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Abstract

The Drell-Yan process is one of the most extensively studied phenomena in particle physics. It involves the production of a vector boson through proton-proton or proton-antiproton scattering, followed by the boson's decay into a pair of leptons. This process features two distinct energy scales: the vector boson invariant mass $Q \sim 90$ GeV and the inverse of the characteristic size of the (anti-)proton, $\Lambda_{\text{QCD}} \sim 1 \text{ fm}^{-1} \sim 1$ GeV. The scattering of a (anti-)quark or gluon from one proton with a (anti-)quark or gluon from the other proton at the electroweak vertex occurs over a much shorter timescale than the interactions among (anti-)quarks and gluons within the same proton. The former process can be calculated using standard perturbative QCD, while the latter must be treated non-perturbatively. To effectively address this multi-scale problem, factorization theorems are typically employed, allowing for a systematic separation of perturbative and non-perturbative contributions. The primary field-theoretic tool used in the derivation and proof of these factorization theorems is the Operator Product Expansion (OPE). This approach involves expressing the product of several interacting fields as an infinite series of low-energy effective operators, each multiplied by perturbatively calculable coefficients encoding short-distance physics. In the context of this thesis, the focus is on calculating matrix elements of the electroweak current between two-proton states. By applying the OPE to the electroweak vertex involving QCD fields, one obtains a series of non-local composite operators, whose dependence on the spatial separation ultimately defines the factorization theorem and the non-perturbative matrix elements of the low-energy effective operators. The assumption that free (anti-)quarks and gluons parallel to the proton beam scatter off free (anti-)quarks and gluons leads to collinear factorization and the definition of Parton Distribution Functions (PDFs) as non-perturbative matrix elements. When instead considering a small spatial separation in the transverse plane defined by the proton beam, one arrives at the Transverse Momentum Dependent Parton Distribution Functions (TMD-PDFs) as the corresponding non-perturbative distribution functions. In this thesis, we present the calculation of power corrections in Λ_{QCD}/Q to the OPE expansion within the TMDPDF factorization theorem for the Drell-Yan process. In particular we focus on kinematic power corrections that arise from expanding the low-energy effective operators around the light-cone direction, with emphasis on the geometrical twist-2 contributions, whose entire series can be resummed exactly. Additionally, we incorporate real radiative effects in the computation of the perturbative OPE coefficients. These fixed-order computations are carried out using the background field method, which allows us to decompose each QCD field into two components: a fast-moving (quantum) field, which is integrated out to yield the OPE coefficients, and a slow-moving (classical) field, which contributes to the definition of the low-energy effective operators. The background field method is particularly advantageous because it allows us to apply different gauge-fixing conditions to the quantum and classical fields, simplifying the intermediate steps. We initially present a position-space calculation demonstrating that, under suitable assumptions for the non-perturbative distributions, the convolution of the OPE coefficients with the effective operators is well-defined. However, due to the cumbersome nature

of this approach and its inefficiency for higher-order computations, we ultimately carry out the entire calculation in momentum space. To validate our approach, we have carried out the computation at next-to-leading order and, for certain contributions, at next-to-next-to-leading order. Our results demonstrate a successful cancellation of the infrared poles, and the elimination the spurious double counting of soft and collinear regions. Our numerical analysis of kinematic power corrections reveals that these are 2% to 4% larger than the electroweak power corrections reported in the literature.

Zusammenfassung

Der Drell-Yan-Prozess ist eines der am eingehendsten untersuchten Phänomene in der Teilchenphysik. Er umfasst die Erzeugung eines Vektorbosons durch Proton-Proton- oder Proton-Antiproton-Streuung, gefolgt vom Zerfall des Bosons in ein Paar von Leptonen. Dieser Prozess weist zwei unterschiedliche Energieskalen auf: die invariante Masse des Vektorbosons $Q \sim 90 \text{ GeV}$ und das Inverse der charakteristischen Größe des (Anti-)Protons, $\Lambda_{\text{QCD}} \sim 1 \text{ fm}^{-1} \sim 1 \text{ GeV}$. Die Streuung eines (Anti-)Quarks oder Gluons aus einem Proton mit einem (Anti-)Quark oder Gluon aus dem anderen Proton am elektroschwachen Vertex erfolgt auf einer viel kürzeren Zeitskala als die Wechselwirkungen zwischen (Anti-)Quarks und Gluonen innerhalb desselben Protons. Ersterer Prozess kann mit der Standard-Methodik der perturbativen QCD berechnet werden, während letzterer nicht-perturbativ behandelt werden muss. Um dieses Multiskalenproblem effektiv zu lösen, werden in der Regel Faktorisierungstheoreme verwendet, die eine systematische Trennung von perturbativen und nicht-perturbativen Beiträgen ermöglichen. Das primäre feldtheoretische Werkzeug bei der Ableitung und dem Beweis dieser Faktorisierungstheoreme ist die Operatorproduktentwicklung (OPE). Dieser Ansatz beinhaltet die Darstellung des Produkts mehrerer wechselwirkender Felder als unendliche Reihe von niederenergetischen effektiven Operatoren, die jeweils mit perturbativ berechenbaren Koeffizienten multipliziert sind, welche die Physik auf kleinen Distanzen kodieren. Im Kontext dieser Arbeit liegt der Fokus auf der Berechnung von Matrixelementen des elektroschwachen Stroms zwischen zwei Protonenzuständen. Durch Anwendung der OPE auf den elektroschwachen Vertex unter Beteiligung von QCD-Feldern erhält man eine Serie nicht-lokaler zusammengesetzter Operatoren, deren Abhängigkeit vom räumlichen Abstand letztendlich das Faktorisierungstheorem und die nicht-perturbativen Matrixelemente der niederenergetischen effektiven Operatoren definiert. Die Annahme, dass freie (Anti-)Quarks und Gluonen parallel zum Protonenstrahl mit freien (Anti-)Quarks und Gluonen streuen, führt zur kollinearen Faktorisierung und zur Definition der Partonverteilungsfunktionen (PDFs) als nicht-perturbative Matrixelemente. Betrachtet man stattdessen einen kleinen räumlichen Abstand in der Transversalebene, die durch den Protonenstrahl definiert wird, so erhält man die Transversale Impulsabhängige Partonverteilungsfunktionen (TMDPDFs) als entsprechende nicht-perturbative Verteilungsfunktionen. In dieser Arbeit präsentieren wir die Berechnung von Korrekturen höherer Ordnung der Potenzreihe in Λ_{QCD}/Q zur OPE-Entwicklung im Rahmen des TMDPDF-Faktorisierungstheorems für den Drell-Yan-Prozess. Insbesondere konzentrieren wir uns auf kinematische Potenzkorrekturen, die aus der Entwicklung der niederenergetischen effektiven Operatoren entlang des Lichtkegels entstehen, mit Schwerpunkt auf den geometrischen Twist-2-Beiträgen, deren gesamte Reihe exakt aufsummiert werden kann. Darüber hinaus integrieren wir reale Strahlungseffekte in die Berechnung der perturbativen OPE-Koeffizienten. Diese fixed-order Rechnungen werden mit der Hintergrundfeldmethode durchgeführt, die es uns ermöglicht, jedes QCD-Feld in zwei Komponenten zu zerlegen: ein schnell bewegtes (quantisiertes) Feld, das integriert wird, um die OPE-Koeffizienten zu erhalten, und ein langsam bewegtes (klassisches) Feld, das zur Definition der niederenergetischen

effektiven Operatoren beiträgt. Die Hintergrundfeldmethode ist besonders vorteilhaft, da sie es uns ermöglicht, unterschiedliche Eichfixierungsbedingungen auf die quantisierten und klassischen Felder anzuwenden, was die Zwischenschritte vereinfacht. Zunächst präsentieren wir eine Berechnung im Ortsraum, die zeigt, dass unter geeigneten Annahmen für die nicht-perturbativen Verteilungen die Faltung der OPE-Koeffizienten mit den effektiven Operatoren wohldefiniert ist. Aufgrund der umständlichen Natur dieses Ansatzes und seiner Ineffizienz für höhere Ordnungsberechnungen führen wir jedoch letztlich die gesamte Berechnung im Impulsraum durch. Zur Validierung unseres Ansatzes haben wir die Berechnung in nächstführender Ordnung und, für bestimmte Beiträge, in nächstnächstführender Ordnung durchgeführt. Unsere Ergebnisse zeigen eine erfolgreiche Kanzellierung der Infrarotpole und die Beseitigung der Doppelzählung von weichen und kollinearen Regionen. Unsere numerische Analyse der kinematischen Potenzkorrekturen zeigt, dass diese um 2% bis 4% größer sind als die elektroschwachen Potenzkorrekturen, die in der Literatur angegeben wurden.

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By convention sweet is sweet, by
convention bitter is bitter, by convention
hot is hot, by convention cold is cold, by
convention color is color. But in reality
there are atoms and the void.

Democritus

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1 Introduction

The Drell-Yan process has a long standing history. In the late 1960s, the production of massive lepton pairs in hadron-hadron scattering ($pp \rightarrow Xl^+l^-$ or $p\bar{p} \rightarrow Xl^+l^-$) was proposed as a mechanism to study intermediate vector bosons [1, 2]. The first experimental measurements of high invariant mass $\mu^+\mu^-$ pairs was performed by Christenson *et al.* in 1970 [3]. In the experimental realization, uranium nuclei were used to obtain the proton-proton collisions. The results showed the fall-off of the differential scattering cross section as the muon pair invariant mass increases. In 1970 Sidney David Drell and Tung Mow Yan proposed the process responsible for the high invariant mass lepton pair production [4, 5]. Since QCD as a theory of quark and gluon interaction was formally developed only in the mid 70's [6–11], Drell and Yan used the parton model introduced by Feynman [12] together with the infinite momentum frame analysis used previously by Bjorken in the electron-proton scattering [13, 14]. Within this framework Drell and Yan were capable to compute: the angular distribution of the leptons pair, the scaling behavior of the cross section and the linear dependence on the mass of the target nucleus. Further experimental realizations [15] backed up the Drell-Yan mechanism for high invariant mass lepton pair production. The analysis by Drell and Yan used the concept of factorization and assumed the universality of the Parton Distribution Functions (PDFs) introduced in the parton model. These two ideas have showed to be fundamental concepts in nowadays perturbative QCD computations having deep implications in our understanding of QCD.

The advent of QCD [6–11] combined with the Operator Product Expansion (OPE) techniques formalised by Kenneth Wilson and Wolfhart Zimmermann [16, 17] offered the appropriate tools to study hadronic scattering processes from a formal field theoretic point of view. Using OPE techniques in Deep Inelastic Scattering (DIS) Dokshitzer, Gribov and Lipatov [18] obtained the evolution equations for the Parton Distribution Functions; independently, Altarelli and Parisi [19] obtained similar results using the QCD improved parton model. These two results proven to be key theoretical predictions from perturbative QCD. The strength of the OPE resides in its capability to separate a multi-scale problem into different quantities, each of them with a precise definition in terms of fields of the Lagrangian under consideration. In the case of the Drell-Yan process there are mainly two scales that we can use to identify the perturbative and non-perturbative QCD regimes. The interactions between quarks and gluons at the electroweak vertex that leads to the production of a gauge vector boson take place at energies Q around 90 GeV, thus the vector boson production can be computed within the perturbative regime. On the other hand, the interactions that bind the quarks and gluons inside the protons take place at a energy scale around $\Lambda_{\text{QCD}} \sim 1 \text{ fm}^{-1} \sim 1 \text{ GeV}$, where 1 fm is the characteristic size of the proton, and therefore their description fall to the non-perturbative QCD regime. Since $\Lambda_{\text{QCD}} \ll Q$, the interactions at the electroweak vertex takes place over time periods much shorter than the interactions inside the proton, therefore at the level of the electroweak vertex one can consider that the quark/gluon from one proton that scatter the

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quark/gluon off from the second proton as *free* up to power corrections in Λ_{QCD}/Q . Thus, one can expect the Drell-Yan cross section to have the form

$$d\sigma \sim \mathcal{H}_{ab} \otimes \hat{\phi}_{b \leftarrow B} \otimes \hat{\phi}_{a \leftarrow A} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right),$$

where $\mathcal{H}_{ab}$ is a perturbative quantity that describes the interactions of two incoming partons, i.e. quark and/or gluon, in the electroweak vertex and $\hat{\phi}_{a(b) \leftarrow A(B)}$ are non-perturbative quantities. The precise definition of $\mathcal{H}_{ab}$ and $\hat{\phi}_{a(b) \leftarrow A(B)}$ determines the factorization theorem.

One of the first cases of a rigorous derivation of a factorization formula was for the inclusive Drell-Yan cross section [20–22]. In this case one sums over all final hadronic final states X imposing only that their total invariant mass p_X^2 is bounded from above $p_X^2 \leq E_{\text{C.M.}}^2(1 - \sqrt{Q^2/E_{\text{C.M.}}^2})^2$, where Q^2 is the vector boson invariant mass and $E_{\text{C.M.}}^2$ is the proton-(anti)proton center-of-mass energy. In this case the Drell-Yan cross section amounts to [20–22]

$$\begin{aligned} \frac{d\sigma_{pApB \rightarrow Xl+l^-}}{dq^2 dY} &= \sigma_{\text{Born}} \sum_{i,j} \int \frac{d\xi_a}{\xi_a} \frac{d\xi_b}{\xi_b} H_{ab}\left(\frac{x_a}{\xi_a}, \frac{x_b}{\xi_b}, Q^2, \mu^2\right) \\ &\times f_{a \leftarrow A}(\xi_a, \mu^2) f_{b \leftarrow B}(\xi_b, \mu^2) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q^2}\right), \end{aligned} \quad (1.1)$$

where the integration bounds are $x_a \leq \xi_a \leq 1$ and $x_b \leq \xi_b \leq 1$; $x_{a(b)} = \sqrt{Q^2/E_{\text{C.M.}}} e^{+(-)Y}$, where Y is the vector boson rapidity. The sum runs over the different parton flavours, i.e. $i, j = \{g, d, \bar{d}, u, \bar{u}, \dots\}$ and the hard function H_{ab} can be computed in fixed-order perturbative QCD [23–27]. The parton distribution function $f_{a(b) \leftarrow A(B)}(\xi_{a(b)}, \mu^2)$ can be thought in the parton model language as the probability of finding a (anti)quark/gluon with momentum fraction ξ'_a in the interval $\xi'_a \in [\xi_a, \xi_a + d\xi_a]$. Eq. (1.1) is known as the collinear factorization formula, see [28–33] for its version when the Drell-Yan cross section is also differential in the vector boson transverse momentum q_T . In the threshold limit, $p_X^2 \ll Q^2$ or $Q^2/E_{\text{C.M.}} \rightarrow 1$, tighter constraints are imposed on the final hadronic state X and only soft radiation is allowed. In this case, the factorization formula amounts to [34–42]

$$\begin{aligned} \frac{d\sigma}{dQ^2} &= \sigma_{\text{Born}} \sum_a H_{a\bar{a}}(Q^2, \mu^2) \int \frac{d\xi_a}{\xi_a} \frac{d\xi_b}{\xi_b} f_{a \leftarrow A}(\xi_a, \mu^2) f_{\bar{a} \leftarrow B}(\xi_b, \mu^2) \\ &\times Q S_{\text{thr}}\left[Q\left(1 - \frac{Q^2}{E_{\text{C.M.}}^2 \xi_a \xi_b}\right)\right] + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}, 1 - \frac{Q^2}{E_{\text{C.M.}}^2}\right), \end{aligned} \quad (1.2)$$

where S_{thr} is the threshold soft function and $H_{a\bar{a}}$ can be obtained from the time-like quark form factor. Notice that in eq. (1.2) the sum over parton flavors is restricted to $a, \bar{a} = \{d\bar{d}, u\bar{u}, \dots\}$.

The collinear factorization theorem in eq. (1.1) and its threshold counterpart in eq. (1.2) assume that the partons interacting in the electroweak vertex are incoming states with light-like momentum and parallel to the proton beam. In order to generalize this, one can consider a third intermediate scale $\mu_i^2 = \{|\mathbf{p}_T|^2, p^2\}$. When in addition to the vector boson invariant mass Q^2 (characteristic scale of the perturbative regime) and Λ_{QCD} (characteristic scale of the non-perturbative regime) the parton transverse momentum $|\mathbf{p}_T|^2$ is also considered and

their hierarchy is $\{\Lambda_{\text{QCD}}, |\mathbf{p}_T|\} \ll Q$, eq. (1.2) get modified to TMDPDF factorization [43–48] and the parton distribution functions generalize to Transverse Momentum Dependent Parton Distribution Functions (TMDPDFs) [46, 49–54]. In this case the partons interacting in the electroweak vertex are no longer considered to be parallel to the protons beam and instead have a small amount of momentum on the plane transverse to the protons beam. If instead one consider the parton off-shellness $\mu_i^2 = p^2$, then eq. (1.2) leads to the n -jettiness factorization [55–58] where the parton distribution functions generalize to the beam functions [58, 59]. For the case of n -jettiness there is also a generalization of eq. (1.1), see [57, 60–63], where n -central jets are allowed in the final states. In the present thesis we show that by including a suitable set of power corrections in TMDPDF factorization we can obtain a generalization of eq. (1.1) where the parton distribution functions are replaced by TMDPDFs. We also perform a numerical analysis of our results including power corrections, which is irrelevant for the phenomenological extraction of TMDPDFs from data. In the following chapters we discuss in detail the relevant formalism.

2 TMD vs. Collinear factorization

In the present chapter we will introduce the main object of study: the hadronic tensor $W_{\mu\nu}$. We will start by presenting the well-known expression of the Drell-Yan differential cross section [64–67] thereby introducing the hadronic tensor. Afterwards we will discuss the concept of *factorization* within the parton model. This first qualitative analysis will help us identify the energy scales involved in the problem under study. Moreover, it will help us set up the picture that will guide us along the next chapters where more formal objects (operators, distributions, etc.) will be introduced and more elaborated technical analysis and computations will be performed. We will then introduce two theoretical frameworks relevant for our study, namely, *TMD factorization* and *collinear factorization*. Each of these relies on different physical assumptions, which make them suitable to describe different regions of the vector boson transverse momentum spectrum in Drell-Yan processes. We will conclude the chapter by discussing the region where both approaches at leading power give less accurate results and the set of power corrections that need to be included to improve its description.

2.1 The hadronic tensor

In this section we will present the ingredients in the expression of the differential cross section for the Drell-Yan process, in order to introduce the notation and conventions we will follow in further chapters. The well-known expression for the Drell-Yan cross section is given by (see e.g. [65–67])

$$d\sigma_{p_A p_B \rightarrow l l' + X} = \frac{1}{4\sqrt{P_A \cdot P_B - M_A^2 M_B^2}} \frac{d^3 p_l}{(2\pi)^3 2E_l} \frac{d^3 p_{l'}}{(2\pi)^3 2E_{l'}} \cdot \sum_X (2\pi)^4 \delta^{(4)}(P_A + P_B - k_X - p_l - p_{l'}) |\langle p_{l'}, s'; p_l, s; X | P_A, S_A; P_B, S_B \rangle|^2 \quad (2.1)$$

where $P_{A(B)}$, $S_{A(B)}$ and $M_{A(B)}$ are the momenta, spin and mass of the colliding proton $p_{A(B)}$; $p_l(p_{l'})$ and $s(s')$ are the momenta and spin of the outgoing leptons; k_X is the momentum of unmeasured state $|X\rangle$. At leading order in the electroweak interaction, the transition amplitude $\langle p_{l'}, s'; p_l, s; X | P_A, S_A; P_B, S_B \rangle$ takes the form [65–67]

$$\langle p_{l'}, s'; p_l, s; X | P_A, S_A; P_B, S_B \rangle = -i|e|^2 \sum_G \bar{u}(p_l, s) \Gamma_\mu^G v(p_{l'}, s') \Delta_G(p_l + p_{l'}) \langle X | J_G^\mu(0) | P_A, S_A; P_B, S_B \rangle \quad (2.2)$$

where $G \in \{\gamma^*, Z, W^\pm\}$; Γ_μ^G is the Dirac structure associated to the three-body interaction $l'Gl$, i.e.

$$\Gamma_\mu^G = g_R^G \gamma_\mu (1 + \gamma_5) + g_L^G \gamma_\mu (1 - \gamma_5), \quad (2.3)$$

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$J_G^\mu(0)$ is the electroweak current involving quarks, i.e.

$$J_G^\mu(0) = \bar{q}'(0) \left[g_R^G \gamma^\mu (1 + \gamma_5) + g_L^G \gamma^\mu (1 - \gamma_5) \right] q(0), \quad (2.4)$$

with the electroweak couplings given by

$$g_R^{\gamma^*} = g_L^{\gamma^*} = \frac{e_f}{2}, \quad g_R^Z = \frac{-e_f \sin^2 \theta_W}{2 \sin \theta_W \cos \theta_W}, \quad g_L^Z = \frac{T_3 - e_f \sin^2 \theta_W}{2 \sin \theta_W \cos \theta_W}, \quad g_R^W = 0, \quad g_L^W = \frac{e_f}{2\sqrt{2}} V_{ff'}. \quad (2.5)$$

Here e_f is the electric charge of the fermion f in unit of the absolute value of the electron charge, T_3 is the third projection of the weak isospin and θ_W is the Weinberg's angle; $\Delta(p_l + p_{l'})$ is the gauge boson propagator, i.e.

$$\begin{aligned} \Delta_{\gamma^*}(p_l + p_{l'}) &= \frac{1}{(p_l + p_{l'})^2}, \quad \Delta_Z(p_l + p_{l'}) = \frac{1}{(p_l + p_{l'})^2 - M_Z^2 + iM_Z \Gamma_Z}, \\ \Delta_{W^\pm}(p_l + p_{l'}) &= \frac{1}{(p_l + p_{l'})^2 - M_W^2 + iM_W \Gamma_W}. \end{aligned} \quad (2.6)$$

Summing over the lepton final state polarizations one obtains the leptonic tensor

$$L_{\mu\nu}^{GG'} = 8 \left[(g_R^G g_R^{G'} + g_L^G g_L^{G'}) (p_{l\mu} p_{l'\nu} + p_{l\nu} p_{l'\mu} - p_l \cdot p_{l'} g_{\mu\nu}) - i(g_R^G g_R^{G'} - g_L^G g_L^{G'}) \epsilon_{\mu\nu\alpha\beta} p_l^\alpha p_{l'}^\beta \right], \quad (2.7)$$

while the hadronic tensor for the Drell-Yan process is defined as

$$\begin{aligned} W_{GG'}^{\mu\nu}(P_A, S_A; P_B, S_B; p_l + p_{l'}) &= \sum_X (2\pi)^4 \delta^{(4)}(P_A + P_B - k_X - p_l - p_{l'}) \\ &\times (\langle X | J_G^\nu(0) | P_A P_B \rangle)^\dagger \langle X | J_{G'}^\mu(0) | P_A P_B \rangle, \end{aligned} \quad (2.8)$$

where we average over the number of colors and protons polarizations. The energy-momentum conservation delta function can be written as

$$\delta^{(4)}(P_A + P_B - k_X - p_l - p_{l'}) = \int \frac{d^4 z}{(2\pi)^4} e^{iz \cdot (P_A + P_B - k_X - p_l - p_{l'})}. \quad (2.9)$$

and used in eq. ((2.8)) to obtain

$$\begin{aligned} W_{GG'}^{\mu\nu} &= \sum_X \int d^4 z e^{iz \cdot (P_A + P_B - k_X - p_l - p_{l'})} (\langle X | J_G^\nu(0) | h_A h_B \rangle)^\dagger \langle X | J_{G'}^\mu(0) | P_A P_B \rangle \\ &= \sum_X \int d^4 z e^{iz \cdot (P_A + P_B - k_X - p_l - p_{l'})} \left(\langle X | e^{-iz \cdot \hat{P}} J_G^\nu(z) e^{iz \cdot \hat{P}} | P_A P_B \rangle \right)^\dagger \langle X | J_{G'}^\mu(0) | P_A P_B \rangle \\ &= \sum_X \int d^4 z e^{-iz \cdot (p_l + p_{l'})} (\langle X | J_G^\nu(z) | P_A P_B \rangle)^\dagger \langle X | J_{G'}^\mu(0) | P_A P_B \rangle. \end{aligned} \quad (2.10)$$

In the second line we have used the transformation property of local operators $\hat{O}(z) = e^{iz \cdot \hat{P}} \hat{O}(0) e^{-iz \cdot \hat{P}}$, where z is the position and $\hat{P}$ the momentum operator. Using eqs. (2.7) and (2.10) one can write eq. (2.1) as

$$\begin{aligned} d\sigma_{p_A p_B \rightarrow l l' + X} &= \frac{|e|^4}{4\sqrt{P_A \cdot P_B - M_A^2 M_B^2}} \frac{d^3 p_l}{(2\pi)^3 2E_l} \frac{d^3 p_{l'}}{(2\pi)^3 2E_{l'}} \\ &\sum_{GG'} \Delta_G(p_l + p_{l'}) \Delta_{G'}(p_l + p_{l'}) W_{GG'}^{\mu\nu} L_{\mu\nu}^{GG'}, \end{aligned} \quad (2.11)$$

2.2. FACTORIZATION WITHIN THE PARTON MODEL

and multiplying by

$$1 = \int d^4q \delta^{(4)}(q - p_l - p_{l'}), \quad (2.12)$$

the Drell-Yan cross section differential in the gauge-boson momentum components can be easily derived,

$$\begin{aligned} \frac{d\sigma_{p_A p_B \rightarrow ll' + X}}{d^4q} = & \frac{|e|^4}{4\sqrt{P_A \cdot P_B - M_A^2 M_B^2}} \frac{d^3p_l}{(2\pi)^3 2E_l} \frac{d^3p_{l'}}{(2\pi)^3 2E_{l'}} \delta^{(4)}(q - p_l - p_{l'}) \\ & \sum_{GG'} \Delta_G(q) \Delta_{G'}(q) W_{GG'}^{\mu\nu}(q) L_{\mu\nu}^{GG'}, \end{aligned} \quad (2.13)$$

where the dependence of the hadronic tensor on the vector gauge boson momentum q has been made explicit.

Using the completeness relation $\sum_X |X\rangle\langle X| = 1$, the hadronic tensor in eq. (2.10) can also be written as [45, 64, 68]

$$W_{GG'}^{\mu\nu} = \int d^4z e^{-iz \cdot q} \langle P_B P_A | J_G^{\nu\dagger}(z) J_{G'}^\mu(0) | P_A P_B \rangle, \quad (2.14)$$

Using eq. (2.10) to define the hadronic tensor is convenient when Soft Collinear Effective Theory (SCET) [69–75] techniques are employed, and perturbative computations are performed in momentum space [43, 45, 47]. This is due to the clear and intuitive physical picture it portrays, making it suitable for its study under the QCD improved parton model assumptions [45, 76, 77]. On the other hand, eq. (2.14) appears to be more suitable for an analysis based on the Operator Product Expansion (OPE) and for calculations in position space [78–83, 83]. While the latter may look tedious and cumbersome they can be highly simplified using the *background field* method [79, 84–89].

2.2 Factorization within the parton model

The time-like momentum transfer, $q^\mu = p_l^\mu + p_{l'}^\mu$, in the proton-proton scattering process $p_A(P_A) + p_B(P_B) \rightarrow \gamma^*/Z/W^\pm(q) + X \rightarrow l(p_l) + l'(p_{l'}) + X(k)$ defines the hard scale $Q = \sqrt{q^2}$. The fundamental scattering process takes place at distance scales of the order $1/Q$. In experiments at Fermilab or the LHC typically $Q \sim 100$ GeV, which corresponds to distance scales smaller than 10^{-2} fm, two orders of magnitude smaller than the characteristic size of the proton, which is of the order $1 \text{ fm} \sim \Lambda_{\text{QCD}}^{-1} \sim 1 \text{ GeV}^{-1}$, [90]¹. This separation between long and short distance physics is the core to the *factorization theorems in QCD* [20, 21, 28, 34, 44, 45, 93].

The proton rest frame and the gauge boson rest frame are related through a (large) boost. Consider a parton with momentum k^μ within the parent proton of momentum P^μ . The

¹As pointed out firstly in the context of the parton model, [14, 91], in the boosted gauge boson rest frame, time dilation implies that the elementary interactions involving the gauge boson take place at a much smaller time scale than the interaction responsible for building up the proton [92].

physical assumption is that in the parent proton rest frame its constituents have momenta scaling like $k^\mu = (k^-, k^+, \mathbf{k}_T) \sim \Lambda_{\text{QCD}}(1, 1, \mathbf{1}_T)$, using light-cone coordinates². After applying a large boost along the z axis, the proton momentum $P^\mu = (M/\sqrt{2}, M/\sqrt{2}, \mathbf{0}_T)$ in the rest frame becomes $P'^\mu = (\kappa M/\sqrt{2}, M/(2\kappa), \mathbf{0}_T) = (P^-, M^2/2P^-, \mathbf{0}_T)$ where $\kappa = \sqrt{2}P^-/M$ and $P^- \gg M \sim \Lambda_{\text{QCD}}$ is the large light-cone component. The scaling of the parton momentum is also modified, and in the boosted frame it scales as $k^\mu \sim Q(1, M^2/Q^2, M/Q)$. The boost invariant quantity $\xi = k^-/P^-$ denotes the fractional momentum of the parton whose *probability distribution* depends on the parton flavor (up quark; down quark, gluon, etc.) and encodes information from the proton's bound-state wave function. Without loss of generality, the parton momentum k^μ that participates in the short-distance interaction can be written as $k^\mu = (\xi P^-, \frac{k^2 + \mathbf{k}_T^2}{2\xi P^-}, \mathbf{k}_T)$, where k^2 is its off-shellness and $\mathbf{k}_T$ is its transverse momentum. According to the parton model assumptions, one can approximate $k^2/P^- \simeq 0$ and neglect the transverse component as well. Therefore, in this framework, the short-distance scattering is carried out by massless on-shell particles whose momenta are parallel to the parent proton momentum.

One can see why the parton model is able to capture the main contribution to the hadronic tensor $W_{GG'}^{\mu\nu}$ by considering the vector boson rest frame, $q^\mu = (q^0, 0)$, where eq. (2.14) takes the form

$$W_{GG'}^{\mu\nu} = \int_{-\infty}^{\infty} dz^0 e^{-iz^0 q^0} \int d^3\bar{z} \langle P_B P_A | J_{G'}^\nu(z) J_G^\mu(0) | P_A P_B \rangle. \quad (2.15)$$

In the high energy limit, $Q^2 = (q^0)^2 \rightarrow \infty$, the complex exponential rapidly oscillates, thereby averaging out contributions outside the region $z^0 \simeq 0$. More precisely, the dominant contributions arise for time scales of the order $z^0 \sim 1/q^0$, in agreement with the discussion above.

In light-cone coordinates the high-energy limit $Q^2 \rightarrow \infty$ implies $2q^+q^- \rightarrow \infty$, since $Q^2 = 2q^+q^- - \mathbf{q}_T^2$. However the scaling of the transverse component is not fixed. It is the hierarchy between Λ_{QCD} , $|\mathbf{q}_T|$ and $q^\pm$ that determines which factorization framework should be used, either *TMD* or *collinear* factorization.

2.3 Collinear factorization

Collinear (or high- q_T) factorization is valid for $\Lambda_{\text{QCD}} \ll \{Q, q_T = |\mathbf{q}_T|\}$ [20, 21, 34, 44, 45, 93–99]. Since the transverse momentum of the vector boson is assumed to be much larger than any non-perturbative scale, and the transverse momentum of the partons bound in the parent proton is of the order of Λ_{QCD} , i.e. $|\mathbf{k}_{a(b)T}| \sim \Lambda_{\text{QCD}}$, momentum conservation implies that there must be further radiative processes, at short distance, that are responsible for the large transverse momentum of the gauge boson. A sketch of the main concepts in collinear factorization is given in fig. 2.1, where the short-distance (perturbative) and long-distance

²The light-cone coordinates is defined by two light-like vectors n^μ and $\bar{n}^\mu$ normalized such that $n \cdot \bar{n} = 1$. Therefo any vector v^μ can be decompose as:

$$v^\mu = v^+ n^\mu + v^- \bar{n}^\mu + v_T^\mu,$$

where $v^+ = n \cdot v$, $v^- = \bar{n} \cdot v$ and v_T^μ is orthogonal to the plane defined by n^μ and $\bar{n}^\mu$. Often for simplicity we will write v^μ as $v^\mu = (v^+, v^-, v_T^\mu)$

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(non-perturbative) contributions are factorized.

In the hard perturbative region, real radiation momenta, k_i^μ , scales as

$$\left\{ \frac{\Lambda_{\text{QCD}}}{k_i^+}, \frac{\Lambda_{\text{QCD}}}{k_i^-}, \frac{\Lambda_{\text{QCD}}}{|\mathbf{k}_{iT}|} \right\} \ll 1.$$

Otherwise real radiation has momentum scaling either collinear to the parent proton i.e. $k_c^\mu \sim Q(\lambda^2, 1, \lambda)$, anti-collinear i.e. $k_{\bar{c}}^\mu \sim Q(1, \lambda^2, \lambda)$, or soft i.e. $k_s \sim Q(\lambda, \lambda, \lambda)$. Here $\lambda = \Lambda_{\text{QCD}}/Q$.

In addition to real radiation, virtual corrections to the propagators and interaction vertices have to be considered. Contrary to the real radiation case, where the momenta of the emitted particles have defined scaling, the momenta of the virtual particles are unconstrained i.e. run from non-perturbative scales of order Λ_{QCD} up to infinity. The arising ultraviolet (UV) divergences can be removed by standard renormalization techniques [45, 64, 93, 94]. The infrared (IR) divergences arising from the soft and collinear regimes at a perturbative level get cancelled in the sum of the real and virtual contributions. If hadronic initial states are involved, non-perturbative physics plays an important role when considering the emission of soft and collinear radiation taking place at distance scales longer than the elementary electroweak interaction. This cannot be accounted for by QCD perturbation theory but it is encoded into non-perturbative matrix elements which define the parton distribution functions. The systematic separation of the perturbative coefficient, represented by the blue box in fig. 2.1, from the non-perturbative parton distribution functions, the red boxes in fig. 2.1, for both colliding protons and the soft radiation, the orange partons, is rigorously achieved within collinear factorization.

As shown in Fig. 2.1, the Initial State Radiation (ISR) $|X\rangle$ factorizes as $|X\rangle = |X_{\text{Hard}}\rangle \otimes |X_c\rangle \otimes |X_{\bar{c}}\rangle \otimes |X_{\text{Soft}}\rangle$. The state $|X_{\text{Hard}}\rangle$ represents all the ISR with momentum scaling as $Q(1, 1, 1)$, while $|X_{c(\bar{c})}\rangle$ accounts for the collinear (anti-collinear) and $|X_{\text{Soft}}\rangle$ refers to soft ISR radiation. This trivial state factorization is motivated by the large gap in rapidity existing between $|X_{\text{Hard}}\rangle$ and $|X_{c(\bar{c})}\rangle$. The hard radiation has rapidity $\hat{y}_{\text{Hard}} = \frac{1}{2} \ln \frac{k^-}{k^+} \sim 0$, for the collinear ISR $\hat{y}_c = \frac{1}{2} \ln \frac{k^-}{k^+} \rightarrow \infty$, while for the anti-collinear radiation $\hat{y}_{\bar{c}} = \frac{1}{2} \ln \frac{k^-}{k^+} \rightarrow -\infty$. Soft radiation has also rapidity $\hat{y}_{\text{Soft}} = \frac{1}{2} \ln \frac{k^-}{k^+} \sim 0$ like the hard contribution, however the two states have different invariant mass, $Q^2 \lambda^2$ vs Q^2 , respectively. Hence based on rapidity and invariant mass of the final state, one can identify four different contributions: hard, collinear, anti-collinear and soft [20, 21, 34, 44, 45, 93, 94, 100, 101]. Of course this separation is not unique since it depends on how precisely the different phase-space regions are properly defined, nonetheless the sum of the four contributions must be unique and independent of how this separation into different regions is performed. Notice also that we have not defined an orange box for the soft radiation, contrary to the hard and (anti-)collinear radiation. The reason is that due to the scaling properties of the soft radiation it is allowed to be present in the hard, collinear and anti-collinear regions, since a hard and (anti-collinear) parton can be dressed with an arbitrary number of soft partons and leaving the rapidity and invariant mass of the hard and (anti-)collinear parton unchanged. In a similar fashion the two proton states factorize, corresponding to a large rapidity gap between initial states. A rigorous

implementation of the physical picture here described and represented in fig. 2.1 leads to the collinear factorization [20, 21, 28, 34, 44, 93, 94, 96, 98–100]. Within this framework, the Drell-Yan differential cross section is given by [20, 21, 28, 34, 44, 93, 94, 96, 99, 100]

$$\begin{aligned} \frac{d\sigma_{p_A p_B \rightarrow ll' + X}}{dQ^2 dy d\mathbf{q}_T^2} &= \sigma^{\text{Born}} \sum_{G, G'} \mathcal{P}_{GG'}(\alpha_{EW}(\mu^2), Q^2, y, \mathbf{q}_T^2) \\ &\times \sum_{a, b} \int_{x_A}^1 \frac{d\xi_a}{\xi_a} \int_{x_B}^1 \frac{d\xi_b}{\xi_b} H_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b) \\ &\times f_{a \leftarrow p_A}^{\text{bare}}\left(\alpha_s(\mu^2), \frac{x_A}{\xi_a}\right) f_{b \leftarrow p_B}^{\text{bare}}\left(\alpha_s(\mu^2), \frac{x_B}{\xi_b}\right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{q_T^2}\right), \end{aligned} \quad (2.16)$$

where Q^2 , y and $\mathbf{q}_T^2$ are the invariant mass, rapidity and transverse momentum of the gauge boson respectively; $s = (P_A + P_B)^2$ is the center-of-mass energy square; $q_T = |\mathbf{q}_T|$; $0 \leq x_{A(B)} = e^{\pm} \sqrt{Q^2/s} \leq 1$; $a(b)$ runs over the parton flavors and $f_{a(b) \leftarrow h_{A(B)}}$ is the standard bare collinear parton distribution functions. The coefficient function H_{ab} can be computed perturbatively in QCD, see fig. 2.2 and 2.3 for the Feynman diagrams contributing at leading order (LO) and next-to-leading order (NLO) respectively. The coefficient $\mathcal{P}$ gathers the electroweak contribution and the leptonic tensor. For unpolarized partons and protons the factorization formula in eq. (2.16) is correct up to terms of order $\mathcal{O}(\Lambda_{\text{QCD}}^2/q_T^2)$.

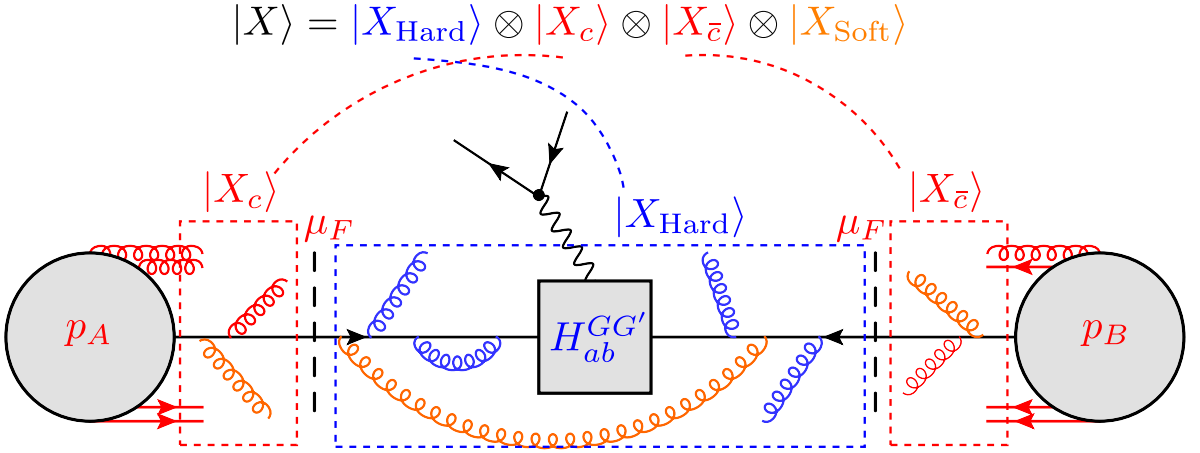


Figure 2.1: Schematic representation of collinear or high- q_T factorization. From each proton there is a massless on-shell emerging parton (quark, anti-quark, gluon). The Initial State Radiation (ISR), $|X\rangle$ is factorized into four independent contributions: $|X_{\text{Hard}}\rangle$, which interacts with the emerging partons; $|X_c\rangle$, which interacts only with the bound partons in the parent proton P_A ; $|X_{\bar{c}}\rangle$, which interacts only with bound partons in the parent hadron P_B ; $|X_{\text{Soft}}\rangle$, which is can be present in the three previous regions. The collinear and anti-collinear emerging partons have momentum $k_a^\mu = \xi_a P_A^- n^\mu$ and $k_b^\mu = \xi_b P_B^+ \bar{n}^\mu$ respectively. The scale μ_F has unit of energy and is known as factorization scale. It sets the separation between short and long distance interaction.

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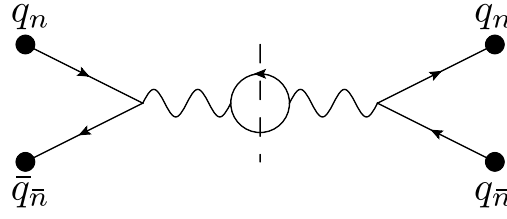


Figure 2.2: LO contribution to the partonic coefficient function $H_{ab}^{GG'}$. The quark, q_n , and anti-quark, $\bar{q}_n$, fields act as source of a emerging collinear and anti-collinear parton respectively. The cut lines represent on-shell propagators.

For kinematical configurations where $q_T^2/Q^2 \ll 1$ the hard radiation is suppressed by momentum conservation, only (anti-)collinear and soft radiation is allowed. In this situation only virtual hard modes can propagate, thus only virtual corrections to the electroweak vertex are allowed, they are represented square blob in fig. 2.1. In the lime $q_T^2/Q^2 \ll 1$ the perturbative coefficient function $H_{ab}^{GG'}$ in eq. (2.16) becomes [28, 100, 101]

$$\begin{aligned} H_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b) &= \mathcal{H}_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \xi_a, \xi_b) \delta^{(2)}(\mathbf{q}_T) \\ &+ \sum_{n=0} \alpha_s(\mu^2)^n \sum_{m=0}^{2n-1} H_{ab}^{GG'[n,m]}(Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b) \left[\frac{1}{\mathbf{q}_T^2} \log^m \frac{Q^2}{\mathbf{q}_T^2} \right]_+ \\ &+ R_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b), \end{aligned} \quad (2.17)$$

where $[f(\mathbf{q}_T^2)]_+$ denotes the usual plus-distribution of the function $f(\mathbf{q}_T^2)$ and $R_{ab}^{GG'}$ is a coefficient function less singular than $\log[Q^2/\mathbf{q}_T^2]/\mathbf{q}_T^2$ and $\delta^{(2)}(\mathbf{q}_T)$ when $\mathbf{q}_T^2/Q^2 \ll 1$ in a distributional sense i.e.

$$\lim_{\mathbf{q}_T^2/Q^2 \rightarrow 0} \int_{x_A}^1 \frac{d\xi_a}{\xi_a} \int_{x_B}^1 \frac{d\xi_b}{\xi_b} R_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b) g\left(\frac{x_A}{\xi_a}, \frac{x_B}{\xi_b}\right) \sim \text{constant} + \mathcal{O}\left(\frac{\mathbf{q}_T^2}{Q^2} \log \frac{Q^2}{\mathbf{q}_T^2}\right), \quad (2.18)$$

where $g(x_A/\xi_a, x_B/\xi_b)$ is a test function with the appropriate analytic properties. The most singular pieces in eq. (2.17) contains large logarithmic contribution of the form $\log^k Q^2/\mathbf{q}_T^2$ which can spoil the convergence properties and therefore they need to be resummed to all orders. This issue has been addressed intensively in the literature [20, 28, 45, 96, 100–103] and currently it is well understood at leading power in Λ_{QCD}/Q .

The hard function $H_{ab}^{GG'}$ in eq. (2.16) exhibits initial-state collinear singularities when massless on-shell partons are used to compute the short-distance scattering process. These divergences are absorbed into the bare parton distribution functions (PDFs), $f_{i \leftarrow h}^{\text{bare}}$. The relation between the renormalized, $\tilde{f}_{i \leftarrow p}$, and bare, $f_{i \leftarrow p}$ PDFs is given by [26, 101, 104, 105]

$$\tilde{f}_{i \leftarrow p}(x) = \sum_j \int_x^1 \frac{dz}{z} \Gamma_{i \leftarrow j}(z) f_{j \leftarrow p}\left(\frac{x}{z}\right) \quad (2.19)$$

where the index j runs over parton flavors and the kernels $\Gamma_{i \leftarrow j}$ are perturbatively calculable

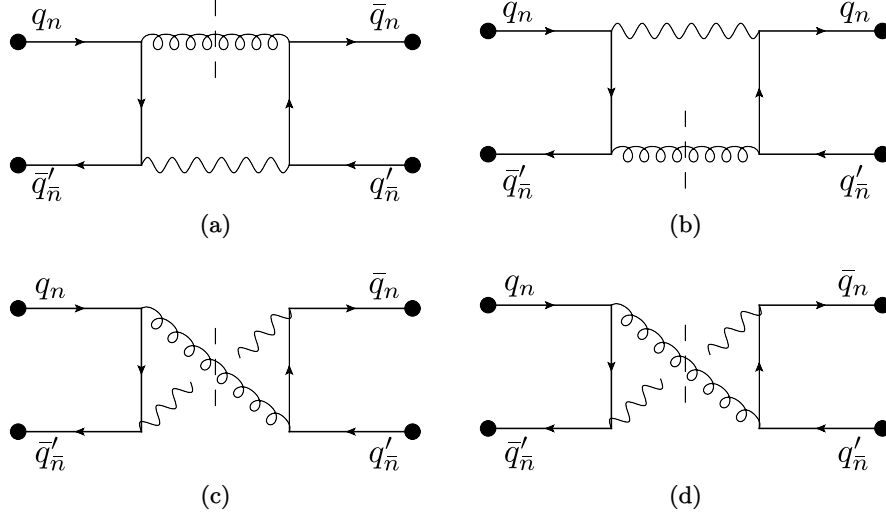


Figure 2.3: Real NLO contribution to the partonic coefficient function $H_{ab}^{GG'}$. In the crossed diagrams, (c) and (d), it is understood that the gauge boson is off-shell. For the sake of simplicity the lepton final state is not shown.

functions. In the $\overline{\text{MS}}$ -scheme these are given by [26, 97, 101, 104, 105]

$$\begin{aligned} \Gamma_{i \leftarrow j}(x) &= \delta_{ij} \delta(1-x) - \frac{\alpha_s(\mu^2)}{4\pi} \frac{P_{i \leftarrow j}^{[0]}(x)}{\epsilon} \\ &+ \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^2 \left\{ \left[\frac{1}{2\epsilon^2} \sum_k \left(P_{i \leftarrow k}^{[0]} \otimes P_{k \leftarrow j}^{[0]} \right)(x) + \beta_0 P_{i \leftarrow j}^{[0]}(x) \right] - \frac{1}{2\epsilon} P_{i \leftarrow j}^{[1]}(x) \right\} + \mathcal{O}(\alpha_s^3) \end{aligned} \quad (2.20)$$

where β_0 is the leading term in the QCD β -function; ϵ is the dimensional regularization parameter, $\epsilon = (4-d)/2$, where d is the space-time dimension; μ is the energy scale associated with dimensional regularization. The splitting functions $P_{i \leftarrow j}^{[n]}$ are the DGLAP evolution kernels [19, 94] and the Mellin convolution $\otimes$ is defined as

$$(g \otimes f)(x) = \int_x^1 \frac{dz}{z} g(z) f\left(\frac{x}{z}\right). \quad (2.21)$$

The physical interpretation of eq. (2.19) is tied to Figs. 2.1 and 2.4. When the integration over Initial State Radiation (ISR) phase-space is performed, there is a contribution that can be understood not as the radiative process off a free parton (blue box in fig. 2.1) but rather as an interaction taking place while the parton is still in a bound state (red box in fig. 2.1). As already mentioned, the separation into collinear and hard regions is not unique. There is a degree of arbitrariness captured by the arbitrary energy scale μ_F , which is known as factorization scale. Since the differential cross section must be independent of this scale, the μ_F -dependence in the PDFs must fulfill certain conditions. These are the well-known DGLAP evolution equations for the PDFs [19, 93, 94]. Thus, radiation that a certain scale μ_F is considered as a contribution to the hard scattering process, at a different scale μ'_F is absorbed into the PDF definition, see Fig. 2.4. In the following we identify μ_F with μ from dimensional

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regularization.

Using a systematic and scale-independent procedure to reabsorb the initial state collinear divergences into the PDFs, the differential Drell-Yan cross section in eq. 2.16 can be written in term of renormalized PDFs, $\tilde{f}_{i \leftarrow j}$, and the sort-distance scattering coefficient $\tilde{H}_{ij}^{GG'}$ (also known as renormalized differential partonic cross section)

$$\begin{aligned} \frac{d\sigma_{p_A p_B \rightarrow ll' + X}}{dQ^2 dy d\mathbf{q}_T^2} &= \sigma^{\text{Born}} \sum_{G, G'} \mathcal{P}_{GG'}(\alpha_{EW}(\mu^2), Q^2, y, \mathbf{q}_T^2) \\ &\times \sum_{a, b} \int_{x_A}^1 \frac{d\xi_a}{\xi_a} \int_{x_B}^1 \frac{d\xi_b}{\xi_b} \tilde{H}_{ab}^{GG'}(\alpha_s(\mu^2), Q^2, y, \mathbf{q}_T^2, \xi_a, \xi_b) \\ &\times \tilde{f}_{a \leftarrow p_A}\left(\alpha_s(\mu^2), \frac{x_A}{\xi_a}\right) \tilde{f}_{b \leftarrow p_B}\left(\alpha_s(\mu^2), \frac{x_B}{\xi_b}\right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right). \end{aligned} \quad (2.22)$$

Comparing eq. (2.16) with eq. (2.22) and using eq. (2.19) one obtains $\tilde{H}_{ij}^{GG'}$ is implicitly defined by [26, 101, 104, 105]

$$H_{ij}^{GG'}(z_i, z_j) = \sum_{a, b} \int_{z_i}^1 \frac{d\xi_a}{\xi_a} \int_{z_j}^1 \frac{d\xi_b}{\xi_b} \Gamma_{a \leftarrow i}\left(\frac{z_i}{\xi_a}\right) \tilde{H}_{ab}^{GG'}(\xi_a, \xi_b) \Gamma_{b \leftarrow j}\left(\frac{z_j}{\xi_b}\right), \quad (2.23)$$

where the left hand-side of eq. (2.23) is determined by diagrams like the ones shown in fig. 2.3.

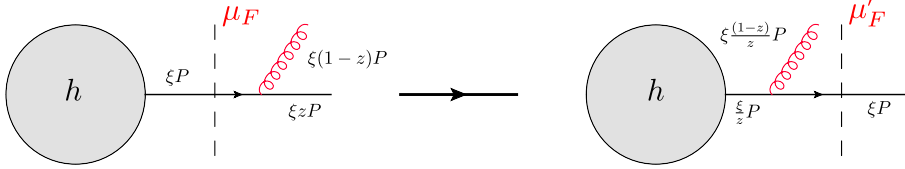


Figure 2.4: PDF renormalization. The emitted collinear gluon can be regarded as originating from an interaction with the parton as a bound state and absorbed therefore into the definition of the renormalized parton distribution function (PDFs), see eq. (2.19).

As we will discuss in details in Chapter 4 the $\mathcal{O}(\Lambda_{\text{QCD}}^2/Q^2)$ contributions to eqs. (2.16) and (2.22)³ originate from additional partonic fields that leads to effective electroweak currents involving more fields and/or their derivatives other than just the collinear and anti-collinear quark fields.

³Order $\mathcal{O}(\Lambda_{\text{QCD}}/Q)$ if we consider polarized protons.

⁴Indeed, we will show in Section 4.1 that the non-locality is responsible of the characteristic soft double pole at NLO.

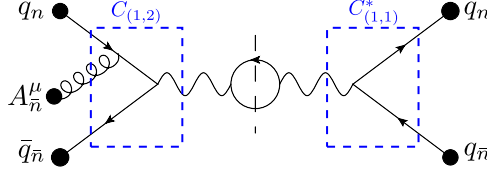


Figure 2.5: Twist-3 contribution to the partonic cross section. In the effective theory the electroweak current on the left-hand side is given by a non-local interaction vertex involving a collinear quark q_n , anti-collinear gluon $A_{\bar{n}}$ and anti-quark $\bar{q}_{\bar{n}}$ fields. The right-hand side is the h.c. of the standard collinear quark q_n and anti-collinear anti-quark $\bar{q}_{\bar{n}}$ non-local vertex interaction⁴.

2.4 A first look into TMD factorization

TMD factorization is defined by the scale hierarchy $\{\Lambda_{\text{QCD}}, q_T\} \ll Q^5$, [28, 43, 47, 53, 99, 106–111]. Contrary to collinear factorization, the transverse momentum of the vector boson can be of the same order as Λ_{QCD} and therefore comparable to the transverse momentum of the partons bound inside the proton. Thus the q_T -spectrum of the vector boson in TMD factorization is regarded as non-perturbative while in collinear factorization it is fully perturbative. This non-perturbative nature of the vector boson transverse momentum, considered to be originated by the transverse motion of the partons inside the proton before they can be regarded as free pseudo-particles, leads to the need to defined non-perturbative distributions known as Transverse Momentum Dependent Parton Distribution Functions (TMD-PDFs), $F_{a \leftarrow p_A}(\xi, \mathbf{k}_{aT}; \mu, \zeta_a)$ [50, 106, 110, 112, 113]. In a parton model analogy a TMDPDF can be interpreted as *the probability of finding a parton of type a in the proton p_A with ξ_a fraction momentum parallel to the proton momentum and a small amount of transverse component $\mathbf{k}_{aT}$ orthogonal to it*. Despite a proton model interpretation at first sight may look tempting in order to get some physical intuition, we must promptly abandon it since TMD factorization strictly speaking make no sense in the traditional parton model. The reason is simple: the massless partons involved in the electroweak interaction are no longer on-shell, since the presence of non-vanishing transverse momentum introduces a virtuality. A consequence of the assumption $q_T/Q \ll 1$ is that real emission processes at the short interaction vertex are forbidden at leading power in q_T/Q since they can dynamically generate $q_T \sim Q$. This is schematically represented in fig. 2.6, where any radiative process takes places below the factorization scale. Above this scale only virtual hard contributions are present, i.e. there are no hard modes as external states since we have integrated them out from our theory leading to the Drell-Yan matching coefficient represented by the square blob in fig. 2.6. The presence of these virtual corrections leads to UV and IR (collinear and soft) poles that are regularized using dimensional regularization. The UV divergences can be removed using standard renormalization methods, while the IR divergences are reabsorbed into the TMDPDFs. It is this redefinition of the TMDPDFs that introduces a dependence on the factorization scale μ_F that is typically set equal to the energy scale μ used in dimensional regularization [45, 51, 110, 112, 114–118].

Another consequence of considering a non-vanishing transverse momentum for the partons

⁵As in the collinear case, we define $q_T = |\mathbf{q}_T|$.

2.4. A FIRST LOOK INTO TMD FACTORIZATION

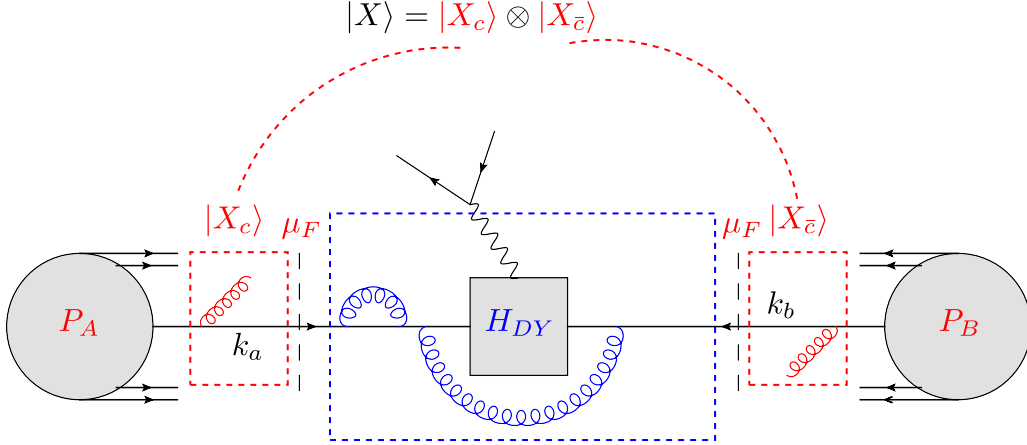


Figure 2.6: Schematic representation TMD factorization. Only virtual hard particles are allowed to interact with the effective vertex (square blob), leading to the Drell-Yan matching between the short distance scale and the non-perturbative contributions. Above the factorization scale μ_F , no radiative process takes place. The partons a and b have momentum $k_a^\mu = \xi_a P_A^- n^\mu + \mathbf{k}_{aT}^\mu$ and $k_b^\mu = \xi_b P_B^+ \bar{n}^\mu + \mathbf{k}_{bT}^\mu$, respectively. The collinear $|X_c\rangle$ and anti-collinear state $|X_{\bar{c}}\rangle$ also include soft radiation.

involved in the short distance scattering is the presence of rapidity divergences [48, 106, 119, 120]. In collinear factorization the soft radiation has a transverse momentum much smaller than the measured $\mathbf{q}_T$, therefore no constraint is imposed in transverse component in the soft phase space integrals. For instance, an emission of a real soft gluon at leading power (LP) leads to a phase space integral of the form

$$I_{\text{Soft}}^{\text{high-}q_T} = \int_0^\infty dk^+ dk^- \int d^{d-2} \mathbf{k}_T \frac{1}{k^+} \frac{1}{k^-} \delta(k^2) = \frac{\pi^{\frac{d-2}{2}}}{\Gamma[\frac{d-2}{2}]} \mu_F^{2\epsilon} \int_0^\infty dk^+ dk^- \frac{(2k^+ k^-)^{-\epsilon}}{k^- k^+} = 0 \quad (2.24)$$

since it is a scaleless integral. At variance, in TMD factorization the measured $\mathbf{q}_T$ can be of the same order as the transverse momentum of the soft and collinear partons, therefore a measurement function is introduced in the soft phase-space integrals. In the case of one soft gluon emission at LP we have a contribution of the form

$$\begin{aligned} I_{\text{Soft}}^{\text{TMD}} &= \int_0^\infty dk^+ dk^- \int d^{d-2} \mathbf{k}_T \frac{1}{k^+} \frac{1}{k^-} \delta(k^2) \delta^{(d-2)}(\mathbf{k}_T + \mathbf{q}_T) \\ &= \frac{1}{\mathbf{q}_T^2} \int_0^\infty dk^+ \frac{1}{k^+}, \end{aligned} \quad (2.25)$$

which is ill defined. In this case dimensional regularization is not enough to regularize the UV and IR poles. This issue was noticed already in the pioneering work of Collins, Soper and Sterman [20, 28, 34, 45, 93, 121], being the main problem that hindered the development of a TMD factorization theorem at LP. The issue is now well understood and calculations have been performed using multiple regularization schemes [28, 47, 50, 122, 122, 123] and currently the soft contribution at LP to the Drell-Yan process is known up to N4LO [124–126]. Employing SCET terminology, the collinear factorization deals with SCET-I observables while

CHAPTER 2. TMD VS. COLLINEAR FACTORIZATION

TMD factorization is a SCET-II problem [43, 47, 48, 106]. Either way, the need to introduce a new regulator for rapidity divergences leads to TMDPDFs that depend on the additional scale ζ with a characteristic double scale evolution [51, 114].

The schematic picture in fig. 2.6 can be formalized into a factorization theorem at leading power (LP) either using Green functions and diagrammatic techniques [28, 45, 99] or SCET based methods [43, 47, 107]. In both cases the factorization formula at LP amounts to

$$\begin{aligned} \frac{d\sigma_{p_A p_B \rightarrow l l' + X}}{dQ^2 dy d\mathbf{q}_T^2} &= \sigma^{\text{Born}} H_{\text{DY}} \left(\alpha_s(\mu^2), \log \frac{Q^2}{\mu^2} \right) \\ &\times \sum_{G, G'} \mathcal{P}_{GG'}(\alpha_{EW}(\mu^2), Q^2, y, \mathbf{q}_T^2) \sum_{a, b} \int \frac{d^2 \mathbf{b}_T}{(2\pi)^2} e^{i \mathbf{b}_T \cdot \mathbf{q}_T} \\ &\times F_{a \leftarrow p_A}(\alpha_s(\mu^2), x_A, \mathbf{b}_T, \mu^2, \zeta_A) F_{b \leftarrow p_B}(\alpha_s(\mu^2), x_B, \mathbf{b}_T, \mu^2, \zeta_B) + \mathcal{O} \left(\frac{q_T^2}{Q^2} \right), \end{aligned} \quad (2.26)$$

where Q , y and $\mathbf{q}_T$ are the vector boson invariant mass, rapidity and transverse momentum respectively; $x_{A(B)} = \sqrt{Q^2 + \mathbf{q}_T^2} / \sqrt{s} e^{\pm y}$, where s is the center of mass energy; $\mathbf{b}_T$ is the impact parameter; $\zeta_{A(B)}$ is the energy scale associated with the renormalization of rapidity divergences and $\mathcal{O}(q_T^2/Q^2)$ stands for higher-order corrections to the unpolarized Drell-Yan differential cross section. The coefficient function H_{DY} gather the short-distance virtual corrections to the electroweak vertex, leading to the effective vertex. The functions $F_{a(b) \leftarrow P_{A(B)}}$ are the renormalized TMDPDFs and we gather in $\mathcal{P}$ the leptonic tensor together with the electroweak prefactors.

In the case $|\mathbf{b}_T| \Lambda_{\text{QCD}} \ll 1$ (or in momentum space $\Lambda_{\text{QCD}}/|\mathbf{q}_T| \ll 1$) the TMDPDFs can be further matched onto standard collinear PDFs,

$$F_{a \leftarrow p_A}(\alpha_s(\mu^2), x_A, \mathbf{b}_T, \mu, \zeta_A) = \sum_i \int_{x_A}^1 \frac{d\xi}{\xi} C_{a \leftarrow i}(\alpha_s(\mu^2), \xi, \mathbf{b}_T, \mu, \zeta_A) f_{i \leftarrow p_A} \left(\alpha_s(\mu^2), \frac{\xi}{x_A} \right) + \mathcal{O}(\Lambda_{\text{QCD}} |\mathbf{b}_T|) \quad (2.27)$$

and similarly for $F_{b \leftarrow p_B}$. The matching coefficient in eq. (2.27) have been computed by several groups at NNLO using different rapidity regulators [50, 127, 128] and currently are known up to N3LO [129]. In Chapter 4 we will calculate in detail the matching procedure at LP and next-to-next leading power (NNLP), being this last one a new result up to our knowledge.

2.5 TMD vs. Collinear factorization

According to the previous discussions, collinear (or high- q_T) factorization is expected to provide reliable theory predictions in the kinematic region $\Lambda_{\text{QCD}} \ll \{|\mathbf{q}_T|, Q\}$, while the TMD factorization range of applicability is for $\{\Lambda_{\text{QCD}}, |\mathbf{q}_T|\} \ll Q$. We must notice that nothing is assumed about the relation either between $|\mathbf{q}_T|$ and Q in collinear factorization or Λ_{QCD} and $|\mathbf{q}_T|$ in TMD factorization. Therefore one open the question remains: what factorization theorem is appropriate in the intermediate region $\Lambda_{\text{QCD}} \ll |\mathbf{q}_T| \ll Q$ [53]? In this section we will analyze different methods used to combine the two factorization frameworks in order to

2.5. TMD VS. COLLINEAR FACTORIZATION

obtain a unique prediction in the intermediate region.

Eq. (2.16) in collinear factorization is correct up to power corrections in Λ_{QCD}/q_T . In general the structure functions have an expansion of the form

$$W_i = \frac{1}{M^2} \sum_{n=2}^{\infty} \left[\frac{M}{q_T} \right]^n \tilde{F}^{[n]} \left(\frac{q_T}{Q} \right), \quad (2.28)$$

where W_i is the projection of the hadronic tensor $W^{\mu\nu}$ on a certain Lorentz structure $T_i^{\mu\nu}$, where $\{T_i^{\mu\nu}\}$ form a basis (not necessarily orthogonal); M is the proton mass ($M \sim \Lambda_{\text{QCD}}$) and $\tilde{F}^{[n]}$ is a dimensionless structure function (without the $1/M^2$ factor above it would have unit of inverse mass squared). For sake of simplicity we omit Lorentz and electroweak indices and any other dependence on $\alpha_s(\mu^2)$. The $n = 2$ term corresponds with the hadronic tensor contribution in eq. (2.16), where the NLO real contributions to $\tilde{F}^{[2]}$ are shown in fig. 2.3. The expansion in powers of M/q_T generates logarithmically enhanced contributions, i.e. terms of the type $\log^l M^2/q_T^2$ with $n - 1 \geq l \geq 0$. These contributions can be properly resummed through a matching procedure onto collinear PDFs: the structure function $\tilde{F}^{[2]}$ can be written as the Mellin convolution of a perturbative partonic cross section times the standard collinear PDFs [20, 21, 44, 45, 53, 95–97], where the logarithmic contributions are resummed evolving the PDFs according to the DGLAP equations [19]. Higher-order terms, $n \geq 3$, have two origins: higher-twist⁶ collinear PDF [79, 130–133] and derivatives of lower-twist collinear PDF. Therefore the structure function $\tilde{F}^{[3]}$ has two contributions: one is purely twist-3, i.e. it is the convolution of a perturbative coefficient times a genuine twist-3 collinear PDF with whole new evolution equations [79, 84]; the second contribution is the result of the matching onto a derivative of a twist-2 collinear PDF. Notice that while at $n = 2$ we also use the term *partonic cross section* for the perturbative matching coefficient, as reminiscent of historical developments [19], for $n \geq 3$ we avoid such terminology, since a parton interpretation is no longer possible. If the transverse momentum q_T becomes small compared to the invariant mass Q , $\Lambda_{\text{QCD}} \sim M \ll q_T \ll Q$, eq. (2.28) admits a second expansion in powers of q_T/Q , known as the collinear expansion

$$W = \frac{1}{M^2} \sum_{n=2}^{\infty} \left[\frac{M}{q_T} \right]^n \sum_{k=2}^{\infty} \tilde{F}^{[n,k]} \left[\frac{q_T}{Q} \right]^{k-2}, \quad (2.29)$$

where k denotes the collinear twist. As a consequence of the expansion in powers of $q_T/Q \ll 1$, the coefficients $\tilde{F}^{[n,k]}$ is affected by logarithmically enhanced contributions of the form $\log^m \frac{q_T^2}{Q^2}$, where $k - 1 \geq m \geq 0$. These contributions cannot be resummed by means of the DGLAP evolution equations, and a different resummation procedure needs to be adopted. For the leading contribution in $\tilde{F}^{[2,2]}$ it is the Sudakov factor that is responsible for the resummation of these large logarithms [20, 34, 45, 93, 94, 97, 101–103, 134]. The question whether for $\tilde{F}^{[2,k]}$ with $k \geq 3$ it is still possible to resum through a Sudakov-like factor is subject of intense studies, see [135–144] for the case of the Drell-Yan process at threshold.

The TMD factorization formula in eq. (2.26) is correct up to power corrections in q_T^2/Q^2 [28,

⁶We will define more precisely the word *twist* in Sections 3.2 and ??.

43, 47, 53, 106–109]⁷. Therefore the general expansion of a structure function W_i ⁸ in powers of q_T/Q takes the form

$$W_i = \frac{1}{M^2} \sum_{k=2}^{\infty} \left[\frac{q_T}{Q} \right]^{k-2} F^{[k]} \left(\frac{M}{q_T} \right) \quad (2.30)$$

where $F^{[k]}$ is a dimensionless structure function. The index k denotes the order of the collinear twist. The expansion in powers of q_T/Q leads to large logarithms of the form $\log^m \frac{q_T^2}{Q^2}$, which can spoil the pattern of convergence in powers of $\alpha_s(\mu^2)$. In order to resum these contributions, the structure functions $F^{[k]}$ are factorized into two terms: a hard matching coefficient and the TMDPDFs. For instance, eq. (2.26) represents the collinear twist-2 ($k=2$) case, where the TMDPDFs are the matrix elements of collinear twist-2 operators. If the transverse momentum q_T becomes much larger than Λ_{QCD} , the structure functions $F^{[k]}$ admit a second expansion, so that in the region $\Lambda_{\text{QCD}} \ll q_T \ll Q$ the structure function W_i takes the general form

$$W_i = \frac{1}{M^2} \sum_{k=2}^{\infty} \left[\frac{q_T}{Q} \right]^{k-2} \sum_{n=2}^{\infty} F^{[k,n]} \left[\frac{M}{q_T} \right]^n. \quad (2.31)$$

In large q_T -limit, $M/q_T \ll 1$, the TMDPDF matrix elements are matched onto collinear PDFs. This expansion generates logarithmically enhanced terms which are resummed through the PDF evolution equations. In the case of $n=2$ the TMDPDF matrix elements match onto standard collinear PDFs of twist-2. Since both factorization frameworks match onto collinear PDFs in the intermediate region $M \ll q_T \ll Q$, one can compare $\tilde{F}^{[n,k]}$ with $F^{[k,n]}$ in order to determine under which conditions there is a smooth transition between both factorizations theorems and whether it is possible to combine both into a single formula in order to describe the q_T -spectrum in Drell-Yan.

Let us start by writing explicitly the first few terms in eq. (2.29)

$$\begin{aligned} W_i = & \frac{1}{q_T^2} \left[\tilde{F}^{[2,2]} + \frac{q_T}{Q} \tilde{F}^{[2,3]} + \frac{q_T^2}{Q^2} \tilde{F}^{[2,4]} + \dots \right] \\ & + \frac{1}{q_T^2} \frac{M}{q_T} \left[\tilde{F}^{[3,2]} + \frac{q_T}{Q} \tilde{F}^{[3,3]} + \frac{q_T^2}{Q^2} \tilde{F}^{[3,4]} + \dots \right] \\ & + \frac{1}{q_T^2} \frac{M^2}{q_T^2} \left[\tilde{F}^{[4,2]} + \frac{q_T}{Q} \tilde{F}^{[4,3]} + \frac{q_T^2}{Q^2} \tilde{F}^{[4,4]} + \dots \right] \\ & + \dots, \end{aligned} \quad (2.32)$$

⁷In case the proton, partons or both are polarized the first corrections occur at q_T/Q .

⁸Here W_i must be regarded as the same presented in eq. (2.28).

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which can be rearranged as

$$\begin{aligned}
W_i = & \frac{1}{q_T^2} \left[\tilde{F}^{[2,2]} + \frac{M}{q_T} \tilde{F}^{[3,2]} + \frac{M^2}{q_T^2} \tilde{F}^{[4,2]} + \dots \right] \\
& + \frac{1}{q_T^2} \frac{q_T}{Q} \left[\tilde{F}^{[2,3]} + \frac{M}{Q} \tilde{F}^{[3,3]} + \frac{M^2}{q_T^2} \tilde{F}^{[4,3]} + \dots \right] \\
& + \frac{1}{q_T^2} \frac{q_T^2}{Q^2} \left[\tilde{F}^{[2,4]} + \frac{M}{q_T} \tilde{F}^{[3,4]} + \frac{M^2}{q_T^2} \tilde{F}^{[4,4]} + \dots \right] \\
& + \dots .
\end{aligned} \tag{2.33}$$

The first line in eq. (2.33) represents the contributions of collinear twist-2, $k = 2$, and different collinear PDFs twist. For instance the term $\tilde{F}^{[3,2]}$, has two contributions: one is given by the matching of a coefficient function of collinear twist-2 onto a PDFs of twist-3 and the other contribution is due to the derivative of a twist-2 PDFs present in $\tilde{F}^{[2,2]}$. A similar analysis applies to the second, third and further lines in eq. (2.33) with the only difference that the collinear twist increases with each line.

The expansion in eq. (2.31) takes an analogous form to eq. (2.33)

$$\begin{aligned}
W = & \frac{1}{q_T^2} \left[F^{[2,2]} + \frac{M}{q_T} F^{[2,3]} + \frac{M^2}{q_T^2} F^{[2,4]} + \dots \right] \\
& + \frac{1}{q_T^2} \frac{q_T}{Q} \left[F^{[3,2]} + \frac{M}{q_T} F^{[3,3]} + \frac{M^2}{q_T^2} F^{[3,4]} + \dots \right] \\
& + \frac{1}{q_T^2} \frac{q_T^2}{Q^2} \left[F^{[4,2]} + \frac{M}{q_T} F^{[4,3]} + \frac{M^2}{q_T^2} F^{[4,4]} + \dots \right] \\
& + \dots ,
\end{aligned} \tag{2.34}$$

where the first coefficient on the first line, $F^{[2,2]}$, corresponds to the hadronic contribution in eq. (2.26). The second term on the first line gathers two contributions: the matching onto PDFs of twist-3 and a derivative of $F^{[2,2]}$. The coefficient $F^{[2,4]}$ gathers three contributions: the matching onto PDFs of twist-4, a derivative of $F^{[2,3]}$ and a second derivative of $F^{[2,2]}$. A similar analysis applies to the second, third and further lines, with the only difference that the collinear twist increases by one unit in each further line.

The analysis in power of q_T/Q and M/q_T in the intermediate region, $\Lambda_{\text{QCD}} \ll q_T \ll Q$ is completely analogous in eqs. (2.33) and (2.34). Thus both factorization theorems agree and there exists a smooth interpolation between them if and only if the coefficient functions $\tilde{F}^{[n,k]} = F^{[k,n]}$. However the comparison of both factorization theorems is not straightforward, this is due to two issues: the gauge fixing and the present of pinch singularities (i.e. rapidity singularities due to soft radiation). Both issues are indeed entangled since the choice of a physical gauge (like the axial gauge or the light-cone gauge) or a covariant one (like the Feynman-Fermi or Landau gauge) affects the singularities structure and therefore different gauges requires different regularization procedure of the rapidity singularities (pinch singularities). Nonetheless the final expression for the cross section must be independent of what

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gauge and regularization procedure is used. All these issues are discussed in details in [53], where is shown at NLO that after a consistent treatment of the gauge fixing and rapidity singularities the coefficient functions $\tilde{F}^{[n,k]}$ and $F^{[k,n]}$ agree. In particular, the transition from TMDPDF to collinear factorization region in the case of unpolarized partons in unpolarized protons has been studied at NNLO+NNLL and N3LO+N3LL [145–148] through rather different numerical matching procedures. The common conclusion of all these works is that it is possible to obtain a smooth transition between the two regions in the q_T -spectrum in Drell-Yan.

3 QCD, Background field method and OPE

In this chapter, we present the QCD Lagrangian and its modifications in the background field method. A thorough derivation of the QCD Lagrangian in the background field method it is presented in [85–89]. The construction of QCD Lagrangian is a standard textbook matter [64, 68, 149]; here we follow the notation from [64]. Finally we introduce the main ideas of the short-distance and light-cone Operator Product Expansion (OPE) and we present the computation of the short-distance contribution to the quark propagator in the presence of a background gauge field and how at leading power the propagator acquires a non-local gauge factor i.e. the Wilson line.

3.1 QCD Lagrangian

The basic building blocks of the QCD Lagrangian are the quark $\psi_q^a(z)$ and gluon $B_\mu^a(z)$ fields.¹ In the construction of QCD there are two major constraints: local gauge invariance and covariance. Covariance requires to embed gluons, which are an irreducible representation of the inhomogeneous Lorentz group or Poincaré group with helicity one and therefore two physical degrees of freedom, into a vector field of four components. On the other hand, local gauge invariance requires that the Lagrangian is invariant under the finite local gauge transformation

$$\psi_q(z) \rightarrow U(\theta(z))\psi_q(z), \quad (3.1)$$

$$B_\mu(z) \rightarrow U(\theta(z))B_\mu(z)U^\dagger(\theta(z)) + \frac{i}{g}U(\partial_\mu U^\dagger(z)), \quad (3.2)$$

where $U(\theta(z))$ is a unitary transformation given by

$$U(\theta(z)) = \exp \left\{ -ig \sum_{a=1}^8 \theta^a(z) T^a \right\} \quad (3.3)$$

where g is the coupling constant, $\theta_a(z)$ are the transformation parameters which depend on the space-time position and T^a are $SU(3)$ generators. In the fundamental representation they are $(T^a)_{bc} = (\lambda^a)_{bc}/2$, where λ^a are the Gell-Mann matrices. In the adjoint representation, $(T^a)_{bc} = -if^{abc} = -if_{abc}$, where f^{abc} are the group structure constants. The quark fields are given in the fundamental representation, while the gluon fields are given in the adjoint². For

¹The superscript a and subscript q are color and flavor indices respectively.

²Notice that the gauge transformation for the gluons fields in eq. (3.2) can generate a gauge field when initially $B_\mu = 0$, (pure gauge configuration).

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an infinitesimal value of $\theta_a(z)$ the local gauge transformation becomes

$$\begin{aligned}\psi^a(z) &\rightarrow \psi^a(z) - ig \sum_{b=1}^8 \sum_{c=1}^3 \theta_a(z) t_{ac}^b \psi^c(z), \\ B_\mu^a(z) &\rightarrow B_\mu^a(z) + g \sum_{b,c=1}^{N^2-1} f_{abc} \theta_b(z) B_\mu^c(z) - \partial_\mu \theta_a(z).\end{aligned}\tag{3.4}$$

The most general Lorentz-invariant and renormalizable Lagrangian invariant under the transformations. (3.4) is [64, 68, 149]

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \sum_q \{i\bar{\psi}_q(z) \not{D} \psi_q(z) - m_q \bar{\psi}_q \psi_q(z)\} - \frac{1}{4} \sum_{a=1}^8 F_a^{\mu\nu}(z) F_{\mu\nu}^a(z) \\ &\quad + \bar{\theta} \epsilon^{\mu\nu\alpha\beta} \sum_{a=1}^8 F_{\mu\nu}^a F_{\alpha\beta}^a\end{aligned}\tag{3.5}$$

where q runs over the quark flavors, m_q is the mass of the quark with flavor q , $\bar{\psi}_q = \psi_q^\dagger \gamma^0$ is the Feynman conjugate, $F_{\mu\nu}^a$ is the field strength tensor

$$F_{\mu\nu}^a(z) \equiv \partial_\mu B_\nu^a(z) - \partial_\nu B_\mu^a(z) + g \sum_{b,c=1}^{N^2-1} f_{abc} B_\mu^b(z) B_\nu^c(z),\tag{3.6}$$

D_μ is the covariant derivative in the fundamental representation of $SU(3)$

$$\begin{aligned}(D_\mu)_{bc} &= \delta_{bc} \partial_\mu - ig \sum_{a=1}^8 B_\mu^a(z) t_{bc}^a, \\ D_\mu \psi^b(z) &= \sum_{c=1}^3 \{\delta_{bc} \partial_\mu - ig \sum_{a=1}^8 B_\mu^a(z) t_{bc}^a\} \psi^c(z),\end{aligned}\tag{3.7}$$

The last term in eq. (3.5) is known as the strong CP term. It amounts to a total derivative $\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^a F_{\alpha\beta}^a = \partial_\mu (\epsilon^{\mu\nu\alpha\beta} A_\nu^a F_{\alpha\beta}^a)$ [64, 149] and will be neglected in the following discussion, which focus on QCD perturbation theory.

Non-Abelian local gauge invariance is an extremely important property. Firstly, it demands the coupling constant between quarks and gluons to be the same as the one of the gluons among themselves. Secondly, 't Hooft [6] proved that gauge invariant non-Abelian gauge theories are renormalizable. Furthermore, thanks to non-abelian gauge invariance, QCD is asymptotically free, Coleman and Gross [150]. The equations of motions of QCD can be obtained from eq. (3.5) by performing variations of the actions, $S_{\text{QCD}} = \int d^4z \mathcal{L}_{\text{QCD}}$, with respect a given field. For the quarks one obtains

$$\bar{\psi}_q(z) (i \overleftarrow{D} + m_q) = 0, \quad (i \not{D} - m_q) \psi_q(z) = 0,\tag{3.8}$$

where $(\overleftarrow{D}_\mu)_{bc} = \delta_{bc} \overleftarrow{\partial}_\mu + ig \sum_a t_{bc}^a B_\mu^a(z)$, while for the gluons one obtains

$$D_\mu F_a^{\mu\nu} \equiv \partial_\mu F_a^{\mu\nu} + g \sum_{bc=1}^{N^2-1} f_{abc} B_{b\mu}(z) F_c^{\mu\nu}(z) = 0,\tag{3.9}$$

where the covariant derivative is in the adjoint representation.

The redundancies in the field B_μ due to the gauge freedom lead to well-known subtleties at the level of the quantization of the theory. One can use covariant gauge fixing conditions (like the Feynman gauge or the Landau gauge) in order to preserve the explicit covariance of eq. (3.5), but the unitarity of the S matrix and the gauge invariance of eq. (3.5) is lost. This issue was first noticed by DeWitt [151] and Feynman [152] and was later solved by Faddeev and Popov [153], by introducing additional non-physical fields (ghost) to restore them the unitarity of the S matrix and the local gauge symmetry of eq. 3.5. Alternatively, one could work with physical gauges which are free of ghost fields, but the QCD Lagrangian would no longer be manifestly covariant. Some examples are: the Fock-Schwinger gauge $z_\mu B^\mu(z) = 0$ [154], which breaks translational invariance, and the *axial gauge* $n \cdot B = 0$, where $n^2 \leq 0$ [155–159]; in the particular case of $n^2 = 0$ it is refereed as *light-like gauge* [160]. The vector n^μ must be understood as an *external vector* i.e. it is the same regardless what frame we are³.

In the present work we employ covariant gauge fixing conditions and the QCD lagrangian is given by [64, 68, 149]

$$\begin{aligned} \mathcal{L}_{\text{QCD}} = & \sum_q \{i\bar{\psi}_q(z)\not{D}\psi_q(z) - m_q\bar{\psi}_q\psi_q(z)\} - \frac{1}{4} \sum_{a=1}^8 F_a^{\mu\nu}(z)F_{\mu\nu}^a(z) \\ & - \frac{1}{2\xi} \sum_{a=1}^8 (\partial_\mu B_a^\mu(z))^2 + \sum_{a,b,c=1}^8 (\partial^\mu \bar{\omega}^a(z))(\delta_{ab}\partial_\mu + g f_{acb}B_\mu^c(z))\omega^b(z) \\ & + \bar{\theta}\epsilon^{\mu\nu\alpha\beta} \sum_{a=1}^8 F_{\mu\nu}^a F_{\alpha\beta}^a \end{aligned} \quad (3.10)$$

where the first and second term on the second line of eq. (3.10) are the covariant gauge-fixing and the ghost Lagrangian respectively, ω and $\bar{\omega}$ are the ghost fields in the adjoint representation. From eq. (3.10) one can obtain the following Feynman rules in momentum space which are relevant for the perturbative calculations in Chapter. 4

³One of the advantages of the background field method is the possibility of choosing different gauge fixing conditions for the quantum fields and the background classical fields.

External lines:

$$\text{ingoing quark} = \longrightarrow^k \bullet = u(k, s)$$

$$\text{ingoing anti-quark} = \xleftarrow{-k} \bullet = \bar{v}(k, s)$$

$$\text{outgoing quark} = \bullet \longrightarrow^k = \bar{u}(k, s)$$

$$\text{outgoing anti-quark} = \bullet \xleftarrow{-k} = v(k, s)$$

$$\text{ingoing gluon} = \curvearrowright^k \bullet = \epsilon^\mu(k, \lambda)$$

$$\text{outgoing gluon} = \bullet \curvearrowleft^k = \epsilon^{\mu*}(k, \lambda)$$

(3.11)

Propagators:

$$\text{gluon propagator} = \bullet \text{oooooo} \bullet \stackrel{b, \nu}{\underset{a, \mu}{\xrightarrow{k}}} = \frac{-i\delta_{ab} \left[g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right]}{k^2 + i0}, \quad \xi = \begin{cases} 1, & \text{Feynman-Fermi gauge} \\ 0, & \text{Landau gauge} \end{cases}$$

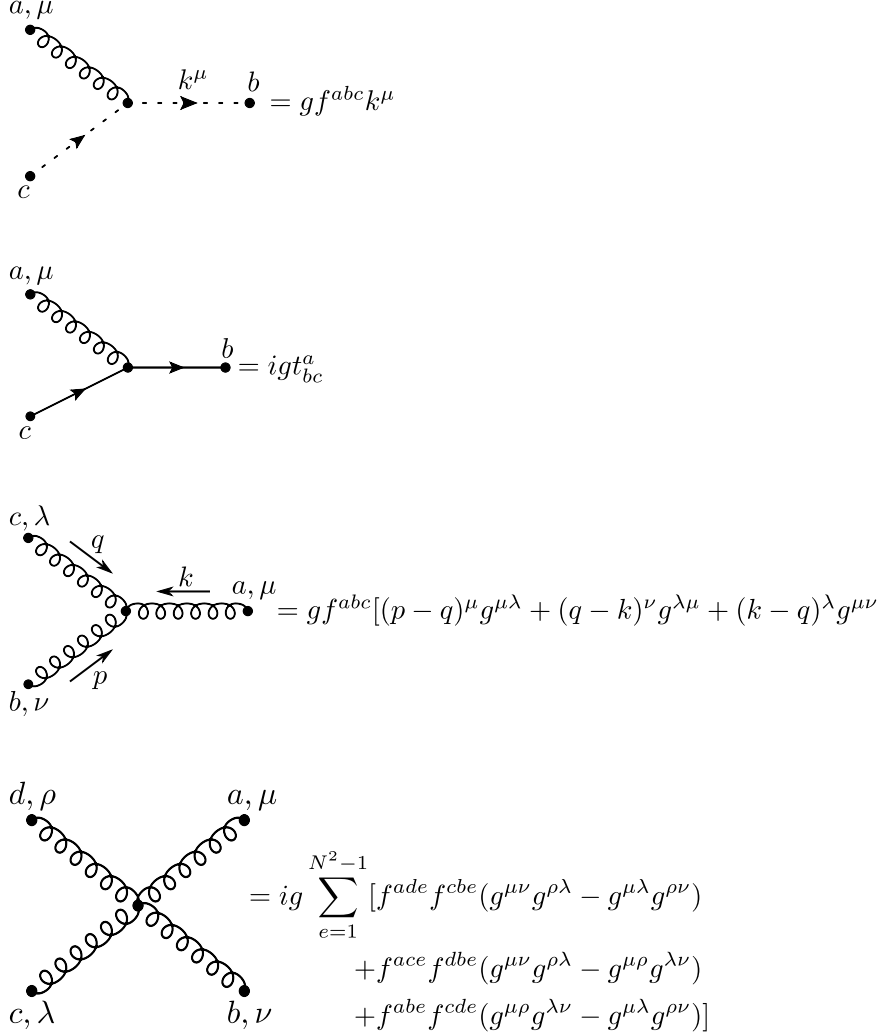
$$\text{quark propagator} = \bullet \xrightarrow{k} \bullet \stackrel{b}{\underset{a}{\xrightarrow{k}}} = \frac{i\delta_{ab}}{\not{k} - m + i0}$$

$$\text{ghost propagator} = \bullet \cdots \xrightarrow{k} \cdots \bullet \stackrel{b}{\underset{a}{\xrightarrow{k}}} = \frac{i\delta_{ab}}{k^2 + i0}$$

(3.12)

3.2. OPERATOR PRODUCT EXPANSION (OPE)

Vertices:



The diagrams show the following vertices:

- Three-point vertex:** A vertex with two incoming gluon lines (curly) labeled a, μ and c , and one outgoing ghost line (dashed) labeled b . The momentum of the ghost line is k^μ . The vertex factor is $gf^{abc}k^\mu$.
- Two-point vertex:** A vertex with one incoming gluon line (curly) labeled a, μ and one incoming fermion line (solid) labeled c , and one outgoing fermion line labeled b . The vertex factor is igt_{bc}^a .
- Four-point vertex:** A vertex with four incoming gluon lines (curly) labeled a, μ , b, ν , c, λ , and d, ρ . The momenta of the lines are k , p , q , and k respectively. The vertex factor is $gf^{abc}[(p-q)^\mu g^{\mu\lambda} + (q-k)^\nu g^{\lambda\mu} + (k-q)^\lambda g^{\mu\nu}]$.

(3.13)

$$\begin{aligned}
 &= ig \sum_{e=1}^{N^2-1} [f^{ade} f^{cbe} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\lambda} g^{\rho\nu}) \\
 &\quad + f^{ace} f^{dbe} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\lambda\nu}) \\
 &\quad + f^{abe} f^{cde} (g^{\mu\rho} g^{\lambda\nu} - g^{\mu\lambda} g^{\rho\nu})]
 \end{aligned}$$

3.2 Operator Product Expansion (OPE)

The short-distance Operator Product Expansion (OPE) was firstly introduced by Wilson and Zimmerman [16, 17, 161], and was later adapted to the case of light-like separations [80, 81, 162]. This method quickly found applications in Deeply Inelastic Scattering (DIS) [163] leading to the DGLAP evolution equations [18]. It was also successfully used to study asymptotic freedom in QCD [10, 11, 164]. During the last forty years it has been also applied to study higher twist effects in (SI-)DIS [130–133, 165], Deeply Virtual Compton Scattering (DVCS) [166–172], e^+e^- annihilation [84, 173–181] as well TMD factorization [182, 183] and Drell-Yan process [184, 185], which are the subjects of this work. In this section, we will discuss first the short-distance OPE and light-cone OPE of composite operators, relevant for the Drell-Yan process.

