the additive notation when dealing with the operations in G), *and second countable.* Second countability of G assures us that G, and hence $G \times G$ will have a σ-finite Haar measure, which will readily give us the Tonelli/Fubini theorems for free (i.e. we can switch integrals at will), along with the fact that $L^p(G)$ for each $1 \leqslant p < \infty$ is separable [Bre11, Theorem 4.13]. Therefore, whenever we deal with density statements involving these spaces, we can comfortably think of the associated nets as just sequences. In particular, we can invoke Lebesgue dominated convergence theorem. For our purposes, nothing much is lost when we make this additional assumption, and in fact we welcome this as we expect "lattices", which are discrete subgroups of the time-frequency plane $G \times \widehat{G}$, to be actually countable[8]

It still takes some time to develop the following result even with Tonelli/Fubini fully available to us. The following result can be found in [Fol15, Proposition 2.40].

Proposition 3.8. *$L^1(G)$ is a commutative Banach $*$-algebra with the following multiplication and involution respectively:*

$$\begin{aligned} (f * g)(s) &= \int_G f(r)g(s-r)d\mu(r) \\ f^*(s) &= \overline{f(-s)}, \end{aligned} \tag{3.5}$$

for all $f, g \in L^1(G)$ and $s \in G$.

Definition 3.8. A net $\{u_i\}_i$ of self-adjoint elements in $L^1(G)$ whose norm is bounded by 1 is called an **approximate identity** for $L^1(G)$ if for all $f \in L^1(G)$, $f * u_i \to f$ in $\|\cdot\|_1$-norm.

Proposition 3.9 ([Wil07, Lemma 1.72])**.** *$L^1(G)$ has an approximate identity in $C_c(G)$.*

It would be wise to recall one of the fundamental results regarding the Gelfand transform on commutative Banach algebras. Recall that if A is a commutative Banach algebra, we denote by

$$\mathcal{M}_A := \{h : A \to \mathbb{C} \mid h \text{ is a non-zero complex algebra homomorphism}\}$$

[8] There exists pathological spaces where a discrete subspace is not necessarily countable.

its **maximal ideal space**. We can equip $\mathcal{M}_A$ with the relative topology it inherits from the continuous dual space A^* equipped with the weak-$*$ topology (it can be shown that the functionals in $\mathcal{M}_A$ are automatically continuous).

Theorem 3.4 ([Mur90])**.** *Given a commutative Banach algebra A, the maximal ideal space $\mathcal{M}_A$ is a locally compact space, and the Gelfand transform $\Phi : A \to C_0(\mathcal{M}_A)$ defined via*

$$\Phi(a) = \widehat{a}, \quad \textit{where}$$
$$\widehat{a}(h) = h(a)$$

for all $a \in A$ and $h \in \mathcal{M}_A$ is a norm-decreasing homomorphism whose image is dense in $C_0(\mathcal{M}_A)$. In the case the A is a C^*-algebra, then Φ becomes an isometric $*$-isomorphism [9].

Definition 3.9. We define $\widehat{G}$ to be the set of continuous homomorphisms from G to $\mathbb{T}$. Under pointwise multiplication $\widehat{G}$ is a group called the **Pontryagin dual** of G (a.k.a **character group**).

The problem of endowment of topology to the Pontryagin dual $\widehat{G}$ is related to the maximal ideal space $\mathcal{M}_{L^1(G)}$. It turns out that [Wil07, Theorem 1.76] the mapping

Lemma 3.15. *The map $\omega \mapsto h_\omega$ where*

$$h_\omega(f) = \int_G f(s)\omega(s)d\mu(s), \qquad \forall f \in L^1(G) \tag{3.6}$$

defines a bijective function from $\widehat{G}$ onto $\mathcal{M}_{L^1(G)}$. The inverse $\mathcal{M}_{L^1(G)} \to \widehat{G}$ is denoted by $h \mapsto \omega_h$.

The bijection between the Pontryagin dual $\widehat{G}$ and the maximal ideal space $\mathcal{M}_{L^1(G)}$ allows us to define a topology on $\widehat{G}$ by identifying all open sets of $\widehat{G}$ with the open sets of $\mathcal{M}_{L^1(G)}$ using the bijection (3.6) (i.e. the initial (resp. final) topology with respect to $\omega \mapsto h_\omega$ (resp. $h \mapsto \omega_h$)). In terms of nets, we have the following equivalences:

$$\omega_i \to \omega \text{ in } \widehat{G} \iff h_{\omega_i} \to h_\omega \text{ in } \mathcal{M}_{L^1(G)}. \tag{3.7}$$

Equivalently,

$$h_i \to h \text{ in } \mathcal{M}_{L^1(G)} \iff \omega_{h_i} \to \omega_h \text{ in } \widehat{G}. \tag{3.8}$$

[9] One has to redefine $\mathcal{M}_A$ in this case to be the collection of all $\varphi : A \to \mathbb{C}$ of algebra $*$-homomorphisms

There is, however, a simpler way to describe the topology on $\widehat{G}$ [Wil07, Corollary 1.79].

Lemma 3.16. *The topology on $\widehat{G}$ identified with $\mathcal{M}_{L^1(G)}$ coincides with the relative compact-open topology on $\widehat{G} \subseteq C(G, \mathbb{T})$*

We could have just easily defined the topology on $\widehat{G}$ to be the relative compact-open topology on $\widehat{G}$ in consideration of Lemma 3.16. However, our original definition synergizes well with the upcoming definition of the *Fourier-transform*, which makes the explicit connection between $\widehat{G}$ and $\mathcal{M}_{L^1(G)}$ pay-off. Furthermore, it also captures a deeper truth about the representation theory of locally compact (non necessarily abelian) groups. The characters on G can be seen as *irreducible* unitary representations $\rho : G \to U\mathcal{L}(H)$ for some Hilbert space H, since in the case that G is abelian, we necessarily have that $\mathcal{L}(H) \cong \mathbb{C}$ (i.e. ρ is one-dimensional) hence $U\mathcal{L}(H) = \mathbb{T}$. Therefore the study of the dual group $\widehat{G}$ generalizes to the study of the irreducible unitary representations of G, henceforth denoted by $\mathrm{Irr}(G)$ in the non-abelian case. The problem now is that we no longer have the accessible compact-open topology when working with representations, however there is still a way to map $\mathrm{Irr}(G)$ to $*$-representations of $L^1(G)$ that parallels Equation (3.6). In fact this is implemented by taking the *integrated form* of ρ. In any case, there is a way to define an analogous space $\mathcal{M}_{L^1(G)}$ equipped with its own topology called the *Jacobson topology*, when G is non-abelian. There also is a map from unitary irreducible representations of G to this '$\mathcal{M}_{L^1(G)}$', the topology on $\mathrm{Irr}(G)$ is now the initial topology induced by the map $\mathrm{Irr}(G) \to \mathcal{M}_{L^1(G)}$, called *Fell's topology.* While interesting , we do not find concrete use of this generality in our current work aside from the ideas surrounding it, like the construction of *group C^*-algebras* (see Section 5.2, Chapter 5).

Remark 3.6. We shall also call any element in $\widehat{G}$, a **character** of G. This ties up nicely with the identification of $\widehat{G}$ with $\mathcal{M}_{L^1(G)}$ as they are also called the characters of $L^1(G)$ in the literature.

Corollary 3.6. *$\widehat{G}$ is a locally compact group.*

Proof. This follows from the trivial homeomorphism $\widehat{G} \cong \mathcal{M}_{L^1(G)}$ and Gelfand's Theorem 3.4. □

Proposition 3.10. *We have the following elementary properties of the dual space In the following, G and $\{G_i\}_{i=1}^n$ are all LCA groups.*

1. *If G is compact, then $\widehat{G}$ is discrete.*

2. *If G is finite, then $G \cong \widehat{G}$.*

3. $\widehat{\prod_{k=1}^{n} G_k} \cong \prod_{k=1}^{n} \widehat{G_k}$.

Proof.

1. Let us assume that G is compact. We shall use the compact-open topology in $\widehat{G}$ (c.f. Lemma 3.16). To show discreteness, we fix $\chi_0 \in \widehat{G}$ and show that there exists an open neighborhood of χ_0 not containing any other point in $\widehat{G}$. First, we start by proving that for any $\varepsilon > 0$, the set

$$U(G, \chi_0, \varepsilon) := \left\{ \chi \in \widehat{G} : \sup_{g \in G} |\chi(g) - \chi_0(g)| < \varepsilon \right\} \tag{3.9}$$

is an open set in the compact-open topology. Fix $\chi \in U(G, \chi_0, \varepsilon)$. Using continuity of χ, we have that for each $g \in G$, there exists an open neighborhood $V_g \in G$ of g such that:

$$\chi(\overline{V_g}) \subseteq \overline{\chi(V_g)} \subseteq \overline{B_{\varepsilon/4}(\chi(g))} \subseteq B_{\varepsilon/3}(\chi(g)) \tag{3.10}$$

where we use the notation $B_\delta(z)$ for the open ball of radius $\delta > 0$ centered at $z \in \mathbb{C}$. By compactness of G, there exists $g_1, ..., g_n$ with corresponding open sets $V_{g_1}, ..., V_{g_n}$ covering G that possess Property (3.10). Trivially, $\chi \in \bigcap_{i=1}^{n} U(\overline{V_{g_i}}, B_{\varepsilon/3}(\chi(g_i))$. Furthermore, if $\eta \in \bigcap_{i=1}^{n} U(\overline{V_{g_i}}, B_{\varepsilon/3}(\chi(g_i))$, it follows from the fact that $\{V_{g_i}\}_{i=1}^{n}$ covers G that for any $g \in G$

$$|\eta(g) - \chi_0(g)| \leqslant |\eta(g) - \chi(g_i)| + |\chi(g_i) - \chi_0(g)| < \frac{2\varepsilon}{3} < \varepsilon \tag{3.11}$$

for some $i = 1, ..., n$ chosen such that $g \in V_{g_i}$. It follows that $\chi \in \bigcap_{i=1}^{n} U(\overline{V_{g_i}}, B_{\varepsilon/3}(\chi(g_i)) \subseteq U(G, \chi_0, \varepsilon)$ therefore $U(G, \chi_0, \varepsilon)$ is open in compact-open topology of $\widehat{G}$ for any $\varepsilon > 0$. Now take $\varepsilon = 1$, and suppose for contradiction that $U(G, \chi_0, 1) \neq \{\chi_0\}$, then there exists $g_0 \in G$ such that $\chi_0(g_0) \neq 1$. However, this implies that there exists an $n \in \mathbb{N}$ such that

$$1 < |\chi_0(g_0)^n - 1| = |\chi_0(g_0^n) - \mathrm{id}_{\widehat{G}}(g_0^{-n})| = |\chi_0(g_0^n) \left(\overline{\chi_0}(g_0^n) - \chi_0(g_0^{-n})\right)| = 0,$$

which is a contradiction. Therefore $U(G, \chi_0, 1) = \{\chi_0\}$, which shows that $\widehat{G}$ is isolated.

2. If G is a finite abelian group, then by the structure theorem $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ for $n_1, ..., n_k \in \mathbb{N}$. For any homomorphism $\chi : \mathbb{Z}_n \to \mathbb{T}$ we may always find some $k =$

$0, 1, ..., n-1$ such that $\chi(m) = \chi_k(m) = e^{2\pi i \frac{km}{n}}$. The map $k \mapsto \chi_k$ is an isomorphism from $\mathbb{Z}_n$ to $\widehat{\mathbb{Z}}_n$. It now follows from the structure theorem that $G \cong \widehat{G}$.

3. The isomorphism is given by $\prod_{k=1}^n \widehat{G}_k \ni (\omega_1, ..., \omega_k, ...\omega_n) \mapsto \prod_{k=1}^n \omega_k \in \widehat{\prod_{k=1}^n G_k}$ where

$$\left(\prod_{j=1}^n \omega_k\right)(s_1, ..., s_k, ..., s_n) := \omega_1(s_1) \cdot ... \cdot \omega_k(s_k) \cdot ... \cdot \omega_n(s_n),$$

for all $(s_1, ..., s_k, ..., s_n) \in \prod_{k=1}^n G_k$. It is not hard to see that the mapping is injective, consider the equality: $\prod_{j=1}^n \omega_k^1 = \prod_{j=1}^n \omega_k^2$, then if e_j is the neutral element of G_j and $s_k \in G_k$ we obtain

$$\omega_k^1(s_k) = \prod_{j=1}^n \omega_k^1(e_1, ..., s_k, ..., e_n) = \prod_{j=1}^n \omega_k^2(e_1, ..., s_k, ..., e_n) = \omega_k^2(s_k)$$

for all $k = 1, ..., n$. This proves injectivity. To show surjectivity, let $\chi \in \widehat{\prod_{j=1}^n G_j}$, then from the fact that $\chi : \prod_{j=1}^n G_j \to \mathbb{C}$ is a homomorphism, it follows that

$$\chi = \prod_{j=1}^n \omega_j, \qquad \omega_j \in \widehat{G}_j, \text{ for } j = 1, ..., n$$

where $\omega_k(s_k) = \chi(e_1, ..., s_k, ..., e_n)$, for each $s_k \in G_k$. It is not hard to check that this defines a group homomorphism, furthermore the continuity of the map and its inverse follow quickly from considering net-convergence. Hence we have an isomorphism of topological groups.

$\square$

Proposition 3.11. *Let G be a compact abelian group with Haar measure μ. Then for any nontrivial character $\omega \in \widehat{G}$,*

$$\int_G \omega(s) d\mu(s) = 0.$$

Proof. Let ω be a nontrivial character of G and choose $h \in G$ such that $\omega(-x) \neq 1$ (which will always exist by nontriviality). Suppose for contradiction that $\int_G \omega(s) d\mu(s) \neq 0$. Then by translation invariance of the Haar measure:

$$\int_G \omega(s) d\mu(s) = \int_G \omega(s-x) d\mu(s) = \omega(-x) \int_G \omega(s) d\mu(s).$$

This implies $1 = \omega(-x)$, which is a contradiction. $\square$

3.6 The Fourier Transform

We define the Fourier transform to be a map whose domain is $L^1(G)$ and denote

$$\mathcal{F}(f) = \widehat{f},$$

where $\widehat{f}$ is exactly the Gelfand transform of f, so that

$$\widehat{f}(h) = \widehat{f}(h_{\overline{\omega}}) = \int_G f(s)\overline{\omega}(s)d\mu(s), \qquad \forall h \in \mathcal{M}_{L^1(G)}$$

where we use the bijection in Proposition 3.6 to have the representation $h = h_{\overline{\omega}}$, for a unique $\omega \in \widehat{G}$, for each $h \in \mathcal{M}_{L^1(G)}$. The following is immediate from the homeomorphism $\mathcal{M}_{L^1(G)} \cong \widehat{G}$ and from Gelfand's Theorem 3.4.

Theorem 3.5. *The Fourier transform $\mathcal{F}$ is a norm-decreasing $*$-homomorphism from $L^1(G)$ to $C_0(\widehat{G})$, and can be defined via:*

$$(\mathcal{F}f)(\omega) = \int_G f(s)\overline{\omega}(s)d\mu(s), \qquad \forall \omega \in \widehat{G}.$$

Furthermore, $\mathcal{F}(L^1(G))$ is dense in $C_0(\widehat{G})$.

One can see from the mapping $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$, that indeed the Fourier transform as it appears above is the generalized version of Riemman-Lebesgue lemma [Rud86, Theorem 9.6] since the dual space of $\mathbb{R}^d$ is itself by the identification $\mathbb{R}^n \ni x \mapsto \widehat{x} \in \mathbb{R}^n$ where $\widehat{x}(y) = e^{2\pi i x\cdot y}$ for all $y \in \mathbb{R}^n$ [Fol15, See pg. 107]. Furthermore, the fact that $\mathcal{F}$ is a $*$-homomorphism that we get for free from Gelfand's theorem gives us the familiar convolution formula:

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}, \qquad \forall f, g \in L^1(G). \tag{3.12}$$

For the rest of the document we shall interchangeably use either $\mathcal{F}f$ or $\widehat{f}$ for the Fourier-transform of f.

We would like to eventually be able to invert the Fourier transform, or at least make it precise when this is possible. To do this, we need to introduce the **measure algebra** $M(G)$

which is the set of all *complex* Radon measures[10] on the locally compact group G. Note that there is an extension of Riesz-Markov-Kakutani 3.3 that associates the set of complex Radon measures on G with continuous linear functionals on $C_0(G)$ [Rud86, Theorem 6.19], furthermore if $\nu \in M(G)$ and $\psi_\nu \in C_0(G) \to \mathbb{C}$ is the associated Riesz-Markov-Kakutani linear functional, then $\|\nu\| = \|\psi_\nu\|$, where $\|\nu\| := \sup \sum_{n=1}^{\infty} |\nu(E_n)|$ with the supremum running over all disjoint measurable sequences $\{E_n\}_{n=1}^{\infty}$ that covers G. $\|\nu\|$ is called the *total variation* of ν (it is important to recall from the definition of complex measures that $\|\nu\| < \infty$ by fiat, so the total variation is well-defined). The total variation endows $M(G)$ a norm, and in fact makes it a Banach space with the obvious operations. We can use this extended Riesz-Markov-Kakutani to show more: suppose $\nu, \rho \in M(G)$, then the map that sends

$$\phi \mapsto \int_G \int_G \phi(s+r) d\nu(s) d\rho(r) \tag{3.13}$$

is a linear functional that satisfies

$$\left| \int_G \int_G \phi(s+r) d\nu(s) d\rho(r) \right| \leqslant \|\phi\|_\infty \|\nu\| \|\rho\|. \tag{3.14}$$

Therefore we have found a continuous linear functional on $C_0(G)$, and we call the corresponding measure to be the convolution $\nu * \rho \in M(G)$. The estimate in (3.14) shows that $\|\nu * \rho\| \leqslant \|\nu\| \|\rho\|$, so barring a basic associativity result (using Fubini) of the convolution operation proof, $M(G)$ can be seen to be a Banach algebra. As a remark, observe that we can define $M(G)$ for any locally compact group by simply replacing the addition notation in (3.14) with the multiplicative notation, but for our purposes, because G is an LCA group, $M(G)$ is a commutative Banach algebra. The main point here is that $L^1(G)$ can actually be embedded into $M(G)$, explicitly, if $f \in L^1(G)$ and μ is the (modulo the scale) Haar measure on G, then

$$\phi \mapsto \int_G \phi(s) f(s) d\mu(s)$$

defines a linear functional on $C_0(G)$, with

$$\left| \int_G \phi(s) f(s) d\mu(s) \right| \leqslant \|\phi\|_\infty \|f\|_1,$$

so the mapping is also a continuous linear function on $C_0(G)$, the corresponding measure is

[10] See Chapter Six of [Rud86]

identified with f. We check a bit more to see if the identification respects the actual convolution in $L^1(G)$: let $f, g \in L^1(G)$ then for $\phi \in C_0(G)$:

$$\begin{aligned}\int_G \phi(s)(f * g)(s)d\mu(s) &= \int_G \phi(s)\left[\int_G f(r)g(s-r)d\mu(r)\right]d\mu(s)\\ &= \int_G\int_G \phi(s+r)f(r)g(s)d\mu(r)d\mu(s),\end{aligned}$$

which is exactly what we expect looking at the defining linear functional (3.13). An extension of the Fourier-transform can now then be defined on $M(G)$, called the **Fourier-Stieltjes** transform: given $\nu \in M(G)$, we denote its Fourier-Stieltjes transform with $\widehat{\nu} : \widehat{G} \to \mathbb{C}$, and define it via

$$\widehat{\nu}(\omega) = \int_G \overline{\omega(s)}d\nu(s), \qquad \forall \omega \in \widehat{G}$$

To see that this is a true extension of the Fourier transform on $L^1(G)$, consider $f \in L^1(G)$ as an element of $M(G)$, then for all $\omega \in \widehat{G}$

$$\begin{aligned}\widehat{f}(\omega) &= \int_G \overline{\omega(s)}(f(s))d\mu(s)\\ &= \int_G f(s)\overline{\omega(s)}d\mu(s),\end{aligned}$$

which is exactly $\mathcal{F}f$ in Theorem 3.5. As an added bonus, we also extend the convolution formula: let $\nu, \rho \in M(G)$, then from the fact that each $\omega \in \widehat{G}$ is a homomorphism:

$$\begin{aligned}\widehat{(\nu * \rho)}(\omega) &= \int_G \overline{\omega(s)}d(\nu * \rho)(s)\\ &= \int_G\int_G \overline{\omega(s+r)}d\nu(s)d\rho(r)\\ &= \int_G \overline{\omega(s)}d\nu(s) \cdot \int_G \overline{\omega(r)}d\rho(r)\\ &= \widehat{\nu}(\omega) \cdot \widehat{\rho}(\omega)\end{aligned}$$

as expected. However, we are more interested in defining the Fourier-Stieltjes transform for the dual group $\widehat{G}$. As a preview to duality, we make an important note here,

Lemma 3.17. *If $g \in G$, then for each $g \in G$, the map* $\mathrm{ev}(g) : \widehat{G} \to \mathbb{C}$

$$\mathrm{ev}(g) : \omega \mapsto \omega(g)$$

is in $\widehat{\widehat{G}}$. *Furthermore,* $\mathrm{ev} : G \to \widehat{\widehat{G}}$ *defines a homomorphism of groups.*

Proof. Consider the net convergence $\omega_i \to \omega$ in $\widehat{G}$. By convergence in compact-open topology, and the fact that singletons are compact in any topological space, we have that for all $g \in G$: $\mathrm{ev}(g)(\omega_i) = \omega_i(g) \to \mathrm{ev}(g)(\omega) = \omega(g)$[11]. We have shown that each $\mathrm{ev}(g) : \widehat{G} \to \mathbb{C}$ is continuous, and it is obvious that it is a nonzero homomorphism. That ev defines a group homomorphism is obvious by elementary computations. □

Definition 3.10. Given G, the Fourier-Stieltjes transform of a measure ν in $M(\widehat{G})$ is a function $\widehat{\nu} : G \to \mathbb{C}$, and is given by

$$\begin{aligned}\widehat{\nu}(g) &:= \int_{\widehat{G}} \mathrm{ev}(g)(\omega) d\nu(\omega) \\ &= \int_{\widehat{G}} \omega(g) d\nu(\omega)\end{aligned}$$

Remark 3.7. The Fourier-Stieltjes transform on $M(\widehat{G})$ above is well-defind because the integrand in the definition is continuous owing to Lemma 3.10. Furthermore, the map $g \mapsto \widehat{\nu}(g)$ is continuous since net-convergence $g_i \to g$ in G implies $\omega(g_i) \to \omega(g)$ in $\mathbb{T} \subseteq \mathbb{C}$, which then implies $\int_G |\omega(g_i) - \omega(g)| d\nu(\omega) \to 0$.

Proposition 3.12 ([Fol15, Proposition 4.15])**.** *The Fourier-Stieltjes transform is a norm-decreasing linear injective map from* $M(\widehat{G})$ *to* $C_b(G)$

Proof. Linearity is trivial (follows from linear structure of $M(\widehat{G})$). We have the following estimate for each $\nu \in M(\widehat{G})$:

$$\begin{aligned}\|\widehat{\nu}\|_\infty &:= \sup_{g \in G} \left| \int_{\widehat{G}} \omega(g) d\nu(\omega) \right| \\ &\leqslant \sup_{g \in G} |\omega(g)| \cdot \left| \int_{\widehat{G}} d\nu(\omega) \right| \\ &\leqslant \|\nu\|.\end{aligned}$$

This simultaneously shows that $\widehat{\nu} \in C_b(G)$, and that the Fourier-Stieltjes is norm-decreasing. Lastly, we want to show injectivity: suppose $\widehat{\nu} = 0$ and $f \in L^1(G)$, if μ is the Haar measure in

[11] In retrospect, we can actually use this argument to show that convergence in compact-open topology implies pointwise convergence

G, then

$$\begin{aligned}
0 &= \int_G f(g)\widehat{\nu}(g)d\mu(g) \\
&= \int_G \int_{\widehat{G}} f(g)\omega(g)d\nu(\omega)d\mu(g) \\
&= \int_{\widehat{G}} \left[\int_G f(g)\omega(g)d\mu(g)\right] d\nu(\omega) \\
&= \int_{\widehat{G}} \widehat{f}(-\omega)d\nu(\omega) \\
&= \int_{\widehat{G}} \widehat{f}(\omega)d\nu(\omega).
\end{aligned}$$

Now by the fact that $\mathcal{F}(L^1(G))$ is dense in $C_0(\widehat{G})$, then the equation above extends to

$$\int_{\widehat{G}} \varphi(\omega)d\nu(\omega), \qquad \forall\varphi \in C_0(\widehat{G}),$$

finally by Riesz-Markov-Kakutani, $\nu = 0$. □

Along with the measure space $M(\widehat{G})$, we need another recipe for the proof of Fourier inversion, we introduce the positive-type functions.

Definition 3.11. A function $\phi \in L^\infty(G)$ is of **positive-type** if for all $f \in L^1(G)$

$$\int_G (f^* * f)(s)\phi(s)d\mu(s) \geqslant 0. \tag{3.15}$$

Furthermore, we denote by $\mathcal{P}(G)$ the space of all continuous positive-type functions, i.e.

$$\mathcal{P}(G) := \{\phi \in C(G) : \phi \text{ is of positive-type}\}.$$

Remark 3.8. Writing the condition (3.15) explicitly, ϕ is positive-type if and only if for all $f \in L^1(G)$:

$$\begin{aligned}
\int_G (f^* * f)(s)\phi(s)d\mu(s) &= \int_G \left[\int_G \overline{f(-r)}f(s-r)d\mu(r)\right]\phi(s)d\mu(s) \\
&= \int_G \int_G \overline{f(r)}f(s+r)\phi(s)d\mu(r)d\mu(s) \\
&= \int_G \int_G \overline{f(r)}f(s)\phi(s-r)d\mu(r)d\mu(s) \geqslant 0.
\end{aligned}$$

Lemma 3.18. *If $g \in L^2(G)$, then $g^* * g \in \mathcal{P}(G)$.*

Proof. We can use Hölder's inequality and translation invariance of the Haar measure to show the following for $g \in L^2(G)$, and $s \in G$:

$$|g^* * g(s)| \leqslant \|\overline{g} \cdot R_s g\|_1 \leqslant \|\overline{g}\|_2 \cdot \|R_s g\|_2 = \|g\|_2^2 < \infty.$$

Therefore $g^* * g \in L^\infty(G)$. Also, for a fixed $g \in L^2(G)$, the map from G to $L^2(G)$ given by

$$s \mapsto R_s g$$

is continuous. Next, the inner product is jointly continuous on $L^2(G) \times L^2(G)$, therefore the map

$$s \mapsto \langle R_s g, g \rangle$$

is continuous from G to $\mathbb{C}$. Let us compute the image of the last mapping,

$$\begin{aligned}
\langle R_s g, g \rangle &= \int_G g(r+s)\overline{g}(r) d\mu(r) \\
&= \int_G \overline{g(-r)} g(s-r) d\mu(r) \\
&= g^* * g(s).
\end{aligned}$$

We have already shown continuity of $g^* * g$ then, all we need to prove is that it satisfies the condition given in (3.15). Let $f \in L^1(G)$ then we compute[12]

$$\begin{aligned}
&\int_G \int_G \overline{f(r)} f(s) (g^* * g)(s-r) d\mu(r) d\mu(s) \\
&= \int_G \int_G \overline{f(r)} f(s) \langle R_{s-r} g, g \rangle d\mu(r) d\mu(s) \\
&= \left\langle \int_G f(s) R_s g d\mu(s), \int_G f(r) R_r g d\mu(s) \right\rangle \\
&= \|R_s(f) g\|_2 \geqslant 0,
\end{aligned}$$

[12] That we can do such an interchange between the integral and the inner product will be justified later. What we have below is what is really called a vector-valued integral: Notice that $\int_G f(s) R_s d\mu(s)$ can be interpreted as a bounded linear operator on $L^2(G)$ that sends g to $\int_G f(s) R_r g d\mu(s)$. In fact the said operator is nothing but the 'integrated form' of the unitary representation $R : G \to U(L^2(G))$ sending r to R_r.

where $R_s(f)g \in L^2(G)$ with value

$$x \mapsto \int_G f(s)R_s g(x)d\mu(s) = \int_G f(s)g(x+s)d\mu(s).$$

To see that it is a bonafide $L^2(G)$ function, we just need to use Young's convolution inequality. $\square$

Lemma 3.19 ([Fol15, Proposition 3.33])**.** *$\mathcal{L}_c(G) := \mathrm{span}_{\mathbb{C}}(C_c(G) \cap \mathcal{P}(G))$ includes functions of the form $f * g$ with $f, g \in C_c(G)$. $\mathcal{L}_c(G)$ is dense in $C_c(G)$, so as a corollary, it is also dense in $L^p(G)$ for each $1 \leqslant p < \infty$.*

Proof. Notice that the convolution $(f,g) \mapsto f * g^*$ is a sesquilinear form, hence we can apply the usual polarization identity on it. In particular, if $f, g \in C_c(G)$ we find that

$$\begin{aligned} f * g &= f * (g^*)^* \\ &= \frac{1}{4}\left[(f+g^*)(f+g^*)^* - (f-g^*)(f-g^*)^* + i(f+ig^*)(f+ig^*)^* - i(f-ig^*)(f-ig^*)^*\right]. \end{aligned}$$

Since $f * g$ can be rewritten as a complex linear span of functions of the form $h * h^*$ with $h \in C_c(G) \subseteq L^2(G)$, then it follows from Lemma 3.18 that $f * g \in \mathcal{L}_c(G)$. It follows from the fact that we can find approximate identities in $C_c(G)$ for $C_c(G) (\subseteq L^1(G))$ via Proposition 3.9 that $\mathcal{L}_c(G)$ is dense in $C_c(G)$. We are done. $\square$

We give below, from [Fol15, Theorem 4.19], a correspondence between positive-type operators and complex Radon measures in $M(\widehat{G})$. This will be a useful tool in elucidating the proof of Fourier-inversion theorem.

Theorem 3.6 (Bochner)**.** *Let G be a locally compact Abealian group. A function ϕ on G is of positive type in $\mathcal{P}(G)$ if and only if there is a unique nonnegative measure $\nu \in M(\widehat{G})$ such that*

$$\phi = \widehat{\nu}.$$

We now define the space $\mathcal{B}(G) := \{\phi \in C_b(G) : \phi = \widehat{\nu},\ \nu \in M(\widehat{G})\}$. Take $\nu \in M(\widehat{G})$ (not necessarily positive measure), and by splitting ν into its real and imaginary parts, and then using Hahn-Jordan decomposition [Rud86, 6.6], we find positive Radon measures $\nu_1, \nu_2, \nu_3, \nu_4 \in M(\widehat{G})$ such that

$$\nu = (\nu_1 - \nu_2) + i(\nu_3 - \nu_4).$$

Remark 3.9. Since each positive measure correspond to a positive-type function by Theorem 3.6, we have

$$\mathcal{B}(G) = \operatorname{span}_{\mathbb{C}} \mathcal{P}(G). \tag{3.16}$$

So any $\nu \in M(\widehat{G})$ corresponds trivially to $\mathcal{B}(G)$ via the Fourier-Stieltjes transform $\widehat{\nu} =: \phi_\nu$. This correspondence is invertible via Theorem 3.6 and Equation (3.6); if $\phi \in \mathcal{B}(G)$, then there is a unique corresponding $\nu_\phi \in M(\widehat{G})$ with the defining property that $\widehat{\nu_\phi} = \phi \in \mathcal{B}(G)$. We will use these notations from now on.

$$\mathcal{B}(G) \ni \phi \longleftrightarrow \nu_\phi \in M(\widehat{G})$$
$$M(\widehat{G}) \ni \nu \longleftrightarrow \phi_\nu \in \mathcal{B}(G).$$

We shall put our notations above in practice immediately.

Lemma 3.20. *If $f, g \in \mathcal{B}(G) \cap L^1(G)$, then for all $\varphi \in C_0(\widehat{G})$*

$$\int_{\widehat{G}} \varphi(\omega)\widehat{f}(\omega)d\nu_g(\omega) = \int_{\widehat{G}} \varphi(\omega)\widehat{g}(\omega)d\nu_f(\omega).$$

As a corollary, the complex measures $\widehat{f}d\nu_g$ and $\widehat{g}d\nu_f$ coincide.

Proof. Let $h \in L^1(G)$, so that

$$\begin{aligned}
\int_{\widehat{G}} \widehat{h}(\omega)d\nu_f(\omega) &= \int_{\widehat{G}} \left[\int_G h(s)\overline{\omega(s)}d\mu(s)\right] d\nu_f(\omega) \\
&= \int_G h(s) \left[\int_{\widehat{G}} \overline{\omega}(s)d\nu_f(\omega)\right] d\mu(s) \\
&= \int_G h(s)\widehat{\nu_f(-s)}d\mu(s) \\
&= \int_G h(s)f(-s)d\mu(s) = (h * f)(0).
\end{aligned}$$

Now then, we do the replacement $h \rightsquigarrow h * g$ in the computation to obtain:

$$\int_{\widehat{G}} \widehat{h}(\omega)\widehat{g}(\omega)d\nu_f(\omega) = ((h * g) * f)(0).$$

In the same computation, we we do the replacements $h \rightsquigarrow h * f$ and $f \rightsquigarrow g$ and obtain

$$\int_{\widehat{G}} \widehat{h}(\omega)\widehat{f}(\omega)d\nu_g(\omega) = ((h * f) * g)(0).$$

By commutativity and associativity of convolution in $L^1(G)$, we have that

$$\int_{\widehat{G}} \widehat{h}(\omega)\widehat{g}(\omega)d\nu_f(\omega) = \int_{\widehat{G}} \widehat{h}(\omega)\widehat{f}(\omega)d\nu_g(\omega) \tag{3.17}$$

From the above equation, if $\varphi \in C_0(\widehat{G})$, then by density of $\mathcal{F}(L^1(G))$ in $C_0(\widehat{G})$, we can always find a sequence $\{h_n\}_n \subseteq L^1(G)$ such that $\widehat{h_n} \to \varphi$ uniformly, therefore along with Equation (3.17), we find that[13]

$$\int_{\widehat{G}} \varphi(\omega)\widehat{g}(\omega)d\nu_f(\omega) = \int_{\widehat{G}} \varphi(\omega)\widehat{f}(\omega)d\nu_g(\omega).$$

φ is arbitrary, so by Riesz-Markov-Kakutani for complex measures, we conclude that $\widehat{g}d\nu_f = \widehat{f}d\nu_g$. ☐

We cite a technical result from [Fol15, Lemma 4.20], which shows that it is always possible to construct 'nicely localized' positive functions.

Lemma 3.21. *If $K \subseteq \widehat{G}$ is compact, then there exists a compactly supported function $f \in \mathcal{P}(G)$ such that $\widehat{f} \geqslant 0$ on $\widehat{G}$, and $\widehat{f} > 0$ on K.*

Here, we can now give a Fourier inversion formula for functions in $f \in \mathcal{B}(G) \cap L^1(G)$.

Theorem 3.7 ([Fol15, Theorem 4.22])**.** *If $f \in \mathcal{B}(G) \cap L^1(G)$, then $\widehat{f} \in L^1(\widehat{G})$. Furthermore, if the Haar measure $\widehat{\mu}$ in $\widehat{G}$ is suitably scaled with respect to the Haar measure μ in G, then we have the following formula:*

$$f(s) = \int_{\widehat{G}} \widehat{f}(\omega)\omega(s)d\widehat{\mu}(\omega). \tag{3.18}$$

Proof. We are going to exploit Riesz-Markov-Kakutani 3.3 again and construct a linear functional on $C_c(\widehat{G})$ that would make Equation (3.18) true. Let $\psi \in C_c(\widehat{G})$, then $K := \operatorname{supp}(\psi)$ is compact, hence by Lemma 3.21 there exists a function $g \in \mathcal{P}(G)$ such that $\widehat{g} \geqslant 0$ on $\widehat{G}$ and $\widehat{g} > 0$ on K.

[13] Recall the uniform convergence theorem for integrals.

Define

$$I(\psi) = \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)} d\nu_g(\omega)$$

We would first like to show that this linear-functional does not depend on which g we choose, suppose $h \in \mathcal{P}(G)$ is another such function having the same properties as g, then both $g, h \in \mathcal{P}(G) \cap L^1(G)$ and thus by Lemma 3.20:

$$\begin{aligned}
\int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)} d\nu_g(\omega) &= \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)\widehat{h}(\omega)} \widehat{h}(\omega) d\nu_g(\omega) \\
&= \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)\widehat{h}(\omega)} \widehat{g}(\omega) d\nu_h(\omega) \\
&= \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{h}(\omega)} d\nu_h(\omega).
\end{aligned} \tag{3.19}$$

So indeed I does not depend on which g that satisfies Lemma 3.21 is used to define it. What we have defined here is a nontrivial positive functional, and so our next step is to show that our functional is translation invariant, i.e. $I(L_\eta \psi) = I(\psi)$ for all $\eta \in \widehat{G}$ and $\psi \in C_c(\widehat{G})$. Given $\eta \in \widehat{G}$, let us first compute the Fourier-Stieltjes transform of the η-translated measure $L_\eta \nu_g \in M(\widehat{G})$

$$\begin{aligned}
\widehat{L_\eta \nu_g}(s) := \int_{\widehat{G}} \omega(s) d(L_\eta \nu_g)(\omega) &:= \int_{\widehat{G}} (\overline{\eta}\omega)(s) d\nu_g(\omega) \\
= \overline{\eta}(s) \int_{\widehat{G}} \omega(s) d\nu_g(\omega) &= \overline{\eta}(s) \widehat{\nu_g}(s) \\
= \overline{\eta}(s) g(s) = (\overline{\eta} g)(s).
\end{aligned}$$

It follows from Bochner's correspondence that $\widehat{L_\eta \nu_g} = \widehat{\nu_{\overline{\eta} g}}$, and by injectivity of the Fourier-Stieltjes transform, we have that $L_\eta \nu_g = \nu_{\overline{\eta} g}$. We shall also use the fact that the usual Fourier transform satisfies $\widehat{g}(\eta\omega) = \widehat{\overline{\eta} g}(\omega)$. Now we compute I with g re-picked so that $\widehat{g} > 0$ on the compact set $K := \operatorname{supp}(\psi) \cup \operatorname{supp}(L_\eta \psi)$ instead.

$$\begin{aligned}
I(L_\eta \psi) &= \int_{\widehat{G}} \frac{\psi(\overline{\eta}\omega)}{\widehat{g}(\omega)} d\nu_g(\omega) = \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\eta\omega)} d(L_\eta \nu_g)(\omega) \\
&= \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\eta\omega)} d(\nu_{\overline{\eta} g})(\omega) = \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{\overline{\eta} g}(\omega)} d(\nu_{\overline{\eta} g})(\omega) = I(\psi).
\end{aligned}$$

Therefore we have constructed a translation invariant nontrivial positive linear functional, hence I corresponds to a Haar measure on $\widehat{G}$. Now, define $\widehat{\mu}$ to be the Haar measure properly scaled that corresponds to I, then we can take any $f \in \mathcal{B}(G) \cap L^1(G)$, so that for any $\psi \in C_c(\widehat{G})$, we can proceed as in the computation in Equation (3.19)

$$\begin{aligned}\int_{\widehat{G}} \psi(\omega)\widehat{f}(\omega)d\widehat{\mu}(\omega) &= I(\psi\widehat{f}) = \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)}\widehat{f}d\nu_g(\omega) \\ &= \int_{\widehat{G}} \frac{\psi(\omega)}{\widehat{g}(\omega)}\widehat{g}(\omega)d\nu_f(\omega) = \int_{\widehat{G}} \psi(\omega)d\nu_f(\omega).\end{aligned} \tag{3.20}$$

Taking $\psi \equiv 1$ in $C_c(\widehat{G})$, we find that $\widehat{f} \in L^1(\widehat{G})$. Finally, if we take $\psi : \widehat{G} \to \mathbb{C}$ to be the one induced by the evaluation map $\psi = \mathrm{ev}(s)$ for $s \in G$ we obtain from Equation (3.20) that

$$f(s) = \widehat{\mu_f}(s) = \int_{\widehat{G}} \omega(s)d\nu_f(\omega) = \int_{\widehat{G}} \omega(s)\widehat{f}(\omega)d\widehat{\mu}(\omega)$$

as required. □

Henceforth we shall always choose the Haar measure $\widehat{\mu}$ on the dual group $\widehat{G}$ to be the one that makes Theorem 3.18 true, in this case $\widehat{\mu}$ is referred to as the **dual measure** on $\widehat{G}$.

We would like to further refine the Fourier inversion given by Theorem 3.18 so that we could hide all of the scaffolding that we've put up using the space $\mathcal{B}(G)$ and Bochner's theorem. Here is one corollary around this idea (c.f. [Fol15, Theorem 4.26])

Theorem 3.8 (Plancherel)**.** *The Fourier transform extends to a unitary map* $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$.

Proof. The idea is to consider the Fourier transform on the dense subspace $L^1(G) \cap L^2(G)$ (this is truly dense because it contains $C_c(G)$ in particular). Indeed let $f \in L^1(G) \cap L^2(G)$, then by Lemma 3.18, $f * f^* \in L^1(G) \cap \mathcal{P}(G) \subseteq L^1(G) \cap \mathcal{B}(G)$, therefore by the inversion formula of Theorem 3.18 applied to $f * f^*$:

$$\int_G |f(s)|^2 d\mu(s) = (f * f^*)(0) = \int_{\widehat{G}} \omega(0)(\widehat{f * f^*})(\omega)d\widehat{\mu}(\omega) = \int_{\widehat{G}} |\widehat{f}(\omega)|^2 d\widehat{\mu}(\omega).$$

We see that $\mathcal{F}$ is an isometry on $L^1(G) \cap L^2(G)$ with respect to the L^2-norm. Therefore the completion of the Fourier transform gives us an isometry $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$. We only need to show that the operator $\mathcal{F}$ as we have extended is actually surjective. Let $g \in L^2(\widehat{G})$ such that $g \in \left(\mathcal{F}(L^1(G) \cap L^2(G)\right)^\perp$ then for any $f \in L^1(G) \cap L^2(G)$, note that $L_y f \in L^1(G) \cap L^2(G)$ still, and so for all $y \in G$

$$0 = \left\langle g, \widehat{L_y f} \right\rangle = \int_{\widehat{G}} g(\omega)\overline{\widehat{L_y f}}(\omega)d\widehat{\mu}(\omega) = \int_{\widehat{G}} \omega(y)g(\omega)\overline{\widehat{f}}(\omega)d\widehat{\mu}(\omega) = \widehat{(g \cdot \overline{\widehat{f}}d\widehat{\mu})}(y),$$

where the last 'hat' is the Fourier-Stieltjes transform, owing to the fact that by Cauchy-Schwarz $g, \widehat{f} \in L^2(\widehat{G}) \implies g \cdot \widehat{f} \in L^1(G) \implies g \cdot \overline{\widehat{f}} d\widehat{\mu} \in M(\widehat{G})$. It now follows from injectivity of the Fourier-Stieltjes transform 3.12 that $0 = g \cdot \overline{\widehat{f}} d\widehat{\mu}$, which implies that $g \cdot \overline{\widehat{f}} = 0$ a.e. for all $f \in L^1(G) \cap L^2(G)$, but by constructing proper f's on a cover of $\widehat{G}$ consisting entirely of compacts subsets via Lemma 3.21, we have in fact $g = 0$ a.e. on $\widehat{G}$. What we have shown is that

$$\begin{aligned}
\{0\} &= \big(\mathcal{F}(L^1(G) \cap L^2(G))\big)^{\perp} \iff \\
L^2(G) &= \big(\mathcal{F}(L^1(G) \cap L^2(G))\big)^{\perp\perp} \\
&= \overline{\mathcal{F}(L^1(G) \cap L^2(G))} \\
&= \mathcal{F}(\overline{L^1(G) \cap L^2(G)}) = \mathcal{F}(L^2(G)),
\end{aligned}$$

as required. □

Corollary 3.7 ([Fol15, Lemma 4.30]). *If $\phi_1, \phi_2 \in C_c(\widehat{G})$, then $\phi_1 * \phi_2 = \widehat{g}$ for some $g \in L^1(G) \cap \mathcal{B}(G)$. As a corollary, $\mathcal{F}(L^1(G) \cap \mathcal{B}(G))$ is dense in $L^p(G)$ for each $1 \leqslant p < \infty$.*

Proof. Using the Fourier-Stieltjes transform, let us define the following $g_1, g_2, g \in \mathcal{B}(G)$ via

$$\begin{aligned}
g_1 &:= \widehat{\phi_1 d\widehat{\mu}} \\
g_2 &:= \widehat{\phi_2 d\widehat{\mu}} \\
g &:= \widehat{(\phi_1 * \phi_2) d\widehat{\mu}}.
\end{aligned}$$

To show that $g_i \in L^2(G)$ for each $i = 1, 2$, let $h \in L^1(G) \cap L^2(G)\backslash\{0\}$, then

$$\begin{aligned}
\left|\int_G g_i(s)\overline{h}(s) d\mu(s)\right| &= \left|\int_G \int_{\widehat{G}} \omega(s)\phi_i(\omega)\overline{h}(s) d\widehat{\mu}(\omega) d\mu(s)\right| \\
&= \left|\int_{\widehat{G}} \phi_i(\omega)\overline{\widehat{h}}(\omega) d\widehat{\mu}(\omega)\right| \\
&\leqslant \|\phi\|_2 \|\widehat{h}\|_2 \\
&= \|\phi\|_2 \|h\|_2 < \infty.
\end{aligned} \tag{3.21}$$

Since $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, we have that $\|g_i\|_2 = \sup_{h \in L^2(G)\backslash\{0\}} |\langle g_i, h\rangle| < \infty$, so $g_i \in L^2(G)$ for each $i = 1, 2$. Now we have

$$g(s) = \widehat{(\phi_1 * \phi_2) d\widehat{\mu}}(s) = \int_{\widehat{G}} \omega(s)(\phi_1 * \phi_2)(\omega) d\widehat{\mu}(\omega)$$

$$
\begin{aligned}
&= \int_{\widehat{G}} \omega(s) \left[\int_{\widehat{G}} \phi_1(\eta)\phi_2(\overline{\eta}\omega) d\widehat{\mu}(\eta) \right] d\widehat{\mu}(\omega) \\
&= \int_{\widehat{G}} \int_{\widehat{G}} \omega(s)\eta(s)\phi_1(\eta)\phi_2(\omega) d\widehat{\mu}(\eta) d\widehat{\mu}(\omega) \\
&= (\widehat{\phi_1 d\widehat{\mu}})(s) \cdot (\widehat{\phi_2 d\widehat{\mu}})(s) \\
&= g_1(s) \cdot g_2(s).
\end{aligned}
$$

It then follows from Cauchy-Schwarz that $g \in L^1(G)$. Furthermore, $g \in L^1(G) \cap \mathcal{B}(G)$, and so by Fourier-inversion $g(s) = \displaystyle\int_{\widehat{G}} \omega(s)\widehat{g}(\omega) d\widehat{\mu}(\omega)$, comparing this with the first line of Equation (3.21), along with injectivity of the Fourier-Stieltjes transform, we find that $\widehat{g} = \phi_1 * \phi_2$ a.e., as required. □

We now have most of the required tools needed to prove one of the main results of abstract harmonic analysis called the Pontryagin duality theorem, which states that any LCA group G is isomorphic to its own double Pontryagin dual group, i.e $G \cong \widehat{\widehat{G}}$. Before we prove this, we need one result coming mainly from the representation theory of locally compact groups, however we will not focus on this, and we shall only state it in the form that is immediately useful for us. We have slightly touched the theory of unitary representations of G before, we will now give a detailed definition.

Definition 3.12. Let G be a locally compact (not-necessarily abelian) group and H a Hilbert space. A pair (H, U) where U is a map $U : G \to U\mathcal{L}(H)$, is said to be a representation[14] of G on H if for all $s, t \in G$:

1. $UsU_t = U_{st}$;
2. $U_{s^{-1}} = U_s^* = U_s^{-1}$;
3. $U : G \to U\mathcal{L}(H)$ is strongly continuous.

Note that we use the notation $U_s := U(s)$. For a subspace $V \subseteq H$, we say that V is an **invariant** subspace of H with respect to U if $U_s V \subseteq V$ for all $s \in G$. We say that U is an **irreducible representation** of G if for the only closed invariant subspace with respect to (H, U) are $\{0\}$ or H.

[14] For groups, they will always be *unitary* representations.

Below, we have two important results pertaining to the representation theory of LCA groups: we have Schur's lemma and Gelfand-Raikov Theorem. Both can be found in most texts dealing with abstract harmonic analysis [Fol15, Rud17].

Lemma 3.22 (Schur)**.** *Let G be an LCA group, then (H, U) is an irreducible representation of G if and only if $U = \mu \, \mathrm{Id}_H$ for some $\mu \in \mathbb{C}$.*

So in the LCA case, as a corollary, for each irreducible representation U, we have the following identification: $U(G) \hookrightarrow \mathbb{C}$. Since the unitary group of $\mathbb{C}$ is $\mathbb{T}$, we can see an irreducible representation U as a homomorphism $U : G \to \mathbb{T}$. In summary, irreducible representations of G are identifiable with its characters.

Theorem 3.9 (Gelfand-Raikov)**.** *If G is any LCA group, then its character group $\widehat{G}$ separate points. Explicitly, if $s \neq r \in G$, then there exists $\omega \in \widehat{G}$ such that $\omega(s) \neq \omega(r)$. This careful study of the representation theory of G implies the Pontryagin duality.*

If one is familiar with the representation theory of C^*-algebras, then we can perhaps traipse around it to obtain this result. We use the known result that the collection of irreducible $*$-representations of any arbitrary C^*-algebra seperate points. Indeed, we apply the fact that the representation theory of of a Banach $*$-algebra, and its C^*-enveloping algebra (see Example A.1 for definition) are 'similar' [FD88, Proposition 10.5]. In particular, there is a one-to-one correspondence between irreducible $*$-representations of the Banach $*$-algebra $L^1(G)$ and the $*$-representations of its C^*-envelope. Since irreducible $*$-representations of C^*-algebras separate points, then we deduce that the irreducible $*$-representations of $L^1(G)$ must also separate points. Lastly, there also is a one-to-one correspondence between the irreducible $*$-representations of $L^1(G)$ and the irreducible (unitary) representations of G (this result is subsumed by a more general result, c.f. Theorem 5.1). However, the irreducible representations of G are exactly its characters, as a consequence of Lemma 3.22. Hence we obtain Gelfand-Raikov.

Lemma 3.23. *Consider the evaluation map $ev : G \to \widehat{\widehat{G}}$. Let $\{s_i\}_{i \in I}$ be a net in G. The following are equivalent:*

1. *$s_i \to s$ in G;*
2. *$f(s_i) \to f(s)$ for all $f \in L^1(G) \cap \mathcal{B}(G)$;*
3. *$\int_{\widehat{G}} \omega(s_i) \widehat{f}(\omega) d\widehat{\mu}(\omega) \to \int_{\widehat{G}} \omega(s) \widehat{f}(\omega) d\widehat{\mu}(\omega)$ for all $f \in L^1(G) \cap \mathcal{B}(G)$;*

4. $\mathrm{ev}(s_i) \to \mathrm{ev}(s)$ *in* $\widehat{\widehat{G}}$.

Proof. ($1 \iff 2$). $1 \implies 2$ is trivial since each function in $L^1(G) \cap \mathcal{B}(G)$ is continuous. We shall prove the converse by contraposition. Suppose that $s_i \nrightarrow s$, then there exists a neighborhood U of s and a subnet $\{s'_j\}_{j\in J}$ if $\{s_i\}_{i\in I}$ such that $s'_j \notin U$ for all $j \in J$. We can construct, by Proposition 3.19, a function $f \in L^1(G) \cap \mathcal{B}(G)$ such that $\operatorname{supp} f \subseteq U$ and $f(s) \neq 0$. Then $f(s'_j) \nrightarrow f(s)$, because otherwise we have $0 = f(s'_j) \to f(s) \neq 0$. We have found a subnet of $\{f(s_i)\}_{i\in I}$ that does not converge to $f(s)$, so by Proposition (B.3), $f(s_i) \nrightarrow f(s)$ as required.

($2 \iff 3$). This is trivial in light of the Fourier-inversion theorem 3.18.

($3 \iff 4$). We use the net-convergence criteria in (3.7). We see that $\mathrm{ev}(s_i) \to \mathrm{ev}(s)$ in $\widehat{\widehat{G}}$ if and only if $h_{\mathrm{ev(s_i)}} \to h_{\mathrm{ev(s)}}$ in $\mathcal{M}_{L^1(\widehat{G})}$ in the relative weak $*$-topology, that is, for all $\widetilde{f} \in L^1(\widehat{G})$:

$$\begin{aligned}
&\int_{\widehat{G}} \mathrm{ev}(s_i)(\omega)\widetilde{f}(\omega)d\widehat{\mu}(\omega) \to \int_{\widehat{G}} \mathrm{ev}(s)(\omega)\widetilde{f}(\omega)d\widehat{\mu}(\omega) \iff \\
&\int_{\widehat{G}} \omega(s_i)\widetilde{f}(\omega)d\widehat{\mu}(\omega) \to \int_{\widehat{G}} \omega(s)\widetilde{f}(\omega)d\widehat{\mu}(\omega).
\end{aligned}$$

the equality of 3 and 4 now follows from the fact that $\mathcal{F}(L^1(G) \cap \mathcal{B}(G))$ is dense in $L^1(\widehat{G})$ via Corollary 3.7. $\square$

Theorem 3.10 (Pontryagin Duality)**.** *The evaluation map ev* : $G \to \widehat{\widehat{G}}$ *via* $\mathrm{ev}(g) : \omega \mapsto \omega(g)$ *is a topological homeomorphism and group isomorphism at the same time. That is,* $G \cong \widehat{\widehat{G}}$ *as LCA groups.*

Proof. Let $s, r \in \widehat{G}$ such that $\mathrm{ev}(s) = \mathrm{ev}(r)$, then for all $\omega \in \widehat{G}$ we have $\omega(s) = \omega(r)$, but by Gelfand-Raikov 3.9, this means $s = r$. We have shown that ev is injective. It now follows from Lemma 3.23 that ev is a homeomorphism to its image. We now only need to show that $\mathrm{ev}(G) = \widehat{\widehat{G}}$. We know that $G \cong \mathrm{ev}(G)$ topologically, and so $\mathrm{ev}(G)$ is a locally compact group. It follows from Corollary 3.2 that $\mathrm{ev}(G)$ is a closed subgroup of $\widehat{\widehat{G}}$. Suppose for contradiction that there exists $\widetilde{s} \in \widehat{\widehat{G}} \backslash \mathrm{ev}(G)$. Because $\widehat{\widehat{G}} \backslash \mathrm{ev}(G)$ must be open, we can construct[15] V, a neighborhood of the neutral element $\widetilde{e}$ of $\widehat{\widehat{G}}$ such that $V = V^{-1}$ and $\widetilde{s}V \cdot V \subseteq \widehat{\widehat{G}} \backslash \mathrm{ev}(G)$, which implies $\widetilde{s}V \cdot V \cap \mathrm{ev}(G) = \varnothing$. Construct $\phi_1, \phi_2 \in C_c(\widehat{\widehat{G}})$ such that $\operatorname{supp} \phi \subseteq \widetilde{s}V$ and $\operatorname{supp} \phi_2 \subseteq V$, so that $\phi_1 * \phi_2 \not\equiv 0$ since $\operatorname{supp}(\phi_1 * \phi_2) \subseteq \operatorname{supp} \phi_1 \cdot \operatorname{supp} \phi_2 \subseteq \widetilde{s}V \cdot V$. Hence $\operatorname{supp}(\phi_1 * \phi_2) \cap \mathrm{ev}(G) \subseteq \widetilde{s}V \cdot V \cap \mathrm{ev}(G) = \varnothing$. By Corollary 3.7 applied on the double dual $\widehat{\widehat{G}}$, there must be some $g \in L^1(\widehat{G}) \cap \mathcal{B}(\widehat{G})$ such that

[15] These are so-called ‘symmetric’ neighborhoods of the neutral element, they can easily be constructed using the continuity of the the multiplication and inversion maps.

$\phi_1 * \phi_2 = \widehat{g}$. Now, if $s \in G$ we find that

$$0 = \widehat{g}(\mathrm{ev}(s)) = \int_{\widehat{G}} \mathrm{ev}(s)(\omega)h(\omega)d\widehat{\mu}(\omega) = \int_{\widehat{G}} \omega(s)h(\omega)d\widehat{\mu}(\omega) = \widehat{hd\widehat{\mu}}(s).$$

By the injectivity of the Fourier-Stieltjes transform, it follows that $h \equiv 0$ a.e. on $\widehat{G}$, which contradicts the choice of $\phi_1 * \phi_2 = \widehat{h} \not\equiv 0$, therefore ev must be surjective. It follows that it is a topological and group-theoretic (c.f. Lemma 3.17) isomorphism. □

We can strightforwardly apply the Pontryagin duality to the results of Proposition 3.10.

Corollary 3.8. *If G and $\{G_i\}_{i=1}^n$ are all LCA groups, then*

1. *If $\widehat{G}$ is discrete, then G is compact. If $\widehat{G}$ is compact, then G is discrete.*

2. *If $\widehat{G}$ is finite, then $G \cong \widehat{G}$.*

3. $\widehat{\widehat{\prod_{i=1}^n G_i}} \cong \prod_{i=1}^n G_i$.

Remark 3.10. This is an important result, and from hereon out, by the power vested to us by the Pontryagin duality, we shall make the following identifications with (almost) reckless abandon: if $h : \widehat{\widehat{G}} \to \mathbb{C}$, then we shall always identify it as a function $h : G \to \mathbb{C}$, which we may explicitly see using the following identification:

$$h \rightsquigarrow h \circ \mathrm{ev}\,. \tag{3.22}$$

This identification synergizes well with the pushforward measure (see [Tao11] for definition) for $\widehat{\widehat{G}}$ induced by the Haar measure μ of G and the evaluation map $\mathrm{ev} : G \to \widehat{\widehat{G}}$, denoted by $\mathrm{ev}_*(\mu)$ where if $\widetilde{E} \in \mathrm{Bor}(\widehat{\widehat{G}})$, we define

$$\mathrm{ev}_*(\mu)(\widetilde{E}) := \mu(\mathrm{ev}^{-1}(\widetilde{E})).$$

As we are just identifying the Borel measures of $\widehat{\widehat{G}}$ with those of G, this is also a Haar measure on $\widehat{\widehat{G}}$. More importantly, the pushforward measure gives us a change-of-variables formula. If $h : \widehat{\widehat{G}} \to \mathbb{C}$ is measurable, then

$$\int_{\widehat{\widehat{G}}} h(\widetilde{s})d(\mathrm{ev}_*(\mu))(\widetilde{s}) = \int_G (h \circ \mathrm{ev})(s)d\mu(s), \tag{3.23}$$

which is consistent with (3.22), and so we take all integrals against all such functions h to be the right-hand side of (3.23).

We will in fact use the pushforward measure whenever we come in contact between two isomorphic LCA groups. In general if $G_1 \cong G_2$ implemented by $\Phi : G_1 \to G_2$, and if μ_{G_1} is a Haar-measure on G_1, then whenever we say that we 'identify' G_1 with G_2, we shall always have G_2 come automatically with the pushforward measure for G_2: $\mu_{G_2} = \Phi_*(\mu_{G_1})$. Of course this goes both ways, since Φ is an isomorphism: $\mu_{G_1} = \Phi_*^{-1}(\mu_{G_2})$.

We now obtain as an easy collary, a stronger version of the inversion theorem (c.f. [Fol15, Theorem 4.33]) from (3.18), one that does not make explicit mention of the space $\mathcal{B}(G)$ in its statement (though one can't seem to avoid it in the proof).

Theorem 3.11 (Fourier Inversion Theorem)**.** *Let $f \in L^1(G)$ such that $f \in L^1(\widehat{G})$, then $\widehat{\widehat{f}} \in L^1(\widehat{\widehat{G}})$ can be identified a.e. with the reflection $f^r \in L^1(G)$ defined via $f^r(s) = f(-s)$ for all $s \in G$. Furthermore, for a suitably rescaled Haar measure $\widehat{\mu}$ on $\widehat{G}$, we have the following formula:*

$$f(s) = \int_{\widehat{G}} \omega(s)\widehat{f}(\omega)d\widehat{\mu}(\omega), \qquad a.e.\ s \in G. \tag{3.24}$$

Proof. If f is chosen as in the hypothesis, then $\widehat{f} \in L^1(\widehat{G})$ but also

$$\widehat{f}(\omega) = \int_G \overline{\omega}(s)f(s)d\mu(s) = \int_G \omega(s)f(-s)d\mu(s) = \int_G \omega(s)f^r(s)d\mu(s).$$

We obtain via Fourier-Stieltjes transform

$$\widehat{f}(\omega) = \widehat{f^r d\mu}(\omega).$$

Therefore $\widehat{f} \in \mathcal{B}(\widehat{G}) \cap L^1(\widehat{G})$. We can then apply the first Fourier inversion result form 3.18, and use the identifications from (3.22) and (3.23) to obtain

$$\widehat{f}(\omega) = \int_G \omega(s)\widehat{\widehat{f}}(s)d\mu(s).$$

We then obtain

$$0 = \int_G \omega(s)\left[f^r(s) - \widehat{\widehat{f}}(s)\right]d\mu(s), \qquad \forall \omega \in \widehat{G}. \tag{3.25}$$

Take the neutral character $\omega \equiv 1$, and we find that $\widehat{\widehat{f}} = f^r$ a.e. on G. Writing out the equivalence,

$$f(s) = f^r(-s) = \widehat{\widehat{f}}(-s) = \int_{\widehat{G}} \overline{\omega(-s)} \widehat{f}(\omega) d\widehat{\mu}(\omega) = \int_{\widehat{G}} \omega(s) \widehat{f}(\omega) d\widehat{\mu}(\omega)$$

as required. □

3.7 The Annihilator Subgroup

In the upcoming Chapter 4, we shall see that the *annihilator subgroup* of a lattice in the time-frequency plane is identifiable with the so-called *adjoint lattice*. The adjoint lattice plays an important role in time-frequency analysis, indeed it will be a crucial recipe in the proof of one of our major results: Theorem 7.5. We note that some of the results in this section, especially those pertaining the Weil's formula, were based on the classic text [RS00].

Definition 3.13. Let H be a closed subgroup of G, then the **annihilator subgroup** denoted by $H^\perp \subseteq \widehat{G}$ is defined via $H^\perp := \{\omega \in \widehat{G} : \omega(s) = 1, \text{ for all } s \in H\}$.

Proposition 3.13. *If H is a closed subgroup of G, then $H^{\perp\perp} = H$.*

Proof. We can use the Pontryagin duality to see that

$$\begin{aligned} H^{\perp\perp} &= \{s \in G : \mathrm{ev}(s)(\omega) = 1, \text{ for all } \omega \in H^\perp\} \\ &= \{s \in G : \omega(s) = 1, \text{ for all } \omega \in H^\perp\}. \end{aligned}$$

From this we can see that $H \subseteq H^{\perp\perp}$. We still need to prove the reverse inclusion, and we shall do so by contraposition. Let $q_H : G \to G/H$ be the canonical quotient map, let $e_{G/H}$ be the neutral element in G/H, and let $s \notin H$, so that $q_H(s) \neq e_{G/H}$. Because H is closed, G/H is an LCA group (c.f. Corollary 3.4) we know by Gelfand-Raikov 3.9 that there exists a character $\xi \in \widehat{G/H}$ such that $\xi(q_H(s)) \neq \xi(e_{G/H}) = 1$, since $\xi \circ q_H \in H^\perp$, then we have shown that $s \notin H^{\perp\perp}$. □

Proposition 3.14. *The annihilator subgroup $H^\perp$ is a closed subgroup of $\widehat{G}$. Furthermore, we have the following isomorphisms:*

1. $\widehat{G/H} \cong H^\perp$;

2. $\widehat{G}/H^\perp \cong \widehat{H}$.

Proof. As we have pointed out in the proof of Lemma 3.10, convergence in compact-open topology on $\widehat{G}$ implies pointwise convergence. Therefore if $\{\omega_i\}_i$ is a net in $H^\perp$ such that $\omega_i \to \omega$ in $\widehat{G}$, then for all $h \in H$

$$1 = \omega_i(h) \to \omega(h),$$

implies $\omega(h) = 1$, therefore $\omega \in H^\perp$ as well. This shows $H^\perp$ is closed. Let $q_H : G \to G/H$ the canonical quotient map. We introduce two maps:

$$\Phi_1 : \widehat{G/H} \to H^\perp$$
$$\Phi_2 : \widehat{G}/H^\perp \to \widehat{H}.$$

Defined by $\Phi_1(\xi) = \xi \circ q_H$ for each $\xi \in \widehat{G/H}$, while $\Phi_2(\omega H^\perp) = \omega_{|H}$ for each $\omega \in \widehat{G}$. It is not hard to see that Φ_1 is well-defined, a group homomorphism, and its image does land in the intended space. For Φ_2 however, we may need to think a little bit to verify that it is indeed well-defined. Suppose $\overline{\eta}\omega \in H^\perp$, then for all $h \in H$, $(\overline{\eta}\omega)(h) = 1 \iff \omega(h) = \eta(h)$ so indeed $\Phi_2(\omega H^\perp) = \Phi_2(\eta H^\perp)$, hence Φ_2 is well-defined. It is also quite easy to see from the definition that it is a group homomorphism. Now, the strategy of course, is to show that Φ_1 and Φ_2 are isomorphisms between topological groups. We start with Φ_1. It is a group isomorphism, whose inverse is easily calculated to be $\Phi^{-1}(\omega) = \widetilde{\omega} \in \widehat{G/H}$ for each $\omega \in H^\perp$, via $\widetilde{\omega}(s + H) = (\widetilde{w} \circ q_H)(s) = \omega(s)$. We only need to show that it is a homeomorphism. Consider the net-convergence $\xi_i \to \xi$ in $\widehat{G/H}$. If $s_i \to s$ is another net in G, then by continuity of q_H, $q_H(s_i) \to q_H(s)$ in G/H. Since $\xi_i \to \xi$ in compact-open topology, then it follows from Proposition 3.5 that $\xi_i(q(s_i)) \to \xi(q(s))$ in $\mathbb{T}$, but this is equivalent to $\Phi_1(\xi_1)(s_i) \to \Phi_1(\xi)(s)$, since $\{\xi_i\}_i$ and ξ are arbitrary, then another application of Proposition 3.5 implies that $\Phi(\xi_i) \to \Phi(\xi)$ converges in the relative compact-open topology in $H^\perp$. Conversely, suppose that $\omega_i \to \omega$ in $H^\perp$ with the relative compact-open topology. Fix $s_i + H \to s + H$ in G/H. Consider the net $\{\widetilde{\omega}_i(s_i + H)\}_i$, where we pass to a subnet and relabel and call it $\{\widetilde{\omega}_i(s_i + H)\}_i$ still. It follows from Lemma 3.10 that there is a subnet $\{s'_j\}_j$ of $\{s_i\}_i$ and another net $\{h_j\}_j$ such that $s'_j + h_j \to s$. We see that $\omega_j(s'_j + h_j) = \omega_j(s'_j) \to \omega(s)$, but this is equivalent to $\widetilde{\omega}_j(s'_j + H) \to \widetilde{\omega}(s + H)$. Therefore it follows from Proposition B.9 that $\widetilde{\omega}_i(s_i + H) \to \widetilde{\omega}(s + H)$, which also implies that $\Phi_1^{-1}(\omega_i) \to \Phi_1^{-1}(\omega)$ in $\widehat{G/H}$. Therefore Φ is an isomorphism of LCA groups.

We can use $\widehat{G/H} \cong H^\perp$ to prove the second isomorphism. We have $(\widehat{G}/H^\perp)^\wedge \cong H^{\perp\perp} = H$

from Proposition 3.13, then by Pontryagin duality

$$\widehat{G}/H^{\perp} \cong ((\widehat{G}/H^{\perp})^{\wedge})^{\wedge} \cong \widehat{H},$$

if we follow the maps implementing this isomorphism, then we should find that it must be Φ_2. $\square$

We have a subtle corollary coming from the fact that $\Phi_2 : \widehat{G}/H^{\perp} \to \widehat{H}$ is invertible.

Corollary 3.9. *Every character in $\widehat{H}$ is a restriction of character from $\widehat{G}$, and conversely, every character in $\widehat{H}$ extends to a character in $\widehat{G}$. Therefore, for a fixed Haar measure μ_H in H, if $f \in L^1(G)$, then $\widehat{f|_{|H}} \in C_0(\widehat{H})$ and we can always write*

$$\widehat{f_{|H}}(\omega) = \int_{\widehat{H}} \overline{\omega}(h) f(h) d\mu_H(h) \tag{3.26}$$

for $\omega \in \widehat{G}$.

So far we have not talked much about the fact that for a closed subgroup H, both H and G/H are also LCA groups, and therefore have their own Haar measures. Given a Haar measure μ in G, we have a 'canonical' rescaling for the Haar measure $\widehat{\mu}$ of the Pontryagin group $\widehat{G}$, i.e. one that allows us to do a clean Fourier inversion as in either Theorems 3.18 or 3.11. It turns out we can also properly scale the Haar measures for H and G/H so that we can have what is called 'Weil's formula', and subsequently, the 'Poisson summation formula'.

We fix H to be a closed subgroup of G, μ_H to be a Haar measure on H, and use $q_H : G \to G/H$ for the canonical quotient map.

Proposition 3.15. *We denote the H-periodization map by P_H and is defined via*

$$(P_H f)(s + H) = \int_H f(s + h) d\mu_H(h), \qquad \forall f \in C_c(G).$$

The H-periodization is well-defined and $P_H : C_c(G) \to C_c(G/H)$. Furthermore, if $\phi \in C_c(G/H)$, then

$$P_H((\phi \circ q_H) \cdot f) = \phi \cdot P_H f, \qquad \forall f \in C_c(G).$$

Proof. Note that if $f \in C_c(G)$, then $(R_s f)_{|H} \in C_c(H)$, therefore

$$\int_H (R_s f)_{|H}(h) d\mu_H(h) = \int_H f(s + h) d\mu_H(h) =: (P_H f)(s + H)$$

is a well-defined integral. To see that $P_H f : G/H \to \mathbb{C}$ is a well-defined function, let $s_1 - s_2 \in H$, then by translation invariance of μ_H, we have

$$\begin{aligned}(P_H f)(s_2 + H) &= \int_H f(s_2 + h) d\mu_H(h) \\ &= \int_H f(s_2 + (s_1 - s_2) + h) d\mu_H(h) \\ &= \int_H f(s_1 + h) d\mu_H(h) = (P_H f)(s_1 + H).\end{aligned}$$

We only need to show that $P_H f \in C_c(G/H)$. We can use the continuity of the translation maps as in Corollary 3.5, the continuity of the restriction map (i.e. $\|f_{|H}\|_\infty \leqslant \|f\|_\infty$), and the continuity of the integral to show that $P_H f : G/H \to \mathbb{C}$ is continuous. Since $\operatorname{supp}(P_H f) \subseteq q_H(\operatorname{supp} f)$, and q_H is a continuous function, then $P_H f$ is compactly supported. We prove the last assertion, take $\phi \in C_c(G/H)$, then

$$\begin{aligned}P_H((\phi \circ q_H) \cdot f)(s + H) &= \int_H (\phi \circ q_H)(s + h) \cdot f(s + h) d\mu_H(h) \\ &= \phi(q_H(s)) \cdot \int_H f(s + h) d\mu_H(h) \\ &= \phi(s + H) \cdot (P_H f)(s + H)\end{aligned}$$

as required. □

We can now use the well-behaved-ness of the periodization operator to pick a 'canonical' Haar measure on G/H.

Proposition 3.16 (Weil's Formula)**.** *For a fixed Haar masure μ for G and μ_H for H, there is a uniquely scaled Haar measure $\mu_{G/H}$ on G/H that satisfies the following, for all $f \in C_c(G)$:*

$$\int_G f(s) d\mu(s) = \int_{G/H} (P_H f)(s + H) d\mu_H(s + H) = \int_{G/H} \int_H f(s + h) d\mu_H(h) d\mu_{G/H}(\dot{s}),$$

where we use the abbreviation $\dot{s} = q_H(s) = s + H$.

Proof. Let $\mu'_{G/H}$ be any Haar measure on G/H, and $f \in C_c(G)$, then the map

$$f \mapsto \int_{G/H} (P_H f)(s + H) d\mu'_{G/H}(s + H) \tag{3.27}$$

is a positive linear functional on $C_c(G)$. Therefore by Riesz-Markov-Kakutani, there must exist a unique $\lambda > 0$ such that

$$\int_G f(s)d\mu(s) = \lambda \int_{G/H} (P_H f)(s+H)d\mu'_{G/H}(s+H).$$

By choosing $\mu_{G/H} := \frac{1}{\lambda}\mu'_{G/H}$, we obtain our desired formula. □

If for fixed Haar measures μ and μ_H on G and H respectively, we choose the Haar measure $\mu_{G/H}$ on G/H such that Weil's formula is true, then we say that these three measures are **canonically related**.

One can actually extend everything to L^1, the following is due to [RS00, Theorem 3.4.6]

Corollary 3.10. *Let μ, μ_H, and $\mu_{G/H}$ be canonically related, then the periodization operator $P_H : C_c(G) \to C_c(G/H)$ extends to a surjective, linear, and bounded operator $P_H : L^1(G) \to L^1(G/H)$ satisfying the estimate $\|P_H f\|_1 \leqslant \|f\|_1$ for each $f \in L^1(G)$.*

Corollary 3.11 (Poisson Summation Formula)**.** *Suppose μ, μ_H, and $\mu_{G/H}$ are canonically related. If $f \in L^1(G)$, then for all $\xi \in \widehat{G/H}$*

$$\widehat{P_H f}(\xi) = \widehat{f}(\xi \circ q_H). \tag{3.28}$$

Therefore $\widehat{P_H f} \in L^1(\widehat{G/H})$ if and only if the map $\xi \mapsto \widehat{f}(\xi \circ q_H)$ is in $L^1(\widehat{G/H})$. In this case, we find that if $\widehat{\mu}_{\widehat{G/H}}$ is the dual measure for $\widehat{G/H}$, then for all $s \in G$

$$P_H f(s+H) = \int_H f(s+h)d\mu_H(h) = \int_{\widehat{G/H}} \xi(s+H)\widehat{f}(\xi \circ q_H)d\widehat{\mu}_{\widehat{G/H}}(\xi). \tag{3.29}$$

The formula above allows us to identify $\widehat{P_H f}$ with $\widehat{f}_{|H^\perp}$ given that we have the isomorphism $\Phi_1 : \widehat{G/H} \to H^\perp$ from Proposition 3.14. We choose the Haar measure on $\widehat{\mu}_{H^\perp}$ to be identified with that of $\widehat{\mu}_{\widehat{G/H}}$. Using these identifications, we can rewrite Equation (3.28) into $\widehat{P_H f} = \widehat{f}_{|H^\perp}$, and if $\widehat{f}_{|H^\perp} \in L^1(H^\perp)$, we can rewrite (3.29) into

$$\int_H f(s+h)d\mu_H(h) = \int_{H^\perp} \omega^\perp(s)\widehat{f}(\omega^\perp)d\widehat{\mu}_{H^\perp}(\omega^\perp), \qquad a.e.\ s \in G. \tag{3.30}$$

*By taking $s \in H$ in (3.30), we obtain what is called **Poisson Summation Formula**:*

$$\int_H f(h)d\mu_H(h) = \int_{H^\perp} \widehat{f}(\omega^\perp)d\widehat{\mu}_{H^\perp}(\omega^\perp). \tag{3.31}$$

Proof. We only need to prove Equation (3.28) and (3.29) since the rest follows from isomorphism Φ_1 of Proposition 3.14. Let $f \in L^1(G)$, then we know from Corollary 3.10 that $P_H \in L^1(G/H)$, we can compute its Fourier transform and find that for $\xi \in \widehat{G/H}$

$$\begin{aligned}
\widehat{P_H f}(\xi) &= \int_{G/H} \overline{\xi(s+H)}(P_H f)(s+H)d\mu_{G/H}(\cdot s) \\
&= \int_{G/H}\int_H \overline{\xi}(q_H(s+h))f(s+h)d\mu_H(h)d\mu_{G/H}(\cdot s) \\
&= \int_G \overline{(\xi \circ q_H)}(s)f(s)d\mu(s) = \widehat{f}(\xi \circ q_H).
\end{aligned}$$

Next, if $\xi \mapsto \widehat{\xi \circ q_H}$ is in $L^1(\widehat{G/H})$ so that equivalently $\widehat{P_H f} \in L^1(\widehat{G/H})$ as well, then we can use the Fourier Inversion Theorem 3.11 so that

$$(P_H f)(s+H) = \int_{\widehat{G/H}} \xi(s+H)\widehat{P_H f}(\xi)d\mu_{\widehat{G/H}}(\xi) = \int_{\widehat{G/H}} \xi(s+H)\widehat{f}(\xi \circ q_H)d\widehat{\mu}_{\widehat{G/H}}(\xi),$$

which proves Equation (3.29). □

Lemma 3.24. *Suppose μ, μ_H, and $\mu_{G/H}$ are canonically related. If $\widehat{\mu}_{\widehat{G/H}}$ is the dual measure for $\widehat{G/H}$, then for all $f \in L^1(G)$ such that $\widehat{P_H f} \in L^1(\widehat{G/H})$, then for all $\omega \in \widehat{G}$*

$$\widehat{f_{|H}}(\omega) = \int_{\widehat{G/H}} T_\omega \widehat{f}(\xi \circ q_H)d\widehat{\mu}_{\widehat{G/H}}(\xi).$$

Using the isomorphism $\Phi_1 : \widehat{G/H} \to H^\perp$ gives us

$$\widehat{f_{|H}}(\omega) = \int_{H^\perp} \widehat{f}(\omega^\perp \cdot \overline{\omega})d\mu_{H^\perp}(\omega^\perp) = P_{H^\perp}\widehat{f}(\overline{\omega}). \tag{3.32}$$

Proof. The result follows from the following computation:

$$\begin{aligned}
\widehat{f_{|H}}(\omega) &= \int_H \overline{\omega}(h)f(h)d\mu_H(h) \\
&= \int_H (M_\omega f)(h)d\mu_H(h) \\
&\overbrace{=}^{3.29} \int_{\widehat{G/H}} \widehat{M_\omega f}(\xi \circ q_H)d\widehat{\mu}_{\widehat{G/H}}(\xi) \\
&= \int_{\widehat{G/H}} T_\omega \widehat{f}(\xi \circ q_H)d\widehat{\mu}_{\widehat{G/H}}(\xi),
\end{aligned}$$

furthermore $\widehat{P_H f}$ if and only if $P_{H^\perp}\widehat{f} \in L^1(\widehat{G}/H^\perp)$. □

It is natural to ask if there is such a canonical choice (as in Weil's formula) of Haar measures on the 'dual' side which synergizes with the isomorphism result of Proposition 3.14.

Proposition 3.17. *If we start with canonically related measures μ, μ_H and $\mu_{G/H}$ on G, H and G/H respectively, and we choose the associated dual measures $\widehat{\mu}, \widehat{\mu}_{\widehat{H}}$ and $\widehat{\mu}_{\widehat{G/H}}$ on $\widehat{G}$, $\widehat{H} \cong \widehat{G}/H^{\perp}$, and $\widehat{G/H} \cong H^{\perp}$ respectively. Then identifying the measures $\widehat{\mu}_{H^{\perp}} \rightsquigarrow \widehat{\mu}_{\widehat{G/H}}$, and $\widehat{\mu}_{\widehat{G}/H^{\perp}} \rightsquigarrow \widehat{\mu}_{\widehat{H}}$, gives us a triple $\widehat{\mu}, \widehat{\mu}_{H^{\perp}}$, and $\widehat{\mu}_{\widehat{G}/H^{\perp}}$ that are all canonically related as Haar measures for $\widehat{G}$, $H^{\perp}$, and $\widehat{G}/H^{\perp}$.*

Proof. If $f \in L^1(\widehat{G}) \cap \mathcal{B}(G)$, then $\widehat{f} \in L^1(\widehat{G})$ by Theorem 3.18, and is enough to show that Weil's formula

$$\int_{\widehat{G}} \widehat{f}(\omega) d\widehat{\mu}(\omega) = \int_{\widehat{G}/H^{\perp}} \int_{H^{\perp}} \widehat{f}(\omega \cdot \omega^{\perp}) d\widehat{\mu}_{H^{\perp}}(\omega^{\perp}) d\widehat{\mu}_{\widehat{G}/H^{\perp}}(\dot{\omega}), \qquad \text{where } \dot{\omega} = \omega H^{\perp}.$$

holds for $\widehat{f}$ since we have Corollary 3.7. We use the fact that $\widehat{\mu}_{\widehat{H}}$ is a dual measure for $\widehat{H}$, and that we have an isomorphism $\Phi_2 : \widehat{G}/H^{\perp} \to \widehat{H}$, so that using the inverse Fourier-transform on $f_{|H} \in L^1(H)$ we have, for the neutral element $e_G \in H \subseteq G$

$$f_{|H}(e_G) = \int_{\widehat{H}} \widehat{f_H}(\omega) d\widehat{\mu}_{\widehat{H}}(\omega) = \int_{\widehat{H}} \widehat{f_H}(\overline{\omega}) d\widehat{\mu}_{\widehat{H}}(\omega) = \int_{\widehat{G}/H^{\perp}} \left[f_{|H}(\Phi_2(\overline{\omega} H^{\perp})) \right] d\widehat{\mu}_{\widehat{G}/H^{\perp}}(\omega H^{\perp}).$$

We pause here and check: we started with $\widehat{f} \in L^1(\widehat{G})$, and therefore $P_{H^{\perp}} \widehat{f} \in L^1(\widehat{G}/H^{\perp})$, which by Lemma 3.24, equivalently means $\widehat{P_H f} \in L^1(\widehat{G/H})$. Lastly we have also chosen $\widehat{\mu}_{\widehat{G/H}}$ to be the dual measure on $\widehat{G/H}$, and so we can apply Equation (3.32) of Lemma 3.24 hence by noting that $\Phi_2^{-1}(\overline{\omega} H^{\perp}) = \overline{\omega} \in \widehat{H}$ for all $\omega \in \widehat{G}$:

$$\begin{aligned} f_{|H}(e_G) &= \int_{\widehat{G}/H^{\perp}} \left[(P_{H^{\perp}} \widehat{f})(\Phi_2(\omega H^{\perp})) \right] d\widehat{\mu}_{\widehat{G}/H^{\perp}}(\omega H^{\perp}) \\ &= \int_{\widehat{G}/H^{\perp}} \int_{H^{\perp}} \widehat{f}(\omega \cdot \omega^{\perp}) d\widehat{\mu}(\omega^{\perp}) d\widehat{\mu}_{\widehat{G}/H^{\perp}}(\dot{\omega}). \end{aligned}$$

However, using Fourier inversion on $f \in L^1(G)$, we find that

$$\int_{\widehat{G}} \widehat{f}(\omega) d\widehat{\mu}(\omega) = f(e_G) = f_{|H}(e_G) = \int_{\widehat{G}/H^{\perp}} \int_{H^{\perp}} \widehat{f}(\omega \cdot \omega^{\perp}) d\widehat{\mu}(\omega^{\perp}) d\widehat{\mu}_{\widehat{G}/H^{\perp}}(\dot{\omega})$$

as required. □

We have now finished our lengthy exposition on basic abstract harmonic analysis, we shall freely make references from this chapter throughout the rest of the document.

