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Sampling in Spaces of Variable Bandwidth

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This thesis contains results that have been obtained by the author and her collaborators. Parts of this thesis can be found verbatim in original research papers by the author and her collaborator, that is in particular in the articles by Andreolli and Gröchenig [6, 7]. The plots and figures in this work were created with MATLAB. The author was supported by the Austrian Science Fund (FWF) projects 10.55776/P31887 and 10.55776/P33217.

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Abstract

This doctoral thesis delves into the domain of time-frequency analysis and sampling theory, focusing on the development of novel function spaces characterized by a variable bandwidth. The notion of variable bandwidth arises from the recognition that in nature it is meaningful to allocate distinct local bandwidths to various segments of a signal. However, giving a mathematical definition of variable bandwidth is challenging, since the bandwidth is a global property of a signal and a rigorous definition conflicts with the uncertainty principle when attempting to define a space of variable bandwidth.

Three constructions of spaces of variable bandwidth that use time-frequency tools are presented. The first space of variable bandwidth is constructed with a Gabor system. The second formulation is motivated by the physics of gravitational waves, and the space of variable bandwidth is built with Wilson bases. The third variation of the space of variable bandwidth is motivated by signal processing, with a space constructed with a local Fourier basis.

All these function spaces are built by using a discrete version of a variable bandwidth space introduced by Aceska and Feichtinger. They use a time-varying frequency truncation of the short-time Fourier transform and obtain an equivalent norm for the standard Sobolev space. The resulting spaces lack the main features of a variable bandwidth space, namely sampling theorems and necessary density conditions for reconstruction.

In our definitions, the short-time Fourier transform is replaced by a frequency truncation of a Gabor system, a Wilson orthonormal basis, or a local Fourier basis.

A key contribution of the thesis is the formulation of sufficient conditions for sampling for these spaces of variable bandwidth. Sampling inequalities are established following the adaptive weights method, and they provide insights into the reconstruction of functions from discrete samples. The exploration of nonuniform sampling scenarios offers a quantitative analysis of local sampling density requirements. The results of the sufficient condition for sampling theorems are tested in numerical simulations.

Zusammenfassung

Diese Doktorarbeit befasst sich mit der Zeit-Frequenz-Analyse und der Abtasttheorie und konzentriert sich auf die Entwicklung neuer Funktionenräume, die durch eine variable Bandbreite charakterisiert sind. Die Idee der variablen Bandbreite entsteht aus der Erkenntnis, dass es in der Natur sinnvoll ist, verschiedenen Segmenten eines Signals unterschiedliche lokale Bandbreiten zuzuweisen. Die Herausforderung bei der Definition eines solchen Konzepts besteht jedoch darin, dass die Bandbreite eine globale Eigenschaft eines Signals ist und somit mit dem Unschärfeprinzip in Konflikt gerät, wenn versucht wird, lokale Bandbreiten festzulegen.

Es werden drei Konstruktionen von Räumen mit variabler Bandbreite vorgestellt, die Zeit-Frequenz-Werkzeuge verwenden. Der erste Raum mit variabler Bandbreite wird mit einem Gaborsystem konstruiert. Die zweite Formulierung wird durch die Physik der Gravitationswellen motiviert, und der Raum mit variabler Bandbreite wird mit Wilson-Basen aufgebaut. Die dritte Variante des Raums mit variabler Bandbreite ist durch die Signalverarbeitung motiviert und wird mit einer lokalen Fourier-Basis konstruiert.

All diese Funktionenräume werden unter Verwendung einer diskreten Version eines variablen Bandbreiteraums konstruiert, der von Aceska und Feichtinger eingeführt wurde. Sie nutzen ein zeitabhängiges Abschneiden der Frequenz der Kurzzeit-Fourier-Transformation und erhalten eine äquivalente Norm für den standardmäßigen Sobolev-Raum. Den resultierenden Räumen fehlen die Hauptmerkmale eines variablen Bandbreitenraums, nämlich Abtasttheoreme und notwendige Dichtebedingungen für die Rekonstruktion.

In unseren Definitionen wird die Kurzzeit-Fourier-Transformation durch ein Abschneiden der Frequenz eines Gaborsystems, einer Wilson-Orthonormalbasis oder einer lokalen Fourier-Basis ersetzt.

Ein zentraler Beitrag der Arbeit ist die Formulierung hinreichender Bedingungen für das Abtasten in diesen Räumen variabler Bandbreite. Abtastungleichungen werden nach der Methode der adaptiven Gewichte aufgestellt und bieten Einblicke in die Rekonstruktion von Funktionen aus diskreten Abtastwerten. Die Untersuchung von nicht-uniformen Abtastszenarien bietet eine quantitative Analyse der lokalen Abtastdichte. Die Ergebnisse der hinreichenden Bedingungen für Abtasttheoreme werden in numerischen Simulationen getestet.

Contents

Introduction

Sampling problem

One of the most extensively researched and influential sampling theorems in the fields of signal processing, telecommunications, and data compression is the Shannon-Nyquist-Kotelnikov-Whittaker sampling theorem. This theorem made its initial appearance in the area of information theory through the work of Claude Shannon [66]. However, even before Shannon's contribution, Nyquist had presented it to the community of communication engineers [62]. In the Russian literature, the theorem was introduced by Kotelnikov [52], and in the context of complex analysis, it was developed by Whittaker [73].

The sampling problem addresses the mathematical principles behind the representation and reconstruction of continuous signals through discrete samples. The main question consists of how to reconstruct a function f from a discrete set of function values, denoted by $\{f(\lambda)\}_{\lambda\in\Lambda}$. However, when we frame the problem this way, it becomes ill-posed. In fact, there is an infinite number of distinct functions that could produce the same values by evaluating them at a countable set of points. In other words, we lack enough information to uniquely determine f. To make this problem well-defined, it is a common practice to impose certain conditions on the function. These conditions are typically based on requiring f to be a member of a particular function space. The problem is known to be well-posed in case the function space is the Paley-Wiener space

$$PW_\Omega = \big\{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-\Omega, \Omega]\big\}. \tag{0.0.1}$$

The functions in PW_Ω are said to have **bandwidth** Ω since their Fourier transform vanishes outside the interval $[-\Omega, \Omega]$ and are also called **band-limited** functions. The Shannon-Nyquist-Kotelnikov-Whittaker sampling theorem states that any continuous function in $PW_{1/2}$ can be fully recovered from its samples at the integers.

Theorem 0.1 (Shannon-Nyquist-Kotelnikov-Whittaker sampling theorem)**.** *If $f \in PW_{1/2}$ is continuous, then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(x-k),$$

with convergence of the symmetric partial sums in $L^2(\mathbb{R})$ and pointwise for all $x \in \mathbb{R}$.

The result can be extended to functions with arbitrary finite bandwidth Ω by applying a suitable scaling. We refer to the books by Eldar [31] and by Vetterli, Kovačević, and Goyal [70] for an exhaustive introduction to sampling theory.

Spaces of variable bandwidth

The concept of variable bandwidth for the representation of signals has always been of interest in signal processing, engineering, and physics and it has recently captured the attention of mathematicians in time-frequency analysis.

The Paley-Wiener space (0.0.1) is undoubtedly the most popular signal model in signal processing thanks to its fundamental connection with the problem of signal reconstruction from samples. However, it is important to recognize that functions in the Paley-Wiener space are entire functions of time, whereas real-world signals typically have a finite duration. Therefore, it might be helpful to define different spaces to represent non-bandlimited signals which might consist of several bursts or pulses of varying duration, intensity, and frequency. With these considerations, the concept of variable bandwidth arises naturally and it is even more intuitive when we think about music. In a piece of music, the perceived highest frequency, i.e., the note, is time-varying. This observation provides a reasonable argument for the assignment of different local bandwidths to different segments of a signal when representing it mathematically. However, producing a rigorous definition of variable bandwidth is a challenging task, since bandwidth is global by definition and the assignment of a local bandwidth meets an obstruction in the uncertainty principle [28].

Recently, numerous researchers have proposed various definitions of variable bandwidth, yet no consensus has been reached on the right one. Refer to the discussion below for more details.

The goal of this thesis is to introduce a novel approach using time-frequency methods. Time-frequency analysis is inherently suited for variable bandwidth analysis, as it aims to analyze functions by using representations that combine the temporal behavior f and the frequency behavior $\hat{f}$ into a unified framework. Our approach begins with a discrete time-frequency representation, enabling us to write any function f as a series expansion of time-frequency atoms. By applying a time-varying frequency truncation, we can create a space characterized by a specific variable bandwidth. We present three implementations of this concept using Gabor Riesz sequences, Wilson bases, and local Fourier bases.

Other interpretations of variable bandwidth

In academic works, various strategies for handling variable bandwidth are documented. While these approaches share a similar objective, the mathematical techniques used in each are distinct from our method.

- A concept of variable bandwidth based on time-frequency methods, namely the truncation of the short-time Fourier transform using a time-varying frequency cut-off was proposed by Aceska and Feichtinger [3, 4]. The resulting function spaces, however, coincide with the standard Sobolev spaces endowed with an equivalent norm and therefore lack the characteristic properties of Paley-Wiener spaces. Thus, these are not variable bandwidth spaces in our sense since they admit neither a sampling theorem nor a Nyquist density.

- The definition of variable bandwidth proposed by Celiz, Gröchenig, and Klotz is based on the spectral theory of a Sturm-Liouville operator $A_p f = -\frac{d}{dx}(p(x)\frac{d}{dx})f$ on $L^2(\mathbb{R})$, where $p > 0$ is a strictly positive function [21, 40, 41]. Denote by $c_\Lambda(A_p)$ the spectral projection of A_p corresponding to A_p in $\Lambda \subset \mathbb{R}^+$, then the range of this projection-valued measure $c_\Lambda(A_p)$ is the space of functions of variable bandwidth with spectral set in Λ. A (nonuniform) sampling theorem and necessary density conditions were proved for this space of variable bandwidth. The main insight of this model is that, for a spectrum $\Lambda = [0, \Omega]$, a function of variable bandwidth behaves like a bandlimited function with local bandwidth $(\Omega/p(x))^{1/2}$ in a neighborhood of $x \in \mathbb{R}$.

- Kempf and his colleagues introduced a procedural concept of variable bandwidth, as outlined in their works [46, 50, 51]. A formal definition of this concept was subsequently proposed in [60]. Their approach involved constructing a symmetric operator and a corresponding reproducing kernel Hilbert space based on a sequence of sampling points. This space is identified as the variable bandwidth space. They parametrized the self-adjoint extensions of the operator and obtained several sampling theorems at the critical density.

- Another interesting interpretation of variable bandwidth in signal processing involves warping functions [19, 24, 48, 67, 72]. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a homeomorphism, called a warping function and let $g \in L^2(\mathbb{R})$ with $\mathrm{supp}(\hat{g}) \subseteq [-\Omega, \Omega]$ be a bandlimited function. A function f possesses variable bandwidth with respect to γ if $f = g \circ \gamma$. The derivative $1/\gamma'(\gamma^{-1}(x))$ of the warping function is interpreted as the local bandwidth of f at x.

Time-frequency analysis tools

This doctoral thesis offers an extensive exploration of variable bandwidth function spaces, built upon discrete time-frequency representations. These representations consist of time-frequency atoms characterized by precise localization in both temporal and frequency domains. The main idea is to build spaces of variable bandwidth by using a discrete version of the spaces of Aceska and Feichtinger. They consider a function $b(x) \geq 0$ that represents the time-varying broadness of a strip ST_b in the time-frequency plane

$$ST_b = \big\{(x, \omega) \in \mathbb{R}^2 : |\omega| \leq b(x)\big\}$$

and the vertical distance

$$d_b(x, \omega) = \begin{cases} 0, & \text{if } (x, \omega) \in ST_b, \\ |\omega| - b(x), & \text{if } (x, \omega) \notin ST_b. \end{cases}$$

Then $m_{b,s}(x, \omega) = (1 + d_b(x, \omega))^s$, $s > 0$ is a variable bandwidth weight on the time-frequency plane. They define a function f of variable bandwidth b as a function belonging to a modulation space

$$M^{p,q}_{m_{b,s}}(\mathbb{R}) = \Big\{ f \in \mathcal{S}'(\mathbb{R}) : \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |V_g f(x,\omega)|^p m_{b,s}(x,\omega)^p dx \Big)^{\frac{q}{p}} d\omega \Big)^{\frac{1}{q}} < \infty \Big\}$$

for some $1 \leq p, q \leq \infty$. The application of these weights implies that a graded weight is added outside the strip ST_b and it gives the possibility to locally describe the decay of the short-time Fourier transform $V_g f$. For $p = q = 2$ they obtain a Sobolev space endowed with an equivalent norm. Therefore, these spaces admit neither a sampling theorem in the style of Theorem 0.1 nor a Nyquist density.

We want to define spaces of variable bandwidth for which sampling theorems and necessary density conditions can be provided. In our approach we replace the (continuous) short-time Fourier transform with specific discrete time-frequency representations.

The starting attempt to define new spaces of variable bandwidth is done by using Gabor systems, in particular Gabor Riesz sequences. A Gabor system is a sequence of the type

$$\{e^{2\pi i\beta lx} g(x - \alpha n)\}_{n,l\in\mathbb{Z}} = \{M_{\beta l} T_{\alpha n} g\}_{n,l\in\mathbb{Z}}\,, \tag{0.0.2}$$

for $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R})$. This representation was introduced by Gabor in 1946 with $\alpha = \beta = 1$ and $g = e^{-\pi x^2}$ [35]. The representation of a signal in terms of Gabor systems provides a unified way to capture both the temporal and frequency behavior of the functions, making Gabor systems a very flexible and convenient tool.

For this reason, a starting attempt to define spaces of variable bandwidth has been made by using a Gabor Riesz sequence and the space of variable bandwidth is defined by

$$V^2_b(g, \mathbb{R}) \coloneqq \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{|l|\leq b(n)} c_{n,l} M_{\beta l} T_{\alpha n} g,\ c \in \ell^2(\mathbb{Z}\times\mathbb{Z}) \Big\},$$

where the sequence b describes the time-varying frequency truncation. If we assume g to be compactly supported, we have that the parameter $\beta b(n)$ represents the local bandwidth of the function f on an interval centered at $\alpha n + \alpha/2$. This intuition can be further understood by using the Gabor orthonormal basis $\{e^{2\pi i l x}\chi_{[0,1]}(x - n) : n, l \in \mathbb{Z}\}$ instead of a Gabor Riesz sequence. In fact, for the Gabor orthonormal basis, the highest frequency component within the interval $[0, 1]$ is $b(n)$. Consequently, to faithfully reconstruct a function in the interval $[0, 1]$ a minimum of $2b(n) + 1$ samples are required. In the case of a Gabor Riesz sequence, the number of samples has to be refined to take care of the multiple translations of the function g within each interval $[\alpha n, \alpha(n + 1)]$.

A second and third attempt to define spaces of variable bandwidth use time-frequency representations which were introduced to solve the problem of finding orthonormal expansions for signals with good time-frequency localization. The Balian-Low theorem in the 1980s [11, 56] established that if a Gabor system of the form (0.0.2) constitute an orthonormal basis then either

$$xg \notin L^2(\mathbb{R}) \text{ or } \omega\hat{g} \notin L^2(\mathbb{R}).$$

This implies that a function g generating a Gabor orthonormal basis cannot be well-localized in both the time and the frequency domain.

Additionally, the density theorem for Gabor frames introduced necessary conditions for a Gabor system of the type (0.0.2) to form a frame. In fact, a system $\{e^{2\pi i\beta lx}g(x-\alpha n)\}_{n,l\in\mathbb{Z}}$ that forms a frame for $L^2(\mathbb{R})$, implies that $\alpha\beta \leq 1$, and at $\alpha\beta = 1$, a normalized tight Gabor frame serves as an orthonormal basis. Thus, if we relax the requirement for the system to be a basis to enhance time-frequency localization, we get a highly redundant expansion with $\alpha\beta < 1$.

Numerous researchers have endeavored to mitigate the negative implications of the Balian-Low theorem, trying to produce orthonormal bases for $L^2(\mathbb{R})$ with a good time-frequency localization. Two distinct constructions, motivated by different purposes, stand out and have been used as the building blocks to form additional two spaces of variable bandwidth.

The first construction emerged from quantum mechanics and was inspired by K. Wilson's idea [74]. In 1987, he proposed to substitute the exponential $e^{2\pi i\beta lx}$ in (0.0.2) with sines and cosines, enabling the functions' localization around two frequencies with opposite signs. This construction was further refined by Ingrid Daubechies, Stéphane Jaffard, and Jean-Lin Journée in [26] to yield what is now known as a Wilson basis:

$$\psi_{n,l}(x) = \begin{cases} g(x-n), & l = 0, \\ \frac{1}{\sqrt{2}}(e^{2\pi ilx} + (-1)^{l+n}e^{-2\pi ilx})g(x-n/2), & l \neq 0, \end{cases}$$

for $n, l \in \mathbb{Z}$ and $l \geq 0$.