3.2.1 Short distance OPE

In general, the time-ordered product of two local operators $\hat{A}$ and $\hat{B}$ (either elementary or composite) in an interacting quantum field theory can be written as

$$T \left\{ \hat{A}^I(y) \hat{B}^I(x) \right\} = \sum_n \frac{i^n}{n!} \int d^d z_1 \cdots d^d z_n T \left\{ \hat{A}^0(y) \hat{B}^0(y) \mathcal{L}_{\text{int}}^0(z_1) \cdots \mathcal{L}_{\text{int}}^0(z_n) \right\}, \quad (3.14)$$

where the superscript I labels the interacting fields while 0 refers to free fields in the Heisenberg picture. The iterative applications of Wick's theorem leads to the short-distance (or Wilson) expansion [16]

$$T \left\{ \hat{A}^I(y) \hat{B}^I(x) \right\} = \sum_i C_i(y-x) N_i^0(y, x), \quad (3.15)$$

where the functions $C_i(y-x)$, called Wilson coefficients contains singularities as $y \rightarrow x$, and $N_i^0(y, x)$ is a bi-local normal ordered product of free fields which is well-defined in the limit $y \rightarrow x$. If the operators $\hat{A}(y)$, $\hat{B}(x)$ and $N_i(y, x)$ have mass dimension m_A , m_B and m_{N_i} respectively, then in the limit $y \rightarrow x$, [16, 64, 68, 149]

$$\lim_{y \rightarrow x} C_i(y-x) \simeq \frac{1}{[-(y-x)^2 + i0]^{\frac{n_A + n_B - m_{N_i}}{2}}}, \quad (3.16)$$

in the presence of interactions this relations gets logarithmic corrections.

Since the operators $N_i(y, x)$ are regular at $y \rightarrow x$ one can perform an expansion around $y = x$

$$N_i(y, x) = \sum_n \frac{(y-x)^{\mu_1} \cdots (y-x)^{\mu_n}}{n!} \partial_{\mu_1} \cdots \partial_{\mu_n} N_i(x, x), \quad (3.17)$$

where ∂_μ is the partial derivative with respect y^μ . Therefore the short-distance expansion in eq. (3.15) can be rewritten as

$$T \left\{ \hat{A}^I(y) \hat{B}^I(x) \right\} = \sum_i C_i(y-x) \sum_n \frac{(y-x)^{\mu_1} \cdots (y-x)^{\mu_n}}{n!} \partial_{\mu_1} \cdots \partial_{\mu_n} N_i(x, x). \quad (3.18)$$

The derivative terms lead (in principle) to sub-leading contributions in the limit $y \rightarrow x$ due to the factors $(y-x)^\mu$ in the numerator. In the next section we will see that this is not always true in the case of a the light-cone OPE, where we are interested in the behavior of product of operators (elementary or composite) in the limit $(y-x)^2 \rightarrow 0$ for an arbitrary $y^\mu - x^\mu$.

In the case of a gauge theory like QCD, in the short-distance expansion in eq. (3.18) the partial derivatives must be replaced by covariant derivative, i.e. $\partial_\mu \rightarrow D_\mu - igA_\mu$ [186, 187]. The interacting propagator of a fermion in the presence of a gluon field satisfies the equation

$$(i\not{D} - m)S_{\text{int}}(y, x) = i\delta^{(4)}(y-x), \quad (3.19)$$

where $D_\mu = \partial_\mu - igA_\mu$ and m is the fermion mass. Keeping only the most singular terms (leading twist approximation), eq. (3.19) has the solution

$$S_{\text{int}}(y, x) \simeq P \exp \left\{ ig \int_x^y dz_\mu A^\mu(z) \right\} S(y-x), \quad (3.20)$$

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where $S(y - x)$ is the fermion free propagator and P is the path-ordering operator. The path-ordered exponential is called Wilson line [17, 186]. Taking the limit $y \rightarrow x$ of the normal ordered product involving two fermions fields,

$$: \bar{\psi}(y) P \exp \left\{ ig \int_x^y dz_\mu A^\mu(z) \right\} \psi(x) :, \quad (3.21)$$

the path-ordered exponential can be expanded up to next order in $y^\mu - x^\mu$,

$$P \exp \left\{ ig \int_x^y dz_\mu A^\mu(z) \right\} = 1 + ig(y-x) \cdot A \left(\frac{x+y}{2} \right) + \mathcal{O}((y-x)^2) = 1 + ig(y-x) \cdot A(x) + \mathcal{O}((y-x)^2), \quad (3.22)$$

which leads to

$$: \bar{\psi}(y) P \exp \left\{ ig \int_x^y dz_\mu A^\mu(z) \right\} \psi(x) := \sum_n \frac{(y-x)^{\mu_1} \cdots (y-x)^{\mu_n}}{n!} : \bar{\psi}(x) \overleftarrow{D}_{\mu_1} \cdots \overleftarrow{D}_{\mu_n} \psi(x) :, \quad (3.23)$$

where $\overleftarrow{D} = \overleftarrow{\partial}_\mu + ig A_\mu(x)$.

Hence, in the case of a gauge theory like QCD, the short-distance expansion amount to

$$T \{ \hat{A}^I(y) \hat{B}^I(x) \} = \sum_i C_i(y-x) \sum_n \frac{(y-x)^{\mu_1} \cdots (y-x)^{\mu_n}}{n!} D_{\mu_1} \cdots D_{\mu_n} N_i(x, x). \quad (3.24)$$

This expression is, however, highly impractical for several reasons. First, the renormalization of an operator $\mathcal{O}_{i, \mu_1 \cdots \mu_n}^{\text{bare}} = D_{\mu_1} \cdots D_{\mu_n} N_i(x, x)$ can lead to mixing with other operators $\mathcal{O}_{j, p}$

$$\mathcal{O}_{i, \mu_1 \cdots \mu_n} = \sum_{j, p} Z_{j, m} \otimes \mathcal{O}_{j, \mu_1 \cdots \mu_m} \quad (3.25)$$

where $Z_{j, m}$ is not proportional to the unit matrix and $\otimes$ denotes a convolution in position space. This intricate structure in the renormalization procedure translates into a system of coupled evolution equations for the operators. The second reason concerns the physical interpretation of the matrix elements of the operators $\mathcal{O}_{i, \mu_1 \cdots \mu_n}$. In the present work we are interested in the case where $\hat{A}$ and $\hat{B}$ in eq. (3.24) are the electroweak quark currents and the time-ordered product and two proton initial and final states, as shown in eq. (2.14) in Chapter 2. Matrix elements of the form $\langle P_A P_B | \mathcal{O}_{i, \mu_1 \cdots \mu_n} | P_A P_B \rangle$ with generic Lorentz indices μ_j lack a physical interpretation. This is because the operator $\mathcal{O}_{i, \mu_1 \cdots \mu_n}$, from the point of the rotation group $SO(3)$ is a tensor with n indices. Therefore one can decompose it into a direct sum of irreducible representations under $SO(3)$, i.e. we can write $\mathcal{O}_{i, \mu_1 \cdots \mu_n}$ as the sum of operators of different spin. It is the matrix elements of the irreducible representations of $\mathcal{O}_{i, \mu_1 \cdots \mu_n}$ what can be interpreted as distributions of polarized partons inside a proton with a given polarization. It is for these two reasons that the concept of *geometrical twist*, which is defined as the mass dimension minus the spin, is introduced to classify the operators in the short-distance OPE. Since operators with different spin belongs to different irreducible representations of the Lorentz group, they will not mix in the renormalization and evolution procedure. Hence, eq. (3.24), is more conveniently rewritten as

$$T \{ \hat{A}^I(y) \hat{B}^I(x) \} = \sum_{i, n, \tau} C_i(y-x) \frac{(y-x)^{\mu_1} \cdots (y-x)^{\mu_n}}{n!} \mathcal{O}_{i, \mu_1 \cdots \mu_n}^{[\tau]}, \quad (3.26)$$

where $\tau = \text{mass dimension} - \text{spin}$.

Let us now focus on the case of the time-ordered product of two electroweak currents i.e. $T\{J^\nu(x)J^\nu(0)\}$, where $J^\mu(x) =: \bar{\psi}\Gamma^\mu\psi(x) :$, Γ^μ is some Dirac structure and the sum over color and Dirac indices is implicit. This example should be enough to illustrate the ideas behind the short distance OPE and the decomposition onto operators with definite geometrical twist. We will see that already at the tree level result without virtual corrections shows a considerable algebraic complexity in its manipulation. This is one of the reason the background field method that we will introduce in the next section plays a crucial role in the simplification of the calculations. Applying the Wick's theorem,

$$\begin{aligned}
 T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\} = & -\langle 0|T\{\psi_s(0)\bar{\psi}_i(x)\}|0\rangle(\Gamma^\nu)_{ij}\langle 0|T\{\psi_j(x)\bar{\psi}_r(0)\}|0\rangle(\Gamma^\mu)_{rs} \\
 & +: \bar{\psi}(x)\Gamma^\nu\langle 0|T\{\psi(x)\bar{\psi}(0)\}|0\rangle\Gamma^\mu\psi(0): \\
 & +: \bar{\psi}_i(0)(\Gamma^\mu)_{ij}\langle 0|T\{\psi_j(0)\bar{\psi}_r(x)\}|0\rangle(\Gamma^\nu)_{rs}\psi_s(x): \\
 & +: \bar{\psi}(x)\Gamma^\mu\psi(x)\bar{\psi}(0)\Gamma^\nu\psi(0):,
 \end{aligned} \tag{3.27}$$

where on the first and third lines we have made explicit the sum over Dirac indices and used the equal time anti-commutation relations for fermionic fields.⁴ The contribution on the first line of eq. (3.27) is the most singular in the limit $x \rightarrow 0$ and gives the leading contribution at high-mass transfer in processes like inclusive e^+e^- annihilation [84, 173], where one is interested in the matrix element $\langle 0|T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\}|0\rangle$. However, for process like DIS [18, 79, 163] and DVCS [170] where the matrix elements of eq. (3.27) with a nucleon states are involved, the first line gives a disconnected terms and therefore does not contribute to the cross section. The leading contribution is given by the second and third lines. For the Drell-Yan process [182, 184, 185], SIDIS [53, 165], and e^+e^- annihilation to hadrons [174–179, 181] the leading contribution is given by the forth line when one evaluates the matrix element $\langle P_A P_B | T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\} | P_A P_B \rangle$ which is the first to give a connected term. By expanding around $x = 0$, one obtains

$$\begin{aligned}
 T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\} = & \bar{\psi}(0)\Gamma^\mu\psi(0)\bar{\psi}(0)\Gamma^\nu\psi(0): \\
 & + x^\alpha : \{\bar{\psi}(0)(\overleftarrow{D}_\alpha + D_\alpha)\Gamma^\mu\psi(0)\}\bar{\psi}\Gamma^\nu\psi(0): \\
 & + \frac{1}{2}x^\alpha x^\beta : \{\bar{\psi}(0)(\overleftarrow{D}_\alpha\overleftarrow{D}_\beta + 2\overleftarrow{D}_\alpha D_\beta + D_\beta D_\alpha)\Gamma^\mu\psi(0)\}\bar{\psi}\Gamma^\nu\psi(0): \\
 & + \dots,
 \end{aligned} \tag{3.28}$$

where $\dots$ stands for higher-order contributions. The covariant derivatives are in the fundamental representation of $SU(3)$ and it is understood that one first performs the derivative and then evaluates it at zero. Using the Fierz identity, see Appendix B.2, eq. (3.28) can be

⁴ $\{\psi_a(\mathbf{y}), \psi_b^\dagger(\mathbf{x})\} = \delta^{(3)}(\mathbf{y} - \mathbf{x})\delta_{ab}$ and $\{\psi_a(\mathbf{y}), \psi_b(\mathbf{x})\} = \{\psi_a^\dagger(\mathbf{y}), \psi_b^\dagger(\mathbf{x})\} = 0$.

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rewritten as

$$\begin{aligned}
T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\} &= - \sum_{\{\Gamma^A, \Gamma^B\}} \text{Tr}[\Gamma^\mu \bar{\Gamma}_B \Gamma^\nu \bar{\Gamma}_A] [:\bar{\psi}_j(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(\Gamma^B)_{rs}\psi_s:(0) \\
&+ x^\alpha : \{\bar{\psi}_j \overleftarrow{D}_\alpha\}(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(\Gamma^B)_{rs}\psi_s : + x^\alpha : \bar{\psi}_j(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(\Gamma^B)_{rs}\{D_\alpha\psi_s\}:(0) \\
&+ \frac{1}{2}x^\alpha x^\beta : \{\bar{\psi}_j(0)\overleftarrow{D}_\alpha\overleftarrow{D}_\beta\}(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(0)(\Gamma^B)_{rs}\psi_s : + \frac{1}{2}x^\alpha x^\beta : \bar{\psi}_j(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(\Gamma^B)_{rs}\{D_\beta D_\alpha\psi_s\}:(0) \\
&+ x^\alpha x^\beta : \{\bar{\psi}_j \overleftarrow{D}_\alpha\}(\Gamma^A)_{ji}\psi_i\bar{\psi}_r(\Gamma^B)_{rs}\{D_\beta\psi_s\}:(0)] + \dots,
\end{aligned} \tag{3.29}$$

where the covariant derivatives are in the fundamental representation, see definition in Section 3.1, and $:\cdot:(0)$ means that all the operators inside the normal ordered product are evaluated at zero. In Drell-Yan, one needs the forward matrix element of eq. (3.29) using the state of two incoming protons $|P_AP_B\rangle$, i.e.

$$\begin{aligned}
\langle P_AP_B|T\{:\bar{\psi}(x)\Gamma^\mu\psi(x)::\bar{\psi}(0)\Gamma^\mu\psi(0):\}|P_AP_B\rangle &= \langle P_AP_B|:\bar{\psi}(x)\Gamma^\mu\psi(x)\bar{\psi}(0)\Gamma^\nu\psi(0):|P_AP_B\rangle \\
&= - \sum_{\{\Gamma^C, \Gamma^D\}} \text{Tr}[\Gamma^\mu \bar{\Gamma}_D \Gamma^\nu \bar{\Gamma}_C] \langle P_A|:\bar{\psi}_j(x)(\Gamma^C)_{ji}\psi_i(0):|P_A\rangle \langle P_B|:\bar{\psi}_r(0)(\Gamma^D)_{rs}\psi_s(x):|P_B\rangle \\
&= - \sum_{\{\Gamma^C, \Gamma^D\}} \text{Tr}[\Gamma^\mu \bar{\Gamma}_D \Gamma^\nu \bar{\Gamma}_C] [\langle P_A|:\bar{\psi}_j(\Gamma^C)_{ji}\psi_i:(0)|P_A\rangle \langle P_B|:\bar{\psi}_r(\Gamma^D)_{rs}\psi_s:(0)|P_B\rangle \\
&+ x^\alpha \langle P_A|:\{\bar{\psi}_j \overleftarrow{D}_\alpha\}(\Gamma^C)_{ji}\psi_i:(0)|P_A\rangle \langle P_B|:\bar{\psi}_r(\Gamma^D)_{rs}\psi_s:(0)|P_B\rangle \\
&+ x^\alpha \langle P_A|:\bar{\psi}_j(\Gamma^C)_{ji}\psi_i:(0)|P_A\rangle \langle P_B|:\bar{\psi}_r(\Gamma^D)_{rs}\{D_\alpha\psi_s\}:(0)|P_B\rangle \\
&+ \frac{1}{2}x^\alpha x^\beta \langle P_A|:\{\bar{\psi}_j \overleftarrow{D}_\alpha\overleftarrow{D}_\beta\}(\Gamma^C)_{ji}\psi_i:(0)|P_A\rangle \langle P_B|:\bar{\psi}_r(\Gamma^D)_{rs}\psi_s:(0)|P_B\rangle \\
&+ \frac{1}{2}x^\alpha x^\beta \langle P_A|:\bar{\psi}_j(\Gamma^A)_{ji}\psi_i:(0)|P_A\rangle \langle P_B|:\bar{\psi}_r(\Gamma^D)_{rs}\{D_\beta D_\alpha\psi_s\}:(0)|P_B\rangle] + \dots.
\end{aligned} \tag{3.30}$$

One may be concerned why there are not matrix elements involving an odd number of fermionic fields such as $\langle P_B|:\psi_s(x):|P_B\rangle$ or $\langle P_A|:\bar{\psi}_j(x)\bar{\psi}_r(0)\psi_i(0):|P_A\rangle$ or why we can simply assume the factorization of the two proton state $|P_AP_B\rangle = |P_A\rangle \otimes |P_B\rangle$. These issues will be discussed in details in Chapter 4. At this stage we will say that the large gap in rapidity between the protons makes it *natural* to assume the factorization of the two-proton state is valid and matrix elements with odd number of fermionic fields violate energy and momentum conservation⁵. The matrix elements in eq. (3.30) have no definite spin, i.e. they are a linear combinations of tensors, each one in a different irreducible representation of the Lorentz group. For example

⁵Also notice that they are not color neutral. $\langle P_B|\psi_s(x)|P_B\rangle$ transforms as a triplet under $SU(3)$; while $\langle P_A|:\bar{\psi}_j(x)\bar{\psi}_r(0)\psi_i(0):|P_A\rangle$ does not belong to a irreducible representation of $SU(3)$, instead it transforms according to the tensor product $\bar{3} \otimes \bar{3} \otimes 3$. Thus, these kind of matrix elements cannot correspond to physical quantities or distributions.

the operators $D_\beta D_\alpha$ and $\overleftarrow{D}_\alpha \overleftarrow{D}_\beta$ can be written as

$$\begin{aligned}
 D_\beta D_\alpha &= \frac{1}{2}(D_\beta D_\alpha + D_\alpha D_\beta - \frac{2}{d}D^2 g_{\alpha\beta}) + \frac{1}{2}(D_\beta D_\alpha - D_\alpha D_\beta) + \frac{1}{d}D^2 g_{\alpha\beta} \\
 &= \frac{1}{2}(D_\beta D_\alpha + D_\alpha D_\beta - \frac{2}{d}D^2 g_{\alpha\beta}) - \frac{ig}{2}F_{\alpha\beta} + D^2 g_{\alpha\beta}, \\
 \overleftarrow{D}_\alpha \overleftarrow{D}_\beta &= \frac{1}{2}(\overleftarrow{D}_\alpha \overleftarrow{D}_\beta + \overleftarrow{D}_\beta \overleftarrow{D}_\alpha - \frac{2}{d}\overleftarrow{D}^2 g_{\alpha\beta}) + \frac{1}{2}(\overleftarrow{D}_\alpha \overleftarrow{D}_\beta - \overleftarrow{D}_\beta \overleftarrow{D}_\alpha) + \overleftarrow{D}^2 g_{\alpha\beta} \\
 &= \frac{1}{2}(\overleftarrow{D}_\alpha \overleftarrow{D}_\beta + \overleftarrow{D}_\beta \overleftarrow{D}_\alpha - \frac{2}{d}\overleftarrow{D}^2 g_{\alpha\beta}) - \frac{ig}{2}F_{\alpha\beta} + \frac{2}{d}\overleftarrow{D}^2 g_{\alpha\beta},
 \end{aligned} \tag{3.31}$$

where d is the space-time dimension and we have used $[D_\alpha, D_\beta] = -igF_{\alpha\beta}$ and $[\overleftarrow{D}_\alpha, \overleftarrow{D}_\beta] = -igF_{\alpha\beta}$. The symmetric traceless term in eq. (3.31) has spin 2, the anti-symmetric term has spin 1, while the term proportional to the trace is a scalar i.e. spin 0. Using eq. (3.31) in eq. (3.30) the decomposition into irreducible representation of the inhomogeneous Lorentz group is completed up to the value of the matrices $\Gamma^{C(D)}$. Let us consider two cases: $\Gamma^C = \Gamma^D = \mathbb{1}$ and $\Gamma^{C(D)} = \gamma^{\lambda(\rho)}$.

Case: $\Gamma^C = \Gamma^D = \mathbb{1}$

It is the simplest case, just using eq. (3.31) in eq. (3.30) is enough to rewrite the short distance OPE of the time-ordered product of two electroweak currents into a sum of operators with definite geometrical twist. The leading geometrical twist will be twist-3 from terms like $\langle P_B | : \bar{\psi}_r(\mathbb{1})_{rs} \psi_s : (0) | P_B \rangle$, $\langle P_A | : \{ \bar{\psi}_j \overleftarrow{D}_\alpha \} (\mathbb{1})_{ji} \psi_i : (0) | P_A \rangle$ or $\langle P_B | : \bar{\psi}_r(\mathbb{1})_{rs} \{ S_{\alpha\beta} \psi_s \} : (0) | P_B \rangle$ where we have defined

$$S_{\alpha\beta} = \frac{1}{2}(D_\beta D_\alpha + D_\alpha D_\beta - \frac{2}{d}D^2 g_{\alpha\beta}). \tag{3.32}$$

Contributions with the gluon field strength tensor have geometrical twist-4; terms with proportional to the trace, $D^2 g^{\alpha\beta}$, have twist-5⁶.

Case $\Gamma^{C(D)} = \gamma^{\lambda(\rho)}$

In these case the leading contribution is given by operators of geometrical twist-2 like $\langle P_B | : \bar{\psi}_r(\gamma^\rho)_{rs} \psi_s : (0) | P_B \rangle$, $\langle P_B | : \bar{\psi}_r \{ (S_\gamma^{\rho\alpha})_{rs} \psi_s \} : (0) | P_B \rangle$, $\langle P_B | : \bar{\psi}_r \{ (S_\gamma^{\rho\alpha\beta})_{rs} \psi_s \} : (0) | P_B \rangle$, where

$$S_\gamma^{\rho\alpha} = \frac{1}{2}(\gamma^\rho D_\alpha + \gamma^\alpha D^\rho - 2\not{D}g^{\alpha\rho}), \tag{3.33}$$

and

$$S_\gamma^{\rho\alpha\beta} = \frac{1}{6}(\gamma^\rho D_\beta D_\alpha + \text{all permutations of } \rho, \alpha, \beta) - \frac{1}{d+2} \text{traces}. \tag{3.34}$$

The next contribution will be twist-3, and it comes when $\gamma^{\lambda(\rho)} D^\alpha$ are anti-symmetrized or when the combination $\gamma^{\lambda(\rho)} D^\beta D^\alpha$ is completely symmetric in two indices but anti-symmetric in the third ones e.g. it can be symmetric in the pairs of indices α, β and $\{\alpha, \rho\}$ but anti-symmetric in the pair $\{\beta, \rho\}$.

⁶Notice that we do not consider the spin of the fermionic field to compute the geometrical twist, since it is the same always.

3.2.2 Light-cone OPE

The basic concepts discussed in the case of short-distance expansion can be easily translated to the light-cone expansion, the main difference being that while in the former one considers the limit $x^\mu \rightarrow 0$ (where x^μ is the separation between two local operators) in the latter the relevant limit is for vanishing light-like separation $x^2 \rightarrow 0$ [64, 68, 80, 81, 83, 188]. This leads to two mayor differences: the light-cone expansion for $q^2 \rightarrow \infty$ rather than $q^\mu \rightarrow \infty$ and there are derivative terms which are not power suppressed (they are the collinear and anti-collinear derivatives, depending on the field they act on). Furthermore, in the light-cone expansion is that the Wilson lines introduced previously to make normal-ordered composite operators gauge invariant does not vanish, since $y^\mu - x^\mu$ approaches a light-like vector.

Our previous example, given in eq. (3.29), becomes in the light-cone OPE

$$\begin{aligned}
 T\{\bar{\psi}(x)\Gamma^\mu\psi(x) :: \bar{\psi}(0)\Gamma^\mu\psi(0) : \} &= \sum \{\Gamma^C, \Gamma^D\} \text{Tr}[\Gamma^\mu \bar{\Gamma}_D \Gamma^\nu \bar{\Gamma}_C] \\
 &\times [\langle P_A | : \bar{\psi}_j(x^+) [x^+, 0] (\Gamma^C)_{ji} \psi_i(0) : | P_A \rangle \langle P_B | : \bar{\psi}_r(0) (\Gamma^D)_{rs} [0, x^-] \psi_s(x^-) : | P_B \rangle \\
 &+ (x_T^\alpha + n^\alpha x^-) \langle P_A | : \{ \bar{\psi}_j(x^+) (\overleftarrow{D}_{T\alpha} + \bar{n}_\alpha \overleftarrow{D}_+) \} [x^+, 0] (\Gamma^C)_{ji} \psi_i(0) : | P_A \rangle \\
 &\times \langle P_B | : \bar{\psi}_r(0) (\Gamma^D)_{rs} [0, x^-] \psi_s(x^-) : | P_B \rangle \\
 &+ (x_T^\alpha + \bar{n}^\alpha x^+) \langle P_A | : \bar{\psi}_j(x^+) [x^+, 0] (\Gamma^C)_{ji} \psi_i(0) : | P_A \rangle \\
 &\times \langle P_B | : \bar{\psi}_r(0) (\Gamma^D)_{rs} [0, x^-] \{ (D_{T\alpha} + n_\alpha D_-) \psi_s(x^-) \} : | P_B \rangle \\
 &+ \dots,
 \end{aligned} \tag{3.35}$$

where n^α and $\bar{n}^\alpha$ are light-like vectors similar to the ones introduces in Chapter 2 and the subscripts T denotes a orthogonal subspace to them. Notice that we have considered that the separation between fields acting on the proton state $|P_A\rangle$ approaches to a light-like vector $\bar{n}^\alpha x^+$ i.e. $x^\alpha \rightarrow \bar{n}^\alpha x^+$, while for the case of fields acting on the proton state $|P_B\rangle$ it is assumed that the separation approaches to the light-like vector $n^\alpha x^-$. We have denoted by $[a + z^\alpha, b + z^\alpha]$ the collinear Wilson line

$$[a + z^\alpha, b + z^\alpha] = P \exp \left\{ ig \int_b^a ds \bar{n} \cdot B(s\bar{n} + z^\alpha) \right\}, \tag{3.36}$$

for the case of the matrix element computed with the state $|P_A\rangle$, while for the case of $|P_B\rangle$ it is given by

$$[a + z^\alpha, b + z^\alpha] = P \exp \left\{ ig \int_b^a ds n \cdot B(sn + z^\alpha) \right\}. \tag{3.37}$$

The assumption for not performing also a expansion around $x^{(+)} = 0$ in the matrix element computed with the proton state $|P_A\rangle$, or $x^- = 0$ in the case of $|P_B\rangle$, is that the derivatives contributions are not power suppressed in the limit $x^+ \rightarrow 0$. Thus the whole series of derivatives would be needed. The physical meaning is that fields acting on the proton state $|P_A\rangle$ have a momentum that scales in light-cone coordinates as $k^\alpha \sim Q(\lambda^2, 1, \lambda)$, where Q is some hard reference energy scale and $\lambda \ll 1$. This means that derivatives along the transverse and anti-collinear component scale as $Q\lambda$ and $Q\lambda^2$ respectively.⁷, see Chapter 2. The light-cone

⁷Notice that

$$\partial_+ = n \cdot \partial = \frac{\partial}{\partial x^-}, \quad \partial_- = \bar{n} \cdot \partial = \frac{\partial}{\partial x^+}$$

OPE appears to be an excellent tool to access sub-leading contribution beyond parton model interpretation. Notice that not only it provides a first principle theoretical foundation to the collinear factorization and the parton model picture, but it also enables TMD factorization since in the previous expression an explicit dependence on the transverse separation of the fields could be retained. As for the short-distance OPE, a twist decomposition of the operators in eq. (3.35) is appropriate. The reasons are exactly the same: its simplify the renormalization procedure allowing to be just multiplicative procedure and assigning a definite spin to the matrix elements in eq. 3.35 permits its identification with physical spin and helicity distributions of quarks and gluons inside the protons. It is also worth comment the color structure in eq. (3.35). Since $:\bar{\psi}(x)\Gamma^\mu\psi(x):$ is an electroweak current it is implicitly understood that the two fermions fields form a color singlet, i.e. $\bar{\psi}_i^a\psi_j^a$, in order to be color neutral. Therefore the fields combination in the matrix elements of eq. (3.35) are as well color neutral.

3.3 Background field method

We discussed in Chapter 2 the different energy scale hierarchies involved in the Drell-Yan process and presented without formal proof two factorization theorems: collinear or high- q_T and TMD factorization. These provide reliable theoretical predictions in different kinematic regimens in the Drell-Yan process and are concerned with different physical regimes: collinear factorization applies perturbative region of the transverse momentum spectrum in differential Drell-Yan, while the main focus in TMD factorization is the non-perturbative contributions to the q_T -spectrum. The proof of both factorization theorems relies on the same idea: the systematic separation of short-and long-distance contributions to the scattering process. To derive these factorization formulae one can either use Effective Field Theories (EFT) or a light-cone OPE with background fields ⁸. In this section we introduce the background field method as a procedure to compute the effective action for a gauge theory.

3.3.1 Effective action

In the previous section we analysed the short-distance and light-cone OPE of time orderer product of quark electroweak currents. We showed that already at LO, without quantum corrections, the process is algebraically tedious. The OPE process (for a gauge theory like QCD) involves covariant derivatives of the fields, see eqs. (3.24) and (3.35), at LO the fields can be regarded as classical [182,185] and one can simply compute their derivatives. However, when we start considering quantum effects, i.e. real and virtual radiative corrections, the fields are operators and it is less clear if one can simply perform the covariant derivatives. Therefore, we would like to replace an operator like $J^\mu(z) = T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}$, whose matrix element evaluation contains quantum corrections by and series of effective operator $\mathcal{J}_i^\mu$, whose matrix elements evaluation is free of quantum corrections and can be done at LO classically, see fig. 3.1. Thus, one can formally perform a Taylor expansion of the classical fields in the effective operator. In order to obtain the matching of $T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}$ one can compute its matrix element considering some arbitrary initial state composed of quarks, anti-quarks and

⁸Indeed the background field method is the foundation behind any EFT.

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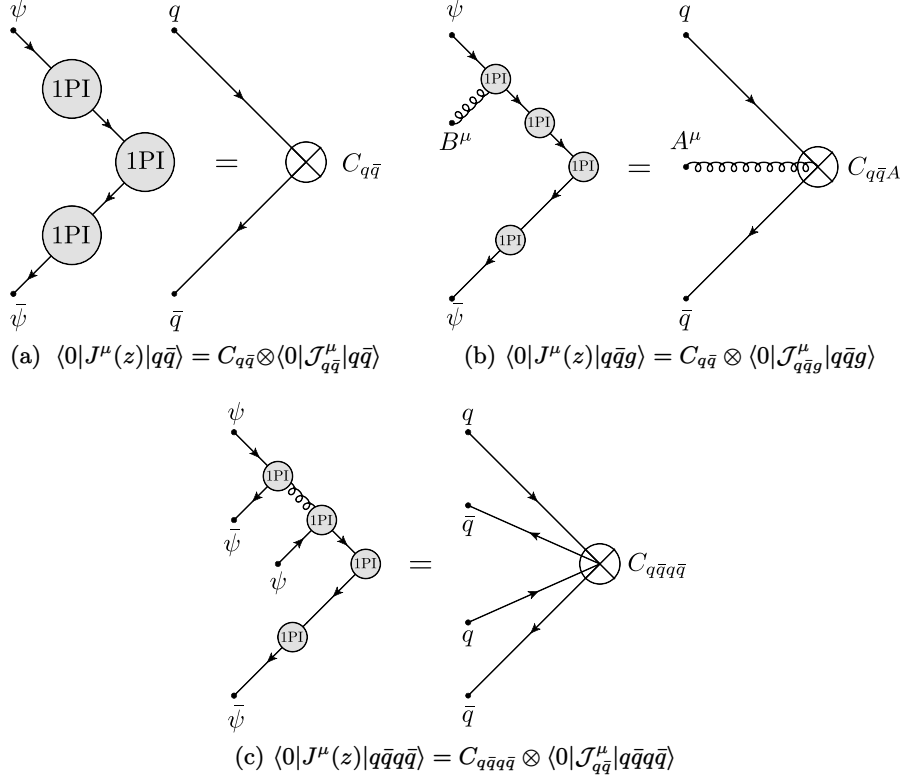


Figure 3.1: Matching procedure of the operator $J^\mu(z) = T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}$ [75, 84, 141, 142, 173, 184, 189, 190], $\otimes$ denotes a integral convolution in position space. (a) shows the matching of the matrix element of J^μ when we consider a quark and anti-quark as incoming states, the blobs denote 1PI diagrams containing the virtual corrections to the propagators and vertices. On the right-hand side the effective operator is simply composed of the effective quark and anti-quark fields, whose matrix element are evaluated at tree level and convoluted with the Wilson coefficient $C_{q\bar{q}}$. (b) shows the matching when the initial states is composed of a quark, an anti-quark and a gluon, $|q\bar{q}g\rangle$; (c) shows the matching when we use as incoming state two quarks and anti-quarks, $|q\bar{q}q\bar{q}\rangle$.

gluons and the vacuum as final state, i.e. $\langle 0|T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}|q\cdots\bar{q}\cdots g\rangle$, [75, 84, 141, 142, 173, 184, 189, 190]. The first non-zero contribution⁹ to the matching procedure is coming from considering the matrix element $\langle 0|T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}|q\bar{q}\rangle$, see fig. 3.1a. Schematically it is given by

$$T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\} = \int dy_1 dy_2 C_{q\bar{q}}(y_1, y_2) \bar{q}(z + y_2) \Gamma_{q\bar{q}}^\mu q(z + y_1) + \cdots, \quad (3.38)$$

⁹The vacuum expectation value $\langle 0|T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\}|0\rangle$ gives disconnected contributions and it does not contribute to the Drell-Yan process.

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where $\dots$ denotes further operators and the Dirac structure in the full theory, Γ^μ , is not necessarily the same than in the effective operator $\Gamma_{q\bar{q}}^\mu$ ¹⁰. If we also consider a gluon in the initial state the matrix element of the effective operator will be zero, however the matrix element of the operator in the full theory is not zero, therefore one needs to consider further effective operators see figs. 3.1b and 3.1c

$$T\{\bar{\psi}(z)\Gamma^\mu\psi(z)\} = C_{q\bar{q}} \otimes \mathcal{J}_{a\bar{q}}^\mu(z, z) + C_{q\bar{q}g} \otimes \mathcal{J}_{q\bar{q}g}^\mu + C_{q\bar{q}q\bar{q}} \otimes \mathcal{J}_{q\bar{q}q\bar{q}}^\mu \dots, \quad (3.39)$$

In principle one must consider an infinite series of effective operator, nevertheless in most of the physical applications the effective fields follow a certain scaling with respect a dimensionless parameter $\lambda \ll 1$. Therefore one can formally organize the effective operators in powers of λ . Depending of the kinematic configuration and the physical assumptions that are made one could not be necessary to consider the whole tower of effective operators and simple the first few contributions can give reasonable results. In Chapter. 4 the effective fields will follow the scaling already discussed Chapter. 2 what allows us to perform a power counting of the different contribution coming from the matrix element of effective operators.

The next question we have to address is how exactly can we construct the effective operators $\mathcal{J}_i^\mu$, since one we have a precise definition for them the computation of the Wilson coefficient C_i is formally straight forward. Since the perturbative evaluation of matrix elements of an operator is performed using the LSZ reduction formula, the evaluation of $\langle 0|T\{\psi(z)\Gamma^\mu\psi(z)\}|q\dots\bar{q}\dots g\dots\rangle$ turns in the computation of connected n -points connected Green functions. Therefore, we need to find a functional that performing a tree level evaluation of the matrix element of a effective operator reproduces the whole quantum correction coming from n -points connected Green function, this functional is the well-known *effective action*, $\mathcal{S}_{\text{eff}}$. Therefore we need to find a method to obtain the effective action starting from the QCD action. Let us quickly reminder of how a general effective action, $\mathcal{S}$, is defined stating from some full theory action S . In the path integral approach the n -point connected Green functions are computed by performing functional derivatives with respect to an external classical source J of the generating functional [64, 68, 149, 191]

$$W[J] = -i \ln Z[J], \quad Z[J] = \int [d\phi] e^{iS[\phi] + \int d^4z J(z)\phi(z)}, \quad (3.40)$$

u where ϕ denotes some arbitrary field i.e. scalar, fermionic or vector. Using eq. (3.40) the connected Green functions are given by [64, 68, 149, 191]

$$\langle 0|\hat{\phi}(y_1)\dots\hat{\phi}(y_n)|0\rangle_c = (-i\hbar)^{n-1} \frac{\delta^n W[J]}{\delta J(y_1)\dots\delta J(y_n)} \Big|_{J=0}, \quad (3.41)$$

where c stands for connected Green functions. Their computation can be simplified further by expressing them in terms of 1 Particle Irreducible (1PI) diagrams, as in fig. 3.2. This can be achieved by working with the Legendre transform of $W[J]$ rather than with $W[J]$ itself [85, 191–193]

$$\Gamma[\bar{\phi}] = W[J_{\bar{\phi}}] - \int d^4z J_{\bar{\phi}}(z)\bar{\phi}(z) \quad (3.42)$$

¹⁰Indeed, we will see that in the Drell-Yan case each effective operator breaks the Ward identity, and it is only the sum of all of them what restores it.

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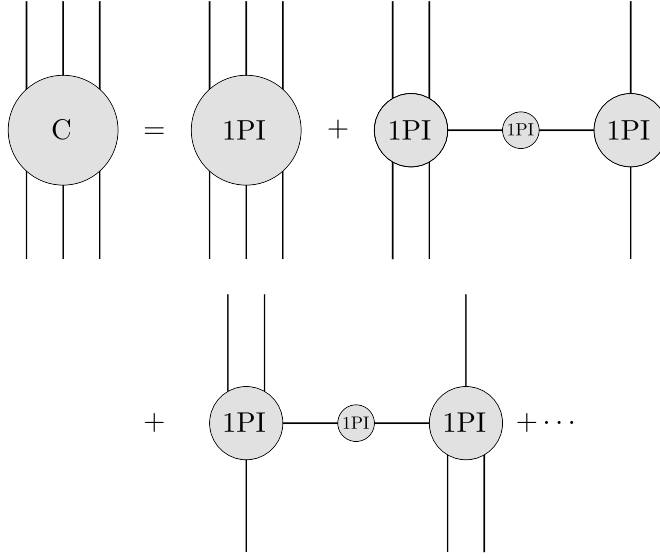


Figure 3.2: n -points connected (C) Green's function expressed as 1PI contributions joint together.

where $\bar{\phi}$ is given by

$$\bar{\phi} = \frac{\delta W[J]}{\delta J(z)}, \quad (3.43)$$

and $J_{\bar{\phi}}$ must be understood as the classical current J written as a functional of $\bar{\phi}(z)$ i.e. $J_{\bar{\phi}}(z) = J[\phi(z)]$, where eq. (3.43) is the defining condition of $J_{\bar{\phi}}$. It can be understood as: for classical field configuration $\phi_c(z)$ and current J_{ϕ_c} , when computing the expected value of $\hat{\phi}$ in present of $J_{\bar{\phi}}$ one obtains $\phi_c = \bar{\phi}$ i.e. $\phi_c = \bar{\phi} = \langle J_{\bar{\phi}} | \hat{\phi} | J_{\bar{\phi}} \rangle$ [149, 191], the classical field configuration is just the expected value of $\hat{\phi}$ in present of $J_{\bar{\phi}}$. The new functional $\Gamma[\bar{\phi}]$ is known as the effective action [64, 68, 149, 191] and it is to a quantum field theory what the classical action $S[\phi]$ is to a classical field theory. Notice that

$$\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}(z)} = \int d^4y \frac{\delta W[J_{\bar{\phi}}]}{\delta J_{\bar{\phi}}(y)} \frac{\delta J_{\bar{\phi}}(y)}{\delta \bar{\phi}(z)} - \int d^4y \left(\frac{\delta J_{\bar{\phi}}(y)}{\delta \bar{\phi}(z)} \bar{\phi}(y) + J_{\bar{\phi}}(y) \frac{\delta \bar{\phi}(y)}{\delta \bar{\phi}(z)} \right) = -J_{\bar{\phi}}(z), \quad (3.44)$$

which is the quantum mechanical field equation for $\bar{\phi}$ and replaces the classical field equation $\delta S[\phi]/\delta \phi = -J$.

Consider the second derivative of the effective action in eq. (3.42) with respect $\bar{\phi}$, one obtains

$$\frac{\delta^2 \Gamma[\bar{\phi}]}{\delta \bar{\phi}^2} = -\frac{\delta J_{\bar{\phi}}}{\delta \bar{\phi}} = \left[-\frac{\delta \bar{\phi}}{\delta J_{\bar{\phi}}} \right]^{-1} = \left[-\frac{\delta^2 W[J]}{\delta J^2} \right]^{-1} = [(-i)G_c^{(2)}]^{-1}, \quad (3.45)$$

where $G_c^{(2)}$ denotes the connected two points Green function and we have used the definition of $\bar{\phi}$ given by eq. (3.43). Eq. (3.45) can be inverted multiplying by both side with $\delta^2 W/\delta J^2$, it leads to

$$\left[(-i)^{-1} G_c^{(2)} \right] \frac{\delta^2 \Gamma[\bar{\phi}]}{\delta \bar{\phi}^2} \left[(-i)^{-1} G_c^{(2)} \right] = -(-i)^{-1} G_c^{(2)}, \quad (3.46)$$

which tells us that the full two points connected Green function can be obtained by computing the 1PI two points function and dressing up the two external lines with connected two points correlators. Obviously it is a functional equation that has to be solved order by order in the coupling constant¹¹. Notice that $\delta/\delta J$ and $\delta/\delta\bar{\phi}$ are related by

$$\frac{\delta}{\delta\bar{\phi}} = \frac{\delta J}{\delta\bar{\phi}} \frac{\delta}{\delta J} = \left[\frac{\delta^2 W[J]}{\delta J^2} \right]^{-1} \frac{\delta}{\delta J} = [(-i)^{-1} G_c^{(2)}]^{-1} \frac{\delta}{\delta J}, \quad (3.47)$$

which has the following meaning: acting with $\delta/\delta\bar{\phi}$ over $\Gamma[\bar{\phi}]$ removes a propagator from a external line, while $\delta/\delta J$ adds a external line to the Green function when it acts on $W[J]$. Let us consider another example: $\delta^3 \Gamma / \delta\bar{\phi}^2$

$$\frac{\delta^3 \Gamma[\bar{\phi}]}{\delta\bar{\phi}^3} = \frac{\delta}{\delta\bar{\phi}} \frac{\delta^2 \Gamma[\bar{\phi}]}{\delta\bar{\phi}^2} = [(-i)^{-1} G_c^{(2)}]^{-1} \frac{\delta}{\delta J} \left[-\frac{\delta^2 W[J]}{\delta J^2} \right]^{-1} = [(-i)^{-1} G_c^{(2)}]^{-3} \frac{\delta^3 W[J]}{\delta J^3}, \quad (3.48)$$

which can be written as

$$[(-i)^{-1} G_c^{(2)}]^3 \frac{\delta^3 \Gamma[\bar{\phi}]}{\delta\bar{\phi}^3} = (-i)^{-2} G_c^{(3)}. \quad (3.49)$$

Therefore, we see that the n -point connected Green function can be computed performing functional derivatives of the *classical effective action*, $\Gamma[\bar{\phi}]$, with respect $\bar{\phi}$, that we regard as the classical counterpart of the quantum field ϕ . In case the dressed external lines correspond to on-shell states, e.g. in the evaluation of a scattering amplitude or in general the matrix elements of some generic operator, in virtue of the LSZ formula we need the amputated n -point connected Green function, i.e. we need the residue of the 2-points connected Green function $G_c^{(2)}$ rather than its full form. Therefore, the effective action allows us to translate the evaluation of the matrix elements of a certain operator, including all its quantum corrections, into a tree level evaluation with *classical* fields, which was precisely our initial aim. However, it is still not clear how exactly obtain the effective action, since its whole calculation depends on eq. (3.43), which is far from trivial. Firstly, there is no guaranty we can obtain a closed form for $W[J]$ and in case we were able to obtain one, we could not be able to invert the resulting equation and write J as a functional of $\bar{\phi}$. Indeed there are few cases where this is possible, one of them being for a real free scalar field. Therefore, we can expect that for a theory with the level of complexity of QCD such a procedure is not possible, instead the best that can be done is to use perturbative methods to compute the effective action introducing a *classical background field*.

Let us consider again the functional $Z[J]$ in eq. (3.40) and replace the field ϕ in the action $S[\phi]$ by the shifted field $\phi \rightarrow \phi + \phi_B$, where ϕ_B is an arbitrary classical background field

$$\tilde{Z}[J, \phi_B] = \int [d\phi] \exp \left\{ iS[\phi + \phi_B] + i \int d^4 z J(z) \phi(z) \right\}, \quad (3.50)$$

the functional $\tilde{Z}$ depends on both, the classical current J and the on the background field, which can be thought as another source. Analogously to eqs. (3.40) and (3.42) one can define

$$\tilde{W}[J, \phi_B] = -i \ln \tilde{Z}[J, \phi_B], \quad (3.51)$$

¹¹We are being a bit sloppy in the notation, since it is true that in momentum space the 1PI and connected Green function are simply multiplied, in position space it must be understood as a convolution in coordinates.

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$$\tilde{\phi} = \frac{\delta \tilde{W}[J, \phi_B]}{\delta J}, \quad (3.52)$$

and the background effective action

$$\tilde{\Gamma}[\tilde{\phi}, \phi_B] = \tilde{W}[J, \phi_B] - \int d^4z J_{\tilde{\phi}}(z) \tilde{\phi}(z). \quad (3.53)$$

The generating functionals $\tilde{Z}[J, \phi_B]$ and $Z[J]$ can be related by doing shift $\phi \rightarrow \phi - \phi_B$ in eq. (3.50), one finds

$$\tilde{Z}[J, \phi_B] = \int [d\phi] \exp \left\{ iS[\phi] + i \int d^4z J(z) (\phi(z) - \phi_B(z)) \right\} = Z[J] \exp \left\{ -i \int d^4z J(z) \phi_B(z) \right\}, \quad (3.54)$$

and $\tilde{W}[J, \phi_B]$ is given by

$$\tilde{W}[J, \phi_B] = -i \ln \tilde{Z}[J, \phi_B] = W[J] - \int d^4z J(z) \phi_B(z). \quad (3.55)$$

Performing the functional derivative with respect $J(z)$ one obtains

$$\tilde{\phi} = \bar{\phi} - \phi_B, \quad (3.56)$$

finally replacing $\tilde{W}[J, \phi_B]$ and $\tilde{\phi}$ in eq. (3.53) by their expression given in eqs. (3.55) and (3.56) respectively one arrives

$$\tilde{\Gamma}[\tilde{\phi}, \phi_B] = W[J] - \int d^4z J(z) \phi_B(z) = \Gamma[\bar{\phi}], \quad (3.57)$$

but $\bar{\phi} = \tilde{\phi} + \phi_B$, therefore

$$\tilde{\Gamma}[\tilde{\phi}, \phi_B] = \Gamma[\tilde{\phi} + \phi_B], \quad (3.58)$$

therefore the effective action $\Gamma[\phi_B]$ can be computed by taking $\tilde{\phi} = 0$.

The background field effective action $\tilde{\Gamma}[\tilde{\phi}, \phi_B]$ is just a usual effective action computed in the present of a classical background field ϕ_B . This means it is composed of all 1PI diagrams that contribute to Green's functions, which are generated by taking functional derivatives of the effective action with respect the field. In the case of $\tilde{\Gamma}[\tilde{\phi}, \phi_B]$ the functional derivatives are with respect $\tilde{\phi}$, which will generate 1PI Green's functions in the present of a background field ϕ_B . If one set $\tilde{\phi} = 0$ the effective action $\tilde{\Gamma}$ has no dependence on $\tilde{\phi}$, therefore no diagrams with external lines are generated. Instead, $\tilde{\Gamma}[0, \phi_B]$ is the sum of all 1PI vacuum to vacuum transitions of the quantum field in the present of a background field ϕ_B . This is a great advantages, since no reference is made to external states in the calculation of the effective action.¹² The condition $\tilde{\phi} = 0$ can be ensured by choosing the current $J(z)$ in eq. (3.50) as [194]

$$\tilde{J} := - \left. \frac{\delta S[\phi + \phi_B]}{\delta \phi} \right|_{\phi=0}, \quad (3.59)$$

¹²This feature will show to be extremely powerful in order to rigorous and systematically access to sub-leading contributions in Drell-Yan process.

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this is, the expectation value of the quantum field $\hat{\phi}$ vanish in the present of a current of this choice. Thus one can write

$$\exp\{i\tilde{W}[\tilde{J}, \phi_B]\} = \int [d\phi] \exp\left\{iS[\phi + \phi_B] + i \int d^4z \tilde{J}(z)\phi(z)\right\} \quad (3.60)$$

or explicitly replacing $\tilde{J}$ by its definition in eq. (3.59) one obtain

$$\tilde{W}[\tilde{J}, \phi_B] = \tilde{\Gamma}[\tilde{\phi}, \phi_B] + \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \tilde{\phi}(z) = \tilde{\Gamma}[0, \phi_B] = \Gamma[\phi_B], \quad (3.61)$$

therefore eq. (3.60) amounts to

$$\exp\{i\Gamma[\phi_B]\} = \int [d\phi] \exp\left\{iS[\phi + \phi_B] - i \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \phi(z)\right\}. \quad (3.62)$$

There are two approaches to compute $\tilde{\Gamma}[0, \phi_B]$. The first is consider the dependence on the fields ϕ_B exactly. One start by computing all the 1PI vacuum diagrams using the action

$$S[\phi + \phi_B] - \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \phi(z),$$

to generate the Feynman rules, where the propagator for the field ϕ has a exact dependence on the background field ϕ_B . Then one compute the usual diagrams for the field ϕ with vertices that depends on ϕ_B . It is clear that this approach seems feasible for a very limited types of background fields ϕ_B . Some cases are the computation of the effective potential in scalar field theories when ϕ_B is a constant field [194, 195]. The second approach is the one sued in along this work. It consists in treating the background field ϕ_B perturbatively. Thus, one just need to consider 1PI vacuum diagrams of the ϕ field with ϕ_B fields appearing as external lines. In this approaches the background field is arbitrary and it does not need to be specified. The way to proceed is relatively simple, again one obtains the Feynman rules from

$$S[\phi + \phi_B] - \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \phi(z),$$

but for the propagator of the field ϕ only the quadratic term in ϕ of $S[\phi + \phi_B]$ is consider and every thing else is regarded as a perturbation¹³. Interactions between ϕ fields are used as internal vertex in the diagrams while interactions between ϕ and ϕ_B fields are responsible of the external lines. From a practical point of view in a background field theory one computes Feynman diagrams in exactly the same manner that any other approach, the only different is that the Feynman rules for vertices inside the diagrams may differ from those connected to external lines. One can write the exponential factor in the right hand-side of eq. (3.60) as

$$S[\phi + \phi_B] - \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \phi(z) = S[\phi_B] + S[\phi] + S_{\text{int}}[\phi, \phi_B], \quad (3.63)$$

¹³Since there is no functional integral over the background field ϕ_B is does not propagate and therefore it only appears as external line, never internal.

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where by definition $S_{\text{int}}[\phi, \phi_B]$ is

$$S_{\text{int}}[\phi, \phi_B] := S[\phi + \phi_B] - \int d^4z \frac{\delta S[\phi + \phi_B]}{\delta \phi(z)} \Big|_{\phi=0} \phi(z) - S[\phi_B] - S[\phi], \quad (3.64)$$

hence, the functional $\tilde{Z}[\phi, \phi_B]$ becomes

$$\tilde{Z}[\phi, \phi_B] = \exp\{iS[\phi_B]\} \times \int [d\phi] \exp\{iS[\phi] + S_{\text{int}}[\phi, \phi_B]\}, \quad (3.65)$$

where we can pull out of the functional integral the phase factor $\exp\{iS[\phi_B]\}$ since it has no dependence on the quantum fields ϕ . Thus the effective action amounts to

$$\Gamma[\phi_B] = -i \ln \tilde{Z}[\phi, \phi_B] = S[\phi_B] - i \ln \int [d\phi] \exp\{iS[\phi] + S_{\text{int}}[\phi, \phi_B]\}, \quad (3.66)$$

from eq. (3.66) it is self-evident that the corrections to the classical action $S[\phi_B]$ are 1PI virtual diagrams in presence of external fields ϕ_B . A perturbative evaluation of the second term on the right hand-side leads to the effective action $\Gamma[\phi_B]$ to take the form [64, 87, 149]

$$\Gamma[\phi_B] = S[\phi_B] + \sum_n \int d^4z_1 \cdots d^4z_n \Gamma^{[n]}(z_1, \dots, z_n) \phi_B(z_1) \cdots \phi_B(z_n), \quad (3.67)$$

where the coefficient $\Gamma^{[n]}(z_1, \dots, z_n)$ are the 1PI Green's functions. Notice the non-local nature of the quantum corrections in position space, also it is not a coincidence that eqs. (3.67) and (3.14) show some sort of similarities, since both equations aim to a similar purpose: separates degrees of freedom associated to different energy scales.

3.3.2 Composite background field method in a gauge theory

Yang-Mills lagrangian We start considering the background field method for a pure Yang-Mills lagrangian, which is the most challenging due to the gauge fixing condition; the Dirac lagrangian will show to have a fairly simple extension. The background generating functional for a pure Yang-Mills theory is [64, 68, 85, 87, 149]

$$\begin{aligned} \tilde{Z}[J, A] = & \int [dB] \det \left[\frac{\delta \tilde{G}^a[B, A]}{\delta \theta^a} \right] \\ & \times \exp \left\{ i \int d^4z \mathcal{L}_{\text{YM}}[A + B] + i \int d^4z \mathcal{L}_{\text{GF}}[\tilde{G}] + i \int d^4z \sum_{a=1}^{N^2-1} J_\mu^a(z) B_\mu^a(z) \right\}, \end{aligned} \quad (3.68)$$

where B^μ and A^μ are the quantum (dynamical) and background gauge fields respectively; $\mathcal{L}_{\text{GF}}[\tilde{G}]$ and $\tilde{G}^a[B, A]$ are the gauge fixing lagrangian and condition respectively; $\mathcal{L}_{\text{YM}}[A + B]$ is the Yang-Mills lagrangian [64, 68, 85, 149]

$$\mathcal{L}_{\text{YM}}[A + B] = -\frac{1}{4} \sum_{a=1}^{N^2-1} F_{\mu\nu}^a[A + B] F_a^{\mu\nu}[A + B], \quad (3.69)$$

CHAPTER 3. QCD, BACKGROUND FIELD METHOD AND OPE

where

$$F_{\mu\nu}^a[A+B] = \partial_\mu(A_\nu^a(z) + B_\nu^a(z)) - \partial_\nu(A_\mu^a(z) + B_\mu^a(z)) + g \sum_{b,c=1}^{N^2-1} f^{abc}(A_\mu^b(z) + B_\mu^b(z))(A_\nu^c(z) + B_\nu^c(z)); \quad (3.70)$$

and $\delta\tilde{G}^a/\delta\theta^a$ is the functional derivative of the gauge fixing condition under a infinitesimal gauge transformation

$$B_\mu^a(z) \rightarrow B_\mu^a(z) + g \sum_{b,c} f^{abc}\theta^b(z)(B_\mu^c(z) + A_\mu^c(z)) - \partial_\mu\theta^a(z). \quad (3.71)$$

The Yang-Mills term is invariant under two set of infinitesimal transformations:

$$B_\mu^a(z) \rightarrow B_\mu^a(z) + g \sum_{b,c} f^{abc}\theta^b(z)B_\mu^c(z) - \partial_\mu\theta^a(z), \quad A_\mu^a(z) \rightarrow A_\mu^a(z) + g \sum_{b,c=1}^{N^2-1} f^{abc}\theta^b(z)A_\mu^c(z) \quad (3.72)$$

and

$$A_\mu^a(z) \rightarrow A_\mu^a(z) + g \sum_{b,c} f^{abc}\theta^b(z)A_\mu^c(z) - \partial_\mu\theta^a(z), \quad B_\mu^a(z) \rightarrow B_\mu^a(z) + g \sum_{b,c=1}^{N^2-1} f^{abc}\theta^b(z)B_\mu^c(z) \quad (3.73)$$

since $\mathcal{L}_{\text{YM}}[B+A]$ is symmetric under $A \leftrightarrow B$. The former is a rotation in color space of the background field A_μ^a plus a infinitesimal gauge transformation of the dynamical gauge field B_μ^a , while in the latter the infinitesimal gauge transformation is applied to the background field A_μ^a and the dynamical field B_μ^a is simply rotated in color space. It is not possible to keep both symmetries manifest at the same time and one is forced to choose. The advantages of the background field method for gauge theories is based on the explicit gauge invariance for the background field. Thus one decides to maintain explicit gauge invariant under the set of transformation given by eq. (3.73), since A_μ^a is not a dynamical field and there is not need to define its propagator.

Once noes choose to have lagrangian that preserves a explicit gauge invariant form for the background field A_μ^a , the gauge fixing term $\mathcal{L}_{\text{GF}}$ can be discussed. It can be shown that a gauge fixing terms like

$$\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = -\frac{1}{2\xi} \sum_{a=1}^{N^2-1} (\partial_\mu B_\mu^a(z))^2 + \sum_{a,b,c=1}^{N^2-1} (\partial^\mu \bar{\omega}^a(z))(\delta_{ab}\partial_\mu + g f_{acb}B_\mu^c(z))\omega^b(z) \quad (3.74)$$

are no longer adequate since they are not gauge invariant under the set of transformation given in eq. (3.73). The reasoning is analogous to the one presented in the replacement of partial derivatives by covariant derivatives in a OPE, see Section 3.2. It is again due to the non local nature of the partial derivative operator. Therefore we need to parallel transport the dynamical field B_μ^a and the ghost fields ω and $\bar{\omega}$. However in this case the gauge field present in the Wilson line is going to be the background field A_μ^a , therefore one can simply do the replacement

$$\delta_{ac}\partial_\mu \rightarrow (\tilde{D}_\mu[A])_{ac} = \delta_{ac}\partial_\mu + g \sum_{b=1}^{N^2-1} f_{abc}A_\mu^b(z). \quad (3.75)$$

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Therefore we will use as gauge fixing condition $\tilde{G}^a$

$$\tilde{G}^a[A, B] = \sum_{c=1}^{N^2-1} (\tilde{D}_\mu[A])_{ac} B_c^\mu(z) = \sum_{c=1}^{N^2-1} \left\{ \delta_{ac} \partial_\mu + g \sum_{b=1}^{N^2-1} f_{abc} A_\mu^b(z) \right\} B_c^\mu(z), \quad (3.76)$$

the gauge fixing lagrangian amounts to [64, 85, 149]

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} \sum_{a=1}^{N^2-1} \tilde{G}^a[A, B] \tilde{G}_a[A, B] = -\frac{1}{2\xi} \sum_{a,b,c=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ab} B_b^\mu(z)\} \{(\tilde{D}_\nu[A])_{ac} B_c^\nu(z)\}, \quad (3.77)$$

and the Faddeev–Popov lagrangian is

$$\mathcal{L}_{\text{FP}} = \sum_{a,c,d=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ac} \bar{\omega}^c(z)\} \{(\tilde{D}^\mu[A])_{ad} + g \sum_{e=1}^{N^2-1} f_{aed} B_e^\mu(z)\} \omega^d(z), \quad (3.78)$$

where we used $\{\cdot\}$ to makes explicit that the covariant derivative only acts on the fields that are inside it. The gauge fixing condition in eq. (3.76) is known as background-field gauge [64, 85, 149].

Putting together the Yang–Mills, gauge fixing and Faddeev–Popov lagrangians in eqs. (3.69), (3.77) and (3.78) respectively one obtains

$$\begin{aligned} \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = & -\frac{1}{4} \sum_{a=1}^{N^2-1} F_{\mu\nu}^a[A, B] F_a^{\mu\nu}[A, B] \\ & -\frac{1}{2\xi} \sum_{a,b,c=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ab} B_b^\mu(z)\} \{(\tilde{D}_\nu[A])_{ac} B_c^\nu(z)\} \\ & + \sum_{a,c,d=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ac} \bar{\omega}^c(z)\} \{(\tilde{D}^\mu[A])_{ad} + g \sum_{e=1}^{N^2-1} f_{aed} B_e^\mu(z)\} \omega^d(z) \end{aligned} \quad (3.79)$$

where $F_{\mu\nu}^a[A, B]$ is given by

$$\begin{aligned} F_{\mu\nu}^a[A, B] = & F_{\mu\nu}^a[A] + \sum_{b=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ab} B_\nu^b(z) - (\tilde{D}_\nu[A])_{ab} B_\mu^b(z)\} + g \sum_{c,b=1}^{N^2-1} f^{abc} B_\mu^b(z) B_\nu^c(z) \\ F_{\mu\nu}^a[A] = & \partial_\mu A_\nu^a(z) - \partial_\nu A_\mu^a(z) + g \sum_{b,c=1}^{N^2-1} f^{abc} A_\mu^b(z) A_\nu^c(z), \end{aligned} \quad (3.80)$$

and under the infinitesimal gauge transformation in eq. (3.73) each term in $F_{\mu\nu}^a[A, B]$ trans-

forms as

$$\begin{aligned}
 F_{\mu\nu}^a[A] &\rightarrow F_{\mu\nu}^a[A] + g \sum_{b,c=1}^{N^2-1} f^{abc} \theta_b(z) F_{\mu\nu}^c[A], \\
 \sum_{b=1}^{N^2-1} (\tilde{D}_\mu[A])_{ab} B_\nu^b(z) &\rightarrow \sum_{b=1}^{N^2-1} (\tilde{D}_\mu[A])_{ab} B_\nu^b(z) + g \sum_{c,d,b=1}^{N^2-1} f^{acd} \theta_c(z) (\tilde{D}_\mu[A])_{db} B_\nu^b(z), \\
 \sum_{b,c=1}^{N^2-1} f^{abc} B_\mu^b(z) B_\nu^c(z) &\rightarrow \sum_{b,c=1}^{N^2-1} f^{abc} B_\mu^b(z) B_\nu^c(z) + g \sum_{b,c,d,e} f^{ade} f^{ebc} \theta_d(z) B_\mu^b(z) B_\nu^c(z),
 \end{aligned} \tag{3.81}$$

therefore $F_{\mu\nu}^a[A, B]$ transforms as

$$F_{\mu\nu}^a[A, B] \rightarrow F_{\mu\nu}^a[A, B] + g \sum_{b,c=1}^{N^2-1} f^{abc} \theta_b(z) F_{\mu\nu}^c[A, B]. \tag{3.82}$$

Dirac lagrangian In the case of the Dirac lagrangian the procedure is rather straightforward, one just need to do the replacement $B_\mu^a \rightarrow A_\mu^a + B_\mu^a$, $\psi^a \rightarrow q^a + \psi$ and $\bar{\psi}^a \rightarrow \bar{q}^a + \bar{\psi}^a$; where q^a and $\bar{q}^a$ are the background classical quark fields, ψ^a and $\bar{\psi}^a$ are the dynamical or quantum quark fields and as before A_μ^a and B_μ^a are the background and dynamical gluon field respectively. The Dirac action with the standard minimal coupling takes the form

$$\begin{aligned}
 \mathcal{L}_{\text{Dirac}}[\bar{q} + \bar{\psi}, q + \psi, A + B] &= \sum_{a,b=1}^N (\bar{q}^a(z) + \bar{\psi}^a(z)) (\delta_{ab} \{i\not{D} - m\} \\
 &\quad + g \sum_{c=1}^{N^2-1} t_{ab}^c \{A_c(z) + B_c(z)\}) (q^b(z) + \psi^b(z)) \\
 &= \sum_{a,b=1}^N (\bar{q}^a(z) + \bar{\psi}^a(z)) (i\gamma^\mu \tilde{D}_\mu[A])_{ab} (q^b(z) + \psi^b(z)) \\
 &\quad + \sum_{a,b=1}^N (\bar{q}^a(z) + \bar{\psi}^a(z)) \left\{ g \sum_{c=1}^{N^2-1} t_{ab}^c B_c(z) - \delta_{ab} m \right\} (q^b(z) + \psi^b(z)),
 \end{aligned} \tag{3.83}$$

where $(\tilde{D}_\mu[A])_{ab}$ is in the fundamental representation, i.e. $(\tilde{D}_\mu[A])_{ab} = \delta_{ab} \partial_\mu - ig \sum_c t_{ab}^c A_\mu^c(z)$. The quantum ψ^a and classical background fermion fields under infinitesimal gauge transformation transform as

$$\begin{aligned}
 q^a(z) &\rightarrow q^a(z) - ig \sum_{c=1}^N \sum_{b=1}^{N^2-1} t_{ac}^b \theta^b(z) q^c(z), \\
 \psi^a(z) &\rightarrow \psi^a(z) - ig \sum_{c=1}^N \sum_{b=1}^{N^2-1} t_{ac}^b \theta^b(z) \psi^c(z).
 \end{aligned} \tag{3.84}$$

The second and third lines in eq. (3.83) are invariant under the set of gauge transformation defined by the eqs. (3.73) and (3.84).

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The next step is to remove from the Dirac lagrangian in eq. (3.83) the contribution coming from the equations of motion, as we discussed for the scalar case in Section 3.3.1 it is achieved by subtraction the contribution given by $\phi \frac{\delta S[\phi+\phi_B]}{\delta \phi} \Big|_{\phi=0}$. In the case of eq. (3.83) one need to perform two functional derivatives with respect $\bar{\psi}$ and ψ , one obtains¹⁴

$$\begin{aligned} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \bar{\psi}^a} \Big|_{\bar{\psi}=0, \psi=0, B=0} &= \sum_{b=1}^N \{i\gamma^\mu (\tilde{D}_\mu[A])_{ab} - m\delta_{ab}\} q^b(z), \\ \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \psi^a} \Big|_{\bar{\psi}=0, \psi=0, B=0} &= - \sum_{a=1}^N \bar{q}^a(z) \{i\gamma^\mu (\overleftarrow{\tilde{D}}_\mu[A])_{ab} + \delta_{ab}m\}, \end{aligned} \quad (3.85)$$

where $(\overleftarrow{\tilde{D}}_\mu[A])_{ab} = \delta_{ab} \overleftarrow{\partial}_\mu + ig \sum_{c=1}^{N^2-1} t_{ab}^c A_\mu^c(z)$ ¹⁵. The Dirac lagrangian in eq. (3.83) minus the contribution from e.o.m in eq. (3.85) gives

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{Dirac}} &= \mathcal{L}_{\text{Dirac}} - \sum_{a=1}^N \bar{\psi}^a(z) \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \bar{\psi}^a} \Big|_{\bar{\psi}=0, \psi=0, B=0} - \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \psi^a} \Big|_{\bar{\psi}=0, \psi=0, B=0} \psi^a(z) \\ &= \mathcal{L}_{\text{Dirac}}[\bar{q}, q, A] + \mathcal{L}_{\text{Dirac}}[\bar{\psi}, \psi, B] + g \sum_{a,b=1}^N \sum_{c=1}^{N^2-1} \{ \bar{q}^a t_{ab}^c \not{B}_c q^b + \bar{\psi}^a t_{ab}^c \not{A}_c \psi^b \\ &\quad + \bar{\psi}^a t_{ab}^c \not{B}_c q^b + \bar{q}^a t_{ab}^c \not{B}_c \psi^b \} + \text{total derivative} + \text{mass terms}. \end{aligned} \quad (3.86)$$

Notice that removing term proportional to the e.o.m terms like $\bar{\psi}^a \partial q^b$ or $\bar{q}^a \partial_\mu \psi^b$ are no longer present. Terms like these would not be allowed since they leads to disconnected graphs. However eq. (3.86) still contains terms that leads to disconnected contributions like $\bar{q}^a \not{B}_c q^a$, it will be removed once the contribution from gluon e.o.m are subtracted.

Contribution from gluon e.o.m The last piece one needs to remove is the contribution from gluon e.o.m, which are given by the sum of two terms

$$\text{gluon e.o.m} = \frac{\delta \tilde{\mathcal{L}}_{\text{Dirac}}}{\delta B_\mu^a} \Big|_{B=0, \bar{\psi}=0, \psi=0} + \frac{\delta \mathcal{L}_{\text{YM}}}{\delta B_\mu^a} \Big|_{B=0, \bar{\psi}=0, \psi=0}, \quad (3.87)$$

¹⁴We remind that the equations of motions (e.o.m) are obtained performing variations over the action $S = \int d^4z \mathcal{L}[\phi, \partial_\mu \phi]$, where ϕ represents either a bosonic or fermionic field

$$\delta S[\phi, \partial_\mu \phi] = \int d^4z \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi + \text{boundary terms},$$

where the boundary terms are typically dropped. Therefore writing $\frac{\delta \mathcal{L}}{\delta \phi}$ we mean

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}.$$

¹⁵The relative minus sign between both e.o.m is a straight consequence of the differentiation of Grassmann variables

$$\frac{\partial}{\partial \theta_1}(\theta_1 \theta_2) = \theta_2, \quad \frac{\partial}{\partial \theta_2}(\theta_1 \theta_2) = -\theta_1.$$

each one separately gives the contribution

$$\left. \frac{\delta \tilde{\mathcal{L}}_{\text{Dirac}}}{\delta B_\mu^d} \right|_{B=0, \bar{\psi}=0, \psi=0} = g\gamma^\mu \sum_{a,b=1}^N \bar{q}^a t_{ab}^d q^b, \quad (3.88)$$

$$\left. \frac{\delta \mathcal{L}_{\text{YM}}}{\delta B_\mu^d} \right|_{B=0, \bar{\psi}=0, \psi=0} = \sum_{a=1}^{N^2-1} (\tilde{D}_\nu[A])_{da} F_a^{\nu\mu}[A], \quad (3.89)$$

therefore, the lagrangian density in the composite background field method removing contributions proportional to gluon e.o.m (and having previously removed the contributions from quark e.o.m) becomes

$$\tilde{\mathcal{L}}_{\text{BG}} = \tilde{\mathcal{L}}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} + \tilde{\mathcal{L}}_{\text{Dirac}} - g \sum_{c=1}^{N^2-1} \sum_{a,b=1}^N \bar{q}^a \not{B}_c t_{ab}^c q^b, \quad (3.90)$$

where $\tilde{\mathcal{L}}_{\text{YM}}$ is given by

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{YM}} &= \mathcal{L}_{\text{YM}} - \sum_{d=1}^{N^2-1} B_\mu^d \left. \frac{\delta \mathcal{L}_{\text{YM}}}{\delta B_\mu^d} \right|_{B=0, \bar{\psi}=0, \psi=0} \\ &= -\frac{1}{4} \sum_{a=1}^{N^2-1} F_{\mu\nu}^a[A] F_a^{\mu\nu}[A] - \frac{1}{4} \sum_{a=1}^{N^2-1} F_{\mu\nu}^a[B] F_a^{\mu\nu}[B] - \frac{g}{2} \sum_{a,b,c=1}^{N^2-1} F_{\mu\nu}^a[A] f^{abc} B_b^\mu B_c^\nu \\ &\quad - g^2 \sum_{a,b,c,d,e=1}^{N^2-1} f^{acb} f^{ade} A_\mu^c B_\nu^b B_d^\mu B_e^\nu - \frac{g^2}{2} \sum_{a,b,c,d,e} f^{acb} f^{ade} A_\mu^c B_\nu^b (A_d^\mu B_e^\nu - A_d^\nu B_e^\mu) \\ &\quad - g \sum_{a,b,c=1}^{N^2-1} f^{acb} (\partial_\mu B_\nu^a) (A_c^\mu B_b^\nu - A_c^\nu B_b^\mu) + \text{total derivatives}, \end{aligned} \quad (3.91)$$

and the relation

$$\sum_{a,b=1}^{N^2-1} F_a^{\mu\nu}[A] \{(\tilde{D}_\mu[A])_{ab} B_\nu^b\} = - \sum_{a,b=1}^{N^2-1} \{(\tilde{D}_\mu[A])_{ab} F_b^{\mu\nu}[A]\} B_\nu^a + \text{total derivatives}, \quad (3.92)$$

has been used to cancel out contributions proportional to the e.o.m.

In Chapter. 4 a slightly modified version of eq. (3.90) will be used in order to obtain the effective electroweak vertex at sub-leading power.

4 TMD Factorization

In this chapter we present how the background field formalism and light-cone OPE introduced in Chapter 3 apply to the derivation of TMDPDF factorization of the hadronic tensor in the Drell-Yan process. In Section 4.1 we present the factorization at leading power (LP) showing how the standard leading power definition of the collinear unpolarized TMDPDFs arise. Along this derivation we will identify which power corrections are drop at every step. In Section. 4.2 we focus on a particular set of power corrections that were neglected during LP factorization: Kinematic Power Corrections (KPC) and collinear twist power corrections. To conclude, in Section 4.3 we present the effect of KPC and collinear twist power corrections when a real hard parton is emitted (process that in the literature is sometimes referred as F +Jet, where F is the vector gauge boson, see [26, 96, 102–105, 134]).

4.1 TMD Factorization in the background field method

In Section 2.1 we introduced the hadronic tensor for the Drell-Yan differential cross section, we saw that there are two ways it can be written as long we consider only Initial State Radiation (ISR) and connected diagrams. Depending upon whether we are interested in real, eq. (2.10), or virtual corrections, eq. (2.14), we use one form or the other. Nevertheless, neither of the two electroweak currents in eq. (2.10) and eq. (2.14) are gathered in a single-time ordered product. This can be seen explicitly in eq. (2.10) where the current evaluated at zero is a monomial of time ordered fields, while in the hermitian conjugate current the fields are anti-time ordered. One can still perform an OPE of the product of the two electroweak current with only one caveat: to distinguish whether two free fields are gather in a time-ordered product or not [161]. In the case two free fields are under the same time-order operator, then one can use either eq. (D.1) for gluons or eq. (D.2) for fermions in order to split the operator contribution into a regular part at short-distance, given by the normal-ordered product, and a singular contribution capture by the Feynman propagators. In the case when the two free fields are not time ordered then one must use eq. (D.3) in the case of gluons and eq. (D.5) for fermions [84, 161, 173, 196]¹.

As we saw in Sec. 3.2 the OPE is a tedious process, involving covariant derivatives in the case of a gauge theory and on top of that Wilson lines if we are performing a light-cone OPE rather than a short-distance OPE. The reason is due to all the fields are in the same gauge, however the situation can highly simplify if we consider the short-distance iteration of quantum fields under the present of *classical* background fields that act as sources. In this case we can use the background field method presented in Sec. 3.3 to compute the the short-distance Wilson coefficient to the product of two electroweak current, even though they are not under the same time-ordered product. The basic assumption is that the quantum fields do not act

¹An alternative approaches is using functional methods and the Keldysh's formalism [196].

CHAPTER 4. TMD FACTORIZATION

on the external states, therefore regarding the quantum fields we are always computing the n -points connected Green functions in the present of some classical external sources. This assumption is always motivated by a different scaling of the momentum of quantum field and the momentum of the external states. Usually, the external states are composed of low energy degrees of freedom with respect to the quantum fields. In our case, we are interested in the forward matrix element of two proton states of the product of two electroweak currents, see. (2.14). We assume that the electroweak current $J^\mu(z)$ as a composite operator of two types of fields: *fast* and *slow* fields. From the point of view of the fast fields, the slow ones behave classically, since their characteristic wave length is much larger than the fast fields one. We also assume that fast fields do not act on the proton state $|P_A P_B\rangle$, only slow fields do. Thus one can write $|P_{A,n}\rangle \otimes |P_{B,\bar{n}}\rangle \otimes |0_h\rangle$, where $|0_h\rangle$ is the vacuum state of the fast fields and we have factorized further the state $|P_A P_B\rangle$ into a collinear and anti-collinear proton states $|P_{A,n}\rangle$ and $|P_{B,\bar{n}}\rangle$ respectively, due to the large gap in the rapidity of both states. The next step is to identify this abstract fast and slow fields with modes present in our problem. As we already discussed in Chapter 2 the bound quark and gluon fields inside the protons can be regarded as propagating almost parallel with respect to the parent proton. For instance, in the light-cone coordinates the proton p_A has momentum $P_A^\mu = P_A^- n^\mu + \frac{M^2}{2P_A^+} \bar{n}^\mu$ then the momentum of quarks and gluons interacting inside it scale as $k \sim Q(\lambda^2, 1, \lambda)$, where $\lambda = \Lambda_{\text{QCD}}/Q \ll 1$ and Q is the invariant mass of the gauge boson that we identify with the hard scales in our problem. Hence, it makes sense to identify the slow fields with collinear and anti-collinear (anti-)quark and gluon fields that scale as

$$\begin{aligned} (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) \bar{q}_n(z) &\lesssim Q(\lambda^2, 1, \lambda) \bar{q}_n(z), & (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) \bar{q}_{\bar{n}}(z) &\lesssim Q(1, \lambda^2, \lambda) \bar{q}_{\bar{n}}(z), \\ (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) q_n(z) &\lesssim Q(\lambda^2, 1, \lambda) q_n(z), & (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) q_{\bar{n}}(z) &\lesssim Q(1, \lambda^2, \lambda) q_{\bar{n}}(z), \\ (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) A_n^\nu(z) &\lesssim Q(\lambda^2, 1, \lambda) A_n^\nu(z), & (n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu) A_{\bar{n}}^\nu(z) &\lesssim Q(1, \lambda^2, \lambda) A_{\bar{n}}^\nu(z), \end{aligned} \quad (4.1)$$

where we denote collinear and anti-collinear fields with the subscripts n and $\bar{n}$ respectively. We want to stress the presence of the symbol $\lesssim$ in eq. (4.1), contrary to the SCET literature [72–75, 197] where the field modes are assumed to be localized in momentum space around $p_n \sim Q(\lambda^2, 1, \lambda)$ and $p_{\bar{n}} \sim Q(1, \lambda^2, \lambda)$. We defined the fast modes as the complementary to collinear and anti-collinear modes i.e. everything that is not collinear or anti-collinear, see fig. 4.1, therefore it is natural to identify the fast modes with the hard modes to be integrated out. We replace every quark, anti-quark and gluon fields by a hard, collinear and anti-collinear field

$$\begin{aligned} \psi(z) &\rightarrow \psi_h(z) + q_n(z) + q_{\bar{n}}(z) \\ \bar{\psi}(z) &\rightarrow \bar{\psi}_h(z) + \bar{q}_n(z) + \bar{q}_{\bar{n}}(z) \\ B^\mu(z) &\rightarrow B_h^\mu(z) + A_n(z) + A_{\bar{n}}(z), \end{aligned} \quad (4.2)$$

where we use subscripts h to identify the hard fields.

Performing the replacement given in eq. (4.2) in the QCD lagrangian, eq. (3.10), one obtains

$$\begin{aligned} \mathcal{L}_{\text{QCD}}[\bar{\psi}, \psi, B] &\rightarrow \mathcal{L}_{\text{Dirac}}[\bar{\psi}_h + \bar{q}_n + \bar{q}_{\bar{n}}, \psi_h + q_n + q_{\bar{n}}, B + A_n + A_{\bar{n}}] \\ &\quad + \mathcal{L}_{\text{YM}}[\bar{\psi}_h + \bar{q}_n + \bar{q}_{\bar{n}}, \psi_h + q_n + q_{\bar{n}}, B + A_n + A_{\bar{n}}] \\ &\quad + \mathcal{L}_{\text{GF}}[A_n + A_{\bar{n}}] + \mathcal{L}_{\text{FP}}[A_n + A_{\bar{n}}], \end{aligned} \quad (4.3)$$

4.1. TMD FACTORIZATION IN THE BACKGROUND FIELD METHOD

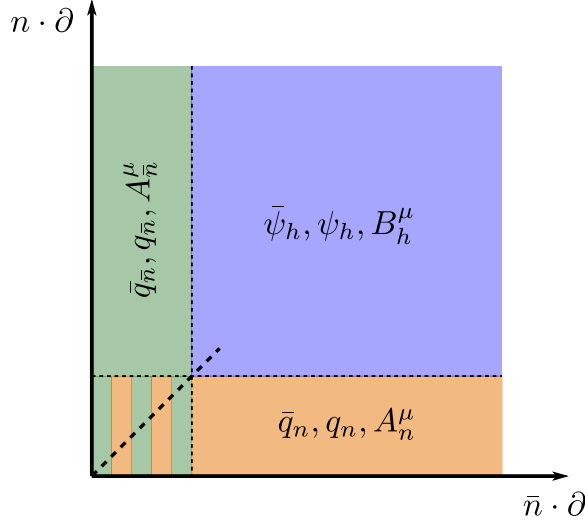


Figure 4.1: Domains of collinear, anti-collinear and hard fields in the light-cone momenta component. By definition the domain of the hard quantum fields is the complementary to the domain of background collinear and anti-collinear fields.

where the gauge fixing $\mathcal{L}_{\text{GF}}$ and Faddeev-Popov $\mathcal{L}_{\text{FP}}$ lagrangian are the same ones given in eqs. (3.77) and (3.78) but the background gauge fields in the covariant derivatives is $A^\mu = A_n^\mu + A_{\bar{n}}^\mu$. Since there are two *copies* of background fields for each hard or quantum field the subtraction of contributions proportional to the equations of motions is slightly more tedious and one need to do it in a symmetric form. We start writing the Dirac and Yang-Mills lagrangians as

$$\begin{aligned} \mathcal{L}_{\text{Dirac}}[\bar{\psi}_h + \bar{q}_n + \bar{q}_{\bar{n}}, \psi_h + q_n + q_{\bar{n}}, B + A_n + A_{\bar{n}}] &= \frac{1}{2} \mathcal{L}_{\text{Dirac}}[\bar{\psi}_n + \bar{q}_{\bar{n}}, \psi_n + q_{\bar{n}}, B_n + A_{\bar{n}}] \\ &+ \frac{1}{2} \mathcal{L}_{\text{Dirac}}[\bar{\psi}_{\bar{n}} + \bar{q}_n, \psi_{\bar{n}} + q_n, B_{\bar{n}} + A_n], \end{aligned} \quad (4.4)$$

$$\mathcal{L}_{\text{YM}}[B + A_n + A_{\bar{n}}] = \frac{1}{2} \mathcal{L}_{\text{YM}}[B_n + A_{\bar{n}}] + \frac{1}{2} \mathcal{L}_{\text{YM}}[B_{\bar{n}} + A_n], \quad (4.5)$$

where we have defined $\bar{\psi}_{n(\bar{n})} = \bar{\psi} + \bar{q}_{n(\bar{n})}$, $\psi_{n(\bar{n})} = \psi + q_{n(\bar{n})}$ and $B_{n(\bar{n})}^\mu = B^\mu + A_{n(\bar{n})}^\mu$. The idea is to consider the fields $\bar{q}_n$, q_n and A_n^μ as classical compare to $\bar{\psi}_{\bar{n}}$, $\psi_{\bar{n}}$ and $B_{\bar{n}}^\mu$; analogously $\bar{q}_{\bar{n}}$, $q_{\bar{n}}$ and $A_{\bar{n}}^\mu$ are classical compare to $\bar{\psi}_n$, ψ_n and B_n^μ . Therefore, for the lagrangian density that is a functional of $\bar{\psi}_{\bar{n}}$, $\psi_{\bar{n}}$, $B_{\bar{n}}^\mu$ one removes the equations of motions for $\bar{q}_n$, q_n and A_n^μ

$$\begin{aligned} \mathcal{L}_{\bar{n}} &= \frac{1}{2} \mathcal{L}_{\text{Dirac}}[\bar{\psi}_{\bar{n}} + \bar{q}_n, \psi_{\bar{n}} + q_n, B_{\bar{n}} + A_n] + \frac{1}{2} \mathcal{L}_{\text{YM}}[B_{\bar{n}} + A_n] \\ &- \frac{1}{2} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta B_{\bar{n}}^\mu} B_{\bar{n}}^\mu - \frac{1}{2} \frac{\delta \mathcal{L}_{\text{YM}}}{\delta B_{\bar{n}}^\mu} B_{\bar{n}}^\mu - \frac{1}{2} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \psi_{\bar{n}}} \psi_{\bar{n}} - \frac{1}{2} \bar{\psi}_{\bar{n}} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \bar{\psi}_{\bar{n}}}, \end{aligned} \quad (4.6)$$

where the functional derivatives are evaluated at $\bar{\psi}_{\bar{n}}, \psi_{\bar{n}}, B_{\bar{n}}^\mu = 0$, while for the case of la-

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grangian depending on the fields $\bar{\psi}_n, \psi_n, B_n^\mu$ one obtain

$$\begin{aligned} \mathcal{L}_n = & \frac{1}{2} \mathcal{L}_{\text{Dirac}}[\bar{\psi}_n + \bar{q}_{\bar{n}}, \psi_n + q_{\bar{n}}, B_n + A_{\bar{n}}] + \frac{1}{2} \mathcal{L}_{\text{YM}}[B_n + A_{\bar{n}}] \\ & - \frac{1}{2} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta B_n^\mu} B_n^\mu - \frac{1}{2} \frac{\delta \mathcal{L}_{\text{YM}}}{\delta B_n^\mu} B_n^\mu - \frac{1}{2} \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \psi_n} \psi_n - \frac{1}{2} \bar{\psi}_n \frac{\delta \mathcal{L}_{\text{Dirac}}}{\delta \bar{\psi}_n}, \end{aligned} \quad (4.7)$$

where the functional derivatives must be understood as evaluated at $\bar{\psi}_n, \psi_n, B_n^\mu = 0$. Putting together $\mathcal{L}_{\bar{n}}, \mathcal{L}_n, \mathcal{L}_{GF}$ and $\mathcal{L}_{FP}$ one obtains [85, 87, 184]

$$\begin{aligned} \mathcal{L}_{\text{QCD C.B.G}} = \mathcal{L}_n + \mathcal{L}_{\bar{n}} + \mathcal{L}_{GF} + \mathcal{L}_{FP} = & \mathcal{L}_{\text{QCD, cov G.F.}}[\bar{\psi}_h, \psi_h, B_h] \\ & + \mathcal{L}_{\text{QCD}}[\bar{q}_n, q_n, A_n] + \mathcal{L}_{\text{QCD}}[\bar{q}_{\bar{n}}, q_{\bar{n}}, A_{\bar{n}}] + \mathcal{L}_{\text{int}}, \end{aligned} \quad (4.8)$$

where QCD C.B.G stands for QCD in the composite background, we have written explicitly everything in terms of hard fields, collinear and anti-collinear. $\mathcal{L}_{\text{QCD, cov G.F.}}[\bar{\psi}_h, \psi_h, B_h]$ is given by eq. (3.10) and the subscripts cov G.F. denotes a covariant gauge fixing; $\mathcal{L}_{\text{QCD}}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}]$ are the sum of Dirac and Yang-Mills lagrangians; the last term $\mathcal{L}_{\text{int}}$ includes interaction terms of hard with (anti-)collinear fields and anti-collinear with collinear fields. Its explicit form is rather lengthy and it is given in Appendix C.1. Once one have obtained the QCD lagrangian in the composite background field method, eq. (4.8), the product of effective electroweak currents can be written as

$$\begin{aligned} & \mathcal{J}^{\dagger \nu}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}] \left(\frac{z}{2} \right) \mathcal{J}^\mu[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}] \left(-\frac{z}{2} \right) \\ & = \bar{T} \left\{ J^{\nu \dagger} \left(\frac{z}{2} \right) e^{-iS_{\text{QCD, cov G.F.}} - iS_{hn} - iS_{h\bar{n}} - iS_{hn\bar{n}}} \right\} T \left\{ J^\mu \left(-\frac{z}{2} \right) e^{iS_{\text{QCD, cov G.F.}} + iS_{hn} + iS_{h\bar{n}} + iS_{hn\bar{n}}} \right\}, \end{aligned} \quad (4.9)$$

on the right-hand side it is understood that the product of hard (or quantum) fields is split into a short distance divergent contribution plus it normal-ordered; notice also that for the anti-time-ordered product the complex exponential have an overall minus sign. As we have said, we assume that the hard fields do no act on the proton states, therefore when we take the forward matrix element of eq. (4.9) the normal-ordered hard fields vanishes since they act on the free theory vacuum state. The short-distance contribution inter in loop integral in the process of integrating out the hard fields, in principle one should introduce a cut-off to avoid the collinear and anti-collinear sectors, see fig. 4.1. Nevertheless this could hinder the perturbative expansion, instead we perform the loop integrals without any constrain over the Feynman integrals. This inconsistency in the treatment of the hard fields will amount to infrared divergences. For the case of virtual contributions to the electroweak current such a infrared divergences are reabsorbed into the TMDPDF, leading to the definition of renormalized TMDPDF, see. [183, 184, 198]. For the case of real hard modes, i.e. the expansion of two fields that are not under the same time-ordered product, the situation is more involved, being this case the main topic and novelty of the present work. We find that making some assumptions on the TMDPDFs matrix elements one can show that the integrals are finite; nevertheless, obtain explicit expression is extremely cumbersome. It is for this reason that we perform the computation in momentum space, where expression simplify. The drawback is that one need to interchange the order in which the integrals are performed what leads to collinear and soft divergences. It is for this reason that we introduce a multiplicative factor

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that we call double counting subtraction, whose definition will be presented later.

While $S_{\text{QCD},n(\bar{n})}$ contains exclusively (anti-)collinear terms, the interaction terms between the collinear and anti-collinear sector gather in $S_{n\bar{n}}^{(\pm)}$, see (C.5) for its explicit form, hinder the factorization into two well-defined sectors. This could be foreseen already from the fig. 4.1 where there is an overlap of the collinear and anti-collinear sectors when the fields modes scales as $\{n \cdot \partial, \bar{n} \cdot \partial, \partial_T^\mu\} \bar{q}_{n(\bar{n})}, q_{n(\bar{n})}^{(\pm)}, A_{n(\bar{n})} \lesssim \{\lambda^2, \lambda^2, \lambda\} \bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}$. In SCET terminology this region is known as Gluaberr region, in the present work we will refer to it as *soft* region following the notation and terminology of [184]. In order to establish a consistent and free of divergences definition of the TMDPDFs as proton matrix element of exclusively collinear or anti-collinear fields one need to remove the double counting of this region present in hadronic matrix elements. This is one of the most critical point since only after removing the double counting one can obtain a renormalize TMDPDFs together with its corresponding renormalization constant that will cancel out the IR poles that arise in the process of integrating out the hard fields, i.e. in the process of mapping the IR divergences of the full theory onto UV divergences in the effective theory. In the current literature one can find several solution to the double counting of the soft region by the (anti-)collinear fields. Just to mention a few of them:

- (a) One can introduce a rapidity cutoff in the collinear and anti-collinear sector leading to a change in the definition of the background fields. Each TMDPDF acquires a dependence on the introduced cutoff that is compensated in the product of both distribution at the level of the factorized differential cross section. This approach is used in [106, 199, 200].
- (b) The functional integration measure can be redefined by introducing a normalization factor that remove the soft modes. This is a valid approach as long as one assumes that the proton wave function does not depend on the soft fields, which is reasonable in the non-small x limit. The normalization factor responsible for removing the double counting is known as the soft function [20, 43–45, 201]. Currently this is the standard approaches for leading-power calculations. In this work we show that the same simple multiplicative structure also holds for NNLP in the matching onto collinear distributions of geometrical twist-2.
- (c) One can ignore the issue of double counting and obtain the effective operators, demanding later that they must be well-defined. This procedure leads to reconstruct the rapidity renormalization constant. This approach is the one taken in SCET-I [47, 107, 202].