3.8 Some notes on Vector-Valued Integrals

While it can be argued that measure and integration theory on vector-valued functions are not exactly within the realm of abstract harmonic analysis, I choose to include a short section on it as I believe it is still thematically relevant. After all, in the sequel, all that we will be interested in will be integrating vector-valued functions over locally compact groups. In which case, we will only consider the generated measure space of a (second countable) locally compact group G, i.e. $(G, \mathrm{Bor}(G), \mu_G)$ where $\mathrm{Bor}(G)$ is the Borel σ-algebra generated by G, and μ_G is a Haar measure on G. For the rest of this section, we shall fix $(G, \mathrm{Bor}(G), \mu_G)$, but note that the definition and properties of both the integrals-to-be-defined below apply to any measure space. We shall fix B to be a Banach space. Our main reference in this section is [DU77].

There are two notions of integration for vector-valued functions, one is the Bochner integral, while the other is called the Pettis integral. In the literature, sometimes it is not exactly clear which of these two notions of integration is in use. Our goal, primarily, is to establish a definition for all the vector-valued integrals that will appear in this work. Secondarily, we would like to clear up when these two notions of integration coincide, which would further allow us to scrutinize particular results in the literature where this equivalence is not exactly mentioned.

We shall start with the so-called Pettis integral, also known as the weak integral for vector-valued functions. To this end, we shall first define a suitable notion of weak μ_G-measurability.

Definition 3.14. A function $f : G \to B$ is said to be **weakly $\mu_{\mathbf{G}}$-measure** if for all $\varphi \in B^*$, $\varphi \circ f : G \to \mathbb{C}$ is μ_G-measurable i.e. $(\varphi \circ f)^{-1}(O) \in \mathrm{Bor}(G)$ for all open set $O \subseteq G$.

Equipped with a notion of weak-measurability, we can now define weak-integrability in an expected way.

Definition 3.15. A weakly μ_G-measurable function $f : G \to B$ is said to be **weakly $\mu_{\mathbf{G}}$-integrable** if for all $\varphi \in B^*$, $\varphi \circ f \in L^1(G)$.

We might expect that this would be enough to obtain Pettis integral of a vector-valued function. However, we only obtain an integral of an f living in the double dual B^{**}, as described by the following Lemma.

Lemma 3.25 ([DU77, Lemma 2.3.1])**.** *If $f : G \to B$ is a weakly μ_G-integrable function, then for each $E \in \mathrm{Bor}(G)$, there exists a (unique) element in B^{**}, which we shall denote by $(D)\int_E f(s)d\mu_G(s) \in B^{**}$ such that:*

$$\left((D)\int_E f(s)d\mu_G(s)\right)(\varphi) = \int_E (\varphi \circ f)(s)d\mu_G(s), \qquad \forall \varphi \in B^*. \tag{3.33}$$

*The integral $(D)\int_E f(s)d\mu_G(s)$ is called the **Dunford Integral of f over E**.*

The Dunford integral only affords us an integral in B^{**}, while we would expect our weak integral to be in B. This can be easily remedied in light of the canonical isometric embedding $B \hookrightarrow B^{**}$. It is given by $b \mapsto b^{**}$ where $b^{**}(\varphi) = \varphi(b)$ for all $\varphi \in B^*$ and $b \in B$. We therefore make the following definition.

Definition 3.16. A weakly μ_G-integrable function $f : G \to B$ is said to be **Pettis Integrable** if for all $E \in \mathrm{Bor}(G)$, $(D)\int_E f(s)d\mu_G(s) \in B$. In which case, we also write:

$$(P)\int_E f(s)d\mu_G(s) := (D)\int_E f(s)d\mu_G(s) \in B, \tag{3.34}$$

and call $(P)\int_E f(s)d\mu_G(s)$ to be the **Pettis Integral of f over E**.

This notion of integration over vector vector spaces was introduce by Pettis in [Pet38]. We also see that the term "weak-integral" for the Pettis integral is appropriate, given we that can rewrite the defining property (3.33) into:

$$\varphi\left((P)\int_E f(s)d\mu_G(s)\right) = \int_E (\varphi \circ f)(s)d\mu_G(s). \tag{3.35}$$

Note the it follows quite easily as a corollary of Equation (3.33), Hahn-Banach theorem, and the fact that $\sup\{|\varphi(b)| : \varphi \in B^*, \ \|\varphi\| \leqslant 1\} = \|b\|_B$ that we have the usual integral inequality.

Corollary 3.12. *For any Pettis integrable $f : G \to B$, and $E \in \mathrm{Bor}(G)$, we have the following inequality:*

$$\left\|(P)\int_E f(s)d\mu_G(s)\right\|_B \leqslant \int_E \|f(s)\|_B d\mu_G(s). \tag{3.36}$$

The Dunford integral does not perfectly generalize the usual properties of a classical integral as it fails the usual countable additivity property. The Pettis integral now shows its value, it turns out that one can use Pettis integrability to characterize Dunford integrals with countable additivity. A weakly μ_G-integrable function has a countably additive Dunford integral over E if and only if it is Pettis integrable over E [DU77, Theorem 2.3.5].

In principle, all that we'll really need in this work is the notion of Pettis integral, and indeed most of the relevant literature (implicitly) uses this weak definition of measurability and integrability [Fol15, Wil07] although some paper would mention Bochner integrals in passing when discussing their notion of vector-valued integral (take [PR89] in particular). We shall show that for all intents and purposes, these two are 'equivalent'. For now, we stick with Pettis integration, where we have the following result from [Wil07, Lemma 1.91]. Note that this result has been similarly discussed in Folland's [Fol15, Theorem A.22], though it is slightly less general, however the proof should still go through *mutatis mutandis.*

Proposition 3.18. *If $f : G \to B$ is a function in $C_c(G, B)$, then f is Pettis integrable. Furthermore, if Y is another Banach space, $f \in C_c(G, B)$, and $T \in \mathcal{L}(B, Y)$, then $L \circ f \in C_c(G, Y)$. We also have the following formula:*

$$T\left((P)\int_G f(s)d\mu_G(s)\right) = (P)\int_G (T \circ f)(s)d\mu_G(s). \tag{3.37}$$

Indeed, we can build up the theory of twisted crossed products of Section 5.1 purely in terms of $C_c(G, A)$ when A is a C^*-algebra using Proposition 3.18 alone. In particular, integrated forms of a strongly continuous unitary 'representation' (c.f. Proposition 5.2) can be deduced from Proposition 3.18. To be more explicit, we first discuss the problem with Proposition 3.18. Fix a Hilbert space H, it is not defined when we want to integrate over a map $\pi : G \to \mathcal{L}(H)$ that is compactly supported but is only continuous with respect to the strong operator topology on $\mathcal{L}(H)$ (the original proposition requires continuity with respect to the norm-topology). Let us denote by $\mathcal{L}_s(H)$ the operator space $\mathcal{L}(H)$ equipped with strong operator topology. Suppose $\pi \in C_s(G, \mathcal{L}_s(H))$, then for all $h \in H$, $s \mapsto \pi(s)h$ is in $C_c(G, H)$. It follows from Proposition 3.18 that $s \mapsto C_c(G, H)$ is Pettis integrable, therefore the map from H to H denoted via:

$$\left(\int_G \pi(s)d\mu_G(s)\right) : h \mapsto (P)\int_G \pi(s)hd\mu_G(s) \tag{3.38}$$

is well-defined. The next proposition due to [RW98, Lemma C.11] says that this definition actually gives us a continuous map in H.

Proposition 3.19. *The map* $\pi \mapsto \int_G \pi(s)d\mu_G(s)$ *defines a continuous map from* $C_c(G, \mathcal{L}_s(H))$ *to* $\mathcal{L}(H)$ *satisfying the following estimate:*

$$\left\| \int_G \pi(s)d\mu_G(s) \right\| \leqslant \|\pi\|_\infty \cdot \mu_G(\mathrm{supp}(\pi)). \tag{3.39}$$

In this case, we can write $\int_G \pi(s)d\mu_G(s) \in \mathcal{L}(H)$*, and*

$$\left(\int_G \pi(s)d\mu_G(s) \right) h = (P) \int_G \pi(s)hd\mu_G(s), \qquad \forall h \in H. \tag{3.40}$$

With this in place, we end with a dicsussion of a few more auxiliary results on the Pettis integral, in particular, a version of Fubini's theorem holds. Fixing another locally compact group G' with Haar measure $\mu_{G'}$, the following result can be found in [Wil07, Corollary 1.104].

Proposition 3.20. *Let* $F \in C_c(G \times G', B)$*, then the following maps:*

$$s \mapsto (P) \int_{G'} F(s,r)d\mu_{G'}(r) \quad \textit{and} \quad r \mapsto (P) \int_G F(s,r)d\mu_G(s)$$

belong to $C_c(G, B)$ *and* $C_c(G', B)$ *respectively.*

Let us now give a very brief introduction to the Bochner integral, which was introduced Bochner in [Boc33]. It can be thought of as a straightforward generalization of integration to Banach space valued functions. We say that a function $f : G \to B$ is **simple** if there exists finite Borel sets $\{E_1, ..., E_n\} \subseteq \mathrm{Bor}(G)$ and $b_1, .., b_n \in B$ such that we can write $f = \sum_{i=1}^n b_i 1_{E_i}$ where 1_{E_i} is the characteristic function on E_i. The **Bochner integral of a simple function** f is

$$(B) \int_G f(s)d\mu_G(s) = \sum_{i=1}^n b_i \mu(E_i) \in B. \tag{3.41}$$

A strongly μ_G-measurable function, or Bochner μ_G-measurable function is now defined reminiscent of the simple approximation Lemma from measure theory[16] in $\mathbb{C}$.

Definition 3.17. A function $f : G \to B$ is said to be **strongly μ_G-measurable** or **Bochner μ_G-measurable** if there exists a sequence of simple functions $\{f_n\}_{n \in \mathbb{N}}$ such that:

$$\|f(s) - f_n(s)\|_B \to 0, \qquad \forall s \in G \text{ a.e.}$$

[16] Recall that a map $f : G \to \mathbb{C}$ is measurable if and only if it can be approximated by simple functions

We call $\{f_n\}_{n\in\mathbb{N}}$ a **simple approximation of f**.

The so-called Pettis's-measurability theorem, which we give below, shows that the notion of *strong* measurability is appropriate.

Theorem 3.12 ([DU77, Theorem II.I.2])**.** *A function $f : G \to B$ is strongly μ_G-measurable if and only if*

1. *There exists a Borel set $E \in \mathrm{Bor}(G)$ with $\mu_G(E) = 0$ such that $f(G\backslash E)$ is a norm-separable subset of B. In which case, we say that f is μ_G-essentially separably valued;*

2. *f is weakly μ_G-measurable.*

We see that every strongly μ_G-measurable function is automatically weakly μ_G-measurable function. The converse is not true, as there exist weakly μ_G-measurable functions that are not μ_G-essentially separably valued [DU77, Example 2.1.5]. On the other hand, we see that Pettis's measurablity theorem imply that weak and strong μ_G-measurability are equivalent when B is a separable Banach space! In [PR89], this is the reason why they assume separability in their C^*-algebras.

We are starting to see the relationship between these two different notions of measurability. However, we are interested in the integrals themselves. To see this, we first need to define what a Bochner integral is.

Definition 3.18. A strongly μ_G-measurable function $f : G \to B$ is said to be **strongly integrable** or **Bochner integrable** if $\displaystyle\int_G \|f(s) - f_n(s)\|_B \to 0$ for a simple approximation $\{f_n\}_{n\in\mathbb{N}}$ of f. Consequently, we define the Bochner integral of f to be:

$$(B)\int_G f(s)d\mu_G(s) = (B)\lim_{n\to\infty}\int f_n(s)d\mu_G(s), \tag{3.42}$$

where the right-hand side in the equation above is defined as a limit of limit of Bochner integral of simple functions via (3.41).

The limit in Equation (3.42) since B is a Banach space, hence complete. A proof of independence of the Bochner-integral to any simple approximation is also warranted, but one can furnish a proof similar from Lebesgue integration.

More importantly, the Bochner integral possesses the following properties (c.f. [DU77]).

Proposition 3.21. *Consider a strongly μ_G-mesurable function $f : G \to B$. Then:*

1. *f is Bochner integrable if and only if* $\int_G \|f(s)\|_B d\mu_G(s) < \infty$*;*

2. *If f is Bochner integrable, then*

$$\left\|(B)\int_G f(s)d\mu_G(s)\right\|_B \leqslant \int_G \|f(s)\|_B d\mu_G(s). \tag{3.43}$$

3. *Suppose Y is another Banach space, and $T \in \mathcal{L}(B, Y)$. If $f : G \to B$ is Bochner integrable, then so is $T \circ f : G \to Y$. Consequently:*

$$T\left((B)\int_G f(s)d\mu_G(s)\right) = (B)\int_G (T \circ f)(s)d\mu_G(s). \tag{3.44}$$

4. *If $f, g : G \to B$ are both Bochner integrable functions, we have: if* $(B)\int_G f(s)d\mu_G(s) = (B)\int_G g(s)d\mu_G(s)$*, then $f = g$ a.e.*[17]

Remark 3.11. If $B = \mathbb{C}$, everything about Bochner integral above (with the possible exception of Item 3 in Proposition 3.21) becomes nothing more than the usual integral definition and properties.

We now fully see, how Pettis and Bochner integrals are related.

Proposition 3.22. *Let $f : G \to B$ be Bochner integrable, then f is also Pettis integrable, and the Bochner and Pettis integrals coincide: for all $E \in \mathrm{Bor}(G)$*

$$(B)\int_G (f \cdot 1_E)(s)d\mu_G(s) = (P)\int_E f(s)d\mu_G(s). \tag{3.45}$$

Proof. If f is Bochner integrable, then it is by definition strongly μ_G-measurable. Therefore f must be weakly μ_G-measurable as well from Pettis's measurability theorem. For any $E \in \mathrm{Bor}(G)$, $f \cdot 1_E$ is also Bochner integrable due to Item 1 and 2 of Proposition 3.21 and $\int_E (f \cdot 1_E)(s)d\mu_G(s) \in B$.

Furthermore, it follows from Item 3 Proposition 3.21 that for any $E \in \mathrm{Bor}(G)$ and linear functional in B^*:

$$\varphi\left((B)\int_G f(s) \cdot 1_E d\mu_G(s)\right) = \int_E (\varphi \circ f)(s)d\mu_G(s).$$

[17] Equivalence almost everywhere for vector-valued functions also define an equivalence relation.

Since $\int_G (f \cdot 1_E)(s) d\mu_G(s)$ for any $E \in \mathrm{Bor}(G)$, it follows from Definition 3.16 that f is Pettis-integrable. It also follows from the defining property of Pettis integral (3.35) that for all $\varphi \in B^*$

$$\varphi\left((P)\int_E f(s) d\mu_G(s)\right) = \int_E (\varphi \circ f)(s) d\mu_G(s) = \varphi\left((B)\int_G (f \cdot 1_E)(s) d\mu_G(s)\right).$$

Therefore, we can deduce from the Hahn-Banach theorem that

$$(P)\int_E f(s) d\mu_G(s) = (B)\int_G (f \cdot 1_E)(s) d\mu_G(s)$$

. □

In light of Proposition 3.22, we will now be dropping the 'markers' in front of the integrals for vector-valued functions. That is, we no longer distinguish between Bochner and Pettis integrals (we won't be dealing with proper Dunford integrals in this work), *as all vector-valued integrals henceforth will now be interpreted as Pettis integrals.*

We shall end this section with a definition of L^p-spaces for vector-valued functions. For a strongly μ_G-integrable function $f : G \to B$ and $1 \leqslant p < \infty$, we define the p-norm via:

$$\|f\|_p = \left(\int_G \|f(s)\|_B d\mu_G(s)\right)^{1/p}. \tag{3.46}$$

We say that f is p-integrable if $\|f\|_p < \infty$. As in ordinary measure theory, under the p-norm, the difference of two functions whose p-norm vanishes, are indistinguishable almost everywhere. Therefore we can define a normed space, in fact a Banach space $L^p(G, B)$, whose elements are the equivalence classes of p-integrable functions modulo equivalence almost everywhere. Note again that for $p = 2$, $L^2(G, B)$ is a Hilbert space. We will also use the following approximation theorem, so that we can always 'concretize' an $L^1(G, B)$ 'function' in terms of compactly supported continuous functions. A proof of the following can be found in [Wil07, Proposition B.33]

Proposition 3.23. *$C_c(G, B)$ is dense densely embedded $L^1(G, B)$. That is, for any $f \in L^1(G, B)$, there exists a net $\{f_i\}_{i \in I} \subseteq C_c(G, B)$ such that* $\lim_i \|f - f_i\| \to 0$.

We call $L^1(G, B)$ the integrable B-valued 'functions', and in Section 5.1 of Chapter 5 we will invoke Proposition 3.23 to (densely) define a twisted version of $L^1(G, B)$, along with its twisted representations.

Chapter 4

Analysis on the Time-Frequency Plane

This chapter is a direct sequel to Chapter 3, as we continue our study of abstract harmonic analysis, with a particular emphasis on performing the analysis on the time-frequency plane. We open the first section by describing the fundamental operators in time-frequency analysis—the translation and modulation operators. We then proceed with an exploration of Fourier analysis on lattices in the time-frequency plane. The second section introduces Gabor analysis, which we shall approach from the framework of frame theory. Afterwards in the third section, we provide an overview of Feichtinger's algebra, not only aiming to convince the reader that it is the ideal function space for Gabor analysis, but also explaining how its properties motivated this thesis. Lastly, we close the chapter with a section detailing the theory of operator-quantization based on the kernel theorems afforded by the Feichtinger's algebra. This will be done in the spirit of Feichtinger and Kozek [FK04]. We verify that their results also hold for the general LCA groups.

The results of Sections 4.1 and 4.2 are primarily my personal verification of the presentation of abstract harmonic analysis on the time-frequency plane and Gabor analysis as discussed in [Jak18, JL20, Grö98, FK04]. For our overview of the Feichtinger's algebra on LCA groups, our primary reference is the survey paper [Jak18], and the seminal paper [Fei81] by Feichtinger. The last section, of course, is another telling of [FK04] using all of the tools that we have built up from previous sections. So there will be some differences in the presentation of proofs compared to the original paper.

4.1 The Time-Frequency Plane

Building up from the language that we have developed in Chapter 3: we always assume that G is any second countable LCA group equipped with a Haar measure μ, the dual space of characters $\widehat{G}$ is then equipped with the dual Haar measure $\widehat{\mu}$ wherein the inverse Fourier transform 3.18 and 3.11 holds.

For $s \in G$, $\omega \in \widehat{G}$, and a complex function $f : G \to \mathbb{C}$, the fundamental operators of time-frequency analysis are the **translation** and **modulation** operators, defined respectively:

$$\begin{aligned} (T_s f)(r) &:= f(r-s); \\ (M_\omega f)(r) &:= \omega(r) f(r). \end{aligned} \tag{4.1}$$

Given the Fourier transform $\mathcal{F}$ on G, it follows from a straightforward computation that the translation and modulation maps also satisfy the following commutation relations:

$$\begin{aligned} M_\omega T_x &= \omega(x) T_x M_\omega; \\ \mathcal{F} T_x &= M_{-x} \mathcal{F}; \\ \mathcal{F} M_\omega &= T_\omega \mathcal{F}. \end{aligned} \tag{4.2}$$

It is not hard to see that translation and modulation: $s \mapsto T_s$ and $\omega \mapsto M_\omega$ defines isometric maps from $G \to \mathcal{L}(L^p(G))$ and $\widehat{G} \to \mathcal{L}(L^p(G))$ respectively. In particular, this means that these maps also define a unitary map $G \to U\mathcal{L}(L^2(G))$ and $\widehat{G} \to U\mathcal{L}(L^2(G))$. Finally, we can easily compute for $s, s_1, s_2 \in G$ and $\omega, \omega_1, \omega_2 \in \widehat{G}$ that:

$$\begin{aligned} T_{s_1} T_{s_2} &= T_{s_1+s_2} \quad \text{and} \quad T_{-s} = T_s^{-1}, \\ M_{\omega_1} M_{\omega_2} &= M_{\omega_1 \omega_2} \quad \text{and} \quad M_{\overline{\omega}} = M_\omega^{-1}. \end{aligned}$$

We have the following result; note that we are using 0 to denote both the neutral element in G and $\widehat{G}$ for the rest of the document.

Proposition 4.1. *The translation and modulation operators define unitary representations of G on the Hilbert space $L^2(G)$. Given by: $s \mapsto T_s : G \to U\mathcal{L}(L^2(G))$, and $\omega \mapsto M_\omega : \widehat{G} \to U\mathcal{L}(L^2(G))$ respectively.*

Proof. We only need to prove strong continuity of the maps. We shall show that:

$$\lim_{s\to 0} \|T_s f - f\|_2 \to 0$$
$$\lim_{\omega\to 0} \|M_\omega f - f\|_2 \to 0$$

for all $f \in C_c(G)$. because $C_c(G)$ is dense in $L^2(G)$. This is sufficient because $C_c(G)$ is dense in $L^2(G)$. Fix $f \in C_c(G)$. To prove the first limit, we first note that for all $s, s' \in G$: $|f(s' - s) - f(s')|^2 \leqslant 4\|f\|_\infty^2 \chi_{\operatorname{supp} f}(s')$. Therefore we can use Lebesgue's dominated convergence theorem (LDCT) to justify the computation below:

$$\lim_{s\to 0} \|T_s f - f\|_2 = \left(\int_G \lim_{s\to 0} |f(s' - s) - f(s')|^2 d\mu_G(s) \right)^{1/2} = 0. \tag{4.3}$$

We proceed similarly to prove the second limit. Note that for all $\omega \in \widehat{G}$ and $s' \in G$ that $|\omega(s')f(s') - f(s')|^2 = |\omega(s') - 1|^2 |f(s')|^2 \leqslant 4|f(s')|$. Therefore we can again use LDCT to justify the computation below:

$$\lim_{\omega\to 0} \|M_\omega f - f\|_2 = \left(\int_G \lim_{\omega\to 0} |\omega(s') - 1|^2 |f(s')|^2 \right)^{1/2} = 0.$$

□

In the dual side, i.e. when we consider some $g : \widehat{G} \to \mathbb{C}$, then by Pontryagin duality there is an obvious analogue of the translation and modulation operators for $\varsigma, \omega \in \widehat{G}$ and $s \in G$, given respectively by

$$\begin{aligned} (T_\varsigma g)(\omega) &:= g(\omega\overline{\varsigma}); \\ (M_s g)(\omega) &:= \operatorname{ev}(s)(\omega) g(\omega) = \omega(s) g(\omega). \end{aligned} \tag{4.4}$$

From the translation and modulation operators, the composition

$$\pi(x, \omega) := M_\omega T_x \tag{4.5}$$

is called the **time-frequency** shift operator. It follows from the definition that it is also an isometry on the $L^p(G)$ spaces for $1 \leqslant p \leqslant \infty$, and in particular a unitary map in $\mathcal{L}(L^2(G))$.

The product $G \times \widehat{G}$ is interpreted as the **time-frequency plane**, and its elements are denoted either simply by a single symbol like $\chi \in G \times \widehat{G}$, or we may explicitly denote an element by $(x, \omega) \in G \times \widehat{G}$. In either case, the negative elements are denoted by $-\chi$ or $-(x, \omega) = (-x, \overline{\omega})$. One is also usually interested in sampling on $G \times \widehat{G}$, for this we introduce $\Lambda \subseteq G \times \widehat{G}$ to be a discrete subgroup. We endow $G \times \widehat{G}$ the product measure $\mu_{G\times\widehat{G}}$, and since Λ is countable (G is assumed to be second countable), we give it the counting measure μ_Λ. Finally we choose the measure $\mu_{(G\times\widehat{G})/\Lambda)}$ on $(G \times \widehat{G})/\Lambda$ that would make $\mu_{G\times\widehat{G}}, \mu_\Lambda$ and $\mu_{(G\times\widehat{G})/\Lambda}$ canonically related, that is, we have Weil's formula from Proposition 3.10: for each $F \in L^1(G \times \widehat{G})$

$$\begin{aligned}\int_{G\times\widehat{G}} F(\chi) d\mu_{G\times\widehat{G}}(\chi) &= \int_{(G\times\widehat{G})/\Lambda} \int_\Lambda F(\chi + \lambda) d\mu_\Lambda(\lambda) d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\ &= \int_{(G\times\widehat{G})/\Lambda} \sum_{\lambda\in\Lambda} F(\chi + \lambda) d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}).\end{aligned}$$

We give one more condition on the discrete subgroup $\Lambda \subseteq G \times \widehat{G}$, and make it a definition.

Definition 4.1. A discrete subgroup $\Lambda \subseteq G \times \widehat{G}$ is called a **lattice** (with respect to $G \times \widehat{G}$) if the quotient subgroup $(G \times \widehat{G})/\Lambda$ is compact.

Defining lattices this way gives us some of the expected results from time-frequency analysis on $G = \mathbb{R}^d$. In [Grö01] lattices of the the time-frequency plane $G \times \widehat{G} = \mathbb{R}^d \times \widehat{\mathbb{R}^d} \cong \mathbb{R}^{2d}$ are exactly of the form $L\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$ for some invertible matrix $L \in GL_{2d}(\mathbb{R})$, and quotienting out the lattice: $\mathbb{R}^{2d}/L\mathbb{Z}^{2d} \cong (G \times \widehat{G})/\Lambda$ will always give us a compact subgroup since the quotient will be homeomorphic to a $2d$-product of closed and bounded intervals in $\mathbb{R}$. While subgroups of the form $L\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$ satisfy the definition of a lattice, it is in fact a characterization because every lattice in $\mathbb{R}^{2d}$ are exactly of such form [Grö98].

We shall give a result that characterizes a lattice, but we need an additional lemma for its proof. It can be seen as a converse to Proposition 3.6:

Proposition 4.2. *Let G be a locally compact group with neutral element e, and Haar measure μ.*

1. *$\mu(\{e\}) > 0$ if and only if G is discrete;*
2. *$\mu(G) < \infty$ if and only if G is compact.*

Proof.

1. If G is discrete, then $\{e\}$ us open, therefore by Proposition 3.6, $\mu(\{e\}) > 0$. Conversely, suppose $\mu(\{e\}) > 0$, since tranlation is a homeomorphism in G, this implies $\mu(\{e\}) =$

$\mu(\{g\}) > 0$ for all $g \in G$. By outer regularity of μ, for $\varepsilon := \mu(\{e\})$, there exists an open set $U \ni e$ such that

$$\mu(U) < \mu(\{e\}) + \varepsilon = 2\mu(\{e\}). \tag{4.6}$$

Assume for contradiction that G is not discrete, then it follows that there exists a point in G that is not isolated. Again by appealing to the fact that translation is a homeomorphism, all points in G must not be isolated. In particular, the open set U of e must intersect a $g \neq e$ in G. Therefore $U \supseteq \{e, g\}$, which implies that

$$\mu(U) \geqslant \mu(\{e, g\}) = \mu(\{e\}) + \mu(\{g\}) = 2\mu(\{e\}).$$

The inequality above contradict Estimate 4.6, therefore G must be discrete.

2. If G is compact, we already know from Proposition 3.6 that $\mu(G) < \infty$. For the converse, suppose $\mu(G) < \infty$. Assume for contradiction that G is not compact (so locally compact but not compact), then there exists an open set $U \ni e$ such that $\overline{U}$ is compact. By assumption $\overline{U} \neq G$ (otherwise G will be compact). Since $\{gU : g \in G\}$ is an open cover of G, then it follows by non-compactness of G that no finite subfamily of $\{gU : g \in G\}$ can cover G. Set $g_0 = e$, then we can always find $g_1 \in G\backslash g_0 U = G\backslash U$. Similarly, we can also find $g_1 = G\backslash (g_0 U \cup g_1 U)$. In general, we can find a sequence $\{g_n\}_{n=0}^{\infty} \subseteq G$ with $g_{n+1} \in G\backslash (\bigcup_{i=0}^{n} g_i U)$ for each $n \in \mathbb{N}$. It is not hard to see that the collection $\{g_i U\}_{i=1}^{\infty}$ consists of pairwise disjoint sets. Therefore, since $\mu(U) > 0$:

$$\mu(G) = \mu\left(\bigcup_{g \in G} gU\right) \geqslant \mu\left(\bigsqcup_{i=1}^{\infty} g_i U\right) = \sum_{i=1}^{\infty} \mu(g_i U) = \sum_{i=1}^{\infty} \mu(U) = \infty,$$

which is a contradiction to the fact that $\mu(G) < \infty$. Therefore G must be compact.

□

Let us also take the opportunity to discuss the commutation relations between time-frequency shifts themselves. The so-called 2-cocycles that appear here will have a very important role in the construction of the Heisenberg module.

Definition 4.2. A continuous map $c : (G \times \widehat{G}) \times (G \times \widehat{G}) \to \mathbb{T}$ is called a **normalized 2-cocycle** on $G \times \widehat{G}$ if it satisfies the following:

1. If 0 is the identity element in $G \times \widehat{G}$, then $c(0, \chi) = c(\chi, 0) = 1$;

2. $c(\chi_1, \chi_2)c(\chi_1 + \chi_2, \chi_3) = c(\chi_1, \chi_2 + \chi_3)c(\chi_2, \chi_3)$.

The natural 2-cocycle in time-frequency analysis is the so-called **Heisenberg 2-cocycle** $c : (G \times \widehat{G}) \times (G \times \widehat{G}) \to \mathbb{T}$ given by:

$$c((x, \omega), (t, \nu)) = \overline{\nu}(x) \tag{4.7}$$

For the rest of this chapter, we take c to take on this particular form. We can deduce from a straightforward calculation that the Heisenberg 2-cocycle appears naturally as a "twist" that prevents commutation between time-frequency shifts.

Proposition 4.3. *The time-frequency shift operator define a strongly continuous map $\pi : G \times \widehat{G} \to U\mathcal{L}(L^2(G))$ such that for all $(x, \omega), (t, \nu) \in G \times \widehat{G}$:*

$$\pi(x, \omega)\pi(t, \nu) = c((x, \omega), (t, \nu))\pi((x, \omega) + (t, \nu)). \tag{4.8}$$

Furthermore, the adjoint of the time-frequency shifts also define a strongly continuous map $\pi^* : G \times \widehat{G} \to U\mathcal{L}(L^2(G))$ via $\pi^*(x, \omega) = (\pi(x, \omega))^*$ for $(x, \omega) \in G \times \widehat{G}$ where

$$\pi^*(x, \omega) = c((x, \omega), (x, \omega))\pi(-x, \overline{\omega}). \tag{4.9}$$

Proof. Formulae (4.8) and (4.9) follow from a straighforward calculation. We will only show strong continuity of $\pi : (x, \omega) \mapsto \pi(x, \omega)$. Let $f \in L^2(G)$, and let us denote by 0 the neutral element $(0, 0)$ in $G \times \widehat{G}$. It follows from Proposition 4.1 that

$$\begin{aligned}
\lim_{(x,\omega)\to 0} \|\pi(x, \omega)f - f\|_2 &= \lim_{(x,\omega)\to 0} \|M_\omega T_x f - f\|_2 \\
&\leqslant \lim_{(x,\omega)\to 0} \|M_\omega(T_x f) - (T_x f)\|_2 + \lim_{(x,\omega)\to 0} \|f - T_x f\|_2 = 0 + 0 \\
&= 0,
\end{aligned}$$

as required. □

From the Heisenberg 2-cocycle, we can induce the **symplectic cocycle** $c_s : (G \times \widehat{G}) \times (G \times$

$\widehat{G}) \to \mathbb{T}$ via:

$$c_s((x,\omega),(t,\nu)) = c((x,\omega),(t,\nu)) \cdot \overline{c((t,\nu),(x,\omega))} = \overline{\nu}(x)\omega(t). \tag{4.10}$$

It is not hard to verify that

$$\pi(x,\omega)\pi(t,\nu) = c_s((x,\omega),(t,\nu))\pi(t,\nu)\pi(x,\omega). \tag{4.11}$$

As an easy corollary of this observation, we have the following lemma.

Lemma 4.1. *Let $(x,\omega),(t,\nu) \in G \times \widehat{G}$. The time-frequency shifts $\pi(x,\omega)$ and $\pi(t,\nu)$ commute if and only if $c_s((x,\omega),(t,\nu)) = 1$.*

Before we continue, let us first make some isomorphisms explicit. We know from Proposition 3.10 that the Pontryagin dual of the time-frequency plane satisfies: $\widehat{G \times \widehat{G}} \cong \widehat{G} \times \widehat{\widehat{G}} \cong \widehat{G} \times G \cong G \times \widehat{G}$. That is, the time-frequency plane is self-dual. We could chase all the necessary isomorphisms, and indeed by exactly the isomorphism in Item 3 of Proposition 3.10, the Pontryagin duality, and the obvious isomorphism $G_1 \times G_2 \cong G_2 \times G_1$ between two (topological) groups, we have the following result:

Lemma 4.2. $\widehat{G \times \widehat{G}} = \{\Omega_{\nu,t} : (\nu,t) \in \widehat{G} \times G\}$ *where* $\Omega_{\nu,t}(x,\omega) = \nu(x)\omega(t)$ *for each* $(x,\omega) \in G \times \widehat{G}$. *The map $(t,\nu) \mapsto \Omega_{\nu,t}$ provides an isomorphism from $G \times \widehat{G}$ to $\widehat{G \times \widehat{G}}$.*

Proposition 4.4. *Let Λ be a discrete subgroup of $G \times \widehat{G}$. The following are equivalent:*

1. *Λ is a lattice;*
2. *The annihilator subgroup of Λ, given by*

$$\Lambda^{\perp} := \left\{\Omega \in \widehat{G \times \widehat{G}} : \Omega(t,\nu) = 1, \text{ for all } (t,\nu) \in G \times \widehat{G}\right\}$$

is discrete;

3. *The **adjoint subgroup with respect to** Λ, given by*

$$\Lambda^{\circ} := \{(t,\nu) \in G \times \widehat{G} : \pi(x,\omega)\pi(t,\nu) = \pi(t,\nu)\pi(x,\omega), \text{ for all } (x,\omega) \in \Lambda\}$$

is discrete;

4. *The **size** of Λ, given by* $d(\Lambda) = \displaystyle\int_{(G\times\widehat{G})/\Lambda} 1 d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi})$ *is finite.*

Proof. **(1 $\iff$ 2).** The following equivalence follows from Corollary 3.8: Λ is a lattice if and only if $(G \times \widehat{G})/\Lambda$ is compact if and only if $\Lambda^{\perp} \cong ((G \times \widehat{G})/\Lambda)^{\wedge}$ is discrete.

(1 $\iff$ 4). The following equivalence follows from Item 2 of Proposition 4.2: Λ is a lattice if and only if $((G\times\widehat{G})/\Lambda)$ is compact if and only if $\mu_{(G\times\widehat{G})/\Lambda}((G\times\widehat{G})/\Lambda) = \int_{(G\times\widehat{G})/\Lambda} 1 d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) = d(\Lambda) < \infty$.

The last sufficient equivalence (2 $\iff$ 3) will follow if we can prove that $\Lambda^{\perp} \cong \Lambda^{\circ}$. Let us define the mapping $\mathcal{J} : G \times \widehat{G} \to \widehat{G \times \widehat{G}}$ via

$$\mathcal{J}(t, \nu) = \Omega_{\nu,-t} \tag{4.12}$$

with $\Omega_{\nu,-t}$ given by the same character appearing in Lemma 4.2. By the same lemma, we see that $\mathcal{J}$ is a topological group isomorphism. Now if $(t, \nu) \in \Lambda^{\circ}$, it follows from Lemma 4.1 that for each $(x, \omega) \in G \times \widehat{G}$

$$\mathcal{J}(t, \nu)(x, \omega) = \Omega_{\nu,-t}(x, \omega) = \overline{\omega(t)}\nu(x) = c_s((t, \nu), (x, \omega)) = 1.$$

Consequently, this means $\mathcal{J}(t, \nu) \in \Lambda^{\perp}$, hence $\Lambda^{\circ} \cong \Lambda^{\perp}$ as required. □

We had inadvertently made some definitions along the way as we stated Proposition 4.1, namely the adjoint subgroup Λ°, and the size of the lattice $d(\Lambda)$. Let us expound on their relevance here. Since we shall always work with lattices for Λ, the proposition tells us that the adjoint subgroup $\Lambda^{\circ} \cong \Lambda^{\perp}$ is discrete. However, we know from Proposition 3.13 that $\Lambda^{\perp\perp} = \Lambda$, so $(\Lambda^{\circ})^{\circ} \cong \Lambda^{\perp\perp} = \Lambda$, therefore Λ° is also discrete, which means Λ° is also co-compact. All-in-all, if Λ is a lattice, then Λ° is also a lattice, and so in our case we also refer to it as the **adjoint lattice**. Next, let $\widehat{\mu}_{((G\times\widehat{G})/\Lambda)^{\wedge}}$ and $\widehat{\mu}_{\widehat{\Lambda}}$ be the dual measures on $((G \times \widehat{G})/\Lambda)^{\wedge}$ and $\widehat{\Lambda}$ respectively. We already know that $\mu_{G\times\widehat{G}}, \mu_{\Lambda}$, and $\mu_{(G\times\widehat{G})/\Lambda}$ are all canonically related, therefore using $\Lambda^{\circ} \cong \Lambda^{\perp}$, and by identifying the associated dual measures $\mu_{\Lambda^{\circ}} \rightsquigarrow \widehat{\mu}_{((G\times\widehat{G})/\Lambda)^{\wedge}}$ and $\mu_{(G\times\widehat{G})/\Lambda^{\circ}} \rightsquigarrow \widehat{\mu}_{\widehat{\Lambda}}$. Proposition 3.17 says that $\mu_{G\times\widehat{G}}, \mu_{\Lambda^{\circ}}$, and $\mu_{(G\times\widehat{G})/\Lambda^{\circ}}$ are also canonically related, more specifically, we can do Fourier-inversion on functions in Λ°. We shall make this precise in the following Lemma.

Lemma 4.3. *The symplectic cocycle c_s induces a continuous map $((G \times \widehat{G})/\Lambda) \times \Lambda^{\circ} \to \mathbb{T}$ which* *we shall denote via c_s still, that is:*

$$c_s(\dot{\chi}, \lambda^\circ) := c_s(\chi, \lambda^\circ), \qquad \forall (\chi, \lambda^\circ) \in (G \times \widehat{G}) \times \Lambda^\circ$$

where we used the abbreviation $\dot{\chi} = \chi + \Lambda$ for the cosets. Furthermore, the collections $\{c_s(\,\cdot\,, \lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ}$ and $\{c_s(\dot{\chi},\ \cdot)\}_{\chi \in G \times \widehat{G}}$ can be identified with the characters of $(G \times \widehat{G})/\Lambda$ and Λ° respectively.

Proof. We shall first check for well-definedness: if $\chi_1 - \chi_2 \in \Lambda$, then we have that for any $\lambda^\circ \in \Lambda^\circ$,

$$c_s(\chi_1 - \chi_2, \lambda^\circ) = 1.$$

Equivalently, $c_s(\chi_1, \lambda^\circ) = c_s(\chi_2, \lambda^\circ)$. So the induced map is well-defined, and does not depend on the coset representative. Continuity of the induced map should follow from continuity of c_s itself.

For the next assertion, recall that $((G \times \widehat{G})/\Lambda)^\wedge \cong \Lambda^\circ$, to see this explicitly, note that all characters of $(G \times \widehat{G})/\Lambda$ is induced by characters from $\Lambda^\perp \subseteq \widehat{G \times \widehat{G}}$ (see the inverse of the isomorphism Φ_1 in Proposition 3.14), but we know that all such characters are of the form $\Omega_{\nu,-t}$ for $(\nu, t) \in \widehat{G} \times G$. We have already seen this however. The requirement that $\Omega_{\nu,-t} \in \Lambda^\perp$ implies that $\mathcal{J}(t,\nu)(x,\omega) = 1 = \overline{\mathcal{J}(x,\omega)}$ for all $(x,\omega) \in \Lambda$, hence we conclude that $(t,\nu) \in \Lambda^\circ$. We choose to identify (t,ν) using the inverse isomorphism $\overline{\mathcal{J}}$: it follows that $(t,\nu) \in \Lambda^\circ$ as a character is identified with $\overline{\mathcal{J}}(t,\nu) = \overline{\Omega_{\nu,-t}} = \overline{c_s((t,\nu),\ \cdot\ )} = c_s(\,\cdot\,, (t,\nu)) : (G \times \widehat{G})/\Lambda \to \mathbb{T}$ as required. We can similarly obtain the other assertion if we look at the isomorphism $\widehat{\Lambda^\circ} \cong \frac{G \times \widehat{G}}{\Lambda}$. $\square$

As a corollary, we have an explicit form of the Fourier transform on Λ° and $(G \times \widehat{G})/\Lambda$ that takes into account the duality between these spaces.

Corollary 4.1. *The Fourier transforms $\mathcal{F}_{\Lambda^\circ} : L^1(\Lambda^\circ) \to C((G \times \widehat{G})/\Lambda)$ and $\mathcal{F}_{(G \times \widehat{G})/\Lambda} : L^1((G \times \widehat{G})/\Lambda) \to C_0(\Lambda^\circ)$ can be written as:*

$$(\mathcal{F}_{\Lambda^\circ}\mathbf{b})(\chi) = \int_{\Lambda^\circ} \mathbf{b}(\lambda^\circ)\overline{c_s(\dot{\chi}, \lambda^\circ)}d\mu_{\Lambda^\circ}(\lambda^\circ),$$
$$(\mathcal{F}_{(G \times \widehat{G})/\Lambda}\tau)(\lambda^\circ) = \int_{(G \times \widehat{G})/\Lambda} \tau(\dot{\chi})\overline{c_s(\dot{\chi}, \lambda^\circ)}d\mu_{(G \times \widehat{G})/\Lambda}(\dot{\chi}).$$

For $\mathbf{b} \in L^1(\Lambda^\circ)$ and $\tau \in L^1((G \times \widehat{G})/\Lambda)$.

Note that we know from discreteness of Λ° that the measure in Λ° must be some positive scalar multiple, say $s(\Lambda^\circ)$, of the counting measure. Whence $L^1(\Lambda^\circ) = \ell^1(\Lambda^\circ)$, and

$$\int_{\Lambda^\circ} \mathbf{b}(\lambda^\circ) d\mu_{\Lambda^\circ}(\lambda^\circ) = s(\Lambda^\circ) \sum_{\lambda^\circ \in \Lambda^\circ} b(\lambda^\circ)$$

for $\mathbf{b} = \{b(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ} \in \ell^1(\Lambda^\circ)$. We would like to know given that we chose Λ° to be identifiable with the dual measure on $((G \times \widehat{G})/\Lambda)^\wedge$, what $s(\Lambda^\circ)$ exactly is.

Proposition 4.5. $s(\Lambda^\circ) = \dfrac{1}{d(\Lambda)}$.

Proof. Let $\tau = c_s(\,\cdot\,, 0) \in L^1((G \times \widehat{G})/\Lambda)$. Then it follows from compactness of $(G \times \widehat{G})/\Lambda$, Corollary 4.1, and Proposition 3.11 that

$$\begin{aligned}
(\mathcal{F}_{(G\times\widehat{G})/\Lambda}\tau)(\lambda^\circ) &= \int_{(G\times\widehat{G})/\Lambda} \tau(\dot{\chi})\overline{c_s(\dot{\chi}, \lambda^\circ)} d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \int_{(G\times\widehat{G})/\Lambda} c_s(\dot{\chi}, -\lambda^\circ) d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \begin{cases} \displaystyle\int_{(G\times\widehat{G})/\Lambda} 1 \, d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}), & \text{if } \lambda^\circ = 0 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.13}$$

This shows that $\mathcal{F}_{(G\times\widehat{G})/\Lambda}\tau \in \ell^1(\Lambda^\circ)$. By our choice of measure μ_{Λ°, we know that we have a Fourier-inversion formula for τ, in particular

$$\begin{aligned}
1 = c_s(\dot{0}, 0) = \tau(\dot{0}) &= \int_{\Lambda^\circ} \mathcal{F}_{(G\times\widehat{G})/\Lambda}\tau(\lambda^\circ)\overline{c_s(\dot{0}, \lambda^\circ)} d\mu_{\Lambda^\circ}(\lambda^\circ) \\
&= s(\Lambda^\circ) \sum_{\lambda^\circ \in \Lambda^\circ} (\mathcal{F}_{(G\times\widehat{G})/\Lambda}\tau)(\lambda^\circ)
\end{aligned}$$

It finally follows from $\int_{(G\times\widehat{G})/\Lambda} 1 \, d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) = d(\Lambda)$ and Equation (4.13) that $1 = s(\Lambda^\circ)d(\Lambda)$, from which our conclusion follows. ☐

We lastly discuss an analogue of Corollary 4.1 showing the self-duality of $G \times \widehat{G}$. This is easily obtained by the isomorphism $\mathcal{J} : G \times \widehat{G} \to \widehat{G \times \widehat{G}}$ that we have introduced. Technically, the Fourier-transform on $L^1(G \times \widehat{G})$ is given by $\mathcal{F}_{G\times\widehat{G}} : L^1(G \times \widehat{G}) \to C_0(\widehat{G \times \widehat{G}})$

$$(\mathcal{F}_{G\times\widehat{G}}F)(\Omega) = \int_{G\times\widehat{G}} F(x, \omega)\overline{\Omega(x, \omega)} d\mu_{G\times\widehat{G}}(x, \omega),$$

for each $F \in L^1(G \times \widehat{G})$ and $\Omega \in \widehat{G \times \widehat{G}}$. Of course each Ω is actually of the form $\Omega_{\nu,t}$ for some $(\nu, t) \in \widehat{G} \times G$ so we may as well write $(\mathcal{F}_{G\times\widehat{G}}F)(\nu, t)$, this gives the incarnation $\mathcal{F}_{G\times\widehat{G}}$: $L^1(G \times \widehat{G}) \to C_0(\widehat{G} \times G)$

$$\begin{aligned}(\mathcal{F}_{G\times\widehat{G}}F)(\nu, t) &= \int_{G\times\widehat{G}} F(x, \omega)\overline{\Omega_{\nu,t}(x, \omega)}d\mu_{G\times\widehat{G}}(x, \omega)\\ &= \int_{G\times\widehat{G}} F(x, \omega)\overline{\nu(x)\omega(t)}d\mu_{G\times\widehat{G}}(x, \omega).\end{aligned}$$

We may identify each (ν, t) with (t, ν) and so we find a Fourier transform that maps $L^1(G\times\widehat{G}) \to C_0(G\times\widehat{G})$. However, a more interesting identification is given by identifying (t, ν) with its image with respect to the isomorphism $\mathcal{J}$, we obtain the associated Fourier transform $\mathcal{F}_s : L^1(G\times\widehat{G}) \to C_0(G \times \widehat{G})$ called the **symplectic Fourier-transform**

$$\begin{aligned}(\mathcal{F}_sF)(t, \nu) &:= (\mathcal{F}_{G\times\widehat{G}}F)(\mathcal{J}(t, \nu))\\ &= \int_{G\times\widehat{G}} F(x, \omega)\overline{\mathcal{J}(t, \nu)}(x, \omega)d\mu_{G\times\widehat{G}}(x, \omega)\\ &= \int_{G\times\widehat{G}} F(x, \omega)\overline{\Omega_{\nu,-t}(x, \omega)}d\mu_{G\times\widehat{G}}(x, \omega)\\ &= \int_{G\times\widehat{G}} F(x, \omega)\overline{c_s((t, \nu), (x, \omega))}d\mu_{G\times\widehat{G}}(x, \omega).\end{aligned} \tag{4.14}$$

We can see from the last line above that the **symplectic Fourier-transform** makes use of the **symplectic cocycles** as its characters for this specific incarnation of the Fourier transform. Explicitly, we are identifying each $(t, \nu) \in G\times\widehat{G}$ with the character $c_s((t, \nu),\ \cdot\ )$. As a last remark about the isomorphism $\mathcal{J} : G \times \widehat{G} \to \widehat{G \times \widehat{G}}$, looking at the identification $\widehat{G \times \widehat{G}} \cong \widehat{G} \times G$, and the definition from Equation (4.12), then we can see $\mathcal{J}$ as a coordinate transform:

$$\mathcal{J}(t, \nu) = (\nu, -t) \tag{4.15}$$

for $(t, \nu) \in G \times \widehat{G}$.

Lemma 4.4. *The symplectic Fourier transform extends to a unitary map* $\mathcal{F}_s : L^2(G \times \widehat{G}) \to L^2(G \times \widehat{G})$ *such that* $\mathcal{F}_s = \mathcal{F}_s^* = \mathcal{F}_s^{-1}$, *i.e.* $\mathcal{F}_s$ *is self-inverse.*

Proof. We already know Fourier-transforms always extend to unitary maps, so it only remains to show that $\mathcal{F}_s$ is self-inverse. However, it easily follows from the identity

$$\langle\mathcal{F}_sF_1, F_2\rangle_2 = \langle F_1, \mathcal{F}_sF_2\rangle_2\,, \qquad \forall F_1, F_2 \in L^2(G \times \widehat{G}),$$

which also easily follows from Fubini's theorem and the following identities on the symplectic cocycle:

$$\overline{c_s((t,\nu),(x,\omega))} = c_s((t,\nu),-(x,\omega)) = c_s(-(t,\nu),(x,\omega)) = c_s((x,\omega),(t,\nu)).$$

for all $(t,\nu),(x,\omega) \in G \times \widehat{G}$. $\square$

Following this, we obtain a symplectic version of the relationships in (4.4). We have

$$\begin{aligned} M^s_{\chi'} T_\chi &= c_s(\chi,\chi') T_\chi M_{\chi'} \\ \mathcal{F}_s T_\chi &= M^s_\chi \mathcal{F}_s \\ \mathcal{F}_s M^s_\chi &= T_\chi \mathcal{F}_s, \end{aligned} \tag{4.16}$$

where for $\chi' = (x',\omega'), \chi = (x,\omega) \in G \times \widehat{G}$, we have the usual translation $T_\chi F(t,\nu) = F(t-x,\nu\overline{\omega})$, and with the **symplectic-modulation** $M^s_\chi F(t,\nu) = F(t,\nu)c_s((x,\omega),(t,\nu))$. Note the subtle difference between Equation (4.16) with Equations (4.4).

We shall freely wield all the isomorphisms and identifications that we have done here in the sequel.

4.2 Gabor Analysis

The foundations of Gabor analysis are based on the seminal paper by D. Gabor [Gab47]. The critical idea of Gabor, in terms of our modern mathematical language, is to represent signals modelled as functions in $L^2(\mathbb{R}^d)$ in terms of **Gabor systems**, which are collections of the form:

$$\mathcal{G}(g,\Gamma) := \{\pi(\gamma)g : \gamma \in \Gamma\} \tag{4.17}$$

for a countable subset $\Gamma \subseteq G \times \widehat{G}$ and function $g \in L^2(G)$ called the **Gabor atom**. In a more explicit fashion, Gabor analysis is concerned in trying to find conditions which would allow us to represent every function $f \in L^2(G)$ in terms of time-frequency shifts of the Gabor atom g sampled in Γ. That is,

$$f = \sum_{\gamma\in\Gamma} c_\gamma(f)\pi(\gamma)g$$

where the coefficients $\{c_\gamma(f)\}_{\gamma \in \Gamma} \subseteq \mathbb{C}$ depend on f. We have, of course, already seen this problem. This is exactly the motivation for frame theory, which is the topic of Section 2.4 from Chapter 2. From Example 2.5, $L^2(G)$ can be seen as a Hilbert $\mathbb{C}$-module, and therefore we can use our results from Section 2.4 to do a particular flavor of frame theory on it. Indeed, the modern approach to Gabor analysis is to *frame* it in terms of frame theory. The now classic book of Gröchenig [Grö01] is an indispensable reference for this section on Gabor analysis. We also recognize the works of Jakobsen and Lemvig, who studied Gabor analysis on general locally compact groups [JL16b, JL16a, Jak18].

We start by translating some of the general concepts from (module) frame theory to our setting.

Definition 4.3. We say that the Gabor system $\mathcal{G}(g, \Gamma)$ is a **Gabor frame** if it is a frame for $L^2(G)$. Explicitly, this means there exists $C, D > 0$ such that for all $f \in L^2(G)$:

$$C\|f\|_2^2 \leqslant \sum_{\gamma \in \Gamma} |\langle f, \pi(\gamma)g\rangle_2|^2 \leqslant D\|f\|_2^2. \tag{4.18}$$

We say that $\mathcal{G}(g, \Gamma)$ is a **Bessel system** if only the right-most inequality above is satisfied.

If we suppose $\mathcal{G}(g, \Gamma)$ is at least a Bessel family, then we denote the associated analysis operator $\mathcal{G}(g, \Gamma)$ via $\Phi_{g,\Gamma} : L^2(G) \to \ell^2(\Gamma)$. It is given explicitly by:

$$\Phi_{g,\Gamma}(f) = \{\langle f, \pi(\gamma)g\rangle_2\}_{\gamma \in \Gamma} \tag{4.19}$$

Its adjoint (see Proposition 2.17), the synthesis operator, is denoted by $\Psi_{g,\Gamma} = \Phi_{g,\Gamma}^* : \ell^2(\Gamma) \to L^2(G)$ is given, for all $\mathbf{c} = \{c_\gamma\}_{\gamma \in \Gamma} \in \ell^2(\Gamma)$ by

$$\Psi_{g,\Gamma}(\mathbf{c}) = \sum_{\gamma \in \Gamma} c_\gamma \pi(\gamma) g. \tag{4.20}$$

Definition 4.4. Let $\mathcal{G}(g, \Gamma)$ be a Bessel system, then the associated frame operator is called the **Gabor frame** operator, and is defined by:

$$S_{g,\Gamma} = \Psi_{g,\Gamma} \circ \Phi_{g,\Gamma} \in \mathcal{L}(L^2(G)). \tag{4.21}$$

More generally, we can consider another Bessel system $\mathcal{G}(h, \Gamma)$. The mixed Gabor frame operator is defined by:

$$S_{g,h,\Gamma} = \Psi_{h,\Gamma} \circ \Phi_{g,\Gamma} \in \mathcal{L}(L^2(G)). \tag{4.22}$$

Remark 4.1. Similar to what we have done in Section 2.4 of Chapter 2, we will simply refer to mixed Gabor frame operators as Gabor frame operators. Unless we need to explicitly differentiate between the two, this will be the case for the rest of the manuscript.

Proposition 4.6. *Let $\mathcal{G}(g,\Gamma)$ be a Bessel system for $L^2(G)$, then the frame operator $S_{g,\Gamma}$ is a positive operator in $\mathcal{L}(L^2(G))$ and*

$$\|S_{g,\Gamma}\| = \inf\{D^{1/2} : D \textit{ is a Bessel bound for } \mathcal{G}(g,\Gamma)\} \tag{4.23}$$

Proof. Applying 2.27 to the Bessel system $\mathcal{G}(g,\Gamma)$ gives us that $S_{g,\Gamma}$ is a positive operator. Similarly, applying Proposition 2.28 to $\mathcal{G}(g,\Gamma)$ gives us Equation (4.23) □

In anticipation of the application of the general reconstruction formulae (2.22) using Gabor systems, we would like to ask what possible conditions we can endow $\mathcal{G}(g,\Gamma)$ so that it forms a Gabor frame in $L^2(G)$. This would involve either imposing some structure on the countable subset Γ, or some regularity condition on the Gabor atom g. For this section we shall focus on the former. We now only consider $\Gamma = \Lambda \subseteq G \times \widehat{G}$ where Λ is a lattice (see Definition 4.1) in the time-frequency plane. This means in particular that Λ has a group structure, from which the next result follows.

Proposition 4.7. *Let $\mathcal{G}(g,\Lambda)$ be a Bessel system, then for all $\mu \in \Lambda$,*

$$S_{g,\Lambda}\pi(\mu) = \pi(\mu)S_{g,\Lambda}. \tag{4.24}$$

That is, the Gabor frame operator associated with $\mathcal{G}(g,\Lambda)$ commutes with all the time-frequency shifts on Λ.

Proof. Let $f \in L^2(G)$ and $\mu \in \Lambda$, recall the Heisenberg 2-cocycle c (4.7) from Section 4.1 then:

$$\begin{aligned} S_{g,\Lambda}\pi(\mu)f &= \sum_{\lambda\in\Lambda} \langle \pi(\mu)f, \pi(\lambda)g\rangle_2 \\ &= \sum_{\lambda\in\Lambda} \langle f, \pi^*(\mu)\pi(\lambda)g\rangle_2\, \pi(\lambda)g \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda-\mu)g\rangle_2\, c(\mu,\lambda-\mu)\pi(\lambda)g \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_2\, c(\mu,\lambda)\pi(\lambda+\mu)g \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_2\, \pi(\mu)\pi(\lambda)g \\
&= \pi(\mu)\left(\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_2\, \pi(\lambda)g\right) = \pi(\mu)S_{g,\Lambda}f.
\end{aligned}
$$

Therefore $S_{g,\Lambda}\pi(\mu) = \pi(\mu)S_{g,\Lambda}$ as required. □

The seemingly innocent result above can be rewritten into $\pi(\mu)S_{g,\Lambda}\pi^*(\mu) = S_{g,\Lambda}$ for all $\mu \in \Lambda$. From the point of view of Quantum Harmonic Analysis (QHA), the conjugation by time-frequency shifts can be interpreted as the operator-translation of $S_{g,\Lambda}$ by μ. Therefore we can further interpret the result above as $S_{g,\Lambda}$ being "operator-translation invariant" in Λ (see Definition 4.14). One can then ask if there are other operator-translation invariant operators. Equivalently are there other operators that commute with all the time-frequency shifts in a lattice? Part of this dissertation's major results involve the characterization of all such operators: see Chapter 7 Section 7.2.

Let us now go back to the application of the general reconstruction formulae (2.22) to our setting.

Theorem 4.1 (Reconstruction Formula). *Suppose $\mathcal{G}(g,\Lambda)$ is a Gabor frame, then $S_{g,\Lambda}$ is invertible and for all $f \in L^2(G)$:*

$$
\begin{aligned}
f &= \sum_{\lambda\in\Lambda} \left\langle f, \pi(\lambda)S_{g,\Lambda}^{-1}g\right\rangle_2 \pi(\lambda)g \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_2\, \pi(\lambda)(S_{g,\Lambda}^{-1}g)
\end{aligned}
\tag{4.25}
$$

*Representations of f as in (4.25) are called **Gabor expansions** of f.*

Proof. This is a direct application of Corollary 2.9 to our setting, along with the fact that due to Equation (4.24), we also have that $\pi(\mu)S_{g,\Lambda}^{-1} = \pi(\mu)S_{g,\Lambda}^{-1}$ for all $\mu \in \Lambda$. □

We see from the reconstruction formulae (4.25) for Gabor systems that because the inverse of the Gabor frame operator also commutes with all the time-frequency shifts with respect to Λ

that the canonical dual frame takes on the form

$$\{S^{-1}(\pi(\lambda)g) : \lambda \in \Lambda\} = \{\pi(\lambda)(S_{g,\Lambda}^{-1}g) : \lambda \in \Lambda\} = \mathcal{G}(h,\Lambda),$$

where $h := S_{g,\Lambda}^{-1}g \in L^2(G)$. Therefore the canonical dual frame of a Gabor system, is also a Gabor system. This motivates us to make the following definition.

Definition 4.5. Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame for $L^2(G)$. Then $h := S_{g,\Lambda}^{-1}g$ is called the **canonical dual Gabor atom** with respect to g.

As we have seen, the time-frequency shifts in Λ of the canonical dual Gabor atom of $h = S_{g,\Lambda}^{-1}g$ generates the canonical dual frame for $\mathcal{G}(g,\Lambda)$. What we notice now when we rewrite the Gabor-frame reconstruction formulae (4.25) in terms of the canonical dual Gabor atom is that for all $f \in L^2(G)$:

$$\begin{aligned} f = (S_{g,\Lambda} \circ S_{h,\Lambda})f = (S_{h,g,\Lambda})f &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h\rangle_2 \, \pi(\lambda)g \\ = (S_{h,\Lambda} \circ S_{g,\Lambda})f = (S_{g,h,\Lambda})f &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_2 \, \pi(\lambda)h \end{aligned}$$

Not only does the canonical dual Gabor atom satisfy $\mathrm{Id}_{L^2(G)} = S_{g,h,\Lambda} = S_{h,g,\Lambda}$, but we can also see from the equations above that the coefficients of the Gabor expansion of f can be computed in terms of the following map $\chi \mapsto \langle f, \pi(\chi)g\rangle_2$, which is named, as we will give below.

Definition 4.6. Let $g \in L^2(G)$, the **short-time Fourier transform (STFT)** of f, **windowed** by g is denoted by $\mathcal{V}_g f$ given by:

$$\mathcal{V}_g f(\chi) = \langle f, \pi(\chi)g\rangle_2\,, \qquad \forall \chi \in G \times \widehat{G}. \tag{4.26}$$

The motivation for the name "short-time Fourier transform" can be seen when we explicitly write the inner-product in Equation (4.26):

$$\mathcal{V}_g f(\chi) = \langle f, \pi(\lambda)g\rangle_2 = \int_G f(t)\overline{\omega}(t)\overline{g}(t-x)d\mu_G(t), \qquad \chi = (x,\omega) \in G \times \widehat{G}.$$

$\mathcal{V}_g f(\chi)$ 'is' the Fourier-transform of f at ω, for which the integral is 'smeared' by a function g translated at x. Traditionally, g is taken to be a Gaussian concentrated at the origin 0 (e.g. in the Euclidean case, take $\frac{1}{\sqrt[4]{\pi}}e^{-\frac{|t|^2}{2}}$). So as we can see, $\mathcal{V}_g f(\chi)$ can now be interpreted as a time-frequency 'localized' Fourier transform of f at $\chi = (x, \omega)$ because the window g 'essentially' cuts off parts of f that are far away from x, and then takes the Fourier transform of this 'time-x' concentrated portion of f at 'frequency ω'. With this interpretation of the short-time Fourier transform, we can compare the Gabor expansion of f (4.25) with the Fourier inversion formula of (3.24) and see that (4.25) is a Λ-sampled (or discretized) version of the Fourier-inversion formula that uses $\mathcal{V}_g f(\lambda)$ as time-frequency localized version of Fourier transforms instead. There is in fact a continuous (non-sampled) version of the Gabor expansion (4.25), which we state below without proof. One may consult [Grö01, Theorem 3.2.1] for a proof that one can adapt to the abstract time-frequency plane.

Theorem 4.2. *For any $g, h \in L^2(G)$, $\mathcal{V}_g h \in L^2(G \times \widehat{G})$. Furthermore, we have the following representation:*

$$f = \frac{1}{\langle h, g \rangle_2} \int_{G \times \widehat{G}} (\mathcal{V}_g f)(\chi) \pi(\chi) h \, d\mu_{G \times \widehat{G}}(\chi). \tag{4.27}$$

The STFT is a construction that allows us to do simultaneous time and frequency representations for signals, and in fact is only one such scheme in *time-frequency analysis* (see for example [Grö01, Dau92, Coh95]). Properly recontextualized, we can now see that the main problem of Gabor analysis is to find a Λ-sampled, or Λ-discretized version of the time-frequency representation given by (4.27).

Let us move along and see if we can extract any interesting necessary conditions for $\mathcal{G}(g, \Lambda)$ to be a Gabor frame. The remaining results will be stated without proof, however, a version of them that will be relevant for our original contributions (i.e. Chapter 7) will be presented using operator-algebraic techniques (see Section 5.4 of Chapter 5).

Since we are working with a lattice Λ, we have access to its adjoint lattice Λ° (see Definition 4.4). Here we present the so-called **Wexler-Raz biorthogonality relations** which can be seen as an orthogonality condition between a Gabor atom and any of its dual atom. A proof of the following result, which uses only harmonic analysis techniques can be found in [JL16b, Lemma 4.4], and [Grö01, Theorem 7.3.1] in the Euclidean case.

Theorem 4.3 (Wexler-Raz Biorthogonality Relations)**.** *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with dual* *atom h, then for all $\mu^\circ \in \Lambda^\circ$:*

$$\langle h, \pi(\mu^\circ) g\rangle_2 = \begin{cases} d(\Lambda), & \text{if } \mu^\circ = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.28}$$

The reader may realize that the moniker "Wexler-Raz" has already appeared somewhere in this manuscript, indeed, there is a version of Wexler-Raz relations for imprimitivity bimodules given in Remark 2.10. We shall prove later (c.f. Corollary 5.10) that we can recover Equation (4.28) when g and the dual atom h belong to the *Heisenberg module*.

The following is a fundamental result showing that the lattice Λ must be sufficiently dense to generate a Gabor frame. This result is well-known in the Euclidean case [Grö01, Corollary 7.5.1], while the general case was proven in [JL16b, Theorem 5.6]

Theorem 4.4. *If $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(G)$, then $d(\Lambda) \leqslant 1$.*

There is a relationship between the Gabor atoms of a Gabor system and the density of the lattice Λ, whose discussion we shall defer until Section 5.4 of Chapter 5. For the moment, we shall give the following generalization of a Gabor system on the lattice Λ with finitely many Gabor atoms.

Definition 4.7. Let $g_1, \dots, g_n \in L^2(G)$, the collection $\bigcup_{i=1}^n \mathcal{G}(g_i, \Gamma)$ is called a *multi-window Gabor system*. If it forms a Bessel family, then it is called a **multi-window Bessel system**. If it forms a frame, it is called a **multi-window Gabor frame**.

Remark 4.2. The frame operator associated with a multi-window Gabor system $\bigcup_{i=1}^n \mathcal{G}(g_i, \Gamma)$ is given by $\sum_{i=1}^n S_{g_i,\Gamma}$.

The assumption that $\Gamma = \Lambda$ be a lattice has allowed us to introduce elegant algebraic or 'soft-analytic' techniques (e.g. analysis on LCA *groups* and in the sequel, operator algebras) to Gabor analysis. We shall restrict ourselves to the lattice case for the rest of the manuscript. However, note that there is also quite a lot of interesting research being done with in the case where Γ does not have any algebraic structure– i.e. when Γ is an **irregular sampling set** for $G \times \widehat{G}$ (where G is mostly restricted to the Euclidean case $\mathbb{R}^d$). The techniques involved in this case tend to fall more within the "hard-analysis" variant of mathematical analysis, characterized by explicit estimates and similar methods. Finally, we also recognize that a natural generalization of our setting is to remove the assumption of discreteness on Λ, instead consider Λ to simply be a *closed cocompact subgroup* of the time-frequency plane $G \times \widehat{G}$.

4.3 The Feichtinger's Algebra: Basic Properties

Throughout the section we fix G to be a second countable LCA group. As we have discussed in the previous section, given a lattice $\Lambda \subseteq G \times \widehat{G}$, the central problem of Gabor analysis is to find a $g \in L^2(G)$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. However, the first obstacle in this quest if we hope to even have a well-defined Gabor frame operator, is to look for those g's where $\mathcal{G}(g, \Lambda)$ is a Bessel system. Bearing this in mind, we shall introduce the function space which is nowadays called the "Feichtinger's algebra". Its definition in the general LCA group case was first given by Feichtinger in [Fei81]. A subsequent survey of the Feichtinger's algebra was also recently published [Jak18], where various characterizations of the said algebra can be found. In the first section of this chapter, we shall quickly go over the crucial properties of the Feichtinger's algebra. In the subsequent section, we will mostly focus our attention in revisiting the paper [FK98] of Feichtinger and Kozek on time-frequency quantization of lattice-invariant operators using the Feichtinger's algebra. Through the so-called kernel theorems of Jakobsen [Jak18, FJ22], we update their results and show that they also hold for lattices on LCA groups as well. The discussion here will be a good prelude to Quantum Harmonic Analysis, which is a major component of our main results in Section 7.2 of Chapter 7.

We give an accessible definition of the Feichtinger's algebra in terms of the Short-Time Fourier Transform.

Definition 4.8. Given an arbitrary LCA group G, the Feichtinger's algebra denoted by $\mathbf{S}_0(G)$, is given by

$$\mathbf{S}_0(G) := \left\{ f \in L^2(G) : \int_{G \times \widehat{G}} |\mathcal{V}_f f(x, \omega)| d\mu_{G \times \widehat{G}}(x, \omega) < \infty \right\}.$$

For a fixed (**window function**) $g \in \mathbf{S}_0(G) \backslash \{0\}$, $\mathbf{S}_0(G)$ can be made a Banach space via the norm

$$\|f\|_{\mathbf{S}_0(G),g} := \int_{G \times \widehat{G}} |\mathcal{V}_g f(x, \omega)| d\mu_{G \times \widehat{G}}(x, \omega).$$

Note that any other $g_0 \in \mathbf{S}_0(G) \backslash \{0\}$ induces an equivalent norm on $\mathbf{S}_0(G)$ [Jak18, Proposition 4.10]. In light of this, we shall generally denote the norm of $\mathbf{S}_0(G)$ using the following: $\|\cdot\|_{\mathbf{S}_0(G),g}$, $\|\cdot\|_{\mathbf{S}_0(G)}$, or $\|\cdot\|_{\mathbf{S}_0}$. The first one we have already used in the definition above, it will be used when we want to be explicit about the ambient LCA group G *and* the window function g. The second one will be used when we are only concerned with making the LCA group clear, otherwise we will mostly abbreviate the $\mathbf{S}_0(G)$-norm using the third one. Finally, one can use a fixed window function as a way to characterize $\mathbf{S}_0(G)$.

Proposition 4.8. *Fix $g \in \mathbf{S}_0(G)\backslash\{0\}$, then*

$$\mathbf{S}_0(G) = \left\{f \in L^2(G) : \|f\|_{\mathbf{S}_0(G),g} < \infty\right\}.$$

Before we give a summary of the most important properties of the Feichtinger's algebra, we recall another important function space: the Fourier-algebra $\mathbf{A}(G)$, defined

$$\mathbf{A}(G) = \left\{f \in C_0(G) : \exists g \in L^1(\widehat{G}) \text{ where } \mathcal{F}_{\widehat{G}} g = f\right\} \tag{4.29}$$

All of the results below can be found in [Jak18].

Proposition 4.9. *For general LCA groups G, G_i where $i = 1, 2$, $\mathbf{S}_0$ has the following properties:*

1. *$\mathbf{S}_0(G)$ is continuously and densely embedded in $C_0(G)$, $\mathbf{A}(G)$, and $L^p(G)$ for all $1 \leqslant p \leqslant \infty$.*
2. *For any discrete subgroup $\Delta \subseteq G \times \widehat{G}$, and $g \in \mathbf{S}_0(G)$, $\mathcal{G}(g, \Delta)$ is always a Bessel system in $L^2(G)$;*
3. *$\mathbf{S}_0(G)$ is time-frequency shift invariant. In fact the time-frequency shift operators restrict to an isometry still on $\mathbf{S}_0(G)$: for any $(x, \omega) \in G \times \widehat{G}$, $\|\pi(x, \omega)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$ for all $f \in \mathbf{S}_0(G)$;*
4. **(Periodization).** *For any closed subgroup $H \subseteq G$, for each $f \in \mathbf{S}_0(G)$, and $\dot{x} = x + H$, define the periodization operator via $P_H(f)(\dot{x}) := \int_H f(z+h) d\mu_H(h)$. It is a linear, bounded and surjective map from $\mathbf{S}_0(G)$ to $\mathbf{S}_0(G/H)$.*
5. *If G is a discrete (countable) locally compact abelian group, then $\mathbf{S}_0(G) = \ell^1(G)$. If G is compact, then $\mathbf{S}_0(G) = \mathbf{A}(G)$.*
6. **(Restriction).** *For any closed subgroup $H \subseteq G$, the restriction operator $R_H f = f_{|H}$ is a linear, bounded, and surjective operator from $\mathbf{S}_0(G)$ to $\mathbf{S}_0(H)$.*
7. **(Poisson Summation Formula for Lattices).** *If $F \in \mathbf{S}_0(G \times \widehat{G})$, then*

$$\sum_{\lambda \in \Lambda} F(\lambda) = \frac{1}{s(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_s(F)(\lambda^\circ). \tag{4.30}$$

8. ***(Sufficient Conditions for Isomorphisms).*** *Let $T : L^2(G_1) \to L^2(G_2)$ be a linear and bounded bijection. If there exists a topological group isomorphism $\alpha : G_2 \times \widehat{G}_2 \to G_1 \times \widehat{G}_1$*

and a map $c : G_2 \times \widehat{G}_2 \to \mathbb{T}$ such that for all $\chi_2 \in G_2 \times \widehat{G}_2$

$$\pi(\chi_2)T = c(\chi_2)T\pi(\alpha(\chi_2)),$$

then T (resp. the Hilbert space adjoint T^*) restricts to an isomorphism $T_{|\mathbf{S}_0(G_1)} : \mathbf{S}_0(G_1) \to \mathbf{S}_0(G_2)$ (resp. $T^*_{|\mathbf{S}_0(G_2)} : \mathbf{S}_0(G_2) \to \mathbf{S}_0(G_1)$).

Remark 4.3. We can see from Item 2 above why $\mathbf{S}_0(G)$ is a good candidate as a function space for Gabor analysis, the Gabor frame operator $S_{g,\Lambda}$ will always be a bounded linear operator on $\mathcal{L}(L^2(G))$ for any lattice Λ when $g \in \mathbf{S}_0(G)$.

Note that any isomorphism derived from restriction of a bounded linear bijection $T : L^2(G_1) \to L^2(G_2)$ via Item 8 above will be denoted by T still. The Fourier transform $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is one such important example (see Theorem 3.8). Pick $G_1 = G$, $G_2 = \widehat{G}$, so that by Pontryagin duality we can identify $\widehat{G}_2 = G$, consider the isomorphism $\alpha : G_2 \times \widehat{G}_2 = \widehat{G} \times G \to G \times \widehat{G} = G_1 \times \widehat{G}_1$ via $\alpha(\omega, x) = (-x, \omega)$ (i.e. an incarnation of the $\mathcal{J}$-isomorphism as in Equation (4.15)), then

$$\pi(\omega, x)\mathcal{F} = \omega(x)\mathcal{F}\pi(\alpha(\omega, x)).$$

If we take $c : G_2 \times \widehat{G}_2 = \widehat{G} \times G \to \mathbb{T}$ to be $c(\omega, x) = \omega(x)$, then we find that $\mathcal{F}$ satisfies all condition needed so that we indeed have an isomorphism $\mathcal{F} : \mathbf{S}_0(G) \to \mathbf{S}_0(\widehat{G})$. We shall keep this example in mind and note that we will encounter more isomorphisms on the Feichtinger's algebras in a similar way.

Let G, G_1, and G_2 be arbitrary LCA groups. For $(f_1, f_2) \in \mathbf{S}_0(G_1) \times \mathbf{S}_0(G_2)$ we define their tensor product via $(f_1 \otimes f_2)(x, y) = f_1(x)f_2(y)$ for $(x, y) \in G_1 \times G_2$, and it can be shown that $f_1 \otimes f_2 \in \mathbf{S}_0(G_1 \times G_2)$. In fact we have the following projective tensor product factorization result from [Fei81, Theorem 7].

Proposition 4.10. *$\mathbf{S}_0(G_1 \times G_2) = \mathbf{S}_0(G_1)\widehat{\otimes}\mathbf{S}_0(G_2)$ where $\widehat{\otimes}$ denote the projective tensor product. More explicitly, for each $F \in \mathbf{S}_0(G_1 \times G_2)$, there exist (non-unique) sequences $\{f_i^{(1)}\}_{i=1}^{\infty} \subseteq \mathbf{S}_0(G_1)$ and $\{f_i^{(2)}\}_{i=1}^{\infty} \subseteq \mathbf{S}_0(G_2)$ such that*

$$F = \sum_{i=1}^{\infty} f_i^{(1)} \otimes f_i^{(2)}, \text{ and} \tag{4.31}$$

$$\sum_{i=1}^{\infty} \|f_i^{(1)}\|_{\mathbf{S}_0(G_1)} \|f_i^{(2)}\|_{\mathbf{S}_0(G_2)} < \infty. \tag{4.32}$$

Furthermore,

$$\|F\|_{\mathbf{S}_0(G_1\times G_2)} = \inf\left\{\sum_{i=1}^{\infty}\|f_i^{(1)}\|_{\mathbf{S}_0(G_1)}\|f_i^{(2)}\|_{\mathbf{S}_0(G_2)}\right\}$$

running over all such sequences $\{f_i^{(1)}\}_{i=1}^{\infty} \subseteq \mathbf{S}_0(G_1)$ *and* $\{f_i^{(2)}\}_{i=1}^{\infty} \subseteq \mathbf{S}_0(G_2)$ *where* (4.31) *and* (4.32) *hold.*

Remark 4.4. From hereon out, the adjoint lattice Λ° will make quite an extended appearance. We would like to point out that it is always equipped with the measure μ_{Λ°, which is the counting measure scaled with $\frac{1}{d(\Lambda)}$ (see Proposition 4.5). To avoid confusion, we will write out the sums corresponding to the integral of a particular measure. In particular, this means that if $\mathbf{k} = \{k(\lambda^\circ)\}_{\lambda^\circ\in\Lambda^\circ} \in \ell^1(\Lambda^\circ)$, then

$$\|\mathbf{k}\|_1 = \int_{\Lambda^\circ} |k(\lambda^\circ)| d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}|k(\lambda^\circ)|.$$

Given what we now know about the Feichtinger's algebra, there is more to be said about the symplectic characters in connection with adjoint lattice.

Proposition 4.11. *For each* $\lambda^\circ \in \Lambda^\circ$, *we have* $c_s(\,\cdot\,,\lambda^\circ) \in \mathbf{A}((G\times\widehat{G})/\Lambda)) = \mathbf{S}_0((G\times\widehat{G})/\Lambda)$ *with* $\|c_s(\,\cdot\,,\lambda^\circ)\|_{\mathbf{A}((G\times\widehat{G})/\Lambda)} = 1$.

Proof. Let us recall that this incarnation of the relevant Fourier-algebra takes on the form (see Corollary 4.1)

$$\mathbf{A}((G\times\widehat{G})/\Lambda) = \{\tau \in C((G\times\widehat{G})/\Lambda) : \tau = \mathcal{F}_{\Lambda^\circ}\mathbf{b}, \text{ s.t. } \mathbf{b}\in\ell^1(\Lambda^\circ)\}.$$

The equality with $\mathbf{S}_0((G\times\widehat{G})/\Lambda)$ follows from compactness of $(G\times\widehat{G})/\Lambda$ and Item 5 of Proposition 4.9. Now for a fixed $\lambda^\circ\in\Lambda^\circ$, define the sequence $c_{\lambda^\circ}\in\ell^1(\Lambda^\circ)$ to be

$$c_{\lambda^\circ}(\upsilon^\circ) = \begin{cases} d(\Lambda), & \text{whenever } \upsilon^\circ = -\lambda^\circ \\ 0, & \text{otherwise.}\end{cases} \tag{4.33}$$

We find that

$$\begin{aligned}(\mathcal{F}_{\Lambda^\circ}c_{\lambda^\circ})(\dot{\chi}) &= \int_{\Lambda^\circ} c_{\lambda^\circ}(\upsilon^\circ)\overline{c_s(\dot{\chi},\upsilon^\circ)}d\mu_{\Lambda^\circ}(\upsilon^\circ)\\ &= \frac{1}{d(\Lambda)}\sum_{\upsilon^\circ\in\Lambda^\circ}c_{\lambda^\circ}(\upsilon^\circ)\overline{c_s(\dot{\chi},\upsilon^\circ)} = \frac{d(\Lambda)}{d(\Lambda)}c_s(\dot{\chi},\lambda^\circ)\end{aligned}$$

$$= c_s(\dot{\chi}, \lambda^\circ).$$

Therefore $c_s(\,\cdot\,, \lambda^\circ) \in \mathbf{A}((G \times \widehat{G})/\Lambda)$, with $\|c_s(\,\cdot\,, \lambda^\circ)\|_{\mathbf{A}((G\times\widehat{G})/\Lambda)} = \|c_{\lambda^\circ}\|_1 = 1$ for all $\lambda^\circ \in \Lambda^\circ$ $\square$

We shall make ourselves well acquainted to the dual space of the Feichtinger's algebra, and the maps on it. We shall follow the notation of [Jak18]. In general, we denote the dual space of a Banach space X via X', and define with them the following bilinear form $(\cdot,\cdot)_{X,X'} : X \times X' \to \mathbb{C}$

$$(x, h)_{X,X'} := h(x), \qquad \forall (x, h) \in X \times X'.$$

For a general LCA group G, we denote by $\mathbf{S}_0'(G)$, the dual space of the Feichtinger's algebra $\mathbf{S}_0(G)$. There is a longstanding strand of ideas (e.g. invariance under most 'Fourier-analytic notions' [Jak18], or as a pre-imprimitivity bimodule for the construction of the Heisenberg module [Lue07]) that motivates $\mathbf{S}_0(G)$ to be a good substitute for the Schwartz(-Bruhat) space, and in keeping with this tradition, we shall also refer to the functionals in $\mathbf{S}_0'(G)$ as **distributions**.

Definition 4.9. The so-called **linear complex-conjugate** of a functional in $\mathbf{S}_0'(G)$ is defined via:

$$(f, \overline{\sigma})_{\mathbf{S}_0,\mathbf{S}_0'} = \overline{(\overline{f}, \sigma)}_{\mathbf{S}_0,\mathbf{S}_0'}, \qquad \forall (f, \sigma) \in \mathbf{S}_0(G) \times \mathbf{S}_0'(G) \tag{4.34}$$

The definition above is actually particular to function spaces since it involves the complex-conjugate of a function. The reason why we call it the linear complex-conjugate for $\mathbf{S}_0'(G)$ is because it results in a linear function. One may contrast the linear complex-conjugate to the ordinary complex-conjugate of a linear functional which results to an antilinear functional.

Note that with the notation we are using, we do not make explicit which LCA group we are taking with the bilinear form $(\cdot,\cdot)_{\mathbf{S}_0,\mathbf{S}_0'}$, we leave it to be (easily) figured out from context by the reader.

There are two relevant topologies on $\mathbf{S}_0'(G)$, namely, the norm-topology and the weak $*$-topology. The norm topology on $\mathbf{S}_0(G)$ is the one induced by the operator norm

$$\|\sigma\|_g := \sup_{\|f\|_{\mathbf{S}_0,g}=1} |(f, \sigma)_{\mathbf{S}_0,\mathbf{S}_0'}|$$

for a given window function $g \in \mathbf{S}_0(G)$. Similar to $\mathbf{S}_0(G)$, different choices of window function $g \in \mathbf{S}_0(G)\backslash\{0\}$ result in equivalent operator norms for $\mathbf{S}_0'(G)$. The more relevant topology for

$\mathbf{S}_0'(G)$ is the weak-$*$ topology, which is the topology of pointwise convergence: i.e. the net $\{\sigma_i\}_i \subseteq \mathbf{S}_0'(G)$ converges to $\sigma \in \mathbf{S}_0'(G)$ in the weak-$*$ topology if and only if for all $f \in \mathbf{S}_0(G)$, $\sigma_i(f) \to \sigma(f)$ in $\mathbf{S}_0(G)$-norm (this of course just comes from the general definition of weak-$*$ topology for any topological vector spaces)

Definition 4.10. Given an arbitrary LCA group G, an open set $U \subseteq G$ is said to be an **open annihilation set** of a distribution $\sigma \in \mathbf{S}_0'(G)$ if and only if for all $f \in \mathbf{S}_0(G)$ with $\mathrm{supp}(f) \subseteq U$,

$$(f, \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = 0.$$

The **support of a distribution** $\sigma \in \mathbf{S}_0'(G)$ is defined to be the complement of the largest open annihilator of σ. Equivalently, it is the complement of the union of all the open annihilators of σ.