Here, g can be chosen such that both g and its Fourier transform $\hat{g}$ have exponential decay, or g can be selected to be smooth and compactly supported. Therefore, they found a way to overcome the barrier posed by the Balian-Low theorem and at the same time keep much of the structure of a Gabor system. The space of variable bandwidth constructed with a Wilson basis is

$$PW_b^2(g,\mathbb{R}) := \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l}\psi_{n,l},\ c \in \ell^2(\mathbb{Z}\times\mathbb{N}) \Big\}.$$

When g is chosen to be compactly supported, the sequence b represents the local bandwidth of a function over an interval centered at $n/2$.

Another construction was introduced by Malvar [59] in the context of subband coding theory and independently formulated by Coifman and Meyer [25]. In signal processing, there is sometimes a need to focus on the local properties of a signal f, particularly within a finite interval I. Orthonormal bases for $L^2(I)$, consisting of trigonometric functions multiplied by the characteristic function over I, are of the type: (i) $\sin\big(\frac{2l+1}{2}\frac{\pi}{|I|}x\big)\chi_I(x)$, for $l = 0, 1, 2, \ldots$, (ii) $\sin\big(l\frac{\pi}{|I|}x\big)\chi_I(x)$, for $l = 1, 2, 3\ldots$, (iii) $\cos\big(\frac{2l+1}{2}\frac{\pi}{|I|}x\big)\chi_I(x)$, for $l = 0, 1, 2, \ldots$, and (iv) $\cos\big(l\frac{\pi}{|I|}x\big)\chi_I(x)$, for $l = 0, 1, 2, \ldots$. These bases can be extended to form an orthonormal basis in $L^2(\mathbb{R})$ by employing a partition $\{\alpha_n\}_{n\in\mathbb{Z}}$ of $\mathbb{R}$. While these bases exhibit strong localization in the time domain, the use of the characteristic function of I introduces artificial discontinuities that hinder effective frequency localization. To improve the frequency localization the characteristic function needs to be replaced by a smoother function. Auscher and his collaborators unified the construction of the Malvar bases and the bases of Coifman and Meyer to form the so-called local Fourier bases [10]. A local Fourier basis is a collection of the type

$$\phi_{n,l}(x) = \sqrt{\frac{2}{\ell_n}} g_n(x) \cos\Big(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x-\alpha_n)\Big), \text{ for } n \in \mathbb{Z},\ l \in \mathbb{N} \cup \{0\},$$

where $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, $\ell_n = \alpha_{n+1} - \alpha_n$, and g_n is a smooth and compactly supported function. The space of variable bandwidth constructed with a local Fourier basis is

$$LF_b^2(\mathbb{R}) := \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lceil 2b(n)\ell_n - \frac{1}{2}\rceil} c_{n,l}\phi_{n,l},\ c \in \ell^2(\mathbb{Z}\times\mathbb{N}) \Big\}.$$

Since g_n are compactly supported by the construction of a local Fourier basis, $b(n)$ is a measure of the local frequency on the interval $[\alpha_n, \alpha_{n+1}]$ of the partition.

Main results and outlook

The main objects of this thesis are the spaces of variable bandwidth presented above: $V_b^2(g,\mathbb{R})$ the **variable bandwidth space constructed via a Gabor Riesz sequence**, $PW_b^2(g,\mathbb{R})$ the **variable bandwidth space constructed via a Wilson basis**, and $LF_b^2(\mathbb{R})$ the **variable bandwidth space constructed via a local Fourier basis**.

We show that these spaces of variable bandwidth behave in many ways like the classical Paley-Wiener space (0.0.1). Indeed, in analogy to the Shannon-Nyquist-Kotelnikov-Whittaker sampling theorem and its variations, we prove sampling theorems for the above-mentioned spaces. Moreover, for the space $PW_b^2(g,\mathbb{R})$ we show explicitly when $PW_b^2(g,\mathbb{R})$ contains or is contained in the Paley-Wiener space PW_Ω.

We focus on the **nonuniform sampling problem**: instead of dealing with evenly spaced sampling points $k \in \mathbb{Z}$, we will explore a more general scenario where these sampling points are represented by a sequence $\{\lambda\}_{\lambda\in\Lambda}$, with $\Lambda \subset \mathbb{R}$.

Within the scope of this thesis, we delve into the examination of two fundamental problems:

- **Sufficient conditions for sampling.** We study under which sufficient conditions on a set of discrete points a function in a variable bandwidth space can be reconstructed completely. In general, this is the most difficult aspect of sampling theory: in most cases, only qualitative statements are known requiring "a sufficiently high oversampling rate". Sharp results in this field are rare and highly regarded. Examples include the work of Beurling on spectral subspaces of the Laplacian on $\mathbb{R}^d$ (band-limited functions with a ball as the spectrum) [13, 14], the contributions of Lyubarskiĭ and Seip [57, 64] on sampling in Bargmann-Fock spaces, and the sharp results of Gröchenig in the context of band-limited functions [38], shift-invariant spaces [43], and Gabor frames [45].

For all the spaces of variable bandwidth introduced before, quantitative sufficient conditions for sampling theorems are presented. By following a method introduced by Gröchenig in [38] and then called **adaptive weigths method** in [32], we construct sufficient conditions for sampling theorems for the spaces of variable bandwidth. We validate our initial intuition that $b(n)$ measures the local bandwidth of the function of variable bandwidth. Moreover, we show the dependence of the bandwidth sequence b to the sampling rate at which we have to evaluate the function to obtain a stable and complete reconstruction. The results can be applied to numerical simulations.

- **Necessary density condition for sampling.** For the space of variable bandwidth constructed with a Wilson basis $PW_b^2(g, \mathbb{R})$ we additionally investigate whether there exists a critical density that separates sampling, the complete reconstruction from samples, from interpolation, the approximation of given data by a function with prescribed properties. The existence of a critical density, a so-called Nyquist rate, is a fundamental information-theoretic statement about a function space.

For the Paley-Wiener space, these questions have been investigated deeply for many years [65, 68], with the most important results due to Beurling [15] and Landau [53]. Nowadays, sampling theorems are formulated as sampling inequalities for reproducing kernel Hilbert spaces. A reproducing kernel Hilbert space $\mathcal{K}$ is a closed subspace of $L^2(\mathbb{R})$ for which the point evaluation $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{K}$ and a set of points $\Lambda \subset \mathbb{R}$ is a **sampling set** for $\mathcal{K}$ if an inequality of the type

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathcal{K} \tag{0.0.3}$$

holds. The sampling inequality (0.0.3) indicates that the energy of the function is captured by the evaluation of the function at the samples. We show that our spaces of variable bandwidth are reproducing kernel Hilbert spaces and we prove under which conditions the set Λ is a sampling set for these spaces.

The thesis is structured as follows:

Chapter 1 reviews the key definitions and results in frame theory (Section 1.1) and Gabor analysis (Section 1.2). It also covers Wirtinger's inequality and a generalization of Bernstein's inequality (Section 1.3), as well as an overview of reproducing kernel Hilbert spaces (Section 1.4).

Chapter 2 examines the space of variable bandwidth constructed using a Gabor Riesz sequence. It presents two sufficient conditions for sampling for the space $V_b^2(g, \mathbb{R})$. The first condition involves a specific window function g with a fixed parameter $\alpha = 1$ (Section 2.1), while the second considers a more general compactly supported function g without restrictions on α (Section 2.2). Section 2.3 discusses and analyzes the main results.

Chapter 3 studies the sampling problem for the space of variable bandwidth constructed using Wilson bases. It begins with the main characterizations of a Wilson basis (Section 3.1) and describes a family of compactly supported functions g generating a Wilson orthonormal basis (Section 3.2). Section 3.3 focuses on the necessary density conditions for sampling in $PW_b^2(g,\mathbb{R})$, while Section 3.4 addresses the sufficient conditions for sampling problem. The results in Sections 3.3 and 3.4 are from the joint work [7]. Section 3.5 presents a sufficient condition for sampling using a Wilson Riesz basis, detailed in another collaborative article [6]. Section 3.6 discusses the relationship between the space of variable bandwidth $PW_b^2(g,\mathbb{R})$ and the classical Paley-Wiener space, with a comparison and discussion of the main results in Section 3.7.

Chapter 4 explores the spaces of variable bandwidth constructed using local Fourier bases. The construction of these bases is detailed in Section 4.1. Section 4.2 introduces the space of variable bandwidth $LF_b^2(\mathbb{R})$, showing that it is a reproducing kernel Hilbert space. Sections 4.3 and 4.4 offer two sufficient conditions for sampling for $LF_b^2(\mathbb{R})$. The main results of Chapter 4 and their comparison with the results for $V_b^2(g,\mathbb{R})$ and $PW_b^2(g,\mathbb{R})$ are discussed in Section 4.5.

Chapter 5 showcases the numerical implementations in MATLAB of the theoretical results from Chapter 3. Section 5.1 formally describes the numerical reconstruction of a function from samples in $PW_b^2(g,\mathbb{R})$. Section 5.2 introduces the window function used in the simulations, describes the least squares problem, and demonstrates the reconstruction of a function in $PW_b^2(g,I)$. Section 5.3 analyzes the reconstruction of a function in $PW_b^2(g,\mathbb{R})$ within a restricted interval, while Section 5.4 presents the experiment of reconstructing a chirp. These experiments are contained in the article [7]. Section 5.5 includes two examples of numerical reconstruction in a space of variable bandwidth constructed using Gabor systems.

1 Time-Frequency Analysis

In this chapter, we present the fundamental definitions and theorems within the realm of time-frequency analysis, essential for this thesis. In Section 1.1, we provide the main definitions and outcomes concerning bases and frames. Section 1.2 briefly reviews the notation and results of Gabor analysis. For these first two sections we follow part of the discussion in [23]. Section 1.3 introduces the two main tools for the study of sufficient conditions for sampling, i.e., Wirtinger's inequality and a modification of Bernstein's inequality. In Section 1.4, we focus on reproducing kernel Hilbert spaces and their involvement in sampling theory.

1.1 Bases and Frames

Bases play a pivotal role in the study of Banach and Hilbert spaces. They provide a fundamental framework for representing elements within these spaces. Specifically, a basis, denoted as $\{e_n\}_{n\in\mathbb{Z}}$, in a Banach space $\mathcal{B}$, possesses a crucial property: it allows every function f of the space to be uniquely expressed as a linear combination of its elements. More formally, a **(Schauder) basis** for a Banach space $\mathcal{B}$ is a sequence $\{e_n\}_{n\in\mathbb{Z}}$ in $\mathcal{B}$ such that for every $f \in \mathcal{B}$, there exist unique coefficients $\{c_n(f)\}_{n\in\mathbb{Z}}$ such that

$$f = \sum_{n\in\mathbb{Z}} c_n(f)e_n. \tag{1.1.1}$$

This means that the series $f = \sum_{n\in\mathbb{Z}} c_n(f)e_n$ converges according to the chosen order of the elements. If (1.1.1) converges unconditionally for each $f \in \mathcal{B}$, we refer to $\{e_n\}_{n\in\mathbb{Z}}$ as an **unconditional basis**.

However, satisfying the conditions for a basis can be quite challenging. Therefore, it can be advantageous to introduce the concept of frames.

A frame is essentially a sequence of elements, denoted as $\{f_n\}_{n\in\mathbb{Z}}$, in the Hilbert space $\mathcal{H}$. It satisfies the linear expansion property, similar to that of a basis as expressed in (1.1.1), but in the case of frames, the coefficients might not be unique.

In this context, we will provide a concise overview of the fundamental concepts related to frames and Gabor frames, without delving into detailed proofs. For a more comprehensive understanding, see the book by Christensen [23].

Definition 1.1. A sequence $\{f_n\}_{n\in\mathbb{Z}}$ of elements in a Hilbert space $\mathcal{H}$ is a **frame** for $\mathcal{H}$ if there exist constants $A, B > 0$ such that for every $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \tag{1.1.2}$$

where A, B are called **frame bounds**. If $A = B$, then $\{f_n\}_{n\in\mathbb{Z}}$ is called a **tight frame**. The sequence $\{f_n\}_{n\in\mathbb{Z}}$ is called a **Bessel sequence** if at least the upper frame condition in (1.1.2) holds.

It follows from the definition that if $\{f_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$, then

$$\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{Z}} = \mathcal{H}.$$

Moreover, if $\{f_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}$, then the operators C and C^*, defined by

$$C : \ell^2(\mathbb{Z}) \to \mathcal{H}, \quad C\{c_n\}_{n\in\mathbb{Z}} = \sum_{n\in\mathbb{Z}} c_n f_n$$

and

$$C^* : \mathcal{H} \to \ell^2(\mathbb{Z}), \quad C^* f = \{\langle f, f_n\rangle\}_{n\in\mathbb{Z}},$$

are well-defined and bounded operators and are called **synthesis operator** and **analysis operator** respectively.

By composing C and C^*, we obtain the **frame operator** S defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = CC^* f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle f_n,$$

which is positive, bounded, invertible, and satisfies

$$A\langle f, f\rangle \le \langle Sf, f\rangle \le B\langle f, f\rangle, \qquad \text{for every } f \in \mathcal{H}.$$

Moreover, its inverse S^{-1} is positive and has a self-adjoint square root $\{S^{-\frac{1}{2}}\}$ such that $\{S^{-\frac{1}{2}} f_n\}_{n\in\mathbb{Z}}$ is a tight frame. The sequence $\{S^{-1} f_n\}_{n\in\mathbb{Z}}$ is again a frame with bounds B^{-1} and A^{-1} and is called the **canonical dual frame** for $\{f_n\}_{n\in\mathbb{Z}}$ in the sense that

$$f = \sum_{n\in\mathbb{Z}} \langle f, S^{-1} f_n\rangle f_n = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle S^{-1} f_n, \qquad \text{for all } f \in \mathcal{H}, \tag{1.1.3}$$

and both series converge unconditionally for all $f \in \mathcal{H}$.

A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ is called a **frame sequence** for $\mathcal{H}$ if it is a frame for $\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{Z}}$.

A frame for $\mathcal{H}$ can be viewed as an extension or generalization of a basis. Unlike a basis, a frame comprises vectors that may not necessarily be linearly independent or orthogonal. Thanks to their high flexibility, frames are employed in signal analysis, image processing, and sampling theory for reconstructing information from signals [58, 69].

Definition 1.2. A sequence $\{f_n\}_{n\in\mathbb{Z}}$ of a Hilbert space $\mathcal{H}$ is a **Riesz sequence** if there exist bounds $A, B > 0$ such that for all finite sequences $c \in \ell^2(\mathbb{Z})$,

$$A\|c\|^2 \le \Big\| \sum_{n\in\mathbb{Z}} c_n f_n \Big\|^2 \le B\|c\|^2.$$

A Riesz sequence that generates the whole space $\mathcal{H}$ is called a **Riesz basis** for $\mathcal{H}$. The following result about the Riesz basis will be useful in Chapter 2 and 3, when we discuss the reconstruction of functions in spaces of variable bandwidth generated by a Riesz sequence or a Wilson Riesz basis respectively.

Theorem 1.3. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a Riesz basis for $\mathcal{H}$, then there exists a unique sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ such that for every $f \in \mathcal{H}$*

$$f = \sum_{n\in\mathbb{Z}} \langle f, g_n \rangle f_n = \sum_{n\in\mathbb{Z}} \langle f, f_n \rangle g_n.$$

*$\{g_n\}_{n\in\mathbb{Z}}$ is also a Riesz basis for $\mathcal{H}$, $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ are biorthogonal, i.e., $\langle g_n, f_j \rangle = \delta_{n,j}$, for every $n, j \in \mathbb{Z}$, and $\{g_n\}$ is called the **dual Riesz basis** of $\{f_n\}_{n\in\mathbb{Z}}$ and the series converges unconditionally for all $f \in \mathcal{H}$. Moreover, if $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ are a pair of dual Riesz bases, the following are equivalent:*

(a) $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis with lower bound A.

(b) $\{g_n\}_{n\in\mathbb{Z}}$ is Riesz basis with upper bound A^{-1}.

1.2 Gabor analysis in $L^2(\mathbb{R})$

Gabor analysis, a concept bridging the gap between time analysis and frequency analysis, provides a versatile approach to the representation of functions. When dealing with a signal, primarily characterized by its temporal behavior, we often employ the Fourier transform to unveil the distribution of Fourier coefficients. These coefficients enable us to approximate or reconstruct the signal using trigonometric functions. However, this approach lacks insight into the signal's temporal distribution.

Gabor's ingenious idea was to create a two-dimensional representation of a one-dimensional signal. This representation simultaneously captures information about the signal's behavior in both the time and frequency domains.

Let $g \in L^2(\mathbb{R})$, we define the translation operator and the modulation operator as

$$T_n g(x) = g(x-n) \quad \text{and} \quad M_l g(x) = e^{2\pi i l x} g(x). \tag{1.2.1}$$

The main idea of Gabor was to represent functions $f \in L^2(\mathbb{R})$ as a superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$.

Definition 1.4. Let $g \in L^2(\mathbb{R})$, $\alpha, \beta > 0$, and $x \in \mathbb{R}$, a collection of the form $\{M_{\beta l} T_{\alpha n} g\}_{n,l\in\mathbb{Z}}$, more explicitly

$$M_{\beta l} T_{\alpha n} g(x) = e^{2\pi i l \beta x} g(x - \alpha n),$$

is called a **Gabor system**.