Since we will perform the calculations using light-cone coordinates it is convenient split the (anti-) collinear fermion fields $q_{n(\bar{n})}$ into *good* $\xi_{n(\bar{n})}$ and *bad* $\eta_{n(\bar{n})}$ spin components defined as

$$\begin{aligned} \xi_n &= \frac{\not{n} \not{\bar{n}}}{2} q_n, & \eta_n &= \frac{\not{\bar{n}} \not{n}}{2} q_n, \\ \xi_{\bar{n}} &= \frac{\not{\bar{n}} \not{n}}{2} q_{\bar{n}}, & \eta_{\bar{n}} &= \frac{\not{n} \not{\bar{n}}}{2} q_{\bar{n}}, \end{aligned} \tag{4.10}$$

where as usual n^μ and $\bar{n}^\mu$ are two light-like vectors normalize to one i.e. $n \cdot \bar{n} = 1$. There are three reasons that justify the convenience of the fermion field split in eq. (4.10). The

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first one is their power counting according to $\lambda = \Lambda_{\text{QCD}}/Q$, it can be shown starting from the free quark propagator and projecting out either the *good* or the *bad* component, e.g. $\langle 0 | \frac{\not{n}}{2} T \{ q_n(y) \bar{q}_n(x) \} \frac{\not{n}}{2} | 0 \rangle$ or $\langle 0 | \frac{\not{\bar{n}}}{2} T \{ q_n(y) \bar{q}_n(x) \} \frac{\not{\bar{n}}}{2} | 0 \rangle$ for the collinear quark field, that the good and bad spin components scale as $\xi_{n(\bar{n})} \lesssim Q^{3/2} \lambda$ and $\eta_{n(\bar{n})} \lesssim Q^{3/2} \lambda^2$ respectively [72, 73, 75, 197]. Therefore in the collinear twist expansion, i.e. the expansion in powers of λ , one needs to distinguish between both components. The second reason is due to the quantization of fermionic field after integrating out the hard fields. It can be shown using canonical quantization methods that $\eta_{n(\bar{n})}$ are not independent degrees of freedom and need to be removed using the equations of motions [203], see Appendix C.2. The third reason is that the bad components $\eta_{n(\bar{n})}$ do not have a definite geometrical twist, it can be already seen from the equation of motions in Appendix C.2. For instance, η_n in the light-cone gauge takes the form²

$$\bar{\eta}_n(z) \gamma^- = - \int_{-\infty}^0 ds (\partial_\mu \bar{\xi}_n(s\bar{n} + z) \gamma_T^\mu + i g \bar{\xi}_n(s\bar{n} + z) \not{A}_{n,T}(s\bar{n} + z)), \quad (4.11)$$

which is the sum of two operators, the first one has geometrical twist-1 while the second one has twist-2. The first term will contribute to the matching onto collinear PDFs of geometrical twist-2, while the second leads to contributions proportional to collinear PDFs of twist-3. In the present work we are mainly interested in the first one.

To conclude this section let us just briefly enumerate the steps to follow in order to obtain the matching of eq. (2.14) onto collinear unpolarized TMDPDFs:

- (1) Using eq. (4.9) one can compute using standard perturbation theory for the hard (*fast*) fields the virtual and radiative corrections to the electroweak vertex under a present of (anti-)collinear (*slow*) fields that play a back ground source. From the point of view of the hard fields, the background fields are classical fields, therefore they can be expanded around a certain point in a Taylor series.
- (2) Using Fierz identities the product of two effective electroweak currents can be written as the convolution of two operators, containing only (anti-)collinear fields and a hard partonic scattering process.
- (3) The composite operators of background fields can be classified according to their collinear twist and their hadronic matrix elements parametrized by Transverse Momentum Depend Parton Distribution Functions (TMDPDFs), see. [53, 77, 182, 184, 204]. From the plethora of distributions we focus on unpolarized TMDPDFs. We obtain that operators with collinear twist-2, 3 and 4 contribute to the uTMDPDFs.
- (4) The virtual corrections to the electroweak vertex leads to UV and IR divergences. The UV divergences are removed through the renormalization of the coupling constant and fields. The IR divergences are an indication of the inconsistency in our calculations, since we are integration over soft and collinear modes. The IR poles in the hard region exactly match the UV poles in the renormalization of the TMDPDF, therefore the product of the hard virtual corrections with the bare TMDPDF operators is finite. Thus, one can define the physical renormalized TMDPDF operators and the hard Wilson coefficients

²The light-cone gauge is defined by $\bar{n} \cdot A_n = 0$ and $n \cdot A_{\bar{n}} = 0$

as the objects obtained after subtracting the UV and IR poles from the bare TMDPDF and virtual corrections to the electroweak vertex respectively.

- (5) The case of the short distance contribution given by massless on-shell propagators also leads to IR singularities. As for hard virtual corrections, their origin is due to overlapping of hard and (anti-)collinear modes. In order to remove this double counting we rescale the product of effective currents in eq. (4.9) by a perturbative factor $Z_{ab}(\alpha_s(\mu))$ that will remove the contributions from the regions when the hard fields becomes (anti-)collinear. We obtain that $Z_{ab}(\alpha_s(\mu))$ can be defined as the partonic matrix elements of a TMDPDF-like operator at leading power and collinear twist 2 (sometimes they are referred to as inclusive p_T -dependent beam function, see. [55, 56, 59, 205, 206]).

4.2 Collinear twist expansion and KPC at LO

In this section we start by presenting the algebraic multiplication necessary in order to build from the background fields $q_{n(\bar{n})}$, $\bar{q}_{n(\bar{n})}$ and $A_{n(\bar{n})}^\mu$ gauge invariant operators with definite collinear and geometrical twist. At LO there is neither virtual nor real hard corrections, therefore in the effective current operator is simply obtained by replacing the standard fermion fields in the electroweak current by their (anti-)collinear counterpart, i.e. $\psi \rightarrow q_n + q_{\bar{n}}$. This amounts to

$$\mathcal{J}^\mu \left(\frac{z}{2} \right) = \bar{q}_n \left(\frac{z}{2} \right) \Gamma^\mu q_{\bar{n}} \left(\frac{z}{2} \right) + \bar{q}_{\bar{n}} \left(\frac{z}{2} \right) \Gamma^\mu q_n \left(\frac{z}{2} \right), \quad (4.12)$$

where Γ^μ is some combination of Dirac matrices. Notice there is no term of the form $\bar{q}_{n(\bar{n})} \Gamma^\mu q_{n(\bar{n})}$, which is forbidden by momentum conservation. The product of two effective currents amounts to

$$\mathcal{J}^{\nu\dagger} \left(\frac{z}{2} \right) \mathcal{J}^\mu \left(-\frac{z}{2} \right) = [\bar{q}_n \Gamma^\nu q_{\bar{n}}] \left(\frac{z}{2} \right) [\bar{q}_{\bar{n}} \Gamma^\mu q_n] \left(-\frac{z}{2} \right) + [\bar{q}_{\bar{n}} \Gamma^\nu q_n] \left(\frac{z}{2} \right) [\bar{q}_n \Gamma^\mu q_{\bar{n}}] \left(-\frac{z}{2} \right), \quad (4.13)$$

where terms like $[\bar{q}_{\bar{n}} \Gamma^\nu q_n](z/2)[\bar{q}_{\bar{n}} \Gamma^\mu q_n](-z/2)$ are forbidden. Using Fierz identity (see eq. (B.8)) we can rewrite eq. (4.13) as

$$\begin{aligned} \mathcal{J}^{\nu\dagger} \left(\frac{z}{2} \right) \mathcal{J}^\mu \left(-\frac{z}{2} \right) &= -\frac{1}{16} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \bar{q}_{i,n}^a \left(\frac{z}{2} \right) (\Gamma^C)_{il} q_{l,n}^b \left(-\frac{z}{2} \right) \bar{q}_{k,\bar{n}}^b \left(-\frac{z}{2} \right) (\Gamma^D)_{kj} q_{j,\bar{n}}^a \left(-\frac{z}{2} \right) \\ &= -\frac{1}{16} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \bar{q}_{i,\bar{n}}^a \left(\frac{z}{2} \right) (\Gamma^C)_{il} q_{l,\bar{n}}^b \left(-\frac{z}{2} \right) \bar{q}_{k,n}^b \left(-\frac{z}{2} \right) (\Gamma^D)_{kj} q_{j,n}^a \left(\frac{z}{2} \right), \end{aligned} \quad (4.14)$$

where an implicit sum in the Dirac i, j, k, l and color a, b indices is understood. The minus sign in front comes from an odd number of permutation of the fermionic fields, therefore the minus sign in front. Since the collinear and anti-collinear background fields do not interact, the combinations $q_{i,n(\bar{n})}^a q_{j,n(\bar{n})}^b$ must be color singlets. We perform the replacement [37, 43, 47, 107, 197]

$$q_{i,n(\bar{n})}^a q_{j,n(\bar{n})}^b \rightarrow \frac{\delta_{ab}}{N_c} q_{i,n(\bar{n})}^c q_{j,n(\bar{n})}^c, \quad (4.15)$$

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where we average over color. The product of effective currents in eq. (4.13) thus becomes

$$\begin{aligned} \mathcal{J}^{\nu\dagger}\left(\frac{z}{2}\right)\mathcal{J}^{\mu}\left(\frac{z}{2}\right) = & -\frac{1}{16N_c}\sum_{C,D}\text{Tr}[\Gamma_C\Gamma^\nu\Gamma_D\Gamma^\mu]\bar{q}_n^c\left(\frac{z}{2}\right)\Gamma^Cq_n^c\left(-\frac{z}{2}\right)\bar{q}_n^d\left(-\frac{z}{2}\right)\Gamma^Dq_n^d\left(\frac{z}{2}\right) \\ & -\frac{1}{16N_c}\sum_{C,D}\text{Tr}[\Gamma_C\Gamma^\nu\Gamma_D\Gamma^\mu]\bar{q}_n^c\left(\frac{z}{2}\right)\Gamma^Cq_n^c\left(-\frac{z}{2}\right)\bar{q}_n^d\left(-\frac{z}{2}\right)\Gamma^Dq_n^d\left(\frac{z}{2}\right), \end{aligned} \quad (4.16)$$

where we have used $\sum_{a,b}\delta_{ab}\delta_{ab} = \sum_a\delta_{aa} = N_c$ to cancel one of the $1/N_c$ coming from the replacement given in eq. (4.15). The fields binomial $\bar{q}_n^c(z)\Gamma^\mu q_n^c(-z)$ is a non-local operator evaluated at the space-time points z^μ and $-z^\mu$, and thus it is not gauge invariant. We already discussed this issue in Sec. 3.2.2, where we introduced the Wilson line (see eqs. (3.36) and (3.37)), which leaves non-local operators gauge invariant. Therefore, the field operators given eq. (4.16) after introducing the corresponding Wilson lines become

$$\begin{aligned} \bar{q}_n^c\left(\frac{z}{2}\right)\Gamma^Cq_n^c\left(-\frac{z}{2}\right) & \rightarrow \bar{q}_n^a\left(\frac{z}{2}\right)\Gamma^C[z/2, -z/2]^{ab}q_n^b\left(-\frac{z}{2}\right), \\ \bar{q}_n^c\left(-\frac{z}{2}\right)\Gamma^Cq_n^c\left(\frac{z}{2}\right) & \rightarrow \bar{q}_n^a\left(-\frac{z}{2}\right)\Gamma^C[-z/2, z/2]^{ab}q_n^b\left(\frac{z}{2}\right), \end{aligned} \quad (4.17)$$

where $[y, x]$ is given by

$$[y, x] = P \exp \left\{ ig \int_x^y dz_\mu A^\mu(z) \right\}, \quad (4.18)$$

and P is the symbol for path ordering i.e.

$$P[A^\mu(z_j)A^\nu(z_i)] = A^\mu(z_j)A^\nu(z_i), \quad y^\alpha \geq z_j^\alpha \geq z_i^\alpha \geq x^\alpha. \quad (4.19)$$

The gluon field A^μ is (anti-)collinear if the fields in the non-local operator are (anti-)collinear. The operators given in eq. (4.17) are gauge invariant and the path ordering operator ensures path independence, see. [43, 79, 84, 173, 197, 207–209]. Therefore any parametrization that links the space-time points y^α and x^α is allowed, however some parametrization could be easier to handle than others from the fixed-order computation point of view. For instance, a straight line parametrization, i.e.

$$[y, x] = P \exp \left\{ ig \int_0^1 d\sigma (y-x)^\mu A_\mu(\sigma(y-x) + x) \right\}, \quad (4.20)$$

would be preferable over more complicated one. Nevertheless, a straight line parametrization like the one above is not the most suitable to perform a light-cone OPE, since it contains collinear, anti-collinear and transverse component of the gluon field. One can exploit the path ordering of the Wilson lines and one of its most useful properties, namely path transitivity (sometimes also refereed as causality), see [207, 208]. This means that given three points x^α , y^α and z^α , the Wilson line connecting x^α and y^α can be written as

$$[z, x] = [z, y][y, x], \quad (4.21)$$

provided $[z, y]$ and $[y, x]$ must have the same parametrization, i.e. the path circles back from y^α to z^α in order to cancel the excess contributions introduced by $[y, x]$, see fig. 4.2. Therefore,

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one can split the Wilson line $[y, x]$ with a collinear gluon field, A_n^μ , in the following manner, see fig. 4.3:

$$[y, x] = [y, -\infty n + (y \cdot n)\bar{n} + y_\perp][-\infty n + (y \cdot n)\bar{n} + y_\perp, -\infty n + (y \cdot n)\bar{n} + x_\perp] \times [-\infty n + (y \cdot n)\bar{n} + x_\perp, -\infty n + (x \cdot n)\bar{n} + x_\perp][-\infty n + (x \cdot n)\bar{n} + x_\perp, x]. \quad (4.22)$$

The split for the Wilson line $[y, x]$ presented in eq. (4.22) can be simplified further depending the choice of gauge fixing for the background (anti-)collinear gluon fields $A_{n(\bar{n})}^\mu$. In covariant gauge all the component of the background gluon fields vanish at light-cone infinity, i.e. $A_n^\mu(\pm\infty n) = 0$ and $A_{\bar{n}}^\mu(\pm\infty \bar{n}) = 0$. The family of gauges where all gluon field components vanish are known as regular gauges [43, 106, 184, 197, 210]. The Wilson lines along the transverse and anti-collinear direction amounts to 1

$$\begin{aligned} [-\infty n + (y \cdot n)\bar{n} + x_\perp, -\infty n + (x \cdot n)\bar{n} + x_\perp] &= 1, \\ [-\infty n + (y \cdot n)\bar{n} + y_\perp, -\infty n + (y \cdot n)\bar{n} + x_\perp] &= 1. \end{aligned} \quad (4.23)$$

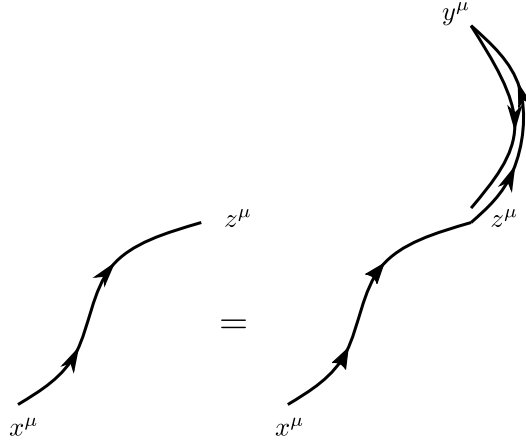


Figure 4.2: Additivity of two Wilson lines. The Wilson line connecting two points x^μ and z^μ can be decomposed as the product of two Wilson lines: one from x^μ to y^μ and a second one from y^μ to z^μ , where y^μ does not necessarily lay on the path joining x^μ and z^μ if and only if the parametrization used in the Wilson line from x^μ to y^μ is the same than from y^μ to z^μ .

Therefore, the gauge-invariant non-local operator with two (anti-)collinear background quark fields are given by

$$\begin{aligned} \mathcal{O}_n^{[\Gamma^C]}(y, x) &= \bar{q}_n^b(y)[y, -\infty n + (y \cdot n)\bar{n} + y_\perp]^{bc} \Gamma^C [-\infty n + (x \cdot n)\bar{n} + x_\perp, x]^{ca} q_n^a(x), \\ \mathcal{O}_{\bar{n}}^{[\Gamma^C]}(y, x) &= \bar{q}_{\bar{n}}^b(y)[y, -\infty \bar{n} + (y \cdot \bar{n})n + y_\perp]^{bc} \Gamma^C [-\infty \bar{n} + (x \cdot \bar{n})n + x_\perp, x]^{ca} q_{\bar{n}}^a(x). \end{aligned} \quad (4.24)$$

Using the operator in eq. (4.24) the product of effective currents is given by

$$\begin{aligned} \mathcal{J}^{\nu\dagger}\left(\frac{z}{2}\right) \mathcal{J}^\mu\left(-\frac{z}{2}\right) &= -\frac{1}{16N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \mathcal{O}_n^{[\Gamma^C]}\left(\frac{z}{2}, -\frac{z}{2}\right) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}\left(-\frac{z}{2}, \frac{z}{2}\right) \\ &\quad - \frac{1}{16N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \mathcal{O}_{\bar{n}}^{[\Gamma^C]}\left(\frac{z}{2}, -\frac{z}{2}\right) \mathcal{O}_n^{[\Gamma^D]}\left(-\frac{z}{2}, \frac{z}{2}\right). \end{aligned} \quad (4.25)$$

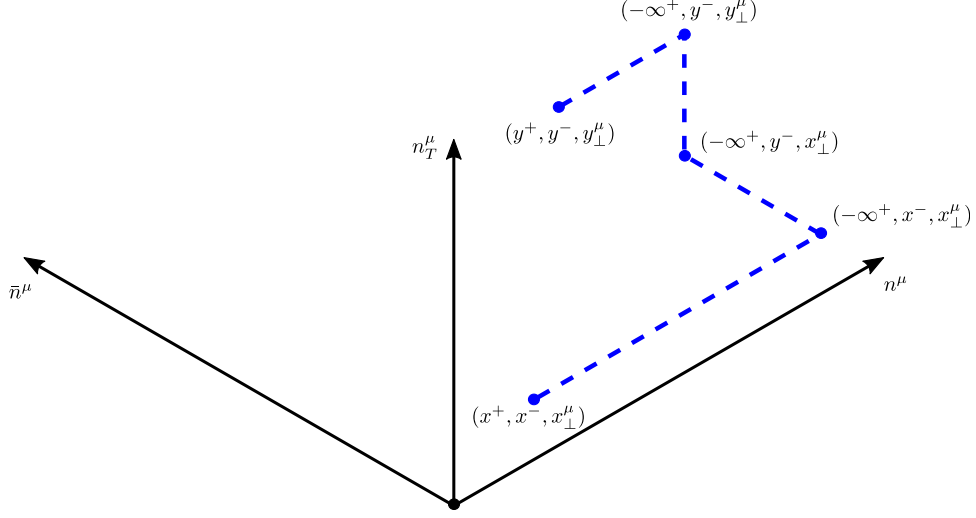


Figure 4.3: Wilson line connecting x^μ and y^μ via successive paths parametrized by straight lines. In a non-singular gauge [41, 210, 211] the Wilson lines with the gluon field evaluated at the light-cone minus infinity amount to 1.

The operators in eq. (4.25) are not proper TMDPDF yet, rather they can be thought of as quark-quark correlators with general field separation. The TMDPDF operators are obtained from the ones in eq. (4.25) by performing an expansion around $z^{+(-)} = 0$ for $\mathcal{O}_{n(\bar{n})}$. The leading-order contribution, $\mathcal{O}_{n(\bar{n})}(z^{+(-)} = 0)$, is referred to as leading-power (LP) contribution; while terms containing derivatives of the operators at $z^{+(-)} = 0$ are known as kinematic power corrections (KPC) [184, 212]. One further needs to project the quark fields $q_{n(\bar{n})}$ onto *good*, $\xi_{n(\bar{n})}$, and *bad*, $\eta_{n(\bar{n})}$, spin component and further write $\eta_{n(\bar{n})}$ as function of $\xi_{n(\bar{n})}$ (and gluon fields), see eqs. (C.24) to (C.27) in Appendix C.2. The expansion of background fields in the effective vertex operators into *good*, $\xi_{n(\bar{n})}$ and *bad*, $\eta_{n\bar{n}}$, (in the sense of [184, 185, 212, 213]) spin field components is known as collinear twist expansion [69, 75, 189, 190, 213]. For every collinear twist we have potentially an infinite number of KPC. In the following sections we present how to sum KCPs for operators of collinear twist-2, 3 and 4. In particular we will focus in the KPC of an operator of collinear twist- n that matches onto collinear PDFs of geometrical twist-2.

4.2.1 Collinear twist-2 at LO

The collinear twist-2 contributions are obtained when all the fields in the effective vertex operator are *good* spin components, i.e $\xi_{n(\bar{n})}$, and thus the product of two effective currents in eq. (4.25) amounts to

$$\begin{aligned} \mathcal{J}_{\text{c-tw } 2}^{\nu\dagger}(z) \mathcal{J}_{\text{c-tw } 2}^\mu(0) &= -\frac{1}{4N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \mathcal{O}_{n,\text{c-tw } 2}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n},\text{c-tw } 2}^{[\Gamma^D]}(0, z) \\ &\quad - \frac{1}{4N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] \mathcal{O}_{\bar{n},\text{c-tw } 2}^{[\Gamma^C]}(z, 0) \mathcal{O}_{n,\text{c-tw } 2}^{[\Gamma^D]}(0, z), \end{aligned} \quad (4.26)$$

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where we have shifted the fields by $z^\mu/2$ since the operators only depends on the relative separations and the subscript "c-tw 2" indicates a collinear twist-2 operator. Furthermore

$$\mathcal{O}_{n,\text{c-tw } 2}^{[\Gamma^C]}(y, x) = \bar{\xi}_n^b(y)[y, -\infty \bar{n} + y^- n + y_\perp]^{bc} \frac{\Gamma^C}{2} [-\infty \bar{n} + x^- n + x_\perp, x]^{ca} \xi_n^a(x), \quad (4.27)$$

$$\mathcal{O}_{\bar{n},\text{c-tw } 2}^{[\Gamma^C]}(y, x) = \bar{\xi}_{\bar{n}}^b(y)[y, -\infty n + y^+ \bar{n} + y_\perp]^{bc} \frac{\Gamma^C}{2} [-\infty n + x^+ \bar{n} + x_\perp, x]^{ca} \xi_{\bar{n}}^a(x). \quad (4.28)$$

Notice that we have included a factor $\frac{1}{2}$ in the definitions of the operators $\mathcal{O}_{\text{c-tw } 2}^{[\Gamma]}$, a sum over the Dirac and color indices is understood. The operators $\mathcal{O}_n$ and $\mathcal{O}_{\bar{n}}$ in eq. (4.26) have to be expanded around $z^- = 0$ and $z^+ = 0$ respectively in order to write them as a sum of TMDPDF operators. Nevertheless the dependence on $z^\pm$ in the Wilson lines hinders the Taylor expansion. There are several alternatives to handle this issue: one can replace the non-local form of the operators in eq. (4.26) by an infinite series of local operators, work with this local representation, perform the corresponding algebraic manipulation to obtain the decomposition and at the end sum back into a non-local representation, see [185, 212, 214]; an other approach is to directly work with the non-local expression but off the light-cone, perform the corresponding manipulations and afterwards take the light-cone limit see [79, 182, 215]. However, we can exploit the freedom of choosing different gauge fixing conditions for hard (or quantum), $B^\mu(z)$, and background (or *classical*), $A_{n(\bar{n})}^\mu(z)$, gluons fields. Instead of working in a regular gauge for the background fields, we can always choose to work in the light-cone gauge [183, 184, 209–211, 216, 217]

$$\bar{n} \cdot A_n^\mu(z) = 0, \quad n \cdot A_{\bar{n}}^\mu(z) = 0, \quad (4.29)$$

where the Wilson lines along the light-like directions in eqs. (4.27) and (4.28) amount to one. Nonetheless, it is well known that in the light-cone gauge not all the gluon fields component vanish in the light-cone infinity³ [207, 209–211, 216, 218–223]. In this case one can mainly proceed in two different ways: one can introduce a Wilson line along the transverse directions in order to restore the gauge invariance of the operators in eqs. (4.26) [210, 211, 216]; or use the residual gauge invariance left by the light-cone gauge regarding transformations independent of $z^{+(-)}$, which can be removed by the appropriate boundary conditions [183, 184]. In this work we choose the second option, implementing the boundary conditions

$$\lim_{z^+ \rightarrow -\infty} A_n^\mu(z^+) = 0, \quad \lim_{z^- \rightarrow -\infty} A_{\bar{n}}^\mu(z^-) = 0. \quad (4.30)$$

After performing the expansion around $z^{-(+)} = 0$ we can always re-write the operators in a form that is explicitly gauge invariant and valid for gauge fixing conditions even though the derivation has been done in the light-cone gauge, see App. C.2 and [183, 184]. In the light-cone gauge the expansion around $z^- = 0$ of the operator $\mathcal{O}_n^{[\Gamma^C]}(-z, z)$ amounts to [212]

$$\mathcal{O}_{n,\text{c-tw } 2}^{[\Gamma^C]}(z, 0) = \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \partial_+^2 \bar{\xi}_n(z) \frac{\Gamma^C}{2} \xi_n(0) \Big|_{z^-=0}. \quad (4.31)$$

³The family of gauges where not all the gluon field components vanish at the light-cone infinity are known as singular gauges.

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Using the equations of motions in the light-cone gauge, see eqs. (C.15), (C.16), (C.17) and (C.18), one can formally write

$$\mathcal{O}_{n,\text{c-tw } 2}^{[\Gamma^C]}(z, 0) = \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right) \bar{\xi}_n(z^+, 0^-, z_{\perp}) \frac{\Gamma^C}{2} \xi_n(0), \quad (4.32)$$

where we have introduced the inverse derivative operator defined as [184, 197, 212, 224]

$$\frac{1}{\partial_-} g(z^+) := \int_{-\infty}^0 ds g(z^+ + s) \quad (4.33)$$

with $g(z^+)$ an analytic function going to zero at infinity rapidly enough. In order for the Fourier transform of the matrix elements of the operator in eq. (4.32) to exist, and therefore be able to defined the TMDPDF in momentum space, some assumptions regarding analyticity and behavior of the fields at infinity are necessary. For this reason we assume that the action of the inverse derivative operator upon the operators in eq. (4.32) is well-defined. In order to proceed further we need to specify the Dirac structure in Γ^C . Since we are interested in unpolarized cases only $\Gamma^C = \gamma^{\alpha}$ gives the relevant contribution

$$\mathcal{O}_{n,\text{c-tw } 2}^{[\gamma^{\alpha}]}(z, 0) = n^{\alpha} \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \bar{\xi}_n(z^+, 0, z_{\perp}) \frac{\gamma^-}{2} \xi_n(0), \quad (4.34)$$

where we have used

$$\frac{\gamma^+ \gamma^-}{2} \xi_n = \xi_n, \quad (4.35)$$

and

$$\frac{\gamma^- \gamma^+}{2} \gamma^{\alpha} \frac{\gamma^+ \gamma^-}{2} = n^{\alpha} \gamma^- \frac{\gamma^- \gamma^+}{2}. \quad (4.36)$$

The differential operator in eq. (4.34) leads to simple expressions momentum space. To this end we first take the matrix element of the eq. (4.34) between proton state with momentum P_A ,

$$\langle P_A | \mathcal{O}_{n,\text{c-tw } 2}^{[\gamma^{\alpha}]}(z, 0) | P_A \rangle = n^{\alpha} \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{\partial_-} \right)^n \langle P_A | \bar{\xi}_n(z^+, 0, z_{\perp}) \frac{\gamma^-}{2} \xi_n(0) | P_A \rangle. \quad (4.37)$$

This matrix element can be written for a generic gauge by introducing the corresponding Wilson lines [183, 184], see also App. C.2:

$$\begin{aligned} \langle P_A | \mathcal{O}_{n,\text{c-tw } 2}^{[\gamma^{\alpha}]}(z, 0) | P_A \rangle &= n^{\alpha} \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \\ &\times \langle P_A | \bar{\xi}_n(z^+, 0, z_{\perp}) [z^+ \bar{n} + z_{\perp}, -\infty \bar{n} + z_{\perp}] \frac{\gamma^-}{2} [-\infty \bar{n}, 0] \xi_n(0) | P_A \rangle. \end{aligned} \quad (4.38)$$

The matrix element on the right-hand side is nothing but the unpolarized TMDPDF at leading collinear and geometrical twist in position space [43, 50, 53, 77, 127, 182, 185]

$$\Phi_{n,q \leftarrow p_A}^{[\gamma^-]}(z^+, \mathbf{z}_T) = \langle P_A | \bar{\xi}_n(z^+, 0, z_{\perp}) [z^+ \bar{n} + z_{\perp}, -\infty \bar{n} + z_{\perp}] \frac{\gamma^-}{2} [-\infty \bar{n}, 0] \xi_n(0) | P_A \rangle, \quad (4.39)$$

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which can be written as the Fourier transform of its counterpart in momentum space,

$$\Phi_{n,q \leftarrow p_A}^{[\gamma^-]}(z^+, \mathbf{z}_T) = P_A^- \int dx d^{d-2} \mathbf{p}_T e^{iz^+ x P_A^- - i \mathbf{z}_T \cdot \mathbf{p}_T} f_{q \leftarrow p_A}(x, \mathbf{p}_T) \quad (4.40)$$

where $z_\perp^\mu = -\mathbf{z}_T^\mu$. Replacing eq. (4.40) in eq. (4.38) one obtains

$$\langle P_A | \mathcal{O}_{n,c\text{-tw } 2}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = n^\alpha P_A^- \int dx d^{d-2} \mathbf{p}_T \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i \mathbf{p}_T^2 z^-}{2x P_A^-} \right)^n e^{iz^+ x P_A^- - i \mathbf{z}_T \cdot \mathbf{p}_T} f_{q \leftarrow p_A}(x, \mathbf{p}_T). \quad (4.41)$$

The sum amounts to an imaginary exponential, $\exp\{i \frac{\mathbf{p}_T^2 z^-}{2x P_A^-}\}$. Introducing the variables $p^- = x P_A^-$ and $p^+ = \frac{\mathbf{p}_T^2}{x P_A^-}$, eq. (4.41) can be written as

$$\langle P_A | \mathcal{O}_{n,c\text{-tw } 2}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = n^\alpha P_A^- \int dx d^d p e^{ip \cdot z} \delta(p^- - x P_A^-) \delta(p^2) 2p^- f_{q \leftarrow p_A}(x, \mathbf{p}_T), \quad (4.42)$$

where $d^d p = dp^+ dp^- d^{d-2} \mathbf{p}_T$, the factor $2p^-$ comes from the Jacobian of $\delta(p^2)$ and $1 \geq x \geq -1$ is the fraction of collinear momentum of the parton with respect the proton.

For the case of the operator $\mathcal{O}_{\bar{n},c\text{-tw } 2}^{[\Gamma^D]}(0, z)$, the steps to follow are analogous. The matrix element of the operator

$$\mathcal{O}_{\bar{n},c\text{-tw } 2}^{[\Gamma^D]}(0, z) = \sum_{n=0}^{\infty} \frac{(z^+)^2}{n!} \bar{\xi}_{\bar{n}}(0) \frac{\Gamma^D}{2} \left(-\frac{\partial_\perp^2}{2\partial_+} \right)^n \xi_{\bar{n}}(0, z^-, z_\perp) \Big|_{z^+=0}, \quad (4.43)$$

between proton state of momentum P_B , is related to the anti-quark unpolarized TMDPDF [43, 50, 53, 77, 127, 183, 184]

$$\begin{aligned} \Phi_{\bar{n},\bar{q} \leftarrow p_B}^{[\gamma^+]}(z^-, \mathbf{z}_T) &= \langle P_B | \frac{(\gamma^+)^{ji}}{2} [-\infty n + z_\perp, z^- n + z_\perp] \xi_{\bar{n},i}(0, z^-, z_\perp) \bar{\xi}_{\bar{n},j}(0) [0, -\infty n] | P_B \rangle \\ &= P_B^+ \int dx d^{d-2} \mathbf{p}_T e^{-ix P_B^+ z^- + i \mathbf{p}_T \cdot \mathbf{z}_T} f_{\bar{q} \leftarrow p_B}(x, \mathbf{p}_T), \end{aligned} \quad (4.44)$$

in the following manner:

$$\langle P_B | \mathcal{O}_{\bar{n},c\text{-tw } 2}^{[\gamma^\alpha]}(0, z) | P_B \rangle = -\bar{n}^\alpha P_B^+ \int dx d^d p e^{-ip \cdot z} \delta(p^+ - x P_B^+) \delta(p^2) 2p^+ f_{\bar{q} \leftarrow p_B}(x, \mathbf{p}_T). \quad (4.45)$$

Here we have introduced the variables $p^+ = x P_B^+$ and $p^- = \frac{\mathbf{p}_T^2}{x P_B^+}$; the integration measure is $dx; d^d p = dx dp^+ dp^- d^{d-2} \mathbf{p}_T$ with $1- \geq x \geq -1$ the anti-collinear momentum fraction of the parton with respect the proton p_B ; the factor $2p^+$ comes from the Jacobian of $\delta(p^2)$ and the minus sign comes from the anti-commutation of the fermionic fields in the definition of the anti-quark TMDPDFs. Notice that in the definition of the unpolarized quark and anti-quark TMDPDFs there is a relative sign in the complex exponential, since the range $1 \geq x \geq 0$ is identified with the quark TMDPDF and $0 \geq x \geq -1$ with the anti-quark TMDPDF [50, 53, 77, 183, 184]. The quark and anti-quark TMDPDFs are related by charge conjugation [53, 77, 183]

$$(\mathcal{O}_{q \leftarrow P})_{ij}^*(p; P) = (-\gamma^1 \gamma^3)_{ik} (\mathcal{O}_{q \leftarrow P})_{kl}(\bar{p}; \bar{P}) (\gamma^1 \gamma^3)_{lj}, \quad (4.46)$$

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where $\bar{v}^\mu = (v^0, -\mathbf{v})$ and

$$\begin{aligned} (\mathcal{O}_{q \leftarrow P})_{ij}(p, P) &= \int \frac{d^4 z}{(2\pi)^4} e^{-ip \cdot z} \langle P | \bar{\xi}_j(z) \xi_i(0) | P \rangle \\ &= \int \frac{d^4 z}{(2\pi)^4} e^{-ip \cdot z} \langle P | \bar{\xi}_j(0) \xi_i(-z) | P \rangle = \int \frac{d^4 z}{(2\pi)^4} e^{ip \cdot z} \langle P | \bar{\xi}_j(0) \xi_i(z) | P \rangle, \end{aligned} \quad (4.47)$$

To go from the first line to the second we have used the fact that forward matrix elements of an operator are invariant under a shift of the fields origin [114, 183–185, 212]⁴, and thus

$$\langle P | \mathcal{O}(z) | P \rangle = \langle P | \mathcal{O}(z + y) | P \rangle, \quad (4.48)$$

We have dropped the Wilson lines, one can perform the algebraic manipulations in the light-cone gauge and any conclusion drawn are also valid in any other general gauge. Explicitly, charge conjugation in eq. (4.46) yields

$$(\mathcal{O}_{q \leftarrow P})_{ij}^*(p; P) = (-\gamma^1 \gamma^3)_{ik} \int \frac{d^4 z}{(2\pi)^4} e^{ip^0 z^+ + i\mathbf{p} \cdot \mathbf{z}} \langle \bar{P} | \bar{\xi}_l(0) \xi_k(z) | \bar{P} \rangle (\gamma^1 \gamma^3)_{lj}. \quad (4.49)$$

Using the time reversal operator $U(T)$ [77, 192, 193]

$$\begin{aligned} U(T) | P \rangle &= | \bar{P} \rangle, \\ U(T) \xi(z) U^\dagger(T) &= \gamma^1 \gamma^3 \xi(-z^0, \mathbf{z}), \\ U(T) \bar{\xi}(z) U^\dagger(T) &= \bar{\xi}(-z^0, \mathbf{z}) \gamma^3 \gamma^1, \end{aligned} \quad (4.50)$$

the charge-conjugate operator for the quark-quark correlator in eq. (4.49) becomes

$$\begin{aligned} (\mathcal{O}_{q \leftarrow P})_{ij}^*(p; P) &= (-\gamma^1 \gamma^3)_{ik} (\gamma^1 \gamma^3)_{lj} \int \frac{d^4 z}{(2\pi)^4} e^{ip^0 z^+ + i\mathbf{p} \cdot \mathbf{z}} \\ &\quad \times \langle \bar{P} | U^\dagger(T) U(T) \bar{\xi}_l(0) U^\dagger(T) U(T) \xi_k(z^0, \mathbf{z}) U^\dagger(T) U(T) | \bar{P} \rangle \\ &= \int \frac{d^4 z}{(2\pi)^4} e^{ip^0 z^+ + i\mathbf{p} \cdot \mathbf{z}} \langle P | \bar{\xi}_j(0) \xi_i(-z^0, \mathbf{z}) | P \rangle. \end{aligned} \quad (4.51)$$

By sending $\mathbf{z} \rightarrow -\mathbf{z}$ and shifting the origin of the fields, we arrive to

$$(\mathcal{O}_{q \leftarrow P})_{ij}^*(p; P) = \int \frac{d^4 z}{(2\pi)^4} e^{ip \cdot z} \langle P | \bar{\xi}_j(z) \xi_i(0) | P \rangle. \quad (4.52)$$

Therefore inverting the eq. (4.52) for the transverse and anti-collinear components leads to

$$(\mathcal{O}_{q \leftarrow P})_{ij}^*(p^-, z^-, \mathbf{z}_T; P) = \int \frac{dz^+}{2\pi} e^{ip^- z^+} \langle P | \bar{\xi}_j(z) [z, -\infty \bar{n} + z^- n + z_\perp] [-\infty \bar{n}, 0] \xi_i(0) | P \rangle, \quad (4.53)$$

⁴This property is also sometime stated as follows: the total derivative of a forward scattering matrix element

$$\langle p_2 | \partial_\mu \mathcal{O} | p_1 \rangle = i(p_2 - p_1) \langle p_2 | \mathcal{O} | p_1 \rangle,$$

is zero due to momentum conservation.

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while the operator without charge conjugation in the mixed momentum-position space is given by

$$(\mathcal{O}_{q \leftarrow P})_{ij}(p^-, z^-, \mathbf{z}_T; P) = \int \frac{dz^+}{2\pi} e^{-ip^- z^+} \langle P | \bar{\xi}_j(z) [z, -\infty \bar{n} + z^- n + z_\perp] [-\infty \bar{n}, 0] \xi_i(0) | P \rangle, \quad (4.54)$$

Notice the sign difference in the complex exponential. Projecting onto the Dirac structure of interest, which in the collinear case is γ^- (while in the anti-collinear it is γ^+), and setting $z^- = 0$ (in the anti-collinear case $z^+ = 0$), we obtain the unpolarized quark distributions at leading power [50, 53, 77, 183, 184]

$$f_{q \leftarrow P}(x, \mathbf{z}_T) = \int \frac{dz^+}{2\pi} e^{-ixP^- z^+} \langle P | \bar{\xi}_{j,n}(z^+, 0, z_\perp) [z^+ \bar{n} + z_\perp, -\infty \bar{n} + z_\perp] \frac{\gamma_{ji}^-}{2} [-\infty \bar{n}, 0] \xi_{i,n}(0) | P \rangle \quad (4.55)$$

while the collinear anti-quark TMDPDF, at leading power, is given by [50, 53, 77, 183, 184]

$$f_{\bar{q} \leftarrow P}(x, \mathbf{z}_T) = \int \frac{dz^+}{2\pi} e^{-ixP^- z^+} \langle P | \frac{\gamma_{ji}^-}{2} [-\infty \bar{n}, 0] \xi_{i,n}(0) \bar{\xi}_{n,j}(z^+, 0, z_\perp) [z^+ \bar{n} + z_\perp, -\infty \bar{n} + z_\perp] | P \rangle. \quad (4.56)$$

Thus

$$f_{\bar{q} \leftarrow P}(x, \mathbf{z}_T) = -\text{Tr}[\gamma_{ji}^- (\mathcal{O}_{q \leftarrow P})_{ij}^*(-p^-, 0, \mathbf{z}_T; P)] = -f_{q \leftarrow P}(-x, \mathbf{z}_T) \quad (4.57)$$

where the minus sign comes from the anti-commutation of the fermionic fields. Therefore we can formally write

$$f_{q \leftarrow P}(x, \mathbf{z}_T) = (\theta(x) + \theta(-x)) f_{q \leftarrow P}(x, \mathbf{z}_T) = \theta(x) f_{q \leftarrow P}(x, \mathbf{z}_T) - \theta(-x) f_{\bar{q} \leftarrow P}(-x, \mathbf{z}_T), \quad (4.58)$$

where we identify negative values of the fraction momentum with anti-quarks. Thus eq. (4.45) amounts to

$$\begin{aligned} \langle P_B | \mathcal{O}_{\bar{n}, \text{c-tw } 2}^{[\gamma^\alpha]}(0, z) | P_B \rangle &= -2P_B^+ \bar{n}^\alpha \int dx d^d p e^{-ip \cdot z} \delta(p^+ - xP_B^+) \delta(p^2) p^+ \theta(-x) f_{\bar{q} \leftarrow P_B}(x, \mathbf{p}_T) \\ &= 2P_B^+ \bar{n}^\alpha \int dx d^d p e^{ip \cdot z} \delta(p^+ - xP_B^+) \delta(p^2) p^+ \theta(x) f_{\bar{q} \leftarrow P_B}(-x, \mathbf{p}_T), \end{aligned} \quad (4.59)$$

where we have sent $x \rightarrow -x$ and $p^+ \rightarrow -p^+$, since in the unpolarized case the orientation of the transverse component is irrelevant. One can interpret eq. (4.58) in the parton model as: the collective result of quark minus anti-quark propagating parallel to the proton momentum is pseudo-particle, i.e. parton, participating in the effective electroweak vertex.

Using eqs. (4.42) and (4.59) the matrix element between two proton states, $|P_A P_B\rangle$, of the effective electroweak currents product in eq. (4.26) amounts to

$$\begin{aligned} \langle P_B P_A | \mathcal{J}_{\text{c-tw } 2}^{\nu\dagger}(z) \mathcal{J}_{\text{c-tw } 2}^\mu(0) | P_A P_B \rangle &= -\frac{P_A^- P_B^+}{N_c} n^\alpha \bar{n}^\beta \sum_{a,b} \text{Tr}[\gamma_\alpha \Gamma^\nu \gamma_\beta \Gamma^\mu] \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] \\ &\times e^{iz \cdot p_a + iz \cdot p_b} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^- - \xi_b P_B^+) p_a^- p_b^+ f_{n, a \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}) \\ &+ n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.60)$$

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where $1 \geq \xi_{a(b)} \geq 0$, $[d^d p] = d^d p \delta(p^2)$ and $\dots$ denotes further Dirac structures that do not contribute to the unpolarized case.

Before we proceed to study the contributions at collinear twist 2 and 4, it is worth adding some comments regarding the computations above. One may be concerned about the integration in momentum space of the inverse derivative operator, since this is required to act on analytic functions that go to zero rapidly enough at the light-cone infinity. However, once we replace the TMDPDF matrix elements in position space by their counterparts in momentum space the inverse derivative operator leads to integrals of the form

$$\int_{-\infty}^0 ds e^{-isp}, \quad (4.61)$$

which is obviously not well-defined if $p \geq 0$ (or $p \leq 0$)⁵. One could use some regularization procedure for eikonal propagators, like the δ -regularization [43, 50, 106, 123, 183, 225] where the above integral is replaced by

$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^0 ds e^{-isp+s\delta}. \quad (4.62)$$

Nevertheless, the sum of all contributions coming from the inverse derivative operator is well-defined and therefore no regularization is needed. It is also worth noticing that in eqs. (4.32) and (4.43) we have only considered contributions of geometrical twist 2. Terms proportional to the gluon fields give contributions of at least of geometrical twist 3 and are thus neglected.

4.2.2 Collinear twist 3 at LO

At collinear twist 3 at least one of the operators in eq. (4.25) contains at least one *bad* spin component of the background fermion fields. The product of the two effective electroweak currents amounts to

$$\begin{aligned} \mathcal{J}_{c\text{-tw } 3}^{\nu\dagger}(z) \mathcal{J}_{c\text{-tw } 3}^{\mu}(0) &= -\frac{1}{4N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] [\mathcal{O}_{n,c\text{-tw } 2}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^D]}(0, z) \\ &\quad + \mathcal{O}_{n,c\text{-tw } 3}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n},c\text{-tw } 2}^{[\Gamma^D]}(0, z) + \mathcal{O}_{n,c\text{-tw } 3}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^D]}(0, z)] \\ &\quad + n \leftrightarrow \bar{n}, \end{aligned} \quad (4.63)$$

where the operators are given by

$$\begin{aligned} \mathcal{O}_{n,c\text{-tw } 3}^{[\Gamma^C]}(z, 0) &= \bar{\eta}_n(z) [z, -\infty \bar{n} + z^- n + z_\perp] \frac{\Gamma^C}{2} [-\infty \bar{n}, 0] \xi_n(0) \\ &\quad + \bar{\xi}_n(z) [z, -\infty \bar{n} + z^+ n + z_\perp] \frac{\Gamma^C}{2} [-\infty \bar{n}, 0] \eta_n(0), \end{aligned} \quad (4.64)$$

⁵If p can take negative and positive values, then the integral is well-defined as a distribution action on a test function. One can show

$$\int_{-\infty}^0 ds e^{-isp} = i\text{P.V.} \left(\frac{1}{p} \right) + \pi \delta(p),$$

where $\text{P.V.}(1/p)$ is the principal value distribution.

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$$\begin{aligned}\mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^C]}(0, z) &= \bar{\eta}_{\bar{n}}(0)[0, -\infty n] \frac{\Gamma^C}{2} [-\infty n + z^+ \bar{n} + z_{\perp}, z] \xi_{\bar{n}}(z) \\ &+ \bar{\xi}_{\bar{n}}(0)[0, -\infty n] \frac{\Gamma^C}{2} [-\infty n + z^+ \bar{n} + z_{\perp}, z] \eta_{\bar{n}}(z).\end{aligned}\quad (4.65)$$

Expanding $\mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^C]}(z, 0)$ around $z^- = 0$ in the light-cone gauge, one obtains

$$\mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^C]}(z, 0) = \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} [\partial_+^n \bar{\eta}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\Gamma^C}{2} \xi_n(0) + \partial_+^n \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\Gamma^C}{2} \eta_n(0)]. \quad (4.66)$$

Using the solutions to the fermion equations of motions given in App. C.2 and neglecting terms containing gluons fields one can write the collinear partial derivative of the bad spin component as

$$\partial_+^n \bar{\eta}_{\bar{n}}(z) = - \int_{-\infty}^0 ds \partial_{\mu} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \bar{\xi}_{\bar{n}}(s\bar{n} + z) \gamma_{\perp}^{\mu} \frac{\not{n}}{2} + \dots, \quad (4.67)$$

where $\dots$ stands for terms that lead to contributions of geometrical twist 3 or higher, we obtain

$$\begin{aligned}\mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\Gamma^C]}(z, 0) &= - \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \left[\frac{\partial_{\mu}}{\partial_-} \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\gamma_{\perp}^{\mu} \not{n} \Gamma^C}{4} \xi_n(0) \right. \\ &\quad \left. + \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\Gamma^C \not{n} \gamma_{\perp}^{\mu}}{4} \frac{\partial_{\mu}}{\partial_-} \xi(0) \right].\end{aligned}\quad (4.68)$$

Only the element $\Gamma^C = \gamma^{\alpha}$ of the Dirac basis contributes to the unpolarized case, therefore eq. (4.68) amounts to

$$\begin{aligned}\mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\gamma^{\alpha}]}(z, 0) &= - \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \left[\frac{\partial_{\mu}}{\partial_-} \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\gamma_{\perp}^{\mu} \not{n} \gamma_{\perp}^{\alpha}}{4} \xi_n(0) \right. \\ &\quad \left. + \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\gamma_{\perp}^{\alpha} \not{n} \gamma_{\perp}^{\mu}}{4} \frac{\partial_{\mu}}{\partial_-} \xi(0) \right],\end{aligned}\quad (4.69)$$

where we have used

$$\frac{\gamma_{\perp}^{\mu} \not{n} \gamma_{\perp}^{\alpha}}{4} \frac{\not{n} \not{n}}{2} = \frac{\gamma_{\perp}^{\mu} \not{n} \gamma_{\perp}^{\alpha}}{4} \frac{\not{n} \not{n}}{2}, \quad \frac{\not{n} \not{n} \gamma^{\alpha} \not{n} \gamma_{\perp}^{\mu}}{2} = \frac{\not{n} \not{n} \gamma_{\perp}^{\alpha} \not{n} \gamma_{\perp}^{\mu}}{2}. \quad (4.70)$$

Taking the forward matrix element using the state of a proton $|P_A\rangle$, eq. (4.69) amounts to

$$\begin{aligned}\langle P_A | \mathcal{O}_{\bar{n},c\text{-tw } 3}^{[\gamma^{\alpha}]}(z, 0) | P_A \rangle &= - \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_{\perp}^2}{2\partial_-} \right)^n \left[\langle P_A | \frac{\partial_{\mu}}{\partial_-} \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\gamma_{\perp}^{\mu} \not{n} \gamma_{\perp}^{\alpha}}{4} \xi_n(0) | P_A \rangle \right. \\ &\quad \left. + \langle P_A | \bar{\xi}_{\bar{n}}(z^+, 0, z_{\perp}) \frac{\gamma_{\perp}^{\alpha} \not{n} \gamma_{\perp}^{\mu}}{4} \frac{\partial_{\mu}}{\partial_-} \xi(0) | P_A \rangle \right].\end{aligned}\quad (4.71)$$

Using that the forward matrix element of a total derivative of an operator is zero i.e.

$$\langle p | \partial_{\mu} \mathcal{O} | p \rangle = 0 \quad (4.72)$$

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we can move the transverse derivative in the second term from ξ_n to $\bar{\xi}_n$:

$$\begin{aligned} \langle P_A | \mathcal{O}_{n,\text{c-tw } 3}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = & - \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_\perp^2}{2\partial_-} \right)^n \left[\langle P_A | \frac{\partial_\mu}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\gamma_\perp^\mu \not{n} \gamma_\perp^\alpha}{4} \xi_n(0) | P_A \rangle \right. \\ & \left. - \langle P_A | \partial_\mu (\bar{\xi}_n(z^+, 0, z_\perp)) \frac{\gamma_\perp^\alpha \not{n} \gamma_\perp^\mu}{4} \frac{1}{\partial_-} \xi(0) | P_A \rangle \right]. \end{aligned} \quad (4.73)$$

Since we can write the Dirac structure in the first term as⁶

$$\frac{\gamma_\perp^\mu \not{n} \gamma_\perp^\alpha}{4} = -g_\perp^{\mu\alpha} \frac{\not{n}}{2} - \frac{\gamma_\perp^\alpha \not{n} \gamma_\perp^\mu}{4}, \quad (4.74)$$

eq. (4.73) becomes

$$\begin{aligned} \langle P_A | \mathcal{O}_{n,\text{c-tw } 3}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = & \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_\perp^2}{2\partial_-} \right)^n \left[\langle P_A | \frac{\partial^\alpha}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\not{n}}{2} \xi_n(0) | P_A \rangle \right. \\ & \left. + \langle P_A | \frac{\partial_\mu}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\gamma_\perp^\alpha \not{n} \gamma_\perp^\mu}{4} \xi_n(0) | P_A \rangle + \langle P_A | \partial_\mu (\bar{\xi}_n(z^+, 0, z_\perp)) \frac{\gamma_\perp^\alpha \not{n} \gamma_\perp^\mu}{4} \frac{1}{\partial_-} \xi(0) | P_A \rangle \right], \end{aligned} \quad (4.75)$$

In momentum space, the inverse derivative operator acting on the first term on the second line gives a factor $(ip^-)^{-1}$ while acting on the second term gives $(-ip^-)^{-1}$. Thus only the term on the first line contributes. As we did for the case of collinear twist 2 we replace the unpolarized TMDPDFs in position space by the Fourier transform of its corresponding parametrization in momentum space, which simplifies the differential operators and allows us to sum the whole series of terms

$$\langle P_A | \mathcal{O}_{n,\text{c-tw } 3}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = 2P_A^- \int dx [d^d p] e^{ip \cdot z} \delta(p^- - xP_A^-) \mathbf{p}_T^\alpha f_{q \leftarrow P_A}(x, \mathbf{p}_T), \quad (4.76)$$

where as in the case of collinear twist-2 we have introduced $p^+ = \frac{\mathbf{p}_T^2}{2p^-}$ as auxiliary variable and $1 \geq x \geq 0$.

For the case of the operator $\mathcal{O}_{\bar{n},\text{c-tw } 3}^{[\Gamma^C]}(0, z)$ the expansion is performed around $z^+ = 0$. The steps to follow are analogous to the previous one. Using the solutions to the fermion equations of motion in App. C.2, $\partial_-^n \eta_{\bar{n}}$ can be written neglecting gluon terms as

$$\partial_-^n \eta_{\bar{n}}(0, z^-, z_\perp) = -\frac{\not{n}}{2} \left(-\frac{\partial_\perp^2}{2\partial_+} \right)^n \int_{-\infty}^0 ds \not{\partial}_\perp \xi_{\bar{n}}(sn + z) + \dots, \quad (4.77)$$

and $\mathcal{O}_{\bar{n},\text{c-tw } 3}^{[\Gamma^C]}(0, z)$ amounts to

$$\begin{aligned} \mathcal{O}_{\bar{n},\text{c-tw } 3}^{[\Gamma^C]}(0, z) = & - \sum_{n=0}^{\infty} \frac{(z^+)^n}{n!} \left[\frac{\partial_\mu}{\partial_+} \bar{\xi}_{\bar{n}}(0) \frac{\gamma_\perp^\mu \Gamma^C}{4} \xi_{\bar{n}}(0, z^-, z_\perp) \right. \\ & \left. + \bar{\xi}_{\bar{n}}(0) \frac{\Gamma^C \not{n} \gamma_\perp^\mu}{4} \partial_\mu \xi_{\bar{n}}(0, z^-, z_\perp) \right] \left(-\frac{\overleftarrow{\partial}_\perp^2}{2\overleftarrow{\partial}_+} \right)^n. \end{aligned} \quad (4.78)$$

⁶The transverse metric tensor $g_\perp^{\mu\nu}$ is given by

$$g_\perp^{\mu\nu} = g^{\mu\nu} - n^\mu \bar{n}^\nu - n^\nu \bar{n}^\mu.$$

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Taking the forward matrix element using the state of a proton p_B with momentum P_B , and restricting ourselves to the unpolarized case ($\Gamma^C = \gamma^\alpha$), one obtains

$$\langle P_B | \mathcal{O}_{\bar{n}, \text{c-tw } 3}^{[\gamma^\alpha]}(0, z) | P_B \rangle = -2P_B^+ \int dx [d^d p] \delta(p^+ - xP_B^+) e^{-iz \cdot p} \mathbf{p}_T^\alpha f_{\bar{n}, \bar{q} \leftarrow p_B}(x, \mathbf{p}_T) \quad (4.79)$$

where $0 \geq x \geq 1$ and the minus sign once again is coming from the anti-commutation of the fermionic fields in the definition of unpolarized anti-quark TMDPDFs. Doing $x \rightarrow -x$ and $p^+ \rightarrow -p^+$ one obtains

$$\langle P_B | \mathcal{O}_{\bar{n}, \text{c-tw } 3}^{[\gamma^\alpha]}(0, z) | P_B \rangle = P_B^+ \int dx [d^d p] \delta(p^+ - xP_B^+) e^{iz \cdot p} 2p^+ \left(-\frac{\mathbf{p}_T^\alpha}{p^+} \right) f_{\bar{n}, \bar{q} \leftarrow p_B}(x, \mathbf{p}_T). \quad (4.80)$$

Using eqs. (4.76) and (4.78) the matrix element of the product of effective currents in eq. (4.63) evaluated on the two-proton state $|P_A P_B\rangle = |P_A\rangle \otimes |P_B\rangle$ becomes

$$\begin{aligned} \langle P_A P_B | \mathcal{J}_{\text{c-tw } 3}^{\nu\dagger}(z) \mathcal{J}_{\text{c-tw } 3}^\mu(0) | P_A P_B \rangle &= -\frac{P_A^- P_B^+}{N_c} \sum_{a,b} \text{Tr}[\gamma_\alpha \Gamma^\nu \gamma_\beta \Gamma^\mu] \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] \\ &\times e^{iz \cdot p_a + iz \cdot p_b} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^- - \xi_b P_B^-) f_{n, a \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}) \\ &\times [\mathbf{p}_{aT}^\alpha \mathbf{p}_{bT}^\beta + \mathbf{p}_{aT}^\alpha p_b^+ \bar{n}^\beta + \mathbf{p}_{bT}^\beta p_a^- n^\alpha] \\ &+ n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.81)$$

Before we move on to discuss the corrections at collinear twist-4, it is important to stress that in eqs. (4.75) and (4.78) is the total derivative of a non-local operator, i.e. $\mathcal{O}(y, x)$. It refers to changes of the operator by an infinitesimal shift of the coordinates y^μ and x^μ but the relative distance, i.e. $y^\mu - x^\mu$ remains fixed. This should not be confused with the expansion of the operator when this distance goes to zero, that is what we do for $z^-(+) \rightarrow 0$.

4.2.3 Collinear twist 4 at LO

At collinear twist 4 at least one of the operators in eq. (4.25) contains at least two bad spin component of the background fermion fields. The product of two effective electroweak currents amounts to

$$\begin{aligned} \mathcal{J}_{\text{c-tw } 4}^{\nu\dagger}(z) \mathcal{J}_{\text{c-tw } 4}^\mu(0) &= -\frac{1}{4N_c} \sum_{C,D} \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \Gamma^\mu] [\mathcal{O}_{n, \text{c-tw } 4}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n}, \text{c-tw } 4}^{[\Gamma^D]}(0, z) \\ &+ \mathcal{O}_{n, \text{c-tw } 2}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n}, \text{c-tw } 4}^{[\Gamma^D]}(0, z) + \mathcal{O}_{n, \text{c-tw } 4}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n}, \text{c-tw } 2}^{[\Gamma^D]}(0, z) \\ &+ \mathcal{O}_{n, \text{c-tw } 3}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n}, \text{c-tw } 4}^{[\Gamma^D]}(0, z) + \mathcal{O}_{n, \text{c-tw } 4}^{[\Gamma^C]}(z, 0) \mathcal{O}_{\bar{n}, \text{c-tw } 3}^{[\Gamma^D]}(0, z)] \\ &+ n \leftrightarrow \bar{n}, \end{aligned} \quad (4.82)$$

where the operators are given by

$$\mathcal{O}_{n, \text{c-tw } 4}^{[\Gamma^C]}(z, 0) = \bar{\eta}_n(z) [z, -\infty \bar{n} + z^- n + z_\perp] \frac{\Gamma^C}{2} [-\infty \bar{n}, 0] \eta_n(0), \quad (4.83)$$

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$$\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\Gamma^C]}(0, z) = \bar{\eta}_{\bar{n}}(0)[0, -\infty n] \frac{\Gamma^C}{2} [-\infty n + z^+ \bar{n} + z_\perp, z] \eta_{\bar{n}}(z). \quad (4.84)$$

The expansion of $\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\Gamma^C]}(z, 0)$ in eq. (4.82) around $z^- = 0$ amounts to

$$\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\Gamma^C]}(z, 0) = \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_\perp^2}{2\partial_-} \right)^n \frac{\partial_\nu}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\gamma_\perp^\nu \not{n} \Gamma^C \not{n} \gamma_\perp^\mu}{8} \frac{\partial_\mu}{\partial_-} \xi_n(0). \quad (4.85)$$

In the unpolarized case ($\Gamma^C = \gamma^\alpha$) the Dirac structure simplifies to

$$\frac{\gamma_\perp^\nu \not{n} \gamma^\alpha \not{n} \gamma_\perp^\mu}{8} = -\bar{n}^\alpha \frac{\gamma_\perp^\nu \gamma_\perp^\mu \not{n}}{4} \frac{\not{n}}{2}, \quad (4.86)$$

and eq. (4.85) becomes

$$\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\gamma^\alpha]}(z, 0) = -\bar{n}^\alpha \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_\perp^2}{2\partial_-} \right)^n \frac{\partial_\nu}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\gamma_\perp^\nu \gamma_\perp^\mu \not{n}}{4} \frac{\partial_\mu}{\partial_-} \xi_n(0), \quad (4.87)$$

Since the forward matrix element of a total derivative operator is zero, we can move the transverse derivative from ξ_n to $\bar{\xi}_n$,

$$\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\gamma^\alpha]}(z, 0) = \bar{n}^\alpha \sum_{n=0}^{\infty} \frac{(z^-)^n}{n!} \left(-\frac{\partial_\perp^2}{2\partial_-} \right)^n \frac{\partial_\perp^2}{\partial_-} \bar{\xi}_n(z^+, 0, z_\perp) \frac{\not{n}}{4} \frac{1}{\partial_-} \xi_n(0) \quad (4.88)$$

and the forward matrix element can be written as

$$\langle P_A | \mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\gamma^\alpha]}(z, 0) | P_A \rangle = 2P_A^- \bar{n}^\alpha \int dx [d^d p] e^{ip \cdot z} \delta(p^- - xP_A^-) \frac{\mathbf{p}_T^2}{2p^-} f_{n,q \leftarrow p_A}(x, \mathbf{p}_T), \quad (4.89)$$

where $1 \geq x \geq 0$. For the operator $\mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\Gamma^C]}(0, z)$ in eq. (4.84) whose forward matrix element we identify with the anti-quark distribution, we obtain

$$\langle P_B | \mathcal{O}_{\bar{n},c\text{-tw } 4}^{[\gamma^\alpha]}(0, z) | P_B \rangle = 2P_B^+ n^\alpha \int dx [d^d p] e^{ip \cdot z} \frac{\mathbf{p}_T^2}{2p^+} f_{\bar{n},\bar{q} \leftarrow p_B}(x, \mathbf{p}_T). \quad (4.90)$$

Using eqs. (4.89) and (4.90) the two-proton-state matrix element of the product of effective currents in eq. (4.82) amounts to

$$\begin{aligned} \langle P_A P_B | \mathcal{J}_{c\text{-tw } 4}^{\nu\dagger}(z) \mathcal{J}_{c\text{-tw } 4}^\mu(0) | P_A P_B \rangle &= -\frac{P_A^- P_B^+}{N_c} \sum_{a,b} \text{Tr}[\gamma_\alpha \Gamma^\nu \gamma_\beta \Gamma^\mu] \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] \\ &\times e^{iz \cdot p_a + iz \cdot p_b} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^- - \xi_b P_B^-) f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\ &\times \left[\bar{n}^\alpha \bar{n}^\beta \frac{p_b^+ \mathbf{p}_{aT}^2}{2p_a^-} + n^\alpha n^\beta \frac{p_a^- \mathbf{p}_{bT}^2}{2p_b^+} + \bar{n}^\alpha \frac{\mathbf{p}_{bT}^\beta \mathbf{p}_{aT}^2}{2p_a^-} + n^\beta \frac{\mathbf{p}_{aT}^\alpha \mathbf{p}_{bT}^2}{2p_b^+} + \bar{n}^\alpha n^\beta \frac{\mathbf{p}_{aT}^2 \mathbf{p}_{bT}^2}{4p_a^- p_b^+} \right] \\ &+ n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.91)$$

4.2.4 Sum of collinear twist 2, 3 and 4 contributions

Putting together eqs. (4.60), (4.81) and (4.91), the matrix element of the product of two effective electroweak currents at LO amounts to

$$\begin{aligned}
 \langle P_A P_B | \mathcal{J}^{\nu\dagger}(z) \mathcal{J}^\mu(0) | P_A P_B \rangle &= \frac{P_A^- P_B^+}{N_c} \sum_{a,b} \text{Tr}[\gamma_\alpha \Gamma^\nu \gamma_\beta \Gamma^\mu] \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] \\
 &\times e^{iz \cdot p_a + iz \cdot p_b} \delta(p_a^- - \xi_a P_A^-) \delta(p_a^2) \delta(p_b^- - \xi_b P_B^-) \delta(p_b^2) p_a^\alpha p_b^\beta f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
 &+ n \leftrightarrow \bar{n} + \dots
 \end{aligned} \tag{4.92}$$

where $\dots$ denotes higher geometrical twist corrections, $1 \geq \xi_{a,b} \geq 0$ and

$$\begin{aligned}
 p_a^\alpha &= n^\alpha p_a^- + \bar{n}^\alpha \frac{\mathbf{p}_{aT}^2}{2p_a^-} + \mathbf{p}_{aT}^\alpha, \\
 p_b^\beta &= \bar{n}^\beta p_b^+ + n^\beta \frac{\mathbf{p}_{bT}^2}{2p_b^+} + \mathbf{p}_{bT}^\beta.
 \end{aligned}$$

The Dirac structures Γ^μ and Γ^ν amounts to [65, 149]

$$\begin{aligned}
 \Gamma_G^\mu &= \gamma^\mu \frac{c_L^G}{2} (\mathbb{1} - \gamma^5) + \gamma^\mu \frac{c_R^G}{2} (\mathbb{1} + \gamma^5), \\
 \Gamma_{G'}^\nu &= \gamma^\nu \frac{c_L^{G'}}{2} (\mathbb{1} - \gamma^5) + \gamma^\nu \frac{c_R^{G'}}{2} (\mathbb{1} + \gamma^5),
 \end{aligned} \tag{4.93}$$

where c_L and c_R are given by [65, 149]

$$\begin{cases} c_L^G(f, f') = c_R^G(f, f') = e Q_f \delta_{\bar{f} f'}, & G = \gamma^* \\ c_L^G(f, f') = \frac{2e[(T_3^f)_L - \sin^2 \theta_W Q_f]}{\sin 2\theta_W} \delta_{\bar{f} f'}, & c_R^G(f, f') = -\frac{-e \sin \theta_W Q_f}{\cos \theta_W} \delta_{\bar{f} f'}, & G = Z \\ c_L^G(f, f') = \frac{e V_{ff'}}{\sqrt{2} \sin \theta_W}, & c_R^G = 0, & G = W^\pm \end{cases} \tag{4.94}$$

Here e is the absolute value of the electron charge, f and f' are flavor indices, Q_f is the electric charge of the particle f in unit of e , T_3 is the third projection of the weak isospin, θ_W is the Weinberg angle and $V_{ff'}$ is the CKM matrix. For the case photon plus Z boson (γ^*/Z) eq. (4.92) gives

$$\begin{aligned}
 \langle P_A P_B | \mathcal{J}^{\nu\dagger}(z) \mathcal{J}^\mu(0) | P_A P_B \rangle &= -\frac{2P_A^- P_B^+}{N_c} \sum_{a,b} \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] e^{iz \cdot p_a + iz \cdot p_b} p_a \cdot p_b \\
 &\times \left[\left(c_L^G(a, b) c_L^{G'}(a, b) + c_R^G(a, b) c_R^{G'}(a, b) \right) \left[g^{\mu\nu} - \frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{p_a \cdot p_b} \right] \right. \\
 &+ i \frac{c_L^G(a, b) c_L^{G'}(a, b) - c_R^G(a, b) c_R^{G'}(a, b)}{p_a \cdot p_b} \epsilon^{\mu\nu\alpha\beta} p_{a\alpha} p_{b\beta} \left. \right] \\
 &\times \delta(p_a^- - \xi_a P_A^-) \delta(p_b^- - \xi_b P_B^-) f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
 &+ n \leftrightarrow \bar{n} + \dots,
 \end{aligned} \tag{4.95}$$

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where $G, G' \in \{\gamma^*, Z\}$. For the case of $W^\pm$, eq. (4.92) amounts to

$$\begin{aligned} \langle P_A P_B | \mathcal{J}^{\nu\dagger}(z) \mathcal{J}^\mu(0) | P_A P_B \rangle &= -\frac{P_A^- P_B^+}{N_c} \sum_{a,b} \int d\xi_a [d^d p_a] d\xi_b [d^d p_b] e^{iz \cdot p_a + iz \cdot p_b} \frac{e^2 |V_{ab}|^2 p_a \cdot p_b}{\sin^2 \theta_W} \\ &\times \left[g^{\mu\nu} - \frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{p_a \cdot p_b} + i \frac{\epsilon^{\mu\nu\alpha\beta} p_{a\alpha} p_{b\beta}}{p_a \cdot p_b} \right] \delta(p_a^- - \xi_a P_A^-) \delta(p_b^- - \xi_b P_B^-) f_{n,a \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}) \\ &+ n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.96)$$

The hadronic tensor at LO is given by the Fourier transform of eq. (4.92)

$$W_{GG'}^{\mu\nu,[0]}(q) = \int \frac{d^4 z}{(2\pi)^4} e^{-iq \cdot z} \langle P_A P_B | \mathcal{J}_{G'}^{\nu\dagger}(z) \mathcal{J}_G^\mu(0) | P_A P_B \rangle + \dots, \quad (4.97)$$

where the subscripts G and G' label to which vector boson each effective electroweak current correspond. Combining with the leptonic tensor $L_{GG'}^{\mu\nu}(q, \theta(\text{cuts}))$, one obtains the Drell-Yan cross section [37, 43, 47, 107]

$$\sigma_{\text{DY}} = \int d^4 q \sum_{G,G'} L_{\mu\nu}^{GG'}(q, \theta(\text{cuts})) W_{GG'}^{\mu\nu}(q) \Delta_G(q^2) \Delta_{G'}^*(q^2), \quad (4.98)$$

where $\Delta_G(q^2)$ denotes the gauge vector boson propagator

$$\Delta_G(q^2) = \frac{\delta_{\gamma^* G}}{q^2} + \frac{\delta_{ZG}}{q^2 - M_Z^2 + i\Gamma_Z M_Z} + \frac{\delta_{WG}}{q^2 - M_W^2 + i\Gamma_W M_W}, \quad (4.99)$$

and $\theta(\text{cuts})$ denotes possible dependence on experimental cuts over the leptonic phase space. In this work we are interested in the Drell-Yan cross section differential in the vector boson invariant mass (Q^2), rapidity (y) and modulus of the transverse momentum with respect the axis defined by the proton beams (q_T). The method to turn the inclusive cross section in eq. (4.98) into a differential one is standard. One starts by introducing measurement functions, which in our case are given through

$$\begin{aligned} 1 &= \int dQ^2 \delta(Q^2 - q^2), \\ 1 &= \int dy \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right), \\ 1 &= \int dq_T 2q_T \delta(q_T^2 + q^2 - 2(\bar{n} \cdot q)(n \cdot q)), \end{aligned} \quad (4.100)$$

thereby allowing us to write the measured quantity as a function of the particle momentum involved in the process. Inserting eq. (4.100) into eq. (4.98) one obtains

$$\begin{aligned} \sigma_{\text{DY}} &= \int dQ^2 dy dq_T d^4 q 2q_T \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 - q^2 + 2(\bar{n} \cdot q)(n \cdot q)) \\ &\times \sum_{G,G'} L_{\mu\nu}^{GG'}(q, \theta(\text{cuts})) W_{GG'}^{\mu\nu}(q) \Delta_G(q^2) \Delta_{G'}^*(q^2), \end{aligned} \quad (4.101)$$

4.2. COLLINEAR TWIST EXPANSION AND KPC AT LO

therefore the differential Drell-Yan cross section is given by

$$\begin{aligned} \frac{d\sigma_{\text{DY}}}{dQ^2 dy dq_T} &= \int d^4q \, 2q_T \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 - q^2 + 2(\bar{n} \cdot q)(n \cdot q)) \\ &\times \sum_{G, G'} L_{\mu\nu}^{GG'}(q, \theta(\text{cuts})) W_{GG'}^{\mu\nu}(q) \Delta_G(q^2) \Delta_{G'}^*(q^2). \end{aligned} \quad (4.102)$$

In phenomenological applications one is typically interested in the cumulative version of eq. (4.102), i.e its integral over some finite range of Q^2 , y and q_T , rather than its value at a certain point (Q^2, y, q_T) :

$$\sigma_{\text{DY}}(\{Q_{\min}^2, Q_{\max}^2, y_{\min}, y_{\max}, q_{T\min}, q_{T\max}\}) = \int_{Q_{\min}^2}^{Q_{\max}^2} dQ^2 \int_{y_{\min}}^{y_{\max}} dy \int_{q_{T\min}}^{q_{T\max}} dq_T \frac{d\sigma_{\text{DY}}}{dQ^2 dy dq_T}, \quad (4.103)$$

where the integrals over Q^2 , y and q_T can be solved using the measurements functions. Thus

$$\begin{aligned} \sigma_{\text{DY}}(\{Q_{\min}^2, Q_{\max}^2, y_{\min}, y_{\max}, q_{T\min}, q_{T\max}\}) &= \\ &\int d^4q \, \Theta(Q^2, y, q_T^2) \sum_{G, G'} L_{\mu\nu}^{GG'}(q, \theta(\text{cuts})) W_{GG'}^{\mu\nu}(q) \Delta_G(q^2) \Delta_{G'}^*(q^2), \end{aligned} \quad (4.104)$$

where we have defined $\Theta(Q^2, y, q_T^2)$ as

$$\Theta(Q^2, y, q_T^2) = \theta(Q_{\max}^2 - Q^2) \theta(Q^2 - Q_{\min}^2) \theta(y_{\max} - y) \theta(y - y_{\min}) \theta(q_{T\max}^2 - q_T^2) \theta(q_T^2 - q_{T\min}^2), \quad (4.105)$$

where Q^2 , y and q_T^2 are given by

$$\begin{aligned} Q^2 &= q^2, \\ y &= \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}, \\ q_T^2 &= 2(\bar{n} \cdot q)(n \cdot q) - q^2. \end{aligned} \quad (4.106)$$

From eq. (4.104) we define a cumulant version of the hadronic tensor in momentum space at LO

$$\begin{aligned} W_{GG'}^{\mu\nu, [\text{LO}]}(q) \Theta(Q^2, y, q_T^2) &= \frac{P_A^- P_B^+}{N_c} \sum_{a, b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta^{(4)}(q - p_a - p_b) \Theta(Q^2, y, q_T^2) \\ &\times \text{Tr}[\gamma_\alpha \Gamma_{G'}^\nu \gamma_\beta \Gamma_G^\mu] p_a^\alpha p_b^\beta \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) f_{n, a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\ &+ n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.107)$$

From eq. (4.107) we obtain that summing the whole series of kinematic power corrections leads to an effective hard scattering process where the incoming partons can be thought as massless on-shell particles with a finite transverse momentum, which is given by the unpolarized (anti-)quark TMDPDFs. This is a fundamental difference compared to collinear factorization, where the incoming partons participating in the hard scattering process are regarded as purely collinear particles, with momentum parallel to the proton momentum.

CHAPTER 4. TMD FACTORIZATION

Before introducing the NLO virtual and real corrections to the effective electroweak currents we must comment on the issue of defining γ^5 . At LO all the computations can be performed four space-time dimensions, one can use [65, 192, 193]

$$\begin{aligned}\{\gamma^5, \gamma^\mu\} &= 0, \\ \text{Tr}[\gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] &= 4i\epsilon^{\mu\nu\alpha\beta}, \\ \text{Tr}[\Gamma_i \Gamma_j] &= \text{Tr}[\Gamma_j \Gamma_i].\end{aligned}\tag{4.108}$$

However, since in the computation of virtual and real corrections we work in dimensional regularization with $d = 4 - 2\epsilon$ we need to choose a scheme to handle γ^5 in d dimensions. In this work we use the Larin scheme [226–228]

$$\bar{\psi} \gamma^\mu \gamma^5 \psi \rightarrow -Z_5^{\text{s(ns)}} \frac{i\epsilon^{\mu\alpha_1\alpha_2\alpha_3}}{3!} \bar{\psi} \gamma_{\alpha_1} \gamma_{\alpha_2} \gamma_{\alpha_3} \psi,\tag{4.109}$$

where we can consider the Dirac γ_{α_i} as d -dimensional objects, perform the computation with open Lorentz indices, remove the infrared poles and finally contract with the Levi-Civita symbol in four dimensions once we have rendered our calculation finite. The presence of Z_5 is amend the violation of the Ward identity relating the renormalization of the vector and axial currents since γ^5 defined by eq. (4.109) does not anti-commute with all the γ^μ [226, 227], therefore it is necessary introduce a finite renormalization constant that restores the Ward identity. For the singlet case (i.e. ψ and $\bar{\psi}$ are fermions of same flavor), Z_5^{s} is given at NNLO by [226]

$$Z_5^{\text{s}}(\alpha_s(\mu^2)) = 1 - \alpha_s(\mu^2) \frac{C_F}{\pi} + \frac{\alpha_s^2(\mu^2)}{16\pi^2} \left(22C_F^2 - \frac{107}{9}C_F C_A + \frac{31}{18}C_F n_f \right) + \mathcal{O}(\alpha_s^3(\mu^2)),\tag{4.110}$$

and for the non-singlet case (i.e. $\bar{\psi}$ and ψ are fermions of different flavor), Z_5^{ns} is given at NNLO by [226]

$$Z_5^{\text{ns}}(\alpha_s(\mu^2)) = 1 - \alpha_s(\mu^2) \frac{C_F}{\pi} + \frac{\alpha_s^2(\mu^2)}{16\pi^2} \left(22C_F^2 - \frac{107}{9}C_F C_A + \frac{2}{9}C_F n_f \right) + \mathcal{O}(\alpha_s^3(\mu^2)).\tag{4.111}$$

4.3 Collinear twist expansion at NLO

In the previous section we have obtained the effective hadronic tensor including the whole series of KPC of geometrical twist 2 operator at collinear twist 2, 3 and 4 at LO. By working at order $\alpha_s^0(\mu)$, we have relaxed the notation not writing (anti-)time order operator or specifying whether the backgrounds fields are (anti-)time ordered. In this section we present the NLO corrections in $\alpha_s(\mu^2)$, which means that the notation necessarily becomes more involved. The virtual corrections to the effective electroweak currents have been already computed [184, 212]; the corrections with one real emission is the novelty of the present work.

Figure 4.4: Virtual corrections at NLO in $\alpha_s(\mu^2)$ to the electroweak effective current.

4.3.1 Virtual corrections

The virtual vertex correction at NLO to eq. (4.9) amounts to

$$\begin{aligned} \mathcal{J}^{\dagger\nu}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_2) \mathcal{J}^\mu[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_1) &= \bar{T} \left\{ \bar{q}_n(z_2) \Gamma^\nu q_{\bar{n}}(z_2) \right\} \\ &\times (ig)^2 \int d^d y_1 d^d y_2 T \{ \overbrace{[\bar{q}_{\bar{n}} \not{B} \psi](y_2) :: [\bar{\psi} \Gamma^\mu \psi](z_1) :: [\psi \not{B} q_n](y_1)} : \}. \end{aligned} \quad (4.112)$$

$+ n \leftrightarrow \bar{n}$

Replacing the fields contractions by the hard gluon and quark Feynman propagators, eqs. (D.8) and (D.11), one obtains, see fig. 4.4,

$$\begin{aligned} \mathcal{J}^{\dagger\nu}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_2) \mathcal{J}^\mu[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_1) &= \bar{T} \left\{ \bar{q}_n(z_2) \Gamma^\nu q_{\bar{n}}(z_2) \right\} \\ &\times (ig)^2 \int d^d y_1 d^d y_2 T \{ \bar{q}_{j,\bar{n}}(y_2) q_{i,n}(y_1) \} (\gamma_\beta S(y_2 - z_1) \Gamma^\mu S(z_1 - y_1) \gamma_\alpha)_{ji} \Delta^{\alpha\beta}(y_2 - y_1), \end{aligned} \quad (4.113)$$

$+ n \leftrightarrow \bar{n},$

where $S(y - z)$ and $\Delta^{\alpha\beta}(y - z)$ are the hard fermion and gluon propagator respectively. We are interested in the limit $y_{1(2)\perp}^\mu \rightarrow z_{1\perp}^\mu$, $y_1^- \rightarrow z_1^-$ and $y_2^+ \rightarrow z_1^+$ (which also implies $y_{2\perp}^\mu - y_{1\perp}^\mu \rightarrow 0$), i.e. the limit where the quantum correction to the electroweak currents takes places at short distances (in the transverse plane) and the background quark fields *see* the effective electroweak current as point like in the transverse coordinate. Since, the scaling of the background field $q_n(\bar{q}_{bn})$ along the (anti-)collinear direction $n^\mu(\bar{n}^\mu)$ is comparable to the hard fields. Thus the background fields *see* the effective electroweak current as an extended object along these directions, which leads to a convolution of the 3-points Green function with the background fields. Performing the shift of coordinates $y_{1(2)}^\mu \rightarrow y_{1(2)}^\mu + z_1^\mu$ and using Fierz identity, eq. (B.8), eq. (4.113) leads to

$$\begin{aligned} \mathcal{J}^{\dagger\nu}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_2) \mathcal{J}^\mu[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}](z_1) \\ = -\frac{(ig)^2}{4N_c} \sum_{C,D} \int d^d y_1 d^d y_2 \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_2 + z_1, z_2) \\ \times \text{Tr}[\Gamma_C \Gamma^\nu \Gamma_D \gamma_\beta S(y_2) \Gamma^\mu S(-y_1) \gamma_\alpha] \Delta^{\alpha\beta}(y_2 - y_1) + n \leftrightarrow \bar{n}, \end{aligned} \quad (4.114)$$

where the minus sign comes from the anti-commutation of fermionic fields and the factor $1/N_c$ from color reconnection as in the LO cases. The operators $\mathcal{O}_{n(\bar{n})}^{[\Gamma]}$ in the light-cone gauge are given by

$$\mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) = \bar{T} \{ \bar{q}_{\bar{n}}^a(z_2) \} \frac{\Gamma^C}{2} T \{ q_n^a(y_1 + z_1) \}, \quad (4.115)$$

$$\mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_2 + z_1, z_2) = T \{ \bar{q}_{\bar{n}}^a(y_2 + z_1) \} \frac{\Gamma^D}{2} \bar{T} \{ q_n^a(z_2) \}, \quad (4.116)$$

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where a is a color index. The dependence on $y_{1(2)}$ in the operator $\mathcal{O}_{n(\bar{n})}$ hinders the computation of integrals. Nevertheless, we can use that from the point of view of the background fields the interaction at the electroweak vertex is point like (at least on the transverse plane and along the (anti-)collinear directions), thus one can expand the background field around $y_{1\perp}^\mu = y_{2\perp}^\mu = 0$, $y_1^- = 0$ and $y_2^+ = 0$. In order to simplify the Taylor expansion of the background fields we introduce the momentum operators $(\hat{p}_{q_n})_\mu = -i\partial_\mu$ and $(\hat{p}_{\bar{q}_n})_\mu = -i\partial_\mu$, where $(\hat{p}_{q_n(\bar{q}_n)})_\mu$ acts on the (anti-)collinear (anti-)quark fermion field. Thus, the background fields $q_n(z_1 + y_1)$ and $\bar{q}_n(y_2 + z_1)$ can be written as⁷

$$\begin{aligned} q_n(y_1 + z_1) &= e^{iy_1 \cdot (\hat{p}_{q_n})} q_n(z_1), \\ \bar{q}_n(y_2 + z_1) &= e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \bar{q}_n(z_1), \end{aligned} \quad (4.117)$$

which allows us to write eqs. (4.115) and eqs. (4.116) as

$$\mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) = \bar{T}\{\bar{q}_n^a(z_2)\} \frac{\Gamma^C}{2} T\{e^{iy_1 \cdot (\hat{p}_{q_n})} q_n^a(z_1)\}, \quad (4.118)$$

$$\mathcal{O}_n^{[\Gamma^D]}(y_2 + z_1, z_2) = T\{e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \bar{q}_n^a(z_1)\} \frac{\Gamma^D}{2} \bar{T}\{q_n^a(z_2)\}. \quad (4.119)$$

The next step is to split the background fermion field, $q_{n(\bar{n})}$ into *good*, $\xi_{n(\bar{n})}$ and *bad*, $\eta_{n(\bar{n})}$, spin components and use the equation of motions (see App. C.2 and Sec. 4.2) to integrate out $\eta_{n(\bar{n})}$

$$\begin{aligned} \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) &= \bar{T}\{\bar{\xi}_n^a(z_2)\} \frac{\Gamma^C}{2} T\{e^{iy_1 \cdot (\hat{p}_{q_n})} \xi_n^a(z_1)\} \\ &\quad - \bar{T}\left\{\frac{\partial_\mu}{\partial_-} \bar{\xi}_n^a(z_2)\right\} \frac{\gamma_\perp^\mu \not{n} \Gamma^C}{4} T\left\{e^{iy_1 \cdot (\hat{p}_{q_n})} \xi_n^a(z_1)\right\} \\ &\quad - \bar{T}\left\{\bar{\xi}_n^a(z_2)\right\} \frac{\Gamma^C \not{n} \gamma_\perp^\mu}{4} T\left\{e^{iy_1 \cdot (\hat{p}_{q_n})} \left(\frac{\partial_\mu}{\partial_-} \xi_n^a(z_1)\right)\right\} \\ &\quad + \bar{T}\left\{\frac{\partial_\nu}{\partial_-} \bar{\xi}_n^a(z_2)\right\} \frac{\gamma_\perp^\nu \not{n} \Gamma^C \not{n} \gamma_\perp^\mu}{8} T\left\{e^{iy_1 \cdot (\hat{p}_{q_n})} \left(\frac{\partial_\mu}{\partial_-} \xi_n^a(z_1)\right)\right\} + \dots, \end{aligned} \quad (4.120)$$

$$\begin{aligned} \mathcal{O}_n^{[\Gamma^D]}(y_2 + z_1, z_2) &= T\{e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \bar{\xi}_n^a(z_1)\} \frac{\Gamma^D}{2} \bar{T}\{\xi_n^a(z_2)\} \\ &\quad - T\left\{e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \left(\frac{\partial_\mu}{\partial_+} \bar{\xi}_n^a(z_1)\right)\right\} \frac{\gamma_\perp^\mu \not{n} \Gamma^D}{4} \bar{T}\{\xi_n^a(z_2)\} \\ &\quad - T\left\{e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \bar{\xi}_n^a(z_1)\right\} \frac{\Gamma^D \not{n} \gamma_\perp^\mu}{4} \bar{T}\left\{\frac{\partial_\mu}{\partial_+} \xi_n^a(z_2)\right\} \\ &\quad + T\left\{e^{iy_2 \cdot (\hat{p}_{\bar{q}_n})} \left(\frac{\partial_\nu}{\partial_+} \bar{\xi}_n^a(z_1)\right)\right\} \frac{\gamma_\perp^\nu \not{n} \Gamma^D \not{n} \gamma_\perp^\mu}{8} \bar{T}\left\{\frac{\partial_\mu}{\partial_+} \xi_n^a(z_2)\right\} + \dots, \end{aligned} \quad (4.121)$$

⁷The exponential operator must be understood in the sense of Fourier multipliers [229]

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where ... stands for higher operators containing transverse gluon field components, which are at least geometrical twist 3. Since at the very end we are interested in the forward matrix elements of the operators, we can perform a global shift of the fields origin and write

$$\begin{aligned}\langle P_A | \mathcal{O}_n^{[\Gamma^C]}(z_2, z_1 + y_1) | P_A \rangle &= \langle P_A | \mathcal{O}_n^{[\Gamma^C]}(z_2 - z_1, y_1) | P_A \rangle, \\ \langle P_B | \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1 + y_2, z_2) | P_B \rangle &= \langle P_B | \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_2, z_2 - z_1) | P_B \rangle.\end{aligned}\quad (4.122)$$

Taking the forward matrix element in eqs. (4.120) and (4.121) we obtain

$$\begin{aligned}\langle P_A | \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) | P_A \rangle &= e^{iy_1 \cdot (\hat{p}_{\xi_n})} \langle P_A | \bar{T} \{ \bar{\xi}_n^a(z_2 - z_1) \} \frac{\Gamma^C}{2} T \{ \xi_n^a(0) \} | P_A \rangle \\ &- e^{iy_1 \cdot (\hat{p}_{\xi_n})} \langle P_A | \bar{T} \left\{ \frac{\partial_\mu}{\partial_-} \bar{\xi}_n^a(z_2 - z_1) \right\} \frac{\gamma_\perp^\mu \not{n} \Gamma^C}{4} T \left\{ \xi_n^a(0) \right\} | P_A \rangle \\ &- e^{iy_1 \cdot (\hat{p}_{\xi_n})} \langle P_A | \bar{T} \left\{ \bar{\xi}_n^a(z_2 - z_1) \right\} \frac{\Gamma^C \not{n} \gamma_\perp^\mu}{4} T \left\{ \left(\frac{\partial_\mu}{\partial_-} \xi_n^a(0) \right) \right\} | P_A \rangle \\ &+ e^{iy_1 \cdot (\hat{p}_{\xi_n})} \langle P_A | \bar{T} \left\{ \frac{\partial_\nu}{\partial_-} \bar{\xi}_n^a(z_2 - z_1) \right\} \frac{\gamma_\perp^\nu \not{n} \Gamma^C \not{n} \gamma_\perp^\mu}{8} T \left\{ \left(\frac{\partial_\mu}{\partial_-} \xi_n^a(0) \right) \right\} | P_A \rangle + \dots,\end{aligned}\quad (4.123)$$

$$\begin{aligned}\langle P_B | \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_2 + z_1, z_2) | P_B \rangle &= -e^{iy_2 \cdot (\hat{p}_{\bar{\xi}_n})} \langle P_B | \bar{T} \{ \xi_{\bar{n},i}^a(z_2 - z_1) \} \frac{(\Gamma^D)_{ji}}{2} T \{ \bar{\xi}_{\bar{n},j}^a(0) \} | P_B \rangle \\ &+ e^{iy_2 \cdot (\hat{p}_{\bar{\xi}_n})} \langle P_B | \bar{T} \left\{ \xi_{\bar{n},i}^a(z_2 - z_1) \right\} \frac{(\gamma_\perp^\mu \not{n} \Gamma^D)_{ij}}{4} T \left\{ \left(\frac{\partial_\mu}{\partial_+} \bar{\xi}_{\bar{n},j}^a(0) \right) \right\} | P_B \rangle \\ &+ e^{iy_2 \cdot (\hat{p}_{\bar{\xi}_n})} \langle P_B | \bar{T} \left\{ \frac{\partial_\mu}{\partial_+} \xi_{\bar{n},i}^a(z_2 - z_1) \right\} \frac{(\Gamma^D \not{n} \gamma_\perp^\mu)_{ij}}{4} T \left\{ \bar{\xi}_{\bar{n},j}^a(0) \right\} | P_B \rangle \\ &- e^{iy_2 \cdot (\hat{p}_{\bar{\xi}_n})} \langle P_B | \bar{T} \left\{ \frac{\partial_\nu}{\partial_+} \xi_{\bar{n},i}^a(z_2 - z_1) \right\} \frac{(\gamma_\perp^\nu \not{n} \Gamma^D \not{n} \gamma_\perp^\mu)_{ij}}{8} T \left\{ \left(\frac{\partial_\nu}{\partial_+} \bar{\xi}_{\bar{n},j}^a(0) \right) \right\} | P_B \rangle + \dots,\end{aligned}\quad (4.124)$$

where the minus sign in eq. (4.124) comes from the anti-commutation of the fermionic fields. The complex exponential with the momentum operator return the momentum associated to the background (anti-)collinear fermion field $\xi_n(\bar{\xi}_n)$. In case we work in a general gauge, we must include a Wilson line accompanying each field, therefore the complex exponential with the momentum operator amount to the momentum associated to the fermion field together with the Wilson line. As we have seen in the LO computation in Sec. 4.2 at leading geometrical twist and considering the whole series of KPC the momentum associated to the operator amount to the one of a massless on-shell parton, whose orientation is not completely parallel to the proton beam being the relative angle proportional to the transverse momentum. Thus one can formally write eqs. (4.123) and (4.124) as [55, 56, 72–74, 183]

$$\langle P_A | \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1 + z_1) | P_A \rangle = e^{iy_1 \cdot (\hat{p}_{\xi_n})} \langle P_A | \mathcal{O}_n^{[\Gamma^C]}(z_2 - z_1, 0) | P_A \rangle, \quad (4.125)$$

$$\langle P_B | \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_2 + z_1, z_2) | P_B \rangle = e^{iy_2 \cdot (\hat{p}_{\bar{\xi}_n})} \langle P_B | \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(0, z_2 - z_1) | P_B \rangle. \quad (4.126)$$

The matrix element on the right hand-side in eqs. (4.125) and (4.126) are nothing but the forward matrix element we have evaluated in term of TMDPDFs in the LO case, see Sec. 4.2.