Given a bounded and linear operator $T : X \to Y$ for Banach spaces X and Y, there exists a unique map $T^\times : Y' \to X'$, defined by the following formula

$$(x, T^\times y)_{X,X'} = (Tx, y)_{Y,Y'}$$

that is both weak-$*$-weak-$*$ and norm-norm continuous to the dual spaces, called the **Banach space adjoint** operator induced from T [Kre91].

Definition 4.11. Let G be an arbitrary LCA group with closed subgroup H. The distribution $\sigma \in \mathbf{S}_0'(G)$ is said to be H**-periodic** if for all $h \in H$, $T_h^\times \sigma = \sigma$.

The following result summarizes all the relevant properties of the dual space of the Feichtinger's algebra, all of which are referenced from [Jak18].

Proposition 4.12. *For general LCA groups G, G_i where $i = 1, 2$, $\mathbf{S}_0'$ has the following properties:*

1. *$\mathbf{S}_0(G)$ and $L^p(G)$ for $1 \leqslant p \leqslant \infty$ are both continuously embedded in $\mathbf{S}_0'(G)$. The embedding is explicitly given by the map $\iota : \mathbf{S}_0(G)$ (or $L^p(G)$) $\hookrightarrow \mathbf{S}_0'(G)$ via: $(f, \iota(h))_{\mathbf{S}_0,\mathbf{S}_0'} := \int_G f(x)h(x)d\mu_G(x)$. In light of this, we abbreviate via $(f, \iota(h))_{\mathbf{S}_0,\mathbf{S}_0'} =: (f, h)_{\mathbf{S}_0,\mathbf{S}_0'}$. Furthermore, the embedding of $\mathbf{S}_0(G)$ in $\mathbf{S}_0'(G)$ is weak-$*$ dense.*

2. *For each $s \in G$, the dirac-masses $\delta_s(f) = f(s)$ for $f \in \mathbf{S}_0(G)$ are all in $\mathbf{S}_0'(G)$;*

3. *Each $\omega \in \widehat{G}$ induces an element $e^{\omega} \in \mathbf{S}_0'(G)$ via*

$$e^{\omega}(f) = \int_G f(s)\omega(s)d\mu(s) = (\mathcal{F}f)(\overline{\omega}). \tag{4.35}$$

4. **(Restriction)**. *For a closed subgroup $H \subseteq G$, we have the following isomorphism through the Banach space adjoint operator of the restriction map $R_H^{\times} : \mathbf{S}_0'(H) \to \mathbf{S}_0'(G)$*

$$\operatorname{Im} R_H^{\times} = \{\sigma \in \mathbf{S}_0'(G) : \operatorname{supp} \sigma \subseteq H\};$$

5. **(Periodization)**. *For a closed subgroup $H \subseteq G$, we have the following isomorphism through the Banach space adjoint operator of the periodization map $P_H^{\times} : \mathbf{S}_0'(G/H) \to \mathbf{S}_0'(G)$:*

$$\operatorname{Im} P_H^{\times} = \{\sigma \in \mathbf{S}_0'(G) : \sigma \textit{ is } H\textit{-periodic}\}.$$

6. ***(Sufficient Conditions for Isomorphisms).*** *If $T \in \mathcal{L}(L^2(G_1), L^2(G_2))$, then $\iota \circ T : L^2(G_1) \to \mathbf{S}_0'(G_2)$ uniquely extends to a weak-$*$-weak-$*$ (hence norm-norm) $T : \mathbf{S}_0'(G_1) \to \mathbf{S}_0'(G_2)$ that satisfies $T(\iota(f)) = \iota(T(f))$ for all $f \in L^2(G_1)$ if and only if $T^*_{|\mathbf{S}_0(G_2)} \in \mathcal{L}(\mathbf{S}_0(G_2), \mathbf{S}_0(G_1))$. In which case:*

$$(f_2, T\sigma_1)_{\mathbf{S}_0, \mathbf{S}_0'} = (\overline{T^*\overline{f_2}}, \sigma_1)_{\mathbf{S}_0, \mathbf{S}_0'}, \qquad \forall (f_2, \sigma_1) \in \mathbf{S}_0(G_2) \times \mathbf{S}_0'(G_1).$$

Furthermore, if T is a unitary operator in $\mathcal{L}(L^2(G_1), L^2(G_2))$, then the extension above is an isomorphism.

Remark 4.5. Note that as we can see above, any operator from $\mathbf{S}_0'(G_1)$ to $\mathbf{S}_0'(G_2)$ that is obtained via extension of some $T : L^2(G_1) \to L^2(G_2)$ as in Item 6 of Proposition 4.12 is denoted simply by T. Furthermore, if $T : L^2(G_1) \to L^2(G_2)$ such that $T_{|\mathbf{S}_0(G_1)} \in \mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0(G_2))$, then we have the extension $T^* : \mathbf{S}_0'(G_2) \to \mathbf{S}_0'(G_1)$ coincides with the Banach space adjoint $S^{\times} : \mathbf{S}_0'(G_2) \to \mathbf{S}_0'(G_1)$ of some $S \in \mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0(G_2))$ if and only if $\overline{T\overline{f_1}} = Sf_1$ for all $f_1 \in \mathbf{S}_0(G_1)$.

Example 4.1 (Translation). An example of the above discussion would be the translation operator $T_s : L^2(G) \to L^2(G)$ for any $s \in G$, which is a unitary operator in $L^2(G)$ such that $T_s^* = T_{-s}$ restricts to an isometry in $\mathbf{S}_0(G)$. We can see that the extension $T_s^* : \mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$ satisfies for all $(f, \sigma) \in \mathbf{S}_0(G) \times \mathbf{S}_0'(G)$ the following formula:

$$(f, T_s^* \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (\overline{T_s \overline{f}}, \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (T_s f, \sigma) = (f, T_s^\times \sigma)_{\mathbf{S}_0, \mathbf{S}_0'}$$

In other words, $T_s^\times = T_s^* = T_{-s}$. Note that we can use this fact to redefine H-periodicity of a distribution for a closed subgroup H of G, so it does not matter if we use the Banach space adjoint $T_h^\times$ or the extended unitary map $T_h : \mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$ to define H-periodicity.

Corollary 4.2. *Let H be a closed subgroup of G. A distribution $\sigma \in \mathbf{S}_0'(G)$ is H-periodic if and only if for all $h \in H$, $T_h \sigma = \sigma$.*

Proof. Note that by the example above, $T_h^\times = T_h^* = T_{-h}$ for all $h \in H$. Therefore σ is H-periodic if and only if

$$(f, \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (f, T_h^\times \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (T_h f, \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (\overline{T_{-h} \overline{h}}, \sigma)_{\mathbf{S}_0, \mathbf{S}_0'} = (f, T_{-h} \sigma)_{\mathbf{S}_0, \mathbf{S}_0'}.$$

for all $h \in H$. This proves the desired result. □

Example 4.2 (Modulation). On the other hand, for $\omega \in \widehat{G}$, we have the modulation operator $M_\omega : L^2(G) \to L^2(G)$, a unitary operator in $L^2(G)$ such that $M_\omega^* = M_{\overline{\omega}}$ restricts to an isometry in $\mathbf{S}_0(G)$. Therefore it also extends to a $\mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$ operator. However, because $\overline{M_\omega \overline{f}} = M_{\overline{\omega}} f$ for all $f \in \mathbf{S}_0(G)$, the Banach space adjoint $M_{\overline{\omega}}^\times$ coincides with the extension of the Hilbert space adjoint $M_\omega^* = M_{\overline{\omega}}$. Equivalently $M_\omega^\times = M_\omega$, which seems to be one of the rare cases where for an operator $S \in \mathcal{L}(L^2(G))$ s.t. $S_{|\mathbf{S}_0(G)} \in \mathcal{L}(\mathbf{S}_0(G))$, that an extension $S : \mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$ coincides with the Banach space adjoint $S^\times : \mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$.

Example 4.3 (Time-Frequency shifts). The time-frequency shifts $\pi(\chi) = M_\omega T_x$ for $(x, \omega) = \chi \in G \times \widehat{G}$ are unitary maps in $\mathcal{L}(L^2(G))$ such that $\pi^*(\chi)_{|\mathbf{S}_0(G)} : \mathbf{S}_0(G) \to \mathbf{S}_0(G)$, hence time-frequency shifts also have an extension as in Item 6 of Proposition 4.12.

Example 4.4 (Fourier Transform). There are two Fourier-transforms acting on the $\mathbf{S}_0'(G)$, one is given by the Banach space adjoint $\mathcal{F}_{\widehat{G}}^\times : \mathbf{S}_0'(G) \to \mathbf{S}_0'(\widehat{G})$, and the other is the extension of the usual unitary Fourier transform $\mathcal{F}_G : \mathbf{S}_0'(G) \to \mathbf{S}_0'(\widehat{G})$. We would like to know how these two are related. However, it follows by straightforward computation that for all $h \in \mathbf{S}_0(\widehat{G})$ that

$$\overline{\mathcal{F}_{\widehat{G}}^* \overline{h}}(x) = \overline{\int_{\widehat{G}} \overline{h}(x) \omega(x) d\widehat{\mu}(\omega)}$$

$$
\begin{aligned}
&= \int_{\widehat{G}} h(\omega)\overline{\omega}(x) d\widehat{\mu}(\omega) \\
&= (\mathcal{F}_{\widehat{G}} h)(x)
\end{aligned}
$$

for all $x \in G$. Therefore it follows from Remark 4.5 that $\mathcal{F}_G = \mathcal{F}^\times_{\widehat{G}}$ on $\mathbf{S}_0'(G)$. From this observation, we have the following corollary.

Corollary 4.3. *Let* $\sigma \in \mathbf{S}_0'(G)$, *then* $\operatorname{supp}(\mathcal{F}_G\sigma) = \operatorname{supp}(\mathcal{F}^\times_{\widehat{G}}\sigma)$

Example 4.5. Let $\chi \in G \times \widehat{G}$, consider the delta-function δ_χ, and we know that $\delta_\chi \in \mathbf{S}_0(G \times \widehat{G})$. The symplectic Fourier-transform $\mathcal{F}_s$ (4.14) also extends to a unitary map in $\mathcal{L}(L^2(G \times \widehat{G}))$, and whose Hilbert space adjoint restricts to $\mathcal{L}(\mathbf{S}_0(G \times \widehat{G}))$ (this is similar to the discussion regarding the Fourier-transform in the paragraph after Remark 4.3). We compute the following for $F \in \mathbf{S}_0(G \times \widehat{G})$

$$
(F, \mathcal{F}_s\delta_\chi)_{\mathbf{S}_0,\mathbf{S}_0'} = (\overline{\mathcal{F}_s\overline{F}}, \delta_\chi)_{\mathbf{S}_0,\mathbf{S}_0'} = (\mathcal{F}_sF)(-\chi). \tag{4.36}
$$

What we find find then, is a symplectic version of (4.35), where we have $e_s^\chi \in \mathbf{S}_0'(G \times \widehat{G})$ defined via

$$
(F, e_s^\chi)_{\mathbf{S}_0,\mathbf{S}_0'} := \int_{G\times\widehat{G}} F(\chi')c_s(\chi,\chi')d\mu_{G\times\widehat{G}}(\chi') = (\mathcal{F}_sF)(-\chi), \qquad \forall \chi \in G \times \widehat{G}.
$$

So from (4.36), we obtain $\mathcal{F}_s(\delta_\chi) = e_s^\chi$ for all $\chi \in G\times\widehat{G}$. It is not hard to see that for each $\lambda^\circ \in \Lambda^\circ$, $e_s^{\lambda^\circ}$ is a Λ-periodic distribution, therefore by Item 5 of Proposition 4.12, we know that $e_s^{\lambda^\circ}$ is in the image of the Banach space adjoint of the Periodization operator $P_\Lambda^\times : \mathbf{S}_0'((G\times\widehat{G})/\Lambda) \to \mathbf{S}_0'(G\times\widehat{G})$. In fact, it follows from Weil's formula that

$$
e_s^{\lambda^\circ} = P_\Lambda^\times(c_s(\,\cdot\,,\lambda^\circ)). \tag{4.37}
$$

The formula above will be important for our calculations in the sequel, it says that each of $e_s^{\lambda^\circ}$ is the Banach-space-adjoint-periodization of a distribution induced by a concrete $\mathbf{S}_0((G \times \widehat{G})/\Lambda)$ function. In general, we shall make use of the following notation, if $\sigma \in \mathbf{S}_0'(G \times \widehat{G})$ is a Λ-Periodic distribution, then it can be identified with $\mathbf{S}_0'((G \times \widehat{G})/\Lambda)$ by the $P_\Lambda^\times$ operator, we denote this identification by

$$
\widetilde{\sigma} \in \mathbf{S}_0((G \times \widehat{G})/\Lambda). \tag{4.38}
$$

Explicitly, this just means: $\sigma = P_\Lambda^\times \widetilde{\sigma}$. Therefore by our notation, we have $\widetilde{e_s^{\lambda^\circ}} = c_s(\,\cdot\,,\lambda^\circ)$.

We have one more result which we cite from [Ber03], obtained using Bounded Uniform Partitions of Unity (BUPU) centric arguments, showing the 'duality' between periodization and restriction.

Proposition 4.13. *For a closed subgroup H of an arbitrary locally compact group G, $\sigma \in \mathbf{S}_0'(G)$ is H-periodic if and only if* $\operatorname{supp}(\mathcal{F}_G\sigma) \subseteq H^\perp$.

We have symplectic form of the previous result as a corollary.

Corollary 4.4. *If $\sigma \in \mathbf{S}_0'(G \times \widehat{G})$, then* $\operatorname{supp}(\mathcal{F}_s\sigma) = \mathcal{J}^{-1}((\operatorname{supp}\mathcal{F}_{G\times\widehat{G}}\sigma))$. *Therefore* $\operatorname{supp}(\mathcal{F}_{G\times\widehat{G}}\sigma) \subseteq \Lambda^\perp$ *if and only if* $\operatorname{supp}(\mathcal{F}_s\sigma) \subseteq \Lambda^\circ$.

Proof. Let $F \in \mathbf{S}_0(G \times \widehat{G})$, then it follows from definition of the symplectic Fourier-transform (4.14) that $(\mathcal{F}_sF)(x,\omega) = 0$ if and only if $(\mathcal{F}_{G\times\widehat{G}}F\mathcal{J})(x,\omega) = \mathcal{F}_{G\times\widehat{G}}(\omega,-x) = 0$. This proves that $\operatorname{supp}(\mathcal{F}_sF) = \mathcal{J}^{-1}((\operatorname{supp}\mathcal{F}_{G\times\widehat{G}}F))$ for any $F \in \mathbf{S}_0(G \times \widehat{G})$, but this is a weak-$*$ dense subspace of $\mathbf{S}_0'(G \times \widehat{G})$, so what we have done is sufficient to show the first assertion. The last assertion follows directly from what we have just shown, from Proposition 4.13, and from the fact that $\mathcal{J}$ is a group isomorphism such that $\mathcal{J}^{-1}(\Lambda^\perp) = \Lambda^\circ$ (see the proof of Proposition 4.4). $\square$

We end our survey of Feichtinger's algebra with the two motivating results for *Operator-Quantization and Deformation of Heisenberg Modules*. We start with the Gabor stability of linear deformations of Gabor frames [FK04, Thm. 3.8] with windows on Feichtinger's algebras, on Euclidean spaces. In this particular case, we have invertible matrices in $\mathrm{GL}_{2d}(\mathbb{R}^d)$, and any lattice in $\mathbb{R}^{2d}$ is given by $\Lambda = L\mathbb{Z}^{2d}$ for some $L \in \mathrm{GL}_{2d}(\mathbb{R})$. Feichtinger and Kaiblinger's stability result now reads:

Theorem 4.5. *Let $g_0 \in \mathbf{S}_0(\mathbb{R}^d)$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$ such that $\mathcal{G}(g_0, L_0\mathbb{Z}^{2d})$ is a Gabor frame, then there exist open sets $U_0 \ni g_0$ in $\mathbf{S}_0(\mathbb{R}^d)$ and $V_0 \ni L_0$ in $\mathrm{GL}_{2d}(\mathbb{R})$ such that $\mathcal{G}(g, L\mathbb{Z}^{2d})$ is a Gabor frame for all $(g, L) \in U_0 \times V_0$.*

This type of deformation is termed *linear* because the lattice $\Lambda_0 = L_0\mathbb{Z}^{2d}$ (or its generating matrix L_0) is perturbed by a linear transformation, namely multiplication with $LL_0^{-1} \in \mathrm{GL}_{2d}$. A careful reading of the statement, gives us a hint on why the term Gabor stability is used. It says that once it is shown that $\mathcal{G}(g_0, \Lambda_0)$ is a Gabor frame, then it is stable in the sense that the neighborhood around (g_0, Λ_0) still keeps the Gabor frame property. Our original contribution in

Section 7.1 of Chapter 7 deals with extending the deformation result of Theorem 4.5 above to the Heisenberg modules, however, there are technical problems in doing this extension. Chiefly, that the Heisenberg modules themselves depend on the lattices, hence we end up with a family of spaces that needs to be continuously deformed. This further motivates us to introduce the theory of Banach bundles, which is the subject of Chapter 6.

The next result are the kernel theorems for the Feichtinger's algebras. The outer kernel theorem is well-known for the Euclidean case [Grö01, 14.4.1]. A refinement and a statement of the outer and inner kernel theorems, respectively, to *elementary LCA groups* can be found in [FK98, Theorem 7.4.1 & Theorem 7.4.2]. Finally, the result which we will cite, are the general statements for *any* LCA groups, due to [Jak18, FJ22].

Proposition 4.14 (The Kernel Theorems). *Let G_1, and G_2 be arbitrary LCA groups.*

1. *(Outer Kernel Theorem). For each $T \in \mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0'(G_2))$, there exists a unique (outer kernel) $\kappa(T) \in \mathbf{S}_0'(G_2 \times G_1)$ such that*

$$(f_1, Tf_2)_{\mathbf{S}_0,\mathbf{S}_0'} = (f_1 \otimes f_2, \kappa(T))_{\mathbf{S}_0,\mathbf{S}_0'}, \qquad \forall (f_1, f_2) \in \mathbf{S}_0(G_1) \times \mathbf{S}_0(G_2).$$

The map $T \mapsto \kappa(T)$ defines a Banach space isomorphism between $\mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0'(G_2))$ and $\mathbf{S}_0'(G_1 \times G_2)$.

2. *(Inner Kernel Theorem). Let $\mathcal{L}_{*-\|\cdot\|}(\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2))$ be the subspace of $\mathcal{L}(\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2))$ that sends $\mathbf{S}_0'(G_1)$-norm bounded weak-$*$ convergent nets to norm-convergent nets in $\mathbf{S}_0(G_2)$. If $T \in \mathcal{L}_{*-\|\cdot\|}, (\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2))$, then there exists a unique (inner kernel) $\kappa(T) \in \mathbf{S}_0(G_1 \times G_2)$ such that*

$$(T\sigma_1, \sigma_2)_{\mathbf{S}_0,\mathbf{S}_0'} = (\kappa(T), \sigma_1 \otimes \sigma_2)_{\mathbf{S}_0,\mathbf{S}_0'}, \qquad \forall (\sigma_1, \sigma_2) \in \mathbf{S}_0'(G_1) \times \mathbf{S}_0'(G_2).$$

The map $T \mapsto \kappa(T)$ defines a Banach space isomorphism between $\mathcal{L}_{*-\|\cdot\|}(\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2))$ and $\mathbf{S}_0(G_1 \times G_2)$.

The kernel theorems are our starting point for Quantum Harmonic Analysis as envisioned by Werner [Wer84] since we have access to a one-to-one correspondence between functions (technically these are distributions) and operators. We shall follow the ideas which introduced time-frequency and Gabor analytic methods to QHA by Luef and Skrettingland in [LS18, Skr20, Skr21b]. Our original contribution in Section 7.2 of Chapter 7 deals with establishing the role of Heisenberg modules to the so-called "lattice translation invariant operators", introduced by Feichtinger and Kozek in [FK98] via kernel theorems. We will also introduce and study via Heisenberg modules, a corresponding notion called "lattice-modulation invariant operators", to complement translation invariant operators.

4.4 Operator Quantization

In this section we revisit in detail the techniques and ideas of Feichtinger and Kozek presented in [FK98]. We generalize their results to lattices on the abstract time-frequency plane $G \times \widehat{G}$. Recall the standard kernel theorem for Hilbert-Schmidt operators on the L^2-spaces denoted by $\mathcal{HS}(L^2(G), L^2(G)) \subseteq \mathcal{L}(L^2(G))$, which says that

$$L^2(G_1 \times G_2) \cong \mathcal{HS}(L^2(G_1), L^2(G_2)).$$

Along with Proposition 4.14, these kernel theorems implement an isomorphism of the triple operator spaces

$$\mathcal{L}_{*-\|\cdot\|}(\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2)) \hookrightarrow \mathcal{HS}(L^2(G_1), L^2(G_2)) \hookrightarrow \mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0'(G_2))$$

with the so-called **Banach Gelfand Triple**[1] [GS64, Rob66]:

$$\mathbf{S}_0(G_1 \times G_2) \hookrightarrow L^2(G_1 \times G_2) \hookrightarrow \mathbf{S}_0'(G_1 \times G_2).$$

In general, we denote Banach Gelfand Triples based on the Feichtinger's algebra via $(\mathbf{S}_0, L^2, \mathbf{S}_0')(G)$, for any LCA group G. Any Banach Gelfand triple isomorphisms would take on this format. For example, the kernel theorem $T \mapsto \kappa(T)$ implements a Banach Gelfand Triple isomorphism on the Banach Gelfand Triples $(\mathcal{L}_{*-\|\cdot\|}, \mathcal{HS}, \mathcal{L})(G_1, G_2)$ to $(\mathbf{S}_0, L^2, \mathbf{S}_0')(G_1 \times G_2)$ in the following sense:

1. At the inner level: $\mathcal{L}_{*-\|\cdot\|}(\mathbf{S}_0'(G_1), \mathbf{S}_0(G_2)) \xrightarrow{\sim} \mathbf{S}_0(G_1 \times G_2)$ (via inner kernel theorem);
2. At the middle level:$\mathcal{HS}(L^2(G_1), L^2(G_2)) \xrightarrow{\sim} L^2(G_1 \times G_2)$ (via regular kernel theorem);
3. At the outer level: $\mathcal{L}(\mathbf{S}_0(G_1), \mathbf{S}_0'(G_2)) \xrightarrow{\sim} \mathbf{S}_0'(G_1 \times G_2)$ (via outer kernel theorem)

[1] The classical notion is that of a Fréchet Gelfand Triple, where instead of the Feichtinger's algebra (a Banach space), one considers the Schwartz(-Bruhat) space, a Fréchet space. These triples also go by the name "Rigged Hilbert spaces".

In general, whenever two Banach Gelfand triples appear, morphisms between them means that the map must respect the corresponding three levels as in the example above.

We again reiterate that Proposition 4.14 was only previously known to hold for *elementary* LCA groups as seen in [FK98]. We now have a way of doing pseudodifferential calculus on the Feichtinger algebras on general LCA groups since we can always apply isomorphisms on the kernel of operators. We shall give an account of their relevant results in terms purely of the kernel theorems 4.14. We must expect that results about the Feichtinger's algebra must have corresponding results on the operator side.

We now take note of the necessary maps that will act on the kernels to switch between calculi. We have the following technical lemma which can be deduced from Item 8 of Proposition 4.9 and Item 6 of Proposition 4.12.

Lemma 4.5. *On an arbitrary LCA group G, the* ***asymmetric coordinate transform*** $\tau_a : L^2(G \times G) \to L^2(G \times G)$*, the coordinate transform* $A_0 : L^2(G \times G) \to L^2(G \times G)$*, and the* ***partial Fourier transform*** $\mathcal{F}_2 : L^2(G \times G) \to L^2(G \times \widehat{G})$ *are unitary maps. Explicitly, for* $F \in L^2(G \times \widehat{G})$ *they are defined via:*

1. $(\tau_a F)(x, y) = F(y, y - x)$
2. $(A_0 F)(x, y) = F(x, x - y)$
3. $(\mathcal{F}_2 F)(x, \omega) = \int_G F(x, y)\overline{\omega}(y) d\mu(y).$

These maps restrict and extend to isomorphisms on the relevant $\mathbf{S}_0$*, and* $\mathbf{S}_0'$ *spaces respectively.*

Remark 4.6. The STFT can be written in terms of the two transform given above. Let $g, h \in L^2(G)$, then it can easily be verified that:

$$\mathcal{V}_h g = (\mathcal{F}_2 \circ \tau_a)\left(g \otimes \overline{h}\right). \tag{4.39}$$

As a Corollary of Lemma 4.5, we obtain that $\mathcal{V}_g h \in \mathbf{S}_0(G \times \widehat{G})$.

Definition 4.12. Given $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$, its **spreading function** is denoted by $\eta(T) \in \mathbf{S}_0'(G \times \widehat{G})$ defined by

$$\eta(T) = (\mathcal{F}_2 \circ \tau_a)(\kappa(T)).$$

The **generalized Kohn-Nirenberg symbol** of T, denoted by $\zeta(T) \in \mathbf{S}_0'(G \times \widehat{G})$ is defined to be

$$\zeta(T) := (\mathcal{F}_2 \circ A_0)(\kappa(T)).$$

Remark 4.7. Using the the identity: $\mathcal{F}_s \circ \mathcal{F}_2 \circ \tau_a = \mathcal{F}_2 \circ A_0$, we can compute the generalized Kohn-Nirenberg symbol of T via $\zeta(T) = (\mathcal{F}_s)(\eta(T))$, and conversely, $\eta(T) = (\mathcal{F}_s)(\zeta(T))$ using the fact that the symplectic Fourier-transform is self-inverse in $L^2(G)$ (Lemma 4.4).

Example 4.6. For each $(x, \omega) \in G \times \widehat{G}$, the time-frequency shift can be seen as a norm-norm continuous operator $\pi(x, \omega) : \mathbf{S}_0(G) \to \mathbf{S}_0'(G)$ via Item 3 of Proposition 4.9. For $f, g \in \mathbf{S}_0(G)$,

$$\begin{aligned} \left\langle f, \overline{\pi(x,\omega)g} \right\rangle_2 &=: (f, \pi(x,\omega)g)_{\mathbf{S}_0,\mathbf{S}_0'} \\ &= (f \otimes g, \kappa(\pi(x,\omega)))_{\mathbf{S}_0,\mathbf{S}_0'} \end{aligned} \tag{4.40}$$

Observe that substituting $\kappa(\pi(x,\omega)) = \tau_a^{-1} \circ \mathcal{F}_2^{-1}\delta_{(x,\omega)}$ satisfies the equation (4.40) above, owing to the fact that $\left\langle f, \overline{\pi(x,\omega)g} \right\rangle_2 = \overline{\mathcal{F}_2 \circ \tau_a(\overline{f \otimes g})}(x,\omega)$. Therefore by uniqueness of kernels, it follows that $\kappa(\pi(x,\omega)) = (\tau_a^{-1} \circ \mathcal{F}_2^{-1})\delta_{(x,\omega)}$. Consequently, we have $\eta(\pi(x,\omega)) = (\mathcal{F}_2 \circ \tau_a)(\kappa(\pi(x,\omega))) = \delta_{(x,\omega)}$. Therefore, the spreading function of the time-frequency shift at $(x,\omega) \in G \times \widehat{G}$ is the dirac-mass at $(x,\omega) \in G \times \widehat{G}$.

Here we give a direct proof of the so-called shift-covariance formula using the outer kernel theorem (O.K.T.).

Theorem 4.6. *For each* $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ *and* $\lambda = (x,\omega) \in G \times \widehat{G}$, *we have:* $\pi(\lambda) \circ T \circ \pi^*(\lambda) \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$. *Furthermore,*

1. $\kappa(\pi(\lambda) \circ T \circ \pi^*(\lambda)) = M_{(\omega,\overline{\omega})}T_{(x,x)}\kappa(T)$
2. (Shift-Covariance Formula). $\zeta(\pi(\lambda) \circ T \circ \pi^*(\lambda)) = T_\lambda(\zeta(T))$.

Proof. By elementary computation, given $f, g \in \mathbf{S}_0(G)$:

$$\begin{aligned} \overline{\pi^*(\lambda)\overline{f}} \otimes \pi^*(\lambda)g &= T_{(-x,-x)}M_{(\omega,\overline{\omega})}(f \otimes g) \\ &= \overline{T_{(-x,-x)}M_{(\overline{\omega},\omega)}(\overline{f \otimes g})} \\ &= \overline{(M_{(\omega,\overline{\omega})}T_{(x,x)})^*(\overline{f \otimes g})}. \end{aligned}$$

Therefore

$$
\begin{aligned}
(f, \pi(\lambda) \circ T \circ \pi^*(\lambda)g)_{\mathbf{S}_0,\mathbf{S}_0'} &= (\overline{\pi^*(\lambda)\overline{f}}, T \circ \pi^*(\lambda)g)_{\mathbf{S}_0,\mathbf{S}_0'} \\
&\overset{\text{O.K.T.}}{=} (\overline{\pi^*(\lambda)\overline{f}} \otimes \pi^*(\lambda)g, \kappa(T))_{\mathbf{S}_0,\mathbf{S}_0'} \\
&= (\overline{(M_{(\omega,\overline{\omega})}T_{(x,x)})^*(\overline{f} \otimes g)}, \kappa(T))_{\mathbf{S}_0,\mathbf{S}_0'} \\
&= (f \otimes g, M_{(\omega,\overline{\omega})}T_{(x,x)}\kappa(T))_{\mathbf{S}_0,\mathbf{S}_0'}.
\end{aligned}
$$

By uniqueness of kernels, we have $\kappa(\pi(\lambda) \circ T \circ \pi^*(\lambda)) = M_{(\omega,\overline{\omega})}T_{(x,x)}\kappa(T)$. Therefore, using what we have just shown, the shift-covariance formula follows if we can show that: $\mathcal{F}_2 \circ A_0 \circ M_{(\omega,\overline{\omega})} \circ T_{(x,x)} = T_{(x,\omega)} \circ \mathcal{F}_2 \circ A_0$. Let $F \in L^2(G \times G)$

$$
\begin{aligned}
\mathcal{F}_2(A_0 \circ M_{(\omega,\overline{\omega})} \circ T_{(x,x)}F)(t,\xi) &= \int_G (A_0 \circ M_{(\omega,\overline{\omega})} \circ T_{(x,x)}F)(t,y)\overline{\xi}(y)d\mu(y) \\
&= \int_G (M_{(\omega,\overline{\omega})} \circ T_{(x,x)}F)(t,t-y)\overline{\xi(y)}d\mu(y) \\
&= \int_G F(t-x,t-y-x)\omega(t)\overline{\omega(t-y)\xi(y)}d\mu(y) \\
&= \int_G F(t-x,t-y-x)\omega(y)\overline{\xi(y)}d\mu(y).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(T_{(x,\omega)}(\mathcal{F}_2 \circ A_0F))(t,\xi) &= (\mathcal{F}_2 \circ A_0F)(t-x,\xi \cdot \overline{\omega}) \\
&= \int_G (A_0F)(t-x,y)\overline{(\xi \cdot \overline{\omega})(y)}d\mu(y) \\
&= \int_G F(t-x,t-x-y)\omega(y)\overline{\xi(y)}d\mu(y)
\end{aligned}
$$

Hence we see that indeed $\mathcal{F}_2 \circ A_0 \circ M_{(\omega,\overline{\omega})} \circ T_{(x,x)} = T_{(x,\omega)} \circ \mathcal{F}_2 \circ A_0$ for all $F \in L^2(G \times G)$. Since all the involved maps are unitaries in L^2 that satisfies the hypotheses of Item 3 in Proposition 4.12, the formula uniquely extends to $\mathbf{S}_0'(G \times G)$. Finally we obtain:

$$
\begin{aligned}
\zeta(\pi(\lambda) \circ T \circ \pi^*(\lambda)) &= (\mathcal{F}_2 \circ A_0)(\kappa(\pi(\lambda) \circ T \circ \pi^*(\lambda))) \\
&= (\mathcal{F}_2 \circ A_0 \circ M_{(\omega,\overline{\omega})} \circ T_{(x,x)})(\kappa(T)) \\
&= (T_{(x,\omega)} \circ \mathcal{F}_2 \circ A_0)(\kappa(T)) \\
&= T_\lambda(\zeta(T))
\end{aligned}
$$

as required. $\square$

The shift-covariance formula (4.6), as noted by Feichtinger and Kozek, motivates us to give the following definition of translation of an operator in $\mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$

Definition 4.13. Let $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ and $\lambda \in G \times \widehat{G}$, we define the **operator-translation** of T by $\chi \in G \times \widehat{G}$ by:

$$\alpha_\chi(T) = \pi(\chi) \circ T \circ \pi^*(\chi).$$

We see then that the shift-covariance formula tells us that there is a correspondence between the translation of a function and its associated operator via Kohn-Nirenberg quantization scheme. Explicitly,

$$\zeta(\alpha_\lambda(T)) = T_\lambda(\zeta(T)), \tag{4.41}$$

so that the Kohn-Nirenberg symbol of the translation of T is the same as the translation of the Kohn-Nirenberg symbol of T.

Definition 4.14. Given a lattice Λ of $G \times \widehat{G}$, an operator $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ is said to be **Λ-translation invariant** if for all $\lambda \in \Lambda$, $T\pi(\lambda) = \pi(\lambda)T$, or equivalently, $\alpha_\lambda(T) = T$ for all $\lambda \in \Lambda$.

Remark 4.8. It is important to note that the space of all Λ-invariant operators is in general a closed subalgebra of the Banach algebra $\mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ owing from the fact that operator composition is a continuous map.

Remark 4.9. Let us try to take a pause and make sense of Definition 4.14. Because $\pi(\lambda) : \mathbf{S}_0(G) \to \mathbf{S}_0(G)$ extends to $\pi(\lambda) : \mathbf{S}_0'(G) \to \mathbf{S}_0'(G)$, then T is Λ-translation invariant if and only if the following diagram commutes:

$$\begin{tikzcd}
\mathbf{S}_0(G) \arrow[r, "T"] \arrow[d, "\pi"'] & \mathbf{S}_0'(G) \arrow[d, "\pi"] \\
\mathbf{S}_0(G) \arrow[r, "T"] & \mathbf{S}_0'(G)
\end{tikzcd}$$

We now present a version of a representation for Λ-translation invariant operators of Feichtinger and Kozek valid for general LCA groups. Crucial to our calculations is the fact that the Banach space adjoint of the Periodization operator $P_\Lambda^\times : \mathbf{S}_0'((G \times \widehat{G})/\Lambda) \to \mathbf{S}_0'(G \times \widehat{G})$ is invertible when its codomain is restricted to Λ-periodic distributions (Proposition 4.12 Item 5).

Theorem 4.7. *If $T \in \mathcal{L}_\Lambda(\mathbf{S}_0(G), \mathbf{S}_0'(G))$, then $\zeta(T)$ is Λ-periodic, $\mathrm{supp}(\eta(T)) \subseteq \Lambda^\circ$, and there exists a unique sequence $\mathbf{k} = \{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ} \in \ell^\infty(\Lambda^\circ)$ such that we have the following weak representation of T:*

$$\begin{aligned}(g, Tf)_{\mathbf{S}_0,\mathbf{S}_0'} &= \int_{\Lambda^\circ} k(\lambda^\circ)(g, \pi(\lambda^\circ)f)_{\mathbf{S}_0,\mathbf{S}_0'} d\mu_{\Lambda^\circ}(\lambda^\circ) \\ &= \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)(g, \pi(\lambda^\circ)f)_{\mathbf{S}_0,\mathbf{S}_0'}\end{aligned}$$

for all $f, g \in \mathbf{S}_0(G)$. This weak representation gives an explicit form of the spreading and Kohn-Nirenberg symbols of T, we respectively have:

$$\eta(T) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)\delta_{\lambda^\circ}, \qquad \zeta(T) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)e_s^{\lambda^\circ} \tag{4.42}$$

Proof. Λ-invariance of T implies $\pi(\lambda) \circ T \circ \pi^*(\lambda) = T$ for all $\lambda \in \Lambda$, therefore Shift-Covariance from Theorem 4.6 implies that $\zeta(T) = T_\lambda(\zeta(T))$ for all $\lambda \in \Lambda$. In other words, $\zeta(T)$ is a Λ-periodic distribution, so it follows from Corollary 4.2 and Proposition 4.13 that $\mathrm{supp}(\mathcal{F}_{G\times\widehat{G}}\zeta(T)) \subseteq \Lambda^\perp$. Consequently, it also follows from Corollary 4.4 that $\mathrm{supp}(\mathcal{F}_s\zeta(T)) = \mathrm{supp}(\eta(T)) \subseteq \Lambda^\circ$. Therefore from the Restriction property of Proposition 4.12, $\eta(T) = R_{\Lambda^\circ}^\times(\{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ})$, for a unique $\mathbf{k} = \{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ} \in \ell^\infty(\Lambda^\circ) \cong \mathbf{S}_0'(\Lambda^\circ)$ we have

$$\begin{aligned}(F, \eta(T))_{\mathbf{S}_0,\mathbf{S}_0'} &= (R_{\Lambda^\circ}F, \{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ})_{\mathbf{S}_0,\mathbf{S}_0'} \\ &= \int_{\Lambda^\circ} k(\lambda^\circ)F(\lambda^\circ) d\mu_{\Lambda^\circ}(\lambda^\circ) \\ &= \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)F(\lambda^\circ) \\ &= \left(F, \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)\delta_{\lambda^\circ}\right)_{\mathbf{S}_0,\mathbf{S}_0'}\end{aligned}$$

for all $F \in \mathbf{S}_0(G \times \widehat{G})$, therefore $\eta(T) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)\delta_{\lambda^\circ}$. Taking the symplectic Fourier transform of $\eta(T)$, along with Example 4.5, we obtain $\zeta(T) = \dfrac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)e_s^{\lambda^\circ}$. From Example 4.6, we obtain

$$\eta\left(\frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)\pi(\lambda^\circ)\right) = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)\delta_{\lambda^\circ}.$$
$$= \eta(T)$$

Now it follow from uniqueness of kernels, and that $\tau_a, \mathcal{F}_2, A_0$ are all isomorphisms in their respective $\mathbf{S}_0'$ spaces that $T = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)\pi(\lambda^\circ)$ (weakly) as required. □

Remark 4.10. If T is Λ-invariant, then the fact that its Kohn-Nirenberg symbol $\zeta(T)$ is Λ-periodic means that there is a unique corresponding $\widetilde{\zeta}(T) \in \mathbf{S}_0'((G\times\widehat{G})/\Lambda)$ such that $\zeta(T) = P_\Lambda^\times(\widetilde{\zeta}(T))$. Note that in the original paper, this correspondence was not explicitly pointed out, and the authors simply treat $\zeta(T)$ and $\widetilde{\zeta}(T)$ interchangeably. One advantage of a careful tracking of these correspondence is that we have an explicit expression for $\widetilde{\zeta}(T)$. We know that $\zeta(T) = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)e_s^{\lambda^\circ}$, and therefore from Equation (4.37), we find that

$$\widetilde{\zeta}(T) = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)c_s(\,\cdot\,,\lambda^\circ). \tag{4.43}$$

Theorem 4.8. *Keeping the setting and hypotheses from Theorem 4.7, the following are equivalent:*

1. $\mathbf{k}\in\ell^1(\Lambda^\circ)$;
2. $\widetilde{\zeta}(T)\in\mathbf{S}_0((G\times\widehat{G})/\Lambda)$.

If any of the above is true, then we have $\mathrm{Im}(T)\subseteq\mathbf{S}_0(G)$, *with the following strong representation for* T

$$T = \int_{\Lambda^\circ}k(\lambda^\circ)\pi(\lambda^\circ)d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)\pi(\lambda^\circ), \tag{4.44}$$

where the coefficients can be computed via:

$$k(\lambda^\circ) = \int_{(G\times\widehat{G})/\Lambda}\widetilde{\zeta}(T)(\dot{\chi})\overline{c_s(\dot{\chi},\lambda^\circ)}d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}). \tag{4.45}$$

We also have the following operator norm estimate:

$$\|T\|_{\mathcal{L}(\mathbf{S}_0(G))}\leqslant\|\mathbf{k}\|_1.$$

Proof. We obtain from Equation (4.37) that $\widetilde{\zeta}(T) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ) c_s(\,\cdot\,, \lambda^\circ)$. If we assume 1, then Proposition 4.11 implies

$$\|\widetilde{\zeta}(T)\|_{\mathbf{A}((G\times\widehat{G})/\Lambda)} \leqslant \sum_{\lambda^\circ \in \Lambda^\circ} |k(\lambda^\circ)| \|c_s(\,\cdot\,, \lambda^\circ)\|_{\mathbf{A}((G\times\widehat{G})/\Lambda)} = \sum_{\lambda \in \Lambda^\circ} |k(\lambda^\circ)| < \infty.$$

So $\widetilde{\zeta}(T) \in \mathbf{S}_0((G\times\widehat{G})/\Lambda) = \mathbf{A}((G\times\widehat{G})/\Lambda)$. Conversely, if $\widetilde{\zeta}(T) \in \mathbf{S}_0((G\times\widehat{G})/\Lambda)$, then by the fact that the Fourier transform $\mathcal{F}_{(G\times\widehat{G})/\Lambda} : \mathbf{S}_0((G\times\widehat{G})) \to \ell^1(\Lambda^\circ)$ is an isomorphism, we have from Corollary 4.1, computation (4.13), and Equation (4.43) that for λ°:

$$\begin{aligned}
(\mathcal{F}_{(G\times\widehat{G})/\Lambda}\widetilde{\zeta}(T))(\lambda^\circ) &= \int_{(G\times\widehat{G})/\Lambda} \left(\frac{1}{d(\Lambda)} \sum_{\upsilon^\circ \in \Lambda^\circ} k(\upsilon^\circ) c_s(\dot{\chi}, \upsilon^\circ) \right) \overline{c_s(\dot{\chi}, \lambda^\circ)} d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \frac{1}{d(\Lambda)} \sum_{\upsilon^\circ \in \Lambda^\circ} k(\upsilon^\circ) \int_{(G\times\widehat{G})/\Lambda} c_s(\dot{\chi}, \upsilon^\circ - \lambda^\circ) d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \frac{1}{d(\Lambda)} k(\lambda^\circ) d(\Lambda) = k(\lambda^\circ).
\end{aligned}$$

From this we obtain the formula $\{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ} = \mathcal{F}_{(G\times\widehat{G})/\Lambda}\widetilde{\zeta}(T) \in \ell^1(\Lambda^\circ)$. This shows equivalence of 1 and 2, and also the coefficient formula (4.45). Next, we shall show that 1 implies $\operatorname{Im}(T) \subseteq \mathbf{S}_0(G)$. To do this, fix $g \in \mathbf{S}_0(G)$, we compute for $f \in \mathbf{S}_0(G)$:

$$\begin{aligned}
&\int_{G\times\widehat{G}} |(\pi(x,\omega)g, Tf)_{\mathbf{S}_0,\mathbf{S}_0'}| d\mu_{G\times\widehat{G}}(x,\omega) \\
&\leqslant \frac{1}{d(\Lambda)} \int_{G\times\widehat{G}} \sum_{(x^\circ,\omega^\circ) \in \Lambda^\circ} |k(x^\circ,\omega^\circ)| \, |(\pi(x,\omega)g, \pi(x^\circ,\omega^\circ)f)_{\mathbf{S}_0,\mathbf{S}_0'}| d\mu_{G\times\widehat{G}}(x,\omega) \\
&= \frac{1}{d(\Lambda)} \sum_{(x^\circ,\omega^\circ) \in \Lambda^\circ} |k(x^\circ,\omega^\circ)| \int_{G\times\widehat{G}} \left| \left\langle \pi(x,\omega)g, \overline{\pi(x^\circ,\omega^\circ)f} \right\rangle_2 \right| d\mu_{G\times\widehat{G}}(x,\omega) \\
&= \frac{1}{d(\Lambda)} \sum_{(x^\circ,\omega^\circ) \in \Lambda^\circ} |k(x^\circ,\omega^\circ)| \int_{G\times\widehat{G}} \left| \langle \pi(x,\omega)g, \pi(x^\circ,\overline{\omega}^\circ)\overline{f} \rangle_2 \right| d\mu_{G\times\widehat{G}}(x,\omega) \\
&= \frac{1}{d(\Lambda)} \sum_{(x^\circ,\omega^\circ) \in \Lambda^\circ} |k(x^\circ,\omega^\circ)| \int_{G\times\widehat{G}} \left| \langle \pi(x - x^\circ,\omega\omega^\circ)g, \overline{f} \rangle_2 \right| d\mu_{G\times\widehat{G}}(x,\omega) \\
&= \frac{1}{d(\Lambda)} \sum_{(x^\circ,\omega^\circ) \in \Lambda^\circ} |k(x^\circ,\omega^\circ)| \int_{G\times\widehat{G}} \left| \langle \pi(x,\omega)g, \overline{f} \rangle_2 \right| d\mu_{G\times\widehat{G}}(x,\omega) \\
&= \|\mathbf{k}\|_1 \|f\|_{\mathbf{S}_0}.
\end{aligned}$$

By the characterization [Jak18, Theorem 6.12], $Tf \in \mathbf{S}_0(G)$ if and only if the left-hand most side of the computation above is finite. We find from the above estimate that a sufficient condition is $\|\mathbf{k}\|_1 < \infty$. Now, notice that more is true from the computation, the strong representation of

T follows from it, and also the following estimate: $\|T\|_{\mathcal{L}(\mathbf{S}_0(G))} \leqslant \|\mathbf{k}\|_1$. $\square$

The general principle of QHA is to build-up operator-versions of classical notions that hold for functions on the time-frequency plane $G \times \widehat{G}$. Take for example, the periodization operator (Item 4 Proposition 4.9) on the time-frequency plane $P_\Lambda : \mathbf{S}_0(G \times \widehat{G}) \to \mathbf{S}_0((G \times \widehat{G})/\Lambda)$ which can be written as:

$$P_\Lambda F = \int_\Lambda T_\lambda F d\mu_\Lambda(\lambda) = \sum_{\lambda \in \Lambda} T_\lambda F, \qquad \forall F \in \mathbf{S}_0(G \times \widehat{G}).$$

Building up from this and the notion of operator-translation, we have the following.

Definition 4.15. Let $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$, the **operator-periodization** of T with respect to the lattice $\Lambda \subseteq G \times \widehat{G}$ is

$$\sum_{\lambda \in \Lambda} \alpha_\lambda(T) \in \mathcal{L}(L^2(G)). \tag{4.46}$$

Therefore one can always see $P_\Lambda T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$.

The verification that operator-periodization lands in $\mathcal{L}(L^2(G))$ can be deduced via duality i.e. by weakly-defining (4.46). An explicit proof of this can be found, albeit only in the Euclidean case, in [Skr21a][2]. For the application that we have in mind though, we do not really have to deal with the subtleties of this generality.

Operator-periodization extends the following intuition to operators: "*Every Λ-periodic function is Λ-translation invariant*". We see this in the following computation, for $\lambda' \in \Lambda$:

$$\begin{aligned}
\alpha_{\lambda'} P_\Lambda T &= \alpha_{\lambda'} \left(\sum_{\lambda \in \Lambda} \alpha_\lambda T \right) \\
&= \sum_{\lambda \in \Lambda} \pi(\lambda')\pi(\lambda) T \pi^*(\lambda)\pi^*(\lambda') \\
&= \sum_{\lambda \in \Lambda} \pi(\lambda' + \lambda) T \pi^*(\lambda' + \lambda) c(\lambda', \lambda)\overline{c(\lambda', \lambda)} \\
&= \sum_{\lambda \in \Lambda} \alpha_\lambda T = P_\Lambda T.
\end{aligned} \tag{4.47}$$

Corollary 4.5. *Let $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ be a Λ-translation invariant operator that satisfies any of the equivalent hypotheses in Theorem 4.8. Then there exits a "prototype" operator*

[2] It is important to recognize the author of the paper proved this result by a careful consideration of the duality theory on $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$. Furthermore, they have also integrated the theory of operator-Fourier transforms.

$Q \in \mathcal{L}(\mathbf{S_0}(G), \mathbf{S}_0'(G))$ such that $P_\Lambda Q \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ and $\widetilde{\zeta}(T) = P_\Lambda \zeta(Q) = P_\Lambda^\times(\zeta(P_\Lambda Q))$. Furthermore, $T = P_\Lambda Q$.

Furthermore, the strong-representation (4.44) can be written in terms of the spreading function $\eta(Q) \in \mathbf{S}_0(G \times \widehat{G})$:

$$T = \int_{\Lambda^\circ} \eta(Q)(\lambda^\circ)\pi(\lambda)d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)} \sum_{\lambda \in \Lambda^\circ} \eta(Q)(\lambda^\circ)\pi(\lambda^\circ). \tag{4.48}$$

That is, the coefficients $\mathbf{k}$ that appear in Theorem 4.8 satisfy $\mathbf{k} = R_{\Lambda^\circ}(\eta(Q)) \in \ell^1(\Lambda^\circ)$.

Proof. We first show that $T = P_\Lambda Q = \sum_{\lambda \in \Lambda} \alpha_\lambda(Q)$ for some $Q \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$. To do this, we use the hypothesis $\widetilde{\zeta}(T) \in \mathbf{S}_0((G \times \widehat{G})/\Lambda)$. It follows from the surjectivity of the periodization operator and the Outer Kernel Theorem (c.f. Item 4 Proposition 4.9 and Theorem 4.14) that there exists a $Q \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ such that $\widetilde{\zeta}(T) = P_\Lambda(\zeta(Q))$. We can now use the shift-covariance formula (4.6) to conclude that $\widetilde{\zeta}(T) = P_\Lambda(\zeta(Q)) = P_\Lambda^\times(\zeta(P_\Lambda Q))$. Finally, since $P_\Lambda^\times$ is an isomorphism to Λ-periodic distributions in $\mathbf{S}_0'(G \times \widehat{G})$, we find from the equation $P^\times(\zeta(T)) = \widetilde{\zeta}(T) = P^\times(\zeta(P_\Lambda Q))$ that $T = P_\Lambda Q$.

On the other hand, the computation of the coefficients in Theorem 4.8, along with Poisson's summation formula (3.31) show that

$$\begin{aligned}
k(\lambda^\circ) &= \int_{(G \times \widehat{G})/\Lambda} \widetilde{\zeta}(\dot{\chi})\overline{c_s(\lambda^\circ, \dot{\chi})}d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \int_{(G \times \widehat{G})/\Lambda} P_\Lambda(\zeta(Q))(\dot{\chi})\overline{c_s(\lambda^\circ, \dot{\chi})}d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \int_{(G \times \widehat{G})/\Lambda} \sum_{\lambda \in \Lambda} T_\lambda \zeta(Q)(\chi)\overline{c_s(\lambda^\circ, \dot{\chi})}d\mu_{(G\times\widehat{G})/\Lambda}(\dot{\chi}) \\
&= \int_{(G \times \widehat{G})/\Lambda} \sum_{\lambda \in \Lambda} T_\lambda \zeta(Q)(\chi)\overline{c_s(\lambda^\circ, \dot{\chi})}d\mu_{(G\times\widehat{G})/\Lambda} \\
&= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} (F_s\zeta(Q))(\mu^\circ) \int_{(G \times \widehat{G})/\Lambda} c_s(\lambda^\circ, \dot{\chi})d\mu_{(G\times\widehat{G})}(\dot{\chi}) \\
&= \frac{1}{d(\Lambda)} \eta(\zeta(Q))(\lambda^\circ)d(\Lambda) \\
&= \eta(Q)(\lambda^\circ).
\end{aligned}$$

It follows that $\mathbf{k} = R_{\Lambda^\circ}(\eta(Q))$, from which Equation (4.48) further follows. $\square$

We denote the rank-one operators generated by $g, h \in \mathbf{S}_0(G)$ by $g \otimes h \in \mathcal{L}(L^2(G))$, and will be defined in terms of Jakobsen's notation[3]:

$$(g \otimes h)f = (f, h)_{\mathbf{S}_0,\mathbf{S}_0'} g = \langle f, \overline{h} \rangle_2 \, g, \qquad \forall f \in L^2(G). \tag{4.49}$$

It also follows from a straightforward computation that the adjoint of a rank-one operator satisfies: $(g \otimes h)^* = h \otimes g$. One may realize that we are using the tensor-product notation for rank-one operators. It is in general ambiguous whether we mean the function tensor-product $g \otimes h \in \mathbf{S}_0(G \times G)$, or if we mean the operator tensor-product $g \otimes h \in \mathcal{L}(L^2(G))$, we shall leave it to the reader to figure out by context. We will however, motivate the use of this notation using the following result.

Proposition 4.15. *The rank-one operator satisfies* $\kappa(g \otimes h) = g \otimes h \in \mathbf{S}_0(G \times G)$ *for all* $g, h \in \mathbf{S}_0(G)$.

Proof. The proof follows from simple computation:

$$(f_1, (g \otimes h)f_2)_{\mathbf{S}_0,\mathbf{S}_0'} = (f_1, (f_2, h)_{\mathbf{S}_0,\mathbf{S}_0'} g)_{\mathbf{S}_0,\mathbf{S}_0'} = (f_1, h)_{\mathbf{S}_0,\mathbf{S}_0'} \cdot (f_2, g)_{\mathbf{S}_0,\mathbf{S}_0'} = (f_1 \otimes f_2, g \otimes h)_{\mathbf{S}_0,\mathbf{S}_0'}.$$

Hence $\kappa(g \otimes h) = g \otimes h$. □

Applications of QHA on Gabor analysis can now be seen by the simple observation that Gabor frame operators can be written as operator-periodized rank-one operators.

Proposition 4.16. *Let* $g, h \in \mathbf{S}_0(G)$, *then*

$$S_{g,h,\Lambda} = \sum_{\lambda \in \Lambda} \alpha_\lambda((\overline{g} \otimes h)^*) = P_\Lambda((\overline{g} \otimes h)^*) = P_\Lambda(h \otimes \overline{g}). \tag{4.50}$$

Proof. For $f \in L^2(G)$:

$$\begin{aligned} P_\Lambda(h \otimes \overline{g})f &= \sum_{\lambda \in \Lambda} \alpha_\lambda(h \otimes \overline{g})f \\ &= \sum_{\lambda \in \Lambda} \pi(\lambda)\,(h \otimes \overline{g})\,\pi^*(\lambda)f \\ &= \sum_{\lambda \in \Lambda} \pi(\lambda)\,(\langle \pi^*(\lambda)f, g \rangle_2\, h) \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle_2\, \pi(\lambda)h \end{aligned}$$

[3]This generalizes to $g, h \in L^2(G)$ obviously, by using the rightmost formula.

$$= S_{g,h,\Lambda}.$$

□

As a corollary of Proposition 4.16, every Gabor frame operator $S_{g,h,\Lambda}$ with $g, h \in \mathbf{S}_0(G)$ must be Λ-translation invariant and it satisfies the hypotheses of Corollary 4.5– of course this is consistent with what we already know from Proposition 4.7. Moreover, $S_{g,h,\Lambda}$ can be written as the periodization of the "prototype" operator $(\overline{g} \otimes h)^*$ as called by Feichtinger and Kozek [FK98].

Theorem 4.9 (Janssen's Representation)**.** *For all $g, h \in \mathbf{S}_0(G)$, we have:*

$$S_{g,h,\Lambda} = \int_{\Lambda^\circ} \langle h, \pi(\lambda^\circ) g\rangle_2 \, \pi(\lambda) d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} \langle h, \pi(\lambda^\circ) g\rangle_2 \, \pi(\lambda^\circ). \tag{4.51}$$

Proof. Given that $(\overline{g} \otimes h)^* = (h \otimes \overline{g})$ is the "prototype" operator for $S_{g,h,\Lambda}$ in the sense of Corollary 4.5, it follows from the same result that:

$$S_{g,h,\Lambda} = \int_{\Lambda^\circ} \eta(h \otimes \overline{g})(\lambda^\circ) \pi(\lambda) d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)} \sum_{\lambda \in \Lambda^\circ} \eta(h \otimes \overline{g})(\lambda^\circ) \pi(\lambda^\circ).$$

All that remains then, is to compute the spreading function of the rank-one operator $h \otimes \overline{g}$. However, it follows from Proposition 4.15 and Equation (4.39) that $\eta(h \otimes \overline{g})(\lambda^\circ) = (\mathcal{F}_2 \circ \tau_a)(\kappa(h \otimes \overline{g}))(\lambda^\circ) = (\mathcal{F}_2 \circ \tau_a)(h \otimes \overline{g})(\lambda^\circ) = \langle h, \pi(\lambda^\circ) g\rangle_2$, which proves our desired result. □

We also obtain from Janssen's representation, the equivalent form, called the **Fundamental Identity of Gabor Analysis** (FIGA).

Corollary 4.6 (FIGA)**.** *For all $g, h, f \in \mathbf{S}_0(G)$:*

$$S_{g,h,\Lambda} f = \frac{1}{d(\Lambda)} S_{g,f,\Lambda^\circ} h. \tag{4.52}$$

Proof. Let $\varphi \in L^2(G)$, then due to Equation (4.51) we have:

$$\begin{aligned} \langle S_{g,h,\Lambda} f, \varphi\rangle_2 &= \left\langle \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} \langle h, \pi(\lambda^\circ) g\rangle_2 \, \pi(\lambda^\circ) f, \varphi \right\rangle_2 \\ &= \left\langle \frac{1}{d(\Lambda)} S_{g,f,\Lambda^\circ} h, \varphi \right\rangle_2 . \end{aligned}$$

Since $\varphi \in L^2(G)$ is arbitrary, then (4.52) follows. $\square$

Equation (4.51) is the celebrated **Janssen's representation**, originally proven by Janssen in [Jan95] in the Euclidean case. Rieffel has also proven as similar representation in [Rie88] which holds for lattices in abstract time-frequency plane $G \times \widehat{G}$. However, the Gabor frame operators in this paper by Rieffel only involve those with windows coming from the Schwartz-Bruhat space $\mathcal{S}(G) \subseteq \mathbf{S}_0(G)$. Our proof above is largely parallel to the ideas of Feichtinger and Kozek, although the proof shown above is verification that there is a time-frequency analytic proof that is valid for lattices in the abstract time-frequency planes, and for Gabor frames with windows from $\mathbf{S}_0(G)$.

Remark 4.11. We can repeat Proposition 4.16 using $\sum_{i=1}^{n}(\overline{g}_i \otimes h_i)^* = \sum_{i=1}^{n} h_i \otimes \overline{g}_i$ to obtain a multi-window versions of Janssen's Representation and FIGA.

While we have only developed enough to prove Janssen's representation, we can actually slightly refine all our results and repackage them in terms of a Banach Gelfand triples.

Proposition 4.17. *For any Λ-translation invariant $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ we have $\widetilde{\zeta}(T) \in (\mathbf{S}_0, L^2, \mathbf{S}_0')((G \times \widehat{G})/\Lambda)$, and there exists a unique coefficient $\mathbf{k} \in (\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$ such that*

$$T = \int_{\Lambda^\circ} k(\lambda^\circ)\pi(\lambda^\circ) d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ)\pi(\lambda^\circ)$$

where the convergence is in the weak sense when $\widetilde{\zeta}(T) \in \mathbf{S}_0'((G \times \widetilde{G})/\Lambda)$.

Remark 4.12. The theorem says that if we have a Λ-translation invariant operator T as in above, the association $\widetilde{\zeta}(T) \mapsto \mathbf{k}$ is a Banach Gelfand triple isomorphism. The outer isomorphism is Theorem 4.7, the inner isomorphism is Theorem 4.8. While we have skipped the proof for the middle isomorphism, though it can be done in a very similar fashion to the middle isomorphism.

In the next chapter, we shall see that Janssen's representation will be a crucial ingredient in the construction of the Heisenberg modules.

Chapter 5

The Heisenberg Module

We shall introduce the general framework from which we construct the C^*-algebras most relevant to us. We shall study the so-called **twisted C*-dynamical systems** introduced and developed by Busby, Smith, Packer and Raeburn in [BS70, PR89, PR90]. There also are connections with the older paper of Green [Gre78] on "twisted covariant algebras" as explicitly pointed out in [PR89]. However we shall not see much use of twisted covariance algebras in this document as the "twisted covariant systems" from which such algebras are generated purposefully avoid working with cocycles. Since we will eventually want to deal with the algebras generated by the time-frequency shifts which obey a "projective representation" condition given by (4.8) from which the Heisenberg 2-cocycles appear, then we naturally gravitative towards a theory wherein the cocycles are explicitly included. We shall follow the development of twisted C^*-dynamical systems outlined in [MPR05], wherein while the starting definitions are mostly based from the original references [PR89, PR90], they have chosen to treat their twisted actions as continuous maps instead of merely Borel maps. Furthermore, the induced $*$-algebras from these twisted C^*-dynamical systems are chosen to be more concretely defined in the spirit of [Wil07] and [BS70]. Our purpose in Section 5.2 is to study twisted group C^*-algebras, and we obtain them as an instance of the twisted crossed-product of a twsited C^*-dynamical system with $A = \mathbb{C}$.

Equipped with a good understanding of twisted group C^*-algebras, we can finally construct the Heisenberg module in Section 5.3. We will combine insights from the original paper [Rie88] of Rieffel and [Lue07] of Luef, where the connections of the Heisenberg module with Gabor analysis was first documented. We note that that we present an original result for the fullness condition on the Heisenberg module, using only time-frequency analysis and our knowledge of the Feichtinger's algebra (see Proposition 5.8). We close the chapter with a detailed exposition of the localization scheme of Austad and Enstad [AE20], allowing us to embed the Heisenberg modules into the space of square-integrable functions.

5.1 Twisted Crossed-Products

Definition 5.1. A C^*-dynamical system is a triple (A, G, α) such that A is a C^*-algebra, G is a locally compact group, and α is a continuous (group) homomorphism $\alpha : G \to \mathrm{Aut}(A)$ where $\mathrm{Aut}(A)$ is equipped with the pointwise convergence topology. The map α is also referred to as a **group action** of G on A.

Remark 5.1. Note that $\mathrm{Aut}(A)$ is the group of all automorphisms on A, and we can characterize the pointwise-convergence topology on it by saying that a net ϕ_i converges to ϕ in $\mathrm{Aut}(A)$ with the pointwise topology if and only if for all $a \in A$, the net $\phi_i(a) \to \phi(a)$ in A. Furthermore, given a C^*-dynamical system (A, G, α), we shall also use the notation $\alpha_s := \alpha(s)$ for each $s \in G$. In some of our estimates, we will also make use of the fact that for all $s \in G$, $\|\alpha_s(a)\|_A \leqslant \|a\|_A$, which follows from the fact that all $*$-homomorphisms on C^*-algebras are contractive [FD88, Theorem 3.7].

In the commutative case we have the canonical example.

Example 5.1. Let G be a locally compact abelian group, and $\alpha : G \to \mathrm{Aut}(C_0(G))$ defined by the following: for each $f \in C_0(X)$ and $s \in G$:

$$\alpha_s(f)(x) = f(x - s), \qquad \forall x \in G.$$

It is not hard to see that indeed $\alpha_s \in \mathrm{Aut}(C_0(G))$, with $(\alpha_s)^{-1} = \alpha_{-s}$ and $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \in G$. That α is continuous in the required sense can be found from [Wil07, Lemma 2.5]. Curiously, the continuity statement for α is equivalent to the following statement regarding the translation operator: α is continuous if and only if for all $f \in C_0(X)$

$$\lim_{s \to 0} \|T_s(f) - f\|_\infty \to 0$$

Therefore $(C_0(G), G, \alpha)$ defines a C^*-dynamical system.

Now that we have a good idea of what C^*-dynamical systems look like, we can compare them to their twisted versions. The 'twist' in a C^*-dynamical system is formally implemented by introducing a 2-cocycle and having it interact with the group action in such a way that it generalizes the homomorphism property into a twisted one. Before we proceed, recall the notion of unitary elements (which forms a group) in a C^*-algebra: we denote it by $U(C) = \{cc^* = 1_C = c^*c : c \in C\}$ for a unital C^*-algebra C with identity 1_C.

Definition 5.2. Let A be a C^*-algebra with identity $1_{M(A)}$ for its multiplier algebra, G a locally compact group with identity e, $\alpha : G \to \mathrm{Aut}(A)$ a continuous map, and $c : G \times G \to UM(A)$ a ($*$-strongly) continuous map. We say that (α, c) is a **twisted action** of G on A if they satisfy the following:

$$\alpha_e(a) = a, \ c(s,e) = c(e,s) = 1_{M(A)}, \qquad \forall a \in A, s \in G \tag{5.1}$$

$$c(g,t)c(gt,s) = \alpha_g(c(t,s))c(g,ts), \qquad \forall g,t,s \in G; \tag{5.2}$$

$$(\alpha_s \circ \alpha_t)(a) = c(s,t)\alpha_{st}(a)c^*(s,t), \qquad \forall a \in A. \tag{5.3}$$

A quadruple (A, G, α, c) satisfying the above axioms is called a **twisted C*-dynamical system**.

Remark 5.2. For each $s \in G$, $\alpha_s : A \to A$ can be seen as a nondegenerate $*$-homomorphism (see Definition 2.16). The nondegeneracy can be deduced from the automorphism property of α_s. Therefore it follows from Corollary 2.4 that we have the extension $\overline{\alpha_s} : M(A) \to M(A)$, but for the rest of the manuscript we shall suppress the bar above such extensions, hence we have $\alpha : M(A) \to M(A)$ as it appears in in Equation (5.2).

Remark 5.3. A map $c : G \times G \to UM(A)$ that satisfies (5.2) is called a **2-cocycle**, if it further satisfies the latter half of (5.1), we say that c is **normalized**. Furthermore, observe that if c is related to α via Formula (5.3), then in the case that $c \equiv 1_{M(A)}$ (i.e. we don't have a twist), we find that α will satisfy the homomorphism property $\alpha_t \circ \alpha_s = \alpha_{st}$. Henceforth we see that (A, G, α) then becomes a normal, 'untwisted', C^*-dynamical system. Curiously, we can have a nontrivial 2-cocycle $c : G \times G \to UM(A)$ and still have α satisfy the homomorphism property. In particular, this happens in the case where A is commutative.

We will eventually construct C^*-algebras out of these twisted C^*-dynamical systems, where some of the concepts from Appendix A will be relevant. We outline the general strategy: we shall find a way to induce a Banach $*$-algebra out of (A, G, α, c), then we proceed with completing it with the universal C^*-norm as in Proposition A.1. Hence it only makes sense that we will be interested at the nondegenerate $*$-homomorphisms of this induced Banach $*$-algebra. It turns out however, as we shall see later, that one can induce the relevant nondegenerate $*$-homomorphisms of the resulting $*$-algebra on the level of twisted C^*-dynamical systems (see Proposition 5.1).

Definition 5.3. Let (A, G, α, c) be a twisted C^*-dynamical system, we say that the triple (H, ρ, U) is a **twisted covariant representation** of (A, G, α, c) if they satisfy the following

1. $\rho : A \to \mathcal{L}(H)$ is a nondegenerate $*$-representation of A;

2. $U : G \to U\mathcal{L}(H)$ is a strongly continuous mapping that satisfies a *projective* relation with the 2-cocycle:

$$U_s U_t = \rho(c(s,t)) U_{st}, \qquad \forall s, t \in G. \tag{5.4}$$

   Where we have used the notation $U_s := U(s)$ for all $s \in G$.

3. A *covariant* relation between ρ and α, given by:

$$\rho(\alpha_g(a)) = U_g \circ \rho(a) \circ U_g^* \qquad \forall a \in A. \tag{5.5}$$

We say that (H, ρ, U) is a **nondegenerate** covariant representation if $\rho : A \to \mathcal{L}(H)$ is a nondegenerate $*$-representation. Furthermore, a subspace $V \subseteq H$ is said to be **invariant** with respect to (H, ρ, U) if $\rho(a)V \subseteq V$ and $U_s(V) \subseteq V$ for all $a \in A$ and $s \in G$. We now say that (H, ρ, U) is **irreducible** if its only closed invariant subspace are $\{0\}$ and H. Finally, two twisted covariant representations (H, ρ, U) and (H', ρ', U') are said to be **unitarily equivalent** if there exists a unitary map $\mathcal{U} : H \to H'$ such that $\rho'(a) = \mathcal{U}\rho(a)\mathcal{U}^*$ and $U'_s = \mathcal{U}U_s\mathcal{U}^*$ for all $a \in A$ and $s \in G$.

We shall next give an important example of a covariant representation on (A, G, α, c), but first let us fix μ_G to be a Haar measure on G, and $\Delta_G : G \to \mathbb{R}_{\geqslant 0}$ for the modular function associated with G (see Lemma 3.12).

Example 5.2. Let (A, G, α, c) be a twisted C^*-dynamical system, and (H_ρ, ρ) be a faithful $*$-representation of A. We can induce the so-called **regular covariant representation** $(\widetilde{H}_\rho, \widetilde{\rho}, \widetilde{U})$ for (A, G, α, c) with $\widetilde{H}_\rho = L^2(G, H_\rho)$, $\widetilde{\rho} : A \to \mathcal{L}(\widetilde{H}_\rho)$, and $\widetilde{U} : G \to U\mathcal{L}(\widetilde{H}_\rho)$, where

1. $(\widetilde{\rho}(a)\varphi)(s) = \rho(\alpha_s(a))(\varphi(s))$

2. $(\widetilde{U}_t\varphi)(s) = \rho(c(s,t))\varphi(st)$

for $s, t \in G$ and $\varphi \in \widetilde{H}_\rho$. It follows from a straightforward computation that $\widetilde{U}$ and $\widetilde{\rho}$ satisfy Equations (5.4), and (5.5), which follow from Equations (5.2) and (5.3).

Let us now define the $*$-algebra that will be induced by the twisted C^*-dynamical system (A, G, α, c) as outlined in [Ric14, MPR05], which can be seen to be quite similar to the strategy in [Wil07]. First, we start with the space

$$C_c(G, A) := \{f : G \to A \mid f \text{ is continuous and compactly supported in } G\}.$$

Given that $\|\cdot\|_A$ is the norm on A, we fix a Haar measure μ_G on G, and equip $C_c(G, A)$ with the norm $\|f\|_1 := \int_G \|f(g)\|_A d\mu_G(g)$, and this is indeed well-defined when $f \in C_c(G, A)$. Let us further equip $C_c(G, A)$ with the respective 'twisted' convolution and involution: for each $f_1, f_2 \in C_c(G, A)$, and $s \in G$:

$$(f_1 *_c f_2)(s) = \int_G f(g)\alpha_g(f_2(g^{-1}s))c(g, g^{-1}s)d\mu_G(g); \tag{5.6}$$

$$f_1^{*_c}(s) = c(s, s^{-1})^*\alpha_s(f_1(s^{-1})^*)\Delta_G(s^{-1}). \tag{5.7}$$

We do not delve into the all the tedious details of proving that $C_c(G, A)$ is a $*$-algebra with the prescribed convolution and involution via (5.6) and (5.7) respectively, but we shall sketch some of them here. In principle, the arguments will be similar to the one given in [Wil07] with proper modifications to account for the twist. Since we have $\|c(t, g)\|_{M(A)} = 1$ for all $t, g \in G$ (because the image of c are all unitary elements) and the integral inequality (3.36), then for all $s \in G$, we have the estimates:

$$\|(f_1 *_c f_2)(s)\|_A \leqslant \int_G \|f(g)\|_A \|f_2(g^{-1}s)\|_A d\mu_G(g), \tag{5.8}$$

$$\|f_1^{*_c}(s)\|_A \leqslant \|f_1(s^{-1})\|_A \Delta_G(s^{-1})$$

The inequalities above show that $f_1 *_c f_2$ and f^{*_c} are compactly supported. As an exercise, we shall show the involutive property; take $s \in G$ and $f \in C_c(G, A)$

$$\begin{aligned}
(f^{*_c})^{*_c}(s) &= c(s,s^{-1})^*\alpha_s\left(f^{*_c}(s^{-1})\right)\Delta_G(s^{-1}) \\
&= c(s,s^{-1})^*\alpha_s\left(\left[c(s^{-1},s)^*\alpha_{s^{-1}}\left(f(s)^*\right)\Delta_G(s)\right]^*\right)\Delta_G(s^{-1}) \\
&= c(s,s^{-1})^*\alpha_s(\alpha_{s^{-1}}(f(s))c(s^{-1},s))\Delta_G(s)\Delta_G(s^{-1}) \\
&= c(s,s^{-1})^*(\alpha_s \circ \alpha_{s^{-1}})(f(s))\alpha_s(c(s^{-1},s)) \\
&= c(s,s^{-1})^*c(s,s^{-1})f(s)c(s,s^{-1})^*\alpha_s(c(s^{-1},s)) \\
&= f(s)c(s,s^{-1})^*\alpha_s(c(s^{-1},s)) \\
&= f(s),
\end{aligned}$$

where the last equality follows from the 2-cocycle condition (5.2). It is also not hard to see that $s \mapsto f^{*_c}(s)$ must be continuous given the assumptions on f_1 and α, and c. Similarly, the map $(s,g) \mapsto f_1(g)\alpha_g(f_2(g^{-1}s))c(g,g^{-1}s)$ is continuous from $G \times G$ to A, and therefore it follows from Proposition 3.20 that $s \mapsto (f_1 *_c f_2)(s)$ is continuous. The associativity of $*_c$ follows from the 2-cocyle formula (5.2). Finally, by taking the integral of Estimate (5.8) it follows from the invariance of the Haar measure under translations that:

$$\|f_1 *_c f_2\|_1 \leqslant \|f_1\|_1\|f_2\|_1, \tag{5.9}$$

so we obtain submultiplicativity with respect to the $\|\cdot\|_1$-norm. We summarize all that we have discussed in the following proposition.

Proposition 5.1. *Equations (5.6) and (5.7) induce a $*$-algebra structure on $C_c(G, A)$, which we shall now formally denote by $C_c(G, A, \alpha, c)$. Furthermore, when equipped with the norm $\|\cdot\|_1$, $C_c(G, A; \alpha, c)$ becomes a normed $*$-algebra.*

Definition 5.4. We define the Banach $*$-algebra $L^1(G, A; \alpha, c)$ to be the completion of the normed $*$-algebra $C_c(G, A; \alpha, c)$ under the $\|\cdot\|_1$-norm.

Proposition 5.2. *Let (A, G, α, c) be a twisted C^*-dynamical system. If (H, ρ, U) is a covariant representation for (A, G, α, c), then it induces a $*$-representation on $L^1(G, A; \alpha, c)$ denoted by $\rho \rtimes_{\alpha,c} U : L^1(G, A; \alpha, c) \to \mathcal{L}(H)$, defined densely via:*

$$(\rho \rtimes_{\alpha,c} U)f = \int_G \rho(f(g))U_g d\mu_G(g), \qquad \forall f \in C_c(G, A), \tag{5.10}$$

*where the vector-integral above in (5.10) is made meaningful by Proposition 3.19. We call $\rho \rtimes_{\alpha,c} U$ the **integrated form** of (H, ρ, U). Furthermore, $\rho \rtimes_{\alpha,c} U$ is nondegenerate if (H, ρ, U) is.*

Remark 5.4. For each $f \in C_c(G, A)$, we note that the mapping $s \mapsto \rho(f(s))U_g$ belongs to $C_c(G, \mathcal{L}_s(H))$, which justifies the use of Proposition 3.19 in the definition of (5.10).

A proof of Proposition 5.2 would be a good exercise to see if everything is working as intended. We will only sketch the homomorphism property of $\rho \rtimes_{\alpha,c} U$, but one can cook up a detailed proof by mostly following [Wil07, Proposition 2.23], with obvious modifications to accomodate the twist c. A note of this can also be found in [MPR05, Lemma 5.2.5]. To begin the sketch, let $f_1, f_2 \in L^1(G, A; \alpha, c)$, then we have:

$$\begin{aligned}
(\rho \rtimes_{\alpha,c} U)(f_1 *_c f_2) &= \int_G \rho((f_1 *_c f_2)(s))U_s d\mu_G(s) \\
&= \int_G \rho\left(\int_G f_1(t)\alpha_t(f_2(t^{-1}s))c(t, t^{-1}s)d\mu_G(t)\right) U_s d\mu_G(s) \\
&= \int_G \int_G \rho(f_1(t))\rho(\alpha_t(f_2(t^{-1}s)))\rho(c(t, t^{-1}s))U_s d\mu_G(t)d\mu_G(s) \\
&= \int_G \int_G \rho(f_1(t)) \left[U_t\rho(f_2(t^{-1}s))U_t^*\right] \left[U_t U_{t^{-1}s} U_s^*\right] U_s d\mu_G(t)d\mu_G(s) \\
&= \int_G \int_G \rho(f_1(t))U_t\rho(f_2(t^{-1}s))U_{t^{-1}s} d\mu_G(s)d\mu_G(t) \\
&= \int_G \int_G \rho(f_1(t))U_t\rho(f_2(s))U_s d\mu_G(s)d\mu_G(t) \\
&= \int_G (\rho(f_1(t)))U_t d\mu_G(t) \circ \int_G \rho(f_2(s))U_s d\mu_G(s) \\
&= (\rho \rtimes_{\alpha,c} U)f_1 \circ (\rho \rtimes_{\alpha,c} U)f_2.
\end{aligned}$$

Note that we have used various concepts from the theory of vector-valued integration (c.f. Section 3.8 of Chapter 3) for the derivation above, along with formulae (5.4), and (5.5).

We can now give $L^1(G, A, \alpha, c)$ a candidate C^*-norm, which we denote simply by $\|\cdot\|$, which we define below:

$$\|f\| := \sup \|(\rho \rtimes_{\alpha,c} U)f\|, \tag{5.11}$$

where the supremum runs over all nondegenerate covariant representation (H, ρ, U) for (A, G, α, c). It is not immediate that the supremum above even exists since the class of all nondegenerate $*$-representations ρ on A is not a set but a proper class (the unitary 'projective' representations U on G satisfying (5.4) run into the same problem). A way to fix this is outlined in [Phi17], where instead of the collection of all nondegenerate covariant representations (H, ρ, U), we only consider a single fixed Hilbert space H_0 satisfying $\mathrm{card}(H_0) \leqslant \mathrm{card}(G)\,\mathrm{card}(A) =: \mathrm{card}(G \times A)$, in which case the set of all nondegenerate covariant representations on H_0 becomes a set. Afterwards, one should show that any nondegenerate covariant representation of (A, G, α, c) is always 'unitarily equivalent' to some nondegenerate covariant representation (H_0, ρ_0, U_0). Once we have gotten over this set-theoretic hurdle, we can now ask if $\|\cdot\|$ is an actual C^*-norm on $L^1(G, A; \alpha, c)$. Since each $\rho \rtimes_{\alpha,c} U$ is a nondegenerate $*$-representation on $L^1(G, A; \alpha, c)$, it is easy to see that $\|\cdot\|$ satisfies all requirements for a C^*-norm except possibly positive-definiteness.

For each $g \in G$, we introduce the map $i_G(g) : L^1(G, A; \alpha, c) \to L^1(G, A; \alpha, c)$ via

$$(i_G(r)f)(s) := \alpha_r(f(r^{-1}s))c(r, r^{-1}s) \tag{5.12}$$

We now have for any twisted covariant representation (H, ρ, U) of (A, G, α, c) that:

$$\begin{aligned}
(\rho \rtimes_{\alpha,c} U)(i_G(r)f) &= \int_G \rho(i_G(r)f(s))U_s d\mu_G(s) \\
&= \int_G \rho(\alpha_r(f(r^{-1}s))c(r, r^{-1}s))U_s d\mu_G(s) \\
&= \int_G \rho(\alpha_r(f(r^{-1}s)))\rho(c(r, r^{-1}s))U_s d\mu_G(s) \\
&= \int_G \rho(\alpha_r f(s))\rho(c(r, s))U_{rs} d\mu_G(s) \\
&= \int_G (U_r \circ \rho(f(s)) \circ U_r^*) \circ U_r \circ U_s d\mu_G(s) \\
&= U_r \left(\int_G \rho(f(s))U_s d\mu_G(s) \right) \\
&= (U_r \circ (\rho \rtimes_{\alpha,c} U))f
\end{aligned}$$

From the computation above, we can conclude that:

$$\|(\rho \rtimes_{\alpha,c} U)(i_G(r)f)\| = \|(U_r \circ (\rho \rtimes_{\alpha,c} U))f\| = \|(\rho \rtimes_{\alpha,c} U)f\| \tag{5.13}$$

Note as well that $((i_G(g_1) \circ i_G(g_2))f)(s) = c(g_1, g_2)(i_G(g_1g_2)f)(s)$ for all $g_1, g_2, s \in G$.

Lemma 5.1. *Let (A, G, α, c) be a twisted C^*-dynamical system. Suppose $\rho : A \to \mathcal{L}(H_\rho)$ is a faithful $*$-representation of A. Then if $(\widetilde{H}_\rho, \widetilde{\rho}, \widetilde{U})$ is the corresponding regular covariant representation (c.f. Example 5.2), then the integrated form $\widetilde{\rho} \rtimes_{\alpha,c} \widetilde{U} : L^1(G, A; \alpha, c) \to \mathcal{L}(\widetilde{H}_\rho)$ is a faithful $*$-representation of $L^1(G, A; \alpha, c)$.*

Proof. We want to show that if $f \in L^1(G, A; \alpha, c)$ s.t. $f \neq 0$, then $(\tilde{\rho} \rtimes_{\alpha,c} \tilde{U})f \neq 0$. Suppose that $f \neq 0$, then there exists $x \in G$ such that $f(x) \neq 0$ (with x contained in a set of positive measure). It follows from Equation (5.13) and the sentence after it, that by replacing f with the (integrable) map $s \mapsto (i_G(x^{-1})f)(s)$, we may assume WLOG that $x = e$, so that $f(e) \neq 0$. By assumption ρ is faithful, meaning $\rho(f(e)) \neq 0$, so there must exist $h_1, h_2 \in \mathcal{H}_\rho$ such that $\langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \neq 0$. By continuity of the map $(r, s) \mapsto \langle \rho(\alpha_r(f(s))c(s, r))h_1, h_2 \rangle_{H_\rho}$, we can find an open neighborhood $V \ni e$ such that for $s, r \in V$:

$$|\langle \rho(\alpha_r(f(s))c(s, r))h_1, h_2 \rangle_{H_\rho} - \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho}| < \frac{|\langle \rho(f(e))h_1, h_2 \rangle_{H_\rho}|}{3}.$$

Construct $\varphi \in C_c(G)$ to be a positive function such that its support is contained in a smaller open set $W \subseteq V \ni e$ where $W \cdot W \subseteq V$, satisfying $W^{-1} = W$, $W \cdot W \subseteq V$ and

$$\int_G \int_G \varphi(s)\varphi(sr) d\mu_G(r) d\mu_G(s) = 1, \qquad \forall r \in W.$$

Now if we define $\xi_1 \in \tilde{H}_\rho := L^2(G, H_\rho)$, $\xi_2 \in \tilde{H}_\rho := L^2(G, H_\rho)$ via $\xi_1(s) = \varphi(s)h_1$ and $\xi_2(s) = \varphi(s)h_2$ for all $s \in G$, then we have the following estimate:

$$\begin{aligned}
&\left| \left\langle ((\tilde{\rho} \rtimes_{\alpha,c} \tilde{U})f)\xi_1, \xi_2 \right\rangle_{\tilde{H}_\rho} - \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \right| \\
&= \left| \int_G \int_G \langle \rho(\alpha_r(f(r)c(s, r))\xi_1(s), \xi_2(sr) \rangle_{H_\rho} d\mu_G(r) d\mu_G(s) - \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \right| \\
&\leqslant \int_G \int_G \varphi(s)\varphi(sr) \left| \langle \rho(\alpha_r(f(r)c(s, r))h_1, h_2 \rangle_{H_\rho} - \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \right| d\mu_G(r) d\mu_G(s) \\
&< \frac{|\langle \rho(f(e))h_1, h_2 \rangle_{H_\rho}|}{3}.
\end{aligned}$$

Therefore we have shown that $\left\langle ((\tilde{\rho} \rtimes_{\alpha,c} \tilde{U})f)\xi_1, \xi_2 \right\rangle_{\tilde{H}_\rho} \neq 0$, since otherwise we'll have

$$\left| \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \right| < \frac{1}{3} \left| \langle \rho(f(e))h_1, h_2 \rangle_{H_\rho} \right|$$

which cannot happen since $1 \nless \frac{1}{3}$. What we have shown implies that $(\tilde{\rho} \rtimes_{\alpha,c} \tilde{U})f \neq 0$, which is what we want. □

Corollary 5.1. *$\|\cdot\|$ is a C^*-norm on $L^1(G, A; \alpha, c)$ dominated by the $\|\cdot\|_1$-norm*

Proof. Let $f \in L^1(G, A; \alpha, c)$, since each integrated form $\rho \rtimes_{\alpha,c} U : L^1(G, A; \alpha, c) \to \mathcal{L}(H)$ of a nondegenerate covariant representation (H, ρ, U) is an actual $*$-representation on $L^1(A, G, \alpha, c)$ as a Banach $*$-algebra, then they are contractive, meaning $\|(\rho \rtimes_{\alpha,c} U)f\| \leqslant \|f\|_1$, which implies:

$$\|f\| = \sup \|(\rho \rtimes_{\alpha,c} U)f\| \leqslant \|f\|_1.$$

The inequality above not only shows that $\|\cdot\|_1$ dominates $\|\cdot\|$, but that $\|\cdot\|$ is well-defined in $L^1(G, A, \alpha, c)$.