The Gabor system $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ involves translates with parameters αn, $n \in \mathbb{Z}$ and modulations with parameters βl, $l \in \mathbb{Z}$. The points $\{(\alpha n, \beta l)\}_{n,l\in\mathbb{Z}}$ form a **lattice** in $\mathbb{R}^2$. A Gabor system that forms a frame is called a **Gabor frame**.

We state now a density theorem for Gabor systems.

Theorem 1.5 (Theorem 2.5.1, [42]). *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ be given. Then the following hold:*

(i) If $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $\alpha\beta \leq 1$.

(ii) If $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then $\alpha\beta \geq 1$.

(iii) If $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is a Riesz basis if and only if $\alpha\beta = 1$.

Note that a Gabor frame $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is overcomplete if and only if $\alpha\beta < 1$. As a quantitative measure of overcompleteness, we use the **redundancy**, defined by $(\alpha\beta)^{-1}$.

For a Gabor frame $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ with associated frame operator S, the frame decomposition in (1.1.3) can be written as

$$f = \sum_{n,l\in\mathbb{Z}} \langle f, S^{-1}M_{\beta l}T_{\alpha n}g\rangle M_{\beta l}T_{\alpha n}g = \sum_{n,l\in\mathbb{Z}} \langle f, M_{\beta l}T_{\alpha n}S^{-1}g\rangle M_{\beta l}T_{\alpha n}g, \quad \text{for all } f \in L^2(\mathbb{R}).$$

The canonical dual frame $\{S^{-1}M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ has the structure of a Gabor frame and $S^{-1}g$ is called the **canonical dual window** or the **canonical dual generator**. $\{M_{\beta l}T_{\alpha n}S^{-\frac{1}{2}}g\}_{n,l\in\mathbb{Z}}$ is the associated canonical tight frame.

The following necessary condition for a Gabor frame focuses on the relation between the window g and the translation parameter α.

Proposition 1.6 (Proposition 11.3.4, [23]). *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ be given, and assume that $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ is a frame with bounds A, B. Then*

$$\beta A \leq \sum_{n\in\mathbb{Z}} \left|g(x-\alpha n)\right|^2 \leq \beta B, \ \text{a.e. } x \in \mathbb{R}, \tag{1.2.2}$$

and

$$\alpha A \leq \sum_{n\in\mathbb{Z}} \left|\hat{g}(\nu-\beta n)\right|^2 \leq \alpha B, \ \text{a.e. } \nu \in \mathbb{R}.$$

A critical concern revolves around deriving expansions of functions $f \in L^2(\mathbb{R})$ using Gabor systems where g is continuous and has compact support. As the subsequent result demonstrates, these criteria can not align with $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ being a Riesz basis for $L^2(\mathbb{R})$. However, Gabor frames fulfilling these conditions do exist.

Proposition 1.7 (Proposition 4.2.2 [23]). *Let g be a continuous function with compact support. Then the following hold:*

(i) $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *cannot be an orthonormal basis for* $L^2(\mathbb{R})$.

(ii) $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *cannot be a Riesz basis for* $L^2(\mathbb{R})$.

(iii) If $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$, *then* $0 < \alpha\beta < 1$.

After focusing on the necessary conditions, we present some sufficient conditions for a Gabor frame where the function g is continuous and compactly supported. Let

$$G(x) = \sum_{n\in\mathbb{Z}} \left|g(x-\alpha n)\right|^2.$$

For a compactly supported function g, it suffices to ensure that G is bounded below and above for some $\alpha > 0$, to establish that a Gabor system forms a frame for sufficiently small values of β. Furthermore, explicit expressions for the frame operator and its inverse are derived in this scenario.

Proposition 1.8 (Corollary 11.4.5, [23]). *Let* $\alpha, \beta > 0$. *Suppose that* $g \in L^2(\mathbb{R})$ *has support in an open interval of length* $1/\beta$ *and that the function* G *satisfies* (1.2.2) *for some* $A, B > 0$. *Then* $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B. *The frame operator and its inverse are given by*

$$Sf = \frac{G}{\beta}f, \quad S^{-1}f = \frac{\beta}{G}f, \quad f \in L^2(\mathbb{R}).$$

The following sufficient condition is even more explicit.

Proposition 1.9 (Corollary 11.4.6, [23]). *Suppose that* $g \in L^2(\mathbb{R})$ *is a continuous function with support on an interval* I *and that* $|g(x)| > 0$ *on the interior of* I. *Then* $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *is a frame for all* $(\alpha, \beta) \in (0, |I|) \times (0, \frac{1}{|I|})$.

We conclude our excursion through Gabor analysis by considering one of the most significant results in this area, which is commonly referred to as the "duality principle". The duality principle is concerned with the connection between the frame properties of the window function g in relation to the lattice $\{(\alpha n, \beta l)\}_{n,l\in\mathbb{Z}}$, and its counterpart concerning the adjoint lattice $\{(\frac{n}{\beta}, \frac{l}{\alpha})\}_{n,l\in\mathbb{Z}}$. This relation was proved independently between 1995 and 1997 in the work of three distinct groups of researchers: Janssen [49], Daubechies, Landau, and Landau [27], and Ron and Shen [63].

Theorem 1.10 (Duality Principle). *Let* $g \in L^2(\mathbb{R})$ *and* $\alpha, \beta > 0$ *be given. Then the following are equivalent:*

(i) $\{M_{\beta l}T_{\alpha n}g\}_{n,l\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B;

(ii) $\{M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}g\}_{n,l\in\mathbb{Z}}$ *is a Riesz sequence with bounds* $\alpha\beta A, \alpha\beta B$.

1.3 Wirtinger's inequality and Bernstein's inequality

The proofs of the main results of the next sections of this chapter and the forthcoming ones lie in the utilization of two fundamental tools: Wirtinger's inequality and Bernstein's inequality.

The Wirtinger's inequality is a standard tool in the proofs of sampling theorems [8, 38, 71].

Lemma 1.11 (Wirtinger's inequality). *If $f, f' \in L^2(a,b)$, $a < c < b$, and $f(c) = 0$, then*

$$\int_a^b |f(x)|^2 dx \leq \frac{4}{\pi^2} \max\{(b-c)^2, (c-a)^2\} \int_a^b |f'(x)|^2 dx. \tag{1.3.1}$$

Lemma 1.11 follows from [47, p.184], by applying a change of variables.

The following lemma provides a bound for the L^2-norm of a trigonometric polynomial over an interval.

Lemma 1.12. *Let P be a trigonometric polynomial of degree n, i.e., $P(x) = \sum_{|l| \leq n} c_l e^{2\pi i \beta l x}$ with $\beta > 0$ and let $I \subset \mathbb{R}$ be an interval. Then*

$$\|P\chi_I\|_2 \leq \sqrt{\lceil \beta |I| \rceil} \|P\|_2,$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$.

Proof. P is a trigonometric polynomial of period $1/\beta$. Assume $1/\beta \geq |I|$, then $\beta|I| \leq 1$ and $\lceil \beta|I| \rceil = 1$. Hence,

$$\|P\chi_I\|_2^2 = \int_I |P(x)|^2 dx \leq \int_0^{1/\beta} |P(x)|^2 dx = \|P\|_2^2.$$

Assume $1/\beta < |I|$, then $|I|/(1/\beta) \leq \lceil \beta|I| \rceil$ and $\|P\chi_I\|_2 \leq \sqrt{\lceil \beta|I| \rceil}\|P\|_2$. □

Throughout this thesis, whenever we write $\|P\|_2$ the integral is to be considered over the period of the trigonometric polynomial P. In all the other cases, where the integral is taken over a different interval, the interval is explicitly written.

The following lemma is a modification of Bernstein's inequality, which provides a bound for the derivative of a trigonometric polynomial.

Lemma 1.13 (Bernstein's inequality). *Let P be a trigonometric polynomial of degree n, i.e., $P(x) = \sum_{|l| \leq n} c_l e^{2\pi i \beta l x}$ with $\beta > 0$ and let $I \subset \mathbb{R}$ be an interval. Then*

$$\|P'\chi_I\|_2 \leq 2\pi\beta n \sqrt{\lceil \beta|I| \rceil} \|P\|_2. \tag{1.3.2}$$

Proof. By Lemma 1.12, we have $\|P'\chi_I\|_2 \leq \sqrt{\lceil \beta|I| \rceil}\|P'\|_2$. Then

$$\begin{aligned}
\int_I |P'(x)|^2 dx &\leq \lceil \beta|I| \rceil \int_0^{1/\beta} |P'(x)|^2 dx \\
&= \lceil \beta|I| \rceil \int_0^{1/\beta} (2\pi i\beta)(-2\pi i\beta) \sum_{|l|\leq n} \sum_{|l'|\leq n} ll' c_l c_{l'} e^{2\pi i\beta(l-l')x} dx \\
&= 4\pi^2\beta^2 \lceil \beta|I| \rceil \int_0^{1/\beta} \sum_{|l|\leq n} \sum_{|l'|\leq n} ll' c_l c_{l'} e^{2\pi i\beta(l-l')x} dx \\
&\leq 4\pi^2\beta^2 n^2 \lceil \beta|I| \rceil \int_0^{1/\beta} |P(x)|^2 dx \\
&= 4\pi^2\beta^2 n^2 \lceil \beta|I| \rceil \|P\|_2^2.
\end{aligned}$$

□

1.4 Reproducing kernel Hilbert spaces

In this section, we recall the definition of reproducing kernel Hilbert spaces and we prove a sufficient condition for an orthonormal system to generate a reproducing kernel Hilbert space. Moreover, we define a set of sampling and we show two lemmas that we apply to show the sufficient condition for sampling theorems of this dissertation.

Definition 1.14. A closed subspace $\mathcal{K} \subseteq L^2(\mathbb{R})$ is called a **reproducing kernel Hilbert space** if the point evaluation $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{K}$ for every $x \in \mathbb{R}$. In this case there exists a family $k_x \in \mathcal{K}$, $x \in \mathbb{R}$, such that $f(x) = \langle f, k_x \rangle$. The function $k(x,y) = \overline{k_x(y)}$ is called the **reproducing kernel** of $\mathcal{K}$.

If $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{K}$, then we have an explicit formula for the kernel of the type

$$k(x,y) = \sum_{n\in\mathbb{Z}} \overline{e_n(y)} e_n(x),$$

with pointwise convergence of the series.

The following lemma provides a sufficient condition for an orthonormal system to generate a reproducing kernel Hilbert space.

Proposition 1.15. *Let $\{e_n : n \in \mathbb{Z}\}$ be a orthonormal system for $L^2(\mathbb{R})$ such that $e_n \in \mathcal{C}(\mathbb{R})$ for every $n \in \mathbb{Z}$ and $\sum_{n\in\mathbb{Z}} |e_n(x)|^2 \leq N^2$ for every $x \in \mathbb{R}$. Then $\mathcal{K} = \operatorname{span}\{e_n : n \in \mathbb{Z}\}$ forms a reproducing kernel Hilbert space.*

Proof. Since $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal system for $L^2(\mathbb{R})$, then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{K} = \operatorname{span}\{e_n : n \in \mathbb{Z}\}$. Therefore $\mathcal{K}$ is a closed subspace of $L^2(\mathbb{R})$. We show that there exists $N > 0$ such that $|f(x)| \leq N\|f\|_2$ for every $x \in \mathbb{R}$ and every $f \in \mathcal{K}$.

Let $x \in \mathbb{R}$ and $f \in \mathcal{K}$, then expanding f in terms of the basis elements, we get

$$|f(x)| = \Big|\sum_{n\in\mathbb{Z}} c_n e_n(x)\Big| = \Big|\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n(x)\Big|.$$

We apply Cauchy-Schwarz, to obtain

$$|f(x)| \le \Big(\sum_{n\in\mathbb{Z}} \big|\langle f, e_n\rangle\big|^2\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}} \big|e_n(x)\big|^2\Big)^{1/2}.$$

Since $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{K}$, we have that

$$\Big(\sum_{n\in\mathbb{Z}} \big|\langle f, e_n\rangle\big|^2\Big)^{1/2} = \|f\|_2.$$

Moreover, $\sum_{n\in\mathbb{Z}}|e_n(x)|^2 \le N^2$ for every $x \in \mathbb{R}$. Therefore $|f(x)| \le N\|f\|_2$ for every $x \in \mathbb{R}$ and for every $f \in \mathcal{K}$, and $\mathcal{K}$ is a reproducing kernel Hilbert space. □

Reproducing kernel Hilbert spaces turn out to be the right spaces to study sampling theory. Indeed, sampling theorems are nowadays formulated in terms of sampling inequalities in reproducing kernel Hilbert spaces.

Definition 1.16. Let $\mathcal{K} \subset L^2(\mathbb{R})$ be a reproducing kernel Hilbert space. A set $\Lambda \subset \mathbb{R}$ is a **set of (stable) sampling** for $\mathcal{K}$ if there exist constants $A, B > 0$ such that for every $f \in \mathcal{K}$

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda}|f(\lambda)|^2 \le B\|f\|^2. \tag{1.4.1}$$

The sampling inequality (1.4.1) is equivalent of $\{k_\lambda\}_{\lambda\in\Lambda}$ being a frame for $\mathcal{K}$. Moreover, the inequality reveals that the information carried by the function is captured by the evaluation of the function at the samples.

The following lemma is the starting point of any sufficient condition for sampling theorem that is proved in this thesis.

Lemma 1.17. *Let $\mathcal{K} \subseteq L^2(\mathbb{R})$ be a reproducing kernel Hilbert space and $f \in \mathcal{K}$. Let $\{\alpha_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence such that $\lim_{k\to\pm\infty}\alpha_k = \pm\infty$. Let $\Lambda \subset \mathbb{R}$ be the set of points labeled as*

$$\Lambda = \big\{x_{k,j} \in [\alpha_k, \alpha_{k+1}) : \ k \in \mathbb{Z},\ j = 1, \dots, j_{\max(k)}\big\}, \tag{1.4.2}$$

where $x_{k,j} < x_{k,j+1}$, for every $k \in \mathbb{Z}$ and $j = 1, \dots, j_{\max(k)} - 1$, and set

$$\delta_k = \sup_{j=1,\dots,j_{\max(k)}-1} (x_{k,j+1} - x_{k,j}).$$

For each $k \in \mathbb{Z}$, we define

$$y_{k,j} = \frac{x_{k,j} + x_{k,j+1}}{2}, \quad \textit{for } j = 1, \dots, j_{\max(k)} - 1,$$

$y_{k,0} = \alpha_k$, $y_{k,j_{\max(k)}} = \alpha_{k+1}$, and $\chi_{k,j} = \chi_{[y_{k,j-1}, y_{k,j})}$. Let $x_{k,1} - \alpha_k \leq \frac{1}{2}\delta_k$ and $\alpha_{k+1} - x_{k,j_{\max(k)}} \leq \frac{1}{2}\delta_k$ and define the operator A as

$$Af = P\Big(\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} f(x_{k,j})\chi_{k,j}\Big), \tag{1.4.3}$$

where P is the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{K}$. Then for every $f \in \mathcal{K}$

$$\|f - Af\|_2^2 \leq \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx.$$

Figure 1.1 represents an approximation operator of the type (1.4.3).

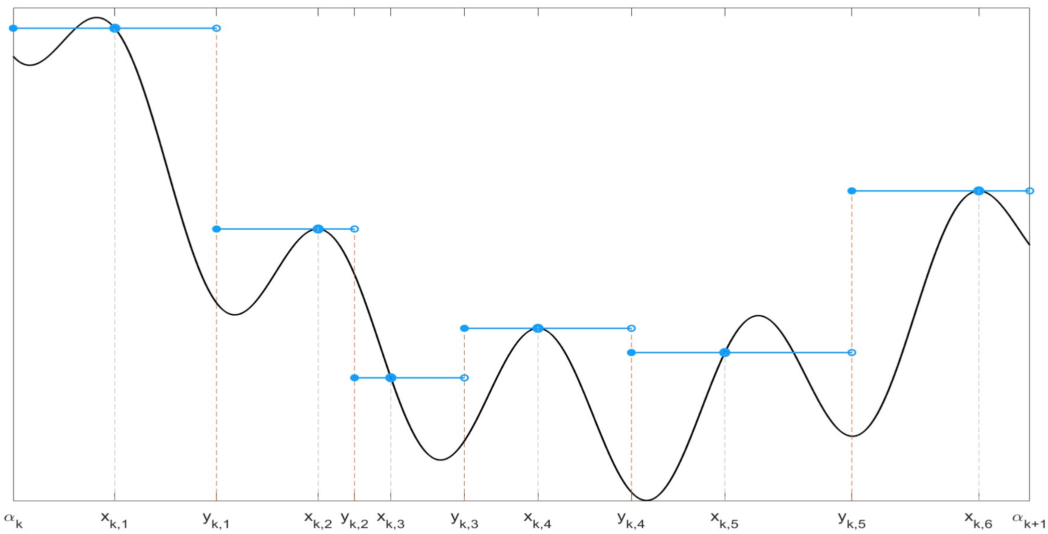


Figure 1.1: Example of an approximation operator of the type (1.4.3). In black is a function f and in blue is drawn the piecewise constant approximation function Af constructed from the samples $\{x_{k,j}\}$.