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Thus, setting $z_2^\mu = -z_1^\mu = z^\mu/2$ eqs. (4.125) and (4.126) for the unpolarized case $\Gamma^{C(D)} = \gamma^{\alpha(\beta)}$ amount to

$$\langle P_A | \mathcal{O}_n^{[\gamma^\alpha]}(z_2, y_1 + z_1) | P_A \rangle = 2P_A^- \int [d^d p_q] d\xi_q e^{ip_a \cdot (z - y_1)} \delta(p_a^- - \xi_a P_A^-) p_q^\alpha f_{n, q \leftarrow P_A}(\xi_q, \mathbf{p}_{qT}), \quad (4.127)$$

$$\langle P_B | \mathcal{O}_{\bar{n}}^{[\gamma^\beta]}(y_2 + z_1, z_2) | P_B \rangle = 2P_B^+ \int [d^d p_{\bar{q}}] d\xi_{\bar{q}} e^{ip_{\bar{q}} \cdot (z - y_2)} \delta(p_{\bar{q}}^+ - \xi_{\bar{q}} P_B^+) p_{\bar{q}}^\beta f_{\bar{n}, \bar{q} \leftarrow P_B}(\xi_{\bar{q}}, \mathbf{p}_{\bar{q}T}), \quad (4.128)$$

where as in the LO case $1 \geq \xi_{q(\bar{q})} \geq 0$. Taking the matrix element in eq. (4.114) and plugging in eqs. (4.127) and (4.128), one obtains

$$\begin{aligned} & \langle P_B P_B | \mathcal{J}^{\dagger\nu}[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}] \left(\frac{z}{2} \right) \mathcal{J}^\mu[\bar{q}_{n(\bar{n})}, q_{n(\bar{n})}, A_{n(\bar{n})}] \left(-\frac{z}{2} \right) | P_A P_B \rangle \\ &= -\frac{4\pi i \alpha_s (\mu^2) P_A^- P_B^+}{N_c} \int [d^d p_a] d\xi_a [d^d p_b] d\xi_b e^{iz \cdot (p_a + p_b)} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ & \times f_{n, q \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}) \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^d k}{(2\pi)^2} \frac{\text{Tr}[\not{p}_a \Gamma^\nu \not{p}_b \gamma^\rho (\not{p}_b + \not{k}) \Gamma^\mu (\not{p}_a - \not{k}) \gamma_\rho]}{[k^2 + i0^+][(p_a - k)^2 + i0^+][(p_b + k)^2 + i0^+]} \\ & + n \leftrightarrow \bar{n} + \dots \end{aligned} \quad (4.129)$$

where ... denotes further Dirac structures and higher geometrical twist contributions. One must also consider the hermitian conjugate of eq. (4.129), i.e. the virtual correction to the anti-time-ordered effective electroweak current. The other contributions to the virtual corrections to the effective currents come from wave function renormalization (WFR), see fig. 4.4. However, since we are considering massless quark the integrals are scaleless and therefore zero in dimensional regularization.

Starting from eq. (4.129), the Drell-Yan hadronic tensor and the differential cross section are obtained following similar steps, see Sec. 4.2.4. The NLO virtual correction to the hadronic tensor is given by

$$\begin{aligned} W_{GG'}^{\mu\nu, [\text{NLO}, \text{V}]}(q) \Theta(Q^2, y, q_T^2) &= \frac{4\pi \alpha_s (\mu^2) P_A^- P_B^+}{N_c} \sum_{a,b} \int [d^d p_a] d\xi_a [d^d p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ & \times f_{n, q \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}) \Theta(Q^2, y, q_T^2) \delta^{(4)}(q - p_a - p_b) \\ & \times \left[\Pi_{V, GG'}^{\mu\nu} \left((p_a + p_b)^2, \log \frac{(p_a + p_b)^2}{\mu^2} + i\pi \right) + \text{h.c.} \right] \\ & + n \leftrightarrow \bar{n}, \end{aligned} \quad (4.130)$$

where $\Pi_{V, GG'}^{\mu\nu}$ is defined as

$$\begin{aligned} & \Pi_{V, GG'}^{\mu\nu} \left((p_a + p_b)^2, \log \frac{\mu^2}{(p_a + p_b)^2} + i\pi \right) = \\ & i \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}[\not{p}_a \Gamma_{G'}^\nu \not{p}_b \gamma^\rho (\not{p}_b + \not{k}) \Gamma_G^\mu (\not{p}_a - \not{k}) \gamma_\rho]}{[k^2 + i0^+][(p_a - k)^2 + i0^+][(p_b + k)^2 + i0^+]}, \end{aligned} \quad (4.131)$$

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and $\Theta(Q^2, y, q_T^2)$ is the one given in eq. (4.105). The explicit expression for $\Pi_{V,GG'}$ in the cases $GG' = \{\gamma^* \gamma^*, ZZ, \gamma^* Z, W^\pm W^\pm\}$ can be found in App. E.1. The hadronic tensor in eq. (4.130) is given in terms of unphysical bare TMDPDFs distributions, which are UV divergent. These divergences can be factorized out into a renormalization constant, leading to the definition of physical TMDPDFs with a dependence on the factorization scale [43, 45, 48, 50, 51, 53, 106, 110, 112, 114, 183, 184, 230]

$$\Phi_{n,q \leftarrow p_A}^{B[\gamma^-]}(z^+, z_\perp) = Z_2(\alpha_s(\mu^2)) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_n}\right) \Phi_{n,q \leftarrow p_A}^{[\gamma^-]}(z^+, z_\perp; \mu^2, \zeta_n) \quad (4.132)$$

, where μ is UV renormalization scale; ζ_n has unit of energy square and is due to the renormalization of the rapidity divergences; Z_2 and Z_{TMD} are the fermion and operator renormalization constants respectively⁸. We have denoted with a superscripts B the bare quantities and written explicitly the dependence on energy scales for their renormalized counterparts. Since the hadronic tensor in eq. (4.130) is given in momentum space, we need to Fourier transform eq. (4.132)

$$\begin{aligned} f_{n,q \leftarrow p_A}(\xi_q, \mathbf{p}_{qT}) &= \int \frac{dz^+ d^2 \mathbf{z}_T}{(2\pi)^3} e^{-i\xi_q P_A^- z^+ + i\mathbf{z}_T \cdot \mathbf{p}_{qT}} \Phi_{n,q \leftarrow p_A}^{B[\gamma^-]}(z^+, z_\perp) \\ &= \int \frac{dz^+ d^2 \mathbf{z}_T}{(2\pi)^3} e^{-i\xi_q P_A^- z^+ + i\mathbf{z}_T \cdot \mathbf{p}_{qT}} Z_2(\alpha_s(\mu^2)) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_n}\right) \Phi_{n,q \leftarrow p_A}^{[\gamma^-]}(z^+, z_\perp; \mu^2, \zeta_n) \\ &= Z_2(\alpha_s(\mu^2)) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_n}\right) f_{n,q \leftarrow p_A}(\xi_q, \mathbf{p}_{qT}; \mu^2, \zeta_n). \end{aligned} \quad (4.133)$$

For the unpolarized anti-quark TMDPDFs the renormalization procedure is analogous. Thus, in terms of renormalized TMDPDFs the hadronic tensor in eq. (4.130) amounts to

$$\begin{aligned} W_{GG'}^{\mu\nu, [\text{NLO}, \text{V}]}(q) \Theta(Q^2, y, q_T^2) &= \frac{4\pi\alpha_s(\mu^2) P_A^- P_B^+}{N_c} \sum_{a,b} \int [d^d p_a] d\xi_a [d^d p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ &\times f_{n,q \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}; \mu^2, \zeta_n) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \Theta(Q^2, y, q_T^2) \delta^{(4)}(q - p_a - p_b) \\ &\times Z_2^2(\alpha_s(\mu^2)) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_n}\right) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_{\bar{n}}}\right) \\ &\times \left[\Pi_{V,GG'}^{\mu\nu} \left((p_a + p_b)^2, \log \frac{(p_a + p_b)^2}{\mu^2} + i\pi \right) + \text{h.c.} \right] \\ &+ n \leftrightarrow \bar{n}. \end{aligned} \quad (4.134)$$

The contribution in square brackets contains IR poles that are cancelled out by the renormalization constant if and only if $\zeta_n \zeta_{\bar{n}} = [(p_a + p_b)^2]^2$ [43, 50, 183, 184]. Therefore, the hadronic tensor can be written either in term of bare quantities, which on their own are ill-defined due to IR and UV poles, or as the combinations of renormalized well-defined objects. Notice that either way the hadronic tensor is always a finite quantity. Nevertheless, writing the hadronic tensor as a convolution of well-defined quantities on their own does have a deep physical

⁸At geometrical twist-3 or higher one would have an integral convolution in the light-cone coordinates between the renormalization constants and the renormalized operator [183, 184, 198]

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meaning. The presence of IR poles in the virtual corrections to the effective vertex is an indication of an inconsistent treatment of the loop integral of the hard fields, where formally modes that do not correspond hard fields are included, see 4.1. It is this contamination from the collinear region in the integration measure that leads to IR poles in the partonic cross section. On the other hand a similar issue happens for the TMDPDFs, the virtual corrections to the TMDPDFs also include contributions from the hard modes. This counting of hard modes amount to the UV renormalization constant $Z_2 Z_{\text{TMD}}$. Nonetheless, this inconsistency in the treatment of virtual corrections in the TMDPDFs cancels out the one from the effective vertex. Thus, only after renormalization of different terms in the hadronic tensor, factorization is properly achieved. The other issue regarding the use or not of renormalize quantity is the concept of universality, we would like that the non-perturbative distributions that we have introduced in the factorization of the hadronic tensor to be universal i.e. be the same of any scattering process containing proton(s) in the initial state like SIDIS [52, 53, 231, 232]. If we were to identify the bare matrix element with the physical distributions, we would be forced to always specify that they must be combined with a hard matching coefficient with exactly the same IR pole structure than the one obtained in Drell-Yan. However, if we could use the same renormalize quantity in different processes containing a proton (or protons) as initial state, this would mean that despite being dealing with different hadronic process the nature of infrared physics is universal across all process, allowing to identify universal quantities at low energy scales. So far, at geometrical twist-2 the study of processes like Drell-Yan and (SI)DIS support the thesis of the universality of the infrared physics in processes containing a proton as initial state [19, 43, 52, 53, 105, 170, 184, 213, 227, 232].

4.3.2 Real corrections

In the case of virtual corrections in Sec. 4.3.1, we showed that the collinear twist expansion together with the expansion of (anti-)collinear background quark fields around $z^{-(+)}$ leads to non-local differential operators which can be highly simplified by replacing the TMDPDF matrix elements in position space by their parametrization in momentum space. One can then Fourier transform the Wilson coefficient from position to momentum space and compute the loop integrals afterwards. Such simplification relies on the assumption that the order of integration can be interchanged, namely we can either first compute the Fourier transform from position to momentum space and then evaluate the loop integrals, or first compute the loop integrals in position space and afterwards perform the Fourier transform to momentum space. This is possible because loop momenta do not enter the measurement function, see. (4.130), and thus the analyticity properties present in position space remain unchanged in momentum space. However, for the case of real radiative corrections the situation is slightly more involved and one must be cautious regarding the order of integration since the momentum of real emitted particles enters the measurement functions through momentum conservation, see. (4.161). The measurement functions introduce cuts on the phase space that alter the analytic structure of the hard scattering process and therefore one cannot naively interchange the order of integration. We will show that by computing the real radiative corrections in position space and then performing the Fourier transform one obtains a finite result, nevertheless this procedure leads to cumbersome expressions where the Fourier transform of transcendental functions need to be computed. While for the NLO the computation it could be doable, for

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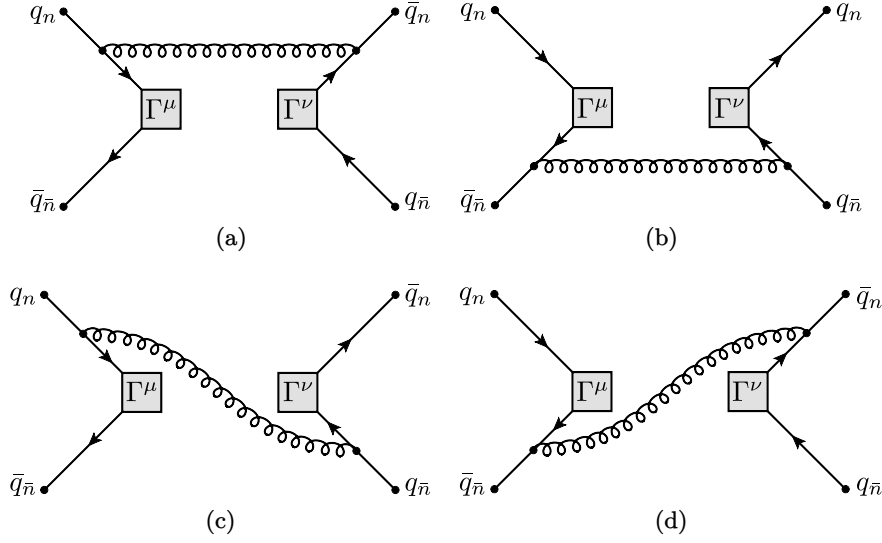


Figure 4.5: Real radiative correction to the VEV of the product of two electroweak currents, $\langle 0 | J^{\nu\dagger}(z_2) J^\mu(z_1) | 0 \rangle$, where the fields in $J^\mu(z_1)$ are time-ordered while in $J^{\nu\dagger}(z_2)$ are anti-time-ordered.

NNLO and higher orders the method is not efficient. For this reason that we present the computation in momentum space. The main advantage of this approach is that the phase space integrals do not need to be evaluated analytically, but can be computed through a Monte-Carlo integration. Nevertheless, as we have already mentioned, the analyticity properties that make the computation finite are altered if one performs the Fourier transform before computing the phase space integrals. The phase space will thus contain spurious soft and collinear divergences that need to be removed in order to obtain meaningful predictions.

4.3.2.1 Computation in position space

At NLO there are four different diagrams that contribute to the real radiative corrections⁹ to the vacuum expectation value (VEV) of the product of two electroweak currents, see eq. (4.9), which are displayed in fig. 4.5. Their computation in position space is rather similar for all of them. We thus present in detail the calculation of diagram (a). From the collinear-hard interaction Lagrangian in eq. (C.2), we pull down the interaction vertex between a collinear quark, hard quark and hard gluon fields and obtain

⁹We refer to them as real since the singular contribution in the short-distance limit is the propagator of an on-shell particle, see eq. (D.3).

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$$\begin{aligned} \langle 0_h | J^{\nu\dagger}(z_2) J^\mu(z_1) | 0_h \rangle_{\text{diag (a)}} &= -(ig)^2 \int d^d y_1 d^d y_2 \langle 0 | \bar{T} \{ [\bar{q}_n \not{B} \psi](y_2) [\bar{\psi} \Gamma^\nu q_{\bar{n}}](z_2) \} \\ &\quad \times T \{ [\bar{q}_{\bar{n}} \Gamma^\mu \psi](z_1) [\bar{\psi} \not{B} q_n](y_1) \} | 0 \rangle, \end{aligned} \quad (4.135)$$

Notice that in the anti-time ordered product of operators the interaction vertex has a factor $-i$ instead of i . Using eqs. (D.2) and (D.3) one obtains

$$\begin{aligned} \langle 0 | J^{\nu\dagger}(z_2) J^\mu(z_1) | 0 \rangle_{\text{diag (a)}} &= \\ &- (ig)^2 \int d^d y_1 d^d y_2 \langle 0 | \bar{T} \{ [\bar{q}_n \not{B} \psi](y_2) [\bar{\psi} \Gamma^\nu q_{\bar{n}}](z_2) \} T \{ [\bar{q}_{\bar{n}} \Gamma^\mu \psi](z_1) [\bar{\psi} \not{B} q_n](y_1) \} | 0 \rangle \\ &= -(ig)^2 \int d^d y_1 d^d y_2 \bar{T} \{ \bar{q}_n(y_2) \gamma_\alpha t^a S(y_2 - z_2) \Gamma^\nu q_{\bar{n}}(z_2) \} \\ &\quad \times T \{ \bar{q}_{\bar{n}}(z_1) \Gamma^\mu S(z_1 - y_1) \gamma_\beta t^b q_n(y_1) \} \Delta_{ab,+}^{\alpha\beta}(y_2 - y_1), \end{aligned} \quad (4.136)$$

where we have used the fact that the VEV of normal-ordered product of fields is zero. Using the Fierz identity (see eq. (B.8)), we can write eq. (4.136) as

$$\begin{aligned} \langle 0 | J^{\nu\dagger}(z_2) J^\mu(z_1) | 0 \rangle_{\text{diag (a)}} &= -\frac{4\pi\alpha_s(\mu^2)C_F}{4N_c} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \\ &\quad \times \sum_{C,D} \int d^d y_1 d^d y_2 \text{Tr}[\Gamma_C \gamma_\alpha S(y_2 - z_2) \Gamma^\nu \Gamma_D \Gamma^\mu S(z_1 - y_1) \gamma_\beta] \Delta_+^{\alpha\beta}(y_2 - y_1) \\ &\quad \times \bar{T} \{ \bar{q}_n(y_2) \} \frac{\Gamma^C}{2} T \{ q_n(y_1) \} T \{ \bar{q}_{\bar{n}}(z_1) \} \frac{\Gamma^D}{2} \bar{T} \{ q_{\bar{n}}(z_2) \} \end{aligned} \quad (4.137)$$

where we have performed the color recombination¹⁰ in order to define the TMDPDF operators as color singlets and the minus sign in front is due to the anti-commutation of fermionic fields. As we discussed in Sec. 4.1, the momentum of the collinear background fields scales as $p_n^\mu \sim Q(\lambda^2, 1, \lambda)$ while for the hard fields it scales as $p_h^\mu \sim Q(1, 1, 1)$, which means that along the anti-collinear direction $\bar{n}^\mu$ and on the transverse plane the leading contribution is given by the short-distance interaction of the hard field at the electroweak vertex. Therefore, the leading contribution in eq. (4.137) arises from the limits $y_{1(2)}^- \rightarrow z_{1(2)}^-$ and $y_{1(2)\perp}^\mu \rightarrow z_{1(2)\perp}^\mu$, i.e. from the point of view of the collinear quark background fields $q_n(y_1)$ and $\bar{q}_{\bar{n}}(y_2)$ the effective electroweak vertex is a local operator along the anti-collinear direction $\bar{n}^\mu$ and on the transverse plane. Thus, one can formally perform a Taylor expansion of the background collinear quarks around $y_{1(2)}^- = z_{1(2)}^-$ and $y_{1(2)\perp}^\mu = z_{1(2)\perp}^\mu$ and integrate in the anti-collinear and transverse coordinates, $y_{1(2)}^-$ and $y_{1(2)\perp}^\mu$, order by order in the Taylor expansion. One can simplify the Taylor expansion of the background fields by suitable shifting coordinates $y_{1(2)}^\mu \rightarrow y_{1(2)}^\mu + z_{1(2)}^\mu$. Then the off-shell fermion propagators $S(y_2 - z_2)$ and $S(z_1 - y_1)$ simplify to $S(y_2)$ and $S(y_2)$ and $S(-y_1)$ respectively, and the on-shell gluon propagator $\Delta_+^{\alpha\beta}(y_2 - y_1)$ acquires a dependence on $z_2^\mu - z_1^\mu$, i.e. $\Delta_+^{\alpha\beta}(y_2 - y_1 - z_2 + z_1)$. Replacing $S(z)$ and $\Delta_+^{\alpha\beta}(z)$ by their explicit expressions, eqs. (D.11) and (D.4) respectively, eq. (4.137) amounts to

¹⁰Notice that $\Delta_+^{\alpha\beta}(y_2 - y_1)$ no longer has color indices.

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$$\begin{aligned}
\langle 0|J^{\nu\dagger}(z_2)J^\mu(z_1)|0\rangle_{\text{diag (a)}} &= \frac{1}{\pi^2} \frac{\alpha_s(\mu^2)C_F}{4N_c} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \sum_{C,D} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) e^{-ik \cdot (z_2 - z_1)} \\
&\times \int d^d y_1 d^d y_2 \text{Tr}[\Gamma_C \gamma_\alpha \gamma_{\sigma_2} \Gamma^\nu \Gamma_D \Gamma^\mu \gamma_{\sigma_1} \gamma^\alpha] e^{-ik \cdot (y_2 - y_1)} \frac{y_1^{\sigma_1}}{[-y_1^2 + i0^+]^2} \frac{y_2^{\sigma_2}}{[-y_2^2 - i0^+]^2} \\
&\times \bar{T}\{\bar{q}_n(y_2 + z_2)\} \frac{\Gamma^C}{2} T\{q_n(y_1 + z_1)\} T\{\bar{q}_n(z_1)\} \frac{\Gamma^D}{2} \bar{T}\{q_n(z_2)\},
\end{aligned} \tag{4.138}$$

where the background fields $q_n(z_1 + y_1)$ and $\bar{q}_n(y_2 + z_2)$ are now expanded around $y_{1(2)}^- = 0$ and $y_{1(2)\perp}^\mu = 0$:

$$\begin{aligned}
q_n(y_1 + z_1) &= \sum_{m_1, m_2=0} \frac{(\bar{n} \cdot y_1)^{m_1}}{m_1! m_2!} (y_{1\perp} \cdot \partial_\perp)^{m_2} (n \cdot \partial)^{m_1} q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp}), \\
\bar{q}_n(y_2 + z_2) &= \sum_{m_1, m_2=0} \frac{(\bar{n} \cdot y_2)^{m_1}}{m_1! m_2!} (y_{2\perp} \cdot \partial_\perp)^{m_2} (n \cdot \partial)^{m_1} \bar{q}_n(y_2^+ + z_2^+, z_2^-, z_{2\perp}).
\end{aligned} \tag{4.139}$$

The Taylor expansion of the fields in eq. (4.139) can be rewritten as [183, 184]

$$\begin{aligned}
q_n(y_1 + z_1) &= e^{iy_1^- (n \cdot \hat{p}_{q_n})} e^{iy_{1\perp} \cdot \hat{p}_{q_n}} q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp}), \\
\bar{q}_n(y_2 + z_2) &= e^{iy_2^- (n \cdot \hat{p}_{\bar{q}_n})} e^{iy_{2\perp} \cdot \hat{p}_{\bar{q}_n}} \bar{q}_n(y_2^+ + z_2^+, z_2^-, z_{2\perp}),
\end{aligned} \tag{4.140}$$

where $(\hat{p}_{q_n})_\mu = -i\partial_\mu$ and $(\hat{p}_{\bar{q}_n})_\mu = -i\partial_\mu$ are differential operators acting on the quark and anti-quark fields respectively¹¹. Using eqs. (D.44) and (D.46) it is straightforward to evaluate the integrals on the transverse variables in eq. (4.138), one obtains

$$\begin{aligned}
\langle 0|J^{\nu\dagger}(z_2)J^\mu(z_1)|0\rangle_{\text{diag (a)}} &= \frac{\alpha_s(\mu^2)C_F}{4N_c} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \sum_{C,D} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) e^{-ik \cdot (z_2 - z_1)} \\
&\times \int dy_1^+ y_1^- dy_2^+ dy_2^- \text{Tr}[\Gamma_C \gamma_\alpha \gamma_{\sigma_2} \Gamma^\nu \Gamma_D \Gamma^\mu \gamma_{\sigma_1} \gamma^\alpha] e^{-ik \cdot (y_2^+ - y_1^+) - i(k^+ - \hat{p}_{q_n}^+) y_2^- + i(k^+ + \hat{p}_{q_n}^+) y_1^-} \\
&\times \left(\frac{(\hat{q}_{q_n})_T^2}{-2y_1^+ y_1^- + i0^+} \right)^{\frac{1+\epsilon}{2}} \left(\frac{(\hat{q}_{\bar{q}_n})_T^2}{-2y_2^+ y_2^- - i0^+} \right)^{\frac{1+\epsilon}{2}} \left[K_{-1-\epsilon} \left(\sqrt{(\hat{q}_{q_n})_T^2 (-2y_1^+ y_1^- + i0^+)} \right) (y_1^+ \bar{n}^{\sigma_1} + y_1^- n^{\sigma_1}) \right. \\
&- i(\hat{q}_{q_n})_T^{\sigma_1} \sqrt{\frac{-2y_1^+ y_1^- + i0^+}{(\hat{q}_{q_n})_T^2}} K_{-\epsilon} \left(\sqrt{(\hat{q}_{q_n})_T^2 (-2y_1^+ y_1^- + i0^+)} \right) \\
&\times \left[K_{-1-\epsilon} \left(\sqrt{(\hat{q}_{\bar{q}_n})_T^2 (-2y_2^+ y_2^- - i0^+)} \right) (y_2^+ \bar{n}^{\sigma_2} + y_2^- n^{\sigma_2}) \right. \\
&+ i(\hat{q}_{\bar{q}_n})_T^{\sigma_2} \sqrt{\frac{-2y_2^+ y_2^- - i0^+}{(\hat{q}_{\bar{q}_n})_T^2}} K_{-\epsilon} \left(\sqrt{(\hat{q}_{\bar{q}_n})_T^2 (-2y_2^+ y_2^- - i0^+)} \right) \\
&\times \bar{T}\{\bar{q}_n(y_2^+ + z_2^+, z_2^-, z_{2\perp})\} \frac{\Gamma^C}{2} T\{q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp})\} T\{\bar{q}_n(z_1)\} \frac{\Gamma^D}{2} \bar{T}\{q_n(z_2)\},
\end{aligned} \tag{4.141}$$

¹¹The operators $e^{iy_{1(2)}^- (n \cdot \hat{p}_{q_n(\bar{q}_n)})}$ and $e^{iy_{1(2)\perp} \cdot \hat{p}_{q_n(\bar{q}_n)}}$ must be understood in the sense of Fourier multipliers [229].

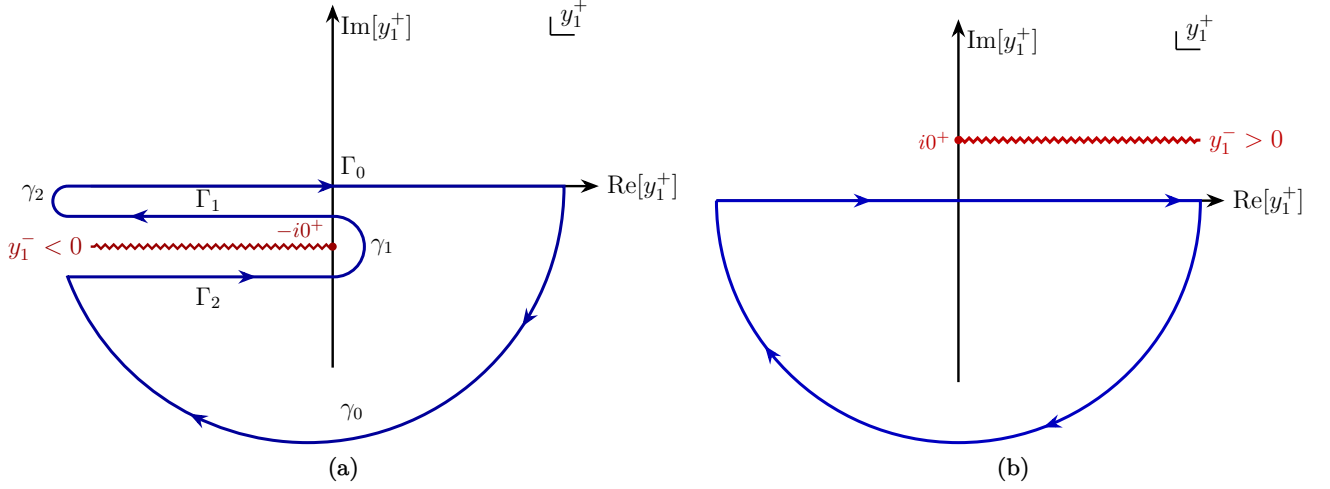


Figure 4.6: Closed integration contour for y_1^+ : (a) the branch cut (jagged red line) is placed on the lower-half plane, the integration contour has to be deformed to go around it; (b) the integration contour does not enclose the branch cut, which lies on the upper-half plane (jagged red line).

where we have defined the operators $(\hat{q}_{q_n})^{\sigma_1} = k^{\sigma_1} + (\hat{p}_{q_m})^{\sigma_1}$ and $(\hat{q}_{\bar{q}_n}) = (\hat{p})_{\bar{q}_n}^{\sigma_2} - k^{\sigma_2}$. We can see the complexity of performing the computation in position space, where we are left with the Fourier transform of transcendental functions that depends on differential operators acting on the (anti-)quark fields. One could try to expand the Bessel functions $K_{-1-\epsilon}$ and $K_{-\epsilon}$ around $(\mathbf{k}_T + (\hat{p}_{q_n})_T)^2 \rightarrow 0$ and $(\mathbf{k}_T - (\hat{p}_{\bar{q}_n})_T)^2 \rightarrow 0$, which physically correspond to the limit of a parton with small transverse momentum entering the effective electroweak vertex. However this would spoil the analytic structure of the integrand, leading to IR divergences at $y^- \rightarrow \pm\infty$. On the other hand, the expansion in the limit $(\mathbf{k}_T + (\hat{p}_{q_n})_T)^2 \rightarrow \infty$ and $(\mathbf{k}_T - (\hat{p}_{\bar{q}_n})_T)^2 \rightarrow \infty$, i.e. large transverse momentum entering the effective vertex, leads to singularities in the limit $y^\pm \rightarrow 0$, i.e. UV divergences. Therefore, expanding the Bessel function in eq. (4.141) one loses the analytic properties that make the integral finite. In order to show that the integrand is an analytic function and the integral is finite we start by pointing out that the integrand contains branch cuts for $y_{1(2)}^+$. Let us consider y_1^+ (for y_2^+ the analysis is completely analogous). Using the convention that $\pi \geq \arg(y_1^+) > -\pi$, i.e. the phase of y_1^+ is contained between π and $-\pi$, one can identify two branch cuts: for $y_1^- > 0$ the cut is placed on the upper-half plane and goes from $y_1^+ = i0^+$ up to $y_1^+ = \infty + i0^+$. For $y_1^- < 0$ the cut is on the lower-half plane and goes from $y_1^+ = -i0^+$ up to $y_1^+ = -\infty - i0^+$, see fig. 4.6. The branch cuts for $y_{1(2)}^+$ depending upon whether $y_{1(2)}^-$ is positive or negative thus

$$y_1^+ \in \begin{cases} (i0^+, \infty + i0^+), & y_1^- > 0, \quad \pi \geq \arg(y_1^+) > -\pi \\ (-i0^+, -\infty - i0^+), & y_1^- < 0, \quad \pi \geq \arg(y_1^+) > -\pi \end{cases} \quad (4.142)$$

$$y_2^+ \in \begin{cases} (-i0^+, \infty - i0^+), & y_2^- > 0, \quad \pi \geq \arg(y_2^+) > -\pi \\ (i0^+, -\infty + i0^+), & y_2^- < 0, \quad \pi \geq \arg(y_2^+) > -\pi \end{cases} \quad (4.143)$$

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As done in Sec.4.3.1, the operators on the last line in eq. (4.141) have to be written in terms of the good spin components, $\xi_{n(\bar{n})}$, using the equations of motion, see App. C.2. Afterwards, one need to expand the resulting operators around $z_2^{-(+)} - z_1^{-(+)} = 0$ in the (anti-)collinear case. These can be then identified with the TMDPDF operators and their matrix elements with the TMDPDF distributions. Since we are working at order α_s , we can identify the matrix elements of the TMDPDF operators in eq. (4.141) with the renormalized ones. In order to their Fourier transform exists and therefore relate their matrix elements in position space to physical distributions in momentum spaces if certain analytic properties and behavior at spatial infinity are satisfied; we assume that the TMDPDF matrix elements are functions of the Schwartz functions space $\mathcal{S}$. [229,233]¹² We emphasize that this assumption applies to the renormalized matrix elements. Under these conditions, it can be shown that the integrals in $y_1^\pm$ and $y_2^\pm$ in eq. (4.141) amounts to a finite quantity. We prove it for the integration in $y_1^\pm$ (for the integration in $y_2^\pm$ the steps to follow are analogous).

Let us first consider the case of $y_1^- > 0$. The branch cut is on the upper-half plane, and we can thus y_1^+ along the closed contour shown in fig. 4.6b. Since the contour does not contain any pole in its interior the integral along the closed path is zero, thus the contribution from the integration along the real line must be equal to minus the contribution from the integration along the semi-circle. The semi-circle integral vanishes in the limit of infinite radius,¹³ what means that the integral along the real line from $-\infty$ to ∞ must be zero. The case of $y_1^- < 0$ is slightly more involved due to the presence of a branch cut in the lower-half plane. In this case one needs to deform the integration contour to go around the branch cut, see fig. 4.6a. Since the integration contour does not enclose any pole the sum of the integrals along Γ_0 , Γ_1 , Γ_2 , γ_0 , γ_1 and γ_3 must be equal to zero. In the limit of infinite radius the integration along the arc γ_0 vanishes, the contribution from the integration along the arc γ_2 vanishes as well, since it is centered at $-\infty$ and we assume that the background fields goes to zero rapidly enough in order to guarantee the existence of the Fourier transform and well-defined TMDPDF matrix elements. Along the semi-circle γ_1 we parametrize y_1^+ as $y_1^+ \mapsto -i0 + i\delta e^{-i\theta}$; in the limit $\delta \rightarrow 0$, i.e. we shrink the semi-circle around the point $y_1^+ = -i0^+$, the integrand in eq. (4.141) goes to zero as $\delta^{-\epsilon}$, which vanishes in dimensional regularization for $\epsilon < 0$. Therefore we conclude that the integral along Γ_0 must be equal to minus the sum of the integration along $\Gamma_1 + \Gamma_2$. For the integral along Γ_1 we parametrize y_1^+ as $y_1^+ \mapsto y_1^+ - i0^+(1 - \delta)$, where $y_1^+ \in (0, -\infty]$ and $1 > \delta \geq 0$. In integral along Γ_2 we parametrize y_1^+ as $y_1^+ \mapsto y_1^+ - i0^+(1 + \delta)$, where

¹²The Schwartz space or space of rapidly decreasing functions is defined as [229, 233]: the function space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of all infinitely differentiable functions in $\mathbb{R}^n$ that decrease together with all their derivatives, as $|z| \rightarrow \infty$, faster than any power of $|z|^{-1}$.

¹³One may be concerned about the complex exponential $e^{ik^- y_1^+}$ in eq. (4.141). However the field $q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp})$ can be written as

$$q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp}) = e^{iy_1^+ \hat{p}_{q_n}^-} q_n(z_1^+, z_1^-, z_{1\perp}),$$

and the exponential factor $\exp\{iy_1^+(k^- + \hat{p}_{q_n}^-)\}$ acting on the TMDPDF matrix elements amounts to $\exp\{-iy_1^+ q_{q_n}^-\}$ where $q_{q_n}^- \geq 0$.

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$y_1^+ \in [-\infty, 0)$ and $1 > \delta \geq 0$. Thus, $(y_1^+ + i0^+)^\lambda$ amounts to

$$(y_1^+ + i0^+)^\lambda = \begin{cases} |y_1^+ + i0^+ \delta|^\lambda e^{i\pi\lambda - i\lambda \arctan\left(\frac{0^+\delta}{-y_1^+}\right)}, & y_1^+ \in \Gamma_1 \\ |y_1^+ - i0^+ \delta|^\lambda e^{-i\pi\lambda + i\lambda \arctan\left(\frac{0^+\delta}{-y_1^+}\right)}, & y_1^+ \in \Gamma_2 \end{cases} \quad (4.144)$$

where $\lambda \in \mathbb{R}$. In the limit $\delta \rightarrow 0$ the integration along Γ_1 and Γ_2 amounts to

$$\begin{aligned} \langle 0 | J^{\nu\dagger}(z_2) J^\mu(z_1) | 0 \rangle_{\text{diag (a)}, \Gamma_1 + \Gamma_2} &= -\frac{\alpha_s(\mu^2) C_F}{4N_c} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \sum_{C,D} \int \frac{[d^d k]}{(2\pi)^{d-1}} \theta(k^0) e^{-ik \cdot (z_2 - z_1)} \\ &\times \int dy_2^+ dy_2^- h_n^{\sigma_2}(y_2^+, y_2^-, (\hat{q}_{\bar{q}_n})_T) e^{-ik^- y_2^+ + i(n \cdot \hat{q}_{\bar{q}_n}) y_2^-} \text{Tr}[\Gamma_C \gamma_\alpha \gamma_{\sigma_2} \Gamma^\nu \Gamma_D \Gamma^\mu \gamma_{\sigma_1} \gamma^\alpha] \\ &\times \int_{-\infty}^0 dy_1^- \int_{-\infty}^0 dy_1^+ g_n^{\sigma_1}(y_1^+, y_1^-, (\hat{q}_{q_n})_T) e^{ik^- y_1^+ + i(n \cdot \hat{q}_{q_n}) y_1^-} \\ &\times \mathcal{O}_n^{\Gamma^C}(z_2 + y_2^+ - z_1 - y_1^+) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1 - z_2) \end{aligned} \quad (4.145)$$

where $[d^d k] = d^d k \delta(k^2)$ and the operators $\mathcal{O}_{n(\bar{n})}^{[\Gamma^{C(D)}]}$ are defined as in the LO and NLO virtual contributions

$$\begin{aligned} \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+) &= \bar{T} \{ \bar{q}_n(y_2^+ + z_2^+, z_2^-, z_{2\perp}) \} \frac{\Gamma^C}{2} T \{ q_n(y_1^+ + z_1^+, z_1^-, z_{1\perp}) \}, \\ \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1 - z_2) &= T \{ \bar{q}_{\bar{n}}(z_1) \} \frac{\Gamma^D}{2} \bar{T} \{ q_{\bar{n}}(z_2) \}, \end{aligned} \quad (4.146)$$

and the functions $g_n^{\sigma_1}$ and $h_n^{\sigma_2}$ are given by

$$\begin{aligned} g_n^{\sigma_1}(y_1^+, y_1^-, (\hat{q}_{q_n})_T) &= (y_1^+ \bar{n}^{\sigma_1} + y_1^- n^{\sigma_1}) \left(\frac{(\hat{q}_{q_n})^2}{2|y_1^+||y_1^-|} \right)^{\frac{1+\epsilon}{2}} \\ &\times \left\{ e^{-\frac{i\pi(1+\epsilon)}{2}} K_{-1-\epsilon} \left(i\sqrt{2|y_1^+||y_1^-|} (\hat{q}_{q_n})_T^2 \right) - e^{\frac{i\pi(1+\epsilon)}{2}} K_{-1-\epsilon} \left(-i\sqrt{2|y_1^+||y_1^-|} (\hat{q}_{q_n})_T^2 \right) \right\} \\ &- i(\hat{q}_{q_n})_T^{\sigma_1} \left(\frac{(\hat{q}_{q_n})^2}{2|y_1^+||y_1^-|} \right)^{\frac{\epsilon}{2}} \\ &\times \left\{ e^{-\frac{i\pi\epsilon}{2}} K_{-\epsilon} \left(i\sqrt{2|y_1^+||y_1^-|} (\hat{q}_{q_n})_T^2 \right) - e^{\frac{i\pi\epsilon}{2}} K_{-\epsilon} \left(-i\sqrt{2|y_1^+||y_1^-|} (\hat{q}_{q_n})_T^2 \right) \right\}, \end{aligned} \quad (4.147)$$

$$\begin{aligned} h_n^{\sigma_2}(y_2^+, y_2^-, (\hat{q}_{\bar{q}_n})_T) &= (y_2^+ \bar{n}^{\sigma_2} + y_2^- n^{\sigma_2}) \left(\frac{(\hat{q}_{\bar{q}_n})_T^2}{-2y_2^+ y_2^- - i0^+} \right)^{\frac{1+\epsilon}{2}} K_{-1-\epsilon} \left(\sqrt{(\hat{q}_{\bar{q}_n})_T^2 (-2y_2^+ y_2^- - i0^+)} \right) \\ &+ i(\hat{q}_{\bar{q}_n})_T^{\sigma_2} \left(\frac{(\hat{q}_{\bar{q}_n})_T^2}{-2y_2^+ y_2^- - i0^+} \right)^{\frac{\epsilon}{2}} K_{-\epsilon} \left(\sqrt{(\hat{q}_{\bar{q}_n})_T^2 (-2y_2^+ y_2^- - i0^+)} \right). \end{aligned} \quad (4.148)$$

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The overall minus sign in eq. (4.145) is due to the fact that the real line integration in y_1^+ is equal to minus the line integration along $\Gamma_1 + \Gamma_2$. The relative minus sign between the Bessel functions in $g_n^{\sigma_1}$ is due to the different orientation between the line integral along Γ_1 , which is from zero to minus infinity, and the integration along Γ_2 , which is from minus infinity to zero.

In order to prove that the integral in y_1^+ and y_1^- is finite. i.e. free of UV and IR poles, we start by splitting the integration domain, $(y_1^+, y_1^-) \in (-\infty, 0] \times (-\infty, 0]$, into two contributions: $y_1^+ \geq y_1^-$ and $y_1^- \geq y_1^+$. The contribution from $y_1^- \geq y_1^+$ amounts to

$$I_n^{\sigma_1}(z_2 - z_1, (\hat{q}_{q_n})_T; y_1^- \geq y_1^+) = \int_{-\infty}^0 dy_1^+ \int_{y_1^+}^0 dy_1^- g_n^{\sigma_1}(y_1^+, y_1^-, (\hat{q}_{q_n})_T) \\ \times \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+) \mathcal{O}_n^{[\Gamma^D]}(z_1 - z_2) e^{ik^- y_1^+ + i(n \cdot \hat{q}_{q_n}) y_1^-}, \quad (4.149)$$

with the change of variables $y_1^- \rightarrow y_1^- y_1^+$ one obtains

$$I_n^{\sigma_1}(z_2 - z_1, (\hat{q}_{q_n})_T; y_1^- \geq y_1^+) = - \int_{-\infty}^0 dy_1^+ \int_0^1 dy_1^- g_n^{\sigma_1}(y_1^+, y_1^- y_1^+, (\hat{q}_{q_n})_T) y_1^+ \\ \times \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+) \mathcal{O}_n^{[\Gamma^D]}(z_1 - z_2) e^{ik^- y_1^+ + i(n \cdot \hat{q}_{q_n}) y_1^- y_1^+}, \quad (4.150)$$

Notice that now y_1^- is a dimensionless variable. Our initial assumption was that the matrix elements of the operator $\mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+)$ are analytic functions on the lower-half plane, and thus the only term that can contain any singularity is $g_n^{\sigma_1}(y_1^+, y_1^- y_1^+, (\hat{q}_{q_n})_T) y_1^+$. In the limit $y_1^+ \rightarrow 0^-$ one obtains

$$\lim_{y_1^+ \rightarrow 0^-} g_n^{\sigma_1}(y_1^+, y_1^- y_1^+, (\hat{q}_{q_n})_T) y_1^+ = - \frac{2^{1+\epsilon} i \pi (y_1^+)^{-2\epsilon}}{(y_1^-)^{1+\epsilon} \Gamma[-\epsilon]} \bar{n}^{\sigma_1} + \mathcal{O}((y_1^+)^{2-2\epsilon} (y_1^-)^{-\epsilon}), \quad (4.151)$$

while in the limit $y_1^+ \rightarrow -\infty$ the matrix elements of $\mathcal{O}_n^{[\Gamma]}$ go to zero rapidly enough to make integral finite. There are no further points in $y_1^+ \in (-\infty, 0]$ that could lead to singularities, thus the integration in y_1^+ is well-defined and finite. Nevertheless, for the integration in y_1^- there is a singularity at $y_1^- \rightarrow 0$. We isolate it rewriting $g_n^{\sigma_1}(y_1^+, y_1^- y_1^+) y_1^+$ as

$$g_n^{\sigma_1}(y_1^+, y_1^- y_1^+, (\hat{q}_{q_n})_T) y_1^+ = - \frac{2^{1+\epsilon} i \pi (y_1^+)^{-2\epsilon}}{(y_1^-)^{1+\epsilon} \Gamma[-\epsilon]} \bar{n}^{\sigma_1} + \left\{ g_n^{\sigma_1}(y_1^+, y_1^- y_1^+, (\hat{q}_{q_n})_T) y_1^+ + \frac{2^{1+\epsilon} i \pi (y_1^+)^{-2\epsilon}}{(y_1^-)^{1+\epsilon} \Gamma[-\epsilon]} \bar{n}^{\sigma_1} \right\}. \quad (4.152)$$

The second term in curly brackets is not singular at $y_1^- \rightarrow 0$ nor $y_1^- \rightarrow 1$, therefore its contribution to the integration in y_1^- is finite. The first term on the right-hand side can be expanded in terms of plus and delta distributions, see App. A.1, which are integrated against the matrix elements of $\mathcal{O}_n^\Gamma$. One can see that in the limit $\epsilon \rightarrow 0^-$ only the term proportional to $\delta(y_1^-)/\epsilon$ will contribute.

The integration for $y_1^+ \geq y_1^-$ gives

$$I_n^{\sigma_1}(z_2 - z_1, (\hat{q}_{q_n})_T; y_1^+ \geq y_1^-) = \int_{-\infty}^0 dy_1^- \int_{y_1^-}^0 dy_1^+ g_n^{\sigma_1}(y_1^+, y_1^-, (\hat{q}_{q_n})_T) \\ \times \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+) \mathcal{O}_n^{[\Gamma^D]}(z_1 - z_2) e^{ik^- y_1^+ + i(n \cdot \hat{q}_{q_n}) y_1^-}, \quad (4.153)$$

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and with the change of variable $y_1^+ \rightarrow y_1^+ y_1^-$ one obtains

$$I_n^{\sigma_1}(z_2 - z_1, (\hat{q}_{q_n})_T; y_1^+ \geq y_1^-) = - \int_{-\infty}^0 dy_1^- \int_0^1 dy_1^+ g_n^{\sigma_1}(y_1^+ y_1^-, y_1^-, (\hat{q}_{q_n})_T) y_1^- \\ \times \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^+ y_1^-) \mathcal{O}_n^{[\Gamma^D]}(z_1 - z_2) e^{ik^- y_1^+ y_1^- + i(n \cdot \hat{q}_{q_n}) y_1^-}, \quad (4.154)$$

The integrand does not contain any singularity in the region $y_1^- \in (-\infty, 0]$, since the dependence on y_1^- of the operator $\mathcal{O}_n^{[\Gamma^C]}$ regularizes the integral. However the limit $y_1^+ \rightarrow 0$ is troublesome, due to the fact that the limits $(y_1^+, y_1^-) \rightarrow (0, -\infty)$ do not commute:

$$\lim_{y_1^+ \rightarrow 0} \lim_{y_1^- \rightarrow -\infty} \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^- y_1^+) = \lim_{y_1^+ \rightarrow 0} \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 + \infty y_1^+) = 0, \\ \lim_{y_1^- \rightarrow -\infty} \lim_{y_1^+ \rightarrow 0} \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^- y_1^+) = \lim_{y_1^- \rightarrow -\infty} \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - 0) = \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1), \quad (4.155)$$

Hence taking first the limit $y_1^+ \rightarrow 0$ leads to IR divergences in the y_1^- integration. The limits must then be taken carefully. With this aim, we rewrite $g_n^{\sigma_1}(y_1^+ y_1^-, y_1^-, (\hat{q}_{q_n})_T) y_1^-$ as

$$g_n^{\sigma_1}(y_1^+ y_1^-, y_1^-, (\hat{q}_{q_n})_T) y_1^- = - \frac{2i\pi(y_1^-)^{-2\epsilon}}{(y_1^+)^{1+\epsilon}\Gamma[-\epsilon]} n^{\sigma_1} + \frac{i\pi(y_1^+)^{-\epsilon}(y_1^-)^{2-2\epsilon}}{2\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^2 n^{\sigma_1} \\ + \frac{\pi(y_1^+)^{-\epsilon}(y_1^-)^{1-2\epsilon}}{\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^{\sigma_1} + \left\{ g_n^{\sigma_1}(y_1^+ y_1^-, y_1^-, (\hat{q}_{q_n})_T) y_1^- + \frac{2i\pi(y_1^-)^{-2\epsilon}}{(y_1^+)^{1+\epsilon}\Gamma[-\epsilon]} n^{\sigma_1} \right. \\ \left. - \frac{i\pi(y_1^+)^{-\epsilon}(y_1^-)^{2-2\epsilon}}{2\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^2 n^{\sigma_1} - \frac{\pi(y_1^+)^{-\epsilon}(y_1^-)^{1-2\epsilon}}{\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^{\sigma_1} \right\}. \quad (4.156)$$

The term in curly brackets goes to zero as $(y_1^+)^{2-\epsilon}$ in the limit $y_1^+ \rightarrow 0$, and one can thus interchange the order of integration in y_1^+ and y_1^- . The contributions from the first line must be handled more carefully. We perform two changes of variables in (4.154): first $y_1^- \rightarrow y_1^-/y_1^+$ and afterwards $y_1^+ \rightarrow 1/y_1^+$. One obtains

$$I_n^{\sigma_1}(z_2 - z_1, (\hat{q}_{q_n})_T; y_1^+ \geq y_1^-) = - \int_1^\infty dy_1^+ \int_{-\infty}^0 dy_1^- \\ \times \left\{ - \frac{i\pi(y_1^-)^{-2\epsilon}}{(y_1^+)^{-\epsilon}\Gamma[-\epsilon]} n^{\sigma_1} + \frac{i\pi(y_1^+)^{1-\epsilon}(y_1^-)^{2-2\epsilon}}{2\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^2 n^{\sigma_1} + \frac{\pi(y_1^+)^{-\epsilon}(y_1^-)^{1-2\epsilon}}{\Gamma[1-\epsilon]} (\hat{q}_{q_n})_T^{\sigma_1} \right\} \quad (4.157) \\ \times \mathcal{O}_n^{[\Gamma^C]}(z_2 + y_2^+ - z_1 - y_1^-) \mathcal{O}_n^{[\Gamma^D]}(z_1 - z_2) e^{i(k^- + (n \cdot \hat{q}_{q_n}) y_1^+) y_1^-} + \dots,$$

where ... stands for the contribution given by the second line in eq. (4.156). The terms $(y_1^-)^n$ with $n \geq 0$ are known as multipliers in $\mathcal{S}$. It can be shown that if $f(y_1^-) \in \mathcal{S}$ also $(y_1^-)^n f(y_1^-) \in \mathcal{S}$, see [229, 233] for the proof. The Fourier transform maps $\mathcal{S}$ onto itself in reciprocal one-to-one and continuous fashion, see [229, 233] for the proof. This means that after integration in y_1^- the resulting function, which depends on $k^- + (n \cdot \hat{q}_{q_n}) y_1^+$, is also a function of the space $\mathcal{S}$ and thus the integral in y_1^+ is well-defined.

4.3.2.2 Computation in momentum space

As we have seen, the computation in position space is free of IR and UV divergences. Nevertheless, there are two drawbacks: we need to introduce a suitable parametrization of the TMDPDF matrix elements in position spaces and the integrals must be done in a precise order. The whole procedure is rather cumbersome, and while it shows the finiteness of the integrals is not practical. For this reasons we decided to perform the computation of the diagrams in fig. 4.5 in momentum spaces, i.e. we first perform the Fourier transform and afterwards the phase spaces integrals. This order of perform the computation leads to the present of IR poles in the convolution of the scattering amplitudes given in fig. 4.5 with the TMDPDF matrix elements in momentum spaces. We identify these divergences with a double counting of the soft and (anti-)collinear regions, see fig. 4.1. In the computation in position space fog diagram (a) in fig. 4.5a we assume that the TMDPDF matrix elements in go to zero (at the spatial infinity) rapidly enough, what regularized all the IR divergences and made the integrals finite. In momentum space the IR occurs when the momentum of the gluon goes to zero (soft) or becomes parallel to one of the incoming partons (collinear). The method known as phase-space slicing introduces a phase-space cut-off to avoid soft and (anti-)collinear regions [234–244]. It is straightforward and simple of implement, however studying the dependence on the cut-off parameter(s) is rather cumbersome and can hardly be done analytically. This is computationally very expensive and a finite small value of the cut-off parameter(s) is maintained, which leads to a residual cut-off dependence.

The measurement functions introduced in Sec. 4.2.4 can be modified to discriminate among hard, soft and (anti-)collinear real radiation [55–57, 61, 96, 245–250]. This method may introduce a dependence on non-physical parameters which can lead to large logarithms in the soft and (anti-)collinear regions. The resummation of logarithmically enhanced contributions and the cancellation of non-physical parameter dependence introduced through the measurement functions is typically taken care of using either effective field theory technique [55–57, 59, 205, 249] or a diagrammatic approach [95, 97, 101–103, 134]. In this work we introduce a third method, by identifying the contributions inconsistent with the power counting of the hard modes as a double counting contributions that need to be removed. This situation and approach is rather standard and common in effective field theories like SCET [43, 47, 106, 201]. We will comment more on this after the integration of the real contribution to eq. (4.9).

The NLO real radiative corrections in fig. 4.5 to the product of electroweak currents in

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eq. (4.9) amounts to

$$\begin{aligned}
[J^{\nu\dagger}(z_2)J^\mu(z_1)]^{[NLO]} &= -(ig)^2 \int d^d y_1 d^d y_2 \\
&\times \left[\bar{T}\{[\bar{q}_n \not{B}\psi](y_2)[\bar{\psi}\Gamma^\nu q_{\bar{n}}](z_2)\}T\{[\bar{q}_{\bar{n}}\Gamma^\mu\psi](z_1)[\bar{\psi}\not{B}q_n](y_1)\} \right. \\
&+ \bar{T}\{[\bar{\psi}\not{B}q_{\bar{n}}](y_2)[\bar{q}_n\Gamma^\nu\psi](z_2)\}T\{[\bar{\psi}\Gamma^\mu q_n](z_1)[\bar{q}_{\bar{n}}\not{B}\psi](y_1)\} \\
&+ \bar{T}\{[\bar{q}_n \not{B}\psi](y_2)[\bar{\psi}\Gamma^\nu q_{\bar{n}}](z_2)\}T\{[\bar{\psi}\Gamma^\mu q_n](z_1)[\bar{q}_{\bar{n}}\not{B}\psi](y_1)\} \\
&\left. + \bar{T}\{[\bar{\psi}\not{B}q_{\bar{n}}](y_2)[\bar{q}_n\Gamma^\nu\psi](z_2)\}T\{[\bar{q}_{\bar{n}}\Gamma^\mu\psi](z_1)[\bar{\psi}\not{B}q_n](y_1)\} \right] + n \leftrightarrow \bar{n}
\end{aligned} \tag{4.158}$$

Using eq. (D.2) to perform the short-distance expansion of the product of two free hard fermion fields (under the same time-ordered operator) and eq. (D.3) for the product of two free hard gluon fields one obtains

$$\begin{aligned}
[J^{\nu\dagger}(z_2)J^\mu(z_1)]^{[NLO]} &= -(ig)^2 \int d^d y_1 d^d y_2 \\
&\times \left[\bar{T}\{[\bar{q}_n \not{B}\psi](y_2)[\bar{\psi}\Gamma^\nu q_{\bar{n}}](z_2)\}T\{[\bar{q}_{\bar{n}}\Gamma^\mu\psi](z_1)[\bar{\psi}\not{B}q_n](y_1)\} \right. \\
&+ \bar{T}\{[\bar{\psi}\not{B}q_{\bar{n}}](y_2)[\bar{q}_n\Gamma^\nu\psi](z_2)\}T\{[\bar{\psi}\Gamma^\mu q_n](z_1)[\bar{q}_{\bar{n}}\not{B}\psi](y_1)\} \\
&+ \bar{T}\{[\bar{q}_n \not{B}\psi](y_2)[\bar{\psi}\Gamma^\nu q_{\bar{n}}](z_2)\}T\{[\bar{\psi}\Gamma^\mu q_n](z_1)[\bar{q}_{\bar{n}}\not{B}\psi](y_1)\} \\
&\left. + \bar{T}\{[\bar{\psi}\not{B}q_{\bar{n}}](y_2)[\bar{q}_n\Gamma^\nu\psi](z_2)\}T\{[\bar{q}_{\bar{n}}\Gamma^\mu\psi](z_1)[\bar{\psi}\not{B}q_n](y_1)\} \right] \\
&+ n \leftrightarrow \bar{n}.
\end{aligned} \tag{4.159}$$

One could also consider contributions of the form

$$\begin{aligned}
[J^{\nu\dagger}(z_2)J^\mu(z_1)]^{[NLO]} &= -(ig)^2 \int d^d y_1 d^d y_2 \\
&\bar{T}\{[\bar{q}_n \not{B}\psi](y_2)[\bar{\psi}\Gamma^\nu q_{\bar{n}}](z_2)\}T\{[\bar{q}_{\bar{n}}\Gamma^\mu\psi](z_1)[\bar{\psi}\not{B}q_n](y_1)\} + \dots,
\end{aligned}$$

where we perform the short-distance expansion of fermion fields under different time-ordered operator. Nevertheless, contributions like the one above does not full fill momentum conservation and thus we do not consider them. Replacing each hard field contraction in eq. (4.159) by their corresponding propagator (eq. (D.11) for the fermion and eq. (D.4) for the gluon fields, respectively) and using the Fierz identity one obtains

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$$\begin{aligned}
[\mathcal{J}^{\nu\dagger}(z_2)\mathcal{J}^\mu(z_1)]^{[NLO]} &= -\frac{\pi\alpha_s(\mu^2)}{N_c}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \sum_{C,D} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) \int d^d y_1 d^d y_2 e^{-ik\cdot(y_2-y_1)} \\
&\times \left[\text{Tr}[\Gamma_C \gamma_\beta S(y_2-z_2) \Gamma^\nu \Gamma_D \Gamma^\mu S(z_1-y_1) \gamma^\alpha] \mathcal{O}_n^{[\Gamma^C]}(y_2, y_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1, z_2) \right. \\
&+ \text{Tr}[\Gamma_C \Gamma^\nu S(z_2-y_2) \gamma^\beta \Gamma_D \gamma_\beta S(y_1-z_1) \Gamma^\mu] \mathcal{O}_n^{[\Gamma^C]}(z_2, z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_1, y_2) \\
&+ \text{Tr}[\Gamma_C \gamma_\beta S(y_2-z_2) \Gamma^\nu \Gamma_D \gamma^\beta S(y_1-z_1) \Gamma^\mu] \mathcal{O}_n^{[\Gamma^C]}(y_2, z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_1, z_2) \\
&+ \text{Tr}[\Gamma_C \Gamma^\nu S(z_2-y_2) \gamma_\beta \Gamma_D \Gamma^\mu S(z_1-y_1) \gamma^\beta] \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1, y_2) \Big] \\
&+ n \leftrightarrow \bar{n},
\end{aligned} \tag{4.160}$$

where we have written the effective vertex $\mathcal{J}^\mu$ to make explicit that the fields have been integrated out using the short-distance expansion of the (time-ordered-)product fields. As we have discussed, the leading contribution to eq. (4.160) arises from the limits $y_{1(2)}^\mu \rightarrow z_{1(2)}^\mu$. The operators $\mathcal{O}_{n(\bar{n})}^{[\Gamma^{C(D)}]}$ are defined as in eqs. (4.115) and (4.116). Performing the coordinates shift $y_1 \rightarrow z_1 + y_1$ and $y_2 \rightarrow z_2 + y_2$ eq. (4.160) becomes

$$\begin{aligned}
[\mathcal{J}^{\nu\dagger}(z_2)\mathcal{J}^\mu(z_1)]^{[NLO]} &= -\frac{\pi\alpha_s(\mu^2)}{N_c}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \sum_{C,D} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) \int d^d y_1 d^d y_2 e^{-ik\cdot(y_2+z_2-y_1-z_1)} \\
&\times \left[\text{Tr}[\Gamma_C \gamma_\beta S(y_2) \Gamma^\nu \Gamma_D \Gamma^\mu S(-y_1) \gamma^\alpha] \mathcal{O}_n^{[\Gamma^C]}(y_2+z_2, y_1+z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1, z_2) \right. \\
&+ \text{Tr}[\Gamma_C \Gamma^\nu S(-y_2) \gamma^\beta \Gamma_D \gamma_\beta S(y_1) \Gamma^\mu] \mathcal{O}_n^{[\Gamma^C]}(z_2, z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_1+z_1, y_2+z_2) \\
&+ \text{Tr}[\Gamma_C \gamma_\beta S(y_2) \Gamma^\nu \Gamma_D \gamma^\beta S(y_1) \Gamma^\mu] \mathcal{O}_n^{[\Gamma^C]}(y_2+z_2, z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(y_1+z_1, z_2) \\
&+ \text{Tr}[\Gamma_C \Gamma^\nu S(-y_2) \gamma_\beta \Gamma_D \Gamma^\mu S(-y_1) \gamma^\beta] \mathcal{O}_n^{[\Gamma^C]}(z_2, y_1+z_1) \mathcal{O}_{\bar{n}}^{[\Gamma^D]}(z_1, y_2+z_2) \Big] \\
&+ n \leftrightarrow \bar{n},
\end{aligned} \tag{4.161}$$

where the leading contribution is given now in the limit $y_1^\mu = y_2^\mu \rightarrow 0$. From eq. (4.161) the step to follow are rather similar to the ones described for the virtual corrections, which leads

$$\begin{aligned}
 \langle P_B P_A | \mathcal{J}^{\nu\dagger}(z_2) \mathcal{J}^\mu(z_1) | P_A P_B \rangle^{[NLO]} &= \frac{4\pi P_A^- P_B^+ \alpha_s(\mu^2) C_F}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \\
 &\left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) e^{i(p_a + p_b - k) \cdot (z_2 - z_1)} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
 &\times \left[\frac{\text{Tr}[\not{p}_a \gamma_\alpha (\not{p}_a - \not{k}) \Gamma^\nu \not{p}_b \Gamma^\mu (\not{p}_a - \not{k}) \gamma^\alpha]}{[(p_a - k)^2 - i0][(p_a - k)^2 + i0]} - \frac{\text{Tr}[\not{p}_a \Gamma^\nu (\not{p}_b - \not{k}) \gamma_\alpha \not{p}_b \Gamma^\mu (\not{p}_a - \not{k}) \gamma^\alpha]}{[(p_b - k)^2 - i0][(p_a - k)^2 + i0]} \right. \\
 &- \frac{\text{Tr}[\not{p}_a \gamma_\alpha (\not{p}_a - \not{k}) \Gamma^\nu \not{p}_b \gamma^\alpha (\not{p}_b - \not{k}) \Gamma^\mu]}{[(p_a - k)^2 - i0][(p_b - k)^2 + i0]} + \frac{\text{Tr}[\not{p}_a \Gamma^\nu (\not{p}_b - \not{k}) \gamma_\alpha \not{p}_b \gamma^\alpha (\not{p}_b - \not{k}) \Gamma^\mu]}{[(p_b - k)^2 - i0][(p_b - k)^2 + i0]} \Big] \\
 &\times f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
 &+ n \leftrightarrow \bar{n} + \dots,
 \end{aligned} \tag{4.162}$$

where ... stand for further Dirac structures and higher geometrical twist contributions that are neglected. Setting $z_2^\mu = -z_1^\mu = z^\mu/2$ in eq. (4.162) performing its the Fourier transform with respect z^μ one obtains the real NLO contribution to the Drell-Yan hadronic tensor in the quark-antiquark channel.¹⁴

The differential Drell-Yan cross section is obtained following an analogous procedure than the one presented in Sec. 4.2.4. From eq. (4.162) one can obtained the cumulant version of the hadronic tensor, it amounts to

$$\begin{aligned}
 W_{GG'}^{[NLO,R]}(q) \Theta(Q^2, y, q_T^2) &= \frac{4\pi \alpha_s(\mu^2) P_A^- P_B^+ C_F}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
 &\times \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) \Theta(Q^2, y, q_T^2) \delta^{(d)}(p_a + p_b - q - k) \\
 &\times \mathcal{H}_{ab}^{\mu\nu}(p_a, p_b, q) f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
 &+ n \leftrightarrow \bar{n} + \dots,
 \end{aligned} \tag{4.163}$$

where the scattering amplitude $\mathcal{H}_{ab,GG'}^{\mu\nu}$ is given by

$$\begin{aligned}
 \mathcal{H}_{ab,GG'}^{\mu\nu}(p_a, p_b, k) &= \sum_{G,G'} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left[\frac{\text{Tr}[\not{p}_a \gamma_\beta (\not{p}_a - \not{k}) \Gamma_{G'}^\nu \not{p}_b \Gamma_G^\mu (\not{p}_a - \not{k}) \gamma^\beta]}{[(p_a - k)^2 - i0][(p_a - k)^2 + i0]} + \right. \\
 &- \frac{\text{Tr}[\not{p}_a \Gamma_{G'}^\nu (\not{p}_b - \not{k}) \gamma_\beta \not{p}_b \Gamma_G^\mu (\not{p}_a - \not{k}) \gamma^\beta]}{[(p_b - k)^2 - i0][(p_a - k)^2 + i0]} - \frac{\text{Tr}[\not{p}_a \gamma_\beta (\not{p}_a - \not{k}) \Gamma_{G'}^\nu \not{p}_b \gamma^\beta (\not{p}_b - \not{k}) \Gamma_G^\mu]}{[(p_a - k)^2 - i0][(p_b - k)^2 + i0]} \\
 &\left. + \frac{\text{Tr}[\not{p}_a \Gamma_{G'}^\nu (\not{p}_b - \not{k}) \gamma_\beta \not{p}_b \gamma^\beta (\not{p}_b - \not{k}) \Gamma_G^\mu]}{[(p_b - k)^2 - i0][(p_b - k)^2 + i0]} \right],
 \end{aligned} \tag{4.164}$$

and the function $\Theta(Q^2, y, q_T^2)$ is given by eq. (4.105). Notice that the momentum conservation delta function in eq. (4.163) introduces a dependence on the real emitted gluon in Q^2 , y and q_T^2

¹⁴The other real contribution is in the gluon-(anti)quark channel, where one perform the the short-distance expansion of the product of two non time ordered hard fermion fields and the resulting hard-scattering process is convoluted with one (anti-) quark TMDPDF and a gluon TMDPDF. This kind of process will be subject of study in future work.

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that are defined by eq. (4.106). In App. E.2 we give the explicit expression for eq. (4.164) and thoroughly explain the computation of the gluon momentum integral. Here, we limit ourselves to highlight and introduce the two main technical challenges that we faced in the computation: the selection of an appropriate frame and integration variables, the modes double counting.

Frame selection From eq. (4.164) one can see the presence of collinear poles when k^μ becomes parallel to $p_{a(b)}^\mu$ and soft ones when $k^0 \rightarrow 0$. The main difficulty in isolating these divergent contributions is that in the proton-proton center of mass frame, while the argument of the theta functions in $\Theta(Q^2, y, q_T^2)$ takes a rather simple form, the denominators become cumbersome. This is because the parton momenta p_a^μ and p_b^μ are not back-to-back and both have non-zero components along n^μ , $\bar{n}^\mu$ and on the transverse plane to the proton beams. Therefore, if we naively attempt to write the denominators in this frame, we obtain something like

$$(p_{a(b)} - k)^2 = -2(\bar{n} \cdot p_{a(b)})(n \cdot k) - 2(n \cdot p_{a(b)})(\bar{n} \cdot k) + 2\mathbf{p}_{a(b)T} \cdot \mathbf{k}_T,$$

whose soft and collinear limit are not easily made explicit. Alternatively, by writing the propagators in $d - 1$ dimensional spherical coordinates

$$(p_{a(b)} - k)^2 = -2p_{a(b)}^0 k^0 (1 - \mathbf{n}_k \cdot \mathbf{n}_{a(b)}),$$

where $\mathbf{n}_k$ and $\mathbf{n}_{a(b)}$ are spatial unit vectors pointing along the direction of k^μ and $p_{a(b)}^\mu$. It is straightforward to identify and isolate the soft divergences. Nevertheless the collinear poles still remain hard to isolate since the scalar product $\mathbf{n}_k \cdot \mathbf{n}_{a(b)}$ is an involved combination of angles. It is thus interesting to find a reference frame where $\mathbf{n}_k \cdot \mathbf{n}_{a(b)}$ takes the simplest possible form. If for instance we chose a reference system where the $\hat{z}$ -axis along $\mathbf{p}_a$ then $\mathbf{n}_k \cdot \mathbf{n}_a = \cos \theta_k$ with $\pi \geq \theta_k \geq 0$, see fig. 4.7. However, $\mathbf{n}_k \cdot \mathbf{n}_b$ still remains a cumbersome combination of angles. The situation highly simplifies if p_a^μ and p_b^μ are back-to-back, i.e. $\angle(\mathbf{p}_a, \mathbf{p}_b) = \pi$, therefore $\mathbf{n}_k \cdot \mathbf{p}_a = \cos \theta_k$, $\mathbf{n}_k \cdot \mathbf{n}_b = -\cos \theta_k$ and

$$(p_a - k)^2 = -2p_a^0 k^0 (1 - \cos \theta_k),$$

$$(p_b - k)^2 = -2p_b^0 k^0 (1 + \cos \theta_k),$$

where θ_k is the polar angle between the $\hat{z}$ -axis and $\mathbf{k}$. In this case, it is straightforward to identify and isolate both collinear and soft poles. We can use the on-shell delta function, $\delta(k^2)$, to integrate over the gluon energy and after performing some changes of variables (see App. E.2), the $d - 1$ dimensional gluon integration measure can be written as

$$\frac{d^{d-1}\mathbf{k}}{2k^0} = C ds_{ak} ds_{bk} s_{ak}^{-\epsilon} s_{bk}^{-\epsilon},$$

where $s_{ak} = -(p_a - k)^2$ and $s_{bk} = -(p_b - k)^2$ and C denotes some prefactor irrelevant for the current discussion. After writing the gluon integration measure and the propagators in terms of $s_{a(b)k}$, we can perform an expansion in plus distributions (see App. A.1) of eq. (4.164), which allows us to readily identify soft and collinear poles. Nonetheless, there are two open questions: how do the proton momenta $P_{A(B)}^\mu$ and the light-like vectors n^μ and $\bar{n}^\mu$ change? What is now the integration measure for $p_{a(b)}^\mu$? To answer both questions we show how by

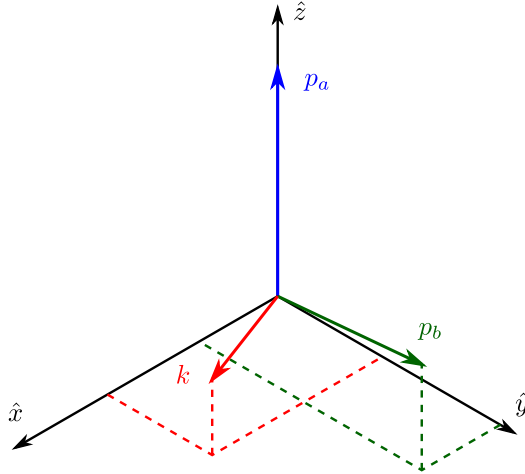


Figure 4.7: Reference frame in which $\mathbf{p}_a$ is parallel to the $\hat{z}$ -axis, $\mathbf{k}$ and $\mathbf{p}_b$ have arbitrary orientation.

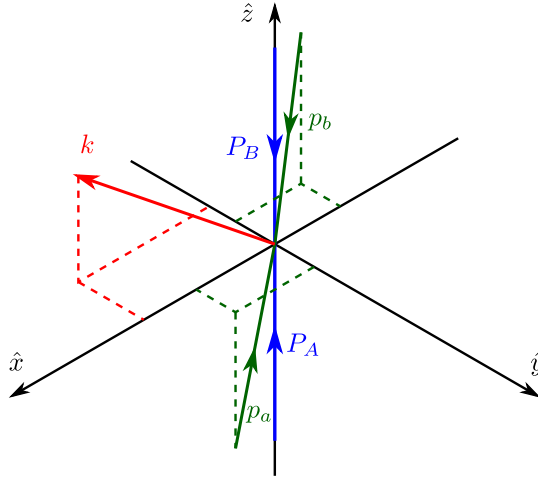


Figure 4.8: Proton center-of-mass frame. The proton beams are back-to-back along the $\hat{z}$ -axis and the partons have arbitrary orientation.

applying a series of rotation and boost transformations we can go from the proton center-of-mass frame to the reference frame where p_a^μ and p_b^μ are back-to-back, namely the partonic center-of-mass frame.

In the proton center-of-mass frame (see. fig 4.8)

$$\begin{aligned}
 P_A^\mu &= (P_A^0, \mathbf{0}_{d-2}, P_A^z), & P_B^\mu &= (P_B^0, \mathbf{0}_{d-2}, P_B^z), \\
 n^\mu &= \frac{1}{\sqrt{2}}(1, \mathbf{0}_{d-2}, 1), & \bar{n}^\mu &= \frac{1}{\sqrt{2}}(1, \mathbf{0}_{d-2}, -1), \\
 p_a^\mu &= p_a^0(1, \sin \theta_a \cos \phi_a, \sin \theta_a \sin \phi_a, \mathbf{0}_{d-4}, \cos \theta_a), \\
 p_b^\mu &= p_b^0(1, \sin \theta_b \cos \phi_b, \sin \theta_b \sin \phi_b, \mathbf{0}_{d-4}, \cos \theta_b),
 \end{aligned} \tag{4.165}$$

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where $\pi \geq \theta_{a(b)}, \phi_{a(b)} \geq 0$, $s = (P_A + P_B)^2$, $P_A^0 = P_B^0 = \frac{\sqrt{s}}{2}$ and $P_A^z = \sqrt{(P_A^0)^2 - M^2} = -P_B^z$. In the parametrization of $p_{a(b)}^\mu$ one could consider non-zero components in the $d-4$ dimensional subspace. However those contributions go to zero in the limit $d \rightarrow 4$. The integration over the $d-4$ dimensional solid angle amounts to a factor 2. Indeed one could already consider $p_{a(b)}^\mu$, $P_{A(B)}^\mu$ and $n^\mu(\bar{n}^\mu)$ in $d=4$, since the integration on $p_{a(b)}^\mu$ in eq. (4.161) is done after we have subtracted the soft and collinear poles and therefore it is safe to take the $\epsilon \rightarrow 0$ limit. The only integration measure that we has to be explicitly considered in $d=4-2\epsilon$ dimension is over the gluon momentum, i.e. $d^d k \delta(k^2)$. Applying a rotation $R^{\hat{z}}(\phi_a)$ around the $\hat{z}$ axis

$$R^{\hat{z}}(\phi_a) = \begin{pmatrix} \cos \phi_a & \sin \phi_a & \mathbf{0}_{1,d-3} \\ -\sin \phi_a & \cos \phi_a & \mathbf{0}_{1,d-3} \\ \mathbf{0}_{d-3,1} & \mathbf{0}_{d-3,1} & \mathbb{1}_{d-3,d-3} \end{pmatrix}, \quad (4.166)$$

where $\mathbb{1}_{d-3,d-3}$, $\mathbf{0}_{1,d-3}$ and $\mathbf{0}_{d-3,1}$ denote the identity matrix, zero row and column vector respectively in a subspace of dimension $d-3$. We can see that $R^{\hat{z}}(\phi_a)$ does not change $P_{A(B)}^\mu$ and $n^\mu(\bar{n}^\mu)$, $p_{a(b)}^\mu$ amount to

$$p_a'^\mu = p_a^0(1, \sin \theta_a, \mathbf{0}_{d-3}, \cos \theta_a), \quad p_b'^\mu = p_b^0(1, \sin \theta_b \cos(\phi_b - \phi_a), \sin \theta_b \sin(\phi_b - \phi_a), \mathbf{0}_{d-4}, \cos \theta_b). \quad (4.167)$$

From $p_b'^\mu$ we can defined new angles

$$\cos \theta'_b = \cos \theta_b, \quad \sin \theta'_b \cos \phi'_b = \sin \theta_b \cos(\phi_b - \phi_a) \quad (4.168)$$

and the integration measure changes as

$$d \cos \phi_b d \cos \theta_b \sin^{-1} \phi_b = d \cos \phi'_b d \cos \theta'_b \sin^{-1} \phi_b |J|^{-1} = d \cos \phi'_b d \cos \theta'_b \sin^{-1} \phi'_b, \quad (4.169)$$

where $|J|$ is the Jacobian given by

$$\left| \frac{d \cos \theta'_b d \cos \phi'_b}{d \cos \theta_b d \cos \phi_b} \right| = |\cos \phi_a - \sin \phi_a \cot \phi_b|. \quad (4.170)$$

Therefore $p_b'^\mu$ can be written as a function of ϕ_b and θ_b or in terms of θ'_b and ϕ'_b as

$$p_b'^\mu = p_b^0(1, \sin \theta'_b \cos \phi'_b, \sin \theta'_b \sin \phi'_b, \mathbf{0}_{d-4}, \cos \theta'_b). \quad (4.171)$$

After applying the rotation $R^{\hat{z}}(\phi_a)$, we perform another rotation $R^{\hat{y}}(\theta_a)$ around $\hat{y}$ -axis

$$R^{\hat{y}}(\theta_a) = \begin{pmatrix} \cos \theta'_a & \mathbf{0}_{1,d-3} & -\sin \theta'_a \\ \mathbf{0}_{d-3,1} & \mathbb{1}_{d-3,d-3} & \mathbf{0}_{d-3,1} \\ \sin \theta'_a & \mathbf{0}_{1,d-3} & \cos \theta'_a \end{pmatrix}, \quad (4.172)$$

such that the new rotated $p_a'^\mu$ has no $\hat{x}$ component. Applying $R^{\hat{y}}(\theta_a)$ to $P_{A(B)}^\mu$, $n^\mu(\bar{n}^\mu)$ and

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$p'_{a(b)}$ one obtains

$$\begin{aligned}
P_A^\mu &= (P_A^0, -|\mathbf{P}_A| \sin \theta_a, \mathbf{0}_{d-3}, |\mathbf{P}_A| \cos \theta_a), \quad P_B^\mu = (P_B^0, |\mathbf{P}_B| \sin \theta_a, \mathbf{0}_{d-3}, -|\mathbf{P}_B| \cos \theta_a), \\
n &= \frac{1}{\sqrt{2}}(1, -\sin \theta_a, \mathbf{0}_{d-3}, \cos \theta_a), \quad \bar{n}^\mu = \frac{1}{\sqrt{2}}(1, \sin \theta_a, \mathbf{0}_{d-3}, -\cos \theta_a), \\
p_a^\mu &= p_a^0(1, \mathbf{0}_{d-2}, 1), \\
p_b^\mu &= p_b^0(1, \cos \theta_a \cos \phi_b \sin \theta_b - \cos \theta_b \sin \theta_a, \sin \theta_b \sin \phi_b, \mathbf{0}_{d-4}, \cos \phi_b \sin \theta_a \sin \theta_b + \cos \theta_a \cos \theta_b),
\end{aligned} \tag{4.173}$$

where we have relabelled the variables without prime in order to lighten notation, and we will repeat this in several steps as long as there is no risk of confusion. Defining the angles

$$\begin{aligned}
\cos \theta'_b &= \cos \phi_b \sin \theta_a \sin \theta_b + \cos \theta_a \cos \theta_b, \\
\cos \phi'_b &= \frac{\cos \theta_a \cos \phi_b \sin \theta_b - \cos \theta_b \sin \theta_a}{\sqrt{1 - \cos^2 \theta'_b}},
\end{aligned} \tag{4.174}$$

where $\pi \geq \theta'_b, \phi'_b \geq 0$, one can perform a change of variables and write p_b^μ as a function of θ'_b and ϕ'_b . The integration measure changes as

$$d \cos \phi_b d \cos \theta_b \sin^{-1} \phi_b = d \cos \phi'_b d \cos \theta'_b |J^{\hat{y}}|^{-1} \sin^{-1} \phi_b = d \cos \phi'_b d \cos \theta'_b \sin^{-1} \phi'_b. \tag{4.175}$$

where we have used

$$\sin \phi_b \left| \frac{d \cos \theta'_b d \cos \phi'_b}{d \cos \theta_b d \cos \phi_b} \right| = \sqrt{1 - \cos^2 \theta'_b} = \sin \phi'_b, \tag{4.176}$$

The next step is to perform a second rotation around the $\hat{z}$ -axis

$$R^{\hat{z}}(\phi'_b) = \begin{pmatrix} \cos \phi'_b & \sin \phi'_b & 0_{1,d-3} \\ -\sin \phi'_b & \cos \phi'_b & 0_{1,d-3} \\ 0_{d-3,1} & 0_{d-3,1} & \mathbb{1}_{d-3,d-3} \end{pmatrix}, \tag{4.177}$$

such that the the rotated p_b^μ has zero $\hat{y}$ -component. The rotated vectors $P_{A(B)}^\mu$, $n^\mu(\bar{n}^\mu)$ and $p'_{a(b)}$ are given by

$$\begin{aligned}
P_A^\mu &= (P_A^0, -|\mathbf{P}_A| \sin \theta_a \cos \phi'_b, |\mathbf{P}_A| \sin \theta_a \sin \phi'_b, \mathbf{0}_{d-4}, |\mathbf{P}_A| \cos \theta_a), \\
P_B^\mu &= (P_B^0, |\mathbf{P}_B| \sin \theta_a \cos \phi'_b, -|\mathbf{P}_B| \sin \theta_a \sin \phi'_b, \mathbf{0}_{d-4}, -|\mathbf{P}_B| \cos \theta_a), \\
n^\mu &= \frac{1}{\sqrt{2}}(1, -\sin \theta_a \cos \phi'_b, \sin \theta_a \sin \phi'_b, \mathbf{0}_{d-4}, \cos \theta_a), \\
\bar{n}^\mu &= \frac{1}{\sqrt{2}}(1, \sin \theta_a \cos \phi'_b, -\sin \theta_a \sin \phi'_b, \mathbf{0}_{d-4}, -\cos \theta_a), \\
p_a^\mu &= p_a^0(1, \mathbf{0}_{d-2}, 1), \quad p_b^\mu = p_b^0(1, \sin \theta'_b, \mathbf{0}_{d-3}, \cos \theta'_b).
\end{aligned} \tag{4.178}$$

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The last rotation we want perform is around the $\hat{y}$ axis again, but in this case we rotate by an angle θ such that the angles between the rotated $\mathbf{p}_{a(b)}$ and the $\hat{z}$ -axis is $\theta(\pi - \theta)$ (see fig. 4.9)

$$R^{\hat{y}}(\theta) = \begin{pmatrix} \cos \theta & \mathbf{0}_{1,d-3} & -\sin \theta \\ \mathbf{0}_{d-3,1} & \mathbb{1}_{d-3,d-3} & \mathbf{0}_{d-3,1} \\ \sin \theta & \mathbf{0}_{1,d-3} & \cos \theta \end{pmatrix}, \quad (4.179)$$

where $2\theta + \theta'_b = \pi$; since $\pi \geq \theta'_b \geq 0$, then $\pi/2 \geq \theta \geq 0$. The rotated vectors $P_{A(B)}^\mu$, $n^\mu(\bar{n}^\mu)$ and $p_{a(b)}^\mu$ are given by

$$\begin{aligned} P_A^\mu &= \left(P_A^0, |\mathbf{P}_A| \left(\cos \theta_a \cos \frac{\theta_b}{2} - \cos \phi_b \sin \theta_a \sin \frac{\theta_b}{2} \right), |\mathbf{P}_A| \sin \theta_a \sin \phi_b, \right. \\ &\quad \left. |\mathbf{P}_A| \left(\cos \theta_a \sin \frac{\theta_b}{2} + \cos \frac{\theta_b}{2} \cos \phi_b \sin \theta_a \right) \right), \\ P_B^\mu &= \left(P_B^0, -|\mathbf{P}_B| \left(\cos \theta_a \cos \frac{\theta_b}{2} - \cos \phi_b \sin \theta_a \sin \frac{\theta_b}{2} \right), -|\mathbf{P}_B| \sin \theta_a \sin \phi_b, \right. \\ &\quad \left. -|\mathbf{P}_B| \left(\cos \theta_a \sin \frac{\theta_b}{2} + \cos \frac{\theta_b}{2} \cos \phi_b \sin \theta_a \right) \right), \\ n^\mu &= \frac{1}{\sqrt{2}} \left(1, \left(\cos \theta_a \cos \frac{\theta_b}{2} - \cos \phi_b \sin \theta_a \sin \frac{\theta_b}{2} \right), \sin \theta_a \sin \phi_b, \right. \\ &\quad \left. \left(\cos \theta_a \sin \frac{\theta_b}{2} + \cos \frac{\theta_b}{2} \cos \phi_b \sin \theta_a \right) \right), \\ \bar{n}^\mu &= \frac{1}{\sqrt{2}} \left(1, -\left(\cos \theta_a \cos \frac{\theta_b}{2} - \cos \phi_b \sin \theta_a \sin \frac{\theta_b}{2} \right), -\sin \theta_a \sin \phi_b, \right. \\ &\quad \left. -\left(\cos \theta_a \sin \frac{\theta_b}{2} + \cos \frac{\theta_b}{2} \cos \phi_b \sin \theta_a \right) \right), \\ p_a^\mu &= p_a^0 \left(\cos \frac{\theta_b}{2}, \mathbf{0}_{d-3}, \sin \frac{\theta_b}{2} \right), \quad p_b^\mu = p_b^0 \left(\cos \frac{\theta_b}{2}, \mathbf{0}_{d-3}, -\sin \frac{\theta_b}{2} \right), \end{aligned} \quad (4.180)$$

where we have again relabelled the angles to remove the primed notation. We see that $P_{A(B)}^\mu$ and $n^\mu(\bar{n}^\mu)$ become rather cumbersome functions of the angles, nevertheless this only affects the measurement functions which are evaluated numerically and for the subtraction of the soft and collinear poles are irrelevant. The important point is that the expressions in eq. (4.180) are suitable to identify and isolate IR divergences (see App. E.2). Before discussing the issue of the modes double counting, we perform a last change of variables to bring $P_{A(B)}^\mu$ and $n^\mu(\bar{n}^\mu)$ in a slightly more simplified form. By defining the angles $\pi \geq \theta'_a \geq 0$ and $2\pi > \phi'_b \geq 0$ implicitly by the equations

$$\begin{aligned} \cos \theta'_a &= \cos \theta_a \sin \frac{\theta_b}{2} + \cos \frac{\theta_b}{2} \cos \phi_b \sin \theta_a, \\ \cos \phi'_b &= \frac{\cos \theta_a \cos \frac{\theta_b}{2} - \cos \phi_b \sin \theta_a \sin \frac{\theta_b}{2}}{\sqrt{1 - \cos^2 \theta'_a}} \end{aligned} \quad (4.181)$$

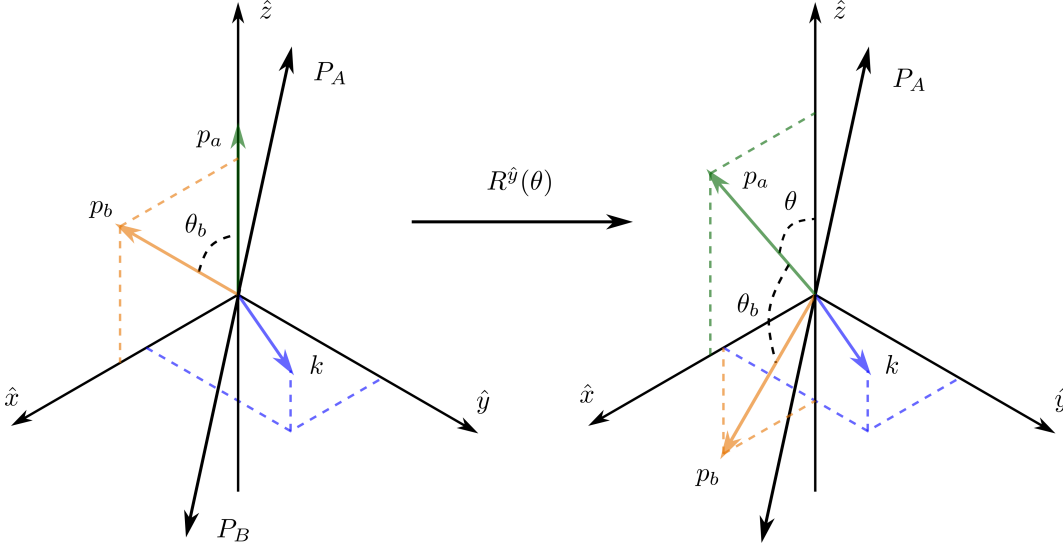


Figure 4.9: Change of reference frame after applying the rotation matrix $R^{\hat{y}}(\theta)$, with $2\theta + \theta_b = \pi$

the new integration measure is given by

$$d \cos \phi_b d \cos \theta_a \sin^{-1} \phi_b = d \cos \phi'_b d \cos \theta'_a \sin^{-1} \phi_b \left| \frac{d \cos \theta'_a d \cos \phi'_b}{d \cos \theta_a d \cos \phi_b} \right|^{-1} = d \cos \phi'_b d \cos \theta'_a \sin^{-1} \phi'_b. \quad (4.182)$$

$P_{A(B)}^\mu$ and $n^\mu(\bar{n}^\mu)$ are given by

$$\begin{aligned} P_A^\mu &= (P_A^0, |\mathbf{P}_A| \sin \theta'_a \cos \phi'_b, |\mathbf{P}_A| \sin \theta'_a \sin \phi'_b, \mathbf{0}_{d-4}, |\mathbf{P}_A| \cos \theta'_a), \\ P_B^\mu &= (P_B^0, -|\mathbf{P}_B| \sin \theta'_a \cos \phi'_b, -|\mathbf{P}_B| \sin \theta'_a \sin \phi'_b, \mathbf{0}_{d-4}, -|\mathbf{P}_B| \cos \theta'_a), \\ n^\mu &= \frac{1}{\sqrt{2}}(1, \sin \theta'_a \cos \phi'_b, \sin \theta'_a \sin \phi'_b, \mathbf{0}_{d-4}, \cos \theta'_a), \\ \bar{n}^\mu &= \frac{1}{\sqrt{2}}(1, -\sin \theta'_a \cos \phi'_b, -\sin \theta'_a \sin \phi'_b, \mathbf{0}_{d-4}, -\cos \theta'_a). \end{aligned} \quad (4.183)$$

Notice that there is no dependence on the azimuthal angle ϕ_a . The last step is to perform a boost such that p_a^μ and p_b^μ become back-to-back, which is explained in detail in App. E.2.