As discussed, we will only further prove positive-definiteness of $\|\cdot\|$. As a corollary of GNS-theorem, there exists a nondegenerate faithful $*$-representation $\rho : A \to \mathcal{L}(H_\rho)$, consequently we can construct the regular covariant representation $(\widetilde{H}_\rho, \widetilde{\rho}, \widetilde{U})$. We know from Lemma 5.1 that $\widetilde{\rho} \rtimes_{\alpha,c} \widetilde{U} : L^1(G, A; \alpha, c) \to \mathcal{L}(\widetilde{H_\rho})$ is a nondegenerate faithful $*$-representation. Now if we suppose that $\|f\| = 0$, then it follows that:

$$\|(\widetilde{\rho} \rtimes_{\alpha,c} \widetilde{U})f\| \leqslant \|f\| = 0,$$

therefore $(\widetilde{\rho} \rtimes_{\alpha,c} \widetilde{U})f = 0$, which implies $f = 0$, proving that $\|\cdot\|$ is positive-definite. $\square$

Definition 5.5. The **twisted crossed-product** induced by (A, G, α, c) is the C^*-algebra denoted by $A \rtimes_{\alpha,c} G$, obtained by completing $L^1(G, A; \alpha, c)$ with respect to the C^*-norm $\|\cdot\|$.

It would be interesting to see how twisted crossed-products can be constructed in terms of universal C^*-algebras. A relevant result, which we have alluded to earlier, is given below without proof.

Theorem 5.1. *Consider the twisted C^*-dynamical system (A, G, α, c). Every nondegenerate $*$-representation on $A \rtimes_{\alpha,c} G$ is an extension of an integrated form $\rho \rtimes_{\alpha,c} U : L^1(G, A, \alpha, c) \to \mathcal{L}(H)$ to $A \rtimes_{\alpha,c} G$ of some nondegenerate covariant representation (H, ρ, U) of (A, G, α, c). More precisely, the mapping $(H, \rho, U) \mapsto \rho \rtimes_{\alpha,c} U$ defines an irreducibility preserving one-to-one correspondence between nondegenerate covariant representations of (A, G, α, c) to nondegenerate $*$-representations of $A \rtimes_{\alpha,c} G$.*

A proof can be deduced by obvious modifications on the development shown in [Wil07, Proposition 2.40], using nondegenerate homomorphisms (c.f. Definition 2.16). This same result was also noted in [MPR05]. In fact, the map i_G that we have introduced in (5.12) densely defines a map $i_G : G \to UM(A \rtimes_{\alpha,c} G)$, along with $i_A : A \to M(A \rtimes_{\alpha,c} G)$ via

$$(i_A(a)f)(s) = af(s), \qquad \forall a \in A, f \in C_c(G, A), s \in G. \tag{5.14}$$

These two define a particular **twisted covariant homomorphism** for (A, G, α, c). That is, they satisfy the following:

1. $i_G : G \to UM(A \rtimes_{\alpha,c} G)$ is a strongly continuous map;

2. $i_A : A \to M(A \rtimes_{\alpha,c} G)$ is a $*$-homomorphism in the sense of Definition 2.16;

3. $i_G(s) \circ i_G(r) = i_A(c(s, r))i_G(sr)$ for all $s, r \in G$;

4. $i_A(\alpha_s(a)) = i_G(s) \circ \rho(a) \circ i_G(s)^*$, for all $s \in G$.

An interesting corollary of Theorem 5.1, in light of universal C^*-constructions, is the following.

Corollary 5.2. *$L^1(G, A; \alpha, c)$ is a $*$-semisimple Banach $*$-algebra, and its enveloping C^*-algebra is the twisted crossed-product $A \rtimes_{\alpha,c} G$.*

Proof. It follows from Proposition A.1 and Theorem 5.1 that the universal-norm $\|\cdot\|_u$ is equal to $\|\cdot\|$. Therefore the reducing ideal I_u of $L^1(G, A; \alpha, c)$ is given by:

$$I_u = \{f \in L^1(G, A; \alpha, c) : \|f\| = 0\} = \{0\},$$

because $\|\cdot\|$ is a norm on $L^1(G, A; \alpha, c)$. Therefore by definition $L^1(G, A; \alpha, c)$ is $*$-semisimple. Since $L^1(G, A; \alpha, c)$ is $*$-semisimple, its enveloping C^*-algebra is simply its completion under the universal C^*-norm, which we know is equivalent to $\|\cdot\|$, therefore this completion is the same as $A \rtimes_{\alpha,c} G$. ∎

5.2 Twisted Group C^*-Algebras

We now reap the benefits of our efforts in describing twisted crossed products. For our applications, we only really need to focus on the particular case $A = \mathbb{C}$.

Let us start with the untwisted case. As we have discussed in Remark 5.3, every C^*-dynamical system is a twisted C^*-dynamical system (A, G, α, c) with trivial 2-cocycle which we denote by 1, i.e. $1 \equiv 1_{M(A)}$. Let us now further particularize and also take $A = \mathbb{C}$, then this will also restrict our choice of the group action $\alpha : G \to \mathrm{Aut}(\mathbb{C})$. Take any $s \in G$, then $\alpha_s : \mathbb{C} \to \mathbb{C}$ should be an automorphism on C^*-algebras[1], and so for any $\mu \in \mathbb{C}$ we obtain $\alpha_s(\mu) = \mu\alpha_s(1_{\mathbb{C}}) = \mu$. We see that if $A = \mathbb{C}$, then the group action α is also trivial, and we shall rename it with $\mathrm{Id} = \alpha$. In this case, if $f_1, f_2 \in L^1(G, \mathbb{C}, \mathrm{Id}, 1)$, then

$$\begin{aligned} (f_1 *_c f_2)(s) &= \int_G f_1(g) f_2(g^{-1}s) d\mu_G(g), \\ f_1^{*_c}(s) &= \Delta_G(s^{-1})\overline{f_1(s^{-1})}. \end{aligned} \tag{5.15}$$

This is nothing but the usual Banach $*$-algebra structure for $L^1(G)$ with G a locally compact (not necessarily abelian) group. We reiterate that in our current setting, $L^1(G, \mathbb{C}; \mathrm{Id}, 1) = L^1(G)$ as a Banach $*$-algebra.

Definition 5.6. The resulting twisted crossed-product $\mathbb{C} \rtimes_{\mathrm{Id},1} G$ is defined to be the **group C*-algebra** associated to G, denoted by $C^*(G)$.

There is more to be said, especially pertaining to the representation theory of $C^*(G)$. In our setting, the group action and the 2-cocycle are both trivial, and in a sense, even the covariant representations on $(\mathbb{C}, G, \mathrm{Id}, 1)$ will be trivial. To see this, let $\rho : \mathbb{C} \to \mathcal{L}(H)$ be a $*$-representation, then it follows that $\rho(\mu) = \mu \, \mathrm{Id}_H$ where $\mathrm{Id}_H \in \mathcal{L}(H)$ is the identity operator. This is true for all $*$-representations of $\mathbb{C}$. We pair this with the fact that in the untwisted case $U : G \to U\mathcal{L}(H)$ satisfying Equation (5.4) just gives us a representation for G. Hence there is a one-to-one correspondence between covariant representations of $(G, \mathbb{C}, \mathrm{Id}, c)$ and representations of G. It then follows from Theorem 5.1 that there is a one-to-one correspondence between representations of G and $*$-representations of $C^*(G) = \mathbb{C} \rtimes_{\mathrm{Id},1} G$, where for each unitary representation $U : G \to U\mathcal{L}(H)$, the corresponding $*$-representation is densely defined on $L^1(G)$ via:

$$L^1(G) \ni f \mapsto \int_G f(g) U_g d\mu_G(g) \in \mathcal{L}(H).$$

As a final note in the untwisted case, we further consider the case when G is abelian. We then have that G must be unimodular: $\Delta_G \equiv 1$. The structure of $L^1(G)$ outlined in Equations (5.15) reduce to Equations (3.5). Our result regarding the correspondence between representations of G and that of $L^1(G)$ in the paragraph after the statement of Gelfand-Raikov Theorem (c.f. 3.9) is consistent with Theorem 5.1: the irreducible representations of G (i.e. characters) correspond

[1] All we really need here is that α_s is an algebra homomorphism.

to the irreducible $*$-representations of $C^*(G)$ or $L^1(G)$. It is also interesting to note that the Fourier-transform $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ densely extends to an isometric $*$-isomorphism $\mathcal{F} : C^*(G) \to C_0(\widehat{G})$ due to Gelfand's theorem 3.4 and Theorem 3.5.

As we have seen, even the untwisted case above welcomes us into the wonderful rabbit hole that leads to the connections between the theory of locally compact groups and C^*-algebras. We resist the temptation of going down this rabbit hole.

We finally study our main object of interest, we take $A = \mathbb{C}$ still, and this again forces us to work with the trivial group action $\alpha = \mathrm{Id} : G \to \mathrm{Aut}(\mathbb{C})$. $\mathbb{C}$ is a unital C^*-algebra, and its unitary group is $\mathbb{T}$, therefore $UM(A) = U(A) = U(\mathbb{C}) = \mathbb{T}$. We take $c : G \times G \to UM(A) = \mathbb{T}$ to be a map such that (Id, c) is a twisted action of G on $\mathbb{C}$ (so c is not necessarily trivial anymore). In particular it follows from Definition 5.2, that $c : G \times G \to \mathbb{T}$ is a continuous continuous map that satisfies the following formulae for all $s, t, g \in G$:

$$\begin{aligned} c(g,t)c(gt,s) &= c(t,s)c(g,ts); \\ c(s,e) &= 1 = c(e,s). \end{aligned} \tag{5.16}$$

Definition 5.7. Let G be a locally compact group and $c : G \times G \to \mathbb{T}$ be a continuous map satisfying 5.16, then $(\mathbb{C}, G, \mathrm{Id}, c)$ is a twisted C^*-dynamical system. The associated twisted crossed-product $\mathbb{C} \rtimes_{\mathrm{Id},c} G$ is called the **c-twisted group C*-algebra** denoted by $C^*(G, c)$ (or just twisted group C^*-algebras when the reference to the 2-cocycle is not relevant).

Remark 5.5. Every group C^*-algebra is a twisted group C^*-algebra, as taking $c \equiv 1$ gives us $C^*(G) = C^*(G, c)$.

It follows from our discussion of the the twisted crossed-products that we can obtain $C^*(G, c)$ by completing the dense $*$-subalgebra $L^1(G, c) := L^1(G, \mathbb{C}; \mathrm{Id}, c)$ under the $\|\cdot\|$-norm (5.11). We also note that $L^1(G, c)$ is the same as $L^1(G)$ equipped with the twisted-convolution and twisted-involution of Equations (5.6) and (5.7), where:

$$\begin{aligned} (f_1 *_c f_2)(s) &= \int_G f_1(g) f_2(g^{-1}s) c(g, g^{-1}s) d\mu_G(g) \\ f_1^{*_c}(s) &= \overline{c(s, s^{-1}) f(s^{-1})} \Delta_G(s^{-1}) \end{aligned} \tag{5.17}$$

for $f_1, f_2 \in L^1(G, c)$. Similar to the untwisted case, because we are working with $A = \mathbb{C}$, the only $*$-homomorphism of $\mathbb{C}$ is the trivial map: $\mathrm{Id}_H : \mathbb{C} \to \mathcal{L}(H)$, for each Hilbert space H. Therefore, all twisted covariant representations of $(G, \mathbb{C}, \mathrm{Id}, c)$ are uniquely determined by those continuous maps $U : G \to U\mathcal{L}(H)$ which satisfy the following version of Equation (5.4):

$$U_sU_t = c(s,t)U_{st}, \qquad \forall s,t \in G. \tag{5.18}$$

It follows from Theorem 5.1 that all $*$-representations of $C^*(G,c)$ are of the form $\mathrm{Id}_H \rtimes_{\mathrm{Id},c} U$. Instead of the unwieldy notation $\mathrm{Id}_H \rtimes_{\mathrm{Id},c} U$ for the integrated form, we will use the following notation moving forward:

$$\overline{U}f := (\mathrm{Id}_H \rtimes_{\mathrm{Id},c} U)f = \int_G f(g)U_g d\mu_G(g) \tag{5.19}$$

densely defined on $f \in L^1(G,c)$. Note that continuous maps satisfying $U : G \to \mathcal{L}(H)$ Equation (5.18) are called also called **c-projective** unitary representations of G.

The following result will be useful in our construction of the Heisenberg module, which follows as a consequence of Corollary 5.2.

Corollary 5.3. *The Banach $*$-algebra $L^1(G,c)$ is a dense $*$-subalgebra of the twisted group C^*-algebra $C^*(G,c)$ under the universal C^*-norm $\|\cdot\|$ densely defined via:*

$$\|f\| = \sup \|\overline{U}f\|, \qquad \forall f \in L^1(G,c) \tag{5.20}$$

where the supremum runs through all continuous maps $U : G \to U\mathcal{L}(H)$ satisfying (5.18).

Proof. Everything follows from Proposition 5.2. All that remains is to check that $\overline{U}$, for each strongly continuous c-projective unitary representations of $U : G \to \mathcal{L}(G)$, defines a *nondegenerate* $*$-representation of $C^*(G,c)$ so that (5.20) does define the universal C^*-norm on $C^*(G,c)$. However, this is trivial in light of Theorem 5.1. Since the map $\mathrm{Id}_H : \mathbb{C} \to \mathcal{L}(H)$ is nondegenerate, then it follows that the integrated form $\overline{U} = \mathrm{Id}_H \rtimes_{\mathrm{Id},c} U$ is also a nondegenerate $*$-representation on $C^*(G,c)$. □

5.3 Construction of the Heisenberg Modules

We now have the necessary prerequisites to finally construct the main object of this dissertation, the Heisenberg modules. Our first assumption, which we shall hold for the rest of this section, is that G is a second countable LCA group. We take $\Lambda \subseteq G \times \widehat{G}$ to be a lattice (see Definition 4.1) in the time-frequency plane $G \times \widehat{G}$. The Heisenberg module associated with Λ, which we shall denote by $\mathcal{E}_\Lambda(G)$, will be an imprimitivity bimodule over certain twisted group C^*-algebras. These twisted C^*-algebras will come from the the Heisenberg 2-cocycle $c : (G \times \widehat{G}) \times (G \times \widehat{G}) \to \mathbb{T}$ as defined in Equation (4.7). Technically, what we need is the restriction of the Heisenberg 2-cocycle on $\Lambda\times\Lambda$, but we shall still denote it via c. We obtain the twisted C^*-dynamical system $(\mathbb{C}, \Lambda, \mathrm{Id}, c)$ since $c : \Lambda \times \Lambda \to \mathbb{T}$ is normalized, and satisfies $c(\lambda_1, \lambda_2)c(\lambda_1 + \lambda_2, \lambda_3) = c(\lambda_1, \lambda_2 + \lambda_3)c(\lambda_2, \lambda_3)$ for $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, which is the commutative version of Equations (5.16). Since Λ is countable, we equip it with the counting measure and define $\ell^1(\Lambda, c) := L^1(\Lambda, c)$, which is nothing but $\ell^1(\Lambda)$ as a set. The Banach $*$-algebra structure of $\ell^1(\Lambda, c)$ is given below, which can be deduced from Equations (5.17):

$$\begin{aligned} \mathbf{a}_1 *_c \mathbf{a}_2(\lambda) &= \sum_{\mu\in\Lambda} a_1(\mu)a_2(\lambda - \mu)c(\mu, \lambda - \mu); \\ \mathbf{a}_1^{*_c}(\lambda) &= \overline{c(\lambda, -\lambda)a_1(-\lambda)}, \end{aligned} \tag{5.21}$$

for all $\lambda \in \Lambda$ and $\mathbf{a}_1, \mathbf{a}_2 \in \ell^1(\Lambda, c)$. Due to Corollary 5.3, we can now simply take the completion of $\ell^1(\Lambda, c)$ in the universal C^*-norm and obtain $C^*(\Lambda, c)$, which as we shall see later, will be the C^*-algebra giving $\mathcal{E}_\Lambda(G)$ a Hilbert left C^*-module structure.

On the other hand, since Λ is a lattice, it follows that its adjoint lattice $\Lambda^\circ \subseteq G \times \widehat{G}$ is also countable (c.f. Proposition 4.4) and that the lattice size $d(\Lambda)$ is finite. Furthermore the complex-conjugate of the Heisenberg 2-cocycle denoted by $\bar{c} : (G \times \widehat{G}) \times (G \times \widehat{G}) \to \mathbb{T}$ defined by

$$\bar{c}((x, \omega), (t, \omega)) = \overline{c((t, \nu), (x, \omega))}, \qquad \forall (x, \omega), (t, \nu) \in G \times \widehat{G} \tag{5.22}$$

satisfies the 2-cocycle formulae (5.16). Similar to the previous discussion, we denote still by $\bar{c}$ the restriction of $\bar{c}$ on $\Lambda^\circ \times \Lambda^\circ$. The quadruple $(G, \Lambda^\circ, \mathrm{Id}, \bar{c})$ defines a twisted C^*-dynamical system. We now choose to equip Λ° with the counting measure scaled by $\frac{1}{d(\Lambda)}$ (the reason is outlined by the paragraph preceding Proposition 4.5). Consider the Banach $*$-algebra $\ell^1(\Lambda^\circ, \bar{c})$ which has the following twisted structure:

$$\begin{aligned} (\mathbf{b}_1 *_{\bar{c}} \mathbf{b}_2)(\lambda^\circ) &= \frac{1}{d(\Lambda)} \sum_{\mu^\circ\in\Lambda^\circ} b_1(\mu^\circ)b_2(\lambda^\circ - \mu^\circ)\bar{c}(\mu^\circ, \lambda^\circ - \mu^\circ); \\ \mathbf{b}_1^{*_{\bar{c}}}(\lambda^\circ) &= \overline{\bar{c}(\lambda^\circ, -\lambda^\circ)b_1(-\lambda^\circ)}, \end{aligned} \tag{5.23}$$

for all λ° and $\mathbf{b}_1, \mathbf{b}_2 \in \ell^1(\Lambda^\circ, \bar{c})$. Similar to $C^*(\Lambda, c)$, the twisted group algebra $C^*(\Lambda^\circ, \bar{c})$ is then the completion of the Banach $*$-algebra $\ell^1(\Lambda^\circ, \bar{c})$ due to Corollary 5.3.

At the risk of excessive repetition, we shall summarize our findings with the following propositions, which follows Corollary 5.3.

Proposition 5.3. *The Banach $*$-algebra $A_0 := \ell^1(\Lambda, c)$ with structure given by Equations (5.21) is a dense $*$-subalgebra of $A_\Lambda := C^*(\Lambda, c)$ under the universal C^*-norm denoted by $\|\cdot\|_{A_\Lambda}$ densely defined via*

$$\|\mathbf{a}\|_{A_\Lambda} := \sup \|\overline{U}\mathbf{a}\| = \sup \left\| \sum_{\mu \in \Lambda} a(\mu) U_\mu \right\|, \qquad \forall \mathbf{a} \in \ell^1(\Lambda, c) \tag{5.24}$$

where the supremum runs over all c-projective maps $U : \Lambda \to U\mathcal{L}(H)$.

Proposition 5.4. *The Banach $*$-algebra $B_0 := \ell^1(\Lambda^\circ, \overline{c})$ with structure given by Equations (5.23) is a dense $*$-subalgebra of $B_\Lambda := C^*(\Lambda^\circ, \overline{c})$ under the universal C^*-norm denoted by $\|\cdot\|_{B_\Lambda}$ densely defined via*

$$\|\mathbf{b}\|_{B_\Lambda} := \sup \|\overline{W}\mathbf{b}\| = \sup \left\| \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\lambda^\circ) W_{\mu^\circ} \right\|, \qquad \forall \mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c}) \tag{5.25}$$

where the supremum runs over all $\overline{c}$-projective maps $W : \Lambda^\circ \to U\mathcal{L}(H)$.

Note that we will keep using the names A_0, A_Λ, B_0, and B_Λ throughout the rest of the chapter.

We may want to further investigate the C^*-norm given by Equations (5.24) and (5.25). For A_Λ, this requires in principle, figuring out all the c-projective representations for Λ. There is, however, a privileged c-projective representation that appears quite ubiquitously in time-frequency analysis. This is none other than the time-frequency shift operator $\pi : G \times \widehat{G} \to U\mathcal{L}(L^2(G))$ satisfying the c-projectivity as what can be seen from Equation (4.8). If we consider the restriction of π on Λ now denoted by π_Λ, then we obtain from (4.8) that:

$$\pi_\Lambda(\lambda)\pi_\Lambda(\mu) = c(\lambda, \mu)\pi_\Lambda(\lambda + \mu), \qquad \forall \lambda, \mu \in \Lambda. \tag{5.26}$$

So the time-frequency shift π_Λ is eligible to be transformed to an integrated form (c.f. (5.19)) $\overline{\pi}_\Lambda : A_\Lambda \to \mathcal{L}(L^2(G))$ densely defined via:

$$\overline{\pi}_\Lambda(\mathbf{a}) = \sum_{\mu \in \Lambda} a(\lambda)\pi(\lambda), \qquad \forall \mathbf{a} \in \ell^1(\Lambda, c). \tag{5.27}$$

We have mentioned that π is a privileged projective representation on Λ, but in what sense? The following result, shown by Rieffel in [Rie88] answers this question.

Theorem 5.2. *The integrated form $\overline{\pi}_\Lambda : A_\Lambda \to \mathcal{L}(L^2(G))$ defines a faithful $*$-representation of A_Λ. In particular, it is an isometry, therefore $\|\mathbf{a}\|_{A_\Lambda} = \|\overline{\pi}_\Lambda(\mathbf{a})\|$ for all $\mathbf{a} \in A_\Lambda$.*

The question for an analogous $*$-representation for for B_Λ is, for our purposes, slightly subtler. What we would really like is a faithful map $B_\Lambda \to \mathcal{L}(L^2(G))$ that is a $*$-antihomomorphism (i.e. it does not preserve the multiplication, but switches their order instead) instead of the usual $*$-homomorphism. The reason roughly, is that we want B_Λ to act to the right of the Heisenberg module through this map. To this end, our candidate is the integrated form of the adjoint of the time-frequency shift $\pi^* : G \times \widehat{G} \to U\mathcal{L}(L^2(G))$, where we denote its restriction to $\Lambda^\circ \times \Lambda^\circ$ via $\pi^*_{\Lambda^\circ}$. The integrated form $\overline{\pi}^*_{\Lambda^\circ} : B_\Lambda \to \mathcal{L}(L^2(G))$ is now densely defined via

$$\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}) = \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\pi^*(\mu^\circ), \qquad \forall \mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c}). \tag{5.28}$$

But we shall check, since we technically cannot rely on Theorem 5.1 anymore, that it is indeed a **$*$-antirepresentation**, meaning: it preserves involution, but $\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_1 *_c \mathbf{b}_2) = \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_2) \circ \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_1)$ for all $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{L}(L^2(G))$. That $\overline{\pi}^*_{\Lambda^\circ}$ preserves the involution on B_Λ follows from Equation (4.9), so we will only check for the antihomomorphism property:

$$\begin{aligned}
\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_1 *_{\overline{c}} \mathbf{b}_2) &= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} (\mathbf{b}_1 *_{\overline{c}} \mathbf{b}_2)(\mu^\circ)\pi^*(\mu^\circ) \\
&= \frac{1}{d(\Lambda)^2} \sum_{\mu \in \Lambda^\circ} \left(\sum_{\lambda^\circ \in \Lambda^\circ} b_1(\lambda^\circ) b_2(\mu^\circ - \lambda^\circ)\overline{c}(\lambda^\circ, \mu^\circ - \lambda^\circ) \right) \pi^*(\mu^\circ) \\
&= \frac{1}{d(\Lambda)^2} \sum_{\mu^\circ, \lambda^\circ \in \Lambda^\circ} b_1(\mu^\circ - \lambda^\circ) b_2(\lambda^\circ)\overline{c}(\mu^\circ - \lambda^\circ, \lambda^\circ)\pi^*(\mu^\circ) \\
&= \frac{1}{d(\Lambda)^2} \sum_{\mu^\circ, \lambda^\circ \in \Lambda^\circ} b_1(\mu^\circ - \lambda^\circ) b_2(\lambda^\circ)\pi^*(\lambda^\circ)\pi^*(\mu^\circ - \lambda^\circ) \\
&= \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} b_2(\lambda^\circ)\pi^*(\lambda^\circ) \circ \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b_1(\mu^\circ)\pi^*(\mu) \\
&= \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_2) \circ \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_1).
\end{aligned}$$

The crucial step in the calculation above occured from the 3rd to 4th equations, where we used the adjoint of Equation (4.8).

Even though $\overline{\pi}^*_{\Lambda^\circ}$ does not exactly fit[2] into the theory of twisted crossed products as we have

[2] I surmise that there must be an equivalent twisted C^*-dynamical systems theory from which its has its

developed in the previous section, it still has an analogue of Theorem 5.2. We mention that the methods employed by Gröchenig and Leinert in [GL04] can be adapted for a more time-frequency analytic proof of both Theorems 5.2 and 5.3.

Theorem 5.3. *The integrated form $\overline{\pi}^*_{\Lambda^\circ} : B_\Lambda \to \mathcal{L}(L^2(G))$ defines a faithful $*$-antirepresentation of B_Λ. In particular, it is an isometry, therefore $\|\mathbf{b}\|_{B_\Lambda} = \|\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})\|$ for all $\mathbf{b} \in B_\Lambda$.*

Remark 5.6. The correct projective representation property for π^* can be seen when we take the adjoint of both sides of Equation (4.8):

$$\begin{aligned}\pi^*(\lambda^\circ)\pi^*(\mu^\circ) &= \overline{c(\mu^\circ, \lambda^\circ)}\pi^*(\lambda^\circ + \mu^\circ) \\ &= \overline{c}(\lambda^\circ, \mu^\circ)\pi^*(\lambda^\circ + \mu^\circ).\end{aligned} \tag{5.29}$$

If we define $c^\circ(\lambda^\circ, \mu^\circ) = \overline{c}(\mu^\circ, \lambda^\circ)$, then the integrated form $\overline{\pi}^*_{\Lambda^\circ}$ becomes a faithful $*$-*representation* for $C^*(\Lambda^\circ, c^\circ)$. This has been discussed in [Rie88, Lue07], where $C^*(\Lambda^\circ, c^\circ)$ has been rightly assessed to be the **opposite**[3] C^*-algebra (and vice-versa) for $C^*(\Lambda, \overline{c})$.

We are now ready[4] to construct the Heisenberg module associated with the lattice Λ. Our strategy here is to exploit the density of $\ell^1(\Lambda, c)$ and $\ell^1(\Lambda^\circ, \overline{c})$ on A_Λ and B_Λ respectively to scaffold the assembly of the Heisenberg module by first defining an $\ell^1(\Lambda, c) - \ell^1(\Lambda^\circ, \overline{c})$-pre-imprimitivity bimodule (c.f. Definition 2.21). Once this is done, then we are essentially finished since the Heisenberg module will be the completion of the said $\ell^1(\Lambda, c) - \ell^1(\Lambda^\circ, \overline{c})$-pre-imprimitivity bimodule in the sense of Corollary 2.6. The question now then, is from which space could we possibly construct this pre-imprimitivity bimodule? It turns out that we can proceed using the Feichtinger's algebra $\mathbf{S}_0(G)$– another object quite ubiquitous in time-frequency analysis. In the following, we shall focus on the *right*[5] $\ell^1(\Lambda^\circ, \overline{c})$ (semi-)inner product structure of $\mathbf{S}_0(G)$; the details for the *left* $\ell^1(\Lambda, c)$ (semi-)inner product structure of $\mathbf{S}_0(G)$ can be obtained similarly. The proof below takes inspiration from the original paper of Luef [Lue07], especially the demonstration of positivity of the module inner-product.

own version of anti-covariant (contravariant?) representations, from which their integrated forms become $*$-antirepresentations for the associated twisted crossed product, with its own version of Theorem 5.1. There certainly is a categorical flavor to this problem, and I hypothesize that everything that we've talked about can be stated in terms of opposite categories. But alas, this will be distracting us too much, and will only make this already large document more unwieldy.

[3] This amounts to $\mathbf{b}_1 *_{c^\circ} \mathbf{b}_2 = \mathbf{b}_2 *_{\overline{c}} \mathbf{b}_1$.

[4] Really!

[5] This is nothing but my personal preference, perhaps owing from the fact that the existing literature has mostly preferred to work on the left structure of the Heisenberg modules. I consider this to be an exercise that would fill-in the (negligible) gap in the literature.

Proposition 5.5. *$\mathbf{S}_0(G)$ is a is a semi-inner product right $\ell^1(\Lambda^\circ, \bar{c})$-module. For $\mathbf{b} \in \ell^1(\Lambda^\circ, \bar{c})$ and $f, f_1, f_2 \in \mathbf{S}_0(G)$:*

1. (Right module action). $f\mathbf{b} = \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\pi^*(\mu^\circ)f \in \mathbf{S}_0(G)$;
2. (Module Inner-Product). *We have* $\langle \cdot\,,\,\cdot \rangle_{\Lambda^\circ} : \mathbf{S}_0(G) \times \mathbf{S}_0(G) \to \ell^1(\Lambda^\circ, \bar{c})$ *via* $\langle f_1, f_2 \rangle_{\Lambda^\circ}(\lambda^\circ) := \langle f_2, \pi^*(\lambda^\circ) f_1 \rangle_2$.

Furthermore, an alternative way to write the right module action is through the integrated form (5.3):

$$\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})f = f\mathbf{b}. \tag{5.30}$$

On the other hand, we can also alternatively write the module inner-product in terms of sampled STFT:

$$\langle f_1, f_2 \rangle_{\Lambda^\circ} = \frac{1}{d(\Lambda)} R_{\Lambda^\circ}\left(\overline{\mathcal{V}_{f_2} f_1}\right) \tag{5.31}$$

Proof. We shall first show that the right module action is well-defined. Note that it follows from a straighforward application of Hölder's inequality that $f\mathbf{b} \in L^2(G)$ at least. Now, fix $g \in \mathbf{S}_0(G)$ and let us compute the following:

$$\begin{aligned}
\int_{G\times\widehat{G}} |\langle f\mathbf{b}, \pi(\chi)g\rangle_2| \, d\mu_{G\times\widehat{G}}(\chi) &\leqslant \int_{G\times\widehat{G}} \frac{1}{d(\Lambda)} \sum_{\mu^\circ\in\Lambda^\circ} |b(\mu^\circ)| |\langle f, \pi(\mu^\circ + \chi)g\rangle_2| \, d\mu_{G\times\widehat{G}}(\chi) \\
&= \frac{1}{d(\Lambda)} \sum_{\mu^\circ\in\Lambda^\circ} |b(\mu^\circ)| \int_{G\times\widehat{G}} \mathcal{V}_g f(\mu^\circ + \chi) d\mu_{G\times\widehat{G}}(\chi) \\
&= \frac{1}{d(\Lambda)} \sum_{\mu^\circ\in\Lambda^\circ} |b(\mu^\circ)| \int_{G\times\widehat{G}} \mathcal{V}_g f(\chi) d\mu_{G\times\widehat{G}}(\chi) \\
&= \|\mathbf{b}\|_1 \|f\|_{\mathbf{S}_{0,g}} < \infty.
\end{aligned}$$

This proves that $f\mathbf{b} \in \mathbf{S}_0(G)$. We shall also show that for $\mathbf{b}_1, \mathbf{b}_2 \in \ell^1(\Lambda^\circ, \bar{c})$, that $(f\mathbf{b}_1)\mathbf{b}_2 = f(\mathbf{b}_1 *_{\bar{c}} \mathbf{b}_2)$. Since we have proven that the right module action takes us to $\mathbf{S}_0(G)$, then we can use the integrated form (5.28) to write $f\mathbf{b} = \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})f$, and so it follows from the fact that $\overline{\pi}^*_{\Lambda^\circ}$ is a $*$-antirepresentation (c.f. Theorem 5.3):

$$(f\mathbf{b}_1)\mathbf{b}_2 = \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_2)(f\mathbf{b}_1) = (\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_1) \circ \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_2))f = \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}_2 *_{\bar{c}} \mathbf{b}_1)f = f(\mathbf{b}_1 *_{\bar{c}} \mathbf{b}_2).$$

Well-definedness of the module inner-product is trivial, and follows from the Remark 4.6 concerning the property of the STFT, and the property of the restriction map R_{Λ° (c.f. Proposition 4.9 Item 6) on $\mathbf{S}_0(G \times \widehat{G})$, and from $\mathbf{S}_0(\Lambda^\circ) = \ell^1(\Lambda^\circ)$.

We should also check the compatibility of the module inner-product with the module-action, i.e. check for Item 2 of Definition 2.18. We compute:

$$
\begin{aligned}
(\langle f_1, f_2\rangle_{\Lambda^\circ} *_{\overline{c}} \mathbf{b})(\lambda^\circ) &= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} \langle f_2, \pi^*(\mu^\circ) f_1\rangle_2\, b(\lambda^\circ - \mu^\circ)\overline{c}(\mu^\circ, \lambda^\circ - \mu^\circ) \\
&= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\lambda^\circ - \mu^\circ)\overline{c}(\mu^\circ, \lambda^\circ - \mu^\circ)\pi(\mu^\circ) f_2, f_1 \right\rangle_2 \\
&= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\overline{c}(\lambda^\circ - \mu^\circ, \mu^\circ)\pi(\lambda^\circ - \mu^\circ) f_2, f_1 \right\rangle_2 \\
&= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\overline{c}(\lambda^\circ - \mu^\circ, \mu^\circ)\overline{c}(\lambda^\circ, -\mu^\circ)\pi(\lambda^\circ)\pi(-\mu^\circ) f_2, f_1 \right\rangle_2 \\
&= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\overline{c}(-\mu^\circ, \mu^\circ)\pi(\mu^{-\circ}) f_2, \pi^*(\lambda^\circ) f_1 \right\rangle_2 \\
&= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ)\pi^*(\mu) f_2, \pi^*(\lambda^\circ) f \right\rangle_2 \\
&= \langle f_2 \mathbf{b}, \pi^*(\lambda^\circ) f_1\rangle_2 \\
&= (\langle f_1, f_2 \mathbf{b}\rangle_{\Lambda^\circ})(\lambda^\circ).
\end{aligned}
$$

As required, we have $\langle f_1, f_2\rangle_{\Lambda^\circ} *_{\overline{c}} \mathbf{b} = \langle f_1, f_2 \mathbf{b}\rangle_{\Lambda^\circ}$. Note that in the computation above, we used Equations (4.8), (4.9), and (5.29).

Next, we shall show that the module inner-product defines a positive element in B_Λ. To do this, we exploit the faithfulness of $\overline{\pi}^*_{\Lambda^\circ} : B_\Lambda \to \mathcal{L}(L^2(G))$ (c.f. Theorem 5.3). If we can show that $\overline{\pi}^*_{\Lambda^\circ}(\langle f, f\rangle_{\Lambda^\circ}) \geqslant 0$ in $\mathcal{L}(L^2(G))$ then we have also shown that $\langle f, f\rangle_{\Lambda^\circ} \geqslant 0$ in B_Λ by faithfulness of $\overline{\pi}^*_{\Lambda^\circ}$. To do this, let $\varphi \in \mathbf{S}_0(G)$, we have:

$$
\begin{aligned}
\left\langle \overline{\pi^*_{\Lambda^\circ}(\langle f, f\rangle_{\Lambda^\circ})\varphi, \varphi} \right\rangle_2 &= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} \overline{\langle f, \pi(\mu^\circ) f\rangle_2}\, \langle \varphi, \pi(\mu^\circ)\varphi\rangle_2 \\
&= \sum_{\mu \in \Lambda} \langle \varphi, \pi(\mu) f\rangle_2\, \overline{\langle \varphi, \pi(\mu) f\rangle_2} \geqslant 0,
\end{aligned}
$$

where we used the Fundamental Identity of Gabor Analysis (FIGA) Corollary 4.6 to obtain the equality above. Since $\varphi \in \mathbf{S}_0(G)$ is arbitrary, and $\mathbf{S}_0(G)$ is dense in $L^2(G)$, then we have shown that $\overline{\pi}^*_{\Lambda^\circ}(\langle f, f\rangle_{\Lambda^\circ}) \geqslant 0$ in $\mathcal{L}(L^2(G))$ as required.

The rest of the semi-inner product module axioms now easily follow from what we have shown here. For reference, once should check Definition 2.18 (except positive-definiteness). $\square$

We have an analogous result pertaining to the module left inner-product structure on $\mathbf{S}_0(G)$.

Proposition 5.6. *$\mathbf{S}_0(G)$ is a semi-inner product left $\ell^1(\Lambda, c)$-module. For $\mathbf{a} \in \ell^1(\Lambda, c)$ and $f, f_1, f_2 \in \mathbf{S}_0(G)$:*

1. (Left module action). $\mathbf{a}f = \sum_{\mu\in\Lambda} a(\mu)\pi(\mu)f \in \mathbf{S}_0(G)$;
2. (Module Inner-Product). *We have* ${}_{\Lambda}\langle \cdot\, , \,\cdot \rangle : \mathbf{S}_0(G)\times\mathbf{S}_0(G) \to \ell^1(\Lambda, c)$ *via* ${}_{\Lambda}\langle f_1, f_2\rangle(\lambda) := \langle f_1, \pi(\lambda)f_2\rangle_2$.

Furthermore, an alternative way to write the left module action is through the integrated representation (5.2)*:*

$$\overline{\pi}_{\Lambda}(\mathbf{a})f = \mathbf{a}f. \tag{5.32}$$

On the other hand, we can also alternatively write the module inner-product in terms of sampled STFT:

$${}_{\Lambda}\langle f_1, f_2\rangle = R_{\Lambda}\left(\mathcal{V}_{f_2}f_1\right) \tag{5.33}$$

For the proof of positivity of the module inner-products above, we have used FIGA 4.6 on the Feichtinger's algebra. As we have discusssed in Section 4.4 of Chapter 4, the FIGA is equivalent to the Janssen's representation on $\mathbf{S}_0(G)$. From the definition of the module inner-products above, we see that Janssen's representation is equivalent to the associativity condition of $\mathbf{S}_0(G)$ as an $\ell^1(\Lambda, c) - \ell^1(\Lambda, \overline{c})$-pre-imprimitivity bimodule. More precisely, we have the following.

Proposition 5.7. *For all $f, g, h \in \mathbf{S}_0(G)$:*

$${}_{\Lambda}\langle f, g\rangle h = f\langle g, h\rangle_{\Lambda^\circ}.$$

Proof. The following follows from Janssen's representation (Equation 4.51) and the cocycle relations (4.9):

$${}_{\Lambda}\langle f, g\rangle h = \sum_{\mu\in\Lambda} \langle f, \pi(\mu)g\rangle_2\, \pi(\mu)h$$

$$
\begin{aligned}
&= S_{g,h,\Lambda} f \\
&= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} \langle h, \pi(\mu^\circ) g \rangle_2 \, \pi(\mu^\circ) f \\
&= \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} \langle h, \pi^*(\mu^\circ) g \rangle_2 \, \pi^*(\mu^\circ) f \\
&= f \, \langle g, h \rangle_{\Lambda^\circ} .
\end{aligned}
$$

□

Corollary 5.4. $\mathbf{S}_0(G)$ *is an* $\ell^1(\Lambda, c) - \ell^1(\Lambda^\circ, \overline{c})$ *bimodule.*

Proof. The bimodule property

$$
\mathbf{a}(f\mathbf{b}) = (\mathbf{a}f)\mathbf{b}, \qquad \forall \mathbf{a} \in \ell^1(\Lambda, c), \mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c}), f \in \mathbf{S}_0(G) \tag{5.34}
$$

directly follows from the commutation property between Λ and its adjoint Λ°, in the sense given by Item 3 Proposition 4.4. □

Next, we would like to show the fullness property: Item 2 Definition 2.21. To the best of my knowledge, this is a new and fully time-frequency analytic proof of the 'fullness' property. This is different from the one presented by Luef and Rieffel in [Lue07] or [Rie88] respectively.

Proposition 5.8. *We have the following:*

1. ${}_{\Lambda}\langle \mathbf{S}_0(G), \mathbf{S}_0(G) \rangle := \operatorname{span}\{{}_{\Lambda}\langle f_1, f_2 \rangle : f_1, f_2 \in \mathbf{S}_0(G)\}$ *is dense in* $\ell^1(\Lambda, c)$*;*

2. $\langle \mathbf{S}_0(G), \mathbf{S}_0(G) \rangle_{\Lambda^\circ} := \operatorname{span}\{\langle f_1, f_2 \rangle_{\Lambda^\circ} : f_1, f_2 \in \mathbf{S}_0(G)\}$ *is dense in* $\ell^1(\Lambda^\circ, \overline{c})$.

Proof. We will only prove Item 2, the proof for Item 1 is similar. Let $\mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c})$, and recall that as a set $\ell^1(\Lambda^\circ, \overline{c}) = \ell^1(\Lambda^\circ)$. Therefore by the the surjectivity of the restriction map $R_{\Lambda^\circ} : \mathbf{S}_0(G \times \widehat{G}) \to \ell^1(\Lambda^\circ)$, there exists an $F \in \mathbf{S}_0(G \times \widehat{G})$ such that $\mathbf{b} = R_{\Lambda^\circ} F$. Now, recall (c.f. Lemma 4.5) that the partial Fourier transform $\mathcal{F}_2 : \mathbf{S}_0(G \times G) \to \mathbf{S}_0(G \times \widehat{G})$, asymmetric coordinate transform $\tau_a : \mathbf{S}_0(G \times G) \to \mathbf{S}_0(G \times G)$, and complex conjugation are all isomorphisms on their respective Feichtinger's algebras. Thus there must exist $G \in \mathbf{S}_0(G \times G)$ such that $F = \frac{1}{d(\Lambda)}(\mathcal{F}_2 \circ \tau_a)(\overline{G})$. Finally, due to the tensor-product factorization result of Proposition 4.10, we can write $G = \sum_{i=1}^{\infty} f_i \otimes \overline{g_i}$ for $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \subseteq \mathbf{S}_0(G)$ where $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}$ satisfies the admissibility condition (4.32). We finally put everything together using Equations (4.39) and

(5.31):

$$\begin{aligned}\mathbf{b} &= \frac{1}{d(\Lambda)} R_{\Lambda^\circ}\left(\mathcal{F}_2 \circ \tau_a\right)(\overline{G}) = \frac{1}{d(\Lambda)} \sum_{i=1}^{\infty} R_{\Lambda^\circ} \overline{\left(\mathcal{F}_2 \circ \tau_a\right)\left(f_i \otimes \overline{g_i}\right)} \\ &= \sum_{i=1}^{\infty} \frac{1}{d(\Lambda)} R_{\Lambda^\circ} \overline{\mathcal{V}_{g_i} f_i} = \sum_{i=1}^{\infty} \langle f_i, g_i \rangle_{\Lambda^\circ}.\end{aligned}$$

Hence $\mathbf{b}$ is in the closure of $\langle \mathbf{S}_0(G), \mathbf{S}_0(G) \rangle_{\Lambda^\circ}$, as required. □

We have shown almost all of the axioms required to show that $\mathbf{S}_0(G)$ is an $\ell^1(\Lambda, c) - \ell^1(\Lambda^\circ, \overline{c})$-pre-imprimitivity bimodule, except for Item 3 of Definition 2.21. Our proof will follow [Lue07, Theorem 26], which was done in the Euclidean case, but easily generalizes to arbitrary LCA G. However, we should first go through a technical lemma which tells us how the module action behave under the L^2-inner product.

Lemma 5.2. *We have the following formulae, for $\mathbf{a} \in \ell^1(\Lambda, c)$, $\mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c})$, and $\varphi_1, \varphi_2 \in \mathbf{S}_0(G)$:*

$$\begin{aligned}\langle \mathbf{a}\varphi_1, \varphi_2 \rangle_2 &= \langle \varphi_1, \mathbf{a}^{*_c} \varphi_2 \rangle_2 \\ \langle \varphi_1 \mathbf{b}, \varphi_2 \rangle_2 &= \langle \varphi_1, \varphi_2 \mathbf{b}^{*_{\overline{c}}} \rangle_2.\end{aligned}$$

Proof. We compute:

$$\begin{aligned}\langle \varphi_1, \varphi_2 \mathbf{b}^{*_{\overline{c}}} \rangle_2 &= \left\langle \varphi_1, \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} \overline{b(-\mu^\circ)} \overline{c}(\mu^\circ, -\mu^\circ) \pi^*(\mu^\circ) \varphi_2 \right\rangle_2 \\ &= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(-\mu^\circ) \overline{c}(\mu^\circ, \mu^\circ) \pi(\mu^\circ) \varphi_1, \varphi_2 \right\rangle_2 \\ &= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ) \overline{c}(\mu^\circ, \mu^\circ) \pi(-\mu^\circ) \varphi_1, \varphi_2 \right\rangle_2 \\ &= \left\langle \frac{1}{d(\Lambda)} \sum_{\mu^\circ \in \Lambda^\circ} b(\mu^\circ) \pi^*(\mu^\circ) \varphi_1, \varphi_2 \right\rangle_2 \\ &= \langle \varphi_1 \mathbf{b}, \varphi_2 \rangle_2.\end{aligned}$$

The proof for the first equality is similar. □

Corollary 5.5. *For all $\mathbf{a} \in \ell^1(\Lambda, c)$, $\mathbf{b} \in \ell^1(\Lambda^\circ, \overline{c})$, and $f \in \mathbf{S}_0(G)$:*

$$\langle \mathbf{a}f, \mathbf{a}f \rangle_{\Lambda^\circ} \leqslant \|\mathbf{a}\|_{A_\Lambda} \langle f, f \rangle_{\Lambda^\circ}$$

$$ {}_{\Lambda}\langle f\mathbf{b}, f\mathbf{b}\rangle \leqslant \|\mathbf{b}\|_{B_\Lambda}\, {}_{\Lambda}\langle f, f\rangle \,. $$

Proof. We will only prove the latter inequality, since as usual, the proof for the former is similar. The strategy here is similar to the proof of positivity of the module inner-product in Proposition (5.6), we shall use the faithfulness of $\overline{\pi}^*_{\Lambda^\circ}$ and density of $\mathbf{S}_0(G)$ in $L^2(G)$ to prove an equivalent statement. Let $\varphi \in \mathbf{S}_0(G)$, then we compute:

$$
\begin{aligned}
\langle \overline{\pi}^*_{\Lambda^\circ}\left({}_{\Lambda}\langle f\mathbf{b}, f\mathbf{b}\rangle\right)\varphi, \varphi\rangle_2 &= \langle {}_{\Lambda}\langle f\mathbf{b}, f\mathbf{b}\rangle \varphi, \varphi\rangle_2 \\
&= \langle f\mathbf{b}\,\langle f\mathbf{b}, \varphi\rangle_{\Lambda^\circ}, \varphi\rangle_2 \\
&= \langle f\mathbf{b}, \varphi\,\langle f\mathbf{b}, \varphi\rangle^{*\overline{c}}_{\Lambda^\circ}\rangle_2 \\
&= \langle f\mathbf{b}, \varphi\,\langle \varphi, f\mathbf{b}\rangle_{\Lambda^\circ}\rangle_2 \\
&= \langle f\mathbf{b}, {}_{\Lambda}\langle \varphi, \varphi\rangle\, f\mathbf{b}\rangle_2 \\
&= \left\langle \left({}_{\Lambda}\langle \varphi, \varphi\rangle\right)^{1/2} f\mathbf{b}, \left({}_{\Lambda}\langle \varphi, \varphi\rangle\right)^{1/2} f\mathbf{b}\right\rangle_2 \\
&= \left\langle \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})\left(\left({}_{\Lambda}\langle \varphi, \varphi\rangle\right)^{1/2} f\right), \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})\left(\left({}_{\Lambda}\langle \varphi, \varphi\rangle\right)^{1/2} f\right)\right\rangle_2 \\
&\leqslant \|\overline{\pi}^*_{\Lambda^\circ}(\mathbf{b})\|\,\langle f, {}_{\Lambda}\langle \varphi, \varphi\rangle\, f\rangle_2 \\
&= \|\mathbf{b}\|_{B_\Lambda}\,\langle {}_{\Lambda}\langle f, f\rangle \varphi, \varphi\rangle_2 \,.
\end{aligned}
$$

This proves ${}_{\Lambda}\langle f\mathbf{b}, f\mathbf{b}\rangle \leqslant \|\mathbf{b}\|_{B_\Lambda}\, {}_{\Lambda}\langle f, f\rangle$. ☐

We now have all the ingredients for the main result of this section.

Theorem 5.4. *$\mathbf{S}_0(G)$ is an $\ell^1(\Lambda, c) - \ell^1(\Lambda^\circ, \overline{c})$-pre-imprimitivity bimodule. Its completion in the sense of Proposition 2.6 is the so-called* ***Heisenberg module*** *associated to Λ, denoted by $\mathcal{E}_\Lambda(G)$, which is an $A_\Lambda - B_\Lambda$-imprimitivity bimodule.*

Remark 5.7. For the rest of the manuscript, the Heisenberg module $\mathcal{E}_\Lambda(G)$ will inherit the notation for its $A_\Lambda - B_\Lambda$-bimodule structure from $\mathbf{S}_0(G)$. In particular, ${}_{\Lambda}\langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{E}_\Lambda(G) \times \mathcal{E}_\Lambda(G) \to A_\Lambda$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\Lambda^\circ} : \mathcal{E}_\Lambda(G) \times \mathcal{E}_\Lambda(G) \to B_\Lambda$ will be the associated module inner-products.

5.4 Localization

In this section we shall follow the paper [AE20] of Austad and Enstad which discusses how to embed the abstractly constructed Heisenberg module $\mathcal{E}_\Lambda(G)$ into $L^2(G)$, thereby allowing us to identify it as an actual function space. This will allow us to do 'ordinary' Gabor analysis on

$\mathcal{E}_\Lambda(G)$, serving as the foundation for the original contributions of this dissertation in Chapter 7. Note that the original paper deals with closed co-compact subspaces $\Lambda \subseteq G \times \widehat{G}$ instead of the special case of lattices, so most of their results actually apply to the case where the associated twisted group C^*-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \bar{c})$ are not necessarily *both* unital. We shall have no need for such generality for our purposes, and we will strive to give a detailed account of their localization results where $\Lambda \subseteq G \times \widehat{G}$ is still a lattice as we have defined in 4.1.

To begin, we shall recall some terminology from C^*-algebras. Suppose B is an arbitrary C^*-algebra, a **trace** on B, denoted by $\operatorname{tr}_B : B \to \mathbb{C}$ is a linear functional on B satisfying the trace property: $\operatorname{tr}_B(b_1 b_2) = \operatorname{tr}_A(b_2 b_1)$ for all $b_1, b_2 \in B$. We say that the trace is **finite** or **bounded** if $\operatorname{tr}_B : B \to \mathbb{C}$ is bounded as a linear functional. We say that tr_B is **positive** if $\operatorname{tr}_B(b) \geqslant 0$ for all $b \geqslant 0$ in B, and **faithful** if $\operatorname{tr}_B(b^* b) = 0 \iff b = 0$. Lastly, in the case where B is unital, we say that tr_B is a **tracial state** if and only if $\operatorname{tr}_\mathrm{B}$ is a positive trace and $\operatorname{tr}_B(1_B) = 1$, where 1_B is the identity element in B. As an elementary exercise, we have the following characterizations of the trace, one can find these in the reference [RLL00]

Proposition 5.9. *Let B be a C^*-algebra, and $\operatorname{tr}_B : B \to \mathbb{C}$, a bounded linear functional. The following are equivalent:*

1. *tr_B is a finite trace on B;*

2. *For all $b \in B$, $\operatorname{tr}_B(bb^*) = \operatorname{tr}_B(b^* b)$.*

Furthermore, in the case that B is unital: tr_B is a tracial state if and only if $\|\operatorname{tr}_B\| = 1$.

We can use finite positive faithful traces on B to embed any right Hilbert B-modules to a Hilbert space. The following is also quite elementary and follows from rudimentary computations.

Proposition 5.10. *If $\mathcal{Z}$ is a right Hilbert B-module and $\operatorname{tr}_B : B \to \mathbb{C}$ is a finite positive faithful trace, then the map*

$$(z, y) \mapsto \operatorname{tr}_\mathrm{B}\left(\langle y, z \rangle_B\right), \qquad \forall (z, y) \in \mathcal{Z} \times \mathcal{Z} \tag{5.35}$$

*defines an inner-product on $\mathcal{Z}$. The completion of $\mathcal{Z}$ with respect to this inner-product is a Hilbert space denoted by $\operatorname{loc}(\mathcal{Z})$ called the **localization** of $\mathcal{Z}$ with respect to tr_B. We shall denote the inner-product and induced norm on $\operatorname{loc}(\mathcal{Z})$ by $\langle \cdot, \cdot \rangle_{\mathrm{loc}}$ and $\| \cdot \|_{\mathrm{loc}}$.*

Corollary 5.6. *If $\mathcal{Z}$ is a right Hilbert B-module, and tr_B is a finite faithful trace, then the localization-inclusion* $\operatorname{loc}: \mathcal{Z} \hookrightarrow \operatorname{loc}(\mathcal{Z})$ *is continuous. The mapping is in fact functorial in the sense that we can induce an isometric continuous map* $\operatorname{loc}_*: \mathcal{L}_B(\mathcal{Z}) \to \mathcal{L}(\operatorname{loc}(\mathcal{Z}))$.

Proof. For each $z \in \mathcal{Z}$, we have

$$\|z\|_{\mathrm{loc}}^2 = \operatorname{tr}_B(\langle z, z\rangle_B) \leqslant \|\operatorname{tr}_B\| \cdot \|\langle z, z\rangle_B\| = \|\operatorname{tr}_B\| \|z\|^2. \tag{5.36}$$

This proves continuity of loc. Next, let $T \in \mathcal{L}_B(\mathcal{Z})$, we have from boundedness of adjointable maps that for all $z \in \mathcal{Z}$

$$\langle Tz, Tz\rangle_B \leqslant \|T\|^2 \cdot \langle z, z\rangle_B$$

Now, the trace is assumed to be positive, and therefore we can pass the inequality above to tr_B to obtain

$$\|Tz\|_{\mathrm{loc}} \leqslant \|T\|\|z\|_{\mathrm{loc}}.$$

The inequality above implies that we can densely extend T to a bounded linear function on the Hilbert space $\operatorname{loc}(\mathcal{Z})$. The extension, we shall denote by $\operatorname{loc}_*(T)$, and if we denote the Hilbert space operator norm of $\operatorname{loc}_*(T)$ by $\|\operatorname{loc}_*(T)\|_{\mathrm{loc}}$, then the same estimate above ensures us that $\|\operatorname{loc}_*(T)\|_{\mathrm{loc}} \leqslant \|T\|$, and so the induced map $\operatorname{loc}_*: \mathcal{L}_B(\mathcal{Z}) \to \mathcal{L}(\operatorname{loc}(\mathcal{Z}))$ is continuous. By the fact that $\operatorname{loc}_*(T_1 \circ T_2)(z) = (T_1 \circ T_2)(z) = (\operatorname{loc}_*(\mathrm{T}_1) \circ \operatorname{loc}_*(T_2))(z)$ for all $T_1, T_2 \in \mathcal{L}_B(\mathcal{Z})$, and $z \in \mathcal{Z}$ implies, by another density argument, that loc_* respect multiplication of operators. Finally if $z, y \in \mathcal{Z}$, we also have

$$\langle Tz, y\rangle_{\mathrm{loc}} = \langle z, T^* y\rangle_{\mathrm{loc}},$$

where T^* is the module adjoint of T. Therefore by another density argument, we find that $\operatorname{loc}_*(T^*) = \operatorname{loc}_*(T)^*$, i.e. loc_* is a $*$-homomorphism between C^*-algebras. Lastly, loc_* is obviously injective, therefore loc_* is actually an injective $*$-homomorphism between C^*-algebras, and therefore it must be isometric as required. □

Remark 5.8. If instead, $\mathcal{Z}$ is a *left* Hilbert A-module with a finite faithful positive trace $\operatorname{tr}_A: A \to \mathbb{C}$, then we shall still recover analogous results as in 5.10, and 5.6. We recall that what we want is an inner-product, i.e. a sequilinear form that is linear in the first component and antilinear in the second-component, then we shall define $\operatorname{loc}(\mathcal{Z})$, the Hilbert space localization of $\mathcal{Z}$ with respect to tr_A via the following densely defined inner-product

$$\langle z, y\rangle_{\mathrm{loc}} = \operatorname{tr}_A({}_A\langle z, y\rangle), \qquad \forall (z, y) \in \mathcal{Z} \times \mathcal{Z}. \tag{5.37}$$

In summary, we shall use the inner-product defined by (5.35) for Hilbert right C^*-modules, while we use (5.37) for Hilbert left C^*-modules.

Now, we want to know what happens with localization when we apply it to the case where $\mathcal{Z}$ is an $A - B$ imprimitivity bimodule between unital C^*-algebras A and B. The proof below uses a straightforward approach using Corollary 5.6. A more general approach is given in [AE20, Proposition 3.1].

Proposition 5.11. *Let $\mathcal{Z}$ be an $A - B$ imprimitivity bimodule for unital C^*-algebras A, B and $\operatorname{tr}_B : B \to \mathbb{C}$ a finite faithful positive trace for B, then there exists a unique finite faithful positive trace $\operatorname{tr}_A : A \to \mathbb{C}$ for A that satisfies the following*

$$\operatorname{tr}_A({}_A\langle z, y\rangle) = \operatorname{tr}_B(\langle y, z\rangle_B), \qquad \forall z, y \in \mathcal{Z}.$$

Furthermore, the Hilbert space localization $\operatorname{loc}(\mathcal{Z})$ of $\mathcal{Z}$ with respect to tr_B coincides with the Hilbert space localization of $\mathcal{Z}$ with respect to the induced trace tr_A when $\mathcal{Z}$ is considered as a left Hilbert A-module (c.f. Remark 5.8). Consequently, loc_* also maps every adjointable operator $\mathcal{L}_A(\mathcal{Z})$ isometrically to $\mathcal{L}(\operatorname{loc}(\mathcal{Z}))$.

Proof. $\mathcal{Z}$ is an $A - B$ imprimivity bimodule between unital C^*-algebras, therefore by fullness: $A = \operatorname{span}\{{}_A\langle z, y\rangle : z, y \in \mathcal{Z}\}$ and $B := \operatorname{span}\{\langle z, y\rangle_B : z, y \in \mathcal{Z}\}$ and so by linearly extending the definition

$$\operatorname{tr}_A({}_A\langle z, y\rangle) := \operatorname{tr}_B(\langle y, z\rangle_B) \tag{5.38}$$

we obtain a linear functional in A. That the induced trace tr_A is indeed finite and faithful follows from Rieffel's result in [Rie81, Proposition 2.2]. Due to Equation (5.38), the localization of $\mathcal{Z}$ with respect to tr_B (as a right B-module) and tr_A (as a left A-module) coincide. □

We now apply the localization result in our setting. Consider the twisted group C^*-algebra

$A_\Lambda = C^*(\Lambda, c)$. We know from Example 2.1 that every C^*-algebra is a (left) Hilbert C^*-module over itself. We know that we can also define a finite, positive, and faithful trace on $C^*(\Lambda, c)$ usually termed as the **canonical trace**, densely defined [BO18] via

$$\operatorname{tr}_{A_\Lambda} : \mathbf{a} \mapsto a(0), \qquad \forall \mathbf{a} \in \ell^1(\Lambda, c).$$

It is important to notice that $\operatorname{tr}_{A_\Lambda}$ is a tracial state in A_Λ, indeed we know that δ_0 is the identity in A_Λ, hence from the following computation:

$$\operatorname{tr}_{A_\Lambda}(\delta_0) = \delta_0(0) = 1,$$

and Item 3 of Proposition 5.9, we find that $\| \operatorname{tr}_{A_\Lambda} \| = 1$.

The following result allows us to embed the seemingly abstract space A_Λ into the square-summable sequence space $\ell^2(\Lambda)$.

Proposition 5.12 ([AE20, Proposition 3.3])**.** *Consider* $\operatorname{loc}(A_\Lambda)$*, the localization of* A_Λ *as a Hilbert left* A_Λ*-module with the canonical trace* $\operatorname{tr}_{A_\Lambda}$*. Then the Hilbert space* $\operatorname{loc}(A_\Lambda)$ *is isometrically isomorphic to* $\ell^2(\Lambda)$*, with the following dense inclusion* $A_\Lambda \hookrightarrow \operatorname{loc}(A_\Lambda)$ *that satisfies the following continuous commutative diagram*

$$\begin{tikzcd} \ell^1(\Lambda, c) \arrow[r, hook] \arrow[d, hook] & A_\Lambda \arrow[d, hook] \\ \ell^2(\Lambda) \arrow[r, hook, "\cong"] & \operatorname{loc}(A_\Lambda) \end{tikzcd}$$

The continuous inclusion satisfies the following estimate:

$$\|\mathbf{a}\|_2 \leqslant \|\mathbf{a}\|_{A_\Lambda}, \qquad \forall \mathbf{a} \in A_\Lambda. \tag{5.39}$$

Proof. Given $\mathbf{a}_1, \mathbf{a}_2 \in \ell^1(\Lambda, c)$, we compute the following:

$$\begin{aligned} \langle \mathbf{a}_1, \mathbf{a}_2 \rangle_{\mathrm{loc}} &= \operatorname{tr}_\Lambda(\mathbf{a}_1 * \mathbf{a}_2^*) = \int_\Lambda a_1(\lambda') a_2^*(-\lambda') c(\lambda', -\lambda') d\mu_\Lambda(\lambda') \\ &= \int_\Lambda a_1(\lambda') \overline{a_2(\lambda')} c(-\lambda', -\lambda') c(\lambda', -\lambda') d\mu_\Lambda(\lambda') = \int_\Lambda a_1(\lambda') \overline{a_2(\lambda')} d\mu_\Lambda(\lambda') = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle . \end{aligned}$$

We find then, that for $\mathbf{a} \in \ell^1(\Lambda, c)$, $\|\mathbf{a}\|_{\mathrm{loc}} = \|\mathbf{a}\|_2$, but $\ell^1(\Lambda)$ is dense in A_Λ. So if $\mathbf{a} \in A_\Lambda$, we find that for $\mathbf{a}_n \to \mathbf{a}$ under the A_Λ-norm with $\{\mathbf{a}_n\}_n \subseteq \ell^1(\Lambda, c)$:

$$\|\mathbf{a}\|_{\mathrm{loc}} = \lim_{n\to\infty} \|\mathbf{a}_n\|_{\mathrm{loc}} = \lim_{n\to\infty} \|\mathbf{a}_n\|_2. \tag{5.40}$$

Therefore, we can identify the localization-norm on A_Λ with the $\ell^2(\Lambda)$-norm. This truly makes sense following the estimate in (5.36):

$$\|\mathbf{a}_n - \mathbf{a}_m\|_2 = \|\mathbf{a}_n - \mathbf{a}_m\|_{\mathrm{loc}} \leqslant \|\operatorname{tr}_{A_\Lambda}\| \|\mathbf{a}_n - \mathbf{a}_m\|_{C^*(\Lambda,c)} \to 0$$

as $m, n \to \infty$, so the right-hand side of (5.40) does converge in the $\ell^2(\Lambda)$-norm, and so we can embed A_Λ inside $\ell^2(\Lambda)$ by identifying $\mathbf{a}$ with the corresponding $\ell^2(\Lambda)$ sequence that $\{\mathbf{a}_n\}_n$ converges to in $\ell^2(\Lambda)$-norm (and denoting it still by $\mathbf{a}$). Using this identification, if $\mathbf{a} \in A_\Lambda$ it follows from the same estimate (5.36) and the fact that $\|\operatorname{tr}_\Lambda\| = 1$ (it is a tracial state) that

$$\|\mathbf{a}\|_2 \leqslant \sqrt{\|\operatorname{tr}_\Lambda\|}\|\mathbf{a}\|_{C^*(\Lambda,c)} \leqslant \|\mathbf{a}\|_{C^*(\Lambda,c)}.$$

That the diagram in the statement does commute follows from what we have just discussed, and the identification of $\ell^2(\Lambda)$ with $\operatorname{loc} A_\Lambda$) follows from the fact that their inner products agree on the dense subspace $\ell^1(\Lambda, c)$. □

We can also consider the twisted C^*-algebra $B_\Lambda = C^*(\Lambda^\circ, \bar{c})$, and we indeed have a result that is analogous to Proposition 5.12. However, we generally lose the estimate as in (5.39) when we equip B_Λ with the canonical trace given densely via

$$\operatorname{tr}_{B_\Lambda} : \mathbf{b} \mapsto b(0), \qquad \forall \mathbf{b} \in \ell^1(\Lambda^\circ, \bar{c}).$$

Looking closely, because of our choice of the measure, the identity element in B_Λ is given by $d(\Lambda)\delta_0$, and so

$$\operatorname{tr}_{B_\Lambda}(d(\Lambda)\delta_0) = d(\Lambda)\delta_0 = d(\Lambda),$$

so $\operatorname{tr}_{B_\Lambda}$ is not a tracial state unless $d(\Lambda) = 1$. What we do find from the previous computation is that the rescaling $\frac{1}{d(\Lambda)} \operatorname{tr}_{B_\Lambda}$ is a tracial state, and therefore

$$\left\| \frac{1}{d(\Lambda)} \operatorname{tr}_{B_\Lambda} \right\| = 1 \implies \|\operatorname{tr}_{\Lambda^\circ}\| = d(\Lambda).$$

As a result, we can run the same program that we did for A_Λ (except this time we treat B_Λ as a Hilbert *right* B_Λ-module) which mostly spits out the same result, but with a different estimate.

Proposition 5.13. *Consider* $\operatorname{loc}(B_\Lambda)$*, the localization of* B_Λ *as a Hilbert right* B_Λ*-module with the canonical trace* $\operatorname{tr}_{B_\Lambda}$*. Then the Hilbert space* $\operatorname{loc}(B_\Lambda)$ *is isometrically isomorphic to* $\ell^2(\Lambda^\circ)$*, with the following dense inclusion* $B_\Lambda \hookrightarrow \operatorname{loc}(B_\Lambda)$ *that satisfies the following continuous commutative diagram*

$$\begin{tikzcd}
\ell^1(\Lambda^\circ, \bar{c}) \arrow[r, hook] \arrow[d, hook] & B_\Lambda \arrow[d, hook] \\
\ell^2(\Lambda^\circ) \arrow[r, hook, "\cong"] & \operatorname{loc}(B_\Lambda)
\end{tikzcd}$$

The continuous inclusion satisfies the following estimate:

$$\|\mathbf{b}\|_2 \leqslant \sqrt{d(\Lambda)}\|\mathbf{b}\|_{B_\Lambda}, \qquad \forall \mathbf{b} \in B_\Lambda. \tag{5.41}$$

We can do a similar thing for the Heisenberg module $\mathcal{E}_\Lambda(G)$, using the result of Proposition 5.11 instead.

Proposition 5.14 ([AE20, Proposition 3.5])**.** *The localization of* $\mathcal{E}_\Lambda(G)$ *with respect to the canonical trace on* $\operatorname{tr}_{B_\Lambda}$ *or the canonical trace* $\operatorname{tr}_{A_\Lambda}$*, denoted by* $\operatorname{loc}(\mathcal{E}_\Lambda(G))$*, is isometrically isomorphic to* $L^2(G)$*. The following diagram of dense embeddings commute:*

$$\begin{tikzcd}
\mathbf{S}_0(G) \arrow[r, hook] \arrow[d, hook] & \mathcal{E}_\Lambda(G) \arrow[d, hook] \\
L^2(G) \arrow[r, hook, "\cong"] & \operatorname{loc}(\mathcal{E}_\Lambda(G))
\end{tikzcd}$$

We have the following estimate on the norms:

$$\|f\|_2 \leqslant \sqrt{d(\Lambda)}\|f\|_{\mathcal{E}_\Lambda(G)}, \qquad \forall f \in \mathcal{E}_\Lambda(G). \tag{5.42}$$

Furthermore, the C^**-algebra of* B_Λ*-adjointable (resp.* A_Λ*-adjointable) maps* $\mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))$ *(resp.* $\mathcal{L}_{B_\Lambda}(\mathcal{E}_\Lambda(G))$*) embed isometrically to the* C^**-algebra* $\mathcal{L}(L^2(G))$ *via extension through localization.*

Proof. We note that given $f, g \in \mathbf{S}_0(G)$,

$$\operatorname{tr}_{A_\Lambda}({}_\Lambda\langle f, g\rangle) = \langle f, \pi(0)g\rangle = \langle f, g\rangle = \langle f, \pi^*(0)g\rangle = \langle g, f\rangle_{\Lambda^\circ} = \operatorname{tr}_{B_\Lambda}(\langle g, f\rangle_{\Lambda^\circ}). \tag{5.43}$$

Therefore by density argument, the unique induced trace $\mathrm{tr} : A_\Lambda \to \mathbb{C}$ by $\mathrm{tr}_{\Lambda^\circ}$ that satisfies

$$\mathrm{tr}({}_\Lambda\langle f, g\rangle) = \mathrm{tr}_{B_\Lambda}(\langle g, f\rangle_{\Lambda^\circ})$$

is exactly tr_{A_Λ}. Now by Proposition 5.11, we find that the localization of $\mathcal{E}_\Lambda(G)$ with respect to either tr_{A_Λ} or tr_{B_Λ} coincide. We can safely denote the localization of $\mathcal{E}_\Lambda(G)$ by $\mathrm{loc}(\mathcal{E}_\Lambda(G))$ without ambiguity. If $f \in \mathcal{E}_\Lambda(G)$ such that $f_n \to f$ in $\mathcal{E}_\Lambda(G)$-norm with $\{f_n\}_n \subseteq \mathbf{S}_0(G)$, we can now further see from Equation (5.43) that

$$\|f\|_{\mathrm{loc}} = \lim_{n\to\infty} \|f\|_{\mathrm{loc}} = \lim_{n\to\infty} \|f_n\|_2.$$

Since

$$\|f_n - f_m\|_2^2 = \langle f_n - f_m, f_n - f_m\rangle_2 = \mathrm{tr}_{B_\Lambda}(\langle f_n - f_m, f_n - f_m\rangle_{\Lambda^\circ}) \leqslant \|\mathrm{tr}_{B_\Lambda}\|^2 \|f_n - f_m\|^2_{\mathcal{E}_\Lambda(G)} \to 0 \tag{5.44}$$

as $n, m \to \infty$. We can therefore identify the localization-norm on $\mathcal{E}_\Lambda(G)$ with the $L^2(G)$-norm, and the previous calculation shows that $\mathcal{E}_\Lambda(G)$ can be embedded inside $L^2(G)$ by identifying f with the limit of $\{f_n\}_n$ in $L^2(G)$-norm, which we know converges by 5.44. Using this identification, we have for $f \in \mathcal{E}_\Lambda(G)$

$$\|f\|_2^2 = \langle f, f\rangle_2 = \mathrm{tr}_{B_\Lambda}(\langle f, f\rangle_{\Lambda^\circ}) \leqslant \|\mathrm{tr}_{B_\Lambda}\| \|f\|^2_{\mathcal{E}_\Lambda(G)} = d(\Lambda)\|f\|^2_{\mathcal{E}_\Lambda(G)}.$$

This also implies that the diagram of dense inclusions above does commute, and the identification of $L^2(G)$ with $\mathrm{loc}(\mathcal{E}_\Lambda(G))$ is valid because $\langle\cdot,\cdot\rangle_{\mathrm{loc}}$ and $\langle\cdot,\cdot\rangle$ coincide in their common dense subspace $\mathbf{S}_0(G)$. The last statement is just the application of the last statement of Proposition 5.11 in the Heisenberg module setting. $\square$

Now that we can comfortably see $\mathcal{E}_\Lambda(G)$ as a dense subspace of $L^2(G)$, we shall see that it inherits much of the properties of the Feichtinger's algebra $\mathbf{S}_0(G)$.

Proposition 5.15 ([AE20, Lemma 3.8])**.** *Let $g, h \in \mathbf{S}_0(G)$, consider $\theta_{g,h} \in \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))$. Then $\mathrm{loc}_*(\theta_{g,h}) = S_{g,h,\Lambda}$, i.e. the (module) frame operator in $\mathcal{E}_\Lambda(G)$ can be be extended by localization to a Gabor frame operator in $L^2(G)$.*

Proof. If $f \in \mathbf{S}_0(G)$, we have

$$\mathrm{loc}_*(\theta_{g,h})(f) = \theta_{g,h}(f) = {}_\Lambda\langle f, g\rangle \cdot h = \int_\Lambda \langle f, \pi(\lambda)g\rangle_2 \, \pi(\lambda)h d\mu_\Lambda(\lambda) = S_{g,h}f.$$

Now take $f \in \mathcal{E}_\Lambda(G)$, and suppose we have $\{f_n\}_n \subseteq \mathbf{S}_0(G)$ such that $f_n \to f$ in the $\mathcal{E}_\Lambda(G)$-norm, but then Estimate (5.42) implies that $f_n \to f$ in $L^2(G)$-norm as well. Therefore

$$\mathrm{loc}_*(\theta_{g,h})(f) = \lim_n \mathrm{loc}_*(\theta_{g,h})(f_n) = \lim_n S_{g,h,\Lambda}(f_n) = S_{g,h,\Lambda}(f).$$

This shows they agree in $\mathcal{E}_\Lambda(G)$, i.e. $\mathrm{loc}_*(\theta_{g,h})_{|\mathcal{E}_\Lambda(G)} = \theta_{g,h} = (S_{g,h,\Lambda})_{|\mathcal{E}_\Lambda(G)}$, therefore $\mathrm{loc}_*(\theta_{g,h}) = S_{g,h,\Lambda}$. ☐

The following is an immediate corollary, when combined with our knowledge of the operator norm of the frame operator.

Corollary 5.7. *For each $g \in \mathbf{S}_0(G)$, we have the following characterization of the $\mathcal{E}_\Lambda(G)$-norm:*

$$\begin{aligned} \|g\|_{\mathcal{E}_\Lambda(G)} &= \|\theta_g\|^{1/2}_{\mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))} && (5.45)\\ &= \|S_{g,\Lambda}\|^{1/2} && (5.46)\\ &= \inf\{B^{1/2} : B \textit{ is a Bessel bound for } \mathcal{G}(g,\Lambda)\}. && (5.47)\end{aligned}$$

Proof. It follows from Proposition 2.29, Proposition 5.15, and the fact that localization on adjointable maps is an isometry that: $\|g\|_{\mathcal{E}_\Lambda(G)} = \|\theta_g\|^{1/2}_{\mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))} = \|S_{g,\Lambda}\|^{1/2}$. All that remains is to show Equation 5.47. However, it easily follows from Equation (4.23). ☐

What we would like to do is to extend these results fully to the Heisenberg module. Firstly, one sees that an extension of Equation (5.47) to the Heisenberg module is essentially the following result.

Proposition 5.16 ([AE20, Proposition 3.11]). *If $g \in \mathcal{E}_\Lambda(G)$, then $\mathcal{G}(g,\Lambda)$ is a Bessel family, and that*

$$\|g\|_{\mathcal{E}_\Lambda(G)} = \inf\{B^{1/2} : B \textit{ is a Bessel bound for } \mathcal{G}(g,\Lambda)\} \qquad (5.48)$$

Consequently, the Gabor analysis, Gabor synthesis, and Gabor frame operators are all bounded *linear operators on* $L^2(G)$*. In particular, Equation* (5.46) *is true when* $g \in \mathcal{E}_\Lambda(G)$ *:*

$$\|g\|_{\mathcal{E}_\Lambda(G)} = \|S_{g,\Lambda}\|^{1/2}. \tag{5.49}$$

Proof. Given $g \in \mathcal{E}_\Lambda(G)$, we only need to prove Equation (5.49), and the rest of the proposition will follow.

To start, let $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathbf{S}_0(G)$ such that $\|g - g_n\|_{\mathcal{E}_\Lambda(G))} \to 0$. It follows that $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}_\Lambda(G)$-norm, hence in conjunction with Equation (5.46), the sequence of operators $\{S_{g_n,\Lambda}\}_{n\in\mathbb{N}}$ is also Cauchy in $\mathcal{L}(L^2(G))$. Since $\mathcal{L}(L^2(G))$ is a Banach space, there exists a $T \in \mathcal{L}(L^2(G))$ such that $T = \lim_n S_{g_n,\Lambda}$. On the other hand, it follows from Estimate (5.42) that $\|g - g_n\|_2 \to 0$ as well, therefore for any $f \in L^2(G)$:

$$\begin{aligned}\|S_{g,\Lambda}f - S_{g_n,\Lambda}f\|_2 &\leqslant \lim_n \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)[g - g_n]\rangle_2| \|g - g_n\|_2 \\ &\leqslant \sum_{\lambda\in\Lambda} \left|\left\langle f, \pi(\lambda) \lim_n [g - g_n]\right\rangle_2\right| \lim_n \|g - g_n\|_2.\end{aligned}$$

Hence $\|S_{g,\Lambda}f - S_{g_n,\Lambda}f\| \to 0$. We can always write:

$$\|Tf - S_{g,\Lambda}f\|_2 \leqslant \|Tf - S_{g_n,\Lambda}f\|_2 + \|S_{g_n,\Lambda}f - S_{g,\Lambda}f\|_2,$$

so by taking the limit of the inequality above, we see that $Tf = S_{g,\Lambda}f$. Since $f \in L^2(G)$ is arbitrary, we conclude that $T = S_{g_n,\Lambda}$. It now follows that $\|g\|_{\mathcal{E}_\Lambda(G))} = \lim_n \|g_n\|_{\mathcal{E}_\Lambda(G)} = \lim_n \|S_{g_n,\Lambda}\|^{1/2} = \|T\|^{1/2} = \|S_{g,\Lambda}\|^{1/2}$ as required. □

The last proposition allows us to fully see $\mathcal{E}_\Lambda(G)$ as a function space inside $L^2(G)$ suitable for time-frequency analysis since we will always get a Bessel family from the time-frequency shifts of elements coming from the Heisenberg module, just like the Feichtinger's algebra. We may even opt to take the definition of $\mathcal{E}_\Lambda(G)$ to be the norm completion of $\mathbf{S}_0(G)$ under the Bessel-bound-norm (i.e. the norm on the right hand side of (5.48)), with the added bonus that is actually an imprimitivity bimodule between $C^*(\Lambda, c)$ and $C^*(\Lambda^\circ, \overline{c})$.

The proofs [AE20, Theorem 3.15 & Proposition 3.18] for the next results two results are similar in principle to Proposition 5.16, where they approximate $\mathcal{E}_\Lambda(G)$ using sequences in $\mathbf{S}_0(G)$. These two are also crucial for the original results of this dissertation (see Chapter 7).

Lemma 5.3. *For* $g, h \in \mathcal{E}_\Lambda(G)$*, then* $\mathrm{loc}_*(\theta_{g,h}) = S_{g,h,\Lambda}$.

Corollary 5.8 (Janssen's Representation for Heisenberg Modules)**.** *For all $g, h \in \mathcal{E}_\Lambda(G)$,*

$$S_{g,h,\Lambda} = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} \langle h, \pi(\lambda^\circ) h \rangle_2 \, \pi(\lambda^\circ) \tag{5.50}$$

Remark 5.9. It is important to realize that (5.50) is a true extension of the Janssen's representation (4.51), and at the same time also a concrete realization of the symmetry between the module inner-products

$$_{\Lambda}\langle \xi, \eta \rangle \cdot \gamma = \xi \cdot \langle \eta, \gamma \rangle_{\Lambda^\circ} , \qquad \forall \xi, \eta, \gamma \in \mathcal{E}_\Lambda(G).$$

We summarize the characterization of the Heisenberg module as a function space.

Theorem 5.5. *We can characterize $\mathcal{E}_\Lambda(G)$ by the following: $\mathcal{E}_\Lambda(G)$ is the completion of $\mathbf{S}_0(G)$ on $L^2(G)$ with respect to the following norm*

$$\|f\|_{\mathcal{E}_\Lambda(G)} = \inf \left\{ C^{1/2} : C \text{ is a Bessel bound for } \mathcal{G}(f, \Lambda) \right\}. \tag{5.51}$$

The characterization also respects the $A_\Lambda - B_\Lambda$-imprimitivity bimodule structure of $\mathcal{E}_\Lambda(G)$, so that for all $\mathbf{a} \in A_\Lambda$, $b \in B_\Lambda$, $f, g \in \mathcal{E}_\Lambda(G)$:

1. $\mathbf{a} \cdot f = \overline{\pi}_\Lambda(\mathbf{a}) f \in \mathcal{E}_\Lambda(G)$;
2. $f \cdot \mathbf{b} = \overline{\pi}^*_{\Lambda^\circ}(\mathbf{b}) f \in \mathcal{E}_\Lambda(G)$;
3. $_{\Lambda}\langle f, g \rangle \in A_\Lambda$;
4. $\langle f, g \rangle_{\Lambda^\circ} \in B_\Lambda$.

Since we chose to work with lattices Λ, it follows that C^*-algebra B_Λ is unital. Therefore it follows from Corollaries 2.7 and 2.8 that $\mathcal{E}_\Lambda(G)$ has further properties, and its adjointable maps can further be characterized.

