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Drinfeld-Sokolov reduction and asymptotic symmetries of higher-spin gauge theories

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Abstract

In this thesis, the framework of the generalized Drinfeld-Sokolov reduction, as derived by B. Khesin and F. Malikov, is examined in the case of the higher-spin algebra. The result of such a reduction is a classical W-algebra on the fields, which one regards as an asymptotic symmetry algebra on a three-dimensional spacetime, that is asymptotically anti-de Sitter, in higher-spin gauge theory. For this, two approaches are discussed: First, an adaptation of the general proof to the higher-spin algebra is considered and second, a generalization of the single-row gauge to the case of the higher-spin algebra is calculated. In order to execute the generalized Drinfeld-Sokolov reduction, the Lie algebra of "complex-size matrices" is required. Representing elements of this algebra via a Verma module leads to certain properties on the resulting matrices, one of which will be slightly adapted for usage in the single-row gauge.

Zusammenfassung

In dieser Arbeit wird die generalisierte Drinfeld-Sokolov Reduktion, welche von B. Khesin und F. Malikov berechnet wurde, im Falle der höheren Spin-Algebra untersucht. Das Ergebnis solch einer Reduktion ist eine klassische W-Algebra in den Feldern, welche man in höherer Spin-Eichtheorie als asymptotische Symmetriealgebra auf einer dreidimensionalen, asymptotischen anti-de Sitter Raumzeit betrachtet. Hierfür werden zwei Ansätze diskutiert: Zunächst wird eine Adaptierung des allgemeinen Beweises für die höhere Spin-Algebra in Betracht gezogen. Anschließend wird eine Generalisierung des Einzelreihen-Eichverfahrens für die höhere Spin-Algebra berechnet. Um die generalisierte Drinfeld-Sokolov Reduktion durchführen zu können, wird die Lie Algebra von „Matrizen komplexer Größe" benötigt. Die Darstellung von Elementen in dieser Algebra, unter Verwendung eines Verma Moduls, führt zu bestimmten Eigenschaften der resultierenden Matrizen, von denen eine leicht für die Verwendung im Einzelreihen-Eichverfahren angepasst wird.

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Contents

1. Introduction

For the longest time physicists have tried to understand fundamental interactions of particles as well as their defining characteristics. One of such a characteristic is the spin of a particle, which crucially affects the behavior of its dynamics in space-time. Higher-spin gauge theories try to capture these dynamics and interactions between particles in the massless case. In particular, they try to couple massless particles (higher-spin fields) of spin $s > 2$ to gravity, which is regarded as a spin-2 theory in this framework. As mentioned in e.g. [6] or [7], in three space-time dimensions one is actually severely restrained by the representations of the little group, which only allows spin-0 and spin-$\frac{1}{2}$ representations, when trying to speak of massless particles of spin $s > 2$. Nonetheless, it is useful to also consider higher-spin gauge fields in three space-time dimensions as there is, in some cases, overlapping behavior with theories of higher dimensions. As it turns out, one can actually develop a higher-spin theory in three space-time dimensions as a Chern-Simons theory on an anti-de Sitter background, which generalizes three-dimensional gravity (refer to [27] for the treatment of gravity in this case) when linearised. Furthermore, upon considering space-times that are asymptotically anti-de Sitter, one obtains specific symmetry transformations that are called asymptotic symmetries. These symmetries form affine Lie algebras and can be reduced to classical W-algebras (refer to [7] and [18]) using the Drinfeld-Sokolov reduction (originally [9]). In the case of symmetry algebras arising from $\mathfrak{sl}_N$ it is well understood how this works, however, subtleties arise when one instead considers the higher-spin algebra $\mathfrak{hs}_\lambda$, which accounts for an unbounded set of spins.

On the other side, the Drinfeld-Sokolov reduction has been generalized by B. Khesin and F. Malikov in their publication [22]. In order to execute this generalization, they require the Lie algebra $\mathfrak{gl}_\lambda$, called the Lie algebra of "complex-size matrices", which was initially introduced in [10]. As a matter of fact, this Lie algebra contains the higher-spin algebra $\mathfrak{hs}_\lambda$ as a subalgebra, which suggests that it should be possible to also perform the reduction in the case of $\mathfrak{hs}_\lambda$. In order to make the connection between this abstract mathematical framework and the theory of asymptotic symmetries in three-dimensional higher-spin gauge theories, we examine two approaches when considering the Drinfeld-Sokolov reduction for $\mathfrak{hs}_\lambda$. On the one hand, one can try to adapt the proof performed in [22] to the case of $\mathfrak{hs}_\lambda$. On the other hand, direct application of the defining properties of a "matrix of a complex size" in the setting of the single-row gauge might also lead to the desired reduction. We will investigate both approaches in this thesis and then draw conclusions afterwards.

The following offers a brief overview over the topics covered. In chapter 2 we recount basic results concerning Lie groups and their respective algebras, as well as some representation theory with focus on coadjoint representations. Chapter 3 then sets the stage for the Drinfeld-Sokolov reduction by presenting Poisson manifolds and the quintessential Hamiltonian reduction thereof. Furthermore, we discuss the Lie group of pseudodifferential symbols and explain how it carries a Poisson structure. After this, in chapter 4, we will finally introduce the concept of complex-size matrices and consider the Hamiltonian reduction on the dual of its (extended) loop algebra. This reduction is what we understand under the term generalized Drinfeld-Sokolov reduction. At this point we reach our first discussion, when considering an adaptation of the proof in [22] to the higher-spin algebra $\mathfrak{hs}_\lambda$. In the final chapter 5, a brief introduction to higher-spin gauge theories is offered. Furthermore, we discuss the arising of asymptotic symmetries and their respective asymptotic symmetry algebras when considering three-dimensional space-times that are asymptotically AdS. Lastly, the single-row gauge is performed for the case of $\mathfrak{hs}_\lambda$, which concludes this thesis.

2. Lie groups and Lie algebras

This chapter aims at giving an introduction into Lie algebras and their representations, but also serves the purpose of presenting notations that will be of importance later. For the most part, we follow the book [24] on infinite-dimensional groups by B. Khesin and R. Wendt. However, we will also refer to the standard literature on representation theory by W. Fulton and J. Harris [13] and to the book by Jones [20] when deemed necessary. For the last part, when discussing the central extension of the affine Lie algebra $\widehat{\mathfrak{gl}}_n^*$, we will also consider the, for this thesis, essential publication [22] by B. Khesin and F. Malikov. The underlying fields will always be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, if not stated otherwise.

2.1. Representation theory

For the readers convenience we quickly recall the following definitions regarding Lie groups and their respective Lie algebras as presented in [24].

Definition 2.1 ([24]). *A Lie group G is a smooth manifold (i.e. infinitely differentiable) which is a group at the same time, such that the multiplication map $\mu : G \times G \to G$ and the inversion map $i : G \to G$ are smooth.*

Since G is smooth as a manifold, one can find tangent spaces T_gG at points $g \in G$ on G and, in particular, the tangent space T_eG at the identity $e \in G$. From the group multiplication an antisymmetric, bilinear map is obtained, which we call the *Lie bracket* in the following. With this, the definition of a Lie algebra can be given.

Definition 2.2 ([24]). *A Lie algebra $\mathfrak{g}$ is a vector space together with an antisymmetric, bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket. Specifically, the Lie algebra $\mathfrak{g}$ is the tangent space T_eG at the identity $e \in G$ and the group mulitplication on the (finite-dimensional) Lie group G gives rise to the Lie bracket $[\ ,\ ]$, which satisfies the Jacobi identity:*

$$\big[[X,Y],Z\big] + \big[[Z,X],Y\big] + \big[[Y,Z],X\big] = 0,\ \ X,Y,Z \in \mathfrak{g}\ . \tag{2.1}$$

As explained in [24], in the case of finite-dimensional Lie groups, we can always relate the Lie group G with its Lie algebra $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \to G$. Consider for example the Lie algebra $\mathfrak{gl}_n$ of $n \times n$-matrices (real or complex) and its associated Lie group GL_n. In this case the exponential map is simply the matrix exponential given by:

$$\exp(A) = \sum_{i=1}^{\infty} \frac{A^n}{n!}, \ A \in \mathfrak{gl}_n \ . \tag{2.2}$$

Having this in mind, one can introduce the notion of Lie group and Lie algebra representations, following [13]. Of particular importance will be the adjoint and coadjoint representation, as the latter plays a major role in the Hamiltonian reduction of Poisson manifolds.

Definition 2.3 ([13])**.** *In general, a representation of a Lie group is a pair (ρ, V), where V is a vector space (real or complex) and ρ is the smooth homomorphism:*

$$\rho : G \to GL(V) \ . \tag{2.3}$$

Here, $GL(V)$ is the general linear group of invertible V-transformations, which can also be seen as the group of all automorphisms $\mathrm{Aut}(V)$ over V. More often than not, the vector space V itself is referred to as the representation as compared to the pair (ρ, V), which is due to abusive notation. There is also an equivalent notion of a group representation, where instead of mapping into $GL(V)$ the smooth linear action [20],

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto gv \ , \end{aligned} \tag{2.4}$$

is considered. Hence, as outlined in [20], Lie group representations are considered as G-modules and the equivalence to the former definition can be seen by simply identifying $\rho(g)v$ with gv. Regarding actions of this kind, there is the concept of group orbits, whose definition shall be given at this point.

Definition 2.4 ([13])**.** *The orbit of an element $v \in V$, regarding the action $G \times V \to V$ of a (Lie) group G, is defined as the set $\{gv | g \in G\}$ in V.*

If there is a subgroup of G that leaves an element $v \in V$ invariant it is called the stabilizer of v. As this becomes important in the Hamiltonian reduction later on, the definition is given explicitly for clarity.

Definition 2.5 ([24])**.** *The stabilizer subgroup $H \subset G$ of an element $v \in V$ regarding the action $G \times V \to V$ is given as the set $H = \{g \in G | gv = v\}$.*

Similarly to how one defines Lie group representations, one can also define Lie algebra representations. However, now there is the requirement that the representation should respect the Lie bracket, that is, the representation should be a homomorphism of Lie algebras.

Definition 2.6 ([13]). *A representation of a Lie algebra $\mathfrak{g}$ is a pair (ψ, V), where V is a vector space (real or complex) and ψ is a homomorphism of Lie algebras,*

$$\psi : \mathfrak{g} \to \mathfrak{gl}(V) \ . \tag{2.5}$$

That is, ψ preserves the Lie bracket:

$$\psi\big([X,Y]\big)(v) = \big[\psi(X), \psi(Y)\big](v), \ \ X, Y \in \mathfrak{g}, \ \ v \in V \ . \tag{2.6}$$

Note that $\mathfrak{gl}(V)$ is the Lie algebra of the space of endomorphisms $\mathrm{End}(V)$ of V, by endowing $\mathrm{End}(V)$ with a Lie bracket. Analogously to the Lie group representation case, a Lie algebra representation (ψ, V) can also be regarded as a $\mathfrak{g}$-module.

As explained in [13] in detail, there is an important connection between the representation (ρ, V) of a Lie group G and the representation (ψ, V) of the corresponding Lie algebra $\mathfrak{g}$ if the Lie group G is connected and simply connected. In this case, the Lie algebra representation can just be obtained by simply differentiating ρ at the identity $e \in G$ to get $\psi = (\mathrm{d}\rho)_e$.

To the end of this introductory section, some further insights, regarding representation theory, are presented for Lie group representations [20]. However, similar definitions apply to Lie algebra representations. To begin, the definition of an invariant subspace with respect to some representation (ρ, V) is given.

Definition 2.7 ([13]). *Let (ρ, V) be a Lie group representation. A subspace $W \subset V$ is called invariant with respect to the representation (ρ, V), if $\rho(g)w \in W$ for all $w \in W$ and all $g \in G$.*

It is important to note, that it might not always be possible to find nontrivial invariant subspaces of a representation, in which case we call the representation irreducible.

Definition 2.8 ([13]). *A Lie group representation (ρ, V) is called irreducible, if the only invariant subspaces are $\{0\}$ and V itself. Otherwise, the representation is called reducible.*

Regarding representations of finite groups, there exists a property called *complete reducibility* (refer to [13] or [20] for an extensive discussion on this), which states that every representation can be decomposed into a direct sum of irreducible representations. While this does not hold true in general for Lie groups, it does hold for semisimple Lie algebras.

2.1.1. Adjoint and Coadjoint representations

For each Lie group G there are two special representations, namely the adjoint representation Ad and the coadjoint representation Ad^* (also referred to as adjoint and coadjoint *action* when speaking of group actions). They are unique in the sense that they define representations on the vector space of automorphisms $\mathrm{Aut}(-)$ of the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ respectively. To see this, remember that the conjugation of group elements $h \in G$ by a group element $g \in G$ defines an isomorphism $G \to G$, as presented in [13]:

$$\begin{aligned} \phi_g : G &\to G \\ h &\mapsto ghg^{-1} \ . \end{aligned} \tag{2.7}$$

Clearly, the identity group element $e \in G$ is left unchanged by the conjugation. This means that we can take the derivative of ϕ_g at the point e, which is an automorphism of the tangent space T_eG, i.e. of the Lie algebra $\mathfrak{g}$. This defines the adjoint representation of the Lie group G.

Definition 2.9 ([24]). *The group adjoint representation Ad of the Lie group G is given by the map $Ad : G \to Aut(\mathfrak{g})$, where $\mathfrak{g}$ is the associated Lie algebra.*

It is helpful to look at the Lie group GL_n once again. In this case, the group adjoint representation is given by the map $\mathrm{Ad} : GL_n \to \mathrm{Aut}(\mathfrak{gl}_n)$ which, according to [24], acts in the following way:

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{gl}_n &\to \mathfrak{gl}_n \\ X &\mapsto gXg^{-1} \ . \end{aligned} \tag{2.8}$$

Therefore, the group adjoint representation is a similarity transformation with respect to a group element $g \in GL_n$ of $n \times n$-matrices in this particular case.

Having seen the group adjoint representation, it is suitable to introduce the group coadjoint representation at this point, following [13, 24], as it is closely related to the former. The construction is quite similar, but instead of considering automorphisms of the Lie algebra $\mathfrak{g}$ one considers automorphisms of the dual space $\mathfrak{g}^*$. In this sense, the group coadjoint representation Ad^* is the dual of the group adjoint representation Ad and the two are connected via the pairing $(\ , \ )$ between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Definition 2.10 ([24]). *The group coadjoint representation Ad^* of the Lie group G is given by the map $Ad^* : G \to Aut(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. Fixing an element $g \in G$, the map $Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ obeys the defining relation:*

$$\big(Ad^*_g(\eta), X\big) := \big(\eta, Ad_{g^{-1}}(X)\big), \ \eta \in \mathfrak{g}^*, \ X \in \mathfrak{g} \ . \tag{2.9}$$

Now suppose, again following [13, 24], that the Lie group G is connected and simply connected. As stated before, a Lie algebra representation ψ can be obtained by differentiating the Lie group representation ρ at the identity $e \in G$. Considering the adjoint representation Ad, this means that the corresponding Lie algebra representation ad, called algebra adjoint representation, is given in the following way:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}), \\ \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto \mathrm{ad}_X(Y) =: [X, Y] \ . \end{aligned} \tag{2.10}$$

It is helpful to take a look at the matrix Lie group GL_n once again, as presented in [13]. In this case, elements of GL_n are connected to the Lie algebra $\mathfrak{gl}_n$ via the matrix exponential (see equation (2.2)). Therefore, the identity $e \in GL_n$ is simply given by $e^{\alpha X}$ at $\alpha = 0$ for $X \in \mathfrak{gl}_n$ and the element X in $\mathfrak{gl}_n$ can be obtained by taking the differential at the identity e, i.e. $X = \frac{\mathrm{d}}{\mathrm{d}\alpha}|_{\alpha=0} e^{\alpha X}$. With this in mind, one can find the differential $\mathrm{ad} = \mathrm{d}(\mathrm{Ad})$ of the group adjoint representation Ad of GL_n by analogously taking the differential at the identity,

$$\mathrm{d}(\mathrm{Ad})_X(Y) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \mathrm{Ad}_{e^{\alpha X}}(Y) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \Big(e^{\alpha X} Y e^{-\alpha X} \Big) = [X, Y] = \mathrm{ad}_X(Y) \ . \tag{2.11}$$

Similarly to how the algebra adjoint representation is defined as the differential of the group adjoint representation Ad, there also exists the notion of an algebra coadjoint representation ad^* as the differential of the group coadjoint representation Ad^* at the identity $e \in G$. Since Ad^* is connected to Ad via the pairing $(\ , \ )$ of $\mathfrak{g}$ and $\mathfrak{g}^*$, ad^* is correspondingly related to ad [24]:

$$\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*) \tag{2.12}$$

$$\text{given by: } \big(\mathrm{ad}^*_Y(\eta), X\big) : = -\big(\eta, \mathrm{ad}_Y(X)\big), \tag{2.13}$$

for some $\eta \in \mathfrak{g}^*, \ X, Y \in \mathfrak{g}$.