Double counting subtraction The other issue regarding real emission diagrams is the double counting. The only constraint on the real gluon momentum is coming from the measurement functions and momentum conservation, in any other regard we are fully inclusive. This means that we are also integrating over regions of phase space where the real gluon is either collinear to the partons entering the effective vertex or soft. However, these two regions of the phase space do not follow the mode power counting of hard (or fast) fields and are covered instead by the background fields. Nevertheless, as we did for the virtual corrections, we do not introduce any cut-off over momentum integration. In the virtual case this issue was solved by the renormalization of TMDPDF matrix elements. In the real case we can reabsorb this

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contributions into TMDPDFs. In order to do that, we follow a diagrammatic approach similar to the one presented in [251, 252]. We introduce some perturbative coefficient functions $C_{a(b)}$ and S_{ab} that account for the contributions in phase space regions where the gluon becomes collinear to the parton $a(b)$ or soft perspective, and subtract them from eq. (4.161). We define the subtraction term for radiation that is collinear to $a(b)$ as the partonic matrix element of operators similar to the unpolarized TMDPDF at leading power (eqs. (4.39) and (4.44) for the quark and anti-quark case respectively). The main modification is that while fields in the unpolarized TMDPDF operator are evaluated at $z^{-(+)} = 0$ in the (anti-)collinear case, in the double counting subtraction term the fields are evaluated at $n_{a(b)} \cdot z = 0$ ¹⁵ where $n_{a(b)}^\mu$ is a light-like vector parallel to $p_{a(b)}$. For the case of radiation collinear to the parton a it is given by

$$\begin{aligned} \hat{f}_{Zb,a}(z_a^+, 0_b^-, z_{ab\perp}; p_a) &= \frac{1}{2n_b \cdot p_a} \sum_X \langle p_a | \bar{T} \{ \bar{\psi}_j^c(z_a^+ n_b + z_{ab\perp}) [z_a^+ n_b + z_{ab\perp}, -\infty n_b + z_{ab\perp}]^{ca} \} | X \rangle \\ &\times \frac{(\not{n}_b)_{ji}}{2} \langle X | T \{ [-\infty n_b, 0]^{ab} \psi_i^b(0) \} | p_a \rangle, \end{aligned} \quad (4.184)$$

when the parton a is a quark, while for an anti-quark it is given by

$$\begin{aligned} \hat{f}_{Zb,a}(z_a^+, 0_b^-, z_{ab\perp}; p_a) &= \frac{1}{2n_b \cdot p_a} \sum_X \langle p_a | \bar{T} \{ [-\infty n_b + z_{ab\perp}, z_a^+ n_b + z_{ab\perp}]^{ac} \psi_i^c(z_a^+ n_b + z_{ab\perp}) \} | X \rangle \\ &\times \frac{(\not{n}_b)_{ji}}{2} \langle X | T \{ \bar{\psi}_j^b(0) [0, -\infty n_b]^{ba} \} | p_a \rangle, \end{aligned} \quad (4.185)$$

where i, j and a, b, c are spin and color indices respectively. The Wilson lines are in the fundamental representation. The factor $(2n_b \cdot p_a)^{-1}$ above is a normalization factor, such that at LO $(\alpha_s^0) \hat{f}_{Zb,a}^{[0]}(p_a, \mu^2) = 1$. See App. E.2.6 for the case of radiation collinear to b . The Wilson lines are written in the chosen coordinates system.

The other phase space region that is double counted is the soft one. Here we use a slightly modified version of the definition of the soft function at LP [43, 50, 106, 123, 183, 184]. Since the field coordinates in the collinear case are expressed in the $n_{a(b)}^\mu$ basis, for the soft radiation we do the same. Therefore we define the soft double counting subtraction term as

$$\begin{aligned} \hat{\phi}_{Zb,ab}(0_a^+, 0_b^-, z_{ab\perp}) &= \sum_X \langle 0 | \bar{T} \{ [-\infty n_a + z_{ab\perp}, z_{ab\perp}]^{ac} [z_{ab\perp}, -\infty n_b + z_{ab\perp}]^{cb} \} | X \rangle \\ &\langle X | T \{ [-\infty n_b, 0]^{bd} [0, -\infty n_a]^{da} \} | 0 \rangle, \end{aligned} \quad (4.186)$$

where a, b, c, d are color indices; the Wilson lines are in the fundamental representation.

Having defined the soft and collinear subtraction term, we can now introduce the double counting subtracting hadronic tensor, or more precisely its differential form with the corre-

¹⁵See App. E.2 for a explicit definition of $n_{a(b)}^\mu$.

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sponding measurement function. It is given by

$$\begin{aligned}
W_{Zb,GG'}^{\mu\nu}(q, Q^2, y, q_T^2) &= \frac{P_A^- P_B^+}{N_c} \sum_{a,b} \int \frac{d^d z}{(2\pi)^d} e^{-iz \cdot q} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
&\times \text{Tr}[\not{p}_a \Gamma_{G'}^\nu \not{p}_b \Gamma_G^\mu] f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
&\times \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 + q^2 - 2(\bar{n} \cdot q)(n \cdot q)) \\
&\times \frac{\hat{f}_{Zb,a}(z_a^+, 0_b^-, z_{ab\perp}; p_a) \hat{f}_{Zb,b}(0_a^+, z_b^-, z_{ab\perp}; p_b)}{\hat{\phi}_{Zb,ab}(0_a^+, 0_b^-, z_{ab\perp})} \\
&+ n \leftrightarrow \bar{n} + \dots,
\end{aligned} \tag{4.187}$$

where $v_a^+ = v \cdot n_a$, $v_b^- = v \cdot n_b$ and $v_{ab\perp}^\mu = v_\nu g_{ab\perp}^{\mu\nu}$

$$g_{ab\perp}^{\mu\nu} = g^{\mu\nu} - \frac{n_a^\mu n_b^\nu + n_a^\nu n_b^\mu}{n_a \cdot n_b}.$$

The presence of the soft contribution in the denominator can be understood from fig. 4.1, $\hat{f}_{Zb,a}$ and $\hat{f}_{Zb,b}$ overlap in the soft region leading to subtracting it twice. The soft factor in the denominator removes this overlap at leading geometrical twist. Explicit expressions of eq. (4.187) can be found in App. E.2.6.

The cumulant counterpart of the hadronic tensor with the double counting subtracted is given by

$$\tilde{W}_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2) = W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2) - W_{Zb,GG'}^{\mu\nu,[1]}(q)\Theta(Q^2, y, q_T^2). \tag{4.188}$$

One can find explicit expressions for the $\tilde{W}_{\gamma^* \gamma^*}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2)$ case in Sec. E.2.7. To the best of our knowledge it is the first time that hard real radiation effects are combined with TMDPDF factorization.

Extension to two real emission processes In order to stress the validity of the computational method presented at NLO for the real emission in momentum spaces and the subtraction of the double contribution we have computed some of the NNLO contributions. In particular, we have performed the evaluation of all contributions to the flavor singlet channel $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$ as well as some of the contributions to $q(\bar{q}) + \bar{q}(q) \rightarrow F + q(\bar{q}) + \bar{q}(q)$, where F is a color singlet state. The details of the computation are presented in App. F. We obtain that the methods presented on Sec. 4.3.2.2 and App. E.2 also work at NNLO, showing to be a suitable form to isolate the poles and cancels out the collinear/soft singularities together with removing spurious double counting of the soft and collinear regions.

4.4 Numerical results at N3LL+NLO

The cumulative Drell-Yan cross section is given by eq. (4.104)

$$\sigma_{\text{DY}}(\{Q_{\text{min}}^2, Q_{\text{max}}^2, y_{\text{min}}, y_{\text{max}}, q_{T\text{min}}, q_{T\text{max}}\}) = \int d^4q \Theta(Q^2, y, q_T^2) \sum_{G, G'} L_{\mu\nu}^{GG'}(q, \theta(\text{cuts})) W_{GG'}^{\mu\nu}(q) \Delta_G(q^2) \Delta_{G'}^*(q^2). \quad (4.189)$$

We showed in Sec. 4.3.1 that the virtual corrections to the hadronic tensor at leading geometrical twist, including contributions up to collinear twist-4 and accounting for the whole series in KPC power corrections amounts to an expression similar to the customary quark form factor at NLO [37, 43, 47, 253]. Thus, for the virtual corrections we use the already known quark form factor up to N3LO [253]. The real contributions at NLO to the hadronic tensor were computed in Sec. 4.3.2. We saw that there was a double counting of collinear and soft modes that needed to be subtracted. Before discussing the numerical results from eq. (4.189), we briefly comment on the unpolarized (anti-)quark TMDPDF distribution and leptonic tensor implementation.

TMDPDFs matching and evolution The unpolarized (anti-)quark TMDPDF distribution is matched onto unpolarized PDFs of geometrical twist-2. In impact parameter space it reads

$$f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta) = \int_{\xi}^1 \frac{dx}{x} \sum_r C_{a\leftarrow r}\left(x, \log \frac{\mu^2 \mathbf{b}_T^2}{4e^{-2\gamma_E}}; \mu, \zeta\right) f_{r\leftarrow p}\left(\frac{\xi}{x}, \mu\right), \quad (4.190)$$

where the matching coefficients $C_{a\leftarrow r}$ are known up to N3LO [129]; μ and ζ are the energy scales associated to the renormalization of UV and rapidity divergences, respectively. The dependence on μ and ζ is governed by the renormalization group equations [43, 49–51, 54, 114]

$$\mu^2 \frac{d}{d\mu^2} f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta) = \frac{1}{2} \gamma_f^a(\mu, \zeta) f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta), \quad (4.191)$$

$$\zeta \frac{d}{d\zeta} f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta) = -\mathcal{D}^a(\mu, \mathbf{b}_T) f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta), \quad (4.192)$$

where the TMDPDF and soft anomalous dimension are given by

$$\gamma_f^a(\mu, \zeta) = \Gamma_{\text{cusp}}^a(\alpha_s(\mu^2)) \log \frac{\mu^2}{\zeta} - \gamma_V^a(\alpha_s(\mu^2)), \quad \mathcal{D}^a(\mu, \mathbf{b}_T) = \sum_{n=1} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^n \sum_{m=0}^n \log \frac{\mu^2 \mathbf{b}_T^2}{4e^{-2\gamma_E}} d_a^{(n,m)}, \quad (4.193)$$

where a is a flavor index. Explicit expressions for Γ_{cusp}^a , γ_V^a and $d_a^{(n,m)}$ up to N3LO can be found in [50], expressions for the rapidity anomalous dimension up to N4LO have been computed in [124, 125]. The soft anomalous, $\mathcal{D}^a(\mu, \mathbf{b}_T)$ also received contributions from the non-perturbative regime, therefore on top of the perturbatively calculable coefficients $d_a^{(n,m)}$ one also needs to introduce a non-perturbative model $\mathcal{D}_{\text{NP}}^a(\mathbf{b}_T)$

$$\mathcal{D}^a(\mu, \mathbf{b}_T) \rightarrow \mathcal{D}^a(\mu, b^*) + \mathcal{D}_{\text{NP}}^a(b^*). \quad (4.194)$$

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The matching procedure in (4.190) also received non-perturbative corrections in the region $|\mathbf{b}_T|\Lambda_{\text{QCD}} \sim 1$, therefore one needs to include some non-perturbative model for $f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta)$

$$f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta) \rightarrow f_{a\leftarrow p}(\xi, \mathbf{b}_T; \mu, \zeta) f_{\text{NP}}^a(\mathbf{b}_T). \quad (4.195)$$

In this work we use the non-perturbative evolution kernel, $\mathcal{D}_{\text{NP}}^a$, and TMDPDF parametrization, f_{NP}^a , from [254]

$$\mathcal{D}_{\text{NP}}^a(b^*) = c_0 |\mathbf{b}_T| b^*(|\mathbf{b}_T|), \quad b^*(|\mathbf{b}_T|) = \frac{|\mathbf{b}_T|}{\sqrt{1 + \frac{\mathbf{b}_T^2}{B_{\text{NP}}^2}}} \quad (4.196)$$

$$f_{\text{NP}}^a(\mathbf{b}_T) = \exp \left(- \frac{(1-x)\lambda_1^a + x\lambda_2^a}{\sqrt{1 + \lambda_0 x^2 \mathbf{b}_T^2}} \mathbf{b}_T^2 \right). \quad (4.197)$$

We use the values of B_{NP} , c_0 , λ_0 and $\lambda_{1,2}^a$ extracted in [254] for the PDF parametrization NNPDF31_nnlo_as_0118 [255] are collected in Table. 4.1.

Parameter [254]	NNPDF31_nnlo_as_0118 [255]
λ_1^u	$0.28^{+0.12}_{-0.10}$
λ_2^u	$2.58^{+1.37}_{-2.05}$
λ_1^d	$0.4^{+0.10}_{-0.22}$
$\lambda_1^{\bar{d}}$	$0.4^{+0.10}_{-0.22}$
$\lambda_1^{\bar{u}} \times 100$	$8.8^{+10.1}_{-8.6}$
$\lambda_2^{\bar{u}}$	$13.^{+28.}_{-6.}$
$\lambda_1^{\bar{d}} \times 100$	$9.8^{+9.4}_{-9.5}$
$\lambda_2^{\bar{d}}$	$6.1^{+16.7}_{-1.6}$
$\lambda_1^{\bar{d}} \times 100$	$0.25^{+0.15}_{-0.12}$
$\lambda_2^{\bar{d}}$	$3.0^{+4.3}_{-3.0}$
λ_0	$181.^{+39.}_{-133.}$
$c_0^{\bar{d}}$	$2.61^{+0.97}_{-0.61}$
B_{NP}	2.

Table 4.1: Values for the TMDPDF non-perturbative parameters. B_{NP} is fixed during the extraction procedure.

Partonic integration measure The partonic tensor in eq. (4.189) contains the integration measure over the momentum p_a and p_b of massless partons entering the hard scattering pro-

cess. In the parton center-of-mass frame we write the integration measure $[d^4 p_a][d^4 p_b]$ as

$$[d^4 p_a][d^4 p_b] = \frac{d^3 \mathbf{p}_a}{2p_a^0} \frac{d^3 \mathbf{p}_b}{2p_b^0} = -\frac{d \cos \theta_a d\phi_b d\theta \sin(2\theta) d\phi dp_a^0 p_a^0 dp_b^0 p_b^0}{2}, \quad (4.198)$$

where $\theta_b = \pi - 2\theta$; ϕ_b and ϕ are azimuthal angles, $\pi \geq \phi, \phi_b \geq -\pi$; θ_a is a polar angle, $\pi \geq \theta_a \geq 0$ and $\pi/2 \geq \theta \geq 0$. Since the hadronic and leptonic tensor are independent of ϕ , it can be integrated trivially. The explicit form of $d^4 q$ is not relevant since we use the momentum conservation delta function in the hadronic tensor to integrate the off-shell vector boson momentum.

The partonic center-of-mass frame was suitable to identify and regularize the collinear and soft divergences. Nevertheless, it has a important drawback since the TMDPDF distribution $f_{a(b) \leftarrow p_{A(B)}}(\xi_{a(b)}, \mathbf{p}_{ab(T)})$ becomes a function of four variables one radial variable and three angles. This can be seen using the delta function $\delta(p_{a(b)}^{-(+)} - \xi_{a(b)} P_{A(B)}^{-(+)})$ in the hadronic tensor to integrate $p_{a(b)}^0$, one obtains

$$p_a^0 = \frac{(\sqrt{s} + \sqrt{s - 4M_A^2}) \xi_a}{2(1 + \cos \theta_a \cos \theta + \cos \phi_b \sin \theta \sin \theta_a)}, \quad (4.199)$$

$$p_b^0 = \frac{(\sqrt{s} + \sqrt{s - 4M_B^2}) \xi_b}{2(1 + \cos \theta_a \cos \theta - \cos \phi_b \sin \theta \sin \theta_a)}. \quad (4.200)$$

Therefore, the modulus square of the transverse components, $\mathbf{p}_{a(b)T}^2$, can be written as

$$\mathbf{p}_{aT}^2 = 2(p_a \cdot n)(p_a \cdot \bar{n}) = \frac{(\sqrt{s} + \sqrt{s - 4M_A^2})^2 \xi_a^2 (1 - \cos \theta_a \cos \theta - \cos \phi_b \sin \theta \sin \theta_a)}{4(1 + \cos \theta_a \cos \theta + \cos \phi_b \sin \theta \sin \theta_a)}, \quad (4.201)$$

$$\mathbf{p}_{bT}^2 = 2(p_b \cdot n)(p_b \cdot \bar{n}) = \frac{(\sqrt{s} + \sqrt{s - 4M_B^2})^2 \xi_b^2 (1 - \cos \theta_a \cos \theta + \cos \phi_b \sin \theta \sin \theta_a)}{4(1 + \cos \theta_a \cos \theta - \cos \phi_b \sin \theta \sin \theta_a)}, \quad (4.202)$$

where we can see that they are functions of one rail component and three angles. This means that during the numerical integration one evaluated them hundred of times what significantly slows down the whole evaluation of the cumulant version of the differential cross section in eq. (4.189). It is for this reason that we will perform some simplifications based on the asymptotic behavior of the TMDPDF distributions in order to speed up the phase-space numerical integration.

The measurement functions in $\Theta(Q^2, y, q_T^2)$ impose lower and upper bound on the different integration variables. Solve analytically all the contained imposed is rather cumbersome. Nevertheless, we can use $\theta(Q_{\max}^2 - q^2)\theta(q^2 - Q_{\min}^2)$, with $q^2 = (p_a + p_b - k)^2$, to obtain some preliminary upper and lower bounds on s_{ak} , s_{bk} and s_{ab}

$$Q_{\max}^2 \geq s_{ab} - s_{ak} - s_{bk} \text{ or } s_{ak} \geq s_{ab} - s_{bk} - Q_{\max}^2, \quad (4.203)$$

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$$s_{ab} - s_{ak} - s_{bk} \geq Q_{\min}^2 \text{ or } s_{ab} - Q_{\min}^2 - s_{bk} \geq s_{ak}, \quad (4.204)$$

since $s_{ak} \geq 0$ we can further impose $s_{ab} - Q_{\min}^2 - s_{bk} \geq 0$; the lower bound will be given by $\max\{0, s_{ab} - s_{bk} - Q_{\max}^2\}$. In order to simplify the phase-space integration we further impose $s_{ab} - Q_{\max}^2 - s_{bk} \geq 0$, which is a tighter bound. Thus, we obtain $s_{ab} - Q_{\max}^2 \geq s_{bk} \geq 0$ rather than $s_{ab} - Q_{\min}^2 \geq s_{bk} \geq 0$. From the $s_{ab} - Q_{\max}^2$ we also obtain $s_{ab} Q_{\max}^2$. Since $s_{ab} = 4p_a^0 p_b^0 \cos^2 \theta$ we obtain

$$\xi_a \geq \xi_a^{\min} = \frac{Q_{\max}^2 [(1 + \cos(\theta_a) \cos(\theta))^2 - \cos^2(\phi_b) \sin^2(\theta_a) \sin^2(\theta)]}{s \xi_b \cos^2(\theta) \left(1 + \sqrt{1 - \frac{4M^2}{s}}\right)^2}, \quad (4.205)$$

$$\xi_b \geq \xi_b^{\min} = \frac{Q_{\max}^2 [(1 + \cos(\theta_a) \cos(\theta))^2 - \cos^2(\phi_b) \sin^2(\theta_a) \sin^2(\theta)]}{s \cos^2(\theta) \left(1 + \sqrt{1 - \frac{4M^2}{s}}\right)^2}, \quad (4.206)$$

where the lower bound on ξ_b is order that the lower bound on ξ_a is smaller than one. Since $1 \geq \xi_{a(b)}$ we can impose further constraints on θ

$$s \cos^2(\theta) \left(1 + \sqrt{1 - \frac{4M^2}{s}}\right) \geq Q_{\max}^2 [(1 + \cos(\theta_a) \cos(\theta))^2 - \cos^2(\phi_b) \sin^2(\theta_a) \sin^2(\theta)] \quad (4.207)$$

solving for $\cos^2(\theta)$ one obtains

$$s \left(1 + \sqrt{1 - \frac{4M^2}{s}}\right) (\cos(\theta) - \cos(\theta)_+) (\cos(\theta) - \cos(\theta)_-) \geq 0, \quad (4.208)$$

where

$$\cos(\theta)_+ = \frac{Q_{\max}^2 \cos(\theta_a) + Q_{\max} \sqrt{\tilde{s} - \cos^2(\phi_b) \sin^2(\theta_a) (\tilde{s} + Q_{\max}^2 \sin^2(\theta_a) \sin^2(\phi_b))}}{\tilde{s} - Q_{\max}^2 (\cos^2 \phi_b - \cos^2(\theta_a) \sin^2(\phi_b))}, \quad (4.209)$$

$$\cos(\theta)_- = \frac{Q_{\max}^2 \cos(\theta_a) - Q_{\max} \sqrt{\tilde{s} - \cos^2(\phi_b) \sin^2(\theta_a) (\tilde{s} + Q_{\max}^2 \sin^2(\theta_a) \sin^2(\phi_b))}}{\tilde{s} - Q_{\max}^2 (\cos^2 \phi_b - \cos^2(\theta_a) \sin^2(\phi_b))}, \quad (4.210)$$

and we have defined

$$\tilde{s} = s \left(1 + \sqrt{1 - \frac{4M^2}{s}}\right)^2. \quad (4.211)$$

Due to $\tilde{s} \gg Q_{\max}^2$ it can be shown that $0 \geq \cos(\theta)_-$ and $\cos(\theta)_+ \geq 0$. Since $\cos(\theta) \geq 0$, the allowed region for $\cos^2(\theta)$ is $1 \geq \cos(\theta) \geq \cos^2(\theta)_+$. Once we have obtained bounds on s_{abk} , $\xi_{a(b)}$ and θ , it is convenient to perform the change of variables

$$\theta_a = \pi \theta'_a, \quad (4.212)$$

$$\theta = \arccos(\cos(\theta)_+) \theta' \quad (4.213)$$

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$$\xi_b = (1 - \xi_b^{\min})\xi'_b + \xi_b^{\min}, \quad (4.214)$$

$$\xi_a = (1 - \xi_a^{\min})\xi'_a + \xi_a^{\min}, \quad (4.215)$$

$$s_{bk} = (s_{ab} - Q_{\max}^2)s'_{bk}, \quad (4.216)$$

$$s_{ak} = (Q_{\max}^2 - Q_{\min}^2)s'_{ak} + s_{ab} - Q_{\max}^2 - s_{bk}, \quad (4.217)$$

where $1 \geq \theta'_a, \theta', \xi_{a(b)'} , s'_{a(b)k} \geq 0$. Notice that $s_{a(b)k}$ is dimensionless, what makes them a more suitable variable for numerical integration since it is pure number.

Leptonic tensor The leptonic tensor at LO in the electroweak interaction is given by

$$\begin{aligned} L_{GG'}^{\mu\nu}(q) = 2 \int \frac{d^3\mathbf{l}_1}{2l_1^0(2\pi)^3} \frac{d^3\mathbf{l}_2}{2l_2^0(2\pi)^3} & \left\{ (c_L^G c_L^{G'} + c_R^G c_R^{G'}) \left(l_1^\mu l_2^\nu + l_1^\nu l_2^\mu - \frac{Q^2}{2} g^{\mu\nu} \right) \right. \\ & \left. - i(c_L^G c_L^{G'} - c_R^G c_R^{G'}) \varepsilon^{\mu\nu\alpha\beta} l_{1\alpha} l_{2\beta} \right\} \delta^{(4)}(l_1 + l_2 - q). \end{aligned} \quad (4.218)$$

In phenomenological applications one could need to consider experimental cuts over the leptonic-pair final state. For instance, in the cases of CMS [256] and ATLAS [257, 258] one constraints the leptonic phase-space by introducing bounds on the transverse momentum and pseudo-rapidity of the leptons. On the other hand, for the cases of CDF [259, 260] and DØ [261, 262] any constraints are imposed over the final states leptons. Thus, in order to include experimental cuts on the lepton transverse momentum and pseudo-rapidity, we modify eq. (4.218) as

$$\begin{aligned} L_{GG'}^{\mu\nu}(q, \theta(\text{cuts})) = 2 \int \frac{d^3\mathbf{l}_1}{2l_1^0(2\pi)^3} \frac{d^3\mathbf{l}_2}{2l_2^0(2\pi)^3} & \left\{ (c_L^G c_L^{G'} + c_R^G c_R^{G'}) \left(l_1^\mu l_2^\nu + l_1^\nu l_2^\mu - \frac{Q^2}{2} g^{\mu\nu} \right) \right. \\ & \left. - i(c_L^G c_L^{G'} - c_R^G c_R^{G'}) \varepsilon^{\mu\nu\alpha\beta} l_{1\alpha} l_{2\beta} \right\} \delta^{(4)}(l_1 + l_2 - q) \\ & \times \theta(\eta_{1\max} - \eta_1) \theta(\eta_1 - \eta_{1\min}) \theta(\eta_{2\max} - \eta_2) \theta(\eta_2 - \eta_{2\min}) \\ & \times \theta(|\mathbf{l}_{1T}| - l_{1T\min}) \theta(|\mathbf{l}_{2T}| - l_{2T\min}). \end{aligned} \quad (4.219)$$

The integration over one of the leptonic momentum can be done trivially using the momentum conservation delta function $\delta^{(4)}(l_1 + l_2 - q)$. We choose to integrate l_2^μ . For the remaining leptonic momentum, l_1^μ , the situation is slightly more involved, for two reasons. Firstly the contraction between the hadronic and leptonic tensors generates terms like $k \cdot l_1$. Since k^μ is in partonic center-of-mass frame we must also consider l_1^μ in that frame. Secondly the dependence on k^μ introduced through the momentum conservation delta function in the hadronic tensor that we use to integrate over q^μ in the cumulative Drell-Yan cross section in eq. (4.189). While in the proton center-of-mass frame the cuts on leptons transverse momenta and pseudo-rapidities take a simple form, in the partonic centre-of-mass frame they turn into rather cumbersome expressions. Also, in the proton center-of-mass frame one can write the

CHAPTER 4. TMD FACTORIZATION

integration measure for l_1^μ in terms of the lepton transverse momentum, azimuthal angle and pseudo-rapidity i.e.

$$\frac{d^3\mathbf{l}_1}{2l_1^0} = \frac{d\phi_{l_1} d|\mathbf{l}_{1T}| |\mathbf{l}_{1T}| d\eta_1}{2},$$

which are the natural variables for the leptonic phase-space. However, in the partonic center-of-mass frame the integration measure is written in spherical coordinates. From the integration measure of the leptonic momentum l_2^μ we have the on-shell condition $\delta(l_2^2) = \delta((q - l_1)^2)$ that we can use to integrate the radial component of l_1^μ . The polar (θ_{l_1}) and azimuthal (ϕ_{l_1}) angles need to be integrated numerically.

Numerical implementation From the leptonic tensor one has to integration variables, ϕ_{l_1} and θ_{l_1} ; the real radiation contributes with another three integration variables, $s_{a(b)k}$ and θ_k ; the initial-states partons contribute with five integration variables, three angles and two radial components. In total there are ten integration variables, thus Monte-Carlo integration techniques is the most suitable approach. One of the aims of the present work is to study the effects of $|\mathbf{p}_{a(b)T}|/q_T$ power corrections on the intermediate- q_T region. These contributions will become more significant as the ratio increases. At the energy scale $\mu = M_Z$ the bulk of the contribution from the unpolarized (anti-)quark TMDPDF distribution comes from the region $0 \text{ GeV} \leq |\mathbf{p}_T| \leq 6 \text{ GeV}$ [49, 51, 54, 263, 264]. Therefore it is interesting to determine the phase-space region for which $|\mathbf{p}_{a(b)T}|$ takes small to intermediate values, for instance $|\mathbf{p}_{a(b)T}| \leq 20 \text{ GeV}$. In figs. 4.10, 4.11, 4.12, 4.13, 4.14 and 4.15 we present the regions in θ'_a, θ'_b for different values of $\xi'_{a(b)}$ and ϕ_b for which $|\mathbf{p}_{a(b)T}| \leq 20 \text{ GeV}$. One can see that $|\mathbf{p}_{a(b)T}| \leq 20 \text{ GeV}$ imposes for $\theta' \leq 0.03$ and $\theta'_a \leq 0.015$. The dependence on ϕ_b changes the shape of the boundary but does not drastically modify the allowed region but in two cases: $\xi'_a = \xi'_b = 0$, $\phi_b = 0$ and $\xi_a = 0$, $\phi_b = -\pi$. In both cases the $|\mathbf{p}_{a(b)T}| \leq 20 \text{ GeV}$ region is contained in a extremely narrow stripe, see figs. 4.16, 4.17 and 4.18. The distance between the upper and lower boundary in θ' is of order 10^{-2} , therefore in this regard is not suppressed compared to the other cases. However, these values are extremely sensitive to changes of ϕ_b . As an example we present how the $|\mathbf{p}_{a(b)T}| \leq 20 \text{ GeV}$ region for $\xi'_a = 0$ and $\xi'_b = 0.6$ changes as we increase ϕ_b , see figs. 4.19 and 4.20. The width of the region in the ϕ_b variable is of order 10^{-2} , which makes it suppressed compared to other values far from $\phi_b = -\pi$. For $\xi'_a = \xi'_b = 0$, $\phi_b = 0$ the analysis is similar. Thus, we can conclude that the main contribution in the phase space integration comes $0.1 \geq \theta'$ and $0.05 > \theta'_a$. For this reason we decide to reduce the integration range for θ' and θ'_a from $\theta' \in [0, 0.1]$ and $\theta'_a \in [0, 0.05]$.

Results We compare the predictions for the q_T -spectrum obtained from eq. (4.189) at small- and intermediate- q_T with the experimental data from [256] and the predictions obtained from TMDPDF factorization at leading power [43, 47, 54, 107, 254, 263]. The numerical evaluation of the TMDPDF is done with the program `artemide` [265] which is publicly available at [266]. The PDF numerical values together with their evolution are taken from [267], we use the PDF set `NNPDF31_nnlo_as_0118` [255].

In fig. 4.21 we present the Drell-Yan cross section differential in q_T computed with eq. (4.189) and the LP TMDPDF factorization, and we superimpose the experimental results. We can

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see that in the limit $q_T \rightarrow 0$ the contribution from the power correction goes to zero and the differential cross section without hard emission, $d\sigma^{\text{LP}}$, and with one hard emission, $d\sigma^{\text{PC}}$. For intermediate values of q_T one obtains $d\sigma^{\text{PC}} > d\sigma^{\text{LP}}$, i.e. the distribution tail from the LP factorization fall off faster than with power corrections. This increase of the cross section at the tails helps improve the agreement between the theoretical prediction and the experimental data. For $q_T \sim 40$ GeV the LP factorization predicts a cross section around 8% smaller than the experimental results, with the inclusion of power corrections we can reduce the disagreement to less than 2%. Overall, the inclusion of the QCD power corrections have an impact of 4% to 5%, bigger than for example the EW power corrections that are around 2% [268, 269]. In figs. 4.22 and 4.23 we present similar plots but for the rapidity ranges $0.8 \leq |y| \leq 1.2$ and $1.2 \leq |y| \leq 1.6$. We can see that as one increases the vector boson rapidity range the effect of power corrections becomes smaller. The reason for this is that the allowed radiation that balances the vector boson momentum becomes more forward (as the gauge vector boson does) with respect the proton beam axis. Therefore, the real radiation at the level of the hard interaction vertex is collinear to the parton entering the effective electroweak vertex. It means that most of its contribution is cancelled by the double counting subtracting term.

Despite the several simplifications we have performed in order to speed up the numerical evaluation of eq. (4.189) we obtain a sizeable contribution from the three power corrections we have included: sub-leading collinear twist, Kinematic Power corrections (KPC) and hard real emission. Thus a more detailed and precise numerical evaluation of eq. (4.189) would be of great interest. We leave this open for future projects. We would also like to emphasize that the transition from the small to intermediate q_T region is purely dictated and controlled by the measurement functions and any kind of profile functions nor weighted sum of fixed order and resummed cross section is used. This is one of the main differences with previous works which combine small and intermediate-large q_T regions to obtain a description of the whole spectrum [96, 102, 146, 147, 243, 247, 250, 270–273].

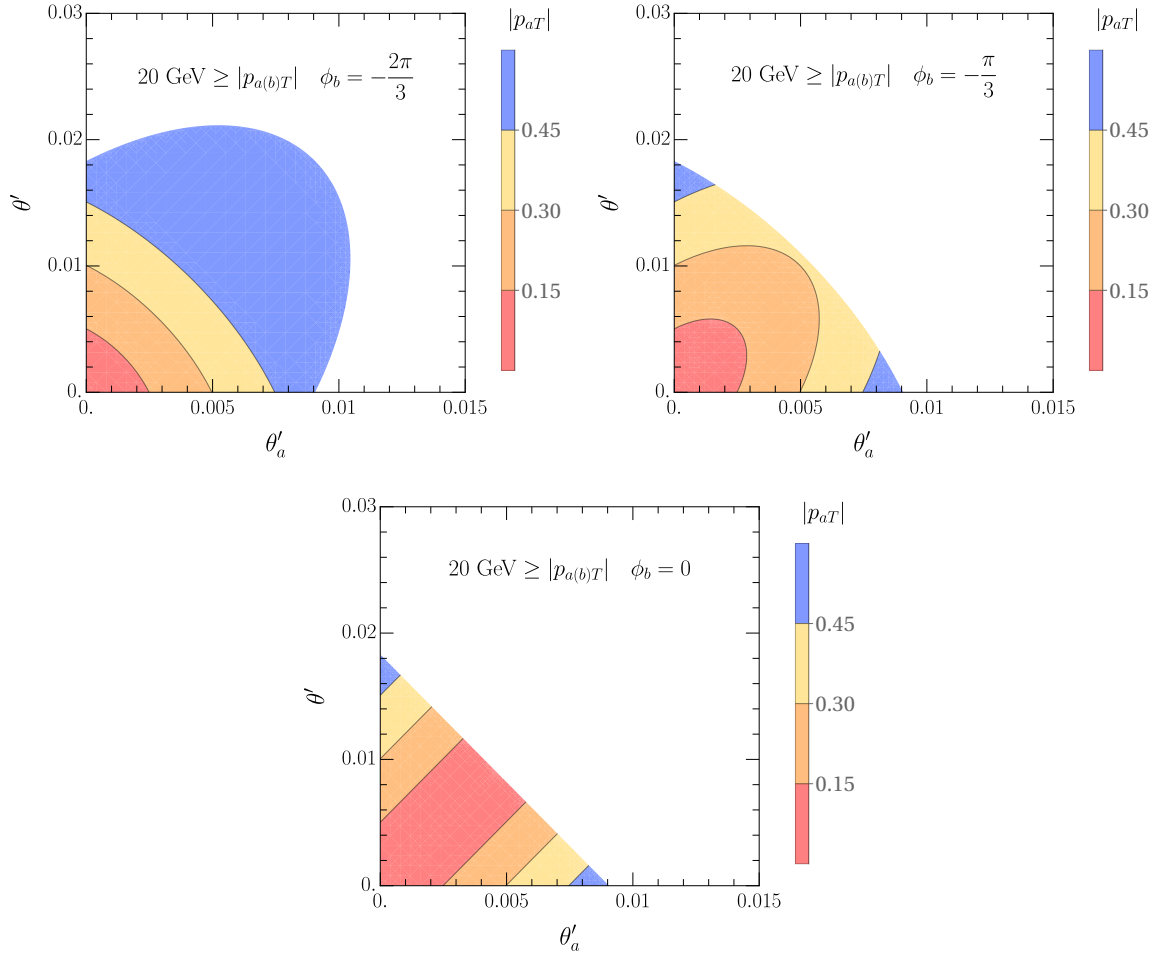


Figure 4.10: Contour plots of $|\mathbf{p}_{aT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0$, $\xi'_b = 0.2$ and $\phi_b = -\pi \times \{\frac{2}{3}, \frac{1}{3}, 0\}$.

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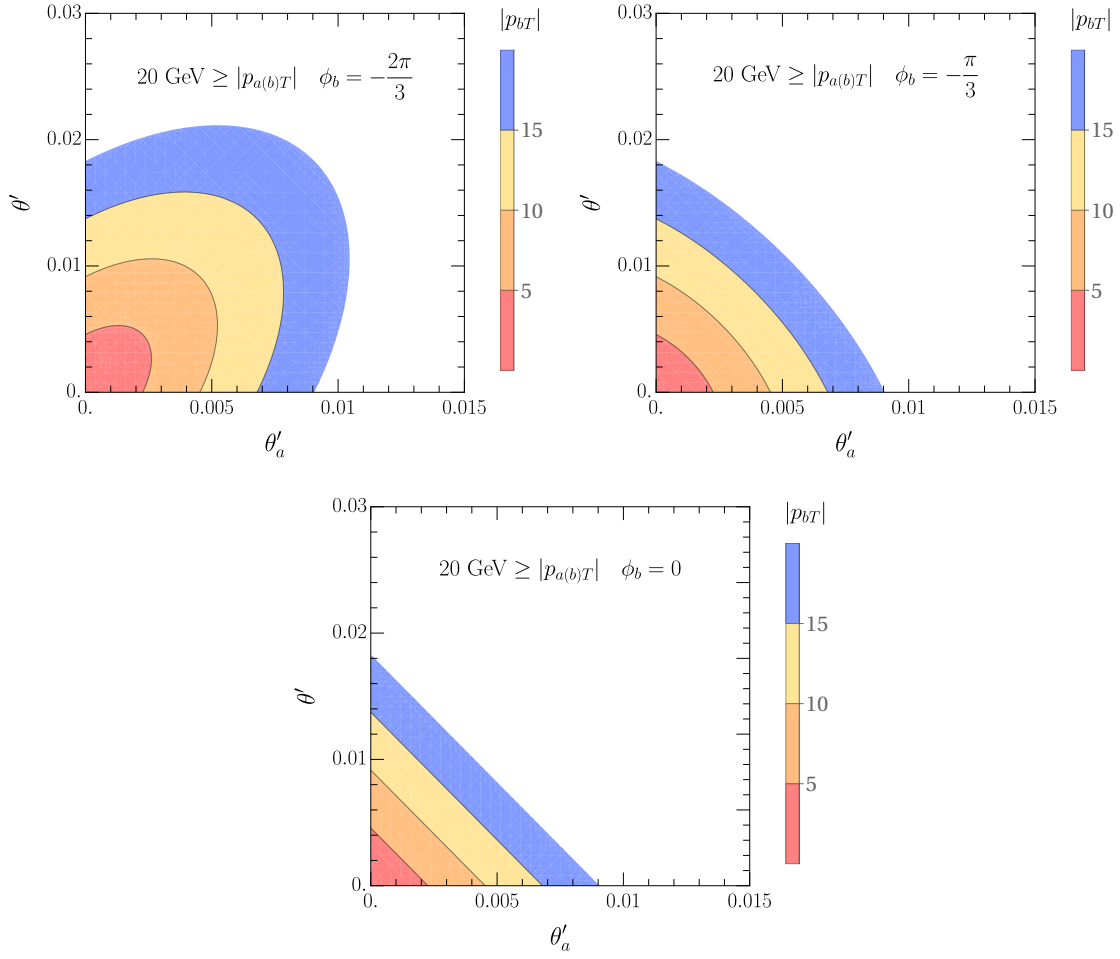


Figure 4.11: Contour plots of $|\mathbf{p}_{bT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0$, $\xi'_b = 0.2$ and $\phi_b = -\pi \times \{\frac{2}{3}, -\frac{1}{3}, 0\}$.

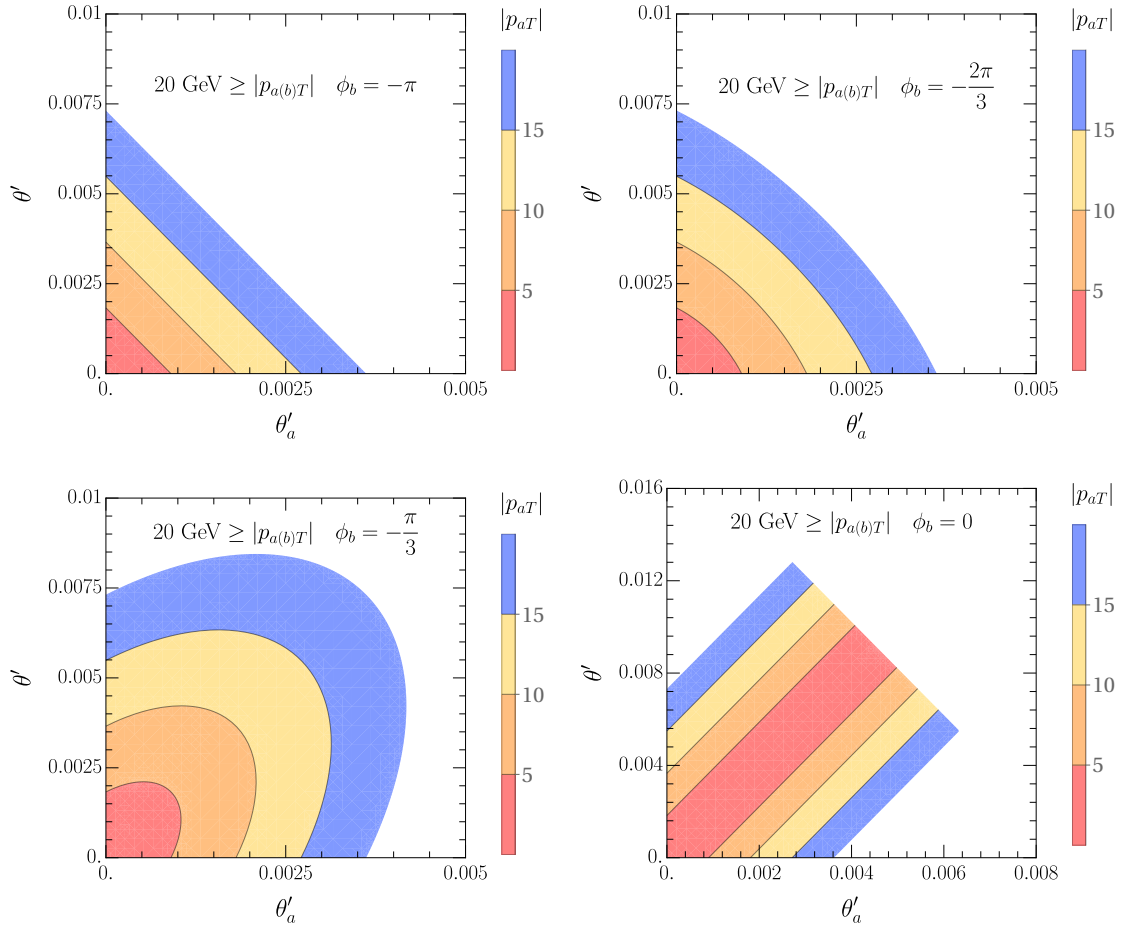


Figure 4.12: Contour plots of $|\mathbf{p}_{aT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0.5$, $\xi'_b = 0.2$ and $\phi_b = -\pi \times \{1, \frac{2}{3}, \frac{1}{3}, 0\}$.

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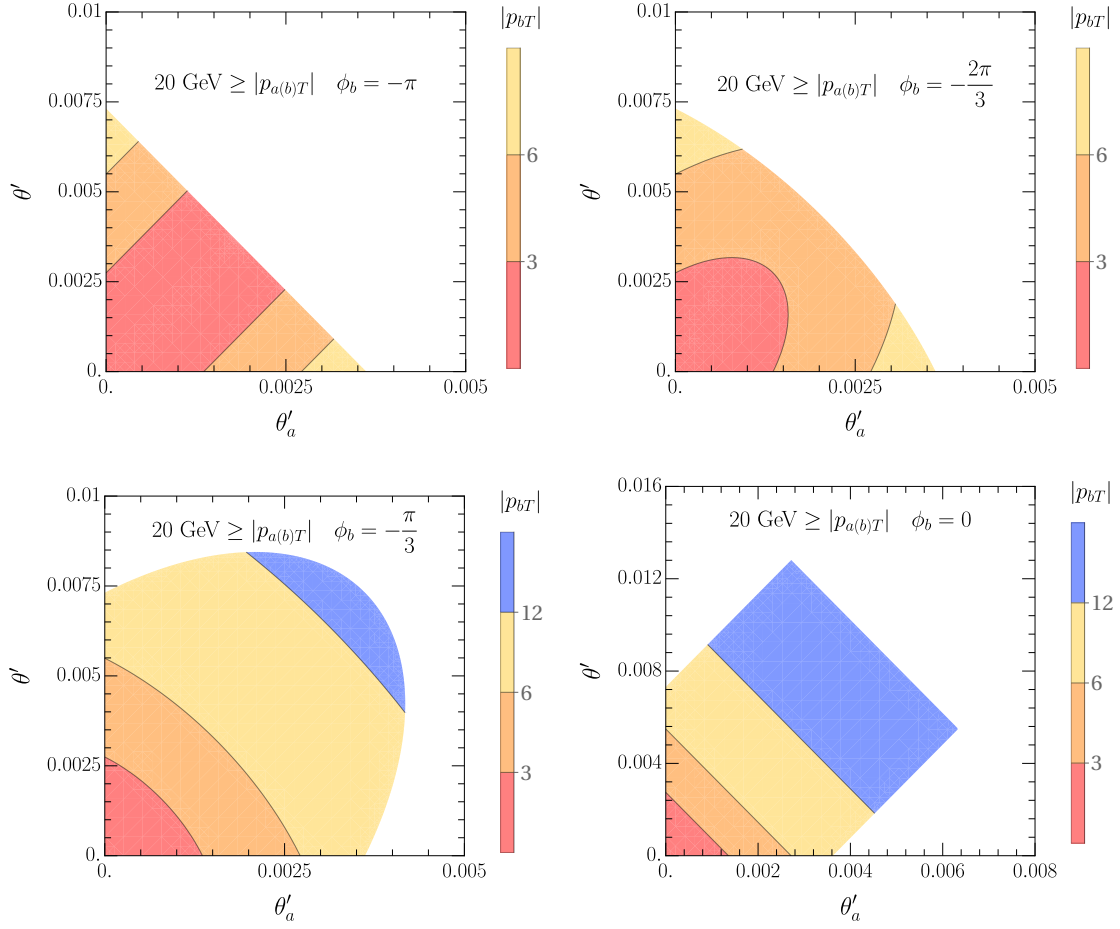


Figure 4.13: Contour plots of $|\mathbf{p}_{bT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0.5$, $\xi'_b = 0.2$ and $\phi_b = -\pi \times \{1, \frac{2}{3}, -\frac{1}{3}, 0\}$.

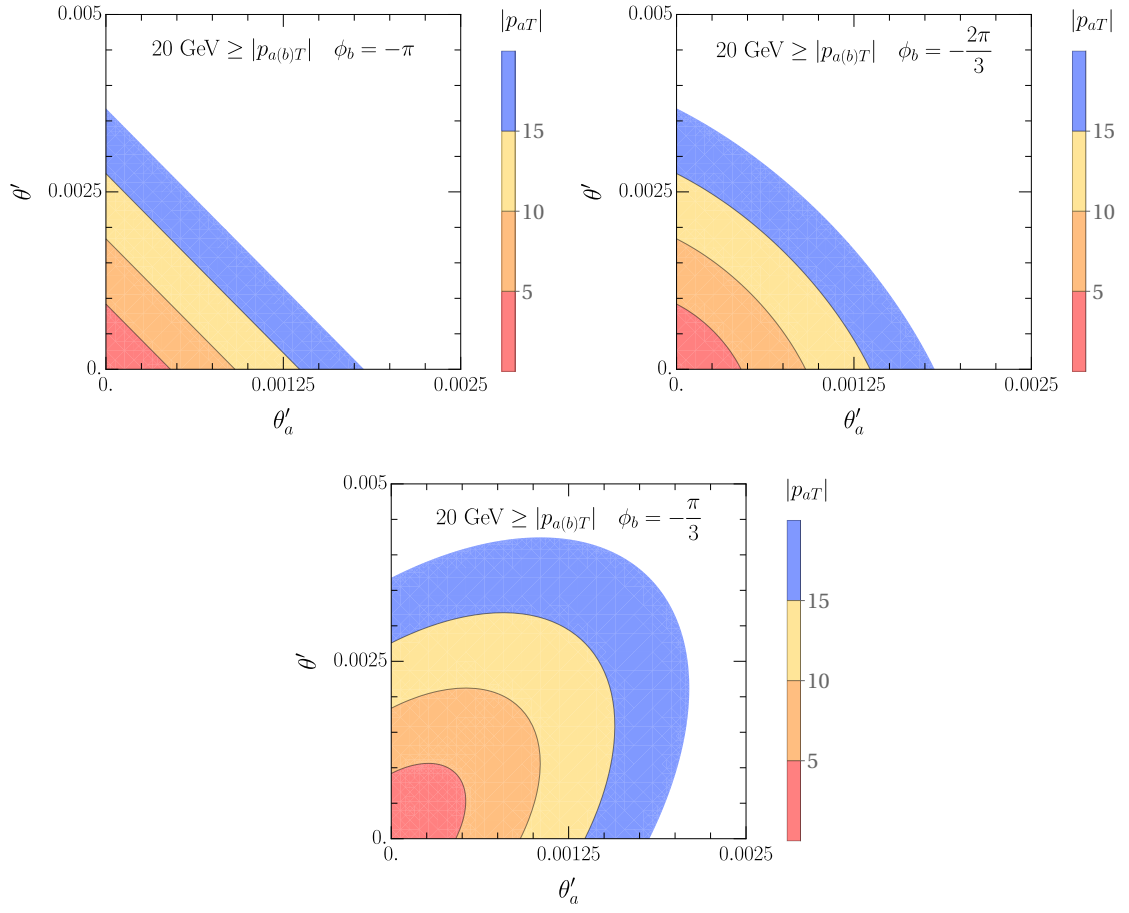


Figure 4.14: Contour plots of $|\mathbf{p}_{aT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0$, $\xi'_b = 0$ and $\phi_b = -\pi \times \{1, \frac{2}{3}, \frac{1}{3}\}$.

4.4. NUMERICAL RESULTS AT N3LL+NLO

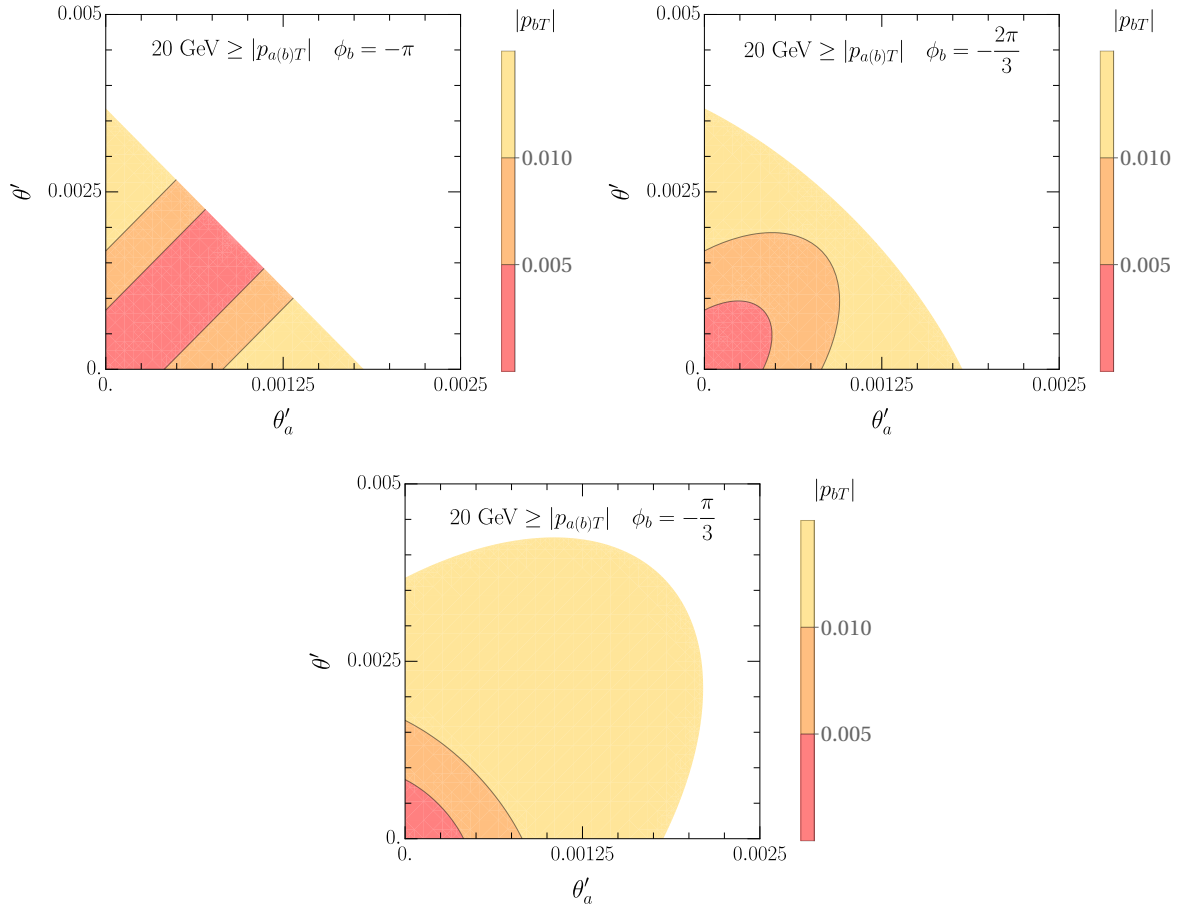


Figure 4.15: Contour plots of $|\mathbf{p}_{bT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0$, $\xi'_b = 0$ and $\phi_b = -\pi \times \{1, \frac{2}{3}, \frac{1}{3}\}$.

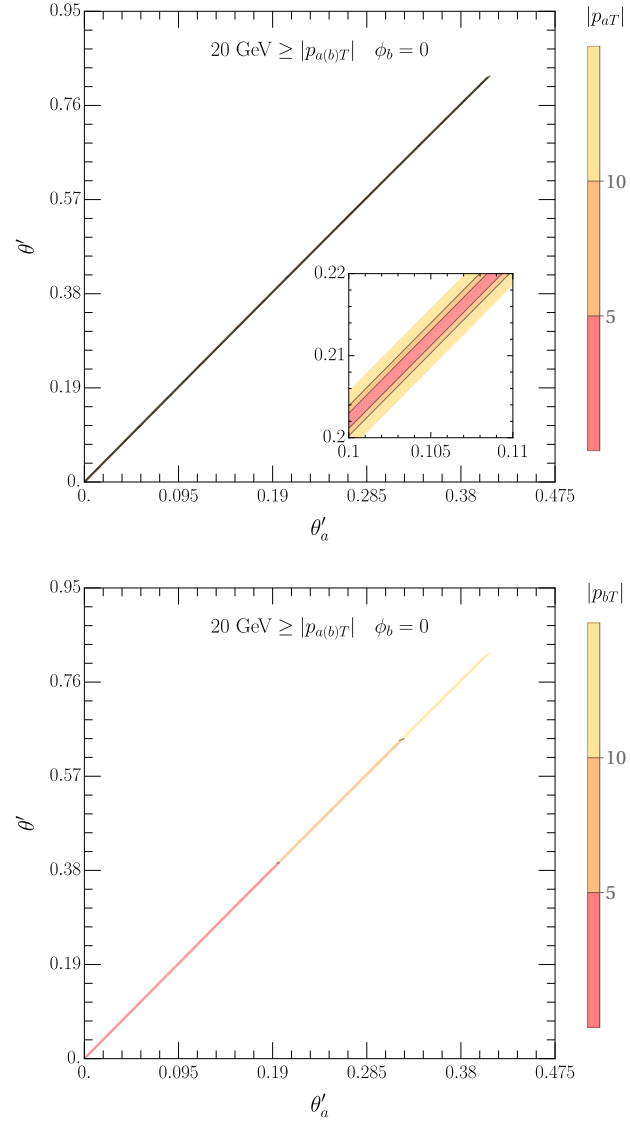


Figure 4.16: Contour plots of $|\mathbf{p}_{aT}|$ (upper) and $|\mathbf{p}_{bT}|$ (lower) in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0, \xi'_b = 0$ and $\phi_b = 0$.

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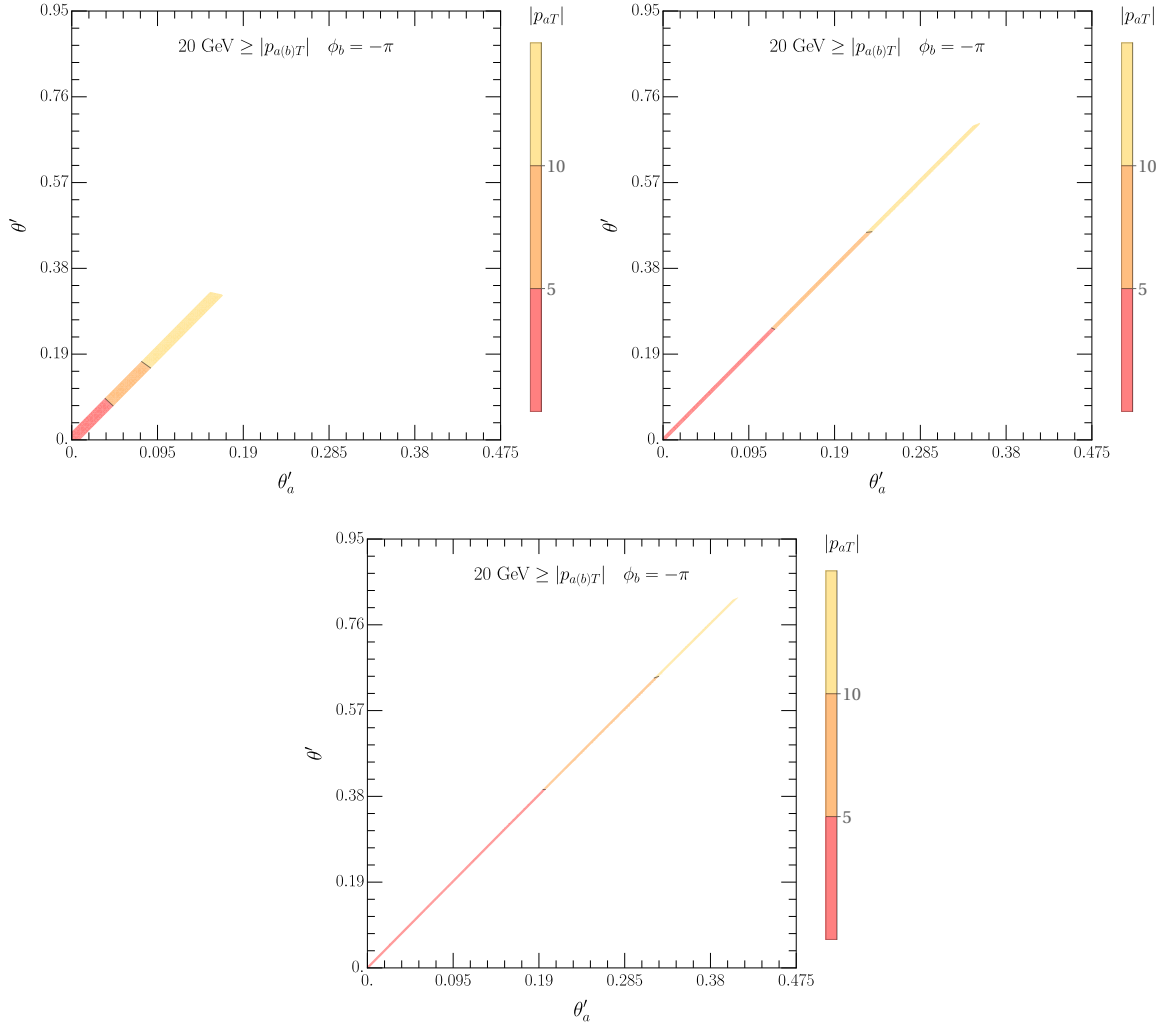


Figure 4.17: Contour plots of $|\mathbf{p}_{aT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0.0$, $\phi'_b = -\pi$ and $\xi'_b = 0.2, 0.6, 1$.

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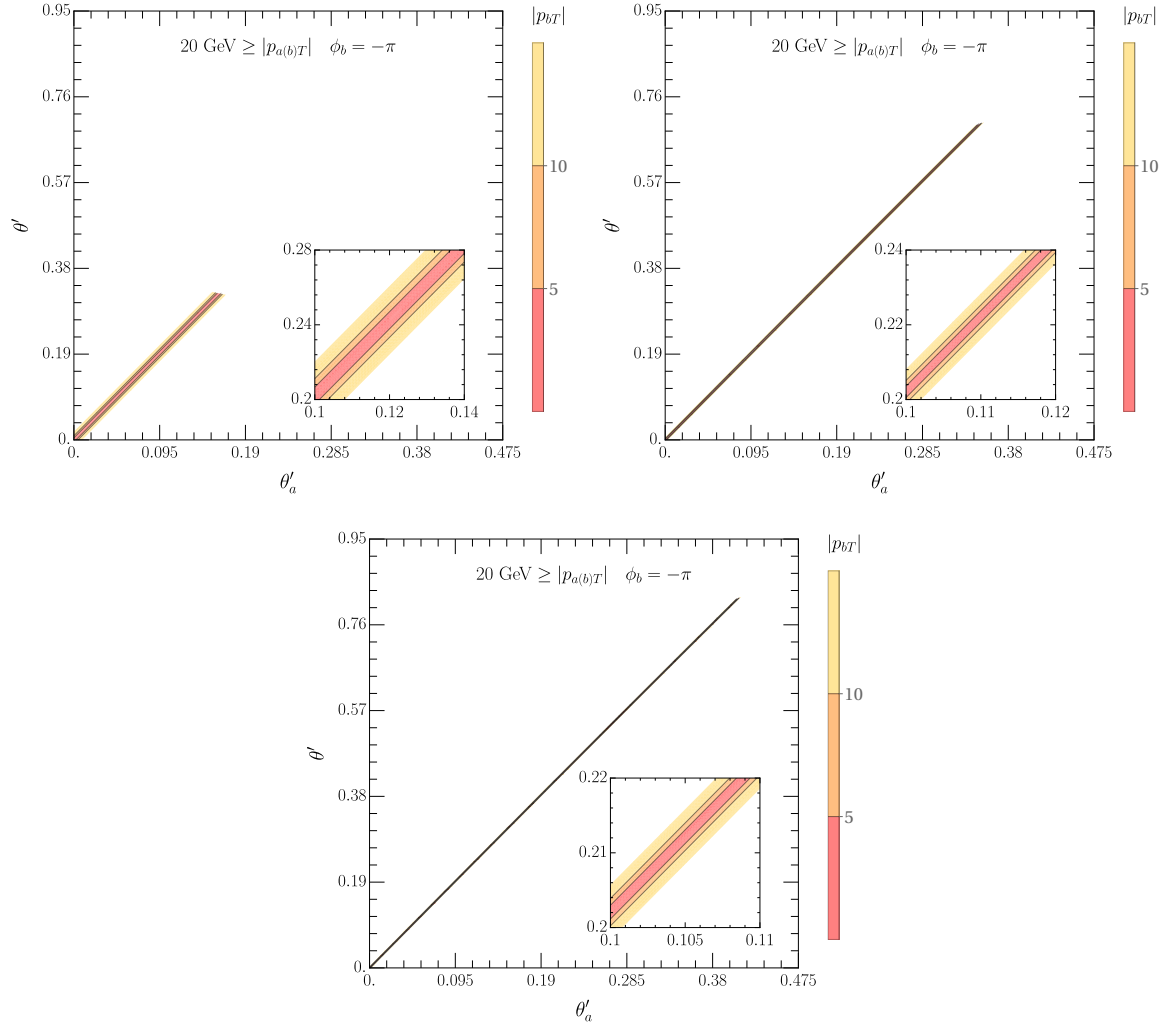


Figure 4.18: Contour plots of $|\mathbf{p}_{bT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0.0$, $\phi'_b = -\pi$ and $\xi'_b = 0.2, 0.6, 1$.

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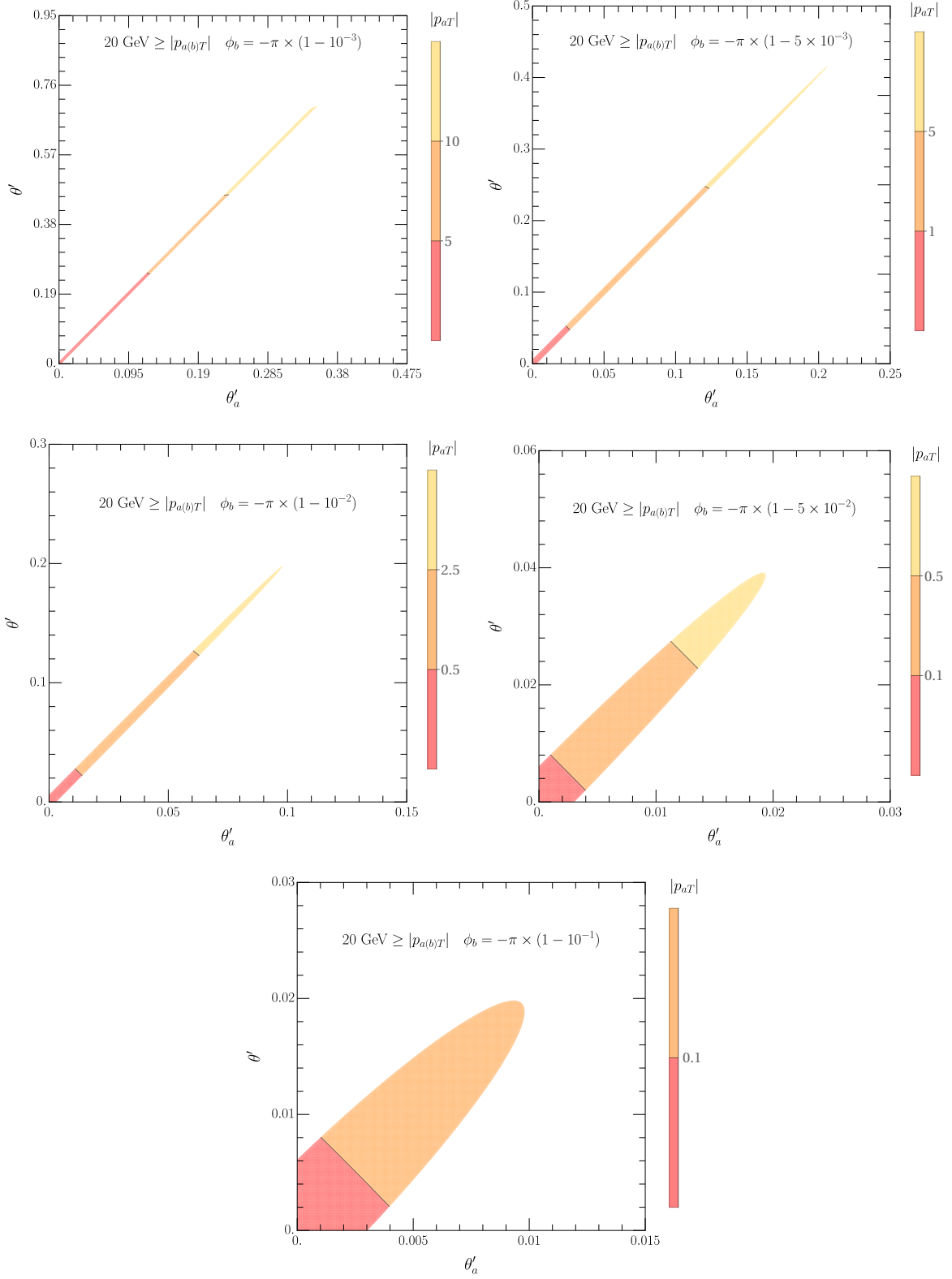


Figure 4.19: Contour plots of $|\mathbf{p}_{aT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0$, $\xi'_b = 0.6$ and $\phi_b = -\pi\{1 - 10^{-3}, 1 - 5 \times 10^{-3}, 1 - 10^{-2}, 1 - 5 \times 10^{-2}, 1 - 10^{-1}\}$.

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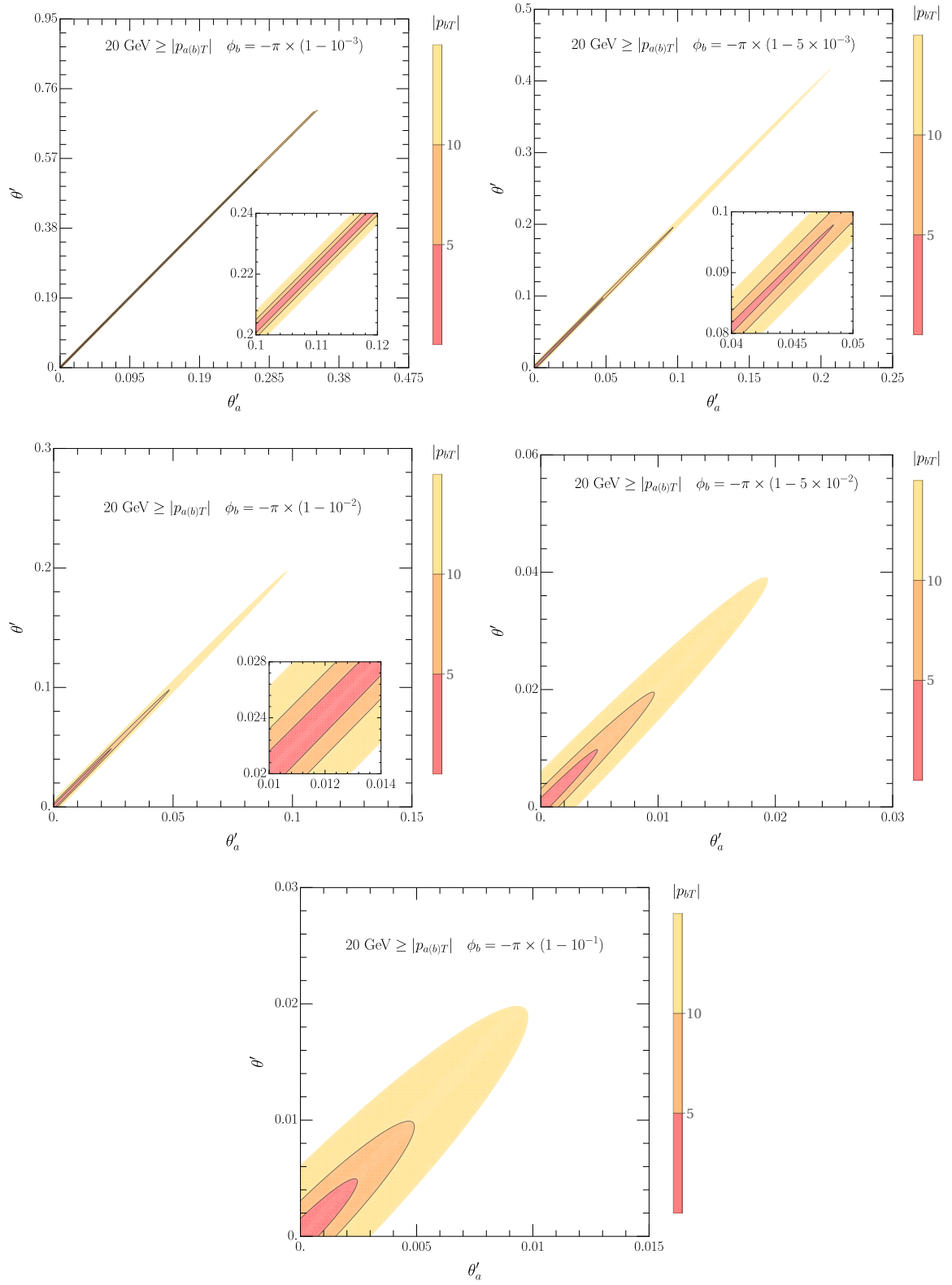


Figure 4.20: Contour plots of $|\mathbf{p}_{bT}|$ in the region $20 \text{ GeV} \geq |\mathbf{p}_{aT}|, |\mathbf{p}_{bT}|$ for $\xi'_a = 0.0$, $\xi'_b = 0.6$ and $\phi_b = -\pi\{1 - 10^{-3}, 1 - 5 \times 10^{-3}, 1 - 10^{-2}, 1 - 5 \times 10^{-2}, 1 - 10^{-1}\}$.

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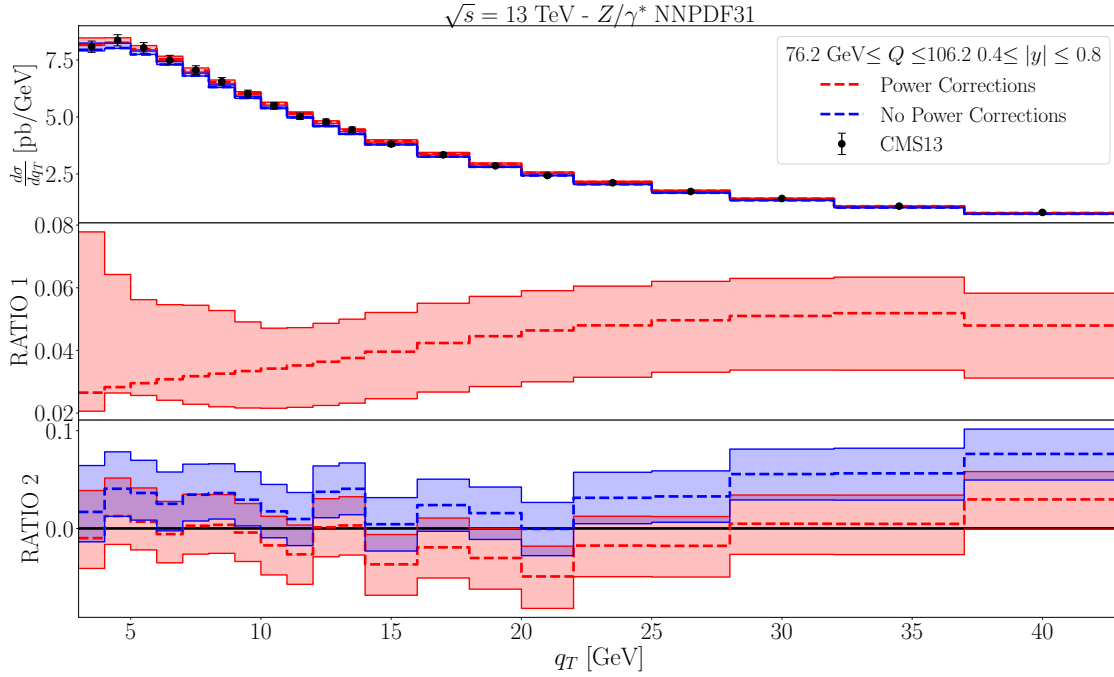


Figure 4.21: Differential Drell-Yan cross section at small- and intermediate- q_T . The vector boson invariant mass, Q , is integrated from 76.2 GeV up to 106.2 GeV; the rapidity lies on the interval $y \in [-0.8, -0.4] \cup [0.4, 0.8]$. $\text{RATIO 1} = 1 - d\sigma^{\text{LP}}/d\sigma^{\text{PC}}$, $\text{RATIO 2} = 1 - d\sigma^{\text{LP(PC)}}/d\sigma^{\text{DATA}}$. The uncertainty band is obtained by performing scale variation of the matching scales at the effective electroweak vertex and on the PDF.

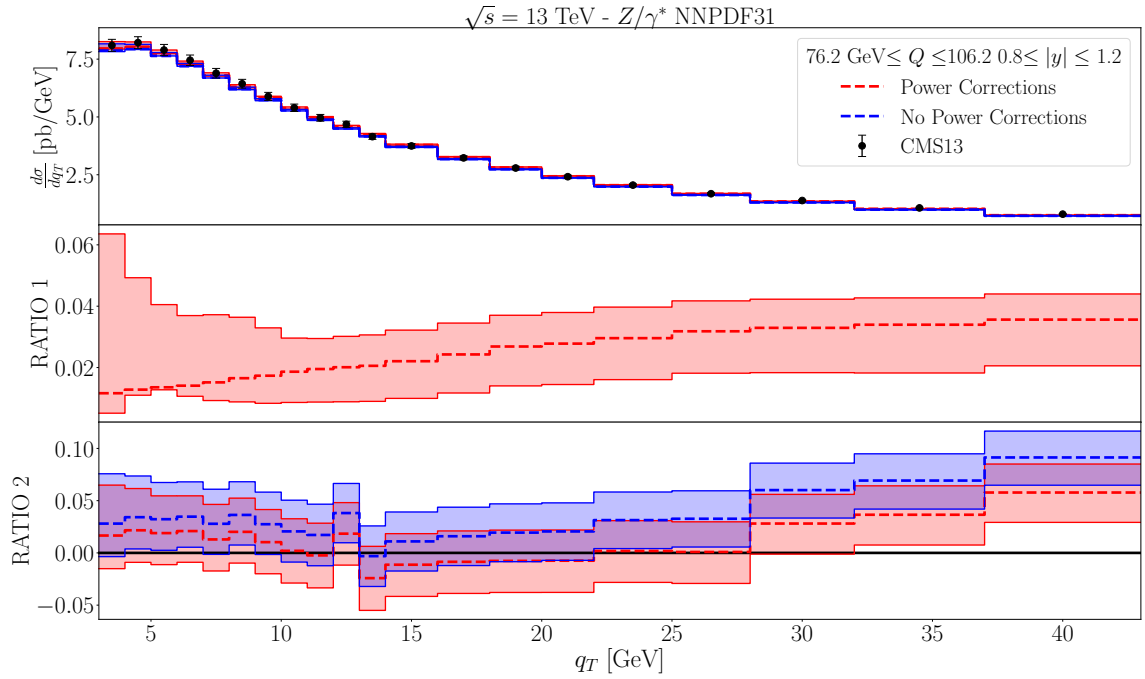


Figure 4.22: Differential Drell-Yan cross section at small- and intermediate- q_T . The vector boson invariant mass, Q , is integrated from 76.2 GeV up to 106.2 GeV; the rapidity lies on the interval $y \in [-1.2, -0.8] \cup [0.8, 1.2]$. $\text{RATIO 1} = 1 - d\sigma^{\text{LP}}/d\sigma^{\text{PC}}$, $\text{RATIO 2} = 1 - d\sigma^{\text{LP(PC)}}/d\sigma^{\text{DATA}}$. The uncertainty band is obtained by performing scale variation of the matching scales at the effective electroweak vertex and on the PDF.

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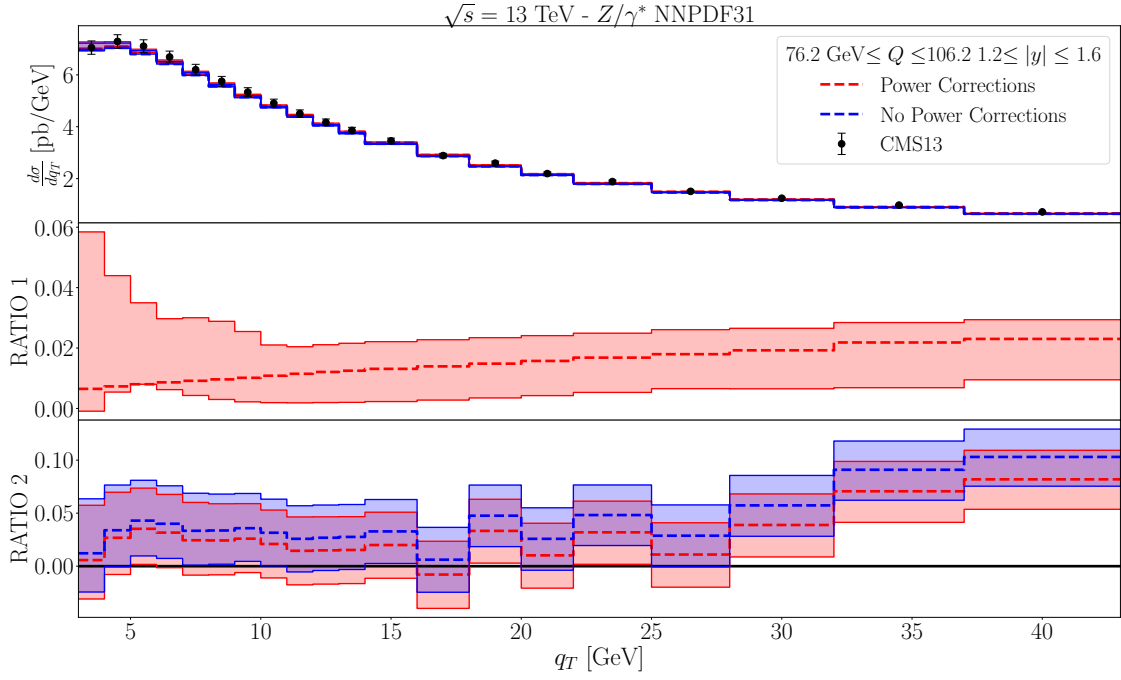


Figure 4.23: Differential Drell-Yan cross section at small- and intermediate- q_T . The vector boson invariant mass, Q , is integrated from 76.2 GeV up to 106.2 GeV; the rapidity lies on the interval $y \in [-1.6, -1.2] \cup [1.2, 1.6]$. $\text{RATIO 1} = 1 - d\sigma^{\text{LP}}/d\sigma^{\text{PC}}$, $\text{RATIO 2} = 1 - d\sigma^{\text{LP(PC)}}/d\sigma^{\text{DATA}}$. The uncertainty band is obtained by performing scale variation of the matching scales at the effective electroweak vertex and on the PDF.

5 Conclusions & Outlook

In this thesis we have presented the NLO computation of different sub-leading contributions to the leading-power TMDPDF factorization formula for the differential Drell-Yan cross section, namely: sub-leading collinear twist contributions, kinematic power corrections (KPC) and hard real radiative corrections to the electroweak vertex, being the last one the main aspect of the present work. The computations of the power suppressed contributions have been done employing an Operator Product Expansion (OPE) and the background field method. The methods presented in this thesis pave the way to the extension of TMDPDF factorization in Drell-Yan when there is hard real radiation at the level of the electroweak vertex. This allows for a smooth transition from the small to intermediate- q_T region without the need to introduce profile functions or any weighted sum of fixed order and resummed computation. Therefore, no further dependence on unphysical scales like jet vetoes or phase space slicing is introduced and the whole transition from the region dominated from q_T resummation to fixed order is completely dictated by measurement functions. This is possible since due to the inclusion of higher collinear twist as well as kinematic power corrections, which introduces a dependence on the $\mathbf{p}_T$ of the partons entering the short-distance interaction in the electroweak vertex.

The basic assumption behind the background field methods amounts to splitting the fields in the QCD Lagrangian into two contributions: a *fast* or quantum field and a *slow* or classical field. The idea is that the quantum fields are the ones responsible for the dynamics and loop corrections in the theory, while the *slow* fields act as a classical background where the interaction between the quantum fields takes place. This is based on the fact that original theory contains some heavy modes that are not part of the initial or final measurable states. Therefore, one can integrate out the contribution of quantum fields, which leads to a Wilson coefficient convoluted with an effective operator. The new effective operators are composed of effective fields whose eigenstates are the external states. In this work we focused on the QCD forward matrix element of the product of two electroweak currents in the two-proton state. Since in the proton center-of-mass frame each proton is boosted along the $\pm\hat{z}$ -direction, the quark and gluon fields inside the proton are also boosted along the same direction with momentum scaling as in eq. (4.1). On the other hand, the QCD loop corrections to the electroweak vertex involve momentum that are arbitrarily large along any direction. Correspondingly, the QCD fields can be split into three parts as in eq. (4.2). The hard fields play the role of the *fast* moving or quantum fields, being the ones responsible for the dynamics and quantum corrections; the (anti-)collinear fields are regarded as classical fields with respect to the hard fields, and thus act as a classical background. Hence we can consider that only the (anti-)collinear fields act on the two-proton states, while the contribution of the hard fields amount to a vacuum expectation value (VEV) under the presence of classical sources that can be computed in perturbation theory. As a result, we obtain an effective electroweak vertex composed of (anti-)collinear fields convoluted with a Wilson coefficient. Which is given by the VEV of the short-distance contribution of the (possible time-ordered) product of two

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hard fields. When the two hard fields are time ordered the short-distance contribution is nothing but the Feynman propagator. When the two hard fields are not time ordered, the short-distance contribution amounts to the on-shell propagator. The case when all hard fields are under the same time ordering operator is referred as virtual contribution; when at least two hard fields are under different time ordering, their short distance contributions amount to the on-shell propagators and we refer to those contributions as real.

One of the advantages of using the background field method is that one can consider different gauge fixing conditions for the quantum and classical fields. Moreover one can integrate out the quantum fields without fixing the gauge for the background fields, since these are not dynamical from the point of view of the hard fields. The computation of the VEV for the hard fields in the presence of (anti-)collinear background fields is done using perturbation theory. We can sort the different contributions to the VEV depending on the number of background fields: in general, the greater the number of (anti-)collinear fields is, the more suppressed the contribution to the VEV will be. This expansion is referred as collinear twist expansion, therefore higher collinear twist are power suppressed. Nevertheless, there is an exception: contributions containing $\bar{n}(n) \cdot A_{n(\bar{n})}$. Since $\bar{n}(n) \cdot A_{n(\bar{n})} \sim Q$, one can insert an arbitrary large number of (anti-)collinear gluon fields along the $\bar{n}(n)$ and the contribution to the VEV is still of order $\sim \lambda^0$, i.e. leading power. This is indeed expected, since these terms give rise to the (anti-)collinear Wilson lines that guarantee the gauge invariance (in a regular gauge) of the effective operators. Therefore, it is convenient to use the (anti-)collinear gluon fields in the light-cone gauge, $\bar{n}(n) \cdot A_{n(\bar{n})} = 0$ from the beginning since it simplifies the algebraic manipulations. One can always restore the background field to an arbitrary regular gauge by introducing the corresponding Wilson lines. Using the equations of motions for the background fermion fields, one can write the effective electroweak vertex in terms of well-defined spin components. From there, one can use the Fierz identity to rewrite the forward matrix element of the product of two electroweak currents as the forward matrix elements of a (anti-)collinear quark-quark correlator. Since in this work we focus on the matching at geometrical twist-2, we neglect contributions containing transverse or $n(\bar{n}) \cdot A_{n(\bar{n})}$ background gluon components since they lead to operators of geometrical twist-3 or higher. From the forward matrix element of the (anti-)collinear quark-quark correlators. One can obtain the standard TMDPDFs in the limit when the separation along the anti-collinear(collinear) coordinate $\bar{n} \cdot z(n \cdot z)$ for the case of collinear(anti-collinear) fermion fields goes to zero. In this limit one can simply expand the quark-quark correlators and use the e.o.m. to rearrange the different contributions. This expansion highly simplifies in momentum space, where the differential operators acting on the forward matrix elements amount to the momentum associated to the operator. This expansion gives rise to the kinematic power corrections (KPC). The inclusion of the whole series of KPC (for effective operators of geometrical twist-2) together with the inclusion of contributions of collinear twist-3 and twist-4 has two consequences. Firstly, the restoration of the Ward identity at the level of the effective electroweak vertex. Secondly, the momentum of the described parton correspond to one of an on-shell massless particle. Thus, we can compute the VEV of hard fields in the presence of background fields through the $p_a + p_b \rightarrow F + X$ scattering process. The fundamental difference with respect to collinear factorization is that the incoming partonic states are not parallel to the proton beams, instead they have a non zero component along the transverse plane and they are not back-to-back, which generalizes the kinematic configuration.

After rearranging the background fields into quark-quark correlators and rewriting these in terms of TMDPDFs, the Drell-Yan hadronic tensor amounts to a VEV of hard fields convoluted with TMDPDFs. The computation of the loop integrals for the VEV of hard fields can be done in position space and the Fourier transform to momentum space can be performed afterwards, or one can first compute the Fourier transform and perform the loop integration in momentum space. For the virtual contributions to the VEV the order is irrelevant since the measurement functions are not sensitive to the momentum of the loop corrections. We performed the computation in momentum space since in this case we can profit from previous results. The IR poles from the VEV are cancelled by the UV renormalization constant of the TMDPDF operators. The case for real emission is more involved, since the momentum of the hard gluon field enter the measurement functions. We obtained that the order in which the loop integration and the Fourier transform are taken matters in order to obtain a finite result. Assuming suitable analyticity conditions and behavior at the spatial infinity for the TMDPDF operators in order to ensure the existence of the Fourier transform, we showed that the convolution of the VEV with the TMDPDF matrix elements in position space amounts to a finite quantity. The analyticity of the TMDPDF allows us to deform the integration contour in order to obtain a quantity free of short-distance (UV) divergences. The behavior at the spatial infinity is crucial to ensure that the convolution of the VEV with the TMDPDF is free of long-distance divergences (IR). Nevertheless, if we first perform the Fourier transform, the dependence of the measurement functions on the gluon momentum spoils the analyticity properties. The IR divergences occur when the gluon momentum becomes collinear or soft, however this limits are not compensated by the behavior of the TMDPDF matrix elements (as it happened in position space), which amounts to some finite number. While one can prove in position space that the convolution of the VEV with TMDPDF operator is well-defined and finite, the computations are rather involved due to the presence of transcendental functions. It is for this reason that we perform the loop integration in momentum space, the price to pay for interchanging loop integration and Fourier transform is the presence of collinear and soft poles. These poles are the consequence of integrating over regions where the gluon momentum does not obey the hard modes scaling. In position space these regions are regularized by the behaviour of the TMDPDF at infinity. In momentum space we define collinear and soft double-counting subtraction terms that remove the contribution coming from the soft and collinear regions. The main advantage of this procedure is that it can be implemented locally at the level of partonic squared amplitude. Also, it does not need the introduction of phase space slicing parameter(s) that discriminate between phase space regions and can lead to unphysical dependence. The only non-physical parameter that we need to introduce is the boundary term for the plus distributions, nevertheless we explicitly show that the hadronic tensor does not depend on it. This is another advantage with respect to phase space slicing, where proving that a cross section is independent of the slicing could be rather involved and often only a numerical check of this independence is possible.