Corollary 5.9. *The Heisenberg module $\mathcal{E}_\Lambda(G)$ is finitely generated and self-dual.*

We shall end this section with the fulfillment of our promises. Recall in Section 4.2 of Chapter 4 that we skipped the proof of Wexler-Raz biorthogonality relations of Theorem 4.3, and the density result of Theorem 4.4. We did so because we will only use their Heisenberg module versions in Chapter 7. Below, we give an operator algebraic proof of these results. Before we start, let us first clarify that another expected result holds: module frames for $\mathcal{E}_\Lambda(G)$ are exactly Gabor frames in $L^2(G)$.

Theorem 5.6 ([AE20, Theorem 3.16]). *The finite set $g_1, ..., g_n \in \mathcal{E}_\Lambda(G)$ generates a (module) frame in $\mathcal{E}_\Lambda(G)$ if and only if $\bigcup_{i=1}^n \mathcal{G}(g_i, \Lambda)$ is a multi-window Gabor frame in $L^2(G)$.*

Proof. The finite set $g_1, .., g_n$ is a (module) frame in $\mathcal{E}_\Lambda(G)$ if and only if given $\mathbf{g} = \{g_1, ..., g_n\}$, the (module) frame operator $\theta_\mathbf{g} \in \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))$ is invertible (c.f. Corollary 2.10). Localization on operators $\mathrm{loc}_* : \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G)) \hookrightarrow \mathcal{L}(L^2(G)))$ is an embedding of unital C^*-algebras (c.f. Proposition 5.11), therefore by inverse-closedness of C^*-algebras, $\theta_\mathbf{g}$ is invertible in $\mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))$ if and only of $\mathrm{loc}_*(\theta_\mathbf{g}) = \sum_{i=1}^n S_{g_n,\Lambda}$ is invertible, equivalently $\bigcup_{i=1}^n \mathcal{G}(g_i, \Lambda)$ is a multi-window Gabor frame (see Definition 4.7 and Remark 4.2). □

Remark 5.10. It follows from Theorem 5.6 and Corollary 5.9 that multi-window Gabor frames always exists for $L^2(G)$.

Corollary 5.10 (Wexler-Raz for Heisenberg Modules). *Let $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(h, \Lambda)$ be dual Gabor frames in $L^2(G)$ with $g, h \in \mathcal{E}_\Lambda(G)$. Then*

$$\langle g, h \rangle_{\Lambda^\circ} = d(\Lambda)\delta_0.$$

Proof. If $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(h, \Lambda)$ are dual Gabor frames in $L^2(G)$ with $g, h \in \mathcal{E}_\Lambda(G)$, then $S_{g,h,\Lambda} = \mathrm{Id}_{L^2(G)}$. By restricting the frame operator $S_{g,h,\Lambda}$ to $\mathcal{E}_\Lambda(G)$, we find that $\theta_{g,h} = \mathrm{Id}_{\mathcal{E}_\Lambda(G)}$. It now follows from Wexler-Raz identity for modules (2.16) that $\langle g, h \rangle_{\Lambda^\circ} = 1_{B_\Lambda}$, where 1_{B_Λ} is the identity element in B_Λ. It follows from definition of B_Λ that $1_{B_\Lambda} = d(\Lambda)\delta_0$. We are done. □

Corollary 5.11. *Let $g \in \mathcal{E}_\Lambda(G)$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame, then $d(\Lambda) \leqslant 1$.*

Proof. Let $g \in \mathcal{E}_\Lambda(G)$ generate a Gabor frame $\mathcal{G}(g, \Lambda)$, then $\theta_g \in \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(G))$ is invertible, in particular, following Remark 2.14 $h := \theta_g^{-1/2} g$ is a Parseval frame. Since the dual frame of a Parseval frame is itself, then it follows from Wexler-Raz identity from Corollary 2.16 that $\langle h, h \rangle_{\Lambda^\circ} = 1_{B_\Lambda}$. Parseval frames correspond to projections in A_Λ, as we shall see below:

$${}_\Lambda\langle h, h \rangle \, {}_\Lambda\langle h, h \rangle = {}_\Lambda\langle {}_\Lambda\langle h, h \rangle h, h \rangle = {}_\Lambda\langle h \langle h, h \rangle_{\Lambda^\circ}, h \rangle = {}_\Lambda\langle h 1_{B_\Lambda}, h \rangle = {}_\Lambda\langle h, h \rangle.$$

Since the identity 1_{A_Λ} is the maximal projection, we have ${}_\Lambda\langle h, h \rangle \leqslant 1_{A_\Lambda}$. The following computation follows from Proposition 5.11 and the fact that positive traces preserve the poset of projections:

$$d(\Lambda) = \mathrm{tr}_{B_\Lambda}(1_{B_\Lambda}) = \mathrm{tr}_{B_\Lambda}(\langle h, h \rangle_{\Lambda^\circ}) = \mathrm{tr}_{A_\Lambda}({}_\Lambda\langle h, h \rangle) \leqslant \mathrm{tr}_{A_\Lambda}(1_{A_\Lambda}) = 1.$$

As required. ☐

Remark 5.11. The original proof in [AJL20, Theorem 3.28] given for Corollary 5.11 actually holds for a much general case, however it would require us to generalize to matrices of Heisenberg modules. In any case, it can be shown that if $\mathcal{G}(g_1, ..., g_n, \Lambda)$ is a multi-window Gabor frame with $g_1, ..., g_n \in \mathcal{E}_\Lambda(G)$ then $d(\Lambda) \leqslant n$.

Chapter 6

Banach and C^*-Bundles

In this chapter, we lay down the necessary tools that would allow us to precisely define what it means to have a continuously varying family of Heisenberg modules. Bundles of vector spaces are ubiquitous in topology and geometry. However, we will be interested in bundles whose fibers are, in general, infinite-dimensional Banach spaces, with some emphasis in the case when they are C^*-algebras. The basics of Banach bundles can be found in the first volume of Fell's tomes [FD88], and the slightly older book by Dixmier [Dix82] where the essential Banach bundles that we are interested in are called "Continuous Fields of Banach Spaces". For a more abstract approach on the C^*-bundle case, one can consult Appendix C of Williams' book [Wil07].

Definition 6.1. A **bundle** is a triple $\mathscr{B} = (\mathcal{B}, \rho, X)$ where $\rho : \mathcal{B} \to X$ is an open continuous surjection from a Hausdorff space $\mathcal{B}$ to a Hausdorff space X. We call X the **base space**, while $\mathcal{B}$ the **bundle space** or **total space**. The map ρ is called the **bundle projection map**. For each $x \in X$, we call $\rho^{-1}(x) := \mathcal{B}_x$ the **fiber** over x. A **section** of the bundle $\mathscr{B} = (\mathcal{B}, \rho, X)$ is any function $f : X \to \mathcal{B}$ such that $\rho \circ f(x) = x$ for all $x \in X$ (which implies that $f(x) \in \rho^{-1}(x) = \mathcal{B}_x$). We say that f **passes through** $s \in \mathcal{B}$ if and only if $s \in \mathrm{Im}(f)$. A bundle $\mathscr{B} = (\mathcal{B}, \rho, X)$ has **enough sections** if for each $s \in \mathcal{B}$, there is a section $f : X \to \mathcal{B}$ that passes through s.

The map $\rho : \mathcal{B} \to X$ is enough to determine a bundle, and we will usually refer to a bundle by the bundle projection map itself. To add context, in general, a **section of a map** $q : X \to Y$ is any other map $\mathfrak{s} : Y \to X$ such that $q \circ \mathfrak{s} = \mathrm{Id}_Y$. We are generally only concerned in sections of bundles though, in this case, when the context is clear we talk about sections $f : X \to \mathcal{B}$ with respect to the bundle projection ρ. That f is a section of ρ ensures us that $(\rho \circ f)(x) = x$, which implies that $f(x) \in \rho^{-1}(x) := \mathcal{B}_x$. Furthermore, note that for $s, t \in X$, we have the following chain of equalities: $\rho^{-1}(t) \cap \rho^{-1}(x) \neq \varnothing$ if and only if $t = x$ if and only if $\rho^{-1}(t) = \rho^{-1}(x)$. In other words $\mathcal{B}_x = \mathcal{B}_t$ if and only if $x = t$. Since ρ is surjective, we see that we can identify $\mathcal{B}$ with the disjoint union $\bigsqcup_{x \in X} \mathcal{B}_x$.

Remark 6.1. Suppose that f is a section of $\rho : \mathcal{B} \to X$ that passes through $s \in \mathcal{B}$. We have some $x \in X$ such that

$$f(x) = s.$$

By definition of a section, this means that $\rho(f(x)) = x = \rho(s)$, therefore

$$f(\rho(s)) = s.$$

That is, f passes through $s \in \mathcal{B}$ if and only if $f(\rho(s)) = s$.

Definition 6.2. A **Banach bundle** $\mathscr{B}$ over X is a bundle $\mathscr{B} = (\mathcal{B}, \rho, X)$ over X, together with operations and norms making each fiber $\mathcal{B}_x$ $(x \in X)$ into a Banach space, and satisfying the following conditions:

1. The map $s \mapsto \|s\|_{\mathcal{B}_x}$ is continuous from $\mathcal{B}$ to $\mathbb{R}$;

2. The operation $+$ is continuous as a function on $\{(s,t) \in \mathcal{B} \times \mathcal{B} : \rho(s) = \rho(t)\}$ to $\mathcal{B}$ for each $s, t \in X$;

3. For each complex λ, the map $s \mapsto \lambda \cdot s$ is continuous from $\mathcal{B}$ to $\mathcal{B}$;

4. If $x \in X$ and $\{s_i\}_i$ is any net of elements in B such that $\|s_i\|_{\mathcal{B}_{x_i}} \to 0$ and $\rho(s_i) \to x$, then $s_i \to 0_x$ where 0_x is the additive identity on $\mathcal{B}_x$ and $s_i \in \mathcal{B}_{x_i}$ for each i.

Furthermore, if each fiber $\mathcal{B}_x$ is a C^*-algbera, then with the added hypothesis that

5. The multiplication $\cdot$ is continuous as a function on $\{(s,t) \in \mathcal{B} \times \mathcal{B} : \rho(s) = \rho(t)\}$ to $\mathcal{B}$;

6. The involution $s \mapsto s^*$ is continuous from $\mathcal{B}$ to $\mathcal{B}$,

then $\mathscr{B}$ is called a **C^*-bundle**. We shall drop the subscripts $\mathcal{B}_x$ when the context is clear that we are taking norms with respect to fibers.

Explicitly, the Banach space structure on the fibers gives our projection map some basic mapping properties, for example, if $s \in \mathcal{B}$ such that $\rho(s) = x$, then it is in the fiber $s \in \mathcal{B}_x$, but in particular $\mathcal{B}_x$ is a linear space, and so $\lambda s + b \in \mathcal{B}_x$ for any $\lambda \in \mathbb{C}$ and $b \in \mathcal{B}_x$ (this is implicitly used to make sense of Axiom 3 above). This linear structure gives us

$$\rho(\lambda s + b) = x = \rho(s) = \rho(b). \tag{6.1}$$

For the rest of the manuscript, given a bundle $\rho : \mathcal{B} \to X$, we denote the collection of continuous sections by $\Gamma(\rho)$. As an immediate exercise on the definitions, we have the following result.

Proposition 6.1. *Let $\rho : X \to \mathcal{B}$ be a Banach bundle, then the zero section that maps $0 : x \mapsto 0_x$ for each $x \in X$ is in $\Gamma(\rho)$.*

Proof. Let $\{x_i\}_i$ be a net in X that converges to x. Consider the net $\{0_{x_i}\}_i$ in $\mathcal{B}$, then trivially $\|0_{x_i}\| \to 0$. Because $0(x_i) = 0_{x_i}$ by definition, then $\rho(0_{x_i}) = \rho(0(x_i)) = x_i \to x$ by hypothesis. However, $\rho : X \to \mathcal{B}$ is a Banach bundle, this means that the Axiom 4 of Definition 6.2 implies that $0_{x_i} = 0(x_i) \to 0_x$. Therefore by net criterion, the zero section $0 : X \to \mathcal{B}$ is continuous, that is, $0 \in \Gamma(\rho)$. ☐

Corollary 6.1. *The collection $\Gamma(\rho)$ is a complex linear space.*

Proof. The zero element is the map $0 : x \mapsto 0_x$, as in the proposition above. The rest of the linear structure axioms immediately follow from Items 2 and 3 in Definition 6.2. ☐

The axioms of the Banach bundle actually afford us a stronger continuity condition as we shall see below.

Proposition 6.2 ([FD88, 13.10])**.** *Let $\rho : X \to \mathcal{B}$ be a Banach bundle, then the map $(\lambda, s) \mapsto \lambda \cdot s$ is continuous from $\mathbb{C} \times \mathcal{B}$ to $\mathcal{B}$.*

Proof. Let $\{\lambda_i\}_i \subseteq \mathbb{C}$, and $\{s_i\}_i \subseteq \mathcal{B}$ be nets that converge to $\lambda \in \mathbb{C}$ and $s \in \mathcal{B}$ respectively. Using Equation (6.1) and continuity of the bundle projection map

$$\rho(\lambda_i s_i - \lambda s_i) = \rho((\lambda_i - \lambda)s_i) = \rho(s_i) \to \rho(s).$$

Furthermore, $\|\lambda_i s_i - \lambda s\| = |\lambda_i - \lambda| \|s_i\| \to 0$ via continuity Axiom 1 in Definition 6.2. Althogether, with Axiom 4 of Definition 6.2, we have shown that $\lambda_i s - \lambda s \to 0_{\rho(s)}$. Finally, combine this with

Axioms 2 and 3 of Definition 6.2, we conclude that

$$\lambda_i s_i = (\lambda_i s_i - \lambda s_i) + \lambda s_i \to 0_{\rho(s)} + \lambda s = \lambda s$$

as required. □

Corollary 6.2. *Let $\rho : X \to \mathcal{B}$ be a Banach bundle, $f \in \Gamma(\rho)$, and $\varphi \in C(X)$. Then $\varphi \cdot f \in \Gamma(\rho)$ where $(\varphi \cdot f)(x) = \varphi(x) \cdot f(x)$.*

Proof. The map $x \mapsto (\varphi(x), f(x))$ is continuous from X to $\mathbb{C} \times \mathcal{B}$ (since the map to each factor is continuous), therefore by Proposition 6.2, the map $x \mapsto \varphi(x) \cdot f(x)$ is also continuous. □

Here we also have a basic result result which would help us verify continuity of a section.

Lemma 6.1. *Let $\rho : \mathcal{B} \to X$ be a Banach bundle. Let $\{s_i\}_{i\in I}$ be a net of elements in $\mathcal{B}$ such that $s \in \mathcal{B}$ with $\rho(s_i) \to \rho(s)$. Suppose further that for each $\varepsilon > 0$, there exists a net $\{u_i\}_{i\in I} \subseteq \mathcal{B}$, and an element $u \in \mathcal{B}$ such that*

1. *$u_i \to u$;*

2. *$\rho(s_i) = \rho(u_i)$;*

3. *$\|s - u\| < \varepsilon$;*

4. *$\|s_i - u_i\| < \varepsilon$ for large enough i,*

then $s_i \to s$ in $\mathcal{B}$.

Proof. Note that Item 2 above imply that the nets $\rho(s_i)$ and $\rho(u_i)$ converge to the same limit, and therefore $\rho(s) = \rho(u)$, which further implies that u and s belong to the same fiber, hence Item 3 above makes sense.

Take any subnet of $\{s_i\}_{i\in I}$, relabel it and call it $\{s_i\}_{i\in I}$ still, using Proposition B.5, it is enough that we find another subnet of $\{s_i\}_{i\in I}$ that converges to s. Now note by hypothesis that $\rho(s_i) \to \rho(s)$. From open-ness of ρ, there exists (here we avoid the 'relabeling' argument to avoid confusion) $\{x_j\}_{j\in J} \subseteq X$, a subnet of $\{\rho(s_i)\}_{i\in I}$ and $\{t_j\}_{j\in J}$ such that

1. $\rho(t_j) = x_j$;

2. $t_j \to s$.

If the subnet above is implemented by the map $N : J \to I$, then we have $\rho(t_j) = x_j = \rho(s_{N(j)}) \to \rho(s)$ as well from Proposition B.3. Now pick an arbitrary $\varepsilon > 0$, and choose the net $\{u_i\}_{i\in I} \subseteq \mathcal{B}$ and $u \in \mathcal{B}$ as in the hypothesis. We find that from continuity of addition in the total space and the fact that all the terms involved below are in the same fibers that:

$$t_j - u_{N(j)} \to s - u.$$

Since $\|s - u\| < \varepsilon$, then $\|t_j - u_{N(j)}\| < \varepsilon$ eventually for large enough $j \in J$. Therefore

$$\|s_{N(j)} - t_j\| \leqslant \|s_{N(j)} - u_{N(j)}\| + \|u_{(j)} - t_{N(j)}\| \leqslant 2\varepsilon,$$

since ε is arbitrary, then $\|s_{N(j)} - t_j\| \to 0$. We use the fact that $\rho(s_{N(j)} - t_j) = x_j \to \rho(s)$, finally we have from Axiom 4 of Definition 6.2 that $s_{N(j)} - t_j \to 0_{\rho(s)}$. From which we can conclude that

$$s_{N(j)} = (s_{N(j)} - t_j) + t_j \to 0_{\rho(s)} + s.$$

Therefore we have found a subnet $\{s_{N(j)}\}_{j\in J}$ of $\{s_i\}_{i\in I}$ that converges to s as required. □

Remark 6.2. A bit of comment regarding the proof above: It is important to notice that $\{t_j\}_{j\in J}$ is not a subnet of $\{s_i\}_{i\in I}$ as it is not even necessarily true that $s_{N(j)} = t_j$ (we even went through the painstaking task of proving that the best that we can do is $s_{N(j)} - t_j \to 0_{\rho(s)}$).

As we have noted above, we can now use Lemma 6.1 to prove that a section that is the uniform limit of continuous sections is itself a continuous section.

Corollary 6.3. *Let $\rho : \mathcal{B} \to X$ be a Banach bundle and $\{f_i\}_i \subseteq \Gamma(\rho)$ a net of continuous sections, if f is a section of ρ such that*

$$\sup_{x\in X} \|f(x) - f_i(x)\| \to 0$$

then $f \in \Gamma(\rho)$.

Proof. Let $(x_j)_j$ be a net in X and $x \in X$ such that $x_j \to x$. We will use Lemma 6.1, set $(s_j)_j = f(x_j)$, and $s = f(x)$. We know that $\rho(s_j) = x_j$ and $\rho(s) = x$ since we are dealing with sections, therefore $\rho(s_j) \to \rho(s)$. For $\varepsilon > 0$, choose β such that

$$\sup_{z \in X} \|f_\beta(z) - f(z)\| < \varepsilon.$$

We set $u_j = f_\beta(x_j)$, with $u = f_\beta(x)$.

1. By continuity of f_β, we know that $u_j \to u$;
2. $\rho(u_j) = \rho(s_j) = x_j$ for all j;
3. $\|f(x) - f_\beta(x)\| = \|s - u\| < \varepsilon$, and;
4. $\|s_j - u_j\| = \|f(x_j) - f_\beta(x_j)\| \leqslant \sup_{z \in X} \|f(z) - f_\beta(z)\| < \varepsilon$.

Therefore by the previous theorem, $f(x_j) = s_j \to s = f(x)$ as required. $\square$

We denote by $\Gamma_0(\rho)$ the Banach space (it is a subspace of $\Gamma(\rho)$) of continuous sections that vanish at infinity, equipped with the supremum norm. Precisely, $f \in \Gamma_0(\rho)$ if and only if for each $\varepsilon > 0$, there exists a compact subset $M_\varepsilon \subseteq X$ such that $\|f(x)\| < \varepsilon$ whenever $x \in X \backslash M_\varepsilon$. Because X is Hausdorff, this is equivalent to: for each $\varepsilon > 0$, the set $\{x \in X : \|f(x)\| \geqslant \varepsilon\}$ is compact. Note that when $\rho : \mathcal{B} \to X$ is a C^*-bundle, then $\Gamma_0(\rho)$ is a C^*-algebra with the obvious operations defined pointwise. We will also make use of the following notation: $\Gamma^x(\rho) = \{\gamma(x) \in \mathcal{B}_x : \gamma \in \Gamma(\rho)\}$, and similarly $\Gamma_0^x(\rho) = \{\gamma(x) \in \mathcal{B}_x : \gamma \in \Gamma_0(\rho)\}$. For the rest of the manuscript, we shall call $\Gamma_0(\rho)$ the **section space**, or the **section C*-algebra**, if $\mathscr{B}$ is a C^*-bundle.

We can now start with the proper discussion of construction of Banach bundles. The following result is from [FD88, Theorem 13.18]. Let us work with the following setting, we let X be a Hausdorff space, $\mathcal{B}$ a generic untopologized set, and $\rho : \mathcal{B} \to X$ a surjective map such that for each $x \in X$, the preimage $\rho^{-1}(x) := \mathcal{B}_x$ is a complex Banach space. We see here that $\rho : \mathcal{B} \to X$ is not a bundle, because $\mathcal{B}$ is untopologized, and hence ρ does not have any coherent notion of open-ness, nor continuity. Our goal here is to endow a topology on $\mathcal{B}$ such that the triple $(\mathcal{B}, \rho, X)$ becomes a Banach bundle. To this end we have the following theorem, which we shall prove for the C^*-bundle case since the proof for the Banach bundle case can easily be modified from the one that we are about to present below.

Theorem 6.1. *If $\rho : \mathcal{B} \to X$ is a surjective map such that $\rho^{-1}(x) := \mathcal{B}_x$ is a C^*-algebra (complex Banach space) for each $x \in X$, and there exists a $*$-algebra (complex linear space) of sections Γ of ρ, with the added properties that*

1. *For all $\gamma \in \Gamma$, the function $x \mapsto \|\gamma(x)\|$ is a continuous function from X to $\mathbb{R}$;*

2. *For each $x \in X$, $\Gamma^x := \{\gamma(x) \mid \gamma \in \Gamma\}$ is dense in $\mathcal{B}_x$.*

Then there exists a unique topology on $\mathcal{B}$ that makes $\rho : \mathcal{B} \to X$ a C^*-bundle (Banach bundle) over X such that the elements of Γ are continuous sections of ρ, i.e. $\Gamma \subseteq \Gamma(\rho)$. Explicitly, a basis for the topology on $\mathcal{B}$ is given by

$$W(f, U, \varepsilon) = \{s \in \mathcal{B} : \rho(s) \in U, \|s - f(\rho(s))\| < \varepsilon\} \tag{6.2}$$

for $f \in \Gamma$, U an open set in X and $\varepsilon > 0$.

Proof. We first show that if we assume that there is indeed a topology for $\mathcal{B}$ such that $\rho : \mathcal{B} \to X$ is a Banach bundle with $\Gamma \subseteq \Gamma(\rho)$, then the topology musst be uniquely determined by Γ. To do this, let us suppose there is some topology τ on $\mathcal{B}$ where our statement holds. It follows from hypothesis and Lemma 6.1 that the net $\{s_i\}_{i\in I}$ converges to $s \in \mathcal{B}$ in the topology τ if and only if $\rho(s_i) \to \rho(s)$ and that for each ε, there is an $f \in \Gamma$ such that $\|f(\rho(s)) - s\| < \varepsilon$ and that $\|f(\rho(s_i)) - s_i\| < \varepsilon$ (since $\|f(\rho(s_i)) - s_i\| \to \|f(\rho(s)) - s\|$) for a large enough i. Notice that one side of this equivalence is independent of whichever topology we have for $\mathcal{B}$, and so if τ_1 and τ_2 are two topologies in $\mathcal{B}$ where our statement holds, then we have the net convergence $s_i \to s$ in τ_1 if and only if it also converges in τ_2.

It still remains to show that such a topology exists and that we have all the property that we want. We will subdivide the rest of the proof into three steps: (1) Show that (6.2) defines a base for a topology of $\mathcal{B}$; (2) Show that this topology makes $\rho : \mathcal{B} \to X$ a C^*-bundle, and; (3) Show that every Γ is continuous in this topology.

1. (Step 1). It is immediate from the properties of Γ that the collection of all $W(f, U, \varepsilon)$ in (6.2) covers $\mathcal{B}$, so it only remains to show that given $W(f_1, U_2, \varepsilon_1)$ and $W(f_2, U_2, \varepsilon_2)$ in the collection, we have that if $s \in W(f_1, U_1, \varepsilon_1) \cap W(f_2, U_2, \varepsilon_2)$, then there exists some $W(f_3, U_3, \varepsilon_3)$ in the collections such that $s \in W(f_3, U_3, \varepsilon_3) \subseteq W(f_1, U_1, \varepsilon_1) \cap W(f_2, U_2, \varepsilon_2)$. We start with $s \in W(f_1, U_1, \varepsilon_1) \cap W(f_2, U_2, \varepsilon_2)$, so that $\|s - (f_1(\rho(s))\| < \varepsilon_1$ and $\|s - f_2(\rho(s))\| < \varepsilon_2$ holds. Choose $0 < \varepsilon_1' < \varepsilon_1$ and $0 < \varepsilon_2' < \varepsilon$ so that both $\|s - f_1(\rho(s))\| < \varepsilon_1'$ and $\|s - f_2(\rho(s))\| < \varepsilon_2'$. Then define $\varepsilon_3 = \frac{1}{2}\min\{\varepsilon_1 - \varepsilon_1', \varepsilon_2 - \varepsilon_2'\}$, so that if we choose $f_3 \in \Gamma$ where $\|s - f_3(\rho(s))\| < \varepsilon_3$, we find that

$$\|h(\rho(s)) - f_1(\rho(s))\| \leqslant \|h(\rho(s)) - s\| + \|s - f_1(\rho(s))\| \leqslant \varepsilon_3 + \varepsilon_1'$$

$$\|h(\rho(s)) - f_2(\rho(s))\| \leqslant \|h(\rho(s)) - s\| + \|s - f_2(\rho(s))\| \leqslant \varepsilon_3 + \varepsilon_2'.$$

$f_3 - f_i \in \Gamma$ for $i = 1, 2$ so that $x \mapsto \|f_3(x) - f_i(x)\|$ is continuous, therefore there exists an open $U_3 \subseteq U_1 \cap U_2$ such that $\rho(s) \in U_3$ and that for all $x \in U_3$:

$$\|f_3(x) - f_1(x)\| < \varepsilon_3 + \varepsilon_1'$$
$$\|f_3(x) - f_2(x)\| < \varepsilon_3 + \varepsilon_2'.$$

We see that $s \in W(f_3, U_3, \varepsilon_3)$. If $b \in W(f_3, U_3, \varepsilon_3)$, we have $\rho(b) \in U_3$, and we find that for $i = 1, 2$

$$\begin{aligned}\|b - f_i(\rho(b))\| &\leqslant \|b - f_3(\rho(s))\| + \|f_3(\rho(s)) - f_i(\rho(s))\| \\ &< \varepsilon_3 + (\varepsilon_3 + \varepsilon_i') \leqslant (\varepsilon_i - \varepsilon_i') + \varepsilon_i < \varepsilon_i\end{aligned}$$

Therefore $s \in W(f_3, U_3, \varepsilon_3) \subseteq W(f_1, U_1, \varepsilon_1) \cap W(f_2, U_2, \varepsilon_2)$ as required, hence we have a base for a topology of $\mathcal{B}$.

2. (Step 2). We shall first show that $\rho : \mathcal{B} \to X$ is an open continuous map in this topology. Consider $W(f, U, \varepsilon)$ in the collection (6.2), we have that if $x \in U$, then $f(x) \in W(f, U, \varepsilon)$ with $x = \rho(f(x))$ so that $U \subseteq \rho(W(f, U, \varepsilon))$. The reverse inclusion obvious from definition of $W(f, U, \varepsilon)$. Therefore $\rho(W(f, U, \varepsilon)) = U$, which shows that ρ is open. Let $s \in \mathcal{B}$ and $\rho(s) \in U$ for some open set U, choose $f \in \Gamma$ such that $\|s - f(\rho(s))\| < 1$, then $\rho(W(f, U, 1)) \subseteq U$ with $s \in W(f, U, 1)$, so ρ is continuous at s, and since $s \in \mathcal{B}$ is arbitrary, $\rho : \mathcal{B} \to X$ must be a continuous function. We have shown that $\rho : \mathcal{B} \to X$ is a bundle projection map. We still have to verify that Items (1)-(6) of Definition 6.2 all hold.

   (a) (Item 1). To show that $s \mapsto \|s\|$ is continuous, fix $s_0 \in \mathcal{B}$ and $\varepsilon > 0$ such that $\|s_0\| < \varepsilon$. Choose $0 < \varepsilon' < \varepsilon$ such that $\|s_0\| < \varepsilon - 2\varepsilon' < \varepsilon$. Let $f \in \Gamma$ such that $\|s_0 - f(\rho(s_0))\| < \varepsilon'$. Therefore $\|f(\rho(s_0))\| \leqslant \|s_0 - f(\rho(s_0))\| + \|s_0\| < \varepsilon' + (\varepsilon - 2\varepsilon') < \varepsilon - \varepsilon'$. Since $x \mapsto \|f(x)\|$ is continuous, we can find an open set $U \ni \rho(s_0)$ such that $\|f(x)\| < \varepsilon - \varepsilon'$. Finally, we have the open set $W(f, U, \varepsilon') \ni s_0$ such that if $s \in W(f, U, \varepsilon')$ we have

$$\|s\| \leqslant \|s - f(\rho(s))\| + \|f(\rho(s))\| < \varepsilon' + \varepsilon - \varepsilon' < \varepsilon.$$

By possibly 'translating' the open sets involved, we have shown that $s \mapsto \|s\|$ is continuous.

(b) (Item 2). We want to show that the addition map $\mathcal{B}_{\text{fiber}} := \{(s,b) \in \mathcal{B} \times \mathcal{B} : \rho(s) = \rho(b)\} \to \mathcal{B}$ is continuous. To set notation, if $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$, then we define $\mathcal{B}_1 + \mathcal{B}_2 = \{s_1 + s_2 \in \mathcal{B} : s_1 \in \mathcal{B}_1\ s_2 \in \mathcal{B}_2 \text{ and } \rho(s_1) = \rho(s_2)\}$. Now let $(s,b) \in \mathcal{B}_{\text{fiber}}$ with $s + b \in W(f, U, \varepsilon)$, and set $\rho(s) = \rho(b) = \rho(s+b) =: x \in X$. Choose $0 < \varepsilon'$ such that $\|(s+b) - f(x)\| < \varepsilon - 2\varepsilon'$. From hypothesis we can also pick $g_s, g_b \in \Gamma$ such that

$$\begin{aligned} \|s - g_s(x)\| &< \frac{\varepsilon'}{2} \\ \|b - g_b(x)\| &< \frac{\varepsilon'}{2}. \end{aligned}$$

So $\|(s+b) - (g_s(x) + g_b(x))\| < \varepsilon'$. Alltogether

$$\begin{aligned} \|f(x) - (g_s(x) + g_b(x))\| &\leqslant \|f(x) - (s+b)\| + \|(s+b) - (g_s(x) + g_b(x))\| \\ &\leqslant (\varepsilon - 2\varepsilon') + \varepsilon = \varepsilon - \varepsilon'. \end{aligned}$$

Since $f - (g_s + g_b) \in \Gamma$, we find that there exists an open neighborhood $V \ni x$ such that $V \subseteq U$ and for all $y \in V$

$$\|f(y) - (g_s(y) + g_b(y))\| < \varepsilon - \varepsilon'.$$

Note that $(s,b) \in W(g_s, V, \frac{\varepsilon'}{2}) \times W(g_b, V, \frac{\varepsilon'}{2})$. Let $(a,c) \in W(g_s, V, \frac{\varepsilon'}{2}) \times W(g_b, V, \frac{\varepsilon'}{2})$ with $\rho(a) = y = \rho(b)$, then

$$\begin{aligned} \|(a+c) - f(y)\| &\leqslant \|(a+c) - (g_s(y) + g_b(y)\| + \|f(y) - (g_s(y) + g_b(y)\| \\ &\leqslant \varepsilon' + (\varepsilon - \varepsilon') = \varepsilon. \end{aligned}$$

Thus we have shown that $s + b \in W(g_s, V, \frac{\varepsilon'}{2}) + W(g_b, V, \frac{\varepsilon'}{2}) \subseteq W(f, U, \varepsilon)$. Therefore fiberwise addition is continuous in the total space $\mathcal{B}$.

(c) (Item 3). If $\lambda \neq 0$, then $s \in W(f, U, \varepsilon)$ if and only if $\lambda s \in W(\lambda f, U, |\lambda|\varepsilon)$. So $s \mapsto \lambda s$ is continuous.

(d) (Item 4). Fix $x \in X$ and a net $\{s_i\}_{i \in I}$ such that $\|s_i\| \to 0$, and $\rho(s_i) \to x$. Take an open set $W(f, U, \varepsilon) \ni 0_x$, we want to show that $\{s_i\}_{i \in I}$ is eventually in $W(f, U, \varepsilon)$. Choose

$0 < \varepsilon' < \varepsilon$ such that $\|f(x)\| < \varepsilon - \varepsilon'$. By hypotheses, for a large enough i: $\{\rho(s_i)\}_{i\in I}$ is eventually in U because $x \in U$, $\|f(\rho(s_i))\| < \varepsilon - \varepsilon'$, and $\|s_i\| < \varepsilon'$. Alltogether, for a large enough i

$$\|s_i - f(\rho(s_i))\| \leqslant \|s_i\| + \|f(\rho(s_i))\| < \varepsilon' + (\varepsilon' - \varepsilon) < \varepsilon.$$

Therefore $\{s_i\}_{i\in I}$ is eventually in $W(f, U, \varepsilon)$ as required.

(e) (Item 5). We want to show that the multiplication map from the space (with relative topology) $\mathcal{B}_{\text{fiber}} = \{(s, b) \in \mathcal{B} \times \mathcal{B} : \rho(s) = \rho(b)\} \to \mathcal{B}$ is continuous. To set notation, if $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$, we define $\mathcal{B}_1\mathcal{B}_2 := \{s_1s_2 \in \mathcal{B} : s_1 \in \mathcal{B}_1,\ s_2 \in \mathcal{B}_2, \text{ and } \rho(s_1) = \rho(s_2)\}$. Now let $(s, b) \in \mathcal{B}_{\text{fiber}}$ such that $sb \in W(f, U, \varepsilon)$. Note that we can set $\rho(s) = \rho(b) = \rho(sb) =: x \in X$. We choose ε' such that

$$\|sb - f(x)\| < \varepsilon - 2\varepsilon'.$$

By hypothesis in Γ, there are $g_s \in \Gamma$ and $g_b \in \Gamma$ such that

$$\begin{aligned} \|s - g_s(x)\| &< \varepsilon_s := \min\left\{\frac{\varepsilon'}{2(\|b\| + 1)}, \frac{1}{2}\right\} \\ \|b - g_b(x)\| &< \varepsilon_b := \min\left\{\frac{\varepsilon'}{2(\|s\| + 1)}, \frac{1}{2}\right\} \end{aligned}$$

These choices give us $\|g_s(x)\| \leqslant \varepsilon_s + \|s\| < \frac{1}{2} + \|s\|$, and similarly $\|g_b\| < \frac{1}{2} + \|b\|$. Alltogether,

$$\begin{aligned} \|sb - g_s(x)g_b(x)\| &\leqslant \|sb - sg_b(x)\| + \|sg_b(x) - g_s(x)g_b(x)\| \\ &\leqslant \|s\|\|b - g_b(x)\| + \|s - g_s(x)\|\|g_b(x)\| \\ &\leqslant \|s\|\left(\frac{\varepsilon'}{2(\|s\| + 1)}\right) + \left(\frac{\varepsilon'}{2(\|b\| + 1)}\right)\left(\frac{1 + \|b\|}{2}\right) \\ &\leqslant \frac{\varepsilon'}{2} + \frac{\varepsilon'}{4} < \varepsilon'. \end{aligned}$$

Therefore

$$\|f(x) - g_s(x)g_b(x)\| \leqslant \|f(x) - sb\| + \|sb - g_s(x)g_b(x)\| < (\varepsilon - 2\varepsilon') + \varepsilon' = \varepsilon - \varepsilon'.$$

Now we use the fact that $g_b, g_s, f - g_sg_b \in \Gamma$ to find an open neighborhood $V \ni x$ such that $V \subseteq U$ and for all $y \in V$

$$\begin{aligned}\|g_b(y)\| &< \frac{1}{2} + \|b\| \\ \|g_s(y)\| &< \frac{1}{2} + \|s\| \\ \|f(y) - g_s(y)g_b(y)\| &< \varepsilon - \varepsilon'.\end{aligned}$$

Note that $(s,b) \in W(g_s, V, \varepsilon_s) \times W(g_b, V, \varepsilon_b)$, and that if $(a,c) \in W(g_s, V, \varepsilon_s) \times W(g_b, V, \varepsilon_b)$ with $\rho(a) = y = \rho(c)$, then we can repeat much of the estimate above to conclude that

$$\begin{aligned}\|ac - f(y)\| &\leqslant \|ac - g_s(y)g_b(y)\| + \|g_s(y)g_b(y) - f(y)\| \\ &\leqslant \varepsilon' + (\varepsilon - \varepsilon') < \varepsilon.\end{aligned}$$

Therefore $sb \in W(g_s, V, \varepsilon_s)W(g_b, V, \varepsilon_b) \subseteq W(f, U, \varepsilon)$, and we have shown that fiber-wise multiplication in the total space is continuous.

(f) (Item 6). Note that $s \in W(f, U, \varepsilon)$ if and only if $s^* \in W(f^*, U, \varepsilon)$, so $s \mapsto s^*$ is continuous.

3. Let $f \in \Gamma$, we want to show that $f : X \to \mathcal{B}$ is continuous. Pick x_0 and an open set $W(g, U, \varepsilon) \ni f(x_0)$, therefore $x \in U$ and $\|f(x_0) - g(\rho(f(x)_0))\| = \|f(x_0) - g(x_0)\| < \varepsilon$. We know that $x \mapsto \|f(x) - g(x)\|$ is continuous, therefore there exists some open set $V \ni x_0$ with $V \subseteq U$ such that $\|f(x) - g(x)\| < \varepsilon$ for all $x \in V$. We now have for each $x \in V$, $\rho(f(x)) = x \in V \subseteq U$ and

$$\|f(x) - g(\rho(f(x)))\| = \|f(x) - g(x)\| < \varepsilon.$$

Therefore $f(x) \in W(g, U, \varepsilon)$. We have shown that f is continuous.

$\square$

Theorem 6.1 above will be the most relevant result for the subsequent Chapter, as all bundles that we shall construct will be done through it.

Definition 6.3. Let $\rho : \mathcal{B} \to X$ be a C^*-bundle (or Banach Bundle) constructed through Theorem 6.1 with initial section $\Gamma \in \Gamma(\rho)$. We call Γ the **local approximating sections** for ρ.

Remark 6.3. Local approximating sections are not unique, it is not hard to see that it is possible to find another family of sections that would induce the same Banach bundle through Theorem 6.1. For an original example, compare Theorems 7.3 and 7.4 with Proposition 7.10 in Chapter 7

The local approximating sections given by Γ above play an important role, as they are used to verify that a section is indeed continuous if only if they can be locally approximated by sections that belong in Γ (see Proposition 6.3 below). Note first that there is a refinement of Proposition 6.2 for Banach bundles constructed from Theorem 6.1, whose proof is essentially the same.

Lemma 6.2. *Let $\rho : X \to \mathcal{B}$ be a Banach bundle via Theorem 6.1 with local approximating sections Γ. If $\varphi \in C(X)$ and $f \in \Gamma$, then $\varphi \cdot f \in \Gamma$.*

Proposition 6.3. *Let $\rho : X \to \mathcal{B}$ be a Banach bundle via Theorem 6.1 with local approximating sections Γ. Then for any section $\gamma : X \to \mathcal{B}$, $\gamma \in \Gamma(\rho)$ if and only if for each $x_0 \in X$ and $\varepsilon > 0$, there exists a $\gamma' \in \Gamma$ such that for some open neighborhood $U_0 \ni x_0$*

$$\|\gamma(x) - \gamma'(x)\| < \varepsilon, \qquad \forall x \in U_0.$$

Proof. To start with, we can apply Theorem 6.1 to the triple $\mathscr{B} = (\mathcal{B}, \rho, X)$ with Γ. Because our Γ satisfies the hypothesis of Theorem 6.1, we have an explicit basis for the topology on $\mathcal{B}$:

$$W(\gamma'', U, \varepsilon) := \{s \in \mathscr{B} : \rho(s) \in U, \|s - \gamma''(\rho(s))\| < \varepsilon\}$$

for $\gamma'' \in \Gamma$, U an open set in X and $\varepsilon > 0$.

If $\gamma \in \Gamma(\mathscr{B})$ i.e., if γ is continuous, then the conclusion is trivial. Suppose now that the converse hypothesis holds for the section $\gamma : X \to \mathcal{B}$. We want to show that γ is continuous at any point, we take $x_0 \in X$ and take a basic open set $W(\gamma'', U_0, \frac{\varepsilon}{2}) \ni \gamma(x_0)$. By hypothesis, there exists $\gamma' \in \Gamma$ and an open set $V_0' \ni x_0$, we have for all $x \in V_0'$:

$$\|\gamma(x) - \gamma'(x)\| < \frac{\varepsilon}{2}.$$

Now, we know that $\gamma' \in \Gamma \subseteq \Gamma(\mathscr{B})$, therefore it is immediate that there exists an open set $V_0'' \ni x_0$ such that $\gamma'(V_0'') \subseteq W(\gamma'', U_0, \frac{\varepsilon}{2})$. Now, take the open set $x_0 \in V_0 := V_0' \cap V_0''$, then we have for all $x \in V_0$:

$$\|\gamma(x) - \gamma''(x)\| \leqslant \|\gamma(x) - \gamma'(x)\| + \|\gamma'(x) - \gamma''(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\gamma(V_0) \subseteq W(\gamma'', U_0, \varepsilon)$, hence γ is continuous. □

Using a simple application of triangle-inequality for Banach bundles defined via Theorem 6.1, we can show that if a section can be locally approximated by a continuous section, then it must also be a continuous section.

Corollary 6.4. *Let $\rho : X \to \mathcal{B}$ be a Banach bundle via Theorem 6.1. We have that $\gamma \in \Gamma(\rho)$ if and only if for each $\varepsilon > 0$ and $x_0 \in X$, there exists a $\gamma' \in \Gamma(\rho)$ such that for some open neighborhood $U_0 \ni x_0$:*

$$\|\gamma(x) - \gamma'(x)\| < \varepsilon, \qquad \forall x \in U_0.$$

Remark 6.4. As we have mentioned, we shall construct all the relevant bundles in the next chapter this way. Practically we start with, again, a Hausdorff space X, and a family of Banach space or C^*-algebras indexed by X, i.e. $\{\mathcal{B}_x\}_{x \in X}$. Hence if

$$\mathcal{B} := \bigsqcup_{x \in X} \mathcal{B}_x := \bigcup_{x \in X} \{(s, x) : s \in B_x\},$$

we simply define $\rho : \mathcal{B} \to X$ to be the usual projection i.e. $\rho(s, x) = x$. In practice, we shall omit the index x in $(s, x) \in \mathcal{B}$ and work with the implicit understanding that each s is an element in some unique $\mathcal{B}_x$. So, we shall write $\rho(s) = x \iff s \in \mathcal{B}_x$. This definition will give us fibers that we want, that is: $\rho^{-1}(x) := \mathcal{B}_x$. A section $\gamma : X \to \mathcal{B}$ with respect to ρ satisfies $\gamma(x) \in \mathcal{B}_x$ for each $x \in X$, and can be succinctly denoted by the product notation: $\gamma \in \prod_{x \in X} \mathcal{B}_x$.

The previous proposition shows that $\gamma : X \to \mathcal{B}$ is a continuous section if and only if we can locally uniformly approximate γ by some continuous section $\gamma' \in \Gamma$. This characterization of continuous sections appears in Dixmier's book [Dix82] (they are called vector fields), and in fact are taken to be part of the definition of the so-called "Continuous Fields of Banach Spaces". A major question whenever one has a Banach bundle is if it has enough continuous sections, it turns out that this is true for bundles via Theorem 6.1. We first recall the following basic result for Banach spaces.

Lemma 6.3. *Let E_0 be a dense linear subspace of a Banach space E, then for each $b \in E$, there exists a sequence $\{b_n\}_n \subseteq E_0$ such that $\sum_{n=0}^{\infty} b_n$ is absolutely convergent and $b = \sum_n b_n$.*

Proof. Indeed let $b \in E$, then for each $n \in \mathbb{N}$, there exists an $e_n \in E_0$ such that

$$\|b - e_n\| < \frac{1}{n^2}.$$

By triangle inequality, we know $\|e_{n+1} - e_n\| < \frac{2}{n^2}$, and therefore the series $\sum_{n=0}^{\infty} b_n$ absolutely converges for $b_n = e_{n+1} - e_n$. The limit of the series is in fact b, we see this in the following computation:

$$b = \lim_{N\to\infty} e_{N+1} = \lim_{N\to\infty} \sum_{n=0}^{N} (e_{k+1} - e_k) = \sum_{k=0}^{\infty} b_n.$$

□

Corollary 6.5 ([Dix82, Proposition 10.1.10]). *Every Banach bundle constructed by Theorem 6.1 will always have enough continuous sections.*

Proof. Let $\rho : X \to \mathcal{B}$ be a Banach bundle constructed by Theorem 6.1 with local approximating sections given by Γ. Let $x_0 \in X$, we want to show that for any $s_0 \in \mathcal{B}_{x_0}$, there exists a continuous section $f : X \to \mathcal{B}$ such that $f(x_0) = s_0$. To construct this, we know from Lemma 6.3 that there exists a sequence of sections $\{\gamma_n\}_n \subseteq \Gamma$ such that $\sum_{n=0}^{\infty} \|\gamma_n(x_0)\| < \infty$ and

$$s_0 = \sum_{n=0}^{\infty} \gamma_n(x_0).$$

Define for each $n \in \mathbb{N}$,

$$\varphi_n := \begin{cases} 1, & \|\gamma_n(x)\| \leqslant \|\gamma_n(x_0)\|, \\ \dfrac{\|\gamma_n(x_0)\|}{\|\gamma_n(x)\|}, & \|\gamma_n(x)\| \geqslant \|\gamma_n(x_0)\|. \end{cases}$$

We can deduce from the Pasting Lemma that $\varphi_n \in C(X)$, and therefore $g_n := \varphi_n \cdot \gamma_n \in \Gamma$ due to Proposition 6.2. Now for each $x \in X$,

$$\sum_{n=0}^{\infty} \|g_n(x)\| \leqslant \sum_{n=0}^{\infty} \|\gamma_n(x_0)\| < \infty.$$

Hence $\sum_{n=0}^{\infty} g_n(x)$ is absolutely convergent, and thus convergent in $\mathcal{B}_x$. We then define $f := \sum_{n=0}^{\infty} g_n$. Certainly $f(x_0) = \sum_{n=0}^{\infty} g_n(x_0) = \sum_{n=0}^{\infty} \gamma_n(x_0) = s_0$, thus we only need to prove that

f is a continuous section, but we know that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left\| f(x) - \sum_{n=0}^{N} g_n(x) \right\| < \varepsilon$$

for all $x \in X$. We know $\sum_{n=0}^{N} g_n \in \Gamma$, therefore by Proposition 6.3, $f \in \Gamma(\rho)$. $\square$

Remark 6.5. Corollary 6.5 is an important result, as it gives us *nontrivial* continuous sections that passes through whichever point in the bundle space that we want. In the sequel, this will have an important application for Gabor frame stability of Heisenberg modules.

While Γ offers a local approximation family for the continuous sections $\Gamma(\rho)$, the next result tells us that if Γ_0 is some linear subspace of the Banach space $\Gamma_0(\rho)$ with certain algebraic and analytic properties, then Γ_0 can always globally approximate $\Gamma_0(\rho)$. Note that this result holds in general even if the Banach bundle is not constructed from Theorem 6.1. The proof given below is similar to the one given in [Wil07, Proposition C.24], although it was originally proven only for C^*-bundles.

Proposition 6.4. *Let $\rho : \mathcal{B} \to X$ be a Banach bundle such that there exists linear subspace Γ_0 of $\Gamma_0(\rho)$ that satisfy:*

1. *For all $\varphi \in C_0(X)$, and $g \in \Gamma_0$, we have $\varphi \cdot f \in \Gamma_0$.*
2. *For each $x \in X$, $\{f(x) : f \in \Gamma_0\}$ is dense in $\mathcal{B}_x$.*

Then Γ_0 is dense in $\Gamma_0(\rho)$.

Proof. Let $f \in \Gamma_0(\rho)$ and $\varepsilon > 0$, we want to show that there exists $g \in \Gamma_0$ such that $\sup_{x \in X} \|f(x) - g(x)\| < \infty$. To do this, let $M = \left\{x \in X : \|f(x)\| \geqslant \frac{\varepsilon}{3}\right\}$, which is a compact set because f vanishes at infinity. As a general property of M, for any $x \in M$ we know from Axiom 2 in the hypothesis that there exits a $g_x \in \Gamma_0$ such that $\|f(x) - g_x(x)\| < \frac{\varepsilon}{3}$. Because the map $y \mapsto \|f(y) - g_x(y)\|$ is continuous about x, we know that there exists an open neighborhood $U_x \ni x$ such that

$$\|f(y) - g_x(y)\| < \frac{\varepsilon}{3}, \qquad \forall y \in U_x.$$

Note that the open neighborhoods $\{U_x\}_{x \in M}$ covers M, and so by compactness of M, there exists finitely many $\{U_i\}_{i=1}^{n} \subseteq \{U_x\}_{x \in M}$ and finitely many $\{g_i\}_{i=1}^{n} \subseteq \Gamma_0$ such that

$$\|f(y) - g_i(y)\| < \frac{\varepsilon}{3}, \qquad \forall y \in U_i.$$

Let $\{\varphi_i\}_{i=1}^n \subseteq C_c(X,[0,1])$ be a family of partition of unity subordinate to $\{U_i\}_{i=1}^n$, so that $\sum_{i=1}^n \varphi_i \equiv 1$ on M, $\sum_{i=1}^n \varphi_i \leqslant 1$ outside M, and the support of φ_i is contained in U_i. We now define $g := \sum_{i=1}^n \varphi_i \cdot g_i$, which is in Γ_0 by hypothesis. Suppose $x \in M$, then

$$\begin{aligned}\|f(x)-g(x)\| = \left\|\sum_{i=1}^n \varphi_i(x)\left(f(x)-g_i(x)\right)\right\| &\leqslant \sum_{i=1}^n \varphi_i(x)\|f(x)-g_i(x)\| \\ &\leqslant \sum_{i=1}^n \varphi_i(x)\cdot\frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

If $x \in X\backslash\bigcup_{i=1}^n U_i$, then $g \equiv 0$, and $\|f(x)-g(x)\| = \|f(x)\| < \frac{\varepsilon}{3} < \varepsilon$. Note that if $x \in U_i\backslash M$ for some i, then $\|g_i(x)\| \leqslant \|g_i(x)-f(x)\| + \|f(x)\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$. Finally if $x \in \bigcup_{i=1}^n U_i\backslash M$, we have

$$\begin{aligned}\|f(x)-g(x)\| &= \left\|f(x)-\sum_{i=1}^n \varphi_i(x)g_i(x)\right\| \\ &\leqslant \|f(x)\| + \sum_{i=1}^n \varphi_i(x)\|g_i(x)\| \\ &\leqslant \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}\cdot\sum_{i=1}^n \varphi_i(x) \\ &< \varepsilon.\end{aligned}$$

Hence we conclude that $\sup_{x\in X}\|f(x)-g(x)\| < \varepsilon$. □

We shall now explore results where the C^*-algebraic structure of a C^*-bundle actually matter. From now on, we shall use the total space $\mathcal{A}$ (hence with fibers $\rho^{-1}(x) = A_x$) instead of $\mathcal{B}$ when we are talking about C^*-bundles in particular. Note that if A is an arbitrary C^*-algebra, and $a \in A$, we shall use $\operatorname{spec}(a)$ to denote the spectrum of the element a. Recall then that if $\rho : \mathcal{A} \to X$ is a C^*-bundle, then the space of $\Gamma(\rho)$ continuous sections is closed with respect to multiplication and involution due to Items 5 and 6 of Definition 6.2. From this observation we have the following result based from [Dix82, Proposition 10.3.3]

Lemma 6.4. *If $P : z \mapsto P(z,\overline{z})$ is a polynomial in $\mathbb{C}$ that has no constant term i.e. $P(0) = 0$, then for each $f \in \Gamma(\rho)$, $P \circ f \in \Gamma(\rho)$ where*

$$(P\circ f)(x) = P(f(x), f(x)^*) \in A_x.$$

Proposition 6.5. *Let $\rho : \mathcal{A} \to X$ be a C^*-bundle. If $f \in \Gamma(\rho)$ such that $ff^* = f^*f$, and $\theta : \mathbb{C}_f \to \mathbb{C}$ is a continuous function where $\mathbb{C}_f \subseteq \mathbb{C}$ is a subset such that $\operatorname{spec}(f(x)) \subseteq \mathbb{C}_f$ for all*

$x \in X$. *Furthermore, suppose that θ can be continuously extended so that $\theta(0) = 0$, then the map $X \ni x \mapsto \theta(f(x)) \in \mathcal{A}_x$ is in $\Gamma(\rho)$ (i.e. a continuous section).*

Proof. The hypotheses on θ allow us to define $\theta(f(x)) \in \mathcal{A}_x$ for each $x \in X$ via continuous functional calculus on arbitrary C^*-algebras. We shall show that $x \mapsto \theta(f(x))$ is pointwise continuous from X to $\mathcal{A}$ at all points.

To start, pick $x_0 \in X$, we know that there exists an open neighborhood U_0 such that there is an $M > 0$ where

$$\|f(x)\| \leqslant M, \qquad \forall x \in U_0.$$

We know from the complex Stone-Weierstrass theorem, that inside the disk $\Omega_{f,M} := \mathbb{C}_f \cap \{z \in \mathbb{C} : |z| \leqslant M\}$, we can find a sequence of complex polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that

$$\sup_{z\in\Omega_{f,M}} \|\theta(z) - p_n(z)\| \to 0.$$

The choice of the disk $\Omega_{f,M}$ means that $\theta(f(x))$ is well-defined for each $x \in U_0$, furthermore

$$\sup_{x\in U_0} \|\theta(f(x)) - p_n(f(x))\| \leqslant \sup_{z\in\Omega_{f,M}} \|\theta(z) - p_n(z)\| \to 0.$$

We know that each $p_n \circ f$ is a continuous section within U_0 by Lemma 6.4, so it follows from Corollary 6.3 that $\theta \circ f$ is a continuous section within U_0, in particular this means $\theta \circ f$ is continuous at the point x_0, but x_0 is arbitrary, so we are done. $\square$

Remark 6.6. A slight modification of the argument above gives us a version of the same result above for the specific case where the section f satisfies $f(x) \geqslant 0$ (instead of just $f(x)f(x)^* = f(x)^*f(x)$) for all $x \in X$, in which case we apply the continuous functional calculus to some continuous function $\theta : \mathbb{R}_f \to \mathbb{C}$ with $\mathbb{R}_f \subseteq \mathbb{R}$ such that $\mathrm{spec}(f(x)) \subseteq \mathbb{R}_f$ extendable so that $\theta(0) = 0$. We also note that we can again modify the argument above to accommodate a result for C^*-bundles that are constructed via 6.1. We write these observations as corollaries below for posterity.

Corollary 6.6. *Let $\rho : \mathcal{A} \to X$ be a C^*-bundle constructed via Theorem 6.1 with local approximating sections given by Γ. If $f \in \Gamma$ such that $ff^* = f^*f$, and $\theta : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $\theta(0) = 0$, then the map $X \ni x \mapsto \theta(f(x)) \in \mathcal{A}_x$ is in Γ.*

Corollary 6.7. *Let $\rho : \mathcal{A} \to X$ be a C^*-bundle, if $f \in \Gamma(\rho)$ such that $f(x) \geqslant 0$ for all $x \in X$, then the section $g : x \mapsto f(x)^{1/2}$ is in $\Gamma(\rho)$, and it satisfies $g = g^*$, $g^2 = f$. This implies that if $f \in \Gamma_0(\rho)$, such that $f(x) \geqslant 0$ for all $x \in X$, then f is a positive element in the C^*-algebra of sections $\Gamma_0(\rho)$*

Proof. We apply a modified version of Proposition 6.5 and use the continuous map $\theta(t) = t^{1/2}$. □

The next result will be useful, it tells us that ideals of the continuous sections that vanish at infinity $\Gamma_0(\rho)$ of a C^*-bundle behave as expected. The following proof is based on [Dix82, Lemma 10.4.2].

Proposition 6.6. *Let $\rho : \mathcal{A} \to X$ be a C^*-bundle. If I is a closed two-sided ideal of $\Gamma_0(\rho)$, and*

$$I_x := \{\gamma(x) \in \mathcal{A}_x : \gamma \in I\}, \qquad \forall x \in X.$$

Then

$$I = \{\gamma \in \Gamma_0(\rho) : \gamma(x) \in I_x\}.$$

Proof. One direction of the equality is trivial, in particular, $I \subseteq \{\gamma \in \Gamma_0(\rho) : \gamma(x) \in I_x\}$. We will work on containment in the other direction. The proof will be reminiscent of the proof of Proposition 6.4. Suppose $\gamma \in \Gamma_0(\rho)$ and $\gamma(x) \in \mathcal{B}_x = I_x$ for each $x \in X$. Fix $\varepsilon > 0$, then the set $M = \{x \in X : \|\gamma(x)\| \geqslant \frac{\varepsilon}{3}\}$ is compact. By definition, we know that for each $y \in X$, there exists an $f_y \in I$ such that $\gamma(y) = f_y(y) \in \mathcal{A}_y$. Since I is a closed two-sided ideal, we know from existence of approximate identities on ideals that there is a $g_y \in \Gamma_0(\rho)$ such that

$$\|f_y - f_y g_y\| < \frac{\varepsilon}{3}.$$

Now note that the map $x \mapsto \|\gamma(x) - f_y(x)g_y(x)\|$ is continuous, in particular, at the point y, and so we can find a neighborhood U_y of y such that

$$\|\gamma(x) - f_y(x)g_y(x)\| < \frac{\varepsilon}{3}, \qquad \forall x \in U_y. \tag{6.3}$$

Since y is arbitrary in the estimate (6.3) above, we can now find a finite open cover $\{U_i\}_{i=1}^n$ of

M, finite collections $\{f_i\}_{i=1}^n \subseteq I$, and $\{g_i\}_{i=1}^n \subseteq \Gamma_0(\rho)$ such that

$$\|\gamma(x) - f_i(x)g_i(x)\| < \frac{\varepsilon}{3}, \qquad \forall x \in U_i.$$

Take the family $\{\varphi_i\}_{i=1}^n \subseteq C(X, [0,1])$ to be a partition of unity subordinate to $\{U_i\}_{i=1}^n$. We can now define the section $f = \sum_{i=1}^n \varphi_i \cdot f_i \cdot g_i$, and we know that this is contained in I from the fact that I is an ideal and from Corollary 6.2. In a completely similar way to the proof of Proposition 6.4, we find that $\|\gamma - f\| < \varepsilon$, which means that $\gamma \in \overline{I} = I$ (since I is closed). We are done. □

Corollary 6.8. *Let $\rho : X \to \mathcal{A}$ be a C^*-bundle. If I is a closed two-sided ideal of $\Gamma_0(\rho)$ such that for each $x \in X$,*

$$I_x := \{\gamma(x) : \gamma \in I\} = \mathcal{A}_x$$

then $I = \Gamma_0(\rho)$.

Proof. We know from Proposition 6.6 that $I = \{\gamma \in \Gamma_0(\rho) : \gamma(x) \in I_x\}$, but $I_x = \mathcal{A}_x$, so $I = \{\gamma \in \Gamma_0(\rho) : \gamma(x) \in \mathcal{A}_x\}$ which is exactly $\Gamma_0(\rho)$. □

Chapter 7

Results and Applications

We present original contributions in this chapter, divided into Section 7.1 and Section 7.2, corresponding to the papers [GLL24] and [LM24], respectively. Our results in Section 7.1 rely heavily on the fact that the Heisenberg module is a Hilbert C^*-module over the higher dimensional noncommutative tori. On the other hand, the integration of operator-quantization and Heisenberg modules in Section 7.2 follows the development of quantum harmonic analysis in which the so-called *Weyl quantization scheme* is the preferred rule for operator-distribution correspondence as opposed to the Kohn-Nirenberg and Spreading schemes (see 4.12). In both cases, our chosen settings necessitate working in the Euclidean space $G = \mathbb{R}^d$.

All results in Sections 7.1 and 7.2 that are presented with proofs but without citations are original contributions.

As mentioned, we will restrict ourselves in the Euclidean case $G = \mathbb{R}^d$ in this chapter. The Haar measure is just the Lebesgue measure (up to a positive constant), and hence we shall denote integrals on $\mathbb{R}^d$ via

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x. \tag{7.1}$$

for $f : \mathbb{R}^d \to \mathbb{C}$. For any $n \in \mathbb{N}_{>0}$, we implicitly identify elements of $\mathbb{R}^n$ as either a column vector or a row vector. The standard inner product of $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y$. In formulae involving matrices, the standard interpretation of vectors in $\mathbb{R}^d$ is as a column vector, row vectors are indicated by a transpose, e.g. $x^T Jy = x \cdot (Jy)$ for $x, y \in \mathbb{R}^d$ and J a $d \times d$-matrix. We shall use the fact that $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ (as LCA groups) by identifying $\tau \in \mathbb{R}^n$, with the character $x \mapsto e^{2\pi i x \cdot \tau}$ for each $x \in \mathbb{R}^n$ [Fol15, See pg. 107]. In this case, the Heisenberg 2-cocycle (4.7) $c : \mathbb{R}^{2d} \times \widehat{\mathbb{R}^{2d}} \to \mathbb{T}$

reads

$$c((x,\omega),(t,\nu)) = e^{-2\pi i x\cdot\nu} = e^{-2\pi i \chi_1\cdot K\chi_2} \tag{7.2}$$

for $\chi_1 = (x,\omega), \chi_2 = (t,\nu) \in \mathbb{R}^{2d}$ and $K = \left(\begin{smallmatrix} 0 & I_d \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_{2d}(\mathbb{R})$. Keeping the foregoing notation, let $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \in \mathrm{GL}_{2d}(\mathbb{R})$, we find that the symplectic cocycle (4.10) can be computed via:

$$c_s((x,\omega),(t,\nu)) = e^{-2\pi i(x\cdot\nu - t\cdot\omega)} = e^{-2\pi i(\chi_1\cdot K\chi_2 - \chi_2\cdot K\chi_1)} = e^{-2\pi i(\chi_1\cdot J\chi_2)}. \tag{7.3}$$

For the sake of completeness, let us also write here the time-frequency shifts operator in the Euclidean case. For $f : \mathbb{R}^d \to \mathbb{C}$ and $\chi = (x,\omega) \in \mathbb{R}^{2d}$, we have:

$$\pi(\chi)f(y) = f(y-x)e^{2\pi i y\cdot\omega}. \tag{7.4}$$

Let us also collect handy explicit formulae for the cocycles and the time-frequency shifts. For $\chi_1, \chi_2 \in \mathbb{R}^{2d}$

$$\overline{c(\chi_1,\chi_2)} = c(-\chi_1,\chi_2) = c(\chi_1,-\chi_2) \tag{7.5}$$

$$\pi(\chi_1)\pi(\chi_2) = c(\chi_1,\chi_2)\pi(\chi_1+\chi_2); \tag{7.6}$$

$$\pi(\chi_1)\pi(\chi_2) = c_s(\chi_1,\chi_2)\pi(\chi_2)\pi(\chi_1). \tag{7.7}$$

The adjoint of the time-frequency shift is related to its reflection via:

$$\pi^*(\chi_1) = c(\chi_1,\chi_1)\pi(-\chi_1). \tag{7.8}$$

These will be relevant in our discussion of noncommutative tori later.

As remarked in Section 4.1 Chapter 4, lattices in $\mathbb{R}^{2d}$ are always of the form $\Lambda = L\mathbb{Z}^{2d}$ for some $L \in \mathrm{GL}_{2d}(\mathbb{R})$ where L is called the **matrix generator** of Λ. Later on, it will be important that we explictly distinguish the spaces generated by different matrix generators. We denote the 2-cocycle on $\mathbb{Z}^{2d}$ associated with the lattice generated by L to be $c_L : \mathbb{Z}^{2d}\times\mathbb{Z}^{2d} \to \mathbb{T}$, it is defined in terms of the Heisenberg 2-cocycle c, and it follows from (7.2) that:

$$c_L(n,m) = c(Ln,Lm) = e^{-2\pi i(Ln)^T K(Lm)}. \tag{7.9}$$

Therefore c_L is essentially the restriction of the Heisenberg 2-cocycle c to the lattice $L\mathbb{Z}^{2d}$. Furthermore, notice that $\ell^1(\Lambda) = \ell^1(L\mathbb{Z}^{2d}) \cong \ell^1(\mathbb{Z}^{2d})$, and so we have that the following twisted group C^*-algebras are isomorphic: $C^*(\Lambda, c) = C^*(L\mathbb{Z}^{2d}, c) \cong C^*(\mathbb{Z}^{2d}, c_L)$.

The adjoint lattice Λ° with respect to $L\mathbb{Z}^{2d}$, can similarly be described in terms of a lattice in $\mathrm{GL}_{2d}(\mathbb{R}^{2d})$. We choose $L^\circ = J(L^{-1})^T J^T$, to denote the associated matrix that generates Λ°, i.e. it can be checked that $L^\circ\mathbb{Z}^{2d} = \Lambda^\circ$. Note that the choice L° is not unique, and Λ° can also be generated by $J(L^{-1})^T$ (see [FK04]). If for $k > 0$, we denote $k\mathbb{Z}^{2d}$ to be $\mathbb{Z}^{2d}$ with the usual counting (Haar) measure scaled by k, then it follows from the fact that $d(\Lambda) = d(L\mathbb{Z}^{2d}) = |\det L|$, that we have $C^*(\Lambda^\circ, \overline{c}) \cong C^*\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c_{L^\circ}}\right)$. We now make explicit the dependence of the associated twisted C^*-algebras to the generating matrices. Using our notation from Chapter 5, we take $A_L := C^*(\mathbb{Z}^{2d}, c_L) \cong A_\Lambda$ and $B_L := C^*(\mathbb{Z}^{2d}, c_{L^\circ}) \cong B_\Lambda$ for $\Lambda = L\mathbb{Z}^{2d}$, $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$.

Let us also collect here the explicit structure on A_L and B_L. Respectively, they are the C^*-enveloping algebra on $\ell^1\left(\mathbb{Z}^{2d}, c_L\right)$ and $\ell^1\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c_{L^\circ}}\right)$. Given $\mathbf{a}_1, \mathbf{a}_2 \in A_L$, and $\mathbf{b}_1, \mathbf{b}_2 \in B_L$ the respective twisted structures on A_L and B_L are given by:

$$\begin{aligned} (a_1 *_{c_L} a_2)(n) &:= \sum_{k\in\mathbb{Z}^{2d}} a_1(k)a_2(n-k)c_L(k, n-k); \\ a_1^{*_{c_L}}(n) &:= \overline{a_1(-n)}c_L(n,n), \end{aligned} \tag{7.10}$$

$$\begin{aligned} (b_1 *_{\overline{c}_{L^\circ}} b_2)(n) &:= \frac{1}{|\det L|}\sum_{k\in\mathbb{Z}^{2d}} b_1(k)b_2(n-k)\overline{c_{L^\circ}(k, n-k)}; \\ b_1^{*_{\overline{c}_{L^\circ}}}(n) &:= \overline{b_1(-n)c_{L^\circ}(n,n)}. \end{aligned} \tag{7.11}$$

For $g, h \in L^2(\mathbb{R}^d)$, Gabor frame operator $S_{g,h,\Lambda}$ will be denoted by $S_{g,h,L}$ instead, given by:

$$S_{g,h,L}f = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle_2\, \pi(Lk)h, \qquad \forall f \in L^2(\mathbb{R}^d). \tag{7.12}$$

If $\Lambda = L\mathbb{Z}^{2d}$, we shall also use the abbreviation

$$\mathcal{G}(g, L) := \mathcal{G}(g, L\mathbb{Z}^{2d}) = \mathcal{G}(g, \Lambda)$$

to emphasize that Gabor systems with repect to a lattice in $\mathbb{R}^{2d}$ also depend on generating matrices.

The faithful $*$-representation and $*$-antirepresentation that appear in Theorems 5.2 & 5.3 will be denoted by $\overline{\pi}_L : A_L \to \mathcal{L}(L^2(\mathbb{R}^d))$ and $\overline{\pi}^*_{L^\circ} : B_L \to \mathcal{L}(L^2(\mathbb{R}^d))$ respectively. Explicitly, they are given by:

$$\begin{aligned}\overline{\pi}_L(\mathbf{a}) &= \sum_{k\in\mathbb{Z}^{2d}} a(k)\pi(Lk)\\ \overline{\pi}^*_{L^\circ}(\mathbf{b}) &= \frac{1}{|\det L|}\sum_{k\in\mathbb{Z}^{2d}} b(k)\pi^*(L^\circ k)\end{aligned} \tag{7.13}$$

for all $\mathbf{a}\in A_L$ and $\mathbf{b}\in B_L$.

Finally, still with $\Lambda = L\mathbb{Z}^{2d}$, we denote the Heisenberg module associated with Λ via $\mathcal{E}_L(\mathbb{R}^d)$. Using the characterization of Theorem 5.5 we have the following.

Proposition 7.1. *$\mathcal{E}_L(\mathbb{R}^d)$ is the completion of $\mathbf{S}_0(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to the following norm*

$$\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} = \inf\left\{C^{1/2} : C \text{ is a Bessel bound for } \mathcal{G}(f,L)\right\}. \tag{7.14}$$

It is an $A_L - B_L$-imprimitivity bimodule where for all $\mathbf{a}\in A_L$, $b\in B_L$, $f,g\in\mathcal{E}_L(\mathbb{R}^d)$:

1. *$\mathbf{a}\cdot f = \overline{\pi}_L(\mathbf{a})f\in\mathcal{E}_L(\mathbb{R}^d)$;*
2. *$f\cdot\mathbf{b} = \overline{\pi}^*_{L^\circ}(\mathbf{b})f\in\mathcal{E}_L(\mathbb{R}^d)$;*
3. *${}_L\langle f,g\rangle\in A_L$;*
4. *$\langle f,g\rangle_{L^\circ}\in B_L$.*

where ${}_L\langle f,g\rangle(n) = \langle f,\pi(Ln)g\rangle_2$ and $\langle f,g\rangle_{L^\circ}(n) = \langle g,\pi^*(L^\circ n)f\rangle_2$ for $n\in\mathbb{Z}^{2d}$.

Remark 7.1. It follows from faithfulness of $\overline{\pi}_L$ and $\overline{\pi}^*_{L^\circ}$ that we can also compute the norm of an arbitrary element $f\in\mathcal{E}_L(\mathbb{R}^d)$ using the formulae below:

$$\begin{aligned}\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} &= \|{}_L\langle f,f\rangle\|_{A_L}^{1/2} = \|\overline{\pi}_L\left({}_L\langle f,f\rangle\right)\|^{1/2} = \left\|\sum_{k\in\mathbb{Z}^{2d}}\langle f,\pi(Lk)f\rangle_2\,\pi(Lk)\right\|^{1/2}\\ &= \|\langle f,f\rangle_{L^\circ}\|_{B_L}^{1/2} = \|\overline{\pi}^*_{L^\circ}\left(\langle f,f\rangle_{L^\circ}\right)\|^{1/2} = \left\|\frac{1}{\det L}\sum_{k\in\mathbb{Z}^{2d}}\langle f,\pi^*(L^\circ k)f\rangle_2\,\pi^*(L^\circ k)\right\|^{1/2}.\end{aligned} \tag{7.15}$$

Remark 7.2. For the rest of the chapter, we will make references to a generic window function $\phi\in\mathbf{S}_0(\mathbb{R}^d)$, so that

$$\mathbf{S}_0(\mathbb{R}^d) := \{f\in L^2(\mathbb{R}^d) : \mathcal{V}_\phi f\in L^1(\mathbb{R}^{2d})\}.$$

7.1 Results Part I: Linear Deformation of Gabor Frames and Heisenberg Modules

In this section, we establish that there is a notion of topological "nearness" between Heisenberg modules. We address this by using the concept of Banach bundles and the fact that Heisenberg modules in the Euclidean case are actually imprimitivity bimodules over higher dimensional noncommutative tori. The strategy now is to use the deformation results on the noncommutative tori, and lift them into deformation results in the Heisenberg modules. This lifting is done by also exploiting the result showing that Heisenberg modules can be seen as Gabor frames for lattices in the time-frequency plane, and from there the seminal results by Feichtinger-Kaiblinger on linear deformations of Gabor frames [FK04] can be integrated into our work. We shall establish that the approach of Feichtinger-Kaiblinger also works for functions in Heisenberg modules, which further relies on the work of Austad and Enstad in [AE20], where they show that the Janssen representation holds for functions in Heisenberg modules.

Recall that for $g \in L^2(\mathbb{R}^d)$ and for a countable set (a.k.a 'sampling set') $\Gamma \subseteq \mathbb{R}^{2d}$, we study Gabor systems $\mathcal{G}(g,\Gamma) = \{\pi(\gamma)g : \gamma \in \Gamma\}$ as a way to represent functions in $L^2(\mathbb{R}^d)$. In case that the sampling set Γ has some additional structure, then the associated Gabor system $\mathcal{G}(g,\Gamma)$ inherits extra features. We are going to study Gabor systems where the sampling set is a lattice Λ in $\mathbb{R}^{2d}$, i.e. $\Lambda = L\mathbb{Z}^{2d}$ for some invertible 2d×2d-matrix L (i.e. $L \in \mathrm{GL}_{2d}(\mathbb{R})$).

In the case that the Gabor atom $g \in \mathbf{S}_0(\mathbb{R}^d)$ is an element of the Feichtinger's algebra $\mathbf{S}_0(\mathbb{R}^d)$, then we have a class of "nice" Gabor systems $\mathcal{G}(g,\Lambda)$. Recall that for any such Gabor system $\mathcal{G}(g,\Lambda)$ is *always* a Bessel family for *any* lattice Λ [FZ98, Theorem 3.3.1]. A myriad of other properties of $\mathbf{S}_0(\mathbb{R}^d)$ (see Proposition 4.9) make it an ideal function space for time-frequency analysis, but we shall focus on Feichtinger and Kaiblinger's stability result [FK04, Thm. 3.8] that allows deformations of the Gabor atom and of the lattice:

Theorem 7.1. *Let $g_0 \in \mathbf{S}_0(\mathbb{R}^d)$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$ such that $\mathcal{G}(g_0, L_0)$ is a Gabor frame, then there exist open sets $U_0 \ni g_0$ in $\mathbf{S}_0(\mathbb{R}^d)$ and $V_0 \ni L_0$ in $\mathrm{GL}_{2d}(\mathbb{R})$ such that $\mathcal{G}(g, L)$ is a Gabor frame for all $(g, L) \in U_0 \times V_0$.*

This type of deformation is termed *linear* because the lattice $\Lambda_0 = L_0\mathbb{Z}^{2d}$ (or its generating matrix L_0) is perturbed by a linear transformation, namely multiplication with $LL_0^{-1} \in \mathrm{GL}_{2d}$. Note that Gabor stability in terms of generalized deformations has been considered. In [GOCR15], a generalized notion of "deformation" of non-uniform sampling sets (e.g. $\Lambda \subseteq \mathbb{R}^{2d}$ possibly not a lattice) which encompasses linear deformations on lattices as a special case, was studied. On the other hand, linear deformations for lattices on the abstract time-frequency plane $G \times \widehat{G}$ for G a locally compact abelian group was studied in [EJLO22] by considering the Braconnier topology on the automorphism group of $G \times \widehat{G}$.

In this paper, our main concern is to extend Theorem 7.1 to the so-called Heisenberg modules. Recently, it was shown by Austad and Enstad in [AE20] that the Heisenberg modules can be seen as function spaces, in that they can be constructed by completing Feichtinger's algebra $\mathbf{S}_0(\mathbb{R}^d)$ in the following norm:

$$\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} := \{C^{\frac{1}{2}} : C \text{ is a Bessel bound for } \mathcal{G}(f, L)\},$$

for each $f \in \mathbf{S}_0(\mathbb{R}^d)$ and $L \in \mathrm{GL}_{2d}(\mathbb{R})$. In this description, it is now apparent that the Heisenberg module, which we denote by $\mathcal{E}_L(\mathbb{R}^d)$, depends on the generating matrix $L \in \mathrm{GL}_{2d}(\mathbb{R})$. Unfortunately, we are now also met with an immediate ambiguity trying to implement our idea of extending Theorem 7.1 to the Heisenberg modules. Suppose $g_{L_0} \in \mathcal{E}_{L_0}(\mathbb{R}^d)$ such that $\mathcal{G}(g_{L_0}, L_0)$ is a frame, it is not clear what we mean for values g such that g is in some Heisenberg module "near" g_{L_0}, where we have that $\mathcal{G}(g, L)$ are all frames in $L^2(\mathbb{R}^d)$ whenever L is close to L_0. In general, because the construction of a Heisenberg module depends on the generating matrix L, we should expect the 'nearby' g's to vary continuously with the matrices $L \in \mathrm{GL}_{2d}(\mathbb{R})$ as well. With this in mind, what we would like to do is to look at families of the Heisenberg modules $\{\mathcal{E}_L(\mathbb{R}^d)\}_L$ for L 'near' L_0 such that there also are families of elements $g_L \in \mathcal{E}_L(\mathbb{R}^d)$ that are 'sufficiently near' $g_{L_0} \in \mathcal{E}_{L_0}(\mathbb{R}^d)$. What we mean by 'nearness' between different Banach spaces is not clear at this point, but it can be made rigorous by the language of bundles. To make it precise, what we want is to be able to define a Banach bundle

$$\kappa\colon \mathcal{E} \to \mathrm{GL}_{2d}(\mathbb{R}), \quad \mathcal{E} = \bigsqcup_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$$

such that, for each $L \in \mathrm{GL}_{2d}(\mathbb{R})$, we have the fiber $\kappa^{-1}(L) = \mathcal{E}_L(\mathbb{R}^d)$ and such that the (constant) sections associated with $f \in \mathbf{S}_0(\mathbb{R}^d) \subset \mathcal{E}_L(\mathbb{R}^d)$ are continuous. The construction of such a Banach bundle is lifted directly from corresponding deformation results on higher dimensional noncommutative tori (in this case, the noncommutative tori generated by the unitaries T_x, M_ω with $(x, \omega) \in \Lambda$). Once $\kappa\colon \mathcal{E} \to \mathrm{GL}_{2d}(\mathbb{R})$ has been constructed, we now have access to its continuous sections $\Gamma(\kappa)$, which are 'continuous' maps $\Upsilon\colon \mathrm{GL}_{2d}(\mathbb{R}^d) \to \mathcal{E}$ with the following property: $\Upsilon(L) \in \mathcal{E}_L(\mathbb{R}^d)$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$. The Gabor-stability theorem for the Heisenberg modules can now be stated in terms of these continuous sections, which is one of our main results (see Theorem 7.5). As a corollary, we shall obtain a no-go Balian-Low type [AFK14] result for Heisenberg modules. We will also study the algebraic properties of the space $\Gamma_0(\kappa)$, the space of continuous section of κ that vanish at infinity. We shall show that we can equip it with a structure which shows that it is an imprimitivity bimodule over certain C^*-algebras.

7.1.1 Intermission: Heisenberg Modules from Weighted Spaces

In the sequel we shall need a technical result which involves weighted spaces. For a fixed $L \in \mathrm{GL}_{2d}(\mathbb{R})$, recall from Section 5.3 of Chapter 5 that one can construct the Heisenberg modules by considering $\mathbf{S}_0(\mathbb{R}^d)$ as a pre-imprimitivity bimodule over $\ell^1(\mathbb{Z}^{2d}, c_L)$ and $\ell^1\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c_{L^\circ}}\right)$. Our goal in this intermission is to discuss that we can also construct the Heisenberg modules as a pre-imprimitivity bimodule over weighted version of the aforementioned spaces.

We denote the ℓ^1-norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by $|x|$, and define the **weight** $\nu \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\nu(x) = 1 + |x|, \qquad x \in \mathbb{R}^n. \tag{7.16}$$

In particular for $n = 2d$, we now obtain the **weighted Feichtinger's algebra $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$**:

$$\mathbf{S}_{0,\nu}(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \mathcal{V}_\phi f \cdot \nu \in L^1(\mathbb{R}^{2d}) \right\}. \tag{7.17}$$

Some of the properties of the unweighted case are inherited by $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$, in particular we have the following [AL21]:

Proposition 7.2. *$\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ is a Banach space with the norm $\|f\|_{\mathbf{S}_{0,\nu}} := \|\mathcal{V}_\phi f \cdot \nu\|_1$. If ϕ is replaced with any $g \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, then we obtain an equivalent norm. Furthermore, $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ is continuously and densely embedded in $\mathbf{S}_0(\mathbb{R}^d)$ satisfying*

$$\|f\|_{\mathbf{S}_0} \leqslant \|f\|_{\mathbf{S}_{0,\nu}}, \qquad \text{for all } \; f \in \mathbf{S}_{0,\nu}(\mathbb{R}^d). \tag{7.18}$$

Next, we introduce the **weighted ℓ^1-space**:

$$\ell^1_\nu(\mathbb{Z}^{2d}) := \left\{ \mathbf{a} = \{a(k)\}_{k \in \mathbb{Z}^{2d}} : \|\mathbf{a}\|_{\ell^1_\nu} := \sum_{k \in \mathbb{Z}^{2d}} |a(k)|\nu(k) < \infty \right\} \tag{7.19}$$

(recall that $\nu(k) = 1 + |k|$). We have a result analogous to Proposition 7.2, which is significantly easier to obtain (i.e. one can prove density in exactly the same way as proving density of finite-sequences $c_{00}(\mathbb{Z}^{2d})$ in $\ell^1(\mathbb{Z}^{2d})$).

Proposition 7.3. *The weighted space $\ell^1_\nu(\mathbb{Z}^{2d})$ is a Banach space with norm $\|\cdot\|_{\ell^1_\nu}$, and is densely continuously embedded in $\ell^1(\mathbb{Z}^{2d})$ satisfying:*

$$\|\mathbf{a}\|_{\ell^1} \leqslant \|\mathbf{a}\|_{\ell^1_\nu}, \qquad \forall \mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d}). \tag{7.20}$$

We can equip $\ell^1_\nu(\mathbb{Z}^{2d})$ with relevant twisted structure such that it becomes a Banach $*$-algebra, whose C^*-enveloping algebra coincide with our C^*-algebras of interest. We summarize this idea in the following results, which can both be found in [AL21].