2.2. Loop Lie algebra $\mathfrak{g}(z)$ and its Central Extension $\widehat{\mathfrak{g}}$

This section focuses on the construction of so-called loop Lie algebras, whose central extensions are often called affine Lie algebras. In particular, the affine Lie algebra of $\mathfrak{gl}_\lambda$ will be of special interest in the Drinfeld-Sokolov reduction. If not stated otherwise, the considered Lie group G is connected and simply connected with the corresponding Lie algebra $\mathfrak{g}$. This whole section again closely follows the book [24].

2.2.1. The Central Extension

According to [24] there are two main definitions of a loop Lie algebra. The first definition is rather geometrical, stating that a loop Lie algebra $\mathfrak{g}(z)$ is the Lie algebra of smooth maps from the circle $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ to the Lie algebra $\mathfrak{g}$. However, in this thesis another notation of the loop Lie algebra will be used, which is more suited for the calculations that follow.

Definition 2.11 ([24]). *A loop Lie algebra $\mathfrak{g}(z)$, associated to a Lie algebra $\mathfrak{g}$, is an infinite-dimensional Lie algebra given by the tensor product $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$.*

Here, $\mathbb{C}[z, z^{-1}]$ is the ring of Laurent polynomials (where only finitely many coefficients of an element in this ring are allowed to be non-zero) and the connection to the other definition of the loop Lie algebra is given by setting $z = e^{i\theta}$, where $\theta \in \mathbb{R}$. We will use, however, the identification $z = e^{-i\theta}$ due to technicalities that arise when performing the single-row gauge in section (5.4). According to [11, 22], it is also possible to extend the loop Lie algebra by alternatively considering the ring of formal Laurent series $\mathbb{C}((z))$, that is, by allowing infinitely many positive exponents in z with non-zero coefficients. Notice, that in this case the identification with smooth maps from S^1 to $\mathfrak{g}$ cannot be realized directly due to the formal sums that are now allowed to appear. Since the properties of the two algebras $\mathfrak{g} \otimes \mathbb{C}\left[z, z^{-1}\right]$ and $\mathfrak{g} \otimes \mathbb{C}((z))$ are somewhat similar and closely connected, the latter is nonetheless identified as a loop Lie algebra, elements of which are $\mathfrak{g}$-valued *formal functions* in z. To simplify notation, we will denote $\mathfrak{g} \otimes \mathbb{C}((z))$ accordingly as $\mathfrak{g}(z)$.

Before trying to obtain affine Lie algebras out of the loop Lie algebra $\mathfrak{g}(z)$, there first is the need to introduce some further notation. In particular, it needs to be clarified what a central extension of a Lie algebra is. To begin, the notion of a 2-cocycle is introduced as given in [24].

Definition 2.12 ([24]). *A map $\gamma : \mathfrak{g} \times \mathfrak{g} \to V$, where $\mathfrak{g}$ is a Lie algebra and V a vector space, is called a 2-cocycle on $\mathfrak{g}$, if:*

$$i)\ \gamma(X, Y) = -\gamma(Y, X) \tag{2.14}$$
$$ii)\ \gamma(\alpha X + \beta Y, Z) = \alpha\gamma(X, Z) + \beta\gamma(Y, Z) \tag{2.15}$$
$$iii)\ \gamma\big([X, Y], Z\big) + \gamma\big([Z, X], Y\big) + \gamma\big([Y, Z], X\big) = 0\ , \tag{2.16}$$

for $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$.

Additionally, a 2-cocycle is set to be a continuous map in the subsequent text. According to [24], a central extension of a Lie algebra $\mathfrak{g}$ by some vector space V is the Lie algebra $\mathfrak{g} \oplus V$ with the Lie bracket:

$$\big[(X, v), (Y, w)\big] = \big([X, Y], \gamma(X, Y)\big)\ . \tag{2.17}$$

Here, γ is some 2-cocycle on $\mathfrak{g}$. Furthermore, notice that this extension is indeed central since V is in the center of $\mathfrak{g} \oplus V$ as the new commutator is independent of V.

This definition will be now applied to the case when $\mathfrak{g}$ is the loop algebra $\mathfrak{g}(z)$.

Proposition 2.1 ([24]). *Suppose there is a symmetric bilinear form $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ that is compatible with the Lie bracket in the sense that $\big\langle [X,Y],Z\big\rangle = \big\langle X,[Y,Z]\big\rangle$, called the invariance of $\langle\ ,\ \rangle$. The central extension of the loop Lie algebra $\mathfrak{g}(z)$ by $\mathbb{C}$ is defined by the 2-cocycle:*

$$\begin{aligned} \gamma_L : \mathfrak{g}(z) \times \mathfrak{g}(z) &\longrightarrow \mathbb{C} \\ \big(X(z),Y(z)\big) &\longmapsto Res\left\langle X(z), \frac{d}{dz}Y(z)\right\rangle \ . \end{aligned} \tag{2.18}$$

The centrally extended loop Lie algebra regarding $\mathfrak{g}(z)$ is denoted by $\widehat{\mathfrak{g}}$ and is referred to as the affine Lie algebra.

From now on, “Res“ will always be understood as taking the residue at $z=0$ unless stated otherwise. Additionally, the derivative with respect to z will be abbreviated by $'$. To see that the above proposition actually gives a (one-dimensional) central extension of the loop Lie algebra $\mathfrak{g}(z)$, it suffices to show that the map γ_L is indeed a 2-cocycle, which is not difficult to see. A proof in the real numbers can be found in e.g. [24].

2.2.2. Coadjoint Representations and $\widehat{\mathfrak{g}}$

Analogously to what was done in subsection 2.1.1, the group and algebra coadjoint representations regarding centrally extended loop Lie algebras are now of interest. As will be seen in chapter 4, it is important to introduce this topic cleanly since the group coadjoint action allows one to find special elements, called Frobenius matrices, in the group coadjoint orbit of a special affine Lie algebra. We will use [22, 24] for the notations.

To begin, one should consider the group and algebra adjoint representation for which the centrally extended loop group $\widehat{G}$ would be needed. However, since the extension of the loop group $G(z)$ is central, it suffices to only consider the loop Lie group $G(z)$ when considering the group adjoint action on $\widehat{\mathfrak{g}}$, according to [24]. This is since the center gives rise to a trivial group adjoint action and is therefore uninteresting. The same applies for the algebra adjoint action. Due to the definitions of the group coadjoint action (refer to definition (2.10)) and the algebra coadjoint action (refer to equation (2.12)) it is enough to introduce the group and algebra adjoint action in this case. First, the group adjoint action of $G(z)$ onto $\widehat{\mathfrak{g}}$ is proposed, which we adapted from [24] to the complex case.

Proposition 2.2 ([24]). *The group adjoint action Ad of the loop Lie group $G(z)$ on the centrally extended loop Lie algebra $\widehat{\mathfrak{g}}$ is given by:*

$$\begin{aligned} &Ad_{h(z)} : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{g}} \\ &\big(X(z),c\big) \longmapsto \Big(Ad_{h(z)}\big(X(z)\big), c - Res\left\langle h^{-1}(z)h'(z), X(z)\right\rangle\Big) \ . \end{aligned} \tag{2.19}$$

Here, h(z) is a fixed group element in $G(z)$, $c \in \mathbb{C}$ and $\langle\ ,\ \rangle$ is an invariant, symmetric bilinear form on $\mathfrak{g}$. To show that this is indeed the correct group adjoint action, one must show that this map is a group action, i.e. a group representation, and that the differential $\mathrm{ad}_{X(z)}\big(Y(z), c\big)$ indeed gives the algebra coadjoint action. This is not hard to proof and can be re-read in e.g. [24].

In order to obtain the coadjoint action, there is the need for the dual Lie algebra $\widehat{\mathfrak{g}}^*$. To get there, first note that, according to [24], there actually is an non-degenerate pairing (which we adapted for our purposes) between $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}^*$ given by

$$\Big(\big(X(z), c\big), \big(Y(z), d\big)\Big) = \mathrm{Res}\big\langle X(z), Y(z)\big\rangle z^{-1} - c \cdot d\ , \tag{2.20}$$

where $\big(X(z), c\big) \in \widehat{\mathfrak{g}}$ and $\big(Y(z), d\big) \in \widehat{\mathfrak{g}}^*$. Observe that the term z^{-1} naturally arises when transforming Fourier polynomials to Laurent ones. As explained in [24], two further things should be remarked: First, only the section of the dual space that consists of smooth 1-forms is considered and second, $Y(z)$ is an element of $\mathfrak{g}(z)$. Having this in mind, the group coadjoint action of $\widehat{G(z)}$ on $\widehat{\mathfrak{g}}^*$ can be introduced, using the conventions established in [22].

Proposition 2.3 ([22])**.** *The group coadjoint action Ad^* of the loop Lie group G(z) on the dual $\widehat{\mathfrak{g}}^*$ of the centrally extended loop Lie algebra $\widehat{\mathfrak{g}}$ is given by:*

$$\begin{aligned} Ad^*_{h(z)} : \widehat{\mathfrak{g}}^* &\longrightarrow \widehat{\mathfrak{g}}^* \\ \big(Y(z), d\big) &\longmapsto \Big(Ad_{h(z)}\big(Y(z)\big) - zd \cdot h'(z)h^{-1}(z), d\Big)\ , \end{aligned} \tag{2.21}$$

where $h(z)$ is a fixed group element and $d \in \mathbb{C}$.

Observe that the i, which would appear in front of the derivative when transforming $\theta \to z$, is absorbed in the d. In order to show that this is indeed the group coadjoint action it suffices to show the defining property of the group coadjoint action (refer to equation (2.9)), where we modified the proof in [24] to the complex case.

Proof. First, the left-hand side of equation (2.9) will be calculated using the pairing defined in equation (2.20). Suppose $\big(Y(z), d\big) \in \widehat{\mathfrak{g}}^*$ and $\big(X(z), c\big) \in \widehat{\mathfrak{g}}$. Then,

$$\begin{aligned} \Big(\mathrm{Ad}^*_{h(z)}\big(Y(z), d\big), \big(X(z), c\big)\Big) = {} & \mathrm{Res}\Big\langle \mathrm{Ad}_{h(z)}\big(Y(z)\big), X(z)\Big\rangle z^{-1} \\ & - d \cdot \mathrm{Res}\Big\langle h'(z)h(z)^{-1}, X(z)\Big\rangle - c \cdot d\ . \end{aligned} \tag{2.22}$$

On the other hand, the right-hand side of equation (2.9) is calculated to be:

$$\Big(\big(Y(z),d\big),\mathrm{Ad}_{h^{-1}(z)}\big(X(z),c\big)\Big) = \mathrm{Res}\Big\langle Y(z),\mathrm{Ad}_{h^{-1}(z)}\big(X(z)\big)\Big\rangle z^{-1} \\ +\ d\cdot\mathrm{Res}\Big\langle h(z)\frac{d}{dz}h^{-1}(z),X(z)\Big\rangle - c\cdot d\ . \tag{2.23}$$

To see that this two expressions are equivalent, note that the first terms in the respective equations represent the property (2.9) for elements of the loop Lie algebra $\mathfrak{g}(z)$ and its dual $\mathfrak{g}^*(z)$ regarding the pairing. What needs some inspection are the middle terms of the two expressions. To show that they are equal some algebraic transformations are required (for better visibility the arguments are left away):

$$h\frac{d}{dz}h^{-1} = -hh^{-2}h' = -h^{-1}h'hh^{-1} = -h'h^{-1}\ . \tag{2.24}$$

In the last step it was used that $h^{-1}h'h$ is just the conjugation of the element $h'\in\mathfrak{g}$. $\square$

By differentiating the group coadjoint action at the identity one finally obtains the algebra coadjoint action of an element $X(z)\in\mathfrak{g}(z)$ onto $\big(Y(z),d\big)\in\widehat{\mathfrak{g}}$ as introduced in [22]:

$$\mathrm{ad}^*_{X(z)}\big(Y(z),d\big) = \Big(\big[X(z),Y(z)\big] - zdX'(z),0\Big)\ . \tag{2.25}$$

In the following subsection this will now be applied to the case when the Lie algebra $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}_n$ of $n\times n$-matrices.

2.2.3. The case $\mathfrak{gl}_n$

To start, the loop Lie algebra of $\mathfrak{gl}_n$ is simply given by $\mathfrak{gl}_n(z)$. Note that there is an invariant, symmetric bilinear form on $\mathfrak{gl}_n$ given by the trace [22]:

$$\big\langle X,Y\big\rangle = \mathrm{Tr}\big(XY\big)\ . \tag{2.26}$$

With this form it is possible to centrally extend the loop Lie algebra $\mathfrak{gl}_n(z)$ to $\widehat{\mathfrak{gl}}_n(z)$ by the 2-cocycle, where we us the notation introduced in [22]:

$$\gamma:\mathfrak{gl}_n(z)\times\mathfrak{gl}_n(z)\longrightarrow\mathbb{C} \\ \big(M(z),N(z)\big)\longmapsto \mathrm{Res}\ \mathrm{Tr}\Big(M(z)N'(z)\Big)\ . \tag{2.27}$$

Since the loop space $\mathfrak{gl}_n(z)$ also carries the trace-product, one finds that the dual space $\mathfrak{gl}_n^*(z)$ is isomorphic to $\mathfrak{gl}_n(z)$ by considering

$$\big(M(z),N(z)\big) = \mathrm{Res}\ \mathrm{Tr}\big(M(z),N(z)\big)z^{-1}\ . \tag{2.28}$$

Accordingly, the affine Lie algebra $\widehat{\mathfrak{gl}}_n$ is isomorphic to $\widehat{\mathfrak{gl}}_n^*$ by means of the pairing

$$\Big(\big(M(z),a\big),\big(N(z),b\big)\Big) = \mathrm{Res}\ \mathrm{Tr}\big(M(z),N(z)\big)z^{-1} + a\cdot b\ , \tag{2.29}$$

where $a, b \in \mathbb{C}$. From this, it can be seen that it is possible to write $\widehat{\mathfrak{gl}}_n^* \cong \mathfrak{gl}_n(z) \oplus \mathbb{C}$ [22].

Now, the attention is shifted to the group and algebra coadjoint action. According to proposition 2.3, the former is given by:

$$\begin{aligned} \mathrm{Ad}^*_{h(z)} : \widehat{\mathfrak{gl}}_n^* &\longrightarrow \widehat{\mathfrak{gl}}_n^* \\ \big(M(z), a\big) &\longmapsto \Big(h(z)M(z)h^{-1}(z) - zah'(z)h^{-1}(z), a\Big) \ , \end{aligned} \tag{2.30}$$

for some $h(z) \in GL_n(z)$ and $a \in \mathbb{C}$. Furthermore, this leads to an algebra coadjoint action, equivalent to the one given in equation (2.25):

$$\mathrm{ad}^*_{N(z)}\big(M(z), a\big) = \Big(\big[N(z), M(z)\big] - zaN'(z), 0\Big) \ , \tag{2.31}$$

where $N(z) \in \mathfrak{gl}_n(z)$ [22].

At this point it is important to make the following crucial observation, as presented in [22]. Suppose there is a matrix-valued differential operator of the form

$$D = -az\frac{d}{dz} + M(z) \ . \tag{2.32}$$

Of course, the matrix representation of the operator d/dz depends on the choice of basis. The space of such operators can be identified with $\widehat{\mathfrak{gl}}_n^*$ by setting,

$$\big(M(z), k\big) \mapsto -az\frac{d}{dz} + M(z) \ . \tag{2.33}$$

Therefore, the coadjoint action on $\widehat{\mathfrak{gl}}_n^*$ is closely connected to the action on such operators.