The more general kinematic configuration of the partons entering the hard scattering process makes the identification of the collinear and soft divergences more involved than in collinear factorization, since the partonic states entering the hard scattering process have some transverse component and are not back-to-back. Nevertheless, after the application of a series of rotations and boosts we are able to write the phase-space integration in a reference frame

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where the incoming partons are back-to-back. we refer to this reference frame as the partonic center-of-mass. Isolating and removing the collinear and soft poles in the partonic center-of-mass frame is rather simple, nevertheless there are two drawbacks. The first one is that the measurement functions and the leptonic tensor take a rather cumbersome form. However, since the measurement functions are IR safe quantities, after removing the IR poles from the double counting of the collinear and soft regions, one can safely evaluate numerically the phase space integrals. The second drawback is that the TMDPDF distributions become functions of four variables: one radial component and three angles, which slows down the numerical evaluation of the Drell-Yan cross section. Nonetheless, based on asymptotic behavior of the TMDPDFs in momentum space, we could perform some simplifications to speed up the numerical evaluation. At the energy scale $\mu = Q = M_Z$ the bulk of the contribution from the TMDPDF is given for transverse momentum in the range $0 \text{ GeV} \leq |\mathbf{p}_T| \leq 6 \text{ GeV}$. Therefore, we can further restrict the phase space integration. In particular, we constraint θ' to be in $\theta' \in [0, 0.1]$ and $\theta'_a \in [0, 0.05]$. These simplifications speed up the numerical integration of the phase space integrals, while the contribution from the discarded phases space region is smaller than the uncertainty due to scale variations. We found that contributions from kinematic power corrections, collinear twist-3 and 4 together with the short distance contribution of real radiation at the level of the effective electroweak vertex systematically increase the value of the differential Drell-Yan cross section on the intermediate-large q_T region. This enhancement of the differential cross section improves the agreement between the experimental results and the theoretical predictions. It is for central rapidity regions, $0.8 \geq |y| \geq 0.4$, that the inclusion of power-suppressed effects have the largest impact. For a transverse momentum $15 \text{ GeV} \leq q_T \leq 40 \text{ GeV}$ the cross section is enhanced by 4% to 5%. For vector boson rapidities in the interval $1.2 \geq |y| \geq 0.8$, the impact of the inclusion of power suppressed term has a milder effect, enhancing the differential cross section by 2% to 4%. For large rapidity regions, $1.6 \geq |y| \geq 1.2$, the effect of the power corrections on the Drell-Yan differential cross section is 2% or smaller. The reason for this behavior is that for large rapidities ranges the bulk of the real emission is forced to be either soft or $p_{a(b)}$ -collinear, however those phase-space regions are removed through the double counting subtraction since they violate the mode scaling associated to hard fields, given by the short-distance expansion of the product of fields. For central regions, $0.8 \geq |y| \geq 0.4$, it is possible to have hard radiation that is not collinear along the direction of the partons emerging from the protons. This kinematic situation allows for a sizeable contribution from the power suppressed terms which is larger than the electroweak corrections. Based on this arguments, one can conclude that in the region $|y| \geq 0.4$ the effect of the power corrections can be even larger.

In order to test the capabilities of the phase-pace parametrization presented at NLO in momentum space for real radiation, we have computed some of the NNLO corrections with two real partons: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$ and $q(\bar{q}) + \bar{q}(q) \rightarrow F + q(\bar{q}) + \bar{q}(q)$. We obtained that complementing the methods presented at NLO with sector decomposition one can systematically isolate and cancel the soft/collinear poles together with the spurious double counting of the soft and collinear regions. In order to be able to perform the whole NNLO computation it is necessary to work out a factorization theorem involving the gluon and (anti-)quark TMDPDF matrix elements, since the NLO channel $q(\bar{q}) + g \rightarrow F + q(\bar{q})$ must be included. The inclusion of such a contributions is left for future works, nevertheless its derivation within the background field method and the light-cone OPE follows similar steps as for the quark

cases. The only mayor challenge will be the inclusion of the KPC. This is simply due to the fact that for the gluon fields the equations of motion do not lead to a simple form of $\partial_{-(+)}$ for (anti-)collinear fields, which makes it harder to sum the whole series KPC at leading geometrical twist.

An other aspect open for further study is the inclusion of $|\mathbf{p}_T|/Q$ (or $1/(|\mathbf{b}_T|Q)$ in position space) power corrections on the TMDPDF matrix evaluation. In this work we have performed the matching onto PDFs of geometrical twist-2 of the operator $\bar{\xi}_n(z^+, 0^-, z_\perp)[z, -\infty][-\infty, 0]\xi_n(0)$. However using the background field method one can also obtain higher collinear twist operators like $\bar{\xi}_n(z^+, 0^-, z_\perp)[z, -\infty][-\infty, 0]A_{n\perp}^\mu(0)\xi_n(0)$ whose matrix elements have matching onto PDFs of geometrical twist-2 and 3. The inclusion of twist-3 leads to the inclusion of power corrections of $|\mathbf{b}_T|\Lambda_{\text{QCD}}$, improving the description of the small- q_T region. Performing the matching onto PDFs of geometrical twist-2 one includes power corrections in $|\mathbf{p}_T|/Q$, whci improves the description of the TMDPDF matrix elements in the intermediate- q_T region. Another potential consequence of the inclusion of $|\mathbf{p}_T|/Q$ corrections is that the decays of the TMDPDF distribution tails are slightly moderated. Nevertheless, the decaying at the tails is mainly dominated by the non-perturbative model used on the TMPPDFs matrix elements, therefore we expect that simplification like in $0.1 \geq \theta' \geq 0$ does not need to be drastically changed.

A Plus distribution: definition and properties

A.1 One dimensional plus distribution

A.1.1 Definition

Let $g(z)$ be a function with support on $z \geq 0$ and less singular than z^{-2} when $z \rightarrow 0$, its plus distribution with boundary condition z_0 is completely defined by the two properties

$$[\theta(z)g(z)]_+^{z_0} = g(z), \quad \text{for } z > 0, \quad (\text{A.1})$$

$$\int_0^{z_0} dz [\theta(z)g(z)]_+^{z_0} = 0. \quad (\text{A.2})$$

Explicitly, it is given by the expression

$$\begin{aligned} [\theta(z)g(z)]_+^{z_0} &= \lim_{\beta \rightarrow 0} \frac{d}{dz} \left[\theta(z - \beta) \int_{z_0}^z dz' g(z') \right] \\ &= \lim_{\beta \rightarrow 0} \left[\theta(z - \beta)g(z) + \delta(z - \beta) \int_{z_0}^z dz' g(z') \right]. \end{aligned} \quad (\text{A.3})$$

Therefore, the action of $[\theta(z)g(z)]_+^{z_0}$ over a test function $f(z)$ analytic around $z = 0$ can be expressed as

$$\begin{aligned} \int_0^\Delta dz [\theta(z)g(z)]_+^{z_0} f(z) &= \lim_{\beta \rightarrow 0} \int_\beta^\Delta dz g(z) f(z) + \lim_{\beta \rightarrow 0} f(\beta) \int_{z_0}^\beta dz (g(z)) \\ &= \int_0^\Delta dz g(z) f(z) - \lim_{\beta \rightarrow 0} \int_0^\beta dz f(z) g(z) + \lim_{\beta \rightarrow 0} f(\beta) \int_{z_0}^\beta dz g(z), \end{aligned} \quad (\text{A.4})$$

since we are under the assumption $f(z)$ is analytic around $z = 0$, one can safely expand it in a Taylor series. Therefore, the second term on the second line can be written as

$$\int_0^\beta dz g(z) f(z) = \int_0^\beta g(z) f(0) + \int_0^\beta dz g(z) \sum_{n=1} \frac{z^n}{n!} f^{(n)}(0), \quad (\text{A.5})$$

since we assume $g(z)$ is less singular than $1/z^2$ for $z \rightarrow 0$ the second term on the right-hand side goes to zero as $\beta^{1-\eta}$ with $1 > \eta \geq 0$ when $\beta \rightarrow 0$. Therefore, one can write

$$\begin{aligned} \int_0^\Delta dz [\theta(z)g(z)]_+^{z_0} f(z) &= \int_0^\Delta dz g(z) f(z) - \lim_{\beta \rightarrow 0} \int_0^\beta dz f(0) g(z) + \lim_{\beta \rightarrow 0} f(\beta) \int_{z_0}^\beta dz g(z) \\ &= \int_0^\Delta dz g(z) f(z) + f(0) \int_{z_0}^0 dz g(z) \\ &= \int_0^\Delta dz g(z) f(z) + f(0) \int_\Delta^0 dz g(z) + f(0) \int_{z_0}^\Delta dz g(z) \\ &= \int_0^\Delta dz g(z) [f(z) - f(0)] + f(0) \int_{z_0}^\Delta dz g(z), \end{aligned} \quad (\text{A.6})$$

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hence

$$\int_0^\Delta dz [\theta(z)g(z)]_+^{z_0} f(z) = \int_0^\Delta dz g(z)[f(z) - f(0)] + f(0) \int_{z_0}^\Delta dz g(z). \quad (\text{A.7})$$

For two different boundary conditions z_0 and z_1 , the respective plus distributions $[\theta(z)g(z)]_+^{z_0}$ and $[\theta(z)g(z)]_+^{z_1}$ are related by

$$\begin{aligned} [\theta(z)g(z)]_+^{z_0} &= \lim_{\beta \rightarrow 0} \left[\theta(z - \beta)g(z) + \delta(z - \beta) \int_{z_0}^z dz' g(z') \right] \\ &= \lim_{\beta \rightarrow 0} \left[\theta(z - \beta)g(z) + \int_{z_1}^z dz' g(z') + \delta(z - \beta) \int_{z_0}^{z_1} dz' g(z') \right] \\ &= [\theta(z)g(z)]_+^{z_1} + \delta(z) \int_{z_0}^{z_1} dz' g(z'). \end{aligned} \quad (\text{A.8})$$

The derivative of a plus distribution with respect its boundary condition is given by

$$\frac{d}{dz_0} [\theta(z)g(z, z_0)]_+^{z_0} = \left[\theta(z) \frac{dg(z, z_0)}{dz_0} \right]_+^{z_0} - \delta(z)g(z_0, z_0), \quad (\text{A.9})$$

where in general $g(z, z_0)$ can depend explicitly on z_0 .

The action of $[(\theta(z))g(z)]_+^{z_0}$ on a test function $f(z)$ can be written as

$$\int_0^\Delta dz [\theta(z)g(z)]_+^{z_0} f(z) = \int_0^\Delta dz [\theta(z)g(z)]_+^{z_1} f(z) + f(0) \int_{z_0}^{z_1} dz g(z), \quad (\text{A.10})$$

therefore in the particular case $z_1 = \Delta$, comparing with eq. (A.7), one obtains

$$\int_0^\Delta dz [\theta(z)g(z)]_+^\Delta f(z) = \int_0^\Delta dz g(z)[f(z) - f(0)]. \quad (\text{A.11})$$

There are two cases of interest in this work: $g(z) = \theta(z)z^{-1-2\epsilon}$ and $g(z) = \theta(1-z)(1-z)^{-1-2\epsilon}$, both for $0 > \text{Re}[\epsilon]$:

Case: $g(z) = \theta(z)z^{-1-2\epsilon}$. If z is a dimensionless variable, the expansion in plus distributions with 1 as boundary condition is given by

$$\begin{aligned} \frac{\theta(z)}{z^{1+2\epsilon}} &= \lim_{\beta \rightarrow 0} \left[\frac{\theta(z - \beta)}{z^{1+2\epsilon}} + \delta(z - \beta) \int_1^z dz' \frac{1}{(z')^{1+2\epsilon}} \right] - \delta(z) \int_1^z dz' \frac{1}{(z')^{1+2\epsilon}} \\ &= \mathcal{L}^\epsilon(z) - \frac{\delta(z)}{2\epsilon} = \sum_{n=0}^{\infty} \frac{(-2\epsilon)^n}{\Gamma[n+1]} \mathcal{L}_n(z) - \frac{\delta(z)}{2\epsilon}, \end{aligned} \quad (\text{A.12})$$

where we have used the standard notation

$$\mathcal{L}^\epsilon(z) = \left[\frac{\theta(z)}{z^{1+2\epsilon}} \right]_+^1, \quad \mathcal{L}_n(z) = \left[\frac{\theta(z) \log^n[z]}{z} \right]_+^1. \quad (\text{A.13})$$

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Case: $g(z) = \theta(z)z^{-1-2\epsilon}$. If z is a dimensionful variable, the expansion in plus distributions with μ as boundary condition is

$$\begin{aligned} \frac{\theta(z)\mu^{2\epsilon}}{z^{1+2\epsilon}} &= \lim_{\beta \rightarrow 0} \left[\frac{\theta(z-\beta)\mu^{2\epsilon}}{z^{1+2\epsilon}} + \delta(z-\beta) \int_{\mu}^z dz' \frac{\mu^{2\epsilon}}{(z')^{1+2\epsilon}} \right] - \delta(z) \int_{\mu}^z dz' \frac{\mu^{2\epsilon}}{(z')^{1+2\epsilon}} \\ &= \mathcal{L}^{\epsilon}(z, \mu) - \frac{\delta(z)}{2\epsilon} = \sum_{n=0}^{\infty} \frac{(-2\epsilon)^n}{\Gamma[n+1]} \mathcal{L}_n(z, \mu) - \frac{\delta(z)}{2\epsilon}, \end{aligned} \quad (\text{A.14})$$

where μ has the same dimensions that z and we have used the standard notation

$$\mathcal{L}^{\epsilon}(z, \mu) = \left[\frac{\theta(z)\mu^{2\epsilon}}{z^{1+2\epsilon}} \right]_{+}^{\mu}, \quad \mathcal{L}_n(z, \mu) = \left[\frac{\theta(z)}{z} \log \frac{z}{\mu} \right]_{+}^{\mu}. \quad (\text{A.15})$$

Case: $g(z) = \theta(1-z)(1-z)^{-1-2\epsilon}$. In the case of z a dimensionless variable, its expansion in plus distributions with boundary condition z_0 is

$$\begin{aligned} \frac{\theta(1-z)}{(1-z)^{1+2\epsilon}} &= \lim_{\beta \rightarrow 0} \left[\frac{\theta(1-z-\beta)}{(1-z)^{1+2\epsilon}} - \delta(1-z-\beta) \int_{z_0}^z dz' \frac{1}{(1-z')^{1+2\epsilon}} \right] + \delta(1-z) \int_{z_0}^z dz' \frac{1}{(1-z')^{1+2\epsilon}} \\ &= \left[\frac{\theta(1-z)}{(1-z)^{1+2\epsilon}} \right]_{+}^{z_0} - \delta(1-z) \frac{(1-z_0)^{-2\epsilon}}{2\epsilon} \\ &= -\delta(1-z) \frac{(1-z_0)^{-2\epsilon}}{2\epsilon} + \left[\frac{\theta(1-z)}{1-z} \right]_{+}^{z_0} - 2\epsilon \left[\frac{\theta(1-z) \log[1-z]}{1-z} \right]_{+}^{z_0} + \dots \end{aligned} \quad (\text{A.16})$$

A.1.2 Fourier transform

The Fourier transform for $\mathcal{L}^{\epsilon}(z, \mu)$ can be obtained from eq. (A.14)

$$\begin{aligned} \mathcal{F}\{\mathcal{L}^{\epsilon}(z, \mu)\} &= \int_{-\infty}^{\infty} dz e^{-ixz} \mathcal{L}^{\epsilon}(z, \mu) = \int_{-\infty}^{\infty} dz e^{-izx} \left[\frac{\theta(z)\mu^{2\epsilon}}{z^{1+2\epsilon}} + \frac{\delta(z)}{2\epsilon} \right] \\ &= \frac{1}{2\epsilon} + \Gamma[-2\epsilon](ix\mu)^{2\epsilon} = -\frac{1}{2\epsilon} \left[e^{-2\epsilon\gamma_E} \Gamma[1-2\epsilon] e^{2\epsilon \log[ix\mu e^{\gamma_E}]} - 1 \right], \end{aligned} \quad (\text{A.17})$$

where $x := x - i0^+$, γ_E is the Euler-Mascheroni constant and we have used

$$\frac{1}{A^n} = \frac{1}{\Gamma[n]} \int_0^{\infty} dz z^{n-1} e^{-zA}, \quad \text{Re}[A] > 0. \quad (\text{A.18})$$

Expanding left and right-hand sides in eq. (A.17) and comparing order by order in ϵ , one obtains the Fourier transforms of $\mathcal{L}_n(z, \mu)$

$$\int_{-\infty}^{\infty} dz e^{-izx} \mathcal{L}_n(z, \mu) = \frac{(-1)^n}{2^n} \frac{d^n}{d\epsilon^n} \left\{ -\frac{1}{2\epsilon} \left[e^{-2\epsilon\gamma_E} \Gamma[1-2\epsilon] e^{2\epsilon \log[ix\mu e^{\gamma_E}]} - 1 \right] \right\} \Big|_{\epsilon=0} \quad (\text{A.19})$$

A.2 One dimensional fractional plus distribution

A.2.1 Definition

In general, power-law divergences of a function $g(z)$ can be isolated by a generalization of the plus distribution presented in the previous section, A.1. Let $g(z)$ be a function with support on $z \geq 0$ and less divergent than $z^{-1-\eta}$ as $z \rightarrow 0$, where $\eta \in \mathbb{R}^+$. Its fractional plus distribution with boundary condition z_0 is completely determined by the two properties

$$\begin{aligned} [g(z)]_{+(\eta)}^{z_0} &= g(z), \quad z > 0, \\ \int_0^{z_0} dz z^m [g(z)]_{+(\eta)}^{z_0} &= 0, \quad \forall m < \eta, \end{aligned} \quad (\text{A.20})$$

An explicit form for $[g(z)]_{+(\eta)}^{z_0}$ can be obtained integrating $g(z)$ against a test function $f(z)$, which is at least $[\eta] - 1$ -times differentiable at $z = 0^1$, and subtracting the divergent contribution

$$\int_0^\Delta dz g(z) f(z) = \int_0^\Delta dz g(z) \left(f(z) - \sum_{n=0}^{[\eta]-1} \frac{z^n}{n!} f^{(n)}(0) \right) + \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_0^\Delta dz z^n g(z), \quad (\text{A.21})$$

where Δ is some arbitrary positive upper bound and $f^{(n)}(0)$ denotes the n -th derivative of $f(z)$ evaluated at $z = 0$. The second integral on the left hand-side can be further split onto two contributions

$$\sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_0^\Delta dz z^n g(z) = \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_{z_0}^\Delta dz z^n g(z) + \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_0^{z_0} dz z^n g(z). \quad (\text{A.22})$$

The first integral on the right hand-side is finite, while the power-law divergences are fully isolated and made explicit in the contribution given by the second integral. Also, the derivatives of the test function $f(z)$ evaluated at zero can be written as

$$f^{(n)}(0) = (-1)^n \int_0^\Delta dz \delta^{(n)}(z) f(z), \quad (\text{A.23})$$

¹ $\lfloor \cdot \rfloor$ denotes the floor function.

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where $\delta^{(n)}$ is the n -th derivative of the Dirac delta distribution. Using eqs. (A.22) and (A.23) in eq. (A.21) one obtains

$$\begin{aligned}
\int_0^\Delta dz g(z) f(z) &= \int_0^\Delta dz g(z) \left(f(z) - \sum_{n=0}^{[\eta]-1} \frac{z^n}{n!} f^{(n)}(0) \right) + \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_{z_0}^\Delta dz z^n g(z) \\
&+ \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_0^{z_0} dz z^n g(z) \\
&= \int_0^\Delta dz g(z) f(z) + \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_{z_0}^0 dz z^n g(z) \\
&+ \sum_{n=0}^{[\eta]-1} \frac{f^{(n)}(0)}{n!} \int_0^{z_0} dz z^n g(z) \\
&= \int_0^\Delta dz g(z) f(z) + \int_0^\Delta dz \sum_{n=0}^{[\eta]-1} \frac{(-1)^n f(z) \delta^{(n)}(z)}{n!} \int_{z_0}^0 d\tilde{z} \tilde{z}^n g(\tilde{z}) \\
&+ \int_0^\Delta dz \sum_{n=0}^{[\eta]-1} \frac{(-1)^n f(z) \delta^{(n)}(z)}{n!} \int_0^{z_0} d\tilde{z} \tilde{z}^n g(\tilde{z}).
\end{aligned} \tag{A.24}$$

Thus, the fractional plus distribution for $g(z)$ with boundary condition z_0 is given explicitly by

$$[g(z)]_{+(\eta)}^{z_0} = \lim_{\beta \rightarrow 0} \left[\theta(z - \beta) g(z) + \sum_{n=0}^{[\eta]-1} \frac{(-1)^n \delta^{(n)}(z - \beta)}{n!} \int_{z_0}^\beta d\tilde{z} \tilde{z}^n g(\tilde{z}) \right]. \tag{A.25}$$

It can be verified that the defining properties of the fractional plus distribution given in eq. (A.20) are present in the explicit form in eq. (A.25).

Case: $g(z) = \theta(z) z^{-\eta-a}$. A particular case of interest in the present work is when $g(z) = \theta(z) z^{-\eta-a}$, where $\eta \in \mathbb{R}^+$ and a is an arbitrary parameter. Eventually we are interested in the limit $a \rightarrow 0^-$. Using eq. (A.25) one can write

$$\begin{aligned}
\frac{\theta(z)}{z^{\eta+a}} &= \left[\frac{\theta(z)}{z^{\eta+a}} \right]_{+(\eta)}^{z_0} + \sum_{n=0}^{[\eta]-1} \frac{(-1)^n \delta^{(n)}(z)}{n!} \int_0^{z_0} dz z^{-\eta-a+n} \\
&= \left[\frac{\theta(z)}{z^{\eta+a}} \right]_{+(\eta)}^{z_0} + \sum_{n=0}^{[\eta]-1} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta-a}}{n+1-\eta-a}.
\end{aligned} \tag{A.26}$$

In going from the first to the second line we have integrated the second term for $-a \geq \eta$. The result in eq. (A.26) can be regarded as an analytic function of a and therefore one can consider its analytical extension to values of a other than $-a \geq \eta$. In particular we are interested in expanding eq. (A.26) around $a = 0$. In order to perform such an expansion two cases must be considered for the second term on the second line: η a positive integer, $[\eta] = \eta$; η is a positive real number with a non-zero decimal part. Nevertheless, for the fractional plus distribution it is irrelevant if η is an integer or not, since the expansion around $a = 0$ is always well defined.

APPENDIX A. PLUS DISTRIBUTION: DEFINITION AND PROPERTIES

Case: $\eta \in \mathbb{N}$. In this case $\lfloor \eta \rfloor = \eta$, and the sum on the second line in eq. (A.26) must thus be split into two pieces

$$\begin{aligned} \sum_{n=0}^{\eta-1} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta-a}}{1+n-\eta-a} &= -\frac{(-1)^{\eta-1} \delta^{(\eta-1)}(z)}{(\eta-1)!} \frac{z_0^{-a}}{a} + \sum_{n=0}^{\eta-2} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta-a}}{1+n-\eta-a} \\ &= -\frac{(-1)^{\eta-1} \delta^{(\eta-1)}(z)}{(\eta-1)!} \frac{1}{a} \sum_{m=0}^{\infty} \frac{(-a)^m \log^m z_0}{m!} \\ &\quad + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\eta-2} \frac{(-a)^m \log^m z_0}{m!} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta}}{1+n-\eta} \left(\frac{a}{1+n-\eta} \right)^k. \end{aligned} \quad (\text{A.27})$$

Plugging eq. (A.27) into eq. (A.26) one obtains

$$\begin{aligned} \frac{\theta(z)}{z^{\eta+a}} &= \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \left[\frac{\theta(z) \log^m z}{z^\eta} \right]_{+(\eta)}^{z_0} - \frac{(-1)^{\eta-1} \delta^{(\eta-1)}(z)}{(\eta-1)!} \frac{1}{a} \sum_{m=0}^{\infty} \frac{(-a)^m \log^m z_0}{m!} \\ &\quad + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\eta-2} \frac{(-a)^m \log^m z_0}{m!} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta}}{1+n-\eta} \left(\frac{a}{1+n-\eta} \right)^k. \end{aligned} \quad (\text{A.28})$$

Case: $\eta \in \mathbb{R}^+$. In this case it is not necessary to split the sum on the second line in eq. (A.26) and one can directly expand around $a = 0$, since there is no value of n that makes the denominator vanish. One obtains

$$\begin{aligned} \frac{\theta(z)}{z^{\eta+a}} &= \sum_{m=0}^{\infty} \frac{(-a)^m}{m!} \left[\frac{\theta(z) \log^m z}{z^\eta} \right]_{+(\eta)}^{z_0} \\ &\quad + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor \eta \rfloor - 1} \frac{(-a)^m \log^m z_0}{m!} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta}}{1+n-\eta} \left(\frac{a}{1+n-\eta} \right)^k. \end{aligned} \quad (\text{A.29})$$

A.2.2 Fourier transform

The procedure to compute the Fourier transform of $[\theta(z) z^{-\eta} \log^m z]_{+(\eta)}^{z_0}$ is analogous to the one presented in Section A.1. One starts from

$$\begin{aligned} \mathcal{F} \left\{ \left[\frac{\theta(z)}{z^{\eta+a}} \right]_{+(\eta)}^{z_0} \right\} &= \int_{-\infty}^{\infty} dz e^{-iyz} \left[\frac{\theta(z)}{z^{\eta+a}} - \sum_{n=0}^{\lfloor \eta \rfloor - 1} \frac{(-1)^n \delta^{(n)}(z)}{n!} \frac{z_0^{1+n-\eta-a}}{n+1-\eta-a} \right] \\ &= \Gamma[1-\eta-a] (iy)^{\eta+a-1} - \sum_{n=0}^{\lfloor \eta \rfloor - 1} \frac{(-1)^n (iy)^n}{n!} \frac{z_0^{1+n-\eta-a}}{n+1-\eta-a} \end{aligned} \quad (\text{A.30})$$

where $y := y - i0^+$ and we have used eq. (A.18) together with

$$\int_{-\infty}^{\infty} dz e^{-iyz} \delta^{(n)}(z) = \int_{-\infty}^{\infty} dz e^{-iyz} \frac{d^n}{dz^n} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{iz\omega} = \int_{-\infty}^{\infty} d\omega (i\omega)^n \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{iz(\omega-y)} = (iy)^n. \quad (\text{A.31})$$

The expansion around $a = 0$ can be performed without need to distinguish if η is either a positive integer or real, both cases leads to the same finite result. Therefore, expanding

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around $a = 0$ the left- and right-hand sides in eq. (A.30) and comparing order by order in a one obtains $\mathcal{F}\{\theta(z)z^{-\eta}\log^m z\}_{+^{(n)}}^{z_0}$. In the present work $m = 0, 1, 2$ are the cases of interest:

$$\mathcal{F}\left\{\left[\frac{\theta(z)}{z^\eta}\right]_{+^{(n)}}^{z_0}\right\} = \Gamma[1-\eta](iy)^{\eta-1} - \sum_{n=0}^{[\eta]-1} \frac{(-1)^n (iy)^n z_0^{1+n-\eta}}{n!} \frac{1}{n+1-\eta}, \quad (\text{A.32})$$

$$\begin{aligned} \mathcal{F}\left\{\left[\frac{\theta(z)\log z}{z^\eta}\right]_{+^{(n)}}^{z_0}\right\} &= (iy)^{\eta-1}\Gamma[1-\eta](\psi(1-\eta) - \log(iy)) \\ &+ \sum_{n=0}^{[\eta]-1} \frac{(-1)^n (iy)^n z_0^{1+n-\eta}}{n!} \left[\frac{1}{1+n-\eta} - \log z_0\right], \end{aligned} \quad (\text{A.33})$$

$$\begin{aligned} \mathcal{F}\left\{\left[\frac{\theta(z)\log^2 z}{z^\eta}\right]_{+^{(n)}}^{z_0}\right\} &= (iy)^{\eta-1}\Gamma[1-\eta](\log(iy) - \psi(1-\eta))^2 + \psi^{(1)}(1-\eta) \\ &- \sum_{n=0}^{[\eta]-1} \frac{2(-1)^n (iy)^n z_0^{1+n-\eta}}{n!(1+n-\eta)} \left[\frac{1}{(1+n-\eta)^2} + \frac{\log^2 z_0}{2} - \frac{\log z_0}{1+n-\eta}\right], \end{aligned} \quad (\text{A.34})$$

where $\psi^{(n)}(z)$ is n -th derivative of the digamma function.

Another distribution of interest is the inverse Fourier transform of $(iy)^{\eta-1}\log^m(iy)$, $\mathcal{F}^{-1}\{(iy)^{\eta-1}\log^m(iy)\}$, which can be written as a linear combination of fractional plus distributions

$$\begin{aligned} \mathcal{F}^{-1}\{(iy)^{\eta-1}\log^m(iy)\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{iyz} (iy)^{\eta-1} \log^m(iy) = \frac{1}{2\pi} \left(\frac{d^m}{da^m} \int_{-\infty}^{\infty} dy e^{iyz} (iy)^{\eta+a-1} \right) \Big|_{a=0} \\ &= \frac{d^m}{da^m} \frac{1}{2\pi\Gamma[1-\eta-a]} \int_{-\infty}^{\infty} dy \int_0^{\infty} dx e^{iyz-iyx} x^{-\eta-a} \Big|_{a=0} \\ &= \frac{d^m}{da^m} \frac{\theta(z)}{\Gamma[1-a-\eta]z^{\eta+a}} \Big|_{a=0}. \end{aligned} \quad (\text{A.35})$$

Before performing the derivatives with respect the parameter a one needs to rewrite the term $\theta(z)z^{-\eta-a}$ as a fraction of plus distribution using eq. (A.26). Afterwards, one can perform the derivatives with respect a and take the limit $a \rightarrow 0$ without need to distinguish if η is either a positive integer or real. As expected, since contributions of the form $(iy)^{\eta-1}\log^m(iy)$ come from the Fourier transform of fractional plus distributions they are analytic functions of a and η .

B Tensor identities and properties

B.1 Color algebra

We follow the convention and definitions of [64] for the color group $SU(3)_c$. Its generators in the fundamental representation are given by $t^a = \lambda^a/2$, where λ^a are the Gell-Mann matrices [274]; the generators in the adjoint representation are given by the matrices $C_{bc}^a = -if_{abc} = -if^{abc}$, where f^{abc} are the totally anti-symmetric structure constants. The matrices t^a and C^a obey the commutation relations

$$\sum_{k=1}^3 [t_{ik}^a, t_{kj}^b] = i \sum_{c=1}^8 f^{abc} t_{ij}^c, \quad \sum_{d=1}^8 [C_{ed}^a, C_{dh}^b] = i \sum_c f^{abc} C_{eh}^c, \quad (\text{B.1})$$

and the anti-commutation relation $\{t^a, t^b\}$ is given by

$$\sum_k \{t_{ik}^a, t_{kj}^b\} = \sum_{c=1}^8 d^{abc} t_{ij}^c + \frac{1}{3} \delta^{ab} \delta_{ij}, \quad (\text{B.2})$$

where $d^{abc} = d_{abc}$ are totally symmetric. Using the fact that the matrices t^a and C^a are orthogonal under the trace inner product, one can invert the previous relations and obtain

$$f^{abc} = -2i \text{Tr}[t^c, [t^a, t^b]], \quad (\text{B.3})$$

and

$$d^{abc} = 2 \text{Tr}[t^c, \{t^a, t^b\}]. \quad (\text{B.4})$$

The invariant C_A , C_F and T_f are defined as

$$\begin{aligned} \delta_{ab} C_A &= \text{Tr}[C^a C^b] = - \sum_{c,c'=1}^8 f^{acc'} f^{bc'e} = - \sum_{c,c'=1}^8 f^{cac'} f^{cc'b} = \sum_{c,c'=1}^8 C_{ac'}^c C_{c'b}^c, \\ \delta_{ij} C_F &= \sum_{a=1}^8 \sum_{k=1}^3 t_{ik}^a t_{kj}^a, \\ \delta_{ab} T_f &= \text{Tr}[t^a t^b] = \sum_{i,j=1}^3 t_{ij}^a t_{ji}^b. \end{aligned} \quad (\text{B.5})$$

T_F and C_F are known as the first and second Casimir operator in the fundamental representation; in the case of the adjoint representation they are the same. For $SU(N)$ they amount to

$$C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_f = \frac{1}{2}, \quad (\text{B.6})$$

APPENDIX B. TENSOR IDENTITIES AND PROPERTIES

where $N = 3$ for $SU(3)$. Some useful relations are

$$\begin{aligned} \sum_{k=1}^3 t_{ik}^a t_{kj}^b &= \frac{1}{2} \left([t^a, t^b] + \{t^a, t^b\} \right)_{ij} = \frac{1}{2} \sum_{c=1}^8 (d^{abc} + if^{abc}) t_{ij}^c + \frac{1}{6} \delta^{ab} \delta_{ij}, \\ \sum_{a=1}^8 t_{ij}^a t_{kl}^a &= -\frac{1}{2N} \delta_{ij} \delta_{kl} + \frac{1}{2} \delta_{il} \delta_{kj}. \end{aligned} \quad (\text{B.7})$$

B.2 Fierz identity

Let be A and B to matrices in Dirac indices. Their tensor product $A \otimes B$ can be decomposed into orthogonal subspaces using the Fierz identity

$$(A)_{ab}(B)_{cd} = \frac{1}{16} \sum_{C,D} \text{Tr}[\Gamma_C A \Gamma_D B] (\Gamma^C)_{ad} (\Gamma^D)_{cb}, \quad (\text{B.8})$$

where Γ^A is an element of the basis of Dirac matrices whose elements obey the orthogonality relation

$$\text{Tr}[\Gamma^C \Gamma_D] = 4\delta^C_D. \quad (\text{B.9})$$

In this work we choose Γ^A and Γ_A to be (see [77]) $\Gamma^A = \{\mathbb{1}, i\gamma_5, \gamma^\mu, \gamma^\mu \gamma_5, i\sigma^{\mu\nu} \gamma_5\}$ and $\Gamma_A = \{\mathbb{1}, -i\gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, -i\gamma_5 \sigma_{\mu\nu}\}$. The choice of this particular bases Γ^A and Γ_A is due to the relation $\Gamma^{A\dagger} = \gamma^0 \Gamma_A \gamma^0$, which highly simplifies the study of hermiticity and parity properties of TMDPDF operators.

C Interaction terms in composite background QCD

C.1 Interaction terms

The interaction term $\mathcal{L}_{\text{int}}$ derived in Section 4.1 can be split into four different contributions [184]

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{hn} + \mathcal{L}_{h\bar{n}} + \mathcal{L}_{hn\bar{n}} + \mathcal{L}_{n\bar{n}}, \quad (\text{C.1})$$

where the $\mathcal{L}_{hn}$ and $\mathcal{L}_{h\bar{n}}$ terms gather the interactions between hard and collinear or anti-collinear fields respectively, $\mathcal{L}_{hn\bar{n}}$ contains all the interaction between hard, collinear and anti-collinear terms, finally $\mathcal{L}_{n\bar{n}}$ takes into account only interactions between the collinear and anti-collinear sector. These Lagrangian densities are given by

$$\begin{aligned} \mathcal{L}_{hn} = & g\bar{q}_n \not{B}_h \psi_h + g\bar{\psi}_h \not{A}_n \psi_h + g\bar{\psi}_h \not{B}_h q_n \\ & + g f^{abc} A_n^{a\mu} \omega^{b\dagger} (\vec{\partial}_\mu - \overleftarrow{\partial}_\mu) \omega^c - g f^{abc} f^{acd} \omega^{a\dagger} (A_n^{b\mu} + B_h^{b\mu}) A_{n\mu}^c \omega^d \\ & - g f^{abc} A_n^{a\mu} [2(\partial_\nu B_{h\mu}^b) B_h^{c\nu} - (\partial_\mu B_h^{b\nu}) B_{h\nu}^c - \frac{1+\xi}{\xi} (\partial_\nu B^{b\nu}) B_{h\mu}^c] \\ & - \frac{g}{2} f^{abr} f^{acd} [A_n^{a\mu} A_n^{b\nu} B_{h\mu}^c B_{h\nu}^d + A_n^{a\mu} B_h^{b\nu} A_{n\mu}^c B_{h\nu}^d + \frac{1}{\xi} A_n^{a\mu} B_{h\mu}^b A_n^{c\nu} B_{h\nu}^d], \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} \mathcal{L}_{h\bar{n}} = & g\bar{q}_{\bar{n}} \not{B}_h \psi_h + g\bar{\psi}_h \not{A}_{\bar{n}} \psi_h + g\bar{\psi}_h \not{B}_h q_{\bar{n}} \\ & + g f^{abc} A_{\bar{n}}^{a\mu} \omega^{b\dagger} (\vec{\partial}_\mu - \overleftarrow{\partial}_\mu) \omega^c - g f^{abc} f^{acd} \omega^{a\dagger} (A_{\bar{n}}^{b\mu} + B_h^{b\mu}) A_{\bar{n}\mu}^c \omega^d \\ & - g f^{abc} A_{\bar{n}}^{a\mu} [2(\partial_\nu B_{h\mu}^b) B_h^{c\nu} - (\partial_\mu B_h^{b\nu}) B_{h\nu}^c - \frac{1+\xi}{\xi} (\partial_\nu B^{b\nu}) B_{h\mu}^c] \\ & - \frac{g}{2} f^{abr} f^{acd} [A_{\bar{n}}^{a\mu} A_{\bar{n}}^{b\nu} B_{h\mu}^c B_{h\nu}^d + A_{\bar{n}}^{a\mu} B_h^{b\nu} A_{\bar{n}\mu}^c B_{h\nu}^d + \frac{1}{\xi} A_{\bar{n}}^{a\mu} B_{h\mu}^b A_{\bar{n}}^{c\nu} B_{h\nu}^d], \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} \mathcal{L}_{hn\bar{n}} = & g\{\bar{q}_n \not{A}_{\bar{n}} \psi_h + \bar{\psi}_h \not{A}_n q_{\bar{n}} + \bar{q}_n \not{B}_h q_{\bar{n}} + \bar{q}_{\bar{n}} \not{A}_n \psi_h + \bar{\psi}_h \not{A}_{\bar{n}} q_n + \bar{q}_{\bar{n}} \not{B}_h q_n \\ & - f^{abe} f^{ecd} \omega^{a\dagger} A_n^{b\mu} A_{\bar{n}\mu}^c \omega^d - f^{abc} [(\partial_\mu A_{n\nu}^a - \partial_\nu A_{n\mu}^a) A_{\bar{n}}^{b\mu} B_{h\nu}^c + A_n^{a\mu} A_{\bar{n}\nu}^b \partial_\mu B_{h\nu}^c] \\ & - f^{abe} f^{ecd} \omega^{a\dagger} A_{\bar{n}}^{b\mu} A_{n\mu}^c \omega^d - f^{abc} [(\partial_\mu A_{\bar{n}\nu}^a - \partial_\nu A_{\bar{n}\mu}^a) A_n^{b\mu} B_{h\nu}^c + A_{\bar{n}}^{a\mu} A_{n\nu}^b \partial_\mu B_{h\nu}^c] \\ & - g f^{abe} f^{ecd} [A_n^{a\mu} A_n^{b\nu} A_{\bar{n}\mu}^c B_{h\nu}^d + A_n^{a\mu} B_h^{b\nu} A_{\bar{n}\mu}^c A_{\bar{n}\nu}^d + B_h^{a\mu} A_n^{b\nu} A_{n\mu}^c A_{\bar{n}\nu}^d] \\ & - g f^{abe} f^{ecd} [A_{\bar{n}}^{a\mu} A_{\bar{n}}^{b\nu} A_{n\mu}^c B_{h\nu}^d + A_{\bar{n}}^{a\mu} B_h^{b\nu} A_{\bar{n}\mu}^c A_{n\nu}^d + B_h^{a\mu} A_{\bar{n}}^{b\nu} A_{\bar{n}\mu}^c A_{n\nu}^d] \\ & - g f^{abe} f^{ecd} [B_h^{a\mu} B_h^{b\nu} A_{n\mu}^c A_{\bar{n}\nu}^d + A_n^{a\mu} B_h^{b\nu} A_{\bar{n}\mu}^c B_{h\nu}^d + B_h^{a\mu} A_n^{b\nu} A_{\bar{n}\mu}^c B_{h\nu}^d + \frac{1}{\xi} A_n^{a\mu} B_h^{b\nu} A_{\bar{n}\mu}^c B_{h\nu}^d]\}, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned}
 \mathcal{L}_{n\bar{n}} = & \frac{g}{2} \{ \bar{q}_n (A_n + A_{\bar{n}}) q_{\bar{n}} + \bar{q}_{\bar{n}} (A_n + A_{\bar{n}}) q_n + \bar{q}_n A_{\bar{n}} q_n + \bar{q}_{\bar{n}} A_n q_{\bar{n}} \\
 & - f^{abc} [(\partial_\mu A_{n\nu}^a) A_{\bar{n}}^{b\mu} A_{\bar{n}}^{c\nu} + (\partial_\mu A_{\bar{n}\nu}^a) A_n^{b\mu} A_{\bar{n}}^{c\nu} + (\partial_\mu A_{n\nu}^a) A_{\bar{n}}^{b\mu} A_n^{c\nu}] \\
 & - f^{abc} [(\partial_\mu A_{\bar{n}\nu}^a) A_n^{b\mu} A_{\bar{n}}^{c\nu} + (\partial_\mu A_{n\nu}^a) A_{\bar{n}}^{b\mu} A_n^{c\nu} + (\partial_\mu A_{\bar{n}\nu}^a) A_n^{b\mu} A_{\bar{n}}^{c\nu}] \\
 & - g f^{abe} f^{ecd} [A_n^{a\mu} A_n^{b\nu} A_{\bar{n}\mu}^c A_{\bar{n}\nu}^d + A_n^{a\mu} A_n^{b\nu} A_{n\mu}^c A_{\bar{n}\nu}^d + A_{\bar{n}}^{a\mu} A_{\bar{n}}^{b\nu} A_{\bar{n}\mu}^c A_{n\nu}^d \\
 & + A_n^{a\mu} A_{\bar{n}}^{b\nu} A_{n\mu}^c A_{\bar{n}\nu}^d + A_{\bar{n}}^{a\mu} A_{\bar{n}}^{b\nu} A_{\bar{n}\mu}^c A_{n\nu}^d] \}.
 \end{aligned} \tag{C.5}$$

Latin letters represent color indices running from 1 to $N^2 - 1$; Greek letters represent Lorentz indices. A sum over repeated indices is understood.

C.2 Fermion equation of motions (e.o.m.)

The e.o.m. for the (anti-)collinear background quarks are given by

$$\not{D}[A_{n(\bar{n})}] q_{n(\bar{n})} = 0, \quad \overleftarrow{\not{D}}[A_{n(\bar{n})}] \bar{q}_{n(\bar{n})} = 0, \tag{C.6}$$

where $(D_\mu[A])_{ij} = \partial_\mu \delta_{ij} - ig A_\mu^a t_{ij}^a$ and $(\overleftarrow{D}_\mu[A])_{ij} = \overleftarrow{\partial}_\mu \delta_{ij} + ig A_\mu^a t_{ij}^a$. Let us consider $\not{D}[A_n] q_n = 0$. Using the projectors $\frac{\not{n}\not{\bar{n}}}{2}$ and $\frac{\not{\bar{n}}\not{n}}{2}$ this can be written as

$$\begin{aligned}
 \mathbb{1} \not{D}[A_n] \mathbb{1} q_n &= \left(\frac{\not{n}\not{\bar{n}}}{2} + \frac{\not{\bar{n}}\not{n}}{2} \right) \not{D}[A_n] \left(\frac{\not{n}\not{\bar{n}}}{2} + \frac{\not{\bar{n}}\not{n}}{2} \right) \left(\frac{\not{n}\not{\bar{n}}}{2} + \frac{\not{\bar{n}}\not{n}}{2} \right) q_n \\
 &= \left(\frac{\not{n}\not{\bar{n}}}{2} + \frac{\not{\bar{n}}\not{n}}{2} \right) \not{D}[A_n] \left(\frac{\not{n}\not{\bar{n}}}{2} + \frac{\not{\bar{n}}\not{n}}{2} \right) (\xi_n + \eta_n) \\
 &= \frac{\not{n}\not{\bar{n}}}{2} (\not{D}_\perp[A_n] \xi_n + \not{n}(\bar{n} \cdot D[A_n]) \eta_n) + \frac{\not{\bar{n}}\not{n}}{2} (\not{D}_\perp[A_n] \eta_n + \not{\bar{n}}(n \cdot D[A_n]) \xi_n) = 0.
 \end{aligned} \tag{C.7}$$

Since $\not{n}\not{\bar{n}}\not{\bar{n}}\not{n} = 0$, each term on the third line belongs to orthogonal subspaces and therefore must vanish individually. This leads to the set of equations

$$\begin{aligned}
 \frac{\not{n}\not{\bar{n}}}{2} (\not{D}_\perp[A_n] \xi_n + \not{n}(\bar{n} \cdot D[A_n]) \eta_n) &= 0, \\
 \frac{\not{\bar{n}}\not{n}}{2} (\not{D}_\perp[A_n] \eta_n + \not{\bar{n}}(n \cdot D[A_n]) \xi_n) &= 0.
 \end{aligned} \tag{C.8}$$

Multiplying the first by $\frac{\not{\bar{n}}}{2}$ and the second by $\frac{\not{n}}{2}$ leads to

$$(\bar{n} \cdot D[A_n]) \eta_n = -\frac{\not{\bar{n}}}{2} \not{D}_\perp[A_n] \xi_n, \tag{C.9}$$

and

$$(n \cdot D[A_n]) \xi_n = -\frac{\not{n}}{2} \not{D}_\perp[A_n] \eta_n. \tag{C.10}$$

In order to determine the solutions to eqs. (C.9) and (C.10) we need to specify the boundary conditions. In this work we choose $\eta_n(z^+ = -\infty) = 0$, $\xi_n(z^- = -\infty) = 0$ and $A_n^\mu(z^+ = -\infty) = 0$. The choice of vanishing fields at the light cone minus-infinity is motivated by

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the analyticity properties of the collinear PDFs, see. [132, 184, 212]. Since the collinear PDFs are the matrix elements of a combination of η , ξ and A fields, one needs to assume some analytical properties in order to guarantee the existence of the Fourier transform and its inverse. If the fields were non-vanishing at light-cone infinity, the analyticity properties would be spoilt, leading to ill-defined operators and matrix elements. Once chosen an appropriate boundary condition, eq. (C.9) can be uniquely solved for η_n . One can simplify it even further by choosing a convenient gauge for the background gluon field A_n^μ . In this case the light-cone gauge¹ $\bar{n} \cdot A_n(z) = 0$ is useful

$$\eta_n(z) = \int_{-\infty}^0 ds \bar{n} \cdot \partial \eta_n(s\bar{n} + z) = -\frac{\not{n}}{2} \int_{-\infty}^0 ds (\not{\partial}_\perp \xi_n(s\bar{n} + z) - ig \mathcal{A}_{n,\perp}(s\bar{n} + z) \xi_n(s\bar{n} + z)), \quad (\text{C.11})$$

where $\bar{n} \cdot \partial = \frac{\partial}{\partial z^+}$. Following a similar procedure (see [73, 75, 184, 190, 197]) one obtains the solution for $\eta_{\bar{n}}$ in the light-cone gauge $n \cdot A_{\bar{n}}^\mu(z) = 0$

$$\eta_{\bar{n}}(z) = \int_{-\infty}^0 ds n \cdot \partial \eta_{\bar{n}}(sn + z) = -\frac{\not{n}}{2} \int_{-\infty}^0 ds (\not{\partial}_\perp \xi_{\bar{n}}(sn + z) - ig \mathcal{A}_{\bar{n},\perp}(sn + z) \xi_{\bar{n}}(sn + z)), \quad (\text{C.12})$$

where $n \cdot \partial = \frac{\partial}{\partial z^-}$. For the field $\bar{\eta}_{n(\bar{n})}$, also in the light-cone gauge $\bar{n}(n) \cdot A_{n(\bar{n})} = 0$, one obtains

$$\bar{\eta}_n(z) = \int_{-\infty}^0 ds \bar{n} \cdot \partial \bar{\eta}_n(s\bar{n} + z) = - \int_{-\infty}^0 ds (\partial_\mu \bar{\xi}_n(s\bar{n} + z) \gamma_\perp^\mu + ig \bar{\xi}_n(s\bar{n} + z) \mathcal{A}_{n,\perp}(s\bar{n} + z)) \frac{\not{n}}{2}, \quad (\text{C.13})$$

$$\bar{\eta}_{\bar{n}}(z) = \int_{-\infty}^0 ds n \cdot \partial \bar{\eta}_{\bar{n}}(sn + z) = - \int_{-\infty}^0 ds (\partial_\mu \bar{\xi}_{\bar{n}}(sn + z) \gamma_\perp^\mu + ig \bar{\xi}_{\bar{n}}(sn + z) \mathcal{A}_{\bar{n},\perp}(sn + z)) \frac{\not{n}}{2}. \quad (\text{C.14})$$

Once we have solved the e.o.m. for $\eta_{n(\bar{n})}$ and $\bar{\eta}_{n\bar{n}}$, eqs. (C.11), (C.12), (C.13) and (C.14), the partial derivative of $\xi_{n(\bar{n})}$ and $\bar{\xi}_{n(\bar{n})}$ along $z^{-(+)}$ can be written as

$$\begin{aligned} n \cdot \partial \xi_n(z) &= ign \cdot A_n(z) \xi_n(z) - \frac{\partial_\perp^2}{2} \int_{-\infty}^0 ds \xi_n(s\bar{n} + z) + \frac{ig}{2} \not{\partial}_\perp \int_{-\infty}^0 ds \mathcal{A}_{n,\perp}(s\bar{n} + z) \xi_n(s\bar{n} + z) \\ &\quad + \frac{ig}{2} \mathcal{A}_{n,\perp}(z) \int_{-\infty}^0 ds \not{\partial}_\perp \xi_n(s\bar{n} + z) + \frac{g^2}{2} \int_{-\infty}^0 ds \mathcal{A}_{n,\perp}(z) \mathcal{A}_{n,\perp}(s\bar{n} + z) \xi_n(s\bar{n} + z), \end{aligned} \quad (\text{C.15})$$

$$\begin{aligned} \bar{n} \cdot \partial \xi_{\bar{n}}(z) &= ig\bar{n} \cdot A_{\bar{n}}(z) \xi_{\bar{n}}(z) - \frac{\partial_\perp^2}{2} \int_{-\infty}^0 ds \xi_{\bar{n}}(sn + z) + \frac{ig}{2} \not{\partial}_\perp \int_{-\infty}^0 ds \mathcal{A}_{\bar{n},\perp}(sn + z) \xi_{\bar{n}}(sn + z) \\ &\quad + \frac{ig}{2} \mathcal{A}_{\bar{n},\perp}(z) \int_{-\infty}^0 ds \not{\partial}_\perp \xi_{\bar{n}}(sn + z) + \frac{g^2}{2} \int_{-\infty}^0 ds \mathcal{A}_{\bar{n},\perp}(z) \mathcal{A}_{\bar{n},\perp}(sn + z) \xi_{\bar{n}}(sn + z), \end{aligned} \quad (\text{C.16})$$

¹Even after imposing the light-cone gauge fixing conditions $\bar{n}(n) \cdot A_{n(\bar{n})}(z) = 0$, there exist a residual gauge redundancy due to $z^{+(-)}$ independent transformations, see [184, 209]. This remaining gauge freedom can be removed by the boundary conditions on the background gluon field component.

APPENDIX C. INTERACTION TERMS IN COMPOSITE BACKGROUND QCD

$$\begin{aligned}
n \cdot \partial \bar{\xi}_n(z) = & -ig \bar{\xi}_n(z) n \cdot A_n(z) - \frac{\partial_\perp^2}{2} \int_{-\infty}^0 ds \bar{\xi}_n(s\bar{n} + z) - \frac{ig}{2} \partial_\mu \int_{-\infty}^0 ds \bar{\xi}_n(s\bar{n} + z) A_{n,\perp}(s\bar{n} + z) \gamma_\perp^\mu \\
& - \frac{ig}{2} A_{n,\mu}(z) \int_{-\infty}^0 ds \partial_\nu \bar{\xi}_n(s\bar{n} + z) \gamma_\perp^\nu \gamma_\perp^\mu \\
& + \frac{g^2}{2} \int_{-\infty}^0 ds \bar{\xi}_n(s\bar{n} + z) A_{n,\mu}(z) A_{n,\perp}(s\bar{n} + z) \gamma_\perp^\mu,
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
\bar{n} \cdot \partial \bar{\xi}_{\bar{n}}(z) = & -ig \bar{\xi}_{\bar{n}}(z) \bar{n} \cdot A_{\bar{n}}(z) - \frac{\partial_\perp^2}{2} \int_{-\infty}^0 ds \bar{\xi}_{\bar{n}}(s\bar{n} + z) - \frac{ig}{2} \partial_\mu \int_{-\infty}^0 ds \bar{\xi}_{\bar{n}}(s\bar{n} + z) A_{\bar{n},\perp}(s\bar{n} + z) \gamma_\perp^\mu \\
& - \frac{ig}{2} A_{\bar{n},\mu}(z) \int_{-\infty}^0 ds \partial_\nu \bar{\xi}_{\bar{n}}(s\bar{n} + z) \gamma_\perp^\nu \gamma_\perp^\mu \\
& + \frac{g^2}{2} \int_{-\infty}^0 ds \bar{\xi}_{\bar{n}}(s\bar{n} + z) A_{\bar{n},\mu}(z) A_{\bar{n},\perp}(s\bar{n} + z) \gamma_\perp^\mu.
\end{aligned} \tag{C.18}$$

Contributions proportional to the transverse components of the (anti-)collinear background fields $A_{n(\bar{n},\perp)}^\mu$ lead to operators whose matching onto collinear PDFs is of geometrical twist-3 or higher. The precision of the current work is at geometrical twist-2, therefore we neglect those contributions.

The solutions in eqs. (C.11), (C.12), (C.13) and (C.14) are given in the light-cone gauge. There are situation where it is convenient to deal with solutions in an arbitrary gauge. To obtain those we first replace the transverse gluon field components $A_{n(\bar{n})\perp}^\mu(z)$ with the field strength tensor $\bar{n}_\nu(n_\nu)F_{n(\bar{n})}^{\mu\nu}(z)$ using the following relations in the light-cone gauge [184]

$$A_n^\mu(z) = - \int_{-\infty}^0 ds \bar{n}_\nu F_n^{\mu\nu}(s\bar{n} + z), \quad A_{\bar{n}}^\mu(z) = - \int_{-\infty}^0 ds n_\nu F_{\bar{n}}^{\mu\nu}(sn + z), \tag{C.19}$$

which can be confirmed by writing explicitly the field strength tensor and using the light-cone gauge $\bar{n}(n) \cdot A_{n(\bar{n})} = 0$ together with the boundary condition $A_{n(\bar{n})}^\mu(z^{+(-)} = -\infty) = 0$. Afterwards, we perform the replacements

$$\begin{aligned}
\xi_n(z) & \rightarrow [-\infty\bar{n} + z, z] \xi_n(z), \\
\bar{\xi}_n(z) & \rightarrow \bar{\xi}_n(z)[z, -\infty\bar{n} + z], \\
F_n^{\mu\nu}(z) & \rightarrow [-\infty\bar{n} + z, z] F_n^{\mu\nu}(z)[z, -\infty\bar{n} + z],
\end{aligned} \tag{C.20}$$

and

$$\begin{aligned}
\xi_{\bar{n}}(z) & \rightarrow [-\infty n + z, z] \xi_{\bar{n}}(z), \\
\bar{\xi}_{\bar{n}}(z) & \rightarrow \bar{\xi}_{\bar{n}}(z)[z, -\infty n + z], \\
F_{\bar{n}}^{\mu\nu}(z) & \rightarrow [-\infty n + z, z] F_{\bar{n}}^{\mu\nu}(z)[z, -\infty n + z],
\end{aligned} \tag{C.21}$$

where each operator is multiplied by semi-infinite Wilson lines. Notice that for the field strength tensor we have two Wilson lines in the fundamental representation, using the iden-

C.2. FERMION EQUATION OF MOTIONS (E.O.M.)

tity [71, 73–75, 197, 224]

$$\sum_{l,k=1}^3 [y, x]^{jk} (t^a)_{kl} [x, y]^{li} = \sum_{b=1}^8 [y, x]_{\text{adj}}^{ba} (t^b)_{ji}, \quad (\text{C.22})$$

where $[y, x]_{\text{adj}}^{ba}$ denotes a Wilson line in the adjoint representation one can write

$$([y, x] F^{\mu\nu}(x) [x, y])_{ji} = \sum_{a=1}^8 \sum_{k,l=1}^3 [y, x]^{jk} F_a^{\mu\nu}(x) (t^a)_{kl} [x, y]^{li} = \sum_{a,b=1}^8 F_a^{\mu\nu}(x) [y, x]_{\text{adj}}^{ba} (t^b)_{ji}, \quad (\text{C.23})$$

and thus the field strength tensor can be written in an explicit gauge-invariant form using either Wilson lines in the fundamental [182, 197, 212, 224] or adjoint [50, 183, 184, 198] representation. Thus, using eq. (C.19) and applying the field redefinition given in eqs. (C.20) and (C.21), the solutions to the equations of motions amount to

$$\begin{aligned} \eta_n(z) = & -\frac{\not{n}}{2} \not{\partial}_\perp \int_{-\infty}^0 ds [-\infty \bar{n} + z, s \bar{n} + z] \xi_n(s \bar{n} + z) \\ & - ig \frac{\not{n}}{2} \gamma_\mu^\perp \int_{-\infty}^0 ds \int_{-\infty}^0 ds' [-\infty \bar{n} + z, (s+s') \bar{n} + z] F_n^{\mu-}((s+s') \bar{n} + z \bar{n}) [(s+s') \bar{n} + z, s \bar{n} + z] \xi_n(s \bar{n} + z), \end{aligned} \quad (\text{C.24})$$

$$\begin{aligned} \eta_{\bar{n}}(z) = & -\frac{\not{n}}{2} \not{\partial}_\perp \int_{-\infty}^0 ds [-\infty n + z, sn + z] \xi_{\bar{n}}(sn + z) \\ & - ig \frac{\not{n}}{2} \gamma_\mu^\perp \int_{-\infty}^0 ds \int_{-\infty}^0 ds' [-\infty n + z, (s+s') n + z] F_n^{\mu+}((s+s') n + z \bar{n}) [(s+s') n + z, sn + z] \xi_{\bar{n}}(sn + z), \end{aligned} \quad (\text{C.25})$$

$$\begin{aligned} \bar{\eta}_n(z) = & -\partial_\mu \int_{-\infty}^0 ds \bar{\xi}_n(s \bar{n} + z) [s \bar{n} + z, -\infty \bar{n} + z] \xi_n(s \bar{n} + z) \gamma_\perp^\mu \frac{\not{n}}{2} \\ & + ig \int_{-\infty}^0 ds \int_{-\infty}^0 ds' \bar{\xi}_n(s \bar{n} + z) [s \bar{n} + z, (s+s') \bar{n} + z] F_n^{\mu-}((s+s') \bar{n} + z \bar{n}) [(s+s') \bar{n} + z, -\infty \bar{n} + z] \gamma_\mu^\perp \frac{\not{n}}{2}, \end{aligned} \quad (\text{C.26})$$

$$\begin{aligned} \bar{\eta}_{\bar{n}}(z) = & -\partial_\mu \int_{-\infty}^0 ds \bar{\xi}_{\bar{n}}(sn + z) [sn + z, -\infty n + z] \gamma_\perp^\mu \frac{\not{n}}{2} \\ & + ig \int_{-\infty}^0 ds \int_{-\infty}^0 ds' \bar{\xi}_{\bar{n}}(sn + z) [sn + z, (s+s') n + z] F_n^{\mu+}((s+s') n + z \bar{n}) [(s+s') n + z, -\infty n + z] \gamma_\mu^\perp \frac{\not{n}}{2}. \end{aligned} \quad (\text{C.27})$$

The solutions to e.o.m. given in eqs. (C.24), (C.25), (C.26) and (C.27) are valid in any gauge.² The contributions proportional to the field strength tensor have geometrical twist 2 and leads to TMDPDF operators of geometrical twist 3. Therefore their matching onto collinear PDFs starts at geometrical twist-3.

²As long as the gauge fixing is not a singular gauge, there is not need to include transverse Wilson lines, see [43, 209, 210]. For the case of light-cone gauge, despite being a singular gauge, one can still avoid the introduction of transverse Wilson lines using the appropriate boundary conditions [183, 184, 209].

D Position Space Formalism

D.1 Product of fields

Time ordered product of two fields Using the Wick's theorem one can show that the time-ordered product of two free vectors fields, $T\{B_a^\mu(x)B_b^\nu(y)\}$, can be written as [64, 192, 193]

$$T\{B_a^\mu(x)B_b^\nu(y)\} =: B_a^\mu(x)B_b^\nu(y) : + \overline{B_a^\mu(x)B_b^\nu(y)} =: B_a^\mu(x)B_b^\nu(y) : + \Delta_{ab}^{\mu\nu}(x-y)\mathbb{1}, \quad (\text{D.1})$$

where $: \cdot :$ is the normal-ordered product of fields, a, b are gauge indices in the adjoint representation, $\mathbb{1}$ is the identity operator and $\Delta_{ab}^{\mu\nu}(x-y)$ is given by eq. (D.8). In the case of anti-time ordered one replaces the $i0^+$ prescription in $\Delta_{ab}^{\mu\nu}(x-y)$ by $-i0^+$. In the case of fermionic free fields one obtains

$$T\{\psi_i(x)\bar{\psi}_j(y)\} =: \psi_i(x)\bar{\psi}_j(y) : + \overline{\psi_i(x)\bar{\psi}_j(y)} =: \psi_i(x)\bar{\psi}_j(y) : + S_{ij}(x-y)\mathbb{1}, \quad (\text{D.2})$$

where $i(j)$ is a gauge index in the (anti-)fundamental representation and $S_{ij}(x-y)$ is given by eq. (D.11). In the case of anti-time-ordered one replaces the $i0^+$ prescription in $S_{ij}(x-y)$ by $-i0^+$.

Product of fields When two free fields are not (anti-)time-ordered it is still possible to write their product as the fields normal-ordered plus the singular contributions when $x^\mu \rightarrow y^\mu$. The product of free gauge vector fields can be written as¹

$$B_a^\mu(x)B_b^\nu(y) =: B_a^\mu(x)B_b^\nu(y) : + \overline{B_a^\mu(x)B_b^\nu(y)} =: B_a^\mu(x)B_b^\nu(y) : + \Delta_{ab,+}^{\mu\nu}(x-y), \quad (\text{D.3})$$

where $\Delta_{ab,+}^{\mu\nu}(x-y)$ is given in the Feynman gauge by

$$\Delta_{ab,+}^{\mu\nu}(x-y) = -g^{\mu\nu}\delta_{ab} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) e^{-ik \cdot (x-y)} = \frac{1}{(2\pi)^2} \frac{ig^{\mu\nu}\delta_{ab}}{[-(x-y)^2 + i(x^0 - y^0)0^+]}, \quad (\text{D.4})$$

¹It is a straight consequence of the free field solution

$$B_b^\mu(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k^0} \sum_\lambda \left(a_b^{(\lambda)}(k) \epsilon_{(\lambda)}^\mu e^{-ik \cdot x} + a_b^{(\lambda)\dagger} \epsilon_{(\lambda)}^{\mu*} e^{ik \cdot x} \right), \quad k^0 = |\mathbf{k}|,$$

and the commutation relations

$$[a_b^{(\lambda)}(k), a_{b'}^{(\lambda')\dagger}(k')] = -2k^0 \delta_{bb'} g^{\lambda\lambda'} \delta^{(3)}(\mathbf{k}' - \mathbf{k}), \quad [a_b^{(\lambda)}(k), a_{b'}^{(\lambda')}(k')] = 0, \quad [a_b^{(\lambda)\dagger}(k), a_{b'}^{(\lambda')\dagger}(k')] = 0.$$

APPENDIX D. POSITION SPACE FORMALISM

which can be confirmed by taking the Fourier transform. For free fermions one obtains²

$$\begin{aligned}\psi_i(x)\bar{\psi}_j(y) &=:\psi_i(x)\bar{\psi}_j(y): + \overbrace{\psi_i(x)\bar{\psi}_j(y)} =:\psi_i(x)\bar{\psi}_j(y): + S_{ij,+}(x-y)\mathbb{1}, \\ \bar{\psi}_j(y)\psi_i(x) &=:\bar{\psi}_j(y)\psi_i(x): + \overbrace{\bar{\psi}_j(y)\psi_i(x)} =:\bar{\psi}_j(y)\psi_i(x): + S_{ij,-}(x-y)\mathbb{1},\end{aligned}\tag{D.5}$$

and $S_{ij,\pm}(x-y)$ are given by

$$\begin{aligned}S_{ij,+}(x-y) &= \delta_{ij} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) \not{k} e^{-ik \cdot (x-y)} = \frac{2}{(2\pi)^2} \frac{i\delta_{ij}(\not{x} - \not{y})}{[-(x-y)^2 + i(x^0 - y^0)0+]^2}, \\ S_{ij,-}(x-y) &= \delta_{ij} \int \frac{d^d k}{(2\pi)^{d-1}} \theta(-k^0) \delta(k^2) \not{k} e^{-ik \cdot (x-y)} = \frac{2}{(2\pi)^2} \frac{i\delta_{ij}(\not{x} - \not{y})}{[-(x-y)^2 - i(x^0 - y^0)0+]^2},\end{aligned}\tag{D.6}$$

which can be verified by taking the Fourier transform.

D.2 Propagators in position space

Hard gluon The free hard gluon propagator in a covariant gauge is determined by the differential equation [64, 192, 193]

$$\left[\partial^\alpha \partial_\alpha g_\mu{}^\beta + \frac{1-\xi}{\xi} \partial_\mu \partial^\beta \right] \Delta_{ab}^{\mu\nu}(x-y) = i g_{\mu\nu} \delta_{ab} \delta^{(d)}(x-y),\tag{D.7}$$

where the partial derivatives are understood with respect x^μ . Its solution in position space amounts to

$$\begin{aligned}\Delta_{ab}^{\mu\nu}(x-y) &= -\frac{1}{(2\pi)^2} \frac{\delta_{ab}}{[-(x-y)^2 + i0^+]} \\ &\quad \times \left[g^{\mu\nu} + (1-\xi) \left\{ 2g^{\mu\nu} + \frac{(x-y)^\mu (x-y)^\nu}{-(x-y)^2 + i0^+} \right\} \right],\end{aligned}\tag{D.8}$$

where

$$\xi = \begin{cases} \xi = 1, & \text{Feynman-Fermi gauge} \\ \xi = 0, & \text{Landau gauge} \end{cases}\tag{D.9}$$

²It is a straight consequence of free fermionic field solutions

$$\begin{aligned}\psi_i(x) &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k^0} \sum_s \left(a_i^{(s)}(k) u^{(s)}(k) e^{-ik \cdot x} + b_i^{(s)}(k) v^{(s)}(k) e^{ik \cdot x} \right), \\ \bar{\psi}_i(x) &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k^0} \sum_s \left(a_i^{(s)\dagger}(k) \bar{u}^{(s)}(k) e^{ik \cdot x} + b_i^{(s)\dagger}(k) \bar{v}^{(s)}(k) e^{-ik \cdot x} \right),\end{aligned}$$

where $k^0 = |\mathbf{k}|$ and the anti-commutation relations

$$\{a_i^{(s)}(k), a_j^{(s')\dagger}(k')\} = \{b_i^{(s)}(k), b_j^{(s')\dagger}(k')\} = (2\pi)^3 2k^0 \delta_{ij} \delta^{ss'} \delta^{(3)}(\mathbf{k} - \mathbf{k}').$$

Hard fermion The free hard quark propagator is determined by the differential equation

$$i\cancel{\partial}_x S_{ij}(x-y) = i\delta_{ij}\delta^{(d)}(x-y), \quad (\text{D.10})$$

whose explicit solution in position space is given by

$$S_{ij}(x-y) = \frac{2}{(2\pi)^2} \frac{i\delta_{ij}(\cancel{x}-\cancel{y})}{[-(x-y)^2 + i0^+]^2}. \quad (\text{D.11})$$

D.3 LSZ reduction formula

Vector bosons For the massless vector bosons

$$\begin{aligned} \langle p_n^{\epsilon_n}, \dots, p_1^{\epsilon_1} | \mathcal{O}(\{z\}) | k_1^{\epsilon_1}, \dots, k_m^{\epsilon_m} \rangle &= \text{disc.} + (-i)^{n+m} Z_3^{\frac{n+m}{2}} \int \prod_{i=1}^n \prod_{j=1}^m d^d x_i d^d y_j e^{ip_i \cdot x_i} e^{-ik_j \cdot y_j} \bar{\epsilon}_{\alpha_i} \epsilon_{\mu_j} \hat{\square}_{x_i}^{\alpha_i \beta_i} \hat{\square}_{y_j}^{\mu_j \nu_j} \\ &\times \langle 0 | T \{ B^{\beta_n}(x_n) \dots B^{\beta_1}(x_1) \mathcal{O}(\{z\}) B^{\nu_1}(y_1) \dots B^{\nu_m}(y_m) \} | 0 \rangle, \end{aligned} \quad (\text{D.12})$$

where disc. denotes the disconnected contributions; $k(p)_i^\epsilon$ and p_i^ϵ denotes a massless vector boson with helicity ϵ_i and momentum $k(p)_i$; the differential operator $\hat{\square}_{x_i}^{\mu_i \nu_i}$ is defined as:

$$\hat{\square}_{x_i}^{\mu_i \nu_i} = \partial^\alpha \partial_\alpha g^{\mu_i \nu_i} + \frac{1-\xi}{\xi} \partial^{\mu_i} \partial^{\nu_i},$$

and the subscript labels over which coordinates the partial derivatives are acting; $\mathcal{O}(\{z\})$ is a monomial of fields which can be evaluated either at the same or different space-time points and $\{z\}$ denotes the set of points at which each field in $\mathcal{O}$ is evaluated.

Fermions For massless fermions

$$\begin{aligned} \langle \bar{p}_1, \dots, \bar{p}_s, p_1, \dots, p_r | \mathcal{O}(\{z\}) | k_1, \dots, k_n, \bar{k}_1, \dots, \bar{k}_m \rangle &= \text{disc.} \\ &+ (-i)^{n+r} i^{m+s} Z_2^{\frac{n+m+r+s}{2}} \int \prod_{i=1}^n \prod_{j=1}^m \prod_{l=1}^r \prod_{q=1}^s d^d \bar{y}_q d^d y_l d^d \bar{x}_i d^d \bar{x}_j e^{-ik_i \cdot x_i} e^{-i\bar{k}_j \cdot \bar{x}_j} e^{ip_l \cdot y_l} e^{i\bar{p}_q \cdot \bar{y}_q} \\ &\times (i\cancel{\partial}_{\bar{x}_j} \bar{v}(\bar{k}_j)) (i\cancel{\partial}_{y_l} \bar{u}(p_l)) \langle 0 | T \{ \bar{\psi}(\bar{y}_q) \dots \bar{\psi}(\bar{y}_s) \psi(y_1) \dots \psi(y_r) \mathcal{O}(\{z\}) \\ &\times \bar{\psi}(x_1) \dots \bar{\psi}(x_n) \psi(\bar{x}_1) \dots \psi(\bar{x}_m) \} | 0 \rangle (-i\overleftarrow{\cancel{\partial}}_{x_i} u(k_i)) (-i\overleftarrow{\cancel{\partial}}_{\bar{y}_q} v(\bar{p}_q)), \end{aligned} \quad (\text{D.13})$$

where disc. denotes the disconnected contributions; $k(\bar{k})_i$ labels a (anti-)fermion; subscripts in the partial derivatives denote over which coordinate they act on and $\mathcal{O}(\{z\})$ is a monomial of fields which can be evaluated either at the same or different space-time points and $\{z\}$ denotes the set of points at which each field in $\mathcal{O}$ is evaluated.

D.4 Useful integrals

D.4.1 Quadratic form diagonalization

Often in the calculation of position space Feynman integrals the denominator takes the form

$$-a_1(y_1 - z_1)^2 - a_2(y_2 - z_2)^2 - a_3(y_2 - y_1)^2 + i0, \quad (\text{D.14})$$

which is just simply a quadratic form. Using matrix notation it can be rewritten as

$$\begin{aligned} -a_1(y_1 - z_1)^2 - a_2(y_2 - z_2)^2 - a_3(y_2 - y_1)^2 + i0 = & - (y_1^\mu, y_2^\mu) \begin{pmatrix} a_1 + a_3 & -a_3 \\ -a_3 & a_2 + a_3 \end{pmatrix} \begin{pmatrix} y_1^\mu \\ y_2^\mu \end{pmatrix} \\ & + 2(a_1 z_1^\mu, a_2 z_2^\mu) \begin{pmatrix} y_1^\mu \\ y_2^\mu \end{pmatrix} - a_1 z_1^2 - a_2 z_2^2 + i0. \end{aligned} \quad (\text{D.15})$$

For the sake of notation simplicity we define the matrix

$$A = \begin{pmatrix} a_1 + a_3 & -a_3 \\ -a_3 & a_2 + a_3 \end{pmatrix}, \quad (\text{D.16})$$

and the *vectors* $\tilde{z}^\mu = (a_1 z_1^\mu, a_2 z_2^\mu)$ and $y = (y_1^\mu, y_2^\mu)$. It just take standard algebra to show

$$\begin{aligned} -a_1(y_1 - z_1)^2 - a_2(y_2 - z_2)^2 - a_3(y_2 - y_1)^2 = & - (y - A^{-1} \tilde{z})^t A (y - A^{-1} \tilde{z}) \\ & + \tilde{z}^t A^{-1} \tilde{z} - a_1 z_1^2 - a_2 z_2^2 + i0 \end{aligned} \quad (\text{D.17})$$

where the *vectors* and matrix multiplication is understood in the Feynman's parameter space and the superscript t denotes the transpose. Performing the change of variables $y \rightarrow y + A^{-1} \tilde{z}$ the quadratic form simplifies to $-y^t A y + \tilde{z}^t A^{-1} \tilde{z} - a_1 z_1^2 - a_2 z_2^2 + i0$, where the inverse matrix A^{-1} is given by

$$A^{-1} = \frac{1}{a_1 a_2 + a_1 a_3 + a_2 a_3} \begin{pmatrix} a_2 + a_3 & a_3 \\ a_3 & a_1 + a_2 \end{pmatrix}. \quad (\text{D.18})$$

The combination $\tilde{z}^t A^{-1} \tilde{z} - a_1 z_1^2 - a_2 z_2^2$ amounts to

$$\tilde{z}^t A^{-1} \tilde{z} - a_1 z_1^2 - a_2 z_2^2 = -\frac{a_1 a_2 a_3 (z_2 - z_1)^2}{a_1 a_2 + a_1 a_3 + a_2 a_3}. \quad (\text{D.19})$$

The last step is to diagonalize the matrix A , i.e. $A = P D P^{-1}$, where D is a diagonal matrix with real entries since A is symmetric. P has the eigenvectors as columns

$$D = \begin{pmatrix} \frac{a_3 - \sqrt{(a_1 - a_2)^2 + 4a_3^2}}{2} & 0 \\ 0 & \frac{a_3 + \sqrt{(a_1 - a_2)^2 + 4a_3^2}}{2} \end{pmatrix}, \quad (\text{D.20})$$

$$P = \begin{pmatrix} -\frac{a_1 - a_2 - \sqrt{(a_1 - a_2)^2 + 4a_3^2}}{\sqrt{8a_3^2 + 2(a_1 - a_2)(a_1 - a_2 - \sqrt{(a_1 - a_2)^2 + 4a_3^2})}} & -\frac{a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + 4a_3^2}}{\sqrt{8a_3^2 + 2(a_1 - a_2)(a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + 4a_3^2})}} \\ \frac{2a_3}{\sqrt{8a_3^2 + 2(a_1 - a_2)(a_1 - a_2 - \sqrt{(a_1 - a_2)^2 + 4a_3^2})}} & \frac{2a_3}{\sqrt{8a_3^2 + 2(a_1 - a_2)(a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + 4a_3^2})}} \end{pmatrix}. \quad (\text{D.21})$$

Therefore the quadratic term can be written as $y^t A y = y^t P D P^{-1} y = (P^t y)^t D P^t y$, where we have used $P^{-1} = P^t$. Doing the change of variables $y \rightarrow P y$ the quadratic form in eq.(D.14) becomes

$$-\frac{(a_3 - \sqrt{(a_1 - a_2)^2 + 4a_3^2})y_1^2}{2} - \frac{(a_3 + \sqrt{(a_1 - a_2)^2 + 4a_3^2})y_2^2}{2} - \frac{a_1 a_2 a_3 (z_2 - z_1)^2}{a_1 a_2 + a_1 a_3 + a_2 a_3} + i0. \quad (\text{D.22})$$

Notice that the jacobian of both change of variables, i.e $y \rightarrow y + A^{-1} \tilde{z}$ first and $y \rightarrow P y$ afterwards, is one. Thus any extra factor appears in the integration measure in the Feynman integrals.

D.4.2 $I_2(n; m_1, m_2)$

The integral denoted by $I(n; m_1, m_3)$ is characteristic of ladder topologies with only two background field sources. It is given by

$$I_2(n; m_1, m_2) = \int_{-\infty}^{\infty} d^d y_1 d^d y_2 \frac{(y_1)^{m_1} (y_2)^{m_2}}{[-\lambda_1 y_1^2 - \lambda_2 y_2^2 - X^2 + i0]^n}. \quad (\text{D.23})$$

Using Schwinger's parametrization the integral over y_1 and y_2 becomes just a d -dimensional gaussian integral

$$\begin{aligned} I(n; m_1, m_2) &= \frac{(-i)^n}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-1} e^{i\alpha(-X^2 + i0)} \int_{-\infty}^{\infty} d^d y_1 d^d y_2 (y_1^2)^{m_1} (y_2^2)^{m_2} e^{-i\alpha\lambda_1 y_1^2 - i\alpha\lambda_2 y_2^2} \\ &= \frac{(-i)^{n+m_1+m_2} i^{d-2} \pi^d}{\Gamma[n]} \frac{\Gamma[\frac{d}{2} + m_1]}{\Gamma[\frac{d}{2}]} \frac{\Gamma[\frac{d}{2} + m_2]}{\Gamma[\frac{d}{2}]} \lambda_1^{-\frac{d}{2}-m_1} \lambda_2^{-\frac{d}{2}-m_2} \\ &\quad \times \int_0^\infty d\alpha \alpha^{n-m_1-m_2-d-1} e^{i\alpha(-X^2 + i0)} \end{aligned} \quad (\text{D.24})$$

To go from the first line to the second line we have used

$$\int_{-\infty}^{\infty} d^d y e^{-\sigma y^2} = \sqrt{\frac{\pi}{\sigma}} \left(\sqrt{\frac{\pi}{-\sigma}} \right)^{d-1} = (-1)^{\frac{1-d}{2}} \left(\frac{\pi}{\sigma} \right)^{\frac{d}{2}}, \quad (\text{D.25})$$

$$\begin{aligned} \int_{-\infty}^{\infty} d^d y (y^2)^m e^{-\sigma y^2} &= (-1)^m \frac{d^d}{d\sigma^m} \int_{-\infty}^{\infty} d^d y e^{-\sigma y^2} = (-1)^m \frac{d^m}{d\sigma^m} \sqrt{\frac{\pi}{\sigma}} \left(\sqrt{\frac{\pi}{-\sigma}} \right)^{d-1} \\ &= (-1)^{m+\frac{1-d}{2}} \frac{d}{2} \left(\frac{d}{2} + 1 \right) \left(\frac{d}{2} + 2 \right) \cdots \left(\frac{d}{2} + m - 1 \right) \pi^{\frac{d}{2}} \sigma^{-\frac{d}{2}-m} \\ &= (-1)^{m+\frac{1-d}{2}} \frac{\Gamma[\frac{d}{2} + m]}{\Gamma[\frac{d}{2}]} \pi^{\frac{d}{2}} \sigma^{-\frac{d}{2}-m}, \end{aligned} \quad (\text{D.26})$$

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and replaced $\sigma = i\alpha$. Performing the integral over α we obtain

$$I_2(n; m_1, m_2) = \frac{(-i)^{2m_1+2m_2+d} i^{d-2} \pi^d}{\Gamma[n]} \frac{\Gamma[\frac{d}{2} + m_1]}{\Gamma[\frac{d}{2}]} \frac{\Gamma[\frac{d}{2} + m_2]}{\Gamma[\frac{d}{2}]} \lambda_1^{-\frac{d}{2}-m_1} \lambda_2^{-\frac{d}{2}-m_2} \times \frac{\Gamma[n - m_1 - m_2 - d]}{[-X^2 + i0]^{n-m_1-m_2-d}}. \quad (\text{D.27})$$

The present of the factor $i0$ is critical in order to not spoil the analytical structure during the integration process. It must be keep in every position space calculations, only at the end when we perform the Fourier transform to momentum space it can be safely removed.

D.4.3 $I_1(n; m)$

The integral $I_1(n; m)$ is characteristic of topologies involving interaction with a Wilson line and the presence of two background fields

$$I_1(n; m) = \int_{-\infty}^{\infty} d^d y \frac{(y^2)^m}{[-\lambda y^2 - X^2 + i0]^n}. \quad (\text{D.28})$$

Using Schwinger's parameters it takes the form of a gaussian integral

$$\begin{aligned} I_1(n; m) &= \frac{(-i)^n}{\Gamma[n]} \int_0^{\infty} d\alpha \alpha^{n-1} e^{i\alpha(-X^2+i0)} \int_{-\infty}^{\infty} d^d y (y^2)^m e^{-i\alpha\lambda y^2} \\ &= \frac{(-i)^{n+m} i^{\frac{d-2}{2}} \pi^{\frac{d}{2}}}{\Gamma[n]} \frac{\Gamma[\frac{d}{2} + m]}{\Gamma[\frac{d}{2}]} \lambda^{-\frac{d}{2}-m} \int_0^{\infty} d\alpha \alpha^{n-m-\frac{d}{2}-1} e^{i\alpha(-X^2+i0)}, \end{aligned} \quad (\text{D.29})$$

in the sept from the first to the second line we have used eq. (D.26). The integral over α leads to the result

$$I_1(n, m) = \frac{(-i)^{2m+\frac{d}{2}} i^{\frac{d-2}{2}} \pi^{\frac{d}{2}}}{\Gamma[n]} \frac{\Gamma[\frac{d}{2} + m]}{\Gamma[\frac{d}{2}]} \lambda^{-\frac{d}{2}-m} \frac{\Gamma[n - m - \frac{d}{2}]}{[-X^2 + i0]^{n-m-\frac{d}{2}}}. \quad (\text{D.30})$$

Again, the $i0$ must be kept along all the position space calculation.

D.5 Useful Fourier's transforms

In this section we present some Fourier's transform integrals that often appear to pass from position to momentum space.

D.5.1 Wilson coefficient at $y \rightarrow x$.

We prove here the statement: *a short distance expansion is a large momentum expansion*. Let us consider the Wilson coefficient $C_i(y - x)$ with units mass to the $m + 2n$ power, where $m, n \in \mathbb{Q}^{+3}$. In the limit $y \rightarrow x$ it behaves as [16]

$$\lim_{y \rightarrow x} C_i(y - x) = \frac{\Lambda^m}{[-(y - x)^2 + i0]^n}, \quad (\text{D.31})$$

³For unpolarized Drell-Yan it simplifies to $m, n \in \mathbb{Z}^+$

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where Λ is a constant with unit of mass. The energy-momentum conservation imposes translational invariance, therefore C_i must be a function of $y^\mu - x^\mu$. Covariant invariance imposes that the dependency on the relative separation must be of the form $[-(y - x)^2 + i0]^n$, where n is a rational number. Translational invariance allows us to set $x = 0$ without loss of generality. The Fourier transform of eq. (D.31) with respect y is given by

$$\mathcal{F} \left[\lim_{y \rightarrow 0} C_i(y) \right] = \int \frac{d^d y}{(2\pi)^d} e^{-iq \cdot y} \frac{\Lambda^m}{[-y^2 + i0]^n} = \frac{(-i)^n}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-1} \int \frac{d^d y}{(2\pi)^d} e^{i\alpha[-y^2 + i0] - iq \cdot y}, \quad (\text{D.32})$$

where in the second equality we used Schwinger's parametrization. Performing the shift $y^\mu \rightarrow y - \frac{q^\mu}{2\alpha}$ one can write

$$\mathcal{F} \left[\lim_{y \rightarrow 0} C_i(y) \right] = \frac{(-i)^n \Lambda^m}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-1} e^{i\frac{q^2 + i0}{4\alpha}} \int \frac{d^d y}{(2\pi)^d} e^{-i\alpha y^2}, \quad (\text{D.33})$$

the integral over y is nothing but a gaussian integral leading to

$$\begin{aligned} \mathcal{F} \left[\lim_{y \rightarrow 0} C_i(y) \right] &= \frac{(-i)^n \Lambda^m \pi^{\frac{d}{2}}}{\Gamma[n] (2\pi)^d} \int_0^\infty d\alpha \alpha^{n-1-\frac{d}{2}} e^{i\frac{q^2 + i0}{4\alpha}} \\ &= i \frac{\Lambda^m}{4^n \pi^{\frac{d}{2}} \Gamma[n]} \frac{\Gamma[\frac{d}{2} - n]}{[-q^2 - i0]^{\frac{d}{2} - n}}, \end{aligned} \quad (\text{D.34})$$

from eq. (D.34) we can conclude that a contribution of the form $[-q^2 - i0]^n$ dominates in momentum space when the leading contribution in position space is given by $[-y^2 + i0]^{-n}$.

D.5.2 Transverse coordinate Fourier transform

We start with the scalar integral

$$\mathcal{F}_{\mathbf{q}_T} \left(\frac{1}{[\mathbf{b}_T^2]^n} \right) = \int_0^\infty \frac{d^{d-2} \mathbf{b}_T}{(2\pi)^{d-2}} \frac{e^{i\mathbf{b}_T \cdot \mathbf{q}_T}}{[\mathbf{b}_T^2]^n}, \quad (\text{D.35})$$

as usual, using Schwinger's parameters it can be written as a standard gaussian integral

$$\begin{aligned} \mathcal{F}_{\mathbf{q}_T} \left(\frac{1}{[\mathbf{b}_T^2]^n} \right) &= \frac{1}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-1} \int_{-\infty}^\infty \frac{d^{d-2} \mathbf{b}_T}{(2\pi)^{d-2}} e^{i\mathbf{b}_T \cdot \mathbf{q}_T - \alpha \mathbf{b}_T^2} \\ &= \frac{\pi^{\frac{d-2}{2}}}{\Gamma[n]} \frac{1}{(2\pi)^{d-2}} \int_0^\infty d\alpha \alpha^{n-1-\frac{d-2}{2}} e^{-\frac{\mathbf{q}_T^2}{4\alpha}} = \frac{4^{\frac{d-2}{2}-n} \pi^{\frac{d-2}{2}} \Gamma[\frac{d-2}{2} - n]}{(2\pi)^{d-2} \Gamma[n]} [\mathbf{q}_T^2]^{n-\frac{d-2}{2}}. \end{aligned} \quad (\text{D.36})$$

The Fourier transform in (D.36) is characteristic of leading power calculations.