Proposition 7.4. *For a fixed $L \in \mathrm{GL}_{2d}(\mathbb{R})$, $\ell^1_\nu(\mathbb{Z}^{2d})$ becomes a Banach $*$-algebra when equipped with twisted convolution and involution as in Equations (7.10), which we denote by $\ell^1_\nu(\mathbb{Z}^{2d}, c_L)$. The C^*-enveloping algebra of $\ell^1_\nu(\mathbb{Z}^{2d}, c_L)$ coincides with A_L.*

Proposition 7.5. *For a fixed $L \in \mathrm{GL}_{2d}(\mathbb{R})$, $\ell^1_\nu(\mathbb{Z}^{2d})$ becomes a Banach $*$-algebra when equipped with twisted convolution and involution as in Equations (7.11), which we denote by $\ell^1_\nu\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c_{L^\circ}}\right)$. The C^*-enveloping algebra of $\ell^1_\nu\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c_{L^\circ}}\right)$ coincides with B_L.*

We finally obtain the result from [AL21, Proposition 3.14], which, along with Proposition 7.1, gives us a concrete version of Heisenberg modules that also takes the weighted spaces into consideration.

Proposition 7.6. *$\mathbf{S}_{0,\nu}(\mathbb{R}^{2d})$ is a $\ell^1(\mathbb{Z}^{2d}, c_L) - \ell^1\left(\frac{1}{|\det L|}\mathbb{Z}^{2d}, \overline{c}_{L^\circ}\right)$ pre-imprimitivity bimodule using the structure defined exactly in Proposition 7.1 whose completion in the sense of Proposision 2.6 coincides with $\mathcal{E}_L(\mathbb{R}^d)$.*

Remark 7.3. The module structure of $\mathcal{E}_L(\mathbb{R}^d)$ respects the continuous dense embeddings

$$\mathbf{S}_{0,\nu}(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d) \hookrightarrow \mathcal{E}_L(\mathbb{R}^d).$$

For example, if $f, g \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$ and $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$, then ${}_L\langle f, g\rangle \in \ell^1_\nu(\mathbb{Z}^{2d})$ and $\mathbf{a}f \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$.

7.1.2 Noncommutative Tori

We denote by $\mathrm{Skew}_n(\mathbb{R})$ the set of skew-symmetric matrices in $\mathrm{M}_n(\mathbb{R})$.

Definition 7.1. Let $\Theta \in \mathrm{Skew}_n(\mathbb{R})$, then the **noncommutative n-torus determined by Θ** is the universal unital C^*-algebra generated by n-generators $U_1(\Theta), \ldots, U_n(\Theta)$ satisfying the relations:

1. $U_i^*(\Theta) = U_i^{-1}(\Theta)$ for all $i = 1, \ldots, n$;

2. $U_i(\Theta)U_j(\Theta) = e^{2\pi i\,\Theta_{ij}} U_j(\Theta)U_i(\Theta)$ for all $i, j = 1, \ldots, n$.

We denote by $\mathcal{A}_\Theta$ the noncommutative n-torus determined by Θ.

Remark 7.4. There is an equivalent way of determining a noncommutative n-torus by just looking at elements of $\mathbb{T}^{n(n-1)/2} \ni \lambda = (\lambda_{i,j})_{1 \leqslant i < j \leqslant n}$, and from here we can define the relations on the generators $U_1(\lambda), \ldots, U_n(\lambda)$ of noncommutative n-torus determined by $\lambda \in \mathbb{T}^{n(n-1)/2}$ to be

1. $U_i(\lambda)^* = U_i^{-1}(\lambda)$ for all $i = 1, \ldots, n$;

2. $U_i(\lambda)U_j(\lambda) = \lambda_{i,j} U_j(\lambda)U_i(\lambda)$ for all $1 \leqslant i < j \leqslant n$.

Similar to the original definition, we denote by $\mathcal{A}_\lambda$ the noncommutative n-torus determined by $\lambda \in \mathbb{T}^{n(n-1)/2}$. Note that one can go from $\Theta \in \mathrm{Skew}_n(\mathbb{R})$ to $\lambda = \mathbb{T}^{n(n-1)/2}$ simply by the map $\Theta \mapsto \lambda = (e^{2\pi i \Theta_{1,2}}, e^{2\pi i \Theta_{1,3}}, \ldots, e^{2\pi i\,\Theta_{n-1,1}})$, so that $\mathcal{A}_\Theta \cong \mathcal{A}_\lambda$. Note however that while $\Theta \mapsto \lambda$ is surjective, we will never find a global inverse for this map since the complex exponential function itself is not globally invertible.

There is yet another way to see these higher dimensional noncommutative tori. Fix the following: Let $\zeta \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ be a normalized 2-cocycle on $\mathbb{Z}^n$ (i.e. it satisfies formulae 5.16 on $\mathbb{Z}^n$), equip $\mathbb{Z}^n$ with the counting measure[1] and construct the twisted group C^*-algebra $C^*(\mathbb{Z}^n, \zeta)$ (see Definition 5.7). Recall that the twisted $*$-algebra structure on $C^*(\mathbb{Z}^n, \zeta)$ is defined densely by:

$$
\begin{aligned}
(a_1 *_\zeta a_2)(m) &:= \sum_{k \in \mathbb{Z}^n} a_1(k) a_2(m-k) \zeta(k, m-k), \\
a_1^{*_\zeta}(m) &:= \overline{a_1(-m)\zeta(m,-m)}
\end{aligned}
$$

for all $k, m \in \mathbb{Z}^n$, and $\mathbf{a}_1, \mathbf{a}_2 \in \ell^1(\mathbb{Z}^n)$.

[1] The construction can be done for any Haar measure on $\mathbb{Z}^n$ of course.

Remark 7.5. For $M \in \mathrm{M}_n(\mathbb{R})$, we shall use the notation $\zeta_M\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ to denote the 2-cocycle induced by M:

$$\zeta_M(k,m) = e^{2\pi i k^T \cdot M m}, \qquad \forall k, m \in \mathbb{Z}^n. \tag{7.21}$$

Before we proceed, we recognize, and we will make references to the work of Gjertsen in [Gje23], who carried out very explicit computations on the noncommutative tori and twisted group C^*-algebras on $\mathbb{Z}^n$. Now, if we have a noncommutative n-torus $\mathcal{A}_\Theta$, then by definition of $\mathcal{A}_\Theta$ as a free $*$-algebra generated by $\{U_i(\Theta)\}_{i=1}^n$ and its inherent commutation relations, a typical element in $\mathcal{A}_\Theta$ consists of

$$\sum_{k\in\mathbb{Z}^n} a(k) U_1^{k_1}(\Theta) \cdot \ldots \cdot U_n^{k_n}(\Theta) \tag{7.22}$$

with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, and $\mathbf{a} = \{a(k)\}_{k\in\mathbb{Z}^n} \in c_{00}(\mathbb{Z}^n)$. If we use the notation $U^k(\Theta) = U_1^{k_1} \cdot \ldots \cdot U_n^{k_n}(\Theta)$, then by a careful book-keeping of the appearing phase-factors, as was done in [Gje23, Lemma 4.2.3], one finds that

$$U^k(\Theta) U^m(\Theta) = \exp\left(2\pi i \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} m_i \Theta_{j,i} k_j\right) U^{k+m}. \tag{7.23}$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} m_i \Theta_{j,i} k_j &= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} m_i \Theta_{i,j} k_j \\ &= -m^T \Theta^{\mathrm{up}} k \\ &= k^T (-\Theta^{\mathrm{up}})^T m = k^T \Theta^{\mathrm{low}} m \end{aligned}$$

where Θ^{up} and Θ^{low} are the strictly upper and lower triangular parts of Θ respectively. We then obtain $\zeta_{\Theta^{\mathrm{low}}}\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$, the 2-cocycle induced by Θ^{low} using the notation of (7.21). We now see an emerging structure on $\mathcal{A}_\Theta$, consider now two typical elements $\mathbf{a}(\Theta) := \sum_{k\in\mathbb{Z}^n} a(k) U^k(\Theta)$ and $\mathbf{b}(\Theta) := \sum_{m\in\mathbb{Z}^n} b(m) U^m(\Theta)$, then we find that

$$\mathbf{a}(\Theta) \cdot \mathbf{b}(\Theta) = \left(\sum_{k\in\mathbb{Z}^n} a(k) U^k(\Theta)\right) \cdot \left(\sum_{k\in\mathbb{Z}^n} b(m) U^m(\Theta)\right)$$

$$= \sum_{k,m\in\mathbb{Z}^n} a(k)b(m)U^k(\Theta)U^m(\Theta)$$
$$= \sum_{k,m\in\mathbb{Z}^n} a(k)b(m)\zeta_{\Theta^{\mathrm{low}}}(k,m)U^{k+m}$$
$$= \sum_{k,m\in\mathbb{Z}^n} a(k)b(m-k)\zeta_{\Theta^{\mathrm{low}}}(k,m-k)U^{m}.$$

The last expression should remind us of twisted convolutions, in fact if we make the identification $\delta_k \mapsto U^k$ for all $k \in \mathbb{Z}^n$ then we find that the multiplication above is exactly the multiplication on the twisted C^*-algebra $C^*(\mathbb{Z}^n, \zeta_{\Theta^{\mathrm{low}}})$. The explicit isomorphism is provided by $\pi_\Theta \colon C^*(\mathbb{Z}^n, \zeta_{\Theta^{\mathrm{low}}}) \to \mathcal{A}_\Theta$, densely defined via:

$$\pi_\Theta(\mathbf{a}) = \sum_{k\in\mathbb{Z}^n} a(k)U^k(\Theta), \qquad \forall \mathbf{a} \in \ell^1(\mathbb{Z}^n) \hookrightarrow C^*(\mathbb{Z}^n, \zeta_{\Theta^{\mathrm{low}}}). \tag{7.24}$$

We can easily verify this by checking that the map is a $*$-homomorphism on the generators. Let $i, j = 1, ..., n$ then we compute:

$$\pi_\Theta(\delta_{e_i} * \delta_{e_j}) = \pi_\Theta(\zeta_{\Theta^{\mathrm{low}}}(e_i, e_j)\delta_{e_i+e_j}) = \zeta_{\Theta^{\mathrm{low}}}(e_i, e_j)\pi_\Theta(\delta_{e_i+e_j}).$$

Suppose $i \leqslant j$, then we find that $\zeta_{\Theta^{\mathrm{low}}}(e_i, e_j) = e^{2\pi i 0} = 1$, and so $\pi_\Theta(\delta_{e_i} * \delta_{e_j}) = 1 \cdot \pi_\Theta(\delta_{e_i+e_j}) = U_iU_j = \pi_\Theta(\delta_{e_i})\pi_\Theta(\delta_{e_j})$. On the other hand, if $i \geqslant j$, we find that $\zeta_{\Theta^{\mathrm{low}}}(e_i, e_j) = e^{2\pi i\Theta_{i,j}}$, and so $\pi_\Theta(\delta_{e_i} * \delta_{e_j}) = e^{2\pi i\Theta_{i,j}}U_jU_i = U_iU_j = \pi_\Theta(\delta_i)\pi_\Theta(\delta_j)$. Furthermore, we also have:

$$\pi_\Theta(\delta_{e_i}^*) = \overline{\zeta_\Theta(e_i, -e_i)}\pi_\Theta(\delta_{-e_i}) = e^{-2\pi i\Theta_{i,i}^{\mathrm{low}}}U_i^{-1} = U_i^* = \pi_\Theta(\delta_{e_i})^*.$$

We summarize what we have in a proposition:

Proposition 7.7. *The map π_Θ of Equation (7.24) extends to an isomorphism $\pi_\Theta \colon C^*(\mathbb{Z}^n, \zeta_{\Theta^{\mathrm{low}}}) \to \mathcal{A}_\Theta$.*

On the other hand, if we consider an arbitrary 2-cocycle $\zeta \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$, it is well-known that $C^*(\mathbb{Z}^n, \zeta)$ is isomorphic to some noncommutative torus. In fact it follows by an elementary computation that:

$$\delta_{e_i} *_\zeta \delta_{e_j} = \zeta(e_i, e_j)\overline{\zeta(e_j, e_i)}\delta_{e_j} *_\zeta \delta_{e_i}, \qquad \forall i, j = 1, ..., n.$$

We can always find $\Theta \in \mathrm{Skew}_n(\mathbb{R})$ such that $\Theta_{i,j} = \zeta(e_i, e_j)\overline{\zeta(e_j, e_i)}$. Merely by looking at the generators $\{\delta_{e_i}\}_{i=1}^n$, we find that $C^*(\mathbb{Z}^n, \zeta) \cong \mathcal{A}_\Theta$. However for our purposes, we want to find an explicit isomorphism that respects the typical presentation (7.22) of an element in a noncommutative torus. By also carefully keeping track of the appearing 2-cocycles, it can be shown, as was done in [Gje23, Lemma 4.2.4] that for any 2-cocycle $\zeta\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$, there exists a unique $P_\zeta\colon \mathbb{Z}^n \to \mathbb{T}$ such that for all $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$

$$\delta_k = P_\zeta(k)\delta_{e_1}^{k_1} *_\zeta \ldots *_\zeta \delta_{e_n}^{k_n}$$

where $\mathbf{a}^{k_i}$ denotes k_i-exponentiation of $\mathbf{a}$ in $C^*(\mathbb{Z}^n, \zeta)$. The map P_ζ is explicitly given by:

$$P_\zeta(k) = \prod_{i=1}^{n-1} \overline{\zeta\left(k_i e_i, \sum_{j=i+1}^{n} k_j e_j\right)} \times \prod_{i=1}^{n}\prod_{j=1}^{k_i-1} \overline{\zeta(e_i, (k_i - j)e_i)} \tag{7.25}$$

and it can be seen, that as expected, $P_\zeta(e_i) = 1$ and $P_\zeta(e_i + e_j) = \overline{\zeta}(e_i, e_j)$ whenever $i < j$ for all $i, j = 1, \ldots, n$. Each $\mathbf{a} \in C^*(\mathbb{Z}^n, \zeta)$ can be written as:

$$\mathbf{a} = \sum_{k \in \mathbb{Z}^n} a(k)P_\zeta(k)\delta_{e_1}^{k_1} *_\zeta \ldots *_\zeta \delta_n^{k_n}. \tag{7.26}$$

The map

$$\mathbf{a} = \sum_{k \in \mathbb{Z}^n} a(k)P_\zeta(k)\delta_{e_1}^{k_1} *_\zeta \ldots *_\zeta \delta_{e_n}^{k_n} \mapsto \sum_{k \in \mathbb{Z}^n} a(k)P_\zeta(k)U^k(\Theta), \qquad \mathbf{a} \in \ell^1(\mathbb{Z}^n) \hookrightarrow C^*(\mathbb{Z}^n, \zeta)$$

extends to a C^*-isomorphism, mapping $C^*(\mathbb{Z}^n, \zeta)$ to $\mathcal{A}_\Theta$. Note that this is consistent with Proposition 7.7, given that in the case $\zeta = \zeta_{\Theta^{\text{low}}}$, we have $P_{\zeta_{\Theta^{\text{low}}}}(k) = 1$ for all $k \in \mathbb{Z}^n$. We do not find much practical use for the general isomorphism $C^*(\mathbb{Z}^n, \zeta) \cong \mathcal{A}_\Theta$ for an arbitrary ζ. It is enough for us that we always have the general presentation (7.26) for twisted C^*-algebras, and that we know Proposition 7.7.

Definition 7.2. Two (continuous) 2-cocycles $\zeta_1, \zeta_2 : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ are said to be **cohomologous**[2], denoted by $\zeta_1 \cong \zeta_2$, if and only if there exists a (continuous) map $\rho : \mathbb{Z}^n \to \mathbb{T}$ called a **1-cochain** such that

$$\zeta_1(k, m) = \zeta_2(k, m)\frac{\rho(k + m)}{\rho(k)\rho(m)}, \qquad \forall (k, m) \in \mathbb{Z}^n \times \mathbb{Z}^n. \tag{7.27}$$

[2] This definition actually extends to any locally compact abelian groups

It is not hard to see that cohomology on 2-cocycles is an equivalence relation. Since we consider normalized 2-cocycles, it follows that 1-cochains satisfy $\rho(0) = 1$. Now, for two normalized 2-cocycles $\zeta_1, \zeta_2 \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$, it turns out that $\zeta_1 \cong \zeta_2$ implies $C^*(\mathbb{Z}^n, \zeta_1) \cong C^*(\mathbb{Z}^n, \zeta_2)$. This is known (e.g. see the discussion on twisted C^*-algebras in [EL08]), and it will be useful for us to write out the exact isomorphism that identifies the two.

Proposition 7.8. *If $\zeta_1 \cong \zeta_2$ by a 1-cochain ρ as in (7.27), define for each $\mathbf{a} \in \ell^1(\mathbb{Z}^n) \hookrightarrow C^*(\mathbb{Z}^n, \zeta_1)$:*

$$\widetilde{\rho}(\mathbf{a}) = \mathbf{a} \cdot \overline{\rho}.$$

Then $\widetilde{\rho}$ extends to a C^*-isomorphism $\widetilde{\rho} : C^*(\mathbb{Z}^n, \zeta_1) \to C^*(\mathbb{Z}^n, \zeta_2)$. Furthermore, for each $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, fix:

$$\rho^k := \rho^{k_1}(e_1) \cdot \ldots \cdot \rho^{k_n}(e_n),$$

then $P_{\zeta_1}(k)\rho(k) = P_{\zeta_2}(k)\rho^k$ for all $k \in \mathbb{Z}^n$.

Proof. $\widetilde{\rho}$ is obviously linear, injective, and surjective as a map from $\ell^1(\mathbb{Z}^n)$ to itself. Next we show that it is a $*$-homomorphism on $\ell^1(\mathbb{Z}^n)$ respecting the relevant twisted structures, and it is enough to show this property on the generators. Let $i, j = 1, \ldots n$. The following computations are valid due to (7.27):

$$\begin{aligned}
\widetilde{\rho}(\delta_{e_i} *_{\zeta_1} \delta_{e_j}) &= \widetilde{\rho}(\zeta_1(e_i, e_j)\delta_{e_i+e_j}) = \zeta_1(e_i, e_j)\delta_{e_i+e_j}\overline{\rho}(e_i + e_j) \\
&= \zeta_1(e_i, e_j)\overline{\zeta_2}(e_i, e_j)\overline{\rho}(e_i + e_j)(\delta_{e_i} *_{\zeta_2} \delta_{e_j}) \\
&= (\overline{\rho}(e_i)\delta_{e_i}) *_{\zeta_2} (\overline{\rho}(e_j)\delta_{e_j}) = \widetilde{\rho}(\delta_{e_i}) *_{\zeta_2} \widetilde{\rho}(\delta_{e_j}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\widetilde{\rho}(\delta_{e_i}^{*_{\zeta_1}}) &= \widetilde{\rho}(\overline{\zeta_1}(e_i, -e_i)\delta_{-e_i}) = \overline{\zeta_1}(e_i, -e_i)\delta_{-e_i}\overline{\rho}(-e_i) \\
&= \overline{\zeta_2}(e_i, -e_i)\rho(e_i)\delta_{-e_i} = \rho(e_i)\delta_{e_i}^{*_{\zeta_2}} \\
&= (\overline{\rho}(e_i)\delta_{e_i})^{*_{\zeta_2}} = \widetilde{\rho}(\delta_{e_i})^{*_{\zeta_2}}.
\end{aligned}$$

Therefore, $\widetilde{\rho}$ extends to an injective and surjective $*$-homomorphism $\widetilde{\rho} : C^*(\mathbb{Z}^n, \zeta_1) \to C^*(\mathbb{Z}^n, \zeta_2)$. Recall that injective $*$-homomorphisms on C^*-algebras are automatically isometric, therefore $\widetilde{\rho}$

is a C^*-isomorphism. Next, we write $\delta_k = P_{\zeta_1}(k)\delta_{e_1}^{k_1} *_{\zeta_1} \ldots *_{\zeta_1} \delta_{e_n}^{k_n}$, hence we have:

$$\widetilde{\rho}(\delta_k) = \delta_k\overline{\rho}(k) = \widetilde{\rho}(P_{\zeta_1}(k)\delta_{e_1}^{k_1} *_{\zeta_1} \ldots *_{\zeta_1} \delta_{e_n}^{k_n}) = P_{\zeta_1}(k)\overline{\rho}^k\delta_{e_1}^{k_1} *_{\zeta_2} \ldots *_{\zeta_2} \delta_{e_n}^{k_n} = P_{\zeta_1}(k)\overline{\rho}^k\overline{P_{\zeta_2}}(k)\delta_k,$$

therefore $P_{\zeta_1}(k)\rho(k) = P_{\zeta_2}(k)\rho^k$. □

Remark 7.6. Regarding cohomology, the 2-cocycle c that we used to give $\ell^1(\mathbb{Z}^{2d})$ a twisted structure is different from the one suggested in some references, e.g [FK04, GL04]. This is because the time-frequency shifts considered in these papers were of the form T_xM_ω, amounting to a cocycle $c'(\chi_1, \chi_2) = e^{2\pi i \chi_2^T K \chi_1}$ instead of the one given by (7.2). The difference however is inconsequential since c and c' are cohomologous via the 1-cochain $\rho(\chi) = e^{-2\pi i \omega \cdot x}$ for $\chi = (x, \omega) \in \mathbb{R}^{2d}$. Therefore they generate isomorphic C^*-algebras.

7.1.3 Deforming the Noncommutative Tori

Consider the noncommutative n-tori determined by skew-symmetric matrices $\Theta \in \mathrm{Skew}_n(\mathbb{R})$ (or by $\lambda \in \mathbb{T}^{n(n-1)/2}$, see Remark 7.4). We see that as we vary Θ (or λ) we find possibly different C^*-algebras, hence it is a reasonable question to ask if families of noncommutative n-tori define a C^*-bundle over the space $\mathrm{Skew}_n(\mathbb{R})$ (or $\mathbb{T}^{n(n-1)/2}$) as in Definition 6.2. The answer is affirmative, but the precise formulation can be quite technical, there are related 'deformation' results in this direction. The case $n = 2$ is quite well studied in the past, note that in this case there really is just one parameter $\lambda \in \mathbb{T}$ fully determining the noncommutative 2-torus $\mathcal{A}_\lambda$. It was noted by Elliot in [Ell82] that the C^*-algebras $\{\mathcal{A}_\lambda\}_{\lambda\in\mathbb{T}}$ do form a C^*-bundle, but the associated C^*-algebra of sections (that vanish at infinity) was not really expounded on. It turns out, due to Anderson and Paschke [AP89], that if H_3 is the so-called discrete 3-dimensional Heisenberg group, then the group C^*-algebra $C^*(H_3)$ is the section C^*-algebra for the C^*-bundle of noncommutative 2-tori. H_3 is an example of a 2-step nilpotent group of rank 3, but more generally we can consider H_n, the 2-step nilpotent group of rank n. As expected $C^*(H_n)$ is the section C^*-algebra of a C^*-bundle over $\mathbb{T}^{n(n-1)/2}$ whose fibers coincide with the noncommutative n-tori $\mathcal{A}_\lambda$ for each $\lambda \in \mathbb{T}^{n(n-1)/2}$ [Oml15].

There are, on the other hand, explicit results regarding $\frac{1}{2}$-Hölder continuity of the unitary generators of the noncommutative n-tori. Let us again go back to the $n = 2$ case, the classic Haagerup and Rørdam result [HR95] is that for a fixed Hilbert H, there exist paths $u, v\colon [0, 1] \to \mathcal{L}(H)$ of unitary operators such that for all $\theta \in [0, 1]$, $u(\theta)v(\theta) = e^{2\pi i\theta}v(\theta)u(\theta)$ and there exists a $C > 0$ where $\max\{\|u(\theta) - u(\theta')\|, \|v(\theta) - v(\theta')\|\} \leqslant C|\theta - \theta'|^{1/2}$. This $\frac{1}{2}$-Hölder continuity generalizes to $n > 2$, as was shown by Gao in [Gao18]. The deformation results on the unitary generators can be lifted to a deformation result of their noncommutative polynomials (see the proof of Theorem 7.2 below), which would allow us to construct a C^*-bundle whose fibers coincide with the higher dimensional noncommutative tori. We shall also borrow techniques from the theory of dilations [GS20].

Let us recall the isomorphism $\pi_\Theta : C^*(\mathbb{Z}^n, \zeta_{\Theta^{\text{low}}}) \to \mathcal{A}_\Theta$ from Equation (7.24), we have the following estimate involving the weighted sequence space $\ell^1_\nu(\mathbb{Z}^n)$ (treated as a dense subspace of $C^*(\mathbb{Z}^n, \zeta_{\Theta^{\text{low}}})$).

Theorem 7.2. *There is a constant $C > 0$ such that for all $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^n)$ and $\Theta, \Theta' \in \mathrm{Skew}_n(\mathbb{R})$:*

$$\big|\|\pi_\Theta(\mathbf{a})\| - \|\pi_{\Theta'}(\mathbf{a})\|\big| \leqslant C\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta - \Theta'\|^{1/2}.$$

Let us present two proofs, a short one with bad control on C and a slightly longer one with $C = 2\sqrt{\pi} < 4$.

Short Proof. By [Gao18, Theorem 1.1], there exist (automatically faithful) representations $U_i(\Theta) \mapsto V_i(\Theta) \in \mathcal{L}(H)$ on the same Hilbert space H such that

$$\|V_i(\Theta) - V_i(\Theta')\| \leqslant \hat{C}\Big(\sum_{j=1}^{n}(\Theta_{ij} - \Theta'_{ij})^{\frac{1}{2}}\Big) \leqslant C\|\Theta - \Theta'\|^{\frac{1}{2}}$$

for some constants $\hat{C}, C \in \mathbb{R}_+$, which may depend on n, and all $\Theta, \Theta' \in \mathrm{Skew}_n(\mathbb{R})$. Let $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^n)$, $\nu \geqslant 1$. Note that $\|\pi_\Theta(\mathbf{a})\| = \|\sum_{k\in\mathbb{Z}^n} a(k)V^k(\Theta)\|$. By an elementary estimate (cf. [GS20, Lemma 4.1]), for each $k \in \mathbb{Z}^n$ we have

$$\|V^k(\Theta) - V^k(\Theta')\| \leqslant |k| \max_i \|V_i(\Theta) - V_i(\Theta')\| \leqslant C|k|\|\Theta - \Theta'\|^{\frac{1}{2}}$$

and therefore

$$\begin{aligned}\Big|\Big\|\sum_{k\in\mathbb{Z}^d} a(k)V^k(\Theta)\Big\| - \Big\|\sum_{k\in\mathbb{Z}^d} a(k)V^k(\Theta')\Big\|\Big| &\leqslant \Big\|\sum_{k\in\mathbb{Z}^d} a(k)V^k(\Theta) - \sum_{k\in\mathbb{Z}^d} a(k)V^k(\Theta')\Big\| \\ &\leqslant \sum_{k\in\mathbb{Z}^d} |a(k)|\|V^k(\Theta) - V^k(\Theta')\| \leqslant C\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta - \Theta'\|^{\frac{1}{2}},\end{aligned}$$

as claimed. □

Proof with $C = 2\sqrt{\pi}$. Fix $\Theta, \Theta' \in \mathrm{Skew}_n(\mathbb{R})$. By [GPSS21, Theorem 5.4], there exist faithful representations $U_i(\Theta) \mapsto W_i(\Theta) \in \mathcal{L}(H)$ of $\mathcal{A}_\Theta$ on some Hilbert space H and $U_i(\Theta') \mapsto W_i(\Theta')$ of $\mathcal{A}_{\Theta'}$ on a Hilbert space H' containing H as a subspace such that compression of $W_i(\Theta')$ to H yields $P_H W_i(\Theta')|_H = cW_\Theta$ with $c = \exp(-\frac{\pi}{2}\|\Theta' - \Theta\|)$.[3] Denote by $R_i := P_{H^\perp} W_i(\Theta')|_{H^\perp} \in B(H^\perp)$ the corresponding compression of $W_i(\Theta')$ to $H^\perp \subset H'$, so that $W_i(\Theta) \oplus R_i \in B(H')$. Note that $\|R_i\| \leqslant \|W_i(\Theta')\| = 1$, so each R_i is a compression. From [GS20, Lemma 4.3] and the elementary inequality $1 - e^{-x} \leqslant x$ for $x \in \mathbb{R}$, we conclude that

$$\|W_i(\Theta') - W_i(\Theta) \oplus R_i\| \leqslant 2\sqrt{1 - c^2} \leqslant 2\sqrt{1 - e^{-\pi\|\Theta - \Theta'\|}} \leqslant 2\sqrt{\pi}\|\Theta - \Theta'\|^{\frac{1}{2}}.$$

Let $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^n)$, $\nu \geqslant 1$. By faithfulness, $\|\pi_\Theta(\mathbf{a})\| = \|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta)\|$ and $\|\pi_{\Theta'}(\mathbf{a})\| = \|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta')\|$. As in the first proof, it follows that for each $k \in \mathbb{Z}^n$ we have

$$\|W^k(\Theta') - W^k(\Theta) \oplus R^k\| \leqslant |k| \max_i \|W_i(\Theta') - W_i(\Theta) \oplus R_i\| \leqslant 2\sqrt{\pi}|k|\|\Theta - \Theta'\|^{\frac{1}{2}}$$

and therefore

$$\Big\|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta') - \sum_{k\in\mathbb{Z}^n} a(k)(W^k(\Theta) \oplus R^k)\Big\| \leqslant 2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta - \Theta'\|^{\frac{1}{2}}$$

This yields

$$\begin{aligned}
\|\pi_\Theta(\mathbf{a})\| &= \Big\|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta)\Big\| \\
&\leqslant \max\left(\Big\|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta)\Big\|, \Big\|\sum_{k\in\mathbb{Z}^n} a(k)R^k\Big\|\right) \\
&= \Big\|\sum_{k\in\mathbb{Z}^n} a(k)(W^k(\Theta) \oplus R^k)\Big\| \\
&\leqslant \Big\|\sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta')\Big\| + \Big\|\sum_{k\in\mathbb{Z}^n} a(k)(W^k(\Theta) \oplus R^k) - \sum_{k\in\mathbb{Z}^n} a(k)W^k(\Theta')\Big\| \\
&\leqslant \|\pi_{\Theta'}(\mathbf{a})\| + 2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta - \Theta'\|^{\frac{1}{2}}.
\end{aligned}$$

Since the roles of Θ and Θ' are interchangeable, we also have

$$\|\pi_{\Theta'}(\mathbf{a})\| \leqslant \|\pi_\Theta(\mathbf{a})\| + 2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta - \Theta'\|^{\frac{1}{2}}$$

[3] In comparison to [GPSS21], there is a discrepancy in the normalization of the matrices Θ parametrizing noncommutative tori by a factor of 2π, which explains the different value of c.

and the proof is complete. □

This result is enough to give us a C^*-bundle over $\mathrm{Skew}_n(\mathbb{R})$ whose fibers coincide with $\mathcal{A}_\Theta$ via Theorem 6.1 with a local approximation section that we can identify with $\ell^1_\nu(\mathbb{Z}^{2d})$ (by seeing sequences inside it as constant sections since we have a dense embedding $\ell^1_\nu(\mathbb{Z}^{2d}) \hookrightarrow \mathcal{A}_\Theta$ for each Θ). However, we have no need for such a general result in the current manuscript, and we would rather defer the relevant constructions to the sequel where the resulting bundles come from noncommutative tori generated by lattices in the time-frequency plane.

Remark 7.7. With the same methods, one can prove a useful estimate for the Hausdorff distance between the spectra $\mathrm{spec}(\pi_\Theta(\mathbf{a}))$ of selfadjoint operators $\pi_\Theta(\mathbf{a})$. Since we have limited use for it, we sketch the argument only roughly and leave the details to the ambitious reader. Suppose that $\Theta, \Theta_0 \in \mathrm{Skew}_n(\mathbb{R})$ and $\mathbf{a}, \mathbf{a}_0 \in \ell^1_\nu(\mathbb{Z}^n)$ such that $\pi_\Theta(\mathbf{a})$ and $\pi_{\Theta_0}(\mathbf{a}_0)$ are selfadjoint (note that rarely happens for $\mathbf{a} = \mathbf{a}_0$). Then, in the notation of the second proof of Theorem 7.2,

$$
\begin{aligned}
\mathrm{spec}(\pi_\Theta(\mathbf{a})) &= \mathrm{spec}\Big(\sum_{k\in\mathbb{Z}^n} a(k) W^k(\Theta)\Big) \\
&\subset \mathrm{spec}\Big(\sum_{k\in\mathbb{Z}^n} a(k)(W^k(\Theta) \oplus R^k)\Big) \\
&\subset \mathrm{spec}(\pi_{\Theta_0}(\mathbf{a}_0)) + [-1,1]\cdot \Big\| \sum_{k\in\mathbb{Z}^n} a(k)(W^k(\Theta)\oplus R^k) - \sum_{k\in\mathbb{Z}^n} a_0(k) W^k(\Theta_0)\Big\| \\
&\subset \mathrm{spec}(\pi_{\Theta_0}(\mathbf{a}_0)) + [-1,1]\cdot (2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta-\Theta_0\|^{\frac{1}{2}} + \|\mathbf{a}-\mathbf{a}_0\|_{\ell^1}).
\end{aligned}
$$

7.1.4 The Noncommutative Tori Generated by Lattices

Our goal here is to generate a skew-symmetric matrix $\Theta_L \in \mathrm{Skew}_{2d}(\mathbb{R})$ such that $\mathcal{A}_{\Theta_L} \cong C^*(\mathbb{Z}^{2d}, c_L) = A_L$. To start, let $\{e_1, e_2, \ldots, e_{2d}\}$ be the standard basis for $\mathbb{R}^{2d}$, consider the family of unitary operators via time-frequency shifts $\{\pi(Le_1), \pi(Le_2), \ldots, \pi(Le_{2d})\} \subseteq \mathcal{L}(L^2(\mathbb{R}^d))$. The commutation-relations (7.7) for this family say that

$$\pi(Le_i)\pi(Le_j) = c_s(Le_i, Le_j)\pi(Le_j)\pi(Le_i), \qquad \forall i,j = 1,\ldots,2d.$$

where $c_s(Le_i, Le_j) = e^{-2\pi i (Le_i)^T J (Le_j)}$, and so we define the skew-symmetric matrix associated to L via $(\Theta_L)_{i,j} = (Le_j)^T(-J)(Le_i) = -e_j^T(L^TJL)e_i$, i.e. $\Theta_L = -L^TJL$. For the rest of the section, we now make the identification $\mathcal{A}_{\Theta_L} = C^*(\{Le_i\}_{i=1}^{2d}) \subseteq \mathcal{L}(L^2(\mathbb{R}^d))$. As we have discussed in Section 7.1.2, by taking $\Theta = \Theta_L$, we know that $\pi_{\Theta_L} : C^*\left(\mathbb{Z}^{2d}, \zeta_{\Theta_L^{\mathrm{low}}}\right) \to \mathcal{A}_{\Theta_L}$ of Equation

(7.24) is a C^*-isomorphism, but for our purposes, we want to be able to lift the deformation results on π_{Θ_L} to the faithful $*$-representations, say $\overline{\pi}_L$. To this end, we have the following result.

Lemma 7.1. *For a fixed $L \in \mathrm{GL}_{2d}(\mathbb{R})$, consider the skew-symmetric matrix $\Theta_L = -L^T JL \in$ $\mathrm{Skew}_{2d}(\mathbb{R})$. The induced 2-cocycle $\zeta_{\Theta_L^{\mathrm{low}}}$ is cohomologous to the 2-cocycle c_L by the 1-cochain $\rho_L : \mathbb{Z}^{2d} \to \mathbb{T}$ given by $\rho_L(k) = \exp(-\pi i k^T(L^T KL + \Theta_L^{\mathrm{low}})k)$ for all $k \in \mathbb{Z}^{2d}$. It satisfies:*

$$c_L(k,m) = \zeta_{\Theta_L^{\mathrm{low}}}(k,m)\frac{\rho_L(k+m)}{\rho_L(k)\rho_L(m)}, \qquad \forall k,m \in \mathbb{Z}^{2d}.$$

Proof. It follows from a straightforward computation that

$$\frac{\rho_L(k+m)}{\rho_L(k)\rho_L(m)} = \exp\left(-\pi i k^T(L^T(K+K^T)L + (\Theta_L + (\Theta_L^{\mathrm{low}})^T))m\right)$$

Therefore we have

$$\begin{aligned}\zeta_{\Theta_L^{\mathrm{low}}}(k,m)\frac{\rho_L(k+m)}{\rho_L(k)\rho_L(m)} &= \exp(2\pi i k^T\Theta_L^{\mathrm{low}}m)\exp\left(-\pi i k^T(L^T(K+K^T)L + (\Theta_L + (\Theta_L^{\mathrm{low}})^T))m\right)\\ &= \exp\left(\pi i k^T(2\Theta_L^{\mathrm{low}} - L^T(K+K^T)L - \Theta_L^{\mathrm{low}} - (\Theta_L^{\mathrm{low}})^T)m\right)\end{aligned}$$

We compute that: $2\Theta_L^{\mathrm{low}} - \Theta_L^{\mathrm{low}} - (\Theta_L^{\mathrm{low}})^T = \Theta_L$ since Θ_L is skew-symmetric. But then, $\Theta_L - L^T(K+K^T)L = -L^TJL - L^T(K+K^T)L = -L^T(J+K+K^T)L = -2L^TKL$. Finally, we obtain

$$\zeta_{\Theta_L^{\mathrm{low}}}(k,m)\frac{\rho_L(k+m)}{\rho_L(k)\rho_L(m)} = e^{\pi i k^T(-2L^TKL)m} = e^{-2\pi i k^T(L^TKL)m} = c_L(n,m).$$

□

Let us now take note of the collected 2-cocycles P_{c_L} and $P_{\zeta_{\Theta_L^{\mathrm{low}}}}$ as in (7.25) generated by c_L and $\zeta_{\Theta_L^{\mathrm{low}}}$. We obtain:

$$P_{c_L}(k) = \prod_{i=1}^{2d-1}\overline{c}\left(Lk_ie_i, \sum_{j=1+1}^{2d} Lk_je_j\right) \times \prod_{i=1}^{2d}\overline{c}(Le_i, Le_i)^{n_i(n_i-1)/2}. \tag{7.28}$$

On the other hand, we have $P_{\zeta_{\Theta_L^{\mathrm{low}}}}(k) = 1$ for all $k \in \mathbb{Z}^{2d}$. Consider now the induced isomorphism $\widetilde{\rho}_L\colon A_L = C^*(\mathbb{Z}^{2d}, c_L) \to C^*(\mathbb{Z}^{2d}, \zeta_{\Theta_L^{\mathrm{low}}})$ (see Proposition 7.8), we have two representations that we want to compare: $\overline{\pi}_L : A_L \to \mathcal{A}_{\Theta_L}$ and $\pi_{\Theta_L} \circ \widetilde{\rho_L}\colon A_L \to \mathcal{A}_{\Theta_L}$. They do not exactly give commutative maps, but they still behave in an expected way. We shall fix the following: $\rho_L^{(\cdot)} : \mathbb{Z}^{2d} \to \mathbb{T}$ is the map $k \mapsto \rho_L^k$.

Proposition 7.9. *Define, for each* $\mathbf{a} \in C^*(\mathbb{Z}^{2d}, c_L) = A_L$*, then*

$$\mathbf{a}^L := \widetilde{\rho_L}(\mathbf{a} \cdot \rho_L^{(\cdot)}) \in C^*(\mathbb{Z}^{2d}, \zeta_{\Theta_L^{\text{low}}}).$$

Explicitly, we have

$$\mathbf{a}^L = \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) (\delta_{e_1}^{k_1} *_{\zeta_{\Theta_L^{\text{low}}}} \cdots *_{\zeta_{\Theta_L^{\text{low}}}} \delta_{e_{2d}}^{k_{2d}}) = \mathbf{a} \cdot P_{c_L} \in C^*(\mathbb{Z}^{2d}, \zeta_{\Theta_L^{\text{low}}}). \tag{7.29}$$

Furthermore,

$$\overline{\pi}_L(\mathbf{a}) = \pi_{\Theta_L}(\mathbf{a}^L) \tag{7.30}$$

Proof. We have

$$\begin{aligned}
\mathbf{a}^L &= \widetilde{\rho_L}(\mathbf{a} \cdot \rho_L^{(\cdot)}) = \widetilde{\rho_L} \left(\sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \rho_L^k \delta_{e_1}^{k_1} *_{c_L} \cdots *_{c_L} \delta_{e_{2d}}^{k_{2d}} \right) \\
&= \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \rho_L^k \widetilde{\rho_L} (\delta_{e_1}^{k_1} *_{c_L} \cdots *_{c_L} \delta_{e_{2d}}^{k_{2d}}) \\
&= \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \rho_L^k \overline{\rho_L}^k (\delta_{e_1}^{k_1} *_{\zeta_{\Theta_L^{\text{low}}}} \cdots *_{\zeta_{\Theta_L^{\text{low}}}} \delta_{e_{2d}}^{k_{2d}}) \\
&= \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) (\delta_{e_1}^{k_1} *_{\zeta_{\Theta_L^{\text{low}}}} \cdots *_{\zeta_{\Theta_L^{\text{low}}}} \delta_{e_{2d}}^{k_{2d}}) \\
&= \sum_{k \in \mathbb{Z}^{2d}} a(k) P c_L(k) \delta_k = \mathbf{a} \cdot P_{c_L} \in C^*(\mathbb{Z}^{2d}, \zeta_{\Theta_L^{\text{low}}})
\end{aligned}$$

Next we compute:

$$\begin{aligned}
\pi_{\Theta_L}(\mathbf{a}^L) &= \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \pi_{\Theta_L} \left(\delta_{e_1}^{k_1} *_{\zeta_{\Theta_L^{\text{low}}}} \cdots *_{\zeta_{\Theta_L^{\text{low}}}} \delta_{e_{2d}}^{k_{2d}} \right) \\
&= \sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \pi(Le_1)^{k_1} \cdot \ldots \cdot \pi(Le_{2d})^{k_{2d}} \\
&= \overline{\pi}_L \left(\sum_{k \in \mathbb{Z}^{2d}} a(k) P_{c_L}(k) \delta_{e_1}^{k_1} *_{c_L} \cdots *_{c_L} \delta_{e_{2d}}^{k_{2d}} \right) \\
&= \overline{\pi}_L(\mathbf{a}).
\end{aligned}$$

as required. $\square$

Lemma 7.2. *For each fixed $n \in \mathbb{Z}^{2d}$, the map $L \mapsto P_{c_L}(n)$ is continuous.*

Proof. Looking at Equation (7.28), $P_{c_L}(n)$ is a finite product of the Heisenberg 2-cocycles of the form $c(Lk, Lm)$ for some fixed $k, m \in \mathbb{Z}^{2d}$. The lemma follows if we can show that $L \mapsto c(Lk, Lm)$ is continuous for any fixed $k, m \in \mathbb{Z}^{2d}$. We have from Equation (7.2) that

$$c(Lk, Lm) = e^{-2\pi i\,(Lk)^T K(Lm)}.$$

Since the complex exponential function is continuous, then it is sufficient to show that $L \mapsto (Ln)^T K(Lm)$ is continuous. Now it follows from a simple application of triangle-inequality that:

$$\begin{aligned}\|(Ln)^T K(Lm) - (L_0 n)K(L_0 m)\| &\leqslant \|n^T(L - L_0)KLm\| + \|n^T L_0 K(L - L_0)m\| \\ &\leqslant \|n\|\|m\|(\|L\| + \|L_0\|)\|L - L_0\|.\end{aligned}$$

Therefore $(Ln)^T K(Lm) \to (L_0 n)^T K(L_0 m)$ as $L \to L_0$ as required. $\square$

Now we are ready to apply Theorem 7.2 to the noncommutative $2d$-tori generated by lattices in $\mathbb{R}^{2d}$. To make sense of our computations below, recall that the weighted sequence space $\ell^1_\nu(\mathbb{Z}^{2d})$ is densely embedded inside all $C^*(\mathbb{Z}^{2d}, \zeta)$ for arbitrary 2-cocycle ζ.

Lemma 7.3. *For each $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$ and $L \in \mathrm{GL}_{2d}(\mathbb{R})$: $\|\mathbf{a}\|_{\ell^1_\nu} = \|\mathbf{a}^L\|_{\ell^1_\nu}$. Moreover, the map $\mathrm{GL}_{2d}(\mathbb{R}) \ni L \mapsto \mathbf{a}^L \in \ell^1_\nu(\mathbb{Z}^{2d})$ is continuous.*

Proof. Since each $P_{c_L}(k) \in \mathbb{T}$, we have that

$$\|\mathbf{a}^L\|_{\ell^1_\nu} = \sum_{k\in\mathbb{Z}^{2d}} |a(k)\nu(k)P_{c_L}(k)| = \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\nu(k) = \|\mathbf{a}\|_{\ell^1_\nu}.$$

Note that for all $L, L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$ and $k \in \mathbb{Z}^{2d}$, $|P_{c_L}(k) - P_{c_{L_0}}(k)| \leqslant 2$, so it follows from the continuity result Lemma 7.2 and the Lebesgue-dominated convergence theorem[4] that

$$\begin{aligned}\lim_{L\to L_0} \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1_\nu} &= \lim_{L\to L_0} \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\nu(k)|P_{c_L}(k) - P_{c_{L_0}}(k)| \\ &= \sum_{k\in\mathbb{Z}^{2d}} |a(k)|\nu(k) \lim_{L\to L_0} |P_{c_L}(k) - P_{c_{L_0}}(k)| = 0\end{aligned}$$

[4]The usual dominated theorem works because $\mathrm{GL}_{2d}(\mathbb{R})$ is second countable, therefore $\lim_{L\to L_0} \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1_\nu} = 0$ if and only if for all sequence $L_n \to L_0$, $\lim_{n\to\infty} \|\mathbf{a}^{L_n} - \mathbf{a}^{L_0}\|_{\ell^1_\nu} = 0$.

as required. $\square$

Remark 7.8. Lemma 7.3 is also true if we consider the unweighted case. That is, $\mathrm{GL}_{2d}(\mathbb{R}) \ni L \mapsto \mathbf{a}^L \in \ell^1(\mathbb{Z}^{2d})$ is continuous for all $\mathbf{a} \in \ell^1(\mathbb{Z}^{2d})$.

Now as a corollary of Theorem 7.2, we have that:

Corollary 7.1. *There is a constant $C > 0$ such that for all $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$ and $L_0, L \in \mathrm{GL}_{2d}(\mathbb{R})$:*

$$\begin{aligned}\big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| &\leqslant C\|\mathbf{a}\|_{\ell^1_\nu}\|L^TJL - L_0^TJL_0\|^{1/2} + \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1} \\ &\leqslant C\|\mathbf{a}\|_{\ell^1_\nu}(\|L\| + \|L_0\|)^{1/2}\|L - L_0\|^{1/2} + \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1}\end{aligned}$$

As a corollary, for each $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$, the map $L \mapsto \|\overline{\pi}_L(\mathbf{a})\|$ is continuous.

Proof. Fix $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$, and $L, L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$. An application of triangle-inequality gives us, along with Equation (7.30):

$$\begin{aligned}\big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| &\leqslant \big|\|\pi_{\Theta_L}(\mathbf{a}^L)\| - \|\pi_{\Theta_{L_0}}(\mathbf{a}^L)\|\big| + \big|\|\pi_{\Theta_{L_0}}(\mathbf{a}^L)\| - \|\pi_{\Theta_{L_0}}(\mathbf{a}^{L_0})\|\big| \\ &\leqslant \big|\|\pi_{\Theta_L}(\mathbf{a}^L)\| - \|\pi_{\Theta_{L_0}}(\mathbf{a}^L)\|\big| + \|\pi_{\Theta_{L_0}}(\mathbf{a}^L - \mathbf{a}^{L_0})\|.\end{aligned}$$

Time-frequency shifts are unitary, so it follows that $\|\pi_{\Theta_{L_0}}(\mathbf{a}^L - \mathbf{a}^{L_0})\| \leqslant \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1}$. On the other hand, an application of Theorem 7.2 implies

$$\big|\|\pi_{\Theta_L}(\mathbf{a}^L)\| - \|\pi_{\Theta_{L_0}}(\mathbf{a}^L)\|\big| \leqslant C\|\mathbf{a}^L\|_{\ell^1_\nu}\|\Theta_L - \Theta_{L_0}\|^{1/2}.$$

Because $\|\mathbf{a}^L\|_{\ell^1_\nu} = \|\mathbf{a}\|_{\ell^1_\nu}$, we finally obtain

$$\big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| \leqslant C\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta_L - \Theta_{L_0}\|^{1/2} + \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1}. \tag{7.31}$$

Similar to what we did in the proof of Lemma 7.2, we can estimate $\|\Theta_L - \Theta_{L_0}\| = \|L^TJL - L_0^TJL_0\| \leqslant (\|L\|+\|L_0\|)\|L-L_0\|$. If we combine this estimate with the continuity result of Lemma 7.3, and Remark 7.8, then we find that as $L \to L_0$, we have $\big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| \to 0$. This proves continuity of $L \mapsto \|\overline{\pi}_L(\mathbf{a})\|$. $\square$

We are now ready to define our C^*-bundle over $\mathrm{GL}_{2d}(\mathbb{R})$ whose fibers coincide with noncommutative tori generated by lattices. To start, define $\mathcal{C} = \bigsqcup_{L \in \mathrm{GL}_{2d}(\mathbb{R})} A_L$, then define the obvious bundle projection map $\rho\colon \mathcal{C} \to \mathrm{GL}_{2d}(\mathbb{R})$, i.e.$\rho(\mathbf{s}) = L \iff \mathbf{s} \in A_L$. Recall that a map

$\mathfrak{a}\colon \mathrm{GL}_{2d}(\mathbb{R}) \to \mathcal{C}$ is a section of ρ if and only if $\mathfrak{a}(L) \in A_L$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, in which case we write $\mathfrak{a} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} A_L$. An important observation is the fact that we have a dense inclusion $\ell^1_\nu(\mathbb{Z}^{2d}) \hookrightarrow A_L$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, this motivates us to identify $\ell^1_\nu(\mathbb{Z}^{2d})$ as a section space for ρ, since for each $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$, we can define the constant section, which we shall also denote by $\mathbf{a}$ so that $\mathbf{a} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} A_L$ via:

$$\mathbf{a}(L) = \mathbf{a} \in \ell^1(\mathbb{Z}^{2d}) \hookrightarrow A_L, \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}). \tag{7.32}$$

Theorem 7.3. *The bundle $\rho\colon \mathcal{C} \to \mathrm{GL}_{2d}$, whose fibers are $\rho^{-1}(L) = A_L$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, defines a C^*-bundle via Theorem 6.1 whose local approximating sections are identifiable with $\ell^1_\nu(\mathbb{Z}^{2d})$. As a consequence, a section $\mathfrak{a} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} A_L$ is continuous, i.e. $\mathfrak{a} \in \Gamma(\rho)$, if and only if for all $\varepsilon > 0$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, there exists $\mathbf{a}_0 \in \ell^1_\nu(\mathbb{Z}^{2d})$ such that for some open neighborhood $U_0 \ni L_0$:*

$$\|\mathfrak{a}(L) - \mathbf{a}_0\|_{A_L} < \varepsilon, \qquad \forall L \in U_0.$$

Proof. We only need to find local approximating sections for a bundle to make it a C^*-bundle (or a Banach bundle) using Theorem 6.1. We identified $\ell^1_\nu(\mathbb{Z}^{2d})$ as the constant sections for ρ defined via (7.32). For $\ell^1_\nu(\mathbb{Z}^{2d})$ to be a space of local approximating sections, we need to show two things:

1. For each $\mathbf{a} \in \ell^1_\nu(\mathbb{Z}^{2d})$, $L \mapsto \|\mathbf{a}\|_{A_L} = \|\overline{\pi}_L(\mathbf{a})\|$ is continuous, and;

2. For all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, $\ell^1_\nu(\mathbb{Z}^{2d})$ is dense in A_L.

We already know that Item 2 is true. Item 1 is covered by Corollary 7.1. The last statement about the continuous sections follows from Proposition 6.3, using $\ell^1_\nu(\mathbb{Z}^{2d})$ as the space of local approximating sections. □

7.1.5 Deforming the Heisenberg Module

The C^*-bundle $\rho\colon \mathcal{C} \to \mathrm{GL}_{2d}(\mathbb{R})$ of Theorem 7.3 can be used to construct a corresponding Banach bundle of Heisenberg modules. In fact, the argument made to construct ρ almost holds mutatis-mutandis for the Heisenberg modules using an analogous observation that for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, we have a dense embedding $\mathbf{S}_{0,\nu}(\mathbb{R}^d) \hookrightarrow \mathcal{E}_L(\mathbb{R}^d)$. To make our construction precise, let $\mathcal{E} = \bigsqcup_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ and define the bundle $\kappa\colon \mathcal{E} \to \mathrm{GL}_{2d}(\mathbb{R})$ using the usual projection map.

We see from our observation that $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ can be identified as a space of constant sections for κ by defining for each $f \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, the section $f \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ such that

$$f(L) := f \in \mathcal{E}_L(\mathbb{R}^d), \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}). \tag{7.33}$$

Theorem 7.4. *The bundle $\kappa \colon \mathcal{E} \to \mathrm{GL}_{2d}(\mathbb{R})$ whose fibers are $\kappa^{-1}(L) = \mathcal{E}_L(\mathbb{R}^d)$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$, defines a Banach bundle via Theorem 6.1 whose local approximating sections are identifiable with $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$. As a consequence, a section $\Upsilon \in \prod_{L \in \mathrm{GL}_{2d}} \mathcal{E}_L(\mathbb{R}^d)$ is continuous, i.e. $\Upsilon \in \Gamma(\kappa)$, if and only if for all $\varepsilon > 0$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, there exists $f_0 \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$ such that for some open neighborhood $U_0 \ni L_0$:*

$$\|\Upsilon(L) - f_0\|_{\mathcal{E}_L(\mathbb{R}^d)} < \varepsilon, \qquad \forall L \in U_0.$$

Proof. As in the proof of Theorem 7.3, we only need to prove that the identification of $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ as sections of κ actually defines a space of local approximating section. We need to show that:

1. For each $f \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, the map $L \mapsto \|f\|_{\mathcal{E}_L(\mathbb{R}^d)}$ is continuous, and;
2. For each $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$, the space $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ is dense in $\mathcal{E}_L(\mathbb{R}^d)$.

We already know that Item 2 is true. Due to (7.15), Item 1 follows if we can show that the map $L \mapsto \|\overline{\pi}_L({}_L\langle f, f\rangle)\|$ is continuous. Using the estimates in Corollary 7.1, we find that:

$$\begin{aligned}
&\big|\|\overline{\pi}_L({}_L\langle f, f\rangle)\| - \|\overline{\pi}_{L_0}({}_{L_0}\langle f, f\rangle)\|\big| \\
&\quad \leqslant \big|\|\overline{\pi}_L({}_L\langle f, f\rangle)\| - \|\overline{\pi}_{L_0}({}_L\langle f, f\rangle)\|\big| + \|\overline{\pi}_{L_0}({}_L\langle f, f\rangle - {}_{L_0}\langle f, f\rangle)\| \\
&\quad \leqslant C \cdot \|{}_L\langle f, f\rangle\|_{\ell^1_\nu} \cdot (\|L\| + \|L_0\|)^{1/2} \cdot \|L - L_0\|^{1/2} \\
&\qquad + \|{}_L\langle f, f\rangle^L - {}_{L_0}\langle f, f\rangle^{L_0}\|_{\ell^1} + \|\,{}_L\langle f, f\rangle - {}_{L_0}\langle f, f\rangle\,\|_{\ell^1}
\end{aligned} \tag{7.34}$$

It can be deduced from the proof [FK04, Lemma 3.5] that, for any $h \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, $L \mapsto {}_L\langle h, h\rangle = \sum_{k \in \mathbb{Z}^{2d}} \mathcal{V}_h h(Lk)$ is continuous with respect to both the ℓ^1 and ℓ^1_ν norm. It follows from a standard triangle-inequality argument that $L \mapsto {}_L\langle f, f\rangle^L$ is also continuous in ℓ^1 norm. All these imply that, as $L \to L_0$, the right-most expression in (7.34) approaches 0. This proves continuity of the map $L \mapsto \|f\|_{\mathcal{E}_L(\mathbb{R}^d)}$. On the other hand, the proof for Item 2 is trivial since $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ is dense in $\mathcal{E}_L(\mathbb{R}^d)$.

The statement about the continuous sections follows from Proposition 6.3, using $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ as the space of local approximating sections. □

So far we have exploited the results involving the weighted spaces $\ell^1_\nu(\mathbb{Z}^{2d})$ and $\mathbf{S}_{0,\nu}(\mathbb{R}^d)$ for an initial construction of the C^*- and Banach bundles ρ and κ respectively. The following result shows that the uweighted spaces $\ell^1(\mathbb{Z}^{2d})$ and $\mathbf{S}_0(\mathbb{R}^d)$ can also be identified as spaces of continuous (constant) sections for ρ and κ respectively. We first give a technical lemma that we can use throughout our computations.

Lemma 7.4. *Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then for any window $\phi \in \mathbf{S}_0(\mathbb{R}^d)$, we have the following estimate for any $L \in \mathrm{GL}_{2d}(\mathbb{R})$:*

$$\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} \leqslant \|\phi\|_2^{-1}\|f\|_{\mathbf{S}_{0,\phi}}. \tag{7.35}$$

Proof. The result follows from a straightforward computation along with [Jak18, Proposition 4.10]:

$$\begin{aligned}
\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} &= \|\overline{\pi}_L\left({}_L\langle f, f\rangle\right)\|^{1/2} \\
&\leqslant \|{}_L\langle f, f\rangle\|_{\ell^1}^{1/2} \\
&= \left(\sum_{k\in\mathbb{Z}^{2d}} |\mathcal{V}_f f|(Lk)\right)^{1/2} \\
&\leqslant \left(\int_{\mathbb{R}^{2d}} |\mathcal{V}_f f|(x,\omega)\,\mathrm{d}x\,\mathrm{d}\omega\right)^{1/2} \\
&= \|f\|_{\mathbf{S}_{0,f}}^{1/2} \\
&\leqslant \left(\|f\|_{\mathbf{S}_{0,\phi}} \cdot \|\phi\|_2^{-2}\|f\|_{\mathbf{S}_{0,\phi}}\right)^{1/2} \\
&\leqslant \|\phi\|_2^{-1}\|f\|_{\mathbf{S}_{0,\phi}}
\end{aligned}$$

as required. □

Proposition 7.10. *$\ell^1(\mathbb{Z}^{2d}) \subseteq \Gamma(\rho)$ and $\mathbf{S}_0(\mathbb{R}^d) \subseteq \Gamma(\kappa)$.*

Proof. We shall prove $\mathbf{S}_0(\mathbb{R}^d) \subseteq \Gamma(\kappa)$, the proof for $\ell^1(\mathbb{Z}^{2d}) \subseteq \Gamma(\rho)$ is similar. Suppose $f \in \mathbf{S}_0(\mathbb{R}^d)$, and $\varepsilon > 0$. For this problem, we use a particular window $\phi \in \mathbf{S}_0(\mathbb{R}^d)$ for our $\mathbf{S}_0(\mathbb{R}^d)$-norm. It follows from Proposition 7.2 that there exists $f_0 \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$ such that $\|f - f_0\|_{\mathbf{S}_{0,\phi}} < \varepsilon \cdot \|\phi\|_2$. Now, for any $L \in \mathrm{GL}_{2d}(\mathbb{R})$, we use Estimate (7.35) to obtain

$$\|f - f_0\|_{\mathcal{E}_L(\mathbb{R}^d)} \leqslant \|\phi\|_2^{-1}\|f - f_0\|_{\mathbf{S}_{0,\phi}} < \varepsilon.$$

Therefore, if we consider $f \in \mathbf{S}_0(\mathbb{R}^d)$ as a constant section in κ with $f(L) := f \in \mathbf{S}_0(\mathbb{R}^d) \hookrightarrow \mathcal{E}_L(\mathbb{R}^d)$, we see from Theorem 7.4 that $f \in \Gamma(\kappa)$. $\square$

Corollary 7.2. *$\mathfrak{a} \in \Gamma(\rho)$ if and only if for any $L_0 \in \mathrm{GL}_{2d}(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists an open set satisfying $U_0 \ni L_0$ and $\mathbf{a}_0 \in \ell^1(\mathbb{Z}^{2d})$ such that $\|\mathfrak{a}(L) - \mathbf{a}_0\|_{A_L} < \varepsilon$ for all $L \in U_0$.*

Similarly, $\Upsilon \in \Gamma(\kappa)$ if and only if for any $L_0 \in \mathrm{GL}_{2d}(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists an open set $U_0 \ni L_0$ and $f_0 \in \mathbf{S}_0(\mathbb{R}^d)$ such that $\|\Upsilon(L) - f_0\|_{\mathcal{E}_L(\mathbb{R}^d)} < \varepsilon$ for all $L \in U_0$.

Proof. The result follows from Corollary 6.4 and Proposition 7.10. $\square$

Corollary 7.2 above shows what we should expect for a map that continuously varies along Heisenberg modules should look like, it should locally behave as if it is any other element in a fixed Heisenberg module – we must be able to approximate it by some function in Feichtinger's algebra. To get a better grasp of what these continuous sections look like, we have the following Lemma regarding the space $C(\mathrm{GL}_{2d}(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}))$ of continuous functions from $\mathrm{GL}_{2d}(\mathbb{R})$ to $\mathbf{S}_0(\mathbb{R}^d)$. Note that any map $\Upsilon \in C(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}))$ will define a section with respect to κ, since its image will always land in $\mathbf{S}_0(\mathbb{R}^d)$, and as we have repeatedly emphasized, is always embedded in *all* $\mathcal{E}_L(\mathbb{R}^d)$, for $L \in \mathrm{GL}_{2d}(\mathbb{R})$. Therefore the norm $\|\Upsilon(L) - \Upsilon(L_0)\|_{\mathcal{E}_L(\mathbb{R}^d)}$ makes sense. In fact, $\|\sum_{i=1}^n \Upsilon(M_i)\|_{\mathcal{E}_L(\mathbb{R}^d)}$ makes sense for all $M_1, \ldots, M_n, L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$.

Lemma 7.5. *We have the following inclusion: $C(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d)) \subseteq \Gamma(\kappa)$.*

Proof. Fix a window function $\phi \in \mathbf{S}_0(\mathbb{R}^d)$. Let $\Upsilon \in C(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$, we must show that Υ satisfies the characterization in Corollary 7.2 to show that $\Upsilon \in \Gamma(\kappa)$. Let $\varepsilon > 0$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, choose $f_0 := \Upsilon(L_0) \in \mathbf{S}_0(\mathbb{R}^d)$. Since $\Upsilon \in C(\mathrm{GL}_{2d}(\mathbb{R}, \mathbf{S}_0(\mathbb{R}^d))$, we can choose an open set U_0 containing L_0 such that $\|\Upsilon(L) - \Upsilon(L_0)\|_{\mathbf{S}_0,\phi} < \|\phi\|_2 \cdot \varepsilon$ for all $L \in U_0$. If we take $L \in U_0$, then by Lemma 7.4:

$$
\begin{aligned}
\|\Upsilon(L) - f_0\|_{\mathcal{E}_L(\mathbb{R}^d)} &= \|\Upsilon(L) - \Upsilon(L_0)\|_{\mathcal{E}_L(\mathbb{R}^d)} \\
&\leqslant \|\phi\|_2^{-1} \|\Upsilon(L) - f_0\|_{\mathbf{S}_0,\phi} < \varepsilon.
\end{aligned}
$$

This shows $\Upsilon \in \Gamma(\kappa)$. $\square$

There is a corresponding statement for the space $\Gamma_0(\kappa)$ of continuous sections that vanish at infinity.

Lemma 7.6. *We have the following inclusion: $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d)) \subseteq \Gamma_0(\kappa)$.*

Proof. Let $\Upsilon \in C_0(\mathrm{GL}_{2d}, \mathbf{S}_0(\mathbb{R}^d))$. We already know that $\Upsilon \in \Gamma(\kappa)$, we only need to show that $L \mapsto \|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}$ vanishes at infinity. However, this follows from the same estimate as in Lemma 7.36. For a window $\phi \in \mathbf{S}_0(\mathbb{R}^d)$:

$$\|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)} \leqslant \|\phi\|_2^{-1}\|\Upsilon(L)\|_{\mathbf{S}_0,\phi}. \tag{7.36}$$

So for any fixed $\varepsilon > 0$, we can find by hypothesis, a compact subset $M_\varepsilon \subseteq \mathrm{GL}_{2d}(\mathbb{R})$ such that $\|\Upsilon(L)\|_{\mathbf{S}_0,\phi} < \|\phi\|_2 \cdot \varepsilon$ whenever $L \in \mathrm{GL}_{2d}(\mathbb{R})\backslash M_\varepsilon$. Therefore

$$\|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)} \leqslant \|\phi\|_2^{-1}\|\Upsilon(L)\|_{\mathbf{S}_0,\phi} < \varepsilon, \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R})\backslash M_\varepsilon.$$

So $L \mapsto \|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}$ vanishes at infinity, as required. □

In the case of the Heisenberg module on a fixed lattice, $\mathbf{S}_0(\mathbb{R}^d)$ serves as space of approximating functions (with respect to the module-norm), and we see in turn that the space $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ takes this role for the Banach space $\Gamma_0(\kappa)$. That is, $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ acts as a *global approximating space* for $\Gamma_0(\kappa)$.

Corollary 7.3. *$C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ is a dense subspace of $\Gamma_0(\kappa)$.*

Proof. It is not hard to see that for any $\varphi \in C_0(\mathrm{GL}_{2d}(\mathbb{R}))$ and $\Upsilon \in C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$, we have that the pointwise product satisfies $\varphi \cdot \Upsilon \in C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$. Furthermore, we shall prove that:

$$\{\Upsilon(L) \in \mathcal{E}_L(\mathbb{R}^d) : \Upsilon \in C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))\} = \mathbf{S}_0(\mathbb{R}^d). \tag{7.37}$$

Fix $f \in \mathbf{S}_0(\mathbb{R}^d)$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$. Let M_0 be a compact neighborhood of L_0 and U_0 be an open neighborhood of L_0 that contains M_0, so we have $L_0 \in M_0 \subseteq U_0$. By Urysohn's Lemma for locally compact spaces [Wil07, Lemma 1.41], there exists a continuous function $\varphi\colon \mathrm{GL}_{2d}(\mathbb{R}) \to [0,1]$ such that $\varphi_{|M_0} \equiv 1$ and $\varphi \equiv 0$ outside U_0. Now define

$$\Upsilon(L) = \varphi(L) \cdot f, \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}).$$

We see that $\Upsilon \in C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ while $\Upsilon(L_0) = f$. Since f and L_0 are both arbitrary, this proves Equation (7.37). We see that $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ satisfies the hypotheses of Proposition 6.4, therefore it must be dense in $\Gamma_0(\kappa)$. □

There are analogues of Lemma 7.5 and 7.6 for ρ, and the proof is easier owing to the fact that the norm of A_L is always dominated by the $\ell^1(\mathbb{Z}^{2d})$-norm for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$.

Lemma 7.7. *$C(\mathrm{GL}_{2d}(\mathbb{R}), \ell^1(\mathbb{Z}^{2d})) \subseteq \Gamma(\rho)$ and $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \ell^1(\mathbb{Z}^{2d})) \subseteq \Gamma_0(\rho)$.*

Notation. Now would be a good time to take a pause to set some notations in place. It would be helpful to review the module action and inner-product on each Heisenberg module given in Proposition 7.1. For any sections (not necessarily continuous) $\Upsilon_1, \Upsilon_2 \in \prod_{L\in\mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ of κ, we define a section ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle \in \prod_{L\in\mathrm{GL}_{2d}(\mathbb{R})} A_L$ via:

$$ {}_\rho\langle \Upsilon_1, \Upsilon_2\rangle (L) := {}_L\langle \Upsilon_1(L), \Upsilon_2(L)\rangle \in A_L, \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}). \tag{7.38} $$

If we also have the section (again, not necessarily continuous) $\mathfrak{a} \in \prod_{L\in\mathrm{GL}_{2d}(\mathbb{R})} A_L$, then we define $\mathfrak{a} \cdot \Upsilon_1 \in \prod_{L\in\mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ via:

$$ (\mathfrak{a} \cdot \Upsilon_1)(L) := \mathfrak{a}(L) \cdot \Upsilon_1(L) \in \mathcal{E}_L(\mathbb{R}^d), \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}), \tag{7.39} $$

Furthermore, we shall fix a notation for the norm on the Banach space $\Gamma_0(\kappa)$. If $\Upsilon \in \Gamma_0(\kappa)$, we write:

$$ \|\Upsilon\|_\kappa := \sup_{L\in\mathrm{GL}_{2d}(\mathbb{R})} \|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}. \tag{7.40} $$

Similarly, we denote by the norm on the C^*-algebra $\Gamma_0(\rho)$ via:

$$ \|\mathfrak{a}\|_\rho := \sup_{L\in\mathrm{GL}_{2d}(\mathbb{R})} \|\mathfrak{a}(L)\|_{A_L}, \tag{7.41} $$

for each $\mathfrak{a} \in \Gamma_0(\rho)$.

Proposition 7.11. *If we again identify $\ell^1(\mathbb{Z}^{2d})$ and $\mathbf{S}_0(\mathbb{R}^d)$ as the space of local approximating sections of ρ and κ respectively, then for all $\mathbf{a} \in \ell^1(\mathbb{Z}^{2d})$, and $f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)$:*

1. ${}_\rho\langle f_1, f_2\rangle \in \Gamma(\rho)$;

2. $\mathbf{a} \cdot f_1 \in \Gamma(\kappa)$.

Consequently, if $\mathfrak{a} \in \Gamma(\rho)$, $\Upsilon_1, \Upsilon_2 \in \Gamma(\kappa)$, then:

3. ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle \in \Gamma(\rho)$;

4. $\mathfrak{a} \cdot \Upsilon_1 \in \Gamma(\kappa)$.

Finally, if $\mathfrak{a} \in \Gamma_0(\rho)$*,* $\Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)$*, then:*

5. ${}_\rho\langle \Upsilon_1, \Upsilon_2 \rangle \in \Gamma_0(\rho)$*;*

6. $\mathfrak{a} \cdot \Upsilon_1 \in \Gamma_0(\kappa)$*.*

Proof. We have already seen the sufficient techniques to prove Items 1 and 2. For Item 1, we need to show that ${}_\rho\langle f, g\rangle$ satisfies the continuous section characterization from Corollary 7.2. Let $\varepsilon > 0$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, choose $\mathbf{a}_0 = {}_\rho\langle f, g\rangle(L_0) := {}_{L_0}\langle f, g\rangle \in \ell^1(\mathbb{Z}^{2d})$. We know from [FK04, Lemma 3.5] that the map $L \mapsto \{\mathcal{V}_g f(Lk)\}_{k\in\mathbb{Z}^{2d}}$ is continuous from $\mathrm{GL}_{2d}(\mathbb{R})$ to $\ell^1(\mathbb{Z}^{2d})$. So choose the open set $U_0 \ni L_0$ of $\mathrm{GL}_{2d}(\mathbb{R})$ such that

$$\| {}_L\langle f, g\rangle - {}_{L_0}\langle f, g\rangle \|_{\ell^1} \leqslant \sum_{k\in\mathbb{Z}^{2d}} |\mathcal{V}_g f(Lk) - \mathcal{V}_g f(L_0 k)| < \varepsilon, \qquad \forall L \in U_0.$$

It follows that for all $L \in U_0$

$$\begin{aligned} \|{}_\rho\langle f, g\rangle(L) - \mathbf{a}_0\|_{A_L} &= \|{}_L\langle f, g\rangle - {}_{L_0}\langle f, g\rangle\|_{A_L} \\ &= \|\overline{\pi}_L({}_L\langle f, g\rangle - {}_{L_0}\langle f, g\rangle)\| \\ &\leqslant \|{}_L\langle f, g\rangle - {}_{L_0}\langle f, g\rangle\|_{\ell^1} \\ &< \varepsilon. \end{aligned}$$

Therefore ${}_\rho\langle f, g\rangle \in \Gamma(\rho)$ as required.

For Item 2, it is sufficient that we show $\mathbf{a} \cdot f \in C(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$, following Lemma 7.5. Let $L, L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, we have:

$$\begin{aligned} \|(\mathbf{a} \cdot f)(L) - (\mathbf{a} \cdot f)(L_0)\|_{\mathbf{S}_0} &:= \left\| \sum_{k\in\mathbb{Z}^{2d}} a(k)\,(\pi(Lk) - \pi(L_0 k))\, f \right\|_{\mathbf{S}_0} \\ &\leqslant \sum_{k\in\mathbb{Z}^{2d}} |a(k)| \|(\pi(Lk) - \pi(L_0 k)) f\|_{\mathbf{S}_0}. \end{aligned}$$

It follows from [FK04, Proposition 2.7 (ii)] that as $L \to L_0$, $\|(\mathbf{a} \cdot f)(L) - (\mathbf{a} \cdot f)(L_0)\|_{\mathbf{S}_0} \to 0$. Therefore $\mathbf{a} \cdot f \in C(\mathrm{GL}_{2d}(\mathbb{R}), \mathbf{S}_0(\mathbb{R}^d))$ as required.

We next prove Item 3. To do this, fix a window function $\phi \in \mathbf{S}_0(\mathbb{R}^d)$, $\varepsilon > 0$, and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$. By local compactness of $\mathrm{GL}_{2d}(\mathbb{R})$, we can fix a compact neighborhood $K \ni L_0$. Since $\Upsilon_2 \colon \mathrm{GL}_{2d}(\mathbb{R}) \to \mathcal{E}$ is a continuous section, then is $L \mapsto \|\Upsilon_2(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}$ continuous, therefore $\sup_{L\in K} \|\Upsilon_2\|_{\mathcal{E}_L(\mathbb{R}^d)} < \infty$. Now, by the continuous section criteria, there exist $f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)$ such that for some open sets $U_1, U_2 \ni L_0$

$$\|\Upsilon_1(L) - f_1\|_{\mathcal{E}_L(\mathbb{R}^d)} < \frac{\varepsilon}{2 \cdot \sup_{L\in K} \|\Upsilon_2\|_{\mathcal{E}_L(\mathbb{R}^d)}}, \qquad \forall L \in U_1$$
$$\|\Upsilon_2(L) - f_2\|_{\mathcal{E}_L(\mathbb{R}^d)} < \frac{\varepsilon}{2\|\phi\|_2^{-1}\|f_1\|_{\mathbf{S}_{0,\phi}}}, \qquad \forall L \in U_2.$$

Using Proposition 2.3 Item 1, we obtain the following estimate for $L \in \operatorname{int}(U_1 \cap U_2 \cap K)$:

$$\begin{aligned}
\|{}_\rho\langle \Upsilon_1, \Upsilon_2\rangle(L) - {}_\rho\langle f_1, f_2\rangle(L)\|_{A_L} &= \|\,{}_L\langle \Upsilon_1(L), \Upsilon_2(L)\rangle - {}_L\langle f_1, f_2\rangle\|_{A_L} \\
&\leqslant \|\,{}_L\langle \Upsilon_1(L) - f_1, \Upsilon_2(L)\rangle\|_{A_L} + \|\,{}_L\langle f_1, \Upsilon_2(L) - f_2\rangle\|_{A_L} \\
&\leqslant \|\Upsilon_1(L) - f_1\|_{\mathcal{E}_L(\mathbb{R}^d)} \cdot \sup_{L\in K} \|\Upsilon_2\|_{\mathcal{E}_L(\mathbb{R}^d)} + \|\phi\|_2^{-1}\|f_1\|_{\mathbf{S}_{0,\phi}}\|\Upsilon_2(L) - f_2\|_{\mathcal{E}_L(\mathbb{R}^d)} \\
&< \varepsilon.
\end{aligned}$$

Since ${}_\rho\langle f_1, f_2\rangle \in \Gamma(\rho)$ by Item 1, it follows from Corollary 6.4 that ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle \in \Gamma(\rho)$.

The proof for Item 4 is basically a rehash of Item 3, where we use the estimate in Proposition 2.3 Item 3 instead.

The proofs for Items 5 and 6 are also similar, we shall show one of them. To start, note that because of Proposition 2.3 Item 1, we have for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$:

$$\begin{aligned}
\|\,{}_\rho\langle \Upsilon_1, \Upsilon_2\rangle(L)\|_{A_L} :&= \|\,{}_L\langle \Upsilon_1(L), \Upsilon_2(L)\rangle\|_{A_L} \\
&\leqslant \|\Upsilon_1(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}\|\Upsilon_2(L)\|_{\mathcal{E}_L(\mathbb{R}^d)}.
\end{aligned}$$

So if both Υ_1 and Υ_2 vanish at infinity, then so does ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle$. Since we already know from Item 3 that ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle \in \Gamma(\rho)$, then it follows from what we have just shown that ${}_\rho\langle \Upsilon_1, \Upsilon_2\rangle \in \Gamma_0(\rho)$. This shows Item 5. □

Corollary 7.4. *The Banach space $\Gamma_0(\kappa)$ defines a full Hilbert left $\Gamma_0(\rho)$-module.*

Proof. We define $\Gamma_0(\kappa)$ as a Hilbert left $\Gamma_0(\kappa)$-module exactly by the module inner-product and action given by Proposition 7.11 Items 5 and 6. The pointwise nature of the definition immediately shows that the Axioms 1, 2, 3 and 5 of Definition 2.20 are satisfied with the defined operations. Now, if $\Upsilon \in \Gamma_0(\kappa)$, then we know that ${}_\rho\langle \Upsilon, \Upsilon\rangle(L) := {}_L\langle \Upsilon(L), \Upsilon(L)\rangle \geqslant 0$ in A_L for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$. It follows from Proposition 6.7 that ${}_\rho\langle \Upsilon, \Upsilon\rangle \geqslant 0$ in the C^*-algebra $\Gamma_0(\rho)$. Now let us look at the induced module norm on $\Gamma_0(\kappa)$, for $\Upsilon \in \Gamma_0(\kappa)$, if we denote it by $\|\cdot\|_{\mathrm{ind}}$, then using Equations 7.40 and 7.41:

$$\begin{aligned}
\|\Upsilon\|_{\text{ind}} &:= \|{}_{\rho}\langle \Upsilon, \Upsilon\rangle\|_{\rho}^{1/2} \\
&= \left(\sup_{L\in \mathrm{GL}_{2d}(\mathbb{R})} \|{}_{\rho}\langle \Upsilon, \Upsilon\rangle(L)\|_{A_L}\right)^{1/2} \\
&= \left(\sup_{L\in \mathrm{GL}_{2d}(\mathbb{R})} \|{}_{L}\langle \Upsilon(L), \Upsilon(L)\rangle\|_{A_L}\right)^{1/2} \\
&= \sup_{L\in \mathrm{GL}_{2d}(\mathbb{R})} \|{}_{L}\langle \Upsilon(L), \Upsilon(L)\rangle\|_{A_L}^{1/2} \\
&= \sup_{L\in \mathrm{GL}_{2d}(\mathbb{R})} \|\Upsilon(L)\|_{\mathcal{E}_L(\mathbb{R}^d)} \\
&=: \|\Upsilon\|_{\kappa}.
\end{aligned}$$

We see that the induced module norm coincides[5] with the norm for $\Gamma_0(\kappa)$ and since $\|\cdot\|_{\kappa}$ is complete, then the induced module norm is also complete. This shows that $\Gamma_0(\kappa)$ is a Hilbert left $\Gamma_0(\rho)$-module. All that remains is to show fullness. Let us first define

$$I := \overline{\operatorname{span}}\left\{{}_{\rho}\langle \Upsilon_1, \Upsilon_2\rangle \in \Gamma_0(\rho) : \Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)\right\}.$$

We know that I is an ideal of $\Gamma_0(\rho)$ by basic Hilbert C^*-module theory. Our goal is to show that $I = \Gamma_0(\rho)$. Define for each $L \in \mathrm{GL}_{2d}(\mathbb{R})$, the ideal $I_L \subseteq A_L$ via:

$$I_L := \{\mathfrak{a}(L) \in A_L : \mathfrak{a} \in I\} = \overline{\operatorname{span}}\left\{{}_{\rho}\langle \Upsilon_1, \Upsilon_2\rangle(L) \in A_L : \Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)\right\}.$$

By Corollary 6.8, we are done if we can show that $I_L = A_L$, because it will follow that $I = \{\mathfrak{a} \in \Gamma_0(\rho) : \mathfrak{a}(L) \in I_L\} = \{\mathfrak{a} \in \Gamma_0(\rho) : \mathfrak{a}(L) \in A_L\} = \Gamma_0(\rho)$. However, it follows from Proposition 5.8 and the density of $\ell^1(\mathbb{Z}^{2d})$ in A_L that under the A_L-norm:

$$A_L = \overline{\operatorname{span}}\left\{{}_{L}\langle f_1, f_2\rangle : f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)\right\}.$$

We can proceed similar to the proof of Corollary 7.3 to show that

$$\left\{{}_{L}\langle f_1, f_2\rangle : f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)\right\} \subseteq \left\{{}_{L}\langle \Upsilon_1(L), \Upsilon_2(L)\rangle : \Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)\right\}.$$

[5] And therefore we shall never make use of the notation $\|\cdot\|_{\text{ind}}$ ever again.