Proof. Let $f \in \mathcal{K}$. Since $f = Pf = P(\sum_{k,j} f\chi_{k,j})$ and the characteristic functions $\chi_{k,j}$ have mutually disjoint support, we have

$$\begin{aligned}
\|f - Af\|_2^2 &= \Big\| P\Big(\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} (f - f(x_{k,j}))\chi_{k,j} \Big) \Big\|_2^2 \\
&\le \Big\| \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} (f - f(x_{k,j}))\chi_{k,j} \Big\|_2^2 \\
&= \int_{\mathbb{R}} \Big| \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} (f(x) - f(x_{k,j}))\chi_{k,j}(x) \Big|^2 dx \\
&= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} \Big| (f(x) - f(x_{k,j}))\chi_{k,j}(x) \Big|^2 dx.
\end{aligned}$$

The sums converge absolutely, hence we can interchange summation and integration. Thus we can write

$$\|f - Af\|_2^2 \le \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} \int_{y_{k,j-1}}^{y_{k,j}} |f(x) - f(x_{k,j})|^2 dx.$$

Since by construction $x_{k,j} \in [y_{k,j-1}, y_{k,j})$ and $|f(x) - f(x_{k,j})| = 0$ for $x = x_{k,j}$, we apply Wirtinger's inequality (1.3.1) to get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} \int_{y_{k,j-1}}^{y_{k,j}} |f(x) - f(x_{k,j})|^2 dx \\
&\qquad \le \frac{4}{\pi^2} \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} \max\Big\{ (x_{k,j} - y_{k,j-1})^2, (y_{k,j} - x_{k,j})^2 \Big\} \int_{y_{k,j-1}}^{y_{k,j}} |f'(x)|^2 dx.
\end{aligned}$$

We note that $y_{k,j} - x_{k,j} \le \delta_k/2$ and $x_{k,j} - y_{k,j-1} \le \delta_k/2$ for every $j = 1, \dots, j_{\max(k)}$. Hence,

$$\begin{aligned}
\|f - Af\|_2^2 &\le \frac{4}{\pi^2} \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} \max\Big\{ (x_{k,j} - y_{k,j-1})^2, (y_{k,j} - x_{k,j})^2 \Big\} \int_{y_{k,j-1}}^{y_{k,j}} |f'(x)|^2 dx \\
&\le \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \sum_{j=1}^{j_{\max(k)}} \int_{y_{k,j-1}}^{y_{k,j}} |f'(x)|^2 dx \\
&= \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx.
\end{aligned}$$

□

The last lemma of this chapter is the final step for all the sufficient conditions for sampling that are proved in this thesis. The lemma uses Neumann series to show that the approximation operator (1.4.3) is invertible and to give a weighted sampling inequality for the reproducing kernel Hilbert space.

Lemma 1.18. *Let $\mathcal{K} \subseteq L^2(\mathbb{R})$ be a reproducing kernel Hilbert space and $f \in \mathcal{K}$. Let Λ be as in* (1.4.2) *and the approximation operator A as in* (1.4.3). *If there exists a constant $\gamma < 1$ such that $\|f - Af\|_2 \le \gamma \|f\|_2$ and $\|f - \sum_{k,j} f(x_{k,j})\chi_{k,j}\|_2 \le \gamma \|f\|_2$, then Λ is a sampling set for $\mathcal{K}$ and satisfies a weighted sampling inequality*

$$(1-\gamma)^2 \|f\|_2^2 \le \sum_{k,j} |f(x_{k,j})|^2 w_{k,j} \le (1+\gamma)^2 \|f\|_2^2, \qquad \text{for all } f \in \mathcal{K}\,, \tag{1.4.4}$$

where $w_{k,j} = \int \chi_{k,j}$. As a consequence, every $f \in \mathcal{K}$ can be reconstructed completely and stably from the samples in Λ.

Proof. The operator A maps $\mathcal{K}$ to $\mathcal{K}$ and the inequality $\|f - Af\|_2 \le \gamma \|f\|_2$ implies that the operator A is invertible on $\mathcal{K}$ with an inverse A^{-1}.

The lower sampling inequality in (1.4.4), equivalently, the stability of the reconstruction, follows from

$$\|f\|_2 = \|A^{-1}Af\|_2^2 \le \|A^{-1}\|_{\mathrm{op}}^2 \|P\|_{\mathrm{op}}^2 \Big\| \sum_{k,j} f\big(x_{k,j}\big)\chi_{k,j} \Big\|_2^2.$$

Since $\|P\|_{\mathrm{op}} \le 1$ and applying Neumann series, we have

$$\|A^{-1}\|_{\mathrm{op}} = \Big\| \sum_{n=0}^{\infty} (\mathrm{Id} - A)^n \Big\|_{\mathrm{op}} \le \sum_{n=0}^{\infty} \|\mathrm{Id} - A\|_{\mathrm{op}}^n = \frac{1}{1-\gamma}.$$

Therefore,

$$\|f\|_2^2 \le \frac{1}{(1-\gamma)^2} \Big\| \sum_{k,j} f\big(x_{k,j}\big)\chi_{k,j} \Big\|_2^2 = \frac{1}{(1-\gamma)^2} \sum_{k,j} |f(x_{k,j})|^2 w_{k,j}\,, \tag{1.4.5}$$

where the weights are given by $w_{k,j} = \int \chi_{k,j}$.

To get the right-hand side of the inequality (1.4.4) we use the last equality in (1.4.5) and triangle inequality to obtain

$$\sum_{k,j} |f(x_{k,j})|^2 w_{k,j} = \Big\| \sum_{k,j} f\big(x_{k,j}\big)\chi_{k,j} \Big\|_2^2 \le \Big(\|f\|_2 + \Big\| \sum_{k,j} f\big(x_{k,j}\big)\chi_{k,j} - f \Big\|_2 \Big)^2.$$

Since $\|f - \sum_{k,j} f(x_{k,j})\chi_{k,j}\|_2 \le \gamma \|f\|_2$, we get

$$\sum_{k,j} |f(x_{k,j})|^2 w_{k,j} \le (1+\gamma)^2 \|f\|_2^2. \tag{1.4.6}$$

□

To study necessary density conditions for sampling we make use of the theory of sampling in reproducing kernel Hilbert spaces and Theorem 1.1 of [34]. We formulate a simplified version of this theorem that is adapted to $\mathbb{R}$ with the Lebesgue measure.

Theorem 1.19 (Theorem 1.1, [34])**.** *Let $\mathcal{K} \subseteq L^2(\mathbb{R})$ be a reproducing kernel Hilbert space with a reproducing kernel $k(x, y)$ satisfying*

(i) $\inf_{x\in\mathbb{R}} k(x, x) > 0$ *and*

(ii) off-diagonal decay of the form $|k(x, y)| \le N(1+|x-y|)^{-1-\epsilon}$ for all $x, y \in \mathbb{R}$ and some $\epsilon > 0$.

If for $\Lambda \subset \mathbb{R}$ there exist $A, B > 0$ such that for every $f \in \mathcal{K}$

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le B\|f\|^2,$$

then

$$D^-(\Lambda) := \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(\Lambda \cap [x-r, x+r])}{2r} \ge \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2r} \int_{x-r}^{x+r} k(y, y)dy.$$

The number $D^-(\Lambda) := \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(\Lambda\cap[x-r,x+r])}{2r}$ is the **Beurling density** of Λ.

2 Variable bandwidth via Gabor Riesz sequences

In this chapter, we introduce the first version of spaces of variable bandwidth. Our approach involves employing a discrete variant of Aceska and Feichtinger's space [3, 4], and substituting the continuous short-time Fourier transform with the truncation of a Gabor Riesz sequence in the frequency plane. In this spirit let us first define our model of variable bandwidth using a Gabor Riesz sequence.

Definition 2.1. Let $g \in L^2(\mathbb{R})$ be such that the Gabor system $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ constitutes a Riesz sequence for $L^2(\mathbb{R})$ with bounds A' and B'. Let $b : \mathbb{Z} \to \mathbb{N}$ be a bounded positive sequence such that $b(n) \leq B < \infty$, for every $n \in \mathbb{Z}$. We define the **space of variable bandwidth generated by the Gabor Riesz sequence** $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ by

$$V_b^2(g,\mathbb{R}) \coloneqq \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{|l|\leq b(n)} c_{n,l} M_{\beta l}T_{\alpha n}g, \text{ where } c \in \ell^2(\mathbb{Z}\times\mathbb{Z}) \Big\}. \qquad (2.0.1)$$

The sequence b represents the time-varying frequency truncation and assuming that the function g has compact support, we can also interpret the parameter $\beta b(n)$ as the local bandwidth of the function f within an interval centered at $\alpha n + \alpha/2$. In other words, the highest frequency component within the interval $[\alpha n, \alpha(n+1)]$ is expected to be $\beta b(n)$. We anticipate that we will require a minimum of $2b(n)+1$ samples to faithfully reconstruct the function f within each interval $[\alpha n, \alpha(n+1)]$. This intuition holds true when we use the Gabor orthonormal basis $\{e^{2\pi i l x}\chi_{[0,1]}(x-n) : n, l \in \mathbb{Z}\}$ in place of a Gabor Riesz sequence. However, we must refine this understanding because the interval $[\alpha n, \alpha(n+1)]$ is contained in the support of several translated versions of the window function g.

Remark. We could have chosen different formulations to construct spaces of variable bandwidth using Gabor systems. Each of them comes with advantages and drawbacks. Assume $V_b^2(g,\mathbb{R})$ is defined such that $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. The truncation of a Gabor frame of the type $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ to the set of indices $\Gamma = \{(n,l) : n \in \mathbb{Z}, |l| \leq b(n)\}$ does not make the system $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},|l|\leq b(n)}$ a frame for $\text{span}\{M_{\beta l}T_{\alpha n}g : n \in \mathbb{Z}, |l| \leq b(n)\}$ and therefore $V_b^2(g,\mathbb{R})$ is not a closed subspace of $L^2(\mathbb{R})$. Hence, $V_b^2(g,\mathbb{R})$ defined in this way is not a reproducing kernel Hilbert space.

On the other hand, if we take $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},|l|\leq b(n)}$ to be a frame for $\overline{\text{span}}\{M_{\beta l}T_{\alpha n}g : n \in \mathbb{Z}, |l| \leq b(n)\}$ and $V_b^2(g,\mathbb{R}) = \overline{\text{span}}\{M_{\beta l}T_{\alpha n}g : n \in \mathbb{Z}, |l| \leq b(n)\}$, we have that $V_b^2(g,\mathbb{R})$ is closed in $L^2(\mathbb{R})$. Unfortunately, the sufficient condition for sampling theorems of this chapter relies on the possibility of having nice window functions g to construct the variable bandwidth space and this is not guaranteed when $V_b^2(g,\mathbb{R})$ is defined in this way.

Therefore, the selection of a Gabor Riesz sequence is not arbitrary; it was motivated by the possibility of constructing a closed subspace of $L^2(\mathbb{R})$ with compactly supported and $\mathcal{C}^1(\mathbb{R})$ window functions g.

Recall that these distinctions are significant in infinite-dimensional spaces but they are not in numerical experiments. In fact, in finite-dimensional spaces, we have that every finite sequence is a frame for its span. In addition, by Linnell [55], the truncation of a Gabor system $\{M_{\beta l}T_{\alpha n}g\}_{(n,l)\in\mathcal{F}}$, where $\mathcal{F}$ is a finite subset of $\mathbb{Z}^2$, is linear independent and we get that a linear independent frame is a Riesz basis for its span. This observation motivates the use of Gabor frames rather than Gabor Riesz sequences to test numerically our results in Section 5.5.

The main goal of this chapter is to investigate the question under which conditions on a set of points Λ a function in $V_b^2(g,\mathbb{R})$ can be reconstructed completely. This implies showing that Λ is a sampling set for $V_b^2(g,\mathbb{R})$, which is equivalent to proving the inequality (1.4.1) for every $f \in V_b^2(g,\mathbb{R})$. To solve this problem, we use a modification of the method proposed in 1992 by Gröchenig [38] and later called the **adaptive weight method** in the work by Feichtinger and Gröchenig [32]. This method relies on the properties of the window function g and, in particular, the requirement of g being compactly supported is essential.

The chapter is organized as follows. Section 2.1 contains the first version of the sufficient condition for sampling theorem for $V_b^2(g,\mathbb{R})$ where the space is generated by a specific compactly supported window function g and by a translation parameter $\alpha = 1$. In Section 2.2, the second variation of the sufficient condition for sampling theorem for $V_b^2(g,\mathbb{R})$ is presented. In this scenario, a more general continuous and compactly supported function g is considered and the translation parameter is not fixed. Section 2.3 provides a discussion and a comparison of the two main results of the chapter.

We start by showing that $V_b^2(g,\mathbb{R})$ is a reproducing kernel Hilbert space.

Lemma 2.2. *Let $g \in \mathcal{C}(\mathbb{R})$ with compact support and $V_b^2(g,\mathbb{R})$ be the space of variable bandwidth generated by the Gabor Riesz sequence $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ defined in* (2.0.1)*. Then $V_b^2(g,\mathbb{R})$ is a reproducing kernel Hilbert space.*

Proof. Since $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},l\in\mathbb{Z}}$ is a Riesz basis for its closed span and therefore the truncated Gabor system $\{M_{\beta l}T_{\alpha n}g\}_{n\in\mathbb{Z},|l|\leq b(n)}$ is a Riesz basis for $\operatorname{span}\{M_{\beta l}T_{\alpha n}g : n \in \mathbb{Z}, |l| \leq b(n)\}$. Hence, $V_b^2(g,\mathbb{R}) = \operatorname{span}\{M_{\beta l}T_{\alpha n}g : n \in \mathbb{Z},\ |l| \leq b(n)\}$ is a closed subspace of $L^2(\mathbb{R})$.

We show that there exists $N > 0$ such that $|f(x)| \leq N\|f\|_2$ for every $x \in \mathbb{R}$ and every $f \in V_b^2(g,\mathbb{R})$. Let $x \in \mathbb{R}$ and $f \in V_b^2(g,\mathbb{R})$, then by Theorem 1.3, there exists a unique biorthogal sequence $\{h_{n,l}\}_{n\in\mathbb{Z},|l|\leq b(n)}$ such that $\{h_{n,l}\}_{n\in\mathbb{Z},|l|\leq b(n)}$ is a Riesz basis for $V_b^2(g,\mathbb{R})$ with upper bound $1/A'$. Then

$$|f(x)| = \Big|\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} c_{n,l}M_{\beta l}T_{\alpha n}g(x)\Big| = \Big|\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} \langle f, h_{n,l}\rangle M_{\beta l}T_{\alpha n}g(x)\Big|.$$

We apply Cauchy-Schwarz, to obtain

$$|f(x)| \leq \Big(\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} \big|\langle f, h_{n,l}\rangle\big|^2\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} \big|M_{\beta l}T_{\alpha n}g(x)\big|^2\Big)^{1/2}. \tag{2.0.2}$$

As a bound for the first sum, we get

$$\Big(\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} \big|\langle f, h_{n,l}\rangle\big|^2\Big)^{1/2} \leq \frac{1}{\sqrt{A'}}\|f\|_2.$$

We find a bound for the second term of (2.0.2). Since g is continuous and compactly supported, there exists a constant $C > 0$ such that $\sum_{n\in\mathbb{Z}}|T_{\alpha n}g(x)|^2 \leq C$. Hence,

$$\sum_{n\in\mathbb{Z}}\sum_{|l|\leq b(n)} \big|M_{\beta l}T_{\alpha n}g(x)\big|^2 \leq \sum_{n\in\mathbb{Z}}(2b(n)+1)\big|T_{\alpha n}g(x)\big|^2 \leq C(2B+1).$$

Then $|f(x)| \leq \sqrt{\frac{C(2B+1)}{A'}}\|f\|_2$ for every $x \in \mathbb{R}$ and for every $f \in V_b^2(g,\mathbb{R})$. Hence, we have shown that $V_b^2(g,\mathbb{R})$ is a reproducing kernel Hilbert space. □

2.1 Sufficient conditions for sampling - first version

In this section, we explore the problem of finding sufficient conditions for sampling for the space $V_b^2(g,\mathbb{R})$. In particular, we show under which conditions on a set of points Λ is a signal uniquely defined by its samples. The sufficient conditions for sampling theorem of this section focuses on a space of variable bandwidth $V_b^2(g,\mathbb{R})$ generated by a class of window functions with a specific compact support and a translation parameter $\alpha = 1$. This specificity highlights the influence of neighboring windows in the reconstruction and it helps to understand the spaces of variable bandwidth.

We start by introducing a convenient labeling for the set of points Λ as follows:

$$\Lambda = \big\{x_{k,j} \in \mathbb{R} : x_{k,j} = k + \eta_{k,j},\ k \in \mathbb{Z}, j = 1,\dots,j_{\max(k)},\ \text{and } \eta_{k,j} \in [0,1)\big\}. \tag{2.1.1}$$

The sampling points in Λ are ordered by magnitude, i.e.,

$$\cdots < x_{k,j} < x_{k,j+1} < \cdots < x_{k+1,j} < x_{k+1,j+1} < \dots, \quad j = 1,\dots,j_{\max(k)} - 1 \quad .$$

The sampling density is measured by the maximal gap between the sample, i.e.,

$$\delta_k = \sup_{j=1,\dots,j_{\max(k)}-1}(x_{k,j+1} - x_{k,j}),$$

where the number $j_{\max(k)}$ depends on the particular interval $[k, k+1)$.