3. Poisson Geometry

Symplectic manifolds are a natural generalization of the phase space of a physical system and, furthermore, they are Poisson manifolds by construction [1]. Because of this, they are of great interest as one can describe the dynamics of a system using this well established framework. This chapter aims at introducing the concepts of Poisson manifolds, the Hamiltonian reduction that arises under certain conditions as well as the group of pseudodifferential symbols. For the most part we follow again the book [24] by B. Khesin and R. Wendt, but also the lecture notes [8] on classical W-algebras by L. Dickey will play an important role. Additionally, we consider the publication [22] (especially when discussing the second Gelfand-Dickey structure), as well as [1] and [5] for remarks concerning physics.

3.1. Poisson manifolds

To begin this section, the notion of Poisson manifold and symplectic manifolds is introduced, using the conventions established in [24]. It shall be remarked, that it is actually more precise to speak of a “manifold with Poisson structure“ , however, this is understood as a Poisson manifold in the following. With this in mind, the definition of a manifold with Poisson structure is given.

Definition 3.1 ([24])**.** *Suppose $\mathcal{M}$ is a (differentiable) manifold and denote by $\mathcal{C}^\infty(\mathcal{M})$ its space of functions. This manifold is called a Poisson manifold if it carries a bracket $\{\ ,\ \}$, which satisfies:*

$$i)\ \{f,g\} = -\{g,f\} \tag{3.1}$$

$$ii)\ \big\{f,\{g,h\}\big\} + \big\{g,\{h,f\}\big\} + \big\{h,\{f,g\}\big\} = 0 \tag{3.2}$$

$$iii)\ \{f,gh\} = \{f,g\}h + \{f,h\}g\ . \tag{3.3}$$

Here, f, g and h are elements of the space $\mathcal{C}^\infty(\mathcal{M})$ and the map $\{\ ,\ \}$ is called Poisson bracket in this case.

Notice, that the first two properties imply that the space $\mathcal{C}^\infty(\mathcal{M})$ can be understood as a Lie algebra, where the Poisson bracket represents the Lie bracket [24].

In the upcoming paragraph we consider the outline presented in [8]. Regarding the above definition, it is actually possible to obtain a Poisson structure on a manifold from a symplectic 2-form. In order to investigate this property, it is appropriate to give a reminder on the description of 2-forms. This will be analysed in the case of finite-dimensional manifolds, however, a generalization to infinite dimensions is possible, as introduced in [8]. Suppose $\mathcal{M}$ is a smooth manifold with the tangent bundle $\mathcal{TM}$, whose sections can be given a Lie algebra structure and elements evaluated at some point $m \in \mathcal{M}$ simply form the tangent space $T_m\mathcal{M}$. Additionally there is the associated dual space $\mathcal{T}^*\mathcal{M}$, elements of which are just 1-forms over $T_m\mathcal{M}$ at some point $m \in \mathcal{M}$. As is well known, for every function f on the function space $\mathcal{C}^\infty(\mathcal{M})$ one can find a 1-form $\mathrm{d}f$ via the pairing between $\mathcal{TM}$ and $\mathcal{T}^*\mathcal{M}$. Note that one can also find 2-forms ω_m over $T_m\mathcal{M}$ by requiring that they are not only linear in both entries but also skew-symmetric, i.e.

$$\omega_m(X_m, Y_m) = -\omega_m(Y_m, X_m) \ , \tag{3.4}$$

for all $X_m, Y_m \in T_m\mathcal{M}$. From a physical standpoint, the focus usually is not on the whole space of 2-forms but rather on those satisfying specific properties. The main interest are so-called symplectic 2-forms, which, together with a manifold, generalize the idea of phase spaces [1].

Definition 3.2 ([8, 22])**.** *A 2-form ω is called symplectic if it is closed ($\mathrm{d}\omega{=}0$) and non-degenerate ($\omega_m(X_m, Y_m) = 0$ for all $X_m \in T_m\mathcal{M}$ implies $Y_m = 0$). A symplectic manifold is a pair $(\mathcal{M}, \omega)$ consisting of a manifold $\mathcal{M}$ and a symplectic 2-form ω.*

One can identify the tangent bundle $\mathcal{TM}$ and the cotangent bundle $\mathcal{T}^*\mathcal{M}$ using non-degenerate 2-forms. To see this, we refer to [8, 22] and first consider the following map $\widetilde{\omega} : \mathcal{TM} \to \mathcal{T}^*\mathcal{M}$

$$\widetilde{\omega}(X)(Y) := -\omega(X, Y) \ , \tag{3.5}$$

for $X, Y \in \mathcal{TM}$ and $\alpha := \widetilde{\omega}(X) \in \mathcal{T}^*\mathcal{M}$. This map is an isomorphism with inverse $\widetilde{\omega}^{-1} : \mathcal{T}^*\mathcal{M} \to \mathcal{TM}$, that is

$$\alpha(Y) = -\omega\Big(\widetilde{\omega}^{-1}(\alpha), Y\Big) \ . \tag{3.6}$$

This means that one can also identify a non-degenerate 2-form with a map $\widetilde{\omega}$. The authors of [8] and [22] introduce some helpful notation that we want to consider in the following: In the case that ω is a symplectic 2-form, one calls $\widetilde{\omega}^{-1}$ a *Hamiltonian mapping*. Furthermore, a vector field X_h preserving the symplectic form ω by the means of the Lie derivative $L_{X_H}\omega = 0$ is called a *Hamiltonian vector field*. This Hamiltonian vector field X_H is generated by a *Hamiltonian function* H in the sense that $X_H = \widetilde{\omega}^{-1}(\mathrm{d}H)$. With all of this in mind, one can then define a Poisson bracket on the space of functions $\mathcal{C}^\infty(\mathcal{M})$, by setting:

$$\{f, g\} = \mathrm{d}g\Big(\widetilde{\omega}^{-1}(\mathrm{d}f)\Big) \ . \tag{3.7}$$

Notice that due to equation (3.6), one can slightly re-write this [8]:

$$\{f, g\} = \omega\big(X_f, X_g\big) \ . \tag{3.8}$$

It is not hard to show that this indeed satisfies the properties (equations (3.1) - (3.3)) of the poisson bracket. In particular, property *ii)* holds only if the non-degenerate 2-form is also closed, i.e. symplectic.

Some comments on the physics of symplectic and Poisson manifolds are in order, for which we refer to [1, 5]. From the above, one first notes that every symplectic manifold is Poisson, whereas the converse is not true in general. As already mentioned, a symplectic manifold can be interpreted as the phase space of some physical system. To understand this, suppose that a manifold $\mathcal{M}$ is the configuration space, that is the space of all possible states, of said physical system. In this case, one sees the tangent bundle $\mathcal{TM}$ as the *velocity phase space*, while the cotangent bundle $\mathcal{T}^*\mathcal{M}$ is regarded as the *momentum phase space*[1]. This stems from the fact, that the cotangent bundle $\mathcal{T}^*\mathcal{M}$ of a smooth manifold $\mathcal{M}$ can be shown to be a symplectic manifold and therefore naturally induces a Poisson bracket, as presented in equation (3.8). For some coordinates $(q^1, ..., q^n, p_1, ..., p_n)$, the symplectic form then takes the form $\omega = \sum \mathrm{d}q^i \wedge \mathrm{d}p_i$ connecting generalized coordinates q^i to conjugate momenta p_i. Generally speaking, a symplectic manifold $(\mathcal{M}, \omega)$ together with a Hamiltonian vector field X_H is called a *Hamiltonian system* (M, ω, X_H) and the vector field X_H gives rise to a differential equation

$$\dot{f}\big(x(t)\big) = \{H, f\}\big(x(t)\big) \ , \tag{3.9}$$

which represents the time evolution of some function $f \in \mathcal{C}^\infty(\mathcal{M})$ on $\mathcal{M}$ [1].

3.1.1. Symplectic leaves

We now wonder whether it is possible to restrict the dynamics of a system on a Poisson manifold to some subspace. This is possible when considering symplectic leaves, for the discussion of which we refer to [22, 24]. As mentioned above, vector fields X_m considered at some point $m \in \mathcal{M}$ on a smooth manifold form the tangent space $T_m\mathcal{M}$. Of course, the set of Hamiltonian vector fields regarding a Poisson manifold $(M, \{\ ,\ \})$ considered at m is then a subspace of $T_m\mathcal{M}$. In particular, there is a subbundle D of the tangent bundle $\mathcal{TM}$ arising from these considerations of Hamiltonian vector fields. The integral submanifolds, i.e. submanifolds whose tangent space equals D at every point, of this subbundle are called symplectic leaves [22]. It can be proven that these symplectic leaves are symplectic manifolds themselves, where the symplectic form ω is simply defined via the Poisson bracket since the tangent space of a leaf only consists of Hamiltonian vector fields. All in all, this means that Poisson manifolds can be represented via symplectic manifolds and from a physics perspective, that it is possible to restrict the dynamics

[1]This is not true in general. For an arbitrary physical system the phase space is not necessarily represented by the cotangent bundle, but can also be represented by e.g. the tangent bundle [1].

of the system to these symplectic leaves. There will be further comments made about symplectic leaves in section 3.2 concerning the Hamiltonian reduction.

Consider now a Lie group G of arbitrary dimension (even infinite-dimensional) and its associated Lie algebra $\mathfrak{g}$ as well as its dual $\mathfrak{g}^*$. As presented in [22, 24], $\mathfrak{g}^*$ possesses a Poisson structure of the following form, called the *Kirillov-Kostant Poisson structure*:

$$\{f, g\}(m) := \big([\mathrm{d}f_m, \mathrm{d}g_m], m\big) \ . \tag{3.10}$$

Here, $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$, $m \in \mathfrak{g}^*$ and $\mathrm{d}f_m$ respectively $\mathrm{d}g_m$ are elements in the Lie algebra $\mathfrak{g}$ itself. Latter interact with elements in $\mathfrak{g}^*$ via the pairing (,). Since the manifold is Poisson, a representation as symplectic leaves is possible as mentioned in the previous paragraph. In fact, one can show (refer to [24, Chapter I.4]) that the symplectic leaves are simply given by the coadjoint orbits of the Lie group G.

3.2. Hamiltonian reduction

The aim of this section is to give an introduction to Hamiltonian reduction, which builds the foundation of the Drinfeld-Sokolov reduction. Roughly speaking, the idea is to obtain a symplectic structure on some quotient manifold of the initial symplectic manifold $(\mathcal{M}, \omega)$ considering some group action on $\mathcal{M}$ [24]. From a physics standpoint, a group action might represent some symmetry in the momentum phase space, which is known to be a symplectic manifold. The Hamiltonian reduction would then lead to a "reduced" phase space, accounting for these symmetries [5].

For the mathmatical outline of the Hamiltonian reduction, we now follow the procedure presented in [22] and [24]. To make the above more clear, suppose $(\mathcal{M}, \omega)$ is a symplectic manifold and G is a Lie group acting on $\mathcal{M}$. If the action of the Lie group preserves the symplectic form ω in the sense that:

$$\forall g \in G : g^*\omega = \omega \ , \tag{3.11}$$

then one calls it *symplectic* [24]. Additionally, if there exist Hamiltonian functions for every X in the Lie algebra $\mathfrak{g}$, such that there is a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{C}^\infty(\mathcal{M})$, one calls the group action *Hamiltonian* [22]. Assume now, that the group action of G on $\mathcal{M}$ is Hamiltonian. Of course, there is also a natural action of the Lie group G on the dual of the Lie algebra $\mathfrak{g}^*$ by means of the coadjoint action. This allows one to connect the manifold $\mathcal{M}$ with $\mathfrak{g}^*$ via a map, called the *momentum map*.

Definition 3.3 ([22], [24])**.** *Suppose G is a Lie group with Hamiltonian group action on a symplectic manifold (M, ω). The momentum map p is a G-equivariant map $p : \mathcal{M} \to \mathfrak{g}^*$, determined by the evaluation of a Hamiltonian function associated to some $X \in \mathfrak{g}$ at a point m on the manifold:*

$$\big(p(m), X\big) := H_X(m) \ . \tag{3.12}$$

Here, $(\ ,\ )$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

The reason for the suggestive name of this map stems from the fact that it encodes conserved quantities. This can be seen by looking at the time evolution (refer to (3.9)) of the momentum map evaluated at some $X \in \mathfrak{g}$: For a Hamiltonian H, if $\{H_X, H\} = 0$, then the momentum map is conserved in time [5].

With this in mind we continue with the Hamiltonian reduction following [22] and [24]. If one fixes an element $\Lambda \in \mathfrak{g}^*$, then denote by G_Λ the subgroup of G, which leaves Λ invariant under the coadjoint action. Since the momentum map is G-equivariant, the inverse image of Λ is just given by a union of G_Λ-orbits. If one then assumes that the set $p^{-1}(\Lambda)$ as well as the quotient $\mathcal{N}_\Lambda = p^{-1}(\Lambda)/G_\Lambda$ are manifolds, one can show that $\mathcal{N}_\Lambda$ is a symplectic manifold with its symplectic form being induced from the symplectic structure of $\mathcal{M}$. This is what is known under the term Hamiltonian reduction and one calls $\mathcal{N}_\Lambda$ the *symplectic quotient* [24].

An important remark should be made, for which we refer to [22]. It is possible to perform this reduction in the more general case of Poisson manifolds. To do this, one first generalizes Hamiltonian group actions by requiring that they, similarly to the case of symplectic manifolds, leave the Poisson structure of some Poisson manifold $(\mathcal{M}, \{\ ,\ \})$ invariant and give rise to a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{C}^\infty(\mathcal{M})$. However, in contrast to the case of symplectic manifolds, there is now also the additional requirement of preserving the symplectic leaves of the Poisson manifold. Due to this preservation of the leaves, the concept of the momentum map still works as one can simply require, that constraining the map to the symplectic leaves gives momentum maps as per definition 3.3. The resulting quotient manifold, obtained by the Hamiltonian reduction, can then be shown to be a Poisson manifold again, as remarked in [22].

3.3. The case of Pseudodifferential Symbols

One example of a Poisson manifold, which is of particular importance in this thesis, is the Lie group $G_{\psi DS}$ of pseudodifferential symbols of highest order $\lambda \in \mathbb{C}$, elements of which are considered as formal Laurent series. Historically, this Poisson structure on the manifold was first found by Adler [2] and then subsequently proven by Gelfand and Dickey [16]. The Lie group itself is constructed by doubly extending (where the first extension is central) the Lie algebra of pseudodifferential symbols, which has elements

$$\sum_{j=-\infty}^{K} a_j(z)\partial^j \text{ where } a_j(z) \in \mathbb{C}((z)) \text{ and } K \in \mathbb{N} \ , \tag{3.13}$$

and then exponentiating the subalgebra of “integral symbols“ of the resulting Lie algebra. A detailed discussion of this procedure can be found in [24, Chapter II.4], we will just consider the resulting infinite-dimensional Lie group and examine its structure for brevity.

To begin, we will review the multiplication of such symbols, using the notations from the book [24], which is simply a generalization of the Leibniz rule:

$$\partial^\alpha \circ f\partial^\beta = \sum_{n=0}^{\infty} \binom{\alpha}{n} f^{(n)} \partial^{\alpha-\beta-n} \ . \tag{3.14}$$

Here, $f' := df/dz$ and if the meaning is understood, the composition sign “$\circ$“ will be often omitted. Notice two things: Firstly, if $\alpha \in \mathbb{N}$ the sum is finite and secondly, this makes sense even if $\alpha \in \mathbb{C}$, since the binomial coefficient is given by [24]

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \ . \tag{3.15}$$

Incidentally, one can also extract the commutator of two symbols out of the Leibniz rule [24]:

$$\left[\partial^\alpha, f\partial^\beta\right] = \sum_{n=1}^{\infty} \binom{\alpha}{n} f^{(n)} \partial^{\alpha+\beta-n} \tag{3.16}$$

Observe, that the commutator is a symbol of highest order $\alpha + \beta - 1$.

Now we also want to fix some notation. Suppose A is an arbitrary operator of the form given in equation (3.13). Denote by A_+ the purely differential part of A and by A_- the purely pseudodifferential part of A:

$$A_+ = \sum_{j=0}^{K} a_j(z)\partial^j \ , \ A_- = \sum_{j=-\infty}^{-1} a_j(z)\partial^j \ . \tag{3.17}$$

Therefore, one can decompose $A = A_+ + A_-$ into A_+ and A_-. Taking this into account, the Lie group $G_{\psi DS}$ of pseudodifferential symbols of highest order $\lambda \in \mathbb{C}$ is defined as [22]

$$G_{\psi DS} = \left\{ \partial^\lambda + \sum_{k=1}^{\infty} u_k(z)\partial^{\lambda-k}, \text{ where } \lambda \in \mathbb{C} \text{ and } u_k(z) \in \mathbb{C}((z)) \right\} \ . \tag{3.18}$$

That this is indeed a Lie group follows from its construction, and the group multiplication is determined by the Leibniz rule. The Poisson structure on this Lie group is called the *second Gelfand-Dickey structure* and some crucial remarks need to be made before introducing the definition [24].