At NnLP it often appears tensorial integrals of the form

$$\mathcal{F} \left(\frac{\mathbf{b}_T^{\mu_1} \mathbf{b}_T^{\mu_2} \dots \mathbf{b}_T^{\mu_m}}{[\mathbf{b}_T^2]^2} \right) = \int_{-\infty}^\infty \frac{d^{d-2} \mathbf{b}_T}{(2\pi)^{d-2}} \frac{\mathbf{b}_T^{\mu_1} \mathbf{b}_T^{\mu_2} \dots \mathbf{b}_T^{\mu_m}}{[\mathbf{b}_T^2]^n} e^{i\mathbf{b}_T \cdot \mathbf{q}_T}, \quad (\text{D.37})$$

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which can be written as an differential tensor operator acting over a scalar integral

$$\begin{aligned}\mathcal{F}\left(\frac{\mathbf{b}_T^{\mu_1}\mathbf{b}_T^{\mu_2}\cdots\mathbf{b}_T^{\mu_m}}{[\mathbf{b}_T^2]}\right) &= (-i)^m \frac{d}{d\mathbf{q}_T^{\mu_1}} \cdots \frac{d}{d\mathbf{q}_T^{\mu_m}} \int_{-\infty}^{\infty} \frac{d^{d-2}\mathbf{b}_T}{(2\pi)^{d-2}} \frac{e^{i\mathbf{b}_T\cdot\mathbf{b}_T}}{[\mathbf{b}_T^2]^n} \\ &= \mathbf{q}_T^{\mu_1} \cdots \mathbf{q}_T^{\mu_m} \frac{(-i)^m 4^{\frac{d-2}{2}-n+2m} \pi^{\frac{d-2}{2}} \Gamma[\frac{d-2}{2}-n]}{(2\pi)^{d-2} \Gamma[n]} [\mathbf{q}_T^2]^{n-m-\frac{d-2}{2}}.\end{aligned}\quad (\text{D.38})$$

Another Fourier transform that we often need to compute is the following

$$\mathcal{F}\left\{\frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}\right\} = \int \frac{d^{d-2}\mathbf{z}_T}{(2\pi)^{d-2}} e^{-i\mathbf{k}\cdot\mathbf{z}_T} \frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}, \quad (\text{D.39})$$

the procedure is similar to the previous ones, using Schwinger's parameters one can write eq. (D.39) as

$$\mathcal{F}\left\{\frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}\right\} = \int_0^\infty d\alpha \frac{(-i)^n \alpha^{n-1}}{\Gamma[n]} \int \frac{d^{d-2}\mathbf{z}_T}{(2\pi)^{d-2}} e^{i\alpha[-M^2 + \mathbf{z}_T^2 + i0^+] - i\mathbf{k}_T\cdot\mathbf{z}_T} z^\mu, \quad (\text{D.40})$$

doing the change of variable $\mathbf{z}_T^\mu \rightarrow \mathbf{z}_T^\mu + \frac{\mathbf{k}_T^\mu}{2\alpha}$ one obtains

$$\begin{aligned}\mathcal{F}\left\{\frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}\right\} &= \int_0^\infty d\alpha \frac{(-i)^n \alpha^{n-1}}{\Gamma[n]} \\ &\times \int \frac{d^{d-2}\mathbf{z}_T}{(2\pi)^{d-2}} e^{i\alpha[-M^2 + i0^+] + i\alpha\mathbf{z}_T^2 - i\frac{\mathbf{k}_T^2}{4\alpha}} \left(z^+ \bar{n}^\mu + z^- n^\mu + \mathbf{z}_T^\mu + \frac{\mathbf{k}_T^\mu}{2\alpha}\right),\end{aligned}\quad (\text{D.41})$$

which is a simply Gaussian integral in $\mathbf{z}_T$, it gives

$$\begin{aligned}\mathcal{F}\left\{\frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}\right\} &= \frac{1}{(4\pi)^{\frac{d-2}{2}}} \int_0^\infty d\alpha \frac{(-i)^{n-\frac{d-2}{2}} \alpha^{n-\frac{d-2}{2}-1}}{\Gamma[n]} e^{i\alpha[-M^2 + i0^+] - i\frac{\mathbf{k}_T^2}{4\alpha}} \\ &\times \left(z^+ \bar{n}^\mu + z^- n^\mu + \frac{\mathbf{k}_T^\mu}{2\alpha}\right),\end{aligned}\quad (\text{D.42})$$

using the integral representation for the modified Bessel function of second kind (sometimes also called McDonald's function)

$$K_m(z) = \frac{1}{2} \left(\frac{z}{2}\right)^m \int_0^\infty dt e^{-t-\frac{z^2}{4t}} t^{-1-m}, \quad (\text{D.43})$$

one can easily evaluate the integral in α in eq. (D.42)

$$\begin{aligned}\mathcal{F}\left\{\frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n}\right\} &= -\frac{2^{1+\frac{d-2}{2}-n}}{\Gamma[n](4\pi)^{\frac{d-2}{2}}} \left(\sqrt{\frac{\mathbf{k}_T^2}{-M^2 + i0^+}}\right)^{-\frac{d-2}{2}+n} \\ &\times \left[(z^+ \bar{n}^\mu + z^- n^\mu) K_{\frac{d-2}{2}-n}\left(\sqrt{\mathbf{k}_T^2(-M^2 + i0^+)}\right) - i\mathbf{k}_T^\mu \sqrt{\frac{-M^2 + i0^+}{\mathbf{k}_T^2}} K_{\frac{d-2}{2}+1-n}\left(\sqrt{\mathbf{k}_T^2(-M^2 + i0^+)}\right)\right].\end{aligned}\quad (\text{D.44})$$

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If one replace the $i0^+$ prescription in eq. (D.39) by $-i0^+$ then eq. (D.42) becomes

$$\mathcal{F} \left\{ \frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 - i0^+]^n} \right\} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \int_0^\infty d\alpha \frac{i^{n-\frac{d-2}{2}} \alpha^{n-\frac{d-2}{2}-1}}{\Gamma[n]} e^{-i\alpha[-M^2-i0^+]+i\frac{\mathbf{k}_T^2}{4\alpha}} \times \left(z^+ \bar{n}^\mu + z^- n^\mu - \frac{\mathbf{k}_T^\mu}{2\alpha} \right), \quad (\text{D.45})$$

notice the change of sign in from of $\mathbf{k}_T^\mu$ since the change of variables is now $\mathbf{z}_T^\mu \rightarrow \mathbf{z}_T^\mu - \frac{\mathbf{k}_T^\mu}{2\alpha}$. Integration in α one obtains

$$\mathcal{F} \left\{ \frac{z^\mu}{[-M^2 + \mathbf{z}_T^2 + i0^+]^n} \right\} = -\frac{2^{1+\frac{d-2}{2}-n}}{\Gamma[n](4\pi)^{\frac{d-2}{2}}} \left(\sqrt{\frac{\mathbf{k}_T^2}{-M^2 - i0^+}} \right)^{-\frac{d-2}{2}+n} \times \left[(z^+ \bar{n}^\mu + z^- n^\mu) K_{\frac{d-2}{2}-n} \left(\sqrt{\mathbf{k}_T^2} (-M^2 - i0^+) \right) + i \mathbf{k}_T^\mu \sqrt{\frac{-M^2 - i0^+}{\mathbf{k}_T^2}} K_{\frac{d-2}{2}+1-n} \left(\sqrt{\mathbf{k}_T^2} (-M^2 - i0^+) \right) \right]. \quad (\text{D.46})$$

D.5.3 Collinear and anti-collinear coordinate Fourier transform

Typically, in the pass from position space to momentum in

$$\mathcal{F} \left(\frac{1}{[-2z^+ z^- + i0]^n} \right) = \int_{-\infty}^\infty dz^+ dz^- \frac{e^{-iq^+ z^- - iq^- z^+}}{[-2z^+ z^- + i0]^n}, \quad (\text{D.47})$$

with the help of Schwinger's parameters it can be written as a 2 dimensional gaussian integral in z^+ and z^- . It takes the form

$$\begin{aligned} \mathcal{F} \left(\frac{1}{[-2z^+ z^- + i0]^n} \right) &= \frac{(-i)^n}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-1} \int_{-\infty}^\infty dz^+ dz^- e^{-2i\alpha z^+ z^- - iq^+ z^- - iq^- z^+} \\ &= \frac{(-i)^n \pi}{\Gamma[n]} \int_0^\infty d\alpha \alpha^{n-2} e^{\frac{i}{4\alpha} [2q^+ q^- + i0]}, \end{aligned} \quad (\text{D.48})$$

from first to second line we wrote the argument in complex exponential as

$$\begin{aligned} -2i\alpha z^+ z^- - iq^+ z^- - iq^- z^+ &= -i\alpha (z^+, z^-) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z^+ \\ z^- \end{pmatrix} - i(q^-, q^+) \begin{pmatrix} z^+ \\ z^- \end{pmatrix} \\ &= -i(z - A^{-1} \tilde{q})^t A (z - A^{-1} \tilde{q}) + i \tilde{q}^t A^{-1} \tilde{q}, \end{aligned} \quad (\text{D.49})$$

where we defined $\tilde{q} = (\frac{q^-}{\alpha}, \frac{q^+}{\alpha})$, $z = z^+, z^-$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\text{D.50})$$

performing the shift $z \rightarrow z + A^{-1} \tilde{q}$ the quadratic form takes the simpler form

$$-i\alpha z^t A z + \frac{i}{4\alpha} [2q^+ q^- + i0], \quad (\text{D.51})$$

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where we have developed $\tilde{q}^t A^{-1} \tilde{q}$ and introduce an infinitesimal positive imaginary part. Once the exponential has been written in a more convenient form we use the gaussian integral

$$\int_{-\infty}^{\infty} dz_1 \cdots dz_m e^{-z^t A z} = \frac{\pi^{\frac{m}{2}}}{\text{Det}(A)}, \quad (\text{D.52})$$

where $z = (z_1, \cdots, z_m)$ and A is a $m \times m$ symmetric matrix. The final step is integrate in α , obtaining

$$\mathcal{F} \left(\frac{1}{[-2z^+ z^- + i0]^n} \right) = -i \frac{(-1)^n 4^{1-n} \pi}{\Gamma[n]} \frac{\Gamma[1-n]}{[-2q^+ q^- - i0]^{1-n}}. \quad (\text{D.53})$$

E Phase space parametrization

E.1 Phase space parametrization with virtual corrections

The cumulated hadronic tensor with virtual corrections, $W_{GG'}^{\mu\nu, [\text{NLO}, \text{V}]}(q)\Theta(Q^2, y, q_T^2)$, considering whole series of kinematic power correction for unpolarized TMDPDF operators at geometrical twist 2 amounts to

$$\begin{aligned} W_{GG'}^{\mu\nu, [\text{NLO}, \text{V}]}(q)\Theta(Q^2, y, q_T^2) &= \frac{4\pi\alpha_s(\mu^2)P_A^-P_B^+}{N_c} \sum_{a,b} \int [d^4p_a]d\xi_a[d^4p_b]d\xi_b \delta(p_a^- - \xi_a P_A^-)\delta(p_b^+ - \xi_b P_B^+) \\ &\times f_{n, q \leftarrow p_A}(\xi_a, \mathbf{p}_{aT})f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT})\Theta(Q^2, y, q_T^2)\delta^{(4)}(q - p_a - p_b) \\ &\times \left[\Pi_{V, GG'}^{\mu\nu} \left((p_a + p_b)^2, \log \frac{\mu^2}{(p_a + p_b)^2} + i\pi \right) + \text{h.c.} \right] \\ &+ n \leftrightarrow \bar{n}, \end{aligned} \quad (\text{E.1})$$

where $[d^4p_{a(b)}] = d^4p_{a(b)}\delta(p_{a(b)}^2)\theta(p_{a(b)}^0)$, $a(b)$ is a flavor index, $\Theta(Q^2, y, q_T^2)$ is given by eq. (4.105)

$$\Theta(Q^2, y, q_T^2) = \theta(Q_{\text{max}}^2 - Q^2)\theta(Q^2 - Q_{\text{min}}^2)\theta(y_{\text{max}} - y)\theta(y - y_{\text{min}})\theta(q_{T\text{max}}^2 - q_T^2)\theta(q_T^2 - q_{T\text{min}}^2), \quad (\text{E.2})$$

the measured vector boson invariant mass Q^2 , rapidity y and transverse momentum q_T^2 are given as functions the quark p_a^μ and anti-quark p_b^μ momentum

$$\begin{aligned} Q^2 &= (p_a + p_b)^2, \\ y &= \frac{1}{2} \log \frac{q^-}{q^+} = \frac{1}{2} \log \frac{n \cdot (p_a + p_b)}{\bar{n} \cdot (p_a + p_b)}, \\ q_T^2 &= 2n \cdot (p_a + p_b)\bar{n} \cdot (p_a + p_b) - Q^2. \end{aligned} \quad (\text{E.3})$$

The NLO virtual corrections to the effective current and its hermitian conjugate, $\Pi_{V, GG'} + \text{h.c.}$, in the Larin scheme [226–228] amounts to

$$\begin{aligned} \Pi_{V, GG'}^{\mu\nu} \left((p_a + p_b)^2, \log \frac{\mu^2}{(p_a + p_b)^2} + i\pi \right) + \text{h.c.} &= \frac{2C_F p_a \cdot p_b (c_L^G c_L^{G'} + c_R^G c_R^{G'})}{\pi} g_{ab, \perp}^{\mu\nu} \\ &\times \left[\frac{1}{\epsilon^2} + \frac{3 + 2 \log \frac{\mu^2}{(p_a + p_b)^2}}{2\epsilon} + 4 - \frac{7\pi^2}{12} + \frac{3}{2} \log \frac{\mu^2}{(p_a + p_b)^2} + \frac{1}{2} \log^2 \frac{\mu^2}{(p_a + p_b)^2} \right] \\ &- iZ_5^{(\text{ns})} \frac{2C_F (c_L^G c_L^{G'} - c_R^G c_R^{G'}) \epsilon^{\mu\nu\alpha\beta} p_{a\alpha} p_{b\beta}}{\pi} \\ &\times \left[\frac{1}{\epsilon^2} + \frac{3 + 2 \log \frac{\mu^2}{(p_a + p_b)^2}}{2\epsilon} + 3 - \frac{7\pi^2}{12} + \frac{3}{2} \log \frac{\mu^2}{(p_a + p_b)^2} + \frac{1}{2} \log^2 \frac{\mu^2}{(p_a + p_b)^2} \right], \end{aligned} \quad (\text{E.4})$$

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where tensor $g_{ab,\perp}^{\mu\nu}$ is given by

$$g_{ab,\perp}^{\mu\nu} = g^{\mu\nu} - \frac{n_a^\mu n_b^\nu + n_a^\nu n_b^\mu}{n_a \cdot n_b}, \quad (\text{E.5})$$

$n_{a(b)}^\mu = (1, \mathbf{n}_{a(b)})$ is a light-like vector and $\mathbf{n}_{a(b)}$ is a unit vector pointing along the direction of $\mathbf{p}_{a(b)}^\mu$. The finite renormalization constant for the singlet case, Z_5^s , at NNLO is given by [226]

$$Z_5^s(\alpha_s(\mu^2)) = 1 - \alpha_s(\mu^2) \frac{C_F}{\pi} + \frac{\alpha_s^2(\mu^2)}{16\pi^2} \left(22C_F^2 - \frac{107}{9}C_FC_A + \frac{31}{18}C_Fn_f \right) + \mathcal{O}(\alpha_s^3(\mu^2)) \quad (\text{E.6})$$

and for the non-singlet case, Z_5^{ns} , at NNLO by [226]

$$Z_5^{\text{ns}}(\alpha_s(\mu^2)) = 1 - \alpha_s(\mu^2) \frac{C_F}{\pi} + \frac{\alpha_s^2(\mu^2)}{16\pi^2} \left(22C_F^2 - \frac{107}{9}C_FC_A + \frac{2}{9}C_Fn_f \right) + \mathcal{O}(\alpha_s^3(\mu^2)), \quad (\text{E.7})$$

where n_f is the number of active.

The sum of LO and the virtual NLO contributions in term of renormalized TMDPDFs amount to

$$\begin{aligned} & [W_{GG'}^{\mu\nu, [\text{LO}]}(q) + W_{GG'}^{\mu\nu, [\text{NLO}, \text{V}]}(q)] \Theta(Q^2, y, q_T^2) \\ &= \frac{P_A^- P_B^+}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \Theta(Q^2, y, q_T^2) \\ &\times f_{n, q \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}; \mu^2, \zeta_n) f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}; \mu^2, \zeta_{\bar{n}}) \delta^{(4)}(q - p_a - p_b) \\ &\times Z_2^2(\alpha_s(\mu^2)) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_n}\right) Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta_{\bar{n}}}\right) \\ &\times \left[2p_a \cdot p_b (c_L^G c_L^{G'} + c_R^G c_R^{G'}) g_{ab,\perp}^{\mu\nu} \left\{ 1 - \frac{\alpha_s(\mu^2) C_F}{\pi} \left(\frac{1}{\epsilon^2} + \frac{3 + 2 \log \frac{\mu^2}{Q^2}}{2\epsilon} + 4 - \frac{7\pi^2}{12} + \frac{3}{2} \log \frac{\mu^2}{Q^2} + \frac{1}{2} \log^2 \frac{\mu^2}{Q^2} \right) \right\} \right. \\ &- 2i(c_L^G c_L^{G'} - c_R^G c_R^{G'}) \epsilon^{\mu\nu\alpha\beta} p_{a\alpha} p_{b\beta} Z_5^{\text{n(ns)}}(\alpha_s(\mu^2)) \\ &\times \left. \left\{ 1 - \frac{\alpha_s(\mu^2) C_F}{\pi} \left(\frac{1}{\epsilon^2} + \frac{3 + 2 \log \frac{\mu^2}{Q^2}}{2\epsilon} + 3 - \frac{7\pi^2}{12} + \frac{3}{2} \log \frac{\mu^2}{Q^2} + \frac{1}{2} \log^2 \frac{\mu^2}{Q^2} \right) \right\} \right] \\ &+ n \leftrightarrow \bar{n}, \end{aligned} \quad (\text{E.8})$$

where the renormalization constant are given by [43, 47, 50, 183, 184]

$$\begin{aligned} Z_2(\alpha_s(\mu^2)) &= 1 - \frac{\alpha_s(\mu^2) C_F}{4\pi\epsilon} + \mathcal{O}(\alpha_s^2(\mu^2)), \\ Z_{\text{TMD}}\left(\alpha_s(\mu^2), \log \frac{\mu^2}{\zeta}\right) &= 1 + \frac{\alpha_s(\mu^2) C_F}{2\pi\epsilon^2} + \frac{\alpha_s(\mu^2) C_F}{2\pi\epsilon} \left(2 + \log \frac{\mu^2}{\zeta} \right) + \mathcal{O}(\alpha_s^2(\mu^2)). \end{aligned} \quad (\text{E.9})$$

From eq. (E.8) one can see that only the combination $\zeta_n \zeta_{\bar{n}} = Q^4$ renders the virtual corrections to the hadronic tensor finite.

E.2 Phase space parametrization with one real emission

The cumulant version of the hadronic tensor with one real gluon and considering whole series of kinematic power correction for unpolarized TMDPDF operators and at geometrical twist 2 is given by

$$\begin{aligned}
 W_{GG'}^{\mu\nu, [\text{NLO}, \text{R}]}(q)\Theta(Q^2, y, q_T^2) &= \frac{4\pi\alpha_s(\mu^2)P_A^- P_B^+ C_F}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
 &\times \int \frac{d^d k}{(2\pi)^{d-1}} \theta(k^0) \delta(k^2) \Theta(Q^2, y, q_T^2) \delta^{(d)}(p_a + p_b - q - k) \\
 &\times \mathcal{H}_{ab}^{\mu\nu}(p_a, p_b, q) f_{n, a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n}, b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
 &+ n \leftrightarrow \bar{n} + \dots,
 \end{aligned} \tag{E.10}$$

where $\Theta(Q^2, y, q_T^2)$ is defined as in eq. (E.2). The measured vector boson rapidity y , invariant mass Q^2 and transverse momentum q_T^2 are functions k^μ , quark p_a^μ and anti-quark p_b^μ momentum

$$\begin{aligned}
 y &= \frac{1}{2} \log \frac{q^-}{q^+} = \frac{1}{2} \log \frac{n \cdot (p_a + p_b - k)}{\bar{n} \cdot (p_a + p_b - k)}, \\
 Q^2 &= (p_a + p_b - k)^2, \\
 q_T^2 &= 2n \cdot (p_a + p_b - k) \bar{n} \cdot (p_a + p_b - k) - Q^2.
 \end{aligned} \tag{E.11}$$

The hard scattering amplitude $\mathcal{H}_{ab}^{\mu\nu}$ is given by

$$\begin{aligned}
 \mathcal{H}_{ab, GG'}^{\mu\nu}(p_a, p_b, k) &= \sum_{G, G'} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left[\frac{\text{Tr}[\not{p}_a \gamma_\beta (\not{p}_a - \not{k}) \Gamma_{G'}^\nu \not{p}_b \Gamma_G^\mu (\not{p}_a - \not{k}) \gamma^\beta]}{[(p_a - k)^2 - i0][(p_a - k)^2 + i0]} + \right. \\
 &- \frac{\text{Tr}[\not{p}_a \Gamma_{G'}^\nu (\not{p}_b - \not{k}) \gamma_\beta \not{p}_b \Gamma_G^\mu (\not{p}_a - \not{k}) \gamma^\beta]}{[(p_b - k)^2 - i0][(p_a - k)^2 + i0]} - \frac{\text{Tr}[\not{p}_a \gamma_\beta (\not{p}_a - \not{k}) \Gamma_{G'}^\nu \not{p}_b \gamma^\beta (\not{p}_b - \not{k}) \Gamma_G^\mu]}{[(p_a - k)^2 - i0][(p_b - k)^2 + i0]} \\
 &\left. + \frac{\text{Tr}[\not{p}_a \Gamma_{G'}^\nu (\not{p}_b - \not{k}) \gamma_\beta \not{p}_b \gamma^\beta (\not{p}_b - \not{k}) \Gamma_G^\mu]}{[(p_b - k)^2 - i0][(p_b - k)^2 + i0]} \right].
 \end{aligned} \tag{E.12}$$

We present in thoroughly detail the isolation of collinear and soft poles in the phase space integrals and their regularization in eq. (E.10) in the case $GG' = \gamma^* \gamma^*$. In the cases $GG' = \{ZZ, W^\pm W^\pm, \gamma^* Z, Z \gamma^*\}$ the eq. (E.10) is rather lengthy and we do not show, nevertheless the procedure for the cancellation of IR divergences is exactly analogous to the cases $GG' = \gamma^* \gamma^*$.

In order to simplify the IR poles analysis we split $\mathcal{H}^{\mu\nu}$ for the case $GG = \gamma^* \gamma^*$ into three contributions

$$\mathcal{H}_{ab, \gamma^* \gamma^*}^{\mu\nu} = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left[\mathcal{H}_{ab, I}^{\mu\nu} + \mathcal{H}_{ab, c_a}^{\mu\nu} + \mathcal{H}_{ab, c_b}^{\mu\nu} \right], \tag{E.13}$$

where

$$\begin{aligned}
 \mathcal{H}_{ab, I}^{\mu\nu} &= - \frac{2(2p_a \cdot p_b)^2}{(p_a \cdot k)(p_b \cdot k)} \left[g^{\mu\nu} - \frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{p_a \cdot p_b} \right] \\
 &- \frac{4p_a \cdot p_b}{(p_a \cdot k)(p_b \cdot k)} \left[(d-4)k^\mu k^\nu + k^\nu p_a^\mu + k^\mu p_a^\nu + k^\nu p_b^\mu + k^\mu p_b^\nu \right],
 \end{aligned} \tag{E.14}$$

APPENDIX E. PHASE SPACE PARAMETRIZATION

$$\mathcal{H}_{ab,c_a}^{\mu\nu} = \frac{8p_a^\mu p_a^\nu + 8p_a \cdot p_b g^{\mu\nu} + 2(d-4)(k^\nu p_a^\mu + k^\mu p_a^\nu) - 2(d-2)p_b \cdot k g^{\mu\nu}}{p_a \cdot k} + \frac{2(d-2)(p_b^\mu k^\nu + p_b^\nu k^\mu) - 4(p_a^\mu p_b^\nu + p_a^\nu p_b^\mu)}{p_a \cdot k}, \quad (\text{E.15})$$

$$\mathcal{H}_{ab,c_b}^{\mu\nu} = \frac{8p_b^\mu p_b^\nu + 8p_a \cdot p_b g^{\mu\nu} + 2(d-4)(k^\nu p_b^\mu + k^\mu p_b^\nu) - 2(d-2)p_a \cdot k g^{\mu\nu}}{p_b \cdot k} + \frac{2(d-2)(p_a^\mu k^\nu + p_a^\nu k^\mu) - 4(p_a^\mu p_b^\nu + p_a^\nu p_b^\mu)}{p_b \cdot k}. \quad (\text{E.16})$$

As we already commented in the main text, depending how one parametrizes $p_{a(b)}^\mu$, k^μ and $n^\mu(\bar{n}^\mu)$ to make explicit the soft, $k^0 \rightarrow 0$, and collinear, $\mathbf{k}$ becomes parallel to $\mathbf{p}_{a(b)}$, divergences can be extremely challenging. The main difficulty is that the measurements are performed in the laboratory center of mass frame where the protons are back-to-back where the light-like vectors n^μ and $\bar{n}^\mu$ takes the simple form: $n^\mu = \frac{1}{\sqrt{2}}(1, 0_\perp, 1)$ and $\bar{n}^\mu = \frac{1}{\sqrt{2}}(1, 0_\perp, -1)$. Nevertheless, the denominators in eq. (E.10) takes a rather cumbersome expression. One could simply use spherical coordinates, this would lead to expressions where is relatively straightforward to identify and isolate the soft divergences, however collinear divergences are difficult to isolate. The whole situation highly simplifies if we were able to find a reference system where p_a^μ and p_b^μ were back-to-back, in such a case the soft and collinear divergences becomes self evident. It is expected that in such a system the proton momentum $P_{A(B)}^\mu$ and the light-like vector $n^\mu(\bar{n}^\mu)$ takes a rather cumbersome expressions.

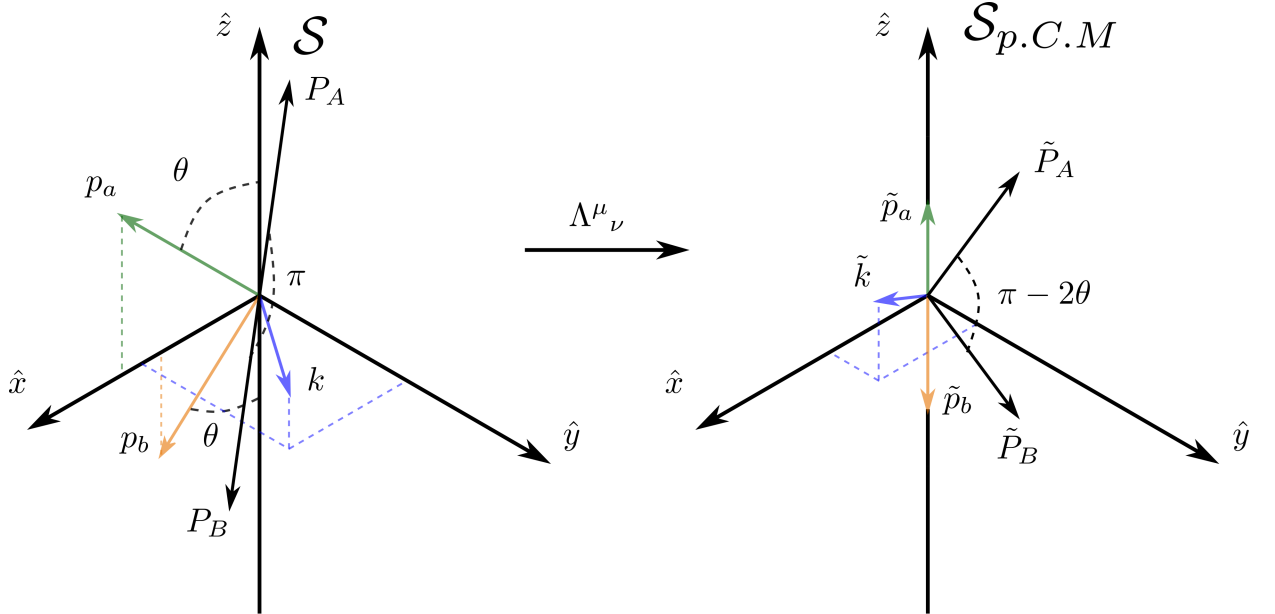


Figure E.1: Change from $\mathcal{S}$ to the partonic center-of-mass frame $\mathcal{S}_{p.C.M.}$.

E.2. PHASE SPACE PARAMETRIZATION WITH ONE REAL EMISSION

We showed in Sec. 4.3.2.2 that applying a series of rotations one can find a reference system where p_a^μ and p_b^μ take the simple form

$$\begin{aligned} p_a^\mu &= p_a^0(1, \sin \theta, \mathbf{0}_{d-3}, \cos \theta), \\ p_b^\mu &= p_b^0(1, \sin \theta, \mathbf{0}_{d-3}, -\cos \theta), \end{aligned} \quad (\text{E.17})$$

where $2\theta + \theta_b = \pi$ (with $\pi/2 \geq \theta \geq 0$) and $\mathbf{0}_{d-3}$ denotes a space-like vector space of dimension $d-3$. In a reference system where p_a^μ and p_b^μ take the simple form given in eq. (E.17), hereafter we refer to it as $\mathcal{S}$, the rest of vectors take the form

$$\begin{aligned} P_A^\mu &= (P_A^0, |\mathbf{P}_A| \sin \theta_a \cos \phi_b, |\mathbf{P}_A| \sin \theta_a \sin \phi_b, \mathbf{0}_{d-4}, |\mathbf{P}_A| \cos \theta_a), \\ P_B^\mu &= (P_B^0, -|\mathbf{P}_B| \sin \theta_a \cos \phi_b, -|\mathbf{P}_B| \sin \theta_a \sin \phi_b, \mathbf{0}_{d-4}, -|\mathbf{P}_B| \cos \theta_a), \\ n^\mu &= \frac{1}{\sqrt{2}}(1, \sin \theta_a \cos \phi_b, \sin \theta_a \sin \phi_b, \mathbf{0}_{d-4}, \cos \theta_a), \\ \bar{n}^\mu &= \frac{1}{\sqrt{2}}(1, -\sin \theta_a \cos \phi_b, -\sin \theta_a \sin \phi_b, \mathbf{0}_{d-4}, -\cos \theta_a), \end{aligned} \quad (\text{E.18})$$

where proton momenta P_A^μ , P_B^μ and the light-like vectors n^μ , $\bar{n}^\mu$ are back-to-back. The next step is to apply a boost along the $\hat{x}$ -axis of the reference system $\mathcal{S}$ such that the resulting $\tilde{p}_a^\mu$ and $\tilde{p}_b^\mu$ vectors are back-to-back, see fig. E.1. To obtain the explicit form of the boost one can start writing how it acts on $p_{a(b)}^\mu$

$$\begin{aligned} \tilde{p}_{a(b)}^0 &= \Lambda^0_{\nu} p_{a(b)}^\nu = p_{a(b)}^0 \gamma (1 - \beta \sin \theta), \\ \tilde{p}_{a(b)}^1 &= \Lambda^1_{\nu} p_{a(b)}^\nu = p_{a(b)}^0 \gamma (\sin \theta - \beta) = 0, \\ \tilde{p}_{a(b)}^i &= \Lambda^i_{\nu} p_{a(b)}^\nu = \delta^i_{\nu} p_{a(b)}^\nu = p_{a(b)}^i, \quad i = 2, \dots, d-1 \end{aligned} \quad (\text{E.19})$$

eq. (E.19) implies $\beta = \sin \theta$, which it is an allowed solution since $\pi/2 \geq \theta \geq 0$. We will refer as partonic center-of-mass frame to the reference system after applying the boost Λ^μ_{ν} defined in eq. (E.19), $\mathcal{S}_{p.C.M.}$, whose components are explicitly

$$\Lambda^\mu_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0_{1,d-2} \\ -\gamma\beta & \gamma & 0_{1,d-2} \\ 0_{d-2,1} & 0_{d-2,1} & \mathbb{1}_{d-2,d-2} \end{pmatrix}. \quad (\text{E.20})$$

One of the consequences of working in the partonic center-of-mass frame is that the proton momenta P_A^μ , P_B^μ and the light-like vector n^μ and $\bar{n}^\mu$ are no longer back-to-back. Once we have obtain an explicit form of $p_{a(b)}^\mu$, $P_{A(B)}^\mu$, $n^\mu(\bar{n}^\mu)$ and k^μ in the partonic center-of-mass frame, we can write the integrand and integration measure in eq. (E.10) in the corresponding reference frame. We start with the integrand, eq. (E.12) transforms as a Lorentz tensor of two indices

$$\mathcal{H}_{ab,\gamma^*\gamma^*}(\tilde{p}_a, \tilde{p}_b, \tilde{k}) = \mathcal{H}_{ab,\gamma^*\gamma^*}(\Lambda p_a, \Lambda p_b, \Lambda k) \quad (\text{E.21})$$

where $\tilde{v}$ denotes the vector v after applying the boost defined in eq. (E.20). The arguments in the measurement function $\Theta(Q^2, y, q_T^2)$ changes as follow

$$y = \frac{1}{2} \log \frac{\tilde{n} \cdot (\tilde{p}_a + \tilde{p}_b - \tilde{k})}{\tilde{n} \cdot (\tilde{p}_a + \tilde{p}_b + \tilde{k})} = \frac{1}{2} \log \frac{\bar{n} \cdot (p_a + p_b) - (\Lambda \bar{n}) \cdot \tilde{k}}{n \cdot (p_a + p_b) - (\Lambda n) \cdot \tilde{k}}, \quad (\text{E.22})$$

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$$Q^2 = (\tilde{p}_a + \tilde{p}_b - \tilde{k})^2 = 2p_a \cdot p_b - 2\tilde{k} \cdot (\Lambda p_a) - 2\tilde{k} \cdot (\Lambda p_b), \quad (\text{E.23})$$

$$\begin{aligned} q_T^2 &= 2\tilde{n} \cdot (\tilde{p}_a + \tilde{p}_b - \tilde{k})\tilde{n} \cdot (\tilde{p}_a + \tilde{p}_b - \tilde{k}) - (\tilde{p}_a + \tilde{p}_b - \tilde{k})^2 \\ &= 2\tilde{n} \cdot (p_a + p_b)n \cdot (p_a + p_b) - 2n \cdot (p_a + p_b)(\Lambda\tilde{n}) \cdot \tilde{k} - 2\tilde{n} \cdot (p_a + p_b)(\Lambda n) \cdot \tilde{k} \\ &\quad + 2(\Lambda\tilde{n}) \cdot \tilde{k}(\Lambda n) \cdot \tilde{k} - 2p_a \cdot p_b + 2\tilde{k} \cdot (\Lambda p_a) + 2\tilde{k} \cdot (\Lambda p_b). \end{aligned} \quad (\text{E.24})$$

The transverse momentum $\mathbf{p}_{a(b)T}^2$ and fraction of (anti-)collinear component $\xi_{a(b)}$ take the form

$$\mathbf{p}_{a(b)T}^2 = 2\tilde{n} \cdot \tilde{p}_{a(b)}\tilde{n} \cdot \tilde{p}_{a(b)} = 2\tilde{n} \cdot p_{a(b)}n \cdot p_{a(b)}, \quad (\text{E.25})$$

$$\begin{aligned} \xi_a &= \frac{\tilde{n} \cdot \tilde{p}_a}{\tilde{n} \cdot \tilde{P}_A} = \frac{\tilde{n} \cdot p_a}{\tilde{n} \cdot P_A}, \\ \xi_b &= \frac{\tilde{n} \cdot \tilde{p}_b}{\tilde{n} \cdot \tilde{P}_B} = \frac{n \cdot p_b}{n \cdot P_B}. \end{aligned} \quad (\text{E.26})$$

The integration measures are invariant under rotations and boost

$$\begin{aligned} \frac{d^{d-1}\tilde{\mathbf{p}}_{a(b)}}{2\tilde{p}_{a(b)}^0} &= \frac{d^{d-1}\mathbf{p}_{a(b)}}{2p_{a(b)}^0}, \\ \frac{d^{d-1}\tilde{\mathbf{k}}}{2\tilde{k}^0} &= \frac{d^{d-1}\mathbf{k}}{2k^0}. \end{aligned} \quad (\text{E.27})$$

For the vectors $p_{a(b)}^\mu$, $n^\mu(\tilde{n}^\mu)$ and $P_{A(B)}^\mu$ we explicitly consider their component in the partonic center-of-mass frame as functions of the corresponding components of the reference system $\mathcal{S}$, while for the momentum of the emitted gluon we simply consider its component in the partonic center-of-mass frame without relating them with the ones in the system $\mathcal{S}$ i.e. we simply parametrize $\tilde{k}^\mu$ as $\tilde{k}^\mu = \tilde{k}^0(1, \sin\theta_{\tilde{k}} \cos\phi_{\tilde{k}}, \sin\theta_{\tilde{k}} \sin\phi_{\tilde{k}}, \mathbf{0}_{d-4}, \cos\theta_{\tilde{k}})$.

Once one has a convenient explicit parametrization for the several momenta present in the integrand of eq. (E.10), it is straight forward to isolate the IR divergences, namely soft and collinear. With this aim we define the variables

$$\begin{aligned} s_{ak} &= 2\tilde{k} \cdot (\Lambda p_a), \\ s_{bk} &= 2\tilde{k} \cdot (\Lambda p_b), \end{aligned} \quad (\text{E.28})$$

using eq. (E.19) one obtain

$$\begin{aligned} \tilde{p}_a^\mu &= p_a^0 \cos\theta(1, \mathbf{0}_{d-2}, 1), \\ \tilde{p}_b^\mu &= p_b^0 \cos\theta(1, \mathbf{0}_{d-2}, -1), \end{aligned} \quad (\text{E.29})$$

and $s_{a(b)k}$ amounts

$$\left. \begin{aligned} s_{ak} &= 2\tilde{k}^0 p_a^0 \cos\theta(1 - \cos\theta_{\tilde{k}}) \\ s_{bk} &= 2\tilde{k}^0 p_b^0 \cos\theta(1 + \cos\theta_{\tilde{k}}) \end{aligned} \right\} \quad \text{for } \frac{\pi}{2} \geq \theta \geq 0, \quad (\text{E.30})$$

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notice that $\infty \geq s_{a(b)k} \geq 0$. Using eqs. (E.30) the integration measure for the emitted gluon can be written as

$$\frac{d^{d-1}\tilde{\mathbf{k}}}{2\tilde{k}^0} = \frac{d\phi_{\tilde{k}} \sin^{-2\epsilon} \phi_{\tilde{k}} d \cos \theta_{\tilde{k}} \sin^{-2\epsilon} \theta_{\tilde{k}} d|\tilde{\mathbf{k}}| |\tilde{\mathbf{k}}|^{d-3}}{2} = d\phi_{\tilde{k}} \sin^{-2\epsilon} \phi_{\tilde{k}} \frac{ds_{ak} ds_{bk}}{4s_{ab}} \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon, \quad (\text{E.31})$$

where $\tilde{k}^0 = |\tilde{\mathbf{k}}|$, $s_{ab} = (p_a + p_b)^2 = 4p_a^0 p_b^0 \cos^2 \theta$ and the Jacobian is

$$\frac{ds_{ak} ds_{bk}}{d|\tilde{\mathbf{k}}| d \cos \theta_{\tilde{k}}} = 2|\tilde{\mathbf{k}}| s_{ab}. \quad (\text{E.32})$$

Once the integration measure for the emitted gluon is written in terms of $s_{a(b)k}$, one can write $\theta(s_{a(b)k}) s_{a(b)k}^{-1-\epsilon}$ as a plus and delta distributions (see Sec. A.1) in order to isolate the soft and collinear poles. Since $s_{a(b)k}$ has unit of momentum square, we need to introduce an energy scale $\mu_{a(b)k}$ such that the ratio $(\mu_{a(b)k}^2/s_{a(b)k})^\epsilon$ is dimensionless and can be expanded in ϵ . Thus, one can write

$$\begin{aligned} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \frac{\theta(s_{ak})\theta(s_{bk})}{s_{ak}s_{bk}} &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \frac{\theta(s_{ak})}{s_{ak}} \left(\frac{\mu_{ak}^2}{s_{ak}} \right)^\epsilon \frac{\theta(s_{bk})}{s_{bk}} \left(\frac{\mu_{bk}^2}{s_{bk}} \right)^\epsilon \\ &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left(\left[\frac{\theta(s_{ak}) \mu_{ak}^{2\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} - \frac{\delta(s_{ak})}{\epsilon} \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \right) \\ &\quad \times \left(\left[\frac{\theta(s_{bk}) \mu_{bk}^{2\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} - \frac{\delta(s_{bk})}{\epsilon} \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \right), \end{aligned} \quad (\text{E.33})$$

$$\begin{aligned} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \frac{\theta(s_{ak})}{s_{ak}} &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 s_{bk}} \right)^\epsilon \frac{\theta(s_{ak})}{s_{ak}} \left(\frac{\mu_{ak}^2}{s_{ak}} \right)^\epsilon \\ &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 s_{bk}} \right)^\epsilon \left(\left[\frac{\theta(s_{ak}) \mu_{ak}^{2\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} - \frac{\delta(s_{ak})}{\epsilon} \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \right), \end{aligned} \quad (\text{E.34})$$

$$\begin{aligned} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \frac{\theta(s_{bk})}{s_{bk}} &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{bk}^2 s_{ak}} \right)^\epsilon \frac{\theta(s_{bk})}{s_{bk}} \left(\frac{\mu_{bk}^2}{s_{bk}} \right)^\epsilon \\ &= \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{bk}^2 s_{ak}} \right)^\epsilon \left(\left[\frac{\theta(s_{bk}) \mu_{bk}^{2\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} - \frac{\delta(s_{bk})}{\epsilon} \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \right), \end{aligned} \quad (\text{E.35})$$

where we have not used a general boundary term $s_{a(b)k}^{\max}$. Formally, the integration in $s_{a(b)k}$ is from 0 up to ∞ ; nevertheless, in practice, the measurement functions in $\Theta(Q^2, y, q_T^2)$ introduce an upper bound in some of the contributions. However, the terms evaluated at $s_{a(b)k} = 0$ have no upper bound (since the measurements functions become independent of $s_{a(b)k}$), we can use the boundary term to remove any contribution coming from above the upper cut introduced by the measurement functions. Let us show the idea with the following simple example: we

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have $\theta(z)z^{-1-\epsilon}$ acting onto the test function $\theta(a-z)f(z)$ with $z \in [0, \infty)$ and $a \geq 0$. Thus, one can write

$$\begin{aligned}
\int_0^\infty dz \frac{\theta(z)\theta(a-z)}{z^{1+\epsilon}} f(z) &= \lim_{z_1 \rightarrow \infty} \int_0^{z_1} dz \frac{\theta(z)\theta(a-z)}{z^{1+\epsilon}} f(z) \\
&= \lim_{z_1 \rightarrow \infty} \left\{ \int_0^{z_1} dz \left[\frac{\theta(z)}{z^{1+\epsilon}} \right]_+^{z_0} \theta(a-z)f(z) - \frac{\theta(a)f(0)}{\epsilon} + \theta(a)f(0) \log z_0 \right\} \\
&= \int_0^a dz \frac{\theta(a-z)f(z) - \theta(a)f(0)}{z^{1+\epsilon}} - \frac{\theta(a)f(0)}{\epsilon} \\
&\quad + \lim_{z_1 \rightarrow 0} \left\{ \int_a^{z_1} dz \frac{\theta(a-z)f(z) - \theta(a)f(0)}{z^{1+\epsilon}} + f(0)\theta(a) \log \frac{z_1}{z_0} + \theta(a)f(0) \log z_0 \right\} \\
&= \int_0^a dz \frac{\theta(a-z)f(z) - \theta(a)f(0)}{z^{1+\epsilon}} - \frac{\theta(a)f(0)}{\epsilon} + \theta(a)f(0) \log a,
\end{aligned} \tag{E.36}$$

which is well-defined (up to the pole in ϵ) and independent of the boundary term. One can also identify the relation

$$\lim_{z_1 \rightarrow \infty} \int_0^{z_1} dz \left[\frac{\theta(z)}{z^{1+\epsilon}} \right]_+^{z_0} \theta(a-z)f(z) = \int_0^a dz \frac{\theta(a-z)f(z) - \theta(a)f(0)}{z^{1+\epsilon}} + \theta(a)f(0) \log \frac{a}{z_0}. \tag{E.37}$$

In the case of eq. (E.10) the Heaviside step functions gathered in $\Theta(Q^2, y, q_T^2)$ lead to logarithmic terms that are functions of $p_{a(b)}^{-(+)}$ and $|\mathbf{p}_{a(b)}|$.

Solving eqs. (E.30) one obtains $|\tilde{\mathbf{k}}|$ and $\cos \theta_{\tilde{k}}$ as functions of $s_{a(b)k}$

$$\left. \begin{aligned} |\tilde{\mathbf{k}}| &= \frac{(p_b^0 s_{ak} + p_a^0 s_{bk}) \cos \theta}{s_{ab}} \\ \cos \theta_{\tilde{k}} &= \frac{p_a^0 s_{bk} - p_b^0 s_{ak}}{p_b^0 s_{ak} + p_a^0 s_{bk}} \end{aligned} \right\} \frac{\pi}{2} \geq \theta \geq 0, \tag{E.38}$$

using eqs. (E.38) the parametrization for $\tilde{k}^\mu$

$$\tilde{k}^\mu = \tilde{k}^0 (1, \sin \theta_{\tilde{k}} \cos \phi_{\tilde{k}}, \sin \theta_{\tilde{k}} \sin \phi_{\tilde{k}}, \mathbf{0}_{d-4}, \cos \theta_{\tilde{k}}),$$

is given by

$$\tilde{k}^\mu = \frac{1}{s_{ab}} ([p_b^0 s_{ak} + p_a^0 s_{bk}] \cos \theta, \sqrt{s_{ab} s_{ak} s_{bk}} \cos \phi_{\tilde{k}}, \sqrt{s_{ab} s_{ak} s_{bk}} \sin \phi_{\tilde{k}}, \mathbf{0}_{d-4}, [p_a^0 s_{bk} - p_b^0 s_{ak}] \cos \theta). \tag{E.39}$$

Using eqs. (E.33), (E.34) and (E.35) together with the parametrization for $\tilde{k}^\mu$ given in eq. (E.39) one can write $\mathcal{H}_{ab,I}^{\mu\nu}$ (eq. (E.14)), $\mathcal{H}_{ab,c_a}^{\mu\nu}$ (eq. (E.15)) and $\mathcal{H}_{ab,c_b}^{\mu\nu}$ (eq. (E.16)) as

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$$\begin{aligned}
& \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \mathcal{H}_{ab,I}^{\mu\nu} = -16\tilde{p}_a \cdot \tilde{p}_b \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left[\frac{\theta(s_{ak}) \mu_{ak}^{2\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk}) \mu_{bk}^{2\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} \\
& \times \left[2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) + (d-4)\tilde{k}^\mu \tilde{k}^\nu + \tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu + \tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu \right] \\
& - 8(2\tilde{p}_a \cdot \tilde{p}_b)^2 \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \frac{\delta(s_{ak})\delta(s_{bk})}{\epsilon^2} \left[g^{\mu\nu} - \frac{\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu}{\tilde{p}_a \cdot \tilde{p}_b} \right] \\
& + 16\tilde{p}_a \cdot \tilde{p}_b \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left[\frac{\theta(s_{ak}) \mu_{ak}^{2\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \frac{\delta(s_{bk})}{\epsilon} \\
& \times \left[2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) + (d-4)\tilde{k}^\mu \tilde{k}^\nu + \tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu + \tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu \right] \\
& + 16\tilde{p}_a \cdot \tilde{p}_b \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left[\frac{\theta(s_{bk}) \mu_{bk}^{2\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \frac{\delta(s_{ak})}{\epsilon} \\
& \times \left[2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) + (d-4)\tilde{k}^\mu \tilde{k}^\nu + \tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu + \tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu \right], \tag{E.40}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \mathcal{H}_{ab,c_a}^{\mu\nu} = 4 \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 s_{bk}} \right)^\epsilon \left[\frac{\theta(s_{ak}) \mu_{ak}^{2\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} \left[4\tilde{p}_a^\mu \tilde{p}_a^\nu + 4\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} \right. \\
& + (d-4)(\tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu) - (d-2)\tilde{p}_b \cdot \tilde{k} g^{\mu\nu} + (d-2)(\tilde{p}_b^\mu \tilde{k}^\nu + \tilde{p}_b^\nu \tilde{k}^\mu) - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) \Big] \\
& - \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 s_{bk}} \right)^\epsilon \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \frac{4\delta(s_{ak})}{\epsilon} \left[4\tilde{p}_a^\mu \tilde{p}_a^\nu + 4\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} + (d-4)(\tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu) \right. \\
& \left. - (d-2)\tilde{p}_b \cdot \tilde{k} g^{\mu\nu} + 2(d-2)(\tilde{p}_b^\mu \tilde{k}^\nu + \tilde{p}_b^\nu \tilde{k}^\mu) - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) \right], \tag{E.41}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left(\frac{s_{ab}}{s_{ak}s_{bk}} \right)^\epsilon \mathcal{H}_{ab,c_b}^{\mu\nu} = 4 \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{bk}^2 s_{ak}} \right)^\epsilon \left[\frac{\theta(s_{bk}) \mu_{bk}^{2\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} \left[4\tilde{p}_b^\mu \tilde{p}_b^\nu + 4\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} \right. \\
& + (d-4)(\tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu) - (d-2)\tilde{p}_a \cdot \tilde{k} g^{\mu\nu} + (d-2)(\tilde{p}_a^\mu \tilde{k}^\nu + \tilde{p}_a^\nu \tilde{k}^\mu) - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) \Big] \\
& - \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{bk}^2 s_{ak}} \right)^\epsilon \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \frac{4\delta(s_{bk})}{\epsilon} \left[4\tilde{p}_b^\mu \tilde{p}_b^\nu + 4\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} + (d-4)(\tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu) \right. \\
& \left. - (d-2)\tilde{p}_a \cdot \tilde{k} g^{\mu\nu} + 2(d-2)(\tilde{p}_a^\mu \tilde{k}^\nu + \tilde{p}_a^\nu \tilde{k}^\mu) - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) \right]. \tag{E.42}
\end{aligned}$$

The different contributions to $\mathcal{H}_{ab,\gamma^*\gamma^*}^{\mu\nu}$, eq. (E.13), can be classified into four categories

- Soft: contributions proportional to $\delta(s_{ak})\delta(s_{bk})$, $S_{ab,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2)$.

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- $\tilde{p}_a$ -collinear: contributions proportional to $\delta(s_{bk})$, $C_{a,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2)$.
- $\tilde{p}_b$ -collinear: contributions proportional to $\delta(s_{ak})$, $C_{b,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2)$.
- Finite $s_{a(b)k}$: everything else that is not proportional neither to $\delta(s_{ak})$ nor to $\delta(s_{bk})$, $C_{ab,GG'}^{\mu\nu,[1]}(\mu_{ak}^2, \mu_{bk}^2)$.

We further write eq. (E.10) as

$$\begin{aligned}
W_{GG'}^{\mu\nu,[\text{NLO,R}]}(q)\Theta(Q^2, y, q_T^2) &= \frac{4\pi C_F \alpha_s(\mu^2) P_A^- P_B^+}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
&\times \sum_{GG'} \left[C_{ab,GG'}^{\mu\nu,[1]}(\mu_{ak}^2, \mu_{bk}^2) + C_{a,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2) + C_{b,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2) + S_{ab,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2) \right] \\
&\times f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\
&+ n \leftrightarrow \bar{n} + \dots,
\end{aligned} \tag{E.43}$$

E.2.1 Soft contribution

The soft contribution, $S_{ab,GG'}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, \mu_{bk}^2)$, can be read from eq. (E.40). It amounts to

$$\begin{aligned}
S_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= -\frac{4\tilde{p}_a \cdot \tilde{p}_b}{\epsilon^2} \left[g^{\mu\nu} - \frac{\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu}{\tilde{p}_a \cdot \tilde{p}_b} \right] \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \left(\frac{\mu_{ak}^2}{s_{ak}^{\max}} \right)^\epsilon \left(\frac{\mu_{bk}^2}{s_{bk}^{\max}} \right)^\epsilon \\
&\times \int \frac{d\Omega_{d-3} ds_{ak} ds_{bk}}{(2\pi)^{3-2\epsilon}} \delta(s_{ak}) \delta(s_{bk}) \Theta(Q^2, y, q_T^2) \delta^{(d)}(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k}),
\end{aligned} \tag{E.44}$$

where $d\Omega_{d-3}$ is the solid angle integration measure in $d-3$ dimensions

$$\int d\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma[\frac{n}{2}]}. \tag{E.45}$$

The integral in $s_{a(b)k}$ can be computed using the delta distribution $\delta(s_{a(b)k})$. Since $\tilde{k}^\mu$ is parametrized as eq. (E.39) the momentum conservation delta, $\delta^{(d)}(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k})$, and measurement functions, $\Theta(Q^2, y, q_T^2)$, become independent of $\tilde{k}^\mu$. Thus, the integrand in eq. (E.44) is independent of the solid angle and it can be easily integrated. Hence, integrating in $s_{a(b)k}$ and the solid angle $d\Omega_{1-2\epsilon}$, one obtains

$$\begin{aligned}
S_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= -\frac{s_{ab} g_{ab,\perp}^{\mu\nu}}{2\pi^2} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{s_{ab} \mu^2}{s_{ak}^{\max} s_{bk}^{\max}} + \frac{1}{2} \log^2 \frac{s_{ab} \mu^2}{s_{ak}^{\max} s_{bk}^{\max}} - \frac{\pi^2}{12} \right\} \\
&\times \Theta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{q}),
\end{aligned} \tag{E.46}$$

where $g_{ab,\perp}^{\mu\nu}$ is given by

$$g_{ab,\perp}^{\mu\nu} = g^{\mu\nu} - \frac{n_a^\mu n_b^\nu + n_b^\mu n_a^\nu}{2}, \tag{E.47}$$

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and $n_{a(b)}^\mu$ are given by

$$\left. \begin{aligned} n_a^\mu &= (1, \mathbf{0}_{d-2}, 1) \\ n_b^\mu &= (1, \mathbf{0}_{d-2}, -1) \end{aligned} \right\} \quad \text{for } \frac{\pi}{2} \geq \theta_a \geq 0. \quad (\text{E.48})$$

As it was already stated along the main text, the soft poles in eq. (E.46) are due to a double counting of soft modes and they are cancelled once a suitable double counting subtracting term for the soft modes is introduced, eq. (E.70). This subtraction procedure will also cancel any dependence on the scale μ^2 .

E.2.2 $\tilde{p}_a^\mu$ -collinear contribution

The contribution proportional to $\delta(s_{bk})$ is obtained from eqs. (E.40) and (E.42)

$$\begin{aligned} C_{a,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= \frac{2g_{ab,\perp}^{\mu\nu}}{\epsilon} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi s_{bk}^{\max}} \right)^\epsilon \int \frac{d\Omega_{d-3} ds_{ak} ds_{bk}}{(2\pi)^{3-2\epsilon}} \delta(s_{bk}) \\ &\times \left\{ \left[\frac{\theta(s_{ak}) s_{ab}^{1+\epsilon}}{s_{ak}^{1+\epsilon}} \right]_+^{s_{ak}^{\max}} - \left(\frac{s_{ab}}{s_{ak}} \right)^\epsilon \left(1 - \frac{(d-2)s_{ak}}{4s_{ab}} \right) \right\} \\ &\times \Theta(Q^2, y, q_T^2) \delta^{(d)}(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k}). \end{aligned} \quad (\text{E.49})$$

where $g_{ab,\perp}^{\mu\nu}$ is the one defined in eq. (E.47). The $d-3$ dimensional solid angle integration can be performed trivially, since $\tilde{k}^\mu$ is independent of them at $s_{bk} = 0$. The integration in s_{bk} is also straightforward using the delta distribution. Expanding in ϵ one obtains

$$\begin{aligned} C_{a,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= \frac{2g_{ab,\perp}^{\mu\nu}}{(2\pi)^2} \int ds_{ak} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{ak}) s_{ab}}{s_{ak}} \right]_+^{s_{ak}^{\max}} - \frac{1}{\epsilon} \left(1 - \frac{(d-2)s_{ak}}{4s_{ab}} \right) \right. \\ &+ \left. \left[\frac{\theta(s_{ak}) s_{ab}}{s_{ak}} \log \frac{\mu^2 s_{ab}}{s_{ak} s_{bk}^{\max}} \right]_+^{s_{ak}^{\max}} - \log \frac{\mu^2 s_{ab}}{s_{ak} s_{bk}^{\max}} \left(1 - \frac{(d-2)s_{ak}}{4s_{ab}} \right) \right\} \\ &\times \Theta(Q^2, y, q_T^2; s_{bk} = 0) \delta^{(4)} \left(\tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ab}} \right) + \tilde{p}_a - \tilde{q} \right), \end{aligned} \quad (\text{E.50})$$

where we have used that $\tilde{k}^\mu(s_{bk} = 0) = s_{ak} \tilde{p}_b^\mu / s_{ab}$, see eq. (E.39). The poles in eq. (E.50) are collinear singularities, its cancellation follows a similar reasoning than for the soft poles in the previous section. They are originated due to a double counting of $\tilde{p}_a$ -collinear modes, therefore after subtracting the appropriate double counting, eq. (E.72), one is left with a finite contribution that is also independent of the scale μ^2 .

E.2.3 $\tilde{p}_b^\mu$ -collinear contribution

The contribution proportional to $\delta(s_{ak})$ is obtained from eqs. (E.40) and (E.41)

$$\begin{aligned}
 C_{b,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= \frac{2g_{ab,\perp}^{\mu\nu}}{\epsilon} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi s_{ak}^{\max}} \right)^\epsilon \int \frac{d\Omega_{d-3} ds_{ak} ds_{bk}}{(2\pi)^{3-2\epsilon}} \delta(s_{ak}) \\
 &\times \left\{ \left[\frac{\theta(s_{bk}) s_{ab}^{1+\epsilon}}{s_{bk}^{1+\epsilon}} \right]_+^{s_{bk}^{\max}} - \left(\frac{s_{ab}}{s_{bk}} \right)^\epsilon \left(1 - \frac{(d-2)s_{bk}}{4s_{ab}} \right) \right\} \\
 &\times \Theta(Q^2, y, q_T^2) \delta^{(d)}(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k}),
 \end{aligned} \tag{E.51}$$

where $g_{ab,\perp}^{\mu\nu}$ is the one defined in eq. (E.47). The $d-3$ dimensional solid angle integration can be performed trivially, since $\tilde{k}^\mu$ is independent of them at $s_{ak} = 0$. The integration in s_{ak} is also straightforward using the delta distribution. Expanding in ϵ one obtains

$$\begin{aligned}
 C_{b,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) &= \frac{2g_{ab,\perp}^{\mu\nu}}{(2\pi)^2} \int ds_{bk} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{bk}) s_{ab}}{s_{bk}} \right]_+^{s_{bk}^{\max}} - \frac{1}{\epsilon} \left(1 - \frac{(d-2)s_{bk}}{4s_{ab}} \right) \right. \\
 &+ \left. \left[\frac{\theta(s_{bk}) s_{ab}}{s_{bk}} \log \frac{\mu^2 s_{ab}}{s_{bk} s_{ak}^{\max}} \right]_+^{s_{bk}^{\max}} - \log \frac{\mu^2 s_{ab}}{s_{bk} s_{ak}^{\max}} \left(1 - \frac{(d-2)s_{bk}}{4s_{ab}} \right) \right\} \\
 &\times \Theta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} \right).
 \end{aligned} \tag{E.52}$$

where we have used that $\tilde{k}^\mu(s_{ak} = 0) = s_{bk} \tilde{p}_a^\mu / s_{ab}$, see eq. (E.39). The poles in eq. (E.50) are collinear singularities, its cancellation follows a similar reasoning than for the soft poles in the previous section. They are originated due to a double counting of $\tilde{p}_a$ -collinear modes, therefore after subtracting the appropriate double counting, eq. (E.79), one is left with a finite contribution that is also independent of the scale μ^2 .

E.2.4 Finite $s_{a(b)k}$ contribution

The finite $s_{a(b)k}$ contribution is given by eqs. (E.40), (E.41) and (E.42). It amounts to

$$\begin{aligned}
 C_{ab}^{\mu\nu,[1]}(s_{ak}^{\max}, s_{bk}^{\max}) &= -2 \int \frac{d\phi_{\tilde{k}} ds_{ak} ds_{bk}}{(2\pi)^3} \left\{ \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \right. \\
 &\times \left[2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - 2(\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) + \tilde{k}^\nu \tilde{p}_a^\mu + \tilde{k}^\mu \tilde{p}_a^\nu + \tilde{k}^\nu \tilde{p}_b^\mu + \tilde{k}^\mu \tilde{p}_b^\nu \right] \\
 &- \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \frac{2\tilde{p}_a^\mu \tilde{p}_a^\nu + 2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - \tilde{p}_b \cdot \tilde{k} g^{\mu\nu} + \tilde{p}_b^\mu \tilde{k}^\nu + \tilde{p}_b^\nu \tilde{k}^\mu - \tilde{p}_a^\mu \tilde{p}_b^\nu - \tilde{p}_a^\nu \tilde{p}_b^\mu}{s_{ab}} \\
 &- \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \frac{2\tilde{p}_b^\mu \tilde{p}_b^\nu + 2\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - \tilde{p}_a \cdot \tilde{k} g^{\mu\nu} + \tilde{p}_a^\mu \tilde{k}^\nu + \tilde{p}_a^\nu \tilde{k}^\mu - \tilde{p}_b^\mu \tilde{p}_a^\nu - \tilde{p}_b^\nu \tilde{p}_a^\mu}{s_{ab}} \Big\} \\
 &\times \Theta(Q^2, y, q_T^2) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k}).
 \end{aligned} \tag{E.53}$$

E.2.5 Boundary scale dependence

The coefficient functions $C_{ab}^{\mu\nu,[1]}$ eq. (E.53), $C_a^{\mu\nu,[1]}$ eq. (E.50), $C_b^{\mu\nu,[1]}$ eq. (E.52) and $S_{ab}^{\mu\nu,[1]}$ eq. (E.46) have separately a non-trivial dependence on the boundary scales $s_{a(b)k}^{\max}$ introduced in the plus distribution expansion process, nevertheless their sum is independent of $s_{a(b)k}^{\max}$. This can be shown performing the derivative with respect $s_{a(b)k}^{\max}$, one obtains it is zero

$$\begin{aligned} \frac{d}{d \log s_{a(b)k}^{\max}} [C_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(s_{ak}^{\max}, s_{bk}^{\max}) + C_{b,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) \\ + C_{a,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) + S_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max})] = 0. \end{aligned} \quad (\text{E.54})$$

Using eq. (A.9) in order to compute the derivative of a plus distribution with respect its boundary term one obtains for $s_{ak}^{\max}$

$$\begin{aligned} \frac{d}{d \log s_{ak}^{\max}} C_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(s_{ak}^{\max}, s_{bk}^{\max}) = \frac{4}{(2\pi)^2} \int ds_{bk} \left\{ \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \right. \\ \times \left[\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} - (\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu) \left(1 - \frac{s_{bk}}{2s_{ab}} \right) \right] \\ \left. - \frac{1}{s_{ab}} \left[\tilde{p}_a \cdot \tilde{p}_b g^{\mu\nu} \left(1 - \frac{s_{bk}}{2s_{ab}} \right) - \frac{\tilde{p}_a^\mu \tilde{p}_b^\nu + \tilde{p}_a^\nu \tilde{p}_b^\mu}{2} \left(1 - \frac{s_{bk}}{s_{ab}} \right) \right] \right\} \\ \times \Theta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} \right), \end{aligned} \quad (\text{E.55})$$

$$\begin{aligned} \frac{d}{d \log s_{ak}^{\max}} C_{b,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) = -\frac{2s_{ab}g_{ab,\perp}^{\mu\nu}}{(2\pi)^2} \int ds_{bk} \left\{ \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} - \frac{1}{s_{ab}} \left(1 - \frac{s_{bk}}{2s_{ab}} \right) \right\} \\ \times \Theta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} \right), \end{aligned} \quad (\text{E.56})$$

$$\begin{aligned} \frac{d}{d \log s_{ak}^{\max}} C_{a,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, s_{ak}^{\max}, s_{bk}^{\max}) = -\frac{2s_{ab}g_{ab,\perp}^{\mu\nu}}{(2\pi)^2} \left\{ \frac{1}{\epsilon} + \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} \right\} \\ \times \Theta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}), \end{aligned} \quad (\text{E.57})$$

$$\begin{aligned} \frac{d}{d \log s_{ak}^{\max}} S_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(\mu^2, \mu_{ak}^2, s_{ak}^{\max}, s_{bk}^{\max}) = \frac{2s_{ab}g_{ab,\perp}^{\mu\nu}}{(2\pi)^2} \left\{ \frac{1}{\epsilon} + \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} \right\} \\ \times \Theta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}), \end{aligned} \quad (\text{E.58})$$

where we have used the $\delta(s_{ak})$ obtained in the derivative to integrate in s_{ak} and since the measurement functions becomes independent of $\phi_{\tilde{k}}$ at $s_{ak} = 0$ the angular integral can be also performed trivially. One can see that the sum of eqs. (E.55), (E.56), (E.57) and (E.58) amounts to zero. For the $s_{bk}^{\max}$ the steps to follow are analogous.

E.2.6 Double counting subtraction

In the computation of a real emission at the hard scale level we must remove the contributions coming from the soft and collinear regions. One possible method to do so is by a phase-space slicing [234–244]. In this procedure cuts in the energies and angles with respect to the other partons are introduced, splitting the whole space in different regions. The main disadvantages of this method is the introduction of unphysical quantities; to prove that the resulting phase space as a whole is independent of those quantities could be highly non-trivial, sometimes it is even not possible to perform an analytical proof. In such a cases, the best one can do is a numerical confirmation that the dependence on a given cut is negligible compare to the aimed numerical precision, nevertheless this process involves taking those parameter to a certain limit that slows down the numerical evaluation of the phase space integrals, see [234]. Other alternative would be to introduce cut-off for the hard field integration measure in the path integral formulation, which can be rewritten as a boundary term for the hard fields. However this formulation may lead to cumbersome Feynman rules that make the perturbative evaluation of the correlation function rather involved. It is for these reasons that in the present work we chose to eliminate soft and collinear mode double counting by the definition of a zero bin and subtracting it from the fixed order evaluation of the correlation function of hard fields. This is motivated for the fact that path integral formulation of a QFT allows to show that the UV divergences are multiplicatively renormalizable, we also consider that we can remove a soft and collinear double counting by re-scaling the correlation function of hard fields by a suitable perturbative factor that we dub zero bin.

p_a -collinear sector We define the double counting bin subtracting term for the p_a -collinear sector as

$$\begin{aligned} \hat{f}_{\text{Zb},a}(z_a^+, 0_b^-, z_{ab\perp}; p_a) &= \frac{1}{2n_b \cdot p_a} \sum_X \langle p_a | \bar{T} \{ \bar{\psi}_j^c(z_a^+ n_b + z_{ab\perp}) [z_a^+ n_b + z_{ab\perp}, -\infty n_b + z_{ab\perp}]^{ca} \} | X \rangle \\ &\times \frac{(\not{p}_b)_{ji}}{2} \langle X | T \{ [-\infty n_b, 0]^{ab} \psi_i^b(0) \} | p_a \rangle, \end{aligned} \quad (\text{E.59})$$

when the parton a is a quark, while for an anti-quark it is given by

$$\begin{aligned} \hat{f}_{\text{Zb},a}(z_a^+, 0_b^-, z_{ab\perp}; p_a) &= \frac{1}{2n_b \cdot p_a} \sum_X \langle p_a | \bar{T} \{ [-\infty n_b + z_{ab\perp}, z_a^+ n_b + z_{ab\perp}]^{ac} \psi_i^c(z_a^+ n_b + z_{ab\perp}) \} | X \rangle \\ &\times \frac{(\not{p}_b)_{ji}}{2} \langle X | T \{ \bar{\psi}_j^b(0) [0, -\infty n_b]^{ba} \} | p_a \rangle, \end{aligned} \quad (\text{E.60})$$

where i, j and a, b, c are spin and color indices respectively. The Wilson lines are in the fundamental representation. The factor $(2n_b \cdot p_a)^{-1}$ above is a normalization factor, such that at LO $(\alpha_s^0) \hat{f}_{\text{Zb},a}^{[0]}(p_a, \mu^2) = \exp\{ip_a \cdot z\}$.

p_b -collinear sector We define the double counting subtracting term for the p_a -collinear sector as

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$$\begin{aligned}\hat{f}_{Zb,b}(0_a^+, z_b^-, z_{ab\perp}; p_b) &= \frac{1}{2n_a \cdot p_b} \sum_X \langle p_b | \bar{T} \{ \bar{\psi}_j^c(z_b^- n_a + z_{ab\perp}) [z_b^- n_a + z_{ab\perp}, -\infty n_a + z_{ab\perp}]^{ca} \} | X \rangle \\ &\times \frac{(\not{p}_a)_{ji}}{2} \langle X | T \{ [-\infty n_a, 0]^{ab} \psi_i^b(0) \} | p_b \rangle,\end{aligned}\tag{E.61}$$

when the parton a is a quark, while for an anti-quark it is given by

$$\begin{aligned}\hat{f}_{Zb,a}(0_a^+, z_b^-, z_{ab\perp}; p_b) &= \frac{1}{2n_a \cdot p_b} \sum_X \langle p_b | \bar{T} \{ [-\infty n_a + z_{ab\perp}, z_b^- n_a + z_{ab\perp}]^{ac} \psi_i^c(z_b^- n_b + z_{ab\perp}) \} | X \rangle \\ &\times \frac{(\not{p}_a)_{ji}}{2} \langle X | T \{ \bar{\psi}_j^b(0) [0, -\infty n_a]^{ba} \} | p_b \rangle,\end{aligned}\tag{E.62}$$

where i, j and a, b, c are spin and color indices respectively. The Wilson lines are in the fundamental representation. The factor $(2n_a \cdot p_b)^{-1}$ above is a normalization factor, such that at LO $(\alpha_s^0) \hat{f}_{Zb,b}^{[0]}(p_b, \mu^2) = \exp\{ip_b \cdot z\}$.

Soft sector We define the double counting subtraction for the soft modes as

$$\begin{aligned}\hat{\phi}_{Zb,ab}(0_a^+, 0_b^-, z_{ab\perp}) &= \sum_X \langle 0 | \bar{T} \{ [-\infty n_a + z_{ab\perp}, z_{ab\perp}]^{ac} [z_{ab\perp}, -\infty n_b + z_{ab\perp}]^{cb} \} | X \rangle \\ &\times \langle X | T \{ [-\infty n_b, 0]^{bd} [0, -\infty n_a]^{da} \} | 0 \rangle,\end{aligned}\tag{E.63}$$

where a, b, c, d are color indices and the Wilson lines are in the fundamental representation.

The vector $n_{a(b)}^\mu$ is a light-like vector whose spatial component is unit vector along the direction dictated by $p_{a(b)}^\mu$ and we have defined $z_a^+ = z \cdot n_a$, $z_b^- = z \cdot n_b$ and $z_{ab\perp}^\mu = g_{ab,\perp}^{\mu\nu} z_\nu$. In the partonic center-of-mass frame they are given by eqs. (E.48), $g_{ab,\perp}^{\mu\nu}$ is given by eq. (E.47).

Double counting subtracting hadronic tensor Once we have defined the double counting subtracting terms for the $p_{a(b)}$ -collinear, $\hat{f}_{Zb,a(b)}$, and for the soft, $\hat{\phi}_{Zb,ab}$, sector we construct the double counting subtraction at the level of the hadronic tensor as

$$\begin{aligned}W_{Zb,GG'}^{\mu\nu}(q, Q^2, y, q_T^2) &= \frac{P_A^- P_B^+}{N_c} \sum_{a,b} \int \frac{d^d z}{(2\pi)^d} e^{-iz \cdot q} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ &\times \text{Tr}[\not{p}_a \Gamma_{G'}^\nu \not{p}_b \Gamma_G^\mu] f_{n,a \leftarrow p_A}(\xi_a, \mathbf{p}_{aT}) f_{\bar{n},b \leftarrow p_B}(\xi_b, \mathbf{p}_{bT}) \\ &\times \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 + q^2 - 2(\bar{n} \cdot q)(n \cdot q)) \\ &\times \frac{\hat{f}_{Zb,a}(p_a, \mu^2; z_a^+, 0_b^-, z_{ab\perp}) \hat{f}_{Zb,b}(p_b, \mu^2; 0_a^+, z_b^-, z_{ab\perp})}{\hat{\phi}_{Zb,ab}(\mu^2, 0_a^+, 0_b^-, z_{ab\perp})} \\ &+ n \leftrightarrow \bar{n} + \dots,\end{aligned}\tag{E.64}$$

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where $[d^4 p_{a(b)}] = d^4 p_{a(b)} \delta(p_{a(b)}^2)$. Expanding up to NLO in α_s for the collinear and soft sectors one obtains

$$\begin{aligned}
W_{Zb,GG'}^{\mu\nu,[1]}(q, Q^2, y, q_T^2) &= \frac{4\pi C_F \alpha_s(\mu^2) P_A^- P_B^+}{N_c} \sum_{a,b} \int_0^1 d\xi_a d\xi_b \int [d^4 p_a][d^4 p_b] \text{Tr}[\not{p}_a \Gamma_{G'}^\nu \not{p}_b \Gamma_G^\mu] \\
&\times f_{a \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}^2, \mu^2, \zeta_a) f_{b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}^2, \mu^2, \zeta_b) \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
&\times \left[C_{Zb,a}(Q^2, y, q_T^2) + C_{Zb,b}(Q^2, y, q_T^2) - S_{Zb}(Q^2, y, q_T^2) \right],
\end{aligned} \tag{E.65}$$

where S_{Zb} denotes the contributions from the partonic soft function defined in eq. (E.63); $C_{Zb,a}$ denotes the partonic p_a -collinear sector given in eqs. (E.59) and (E.60) and $C_{Zb,b}$ refers to the partonic p_b -collinear contribution defined by eqs. (E.61) and (E.62). We have dropped the LO term $\frac{\hat{f}_a^{[0]} \hat{f}_b^{[0]}}{\hat{\phi}_{ab}^{[0]}}$, since it corresponds to the no real emission case. Notice also that the coefficients $C_{Zb,a(b)}$ and S_{Zb} are independent of the effect electroweak vertex structure G , since they account for the IR behavior of a QCD scattering amplitude.

The cumulant counterpart of eq. (E.64) is given by

$$W_{Zb,GG'}^{\mu\nu,[1]}(q) \Theta(Q^2, y, q_T^2) = \int_{Q_{\min}^2}^{Q_{\max}^2} dQ^2 \int_{y_{\min}}^{y_{\max}} dy \int_{q_{T\min}^2}^{q_{T\max}^2} dq_T^2 W_{GG'}^{\mu\nu,[1]}(q, Q^2, y, q_T^2). \tag{E.66}$$

E.2.6.1 Partonic soft double counting

$S_{Zb}(Q^2, y, q_T^2)$ in eq. (E.65) is explicitly given by

$$\begin{aligned}
S_{Zb}(Q^2, y, q_T^2) &= \int \frac{d^d z}{(2\pi)^d} e^{-iz \cdot q + iz \cdot p_a + iz \cdot p_b} \int \frac{d^{d-1} \mathbf{k}}{(2\pi)^{d-1} 2k^0} e^{-ik_{ab\perp} \cdot z_{ab\perp}} \frac{2n_a \cdot n_b}{(n_a \cdot k)(n_b \cdot k)} \\
&\times \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 + q^2 - 2(\bar{n} \cdot q)(n \cdot q)).
\end{aligned} \tag{E.67}$$

As in the previous section, the explicit evaluation of eq. (E.67) is performed in the partonic centre-of-mass frame where the vectors $p_{a(b)}^\mu$ and $n_{a(b)}^\mu$ take a simple form. On the other hand, the measurement functions become rather cumbersome making the analytical integration extremely challenging; nevertheless the integration in $\tilde{k}^\mu$ is performed numerically after the soft singularities are cancelled in the sum of the double counting subtracting defined by eq. (E.67) and the soft contribution given by eq. (E.46). Performing the changes of variables defined in eqs. (E.30), eq. (E.67) amounts

$$\begin{aligned}
S_{Zb}(Q^2, y, q_T^2) &= \frac{1}{(2\pi)^{3-2\epsilon}} \left(\frac{\mu^2 s_{ab} e^{\gamma_E}}{4\pi \mu_{ak}^2 \mu_{bk}^2} \right)^\epsilon \int d\Omega_{d-3} ds_{ak} ds_{bk} \frac{\mu_{ak}^{2\epsilon} \mu_{bk}^{2\epsilon} \theta(s_{ak}) \theta(s_{bk})}{(s_{ak})^{1+\epsilon} (s_{bk})^{1+\epsilon}} \\
&\times \delta(Q^2 - \tilde{q}^2) \delta\left(y - \frac{1}{2} \log \frac{\tilde{n} \cdot \tilde{q}}{\tilde{n} \cdot \tilde{q}}\right) \delta(q_T^2 + \tilde{q}^2 - 2(\tilde{n} \cdot \tilde{q})(\tilde{n} \cdot \tilde{q})) \delta^{(d)}(\tilde{p}_a + \tilde{p}_b - \tilde{q}_{ab} - \tilde{k}_{ab\perp})
\end{aligned} \tag{E.68}$$

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where the integration measure is the one given by eq. (E.31) and the vector boson momentum in the partonic center-of-mass frame amounts to

$$\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \tilde{k}_{ab\perp}^\mu. \quad (\text{E.69})$$

It must be notice that the measurement functions in eq. (E.68) depend only on the orthogonal component to $\tilde{n}_{a(b)}$ of the emitted parton momentum i.e. $\tilde{k}_{ab\perp}^\mu$. This is the main different compare to the computation performed in previous section, where the measurements functions have a full dependence on $\tilde{k}^\mu$, see eqs. (E.68), (E.23) and (E.24). Expanding in plus distributions the denominators in eq. (E.68) one obtains

$$\begin{aligned} S_{\text{Zb}}(Q^2, y, q_T^2) &= \frac{1}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \Delta(Q^2, y, q_T^2) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{k}_{ab\perp} - \tilde{q}) \\ &+ \frac{1}{(2\pi)^2} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} + \frac{1}{2} \log^2 \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} - \frac{\pi^2}{12} \right\} \\ &\times \Delta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\ &- \frac{1}{(2\pi)^2} \Delta(Q^2, y, q_T^2; s_{bk} = 0) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\ &\times \int ds_{ak} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} + \left[\frac{\theta(s_{ak})}{s_{ak}} \log \frac{s_{ab} \mu^2}{s_{bk}^{\max} s_{ak}} \right]_+^{s_{ak}^{\max}} \right\} \\ &- \frac{1}{(2\pi)^2} \Delta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)}(\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\ &\times \int ds_{bk} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} + \left[\frac{\theta(s_{bk})}{s_{bk}} \log \frac{s_{ab} \mu^2}{s_{ak}^{\max} s_{bk}} \right]_+^{s_{bk}^{\max}} \right\}, \end{aligned} \quad (\text{E.70})$$

where we have gathered the measurement functions into $\Delta(Q^2, y, q_T^2)$

$$\Delta(Q^2, y, q_T^2) = \delta(Q^2 - \tilde{q}^2) \delta\left(y - \frac{1}{2} \log \frac{\tilde{n} \cdot \tilde{q}}{\tilde{n} \cdot \tilde{q}}\right) \delta(q_T^2 + \tilde{q}^2 - 2(\tilde{n} \cdot \tilde{q})(\tilde{n} \cdot \tilde{q})).$$

Notice that we have pulled out the measurement functions from the integral on third and fifth lines, since they are independent of $s_{a(b)}$. This is a simply consequence that $\tilde{q}^\mu$ depends only on $\tilde{k}_{ab\perp}^\mu$, which amounts to zero at $s_{a(b)k} = 0$; see eqs. (E.39). Therefore for the fourth and sixth lines the upper bound is undefined, potentially leading to logarithmic UV divergences, however those terms are cancelled out once one includes the contributions from partonic $p_{a(b)}$ -collinear sector in the following sections.

E.2.6.2 Partonic p_a -collinear double counting

$C_{\text{Zb},a}(Q^2, y, q_T^2)$ in eq. (E.65) at NLO amounts to

$$\begin{aligned} C_{\text{Zb},a}(Q^2, y, q_T^2) &= \frac{1}{n_b \cdot p_a} \int \frac{d^d z}{(2\pi)^d} e^{-iz \cdot q + iz \cdot p_b} \int \frac{d^{d-1} \mathbf{k}}{(2\pi)^{d-1} 2k^0} e^{iz \cdot (p_a - k)} \\ &\times \left\{ \frac{(1-\epsilon)k \cdot n_b - 2n_b \cdot p_a}{p_a \cdot k} + \frac{2(n_b \cdot p_a)^2}{(p_a \cdot k)(n_b \cdot k)} \right\} \\ &\times \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\tilde{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 + q^2 - 2(\tilde{n} \cdot q)(n \cdot q)). \end{aligned} \quad (\text{E.71})$$

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Working in the partonic center-of-mass frame and performing the changes of variables defined in eqs. (E.30) the $\tilde{p}_a$ -collinear double counting subtraction integrated against the measurement functions amounts to

$$\begin{aligned}
C_{Zb,a}(Q^2, y, q_T^2) = & -\frac{1}{2(2\pi)^3 s_{ab}^2} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} (2s_{ab} - s_{bk}) \\
& \times \Delta(Q^2, y, q_T^2) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} - \tilde{k}_{ab\perp} \right) \\
& + \frac{1}{2(2\pi)^2 s_{ab}^2} \int ds_{bk} \left\{ (2s_{ab} - s_{bk}) \left(\frac{1}{\epsilon} + \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} \right) + s_{bk} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} \right) \\
& + \frac{1}{(2\pi)^2} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} + \frac{1}{2} \log^2 \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} - \frac{\pi^2}{12} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\
& + \frac{1}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \Delta(Q^2, y, q_T^2) \quad (E.72) \\
& \times \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} - \tilde{k}_{ab\perp} \right) \\
& - \frac{1}{(2\pi)^2} \Delta(Q^2, y, q_T^2; s_{bk} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\
& \times \int ds_{ak} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} + \left[\frac{\theta(s_{ak})}{s_{ak}} \log \frac{\mu^2 s_{ab}}{s_{ak} s_{bk}^{\max}} \right]_+^{s_{ak}^{\max}} \right\} \\
& - \frac{1}{(2\pi)^2} \int ds_{bk} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} + \left[\frac{\theta(s_{bk})}{s_{bk}} \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}} \right]_+^{s_{bk}^{\max}} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} \left(\tilde{p}_a \left(1 - \frac{s_{bk}}{s_{ab}} \right) + \tilde{p}_b - \tilde{q} \right),
\end{aligned}$$

the $\tilde{q}^\mu$ in the measurement functions, $\Delta(Q^2, y, q_T^2)$, is given by

$$\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \frac{n_b \cdot \tilde{k}}{2} n_a^\mu - \tilde{k}_{ab\perp}^\mu = \left(1 - \frac{s_{bk}}{s_{ab}} \right) \tilde{p}_a^\mu + \tilde{p}_b^\mu - \tilde{k}_{ab\perp}^\mu, \quad (E.73)$$

where we have used $n_{a(b)} \cdot \tilde{k} = \tilde{k}^- = \frac{s_{a(b)k}}{2\tilde{p}_{a(b)}^0 \cos \theta}$. Notice that the fifth line in eq. (E.72) cancels the third line in eq. (E.70).

The integration in s_{ak} and s_{bk} on the first line on eq. (E.72) has to be handle cautiously. It is due to the dependence on s_{ak} of the integrand is through the combination $s_{ak}s_{bk}$ rather than s_{ak} . The limit $s_{bk} \rightarrow 0$ makes the measurement functions s_{ak} independent and thus there is no a definite upper bound in the s_{ak} integration; it is for this reason we perform the change of variable: $s_{ak} = k_{ak}^2 s_{ab} / s_{ak}$. The new integration variable, k_{ak}^2 , is independent of the limit $s_{bk} \rightarrow 0$. In order to show that the resulting integral is well-defined in the limit $s_{bk} \rightarrow 0$ we

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consider the integral

$$I(s_{bk}) = \int_0^\infty ds_{ak} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} g(s_{bk}, s_{ak}s_{bk}) \theta(f(s_{bk}, s_{ak}s_{bk})), \quad (\text{E.74})$$

where g and f are analytic functions of their arguments. The function $\theta(f(s_{bk}, s_{bk}s_{ak}))$ can be any of the θ function in $\Theta(Q, y, q_T^2)$ obtained after computing the cross section cumulants. Writing explicitly the definition of the plus distribution one obtains

$$\begin{aligned} I(s_{bk}) &= \lim_{\beta \rightarrow 0} \int_\beta^\infty ds_{ak} \frac{g(s_{bk}, s_{ak}s_{bk}) \theta(f(s_{bk}, s_{ak}s_{bk}))}{s_{ak}} + \lim_{\beta \rightarrow 0} g(s_{bk}, \beta s_{bk}) \theta(f(s_{bk}, \beta s_{bk})) \log \frac{\beta}{s_{ak}^{\max}} \\ &= \lim_{\beta \rightarrow 0} \int_{\frac{\beta s_{bk}}{s_{ab}}}^\infty dk_{ak}^2 \frac{g(s_{bk}, k_{ak}^2 s_{ab}) \theta(f(s_{bk}, k_{ak}^2 s_{ab}))}{k_{ak}^2} + \lim_{\beta \rightarrow 0} g(s_{bk}, \beta s_{bk}) \theta(f(s_{bk}, \beta s_{bk})) \log \frac{\beta}{s_{ak}^{\max}}. \end{aligned} \quad (\text{E.75})$$

We split the integration in k_{ak}^2 into two regions: $\infty > k_{ak}^2 \geq s_{ak}^{\max}$ and $s_{ak}^{\max} \geq k_{ak}^2 \geq \frac{\beta s_{bk}}{s_{ab}}$

$$\begin{aligned} I(s_{bk}) &= \int_{s_{ak}^{\max}}^\infty dk_{ak}^2 \frac{g(s_{bk}, k_{ak}^2 s_{ab}) \theta(f(s_{bk}, k_{ak}^2 s_{ab}))}{k_{ak}^2} + \lim_{\beta \rightarrow 0} \int_{\frac{\beta s_{bk}}{s_{ab}}}^{s_{ak}^{\max}} dk_{ak}^2 \frac{g(s_{bk}, k_{ak}^2 s_{ab}) \theta(f(s_{bk}, k_{ak}^2 s_{ab}))}{k_{ak}^2} \\ &\quad + \lim_{\beta \rightarrow 0} g(s_{bk}, \beta s_{bk}) \theta(f(s_{bk}, \beta s_{bk})) \log \frac{\beta}{s_{ak}^{\max}}. \end{aligned} \quad (\text{E.76})$$

The first integral on the right-hand side is well-defined and $\theta(f(s_{bk}, k_{ak}^2 s_{ab}))$ imposes an upper bound. The second integral on the right-hand side contains a logarithmic divergence in the limit $\beta \rightarrow 0$ that is cancelled out by the contribution on the second line. We can make this cancellation explicit writing the second integral as

$$\begin{aligned} &\lim_{\beta \rightarrow 0} \int_{\frac{\beta s_{bk}}{s_{ab}}}^{s_{ak}^{\max}} dk_{ak}^2 \frac{g(s_{bk}, k_{ak}^2 s_{ab}) \theta(f(s_{bk}, k_{ak}^2 s_{ab}))}{k_{ak}^2} + \lim_{\beta \rightarrow 0} g(s_{bk}, \beta s_{bk}) \theta(f(s_{bk}, \beta s_{bk})) \log \frac{\beta}{s_{ak}^{\max}} \\ &= \int_0^{s_{ak}^{\max}} dk_{ak}^2 \frac{g(s_{bk}, k_{ak}^2 s_{ab}) \theta(f(s_{bk}, k_{ak}^2 s_{ab})) - g(s_{bk}, 0) \theta(f(s_{bk}, 0))}{k_{ak}^2} - g(s_{bk}, 0) \theta(f(s_{bk}, 0)) \log \frac{s_{bk}}{s_{ab}}. \end{aligned} \quad (\text{E.77})$$

The dependence on s_{bk} in the function $g(s_{bk}, 0)$ is polynomial, while in $\theta(f(s_{bk}, 0))$, it is simple enough to exactly solve the constraints. Therefore, one can perform analytically the integration over s_{bk} . This is a great advantage, since the presence of $\log(s_{bk}/s_{ab})$ could slow down a numerical evaluation.

E.2.6.3 Partonic p_b -collinear double counting

$C_{Zb,b}(Q^2, y, q_T^2)$ in eq. (E.65) at NLO amounts to

$$\begin{aligned} C_{Zb,b}(Q^2, y, q_T^2) &= \frac{1}{n_a \cdot p_b} \int \frac{d^d z}{(2\pi)^d} e^{-iz \cdot q + iz \cdot p_a} \int \frac{d^{d-1} \mathbf{k}}{(2\pi)^{d-1} 2k^0} e^{iz \cdot (p_b - k)} \\ &\quad \times \left\{ \frac{(1-\epsilon)k \cdot n_a - 2n_b \cdot p_a}{p_b \cdot k} + \frac{2(n_a \cdot p_b)^2}{(p_b \cdot k)(n_a \cdot k)} \right\} \\ &\quad \times \delta(Q^2 - q^2) \delta\left(y - \frac{1}{2} \log \frac{\bar{n} \cdot q}{n \cdot q}\right) \delta(q_T^2 + q^2 - 2(\bar{n} \cdot q)(n \cdot q)). \end{aligned} \quad (\text{E.78})$$

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Working in the partonic center-of-mass frame and performing the changes of variables defined in eqs. (E.30) the $\tilde{p}_b$ -collinear double counting subtraction integrated against the measurement functions amounts to

$$\begin{aligned}
C_{Zb,b}(Q^2, y, q_T^2) = & -\frac{1}{2(2\pi)^3 s_{ab}^2} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} (2s_{ab} - s_{ak}) \\
& \times \Delta(Q^2, y, q_T^2) \delta^{(4)} \left(\tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ak}} \right) + \tilde{p}_a - \tilde{q} - \tilde{k}_{ab\perp} \right) \\
& + \frac{1}{2(2\pi)^2 s_{ab}^2} \int ds_{ak} \left\{ (2s_{ab} - s_{ak}) \left(\frac{1}{\epsilon} + \log \frac{\mu^2 s_{ab}}{s_{bk}^{\max} s_{ak}} \right) + s_{ak} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{bk} = 0) \delta^{(4)} \left(\tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ak}} \right) + \tilde{p}_a - \tilde{q} \right) \\
& + \frac{1}{(2\pi)^2} \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} + \frac{1}{2} \log^2 \frac{\mu^2 s_{ab}}{s_{ak}^{\max} s_{bk}^{\max}} - \frac{\pi^2}{12} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{a(b)k} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\
& + \frac{1}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \Delta(Q^2, y, q_T^2) \quad (E.79) \\
& \times \delta^{(4)} \left(\tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ak}} \right) + \tilde{p}_a - \tilde{q} - \tilde{k}_{ab\perp} \right) \\
& - \frac{1}{(2\pi)^2} \Delta(Q^2, y, q_T^2; s_{ak} = 0) \delta^{(4)} (\tilde{p}_a + \tilde{p}_b - \tilde{q}) \\
& \times \int ds_{bk} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} + \left[\frac{\theta(s_{bk})}{s_{bk}} \log \frac{\mu^2 s_{ab}}{s_{bk} s_{ak}^{\max}} \right]_+^{s_{bk}^{\max}} \right\} \\
& - \frac{1}{(2\pi)^2} \int ds_{ak} \left\{ \frac{1}{\epsilon} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} + \left[\frac{\theta(s_{ak})}{s_{ak}} \log \frac{\mu^2 s_{ab}}{s_{ak} s_{bk}^{\max}} \right]_+^{s_{bk}^{\max}} \right\} \\
& \times \Delta(Q^2, y, q_T^2; s_{bk} = 0) \delta^{(4)} \left(\tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ak}} \right) + \tilde{p}_a - \tilde{q} \right)
\end{aligned}$$

the $\tilde{q}^\mu$ in the measurement functions, $\Delta(Q^2, y, q_T^2)$, is given by

$$\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \frac{n_a \cdot \tilde{k}}{2} n_b^\mu - \tilde{k}_{ab\perp}^\mu = \tilde{p}_b \left(1 - \frac{s_{ak}}{s_{ab}} \right) + \tilde{p}_a - \tilde{k}_{ab\perp}^\mu, \quad (E.80)$$

where we have used $n_{a(b)} \cdot \tilde{k} = \tilde{k}^- = \frac{s_{a(b)k}}{2\tilde{p}_{a(b)}^0 \cos \theta}$. Notice that the fifth line in eq. (E.79) cancels the forth line in eq. (E.70).