Let $I_k = [k, k+1)$. By the definition above, $x_{k,j} \in I_k$ for every $j = 1,\dots,j_{\max(k)}$. For each $k \in \mathbb{Z}$, we consider a partition of I_k of the type

$$y_{k,j} = \frac{x_{k,j} + x_{k,j+1}}{2}, \quad \text{for } j = 1,\dots,j_{\max(k)} - 1,$$

$y_{k,0} = k$, and $y_{k,j_{\max(k)}} = k + 1$. Set $\chi_{k,j} = \chi_{[y_{k,j-1}, y_{k,j})}$. Note that if the maximal length gap is δ_k, then we get $y_{k,j} - x_{k,j} \leq \delta_k/2$ and $x_{k,j} - y_{k,j-1} \leq \delta_k/2$.

We state the first version of the sufficient condition for sampling theorem for $V_b^2(g, \mathbb{R})$.

Theorem 2.3 (Sufficient conditions for sampling for Gabor Riesz sequences - first version)**.** *Let $g \in \mathcal{C}^1(\mathbb{R})$ with $\operatorname{supp}(g) \subseteq [-\varepsilon, 1+\varepsilon]$ and $0 < \varepsilon < 1/2$ be such that*

(i) $g(x) = 1$, *for* $x \in [\varepsilon, 1-\varepsilon]$.

(ii) $g(x) = 1 - g(-x)$, *for* $x \in [-\varepsilon, \varepsilon]$.

(iii) $g(2-x) = 1 - g(x)$, *for* $x \in [1-\varepsilon, 1+\varepsilon]$.

(iv) g strictly increasing in $[-\varepsilon, \varepsilon]$.

(v) g strictly dencreasing in $[1-\varepsilon, 1+\varepsilon]$.

Assume $\{M_{\beta l}T_n g\}_{n\in\mathbb{Z}, l\in\mathbb{Z}}$ forms a Riesz sequence for $L^2(\mathbb{R})$ with lower bound A'. Let $V_b^2(g,\mathbb{R})$ be the space of variable bandwidth defined by (2.0.1) *and $\Lambda \subseteq \mathbb{R}$ be as in* (2.1.1) *with $x_{k,1} - k \leq \frac{1}{2}\delta_k$ and $k + 1 - x_{k,j_{\max(k)}} \leq \frac{1}{2}\delta_k$. If*

$$\frac{\lceil \beta \rceil}{\beta} \frac{6}{A'} \sup_{k\in\mathbb{Z}} \Big\{ (\delta_{k-1}^2 + 4\delta_k^2 + \delta_{k+1}^2)\Big(\beta^2 b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\Big)\Big\} < 1, \tag{2.1.2}$$

then Λ is a sampling set for $V_b^2(g,\mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} |f(x_{k,j})|^2 w_{k,j} \leq B\|f\|_2^2, \qquad \text{for all } f \in V_b^2(g,\mathbb{R}), \tag{2.1.3}$$

where $w_{k,j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in V_b^2(g,\mathbb{R})$ can be reconstructed completely and stably from the samples in Λ.

Note that the choice of dividing $\mathbb{R}$ into intervals of the type $[k, k+1)$ is naturally given by the translation parameter $\alpha = 1$. In this way, there is always a window that is centered in the interval.

Proof. The main idea of this construction consists of designing an approximation operator Af of f which contains the evaluation of the function at the sampling points $f(x_{k,j})$ and showing that for all $f \in V_b^2(g,\mathbb{R})$, $\|f - Af\|_2 \leq \gamma\|f\|_2$ with $\gamma < 1$. Then f can be recovered from Af via Neumann series.

Let $f \in V_b^2(g,\mathbb{R})$. We apply Lemma 1.17 with $\mathcal{K} = V_b^2(g,\mathbb{R})$, $[\alpha_k, \alpha_{k+1}) = [k, k+1)$ for every $k \in \mathbb{Z}$, and we take as operator A the approximation operator

$$Af = P\Big(\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} f\big(x_{k,j}\big)\chi_{k,j}\Big).$$

With this choice of A, we have

$$\|f - Af\|_2^2 \le \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_k^{k+1} |f'(x)|^2 dx.$$

Step 1. Estimate of local smoothness $\int_k^{k+1} |f'(x)|^2 dx$.
Let

$$P_n(x) = \sum_{|l|\le b(n)} c_{n,l} e^{2\pi i\beta l x} \tag{2.1.4}$$

be the $1/\beta$-periodic trigonometric polynomial of degree $b(n)$.
By Definition 2.1, every $f \in V_b^2(g, \mathbb{R})$ can be written as

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{|l|\le b(n)} c_{n,l} M_{\beta l} T_n g(x) = \sum_{n\in\mathbb{Z}} \sum_{|l|\le b(n)} c_{n,l} e^{2\pi i\beta l x} g(x-n) = \sum_{n\in\mathbb{Z}} P_n(x) g(x-n).$$

By Plancherel's Theorem, we have that the L^2-norm of P_n over a period of length $1/\beta$ is

$$\|P_n\|_2^2 = \int_0^{1/\beta} |P_n(x)|^2 dx = \frac{1}{\beta} \sum_{|l|\le b(n)} |c_{n,l}|^2$$

and by Theorem 1.3, we get that in this representation of $f \in V_b^2(g, \mathbb{R})$

$$\sum_{n\in\mathbb{Z}} \|P_n\|_2^2 = \frac{1}{\beta} \sum_{n\in\mathbb{Z}} \sum_{|l|\le b(n)} |c_{n,l}|^2 \le \frac{1}{A'\beta} \|f\|_2^2\,. \tag{2.1.5}$$

Fix $x \in I_k$ and write $x = k + \eta$, with $\eta \in [0, 1)$. Then

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(k+\eta) g(k+\eta-n).$$

Since g is compactly supported, only a finite number of translates of g are different from 0, i.e., $n = k$ and $n = k \pm 1$. We get

$$f(x) = \sum_{n=k-1}^{k+1} P_n(k+\eta)g(k+\eta-n) = P_k(k+\eta)g(\eta)+P_{k-1}(k+\eta)g(1+\eta)+P_{k+1}(k+\eta)g(-1+\eta)$$

with

- $g(1+\eta) \ne 0, \quad$ only if $\eta \in [0, \varepsilon]$.
- $g(-1+\eta) \ne 0, \quad$ only if $\eta \in [1-\varepsilon, 1]$.

Consider now the derivative of $f(x)$, then

$$\begin{aligned} f'(x) = {} & P_k'(k+\eta)g(\eta) + P_k(k+\eta)g'(\eta)\chi_{[0,\varepsilon]\cup[1-\varepsilon,1]}(\eta) \\ & + P_{k-1}'(k+\eta)g(1+\eta)\chi_{[0,\varepsilon]}(\eta) + P_{k-1}(k+\eta)g'(1+\eta)\chi_{[0,\varepsilon]}(\eta) \\ & + P_{k+1}'(k+\eta)g(-1+\eta)\chi_{[1-\varepsilon,1]}(\eta) + P_{k+1}(k+\eta)g'(-1+\eta)\chi_{[1-\varepsilon,1]}(\eta). \end{aligned}$$

We bound the L^2-norm of each term. We start by finding a bound for the terms containing the derivative of the trigonometric polynomial. Note that by the properties (i), (ii), (iii), (iv), and (v) of g, we have

$$\sup_{x\in[0,1]} |g(x)| = 1 \quad \text{and} \quad \sup_{x\in[0,\varepsilon]} |g(1+x)| = \sup_{x\in[1-\varepsilon,1]} |g(-1+x)| = 1/2.$$

Applying Bernstein's inequality (1.3.2) with $|I| = 1$, we bound the terms containing the derivative of the trigonometric polynomials in the following way:

- $\|P_k' g \chi_{[0,1]}\|_2^2 \le \Big(\sup_{x\in[0,1]} |g(x)|\Big)^2 \|P_k' \chi_{[0,1]}\|_2^2 \le \lceil\beta\rceil (2\pi\beta b(k))^2 \|P_k\|_2^2.$
- $\|P_{k-1}' g(1+\cdot)\chi_{[0,\varepsilon]}\|_2^2 \le \Big(\sup_{x\in[0,\varepsilon]} |g(1+x)|\Big)^2 \|P_{k-1}'\chi_{[0,1]}\|_2^2 \le \dfrac{\lceil\beta\rceil}{4}(2\pi\beta b(k-1))^2 \|P_{k-1}\|_2^2.$
- $$\|P_{k+1}' g(-1+\cdot)\chi_{[1-\varepsilon,1]}\|_2^2 \le \Big(\sup_{x\in[1-\varepsilon,1]} |g(-1+x)|\Big)^2 \|P_{k+1}'\chi_{[0,1]}\|_2^2 \le \frac{\lceil\beta\rceil}{4}(2\pi\beta b(k+1))^2 \|P_{k+1}\|_2^2.$$

We now find a bound for the terms involving the derivative of the window function g.

- $\|P_k g' \chi_{[0,\varepsilon]\cup[1-\varepsilon,1]}\|_2^2 \le \lceil\beta\rceil \|g'\chi_{[0,\varepsilon]\cup[1-\varepsilon,1]}\|_\infty^2 \|P_k\|_2^2 \le \lceil\beta\rceil \|g'\|_\infty^2 \|P_k\|_2^2.$
- $\|P_{k-1} g'(1+\cdot)\chi_{[0,\varepsilon]}\|_2^2 \le \lceil\beta\rceil \|g'\|_\infty^2 \|P_{k-1}\|_2^2.$
- $\|P_{k+1} g'(-1+\cdot)\chi_{[1-\varepsilon,1]}\|_2^2 \le \lceil\beta\rceil \|g'\|_\infty^2 \|P_{k+1}\|_2^2.$

Step 2. Final estimate for $\|f - Af\|_2$.
Recalling that $|\sum_{i\in F} f_i|^2 \le \#F \sum_{i\in F} |f_i|^2$, we have

$$\begin{aligned}
\frac{\delta_k^2}{\pi^2}\int_k^{k+1} |f'(x)|^2 dx \le\ & 6\lceil\beta\rceil \frac{\delta_k^2}{\pi^2}\Big\{ \|P_k\|_2^2 \Big[(2\pi\beta b(k))^2 + \|g'\|_\infty^2\Big] \\
& + \|P_{k-1}\|_2^2 \Big[(\pi\beta b(k-1))^2 + \|g'\|_\infty^2\Big] \\
& + \|P_{k+1}\|_2^2 \Big[(\pi\beta b(k+1))^2 + \|g'\|_\infty^2\Big]\Big\} \\
\le\ & 6\lceil\beta\rceil \frac{\delta_k^2}{\pi^2}\Big\{ 4\|P_k\|_2^2 \Big[(\pi\beta b(k))^2 + \|g'\|_\infty^2\Big] \\
& + \|P_{k-1}\|_2^2 \Big[(\pi\beta b(k-1))^2 + \|g'\|_\infty^2\Big] \\
& + \|P_{k+1}\|_2^2 \Big[(\pi\beta b(k+1))^2 + \|g'\|_\infty^2\Big]\Big\}.
\end{aligned}$$

Summing over all $k \in \mathbb{Z}$, we obtain

$$\|f - Af\|_2^2 \le \sum_{k\in\mathbb{Z}} \Big\{ 6\lceil\beta\rceil (\delta_{k-1}^2 + 4\delta_k^2 + \delta_{k+1}^2)\Big(\beta^2 b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\Big)\|P_k\|_2^2 \Big\}.$$

Define

$$D := \sup_{k\in\mathbb{Z}} \Big\{ 6\lceil\beta\rceil(\delta_{k-1}^2 + 4\delta_k^2 + \delta_{k+1}^2)\Big(\beta^2 b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\Big)\Big\}$$

and recall that $\{M_{\beta l}T_k g\}_{k\in\mathbb{Z},|l|\le b(k)}$ is a Riesz basis for $V_b^2(g,\mathbb{R})$ with lower bound A'.

Then by (2.1.5), we get

$$\|f - Af\|_2^2 \le D\sum_{k\in\mathbb{Z}} \|P_k\|_2^2 \le \frac{D}{\beta A'}\|f\|_2^2.$$

By condition (2.1.2), $\gamma^2 := \frac{D}{\beta A'} < 1$. Therefore,

$$\|f - Af\|_2^2 \le \gamma^2\|f\|_2^2, \quad \text{with } \gamma < 1.$$

It follows that $f = A^{-1}(Af)$ with Af an approximation operator based only on the samples in $\Lambda = \{x_{k,j}\}$. Hence, by Lemma 1.18, f is completely determined by $f(x_{k,j})$ for every $k \in \mathbb{Z}$ and $j = 1, \dots, j_{\max(k)}$, and a weighted sampling inequality of the type (2.1.3) holds with $A = \big(1 - \sqrt{\frac{D}{\beta A'}}\big)^2$ and $B = \big(1 + \sqrt{\frac{D}{\beta A'}}\big)^2$. □

2.2 Sufficient conditions for sampling - second version

An alternative sufficient condition for sampling theorem can be obtained using the same idea for the proof but applying some technical modifications. The adaptive weights method requires the window function g to be differentiable and compactly supported but for the second version of the sufficient condition for sampling theorem, we work with a space of variable bandwidth $V_b^2(g,\mathbb{R})$ generated by a window g with a (possibly) larger support than before. Moreover, we do not require any condition on the translation parameter α.

We show that the sufficient conditions still rely on the properties of g but we see that for a large number of basis elements that play a role in the expansion of a function f at each sample $f(x_{k,j})$, we have that the maximal bandwidth of f over an interval is the correct parameter to determine the density of Λ (or the lowest sampling rate). We present a condition that isolates the maximum gap for each interval of the partition and makes the following result suitable for numerical experiments.

Following the procedure done before, we establish a labeling scheme for the set of points Λ as follows:

$$\Lambda = \big\{x_{k,j} \in \mathbb{R} : x_{k,j} = \alpha k + \alpha\eta_{k,j},\ k \in \mathbb{Z}, j = 1, \dots, j_{\max(k)},\ \text{and } \eta_{k,j} \in [0,1)\big\}. \quad (2.2.1)$$

The points in Λ are arranged in ascending order of magnitude. Since we do not fix a translation parameter α, the partition of $\mathbb{R}$ is in intervals of the type $[\alpha k, \alpha(k+1))$. The density of sampling is defined by the maximal gap length between samples:

$$\delta_k = \sup_{j=1,\dots,j_{\max(k)}-1} (x_{k,j+1} - x_{k,j}),$$

where the $j_{\max(k)}$ depends on the specific interval $[\alpha k, \alpha(k+1))$.

For each $k \in \mathbb{Z}$, we construct a partition of $[\alpha k, \alpha(k+1))$ as follows:

$$y_{k,j} = \frac{x_{k,j} + x_{k,j+1}}{2}, \qquad \text{for } j = 1, \dots, j_{\max(k)} - 1,$$

where $y_{k,0} = \alpha k$ and $y_{k,j_{\max(k)}} = \alpha(k+1)$. Define $\chi_{k,j} = \chi_{[y_{k,j-1}, y_{k,j})}$ and $\chi_k = \chi_{[\alpha k, \alpha(k+1))}$.

It is worth noting that if the maximal gap length is δ_k, then we have $y_{k,j} - x_{k,j} \le \delta_k/2$ and $x_{k,j} - y_{k,j-1} \le \delta_k/2$.

Theorem 2.4 (Sufficient conditions for sampling for Gabor Riesz sequences - second version)**.** *Let $g \in \mathcal{C}^1(\mathbb{R})$ be such that $\operatorname{supp}(g) \subseteq [0, m]$ with $m > \alpha$ and assume $\{M_{\beta l} T_{\alpha n} g\}_{n \in \mathbb{Z}, l \in \mathbb{Z}}$ forms a Riesz sequence for $L^2(\mathbb{R})$ with lower bound A'. Let $V_b^2(g, \mathbb{R})$ be the space of variable bandwidth defined by (2.0.1) and $\Lambda \subseteq \mathbb{R}$ be as in (2.2.1) with $x_{k,1} - \alpha k \le \frac{1}{2}\delta_k$ and $\alpha(k+1) - x_{k,j_{\max(k)}} \le \frac{1}{2}\delta_k$. Define*

$$D := \frac{m}{\alpha} \sqrt{\lceil \alpha\beta \rceil} \max\{2\pi\beta \|g\|_\infty, \|g'\|_\infty\}$$

and

$$\mu_k := \max_{n \in (k - m/\alpha, k+1) \cap \mathbb{Z}} \big(b(n) + 1\big).$$

If

$$\sup_{k \in \mathbb{Z}} \mu_k \delta_k < \frac{\sqrt{A'}\beta\pi}{D}, \tag{2.2.2}$$

then Λ is a sampling set for $V_b^2(g, \mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \le \sum_{k \in \mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} |f(x_{k,j})|^2 w_{k,j} \le B\|f\|_2^2 \qquad \text{for all } f \in V_b^2(g, \mathbb{R})\,, \tag{2.2.3}$$

where $w_{k,j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in V_b^2(g, \mathbb{R})$ can be reconstructed completely and stably from the samples in Λ.