In the following we present the second Gelfand-Dickey structure as it is introduced in [22] and [24]. First, denote by G_λ the set of pseudodifferential symbols of fixed order λ. Obviously, a union of these sets for all $\lambda \in \mathbb{C}$ is the same as $G_{\psi DS}$. Furthermore, the G_λ are actually hyperplanes and one can show (refer to [24, Chapter II.4]) that they are Poisson submanifolds, i.e. it is enough to consider the Poisson structure restricted to the sets G_λ. Additionally, since the G_λ are hyperplanes they also are affine spaces, which means that one can find tangent and cotangent spaces. The tangent space at some point $L \in G_\lambda$ is in this case given by the set [22]

$$T_L G_\lambda = \left\{ \sum_{k=1}^{\infty} \delta u_k(z) \partial^{-k} \circ \partial^{\lambda} \right\} . \tag{3.19}$$

Correspondingly, one finds that the cotangent space $T_L^* G_\lambda$ is defined by [22]

$$T_L^* G_\lambda = \left\{ \partial^{-\lambda} \circ P, \text{ where } P \in \mathrm{DO} \right\} . \tag{3.20}$$

Here, DO is the set of purely differential operators. As given in [22], there also is a pairing between $T_L G_\lambda$ and $T_L^* G_\lambda$ which means that elements in the cotangent space are identified with linear functionals acting on elements in the tangent space, i.e.

$$F_B(\delta L) := \big(B,\ \delta L\big) = \mathrm{Tr}\big(\delta L \circ B\big) , \tag{3.21}$$

where $B \in T_L^* G_\lambda$ and $\delta L \in T_L G_\lambda$. The trace is given by taking the residue at $z = 0$ for the a_{-1} coefficient of $\delta L \circ B$. Keeping this in mind, one defines the second Gelfand-Dickey Poisson structure in the following way (that this indeed gives rise to the correct Poisson structure on $G_{\psi DS}$ is shown in e.g. [24, Chapter II.4]).

Definition 3.4 ([22, 24])**.** *The second Gelfand-Dickey Poisson structure is defined on linear functionals as:*

$$\big\{F_B, F_C\big\}(L) := F_C\Big(L\big(BL\big)_+ - \big(LB\big)_+ L\Big) . \tag{3.22}$$

Observe, as suggested in [24], that one can equivalently represent the second Gelfand-Dickey Poisson structure as:

$$\big\{F_B, F_C\big\}(L) := F_C\Big(\big(LB\big)_- L - L\big(BL\big)_-\Big) , \tag{3.23}$$

because the symbol LBL cancels. Additionally, notice that the term in the bracket on the right-hand side of equations (3.22) and (3.23) is a symbol of highest order $\lambda - 1$ and is therefore understood as an element of the tangent space. This can be easily seen by direct calculation. In particular, this term can be regarded as the image of the following Hamiltonian mapping from $T_L^* G_\lambda$ to $T_L G_\lambda$ [22]:

$$F_B \longmapsto L\big(BL\big)_+ - \big(LB\big)_+ L =: V_{F_B}(L) . \tag{3.24}$$

Furthermore, note that defining the Poisson structure on linear functionals suffices, since smooth functions on a manifold are closely connected to elements on the cotangent space (by building differentials; as described in section 3.1).

As discussed in [23] and [22], it is important to remark, that in physics literature one often calls the resulting Poisson structure (together with the functions) a W-algebra. In our case, one refers to the Poisson structure with the fields u_k as classical $W_{\infty+1}(\lambda)$-algebra ($W_\infty(\lambda)$ if $u_1 = 0$ in L). In similar fashion, when considering the hyperplanes of positive integers $\lambda = N$, the Poisson structure carries the name W_N-algebra, which is nicely explained in e.g. [8]. We will come back to this in chapter 5, where we discuss that these algebras arise as asymptotic symmetry algebras in higher-spin gauge theory.

The Poisson bracket and sequences of functions

It is now important to consider the following identification and resulting filtration procedure concerning the space S of sequences of functions, as presented in [22]. This is essential, as it helps one understand why the Poisson bracket obtained from the Hamiltonian reduction in the generalized Drinfeld-Sokolov reduction is considered as a polynomial in λ later on (refer to sections 4 and 5.4). To begin, according to the authors of [22], one can actually relate G_λ, the subsets of symbols of fixed order λ with the space S of sequences of functions[2] of the form:

$$\big(u_1(z), u_2(z), u_3(z)...\big) \in S \ . \tag{3.25}$$

We want to point out, that this identification does not depend on λ, since the "position" of a coefficient $u_k(z)$ appearing in the sequence is completely determined by the index k. On the other hand, we know that the Poisson structure on $G_{\psi DS}$ is obtained by restricting to the sets G_λ (i.e. by fixation of λ). Therefore, the Poisson structure on $G_{\psi DS}$ gives rise to a Poisson structure on S, treating λ as a parameter rather than a fixed value.

In order to identify the Poisson structure resulting from the Hamiltonian reduction in section 4.2 with the second Gelfand-Dickey bracket (3.22), the authors of [22] filtrate the space S in the following way. First, split S into $S_{k\leq N}$ and $S_{k>N}$ for some arbitrary positive integer N, i.e.:

$$S = S_{k\leq N} \times S_{k>N} \ . \tag{3.26}$$

Here, the subindices are understood as restrictions on the length of the sequence, e.g.

$$\big(u_1(z), u_2(z), ..., u_N(z)\big) \in S_{k\leq N} \ . \tag{3.27}$$

The authors of [22] futhermore observed that for any N this splitting implies a natural embedding of the function space $\mathbf{S}_N := \mathrm{Fun}(S_{k\leq N})$ into the function space $\mathbf{S} := \mathrm{Fun}(S)$, elements of which are regarded as functions rather than sequences of functions. We denote this embedding by ϵ_N and its corresponding image in $\mathbf{S}$ by $\epsilon_N(\mathbf{S}_N)$. The set $\big\{\epsilon_N(\mathbf{S}_N)\big\}$ then gives rise to the filtration of interest:

[2]This identification is homeomorphic according to [22].

$$\epsilon_1(\mathbf{S_1}) \subset \epsilon_2(\mathbf{S_2}) \subset ... \subset \mathbf{S} \ . \tag{3.28}$$

Hence, the union of all the elements in the set $\{\epsilon_N(\mathbf{S}_N)\}$ is the space $\mathbf{S}$ itself [22].

4. Drinfeld-Sokolov reduction

In this chapter, the generalized Drinfeld-Sokolov reduction for the *Lie algebra of complex-size* will be outlined. This reduction was first conjectured by B. Feigin and C. Roger, according to the authors of the paper [22], which provided the proof for this reduction. Originally, the Lie algebra of matrices of complex-size was first introduced by B. Feigin in [10] and, as we will see, the suggestive name stems from the close connection to the Lie algebra $\mathfrak{gl}_n$. For the main part, we will closely follow the notations and conventions used in [22] and make adaptations where necessary.

Before we get to the reduction and all the notations required for its description, we first want to motivate it. Put in simple words, the Drinfeld-Sokolov reduction of interest is a Hamiltonian reduction applied to a certain dual of an affinization (and extension) of the Lie algebra of complex-size matrices. The resulting quotient manifold of this reduction will be a Poisson manifold itself and the main statement of the paper of B. Khesin and F. Malikov is, that its Poisson structure corresponds with the second Gelfand-Dickey structure as given in equation (3.22). Historically, this was first established by Drinfeld and Sokolov [9] for the case of differential operators of highest order n, which appear as a Poisson submanifold of the above Lie group $G_{\psi DS}$ (refer to 3.18). In the case of purely differential operators, the Drinfeld-Sokolov reduction is build on the observation that an n-th order differential equation can be identified with a system of n first order differential equations (also called the *Miura transformation*). A nice presentation of this can be found in these lecture notes by Dickey [8]. To conclude, one notices that the reduction performed by B. Khesin and F. Malikov is a generalization of the Drinfeld-Sokolov reduction, which should arise when restricting λ to positive integers.

4.1. Complex-size matrices

We start the reduction by first introducing the essential Lie algebra of complex-size matrices, following [22]. To do so, consider the well known Lie algebra $\mathfrak{sl}_2$, the generators e, f, h of which satisfy the following commutation relations:

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h \ . \tag{4.1}$$

The universal enveloping algebra $U(\mathfrak{sl}_2)$[1] of $\mathfrak{sl}_2$ possesses all the representations of $\mathfrak{sl}_2$ and has a basis given by the set (also called PBW-basis [17]):

[1]A good introduction to universal enveloping algebras is e.g. offered in [17].

$$\left\{e^i f^j h^k,\ i,j,k \geq 0\right\} . \tag{4.2}$$

Strictly speaking, $U(\mathfrak{sl}_2)$ is an associative algebra and we make it into a Lie algebra by defining the Lie bracket of two elements c, d as:

$$[c,d] := cd - dc . \tag{4.3}$$

This Lie algebra then has a center, generated by the Casimir operator $C := \{e, f\} + \frac{1}{2}h^2$. We follow the construction in [22] and take the quotient of the enveloping algebra $U(\mathfrak{sl}_2)$ by the Lie algebra ideal:

$$\left\langle C - \frac{1}{2}(\lambda^2 - 1)\right\rangle . \tag{4.4}$$

An explanation for this particular choice will be given in the introduction of the Verma module below. The algebra one obtains from this quotient will be denoted by $\mathfrak{gl}_\lambda$ and we call it the Lie algebra of complex-size matrices [22] from now on:

$$\mathfrak{gl}_\lambda = \frac{U(\mathfrak{sl}_2)}{\left\langle C - \frac{1}{2}(\lambda^2 - 1)\right\rangle} . \tag{4.5}$$

We now want to consider the infinite-dimensional space $\{v_1, v_2, ...\}$ and define the following $\mathfrak{sl}_2$ representations, which depend on λ, on it[2]:

$$f v_i = v_{i+1},\ e v_i = (i-1)(\lambda - i + 1)v_{i-1},\ h v_i = (\lambda + 1 - 2i)v_i . \tag{4.6}$$

Hence, the set $\{v_1, v_2, ...\}$ forms a Verma module[3], as observed in [22]. Notice, that due to this definition, the Casimir operator C acts in the following way onto the v_i's:

$$\begin{aligned} C v_i &= \left(ef + fe + \frac{1}{2}h^2\right)v_i && (4.7)\\ &= \left(i(\lambda - i) + (i-1)(\lambda - i + 1) + \frac{1}{2}\left(\lambda + 1 - 2i\right)^2\right)v_i && (4.8)\\ &= \frac{1}{2}\left(\lambda^2 - 1\right)v_i . && (4.9)\end{aligned}$$

This explains the choice taken above in the construction of $\mathfrak{gl}_\lambda$. Additionally, as the authors of [22] noticed, the Verma module is reducible if λ is a positive integer, in the sense that:

$$e v_{\lambda+1} = 0 . \tag{4.10}$$

This implies, that in this case the subspace $V_\lambda := \{v_{\lambda+1}, v_{\lambda+2}, ...\}$ is closed under the actions (4.5). The definition of the Verma module above allows one to identify $U(\mathfrak{sl}_2)$

[2] We have opted for a slightly different convention as presented in [22], starting the index count at one rather than zero.

[3] One often calls a Verma module a "highest-weight representation". They are explained in e.g. [17].

(since a $\mathfrak{sl}_2$-module is also a $U(\mathfrak{sl}_2)$-module by construction) with the algebra $\mathfrak{gl}_\infty$ of linear transformations of the set $\{v_1, v_2, ...\}$. Furthermore, since there is a natural homomorphism between a Lie algebra $\mathfrak{g}$ and its quotient algebra $\mathfrak{g}/\mathfrak{i}$, where $\mathfrak{i}$ is a (Lie) ideal of $\mathfrak{g}$, the identification gives rise to the following embedding[4] of $\mathfrak{gl}_\lambda$ into $\mathfrak{gl}_\infty$ [22]:

$$\mathfrak{gl}_\lambda \hookrightarrow \mathfrak{gl}_\infty \ . \tag{4.11}$$

With this in mind, we are now in a position to give the definitions that restrict the form of matrices accordingly, as presented in [22]. It is important to remark, that we consider λ to be a parameter now instead of taking a specific value, which makes matrix entries into polynomials in λ and therefore gives rise to the map:

$$\eta : U(\mathfrak{sl}_2) \to \mathfrak{gl}_\infty \otimes \mathbb{C}[\lambda] \ . \tag{4.12}$$

Thus, elements in $U(\mathfrak{sl}_2)$, represented as matrices M in $\mathfrak{gl}_\infty \otimes \mathbb{C}[\lambda]$, satisfy the following three properties[5] according to [22]:

i) Matrix entries m_{ii+n} are polynomials in i for every $n \geq 0$. (4.13)

ii) If λ is a positive integer, then $m_{ij} = 0$ if $j > \lambda$ and $i \leq \lambda$. (4.14)

iii) Any matrix M is "bounded from below", i.e. $m_{ij} = 0$ if $i > j + K - 1$, for some $K > 0$. (4.15)

Observe, that the first two properties follow from the definition of the Verma module (4.5) (and because we identify λ as a parameter), while property *iii)* is a general characteristic of an infinite-dimensional matrix in order to preserve a well-defined multiplication operation. In order to execute the single-row gauge carried out in section 5.4, we propose a slightly different interpretation of property *ii)*:

ii) Matrix entries m_{ij} are polynomials in λ, which satisfy the specific root (4.16)
structure $[\lambda - i]_{j-i}$ if $j > \lambda$ and $i \leq \lambda$. (4.17)

Here, $[a]_m$ is the descending Pochhammer symbol, defined as:

$$[a]_m := \prod_{l=0}^{m-1} (a - l) \ . \tag{4.18}$$

This simply means, that we demand the polynomials in λ to have specific roots when considering matrix entries above the main diagonal. Therefore, a generic matrix obeying these three properties should take the form:

[4] That this is indeed an embedding is proven in [22].

[5] Notice, that we do not restrict ourselves to finite linear combinations of basis elements e^i, i.e. matrices are not "bounded from above".

$$\begin{pmatrix} * & (\lambda-1)q_{12}(\lambda) & (\lambda-1)(\lambda-2)q_{13}(\lambda) & (\lambda-1)(\lambda-2)(\lambda-3)q_{14}(\lambda) & \dots \\ * & * & (\lambda-2)q_{23}(\lambda) & (\lambda-2)(\lambda-3)q_{24}(\lambda) & \dots \\ * & * & * & (\lambda-3)q_{34}(\lambda) & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & * & * & * & \dots \\ 0 & 0 & * & * & \dots \\ 0 & 0 & 0 & * & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.19)$$

where q_{ij} are polynomials in λ and i respectively, and the zeros start to appear in the $(K+1)$-th row. From this we can also understand why $\mathfrak{gl}_\lambda$ is called the Lie algebra of complex-size matrices in [22]: If λ is set to some positive integer n, then one "recovers" the Lie algebra of $n \times n$ - matrices $\mathfrak{gl}_n$ by taking the quotient $\mathfrak{gl}_\lambda/\tilde{V}_\lambda$, where $\tilde{V}_\lambda \subset V_\lambda$ is a subset of the submodule V_λ. This can be regarded as truncating the parts of the matrices for which $i \leq n$ and $j > n$ and therefore one recovers $\mathfrak{gl}_n$ up to an isomorphism.

Before considering the Hamiltonian reduction, the authors of [22] introduce some further extensions. In particular, notice that the above restrictions do not allow e.g. matrices that are zero everywhere except at a finite number of rows. However, in the procedure performed by B. Khesin and F. Malikov such matrices are necessary to construct *Frobenius matrices*, as we will see. Therefore, the authors of [22] artificially extend the algebra[6] by such elements. This is done in the following way: Denote by $\mathfrak{gl}$ the algebra, where elements must satisfy property *iii*), and properties *i*) and *ii*) need to be satisfied everywhere except at a finite number of rows. Additionally, call $\mathfrak{g}_\lambda$ the algebra which is obtained from $\mathfrak{gl}$ by, roughly speaking, collapsing[7] the parameter λ to some $\widetilde{\lambda} \in \mathbb{C}$.