Therefore

$$A_L = \overline{\operatorname{span}}\{{}_L\langle \Upsilon_1(L), \Upsilon_2(L)\rangle : \Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)\} := I_L,$$

as required. Hence we have shown that $\Gamma_0(\kappa)$ is a full Hilbert left $\Gamma_0(\rho)$-module. $\square$

Now that we have shown that there is a full Hilbert *left* $\Gamma_0(\rho)$-module structure on the continuous sections $\Gamma_0(\kappa)$ that vanish at infinity, it is natural to ask if $\Gamma_0(\kappa)$ is actually an imprimitivity bimodule implementing a Morita equivalence between $\Gamma_0(\kappa)$ and another C^*-algebra of sections. Knowing that at each $L \in \mathrm{GL}_{2d}(\mathbb{R})$, the corresponding fiber of κ, the Heisenberg module $\kappa^{-1}(L) = \mathcal{E}_L(\mathbb{R}^d)$ is a Morita equivalence between A_L and B_L motivates us to construct $\mathcal{C}^\circ = \bigsqcup_{L\in\mathrm{GL}_{2d}} B_L$ with the corresponding bundle projection map $\rho^\circ \colon \mathcal{C}^\circ \to \mathrm{GL}_{2d}(\mathbb{R})$. We now have a bundle which has the fiber $(\rho^\circ)^{-1}(L) = B_L$ for each $L \in \mathrm{GL}_{2d}(\mathbb{R})$. Our goal now is to prove that ρ° is also a C^*-bundle. There is no need to make this harder than it should be, we already have the tools that we can use from the construction of ρ available to us, and it now all boils down to restating our problem in a familiar way. Let $L \in \mathrm{GL}_{2d}(\mathbb{R})$, with the corresponding matrix $L^\circ = J(L^{-1})^T J^T$ that generates the adjoint lattice, then using the cocycle-adjoint relations (7.8), for each $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$

$$\begin{aligned}
\overline{\pi}^*_{L^\circ}(\mathbf{b}) &= \frac{1}{|\det L|}\sum_{k\in\mathbb{Z}^{2d}} b(k)\pi^*(L^\circ k) \\
&= \frac{1}{|\det L|}\sum_{k\in\mathbb{Z}^{2d}} b(k)c(L^\circ k, L^\circ k)\pi(-L^\circ k) \\
&= \frac{1}{|\det L|}\overline{\pi}_{-L^\circ}\left({}^{L^\circ}\mathbf{b}\right),
\end{aligned}$$

where ${}^{L^\circ}\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$ is the sequence whose terms are: $\left({}^{L^\circ}b\right)(k) = c(L^\circ k, L^\circ k)b(k)$ for each $k \in \mathbb{Z}^{2d}$. The computation above now opens us for an analogue of Corollary 7.1 for ρ°. In fact, all of the results that we have derived for ρ, also have analogues for ρ°, and we shall not repeat the associated proofs for these results. Most of them boil down to the fact that the maps $L \mapsto L^\circ$ and $L \to \det L$ are continuous, with the occassional use of the triangle inequality. If one is truly interested in repeating the proofs for ρ°, then it is also instructive to know that for each $L \in \mathrm{GL}_{2d}(\mathbb{R})$, $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$ and $f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)$ that:

$$\|\mathbf{b}\|_{B_L} \leqslant \frac{1}{|\det L|}\sum_{k\in\mathbb{Z}^{2d}} |b(k)|;$$

$$\langle f_1, f_2 \rangle_{L^\circ}(k) = \frac{1}{|\det L|} \sum_{k \in \mathbb{Z}^{2d}} \overline{\mathcal{V}_{f_2} f_1(L^\circ k)}.$$

Using the same observation that we did for the bundle ρ, we have a dense inclusion: $\ell^1(\mathbb{Z}^{2d}) \hookrightarrow B_L$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, and so we can identify $\ell^1(\mathbb{Z}^{2d})$ as a space of sections for ρ° so that each $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$ can be seen as $\mathbf{b} \in \prod_{L \in \mathrm{GL}_{2d}} B_L$ via

$$\mathbf{b}(L) := \mathbf{b} \in B_L.$$

As a corollary, using exactly the same kind of argument as in Theorem 7.4 and Proposition 7.2, we have that ρ° can be promoted to a C^*-bundle with the following continuous sections:

Corollary 7.5. *The bundle $\rho^\circ \colon \mathcal{C}^\circ \to \mathrm{GL}_{2d}$, whose fibers are $(\rho^\circ)^{-1}(L) = B_L$ for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, defines a C^*-bundle such that every section $\mathfrak{b} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} B_L$ is continuous, i.e. $\mathfrak{b} \in \Gamma(\rho^\circ)$ if and only if for all $\varepsilon > 0$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$, there exists $\mathbf{b}_0 \in \ell^1(\mathbb{Z}^{2d})$ such that for some open neighborhood $U_0 \ni L_0$:*

$$\|\mathfrak{b}(L) - \mathbf{b}_0\|_{B_L} < \varepsilon, \qquad \forall L \in U_0.$$

Proposition 7.12. *$C(\mathrm{GL}_{2d}(\mathbb{R}), \ell^1(\mathbb{Z}^{2d})) \subseteq \Gamma(\rho^\circ)$ and $C_0(\mathrm{GL}_{2d}(\mathbb{R}), \ell^1(\mathbb{Z}^{2d})) \subseteq \Gamma_0(\rho^\circ)$.*

Notation. We now introduce the following notation for the ρ°. For sections (not necessarily continuous) $\Upsilon_1, \Upsilon_2 \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ of κ, we define a section $\langle \Upsilon_1, \Upsilon_2 \rangle_{\rho^\circ} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} B_L$:

$$\langle \Upsilon_1, \Upsilon_2 \rangle_{\rho^\circ}(L) := \langle \Upsilon_1(L), \Upsilon_2(L) \rangle_{L^\circ} \in B_L, \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}). \tag{7.42}$$

If we also have the section $\mathfrak{b} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} B_L$, then we define $\Upsilon_1 \cdot \mathfrak{b} \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \mathcal{E}_L(\mathbb{R}^d)$ via:

$$(\Upsilon_1 \cdot \mathfrak{b})(L) := \Upsilon_1(L) \cdot \mathfrak{b}(L) \in \mathcal{E}_L(\mathbb{R}^d), \qquad \forall L \in \mathrm{GL}_{2d}(\mathbb{R}), \tag{7.43}$$

We denote by the norm on the C^*-algebra $\Gamma_0(\rho^\circ)$ via:

$$\|\mathfrak{b}\|_{\rho^\circ} := \sup_{L \in \mathrm{GL}_{2d}(\mathbb{R})} \|\mathfrak{b}(L)\|_{B_L}, \tag{7.44}$$

for each $\mathfrak{b} \in \Gamma_0(\rho^\circ)$.

Proposition 7.13. *If we again identify $\ell^1(\mathbb{Z}^{2d})$ and $\mathbf{S}_0(\mathbb{R}^d)$ as the space of local approximating sections of ρ° and κ respectively, then for all $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$, and $f_1, f_2 \in \mathbf{S}_0(\mathbb{R}^d)$:*

1. $\langle f_1, f_2 \rangle_{\rho^\circ} \in \Gamma(\rho^\circ)$;
2. $f_1 \cdot \mathbf{b} \in \Gamma(\kappa)$.

Consequently, if $\mathfrak{b} \in \Gamma(\rho)$, $\Upsilon_1, \Upsilon_2 \in \Gamma(\kappa)$, then:

3. $\langle \Upsilon_1, \Upsilon_2 \rangle_{\rho^\circ} \in \Gamma(\rho^\circ)$;
4. $\Upsilon_1 \cdot \mathfrak{b} \in \Gamma(\kappa)$.

Finally, if $\mathfrak{b} \in \Gamma_0(\rho^\circ)$, $\Upsilon_1, \Upsilon_2 \in \Gamma_0(\kappa)$, then:

5. $\langle \Upsilon_1, \Upsilon_2 \rangle_{\rho^\circ} \in \Gamma_0(\rho^\circ)$;
6. $\Upsilon_1 \cdot \mathfrak{b} \in \Gamma_0(\kappa)$.

Corollary 7.6. *$\Gamma_0(\kappa)$ is a full Hilbert right $\Gamma_0(\rho^\circ)$-module with respect to the $\Gamma_0(\rho^\circ)$-valued inner-product and right module action defined by (7.42) and (7.43). As a consequence $\Gamma_0(\kappa)$ is a $\Gamma_0(\rho)$-$\Gamma_0(\rho^\circ)$ imprimitivity bimodule.*

Proof. That $\Gamma_0(\kappa)$ is a full Hilbert right $\Gamma_0(\rho^\circ)$-module follows from the same argument as in Corollary 7.4 using Proposition 7.13. It is almost immediate that $\Gamma_0(\kappa)$ is a $\Gamma_0(\rho)$-$\Gamma_0(\rho^\circ)$ imprimitivity bimodule, but we shall explicitly write what Items 2 and 3 of Definition 2.20 mean in our setting. Let $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \Gamma_0(\kappa)$, $\mathfrak{a} \in \Gamma_0(\rho)$, and $\mathfrak{b} \in \Gamma_0(\rho^\circ)$, we have from the pointwise definition of the section operations and the fact that $\mathcal{E}_L(\mathbb{R}^d)$ is an $A_L - B_L$-imprimitivity bimodule for each $L \in \mathrm{GL}_{2d}(\mathbb{R})$, that:

1. $\langle \mathfrak{a}\Upsilon_1, \Upsilon_2 \rangle_{\rho^\circ} = \langle \Upsilon_1, \mathfrak{a}^*\Upsilon_2 \rangle_{\rho^\circ}$,
2. ${}_\rho\langle \Upsilon_1 \mathfrak{b}, \Upsilon_2 \rangle = {}_\rho\langle \Upsilon_1, \Upsilon_2\ \mathfrak{b}^* \rangle$,
3. ${}_\rho\langle \Upsilon_1, \Upsilon_2 \rangle\ \Upsilon_3 = \Upsilon_1\ \langle \Upsilon_2, \Upsilon_3 \rangle_{\rho^\circ}$. □

7.1.6 Applications to Gabor Analysis

Before we proceed, we note that there is a fundamental obstruction regarding the continuity of the map $\mathrm{GL}_{2d}(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R}))$ given by the integrated representations $L \mapsto \overline{\pi}_L(\mathbf{b})$ (resp. $L \mapsto \overline{\pi}^*_{L^\circ}(\mathbf{b})$) for a fixed $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$. As discussed in [FK98], these maps are in general not globally continuous, and it remains so even if we consider our continuous sections. However, the local continuity of such maps on points of interest remain true in our setting.

Lemma 7.8. *Let $\mathfrak{b} \in \Gamma(\rho^\circ)$. If $\lambda \in \mathbb{C}\backslash\{0\}$ and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$ are such that $\mathfrak{b}(L_0) = \lambda\delta_0 \in \ell^1(\mathbb{Z}^{2d})$, then $L \mapsto \overline{\pi}^*_{L^\circ}(\mathfrak{b}(L))$ is a continuous map at the point L_0.*

Proof. We shall show the proof for continuity of $L \mapsto \overline{\pi}_{L^\circ}(\mathfrak{b}(L))$ at the point L_0. Fix $\varepsilon > 0$, choose an open set $U_1 \ni L_0$ and $\mathbf{b} \in \ell^1(\mathbb{Z}^{2d})$ such that

$$\|\mathfrak{b}(L) - \mathbf{b}\|_{B_L} < \frac{\varepsilon}{4}, \qquad \forall L \in U_1.$$

With these fixed, we take note of the fact that $\mathbf{b} - \mathfrak{b}(L_0) \in \ell^1(\mathbb{Z}^{2d})$ and so the map

$$L \mapsto \|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\|$$

is continuous by Proposition 7.12. In particular, there exists an open set $U_2 \ni L_0$ such that:

$$|\|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\| - \|\overline{\pi}^*_{L_0^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\|| < \frac{\varepsilon}{4}, \qquad \forall L \in U_2.$$

Now, take note of the fact that because $\mathfrak{b}(L_0) = \lambda\delta_0$, we have

$$\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L_0)) - \overline{\pi}^*_{L_0^\circ}(\mathfrak{b}(L_0)) = \lambda\left(\frac{1}{|\det L|} - \frac{1}{|\det L_0|}\right).$$

Hence we finally take the open set $U_3 \ni L_0$ such that

$$\left|\frac{1}{\det L} - \frac{1}{\det L_0}\right| < \frac{\varepsilon}{4d(\Lambda)}, \qquad \forall L \in U_3.$$

Since

$$\begin{aligned}\|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L)) - \overline{\pi}^*_{L_0^\circ}(\mathfrak{b}(L_0))\| \leqslant & \|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L)) - \overline{\pi}^*_{L^\circ}(\mathbf{b})\| + \|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\| \\ & + \|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L_0)) - \overline{\pi}^*_{L_0^\circ}(\mathfrak{b}(L_0))\|,\end{aligned}$$

we can estimate the terms individually, so that if $L \in U_1 \cap U_2 \cap U_3$, then

$$\|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L)) - \overline{\pi}^*_{L^\circ}(\mathbf{b})\| = \|\mathfrak{b}(L) - \mathbf{b}\|_{B_L} < \frac{\varepsilon}{4};$$

and

$$\|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\| \leqslant |\|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\| - \|\overline{\pi}^*_{L_0^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\|| + \|\overline{\pi}^*_{L_0^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\|$$

$$\leqslant |\|\overline{\pi}^*_{L^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\| - \|\overline{\pi}^*_{L_0^\circ}(\mathbf{b} - \mathfrak{b}(L_0))\|| + \|\mathfrak{b}(L_0) - \mathbf{b}\|_{B_{L_0}}$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4};$$

and

$$\|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L_0)) - \overline{\pi}^*_{L_0^\circ}(\mathfrak{b}(L_0))\| = d(\Lambda)\left|\frac{1}{\det L} - \frac{1}{\det L_0}\right| < \frac{\varepsilon}{4}.$$

Therefore, $L \in U_1 \cap U_2 \cap U_3$ implies:

$$\|\overline{\pi}^*_{L^\circ}(\mathfrak{b}(L)) - \overline{\pi}^*_{L_0^\circ}(\mathfrak{b}(L_0))\| < \varepsilon$$

as required. □

Remark 7.9. An analogous result holds for $L \mapsto \overline{\pi}_L(\mathfrak{a}(L))$ with $\mathfrak{a} \in \Gamma(\rho)$.

It is now important that we recall that for each $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$, $\mathcal{E}_L(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ via Proposition 7.1. This means that for any section $\Upsilon_1, \Upsilon_2 \in \prod_{L \in \mathrm{GL}_{2d}(\mathbb{R}^d)} \mathcal{E}_L(\mathbb{R}^d)$, we have for each $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$, that

$$S_{\Upsilon_1(L), \Upsilon_2(L), L} \in \mathcal{L}(L^2(\mathbb{R}^d)).$$

So every section in κ defines a $\mathrm{GL}_{2d}(\mathbb{R}^d)$-indexed family of mixed Gabor frame operators on $L^2(\mathbb{R}^d)$. Now, in the case where Υ_1, Υ_2 are continuous sections that form dual frames on some L_0, we have a crucial result.

Lemma 7.9. *Suppose $\Upsilon_1, \Upsilon_2 \in \Gamma(\kappa)$, and $L_0 \in \mathrm{GL}_{2d}(\mathbb{R}^d)$ such that $\Upsilon_1(L_0)$ and $\Upsilon_2(L_0)$ form dual Gabor frames with respect to the lattice $L_0\mathbb{Z}^{2d}$, then the map $\mathrm{GL}_{2d}(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R}^d))$ via $L \mapsto S_{\Upsilon_1(L), \Upsilon_2(L), L}$ is continuous at the point L_0.*

Proof. It follows from the Janssen's representation for Heisenberg modules, Corollary 5.8, that for all $L \in \mathrm{GL}_{2d}(\mathbb{R}^d)$:

$$S_{\Upsilon_1(L), \Upsilon_2(L), L} = \overline{\pi}^*_{L^\circ}(\langle \Upsilon_2(L), \Upsilon_1(L) \rangle_{L^\circ}) = \overline{\pi}^*_{L^\circ}(\langle \Upsilon_2, \Upsilon_1 \rangle_{\rho^\circ}(L)). \tag{7.45}$$

Since $\Upsilon_1(L_0)$ and $\Upsilon_2(L_0)$ are dual Gabor frames with respect to $L_0\mathbb{Z}^{2d}$, then it follows from the

Wexler-Rax relations for Heisenberg modules, Corollary 5.10, that:

$$\langle \Upsilon_2, \Upsilon_1 \rangle_{\rho^\circ} (L_0) = \det L_0 \delta_0.$$

Since $\langle \Upsilon_2, \Upsilon_1 \rangle_{\rho^\circ} \in \Gamma(\rho^\circ)$ by Proposition 7.13 Item 3, then it follows from Lemma 7.8 that $L \mapsto \overline{\pi}^*_{L^\circ}(\langle \Upsilon_2, \Upsilon_1 \rangle_{\rho^\circ} (L))$ is continuous at L_0. The result now follows from (7.45) □

Theorem 7.5. *Let $L_0 \in \mathrm{GL}_{2d}(\mathbb{R}^d)$, and suppose $f_0 \in \mathcal{E}_{L_0}(\mathbb{R}^d)$ such that $\mathcal{G}(f_0, L_0)$ generates a Gabor frame, then for every continuous section $\Upsilon \in \Gamma(\kappa)$ that passes through f_0, there exists an open set $U_0 \ni L_0$ such that $\mathcal{G}(\Upsilon(L), L)$ generates a Gabor frame for all $L \in U_0$.*

Proof. It follows from Lemma 5.3 that the canonical dual frame $h_0 = S^{-1}_{f_0,L_0} f_0$ is in $\mathcal{E}_{L_0}(\mathbb{R}^d)$. Let $\Upsilon, \Omega \in \Gamma(\kappa)$ such that $\Upsilon(L_0) = f_0$ and $\Omega(L_0) = h_0$. We see that Υ and Ω form dual Gabor frames at the point L_0, and therefore by Lemma 7.9, $L \mapsto S_{\Upsilon(L),\Omega(L),L}$ is continuous at L_0. This means there exists an open set $U_0 \ni L_0$ such that

$$\|S_{\Upsilon(L),\Omega(L),L} - \mathrm{Id}_{L(\mathbb{R}^d)}\| < 1, \qquad \forall L \in U_0.$$

Therefore $S_{\Upsilon(L),\Omega(L),L}$ is invertible for all $L \in U_0$. Since $(S_{\Upsilon(L),\Omega(L),L})_{|\mathcal{E}_L(\mathbb{R}^d)} \colon \mathcal{E}_L(\mathbb{R}^d) \to \mathcal{E}_L(\mathbb{R}^d)$, we define the section:

$$\Phi(L) := S^{-1}_{\Upsilon(L),\Omega(L),L} \Omega(L) \in \mathcal{E}_L(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d), \qquad \forall L \in U_0. \tag{7.46}$$

Therefore if $L \in U_0$, then for all $f \in L^2(\mathbb{R}^d)$:

$$\begin{aligned} S_{\Upsilon(L),\Omega(L),L} S_{\Upsilon(L),\Phi(L),L} f &= S_{\Upsilon(L),\Omega(L),L} \sum_{k \in \mathbb{Z}^{2d}} \langle f, \pi(Lk)\Upsilon(L)\rangle \pi(Lk)\Phi(L) f \\ &= \sum_{k \in \mathbb{Z}^{2d}} \langle f, \pi(Lk)\Upsilon(L)\rangle \pi(Lk) S_{\Upsilon(L),\Omega(L),L} \Phi(L) f \\ &= \sum_{k \in \mathbb{Z}^{2d}} \langle f, \pi(Lk)\Upsilon(L)\rangle \pi(Lk)\Omega(L) f \\ &= S_{\Upsilon(L),\Omega(L),L} f, \end{aligned}$$

hence $S_{\Upsilon(L),\Phi(L),L} = \mathrm{Id}_{L^2(\mathbb{R}^d)}$ for all $L \in U_0$, which implies $\Phi(L)$ is a dual Gabor atom for $\Upsilon(L)$ in $L \in U_0$ and that $\mathcal{G}(\Upsilon(L), L)$ generates a Gabor frame in $L^2(\mathbb{R}^d)$ for all $L \in U_0$. □

The following proposition shows that the 'local section' (it is only defined inside the open set U_0) implementing a local dual Gabor atom for Υ above, denoted by Φ in Equation (7.46), is continuous at the point L_0. Its precise meaning is given below.

Proposition 7.14. *The section $\Phi \in \prod_{L\in U_0} \mathcal{E}_L(\mathbb{R}^d)$ in (7.46) satisfies: for each $\varepsilon > 0$ there exists an open subset $W_0 \ni L_0$ of U_0, and a $g_0 \in \mathbf{S}_0(\mathbb{R}^d)$ such that*

$$\|\Phi(L) - g_0\|_{\mathcal{E}_L(\mathbb{R}^d)} < \varepsilon, \qquad \forall L \in W_0.$$

Proof. Fix $\varepsilon > 0$ and consider a compact neighborhood $K_0 \ni L_0$ inside U_0. Since we have $\| \mathrm{Id}_{L^2(\mathbb{R}^d)} - S_{\Upsilon(L),\Omega(L),L}\| < 1$ for all $L \in U_0$, it follows from the Neumann and geometric series that

$$\|S^{-1}_{\Upsilon(L),\Omega(L),L}\| \leqslant \frac{1}{1 - \| \mathrm{Id}_{L^2(\mathbb{R}^d)} - S_{\Upsilon(L),\Omega(L),L}\|}$$

for all $L \in U_0$. It follows that $\|S^{-1}_{\Upsilon(L),\Omega(L),L}\| \leqslant C$ for some $C > 0$ in K_0. Because $\Omega \in \Gamma(\kappa)$, there exists $W_1 \ni L_0$ an open subset contained in K_0 such that for $L \in W_1$:

$$\|\Omega(L) - g_0\|_{\mathcal{E}_L(\mathbb{R}^d)} < \frac{\varepsilon}{2C}.$$

We know that $L \mapsto S_{\Upsilon(L),\Omega(L),L}$ is continuous at the point L_0, and since operator inversion is continuous in $\mathcal{L}(L^2(\mathbb{R}^d))$, we find that $U_0 \ni L \mapsto S^{-1}_{\Upsilon(L),\Omega(L),L} \in \mathcal{L}(L^2(\mathbb{R}^d))$ is also continuous. Choose $W_2 \ni L_0$ contained in K_0 such that for $L \in W_2$:

$$\|S^{-1}_{\Upsilon(L),\Omega(L),L} - S^{-1}_{\Upsilon(L_0),\Omega(L_0),L_0}\| = \|S^{-1}_{\Upsilon(L),\Omega(L),L} - \mathrm{Id}_{L^2(\mathbb{R}^d)}\| < \frac{\varepsilon}{2\|g_0\|_2}.$$

Finally we can estimate, for $L \in W_0 := W_1 \cap W_2 \subseteq K_0$:

$$\begin{aligned}
\|\Phi(L) - g_0\|_{\mathcal{E}_L(\mathbb{R}^d)} &= \|S^{-1}_{\Upsilon(L),\Omega(L),L}\Omega(L) - g_0\| \\
&\leqslant \|S^{-1}_{\Upsilon(L),\Omega(L),L}(\Omega(L) - g_0)\| + \|S^{-1}_{\Upsilon(L),\Omega(L),L}g_0 - g_0\| \\
&\leqslant C\|\Omega(L) - g_0\|_{\mathcal{E}_L(\mathbb{R}^d)} + \|S^{-1}_{\Upsilon(L),\Omega(L),L} - \mathrm{Id}_{L^2(\mathbb{R}^d)}\|\|g_0\|_2 \\
&< \varepsilon.
\end{aligned}$$

□

As a Corollary of Gabor-stability of Heisenberg modules, we derive that these modules exhibit a version of the Balian-Low Theorem [AFK14] when considered as a function space.

Corollary 7.7 (Balian-Low for Heisenberg Modules)**.** *Let $L_0 \in \mathrm{GL}_{2d}(\mathbb{R})$ such that* $\det L_0 = 1$.

If $f_0 \in \mathcal{E}_{L_0}(\mathbb{R}^d)$, then $\mathcal{G}(f_0, L_0)$ cannot be a Gabor frame for $L^2(\mathbb{R}^d)$.

Proof. Suppose for contradiction that with $f_0 \in \mathcal{E}_{L_0}(\mathbb{R}^d)$ and $\det L_0 = 1$, that $\mathcal{G}(f_0, L_0)$ generates a Gabor frame for $L^2(\mathbb{R}^d)$. Let $\Upsilon \in \Gamma(\kappa)$ be a continuous section that passes through f_0 at L_0. Then it follows from Theorem 7.5 that there exists an $\varepsilon > 0$ such that $\mathcal{G}(\Upsilon(L), L)$ generates a Gabor frame for all $L \in \mathrm{GL}_{2d}(\mathbb{R})$ where $\|L - L_0\| < \varepsilon$. Now take $\delta > 0$ such that $0 < \delta\|L_0\| < \varepsilon$, then $L = (1+\delta)L_0$ satisfies $\|L - L_0\| = \delta\|L_0\| < \varepsilon$, so $\mathcal{G}(\Upsilon(L), L)$ generates a Gabor frame for $L^2(\mathbb{R}^d)$. However, $\det L = \det\left((1+\delta)L_0\right) = (1+\delta)^{2d} > 1$, which contradicts the basic density result given by Corollary 5.11. □

7.1.7 Modulus of Continuity of Local Approximating Sections

In order to establish a Banach bundle structure, it was enough that the the norm varies continuously along the local approximating sections, which we obtained for the sections associated with $f \in \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, where $\nu(k) = 1 + |k|$. However, there are actually much better estimates for sections associated with f in the weighted Feichtinger's algebra

$$\mathbf{S}_{0,\nu_s}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) : \mathcal{V}_\phi f \cdot \nu_s \in L^1(\mathbb{R}^{2d})\},$$

with $\nu_s(k) = (1 + |k|)^s$ for $s \geqslant 2$. The weighted ℓ^1 space $\ell^1_{\nu_s}(\mathbb{Z}^{2d})$ is analogously defined (see Equation (7.19))

Recall that a function $f\colon X \to Y$ between normed spaces X, Y is called α-Hölder continuous (α-Hölder for short) for $0 < \alpha \leqslant 1$ if there is a constant $C > 0$ such that $\|f(x_1) - f(x_2)\| \leqslant C\|x_1 - x_2\|^\alpha$ for all $x_1, x_2 \in X$. More generally, f is called locally α-Hölder if for every $x \in X$ there exists a neighborhood $x \in M \subset X$ such that $f|_M$ is α-Hölder. In the special case of $\alpha = 1$, we say that f is (locally) Lipschitz instead.

Let us summarize some well-known facts on Hölder continuity.

Observation 7.1.

- *if $0 < \alpha < \beta \leqslant 1$, then every locally β-Hölder map is locally α-Hölder;*
- *if $f\colon X \to Y$ is locally α-Hölder and $g\colon Y \to Z$ is locally β-Hölder, then $g \circ f\colon X \to Z$ is locally $(\alpha \cdot \beta)$-Hölder;*
- *if $f\colon S \to B$ with $S \subset \mathbb{R}^n$ and B a Banach space is differentiable, then f is locally Lipschitz.*

Lemma 7.10. *For each fixed $n \in \mathbb{Z}^{2d}$, the map $L \mapsto P_{c_L}(n)$ fulfills the estimate*

$$|P_{c_L}(n) - P_{c_{L_0}}(n)| \leqslant (2d+1)\pi \|K\| \nu_2(n)(\|L\| + \|L_0\|) \cdot \|L - L_0\|.$$

In particular, it is locally Lipschitz.

Proof. Looking at Equation 7.28, we find that

$$P_{c_L}(n) = \exp\left(2\pi i \left(\sum_{j<k} n_j n_k (Le_j)^T K(Le_k) + \sum_k \frac{n_k(n_k-1)}{2}(Le_k)^T K(Le_k)\right)\right)$$

Combining the estimates $|e^{ia} - e^{ib}| \leqslant |a-b|$, $|\frac{m(m-1)}{2}| \leqslant |m|^2$ for all $m \in \mathbb{Z}$, and $\|(Le_j)^T K(Le_k) - (L_0 e_j)^T K(L_0 e_k)\| \leqslant \|K\|(\|L\| + \|L_0\|)\|L - L_0\|$, we obtain

$$|P_{c_L}(n) - P_{c_{L_0}}(n)| \leqslant \pi \|K\|(\|L\| + \|L_0\|)\|L - L_0\| \left(\sum_{j \neq k} |n_j||n_k| + 2\sum_k n_k^2\right).$$

We estimate the last term further

$$\sum_{j \neq k} |n_j||n_k| + 2\sum_k n_k^2 = \begin{pmatrix} |n_1| & \dots & |n_{2d}| \end{pmatrix} \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix} \begin{pmatrix} |n_1| \\ \vdots \\ |n_{2d}| \end{pmatrix}$$

$$\leqslant \|n\|^2 \left\| \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix} \right\| \leqslant \nu_2(n) \cdot (2d+1);$$

indeed, the norm of the symmetric matrix coincides with its largest Eigenvalue $2d+1$ and $\|n\|^2 = \sum n_k^2 \leqslant \nu_2(n)$. □

Lemma 7.11. *For each $\mathbf{a} \in \ell^1_{\nu_2}(\mathbb{Z}^{2d})$ and $L \in \mathrm{GL}_{2d}(\mathbb{R})$: The map $\mathrm{GL}_{2d}(\mathbb{R}) \ni L \mapsto \mathbf{a}^L \in \ell^1(\mathbb{Z}^{2d})$*

fulfills the inequality

$$\|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1} \leqslant (2d+1)\pi\|K\|(\|L\| + \|L_0\|)\|\mathbf{a}\|_{\ell^1_{\nu_2}} \cdot \|L - L_0\|.$$

In particular, it is locally Lipschitz.

Proof.

$$\begin{aligned}
\|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1} &= \sum_{k\in\mathbb{Z}^{2d}} |a(k)||P_{c_L}(k) - P_{c_{L_0}}(k)| \\
&\leqslant \sum_{k\in\mathbb{Z}^{2d}} |a(k)|(2d+1)\pi\|K\|\nu_2(k)(\|L\| + \|L_0\|) \cdot \|L - L_0\| \\
&= (2d+1)\pi\|K\|(\|L\| + \|L_0\|)\|\mathbf{a}\|_{\ell^1_{\nu_2}} \cdot \|L - L_0\|
\end{aligned}$$

□

Lemma 7.12. *For $f, g \in \mathbf{S}_{0,\nu_s}(\mathbb{R}^d)$, $s \geqslant 2r \geqslant 2$, the function $L \mapsto {}_L\langle f, g\rangle$ is continuous as a map to $\ell^1_{\nu_s}(\mathbb{Z}^{2d})$ and r-times differentiable as a map to $\ell^1_{\nu_{s-2r}}(\mathbb{Z}^{2d})$.*

Proof. The continuity statement is contained in [FK04, Lemma 3.5]. Consider the partial derivatives

$$\begin{aligned}
\frac{\partial}{\partial x_\ell}\pi(x,\omega)g(t) &= -e^{2\pi i\,\omega t}(\partial_\ell g)(t-x) = -\pi(x,\omega)(\partial_\ell g)(t), \\
\frac{\partial}{\partial \omega_\ell}\pi(x,\omega)g(t) &= 2\pi i\, t_\ell e^{2\pi i\,\omega t} g(t-x) = \pi(x,\omega)2\pi i(t_\ell + x_\ell)g(t).
\end{aligned}$$

Put

$$z_m := \begin{cases} x_m & 1 \leqslant m \leqslant d, \\ \omega_{m-d} & d+1 \leqslant m \leqslant d; \end{cases} \qquad D_m(g) := \begin{cases} -\partial_m g & 1 \leqslant m \leqslant d, \\ 2\pi i(t_{m-d} + x_{m-d})g & d+1 \leqslant m \leqslant d. \end{cases}$$

so that $z = (x,\omega)$ and $\frac{\partial}{\partial z_m}\pi(z)g = \pi(z)D_m(g)$. Note that $\partial_n g$ and $t_n g$, and therefore all $D_m(g)$, belong to $\mathbf{S}_{0,\nu_{s-1}}(\mathbb{R}^d)$ (the 1-dimensional case is treated in [DJLL21], the multidimensional analogue can be proved along the same lines). By [FK04, Lemma 2.2 (ii)], $z \mapsto \frac{\partial}{\partial z_m}\pi(z)g = \pi(z)D_m(g)$ is a continuous map to $\mathbf{S}_{0,\nu_{s-1}}(\mathbb{R}^d)$. By the chain rule,

$$\begin{aligned}
\frac{\partial}{\partial L_{m,n}}{}_L\langle f, g\rangle(k) = \sum_\ell \left.\frac{\partial\langle f, \pi(z)g\rangle}{\partial z_\ell}\right|_{z=Lk} \frac{\partial(Lk)_\ell}{\partial L_{m,n}} &= \sum_\ell {}_L\langle f, D_\ell(g)\rangle(k)\delta_{m,\ell} \cdot k_n \\
&= {}_L\langle f, D_m(g)\rangle(k) \cdot k_n.
\end{aligned}$$

We know that f and the $D_m(g)$ lie in $\mathbf{S}_{0,\nu_{s-1}}(\mathbb{R}^d)$, therefore

$$L \mapsto {}_L\langle f, D_m(g)\rangle$$

is a continuous map into $\ell^1_{\nu_{s-1}}(\mathbb{Z}^{2d})$ and, consequently,

$$L \mapsto \frac{\partial}{\partial L_{m,n}} {}_L\langle f, g\rangle = ({}_L\langle f, D_m(g)\rangle(k) \cdot k_n)_{k \in \mathbb{Z}^{2d}}$$

is a continuous map into $\ell^1_{\nu_{s-2}}(\mathbb{Z}^{2d})$, i.e. $L \mapsto {}_L\langle f, g\rangle$ is differentiable as a map to $\ell^1_{\nu_{s-2}}(\mathbb{Z}^{2d})$.

The same arguments can be iterated to show that the function $L \mapsto {}_L\langle f, g\rangle$ is r-times differentiable as a map to $\ell^1_{\nu_{s-2r}}(\mathbb{Z}^{2d})$ as long as $s \geqslant 2r$. □

Corollary 7.8. *Let $f, g \in \mathbf{S}_{0,\nu_2}(\mathbb{R}^d)$. Then $L \mapsto \|{}_L\langle f, g\rangle\|_{\ell^1_\nu}$ is continuous and $L \mapsto {}_L\langle f, g\rangle$ is locally Lipschitz as a map $\mathrm{GL}_{2d}(\mathbb{R}) \to \ell^1(\mathbb{Z}^{2d})$.*

Proof. Since $\mathbf{S}_{0,\nu_2}(\mathbb{R}^d) \subseteq \mathbf{S}_{0,\nu}(\mathbb{R}^d)$, $L \mapsto {}_L\langle f, g\rangle$ is continuous as a map to $\ell^1_\nu(\mathbb{Z}^{2d})$ by [FK04, Lemma 3.5] and thus the first statement follows. From the Lemma 7.12 applied to $s = 2$, we conclude that $L \mapsto {}_L\langle f, g\rangle$ is differentiable as a map to $\ell^1(\mathbb{Z}^{2d})$ and, therefore, locally Lipschitz. □

Theorem 7.6. *Let $f, g \in \mathbf{S}_{0,\nu_2}$. Then the map $L \mapsto \|\overline{\pi}_L({}_L\langle f, g\rangle)\|$ is locally $\frac{1}{2}$-Hölder continuous. In particular, the map $L \mapsto \|f\|_{\mathcal{E}_L(\mathbb{R}^d)} = \|\overline{\pi}_L({}_L\langle f, f\rangle)\|^{1/2}$ is locally $\frac{1}{2}$-Hölder continuous.*

Proof. Put $\mathbf{a} := {}_L\langle f, g\rangle$ and $\mathbf{a}_0 := {}_{L_0}\langle f, g\rangle$. Then, using triangle inequality, opposite triangle inequality, and Corollary 7.1

$$\begin{aligned}
&\big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a}_0)\|\big| \\
&\quad \leqslant \big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| + \big|\|\overline{\pi}_{L_0}(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a}_0)\|\big| \\
&\quad \leqslant \big|\|\overline{\pi}_L(\mathbf{a})\| - \|\overline{\pi}_{L_0}(\mathbf{a})\|\big| + \|\overline{\pi}_{L_0}(\mathbf{a} - \mathbf{a}_0)\| \\
&\quad \leqslant C\|\mathbf{a}\|_{\ell^1_\nu}(\|L\| + \|L_0\|)^{1/2}\|L - L_0\|^{1/2} + \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1} + \|\mathbf{a} - \mathbf{a}_0\|_{\ell^1}.
\end{aligned}$$

The first statement thus follows from Lemma 7.11 and Corollary 7.8 and the fact that every locally Lipschitz map is locally $\frac{1}{2}$-Hölder.

The second statement if trivial for $f = 0$. Let $f \neq 0$. For all $L \in \mathrm{GL}_{2d}(\mathbb{R})$, faithfulness of the integrated representation and positive definiteness of the inner product imply that $\overline{\pi}_L({}_L\langle f, f\rangle) \neq$

0, hence $\|f\|_{\mathcal{E}_L(\mathbb{R}^d)} > 0$. Since the square-root function is differentiable on $\mathbb{R}_{>0}$, we get the claimed local Hölder continuity. $\square$

Remark 7.10. By [GL16, Proposition 3.5], $\ell^1_{\nu_2}(\mathbb{Z}^{2d})$ is an inverse-closed subalgebra of A_L, in particular $\operatorname{spec}({}_L\langle f, f\rangle) = \operatorname{spec}(\overline{\pi}_L({}_L\langle f, f\rangle)) = \operatorname{spec}(\pi_{\Theta_L}({}_L\langle f, f\rangle^L))$ for all $f \in \mathbf{S}_{0,\nu_2}(\mathbb{R}^d)$. With the generalization of Theorem 7.2 to spectra mentioned in Remark 7.7 and arguments parallel to the ones we just performed, one can also show that the map $L \mapsto \operatorname{spec}({}_L\langle f, f\rangle)$ is locally $\frac{1}{2}$-Hölder continuous for every $f \in \mathbf{S}_{0,\nu_2}(\mathbb{R}^d)$. Indeed, for $\mathbf{a} = {}_L\langle f, f\rangle$ and $\mathbf{a}_0 = {}_{L_0}\langle f, f\rangle$,

$$\operatorname{spec}(\mathbf{a}) \subset \operatorname{spec}(\mathbf{a}_0) + [-1, 1] \cdot (2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta_L - \Theta_{L_0}\|^{\frac{1}{2}} + \|\mathbf{a}^L - \mathbf{a}_0^{L_0}\|_{\ell^1})$$

by Remark 7.7. Therefore, for the Hausdorff distance d_H we find that

$$\begin{aligned} d_H(\operatorname{spec}(\mathbf{a}), \operatorname{spec}(\mathbf{a_0})) &\leqslant 2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}\|\Theta_L - \Theta_{L_0}\|^{\frac{1}{2}} + \|\mathbf{a}^L - \mathbf{a}_0^{L_0}\|_{\ell^1} \\ &\leqslant 2\sqrt{\pi}\|\mathbf{a}\|_{\ell^1_\nu}(\|L\| + \|L_0\|)^{1/2}\|L - L_0\|^{1/2} + \|\mathbf{a}^L - \mathbf{a}^{L_0}\|_{\ell^1} + \|\mathbf{a} - \mathbf{a}_0\|_{\ell^1} \end{aligned}$$

and the claimed $\frac{1}{2}$-Hölder continuity follows exactly as in the proof of Theorem 7.6.

Of course, everything also works for the right Hilbert module structure and, thus, we get that the map $L \mapsto \operatorname{spec}(\langle f, f\rangle_{L^\circ})$ is $\frac{1}{2}$-Hölder. Finally, recall that according to the Janssen representation (Corollary 5.8), we have $S_{f,L} = \overline{\pi}^*_{L^\circ}(\langle f, f\rangle_{L^\circ})$. Since $L \mapsto L^\circ = -J(L^{-1})^T J$ is locally Lipschitz, we may conclude that for $f \in \mathbf{S}_{0,\nu_2}(\mathbb{R}^d)$ the map $L \mapsto \operatorname{spec}(S_{f,L}) = \operatorname{spec}(\langle f, f\rangle_{L^\circ})$ is locally $\frac{1}{2}$-Hölder.

7.2 Results Part II: On Modulation and Translation Invariant Operators and the Heisenberg Modules

In this section, we establish some connections between the Heisenberg modules and Quantum Harmonic Analysis (QHA). We shall do this by following the recent developments primarily from the papers of Luef and Skrettingland [LS18, Skr20, Skr21b], where time-frequency and Gabor analytic techniques have taken a major role. In light of this, we briefly introduce the Weyl quantization scheme (compare with the Kohn-Nirenberg and Spreading schemes c.f. Definition 4.12), which is the preferred scheme for QHA in the time-frequency plane $\mathbb{R}^{2d}$. There are reasons for this preference, and the case for it will be made stronger in the introduction of *lattice-*

modulation invariant operators. As expected, lattice-modulation invariant operators will be the natural counterpart for lattice-translation invariant operators (c.f. Definition 4.14), and would require a notion of operator-modulation analogous to Definition 4.13. The introduction of operator-modulation subsequently paves the way for "*Quantum Time-Frequency Analysis*" (QTFA), as we now have access to operator time-frequency shifts. An exploration of such ideas has already started with [LM24].

We will be motivated by the simple observation that all adjointable maps of the Heisenberg module associated to the lattice Λ are exactly finite sums of Gabor frame operators on Λ (Lemma 7.7). It then follows that adjointable maps of the Heisenberg module associated to the lattice Λ, are Λ-translation invariant operators. Conversely, we shall use techniques from operator algebras to show that we can characterise Λ-translation invariant operators as limits (under different operator topologies) of periodizations of finite rank operators, generated by functions from the Heisenberg module. Our operator algebraic techniques necessitate that we go beyond the usual Banach Gelfand triple framework for sequences, and in turn, we also obtain characterisation results beyond the usual setting of $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ for Λ-translation invariant operators.

We then introduce the notion of Λ-modulation invariant operators, using the concept of operator modulation similar to the operator translation invariant setting. In examining such operators the parity operator naturally arises as the only $\mathbb{R}^{2d}$-modulation invariant operator. The Parity operator allows us to identify Λ-modulation invariant operators with $\Lambda/2$-translation invariant operators. The identification allows us to formulate (analogous) characterisations of Λ-modulation invariant operators in terms of a new kind of operator-periodization based on operator-modulation, called the "Fourier-Wigner operator-periodization" applied on finite-rank operators generated by the Heisenberg modules. We shall also consider the discrepancy between the densities of the lattices $\Lambda/2$ and Λ, along with its implications on the minimum number of generators required to obtain the aforementioned characterisations for Λ-translation versus Λ-modulation invariant operators.

7.2.1 Intermission: The Weyl Quantization Scheme and QHA

Similar to the Kohn-Nirenberg symbol and the spreading function of Definition 4.12, the so-called **Weyl-symbol** can be obtained by applying an isomorphism to the kernel $\kappa(T)$ of an operator $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$. Let us first define the **symplectic phase-multiplication** map $\mathrm{ph} : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ via:

$$(\mathrm{ph}\, F)(x, \omega) = e^{\pi i x \cdot \omega} F(x, \omega), \qquad \forall (x, \omega) \in \mathbb{R}^{2d}. \tag{7.47}$$

Note that ph is a unitary map on $L^2(\mathbb{R}^{2d})$, satifying

$$(\mathrm{ph}^{-1}\, F)(x, \omega) = e^{-\pi i x \cdot \omega} F(x, \omega).$$

Furthermore, it can be restricted in the Feichtinger's algebra and extended to its dual, both as isomorphisms, using Item 8 of Proposition 4.9 and Item 6 of Proposition 4.12.

Definition 7.3. For each $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, the **Weyl-symbol** of T is given by:

$$\sigma(T) := (\mathcal{F}_s \circ \mathrm{ph})(\eta(T)) \in \mathbf{S}_0'(\mathbb{R}^{2d}), \tag{7.48}$$

where $\eta(T)$ is the spreading function of T.

Recall that the motivation for the spreading function is that the rank-one operators $f \otimes \overline{g}$ give us the STFTs $\mathcal{V}_g f = \eta(f \otimes \overline{g})$ (c.f. (4.39)). This implies that:

$$\mathrm{ph}(\eta(f \otimes \overline{g}))(x, \omega) = e^{\pi i x \cdot \omega} \mathcal{V}_g f(x, \omega) =: A(f, g)(x, \omega), \qquad \forall (x, \omega) \in \mathbb{R}^{2d}.$$

The equation above is the so-called **cross-ambiguity** function for $f, g \in L^2(\mathbb{R}^d)$. The Weyl quantization scheme can now be seen to be the quantum correspondence that allows us to recover the so-called **cross-Wigner distribution** $W(f, g) \in L^2(\mathbb{R}^{2d})$ which is another time-frequency representation defined via:

$$W(f, g)(x, \omega) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t \cdot \omega}\, \mathrm{d}t. \tag{7.49}$$

It is well-known [Grö01, DG11] that the cross-ambiguity function and the cross-Wigner transforms are related via the symplectic-Fourier transform:

$$W(f, g) = \mathcal{F}_s(A(f, g)). \tag{7.50}$$

Therefore it follows that indeed:

$$\sigma(f \otimes \overline{g}) = (\mathcal{F}_s \circ \mathrm{ph})(\eta(f \otimes \overline{g}))) = \mathcal{F}_s(A(f,g)) = W(f,g).$$

On the other hand, the Kohn-Nirenberg symbol of $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ is related to its Weyl symbol via:

$$\sigma(T) = (\mathcal{F}_s \circ \mathrm{ph} \circ \mathcal{F}_s)(\zeta(T)). \tag{7.51}$$

Remark 7.11. Equation (7.51) is obtained through the relationship of the spreading function and the Kohn-Nirenberg symbol. However, one can also obtain the Weyl symbol straight from the kernel $\kappa(T)$ of T by the following formula:

$$\sigma(T) = (\mathcal{F}_2 \circ A_1)(\kappa(T)) \tag{7.52}$$

where A_1 is the coordinate transform $(A_1 F)(x,y) = F\left(x + \frac{y}{2}, x - \frac{y}{2}\right)$ for any $F : \mathbb{R}^{2d} \to \mathbb{C}$. Indeed Equation (7.52) is usually taken to be the definition of the Weyl-symbol, but one can of course show that we obtain Definition 7.3 by the following formulae:

$$\mathcal{F}_2 \circ A_1 = \mathcal{F}_s \circ \mathrm{ph} \circ \mathcal{F}_2 \circ \tau_a.$$

The Weyl correspondence is merely an application of isomorphisms to the kernel of an operator T, so this quantization scheme implements a Banach-Geldand triple isomorphism from the operator space $(\mathcal{L}_{*-\|\cdot\|}, \mathcal{HS}, \mathcal{L})(\mathbb{R}^d, \mathbb{R}^d)$ to $(\mathbf{S}_0, L^2, \mathbf{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$ (see opening paragraph of Section 4.4 of Chapter 4) exactly via $T \mapsto \sigma(T)$. The Weyl correspondence is well-studied, the following gives the inverse to the Weyl-calculus map (see [LS18]).

Proposition 7.15. *For $\sigma \in ((\mathbf{S}_0, L^2, \mathbf{S}_0')(\mathbb{R}^d \times \mathbb{R}^d))$, the **Weyl-transform** L_σ is an operator in $(\mathcal{L}_{*-\|\cdot\|}, \mathcal{HS}, \mathcal{L})(\mathbb{R}^d, \mathbb{R}^d)$ acting as an inverse for the Weyl-calculus $T \mapsto \sigma(T)$. It is explicitly given by the following vector-valued integral:*

$$L_\sigma = \int_{\mathbb{R}^{2d}} \mathcal{F}_s(\sigma)(\chi) e^{-\pi i \chi_1 \cdot \chi_2} \pi(\chi) \, \mathrm{d}\chi. \tag{7.53}$$

Remark 7.12. Note that when $\sigma \in \mathbf{S}_0'(\mathbb{R}^{2d})$ in Equation (7.53), then L_σ is interpreted via duality.

In fact, using the brackets $(\cdot,\cdot)_{\mathbf{S}_0,\mathbf{S}_0'}$

$$(g, L_\sigma f)_{\mathbf{S}_0,\mathbf{S}_0'} = (W(f,g),\sigma)_{\mathbf{S}_0,\mathbf{S}_0'} \tag{7.54}$$

for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$.

The following result is also one of the reasons for the interest in the Weyl-quantization scheme: it possesses a property similar to the Kohn-Nirenberg symbol.

Proposition 7.16 (Shift-Covariance for the Weyl-Symbol)**.** *For any $\chi \in \mathbb{R}^{2d}$:*

$$\sigma(\alpha_\chi T) = T_\chi \sigma(T). \tag{7.55}$$

Proof. Let $\chi \in \mathbb{R}^{2d}$, recall that α_χ denotes the operator-translation in Definition 4.13. It is also straighforward to show that the symplectic modulation commutes with the phase-multiplication map: $\mathrm{ph} \circ M_\chi^s = M_\chi^s \circ \mathrm{ph}$. The following computation now follows from (4.16) and the shift-covariance formula for the Kohn-Nirenberg symbol Equation (4.41) that :

$$\begin{aligned}
\sigma(\alpha_\chi T) &= (\mathcal{F}_s \circ \mathrm{ph} \circ \mathcal{F}_s)(\zeta(\alpha_\chi T)) \\
&= (\mathcal{F}_s \circ \mathrm{ph} \circ \mathcal{F}_s \circ T_\chi)(\zeta(T)) \\
&= (\mathcal{F}_s \circ \mathrm{ph} \circ M_\chi^s \circ \mathcal{F}_s)(\zeta(T)) \\
&= (\mathcal{F}_s \circ M_\chi^s \,\mathrm{ph} \circ \mathcal{F}_s)(\zeta(T)) \\
&= T_\chi(\mathcal{F}_s \circ \mathrm{ph} \circ \mathcal{F}_s)(\zeta(T)) \\
&= T_\chi \sigma(T).
\end{aligned}$$

□

Since the Weyl-symbol also possesses shift-covariance similar to the Kohn-Nirenberg symbol, if $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$, then its Weyl symbol $\sigma(T) \in \mathbf{S}_0'(G \times \widehat{G})$ is also Λ-periodic, meaning there exists $\widetilde{\sigma}(T) \in \mathbf{S}_0'((G \times \widehat{G})/\Lambda)$ such that $\sigma(T) = P_\Lambda^\times(\widetilde{\sigma}(T))$ (see Remark 4.10). It also follows that we can use the Weyl-quantization scheme to reprove Theorem 4.7 and all its subsequent corollaries (including Janssen's representation) for lattice-translation invariant operators. This was properly explored in [Skr21a] where the representation for Λ-translation invariant operators are slightly different. We give an analogue of Proposition 4.17 for the Weyl-quantization scheme.

Proposition 7.17. *Provided a Λ-translation invariant operator $T \in \mathcal{L}(\mathbf{S}_0(G), \mathbf{S}_0'(G))$ we have $\widetilde{\sigma}(T) \in (\mathbf{S}_0, L^2, \mathbf{S}_0')((G \times \widehat{G})/\Lambda)$, and there exists a unique coefficient $\mathbf{k} \in (\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$ such that*

$$T = \int_{\Lambda^\circ} k(\lambda^\circ) e^{\pi i \lambda_1^\circ \cdot \lambda_2^\circ} \pi(\lambda^\circ) d\mu_{\Lambda^\circ}(\lambda^\circ) = \frac{1}{d(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} k(\lambda^\circ) e^{\pi i \lambda_1^\circ \cdot \lambda_2^\circ} \pi(\lambda^\circ) \tag{7.56}$$

where the convergence is in the weak sense when $\widetilde{\sigma}(T) \in \mathbf{S}_0'((G \times \widetilde{G})/\Lambda)$.

Remark 7.13. We shall use the notation $\lambda^\circ = (\lambda_1^\circ, \lambda_2^\circ) \in \Lambda^\circ$ or more generally, $\chi = (\chi_1, \chi_2) \in \mathbb{R}^{2d}$ for the rest of the manuscript. Notice that there is an extra phase-factor representation 7.56. This is in keeping with the interpretation that Λ-translation invariant operators are Λ-periodic, we refer to the unique sequence $\mathbf{k} = \{k(\lambda^\circ)\}_{\lambda^\circ \in \Lambda^\circ}$ in representation (7.56) the **operator Fourier-coefficients of** T.

Example 7.1. Λ-translation invariant operators observe a pleasing symbol calculus, namely if $\frac{1}{ab} = n$ and S, T are Λ-translation invariant operators where $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, then

$$\sigma(S) \cdot \sigma(T) = \sigma(ST). \tag{7.57}$$

A fact that we have avoided discussing until now, since our focus mostly lied on operator-quantization (i.e. correspondence between operators and distributions/functions on the time-frequency plane) is that QHA should necessarily have a version of Fourier transform for operators. Indeed, such a notion exists, it is called the **Fourier-Wigner transform** denoted by $\mathcal{F}_W$. It is defined on the trace-class operators $\mathcal{T}^1$ on $L^2(\mathbb{R}^d)$ so that for $S \in \mathcal{T}^1$, $\mathcal{F}_W(S)$ is a function in the time-frequency plane. Explicitly, if we denote by $\mathrm{tr} : \mathcal{T}^1 \to \mathbb{C}$ the trace on the trace-class operators:

$$\mathcal{F}_W(S)(\chi) = e^{\pi i \chi_1 \cdot \chi_2} \operatorname{tr}(\pi(-\chi)S), \qquad \chi \in \mathbb{R}^{2d}. \tag{7.58}$$

In QHA the trace-class operators $\mathcal{T}^1$ take on the role of $L^1(\mathbb{R}^d)$, and indeed one can define a suitable notion of operator convolution for them. For $S, T \in \mathcal{T}^1$, the operator convolution $S \star T$ is a function on the time-frequency plane $\mathbb{R}^d$ given by:

$$(S \star T)(\chi) = \operatorname{tr}(S \alpha_\chi \check{T}) \tag{7.59}$$

where $\check{T} = PTP$ is called the **operator-parity** with $P : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by the **parity operator** $(Pf)(t) = f(-t)$. Note that (7.58) and (7.59) makes sense because $\mathcal{T}^1$ is an ideal in $\mathcal{L}(L^2(\mathbb{R}^d))$ [Sim05] and $\mathcal{T}^1$ is invariant under time-frequency shifts and operator-parity [LS18].

We have the following results concerning QHA-centric concepts, and the interested reader is referred to [Skr21b, LS18, Skr20]. Note that $\mathcal{T}^p$ appearing below is the **Schatten-p class** of operators [Sim05, BLPY16] in $\mathcal{L}(L^2(\mathbb{R}^d))$ for $1 \leqslant p < \infty$. The Schatten-1 class is the trace-class operators, as the notation suggests.

Proposition 7.18. *For $S, T \in \mathcal{T}^1$, $S \star T \in L^1(\mathbb{R}^d)$. One can also extend operator-convolutions with a Young's inequality type-estimate. If $S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$, then $S \star T \in \mathcal{T}^r$ for $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \leqslant p, q, r \leqslant \infty$. We have the estimate $\|S \star T\|_{\mathcal{T}^r} \leqslant \|S\|_{\mathcal{T}^p} \|T\|_{\mathcal{T}^q}$. The Schatten-2 class is a Hilbert space, which is exactly the Hilbert-Schmidt operator space.*

Proposition 7.19. *The Fourier-Wigner Transform possesses the following properties*

1. *$\mathcal{F}_W(S) \in C_0(\mathbb{R}^d)$ if $S \in \mathcal{T}^1$;*

2. *We can extend the Fourier-Wigner transform (by duality) to any $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ so that $\mathcal{F}_W(S) \in \mathbf{S}_0'(\mathbb{R}^d)$.*

3. *$\mathcal{F}_s(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$ for $S, T \in \mathcal{T}^1$*

4. *The Fourier-Wigner transform and the Weyl-symbol are related via: $\mathcal{F}_W(S) = \mathcal{F}_s(\sigma(S))$ whenever S is well-defined in $\mathcal{F}_W$. As a corollary, the Fourier-Wigner transform is a Gelfand triple isomorphism:*

$$T \in (\mathcal{L}_{*-\|\|}, \mathcal{HS}, \mathcal{L})(\mathbb{R}^d, \mathbb{R}^d) \iff \mathcal{F}_W(S) \in (\mathbf{S}_0, L^2, \mathbf{S}_0')(\mathbb{R}^d).$$

It follows as a consequence of the properties of presented above, that the convolution of two operators, is the convolution of their Weyl-symbols.

We see that certain properties from classical harmonic analysis survive in the QHA set-up, for example, in Proposition 7.19: Item 1 is an analogoue of Riemman-Lebesgue Lemma, Item 3 can be construed as an operator-version of Fourier-convolution product (3.12). On the other hand, the Fourier-Wigner and Weyl-symbol relation will be relevant in our discussion of operator-modulation in Subsection 7.2.3, and indeed is another reason to work with the Weyl quantization scheme.

Before we proceed to the next subsection, let us first recall one of the main motivations for our line of inquiry regarding the Λ-translation invariant operators. The canonical example of Λ-translation invariant operators are the Gabor frame operators with respect to Λ. Recall from Equation that given $f, g \in L^2(\mathbb{R}^d)$: (4.50)

$$S_{g,h,\Lambda} = \sum_{\lambda \in \Lambda} ((\overline{g} \otimes h)^*) = P_\Lambda((\overline{g} \otimes h)^*). \tag{7.60}$$

From the Equation above, Gabor-frame operators can are exactly Λ operator-periodized rank-one operators. In the next subsection, we shall show that Gabor-frame operators and rank-operators occupy a fundamental spot in the space of Λ-translation invariant operators.

7.2.2 Λ-Translation Invariant Operators via Finite-Rank Operators Generated by the Heisenberg Module

In [Skr21b] a Janssen-type representation formula was found for general trace-class operators, from which one can then prove a correspondence between Λ-translation invariant operators (as they should be amenable to an operator-periodization representation) $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ and their operator Fourier coefficients $\mathbf{k}$ as in Theorem 7.17. We further investigate the theory of Λ-translation invariant operators through a combination of this Janssen-type correspondence and the characterisation of adjointable maps of the Heisenberg modules as Gabor frame operators. We shall make the case here that the Gabor frame operators do not only give us 'canonical examples' of Λ-translation invariant operators, but they can be used to topologically characterise all Λ-translation invariant operators not only in $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, but in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ as well. As a corollary of Equation (7.60), we must be able to characterise all Λ-translation invariant operators in $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ and $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ in terms of some topological limit of operator-periodization of finite-rank operators.

For the rest of this manuscript, we will refer to the Heisenberg module as $\mathcal{E}_\Lambda(\mathbb{R}^d)$ as originally notated in Theorem 5.5, with the difference that we work in particular with $G = \mathbb{R}^d$. We go back to this notation moving forward since the generating matrices $\mathrm{GL}_{2d}(\mathbb{R}^d)$ for the lattices do not play any significant role in in this section, unlike the case in Section 7.1.

We first start with making the connection between Λ-translation invariant operators and A_Λ-adjointable maps of the Heisenberg module $\mathcal{E}_\Lambda(\mathbb{R}^d)$ explicit. Let $T \in \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(\mathbb{R}^d))$, then it follows that for all $\mathbf{a} \in \ell^1(\Lambda)$ and $f \in \mathbf{S}_0(\mathbb{R}^d)$ that:

$$\begin{aligned} T(\mathbf{a}f) &= \mathbf{a}(Tf) \\ \iff T\left(\sum_{\lambda\in\Lambda} a(\lambda)\pi(\lambda)f\right) &= \sum_{\lambda\in\Lambda} a(\lambda)\pi(\lambda)(Tf) \end{aligned}$$

If we take $\mathbf{a} = \delta_\lambda$ in particular, then $T\pi(\lambda)f = \pi(\lambda)(Tf)$ for all $f \in \mathbf{S}_0(\mathbb{R}^d)$. If we combine this with the fact that $\mathcal{E}_\Lambda(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0'(\mathbb{R}^d)$, then we obtain that $T_{|\mathbf{S}_0(\mathbb{R}^d)} \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ and is a Λ-translation invariant operator. Subsequently $T_{|\mathbf{S}_0(\mathbb{R}^d)}$ is eligible for a representation as in (7.56). We may ask, where exactly do the operator Fourier-coefficients $\mathbf{k}$ of $T_{|M^1(\mathbb{R}^d)}$ lie within $\ell^\infty(\Lambda^\circ)$? To answer this, we have the following lemma regarding the A_Λ-adjointable maps of $\mathcal{E}_\Lambda(\mathbb{R}^d)$.

Lemma 7.13. *There exists an $n \in \mathbb{N}$ such that for any $T \in \mathcal{L}_A(\mathcal{E}_\Lambda(\mathbb{R}^d))$, we have $T = \sum_{i=1}^n (S_{g_i,h_i,\Lambda})_{|\mathcal{E}_\Lambda(\mathbb{R}^d)}$, where $g_1, ..., g_n, h_1, ..., h_n \in \mathcal{E}_\Lambda(\mathbb{R}^d)$.*

Proof. Since we are working with a lattice Λ, then as previously noted, the coefficient C^*-algebras A and B are unital. In particular, similar to the proof of Corollary 2.7, the unitality of B and fullness of $\mathcal{E}_\Lambda(\mathbb{R}^d)$ with respect to B implies that there exist an $n \in \mathbb{N}$ and $g_1, ..., g_n, \psi_1, ..., \psi_n \in \mathcal{E}_\Lambda(\mathbb{R}^d)$ such that $\sum_{i=1}^n \langle g_i, \psi_i \rangle_{\Lambda^\circ} = 1_B$ for the identity $1_B \in B$. It follows that if $T \in \mathcal{L}_{A_\Lambda}(\mathcal{E}_\Lambda(\mathbb{R}^d))$ and $f \in \mathcal{E}_\Lambda(\mathbb{R}^d)$, we have $Tf = T(f1_B) = T\left(\sum_{i=1}^n f \langle g_i, \psi_i \rangle_{\Lambda^\circ}\right) = T\left(\sum_{i=1}^n {}_\Lambda\langle f, g_i \rangle \psi_i\right) = \sum_{i=1}^n {}_\Lambda\langle f, g_i \rangle (T\psi_i) = \sum_{i=1}^n \theta_{g_i,h_i} f$ where we took $h_i = T\psi_i$. Finally, it follows from Lemma 5.3 that: $T = \sum_{i=1}^n (S_{g_i,h_i,\Lambda})_{|\mathcal{E}_\Lambda(\mathbb{R}^d)}$ as required. □

We need to introduce another 2-cocyle so that we can make sense of the connection between the adjointable maps and the canonical representation (7.56) of Λ-translation invariant operators. We define $c' : \mathbb{R}^{2d} \times \mathbb{T}^{2d} \to \mathbb{T}$ via:

$$c'((x_1, \omega_1), (x_2, \omega_2)) = e^{\pi i (x_2 \cdot \omega_1 - x_1 \cdot \omega_2)}.$$

We denote by B_Λ' the enveloping C^*-algebra obtained from the Banach $*$-algebra $\ell^1(\Lambda^\circ, c')$.

Lemma 7.14. *Let $\rho : \Lambda^\circ \to \mathbb{T}$ via $\rho(\lambda^\circ) = e^{\pi i \lambda_1^\circ \lambda_2^\circ}$. Then the map $\widetilde{\rho} : \ell^1(\Lambda^\circ) \to \ell^1(\Lambda^\circ)$*

$$\widetilde{\rho}(\mathbf{b})(\lambda^\circ) = b(-\lambda^\circ)\rho(\lambda) = b(-\lambda^\circ)e^{\pi i \lambda_1^\circ \cdot \lambda_2^\circ}, \qquad \forall \lambda^\circ \in \Lambda^\circ \tag{7.61}$$

extends to a C^*-isomorphism $\widetilde{\rho} : B_\Lambda' \to B_\Lambda$.

Proof. $\widetilde{\rho}:\ell^1(\Lambda^\circ)\to\ell^1(\Lambda^\circ)$ is obviously a Banach space automorphism with inverse

$$(\widetilde{\rho})^{-1}(\mathbf{b})(\lambda^\circ)=b(-\lambda^\circ)e^{-\pi i\lambda_1^\circ\cdot\lambda_2^\circ}.$$

It also follows from a straight-forward computation that: $\widetilde{\rho}(\mathbf{b}_1 *_{c'} \mathbf{b}_2)=\widetilde{\rho}(\mathbf{b}_1)*_{\overline{c}}\widetilde{\rho}(\mathbf{b}_2)$ and $\widetilde{\rho}(\mathbf{b}_1^{*_{c'}})=\widetilde{\rho}(\mathbf{b}_1)^{*_{\overline{c}}}$, for each $\mathbf{b}_1,\mathbf{b}_2\in\ell^1(\Lambda^\circ,c')$. Therefore on the common dense subspace $\ell^1(\Lambda^\circ)$, we have that $\widetilde{\rho}:\ell^1(\Lambda,c')\to\ell^1(\Lambda,\overline{c})$ is a Banach $*$-algebra isomorphism, hence $\widetilde{\rho}$ must extend to a C^*-isomorphism. □

Remark 7.14. The isomorphism $B'_\Lambda\cong B_\Lambda$ can be explained by the fact that the 2-cocycles $\overline{c}$ and c' are *cohomologous* as we have discussed in Subsection 7.1.2.

We now reiterate that from the localization result Proposition 5.13, we have the following continuous norm-dense embeddings:

$$\ell^1(\Lambda^\circ)\hookrightarrow B_\Lambda\cong B'_\Lambda\hookrightarrow\ell^2(\Lambda^\circ)\hookrightarrow\ell^\infty(\Lambda^\circ). \tag{7.62}$$

From this observation we obtain the following result.

Theorem 7.7. *$T\in\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))$ is a Λ-translation invariant operator whose Fourier coeffients $\mathbf{k}$ lie in B'_Λ, equivalently $\widetilde{\rho}(\mathbf{k})\in B_\Lambda$, if and only if T extends to an A_Λ-adjointable map. In this case, there exists an $n\in\mathbb{N}$ such that T is a finite sum of n Gabor frame operators, and further extends to a map in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ with $\|T\|_{\mathcal{L}(L^2)}=\|\mathbf{k}\|_{B'_\Lambda}=\|\widetilde{\rho}(\mathbf{k})\|_{B_\Lambda}$.*

Proof. Suppose that $T\in\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))$ is a Λ-translation invariant operator with Fourier coefficients $\mathbf{k}=\{k(\lambda^\circ)\}_{\lambda^\circ\in\Lambda^\circ}\in B'$ so that

$$T=\frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k(\lambda^\circ)e^{-\pi i\lambda_1^\circ\cdot\lambda_2^\circ}\pi(\lambda^\circ)=\frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}k_{-\lambda^\circ}e^{\pi i\lambda_1^\circ\cdot\lambda_2^\circ}\pi^*(\lambda^\circ)=\frac{1}{d(\Lambda)}\sum_{\lambda^\circ\in\Lambda}\widetilde{\rho}(\mathbf{k})(\lambda^\circ)\pi^*(\lambda^\circ).$$

We then obtain that $T=\overline{\pi}^*_{\Lambda^\circ}(\widetilde{\rho}(\mathbf{k}))_{|\mathbf{S}_0(\mathbb{R}^d)}$. Now it follows from the fact that B is unital and $\mathcal{E}_\Lambda(\mathbb{R}^d)$ is full with respect to B, that there exists $g_1,\dots,g_n,h_1,\dots,h_n\in\mathcal{E}_\Lambda(\mathbb{R}^d)$ such that $\widetilde{\rho}(\mathbf{k})=\sum_{i=1}^n\langle g_i,h_i\rangle_{\Lambda^\circ}$. We obtain $T=\sum_{i=1}^n\overline{\pi}^*_{\Lambda^\circ}\left(\langle g_i,h_i\rangle_{\Lambda^\circ}\right)_{|\mathbf{S}_0(\mathbb{R}^d)}$. It now follows from the Janssen's representation for the Heisenberg modules (5.8) that $T=\sum_{i=1}^n(S_{g_i,h_i,\Lambda})_{|\mathbf{S}_0(\mathbb{R}^d)}$, from which we can conclude that T extends to an A_Λ-adjointable map via Lemma 5.3.

The converse easily follow from the uniqueness of the operator Fourier coefficients, Equation (7.56), Lemma 7.13, and the Janssen's representation for the Heisenberg modules (5.8). In particular, $T = \sum_{i=1}^n (S_{g_i,h_i,\Lambda})_{|\mathcal{E}_\Lambda(\mathbb{R}^d)} = \overline{\pi}^*_{\Lambda^\circ}\left(\sum_{i=1}^n \langle g_i, h_i\rangle_{\Lambda^\circ}\right)$ for some $g_1, ..., g_n, h_1, ..., h_n \in \mathcal{E}_\Lambda(\mathbb{R}^d)$ and $\widetilde{\rho}(\mathbf{k}) = \sum_{i=1}^n \langle g_i, h_i\rangle_{\Lambda^\circ}$ for some $\mathbf{k} \in B'_\Lambda$. It follows from Proposition 5.3 that T extends to $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, and that $\|T\|_{\mathcal{L}(L^2)} = \|\overline{\pi}^*_{\Lambda^\circ}(\sum_{i=1}^n \langle g_i, h_i\rangle_{\Lambda^\circ})\|_{\mathcal{L}(L^2)} = \|\widetilde{\rho}(\mathbf{k})\|_{B_\Lambda} = \|\mathbf{k}\|_{B'_\Lambda}$. □

Lemma 7.15. *For each Λ-translation invariant operator $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))$ with Fourier coefficients $\mathbf{k} \in \ell^\infty(\Lambda^\circ)$, we have the following estimate:*

$$\|T\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))} \leqslant \frac{\|\mathbf{k}\|_\infty}{d(\Lambda)}.$$

Proof. We compute

$$\begin{aligned}\|T\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))} &= \sup\left\{\left|(g, Tf)_{\mathbf{S}_0,\mathbf{S}'_0}\right| : \|f\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\}\\ &= \sup\left\{\left|\overline{(\overline{g}, Tf)}_{\mathbf{S}_0,\mathbf{S}'_0}\right| : \|f\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\}\end{aligned}$$

Using Equation (7.56) we obtain:

$$\begin{aligned}\|T\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))} &\leqslant \frac{1}{d(\Lambda)}\|\mathbf{k}\|_\infty \sup\left\{\sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{V}_f g(\lambda^\circ)| : \|f\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\}\\ &\leqslant \frac{1}{d(\Lambda)}\|\mathbf{k}\|_\infty \sup\left\{\int_{\mathbb{R}^{2d}} |\mathcal{V}_f g(x,\omega)|\, \mathrm{d}x\, \mathrm{d}\omega : \|f\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\}\\ &\leqslant \frac{1}{d(\Lambda)}\|\mathbf{k}\|_\infty,\end{aligned}$$

where the last inequality follow from [Jak18, Proposition 4.10]. □

Theorem 7.8. *There exists an $n \in \mathbb{N}$ such that every Λ-translation invariant operator in $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))$ is the norm-limit of operator-periodizations of rank-n operators generated by functions coming from $\mathcal{E}_\Lambda(\mathbb{R}^d)$.*

Proof. Suppose T is a Λ-translation invariant operator in $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))$ with Fourier coefficients $\mathbf{k} \in \ell^\infty(\Lambda^\circ)$. We know from the continuous norm-dense embeddings (7.62) that there exists a sequence $\{\mathbf{k}_m\}_{m\in\mathbb{N}} \subseteq B'_\Lambda$ such that $\|\mathbf{k} - \mathbf{k}_m\|_\infty \to 0$. For each $m \in \mathbb{N}$, define the Λ-translation invariant operator $T_m \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))$ via $T_m := \frac{1}{d(\Lambda)} \sum_{\lambda\in\Lambda^\circ} k_m(\lambda^\circ) e^{-\pi i \lambda_1^\circ \cdot \lambda_2^\circ} \pi(\lambda^\circ)$. It then follows from Lemma 7.15 that $\|T - T_m\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}'_0(\mathbb{R}^d))} \leqslant \frac{\|\mathbf{k}-\mathbf{k}_m\|_\infty}{d(\Lambda)} \to 0$. But then we know from Theorem 7.7 that there exists a fixed $n \in \mathbb{N}$ such that each T_m is in fact a finite sum of Gabor frame operator $T_m = \sum_{i=1}^n (S_{g_i,h_i,\Lambda})_{|\mathbf{S}_0(\mathbb{R}^d)}$ for $g_1, ..., g_n, h_1, ..., h_n \in \mathcal{E}_\Lambda(\mathbb{R}^d)$. By Equation (7.60), $T_m = \sum_{\lambda\in\Lambda} \alpha_\lambda\left(\sum_{i=1}^n (h \otimes \overline{g})^*)\right) = \sum_{\lambda\in\Lambda} \alpha_\lambda\left(\sum_{i=1}^n (\overline{g} \otimes h)\right)$, which is what we wanted to prove. □

We have seen that the C^*-algebraic aspects of the theory coming from the Heisenberg modules has allowed us to fully characterise Λ-translation invariant operators in $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ where we have gone outside the usual confines of the Banach Gelfand triple setting $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$ by considering the intermediate sequence spaces B_Λ and B_Λ'. We now consider the von Neumann algebraic aspect of the theory by going out of the the $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ setting. We denote the space of Λ-invariant operators in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ via:

$$\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) := \left\{T \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right) : T = \alpha_\lambda(T), \forall \lambda \in \Lambda\right\}.$$

The following lemma is immediate.

Lemma 7.16. *$T \in \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ if and only if it is an extension of a Λ-translation invariant operator in $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ such that $\mathrm{Im}(S) \subseteq L^2(\mathbb{R}^d)$. In this case, T can be represented as in (7.56), with operator Fourier-coefficients $\mathbf{k} \in \ell^2(\Lambda^\circ)$.*

Proof. The characterisation for $\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ as stated above is immediate. We only show that $T \in \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ will necessarily have operator Fourier-coefficients satisfying $\mathbf{k} \in \ell^2(\Lambda^\circ)$. We have from (7.56) that for all $f, g \in L^2(\mathbb{R}^d)$

$$\langle Tf, Tg\rangle_2 = \frac{1}{d(\Lambda)^2} \sum_{\lambda^\circ, \nu^\circ \in \Lambda^\circ} k(\lambda^\circ)\overline{k(\nu^\circ)} e^{-\pi i(\lambda_1^\circ \cdot \lambda_2^\circ - \nu_1^\circ \cdot \nu_2^\circ)} \langle \pi(\lambda^\circ) f, \pi(\lambda^\circ) g\rangle_2 .$$

By letting $f = g$, and taking the supremum over all $\|f\|_2 = 1$ in the equation above, we obtain $\infty > \|T\|^2_{\mathcal{L}(L^2)} \geqslant \frac{1}{d(\Lambda)^2} \sum_{\lambda^\circ \in \Lambda^\circ} |k(\lambda^\circ)|^2$, which shows $\mathbf{k} \in \ell^2(\Lambda^\circ)$. $\square$

Remark 7.15. Note that all of our results on Λ-translation invariant operators in this section have an analogue for when we replace Λ with its adjoint Λ°. Relevant to this is the fact that we can construct the Heisenberg module again by switching the roles of the lattice Λ and its lattice Λ°, giving us $\mathcal{E}_{\Lambda^\circ}(\mathbb{R}^d)$. It is in fact also true that $\mathcal{E}_{\Lambda^\circ}(\mathbb{R}^d) = \mathcal{E}_\Lambda(\mathbb{R}^d)$ [AE20, Proposition 3.17].

Next, if $S \subseteq \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, we denote the *commutant* of S in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ via $S' := \{T \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right) : Ts = sT, \ \forall s \in S\}$. The *double commutant* of S is $S'' = (S')'$. We use $\overline{S}, \overline{S}^{\mathrm{SOT}}, \overline{S}^{\mathrm{WOT}}$ to denote the closure of S with respect to the norm, strong-operator, and weak-operator topologies respectively. Finally, von Neumann's double commutant theorem [Dav96, Theorem I.7.1] will feature prominently in the subsequent proofs.

Lemma 7.17. *We have:*

$$\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'' = \pi^*_{\Lambda^\circ}(B_\Lambda)''. \tag{7.63}$$

Proof. Note that $\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))$ is a unital $*$-subalgebra of $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, therefore it follows from the double commutant theorem and the fact that the norm-topology contains the strong operator topology, which further contains the weak operator topology, that:

$$\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ)) \subseteq \overline{\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))} \subseteq \overline{\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))}^{\mathrm{WOT}} = \overline{\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))}^{\mathrm{SOT}} = \pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''. \tag{7.64}$$

However, $\pi^*_{\Lambda^\circ}$ is a $*$-homomorphism, and thus has a closed range, from which it follows that

$$\overline{\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))} = \pi^*_{\Lambda^\circ}(\overline{\ell^1(\Lambda^\circ)}) = \pi^*_{\Lambda^\circ}(B_\Lambda). \tag{7.65}$$

Therefore Equations (7.64) and (7.65) imply that $\pi^*_{\Lambda^\circ}(B_\Lambda) \subseteq \pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''$, from which it follows that $\pi^*_{\Lambda^\circ}(B_\Lambda)'' \subseteq \pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''$. The reverse inclusion follows easily from the fact that $\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ)) \subseteq \pi^*_{\Lambda^\circ}(B_\Lambda)$, whence $\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'' \subseteq \pi^*_{\Lambda^\circ}(B_\Lambda)''$. We have proved $\pi^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'' = \pi^*_{\Lambda^\circ}(B_\Lambda)''$. □

Lemma 7.18. *We have:*

$$\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) = \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)' \tag{7.66}$$

Proof. Suppose $T \in \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)'$, because $\pi(\lambda) \in \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)$ for all $\lambda \in \Lambda$, then T commutes with all $\pi(\lambda) \in \Lambda$, so $T \in \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$. We obtain $\mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)' \subseteq \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$. For the reverse inclusion, we use 7.16 adapted for Λ°-translation invariant operators, so that $T^\circ \in \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)$ can always be written

$$T^\circ = \frac{1}{d(\Lambda^\circ)} \sum_{\lambda \in \Lambda} k_\lambda e^{-\pi i \lambda_1 \cdot \lambda_2} \pi(\lambda) \tag{7.67}$$

for some $\{k_\lambda\}_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$. On the other hand, if $T \in \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$, we find due to (7.67) that $TT^\circ = T^\circ T$, whence $\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) \subseteq \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)'$. We now have $\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) = \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)'$ □

We now obtain a von Neumann algebraic version of the Theorem 7.8.

Theorem 7.9. *We have:*

$$\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) = \overline{\pi}^*_{\Lambda^\circ}(B)''. \tag{7.68}$$

As a corollary, there exists an $n \in \mathbb{N}$ such that every operator in $\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ is a weak-operator, or strong-operator limit of periodizations of rank-n operators generated by functions coming from $\mathcal{E}_\Lambda(\mathbb{R}^d)$.

Proof. Suppose $T \in \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'$, because $\pi(\lambda^\circ) \in \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))$ for all $\lambda^\circ \in \Lambda^\circ$, then T commutes $\pi(\lambda^\circ)$ for all $\lambda \in \Lambda^\circ$. Therefore

$$\overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))' \subseteq \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right).$$

Taking the commutant of the inclusion gives us

$$\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) = \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)' \subseteq \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''.$$

On the other hand, we note that $\pi(\lambda) \in \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'$ for all $\lambda \in \Lambda$, hence if $T \in \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''$, then $T\pi(\lambda) = \pi(\lambda)T$ for all $\lambda \in \Lambda$. We have obtained

$$\overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))'' \subseteq \mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right).$$

All-in-all, we have shown that

$$\mathcal{L}_\Lambda\left(L^2(\mathbb{R}^d)\right) = \mathcal{L}_{\Lambda^\circ}\left(L^2(\mathbb{R}^d)\right)' = \overline{\pi}^*_{\Lambda^\circ}(\ell^1(\Lambda^\circ))''.$$

Equation (7.68) now follows from Equations (7.63) and (7.66).

As a corollary of Equation (7.68) and the double commutant theorem:

$$\mathcal{L}_\Lambda(L^2(G)) = \overline{\pi}^*_{\Lambda^\circ}(B)'' = \overline{\overline{\pi}^*_{\Lambda^\circ}(B)}^{\mathrm{SOT}} = \overline{\overline{\pi}^*_{\Lambda^\circ}(B)}^{\mathrm{WOT}}.$$

As a consequence of unitality of B and fullness of $\mathcal{E}_\Lambda(\mathbb{R}^d)$, there exists a fixed $n \in \mathbb{N}$ such that the operators in $\overline{\pi}^*_{\Lambda^\circ}(B)$ (similar to the proof in Lemma 7.13) are exactly of the form $\sum_{i=1}^n S_{g_i,h_i,\Lambda}$ where $g_1, ..., g_n, h_1, ..., h_n \in \mathcal{E}_\Lambda(\mathbb{R}^d)$. The result now follows from Equation (7.60). $\square$

7.2.3 Operator-Modulation

Let us again look at the shift-covariant formula for the Weyl-symbol:

$$T_\chi \sigma(T) = \sigma(\alpha_\chi T), \qquad \forall \chi \in \mathbb{R}^{2d}.$$

Motivated by shift-covariance, we expect that the correct notion of operator-modulation map β_χ must satisfy:

$$M_\chi^s \sigma(T) = \sigma(\beta_\chi T), \qquad \forall \chi \in \mathbb{R}^{2d} \tag{7.69}$$

where M_χ^s is the symplectic modulation operator that appears in Equations (4.16). Explicitly, we have for $\chi = (x, \omega) \in \mathbb{R}^{2d}$ and $(t, \nu) \in \mathbb{R}^{2d}$:

$$\begin{aligned}(M_\chi^s F)(t,\nu) &= F(t,\nu) c_s((x,\omega),(t,\nu)) \\ &= F(t,\nu) e^{2\pi i (t\cdot\omega - x\cdot\nu)}.\end{aligned}$$

Below, we give a definition of operator-modulation formally defined in [LM24] that would give us the required relation from (7.69).

Definition 7.4. Let $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, we define the **operator-modulation** of T at $\chi \in \mathbb{R}^{2d}$ to be:

$$\begin{aligned}\beta_\chi(T) &:= e^{-\pi i (\chi_1 \cdot \chi_2)/2} \pi\left(\frac{\chi}{2}\right) T \pi\left(\frac{\chi}{2}\right) \\ &= \pi\left(\frac{\chi}{2}\right) T \pi^*\left(-\frac{\chi}{2}\right)\end{aligned} \tag{7.70}$$

We see that the equivalence of the two equations in (7.70) follow quite easily from the fact that $\pi^*\left(-\frac{\chi}{2}\right) = c\left(-\frac{\chi}{2}, -\frac{\chi}{2}\right)\pi\left(\frac{\chi}{2}\right)$ and writing explicitly the value of the appearing Heisenberg 2-cocycle in the terms of the Euclidean coordinates $\chi = (\chi_1, \chi_2)$. A proof that β_χ satisfies (7.69) can be found in [LM24, Proposition 3.11] where they used the Fourier-Wigner transform and its relation to the Weyl-symbol (c.f. Item 4 Proposition 7.19). This essentially boils down to showing that

$$\mathcal{F}_W(\beta_\chi T) = T_\chi(\mathcal{F}_W(T)) \tag{7.71}$$

which is already something that one should expect given that the operator Fourier-transform acting on a modulated operator should give us a translated Fourier-transform of the said operator. In any case, an application of the symplectic Fourier-transform yields:

$$\begin{aligned}
\mathcal{F}_s(\mathcal{F}_W(\beta_\chi T)) &= \mathcal{F}_s(T_\chi \mathcal{F}_W(T)) \\
\implies \sigma(\beta_\chi T) &= M_\chi^s \mathcal{F}_s(\mathcal{F}_W(T)) \\
\implies \sigma(\beta_\chi T) &= M_\chi^s \mathcal{F}_s(\mathcal{F}_s(\sigma(T))) = M_\chi^s \sigma(T)
\end{aligned}$$

as required.

We treat the Fourier-Wigner transform as an ambient tool with which we can re-contextualize our discussion on operator-modulation within the broader framework QHA theory. Note that there should be an alternate proof of Equation (7.69) that does not use the Fourier-Wigner transform, whose idea is similar in spirit to the original shift-covariance formula from Theorem 4.6.

We now introduce the concept of Λ-modulation invariant operators using operator-modulation, analogous to Λ-translation invariant operators.

Definition 7.5. An operator $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ is called Λ-modulation-invariant if

$$T = \beta_\lambda(T) \tag{7.72}$$

for every $\lambda \in \Lambda$.

Remark 7.16. Equivalently, it follows from (7.70) that T is Λ-modulation invariant if and only if $T\pi\left(-\frac{\lambda}{2}\right) = \pi\left(\frac{\lambda}{2}\right)T$.

There are parallels between operator-translation and operator-modulation results that we can, again, re-contextualize as a greater part of QHA theory. In particular, let us revisit the result that says: the operator-periodization with respect to Λ of any operator is a Λ-translation invariant operator (c.f. Computation (4.47)). The following is an analogue involving operator-modulation.