The first line in eq. (E.79) suffers from the similar issue discussed in eq. (E.72). The only difference is that now it is the integration over s_{bk} that must be done cautiously. In this case, we perform the change of variable $s_{bk} = k_{bk}^2 s_{ab}/s_{ak}$. From here, the steps to follow are analogous to those already discussed for s_{ak} .

E.2.7 Real emission with double counting subtracted

Once the double counting subtraction for real emission contributions has been defined and computed in Sec. E.2.6, it is straightforward to remove those contribution from the hadronic tensor given in eq. (E.10). We define the double counting subtracted hadronic tensor with one real emission as follow

$$\tilde{W}_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2) = W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2) - W_{Zb,GG'}^{\mu\nu,[1]}(q)\Theta(Q^2, y, q_T^2), \quad (\text{E.81})$$

notice that in $W_{Zb,GG'}^{\mu\nu,[1]}$ the functions $S_{Zb}(Q^2, y, q_T^2)$, $C_{Zb,a}(Q^2, y, q_T^2)$ and $C_{Zb,b}(Q^2, y, q_T^2)$ have been integrated in Q^2 , y and q_T^2 , thus their measurement functions become $\Theta(Q^2, y, q_T^2)$. It can be verified that eq. (E.81) is free of collinear and soft divergences. For the case of $GG' = \gamma^*\gamma^*$ we have explicitly written down the corresponding coefficients in $W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2)$ and $W_{Zb,GG'}^{\mu\nu,[1]}(q)\Theta(Q^2, y, q_T^2)$. The sum of the soft poles in eqs. (E.70) and (E.46) cancels against the sum of eqs. (E.72) and (E.79). The collinear poles in eqs. (E.50) and (E.52) are cancelled by the remaining poles in the sum of eq. (E.72) together with eq. (E.72). Since one obtains a cancellation of the soft and collinear poles, it follows that the cancellation of logarithmic terms with μ^2 dependence also occurs. An extremely important point to keep in mind is that the measurements functions in $W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2)$ have a full dependence on $\tilde{k}^\mu$ through $\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \tilde{k}^\mu$, while the dependence on $W_{Zb,GG'}^{\mu\nu,[1]}(q)\Theta(Q^2, y, q_T^2)$ depends whether one is considering soft or $p_{a(b)}$ -collinear sector. This differences on the argument of measurements functions is irrelevant in the cancellation of poles and scale dependence since after employing the delta distributions to integrate $s_{a(b)k}$ the measurements functions end up having similar arguments. Thus, it is the finite and scale-independent contributions (up to the obvious dependence in the coupling constant) that these differences on the argument of measurement functions becomes relevant. Hence, after the cancellation of soft and collinear divergences eq. (E.81) for $GG' = \gamma^*\gamma^*$ amounts to

$$\begin{aligned} \tilde{W}_{\gamma^*\gamma^*}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2) &= \frac{4\pi C_F \alpha_s(\mu^2) P_A^- P_B^+}{N_c} \sum_{a,b} \int [d^4 p_a] d\xi_a [d^4 p_b] d\xi_b \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ &\times \left\{ C_{ab,\gamma^*\gamma^*}^{\mu\nu,[1]}(s_{ak}^{\max}, s_{bk}^{\max}) + \frac{2s_{ab}g_{ab,\perp}^{\mu\nu}}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \Theta(Q^2, y, q_T^2) \right. \\ &\times \left[\delta^{(4)}\left(\tilde{p}_a\left(1 - \frac{s_{bk}}{s_{ab}}\right) + \tilde{p}_b - \tilde{q} - \tilde{k}_{ab\perp}\right) + \delta^{(4)}\left(\tilde{p}_b\left(1 - \frac{s_{ak}}{s_{ab}}\right) + \tilde{p}_a - \tilde{q} - \tilde{k}_{ab\perp}\right) - \delta^{(4)}\left(\tilde{p}_a + \tilde{p}_b - \tilde{q} - \tilde{k}_{ab\perp}\right) \right] \\ &- \frac{2}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{ak})}{s_{ak}} \right]_+^{s_{ak}^{\max}} \left(1 - \frac{s_{bk}}{2s_{ab}}\right) \Theta(Q^2, y, q_T^2) \delta^{(4)}\left(\tilde{p}_a\left(1 - \frac{s_{bk}}{s_{ab}}\right) + \tilde{p}_b - \tilde{q} - \tilde{k}_{ab\perp}\right) \\ &- \frac{2}{(2\pi)^3} \int d\phi_{\tilde{k}} ds_{ak} ds_{bk} \left[\frac{\theta(s_{bk})}{s_{bk}} \right]_+^{s_{bk}^{\max}} \left(1 - \frac{s_{ak}}{2s_{ab}}\right) \Theta(Q^2, y, q_T^2) \delta^{(4)}\left(\tilde{p}_b\left(1 - \frac{s_{ak}}{s_{ab}}\right) + \tilde{p}_a - \tilde{q} - \tilde{k}_{ab\perp}\right) \Big\} \\ &\times f_{a\leftarrow P_A}(\alpha_s(\mu^2), \xi_a, \mathbf{p}_{aT}^2, \zeta_a, \mu^2) f_{b\leftarrow P_B}(\alpha_s(\mu^2), \xi_b, \mathbf{p}_{bT}^2, \zeta_b, \mu^2). \end{aligned} \quad (\text{E.82})$$

Notice that in order to obtain a complete cancellation of the soft and collinear poles we must use the same boundary condition in the plus distributions in $W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2)$ and $W_{GG'}^{\mu\nu,[NLO,R]}(q)\Theta(Q^2, y, q_T^2)$. Thus, it is easy to verify that eq. (E.82) is independent of $s_{a(b)}^{\max}$

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i.e.

$$\frac{d}{\log s_{a(b)k}^{\max}} \tilde{W}_{GG'}^{\mu\nu, [NLO, R]}(q) \Theta(Q^2, y, q_T^2) = 0. \quad (\text{E.83})$$

To conclude we discuss the evaluation of the hadronic tensor in eq. (E.82). The integrand is an analytic function of its arguments, hence it can be evaluated numerically and in four dimensions. Due to the large number of integration variables deterministic numerical methods are inefficient, it is for this reason that Monte-Carlo integration seems to be the most suitable approach.

F NNLO Real-Real

F.1 Channel: $q(\vec{q})+q(\vec{q}) \rightarrow F+q(\vec{q})+q(\vec{q})$

The contributions to the hard partonic process $q(\vec{q})(p_a)+q(\vec{q})(p_b) \rightarrow F(q)+q(\vec{q})(k)+q(\vec{q})(l)$ can be classified in two groups:

- Diagrams that the sum amounts to a finite contribution up to a simple collinear pole, these are displayed in fig. F.2.
- Diagrams that the sum amounts to a completely finite contribution, these are displayed in figs. F.3 and F.4.

As we did for the case of one real emission, the phase space is parametrized in the partonic center of mass frame $\mathcal{S}_{\text{p.C.M.}}$, see fig. E.1. Therefore, it is convenient to define the invariant masses

$$\begin{aligned}
 s_{ak} &= (\tilde{p}_a + \tilde{k})^2 = 2\tilde{p}_a \cdot \tilde{k}, & s_{al} &= (\tilde{p}_a + \tilde{l})^2 = 2\tilde{p}_a \cdot \tilde{l}, \\
 s_{bk} &= (\tilde{p}_b + \tilde{k})^2 = 2\tilde{p}_b \cdot \tilde{k}, & s_{bl} &= (\tilde{p}_b + \tilde{l})^2 = 2\tilde{p}_b \cdot \tilde{l}, \\
 s_{aq} &= (\tilde{p}_a - \tilde{q})^2 = (\tilde{k} + \tilde{l} - \tilde{p}_b)^2, & s_{bq} &= (\tilde{p}_b - \tilde{q})^2 = (\tilde{k} + \tilde{l} - \tilde{p}_a)^2, \\
 s_{qk} &= (\tilde{k} + \tilde{q})^2 = (\tilde{p}_a + \tilde{p}_b - \tilde{l})^2, & s_{ql} &= (\tilde{l} + \tilde{q})^2 = (\tilde{p}_a + \tilde{p}_b - \tilde{k})^2, & s_{ak} &= (\tilde{k} + \tilde{l})^2 = 2\tilde{k} \cdot \tilde{l}.
 \end{aligned}
 \tag{F.1}$$

The invariant masses $s_{k(l)q}$ are bounded from below by $Q^2 = \tilde{q}^2$; in the case of invariant masses of the form $s_{a(b)k}$ and $s_{a(b)l}$ they become zero when the emitted parton becomes either collinear to the initial state parton, e.g. $\tilde{k}$ becomes collinear to $\tilde{p}_b$, or soft; for the invariant masses $s_{a(b)q}$ the situation is slightly more involved to determine in which limits they go to zero, using the methods presented in [26, 104, 105] one can conclude that $s_{a(b)q}$ goes to zero only when $s_{b(a)k}$ and $s_{b(a)l}$ go to zero but their ratio is finite, i.e. $s_{b(a)k}/s_{b(a)l} \sim 1$; finally the invariant mass s_{kl} is zero when the two emitted partons are either soft or parallel to each other, it is this type of invariant mass which leads to the most complicated singularities to handle, see [26, 104, 104], nevertheless it never appears in a denominator in the set of diagrams we computed.

Before we start discussing how the different set of diagrams cancel between them the soft and collinear singularities it is worthy to discuss in detail how to isolate the singularities arising from terms like $1/s_{a(b)q}^2$. Consider for instance s_{aq} , it amounts to

$$s_{aq} = s_{kl} - s_{bk} - s_{bl} = 2\tilde{k} \cdot \tilde{l} - s_{bk} - s_{bl} = \tilde{k}^- \tilde{l}^+ + \tilde{k}^+ \tilde{l}^- - 2|\tilde{\mathbf{k}}_\perp||\tilde{\mathbf{l}}_\perp| \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - s_{bk} - s_{bl},
 \tag{F.2}$$

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where $|\tilde{\mathbf{k}}_\perp| = \sqrt{\tilde{k}^- \tilde{k}^+}$, $|\tilde{\mathbf{l}}_\perp| = \sqrt{\tilde{l}^- \tilde{l}^+}$, $v^+ = v \cdot n_a$ and $v^- = v \cdot n_b$. Using eqs. (E.30) one can relate $\tilde{k}^+(\tilde{l}^+)$ and $\tilde{k}^-(\tilde{l}^-)$ with $s_{ak(l)}$ and $s_{bk(l)}$

$$\begin{aligned}\tilde{k}^+ &= \frac{s_{ak}}{2\tilde{p}_a^0 \cos \theta}, & \tilde{l}^+ &= \frac{s_{al}}{2\tilde{p}_a^0 \cos \theta}, \\ \tilde{k}^- &= \frac{s_{bk}}{2\tilde{p}_b^0 \cos \theta}, & \tilde{l}^- &= \frac{s_{bl}}{2\tilde{p}_b^0 \cos \theta},\end{aligned}\tag{F.3}$$

therefore s_{aq} becomes

$$s_{aq} = \frac{s_{ak}s_{bl}}{s_{ab}} + \frac{s_{bk}s_{al}}{s_{ab}} - \frac{2\sqrt{s_{bk}s_{ak}}\sqrt{s_{bl}s_{al}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - s_{bk} - s_{bl}\tag{F.4}$$

where $s_{ab} = (\tilde{p}_a + \tilde{p}_b)^2 = 4\tilde{p}_a^0\tilde{p}_b^0 \cos^2 \theta$. Let us consider now the integral

$$I = \int_0^{s_{ab}} ds_{bk} ds_{bl} \frac{s_{bk}^- s_{bl}^-}{s_{aq}^2} g(s_{bk}, s_{bl}),\tag{F.5}$$

where $g(s_{bk}, s_{bl})$ is an analytic function of $s_{bk(l)}$. As we said, I contains a singularity when $s_{bk(l)}$ goes to zero; expanding $g(s_{bk}, s_{bl})$ around $s_{bk} = s_{bl} = 0$ at leading order one obtains

$$I^{[0]} = g(0, 0) \int_0^{s_{ab}} ds_{bk} ds_{bl} \frac{s_{bk}^- s_{bl}^-}{s_{aq}^2},\tag{F.6}$$

if one naively expands in ϵ before performing the integration in s_{bk} and s_{bl} the result suffers from unregulated logarithmic divergences. For instance, integrating first in s_{bk} it leads

$$I^{[0]} = g(0, 0) \int_0^{s_{ab}} ds_{bl} \frac{1}{s_{bl}} f(s_{ab}, s_{bl}),\tag{F.7}$$

where $f(s_{ab}, s_{bl})$ is analytic in s_{bl} and $f(s_{ab}, 0) \neq 0$. Therefore one needs to keep ϵ finite and only after performing the integral the expansion in ϵ leads to the correct poles in ϵ and its later cancellation in the total sum diagrams. In order to facilitate the computation of I we split the integration domain in two regions: **Region I** $s_{bk} \geq s_{bl}$ and **Region II** $s_{bl} \geq s_{bk}$, see F.1. In the **Region I** we perform the change of variables $s_{bl} = s_{bk}s'_{bl}$, while in **Region II** the convenient change of variables is $s_{bk} = s_{bl}s'_{bk}$. The integral I amounts to

$$\begin{aligned}I &= \int_0^{s_{ab}} ds_{bk} s_{bk}^{-\epsilon} \int_0^{s_{bk}} ds_{bl} \frac{s_{bl}^-}{s_{aq}^2} g(s_{bk}, s_{bl}) + \int_0^{s_{ab}} ds_{bl} s_{bl}^{-\epsilon} \int_0^{s_{bl}} ds_{bk} \frac{s_{bk}^-}{s_{aq}^2} g(s_{bk}, s_{bl}) \\ &= \int_0^{s_{ab}} ds_{bk} s_{bk}^{-1-2\epsilon} \int_0^1 ds'_{bl} \frac{(s'_{bl})^{-\epsilon} g(s_{bk}, s_{bk}s'_{bl})}{\left(\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{al}s_{ak}s'_{bl}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{bl} \right)^2} \\ &\quad + \int_0^{s_{ab}} ds_{bl} s_{bl}^{-1-2\epsilon} \int_0^1 ds'_{bk} \frac{(s'_{bk})^{-\epsilon} g(s_{bl}s'_{bk}, s_{bl})}{\left(\frac{s_{al}s'_{bk}}{s_{ab}} + \frac{s_{ak}}{s_{ab}} - \frac{2\sqrt{s_{al}s_{ak}s'_{bk}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{bk} \right)^2},\end{aligned}\tag{F.8}$$

since $s_{ab} > s_{a(b)k}, s_{a(b)l}$ the denominators are never zero; the only terms that lead to collinear poles are $s_{bk(l)}^{-1-2\epsilon}$, which can be expanded in plus distributions. This procedure will prove itself

F.1. CHANNEL: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$

to be crucial in order correctly isolate the soft and collinear poles in topologies that contain $s_{a(b)q}$ as denominator.

In the partonic centre of mass frame $\tilde{k}^\mu$, $\tilde{l}^\mu$ and $p_{a(b)}^\mu$ can be written as

$$\begin{aligned}\tilde{k}^\mu &= \frac{1}{s_{ab}} ([p_a^0 s_{bk} + p_b^0 s_{ak}] \cos \theta, \sqrt{s_{ab} s_{ak} s_{bk}} \cos \phi_{\tilde{k}}, \sqrt{s_{ab} s_{ak} s_{bk}} \sin \phi_{\tilde{k}}, \mathbf{0}_{d-4}, [p_a^0 s_{bk} - p_b^0 s_{ak}] \cos \theta), \\ \tilde{l}^\mu &= \frac{1}{s_{ab}} ([p_a^0 s_{bl} + p_b^0 s_{al}] \cos \theta, \sqrt{s_{ab} s_{al} s_{bl}} \cos \phi_{\tilde{l}}, \sqrt{s_{ab} s_{al} s_{bl}} \sin \phi_{\tilde{l}}, \mathbf{0}_{d-4}, [p_a^0 s_{bl} - p_b^0 s_{al}] \cos \theta), \\ \tilde{p}_a^\mu &= p_a^0 \cos \theta (1, \mathbf{0}_{d-2}, 1), \\ \tilde{p}_b^\mu &= p_b^0 \cos \theta (1, \mathbf{0}_{d-2}, -1),\end{aligned}\tag{F.9}$$

for $\pi/2 \geq \theta \geq 0$.

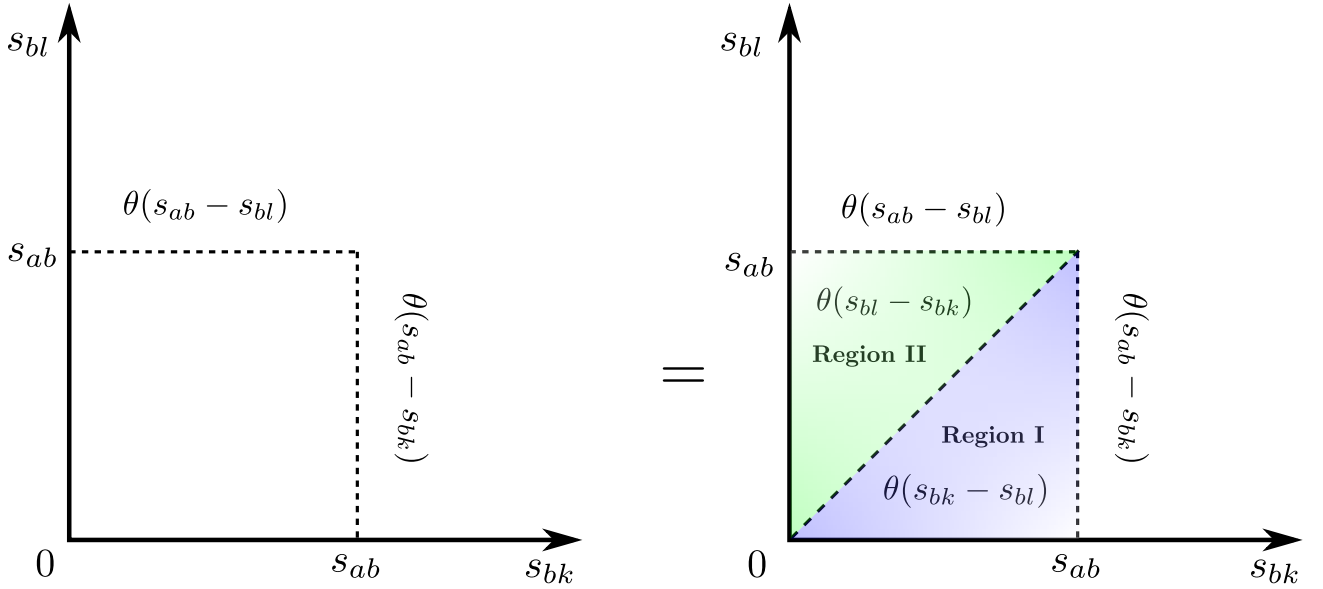


Figure F.1: Integration domain for the invariant mass s_{aq} .

F.1.1 Finite sum up to collinear singularity

Each diagram individually in fig. F.2 contains collinear singularities when the emitted partons becomes parallel to $\tilde{p}_b$, the sum of all them leads to single collinear poles that can be removed by means of the zero bin subtraction defined in E.2.6. It is highly convenient to use the method explained above and split the integration in s_{bk} and s_{bl} into two regions: **Region I** $s_{bk} \geq s_{bl}$ and **Region II** $s_{bl} \geq s_{bk}$. In order to explicitly show how the cancellation of the collinear poles takes place it is easier to start from diagrams $C_4 + \bar{C}_4$. The only propagators that lead to a singularity are $(\tilde{p}_b - \tilde{k})^2 = -s_{bk}$ and $(\tilde{p}_b - \tilde{l})^2 = -s_{bl}$, even there is not $(\tilde{p}_a - \tilde{q})^2$ it is still convenient to split the integration in **Region I** and **Region II** in order to be able to combine with the rest of diagrams.

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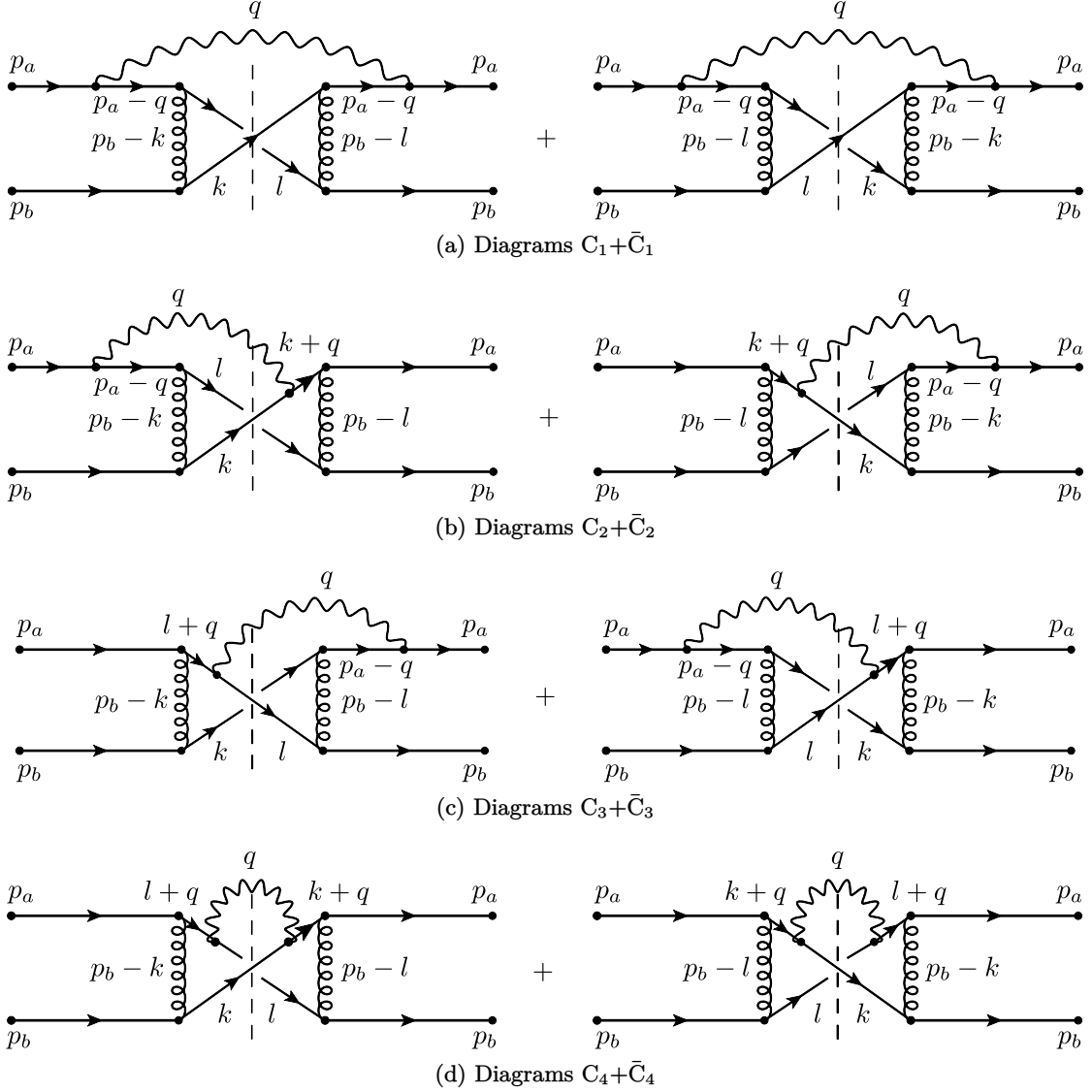


Figure F.2: Diagrams contributing to (anti-)quark (anti-)quark hard scattering square amplitude with identical (anti-)quarks in the final state; each diagram is weighted by a statistical factor $\frac{1}{2}$. The wiggly line represents the off-shell vector gauge boson and lines crossing the dashed line on-shell fermions.

Region I In **Region I** we have an integral of the form

$$I_{C_4 + \bar{C}_4}^{\mathbf{I}} = \int_0^{s_{ab}} ds_{bk} s_{bk}^{-1-2\epsilon} \int_0^1 ds'_{bl} (s'_{bl})^{-1-\epsilon} \frac{N_{C_4 + \bar{C}_4}^{\mu\nu}(s_{bk}, s_{bk} s'_{bl})}{(s_{ab} - s_{ak} - s_{bk})(s_{ab} - s_{al} - s_{bk} s'_{bl})}, \quad (\text{F.10})$$

where $N^{\mu\nu}$ also depends on s_{al} , s_{ak} , $\tilde{p}_{a(b)}^\mu$, $\tilde{k}^\mu$ and $\tilde{l}^\mu$, but we only make explicit its dependence on the relevant variables for our discussion, it is an analytic functions of its arguments; remember that using conservation of momentum we write $\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \tilde{k}^\mu - \tilde{l}^\mu$, therefore

F.1. CHANNEL: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$

$N^{\mu\nu}$ does not depend explicitly on q^μ . The integral given in eq. (F.10) has singularities at $s_{bk} = 0$ and $s'_{bl} = 0$, therefore it is convenient to write $N^{\mu\nu}$ in a way that facilitates to make explicit and isolate the divergences. We start writing $N^{\mu\nu}$ as

$$N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) = N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, 0) + [N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, 0)], \quad (\text{F.11})$$

the term between square brackets goes to zero as $s'_{bl}(s_{bk})$ at the limit $s'_{bl}(s_{bk}) \rightarrow 0(0)$. For the case of s'_{bl} it is by construction; whereas for $s_{bk} \rightarrow 0$ it is a non-trivial result, it can be explicitly checked by replacing $\tilde{k}^\mu$, $\tilde{l}^\mu$ and $\tilde{p}_{a(b)}^\mu$ with the parametrization given in eqs. (F.9) and (??). Therefore $s_{bk}^{-1-2\epsilon}(s'_{bl})^{-1-\epsilon}[N^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) - N^{\mu\nu}(s_{bk}, 0)]$ does not lead to any collinear singularity. The collinear singularities are proportional to $N^{\mu\nu}(s_{bk}, 0)$, we further write it as

$$N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, 0) = N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0) + [N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, 0) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)], \quad (\text{F.12})$$

the double collinear pole at $s_{bk}, s'_{bl} \rightarrow 0$ is proportional to $N^{\mu\nu}(0, 0)$, while the single collinear pole at $s'_{bl} \rightarrow 0$ is proportional to $[N^{\mu\nu}(s_{bk}, 0) - N^{\mu\nu}(0, 0)]$. The double and single collinear pole $C_4+\bar{C}_4$ are cancelled by the contribution given by the diagrams $C_3+\bar{C}_3$, see fig. F.2. In this case it contains integrals of the form

$$I_{C_3+\bar{C}_3}^{\text{I}} = \int_0^{s_{ab}} ds_{bk}^{-2-2\epsilon} \int_0^1 ds'_{bl}(s'_{bl})^{-1-\epsilon} \frac{N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl})}{s_{ab} - s_{ak} - s_{bk}} \frac{1}{\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}}\cos(\phi_{\bar{k}}-\phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bl}}, \quad (\text{F.13})$$

in similar fashion to $N_{C_4+\bar{C}_4}^{\mu\nu}$, we write $N_{C_3+\bar{C}_3}^{\mu\nu}$ as

$$\begin{aligned} N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) &= N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0) + [N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) - N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0)] \\ &= s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0) + [N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0) - s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0)] \\ &\quad + [N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) - N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0)], \end{aligned} \quad (\text{F.14})$$

where we have pulled out a global factor s_{bk} since

$$\lim_{s_{bk} \rightarrow 0} N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) = s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0) + \mathcal{O}(s_{bk}^2). \quad (\text{F.15})$$

The contribution proportional to $s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0)$ contains a double collinear pole that cancelled the one from $I_{C_4+\bar{C}_4}^{\text{I}}$

$$\begin{aligned} \lim_{s_{bk}, s'_{bl} \rightarrow 0} \frac{1}{s_{bk}^{1+\epsilon}(s'_{bl})^{1+\epsilon}} &\left[\frac{N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)}{(s_{ab} - s_{ak} - s_{bk})(s_{ab} - s_{al} - s_{bk}s'_{bl})} \right. \\ &\quad \left. + \frac{N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0)}{s_{ab} - s_{ak} - s_{bk}} \frac{1}{\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}}\cos(\phi_{\bar{k}}-\phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bl}} \right] = 0, \end{aligned} \quad (\text{F.16})$$

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the term $N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0) - s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0)$ cancels the single collinear pole at $s'_{bl} \rightarrow 0$ from $I_{C_4+\bar{C}_4}^I$

$$\lim_{s'_{bl} \rightarrow 0} \frac{1}{s_{bk}^{1+\epsilon}(s'_{bl})^{1+\epsilon}} \frac{1}{s_{ab} - s_{ak} - s_{bk}} \left[\frac{N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bk}, 0) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)}{s_{ab} - s_{al} - s_{bk}s'_{bl}} + \frac{[N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0) - s_{bk}N_{C_3+\bar{C}_3}^{\mu\nu}(0, 0)]}{s_{bk} \left(\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}} \cos(\phi_{\bar{k}} - \phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bl} \right)} \right] = 0, \quad (\text{F.17})$$

notice that by contraction each fraction inside the square brackets is finite at $s_{bk} \rightarrow 0$. One may be concerned about a potential singularity from $(s_{ab} - s_{ak} - s_{bk})^{-1}$ at the limit $s_{bk}(s_{ak}) \rightarrow 0(s_{ab})$; nevertheless we consider the upper bound integration for $s_{a(b)k(l)}$ up to s_{ab} such a value is truly never reached since the measurement functions ruled out such a value, otherwise we would have a situation where $Q^2 \sim 0$. The last singular contribution in $I_{C_3+\bar{C}_3}^I$ is at the limit $s_{bk} \rightarrow 0$ and it is proportional to $N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, s_{bk}s'_{bl}) - N_{C_3+\bar{C}_3}^{\mu\nu}(s_{bk}, 0)$, we identify this collinear pole as a singularity that is only removed after subtraction the double counting.

The cancellation of single and double collinear poles in the sum of diagrams $C_2+\bar{C}_2$ plus $C_1+\bar{C}_1$ follows exactly the same steps that for $C_3+\bar{C}_3$ plus $C_4+\bar{C}_4$, where the final result also contains a single collinear pole at $s_{bk} \rightarrow 0$ which is removed by a suitable double counting subtraction.

Region II In **Region II** we have an integral of the form

$$I_{C_4+\bar{C}_4}^{\text{II}} = \int_0^{s_{ab}} ds_{bl} s_{bk}^{-1-2\epsilon} \int_0^1 ds'_{bk} (s'_{bk})^{-1-\epsilon} \frac{N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl})}{(s_{ab} - s_{ak} - s_{bl}s'_{bk})(s_{ab} - s_{al} - s_{bl})}, \quad (\text{F.18})$$

we decompose the numerator as

$$N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) = N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0) + [N_{C_4+\bar{C}_4}^{\mu\nu}(0, s_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)] + [N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, s_{bl})], \quad (\text{F.19})$$

the double collinear pole at the limit $s'_{bk}, s_{bl} \rightarrow 0$ is proportional to $N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)$ and the single collinear pole at the limit $s'_{bk} \rightarrow 0$ is proportional to $N_{C_4+\bar{C}_4}^{\mu\nu}(0, s_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)$. The term $s_{bl}^{-1-\epsilon}(s'_{bk})^{-1-\epsilon}[N_{C_4+\bar{C}_4}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, s_{bl})]$ does not lead to any collinear singularity. In this case it is the contribution given by diagrams $C_2+\bar{C}_2$, see. F.2 the one that cancels the collinear poles from $C_4+\bar{C}_4$. The steps to follow in order to explicitly obtain the cancellation are rather similar to the one presented above for **Region I**. The contribution from $C_2+\bar{C}_2$ takes the form

$$I_{C_2+\bar{C}_2}^{\text{II}} = \int_0^{s_{ab}} ds_{bl} s_{bl}^{-2-2\epsilon} \int_0^1 ds'_{bk} (s'_{bk})^{-1-\epsilon} \frac{N_{C_2+\bar{C}_2}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl})}{s_{ab} - s_{al} - s_{bl}} \frac{1}{\frac{s_{al}s'_{bk}}{s_{ab}} + \frac{s_{ak}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bk}} \cos(\phi_{\bar{k}} - \phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bk}}, \quad (\text{F.20})$$

F.1. CHANNEL: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$

it is useful to decompose the numerator as

$$N_{C_2+\bar{C}_2}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) = s_{bl}N_{C_2+\bar{C}_2}^{\mu\nu}(0, 0) + [N_{C_2+\bar{C}_2}^{\mu\nu}(0, s_{bl}) - s_{bl}N_{C_2+\bar{C}_2}^{\mu\nu}(0, 0)] \\ + [N_{C_2+\bar{C}_2}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) - N_{C_2+\bar{C}_2}^{\mu\nu}(0, s_{bl})], \quad (\text{F.21})$$

the double collinear pole at the limit $s'_{bk}, s_{bl} \rightarrow 0$ is proportional to $s_{bl}N_{C_2+\bar{C}_2}^{\mu\nu}(0, 0)$ and the single collinear pole at $s'_{bk} \rightarrow 0$ is proportional to $N_{C_2+\bar{C}_2}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) - N_{C_2+\bar{C}_2}^{\mu\nu}(0, s_{bl})$. There is also a single collinear pole at $s_{bl} \rightarrow 0$ proportional to $N_{C_2+\bar{C}_2}^{\mu\nu}(s_{bl}s'_{bk}, s_{bl}) - N_{C_2+\bar{C}_2}^{\mu\nu}(0, s_{bl})$ that is not cancelled by $I_{C_4+\bar{C}_4}^{\text{II}}$ in eq. (F.18) but by subtracting the corresponding double counting. Therefore combining $I_{C_4+\bar{C}_4}^{\text{II}}$ in eq. (F.18) and $I_{C_2+\bar{C}_2}^{\text{II}}$ in eq. (F.20) the following cancellations takes place

$$\lim_{s'_{bk}, s_{bl} \rightarrow 0} \frac{1}{(s'_{bk})^{1+\epsilon} s_{bl}^{1+2\epsilon}} \left[\frac{N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)}{(s_{ab} - s_{ak} - s'_{bk}s_{bl})(s_{ab} - s_{al} - s_{bl})} \right. \\ \left. + \frac{N_{C_2+\bar{C}_2}^{\mu\nu}(0, 0)}{s_{ab} - s_{al} - s_{bl}} \frac{1}{\frac{s_{ak}}{s_{ab}} + \frac{s_{al}s'_{bk}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bk}} \cos(\phi_k - \phi_l)}{s_{ab}} - 1 - s'_{bk}} \right] = 0, \quad (\text{F.22})$$

$$\lim_{s'_{bk} \rightarrow 0} \frac{1}{(s'_{bk})^{1+2\epsilon} s_{bl}^{1+\epsilon}} \frac{1}{s_{ab} - s_{al} - s_{bl}} \left[\frac{N_{C_4+\bar{C}_4}^{\mu\nu}(0, s_{bl}) - N_{C_4+\bar{C}_4}^{\mu\nu}(0, 0)}{s_{ab} - s_{ak} - s_{bl}s'_{bk}} \right. \\ \left. + \frac{[N_{C_2+\bar{C}_2}^{\mu\nu}(0, s_{bl}) - s_{bl}N_{C_2+\bar{C}_2}^{\mu\nu}(0, 0)]}{s_{bl} \left(\frac{s_{ak}}{s_{ab}} + \frac{s_{al}s'_{bk}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bk}} \cos(\phi_k - \phi_l)}{s_{ab}} - 1 - s'_{bk} \right)} \right] = 0. \quad (\text{F.23})$$

The cancellation of single and double collinear poles in the sum of diagrams $C_3+\bar{C}_3$ plus $C_1+\bar{C}_1$ follows exactly the same steps that for $C_2+\bar{C}_2$ plus $C_4+\bar{C}_4$, where the final result also contains a single collinear pole at $s_{bl} \rightarrow 0$ which is remove by a suitable double counting subtraction.

F.1.2 Finite sum

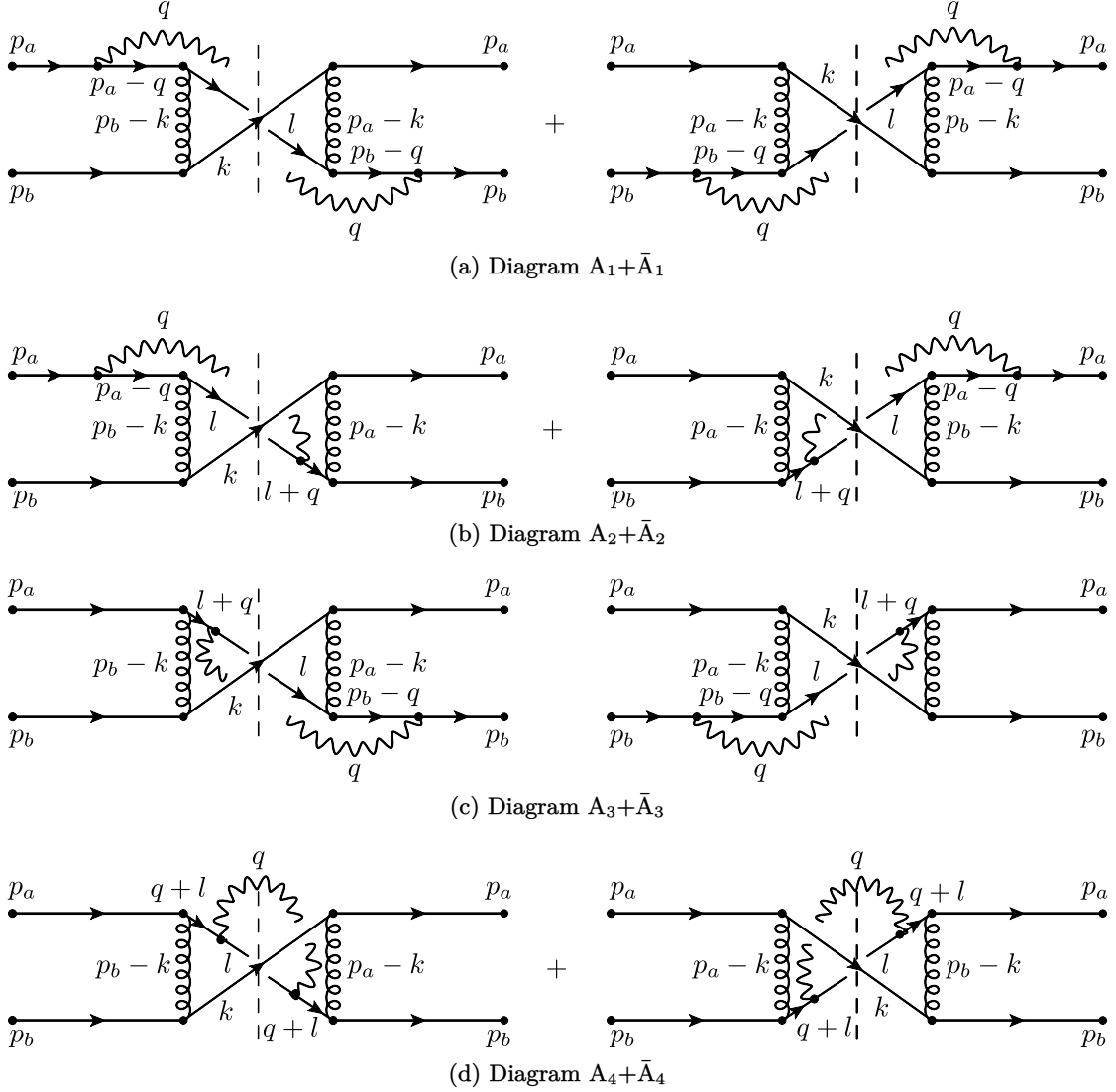
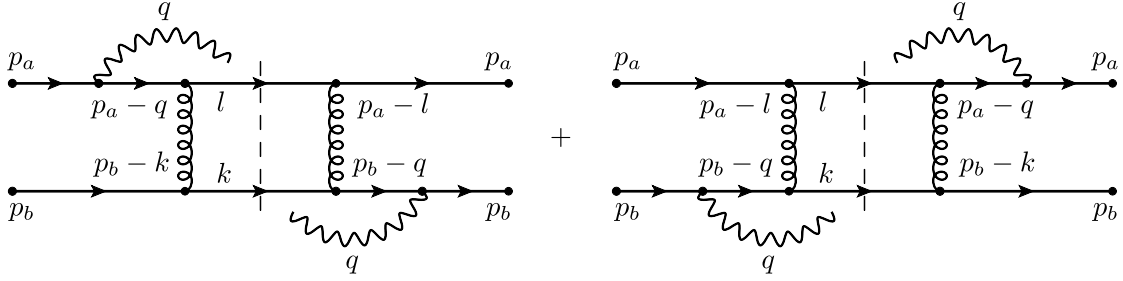
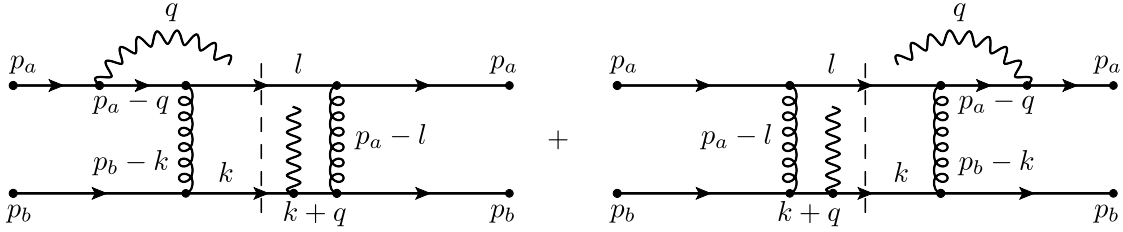


Figure F.3: Diagrams contributing to (anti-)quark-(anti-)quark hard scattering square amplitude with identical (anti-)quarks in the initial state; each diagram is weighted by a statistical factor $\frac{1}{2}$. The wiggly line represents the off-shell vector gauge boson and lines crossing the dashed line on-shell fermions.

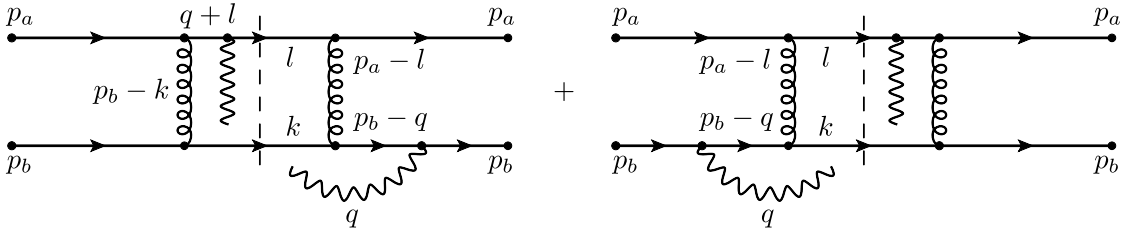
F.1. CHANNEL: $q(\bar{q})+q(\bar{q}) \rightarrow F+q(\bar{q})+q(\bar{q})$



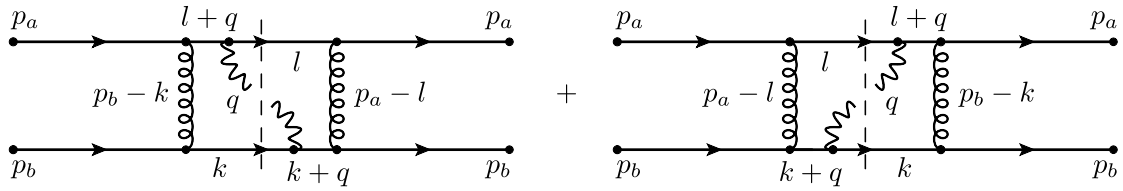
(a) Diagram $B_1 + \bar{B}_1$



(b) Diagram $B_2 + \bar{B}_2$



(c) Diagram $B_3 + \bar{B}_3$



(d) Diagram $B_4 + \bar{B}_4$

Figure F.4: Diagrams contributing to non-identical quark-quark hard scattering square amplitude. The wiggly line represents the off-shell vector gauge boson and lines crossing the dashed line on-shell fermions.

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For the hard scattering processes given in figs. (F.3) and F.4 the situation is slightly more tedious than the case of diagrams in fig. F.2. The reason is simply, in the diagrams given in figs. (F.3) and F.4 the invariant masses s_{aq} and s_{bq} appear as propagators, while in the case of fig. F.2 only s_{aq} appeared. This means that we need to split the integration domain into four different regions

- **Region I** $s_{bk} \geq s_{bl}$ and $s_{ak} \geq s_{al}$
- **Region II** $s_{bl} \geq s_{bk}$ and $s_{ak} \geq s_{al}$
- **Region III** $s_{bk} \geq s_{bl}$ and $s_{al} \geq s_{ak}$
- **Region IV** $s_{bl} \geq s_{bk}$ and $s_{al} \geq s_{ak}$

$$\begin{aligned}
 \int_0^{s_{ab}} ds_{ak} ds_{al} ds_{bk} ds_{bl} g(s_{ak}, s_{al}, s_{bk}, s_{bl}) &= \int_0^{s_{ab}} ds_{ak} \int_0^{s_{ak}} ds_{al} \int_0^{s_{ab}} ds_{bk} \int_0^{s_{bk}} ds_{bl} g(s_{ak}, s_{al}, s_{bk}, s_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{ak} \int_0^{s_{ak}} ds_{al} \int_0^{s_{ab}} ds_{bl} \int_0^{s_{bl}} ds_{bk} g(s_{ak}, s_{al}, s_{bk}, s_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{al} \int_0^{s_{al}} ds_{ak} \int_0^{s_{ab}} ds_{bk} \int_0^{s_{bk}} ds_{bl} g(s_{ak}, s_{al}, s_{bk}, s_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{al} \int_0^{s_{al}} ds_{ak} \int_0^{s_{ab}} ds_{bl} \int_0^{s_{bl}} ds_{bk} g(s_{ak}, s_{al}, s_{bk}, s_{bl}),
 \end{aligned} \tag{F.24}$$

performing the change of variables on: the first line $s_{al} = s'_{al}s_{ak}$ and $s_{bl} = s'_{bl}s_{bk}$, the second line $s_{sl} = s'_{al}s_{ak}$ and $s_{bk} = s'_{bk}s_{bl}$, the third line $s_{ak} = s'_{ak}s_{al}$ and $s_{bl} = s'_{bl}s_{bk}$ and the forth line $s_{ak} = s'_{ak}s_{al}$ and $s_{bk} = s'_{bk}s_{bl}$; one obtains

$$\begin{aligned}
 \int_0^{s_{ab}} ds_{ak} ds_{al} ds_{bk} ds_{bl} g(s_{ak}, s_{al}, s_{bk}, s_{bl}) &= \int_0^{s_{ab}} ds_{ak} \int_0^1 ds'_{al} \int_0^{s_{ab}} ds_{bk} \int_0^1 ds'_{bl} s_{ak} s_{bk} g(s_{ak}, s_{ak}s'_{al}, s_{bk}, s_{bk}s'_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{ak} \int_0^1 ds'_{al} \int_0^{s_{ab}} ds_{bl} \int_0^1 ds'_{bk} s_{ak} s_{bl} g(s_{ak}, s_{ak}s'_{al}, s_{bl}s'_{bk}, s_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{al} \int_0^1 ds'_{ak} \int_0^{s_{ab}} ds_{bk} \int_0^1 ds'_{bl} s_{al} s_{bk} g(s_{al}s'_{ak}, s_{al}, s_{bk}, s_{bk}s'_{bl}) \\
 &+ \int_0^{s_{ab}} ds_{al} \int_0^1 ds'_{ak} \int_0^{s_{ab}} ds_{bl} \int_0^1 ds'_{bk} s_{al} s_{bl} g(s_{al}s'_{ak}, s_{al}, s_{bl}s'_{bk}, s_{bl}).
 \end{aligned} \tag{F.25}$$

Region I The evaluation of diagrams given in fig. F.3 is rather straight forward since each one individually is already finite. On the other hand, the diagrams given in fig. F.4 do contain single collinear pole at the limit $s'_{al} \rightarrow 0$ and only the combinations $B_4 + \bar{B}_4 + B_3 + \bar{B}_3$ and $B_2 + \bar{B}_2 + B_1 + \bar{B}_1$ amount to a singularity free quantities.

F.1. CHANNEL: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$

Region II The diagrams in fig. F.3 contain single collinear singularities at $s'_{bk} \rightarrow 0$, however the combinations $A_4 + \bar{A}_4 + A_2 + \bar{A}_2$ and $A_3 + \bar{A}_3 + A_1 + \bar{A}_1$ amount to a finite quantity. For the case of diagrams given in fig. F.4 the situation is slightly more cumbersome, since there are also double collinear poles at the limit $s'_{al}, s'_{bk} \rightarrow 0$. In order to show how the cancellation of double and single collinear poles takes place we start by considering the contribution given by $B_4 + \bar{B}_4$, it amounts to

$$I_{B_4 + \bar{B}_4}^I = \int_0^{s_{ab}} ds_{ak} ds_{bl} s_{ak}^{-2\epsilon} s_{bl}^{-2\epsilon} \int_0^1 ds'_{al} s'_{bk} (s'_{al})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \times \frac{N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk})}{(s_{ab} - s_{ak} - s_{bl}s'_{bk})(s_{ab} - s_{bl} - s_{ak}s'_{al})}, \quad (F.26)$$

we rewrite the numerator as

$$N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) = [N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) - N_{B_4 + \bar{B}_4}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, 0) + N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)] + [N_{B_4 + \bar{B}_4}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)] + [N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, 0) - N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)] + N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0), \quad (F.27)$$

the term between square brackets on the first line goes to zero as $s'_{al}(s'_{bk})$ at the limit $s'_{al}(s'_{bk}) \rightarrow 0$, what means that contributions proportional to it are completely finite; the collinear pole at $s'_{al} \rightarrow 0$ can only be proportional to $[N_{B_4 + \bar{B}_4}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)]$; the collinear pole at $s'_{bk} \rightarrow 0$ is proportional to $[N_{B_4 + \bar{B}_4}^{\mu\nu}(s_{ak}s'_{al}, 0) - N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)]$; the double collinear pole at $s'_{al}, s'_{bk} \rightarrow 0$ is only proportional to $N_{B_4 + \bar{B}_4}^{\mu\nu}(0, 0)$. The contribution from $B_3 + \bar{B}_3$ amounts to

$$I_{B_3 + \bar{B}_3}^I = \int_0^{s_{ab}} ds_{ak} ds_{bl} s_{ak}^{-1-2\epsilon} s_{bl}^{-2\epsilon} \int_0^1 ds'_{al} s'_{bk} (s'_{al})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \frac{N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk})}{(s_{ab} - s_{ak} - s_{bl}s'_{bk})} \times \frac{1}{\frac{s_{bl}}{s_{ab}} + \frac{s_{bl}s'_{bk}s'_{al}}{s_{ab}} - \frac{2s_{bl}\sqrt{s'_{al}s'_{bk}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{al}}, \quad (F.28)$$

where the numerator can be written as

$$N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) = [N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) - s_{ak}N_{B_3 + \bar{B}_3}^{\mu\nu}(0, s_{bl}s'_{bk}) - s_{ak}N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, 0) + s_{ak}N_{B_3 + \bar{B}_3}^{\mu\nu}(0, 0)] + s_{ak}[N_{B_3 + \bar{B}_3}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_3 + \bar{B}_3}^{\mu\nu}(0, 0)] + s_{ak}[N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, 0) - N_{B_3 + \bar{B}_3}^{\mu\nu}(0, 0)] + s_{ak}N_{B_3 + \bar{B}_3}^{\mu\nu}(0, 0), \quad (F.29)$$

where $N_{B_3 + \bar{B}_3}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) \sim s_{ak}$ as $s_{ak} \rightarrow 0$ and we have pulled out a global factor s_{ak} to emphasize that there is not collinear pole at $s_{ak} \rightarrow 0$. The contribution from $B_3 + \bar{B}_3$ amounts to

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$$I_{B_2+\bar{B}_2}^{\mathbf{I}} = \int_0^{s_{ab}} ds_{ak} ds_{bl} s_{ak}^{-2\epsilon} s_{bl}^{-1-2\epsilon} \int_0^1 ds'_{al} s'_{bk} (s'_{al})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \frac{N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk})}{(s_{ab} - s_{bl} - s_{ak}s'_{al})} \times \frac{1}{\frac{s_{ak}}{s_{ab}} + \frac{s_{ak}s'_{bk}s'_{al}}{s_{ab}} - \frac{2s_{ak}\sqrt{s'_{al}s'_{bk}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{bk}}, \quad (\text{F.30})$$

where the numerator can be written as

$$N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) = [N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) - s_{bl}N_{B_2+\bar{B}_2}^{\mu\nu}(0, s_{bl}s'_{bk}) - s_{bl}N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, 0) + s_{bl}N_{B_2+\bar{B}_2}^{\mu\nu}(0, 0)] + s_{bl}[N_{B_2+\bar{B}_2}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_2+\bar{B}_2}^{\mu\nu}(0, 0)] + s_{bl}[N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, 0) - N_{B_2+\bar{B}_2}^{\mu\nu}(0, 0)] + s_{bl}N_{B_2+\bar{B}_2}^{\mu\nu}(0, 0), \quad (\text{F.31})$$

where $N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) \sim s_{bl}$ as $s_{bl} \rightarrow 0$ and we have pulled out a global factor s_{bl} to emphasize that there is not collinear pole at $s_{bl} \rightarrow 0$. The contribution from $B_1+\bar{B}_1$ amounts to

$$I_{B_1+\bar{B}_1}^{\mathbf{I}} = \int_0^{s_{ab}} ds_{ak} ds_{bl} s_{ak}^{-1-2\epsilon} s_{bl}^{-1-2\epsilon} \int_0^1 ds'_{al} s'_{bk} (s'_{al})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} N_{B_1+\bar{B}_1}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) \times \frac{1}{\frac{s_{bl}}{s_{ab}} + \frac{s_{bl}s'_{bk}s'_{al}}{s_{ab}} - \frac{2s_{bl}\sqrt{s'_{al}s'_{bk}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{al}} \times \frac{1}{\frac{s_{ak}}{s_{ab}} + \frac{s_{ak}s'_{bk}s'_{al}}{s_{ab}} - \frac{2s_{ak}\sqrt{s'_{al}s'_{bk}}}{s_{ab}} \cos(\phi_{\tilde{k}} - \phi_{\tilde{l}}) - 1 - s'_{bk}}, \quad (\text{F.32})$$

where the numerator can be written as

$$N_{B_1+\bar{B}_1}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) = [N_{B_1+\bar{B}_1}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) - s_{bl}s_{ak}N_{B_1+\bar{B}_1}^{\mu\nu}(0, s_{bl}s'_{bk}) - s_{bl}s_{ak}N_{B_1+\bar{B}_1}^{\mu\nu}(s_{ak}s'_{al}, 0) + s_{bl}s_{ak}N_{B_1+\bar{B}_1}^{\mu\nu}(0, 0)] + s_{bl}s_{ak}[N_{B_1+\bar{B}_1}^{\mu\nu}(0, s_{bl}s'_{bk}) - N_{B_1+\bar{B}_1}^{\mu\nu}(0, 0)] + s_{bl}s_{ak}[N_{B_1+\bar{B}_1}^{\mu\nu}(s_{ak}s'_{al}, 0) - N_{B_1+\bar{B}_1}^{\mu\nu}(0, 0)] + s_{bl}s_{ak}N_{B_1+\bar{B}_1}^{\mu\nu}(0, 0), \quad (\text{F.33})$$

where $N_{B_2+\bar{B}_2}^{\mu\nu}(s_{ak}s'_{al}, s_{bl}s'_{bk}) \sim s_{bl}(s_{ak})$ as $s_{bl}(s_{ak}) \rightarrow 0$ and we have pulled out a global factor $s_{bl}s_{ak}$ to emphasize that there is not collinear pole at $s_{bl}(s_{ak}) \rightarrow 0$. The combination $I_{B_1+\bar{B}_1}^{\mathbf{I}} + I_{B_2+\bar{B}_2}^{\mathbf{I}} + I_{B_3+\bar{B}_3}^{\mathbf{I}} + I_{B_4+\bar{B}_4}^{\mathbf{I}}$ cancels the double collinear pole at $s'_{al}, s'_{bk} \rightarrow 0$; the combinations $I_{B_1+\bar{B}_1}^{\mathbf{I}} + I_{B_2+\bar{B}_2}^{\mathbf{I}}$ and $I_{B_3+\bar{B}_3}^{\mathbf{I}} + I_{B_4+\bar{B}_4}^{\mathbf{I}}$ cancel the collinear pole at $s'_{al} \rightarrow 0$; the combinations $I_{B_1+\bar{B}_1}^{\mathbf{I}} + I_{B_3+\bar{B}_3}^{\mathbf{I}}$ and $I_{B_2+\bar{B}_2}^{\mathbf{I}} + I_{B_4+\bar{B}_4}^{\mathbf{I}}$ cancel the collinear pole at $s'_{bk} \rightarrow 0$.

F.1. CHANNEL: $q(\bar{q}) + q(\bar{q}) \rightarrow F + q(\bar{q}) + q(\bar{q})$

Region III In this case the diagrams given in fig. F.4 are each one individually finite already. For the diagrams given in fig. F.3 they contain a single collinear singularity at $s'_{ak} \rightarrow 0$; the combination of diagrams $A_1 + \bar{A}_1 + A_2 + \bar{A}_2$ and $A_3 + \bar{A}_3$ and $A_4 + \bar{A}_4$ are free of the collinear pole at $s'_{ak} \rightarrow 0$.

Region IV In this case the diagrams in fig. F.4 contain a single collinear divergence at $s'_{bk} \rightarrow 0$; the combination of diagrams $B_1 + \bar{B}_1 + B_3 + \bar{B}_3$ and $B_2 + \bar{B}_2 + B_4 + \bar{B}_4$ are free of the collinear divergence at $s'_{bk} \rightarrow 0$. For the diagrams in fig. F.3 the situation is lightly more involved, since there are two single collinear poles at $s'_{ak} \rightarrow 0$ and $s'_{bk} \rightarrow 0$, but there is not a double collinear pole since the numerators in each diagram goes to zero fast enough at the limit $s'_{ak}s'_{bk} \rightarrow 0$. The cancellation of collinear poles is as follow: we start writing the contribution from each pair of diagrams as

$$I_{A_4 + \bar{A}_4}^{\text{IV}} = \int_0^{s_{ab}} ds_{al} ds_{bl} s_{al}^{-2\epsilon} s_{bl}^{-2\epsilon} \int_0^1 ds'_{ak} ds'_{bk} (s'_{ak})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \frac{N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk})}{(s_{ab} - s_{al}s'_{ak} - s_{bl}s'_{bk})^2}, \quad (\text{F.34})$$

where the numerator can be written as

$$\begin{aligned} N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) &= [N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) - N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, 0) - N_{A_4 + \bar{A}_4}^{\mu\nu}(0, s_{bl}s'_{bk})] \\ &\quad + N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, 0) + N_{A_4 + \bar{A}_4}^{\mu\nu}(0, s_{bl}s'_{bk}), \end{aligned} \quad (\text{F.35})$$

the term between square brackets only leads to finite contributions, since it goes to zero as $s'_{ak}(s'_{bk})$ at the limit $s'_{ak}(s'_{bk}) \rightarrow 0$. The collinear singularity at $s'_{ak} \rightarrow 0$ is only proportional to $N_{A_4 + \bar{A}_4}^{\mu\nu}(0, s_{bl}s'_{bk})$, since $N_{A_4 + \bar{A}_4}^{\mu\nu}(0, s_{bl}s'_{bk}) \rightarrow s'_{bk}$ as $s'_{bk} \rightarrow 0$; the collinear singularity at $s'_{bk} \rightarrow 0$ is only proportional to $N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, 0)$, since $N_{A_4 + \bar{A}_4}^{\mu\nu}(s_{al}s'_{ak}, 0) \rightarrow s'_{ak}$ as $s'_{ak} \rightarrow 0$. The other diagrams follow similar decomposition, one obtains

$$\begin{aligned} I_{A_3 + \bar{A}_3}^{\text{III}} &= \int_0^{s_{ab}} ds_{al} ds_{bl} s_{al}^{-1-2\epsilon} s_{bl}^{-2\epsilon} \int_0^1 ds'_{ak} ds'_{bk} (s'_{ak})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \frac{N_{A_3 + \bar{A}_3}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk})}{(s_{ab} - s_{al}s'_{ak} - s_{bl}s'_{bk})} \\ &\quad \times \frac{1}{\frac{s_{bl}s'_{ak}}{s_{ab}} + \frac{s_{bl}s'_{bk}}{s_{ab}} - \frac{2s_{bl}\sqrt{s'_{ak}s'_{bk}}}{s_{ab}} - 1 - s'_{ak}}, \end{aligned} \quad (\text{F.36})$$

where the numerator can be written as

$$\begin{aligned} N_{A_3 + \bar{A}_3}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) &= [N_{A_3 + \bar{A}_3}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) - s_{al}N_{A_3 + \bar{A}_3}^{\mu\nu}(s_{al}s'_{ak}, 0) - s_{al}N_{A_3 + \bar{A}_3}^{\mu\nu}(0, s_{bl}s'_{bk})] \\ &\quad + s_{al}N_{A_3 + \bar{A}_3}^{\mu\nu}(s_{al}s'_{ak}, 0) + s_{al}N_{A_3 + \bar{A}_3}^{\mu\nu}(0, s_{bl}s'_{bk}), \end{aligned} \quad (\text{F.37})$$

$$\begin{aligned} I_{A_2 + \bar{A}_2}^{\text{II}} &= \int_0^{s_{ab}} ds_{al} ds_{bl} s_{al}^{-2\epsilon} s_{bl}^{-1-2\epsilon} \int_0^1 ds'_{ak} ds'_{bk} (s'_{ak})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} \frac{N_{A_2 + \bar{A}_2}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk})}{(s_{ab} - s_{al}s'_{ak} - s_{bl}s'_{bk})} \\ &\quad \times \frac{1}{\frac{s_{al}s'_{ak}}{s_{ab}} + \frac{s_{al}s'_{bk}}{s_{ab}} - \frac{2s_{al}\sqrt{s'_{ak}s'_{bk}}}{s_{ab}} - 1 - s'_{bk}}, \end{aligned} \quad (\text{F.38})$$

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where the numerator can be written as

$$N_{A_2+\bar{A}_2}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) = [N_{A_2+\bar{A}_2}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) - s_{bl}N_{A_2+\bar{A}_2}^{\mu\nu}(s_{al}s'_{ak}, 0) - s_{bl}N_{A_2+\bar{A}_2}^{\mu\nu}(0, s_{bl}s'_{bk})] + s_{bl}N_{A_2+\bar{A}_2}^{\mu\nu}(s_{al}s'_{ak}, 0) + s_{bl}N_{A_2+\bar{A}_2}^{\mu\nu}(0, s_{bl}s'_{bk}), \quad (\text{F.39})$$

$$I_{A_1+\bar{A}_1}^{\text{I}} = \int_0^{s_{ab}} ds_{al} ds_{bl} s_{al}^{-1-2\epsilon} s_{bl}^{-1-2\epsilon} \int_0^1 ds'_{ak} ds'_{bk} (s'_{ak})^{-1-\epsilon} (s'_{bk})^{-1-\epsilon} N_{A_1+\bar{A}_1}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) \times \frac{1}{\frac{s_{bl}s'_{ak}}{s_{ab}} + \frac{s_{bl}s'_{bk}}{s_{ab}} - \frac{2s_{bl}\sqrt{s'_{ak}s'_{bk}}}{s_{ab}} - 1 - s'_{ak} \frac{s_{al}s'_{ak}}{s_{ab}} + \frac{s_{al}s'_{bk}}{s_{ab}} - \frac{2s_{al}\sqrt{s'_{ak}s'_{bk}}}{s_{ab}} - 1 - s'_{bk}}, \quad (\text{F.40})$$

where the numerator can be written as

$$N_{A_1+\bar{A}_1}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) = [N_{A_1+\bar{A}_1}^{\mu\nu}(s_{al}s'_{ak}, s_{bl}s'_{bk}) - s_{bl}s_{bk}N_{A_1+\bar{A}_1}^{\mu\nu}(s_{al}s'_{ak}, 0) - s_{bl}s_{bk}N_{A_1+\bar{A}_1}^{\mu\nu}(0, s_{bl}s'_{bk})] + s_{bl}s_{bk}N_{A_1+\bar{A}_1}^{\mu\nu}(s_{al}s'_{ak}, 0) + s_{bl}s_{bk}N_{A_1+\bar{A}_1}^{\mu\nu}(0, s_{bl}s'_{bk}). \quad (\text{F.41})$$

The diagram combinations $A_4+\bar{A}_4+A_3+\bar{A}_3$ and $A_2+\bar{A}_2+A_1+\bar{A}_1$ are free of the collinear divergence at $s'_{ak} \rightarrow 0$; the combinations $A_4+\bar{A}_4+A_2+\bar{A}_2$ and $A_3+\bar{A}_3+A_1+\bar{A}_1$ are free of the collinear divergence at $s'_{bk} \rightarrow 0$.

F.1.3 Double counting

For the set of diagrams given in fig. F.2 the double counting is purely p_b -collinear, therefore the zero bin is simple given eq. (E.62), where one need to compute the matrix elements at NNLO considering that the state $|\tilde{p}_b\rangle$ is a quark, see fig. F.5. There is neither double counting of soft nor p_a -collinear modes to be subtracted, therefore the soft $\hat{\phi}_{Zb,ab}$ and p_a -collinear $\hat{f}_{Zb,a}$, zero-bin in eq. (E.64) are evaluated at LO.

In a similar fashion we did for the diagrams presented in fig. F.2 we need to split the integration domain in two regions: **Region I** $s_{bk} \geq s_{bl}$ and **Region II** $s_{bl} \geq s_{bk}$.

Region I The diagram $C_{Zb,4}$ in fig. F.5 gives a contribution of the form

$$I_{C_{Zb,4}}^{\text{I}} = \int_0^{s_{ab}} ds_{bk} s_{bk}^{-1-2\epsilon} \int_0^1 ds'_{bl} (s'_{bl})^{-1-\epsilon} \frac{N_{C_{Zb,4}}(s_{bk}, s_{bk}s'_{bl})}{(s_{ab} - s_{al})(s_{ab} - s_{ak})}, \quad (\text{F.42})$$

where the dependence on s_{bk} and s'_{bl} in $N_{C_{Zb,4}}$ is totally contained in the measurement function. For the diagram $C_{Zb,3}$ in fig. F.5 its contribution given by

$$I_{C_{Zb,3}}^{\text{I}} = \int_0^{s_{ab}} ds_{bk} s_{bk}^{-2-2\epsilon} \int_0^1 ds'_{bl} (s'_{bl})^{-1-\epsilon} \frac{N_{C_{Zb,3}}(s_{bk}, s_{bk}s'_{bl})}{s_{ab} - s_{ak}} \times \frac{1}{\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}}\cos(\phi_k - \phi_l)}{s_{ab}} - 1 - s'_{bl}}, \quad (\text{F.43})$$

F.1. CHANNEL: $q(\bar{q})+q(\bar{q}) \rightarrow F+q(\bar{q})+q(\bar{q})$

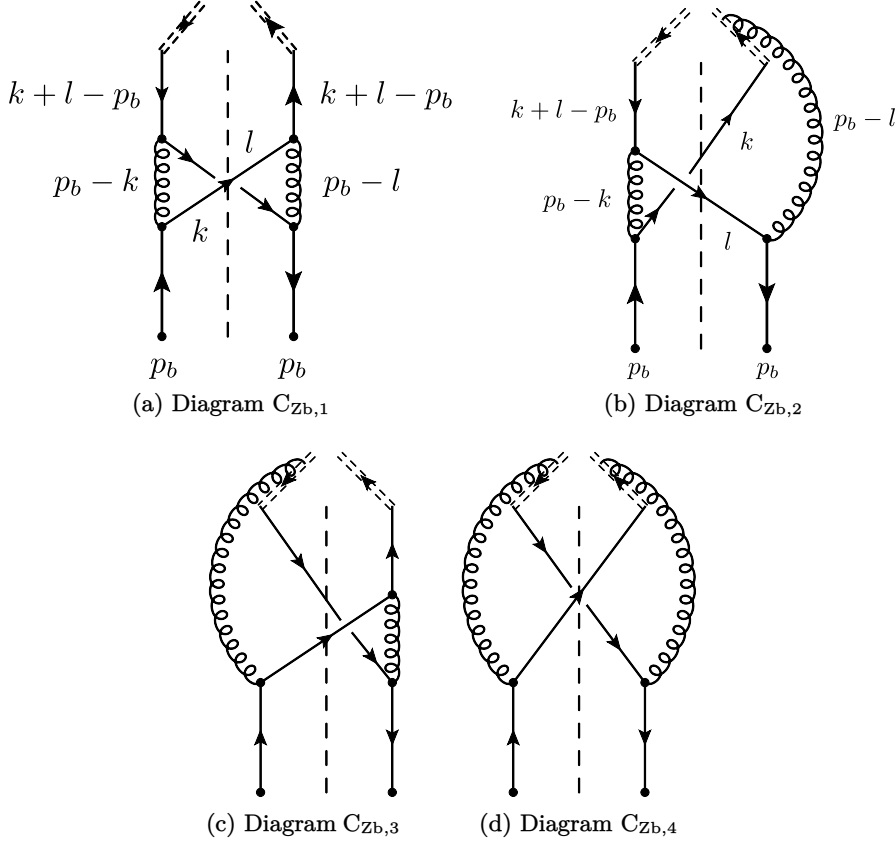


Figure F.5: Diagrams contributing to the zero bin subtraction $C_{Zb,\bar{b}\leftarrow b}$ for the case of scattering with identical quarks in the final state. The double dashed line represent Wilson lines in the fundamental representation, fermion lines crossing the dashed line denotes on-shell particles.

the numerator can be written as

$$N_{C_{Zb,3}}(s_{bk}, s_{bk}s'_{bl}) = s_{bk}N_{C_{Zb,3}}(0, 0) + [N_{C_{Zb,3}}(s_{bk}, s_{bk}s'_{bl}) - s_{bk}N_{C_{Zb,3}}(0, 0)], \quad (F.44)$$

where $s_{bk}N_{C_{Zb,3}}(0, 0)$ is defined as

$$\lim_{s'_{bl} \rightarrow 0} N_{C_{Zb,3}}(s_{bk}, s_{bk}s'_{bl}) = s_{bk}N_{C_{Zb,3}}(0, 0) + \mathcal{O}(s'_{bl}) \quad (F.45)$$

since we can pull out a common factor s_{bk} from $N_{C_{Zb,3}}(s_{bk}, 0)$. The singular contribution at the limit $s_{bk}, s'_{bl} \rightarrow 0$ is proportional to $s_{bk}N_{C_{Zb,3}}(0, 0)$ and it cancels the double collinear pole from $I_{C_{Zb,4}}^I$. The single collinear pole at the limit $s_{bk} \rightarrow 0$ is proportional to $N_{C_{Zb,3}}(s_{bk}, s_{bk}s'_{bl}) - s_{bk}N_{C_{Zb,3}}(0)$, which is cancelled by the single $s_{bk} \rightarrow 0$ pole in $I_{C_3+\bar{C}_3}^I$, see eq. (F.13).

For the case of diagrams $A_{Zb,2}$ and $A_{Zb,1}$, see fig. F.5, the cancellation of collinear poles

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follows a similar pattern. Their contribution is given by the integrals

$$I_{C_{Zb,2}}^I = \int_0^{s_{ab}} ds_{bk} s_{bk}^{-2-2\epsilon} \int_0^1 ds'_{bl} (s'_{bl})^{-1-\epsilon} \frac{N_{C_{Zb,2}}(s_{bk}, s_{bk}s'_{bl})}{s_{ab} - s_{al}} \times \frac{1}{\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}}\cos(\phi_{\bar{k}}-\phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bl}}, \quad (F.46)$$

$$I_{A_{Zb,1}}^I = \int_0^{s_{ab}} ds_{bk} s_{bk}^{-3-2\epsilon} \int_0^1 ds'_{bl} (s'_{bl})^{-1-\epsilon} \frac{N_{C_{Zb,1}}(s_{bk}, s_{bk}s'_{bl})}{\left(\frac{s_{ak}s'_{bl}}{s_{ab}} + \frac{s_{al}}{s_{ab}} - \frac{2\sqrt{s_{ak}s_{al}s'_{bl}}\cos(\phi_{\bar{k}}-\phi_{\bar{l}})}{s_{ab}} - 1 - s'_{bl}\right)^2}, \quad (F.47)$$

the two numerators can be written as

$$N_{C_{Zb,2}}(s_{bk}, s_{bk}s'_{bl}) = s_{bk}N_{A_{Zb,2}}(0,0) + [N_{C_{Zb,2}}(s_{bk}, s_{bk}s'_{bl}) - s_{bk}N_{C_{Zb,2}}(0,0)], \quad (F.48)$$

$$N_{C_{Zb,1}}(s_{bk}, s_{bk}s'_{bl}) = s_{bk}^2N_{C_{Zb,1}}(0,0) + [N_{C_{Zb,1}}(s_{bk}, s_{bk}s'_{bl}) - s_{bk}^2N_{C_{Zb,1}}(0,0)]. \quad (F.49)$$

The double collinear poles cancel in the sum diagrams $C_{Zb,2}$ and $C_{Zb,1}$, while the single collinear pole remaining at $s_{bk} \rightarrow 0$ are cancelled with the ones coming from diagrams $C_2 + \bar{C}_2$ and $C_1 + \bar{C}_1$, i.e. $C_{Zb,2(1)} + C_2(1) + \bar{C}_2(1)$ is free of the pole at $s_{bk} \rightarrow 0$, while $C_{Zb,2} + C_{Zb,1}$ is free of a double pole at s_{bk}, s'_{bl} .

Region II The cancellation of collinear divergences mostly follow the same steps than in **Region I** with only some minor changes. In this case the combinations $C_{Zb,4} + C_{Zb,2}$ and $C_{Zb,3} + C_{Zb,1}$ are the ones that are free of the double collinear divergence at the limit $s'_{bk}, s_{bl} \rightarrow 0$; while the combination $C_{Zb,i} + C_i + \bar{C}_i$ with $i = 1, 2, 3$ is free of the single collinear divergence at $s_{bl} \rightarrow 0$.

In case one replaces the quark initial states in fig. F.2 by anti quarks, then the the zero bin is given by eq. (E.61) where the initial state $|p_b\rangle$ is an anti-quark.

F.1.4 All together

In figs. F.2, F.3 and F.4 we presented the case when initial states are quark, the case when both are anti-quark can be trivially obtained from the previous results, leading to exactly the same contribution. One also needs to consider the case in fig. F.2 when vector boson is emitted and/or absorbed by the parton p_b , i.e. in diagrams $C_1 + \bar{C}_1$ the vector boson is emitted and absorbed by p_b or in diagrams $C_3 + \bar{C}_3$ emitted(absorbed) by the on-shell state k and absorbed(emitted) by p_b . One can obtain such a contributions from fig. F.2 just doing the substitutions $p_a \leftrightarrow p_b$ and $k \leftrightarrow$, it leads to a contribution that amounts to the same quantity than before. In the case of digrams given in figs. F.3 and F.4 one also needs to consider the case where the states k and l are interchanged. The result amounts to exactly the same

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quantity than without interchanging, therefore we just need to multiply by a factor of 2 the results obtained for figs. F.3 and F.4.

Having discussed the cancellation of soft and collinear we are in a position to write the hadronic tensor for the different contributions. We start for the one given by the diagrams in fig. F.2

$$\begin{aligned} \mathcal{W}_{R,Zb}^{\mu\nu,[2,qq,\bar{q}\bar{q}]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \\ = \mathcal{W}_R^{\mu\nu,[2,qq,\bar{q}\bar{q}]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \\ - \mathcal{W}_{Zb}^{\mu\nu,[2,q(\bar{q}) \rightarrow \bar{q}(q)]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}), \end{aligned} \quad (\text{F.50})$$

where

$$\begin{aligned} \mathcal{W}_R^{\mu\nu,[2,qq,\bar{q}\bar{q}]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \\ = \alpha_s^2(\mu^2) \sum_{a,b} \int_0^1 d\xi_a d\xi_b \int [d^4 p_a] [d^4 p_b] \frac{[d^d \tilde{k}]}{(2\pi)^{d-1}} \frac{[d^d \tilde{l}]}{(2\pi)^{d-1}} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\ \times \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left\{ \delta_{ab} \mathcal{H}_{ab, C_{1-4} + \bar{C}_{1-4}}^{\mu\nu} + \delta_{ab} \mathcal{H}_{ab, A_{1-4} + \bar{A}_{1-4}}^{\mu\nu} + \mathcal{H}_{ab, B_{1-4} + \bar{B}_{1-4}}^{\mu\nu} \right\} \\ \times f_{a \leftarrow P_A}(\alpha_s(\mu^2), \xi_a, \mathbf{p}_{aT}^2, \zeta_a, \mu^2) f_{b \leftarrow P_B}(\alpha_s(\mu^2), \xi_b, \mathbf{p}_{bT}^2, \zeta_b, \mu^2) \Theta(Q^2, y, q_T^2), \end{aligned} \quad (\text{F.51})$$

where indices a, b denotes the parton flavor, $\mathcal{H}_{ab, C_{1-4} + \bar{C}_{1-4}}^{\mu\nu}$ denotes the sum of the diagrams given in fig. F.2 plus the corresponding contribution coming from the interchange $p_a \leftrightarrow p_b$ and $k \leftrightarrow l$, $\mathcal{H}_{ab, A_{1-4} + \bar{A}_{1-4}}^{\mu\nu}$ denotes the diagram from fig. F.3 plus the contribution coming from interchange $k \leftrightarrow l$ and $\mathcal{H}_{ab, B_{1-4} + \bar{B}_{1-4}}^{\mu\nu}$ denotes the diagram from fig. F.4 plus the contribution coming from $k \leftrightarrow l$. The function $\Theta(Q^2, y, q_T^2)$ denotes the measurement functions for the gauge vector boson invariant mass Q^2 , rapidity y and transverse momentum q_T^2 with respect the axis defined by the protons beam

$$\Theta(Q^2, y, q_T^2) = \theta(Q_{\max}^2 \geq Q^2 \geq Q_{\min}^2) \theta(y_{\max} \geq y \geq y_{\min}) \theta(q_{T\max}^2 \geq q_T^2 \geq q_{T\min}^2), \quad (\text{F.52})$$

where

$$\begin{aligned} Q^2 &= q^2 \\ y &= \frac{1}{2} \log \frac{\bar{n} \cdot \tilde{q}}{n \cdot \tilde{q}} \\ q_T^2 &= 2(\tilde{q} \cdot n)(\tilde{q} \cdot \bar{n}) - Q^2 \\ \tilde{q}^\mu &= \tilde{p}_a^\mu + \tilde{p}_b^\mu - \tilde{k}^\mu - l^\nu. \end{aligned} \quad (\text{F.53})$$

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The double counting subtracting term is given by

$$\begin{aligned}
& \mathcal{W}_{\text{Zb}}^{\mu\nu, [2, q(\bar{q}) \rightarrow \bar{q}(q)]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \\
&= -\alpha_s^2(\mu^2) \sum_{a,b} \int_0^1 d\xi_a d\xi_b \int [d^4 p_a][d^4 p_b] 4p_a \cdot p_b \left(g^{\mu\nu} - \frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{(p_a \cdot p_b)} \right) \\
&\quad \times f_{a \leftarrow P_A}(\xi_a, \mathbf{p}_{aT}^2, \mu^2, \zeta_a) f_{b \leftarrow P_B}(\xi_b, \mathbf{p}_{bT}^2, \mu^2, \zeta_b) \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
&\quad \times \left[\delta_{ab} C_{\text{Zb}, \bar{b} \leftarrow b}(\{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) + \right. \\
&\quad \left. + \delta_{ab} C_{\text{Zb}, \bar{a} \leftarrow a}(\{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \right],
\end{aligned} \tag{F.54}$$

where the coefficient function $C_{\text{Zb}, \bar{a} \leftarrow a}$ is given by

$$\begin{aligned}
C_{\text{Zb}, \bar{b} \leftarrow b}(\{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{[d^d k]}{(2\pi)^{d-1}} \frac{[d^d l]}{(2\pi)^{d-1}} \Theta(Q^2, y, q_T^2) \\
&\times \left[\frac{N_{\text{CZb},1}}{s_{bk} s_{bl} s_{aq}^2} + \frac{N_{\text{CZb},2}}{s_{bk} s_{bl} s_{aq} (s_{ab} - s_{al})} + \frac{N_{\text{CZb},3}}{s_{bk} s_{bl} s_{aq} (s_{ab} - s_{ak})} + \frac{N_{\text{CZb},4}}{s_{bk} s_{bl} (s_{ab} - s_{al})(s_{ab} - s_{ak})} \right],
\end{aligned} \tag{F.55}$$

where $N_{\text{CZb},i}$ is given by the diagram $\text{C}_{\text{Zb},i}$ from fig. F.5. Notice that the second and third term between square brackets can be merged into a single denominator performing the change of variables $l \leftrightarrow k$. The dependence on $\tilde{r}^\mu$ and $\tilde{l}^\mu$ in the measurements functions of eq. F.55 is slightly different than the one presented in eq. (F.53). When one performs the Fourier from position to momentum space, certain complex exponential lacks the component proportional to n_a^μ or n_b^μ , see eq. (E.71) and (E.78). For the case of $C_{\text{Zb}, \bar{b} \leftarrow b}$ we have

$$\tilde{q}^\mu = \tilde{p}_a^\mu + \tilde{p}_b^\mu - \frac{\tilde{k} \cdot n_a}{2} n_b^\mu - \tilde{k}_{ab\perp}^\mu - \frac{\tilde{l} \cdot n_a}{2} n_b^\mu - \tilde{l}_{ab\perp}^\mu. \tag{F.56}$$

We must point out that while in the measurement functions the momentum components $(n_b \cdot \tilde{k}) n_a^\mu / 2$ and $(n_b \cdot \tilde{l}) n_a^\mu / 2$ are not present, they are in the propagator s_{aq} . Notice also that the measurement functions do depend on the $n_b \cdot \tilde{k}$ and $n_b \cdot \tilde{l}$ through the transverse component.

F.2 Channel: $q + \bar{q} \rightarrow F + q + \bar{q}$

The diagrams whose total sum amounts to a soft and collinear divergences free quantity are given in figs. F.6, F.7 and F.4. In case of the diagrams given in F.4 one just need to replace one initial state quark by an anti-quark. The diagrams displayed in fig. F.6 are free of collinear and soft singularities. In the case of the diagrams presented in fig. F.7 they contain single collinear poles, which cancel out in the total sum; the process to achieve such a cancellation is completely analogous to the one presented above in Sec. F.1.2. The diagrams D_5 and D_6 , see fig. F.7 contain the propagator s_{aq} therefore it is useful to split its integration domain into **Region I** $s_{bk} \geq s_{bl}$ and **Region II** $s_{bl} \geq s_{bk}$. In **Region I** we perform the change of variables $s_{bl} = s'_{bl} s_{bk}$ and we obtain that each diagram is already finite; in **Region II** we perform the change of variable $s_{bk} = s'_{bk} s_{bl}$ and we obtain that the single collinear pole at s'_{bk} is cancelled

in the sum of D_5+D_6 . For the diagrams D_7 and D_8 , see fig. F.7 it is the propagator s_{bq} the one who appears, therefore it is useful to split the integration domain into **Region I** $s_{al} \geq s_{ak}$ and **Region II** $s_{ak} \geq s_{al}$; in **Region I** each diagram is already finite while in **Region II**, after performing $s_{al} = s'_{al}s_{ak}$, the contain a collinear pole at $s'_{al} \rightarrow 0$ that is cancelled in the sum D_7+D_8 .

The hadronic tensor is given by

$$\begin{aligned}
 & \mathcal{W}_R^{\mu\nu,[2,q\bar{q}]}(P_A, P_B; \{y_{\min}, y_{\max}\}, \{Q_{\min}^2, Q_{\max}^2\}, \{q_{T\min}^2, q_{T\max}^2\}) \\
 &= \alpha_s^2(\mu^2) \sum_{a,b} \int_0^1 d\xi_a d\xi_b \int [d^4 p_a][d^4 p_b] \frac{[d^d \tilde{k}]}{(2\pi)^{d-1}} \frac{[d^d \tilde{l}]}{(2\pi)^{d-1}} \delta(p_a^- - \xi_a P_A^-) \delta(p_b^+ - \xi_b P_B^+) \\
 &\times f_{a \leftarrow P_A}(\alpha_s(\mu^2), \xi_a, \mathbf{p}_{aT}^2, \zeta_a, \mu^2) f_{b \leftarrow P_B}(\alpha_s(\mu^2), \xi_b, \mathbf{p}_{bT}^2, \zeta_b, \mu^2) \Theta(Q^2, y, q_T^2) \\
 &\times \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \left\{ \delta_{a\bar{b}} \mathcal{H}_{ab, B_{1-4} + \bar{B}_{1-4}}^{\mu\nu} + \delta_{a\bar{b}} \mathcal{H}_{ab, D_{1-8}}^{\mu\nu} \right\}
 \end{aligned} \tag{F.57}$$

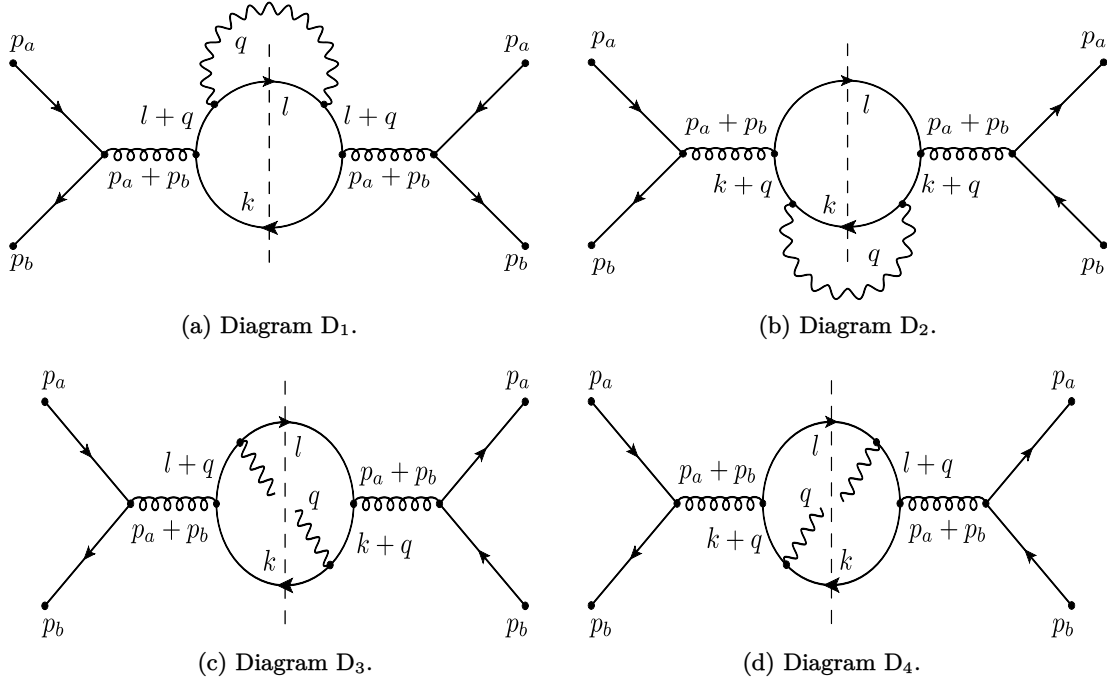


Figure F.6: Diagrams contributing to the quark-ant-quark hard scattering square amplitude. The wiggly line represents the off-shell vector gauge boson and lines crossing the dashed line on-shell fermions. Each diagram is free of collinear and soft singularities, therefore no double counting subtraction is needed.

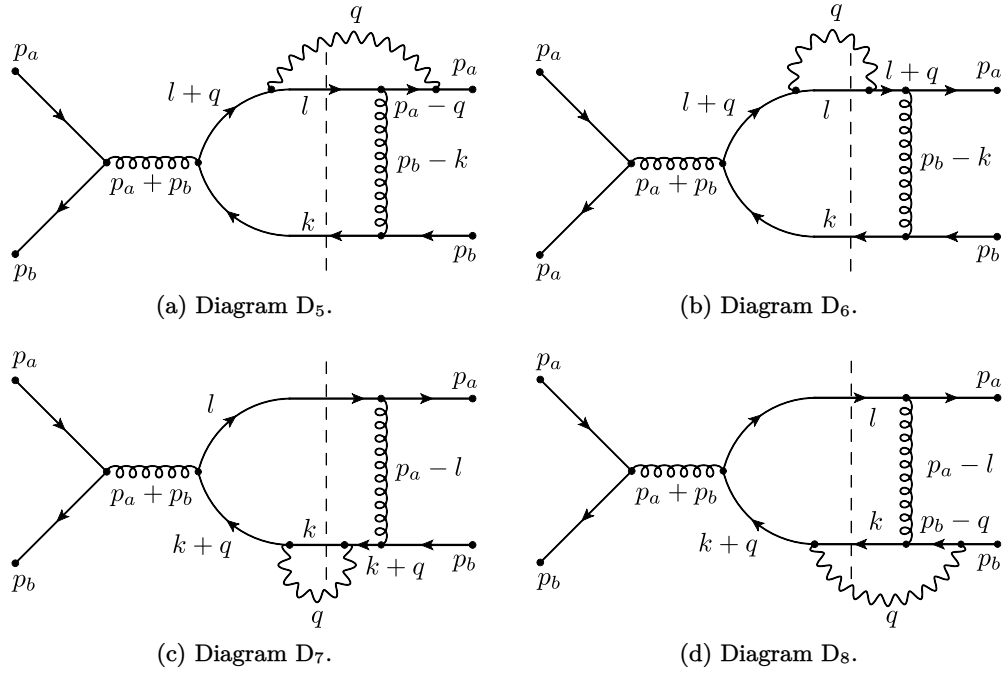


Figure F.7: Diagrams contributing to the quark-ant-quark hard scattering square amplitude. The wiggly line represents the off-shell vector gauge boson and lines crossing the dashed line on-shell fermions. Each diagram contains collinear singularity and only the total sum amounts to a finite quantity, therefore no double counting subtraction is needed.

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