4.2. Hamiltonian reduction and $\mathfrak{gl}_\lambda$

We now want to consider the generalized Drinfeld-Sokolov reduction introduced by B. Khesin and F. Malikov in [22] concerning $\mathfrak{g}_\lambda$. To do so, we first explain the general strategy briefly and then go into some more details afterwards, for proofs refer to the respective proof section [22, Chapter 4]. The starting point of the reduction is the loop algebra $\mathfrak{g}_\lambda(z)$ and the dual $\widehat{\mathfrak{g}}^*_\lambda$ of the corresponding affine Lie algebra. We are interested in the dual algebra since it naturally carries a Poisson structure, the Kirillov-Kostant structure (3.10). One can therefore perform a special Hamiltonian reduction (the (generalized) Drinfeld-Sokolov reduction) on $\widehat{\mathfrak{g}}^*_\lambda$, for which one requires some additional notation, analogous to the notation used in [22]:

[6]It would be interesting to see whether this extension also arises naturally. That is, whether one can find a suitable topology that allows these elements as a limit in some way.

[7]The precise notation introduced in [22] uses a so-called specialization map. That is, mapping $\mathbb{C}[\lambda]$ to $\mathbb{C}[\lambda]/(\lambda-\widetilde{\lambda})\mathbb{C}[\lambda]$ in the image of the map (4.12).

- $\mathfrak{u}_\lambda \subset \mathfrak{g}_\lambda$ is the subalgebra of strictly upper triangular matrices

- $\mathfrak{u}_\lambda^*$ is the dual algebra consisting of strictly lower triangular matrices

- $U_\lambda(z) = \mathfrak{u}_\lambda(z) \oplus \mathbb{1}$ is the loop group associated to the loop algebra $\mathfrak{u}_\lambda(z)$, obtained by exponentiation (refer to the Baker-Campbell-Hausdorff formula (2.2))

With this in mind, we can start the reduction and explain the procedure performed in [22]. First, the authors note that the group $U_\lambda(z)$ acts onto $\widehat{\mathfrak{g}}_\lambda^*$ via coadjoint action, which is Hamiltonian. Projecting $\widehat{\mathfrak{g}}_\lambda^*$ to its strictly lower triangular part (which is not centrally extended) is by construction $U_\lambda(z)$-equivariant and therefore also a momentum map:

$$p_\lambda : \widehat{\mathfrak{g}}_\lambda^* \longrightarrow \mathfrak{u}_\lambda(z)^* \ . \tag{4.20}$$

Upon fixation of an element Λ in $\mathfrak{u}_\lambda(z)^*$, the inverse image $p_\lambda^{-1}(\Lambda)$ of Λ is just a union of $U_\lambda(z)$-orbits (since $U_\lambda(z)$ is upper triangular while $\mathfrak{u}_\lambda(z)^*$ is strictly lower triangular). In particular, the quotient manifold $p_\lambda^{-1}(\Lambda)/U_\lambda(z)$ carries a Poisson structure as a result of the Hamiltonian reduction, which the authors of [22] show to coincide with the second Gelfand-Dickey bracket (3.22) if one chooses:

$$\Lambda := f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ . \tag{4.21}$$

Here, Λ is considered to be the generator f of $\mathfrak{sl}_2$ represented in $\mathfrak{gl}_\infty$, using the map (4.12). It is important to remark, that the authors of [22] actually identify the quotient manifold $p_\lambda^{-1}(\Lambda)/U_\lambda(z)$, which is a Poisson manifold, with the space S of sequences of functions (3.25), which carries a family of second Gelfand-Dickey structures. Therefore, the Poisson bracket obtained by the Hamiltonian reduction is identified with the second Gelfand-Dickey structure when treating λ as a parameter, making the latter into a polynomial in λ (due to equation (3.14)). These considerations are the reason why one views the above Hamiltonian reduction as a generalized version of the Drinfeld-Sokolov reduction. As already mentioned, one can find the usual Drinfeld-Sokolov reduction in e.g. [8] or [24].

It is helpful to further investigate this reduction, using the notations introduced in [22], in order to understand how one would adapt it when considering the higher-spin algebra $\mathfrak{hs}_\lambda$ instead of $\mathfrak{gl}_\lambda$. Therefore, we take a look into the proof section of this publication which constructs so-called Frobenius matrices for the case when λ is treated as a parameter, i.e. by replacing $\mathfrak{g}_\lambda$ with $\mathfrak{gl}$. For this, one requires the affine algebra $\widehat{\mathfrak{gl}}^*$, the subalgebra of strictly upper triangular matrices $\mathfrak{u} \subset \mathfrak{gl}$ and its dual $\mathfrak{u}^*$, the associated loop group $U(z) = \mathfrak{u}(z) \oplus \mathbb{1}$ as well as the group and algebra coadjoint actions with respect to this group. Without going excessively into detail[8], the authors of [22] show that it is enough to define the algebra coadjoint action of $\mathfrak{gl}(z)$ onto $\widehat{\mathfrak{gl}}^*_{q(\lambda)}$, defined as

$$\widehat{\mathfrak{gl}}^*_{q(\lambda)} = \Big\{ \big(q(\lambda)M, q(\lambda)\big), \text{ where } M \in \mathfrak{gl}(z),\ q(\lambda) \in \lambda^{-1}\mathbb{C}\big[[\lambda^{-1}]\big] \Big\} , \tag{4.22}$$

which one identifies with $\mathfrak{gl}(z)$. Here, $\mathbb{C}[[\lambda]]$ is the ring of formal power series in λ and we leave the arguments of $M(\lambda, z)$ away in the following, but emphasize that elements in $\mathfrak{gl}(z)$ are polynomials in λ and z. The algebra coadjoint action then takes a familiar form (refer to (2.25)):

$$\mathrm{ad}^*_X(M) = \big[X, M\big] - zX' , \tag{4.23}$$

where $X \in \mathfrak{gl}(z)$ and $M \in \widehat{\mathfrak{gl}}^*_{q(\lambda)}$. Observe that, without loss of generality, one sets $d = 1$ in the formula of the algebra coadjoint action (refer to the discussion at the beginning of subsection 2.2.2). Via exponentiation the authors of [22] also obtain the group coadjoint action of $U(z)$ onto $\widehat{\mathfrak{gl}}^*_{q(\lambda)}$:

$$\mathrm{Ad}^*_Y(M) = YMY^{-1} - zY'Y^{-1} , \tag{4.24}$$

where $Y \in U(z)$ and $M \in \widehat{\mathfrak{gl}}^*_{q(\lambda)}$. The Hamiltonian reduction is then performed in an analogous way as above, but now one deals with momentum maps of the subsequent kind:

$$p_{q(\lambda)} : \widehat{\mathfrak{gl}}^*_{q(\lambda)} \longrightarrow \mathfrak{u}^*_{q(\lambda)}(z) . \tag{4.25}$$

In order to relate the Poisson structure on the Poisson manifold, resulting from the Hamiltonian reduction, with the family of Poisson structures on the space S of sequences of functions (3.25), the authors of [22] prove that there exists exactly one element (the Frobenius matrix) of the form:

$$\begin{pmatrix} u_1(\lambda,z) & u_2(\lambda,z) & u_3(\lambda,z) & u_4(\lambda,z) & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{4.26}$$

in every $U(z)$-orbit in the inverse image $p^{-1}_{q(\lambda)}(\Lambda)$. Here, we choose the same Λ as defined in equation (4.21). Additionally, they show that the group coadjoint action of $U(z)$ leaves no points in the inverse image of the momentum map invariant. We now discuss, how these proofs might be adapted when one considers $\mathfrak{hs}_\lambda$ instead of $\mathfrak{gl}_\lambda$.

[8] For a detailed discussion on the polynomials $q(\lambda)$, refer to [22, Chapter 4]

4.2.1. The reduction and $\mathfrak{hs}_\lambda$

As will be discussed in section 5.2, the higher-spin algebra $\mathfrak{hs}_\lambda$ arises as a subalgebra of $\mathfrak{gl}_\lambda$ by solely considering the traceless part, i.e.

$$\mathfrak{gl}_\lambda = \mathfrak{hs}_\lambda \oplus \mathbb{C} \ . \tag{4.27}$$

At this point, one can introduce the trace on $\mathfrak{gl}$ in the way it is presented in [22]. Suppose $R \in \mathfrak{gl}$ and denote by Tr the sum over its diagonal entries up to some positive integer N:

$$\mathrm{Tr}(R)_N = \sum_{i=1}^{N} r_{ij} \ . \tag{4.28}$$

Notice, that this is a polynomial in N due to property $i)$ (4.13) of the definition of elements in $\mathfrak{gl}$ (allowing violations in a finite number of rows). Upon replacing N by the parameter λ, one then calls this the trace of the matrix R. With this in mind, we start to examine the proof for the existence and uniqueness of the Frobenius matrices, following [22]. First, consider the group adjoint action of an element Y of the Lie group $U(z)$ onto some $M \in \mathfrak{gl}(z)$ (since one identifies $\widehat{\mathfrak{gl}}^*_{q(\lambda)}$ with $\mathfrak{gl}(z)$) as given in equation (4.24). The group coadjoint action of Y transforms M to a Frobenius matrix F if the following equation is satisfied:

$$YM - zY' = FY \ . \tag{4.29}$$

The authors of [22] noticed that the diagonal elements of this equation correspond to the equations (where we omitted the dependencies on λ and z):

$$i = 1 : m_{11} + y_{12} = u_1 \tag{4.30}$$

$$i > 1 : m_{ii} + y_{ii+1} = y_{i-1i} \ . \tag{4.31}$$

Observe, that one already regards M as an element in the inverse of the momentum map (4.25), elements of which take the approximate form $\Lambda + \mathfrak{t}(z)$, where $\mathfrak{t}(z)$ contains upper triangular matrices. One can insert equation (4.30) into equation (4.31) for $i = 2$ and then insert the result into the equation for $i = 3$ and so on. In the end, one obtains the following expression:

$$y_{ii+1} = -\sum_{j=1}^{i} m_{jj} + u_1 \ . \tag{4.32}$$

As is observed in [22], if one fixes λ to some large enough[9] integer n, y_{nn+1} vanishes, hence allowing one to determine u_1:

[9]Since $\mathfrak{gl}$ is allowed to violate property $ii)$ (4.14) in a finite number of rows.

$$u_1(n,z) = \sum_{j=1}^{n} m_{jj}(n,z) \ . \tag{4.33}$$

But this corresponds to the trace of M upon replacing n by the parameter λ after evaluating the sum. Therefore, if one considers a traceless matrix M (with respect to the trace (4.28), one has that $u_1 = 0$. The rest of the proof for the existence and uniqueness of Frobenius matrices follows analogously as presented in [22]. Hence, this shows that upon considering $\mathfrak{hs}_\lambda$ instead of $\mathfrak{gl}_\lambda$ (and their respective extensions), each $U(z)$-orbit in the inverse image of the momentum map (4.25) contains exactly one Frobenius matrix of the form:

$$\begin{pmatrix} 0 & u_2(\lambda,z) & u_3(\lambda,z) & u_4(\lambda,z) & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{4.34}$$

In the remaining section of the proof outlined in [22], the authors introduce a filtration on the inverse image of the momentum map $p_\lambda^{-1}(\Lambda)$ (4.20) which offers an identification with the filtration on the space S of sequences of functions (as presented in section 3.3). In simple terms, this allows one to identify a Frobenius matrix (regarded as a representative of its respective group orbit) of the form

$$\begin{pmatrix} 0 & u_2 & u_3 & u_4 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{4.35}$$

with the following sequence of functions $\{u_k\}$

$$\big(0, u_2, u_3, u_4, \dots\big) \ . \tag{4.36}$$

In particular, we know that the space of such sequences carries a Poisson structure, to be precise a family of second Gelfand-Dickey structures. Hence, we have finally arrived at the desired result upon extracting the Poisson structure on the fields u_k out of the second Gelfand-Dickey structure (where λ is treated as a parameter which implies that the Poisson bracket on the fields will be of polynomial dependency in λ).

5. Higher-spin gauge theories

This chapter, specifically sections 5.1, 5.2 and 5.3, aims at providing a short introduction into the rich field of higher-spin gauge theories to the reader as well as presenting the results of the single-row gauge, applied for $\mathfrak{hs}_\lambda$ in section 5.4. If there is the need for further information, we will recommend the introductory notes by P. Kessel [21] as well as the lecture notes by S. Fredenhagen and A. Campoleoni [6], where the case of three space-time dimensions is treated. The latter will also be part of the likewise recommended book [26] which, at the time writing this thesis, has not yet been published. Additionally, for an overview over recent developments in the field refer to [3].

5.1. Introduction

In their essence, higher-spin gauge theories aspire to be gauge theories containing massless particles of any spin. The correct description of massless particles of arbitrary (integer) spin in a Minkowski space was first established by Fronsdal [12], incorporated by the following equation of motion (called the *Fronsdal equation*; we use the notations of [6] and [21]):

$$F_{\mu_1\dots\mu_s} = \Box\varphi_{\mu_1\dots\mu_s} - s\partial_{(\mu_1}\partial^\alpha\varphi_{\mu_2\dots\mu_s)\alpha} + \iota(s)\partial_{(\mu_1}\partial_{\mu_2}\varphi_{\mu_3\dots\mu_s)}{}^{\alpha}{}_{\alpha} = 0\ . \tag{5.1}$$

Here, $\iota(s) := \frac{s(s-1)}{2}$ and the tensor $\varphi_{\mu_1\dots\mu_s}$ is sometimes called *Fronsdal field* or *higher-spin field* (for $s > 2$). It is completely symmetric in its indices under interchange. Furthermore, the brackets sandwiching the indices are symmetrisation brackets, where we use the convention presented in [6]:

$$A_{(\mu_1}B_{\mu_2)} := \frac{1}{2}\left(A_{\mu_1}B_{\mu_2} + A_{\mu_2}B_{\mu_1}\right)\ . \tag{5.2}$$

If $s = 1$, Fronsdal's equation simplifies to Maxwell's equations and the Fronsdal field is the spin-1 gauge field A_μ, also known as the electromagnetic four-potential. On the other hand, if $s = 2$, one obtains (linearised) Einstein's equations of motion, where the spin-2 gauge field is given by the perturbation field $h_{\mu\nu}$ of the metric $\eta_{\mu\nu}$. Therefore, we view equation (5.1) as a natural generalization of these theories to particles carrying spin $s > 2$. It is well known that the spin-1 and spin-2 gauge fields admit gauge transformations, i.e. transformations leaving the respective equations of motion unchanged. Similarly, the Fronsdal field allows gauge transformations, which leave equation (5.1) invariant [6]:

$$\delta\varphi_{\mu_1\dots\mu_s} = s\partial_{(\mu_1}\zeta_{\mu_2\dots\mu_s)}\,,\ \text{where}\ \zeta_{\mu_1\dots\mu_{s-3}}{}^{\alpha}{}_{\alpha} = 0\ . \tag{5.3}$$

As is remarked in [6], even though it is possible to define Fronsdal's equation for any dimension D, there are actually no massless higher-spin particles in three space-time dimensions. This is since the little group only allows spin-0 and spin-$\frac{1}{2}$ representations in this case. However, it is still possible to consider higher-spin fields in three dimensions as toy models for higher dimensions.

For the upcoming calculations, it is necessary to extend the above framework to a three-dimensional anti-de Sitter background (AdS_3), where we use the notations presented in [6]. The corresponding Fronsdal equation $\overline{F}_{\mu_1 \dots \mu_s}$ is simply given by:

$$\overline{F}_{\mu_1 \dots \mu_s} := \left(\Box - m_s^2\right)\varphi_{\mu_1 \dots \mu_s} - s\overline{\nabla}^{\alpha}\overline{\nabla}_{(\mu_1}\varphi_{\mu_2 \dots \mu_s)\alpha} + \iota(s)\overline{\nabla}_{(\mu_1}\overline{\nabla}_{\mu_2}\varphi_{\mu_3 \dots \mu_s)}{}^{\alpha}{}_{\alpha} = 0 \ . \tag{5.4}$$

Here, $\overline{\nabla}$ is the covariant derivative and m_s^2 is defined using the AdS_3 radius l:

$$m_s^2 = \frac{2s(s-1)}{l^2} \ . \tag{5.5}$$

Again, this equation of motion is gauge-invariant under gauge transformations (5.3), using the covariant derivative $\overline{\nabla}$ instead of ∂. Requiring that the trace of the gauge parameter vanishes then simply means contracting it with the AdS_3 metric $\overline{g}_{\alpha\beta}$ [6].