Proposition 7.20. *Let $T \in \mathcal{L}(\mathcal{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, then*

$$\sum_{\lambda \in \Lambda} \beta_\lambda(T) \tag{7.73}$$

is a Λ*-modulation invariant operator.*

Proof. Let $\mu \in \Lambda$. We compute:

$$\begin{aligned}
\beta_\mu(T) &= \sum_{\lambda\in\Lambda} e^{-\pi i(\mu_1\mu_2+\lambda_1\lambda_2)/2}\pi\left(\frac{\mu}{2}\right)\pi\left(\frac{\lambda}{2}\right)S\pi\left(\frac{\lambda}{2}\right)\pi\left(\frac{\mu}{2}\right) \\
&= \sum_{\lambda\in\Lambda} e^{-\pi i(\mu_1\mu_2+\lambda_1\lambda_2+\mu_1\lambda_2+\lambda_1\mu_2)/2}\pi\left(\frac{\mu+\lambda}{2}\right)S\pi\left(\frac{\lambda+\mu}{2}\right) \\
&= \sum_{\lambda\in\Lambda} e^{-\pi i(\mu_1+\lambda_1)(\mu_2+\lambda_2)/2}\pi\left(\frac{\mu+\lambda}{2}\right)S\pi\left(\frac{\lambda+\mu}{2}\right) \\
&= T.
\end{aligned}$$

□

Given T, we call the $\sum_{\lambda\in\Lambda}\alpha_\lambda(T)$ the operator-periodization of T since α_λ is interpreted as the operator-translation. In a similar way, we have the following peridization-type concept for operator-modulation.

Definition 7.6. For $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, the **Fourier-Wigner periodization** of T with respect to Λ is given by $\sum_{\lambda\in\Lambda}\beta_\lambda(T)$.

The definition above is motivated by the fact that the Weyl quantization scheme sends $\sum_{\lambda\in\Lambda}\beta_\lambda(T)$ to $\sum_{\lambda\in\Lambda} M_\lambda^s\sigma(T)$ via (7.69), however Equation (7.71) shows that

$$\mathcal{F}_s\left(\sum_{\lambda\in\Lambda} M_\lambda^s\sigma(T)\right) = \sum_{\lambda\in\Lambda} T_\lambda\mathcal{F}_s\sigma(T) = \sum_{\lambda\in\Lambda} T_\lambda\mathcal{F}_W(T), \qquad \forall z \in \mathbb{R}^{2d}.$$

Therefore, $\sum_{\lambda\in\Lambda} M_\lambda^s\sigma(T)$ is exactly the symplectic Fourier transform[6] of the operator-periodization of the Fourier-Wigner transform of T, which corresponds to the operator $\sum_{\lambda\in\Lambda}\beta_\lambda(T)$.

There is an important difference between Λ-translation invariant operators and Λ-modulation invariant operators. It is not hard to see that if S, T are both Λ-translation invariant, that their composition ST is also Λ-translation invariant:

$$\pi(\lambda)ST = S\pi(\lambda)T = ST\pi(\lambda), \qquad \forall\lambda \in \Lambda.$$

This, however, breaks for Λ-modulation invariant operators.

[6]...which is a self-inverse map, so we can consider it as another "correspondence".

Theorem 7.10. *Let S, T be two Λ-modulation invariant operators such that ST is well-defined (e.g.* $\mathrm{Im}(T) \subseteq \mathbf{S}_0(\mathbb{R}^d)$*). Then the composition ST is a $\Lambda/2$-translation invariant operator.*

Proof. Let $S, T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ be Λ-modulation invariant operators such that ST is well-defined. Then for $\lambda \in \Lambda$;

$$\begin{aligned}
\pi\left(\frac{\lambda}{2}\right) ST\pi\left(\frac{\lambda}{2}\right)^* &= \pi\left(\frac{\lambda}{2}\right) S\pi\left(\frac{\lambda}{2}\right)\pi^*\left(\frac{\lambda}{2}\right) T\pi^*\left(\frac{\lambda}{2}\right) \\
&= e^{-\pi i(\lambda_1\cdot\lambda_2)/2}\pi\left(\frac{\lambda}{2}\right) S\pi\left(\frac{\lambda}{2}\right)\pi\left(-\frac{\lambda}{2}\right) T\pi^*\left(\frac{\lambda}{2}\right) \\
&= \left(e^{-\pi i(\lambda_1\cdot\lambda_2)/2}\pi\left(\frac{\lambda}{2}\right) S\pi\left(\frac{\lambda}{2}\right)\right)\left(\pi\left(-\frac{\lambda}{2}\right) T\pi^*\left(\frac{\lambda}{2}\right)\right) \\
&= ST
\end{aligned}$$

where we simply used the unitarity of $\pi(\chi)$ and the definition of Λ-modulation invariant operators. Hence ST is $\Lambda/2$-translation invariant. □

We now turn our attention to a particular operator, the parity operator P which we introduced in the previous subsection. Again, it is defined as the map that sends $f : \mathbb{R}^d \to \mathbb{C}$ to

$$(Pf)(t) = f(-t)$$

The parity operator has the property

$$P\pi(\chi) = \pi(-\chi)P, \tag{7.74}$$

since for $\chi = (x, \omega) \in \mathbb{R}^{2d}$:

$$\begin{aligned}
P\pi(\chi)f(t) &= Pe^{2\pi i\omega\cdot t}f(t-x) \\
&= e^{-2\pi i\omega\cdot t}f(-t-x) \\
&= \pi(-\chi)Pf(t).
\end{aligned}$$

We then find that

$$\alpha_\chi(P) = \pi(\chi)P\pi(\chi)^*$$

$$
\begin{aligned}
&= e^{-2\pi i x\cdot\omega}\pi(\chi)P\pi(-\chi) \\
&= e^{-2\pi i x\cdot\omega}\pi(\chi)\pi(\chi)P \\
&= e^{-4\pi i x\cdot\omega}\pi(2\chi)P. \qquad (7.75)
\end{aligned}
$$

The following Lemma will also help with some of our characterisation results in the sequel:

Lemma 7.19. *The parity operator is a self-inverse unitary map in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, and restricts to an isometric isomorphism $P_{|\mathbf{S}_0(\mathbb{R}^d)} : \mathbf{S}_0(\mathbb{R}^d) \to \mathbf{S}_0(\mathbb{R}^d)$.*

Proof. That $P \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ is self-inverse is obvious, while unitarity follows from:

$$
\int_{\mathbb{R}^{2d}} f(\chi)\,\mathrm{d}\chi = \int_{\mathbb{R}^{2d}} f(-\chi)\,\mathrm{d}\chi, \qquad \forall f \in L^1(\mathbb{R}^d).
$$

That P restricts to an actual isometric isomorphism on $\mathbf{S}_0(\mathbb{R}^d)$ can be found in [Jak18, Equation (4.11)]. ☐

Before we proceed, we note that the symplectic Fourier transform on $\mathbb{R}^{2d}$ can be written in terms of the **symplectic form** $\Omega(\chi,\chi'); = \chi_1'\cdot\chi_2 - \chi_1\cdot\chi_2'$, so that

$$
\mathcal{F}_s(f) = \int_{\mathbb{R}^{2d}} f(\chi')e^{-2\pi i\Omega(\chi,\chi')}\,\mathrm{d}\chi'. \qquad (7.76)
$$

Now we claim the following.

Proposition 7.21. *We have the following equation:*

$$
\int_{\mathbb{R}^{2d}} e^{2\pi i\Omega(\chi,\chi')-i\pi\chi_1'\chi_2'}\pi(\chi')\;\mathrm{d}\chi' = 2^d\alpha_\chi(P).
$$

Proof. To show this we consider the integral weakly:

$$
\begin{aligned}
\left\langle f, \int_{\mathbb{R}^{2d}} e^{2\pi i\Omega(\chi,\chi')-i\pi\chi_1'\chi_2'}\pi(\chi')\;\mathrm{d}\chi'\, g\right\rangle_2 &= \int_{\mathbb{R}^{2d}} e^{-2\pi i\Omega(\chi,\chi')+i\pi\chi_1'\chi_2'}\langle f,\pi(\chi')g\rangle_2\;\mathrm{d}\chi' \\
&= \mathcal{F}_s(A(f,g))(\chi) \\
&= W(f,g)(\chi).
\end{aligned}
$$

Substituting the identity (c.f. [Grö01, Lemma 4.3.1])

$$
W(f,g)(\chi) = 2^d e^{4\pi i x\omega} V_{Pg} f(2\chi),
$$

into the original equation then gives

$$\left\langle f, \int_{\mathbb{R}^{2d}} e^{2\pi i\Omega(\chi,\chi')-i\pi\chi_1'\cdot\chi_2'}\pi(\chi')\,\mathrm{d}\chi', g\right\rangle_2 = 2^d e^{4\pi i x\omega}\langle f, \pi(2z)Pg\rangle_2$$

and the result follows from (7.75). □

From here we find a spreading-type quantization of a Weyl symbol directly, without needing to take the symplectic Fourier transform:

Proposition 7.22. *Given $\sigma \in \mathbf{S}_0'(\mathbb{R}^{2d})$, the Weyl quantization L_σ can be expressed as*

$$L_\sigma = 2^d \int_{\mathbb{R}^{2d}} \sigma(\chi)\alpha_\chi(P)\,\mathrm{d}\chi,$$

or equivalently

$$L_\sigma = \int_{\mathbb{R}^{2d}} \sigma(\tfrac{\chi}{2})e^{-\pi i\chi_1\cdot\chi_2}\pi(\chi)P\,\mathrm{d}\chi.$$

Proof. We know from Equation (7.53) that

$$\begin{aligned}
L_\sigma &= \int_{\mathbb{R}^{2d}} e^{-i\pi\chi_1\cdot\chi_2}\mathcal{F}_s(\sigma)(\chi)\pi(\chi)\,\mathrm{d}\chi \\
&= \int_{\mathbb{R}^{2d}} e^{-i\pi\chi_1\cdot\chi_2}\int_{\mathbb{R}^{2d}} e^{-2\pi i\Omega(\chi,\chi')}\sigma(\chi')\,\mathrm{d}\chi'\,\pi(\chi)\,\mathrm{d}\chi \\
&= \int_{\mathbb{R}^{2d}} \sigma(\chi')\int_{\mathbb{R}^{2d}} e^{2\pi i\Omega(\chi',\chi)-i\pi\chi_1\cdot\chi_2}\pi(\chi)\,\mathrm{d}\chi\,\mathrm{d}\chi'.
\end{aligned}$$

Inserting the result of Proposition 7.21 then gives

$$L_\sigma = 2^d\int_{\mathbb{R}^{2d}} \sigma(\chi')\alpha_{\chi'}(P)d\chi'.$$

The second form follows from the identity (7.75). □

The above representation of an operator is also discussed in [Teo18]. In this manuscript we consider operator-modulations, and Λ-modulation invariant operators. The above quantization intuition serves as a motivation of this approach; while in the case of Λ-translation invariant operators we have spreading quantizations in the form (c.f. [FK98]) of $T = \int_{\mathbb{R}^{2d}} \eta(T)(\chi)\pi(\chi)\,\mathrm{d}\chi$ of functions supported on the lattice Λ, in this work we find that Λ-modulation invariant operators are precisely those operators which are quantizations of the form (7.22) of functions supported on the lattice Λ.

Proposition 7.23. *The parity operator P is a $\mathbb{R}^{2d}$-modulation invariant operator.*

Proof. It follows from Equation 7.74 that for all $\chi \in \mathbb{R}^{2d}$:

$$\begin{aligned} P\pi\left(-\frac{\chi}{2}\right) &= \pi\left(\frac{\chi}{2}\right)P \\ \implies P &= \pi\left(\frac{\chi}{2}\right)P\pi^*\left(-\frac{\chi}{2}\right) = \beta_\chi(P). \end{aligned}$$

□

Since $\Lambda \subseteq \mathbb{R}^{2d}$, it follows that P is also Λ-modulation invariant for any lattice Λ. It follows then that the parity operator is in fact the only $\mathbb{R}^{2d}$-modulation invariant operator (up to a constant):

Proposition 7.24. *If $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ is a $\mathbb{R}^{2d}$-modulation invariant operator, then $S = c \cdot P$ for some constant $c \in \mathbb{C}$.*

Proof. Let $S \in S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ be a $\mathbb{R}^{2d}$-modulation invariant operator. Then (7.71) implies that $\mathcal{F}_W(S)(z) = \mathcal{F}_W(S)(0)$, and so $\mathcal{F}_W(S)(z) = c$ for some $c \in \mathbb{C}$. By the uniqueness of Fourier-Wigner transform, this implies $S = c \cdot P$, since $\mathcal{F}_W(P) = 1$. □

Since $\Lambda \subseteq \mathbb{R}^{2d}$, it follows that P is also Λ-modulation invariant for any lattice Λ. In turn, we have that for any Λ-modulation invariant operator, SP is a $\Lambda/2$-translation invariant operator by Theorem 7.10. This gives us a way to identify each Λ-modulation invariant operator with a $\Lambda/2$-translation invariant operator by composing with the parity operator, and vice versa:

7.2.4 Decompositions of Λ-Modulation Invariant Operators

The periodicity of the Fourier-Wigner transform of Λ-modulation invariant operators means they can be described in terms of a trigonometric decomposition.

Proposition 7.25. *Given an operator $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ which is Λ-modulation invariant, the Fourier-Wigner transform $\mathcal{F}_W(T)$ can be decomposed as*

$$\mathcal{F}_W(T)(\chi) = \sum_{\lambda^\circ \in \Lambda^\circ} \sigma(T)(\lambda^\circ) e^{2\pi i \Omega(\lambda^\circ, \chi)}.$$

Furthermore, the mapping $T \mapsto \{\sigma(T)(\lambda^\circ)\}_{\Lambda^\circ}$ *is a Gelfand triple isomorphism;*

$$\mathcal{F}_W(T) \in \big(\mathbf{S}_0, L^2, \mathbf{S}_0'\big)(\mathbb{R}^{2d}/\Lambda) \iff \{\sigma(T)(\lambda^\circ)\}_{\Lambda^\circ} \in \big(\ell^1, \ell^2, \ell^\infty\big)(\Lambda^\circ).$$

Proof. By the definition of Λ-modulation-invariant along with (7.71), the Fourier-Wigner transform of some Λ-modulation invariant $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ is a Λ-periodic function. By the definition of the Fourier-algebra $\mathbf{A}(\mathbb{R}^{2d}/\Lambda)$ (c.f. (4.29)) and Proposition 4.9 Item 5, the trigonometric decomposition follows. Again, we have: $\mathbf{S}_0(\mathbb{R}^{2d}/\Lambda) = \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$, since $\mathbb{R}^{2d}/\Lambda$ is compact and hence the Fourier series is absolutely convergent, and conversely an absolutely convergent Fourier series implies $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$. The $\mathbf{S}_0(\mathbb{R}^d/\Lambda)$ and $\mathbf{S}_0'(\mathbb{R}^d/\Lambda)$ results then follow. The $L^2(\mathbb{R}^{2d}/\Lambda)$ case is Parseval's identity. □

Using Proposition 7.19 Item 5, we can formulate (7.25) as a statement on the support of the Weyl symbol.

Corollary 7.9. *Given an operator* $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ *which is* Λ*-modulation-invariant, the Weyl symbol* $\sigma(T)$ *is given by*

$$\sigma(T)(\chi) = \sum_{\lambda^\circ \in \Lambda^\circ} \sigma(T)(\lambda^\circ)\delta_{\lambda^\circ}(\chi),$$

where the sum converges in the weak-∗ topology of $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$*. That is, the Weyl symbol is supported on the adjoint lattice.*

Proposition 7.26. *Given an operator* $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ *which is* Λ*-modulation invariant, such that* $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$*, then* T *can be expressed as*

$$T = \sum_{\lambda \in \Lambda} e^{-\pi i \lambda_1 \lambda_2/2} \pi\Big(\frac{\lambda}{2}\Big) S \pi\Big(\frac{\lambda}{2}\Big) = \sum_{\lambda \in \Lambda} \beta_\lambda(S) \tag{7.77}$$

for some $S \in \mathcal{L}_{*-\|\|}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$*.*

Proof. Since T is Λ-modulation-invariant and $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$, the surjectivity of the periodization operator onto $\mathbf{A}(\mathbb{R}^{2d}/\Lambda)$ means there must exist some $g \in \mathbf{S}_0(\mathbb{R}^d)$ such that $\sum_{\lambda \in \Lambda} T_\lambda g = \mathcal{F}_W(T)$. Defining $\mathcal{F}_W(S) = g$ then gives an appropriate operator, where $S \in \mathcal{L}_{*-\|\|}(\mathbf{S}_0'(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}^d))$ due to Proposition 7.25. □

Remark 7.17. Notice that the decomposition result above is the exact analogue of Corollary 4.5 for Λ-modulation invariant operators.

Recalling the definition of the convolution of two operators:

$$T \star S(z) := \operatorname{tr}(T\alpha_z(\check{S})),$$

it is an interesting question in which cases one can reconstruct an operator T from samples of $T \star S$ on, for example, a lattice. This problem appears in several settings: In the case of two rank–one operators $T = f \otimes f$, $S = g \otimes g$ for $f, g \in L^2(\mathbb{R}^d)$, the operator convolution gives the spectrogram;

$$(T \star S)(z) = |V_g f(z)|^2,$$

while for a rank one $S = g \otimes g$ but arbitrary T, the discretisation of $T \star S$ corresponds to the diagonal of the Gabor matrix

$$T \star S(\lambda) = \langle T\pi(\lambda)g, \pi(\lambda)g\rangle_2.$$

In the former case it is known (cf. [GL22]) that there exists no g such that $T \mapsto S \star T|_\Lambda$ is injective for the space of Hilbert-Schmidt operators. On the other hand in the latter case, an operator T with a symbol in some Paley-Wiener space *can* be reconstructed from a discretisation of $S \star T$, for an appropriately chosen g and Λ [GP14].

This line of inquiry motivates Λ-modulation invariant operators, since such operators can be reconstructed from samples of convolution with an appropriate S. In what follows we let K denote a compact neighborhood of the origin not containing any $\lambda^\circ \in \Lambda^\circ$ apart from the origin.

Theorem 7.11. *Let $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ such that $\sigma(S)(0) \neq 0$, and $\operatorname{supp}(\sigma(S)) \subseteq K$. Then given any Λ-modulation invariant operator $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$;*

$$T = \frac{1}{\sigma(S)(0)} \sum_{\lambda^\circ \in \Lambda^\circ} (T \star S)(\lambda^\circ) \cdot e^{-4i\pi\lambda_1^\circ\lambda_2^\circ}\pi(2\lambda^\circ)P$$

Proof. We will use the fact that the convolution of two operators is equal to the convolution of their Weyl symbols (c.f. Proposition 7.19). By Corollary 7.9, the Weyl symbol of T is supported on the lattice Λ°, and can be expressed as

$$\sigma(T)(\chi) = \sum_{\lambda^\circ \in \Lambda^\circ} \sigma(T)(\lambda^\circ)\delta_{\lambda^\circ}(\chi).$$

Given our conditions on S and K, it then follows that

$$\begin{aligned}(T \star S)(\lambda^\circ) &= (\sigma(T) * \sigma(S))(\lambda^\circ) \\ &= \sigma(T)(\lambda^\circ) \cdot \sigma(S)(0),\end{aligned}$$

and so we can rewrite

$$\sigma(T)(\chi) = \frac{1}{\sigma(S)(0)} \sum_{\lambda^\circ \in \Lambda^\circ} (T \star S)(\lambda^\circ)\delta_{\lambda^\circ}(\chi).$$

By the second statement of Proposition 7.22, the Weyl quantisation of δ_χ is the operator $e^{-4i\pi\chi_1\cdot\chi_2}\pi(2\chi)P$, and so

$$T = \frac{1}{\sigma(S)(0)} \sum_{\lambda^\circ \in \Lambda^\circ} (T \star S)(\lambda^\circ) e^{-4i\pi\lambda_1^\circ\cdot\lambda_2^\circ}\pi(2\lambda^\circ)P.$$

□

For the canonical example of Λ-translation invariant operators, the frame operator $S_{g,\Lambda}$, the Janssen's representation can be interpreted as Poisson's summation formula applied to the periodization of the Weyl symbol of $g \otimes g$. Since it can be deduced from Equation (7.77) that the Weyl-symbol of every Λ-modulation invariant operator T with $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$ can be written as the periodization of the Weyl-symbol of some $S \in \mathcal{L}_{*-|||}(\mathbf{S}_0'(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}^d))$, we can follow the same approach for Λ-modulation invariant operators. However, we find a subtle difference in the lattices of reconstruction due to Equation (7.75). We begin by considering the spreading quantisation of $e^{2\pi i\Omega(\chi,\chi')}$.

Lemma 7.20. *For any $\chi \in \mathbb{R}^{2d}$, the Fourier-Wigner transform of $e^{-\pi i\chi_1\cdot\chi_2}\pi(\chi)P$ is the function $f(\chi') = 2^{-d}e^{\pi i\Omega(\chi,\chi')}$.*

Proof. From the second form of Weyl transform in Proposition 7.22, the Weyl symbol of $e^{-\pi i\chi_1\cdot\chi_2}\pi(\chi)P$ is the Dirac delta distribution $\delta_{2\chi}$. Using the correspondence of Weyl symbol and Fourier-Wigner via the symplectic Fourier transform, the Fourier-Wigner transform of $e^{-\pi i\chi_1\cdot\chi_2}\pi(\chi)P$ is thus given by

$$\begin{aligned}\mathcal{F}_W(e^{-\pi i\chi_1\cdot\chi_2}\pi(\chi)P)(\chi') &= \mathcal{F}_s(\delta_{2\chi})(\chi') \\ &= 2^{-d}e^{\pi i\Omega(\chi,\chi')}.\end{aligned}$$

□

Note that for some $\lambda \in \Lambda$, the Fourier-Wigner transform of $e^{-\pi i\lambda_1\cdot\lambda_2}\pi(\lambda)P$ is $\Lambda/2$-periodic as opposed to merely Λ-periodic, as a result of (7.75), and we see the result of this in the Janssen type representation of Λ-modulation invariant operators:

Theorem 7.12. *Let $S \in \mathcal{L}_{*-\|\|}(\mathbf{S}_0'(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}^d))$. Then*

$$\sum_{\lambda\in\Lambda} \beta_\lambda(S) = \frac{2^d}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \sigma(S)(\lambda^\circ) \cdot e^{-4\pi i\lambda_1^\circ\cdot\lambda_2^\circ}\pi(2\lambda^\circ)P,$$

or equivalently

$$\sum_{\lambda\in\Lambda} \beta_\lambda(S) = \frac{2^d}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \sigma(S)(\lambda^\circ) \cdot \alpha_{\lambda^\circ}(P).$$

Proof. We consider the Fourier-Wigner transform of the left hand side sum;

$$\begin{aligned}\mathcal{F}_W\Big(\sum_{\lambda\in\Lambda} \beta_\lambda(S)\Big)(\chi) &= \sum_{\lambda\in\Lambda} \mathcal{F}_W\big(\beta_\lambda(S)\big)(\chi)\\ &= \sum_{\lambda\in\Lambda} T_\lambda\big(\mathcal{F}_W(S)\big)(\chi).\end{aligned}$$

By Proposition 4.9 Item 4, the sum converges absolutely for every $\chi \in \mathbb{R}^{2d}$, for $S \in \mathcal{L}_{*-\|\|}(\mathbf{S}_0'(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}^d))$. The symplectic Poisson summation formula (4.30) then gives

$$\begin{aligned}\sum_{\lambda\in\Lambda} T_\lambda\big(\mathcal{F}_W(S)\big)(\chi) &= \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_s(\mathcal{F}_W(S))(\lambda^\circ)e^{2\pi i\Omega(\lambda^\circ,\chi)}\\ &= \frac{1}{|\Lambda|} \sum_{\lambda^\circ\in\Lambda^\circ} \sigma(S)(\lambda^\circ)e^{2\pi i\Omega(\lambda^\circ,\chi)}.\end{aligned}$$

The result the follows from (7.20), and the equivalent form from (7.75). □

For the next result, we need to introduce *modulation spaces*. By taking a Schwartz function window $g \in \mathscr{S}(\mathbb{R}^d) \subseteq \mathbf{S}_0(\mathbb{R}^d)$, the STFT can be extended to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$ (with $\mathscr{S}(\mathbb{R}^d)$ being the Schwartz space). By taking a $L^2(\mathbb{R}^d)$ normalised Gaussian $\varphi_0 \in \mathbf{S}_0(\mathbb{R}^d)$, the Banach space called **modulation spaces** can then be defined as

$$M^{p,q}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{M^{p,q}} := \|V_{\varphi_0} f\|_{p,q(\mathbb{R}^{2d})} < \infty\}$$

where $\|\cdot\|_{p,q}$ is the norm of the space $L^{p,q}_m(\mathbb{R}^{2d})$, the mixed-norm Lebesgue space (cf. Chapter 11 [Grö01]). All functions in $\mathscr{S}_0(\mathbb{R}^d)$ can be used as windows to define the same spaces with equivalent norms, and the dual of $M^{p,q}(\mathbb{R}^d)$ is $M^{p',q'}(\mathbb{R}^d)$ for $1 \leqslant p,q < \infty$. Furthermore, for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, we have the continuous inclusions [Grö01]: $M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d)$. Finally, we have the equality $\mathbf{S}_0(\mathbb{R}^d) \cong M^{1,1}(\mathbb{R}^d)$, making the Feichtinger's algebra the minimal modulation space. Note that we shall write $M^p(\mathbb{R}^d)$ when $p = q$.

Given that $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$ for all $1 \leqslant p,q \leqslant \infty$ means that all modulation spaces are embedded inside $\mathbf{S}_0'(\mathbb{R}^d)$. In fact we have $M^\infty(\mathbb{R}^d) \cong \mathbf{S}_0'(\mathbb{R}^d)$. This means we can define an operator version of these modulation spaces:

$$\mathcal{M}^{p,q} := \{T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d)) : \kappa(T) \in M^{p,q}(\mathbb{R}^d)\}, \tag{7.78}$$

where $\kappa(T)$ is the (outer) kernel that appears in Theorem 4.14 Item 1. We shall also write $\mathcal{M}^p = \mathcal{M}^{p,q}$ when $p = q$. It follows that

$$(\mathcal{L}_{*-\|\|}, \mathcal{HS}, \mathcal{L})(\mathbb{R}^d, \mathbb{R}^d) = (\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^\infty).$$

Theorem 7.13. *Let $T \in \mathcal{M}^\infty$ be a Λ-modulation-invariant operator, such that $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$. Then T is bounded on all modulation spaces $M^{p,q}(\mathbb{R}^d)$, with*

$$\|T\|_{\mathcal{L}(M^{p,q})} \leqslant C_\Lambda \|\mathcal{F}_W(T)\|_{\mathbf{A}(\mathbb{R}^d/\Lambda)},$$

where $C_\Lambda = \frac{2^d}{|\Lambda|}$.

Proof. Since $\mathcal{F}_W(T) \in \mathbf{A}(\mathbb{R}^{2d}/\Lambda)$, by (7.25) and (7.20) we can write T as

$$T = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} \sigma(T)(\lambda^\circ) \cdot e^{-4\pi i \lambda_1^\circ \cdot \lambda_2^\circ} \pi(2\lambda^\circ) P,$$

where $\{c_{\lambda^\circ}\}_{\Lambda^\circ} \in \ell^1(\Lambda^\circ)$. Since the modulation spaces $M^{p,q}(\mathbb{R}^d)$ are invariant under time-frequency shifts, then for any $f \in M^{p,q}(\mathbb{R}^d)$;

$$\begin{aligned}\|Tf\|_{M^{p,q}} &= C_\Lambda \Big\| \sum_{\lambda^\circ \in \Lambda^\circ} \sigma(T)(\lambda^\circ) \cdot e^{-4\pi i \lambda_1^\circ \cdot \lambda_2^\circ} \pi(2\lambda^\circ) P f \Big\|_{M^{p,q}} \\ &\leqslant C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} |\sigma(T)(\lambda^\circ)| \|\pi(2\lambda^\circ) P f\|_{M^{p,q}}\end{aligned}$$

$$
\begin{aligned}
&= C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} |\sigma(T)(\lambda^\circ)| \|f\|_{M^{p,q}} \\
&= C_\Lambda \|\mathcal{F}_W(T)\|_{\mathbf{A}(\mathbb{R}^d/\Lambda)} \|f\|_{M^{p,q}}.
\end{aligned}
$$

□

7.2.5 A Calculus for Λ-Modulation Invariant Operators

We now consider the composition of Λ-modulation-invariant operators. We reiterate that these will no longer be Λ-modulation invariant, but rather $\Lambda/2$-translation invariant. We also restrict our attention to operators in $\mathcal{L}(L^2(\mathbb{R}^d))$, so that their composition is well defined. We now find the following calculus.

Theorem 7.14. *Let $S, T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0(\mathbb{R}^d))$ be two Λ-modulation invariant operators such that ST is well-defined. Suppose $2\Omega(\Lambda^\circ, \Lambda^\circ) \subset \mathbb{Z}$, that is $2\Omega(\lambda^\circ, \mu^\circ) = n \in \mathbb{Z}$ for every $\lambda^\circ, \mu^\circ \in \Lambda^\circ$. Then S and T satisfy the spreading function calculus*

$$
\mathcal{F}_W(ST)(2z) = 2^{2d} \sigma_S * \check{\sigma}_T(z),
$$

where $*$ is the usual convolution in $L^1(\mathbb{R}^{2d})$.

Proof. Let $S, T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ be Λ-modulation invariant operators where ST is well-defined and $2\Omega(\Lambda^\circ, \Lambda^\circ) \subset \mathbb{Z}$. Then by (7.9) we can consider the discrete representations

$$
\begin{aligned}
\sigma(S) &= \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \delta_{\lambda^\circ} \\
\sigma(T) &= \sum_{\mu^\circ \in \Lambda^\circ} d_{\mu^\circ} \delta_{\mu^\circ}.
\end{aligned}
$$

We then use the second representation given by Proposition 7.22, giving

$$
\begin{aligned}
S &= 2^d \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} e^{-4\pi i \lambda_1^\circ \lambda_2^\circ} \pi(2\lambda^\circ) P \\
T &= 2^d \sum_{\mu^\circ \in \Lambda^\circ} d_{\mu^\circ} e^{-4\pi i \mu_1^\circ \mu_2^\circ} \pi(2\mu^\circ) P.
\end{aligned}
$$

We then calculate the composition to find

$$\begin{aligned} ST &= 2^{2d} \sum_{\lambda^\circ,\mu^\circ\in\Lambda^\circ} c_{\lambda^\circ} d_{\mu^\circ} e^{-4\pi i(\lambda_1^\circ\lambda_2^\circ+\mu_1^\circ\mu_2^\circ)} \pi(2\lambda^\circ) P \pi(2\mu^\circ) P \\ &= 2^{2d} \sum_{\lambda^\circ,\mu^\circ\in\Lambda^\circ} c_{\lambda^\circ} d_{\mu^\circ} e^{-4\pi i(\lambda_1^\circ\lambda_2^\circ+\mu_1^\circ\mu_2^\circ)} \pi(2\lambda^\circ)\pi(-2\mu^\circ) \\ &= 2^{2d} \sum_{\lambda^\circ,\mu^\circ\in\Lambda^\circ} c_{\lambda^\circ} d_{\mu^\circ} e^{-4\pi i(\lambda_1^\circ\lambda_2^\circ+\mu_1^\circ\mu_2^\circ)} e^{8\pi i\lambda_1^\circ\mu_2^\circ} \pi(2\lambda^\circ - 2\mu^\circ). \end{aligned}$$

Note that

$$-4\pi i(\lambda_1^\circ\lambda_2^\circ + \mu_1^\circ\mu_2^\circ) + 8\pi i\lambda_1^\circ\mu_2^\circ = -4\pi i(\lambda_1^\circ - \mu_1^\circ)(\lambda_2^\circ - \mu_2^\circ) - 4\pi i\Omega(\lambda^\circ,\mu^\circ),$$

so by our assumption $2\Omega(\Lambda^\circ,\Lambda^\circ)\subset\mathbb{Z}$, the exponential $e^{-4\pi i\Omega(\lambda^\circ,\mu^\circ)}$ of the last twisting term will be 1. Hence, using the change of variables $\gamma^\circ := \lambda^\circ - \mu^\circ$ we find

$$\begin{aligned} ST &= 2^{2d} \sum_{\lambda^\circ,\mu^\circ\in\Lambda^\circ} c_{\lambda^\circ} d_{\mu^\circ} e^{-4\pi i(\lambda_1^\circ-\mu_1^\circ)(\lambda_2^\circ-\mu_2^\circ)} \pi(2\lambda^\circ - 2\mu^\circ) \\ &= 2^{2d} \sum_{\lambda^\circ,\gamma^\circ\in\Lambda^\circ} c_{\lambda^\circ} d_{\lambda^\circ-\gamma^\circ} e^{-4\pi i\gamma_1^\circ\gamma_2^\circ} \pi(2\gamma^\circ) \end{aligned}$$

It follows from Proposition 7.19 Item 4, Equation (7.48), and the fact that the spreading function of a time-frequency shift is the dirac-delta that: $\mathcal{F}_W(\pi(2\gamma^\circ))(\chi) = e^{\pi i\chi_1\cdot\chi_2}\delta_{2\gamma^\circ}(\chi)$. We now obtain the following formula by taking the Fourier-Wigner transform of ST above:

$$\begin{aligned} \mathcal{F}_W(ST)(2\gamma^\circ) &= 2^{2d} \sum_{\lambda^\circ,\gamma^\circ} c_{\lambda^\circ} d_{\lambda^\circ-\gamma^\circ} \\ &= 2^{2d}\sigma(S) * \breve{\sigma}(T)(\gamma^\circ), \end{aligned}$$

as required. □

Remark 7.18. The above calculus holds in general for $S,T\in\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))$ where $\mathrm{Im}(T)\subset\mathbf{S}_0(\mathbb{R}^d)$ so that ST is well-defined.

We collect some simple corollaries of the perfect spreading function calculus. We collect some simple corollaries of the spreading function calculus. Firstly, since a bounded operator is uniquely determined by its spreading function, we have the following composition property of Λ-modulation invariant operators. Note that the Corollary below follows directly from the formula $\sigma(\check{T}) = \check{\sigma}(T)$ (see for example [LS18])

Corollary 7.10. *If S, T and Λ are as in (7.14), then $S\check{T} = T\check{S}$.*

Secondly, using Proposition 7.19 Item 4 along with the fact the symplectic Fourier transform has its own convolution formula, and $\mathcal{F}_W(\check{T}) = \overline{\mathcal{F}_W(T)}$ (e.g. see [LS18] or [FK98]). We can reformulate (7.14) as a relation of Weyl symbols:

Corollary 7.11. *For S, T and Λ as in (7.14);*

$$\sigma(ST)\left(\frac{\chi}{2}\right) = \left(\mathcal{F}_W(S)\overline{\mathcal{F}_W(T)}\right)(\chi),$$

These characterisations of Λ-modulation invariant operators gives a way to identify each Λ-modulation invariant operator with a $\Lambda/2$-translation invariant operator by composing with the parity operator, and vice versa:

Theorem 7.15. *The map $\Theta : S \mapsto SP$ is a self-inverse isometric isomorphism on $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$. It identifies Λ-modulation invariant operators with $\Lambda/2$-invariant operators. Moreover, in terms of symbols*

$$\mathcal{F}_W(\Theta(S))(2\chi) = 2^d \sigma_S(\chi), \qquad \forall \chi \in \mathbb{R}^{2d}$$

Proof. We will only show that Θ is an isometry on and $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, as this will be enough to show that Θ is a self-inverse isometric isomorphism on these spaces following Lemma 7.19. Let $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$, then it follows from the fact that $P_{|\mathbf{S}_0(\mathbb{R}^d)} : \mathbf{S}_0(\mathbb{R}^d) \to \mathbf{S}_0(\mathbb{R}^d)$ is an isometric isomorphism via Lemma 7.19 that we have the following:

$$\begin{aligned}
\|SP\|_{\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')} &= \sup\left\{\left|\overline{(\overline{g}, SPf)}_{\mathbf{S}_0, \mathbf{S}_0'}\right| : \|f\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\} \\
&= \sup\left\{|\langle Sf, g\rangle_2| : \|Pf\|_{\mathbf{S}_0} = \|g\|_{\mathbf{S}_0} = 1\right\} \\
&= \|S\|_{\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')}.
\end{aligned}$$

It follows that Θ is an isometric isomorphism on $\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$.

Let S be a Λ-modulation invariant operator. We recall that the parity operator P is a $\mathbb{R}^{2d}$-modulation invariant operator, and consequently a Λ-modulation invariant operator for any Λ. Hence by Theorem 7.10, SP is a $\Lambda/2$-translation invariant operator. Conversely, given some

$\Lambda/2$-translation invariant operator T, the operator TP is a Λ-modulation invariant operator, since

$$\begin{aligned} TP\pi\left(\tfrac{\lambda}{2}\right) &= T\pi\left(-\tfrac{\lambda}{2}\right)P \\ &= \pi\left(-\tfrac{\lambda}{2}\right)TP, \end{aligned}$$

and so the result follows from Remark 7.16. Moreover since the Weyl symbol of P is the Dirac delta distribution, by the same argument as the proof of Theorem 7.14 and Remark 7.18, $2^d\sigma(S)(\chi) = \mathcal{F}_W(SP)(2\chi)$ for all $\chi \in \mathbb{R}^{2d}$. □

Remark 7.19. This Λ to $\Lambda/2$ correspondence between the translation invariant operator and modulation invariant operators is one of the reasons why we remain in the Euclidean case $\mathbb{R}^{2d}$, where rescaling is possible. To be more precise, $\chi \mapsto \chi/2$ is an automorphism in $\mathbb{R}^{2d}$. In fact, this aligns well with our choice of working with Weyl quantization, as noted in [FK04], since one encounters difficulties reproducing the properties of Weyl quantization in a general time-frequency plane (a.k.a. phase space) when it cannot be equipped with the said automorphism.

7.2.6 Λ-Modulation Invariant Operators via Finite-Rank Operators Generated by the Heisenberg Module

We revisit the characterisation results of Section 7.2.2 now equipped with the correspondence result of Theorem 7.15. Just like how we consider the Gabor frame operator $S_{g,h,\Lambda}$ as the canonical Λ-translation invariant operator equivalent to the operator-periodization $\sum_{\lambda\in\Lambda}\alpha_\lambda(\overline{g}\otimes h)^*$, we analogously find that $\Lambda/2$-periodization of $(\overline{g}\otimes h)^* = h\otimes\overline{g}$ pre-composed with the parity operator gives us:

$$\sum_{\lambda\in\Lambda}\alpha_{\lambda/2}((\overline{g}\otimes h)^*)P = \sum_{\lambda\in\Lambda}\beta_\lambda(h\otimes P\overline{g}) = S_{g,h,\Lambda/2}P. \tag{7.79}$$

Since P is unitary in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, Equation (7.79) motivates us to consider operators of the form

$$M = \sum_{\lambda\in\Lambda}\beta_\lambda(g\otimes h) \tag{7.80}$$

as the canonical form of Λ-modulation invariant operators. Indeed we shall show, just like in the case of Section 7.2.2, that Λ-modulation invariant operators can be characterised in terms of

Fourier-Wigner periodizations of finite-rank operators with generators coming from the Heisenberg module.

Lemma 7.21. *Fix any lattice $\Lambda \subseteq \mathbb{R}^{2d}$. If $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$, then*

$$S_{P\psi_1,P\psi_2,\Lambda} = PS_{\psi_1,\psi_2,\Lambda}P = \widetilde{S_{\psi_1,\psi_2,\Lambda}}.$$

As a corollary, the parity operator restricts to an isometric isomorphism $P_{|\mathcal{E}_\Lambda(\mathbb{R}^d)} : \mathcal{E}_\Lambda(\mathbb{R}^d) \to \mathcal{E}_\Lambda(\mathbb{R}^d)$ on the Heisenberg module.

Proof. We compute for any $g \in L^2(\mathbb{R}^d)$:

$$\begin{aligned} S_{P\psi_1,P\psi_2,\Lambda} g &= \sum_{\lambda\in\Lambda} \langle g, \pi(\lambda)P\psi_1\rangle_2 \, \pi(\lambda)P\psi_2 \\ &= P\left(\sum_{\lambda\in\Lambda} \langle Pg, \pi(-\lambda)\psi_1\rangle_2 \, \pi(-\lambda)\psi_2\right) \\ &= PS_{\psi_1,\psi_2,\Lambda}Pg = \widetilde{S_{\psi,\Lambda}}g. \end{aligned}$$

Since P is a unitary map in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, it follows in particular that for any $\psi \in L^2(\mathbb{R}^d)$:

$$\begin{aligned} \|S_{P\psi,\Lambda}\|^2 &= \sup_{\|g\|_2=1} \langle PS_{\psi,\Lambda}Pg, PS_{\psi,\Lambda}Pg\rangle_2 \\ &= \sup_{\|g\|_2=1} \langle S_{\psi,\Lambda}Pg, S_{\psi,\Lambda}Pg\rangle_2 \\ &= \sup_{\|g\|_2=1} \langle S_{\psi,\Lambda}g, S_{\psi,\Lambda}g\rangle_2 = \|S_{\psi,\Lambda}\|^2. \end{aligned} \tag{7.81}$$

It then follows from (7.81) and (5.49) that if $f \in \mathcal{E}_\Lambda(\mathbb{R}^d)$, then:

$$\|Pf\|^2_{\mathcal{E}_\Lambda(\mathbb{R}^d)} = \|S_{Pf,\Lambda}\| = \|S_{f,\Lambda}\| = \|f\|^2_{\mathcal{E}_\Lambda(\mathbb{R}^d)}. \tag{7.82}$$

The above only shows that P is an isometry on $\mathcal{E}_\Lambda(\mathbb{R}^d)$, we still have to show that $Pf \in \mathcal{E}_\Lambda(\mathbb{R}^d)$. To do this, we need to find a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathbf{S}_0(\mathbb{R}^d)$ such that $\|Pf - g_n\|_{\mathcal{E}_\Lambda(\mathbb{R}^d)} \to 0$. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbf{S}_0(\mathbb{R}^d)$ such that $\|f - f_n\|_{\mathcal{E}_\Lambda(\mathbb{R}^d)} \to 0$. Due to Lemma 7.19, we can choose $g_n := Pf_n \in \mathbf{S}_0(\mathbb{R}^d)$ for each $n \in \mathbb{N}$. Now it follows from the isometry property (7.82) that $\|Pf - g_n\|_{\mathcal{E}_\Lambda(\mathbb{R}^d)} = \|Pf - Pf_n\|_{\mathcal{E}_\Lambda(\mathbb{R}^d)} = \|f - f_n\|_{\mathcal{E}_\Lambda(\mathbb{R}^d)} \to 0$ as required. $\square$

Theorem 7.16. *There exists an $n_0 \in \mathbb{N}$ such that every Λ-modulation invariant operator in*

$\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ *is the norm-limit of Fourier-Wigner operator-periodization of rank-*n_0 *operators generated by functions coming from* $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$.

Proof. Fix $n_0 \in \mathbb{N}$ as in Theorem 7.8 applied to $\Lambda/2$-translation invariant operators. Let $T \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ be a Λ-modulation invariant operator. It follows from Theorem 7.15 that there exists a $\Lambda/2$-translation invariant operator $S \in \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ such that $T = SP$. It follows from Theorem 7.8 that there exists a sequence of operators $\{S_m\}_{m\in\mathbb{N}} \subseteq \mathcal{L}(\mathbf{S}_0(\mathbb{R}^d), \mathbf{S}_0'(\mathbb{R}^d))$ where $\|S - S_m\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))} \to 0$. Furthermore for each m, $S_m = \sum_{i=1}^{n_0}(S_{\psi_1,h_i,\Lambda/2})$ where $\psi_1, ..., \psi_n, h_1, ..., h_{n_0} \in \mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$. But then it follows from Equation (7.79) that

$$S_m P = \sum_{i=1}^{n_0} \beta_\lambda(h_i \otimes P\overline{\psi_i}) = \sum_{i=1}^{n_0} \beta_\lambda(h_i \otimes \overline{g_i}),$$

where $g_i := \overline{P\overline{\psi_i}} \in \mathcal{E}_\Lambda(\mathbb{R}^d)$ by Lemma 7.21. Therefore each $S_m P$ is the Fourier-Wigner operator-periodization of a rank-n_0 operator generated by functions coming from $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$. Lastly, we have from Theorem 7.15 that $\|T - S_m P\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))} = \|SP - S_m P\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))} = \|S - S_m\|_{\mathcal{L}(\mathbf{S}_0(\mathbb{R}^d),\mathbf{S}_0'(\mathbb{R}^d))} \to 0$. □

Following the results for Λ-translation invariant operators, we shall also consider Λ-modulation invariant operators in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, which we shall denote by

$$\mathcal{M}_\Lambda\left(L^2(\mathbb{R}^d)\right) := \left\{T \in L^2(\mathbb{R}^d) : T = \beta_\lambda(T),\ \forall \lambda \in \Lambda\right\}.$$

Here we have a crucial observation regarding the map Θ of Theorem 7.15, it is well-defined as a map in $L^2(\mathbb{R}^d)$, in fact we have the following result.

Lemma 7.22. Θ *is a self-inverse isometric isomorphism in* $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$. *As a map in* $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ *it is also strong-operator and weak-operator continuous. Finally, it identifies* $\mathcal{M}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ *with* $\mathcal{L}_{\Lambda/2}\left(L^2(\mathbb{R}^d)\right)$.

Proof. We shall only show that Θ is an isometry as a map in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$. Let $S \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$, because $P \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ is unitary, we obtain

$$\|SP\|^2 = \sup_{\|g\|_2=1} \langle SPg, SPg\rangle_2 = \sup_{\|Pg\|_2=1} \langle Sg, Sg\rangle_2 = \|S\|^2.$$

In light of Lemma (7.19), this is enough to show that Θ is an isometric isomorphism on $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$.

Next, let $\{T_i\}_{i\in I}$ be a net in $\mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ that converges to $T \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ in the weak-operator topology, that is: $\langle T_i f, g\rangle_2 \to \langle Tf, g\rangle_2$ for all $f, g \in L^2(\mathbb{R}^d)$. It follows from unitarity of $P \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ that $\langle T_i Pf, g\rangle_2 \to \langle TPf, g\rangle_2$ for all $f, g \in L^2(\mathbb{R}^d)$, equivalently $\Theta(T_i) \to \Theta(T)$ in the weak-operator topology. The proof for continuity of $\Theta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ in strong-operator topology is similar.

The proof showing the identification of $\mathcal{M}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ with $\mathcal{L}_{\Lambda/2}\left(L^2(\mathbb{R}^d)\right)$ via Θ is the same as in Theorem 7.15. □

Theorem 7.17. *There exists an $n_0 \in \mathbb{N}$ such that every operator in $\mathcal{M}_\Lambda\left(L^2(\mathbb{R}^d)\right)$ is a weak-operator, or strong-operator limit of Fourier-Wigner operator-periodizations of rank-n_0 operators generated by functions coming from $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$.*

Proof. Fix $n_0 \in \mathbb{N}$ as in Theorem 7.68 applied to $\Lambda/2$-translation invariant operators. Let $T \in \mathcal{M}_\Lambda\left(L^2(\mathbb{R}^d)\right)$, then by 7.22, there exists a $\Lambda/2$-translation invariance operator $S \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ such that $T = SP = \Theta(S)$. We know from Theorem 7.68 that there exists a sequence $\{S_m\}_{m\in\mathbb{N}} \in \mathcal{L}\left(L^2(\mathbb{R}^d)\right)$ such that $S_m \to S$ in weak-operator or strong-operator topology. We can proceed similarly to the proof of Corollary 7.16 to show that each $S_m P = \Theta(S_m)$ is a Fourier-Wigner operator-periodization of a rank-n_0 operator generated by functions coming from $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$. We are now done since Θ is continuous with respect to the weak-operator and strong-operator topologies, and we obtain $\Theta(S_m) \to \Theta(S) = T$ in either of these topologies. □

Fix a lattice Λ and consider the Heisenberg module $\mathcal{E}_\Lambda(\mathbb{R}^d)$, it actually follows from the proof of Lemma 7.13 that the integer n can be fixed to be the minimum number of elements required to generate $\mathcal{E}_\Lambda(\mathbb{R}^d)$ as a *finitely generated projective* left A-module. We couple this with the fact that the generators of $\mathcal{E}_\Lambda(\mathbb{R}^d)$ as a left A-module are exactly multi-window Gabor frames for $L^2(\mathbb{R}^d)$ [AE20, Theorem 3.16], and that the existence of n-multi-window Gabor frames on $L^2(\mathbb{R}^d)$ with atoms from $\mathcal{E}_\Lambda(\mathbb{R}^d)$ implies that the volume of the lattice Λ must necessarily satisfy: $d(\Lambda) \leqslant n$ (see Remark 5.11); we then have the following Lemma.

Lemma 7.23. *For a fixed Λ, if $n \in \mathbb{N}$ is the minimum number of elements required to generate $\mathcal{E}_\Lambda(\mathbb{R}^d)$ as a left A-module, then $d(\Lambda) \leqslant n$.*

We can interpret this result to mean that the sparser the lattice Λ is (i.e. larger $d(\Lambda)$), the greater number of elements are required to generate $\mathcal{E}_\Lambda(\mathbb{R}^d)$.

Throughout this study, we have seen that there is a fixed integer $n \in \mathbb{N}$ that would allow us to characterise Λ-translation invariant operators as operator-periodizations of rank-n operators.

We have just discussed that this very same n can be fixed to be the minimum number required to generate $\mathcal{E}_\Lambda(\mathbb{R}^d)$. Following Lemma (7.23), we see that there is a discrepancy between the lowest possible rank of operators that we can use to characterise Λ-translation invariant operators via operator-periodization versus that of Λ-modulation invariant operators via Fourier-Wigner operator-periodization.

Proposition 7.27. *For a fixed lattice $\Lambda \subseteq \mathbb{R}^d$, we have the following:*

1. *If n is the minimum number that makes the characterisation of Λ-translation invariant operators as in Theorem 7.8 and Theorem 7.9 true, then $d(\Lambda) \leqslant n$.*

2. *If n_0 is the minimum number the makes the characterisation of Λ-modulation invariant operators as in Theorem 7.16 and Theorem 7.17 true, then $\frac{d(\Lambda)}{4^d} \leqslant n_0$.*

Proof.

1. We know that n is the same as the minimum number of generators required to generate $\mathcal{E}_\Lambda(\mathbb{R}^d)$, therefore it follows from Lemma 7.23 that $d(\Lambda) \leqslant n$.

2. The proof here is similar to Item 1, however n_0 is the minimum number of generators required to generate $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$. Therefore it follows from 7.23 that $|\Lambda/2| \leqslant n_0$. Since we always consider lattices here, there must exist some $L \in \mathrm{GL}_{2d}(\mathbb{R})$ such that $\Lambda = L\mathbb{Z}^{2d}$, therefore $d(\Lambda/2) = |\det(L/2)| = \left(\frac{1}{2}\right)^{2d} |\det L| = \left(\frac{1}{4}\right)^d d(\Lambda)$. We then obtain $\frac{d(\Lambda)}{4^d} \leqslant n_0$.

□

We see from the result above that Λ-modulation invariant operators generally require less generators from the Heisenberg module to be represented as a Fourier-Wigner periodization, compared to representations of Λ-translation invariant operators via operator-periodization. In fact, in one special case, we have an exact result concerning these generators. Let $\{e_1, ..., e_{2d}\}$ be the standard basis for $\mathbb{R}^{2d}$ and $\Omega : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}$ be the standard symplectic form:

$$\Omega((x,\omega),(t,\nu)) = t \cdot \omega - x \cdot \nu, \qquad \forall (x,\omega),(t,\nu) \in \mathbb{R}^{2d}.$$

Note that $c_s((t,\nu),(x,\omega)) = e^{2\pi i \Omega((x,\omega),(t,\nu))}$. We say that a lattice $\Lambda = L\mathbb{Z}^{2d}$, $L \in \mathrm{GL}(2d,\mathbb{R})$ is *non-rational* if there exists $i,j \in \{1,...,2d\}$ such that $\Omega(Le_i, Le_j)$ is not rational. A curious result concerning non-rational lattices due to [JL20, Corollary 5.6] states that the Heisenberg module $\mathcal{E}_\Lambda(\mathbb{R}^d)$ with non-rational lattice Λ satisfying $n-1 \leqslant d(\Lambda) < n$ for $n \in \mathbb{N}$ has exactly n

generators. Therefore, in terms of the integer ceiling function $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, we have the following result, by an argument similar to Proposition 7.27.

Proposition 7.28. *If Λ is non-rational:*

1. *Every Λ-translation invariant operator is characterized as the topological limits of operator-periodization in the sense of Theorem 7.8 and Theorem 7.9, of exactly rank-$\lceil d(\Lambda) \rceil$ operators generated by $\mathcal{E}_\Lambda(\mathbb{R}^d)$.*

2. *Every Λ-modulation invariant operator is characterized as the topological limits of Fourier-Wigner operator-periodization in the sense of Theorem 7.16 and Theorem 7.17, of exactly rank-$\lceil d(\Lambda/2) \rceil$ operators generated by $\mathcal{E}_{\Lambda/2}(\mathbb{R}^d)$.*

7.2.7 Discussion

The interesting appearance of the $\Lambda/2$ lattice in the correspondence between translation and modulation invariant operators seems peculiar at first glance, as one may initially suspect a Λ to Λ type correspondence. However, the $\Lambda/2$ arises directly as a result of using the Weyl quantization, as considered in 7.22. In fact the need for automorphisms of the type $x \mapsto 2x$ can cause problems for Weyl quantization on compact or discrete locally compact abelian groups. However, since in this work we consider phase space as the underlying group, we are free to scale lattices as required. Nonetheless, it is instructive to consider how the concept of modulations looks for other quantization schemes than Weyl. Consider Cohen's class quantizations, that is to say those defined in terms of a convolution with the Wigner distribution:

$$\langle (a \star L_\sigma) f, g \rangle = \langle a * \sigma, W(f,g) \rangle.$$

quantization schemes of this type include τ-Weyl quantization, itself encompassing Kohn-Nirenberg quantization and operators "with right symbol", as well as the Born-Jordan quantization (cf. [BDDO10]). Since these quantization schemes relate to Weyl quantization via a convolution, operator translations given by α_z correspond to translations on the symbols of all of these quantization schemes as a result of the Weyl case;

$$\begin{aligned} \langle \pi(z)(a \star L_\sigma)\pi(z)^* f, g \rangle &= \langle T_z(a * \sigma), W(g,f) \rangle \\ &= \langle a * T_z \sigma, W(g,f) \rangle. \end{aligned}$$

However, considering modulations instead of translation, we clearly lose this universality. For some τ-Weyl quantization where $\tau \neq 1/2$,

$$a_\tau = \frac{2^d}{|2\tau - 1|} e^{2\pi i \frac{2}{2\tau-1} x\cdot\omega}$$

so we have the identity

$$\sigma(S)^\tau = \frac{2^d}{|2\tau - 1|} e^{2\pi i \frac{2}{2\tau-1} x\cdot\omega} * \sigma(S)$$

(cf. [BDDO10]). Modulation in the sense of τ-Weyl quantization must then correspond to a translation of

$$\mathcal{F}_\Omega(\sigma(S)^\tau) = \frac{2^d}{|2\tau - 1|} \mathcal{F}_\Omega(e^{2\pi i \frac{2}{2\tau-1} x\cdot\omega}) \cdot \mathcal{F}_\Omega(\sigma(S)).$$

In this sense Weyl quantization, where the planar wave disappears, can be seen as the natural setting for quantum time–frequency analysis.

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Appendix A

Universal C^*-algebras

The construction of objects in terms of generators and relations is a common technique in algebra. Let us recall the general idea here: using only generic elements in some set $\mathcal{G}$ satisfying certain relations, say $\mathcal{R}$, can we generate an algebraic object from the pair $(\mathcal{G}, \mathcal{R})$? One of the easiest example is from group theory, suppose we have a single generator $\mathcal{G} = \{a\}$, satisfying a single relation $\mathcal{R} = \{a^n = e\}$ for the generic neutral element e. Then the group generated by the pair $(\mathcal{G}, \mathcal{R})$, denoted by $\langle \mathcal{G} | \mathcal{R} \rangle = \langle a | a^n = e \rangle$ is nothing more than the cyclic group of order n, or $\mathbb{Z}_n$. The main idea is that we want $\langle \mathcal{G} | \mathcal{R} \rangle$ to be the 'minimal' group containing $\mathcal{G}$ that also satisfies all the relations stipulated by $\mathcal{R}$. Note that for groups, all relations are essentially equations on $\mathcal{G}$, and can then be always arranged as equating expressions on $\mathcal{G}$ to the neutral element e. Therefore we may as well just denote our set of relations to be expressions in $\mathcal{G}$. So in our example, we have $\langle \mathcal{G} | \mathcal{R} \rangle = \langle a | a^n \rangle \cong \mathbb{Z}$. Staying in the realm of groups, if $\mathcal{R} = \varnothing$, we say that the group $\langle \mathcal{G} | \mathcal{R} \rangle$ is **free** in the sense that the group is simply generated by $\mathcal{G}$ and these elements are free from any stipulated relations between themselves. Still in the single generator case, we have the following example of a free group: $\langle a \rangle := \langle a | \varnothing \rangle \cong \mathbb{Z}$. It is a basic result of group theory that all groups can be described in terms of generators and relations, i.e. $G \cong \langle \mathcal{G} | \mathcal{R} \rangle$, from some generators and relations $\mathcal{G}, \mathcal{R}$ respectively. Equivalently every group is the quotient of some free group i.e. $G \cong \langle \mathcal{G} \rangle / N_{\mathcal{R}}$ for $\langle \mathcal{G} \rangle := \langle \mathcal{G} | \varnothing \rangle$ and $N_{\mathcal{R}}$ a normal subgroup which one can construct from the relations $\mathcal{R}$. As a final example, the cyclic group $\mathbb{Z}_n$ is the quotient of the free group $\mathbb{Z}$, given explicitly by $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z} \cong \langle a | a^n \rangle$

Most algebraic structures (e.g. rings, algebras, etc.) can be constructed in terms of generators and relations as in above, i.e. as a quotient of some **free object**, and we may want to ask if we can also do the same for the C^*-algebras. The problem with simply following this scheme is that C^*-algebras are not purely algebraic objects, and they come equipped with (*unique!*) norms (they even need to be isomorphic to some closed $*$-subalgebra of bounded linear operator on a Hilbert space c.f. GNS Theorem) and so the notion of *free C^*-algebra* via generators $\mathcal{G}$ is not quite well-defined[1]. However, as we shall see, by completing quotients of free $*$-algebras (so no norms) with respect to a prescribed *universal norm*, we may get a useful notion of what C^*-algebras constructed by generators and relations should be.

We shall follow the notes of S. Sundar [Sha20], where the given notion of universal C^*-algebras is general enough to encompass the one presented by [Bla85].

Definition A.1. Let A be a $*$-algebra. A **C*-seminorm** $\rho : A \to [0, \infty)$ on A is a seminorm on A that also satisfies:

1. For all $a \in A$, $\rho(a^*a) = p(a)^2$, (C^*-condition);
2. For all $a, b \in A$, $\rho(ab) \leqslant \rho(a)\rho(b)$, (Submultiplicativity).

Define for each $a \in A$:

$$\|a\|_u := \sup\{\rho(a) : \rho \text{ is a } C^*\text{-seminorm on } A\}.$$

We say that A is **admissible** if $\|a\|_u < \infty$ for all $a \in A$, in which case $\|\cdot\|_u$ is itself a C^*-seminorm on A. For an admissible $*$-algebra A, let us define the ideal (sometimes called the **reducing ideal**) $I_u := \{a \in A : \|a\|_u = 0\}$. If the canonical quotient map is denoted by $q_u : A \to A/I_u$, then the **Universal C^*-algebra** generated by A, denoted by $C^*(A)$, is the completion of the quotient $*$-algebra $q_u(A) = A/I_u$ with respect to the C^*-norm induced by $\|\cdot\|_u$ (i.e. $\|q_u(a)\| = \|a + I_u\|_u := \|a\|_u$ for all $a \in A$). Lastly, we say that A is a **$*$-semisimple** or **reduced** $*$-algebra, if its reducing ideal satisfies $I_u = \{0\}$.

Remark A.1. We see from the definition that $C^*(A)$ exists if and only if A is admissible. In the sequel we shall show examples of non-admissible $*$-algebras.

The definition of the Universal C^*-algebra generated by a $*$-algebra should remind us of the GNS-theorem, in fact this can be made apparent by the following result.

Proposition A.1. *Let A be a $*$-algebra, then the following holds for all $a \in A$:*

$$\|a\|_u = \sup\{\|\pi(a)\| : \pi \textit{ is a } *\textit{-homomorphism from } A \textit{ to a } C^*\textit{-algebra}\}$$

[1] In the language of category theory, there are no free objects in the category of C^*-algebras.

$$= \sup\{\|\pi(a)\| : \pi \textit{ is a nondegenerate } *\textit{-representation of } A\}.$$

Recall that π is a $*$-representation of A if it is a $*$-homomorphism of the form $\pi : A \to \mathcal{L}(H)$ for some Hilbert space H, and it is nondegenerate if $\operatorname{span}\{\pi(a)h : a \in A, h \in H\}$ *is dense in H.*

Proof. For any $*$-homomorphism $\pi : A \to \mathcal{C}$ to a C^*-algebra $\mathcal{C}$, the map $a \mapsto \|\pi(a)\|$ is a C^*-seminorm on A. If we can show the converse, that is, if we can show that every C^*-seminorm on A comes from $*$-homomorphisms to a C^*-algebra, then the first equation is proven. The second equation follows basically from the GNS-construction.

Suppose $\rho : A \to \mathbb{R}$ is a C^*-seminorm, define the ideal (follows from submultiplicativity) $I_\rho := \{a \in A : \rho(a) = 0\}$ on A, then ρ induces a C^*-norm on the quotient A/I_ρ. The completion of A/I_ρ under the induced C^*-norm $\|a + I_\rho\| = \rho(a)$ is a C^*-algebra, which we denote by $C^*_\rho(A)$. We are done since what we have really shown is that every C^*-seminorm ρ sastisfies $\rho(a) = \|\pi_\rho(a)\|$ for some $*$-homomorphism $\pi_\rho : A \to C^*_\rho(A)$ where $\pi_\rho(a) = a + I_\rho$ (i.e. the quotient map). □

Example A.1. One of the ways of generating a C^*-algebra is to start with a Banach $*$-algebra, and take its so-called **C*-enveloping algebra** (c.f. [Dix82]). The C^*-enveloping algebra is exactly the universal C^*-algebra of a given Banach $*$-algebra. Note as well that the C^*-enveloping algebra of a Banach $*$-algebra B is also called the **C*-completion** of B (c.f. [FD88]).

The following fact seems to be exploited in the literature quite commonly, and is usually referred to the 'universal property' of $C^*(A)$.

Proposition A.2. *Let A be a $*$-algebra, and $q_u : A \to A/I_u$ the quotient map as in Definition A.1. Let $\pi : A \to \mathcal{C}$ be a $*$-homomorphism where $\mathcal{C}$ is a C^*-algebra, then there exists a unique $*$-homomorphism $\widetilde{\pi} : C^*(A) \to \mathcal{C}$ such that the following diagram commutes:*

$$\begin{tikzcd} & C^*(A) \arrow[d, dashed, "\widetilde{\pi}"] \\ A \arrow[ur, "q_u"] \arrow[r, "\pi"'] & \mathcal{C} \end{tikzcd} \tag{A.1}$$

that is, $\widetilde{\pi}(q_u(a)) = \pi(a)$ for all $a \in A$.

Proof. The proof is almost trivial. We define $\widetilde{\pi}(q_u(a)) = \pi(a)$ on the dense $*$-subalgebra $q_u(A)$ of $C^*(A)$. All that remains to prove the existence (and well-definedness) is to show that $\widetilde{\pi}$ is bounded on $q_u(A)$. However, because $a \mapsto \|\pi(a)\|$ is already a C^*-seminorm on A, then it follows from the definition of the universal C^*-norm that

$$\|\widetilde{\pi}(q_u(a))\| = \|\pi(a)\| \leqslant \|a\|_u.$$

Therefore $\widetilde{\pi}$ extends to all of $C^*(A)$, and satisfies Diagram A.1. If $\widetilde{\pi}_1 : C^*(A) \to \mathcal{C}$ is another $*$-homomorphism that satisfy:

$$\widetilde{\pi}(q_u(a)) = \pi(a) = \widetilde{\pi}_1(q_u(a)), \qquad \forall a \in A,$$

then they both extend to the same $*$-homomorphism on $C^*(A)$, so $\widetilde{\pi}$ is unique. □

Remark A.2. As an abuse of notation, the canonical quotient map $q_u : A \to A/I_u$ is usually dropped, so that the universal property above reads: $\widetilde{\pi}(a) = \pi(a)$ (as if $A \subseteq C^*(A)$) for all $a \in A$. We shall keep this notation for the rest of the document.

Since the construction of a universal C^*-algebra works on any $*$-algebra (a purely algebraic object), then we may resume our quest of defining C^*-algebras out of generators and relations by the following: let $\mathcal{G}$ be a set of generators and $\mathcal{R}$ a set of $*$-algebra generators on $\mathcal{G}$, we can identify the free $*$-algebra $\mathcal{F}(G)$ to be elements of the form

$$p(g_{i_1},, ..., g_{i_d}, g_{i_1}^*, ..., g_{i_d}^*)$$

for $g_{i_1}, ..., g_{i_d} \in \mathcal{G}$ for any $d \in \mathbb{N}_{\geqslant 0}$, where p is a *noncommutative* polynomial in $2d$-variables (multiplication is essentially noncommutative polynomial multiplication). The set of generators $\mathcal{R}$, we can identify as $\mathcal{R} \subseteq \mathcal{F}(G)$ since they are always of the form:

$$p(g_{i_1}, ..., g_{i_d}, g_{i_1}^*, ..., g_{i_d}^*) = 0$$

for some noncommutative polynomial p on $2d$-variables. Denote by $J(\mathcal{R})$ the two sided ideal (as a $*$-algebra) generated by $\mathcal{R}$ with respect to the $*$-algebra $\mathcal{F}(R)$. Now, when we say that A is a $*$-algebra generated by generators and relations $\mathcal{G}$ and $\mathcal{R}$ respectively, we mean

$$A := \mathcal{F}(\mathcal{G})/J(\mathcal{R}).$$

To mimic the notation from group theory, we denote $\mathcal{F}(\mathcal{G})/J(R) = \langle \mathcal{G} | \mathcal{R} \rangle$, so that we finally have the following definition.

Definition A.2. Let $\mathcal{G}$ and $\mathcal{R}$ be a pair of generators and relations for a $*$-algebra, we define the universal C^*-algebra defined by $\mathcal{G}$ and $\mathcal{R}$ to be $C^*(\langle\mathcal{G}|\mathcal{R}\rangle)$.

We are now led to the question of existence of universal C^*-algebras, first it follows from the definition that if A is a $*$-algebra, then $C^*(A)$ exists if and only if $\|a\|_u < \infty$ for all $a \in A$.

Example A.2 (Enveloping C^*-algebras always exists). If A is a Banach $*$-algebra (not just a $*$-algebra!) with its own norm $\|\cdot\|$. It follows from Proposition A.1 and the fact that all $*$-representations of a Banach $*$-algebra is contractive [FD88, Theorem 3.7] (i.e. $\pi : A \to \mathcal{L}(H)$ satisfies $\|\pi(a)\| \leqslant \|a\|$) that for all $a \in A$:

$$\|a\|_u = \sup\{\|\pi(a)\| : \pi \text{ is a nondegenerate } *\text{-homomorphism of } A\} \leqslant \|a\| < \infty. \tag{A.2}$$

Therefore A is in particular, admissible, as in Definition A.1. Hence $C^*(A)$ exists.

Example A.3 (The Universal C^*-algebra of a C^*-algebra is itself). Perhaps a silly exercise that one may think of is to try and compute the C^*-envelope of a C^*-algebra. If A is indeed a C^*-algebra with C^*-norm $\|\cdot\|$, then it follows from definition that $\|a\| \leqslant \|a\|_u$ for all $a \in A$. In a completely similar way, as in the Banach $*$-algebra case, we also have $\|a\|_u \leqslant \|a\|$ by contractiveness of all C^*-algebra representations, therefore the universal C^*-norm on A coincides with its own C^*-norm. This implies $A = C^*(A)$ as expected.

Example A.4 (A Nonadmissible Example). To give an example of a nonadmissible $*$-algebra, consider $A := \langle a \mid a - a^*\rangle$ (or $\langle a \mid a - a^* = 0\rangle$), i.e. the $*$-algebra generated by a single self-adjoint element. For each $\lambda > 0$, we can find a C^*-algebra $\mathcal{C}_\lambda$ with a self-adjoint element x_λ such that $\|x_\lambda\|_{\mathcal{C}_\lambda} = \lambda$. Then by self-adjointness of a, the map $\pi_\lambda : a \mapsto x_\lambda$ defines a $*$-homomorphism from A to C_λ, but then

$$\|a\|_u = \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-homomorphism from } A \text{ to a } C^*\text{-algebra}\} \geqslant \sup_{\lambda>0} \|\pi_\lambda(a)\|_{\mathcal{C}_\lambda} = \infty, \tag{A.3}$$

so $\|a\|_u \nleqslant \infty$. Therefore A is not admissible.

Remark A.3. Note that while admissibility of A, i.e. $\|a\|_u < \infty$ for all $a \in A$ is enough for the existence of $C^*(A)$, it may so happen that it is trivial, that is $C^*(A) = \{0\}$ (happens when every

C^*-seminorm ρ on A is trivial). To show that one has a non-trivial universal C^*-algebra, we have to construct at least one nontrivial C^*-seminorm on A.

Let us give one more detailed example where we have an admissible $*$-algebra coming from explicitly given generators and relations. Before we proceed let us recall a common notation for C^*-algebras generated 'internally' by subsets. Suppose B is a C^*-algebra, and S is a subset in B, we define

$$C^*_B(S) := \overline{\{p(s_{i_1}, ..., s_{1_d}, s^*_{i_1}, ..., s^*_{1_d}) \in B \mid \{s_{i_k}\}_{k=1}^d \subseteq S,\ d \in \mathbb{N}_{\geqslant 0},\ p \in \mathcal{NCP}_{2d}\}}$$

where $\mathcal{NCP}_{2d}$ is the set of all polynomials of $2d$-noncommuting variables. $C^*(S)$ is a C^*_B-subalgebra of B.

Example A.5. Let $A = \langle u, 1 \mid u^*u - 1, uu^* - 1\rangle$ (or $A = \langle u, 1 \mid u^*u = 1, uu^* = 1\rangle$), that is, A is the unital $*$-algebra generated by a single unitary element. To show that this is nontrivial, let us take any C^*-seminorm $\rho : A \to \mathbb{R}_+$ on A, then we find that $\rho(1)^2 = \rho(1^*1) = \rho(1)$. Since $\rho(1) \geqslant 0$, then either $\rho(1) = 1$ or $\rho(1) = 0$. Therefore $\rho(u) = \sqrt{\rho(u^*u)} = \sqrt{\rho(1)} = 1$ or 0. If we define the $\rho(u) = 1 = \rho(1)$, then we have found a nontrivial C^*-seminorm on A, whence $C^*(A)$ exists and is nontrivial. We shall now proceed to prove the following

$$C^*(A) \cong C(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{C} \mid f \text{ is continuous}\}.$$

Note that the (multiplicative) identity on $C(\mathbb{T})$ is the constant 1 function $1_\mathbb{T} : z \mapsto 1$. Now, we can define a $*$-homomorphism $j : A \to C(\mathbb{T})$ which is fully determined by the following: $j(1) = 1_\mathbb{T}$ and $j(u) = \mathrm{Id}_\mathbb{T}$. From the universal property Proposition A.2, we have a $*$-homomorphism $\widetilde{j} : C^*(A) \to C(\mathbb{T})$ satisfying $j(1) = \widetilde{j}(1) = 1_\mathbb{T}$ and $j(u) = \widetilde{j}(u) = \mathrm{Id}_\mathbb{T}$. We are done if we can construct an inverse for $\widetilde{j}$, but all this requires is a recollection of the continuous functional calculus and the spectrum of a unitary element. Once this recollection is done, then it follows from the fact that unitarity of $\mathrm{Id}_\mathbb{T}$ implies $\mathrm{sp}(\mathrm{Id}_\mathbb{T}) = \mathbb{T}$, so that there is a $*$-isomorphism $\Psi_{\mathrm{Id}_\mathbb{T}} : C(\mathbb{T}) \to C^*_{C(\mathbb{T})}(\mathrm{Id}_\mathbb{T}, 1_\mathbb{T})$ where $\Psi_{\mathrm{Id}_\mathbb{T}}(\mathrm{Id}_\mathbb{T}) = \mathrm{Id}_\mathbb{T}$ and $\Psi_{1_\mathbb{T}} = 1_\mathbb{T}$. Now, the $*$-homomorphism $i : C^*_{C(\mathbb{T})}(\mathrm{Id}_\mathbb{T}, 1_\mathbb{T}) \to C^*(A)$ can be fully determined by $i(\mathrm{Id}_\mathbb{T}) = u$ and $i(1_\mathbb{T}) = 1$. By a simple computation, we find that $(\widetilde{j})^{-1} := i \circ \Psi_{\mathrm{Id}_\mathbb{T}} : C(\mathbb{T}) \to C^*(A)$, as required.

We remark the following regarding the almost-obvious relationship between the universal C^*-algebras and internal C^*-subalgebras generated by subsets.

Lemma A.1. *Let $\mathcal{G}$ be a set of generators and with relations $\mathcal{R}$. Suppose B is a C^*-algebra with subset $S \subseteq B$. If there is a $*$-isomorphism on the free $*$-algebras say $\psi : \mathcal{F}(\mathcal{G}) \to \mathcal{F}(S)$, then $C^*(\langle \mathcal{G} \mid \mathcal{R} \rangle) \cong C^*_B(S)$.*

Let us further discuss the last example, we found that the C^*-algebra $C^*(\langle a \mid a - a^* \rangle)$ is exactly $C(\mathbb{T})$, and with a little more work, we can generalize this result to even more unitary generators satisfying some commutativity relations. In particular for two unitary generators:

$$C^*(\langle a, b \mid a - a^*, b - b^*, ab - ba \rangle) \cong C(\mathbb{T}^2).$$

Through Gelfand's theorem, we see that, $\mathbb{T}^2$, the $2-$torus is categorically dual-equivalent to $C^*(\langle a, b \mid a - a^*, b - b^*, ab - ba \rangle)$. This motivates us to define the noncommutative 2-torus associated with $\theta \in \mathbb{R}$, by replacing the commutativity relations on a, b above: where instead of $ab - ba = 0$, we stipulate $ab - e^{2\pi i\theta} ba = 0$ for $\theta \in \mathbb{R}$. In short, if we denote by A_θ the noncommutative 2-torus associated with θ, we define

$$A_\theta := C^*(\langle a, b \mid a - a^*, b - b^*, ab - e^{2\pi i\theta} ba \rangle),$$

where of course $A_\theta \cong C(\mathbb{T}^2)$ when $\theta \in \mathbb{Z}$. We end this lengthy appendix section here, and for those that want to see the continuation of this story, then please read the main document. Subection 7.1.2 of Chapter 7 is dedicated exactly to the generalization of the noncommutative 2-torus to higher dimensions.

Appendix B

Nets

We give a brief discussion of nets here, as we shall periodically use the general principle of determining topological behavior in terms of net convergence throughout the dissertation. We assume that all ambient topological spaces here are Hausdorff, unless otherwise stated.

Definition B.1. A **directed set** is a nonempty set I together with a reflexive and transitive binary relation $\leqslant$, i.e. a **preorder**, with the additional property that every pair of elements has an upper bound.

Definition B.2. Let I be a directed set with preoder relation $\leqslant$ and X a topological space with topology T. A mapping $f : I \to X$ is said to be a **net**. For a net $f : I \to X$, we also usually write $\{x_i\}_{i\in I}$, which expresses that fact that each $i \in I$ is mapped to x_i. It is also reminiscent of the usual sequence notation.

If Y is a subset of X, we say that the net $\{x_i\}_{i\in I}$ is **eventually in** Y if there exists an $i_0 \in I$ such that for all $i \in I$ with $i_0 \leqslant i$, we have $x_i \in Y$. We can use this as a way to define the limit of a net. We say that the net $\{x_i\}_{i\in I}$ **converges towards**, or has a **limit** x and write

$$\lim_i x_i = x, \text{ or } x_i \to x$$

if for any neighborhood U_x of x, $\{x_i\}_{i\in I}$ is eventually in U_x. We say that a net is **convergent** if it has a limit.

We also have the related notion, we say that $\{x_i\}_{i\in I}$ is **frequently in** Y if for any $i \in I$, there exists a $i_0 \in I$ such that $i \leqslant i_0$ and $x_{i_0} \in Y$. We say that $\{x_i\}_{i\in I}$ **accumulates to**, or has an **accumulation point** x if for any open set U_x of x, $\{x_i\}_{i\in I}$ is frequently in U_x.

Definition B.3. Let X be a topological space, given a net $\{x_i\}_{i\in I}$ and a net $\{x'_j\}_{j\in J}$ in X, we say that $\{x'_j\}_{j\in J}$ is a **subnet** of $\{x_i\}_{i\in I}$ of there exists a function $T : J \to I$ such that

1. For all $j \in J$, $x'_j = x_{T(j)}$;

2. For each $i_2 \leqslant i_1$ in I, $T(i_2) \leqslant T(i_1)$ in J;

3. For each $i_0 \in I$, there is a $j_0 \in J$ such that $j_0 \leqslant j \implies i_0 \leqslant T(j)$.

Following the definitions, we have the following basic result.

Proposition B.1. *Given a net $\{x_i\}_{i\in I}$, every limit point of a net is an accumulation point. Furthermore, if there exists a subnet of $\{x_i\}_{i\in I}$ that converges to x, then it is an accumulation point of $\{x_i\}_{i\in I}$.*

The reason for assuming Hausdorffness of our ambient topological spaces is to preserve the uniqueness of limits for nets. However, unlike the in the case of sequences, the converse turns out to be true for nets.

Proposition B.2. *Let X be a possibly non-Hausdorff topological space. X is Hausdorff if and only if every convergent net in X has a unique limit.*

Proof. ($\implies$). Suppose X is Hausdorff and suppose for contradiction that there exist two distinct limits $x \neq y$ for $\{x_i\}_{i\in I}$. X is Hausdorff, therefore we can find disjoint open sets $x \in U_x$ and $y \in U_y$. It follows that there exists $i_x \in I$ such that $i_x \leqslant i \implies x_i \in U_x$ and $i_y \in I$ such that $i_y \leqslant i \implies x_i \in U_y$. Since I is a directed set, let $i_0 \in I$ be an upper bound for i_x and i_y so that $i_x \leqslant i_0$ and $i_y \leqslant i_0$. However, the choice of i_0 implies $x_{i_0} \in U_x \cap U_y = \varnothing$, which is a contradiction. Therefore the limit of $\{x_i\}_{i\in I}$ must be unique.

($\impliedby$). Suppose every convergent net in X has a unique limit. For contradiction, assume X is not Hausdorff, so that we can find two distinct points $x \neq y$ such that $U \cap W \neq \varnothing$ for any open sets $U \ni x$ and $W \ni y$. Define

$$I := \{(U, W) : U, W \text{ open and } x \in U,\ y \in W\}.$$

A preorder can be endowed to I via $(U_1, W_1) \leqslant (U_2, W_2) \iff U_2 \subseteq U_1$ and $W_2 \subseteq W_1$. With this order, any $(U_1, W_1), (U_2, W_2) \in I$ has an upper bound given by $(U_1 \cap U_2, W_1 \cap W_2)$. Therefore I is a directed set. We again use the fact that each $U \cap W$ with $(U, W) \in I$ is non-empty, therefore there exists a choice-function (c.f. axiom of choice) $f : I \to \bigcup_{(U,W)\in I} (U \cap W)$ with

$f(U, W) \in U \cap W$. We see that this choice-function gives us a net, we can rewrite f as the net $\{x_{U,W}\}_{(U,W) \in I}$. Our claim is that $\{x_{U,W}\}_{(U,W) \in I}$ converges to x and y, which is a contradiction. To show $x_{U,W} \to x$, we can take an arbitrary open set $U_x \ni x$, so that for any $(U, W) \in I$ satisfying $(U_x, X) \leqslant (U, W)$, we have $U \subseteq U_x$ so that:

$$x_{U,W} \in U \cap W \subseteq U_x \cap X \subseteq U_x$$

This shows $\{x_{U,W}\}_{(U,W) \in I}$ is eventually in U_x, hence $x_{U,W} \to x$. The proof for $x_{U,W} \to y$ is similar, and we obtain our contradiction. □

Below, we compile a *sequential criterion*-like characterization of topological properties. We start with something expected.

Proposition B.3. *Let X be a topological space. If a net $\{x_i\}_{i \in I}$ in X converges to x, then every subnet of X converges to x.*

Proof. Let $\{x'_j\}_{j \in J}$ be a subnet of $\{x_i\}_{i \in I}$. Let U_x be an open set of x, then we know that there exists a $i_0 \in I$ such that $i_0 \leqslant i \implies x_i \in U_x$. Let $j_0 \in J$ be the element satisfying Item 3 of Definition B.3. We then obtain that for $j \in J$ such that $j_0 \leqslant j$, then $i_0 \leqslant T(j)$, therefore $x_{T(j)} = x'_j \in U_x$. Therefore $\{x'_j\}_{j \in J}$ converges towards x. □

It also turns out that all accumulation points occur as in the latter part of Proposition B.1.

Proposition B.4. *The net $\{x_i\}_{i \in I}$ accumulates to x if and only if there exists a subnet of $\{x_i\}_{i \in I}$ that converges to x.*

Proposition B.5. *Let X be a topological space. A net $\{x_i\}_{i \in I}$ in X converges to $x \in X$ if and only if every subnet of $\{x_i\}_{i \in I}$ has a subnet that converges to x.*

Proof. The forward direction is trivially true due to Proposition B.3. To prove the converse, we notice that the latter statement, due to Proposition B.4, is equivalent to every subnet of $\{x_i\}_{i \in I}$ accumulating to x. Suppose for contradiction that $\{x_i\}_{i \in I}$ does not converge to x. By carefully negating the definition of convergence, we see that there is an open set $U_x \ni x$ such that for any $i \in I$, there is an $i_0 \in I$ such that $i \leqslant i_0$ and $x_{i_0} \notin U_x$. We can therefore define a nonempty directed set:

$$J := \{j \in I : x_j \notin U_x\},$$

which gives us a subnet $\{x_j\}_{j\in J}$ of $\{x_i\}_{i\in I}$ which does not accumulate to x. A contradiction. $\square$

Proposition B.6 ([Ped01, Proposition 1.3.6]). *Let X be a topological space with subspace Y, then $x \in \overline{Y}$ if and only if there exists a net in Y that converges to x.*

The following then immeditely follows.

Corollary B.1. *A subset Y of a topological space X is closed if and only if every convergent net in Y converges in Y.*

Proposition B.7 ([Ped01, Theorem 1.6.2]). *Let X be a topological space with subset C, then C is compact if and only if every net in C has a convergent subnet.*

Proposition B.8 ([Ped01, Proposition 1.4.3]). *Let $f : X \to Y$ be a map between topological spaces, then f is continuous if and only if for all $x_i \to x$ in X implies $f(x_i) \to f(x)$ in Y.*

Proposition B.9 ([FD88, Proposition 13.2]). *Let X, Y be topological spaces and $f : X \to Y$ a surjective map. The following are equivalent*

1. *f is an open mapping;*
2. *Given a point $x \in X$ and a net $\{y_i\}_{i\in I} \subseteq Y$ such that $y_i \to f(x)$, there exists a subnet $\{y'_j\}_{j\in J}$ based on some directed directed set J, and a net $\{x'_j\}_{j\in J} \subseteq X$ such that*
   (a) *$f(x'_j) = y'_j$ for all $j \in J$;*
   (b) *$x_j \to x$ in X.*