Remark. For this theorem, we have chosen a formulation that separates the properties of the window g, expressed by the constant D, the lower bound A', the local bandwidth μ_k, and the local sampling density δ_k into different constants. In this way, the sufficient condition (2.2.2) looks formally similar to the maximum gap condition for sampling of bandlimited functions, e.g., in [38].

Proof. The main idea of the proof is the same as for Theorem 2.3 but with some technical modifications.

Let $f \in V_b^2(g, \mathbb{R})$. We apply Lemma 1.17 with $\mathcal{K} = V_b^2(g, \mathbb{R})$, a partition in intervals of the type $[\alpha k, \alpha(k+1))$, a set of points Λ as in (2.2.1), and an approximation operator

$$Af = P\Big(\sum_{k \in \mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} f\big(x_{k,j}\big)\chi_{k,j}\Big).$$

We obtain

$$\|f - Af\|_2^2 \leq \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha k}^{\alpha(k+1)} |f'(x)|^2 dx.$$

Step 1. Estimate of local smoothness $\int_{\alpha k}^{\alpha(k+1)} |f'(x)|^2 dx$.
Using the expansion of f in terms of the trigonometric polynomial (2.1.4) we can write

$$f'(x) = \sum_{n\in\mathbb{Z}} \Big(P_n'(x) T_{\alpha n} g(x) + P_n(x) T_{\alpha n} g'(x) \Big).$$

We have that $\mathrm{supp}(g) \subseteq [0, m]$, $x \in [\alpha k, \alpha(k+1))$, and $x - \alpha n \in (0, m)$, then $n \in (k - m/\alpha, k+1) \cap \mathbb{Z}$. We call this set

$$F_k := (k - m/\alpha, k+1) \cap \mathbb{Z}.$$

Then we have

$$\begin{aligned}
&\int_{\alpha k}^{\alpha(k+1)} |f'(x)|^2 dx \\
&\quad = \int_{\alpha k}^{\alpha(k+1)} \Big| \sum_{n\in F_k} \Big(P_n'(x) T_{\alpha n} g(x) + P_n(x) T_{\alpha n} g'(x) \Big) \Big|^2 dx \\
&\quad = \sum_{n,i\in F_k} \int_{\alpha k}^{\alpha(k+1)} \Big(P_n'(x) T_{\alpha n} g(x) + P_n(x) T_{\alpha n} g'(x) \Big) \cdot \overline{\Big(P_i'(x) T_{\alpha i} g(x) + P_i(x) T_{\alpha i} g'(x) \Big)} dx \\
&\quad = \sum_{n,i\in F_k} \int_{\alpha k}^{\alpha(k+1)} \Big(P_n'(x) T_{\alpha n} g(x) \overline{P_i'(x) T_{\alpha i} g(x)} + P_n'(x) T_{\alpha n} g(x) \overline{P_i(x) T_{\alpha i} g'(x)} \\
&\qquad\qquad + P_n(x) T_{\alpha n} g'(x) \overline{P_i'(x) T_{\alpha i} g(x)} + P_n(x) T_{\alpha n} g'(x) \overline{P_i(x) T_{\alpha i} g'(x)} \Big) dx. \qquad (2.2.4)
\end{aligned}$$

We have to bound each term of the equation (2.2.4). Recall that $\chi_k = \chi_{[\alpha k, \alpha(k+1))}$ and consider the first term. We get

$$\int_{\alpha k}^{\alpha(k+1)} \Big(P_n'(x) T_{\alpha n} g(x) \overline{P_i'(x) T_{\alpha i} g(x)} \Big) dx \leq \|T_{\alpha n} g T_{\alpha i} \overline{g}\|_\infty \|P_n' \chi_k\|_2 \|P_i' \chi_k\|_2.$$

We notice that $\|T_{\alpha n} g T_{\alpha i} \overline{g}\|_\infty \leq \|g\|_\infty^2$ and since $|I| = \alpha$, we apply Bernstein's inequality (1.3.2) with $\lceil \beta |I| \rceil = \lceil \alpha\beta \rceil$. Therefore,

$$\begin{aligned}
\int_{\alpha k}^{\alpha(k+1)} \Big(P_n'(x) T_{\alpha n} g(x) \overline{P_i'(x) T_{\alpha i} g(x)} \Big) dx &\leq \|T_{\alpha n} g T_{\alpha i} \overline{g}\|_\infty \|P_n' \chi_k\|_2 \|P_i' \chi_k\|_2 \\
&\leq \|g\|_\infty^2 4\pi^2 \lceil \alpha\beta \rceil \beta^2 b(n) b(i) \|P_n\|_2 \|P_i\|_2.
\end{aligned}$$

Taking the sum over n and i, we obtain

$$\begin{aligned}\sum_{n,i\in F_k}\int_{\alpha k}^{\alpha(k+1)}&\Big(P_n'(x)T_{\alpha n}g(x)\overline{P_i'(x)T_{\alpha i}g(x)}\Big)dx\\&\le 4\pi^2\beta^2\lceil\alpha\beta\rceil\|g\|_\infty^2\sum_{n,i\in F_k}b(n)b(i)\|P_n\|_2\|P_i\|_2\\&=4\pi^2\beta^2\lceil\alpha\beta\rceil\|g\|_\infty^2\Big(\sum_{n\in F_k}b(n)\|P_n\|_2\Big)^2.\end{aligned}$$

Consider now the middle terms of (2.2.4)

$$\begin{aligned}\sum_{n,i\in F_k}\int_{\alpha k}^{\alpha(k+1)}&\Big(P_n'(x)T_{\alpha n}g(x)\overline{P_i(x)T_{\alpha i}g'(x)}+P_n(x)T_{\alpha n}g'(x)\overline{P_i'(x)T_{\alpha i}g(x)}\Big)dx\\&=\sum_{n,i\in F_k}\int_{\alpha k}^{\alpha(k+1)}2\operatorname{Re}\Big\{P_n'(x)T_{\alpha n}g(x)\overline{P_i(x)T_{\alpha i}g'(x)}\Big\}dx.\end{aligned}$$

Applying similar computations as before, we obtain

$$\begin{aligned}2\int_{\alpha k}^{\alpha(k+1)}&\operatorname{Re}\Big\{P_n'(x)T_{\alpha n}g(x)\overline{P_i(x)T_{\alpha i}g'(x)}\Big\}dx\\&\le 2\|T_{\alpha n}gT_{\alpha i}\overline{g'}\|_\infty\|P_n'\chi_k\|_2\|P_i\chi_k\|_2\\&\le 4\pi\beta\lceil\alpha\beta\rceil\|g\|_\infty\|g'\|_\infty b(n)\|P_n\|_2\|P_i\|_2.\end{aligned}$$

Taking the sum over n and i, we have

$$\begin{aligned}\sum_{n,i\in F_k}\int_{\alpha k}^{\alpha(k+1)}&2\operatorname{Re}\Big\{P_n'(x)T_{\alpha n}g(x)\overline{P_i(x)T_{\alpha i}g'(x)}\Big\}dx\\&\le 4\pi\beta\lceil\alpha\beta\rceil\|g\|_\infty\|g'\|_\infty\Big(\sum_{n\in F_k}b(n)\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|P_n\|_2\Big).\end{aligned}$$

A bound for the last term of (2.2.4) is given by

$$\begin{aligned}\sum_{n,i\in F_k}\int_{\alpha k}^{\alpha(k+1)}&\Big(P_n(x)T_{\alpha n}g'(x)\overline{P_i(x)T_{\alpha i}g'(x)}\Big)dx\\&\le\lceil\alpha\beta\rceil\sum_{n,i\in F_k}\|T_{\alpha n}g'T_{\alpha i}\overline{g'}\|_\infty\|P_n\|_2\|P_i\|_2\\&\le\lceil\alpha\beta\rceil\|g'\|_\infty^2\Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2.\end{aligned}$$

We add all terms together.

$$\begin{aligned}\int_{\alpha k}^{\alpha(k+1)} |f'(x)|^2 dx &\le \lceil \alpha\beta \rceil \Big\{ 4\pi^2\beta^2 \|g\|_\infty^2 \Big(\sum_{n\in F_k} b(n)\|P_n\|_2 \Big)^2 + \|g'\|_\infty^2 \Big(\sum_{n\in F_k} \|P_n\|_2 \Big)^2 \\ &\qquad + 4\pi\beta \|g\|_\infty \|g'\|_\infty \Big(\sum_{n\in F_k} b(n)\|P_n\|_2 \Big) \Big(\sum_{n\in F_k} \|P_n\|_2 \Big) \Big\} \\ &= \lceil \alpha\beta \rceil \Big(2\pi\beta \|g\|_\infty \sum_{n\in F_k} b(n)\|P_n\|_2 + \|g'\|_\infty^2 \sum_{n\in F_k} \|P_n\|_2 \Big)^2 \\ &\le \lceil \alpha\beta \rceil \max\{2\pi\beta\|g\|_\infty, \|g'\|_\infty\}^2 \Big(\sum_{n\in F_k} \big(b(n)+1\big)\|P_n\|_2 \Big)^2 .\end{aligned}$$

Summing over all $k \in \mathbb{Z}$, we get

$$\begin{aligned}\|f - Af\|_2^2 &\le \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha k}^{\alpha(k+1)} |f'(x)|^2 dx \\ &\le \frac{\lceil \alpha\beta \rceil}{\pi^2} \max\{2\pi\beta\|g\|_\infty, \|g'\|_\infty\}^2 \sum_{k\in\mathbb{Z}} \delta_k^2 \Big(\sum_{n\in F_k} \|P_n\|_2 \big(b(n)+1\big) \Big)^2 .\end{aligned}$$

We define

$$\mu_k := \max_{n\in(k-m/\alpha,k+1)\cap\mathbb{Z}} \Big(b(n)+1 \Big)$$

and we get

$$\|f - Af\|_2^2 \le \frac{\lceil \alpha\beta \rceil}{\pi^2} \max\{2\pi\beta\|g\|_\infty, \|g'\|_\infty\}^2 \sum_{k\in\mathbb{Z}} \delta_k^2 \mu_k^2 \Big(\sum_{n\in F_k} \|P_n\|_2 \Big)^2 . \tag{2.2.5}$$

Step 2. Final estimate for $\|f - Af\|_2$.
We interpret $\sum_{n\in F_k} \|P_n\|_2$ as a convolution.
Define the sequence $q : \mathbb{Z} \to \mathbb{R}$ and the operator $\chi_{(-1,m/\alpha)}$ as follows:

$$\begin{aligned}&q(n) = \|P_n\|_2, \\ &\chi_{(-1,m/\alpha)} : \mathbb{Z} \to \mathbb{R}, \text{ with } \chi_{(-1,m/\alpha)}(n) = \begin{cases} 1, & n \in (-1, m/\alpha). \\ 0, & \text{elsewhere.} \end{cases}\end{aligned}$$

We have

$$\begin{aligned}\Big(q * \chi_{(-1,m/\alpha)} \Big)(k) &= \sum_{n\in\mathbb{Z}} q(n)\chi_{(-1,m/\alpha)}(k-n) = \sum_{n\in\mathbb{Z}} q(n)\chi_{(k-m/\alpha,k+1)}(n) \\ &= \sum_{n\in F_k} q(n) = \sum_{n\in F_k} \|P_n\|_2 .\end{aligned}$$

We substitute in (2.2.5) to get

$$\|f - Af\|_2^2 \leq \frac{\lceil \alpha\beta \rceil}{\pi^2} \max\{2\pi\beta\|g\|_\infty, \|g'\|_\infty\}^2 \sum_{k\in\mathbb{Z}} \delta_k^2 \mu_k^2 \Big(q * \chi_{(-1,m/\alpha)}(k) \Big)^2.$$

Using Young's inequality and the fact that $\|\chi_{(-1,m/\alpha)}\|_1 = m/\alpha$, the right-hand side can be estimated as

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \delta_k^2 \mu_k^2 \Big(q * \chi_{(-1,m/\alpha)}(k) \Big)^2 &\leq \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \|q * \chi_{(-1,m/\alpha)}\|_2^2 \\ &\leq \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \|\chi_{(-1,m/\alpha)}\|_1^2 \|q\|_2^2 \\ &\leq \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \Big(\frac{m}{\alpha}\Big)^2 \sum_{k\in\mathbb{Z}} \|P_k\|_2^2. \end{aligned}$$

Defining $D := \frac{m}{\alpha}\sqrt{\lceil \alpha\beta \rceil} \max\{2\pi\beta\|g\|_\infty, \|g'\|_\infty\}$, we rewrite (2.2.5) as

$$\|f - Af\|_2^2 \leq \frac{D^2}{\pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \sum_{k\in\mathbb{Z}} \|P_k\|_2^2.$$

We recall that $\{M_{\beta l} T_{\alpha k} g\}_{k\in\mathbb{Z}, |l|\leq b(k)}$ is a Riesz basis for $V_b^2(g,\mathbb{R})$ with lower bound A' and by (2.1.5), we have

$$\|f - Af\|_2^2 \leq \frac{D^2}{\pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \sum_{k\in\mathbb{Z}} \|P_k\|_2^2 \leq \frac{D^2}{\beta A' \pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right) \|f\|_2^2.$$

We set $\gamma^2 := \frac{D^2}{\beta A' \pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right)$ and, by condition (2.2.2), $\gamma < 1$. Therefore, the operator A is invertible on $V_b^2(g,\mathbb{R})$ and every f can be reconstructed completely and stably from the samples in Λ. By Lemma 1.18, the weighted sampling inequality (2.2.3) holds with $A = \big(1 - \sqrt{\frac{D^2}{\beta A' \pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right)}\big)^2$ and $B = \big(1 + \sqrt{\frac{D^2}{\beta A' \pi^2} \sup_{k\in\mathbb{Z}} \left(\delta_k^2 \mu_k^2\right)}\big)^2$. □

2.3 Comparison and conclusions

(i) The main results in Section 2.1 and Section 2.2, namely Theorem 2.3 and Theorem 2.4, present two distinct conditions that ensure that a set of point Λ is a set of sampling for a variable bandwidth space constructed with a Gabor Riesz sequence.

The first theorem, Theorem 2.3, is quite specific. The space of variable bandwidth $V_b^2(g,\mathbb{R})$ is generated by a Gabor Riesz sequence with translation parameter $\alpha = 1$ and a particular window function g with precise support, i.e., $\operatorname{supp}(g) \subseteq [-\varepsilon, 1+\varepsilon]$. This specificity makes it explicit how neighboring windows affect the reconstruction inside each interval $[k, k+1)$ of the partition. Indeed, in this theorem, only three elements of the Riesz sequence play a role in the expansion of a function f at each sample $f(x_{k,j})$. This is highlighted by the influence of nearby bandwidths on the reconstruction in the interval, as described in (2.1.2). Notably, the central window's bandwidth has a more pronounced impact on the sampling rate, as indicated by the constant 4 in (2.1.2), while the windows at the boundaries have a more modest influence on the space's density. Therefore, (2.1.2) offers valuable insights into variable bandwidth spaces and clarifies the role played by the translations of g.

On the other hand, the main modifications in Theorem 2.4 with respect to Theorem 2.3 is that the window function g is allowed to have compact support on a (possibly) larger interval. Here, the sufficient condition still relies on the properties of g, but the key insight is that the expansion of each sample $f(x_{k,j})$, can be expressed as a sum of trigonometric polynomials. Since g is compactly supported, this sum is finite and it forms a trigonometric polynomial with degree the maximum degree among the trigonometric polynomials. As expected, for every interval of the type $[\alpha k, \alpha(k+1))$, the maximal gap (or the lowest sampling rate) depends on the maximal bandwidth of f over the interval. This fact becomes the pivotal element in the condition, as indicated by the presence of the constant μ_k in (2.2.2). It is also worth noting that Theorem 2.4 is particularly useful for numerical experiments. The reason is that it offers separate conditions for all intervals, simplifying the setup for simulations.

(ii) Note that the separation of the points is not a requirement for the proof. In fact, a local variation of the density can be addressed by using the adaptive weights method (sometimes called "density compensation factors" [36]). If the sampling points are separated, i.e., $\inf_{\lambda,\mu\in\Lambda,\lambda\neq\mu}|\lambda-\mu| = \nu > 0$, then $w_{k,j} \geq \nu > 0$ and we obtain the standard sampling inequality (1.4.1)

$$\tilde{A}\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} |f(x_{k,j})|^2 \leq \tilde{B}\|f\|_2^2, \qquad \text{for all } f \in V_b^2(g,\mathbb{R})\,.$$

To get the lower constant $\tilde{A}$, the separation of the points is not required. In fact, from (1.4.5) we have

$$\begin{aligned}\|f\|_2^2 &\leq \frac{1}{(1-\gamma)^2}\Big\|\sum_{k,j} f(x_{k,j})\chi_{k,j}\Big\|_2^2 = \frac{1}{(1-\gamma)^2}\sum_{k,j}|f(x_{k,j})|^2 w_{k,j}\\ &\leq \frac{\sup_{k\in\mathbb{Z}}\delta_k^2}{(1-\gamma)^2}\sum_{k,j}|f(x_{k,j})|^2.\end{aligned}$$

Therefore, $\tilde{A} = \frac{(1-\gamma)^2}{\sup_{k\in\mathbb{Z}}\delta_k^2}$.