In order to realise the above as a Chern-Simons theory, coupling gravity to higher-spin fields, there is the need for further notation. As is observed in [6], one can actually bring Fronsdal's equation (5.4) in the form:

$$T^{a_1 \dots a_{s-1}} = 0, \ R^{a_1 \dots a_{s-1}} = 0 \ . \tag{5.6}$$

Here,

$$T^{a_1 \dots a_{s-1}} = \nabla e^{a_1 \dots a_{s-1}} + (s-1)\varepsilon^{ij(a_1}\overline{e}_i \wedge \omega^{a_2 \dots a_{s-1})}{}_j \tag{5.7}$$

$$R^{a_1 \dots a_{s-1}} = \nabla \omega^{a_1 \dots a_{s-1}} + \frac{s-1}{l^2}\varepsilon^{ij(a_1}\overline{e}_i \wedge e^{a_2 \dots a_{s-1})}{}_j \ , \tag{5.8}$$

using “flat“ indices a_i, the corresponding ∇ and epsilon tensor ε^{ijk} as well as $\overline{e}$ for the AdS background. Furthermore, T and R are called curvatures and they arise as equations of motion of a corresponding action S, the definition of which can be found in [6, Chapter 2.2]. Both T and R are represented via the vielbeins $e^{a_1 \dots a_{s-1}}$ and connections $\omega^{a_1 \dots a_{s-1}}$, the definitions of which are given in [6] as:

$$e^{a_1 \dots a_{s-1}} = e_\nu{}^{a_1 \dots a_{s-1}} dx^\nu, \ \omega^{a_1 \dots a_{s-1}} = \omega_\nu{}^{a_1 \dots a_{s-1}} dx^\nu \ . \tag{5.9}$$

5.2. Chern-Simons theory

As is well known, gravity in three space-time dimensions (on an AdS_3-background) can be beautifully expressed as a Chern-Simons theory with the Einstein Hilbert action S_{EH}

consisting of two copies of the Chern-Simons action S_{CS} (5.11), for one of the original papers we recommend the excellent publication [27] by E. Witten. Now, a Chern-Simons theory is naturally a gauge theory and since Fronsdal's equation (5.1), upon setting $s = 2$, gives rise to (linearised) Einstein's equations of motion one might wonder if there is also a Chern-Simons version of Fronsdal's equation in an AdS$_3$-background. In this section, we want to briefly discuss this. For a detailed discussion we refer to [6] and simply restrict ourselves to the results. In particular, the higher-spin algebra $\mathfrak{hs}_\lambda$ actually arises as a gauge algebra from these considerations, when we are interested in an unbounded set of spin s. As presented in [6], we start by collecting the vielbeins and spin connections in:

$$A^{a_1\dots a_{s-1}} = \Big(\omega^{a_1\dots a_{s-1}} + \frac{1}{l}e^{a_1\dots a_{s-1}}\Big),\ \widetilde{A}^{a_1\dots a_{s-1}} = A^{a_1\dots a_{s-1}} - \frac{2}{l}e^{a_1\dots a_{s-1}}\ . \tag{5.10}$$

For the readers convenience, we quickly recall the three-dimensional Einstein-Hilbert action in the Chern-Simons formulation [6]:

$$S_{EH} = \underbrace{\frac{k}{4\pi}\int \mathrm{Tr}\Big(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\Big)}_{=:S_{CS}(A)} - \underbrace{\frac{k}{4\pi}\int \mathrm{Tr}\Big(\widetilde{A}\wedge \mathrm{d}\widetilde{A} + \frac{2}{3}\widetilde{A}\wedge \widetilde{A}\wedge \widetilde{A}\Big)}_{=:S_{CS}(\widetilde{A})}\ , \tag{5.11}$$

where $k = \frac{l}{4G}$ and G is the gravitational constant. For the following we define $h := \frac{k}{4\pi}$. The symbols A and $\widetilde{A}$ are actually $\mathfrak{sl}_2(\mathbb{R})$-algebra valued elements, defined via the dreibeins e^a and spin connections $\omega^a = \frac{1}{2}\varepsilon^{aij}\omega_{ij}$:

$$A = A^a J_a = \Big(\omega^a + \frac{1}{l}e^a\Big)J_a,\ \widetilde{A} = \Big(A^a - \frac{2}{l}e^a\Big)J_a\ . \tag{5.12}$$

Here, the J_a are generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, which satisfy $\mathrm{Tr}(J_aJ_b) = \eta_{ab}$ for some bilinear form Tr. Similarly to this action, we now also want to have $\mathfrak{g}$-valued fields A and $\widetilde{A}$ for our Chern-Simons formulation of a higher-spin action. In particular, these fields should be a contraction of the above combinations of the vielbeins and spin connections (5.10) with the generators of a Lie algebra $\mathfrak{g}$. As it turns out (refer to [6, Chapter 2.2]), taking two copies $\mathfrak{hs}_\lambda \oplus \mathfrak{hs}_\lambda$ of the higher-spin algebra $\mathfrak{hs}_\lambda$ is exactly what one needs for a Chern-Simons version of a higher-spin gauge theory that couples gravity to fields of (unbounded) spin $s > 2$ in an AdS$_3$-background. In particular, when linearising the resulting action, one obtains a generalization of the Einstein-Hilbert action displayed in equation (5.11).

We will now present a basis for the higher-spin algebra and comment on its connection to the Verma module (4.5), that we defined in section 4.1, following the notations used in [6] and [14]. Denote as $L_{\pm 1}$ and L_0 the $\mathfrak{sl}_2$-generators, satisfying the following commutation relations:

$$\big[L_{+1}, L_{-1}\big] = 2L_0,\ \big[L_{\pm 1}, L_0\big] = \pm L_{\pm 1}\ . \tag{5.13}$$

Furthermore, we introduce the symbols [6]:

$$V^s_{s-1} = \left(L_{+1}\right)^{s-1},\ V^s_{m-1} = -\frac{1}{m+s-1}\left[L_{-1}, V^s_m\right], \tag{5.14}$$

where $V^s_{1-s} = \left(L_{-1}\right)^{s-1}$. Observe, that this corresponds to $2s-1$ generators and one sees from the definition of V^s_{m-1}, that it is a nested commutator. Regarding our $\mathfrak{sl}_2$ elements e, f and h, which we defined in chapter 4, this means that a basis of the higher-spin algebra has generators proportional to:

$$V^s_{s-n} \propto \underbrace{\left[f, \left[f, \left[f, ...\left[f\right.\right.\right.\right.}_{n-1-terms}, e^{s-1}]]]...] \ . \tag{5.15}$$

Therefore, we indeed see that these generators are traceless with respect to the trace (4.28) and the Verma module (4.5), as the elements contributing to the main diagonal will simply consist of commutators of an equal number of $e's$ and $f's$.

5.3. Asymptotic symmetries of higher-spin gauge theories

In this section we finally want to discuss what we actually mean when we speak of asymptotic symmetries, asymptotically anti-de Sitter space-times and the Drinfeld-Sokolov constraint in higher-spin gauge theories. We will follow once again the outline presented in [6] and start by considering asymptotically anti-de Sitter space-times. First, however, we will quickly remind the reader about how one deals with constrained Hamiltonian systems. This is excessively covered in e.g. [19], for a briefer review we refer to [6, Appendix C].

For a system with generalized variables $(q^1, ..., q^n, p_1, ..., p_n)$ the corresponding Hamiltonian takes the form [19]:

$$\mathcal{H} = \sum_{i=1}^{n} \dot{q}^i p_i - \mathcal{L}(q_1, ..., q_n, p_1, ..., p_n) \ , \tag{5.16}$$

where L is the Lagrangian of the system. Implementing m constraints φ_i into the Hamiltonian can be easily achieved by extending the Hamiltonian, i.e. by writing:

$$\mathcal{H}' = \mathcal{H} + \sum_{i=1}^{m} u_i \varphi_i \ . \tag{5.17}$$

The time evolution of the j-th constraint φ_j in the new Hamiltonian is then given by the Poisson bracket,

$$\dot{\varphi}_j = \left\{\varphi_j, \mathcal{H}'\right\} \ . \tag{5.18}$$

We will denote the vanishing of the constraints φ_i on the constraint surface by $\varphi_i \approx 0$. In particular, this should also be true when considering the evolution of a constraint in time regarding $\mathcal{H}'$, and therefore [6]:

$$\dot{\varphi}_j \approx \{\varphi_j, \mathcal{H}\} + \sum_{i=1}^{m} u_i \{\varphi_j, \varphi_i\} \approx 0 \; . \tag{5.19}$$

Furthermore, call Φ a first-class constraint if $\{\Phi, \varphi_i\} \approx 0$ for every i, and a second-class constraint if this is not the case. Additionally, denote by C_{ji} the Poisson bracket of the constraints $\{\varphi_j, \varphi_i\}$ and notice, that the rank of this matrix is maximal exactly if all the constraints are of second-class, which therefore completely determines the u_i. In this case, one is able to equip the space with a specific Poisson bracket, the Dirac bracket [6]:

$$\{f, g\}_D = \{f, g\} - \sum_{i,j} \{f, \varphi_j\} \big(C^{-1}\big)_{ji} \{\varphi_i, g\} \; , \tag{5.20}$$

and considers the system solely on the constraint surface.

With this in hand, we are finally in a position to investigate asymptotic symmetries. Analogously to the presentation in [6] we start by considering a Chern-Simons theory on a cylinder Z, which is informally understood as "setting a boundary at infinity". A very good introduction to the general procedure, called *conformal completion*, can be found e.g. in the book by Penrose [25]. Although it is helpful to discuss a Chern-Simons theory on such a cylinder, one has to be careful when considering asymptotic anti-de Sitter spaces as remarked by the authors of [6]. Keeping this in mind, one can introduce a Chern-Simons action (with some (gauge) group G that we leave undetermined for now) on the cylinder by taking the coordinates $\theta \in [0, 2\pi)$, $\rho \in [0, \infty)$ as well as the time parameter t. Hence, one can express a general Chern-Simons action (refer to equation (5.11) via the fields A_θ, A_ρ and A_t, splitting the action into a boundary and a bulk part [6]:

$$\begin{aligned} S_{CS}(A_\theta, A_\rho, A_t) = h \int_Z & \mathrm{d}t \mathrm{d}\rho \mathrm{d}\theta \; \mathrm{Tr}\big(A_\theta \dot{A}_\rho - A_\rho \dot{A}_\theta + 2 A_t F_{\rho\theta}\big) \\ & - h \int_{\partial Z} \mathrm{d}t \mathrm{d}\theta \; \mathrm{Tr}(A_t A_\theta) \; . \end{aligned} \tag{5.21}$$

Here, A_t is regarded as a Lagrange multiplier that fixes $F_{\rho\theta} = \partial_\rho A_\theta - \partial_\theta A_\rho + \big[A_\rho, A_\theta\big]$ to zero. This is due to the arising of the term $F_{\rho\theta} \delta A_t$ in the variation of the above action. By adapting the boundary integral in (5.21) to:

$$-2h \int_{\partial Z} \mathrm{d}t \mathrm{d}\theta \; \mathrm{Tr}(A_t A_\theta) \; , \tag{5.22}$$

the authors of [6] nullify the boundary contribution of A_θ in the variation of the action. We are now interested in gauge transformations of the fields A and remember, that those are generated by first-class constraints. As it turns out, $F_{\rho\theta}$ is a first-class constraint with respect to the Poisson bracket [6]:

$$\big\{F_1, F_2\big\} = \frac{1}{2h} \int_{D_2} \mathrm{d}\rho \mathrm{d}\theta \; \mathrm{Tr}\left(\frac{\delta F_1}{\delta A_\rho} \frac{\delta F_2}{\delta A_\theta} - \frac{\delta F_1}{\delta A_\theta} \frac{\delta F_2}{\delta A_\rho} \right) \; . \tag{5.23}$$

Here, we consider the bracket for a fixed time parameter t (i.e. by having discs D_2 of equal time) and F_1, F_2 are functionals on both A_ρ and A_θ respectively. One can then introduce the generating objects of gauge transformations of type $\delta A = \mathrm{d}\Gamma + [A, \Gamma]$, denoted by $G(\Gamma)$ [6]:

$$G(\Gamma) = 2h \int_{D_2} \mathrm{d}\rho \mathrm{d}\theta \ \mathrm{Tr}\big(\Gamma F_{\rho\theta}\big) + Q(\Gamma) \ . \tag{5.24}$$

According to [6], the reason for the introduction of the symbol $Q(\Gamma)$ stems from the fact, that one has to make sure that the variation of $G(\Gamma)$ is without terms contributing to the boundary S_1 of the disc. Hence, one has to appropriately choose $Q(\Gamma)$ in order to cancel any of those terms, that would arise from the variation of the first summand. On the other hand, the first summand in the variation of equation (5.24) vanishes upon enforcing $F_{\theta\rho} = 0$, even though a non-trivial $Q(\Gamma)$ does not, which leads to boundary contributions in the resulting variation. In this case (for $Q(\Gamma) \neq 0$) one does not speak of $G(\Gamma)$ as gauge transformations, but rather considers them as global symmetries of the system, which one also calls *asymptotic symmetries* in this context [6, 7].

As explained in [6], one can now execute a gauge-fixing by setting:

$$A_\rho := f^{-1}(\rho)\partial_\rho f(\rho) \ . \tag{5.25}$$

Here, f is a function evaluated in the group G. It shall be noted, that there are still symmetries left after the fixation, which one regards as asymptotic symmetries. They take the form of gauge transformations $\delta A = \partial A + [A, \Gamma]$, which arise as the equations of motion from the variation of the generators $G(\Gamma)$. The above then also restricts the form of A_θ to:

$$A_\theta = f^{-1}(\rho) a_\theta(t, \theta) f(\rho) \ , \tag{5.26}$$

upon collapsing $F_{\rho\theta} = 0$ [6]. Furthermore, this means that the gauge parameter Γ and A_t should look like:

$$\Gamma = f^{-1}(\rho)\gamma(t, \theta) f(\rho), \ A_\theta = f^{-1}(\rho) a_t(t, \theta) f(\rho) \tag{5.27}$$

in order to satisfy the gauge transformations of A_ρ. Shortly, this gauge-fixing will also occur when we discuss asymptotically anti-de Sitter spaces. As examined in [6], the above can then be used to find the Dirac bracket on the constraint space, which naturally depends on $a_\theta(t, \theta)$ (which is unconstrained on the boundary). In particular, the Dirac bracket takes the form of a central extension of the loop Lie algebra $\mathfrak{g}$ and therefore, one indeed observes the emergence of affine Lie algebras as asymptotic symmetry algebras, as discussed in [6].[1]

[1] It shall be remarked, that in the case of the Lie algebra of complex-size matrices $\mathfrak{gl}_\lambda$, the dual of the affine Lie algebra is approximately identified with the affine Lie algebra itself when viewed as the image under the map (4.12) and taking the extension to $\mathfrak{gl}$. For a detailed discussion refer to [22].

We now want to finally investigate asymptotic symmetries of higher-spin gauge theories in asymptotically anti-de Sitter space-times in three dimensions. As the authors of [6] remarked, referring to [4], the above is not a suitable description for asymptotically anti-de Sitter space-times. Nevertheless, the same form of A_ρ and A_θ still arises and one requires that they behave like the AdS$_3$-connections A^{AdS} and $\widetilde{A}^{\text{AdS}}$ at infinity. Therefore, consider first the latter connections (which are regarded as taking elements in $\mathfrak{sl}_2$), where we use the convention presented in [6]:

$$A^{\text{AdS}} = L_0 \mathrm{d}\rho + \left(\frac{\mathrm{d}t}{l} + \mathrm{d}\theta\right)\left(e^{\rho} L_{+1} + \frac{1}{4} e^{-\rho} L_{-1}\right) . \tag{5.28}$$

One obtains $\widetilde{A}^{\text{AdS}}$ by simply replacing $L_{+1} \leftrightarrow L_{-1}$ and inserting two minus signs. One appears in front of the whole term, the other in front of $\mathrm{d}\theta$. With this in mind, we consider A_θ and A_ρ in exactly the same way as before (refer to equations (5.25) and (5.26)) as gauge fixation and make the connection to A^{AdS} by setting $f(\rho) = e^{\rho L_0}$. According to [6], the corresponding $\widetilde{A}_\rho$ and $\widetilde{A}_\theta$ then take the following form:

$$\widetilde{A}_\rho = f^2(\rho) A_\rho f^{-2}(\rho),\ \widetilde{A}_\theta = f(\theta) \widetilde{a}_\theta(t,\theta) f^{-1}(\rho) . \tag{5.29}$$

As noted in [6], the reason why we call the resulting space asymptotically anti-de Sitter arises from the argument, that in the limit $\rho \to \infty$ (i.e. considering ρ asymptotically near the boundary) we want to recover the connections A^{AdS} and $\widetilde{A}^{\text{AdS}}$. The same exact argument applies when we infer the higher-spin connections, presented in equation (5.10). In particular, this condition restricts the connection $a_\theta(t,\theta)$ to [6]:

$$a_\theta(t,\theta) = L_{+1} + \sum_{n=0}^{1} \tau^n(t,\theta) L_{-n} + \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \kappa_{n+1}^m(t,\theta) V_m^{n+1} \quad , \tag{5.30}$$

when considering the higher-spin algebra $\mathfrak{hs}_\lambda$. The other copy $\widetilde{a}_\theta(t,\theta)$ of the connection can be easily obtained by letting $n \to -n$. One calls this condition the Drinfeld-Sokolov constraint [6]. Upon choosing an appropriate representation (e.g. the Verma module we considered in chapter (4), one observes that this restricts the matrix form of the terms appearing in the two sums to either upper or lower triangular matrices, depending on the representation. In particular, suppose we pick a representation such that L_1 is identified as the image of the $\mathfrak{sl}_2$-generator f (refer to equation (4.21). In this case, the two sums are represented by uppertriangular matrices and, specifically, we can choose a matrix that only has entries in the first row (a discussion on this for the spin-3 case can be found in [6]). Then one can consider the most general infinitesimal gauge transformation [6]:

$$\delta_\gamma a = \partial_\theta a + [a, \gamma] , \tag{5.31}$$

such that the resulting matrix is zero everywhere except for the first row. This is what we call a single-row gauge and it will be performed in the upcoming section for the Lie algebra $\mathfrak{hs}_\lambda$. In particular, we will see that the result of this reduction will be a

$W_{\infty(+1)}(\lambda)$ algebra, which is then understood as an asymptotic symmetry algebra. The whole procedure, as we already know, is called the generalized Drinfeld-Sokolov reduction [22].

5.4. Single-row gauge for $\mathfrak{hs}_\lambda$

In order to build a bridge between the mathematical formalism of the generalized Drinfeld-Sokolov reduction in chapter 4 and the infinitesimal gauge transformation (5.31), we identify them with the algebra coadjoint action defined in equation (4.23). In order to fit this into the notations used by the authors of [6], one chooses $d = i$ for the hyperplane of the dual of the affine Lie algebra used in the reduction (equaling an overall rescaling), and by applying the identification $z = e^{-i\theta}$ one obtains that:

$$\mathrm{ad}^*_{X(\theta)}\big(M(\theta)\big) = \big[X(\theta), M(\theta)\big] + X'(\theta) \ . \tag{5.32}$$

Here $X'(\theta) = \frac{d}{d\theta}X(\theta)$ and $X(\theta) \in \mathfrak{gl}(\theta)$ whereas $M(\theta)$ is also viewed as an element in $\mathfrak{gl}(\theta)$. For the sake of brevity, we leave the arguments away in the following. Furthermore, remember that we are dealing with the Lie algebra $\mathfrak{gl}_\lambda$, whose elements we view as the images under the map η (4.12) (modulo the Casimir). In particular, we consider the matrices to be elements in $\mathfrak{gl}$, allowing for violations of properties *i)* (4.13) and *ii)* (4.16), where we use our adapted version, in a finite number of rows. Now, we already know that there exists exactly one Frobenius matrix in every $U(z)$-orbit in the inverse image of the momentum map due to the proposition proven in [22]. Hence, it suffices to choose M to be a Frobenius matrix F of the form:

$$F = \begin{pmatrix} 0 & u_2 & u_3 & u_4 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{5.33}$$

and determine the algebra coadjoint action on it. This is precisely the shape of a matrix satisfying the Drinfeld-Sokolov constraint. In matrix entries, the action is nothing but:

$$\mathrm{ad}^*_X(F)_{ij} = x'_{ij} + u_j x_{i1} + x_{ij+1} - x_{i-1j} - \delta_{i1}\sum_{k=1}^{\infty} u_k x_{kj} \ , \tag{5.34}$$

where $u_1 = 0$. We emphasize once again, that the matrix entries are polynomials in λ and θ. With the above equation in mind, we are now interested in the general form of the matrix entries x_{ij} such that F is transformed to a matrix consisting only of elements in the first row (as already mentioned, hence the name single-row gauge). In order to do so, we can follow the approach introduced in [6] making necessary adaptations along the way. To begin, note that for $i > 1$ equation (5.34) has to vanish, i.e.:

$$x'_{ij} + u_j x_{i1} + x_{ij+1} - x_{i-1j} \stackrel{!}{=} 0 \ . \tag{5.35}$$

Following [6], we introduce the subsequent pseudodifferential operators x_i, x^1, adapted appropriately:

$$x_i = \sum_{j=1}^{\infty} x_{ij} \partial^{\lambda - j}, \ x^1 = \sum_{i=1}^{\infty} \partial^{-\lambda + i - 1} x_{i1} \ . \tag{5.36}$$

At first sight, the second operator x^1 looks problematic seemingly allowing differential operators of infinite order. However, remember property *iii*) (4.15) of matrices in $\mathfrak{gl}$, which states that matrices are "bounded from below", i.e. $x_{i1} = 0$ for all $i > K$. Additionally to the operators x_i and x^1, we introduce the pseudodifferential operator L[2]:

$$L = \partial^{\lambda} + \sum_{k=1}^{\infty} \widetilde{u}_k \partial^{\lambda - k} = \partial^{\lambda} - \sum_{k=1}^{\infty} u_k \partial^{\lambda - k} \ , \tag{5.37}$$

where we made the identification $\widetilde{u}_k := -u_k$ for all k. Keeping these definitions in mind, one can sum equation (5.35) over j from 1 to ∞ after multiplying $\partial^{\lambda - j}$ from the right, hence obtaining [6]:

$$x'_i - x_{i-1} - \sum_{j=1}^{\infty} \left(x_{ij} \partial^{\lambda - j + 1} - x_{ij+1} \partial^{\lambda - j} \right) + x_{i1} \sum_{j=1}^{\infty} u_j \partial^{\lambda - j} \ . \tag{5.38}$$

Here, Leibniz's rule (3.14) was used for the first term. As one can easily see, the sum in the middle simplifies to $x_{i1}\partial^{\lambda}$. Therefore, one obtains the recursive relation below for pseudodifferential operators x_i:

$$x'_i - x_{i-1} - x_{i1} L = 0 \ . \tag{5.39}$$

Solving the recursion

In order to solve the above recursive relation, we make the following Ansatz:

$$x_i = \partial^{-i} F(i, \lambda) - \left(\partial^{\lambda - i} x^1 \right)_+ L \ , \tag{5.40}$$

where $F(i, \lambda)$ signifies a (pseudo)differential symbol with possible dependence on i and λ. To show that this is indeed a viable Ansatz, one has to demonstrate that it satisfies the recursive relation (5.39),

$$\partial^{-i+1} \big(F(i, \lambda) - F(i-1, \lambda) \big) - \partial \left(\partial^{\lambda - i} x^1 \right)_+ L + \left(\partial^{\lambda - i + 1} x^1 \right)_+ L - x_{i1} L \stackrel{!}{=} 0 \ . \tag{5.41}$$

Note that the last three terms vanish due to the following simplification

[2] Notice that the minus arises due to the usage of f as the fixed element. The authors of [6] use $-f$, in which case the sum in the operator L obtains a minus sign again.

$$\partial\big(\partial^{\lambda-i}x^1\big)_+L = \partial\left(\sum_{l=1}^{\infty}\partial^{l-i-1}x_{l1}\right)_+ L = \big(\partial^{\lambda-i+1}x^1\big)_+L - x_{i1}L \ . \tag{5.42}$$

Therefore, what remains in equation (5.41) is a condition on F, that is:

$$F(i,\lambda) \stackrel{!}{=} F(i-1,\lambda) \ . \tag{5.43}$$

This condition implies, that F should be independent of the row index i. However, this information alone is not enough to construct solutions of the recursive relation as there are further restrictions. In particular, the structure of the operator x_i (5.36) on the left-hand side only allows column indices j of highest order -1. Additionally, since the coefficients x_{ij} are matrix entries of a matrix in $\mathfrak{gl}(\theta)$, they need to satisfy the associated properties (refer to equations (4.13) - (4.15) and (4.16)). Keeping this in mind, one can then begin the construction of F.

To begin, observe that the appearance of λ in the exponent of ∂ in x_i restricts F to the form:

$$F = \partial^\lambda G \ . \tag{5.44}$$

Here, G is a general pseudodifferential operator that only possesses integer exponents in ∂, that is

$$G = \sum_{l=-\infty}^{M} g_l\partial^l \ , \tag{5.45}$$

where $M \in \mathbb{N}$ and the g_l are coefficients, which we wish to determine. To do so, we present the following two claims concerning the form of the operator G such that it satisfies the proper structure, as stated above.

$$i)\ G_- = 0 \tag{5.46}$$

$$ii)\ G_+ = \big(x^1L\big)_+ \tag{5.47}$$

The goal is now to prove both of these claims.

Proof. Regarding claim $i)$, first notice that one can rewrite $\partial^{\lambda-i}G_-$, using Leibniz's rule (refer to equation (3.14)), as follows:

$$\partial^{\lambda-i}G_- = \sum_{m=0}^{\infty}\sum_{l=1}^{\infty}\binom{\lambda-i}{m}g_{-l}^{(m)}\partial^{\lambda-i-l-m} \ . \tag{5.48}$$

Denote by $\widetilde{j} := i+l+m$ the index corresponding to the column index j of x_{ij} on the left-hand side of the Ansatz (5.40). Observe that $\widetilde{j} > i$, which satisfies the requirement implied by $j > 0$. Furthermore, this indicates that $\partial^{\lambda-i}G_-$ only contributes to coefficients

x_{ij} above the main diagonal of the matrix X. Therefore, $\partial^{\lambda-i}G_-$ should possess the correct root structure in λ for every l, m everywhere except at a finite number of rows i. As can be seen in equation (5.48), the binomial coefficient actually carries the root structure in λ, but since the sum over l is unbounded the whole term does not possess the correct structure required. This means, that for a fixed $\lambda \geq i$ one always obtains coefficients g_{-l} appearing for $\widetilde{j} > \lambda$, which must not be the case. Note, that this is true for every row because the coefficients are independent of i by construction. Therefore, either the g_{-l} satisfy the correct root structure in λ or $G_- = 0$. The former can never be the case due to the g_{-l} being independent of i, implying $G_- = 0$.

From the above, one concludes that $G = G_+$. In order to determine $G_+ = \left(x^1 L\right)_+$ (and therefore show claim *ii*)) one first has to investigate the other summand appearing in the Ansatz (5.40). The following simplification using Leibniz's rule can be found in Appendix A.1.

$$\left(\partial^{\lambda-i}x^1\right)_+ L = \sum_{\substack{l=1;n,m=0\\ l>n+m+i}}^{\infty} B_1(l,n)B_2(m)\left(x_{l1}^{(n+m)} + \sum_{k=1}^{\infty}\left(x_{l1}\widetilde{u}_k\right)^{(n+m)}\partial^{-k}\right)\partial^{\lambda-\widetilde{j}_1} \tag{5.49}$$

Here, B_1, B_2 are binomial coefficients (refer to equation (A.5)) and the index $\widetilde{j}_1$ is defined as:

$$\widetilde{j}_1 := i + 1 + n + m - l\ . \tag{5.50}$$

Additionally, we set $\widetilde{j}_2 := \widetilde{j}_1 + k$. Due to the condition $l > n + m + i$, the index $\widetilde{j}_1$ is restricted to integers in the set $\{0, -1, -2, ...\}$ which means that the corresponding contributions need to vanish (similarly for $\widetilde{j}_2$) as the column index j is required to be positive and non-zero. Hence, G must be of the general form

$$G = \sum_{l=1;n=0}^{\infty} B_1(l,n)\left(x_{l1}^{(n)} + \sum_{k=1}^{\infty}\left(x_{l1}\widetilde{u}_k\right)^{(n)}\partial^{-k}\right)\partial^{l-1-n} + H_+\ , \tag{5.51}$$

where H_+ represents a generic, purely differential operator that is determined below. From claim *i*) we already know that it suffices to only consider the G_+ part of G, reducing equation (5.51) to:

$$G = \sum_{\substack{l=1;n=0\\ l>n}}^{\infty} B_1(l,n)x_{l1}^{(n)}\partial^{l-1-n} + \sum_{\substack{l,k=1;n=0\\ l>k+n}}^{\infty} B_1(l,n)\left(x_{l1}\widetilde{u}_k\right)^{(n)}\partial^{l-1-n-k} + H_+\ . \tag{5.52}$$

But the first two summands are nothing but $\left(x^1 L\right)_+$ (refer to equation (A.8)), which implies the following form for G:

$$G = \left(x^1 L\right)_+ + H_+\ . \tag{5.53}$$

Inserting this into the Ansatz (5.40), one obtains the subsequent operator,

$$x_i = \partial^{\lambda-i}\big(x^1 L\big)_+ - \big(\partial^{\lambda-i}x^1\big)_+ L + \partial^{\lambda-i}H_+ \ . \tag{5.54}$$

Note, that one can easily rewrite this similarly to how it was done in achieving equation (3.23):

$$x_i = \big(\partial^{\lambda-i}x^1\big)_- L - \partial^{\lambda-i}\big(x^1 L\big)_- + \partial^{\lambda-i}H_+ \ . \tag{5.55}$$

In order to determine H_+, first note that the other terms in the above equation already satisfy the same structure as the x_i on the left-hand side (as examined in Appendix A.2). Therefore, H_+ either does not violate these properties or vanishes, and we generally write:

$$H_+ = \sum_{l=0}^{T} h_l \partial^l \ , \tag{5.56}$$

where $T \in \mathbb{N}$ and the h_l are the coefficients to be determined. To do so, consider the first row x_1,

$$x_1 = \big(\partial^{\lambda-1}x^1\big)_- L - \partial^{\lambda-1}\big(x^1 L\big)_- + \partial^{\lambda-1}\sum_{l=0}^{T} h_l \partial^l \ . \tag{5.57}$$

The highest order term in the last summand is proportional to $\partial^{\lambda+T-1}$. This implies that $T \overset{!}{=} 0$, otherwise there would be contributions to elements of column index $j \leq 0$. Hence, only the h_0 coefficient is allowed to contribute and it needs to vanish. This can be seen from the following fact: Since x_{11} is the sole coefficient associated to $\partial^{\lambda-1}$ on the left-hand side of equation (5.57), the same must be true on the right-hand side. But h_0 also contributes to x_{11} on the right-hand side of equation (5.57), i.e. one has:

$$\big(x_{11} + h_0\big)\partial^{\lambda-1} + \dots \ . \tag{5.58}$$

This implies $h_0 = 0$ and, moreover, $H_+ = 0$. Keeping all of this in mind, the Ansatz (5.40) indeed simplifies to the following solution of the recursion (5.39):

$$x_i = \partial^{\lambda-i}\big(x^1 L\big)_+ - \big(\partial^{\lambda-i}x^1\big)_+ L \tag{5.59}$$

With this solution, both claims (refer to equations (5.46), (5.47)) have been shown. □

Obtaining the second Gelfand-Dickey bracket

Keeping the solution obtained above in mind, we can continue the procedure outlined in [6, Chapter 3.5]. We want to insert this solution into equation (5.34) for $i = 1$, that is for the first row. In this case, equation (5.34) simplifies to:

$$\mathrm{ad}^*_X(F)_{1j} = x'_{1j} + u_j x_{11} + x_{1j+1} - \sum_{k=1}^{\infty} u_k x_{kj} \ . \tag{5.60}$$