On the other hand, the separation of the points provides us with a nice upper constant $\tilde{B}$. Indeed, from (1.4.6), we get

$$\nu\sum_{k,j}|f(x_{k,j})|^2 \leq \sum_{k,j}|f(x_{k,j})|^2 w_{k,j} \leq (1+\gamma)^2\|f\|_2^2.$$

Hence, $\tilde{B} = \frac{(1+\gamma)^2}{\nu}$.

3 Variable bandwidth via Wilson bases

In this chapter, we introduce a second variation to describe variable bandwidth that is in part motivated by the signal processing of gravitational waves. In the same spirit as the previous chapter, we revisit a discrete adaptation of the space introduced by Aceska and Feichtinger [3, 4], employing a Wilson expansion truncated in the frequency domain as a replacement for the continuous short-time Fourier transform.

Definition 3.1. A **Wilson basis** is an orthonormal basis of $L^2(\mathbb{R})$ of the form

$$\psi_{n,l}(x) = \begin{cases} g(x-n), & l = 0, \\ \frac{1}{\sqrt{2}}(e^{2\pi i l x} + (-1)^{l+n} e^{-2\pi i l x}) g(x - n/2), & l \neq 0, \end{cases} \tag{3.0.1}$$

for $n, l \in \mathbb{Z}$ and $l \geq 0$.

Up to normalization, we can rewrite (3.0.1) as

$$\psi_{n,l}(x) = \begin{cases} \sqrt{2} g(x-n/2)\cos(2\pi l x), & \text{if } n+l \text{ is even}, \\ \sqrt{2} g(x-n/2)\sin(2\pi l x), & \text{if } n+l \text{ is odd}. \end{cases} \tag{3.0.2}$$

In their work, Daubechies, Jaffard, and Journé [26] established the existence of such orthonormal bases and the feasibility of selecting a window function g such that both g and $\hat{g}$ have exponential decay or a smooth g with compact support. As a result, each basis element $\psi_{n,l}$ is centered around time $n/2$ and exhibits a frequency within a band near $\pm l$.

These bases have proved to be very useful in time-frequency analysis since they provide a way to overcome the barrier posed by the Balian-Low theorem [12]. The theorem states that a function g, generating a Gabor orthonormal system, cannot be well localized in both time and frequency. In principle, Wilson bases are orthonormal bases that possess good time-frequency localization while keeping much of the structure of a Gabor system. It is still mysterious how they came up with this formula, and the paper they wrote is entirely dedicated to proving that the formulation (3.0.1) has all the properties we are looking for. In addition, they presented a way to construct $\psi_{n,l}$ where both $\psi_{n,l}$ and its Fourier transform $\hat{\psi}_{n,l}$ have exponential decay and can be constructed as a rapidly converging superposition of Gaussians.

We define our model of variable bandwidth using a Wilson basis.

Definition 3.2. Let $b : \mathbb{Z} \to \mathbb{N}$ be a bounded positive sequence such that $b(n) \leq B < \infty$ for every $n \in \mathbb{Z}$ and let $\{\psi_{n,l}\}_{n\in\mathbb{Z}, l\in\mathbb{N}}$ be an orthonormal Wilson basis as in (3.0.1). We define the **space of variable bandwidth generated by a Wilson basis** as

$$PW_b^2(g, \mathbb{R}) := \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l}\psi_{n,l},\ c \in \ell^2(\mathbb{Z}\times\mathbb{N}) \Big\}. \tag{3.0.3}$$

In the same way as for the space $V_b^2(g,\mathbb{R})$, the sequence b expresses the time-varying frequency truncation. When assuming that the function g has compact support, we can interpret the parameter $b(n)$ as the local bandwidth of f within an interval centered at $n/2$. Indeed, by taking the characteristic function $g = \sqrt{2}\chi_{[-1/4,1/4]}$ as the generator of the Wilson basis, then the restriction of a function $f \in PW_b^2(g,\mathbb{R})$ to an interval of the form $[n/2-1/4, n/2+1/4]$ is a trigonometric polynomial of degree $b(n)$, and one needs precisely $b(n)+1$ samples from $[n/2-1/4, n/2+1/4]$ to recover f completely on this interval. If then g is taken to be smooth and with larger support, $b(n)$ is expected to be the maximum frequency on $[n/2-1/4, n/2+1/4]$. While this intuition is basically correct, we will need to refine it because the interval contains several translates $\psi_{n',l}$ for $n' \neq n$.

Since 1991, after the publication of the paper, Wilson bases became a very popular tool in time-frequency analysis and signal processing to study function spaces and time-frequency localization and non-linear approximation [33, 44]. Perhaps the most important application is the detection of the gravitational waves on September 14 2015 [22]. On this day, "Advanced LIGO" became the first detector that physically sensed the gravitational waves caused by a binary black hole system merging to form a single black hole 1.3 billion light-years away [1, 2].

Gravitational waves are "ripples" in space-time produced by some of the most violent and energetic processes in the universe such as colliding black holes, supernovae, and colliding neutron stars. In 1916, Albert Einstein [29, 30] predicted the existence of gravitational waves by showing that massive accelerating objects would modify space-time in such a way that "waves" of undulating space-time would propagate in all directions away from the source. These cosmic signals would travel at the speed of light, carrying with them information about their origins.

Damour, Blanchet and their collaborators [17, 18, 20] were able to calculate the analytic form of a gravitational wave resulting from the coalescence of two neutron stars. They obtained that this function is what in signal processing is called a **chirp** and it is described by the following equation

$$s(t) = c|t-t_0|^{-\frac{1}{4}}\cos(\omega|t-t_0|^{\frac{5}{8}}+\varphi), \qquad \text{for } t < t_0,$$

where c is a constant, $\omega \gg 1$, and t_0 is the time of the coalescence. Since $\omega|t-t_0|^{\frac{5}{8}} = \omega|t-t_0|^{-\frac{3}{8}}|t-t_0|$, the quantity $\omega|t-t_0|^{-\frac{3}{8}}$ represents the (maximal) frequency at time t. More in general, a chirp is a frequency-modulated signal analytically described by $F(t) = A(t)e^{i\phi(t)}$, where $\phi(t)$ is characterized by a strong acceleration, i.e., $|\phi''(t)| \gg 1$, and it is considered a prototype of a function of variable bandwidth.

In practice, the challenge of detecting gravitational waves consists of isolating a chirp signal hidden within a background of noise.

In 2012, Necula, Klimenko, and Mitselmakher [61] proposed employing Wilson bases for gravitational wave detection and wrote the "Coherent WaveBurst" search algorithm, which has proven successful in this endeavor. In fact, the optimal sparsity of the gravitational wave required the use of seven dilated Wilson bases and their quadrature bases, each obtained by a dilation of a factor 2 of the window.

This algorithm has demonstrated efficacy in identifying the merge of black holes and other astrophysical phenomena within gravitational data.

It is worth noting that in this application, the shape of the local bandwidth, $\omega|t-t_0|^{-\frac{3}{8}}$, is predetermined and can be inserted into an appropriate signal space. We propose and will validate through numerical analysis in Chapter 5, that a variable bandwidth Paley-Wiener space $PW_b^2(g,\mathbb{R})$ serves as such a suitable space.

In this chapter, we investigate the basic sampling problems for the space $PW_b^2(g,\mathbb{R})$. The first problem consists of finding the necessary density conditions for sampling: we study whether there exists a critical density that separates sampling (= complete reconstruction from samples) from interpolation (= approximation of given data by a function with prescribed properties). The second challenge involves identifying sufficient conditions for sampling. Similarly to the Gabor Riesz sequence case of Chapter 2, we study under which conditions on a sampling set a function in $PW_b^2(g,\mathbb{R})$ can be reconstructed completely. To solve this problem, we make use again of the adaptive weights method described in [32, 38].

This chapter is organized as follows. In Section 3.1 we revisit the fundamental properties of a Wilson basis and in Section 3.2 we show the existence of a family of compactly supported Wilson bases with $g \in \mathcal{C}^1(\mathbb{R})$ and $\operatorname{supp}(g) \subseteq [-1,1]$. Section 3.3 is dedicated to the proofs of the necessary density conditions for sampling theorems; first, in the context of the finite-dimensional space $PW_b^2(g,I)$ and second, an asymptotic result for $PW_b^2(g,\mathbb{R})$. Section 3.4 faces the challenge of identifying sufficient conditions for sampling for $PW_b^2(g,\mathbb{R})$. In Section 3.5 we provide a sufficient condition for sampling theorem for a space of variable bandwidth constructed using a Wilson Riesz basis. Section 3.6 contains a description of the relation of the Paley-Wiener space with the space of variable bandwidth $PW_b^2(g,\mathbb{R})$. In Section 3.7 we analyze and discuss the main results of the chapter.

The necessary density condition for sampling theorem of Section 3.3 and the sufficient condition for sampling theorem of Section 3.4 are included in the collaborative work [7], jointly written by the author of this thesis and Karlheinz Gröchenig. The results in Section 3.5 are contained in [6] and developed in collaboration with Karlheinz Gröchenig.

3.1 Wilson basis

In this section, we summarize without proof the properties of a Wilson basis.

By defining the usual translation and modulation operator $T_n f(t) = f(t-n)$ and $M_l f(t) = f(t)e^{2\pi i l t}$ as in (1.2.1), we can rewrite the entire collection of functions that forms a Wilson basis (3.0.1) as

$$\Psi_{n,l} = d_l(M_l + (-1)^{l+n} M_{-l}) T_{\frac{n}{2}} g, \quad (n,l) \in \mathbb{Z}^2, l \geq 0,$$

where $d_0 = \frac{1}{2}$ and $d_l = \frac{1}{\sqrt{2}}$, $l \geq 1$. In this form $\Psi_{n,l} = \psi_{n,l}$ for $l \geq 1$ and $n \in \mathbb{Z}$, and $\Psi_{2n,0} = \psi_{n,0} = T_n g$, and $\Psi_{2n+1,0} = 0$. Then every function $f \in PW_b^2(g,\mathbb{R})$, is given by

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l} \Psi_{n,l}(x) = \sum_{n\in\mathbb{Z}} P_n(x) g(x - n/2), \tag{3.1.1}$$

where

$$P_n(x) := \sum_{l=0}^{b(n)} c_{n,l} d_l (M_l + (-1)^{l+n} M_{-l})$$

is a trigonometric polynomial of degree $b(n)$ and period 1.

The following result gives a full characterization of a Wilson basis.

Theorem 3.3 (Corollary 8.5.4, [39]). *Assume $g \in L^2(\mathbb{R})$ to be even and real-valued. Then the following are equivalent:*

(i) the Wilson system $\{\psi_{n,l}\}_{n\in\mathbb{Z},l\in\mathbb{N}}$ as defined by (3.0.1) *is an orthonormal basis for $L^2(\mathbb{R})$.*

(ii) $\|g\|_2 = 1$ and the Gabor system $\{T_{n/2}M_l g : n, l \in \mathbb{Z}\}$ is a tight frame for $L^2(\mathbb{R})$, i.e., there exists a constant $A > 0$ such that for all $f \in L^2(\mathbb{R})$

$$\sum_{n\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\langle f, T_{n/2} M_l g\rangle|^2 = A\|f\|_2^2.$$

(iii) $\sum_{n\in\mathbb{Z}} g(x - k - \frac{n}{2})\overline{g}(x - \frac{n}{2}) = 2\delta_{k0}$ a.e.

(iv) $\langle g, T_n M_{2l} g\rangle = \delta_{n0}\delta_{l0}$.

The following characterization for the Wilson basis in terms of the Fourier transform of the window g has been formulated in [26] and then revised in [9].

Theorem 3.4. *Suppose $\hat{g}$ is a real-valued function that satisfies*

$$|\hat{g}(\omega)| \le C(1 + |\omega|)^{-1-\epsilon} \quad \textit{for some } \epsilon > 0.$$

Then the following are equivalent:

(i) the Wilson system $\{\psi_{n,l}\}_{n\in\mathbb{Z},l\in\mathbb{N}}$ as defined by (3.0.1) *is an orthonormal basis for $L^2(\mathbb{R})$.*

(ii) $\sum_{l\in\mathbb{Z}} \hat{g}(\omega - l)\hat{g}(\omega - l - 2j) = \delta_{0j}, \; j \in \mathbb{Z}, \; \omega \in \mathbb{R}$.

Daubechies, Jaffard and Journé [26] proved that the shift from a Gabor frame of redundancy 2 to a Wilson system has the remarkable effect of eliminating this redundancy and results in an orthonormal basis for $L^2(\mathbb{R})$. A less computational proof, based on the structural properties of Gabor frames can be found in the book of Gröchenig [39, Section 8.5].

The next lemma shows that the variable bandwidth space $PW_b^2(g, \mathbb{R})$ defined using a Wilson basis is a reproducing kernel Hilbert space.

Lemma 3.5. *The space $PW_b^2(g, \mathbb{R})$ of variable bandwidth defined in* (3.0.3) *with $g \in \mathcal{C}(\mathbb{R})$ is a reproducing kernel Hilbert space with reproducing kernel*

$$k(x, y) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} \overline{\psi_{n,l}(y)}\psi_{n,l}(x). \tag{3.1.2}$$

Proof. Recall from the characterization of a Wilson basis the equivalence (iii) of Theorem 3.3, i.e.,

$$\sum_{n\in\mathbb{Z}} \left|g\Big(x-\frac{n}{2}\Big)\right|^2 = 2, \quad \text{a.e.}$$

Therefore, by applying Lemma 1.15 with $\{e_n\}_{n\in\mathbb{Z}} = \{\psi_{n,l}\}_{n\in\mathbb{Z},l=0,\dots,b(n)}$ we have that $PW_b^2(g,\mathbb{R})$ is a reproducing kernel Hilbert space. Moreover, since $\{\psi_{n,l}\}_{n\in\mathbb{Z},l\leq b(n)}$ is an orthonormal basis for $PW_b^2(g,\mathbb{R})$, the reproducing kernel is given by

$$k(x,y) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} \overline{\psi_{n,l}(y)}\psi_{n,l}(x).$$

□

3.2 Window functions generating a Wilson basis

In the literature, one finds many ways to construct a Wilson basis with good time-frequency localization. One construction was introduced already in the seminal paper by Daubechies, Jaffard and Journé [26]. They constructed a Wilson basis such that the window function g and its Fourier transform $\hat{g}$ have both exponential decay by using the Zak transform.

In this section we explore the problem of constructing infinitely many window functions g that generate a Wilson orthonormal basis. In particular, we show the existence of a family of compactly supported functions g, with $\text{supp}(g) \subseteq [-1,1]$ and $g \in \mathcal{C}^1(\mathbb{R})$, that generates a Wilson orthonormal basis.

We follow a constructive approach. Take $a, b \in \mathcal{C}^1(\mathbb{R})$ and let

$$g(x) = a(|x|)\chi_{[0,1/2)}(|x|) + b(|x|)\chi_{[1/2,1)}(|x|). \tag{3.2.1}$$

Then g is even, real-valued and $\text{supp}(g) \subseteq [-1,1]$,

By the characterization (iii) of Theorem 3.3, we need that for every $x \in [0,1/2]$,

$$\sum_{n\in\mathbb{Z}} g\Big(x-k-\frac{n}{2}\Big)g\Big(x-\frac{n}{2}\Big) = 2\delta_{k0}. \tag{3.2.2}$$

Define $\chi_0(x) \coloneqq \chi_{[0,1/2)}(x)$ and $\chi_1(x) \coloneqq \chi_{[1/2,1)}(x)$.

Firstly, we assume $k=0$, then

$$\sum_{n\in\mathbb{Z}} g\Big(x-\frac{n}{2}\Big)^2 = \sum_{n\in\mathbb{Z}} \Big(a(|x-n/2|)\chi_0(|x-n/2|) + b(|x-n/2|)\chi_1(|x-n/2|)\Big)^2.$$

Given that $\chi_0(|x|)$ and $\chi_1(|x|)$ have disjoint support, it follows that the double product $2a(|x-n/2|)\chi_0(|x-n/2|)b(|x-n/2|)\chi_1(|x-n/2|) = 0$. Hence,

$$\sum_{n\in\mathbb{Z}} g\Big(x-\frac{n}{2}\Big)^2 = \sum_{n\in\mathbb{Z}} \Big(a(|x-n/2|)^2\chi_0(|x-n/2|) + b(|x-n/2|)^2\chi_1(|x-n/2|)\Big).$$

We have that $\text{supp}(\chi_0) \subseteq [0,1/2)$, $x \in [0,1/2]$ and

a) $x - \frac{n}{2} \in [0, 1/2)$, then $n = 0$.

b) $\frac{n}{2} - x \in [0, 1/2)$, then $n = 1$.

Similarly, $\operatorname{supp}(\chi_1) \subseteq [1/2, 1)$, $x \in [0, 1/2]$ and

c) $x - \frac{n}{2} \in [1/2, 1)$, then $n = -1$.

d) $\frac{n}{2} - x \in [1/2, 1)$, then $n = 2$.