Similarly to how the recursion relation (refer to equation (5.39)) was obtained, we sum over j and multiply $\partial^{\lambda-j}$ from the right and obtain on the right-hand side of equation (5.60):

$$x_1' - \sum_{j=1}^{\infty}\left(x_{1j}\partial^{\lambda-j+1} - x_{1j+1}\partial^{\lambda-j}\right) + \sum_{j=1}^{\infty} u_j x_{11}\partial^{\lambda-j} - \sum_{k=1}^{\infty} u_k x_k \ . \tag{5.61}$$

Here, the Leibniz rule was used for the first summand in equation (5.60). The terms in the parentheses simply give $x_{11}\partial^{\lambda}$ and one therefore gains, using the definition of L (5.37),

$$\sum_{j=1}^{\infty} \mathrm{ad}^*_X(F)_{1j}\partial^{\lambda-j} = x_1' - x_{11}L - \sum_{k=1}^{\infty} u_k x_k \ . \tag{5.62}$$

This equation is now essential, as we can insert the solution acquired for x_i (refer to equation 5.59) into it, using similar simplifications as used in [6]:

$$\partial^{\lambda}\big(x^1 L\big)_+ - \partial\big(\partial^{\lambda-1}x^1\big)_+ L - x_{11}L - \sum_{k=1}^{\infty} u_k\Big(\partial^{\lambda-k}\big(x^1 L\big)_+ - \big(\partial^{\lambda-k}x^1\big)_+ L\Big) \tag{5.63}$$

$$\Leftrightarrow\ L\big(x^1 L\big)_+ - \big(\partial^{\lambda}x^1\big)_+ L + \sum_{k=1}^{\infty} u_k\big(\partial^{\lambda-k}x^1\big)_+ L \tag{5.64}$$

$$\Leftrightarrow\ L\big(x^1 L\big)_+ - \big(Lx^1\big)_+ L \ . \tag{5.65}$$

One immediately recognises the result as the second Gelfand-Dickey bracket, whose expression was given in definition 3.4, and we denote it the same way as the authors in [6]:

$$\delta_X L := L\big(x^1 L\big)_+ - \big(Lx^1\big)_+ L \ . \tag{5.66}$$

As was mentioned before, the resulting algebra on the fields u_k obtained from the second Gelfand-Dickey bracket is usually called a $W_{\infty+1}(\lambda)$-algebra in physics literature. Note, that this result still holds for the whole Lie algebra $\mathfrak{gl}(\theta)$ (as opposed to its subalgebra $\mathfrak{hs}(\theta)$, the extension of the higher-spin algebra $\mathfrak{hs}_\lambda$) as the condition $\mathrm{ad}^*_X(F)_{11} = 0$ has not been used anywhere so far. We therefore also want to require this condition in the bracket obtained and follow once again the procedure presented in [6, Chapter 3.5]. To begin, observe that the condition simply implies the vanishing of the coefficient associated to $\partial^{\lambda-1}$. In order to determine this, rewrite the second Gelfand-Dickey bracket, as displayed in equation (3.23), to:

$$\delta_X L = \big(Lx^1\big)_- L - L\big(x^1 L\big)_- \ . \tag{5.67}$$

The order of the pseudodifferential operators coming from the parentheses is maximally -1 and the order of L is maximally λ. This means, that the coefficients associated to ∂^{-1}

in the parentheses are the ones we are looking for. We extract them using the residue[3] and obtain a similar expression as given in [6]:

$$\delta_x L = \operatorname{res}\big(Lx^1\big)\partial^{\lambda-1} + ... - \operatorname{res}\big(x^1 L\big)\partial^{\lambda-1} - ... = \operatorname{res}\big[L, x^1\big]\partial^{\lambda-1} + ... \ . \tag{5.68}$$

From this, one therefore gets the condition that $\operatorname{res}\big[L, x^1\big] \overset{!}{=} 0$. By splitting x^1 formally into two parts,

$$x^1 = \partial^{-\lambda} x_{11} + \mathbf{x}^1, \text{ where } \mathbf{x}^1 := \sum_{i=2}^{\infty} \partial^{-\lambda+i-1} x_{i1} \ , \tag{5.69}$$

one then acquires the following equation:

$$\operatorname{res}\big[L, \partial^{-\lambda} x_{11}\big] + \operatorname{res}\big[L, \mathbf{x}^1\big] = 0 \ . \tag{5.70}$$

Using the Leibniz rule (3.14) for the commutator in the first summand, this then simplifies to:

$$\lambda x_{11}' + \operatorname{res}\big[L, \mathbf{x}^1\big] = 0 \ . \tag{5.71}$$

In order to implement this condition into the second Gelfand-Dickey bracket, we analogously split the bracket into two parts (following [6]), one of which contains x_{11}:

$$\delta_X L = \big[L, x_{11}\big] + L\big(\mathbf{x}^1 L\big)_+ - \big(L\mathbf{x}^1\big)_+ L \ . \tag{5.72}$$

The commutator arises from the Leibniz rule (3.14) and the evaluation of the parentheses $(...)_+$. Using the rule for the commutation of symbols (3.16), one can evaluate the commutator to:

$$[L, x_{11}] = \sum_{k=0;n=1}^{\infty} \binom{\lambda - k}{n} \widetilde{u}_k \big(x_{11}'\big)^{(n-1)} \partial^{\lambda-k-n} \ , \tag{5.73}$$

where $\widetilde{u}_0 = 1$. Inserting this into equation (5.72) we obtain:

$$\delta_X L = \sum_{k=0;n=1}^{\infty} \binom{\lambda - k}{n} \widetilde{u}_k \big(x_{11}'\big)^{(n-1)} \partial^{\lambda-k-n} + L\big(\mathbf{x}^1 L\big)_+ - \big(L\mathbf{x}^1\big)_+ L \ . \tag{5.74}$$

Upon reinsertion of x_{11}' (5.71), we have successfully transformed the second Gelfand-Dickey structure, which we obtained from a single-row gauge procedure, to the case where $\operatorname{ad}^*_X(F)_{11} = 0$. This structure now gives what one understands as the classical $W_\infty(\lambda)$-algebra on an infinite number of fields u_k. In principle, one obtains a classical W_n-algebra (which arises from the ordinary Drinfeld-Sokolov reduction) by simply replacing λ by N

[3] Remember that we view pseudodifferential operators of highest order λ as formal Laurent series in ∂. The residue is then simply defined as the coefficient to ∂^{-1}.

and “taking out the ideal“, i.e. truncating the sums to finite sums with upper bound N [6]. To conclude, we indeed see that the classical $W_\infty(\lambda)$ arises as an asymptotic symmetry algebra when considering the higher-spin algebra $\mathfrak{hs}_\lambda$.

Some remarks about the above results are in order. As outlined in [6], observe that in the finite-dimensional case of $\mathfrak{sl}_N$ one can express the first summand in equation (5.74) through L, by successively bringing first order differential operators to the right in the commutator (5.73). However, this strategy is not fruitful when considering $\mathfrak{hs}_\lambda$ as the finite-dimensional case builds on the assumption that this procedure can be performed exactly $N-k$ times, while in our case N is replaced by λ. Nevertheless, a prettier representation of the result (5.74) might be possible via some other strategy. Furthermore, notice that the right-hand side of equation (5.74) seemingly diverges when setting λ to zero, due to the appearance of λ in x'_{11} in the denominator. In the case of $\lambda = 0$, one regards $W_\infty(0)$ as the symmetry algebra of a free fermion theory [15] and, clearly, one has to treat this with extra care in order to avoid trouble. Probably one needs to discuss the conversion procedure separately for $\lambda = 0$, however, we will leave this discussion for another day and simply acknowledge the fact that our bracket is problematic at $\lambda = 0$.

6. Discussion

In this thesis we have considered the generalized Drinfeld-Sokolov reduction, regarding the higher-spin algebra $\mathfrak{hs}_\lambda$, coming from two approaches. First, we have considered a slight adaptation to the proof performed by B. Khesin and F. Malikov [22] and discovered, that the formalism can almost directly be applied to $\mathfrak{hs}_\lambda$. In particular, we have investigated the uniqueness of Frobenius matrices, when coming from a group coadjoint action onto a traceless matrix (with respect to the trace (4.28)), and arrived at an expected result. The rest of the proof can then be applied in an analogous way, by identifying the first rows of Frobenius matrices with the space of sequences, which is known to carry a family of second Gelfand-Dickey structures. However, there is still the general puzzle of the extension, introduced by the authors of [22], of the (Lie) algebra of complex-size matrices $\mathfrak{gl}_\lambda$. As we have discussed in chapter 4, the extension takes the form of a violation of properties *i)* and *ii)* (refer to (4.13) and (4.14) respectively) for matrices, which are regarded as elements in $\mathfrak{gl}_\infty \otimes \mathbb{C}[\lambda]$ under the image of the map η (modulo the Casimir). While the reason for the extension is understood in the sense that it allows matrices with only a finite number of non-zero rows, which we need if we want to construct Frobenius matrices, it is somewhat unclear whether it also arises naturally. Therefore, an interesting question for the future would be trying to obtain such matrices by considering them as some kind of limit in a suitable topology.

In the second part of the thesis we explicitly considered higher-spin gauge theories. In this context, W-algebras (which one identifies with the second Gelfand-Dickey structure on the fields u_k) arise as asymptotic symmetry algebras on asymptotically AdS$_3$-spacetimes, as explained in [6]. In particular, the Drinfeld-Sokolov constraint restricts connections a_θ to Frobenius matrices and their infinitesimal gauge transformations can be identified with the algebra coadjoint action. From this point we followed, for the most part, the procedure explained in [6]. To do so, we considered the algebra coadjoint action of a general matrix X, that lies in the loop algebra of complex-size matrices $\mathfrak{gl}(\theta)$ (setting $u_1 = 0$), onto a Frobenius matrix F. As it turns out, one can perform the single-row gauge in exactly the same manner as for finite $\mathfrak{sl}_N$, but there are some subtleties that required careful examination. Particularly, we viewed property *ii)* (4.16) of the matrices of interest from a different angle for this reason. In the end, upon requiring the first entry of the transformed matrix to vanish, one obtains a $W_\infty(\lambda)$-algebra as a result. Hence we were successful in generalizing the single-row gauge from the case $\mathfrak{sl}_N$ to $\mathfrak{hs}(\lambda)$, allowing for infinite fields. However, since λ can take any value in the complex numbers, the resulting second Gelfand-Dickey structure exhibits obvious problems when setting $\lambda = 0$ ("free fermion theory" [15]). Therefore, it might be necessary to treat this case separately, the discussion of which we set aside for later consideration.

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A. Appendix

This Appendix contains some calculations concerning the single-row gauge procedure performed in section 5.4.

A.1. Expanding $\left(\partial^{\lambda-i}x^1\right)_+ L$

In order to obtain the expansion of the left-hand side of equation (5.49), we first directly write out the terms appearing in $\left(\partial^{\lambda-i}x^1\right)_+ L$:

$$\begin{aligned}
\left(\partial^{\lambda-i}x^1\right)_+ L &= \left(\partial^{\lambda-i}\sum_{l=1}^{\infty}\partial^{-\lambda+l-1}x_{l1}\right)_+ L \qquad &(A.1)\\
&= \sum_{\substack{l=1\\ l>i}}^{\infty}\partial^{l-1-i}x_{l1}\left(\partial^{\lambda}+\sum_{k=1}^{\infty}\tilde{u}_k\partial^{\lambda-k}\right) &(A.2)\\
&= \sum_{\substack{l=1\\ l>i}}^{\infty}\sum_{n=0}^{\infty}\binom{l-1-i}{n}\left(x_{l1}^{(n)}+\sum_{k=1}^{\infty}\left(x_{l1}\tilde{u}_k\right)^{(n)}\partial^{-k}\right)\partial^{\lambda+l-1-i-n}\,. &(A.3)
\end{aligned}$$

Notice, that since $l > i$, the sum over n is finite as $l - 1 - i \geq 0$, which leads to the restriction $l > n + i$. On the other hand, one can split ∂^{l-i-1} to $\partial^{\lambda-i}\partial^{-\lambda+l-1}$ and bring these operators to the right one after the other using Leibniz's rule (3.14).

$$\left(\partial^{\lambda-i}x^1\right)_+ L = \sum_{\substack{l=1\\ l>i}}^{\infty}\sum_{t,m=0}^{\infty}B_1(l,t)B_2(m)\left(x_{l1}^{(t+m)}+\left(x_{l1}\tilde{u}_k\right)^{(t+m)}\partial^{-k}\right)\partial^{\lambda+l-1-i-t-m} \qquad (A.4)$$

Here, we introduce the notation:

$$B_1(l,t) := \binom{l-\lambda-1}{t},\ B_2(m) := \binom{\lambda-i}{m}\,. \qquad (A.5)$$

Now, we want to compare these two results by investigating their orders. Observe that due to the restriction $l > n + i$ in equation (A.3), this immediately implies $l > t + m + i$ by identifying $n \leftrightarrow t + m$. With this, equation (5.49) has been recovered.

A.2. Analysing the solution of the recursion

Even though the solution x_i (refer to equation (5.59)) of the recursion already satisfies the same structure on both sides of the definition by construction, it is still helpful to examine its structure. To do so, first notice that if $i \geq K$ (the "bounding" value appearing in property *iii)* (4.15)), the Ansatz simplifies to:

$$x_i = \partial^{\lambda-i}\big(x^1 L\big)_+ \ . \tag{A.6}$$

This can be easily seen as a consequence of the condition $l > i$ and the restriction $l \leq K$ in equation (A.3). Since the remaining term is the one we constructed, it suffices to check whether this term actually obeys the same structure as the left-hand side. To do so, we first bring all the operators to the right using Leibniz's rule (3.14):

$$\begin{aligned}
\partial^{\lambda-i}\big(x^1 L\big)_+ &= \partial^{\lambda-i}\left(\sum_{l,k=1}^{\infty}\sum_{n=0}^{\infty} B_1(l,n)\Big(x_{l1}^{(n)} + \big(x_{l1}\tilde{u_k}\big)^{(n)}\partial^{-k}\Big)\partial^{l-1-n}\right)_+ &\text{(A.7)}\\
&= \partial^{\lambda-i}\left(\sum_{\substack{l=1;n=0\\l>n}}^{\infty} B_1(l,n) x_{l1}^{(n)}\partial^{l-1-n}\right. &\text{(A.8)}\\
&\quad \left. + \sum_{\substack{l,k=1;n=0\\l>k+n}}^{\infty} B_1(l,n)\big(x_{l1}\tilde{u}_k\big)^{(n)}\partial^{l-1-n-k}\right) \ .
\end{aligned}$$

Note, that it is enough to only check whether the first summand has the correct structure since this analysis works in completely equivalent fashion for the second summand. Define $\tilde{j} := i + 1 + n + m - l$, where m is the index that appears when one brings $\partial^{\lambda-i}$ to the right. To begin, observe that $\tilde{j} > 0$ since $i \geq K$ and $l \leq K$, as required. We even have that $\tilde{j} \geq i + 1 - K$, meaning that the structure satisfies property *iii)* (4.15). Due to the appearance of the binomial coefficient $B_2(m)$ (A.5), the coefficients are polynomials in i and additionally satisfy the correct root structure in λ (since m is the only index that can shift $\tilde{j} > i$). Therefore, the term also correctly satisfies properties *i)* (4.13) and *ii)* (4.16). With this, we see that both summands appearing in equation (A.8) indeed satisfy the same structure as the left-hand side.

It is also interesting to examine the solution for $i \leq K$, in which case the term $\big(\partial^{\lambda-i}x^1\big)_+ L$ does not vanish:

$$x_i = \partial^{\lambda-i}\big(x^1 L\big)_+ - \big(\partial^{\lambda-i}x^1\big)_+ L \ . \tag{A.9}$$

In this case, property *iii)* (4.15) of complex-size matrices is obviously fulfilled by default. Using the $(...)_-$ version of the recursion (equation (5.55) without $\partial^{\lambda-i}H_+$), one additionally observes that the highest possible order is given by $\partial^{\lambda-1}$. Therefore, investigating whether equation (A.9) has a polynomial structure in i (4.13) and the correct root structure in

λ (4.16) is enough. One can show (by a lengthy calculation) that the recursion actually does not satisfy the correct root structure in λ. This, however, causes no problems since the phenomenon only occurs in rows $i \leq K$ and properties *i*) and *ii*) are allowed to be violated in a finite number of rows.