Therefore,

$$\sum_{n\in\mathbb{Z}} g\Big(x - \frac{n}{2}\Big)^2 = a(x)^2 + a\Big(\frac{1}{2} - x\Big)^2 + b\Big(x + \frac{1}{2}\Big)^2 + b(1-x)^2. \tag{3.2.3}$$

Using the characterization (iii) of Theorem 3.3, we impose (3.2.3) to be 2 to get

$$a(x)^2 + a\Big(\frac{1}{2} - x\Big)^2 + b\Big(x + \frac{1}{2}\Big)^2 + b(1-x)^2 = 2, \quad \text{for } x \in [0, 1/2]. \tag{3.2.4}$$

Consider now the case $k = 1$. We rewrite (3.2.2) to get

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} g\Big(x - 1 - \frac{n}{2}\Big) g\Big(x - \frac{n}{2}\Big) \\
&\quad= \sum_{n\in\mathbb{Z}} \Big[a(|x-1-n/2|)\chi_0(|x-1-n/2|) + b(|x-1-n/2|)\chi_1(|x-1-n/2|)\Big]\cdot \\
&\qquad\qquad \cdot \Big[a(|x-n/2|)\chi_0(|x-n/2|) + b(|x-n/2|)\chi_1(|x-n/2|)\Big].
\end{aligned} \tag{3.2.5}$$

Again, $\operatorname{supp}(\chi_0) \subseteq [0, 1/2)$, $x \in [0, 1/2]$ and

e) $x - 1 - \frac{n}{2} \in [0, 1/2)$, then $n = -2$.

f) $1 + \frac{n}{2} - x \in [0, 1/2)$, then $n = -1$.

Furthermore, $\operatorname{supp}(\chi_1) \subseteq [1/2, 1)$, $x \in [0, 1/2]$ and

g) $x - 1 - \frac{n}{2} \in [1/2, 1)$, then $n = -3$.

h) $1 + \frac{n}{2} - x \in [1/2, 1)$, then $n = 0$.

Combining a)-h) and considering the disjoint supports of χ_0 and χ_1, the equation (3.2.5) becomes

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} g\Big(x - 1 - \frac{n}{2}\Big) g\Big(x - \frac{n}{2}\Big) \\
&\quad= a(|x-1/2|)\chi_0(|x-1/2|)b(|x+1/2|)\chi_1(|x+1/2|) + a(|x|)\chi_0(|x|)b(|x-1|)\chi_1(|x-1|) \\
&\quad= a\Big(\frac{1}{2} - x\Big) b\Big(x + \frac{1}{2}\Big) + a(x)b(1-x).
\end{aligned}$$

Applying (3.2.2), we impose

$$a\Big(\frac{1}{2}-x\Big)b\Big(x+\frac{1}{2}\Big)+a(x)b(1-x)=0, \quad \text{for } x\in[0,1/2]. \tag{3.2.6}$$

Set $\tilde{b}(x) := b(x+1/2)$, then $\tilde{b}(1/2-x)=b(1-x)$. We rewrite the two conditions (3.2.4) and (3.2.6), to get for $x\in[0,1/2]$

$$a\Big(\frac{1}{2}-x\Big)\tilde{b}(x)+a(x)\tilde{b}\Big(\frac{1}{2}-x\Big)=0, \tag{3.2.7}$$

$$a(x)^2+a\Big(\frac{1}{2}-x\Big)^2+\tilde{b}(x)^2+\tilde{b}\Big(\frac{1}{2}-x\Big)^2=2. \tag{3.2.8}$$

Define

$$\phi(x)^2 := a(x)^2+\tilde{b}(x)^2 \tag{3.2.9}$$

then $\phi(1/2-x)^2 = a(1/2-x)^2+\tilde{b}(1/2-x)^2$ and from (3.2.8) we have

$$\phi(x)^2+\phi\Big(\frac{1}{2}-x\Big)^2=2, \quad \text{for } x\in[0,1/2]. \tag{3.2.10}$$

Using (3.2.9), we define for $x\in[0,1/2]$

$$\begin{aligned} a(x) &= \phi(x)\cos(\xi(x)), && (3.2.11)\\ b(x) &= \phi(x-1/2)\sin(\xi(x-1/2)) && (3.2.12)\\ \tilde{b}(x) &= \phi(x)\sin(\xi(x)). && (3.2.13)\end{aligned}$$

We rewrite (3.2.7) using (3.2.11) and (3.2.13), to obtain

$$\begin{aligned} 0 &= a\Big(\frac{1}{2}-x\Big)\tilde{b}(x)+a(x)\tilde{b}\Big(\frac{1}{2}-x\Big)\\ &= \phi(x)\phi\Big(\frac{1}{2}-x\Big)\cos\Big(\xi\Big(\frac{1}{2}-x\Big)\Big)\sin(\xi(x))+\phi(x)\phi\Big(\frac{1}{2}-x\Big)\cos(\xi(x))\sin\Big(\xi\Big(\frac{1}{2}-x\Big)\Big)\\ &= \phi(x)\phi\Big(\frac{1}{2}-x\Big)\sin\Big(\xi(x)+\xi\Big(\frac{1}{2}-x\Big)\Big).\end{aligned}$$

We assume $\xi(x)+\xi(1/2-x)=0$ for every $x\in[0,1/2]$. This means that ξ is an odd function around $1/4$, i.e.,

$$\xi\Big(\frac{1}{4}+x\Big)=-\xi\Big(\frac{1}{4}-x\Big), \quad \text{for } |x|\le\frac{1}{4}. \tag{3.2.14}$$

We consider the condition (3.2.10) and we choose ϕ such that $|\phi(x)|\le\sqrt{2}$ arbitrary on $[0,1/4]$ and define

$$\phi\Big(\frac{1}{2}-x\Big) := \pm\sqrt{2-\phi(x)^2}, \quad \text{for } x\in[0,1/4]. \tag{3.2.15}$$

We want the function g to be $\mathcal{C}^1(\mathbb{R})$, which means that we need to impose continuity at $x = 1/2$ and $x = 1$ and differentiability at $x = 0$, $x = 1/2$ and $x = 1$.

1) Continuity at $x = 1/2$.
We impose

$$a\left(\frac{1}{2}\right) = b\left(\frac{1}{2}\right) = \tilde{b}(0).$$

Applying the definitions (3.2.11) and (3.2.13) of a and $\tilde{b}$, we have

$$\phi\left(\frac{1}{2}\right) \cos\left(\xi\left(\frac{1}{2}\right)\right) = \phi(0) \sin(\xi(0)). \tag{3.2.16}$$

Since ξ is an odd function around 1/4 and using (3.2.14), we obtain

$$\xi\left(\frac{1}{2}\right) = \xi\left(\frac{1}{4} + \frac{1}{4}\right) = -\xi\left(\frac{1}{4} - \frac{1}{4}\right) = -\xi(0).$$

Therefore,

$$\phi\left(\frac{1}{2}\right) \cos(\xi(0)) = \phi(0) \sin(\xi(0)).$$

Applying (3.2.15), we have $\phi(1/2) = \pm\sqrt{2 - \phi(0)^2}$ and

$$\tan(\xi(0)) = \frac{\sin(\xi(0))}{\cos(\xi(0))} = \pm\frac{\sqrt{2 - \phi(0)^2}}{\phi(0)}. \tag{3.2.17}$$

2) Continuity at $x = 1$.
Rewriting the definition of g (3.2.1), with (3.2.11) and (3.2.12), we have

$$g(x) = \phi(|x|) \cos(\xi(|x|))\chi_0(|x|) + \phi(|x| - 1/2) \sin(\xi(|x| - 1/2))\chi_1(|x|).$$

We impose $g(1) = 0$, which implies

$$\phi\left(\frac{1}{2}\right) \sin\left(\xi\left(\frac{1}{2}\right)\right) = 0.$$

Therefore, either

$$\phi\left(\frac{1}{2}\right) = 0 \quad \text{or} \tag{3.2.18}$$

$$\xi\left(\frac{1}{2}\right) = -\xi(0) \in \pi\mathbb{Z}. \tag{3.2.19}$$

We show that the two conditions (3.2.18) and (3.2.19) are equivalent.

First, assume $\phi(1/2) = 0$, and consider the equation (3.2.16). We substitute $\phi(1/2) = 0$ on the left hand side, to get

$$\phi(0) \sin(\xi(0)) = 0.$$

By the definition (3.2.15) we have $\phi(0) = \pm\sqrt{2} \neq 0$, therefore $\xi(0) = -\xi(1/2) \in \pi\mathbb{Z}$.

Let now $\xi(0) = -\xi(1/2) \in \pi\mathbb{Z}$. From the equation of the tangent of ξ (3.2.17) we get $\sqrt{2-\phi(0)^2} = 0$, which implies $\phi(0) = \pm\sqrt{2}$. Again by the definition (3.2.15), we obtain $\phi(1/2) = 0$. Hence, the two conditions (3.2.18) and (3.2.19) are equivalent.

3) Differentiability at $x = 0$.
Impose $\lim_{x\to 0} a'(x) = \lim_{x\to 0} a'(-x)$. We have

$$a'(x) = \phi'(x)\cos(\xi(x)) - \phi(x)\sin(\xi(x))\xi'(x) \tag{3.2.20}$$

and

$$a'(-x) = -\phi'(-x)\cos(\xi(-x)) + \phi(-x)\sin(\xi(-x))\xi'(-x).$$

Taking the limit for $x \to 0$ we have

$$\phi'(0)\cos(\xi(0)) - \phi(0)\sin(\xi(0))\xi'(0) = -\phi'(0)\cos(\xi(0)) + \phi(0)\sin(\xi(0))\xi'(0).$$

Since $\xi(0) = -\xi(\frac{1}{2}) \in \pi\mathbb{Z}$ and $|\cos(\xi(0))| = 1$, we get

$$\phi'(0)\cos(\xi(0)) = 0$$

which implies

$$\phi'(0) = 0.$$

4) Differentiability at $x = 1/2$.
We impose $a'(1/2) = b'(1/2) = \tilde{b}'(0)$. Using (3.2.20) and

$$\tilde{b}'(x) = \phi'(x)\sin(\xi(x)) + \phi(x)\cos(\xi(x))\xi'(x) \tag{3.2.21}$$

we obtain

$$\phi'(\tfrac{1}{2})\cos(\xi(\tfrac{1}{2})) - \phi(\tfrac{1}{2})\sin(\xi(\tfrac{1}{2}))\xi'(\tfrac{1}{2}) = \phi'(0)\sin(\xi(0)) + \phi(0)\cos(\xi(0))\xi'(0).$$

Since $\xi(0) = -\xi(\frac{1}{2}) \in \pi\mathbb{Z}$, we have

$$\phi'(\tfrac{1}{2})\cos(\xi(\tfrac{1}{2})) = \phi(0)\cos(\xi(0))\xi'(0).$$

Moreover, $\cos(\xi(\frac{1}{2})) = \cos(\xi(0))$ and, by (3.2.15), $\phi(0) = \pm\sqrt{2} \neq 0$, therefore

$$\xi'(0) = \frac{\phi'\left(\frac{1}{2}\right)}{\phi(0)} = \pm\frac{\phi'\left(\frac{1}{2}\right)}{\sqrt{2}}.$$

The derivative of an odd function is an even function, which means that $\xi'(\frac{1}{4}+x) = \xi'(\frac{1}{4}-x)$ for $|x| \leq 1/4$, i.e.,

$$\xi'(0) = \xi'\left(\tfrac{1}{2}\right) = \pm\frac{\phi'\left(\frac{1}{2}\right)}{\sqrt{2}}.$$

5) Differentiability at $x = 1$.
Let $g'(1) = 0$, then $b'(1) = \tilde{b}'(\frac{1}{2}) = 0$. Substituting in (3.2.21), we obtain

$$\phi'(\tfrac{1}{2})\sin(\xi(\tfrac{1}{2})) + \phi(\tfrac{1}{2})\cos(\xi(\tfrac{1}{2}))\xi'(\tfrac{1}{2}) = 0.$$

Since $\phi(\frac{1}{2}) = 0$ and $\xi(0) = -\xi(\frac{1}{2}) \in \pi\mathbb{Z}$, this condition is already satisfied.

We can summarize the previous results and computations in the following theorem.

Theorem 3.6. *Let* $\phi, \xi \in \mathcal{C}^1(\mathbb{R})$ *and*

$$g(x) = \phi(|x|)\cos(\xi(|x|))\chi_{[0,1/2)}(|x|) + \phi(|x|-1/2)\sin(\xi(|x|-1/2))\chi_{[1/2,1)}(|x|)$$

be such that:

i) $\phi^2(x) + \phi^2(\frac{1}{2}-x) = 2,\quad$ *for every* $x \in [0,1/2]$;

ii) $\phi(\frac{1}{2}) = 0$;

iii) $\phi'(0) = 0$;

iv) ξ *is odd around* $1/4$, *i.e.,* $\xi(\frac{1}{4}+x) = -\xi(\frac{1}{4}-x)$ *for* $|x| \leq 1/4$;

v) $\xi(0) = -\xi(\frac{1}{2}) \in \pi\mathbb{Z}$;

vi) $\xi'(\frac{1}{2}) = \xi'(0) = \phi'(\frac{1}{2})/\phi(0)$.

Then $g \in \mathcal{C}^1(\mathbb{R})$, $\operatorname{supp}(g) \subseteq [-1,1]$, *and* g *generates a Wilson orthonormal basis.*

We provide an example of a function g that satisfies the conditions of Theorem 3.6.

Example A. Construction of $\boldsymbol{\phi}$. Consider $\phi(x) = \sqrt{2}\cos(\pi x)$. We check that (i)-(iii) are satisfied.

i) $2\cos^2(\pi x) + 2\cos^2(\pi(\frac{1}{2}-x)) = 2$.

ii) $\phi(\frac{1}{2}) = \sqrt{2}\cos(\frac{\pi}{2}) = 0$.

iii) $\phi'(0) = -\sqrt{2}\pi\sin(0) = 0$.

Moreover, we have that $\phi(0) = \sqrt{2}$ and $\phi'(\frac{1}{2}) = -\sqrt{2}\pi$.

Construction of $\boldsymbol{\xi}$. Consider $\xi(x) = \frac{1}{4}\sin(4\pi(x-\frac{1}{4}))$. We check that (iv)-(vi) are satisfied.

iv) $\xi(\frac{1}{4}+x) = \frac{1}{4}\sin(4\pi x) = -\frac{1}{4}\sin(-4\pi x) = -\xi(\frac{1}{4}-x)$.

v) $\xi(0) = -\xi(\frac{1}{2}) = 0 \in \pi\mathbb{Z}$.

vi) $\xi'(x) = \pi\cos(4\pi(x-\frac{1}{4}))$. Then $\xi'(0) = \xi'(\frac{1}{2}) = -\pi = \phi'(\frac{1}{2})/\phi(0)$.

Therefore the function

$$\begin{aligned} g(x) = &\sqrt{2}\cos(\pi|x|)\cos(\tfrac{1}{4}\sin(4\pi(|x|-\tfrac{1}{4})))\chi_{[0,1/2)}(|x|) \\ &+ \sqrt{2}\cos(\pi(|x|-\tfrac{1}{2}))\sin(\tfrac{1}{4}\sin(4\pi(|x|-\tfrac{1}{4}-\tfrac{1}{2})))\chi_{[1/2,1)}(|x|) \end{aligned} \tag{3.2.22}$$

generates a Wilson orthonormal basis.

Figure 3.1 represents the function g that we have constructed.

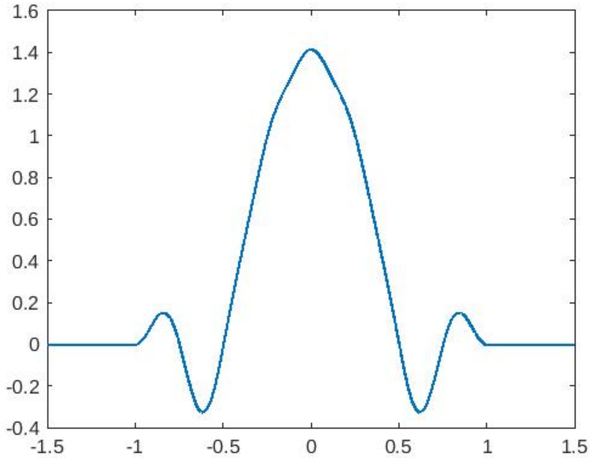


Figure 3.1: Example of a window function $g \in \mathcal{C}^1(\mathbb{R})$ with $\operatorname{supp}(g) \subseteq [-1,1]$ that generates a Wilson orthonormal basis with explicit formula (3.2.22).

3.3 Necessary density condition for sampling

In this section, we prove necessary density conditions for the spaces of variable bandwidth generated by a Wilson basis. As $PW_b^2(g,\mathbb{R})$ mirrors the Paley-Wiener space of bandlimited functions, we anticipate the presence of a critical density for $PW_b^2(g,\mathbb{R})$ too. This critical density represents a threshold for reconstructing a function from its samples, delineating the required number of samples. Additionally, it serves as a benchmark for evaluating constructive approaches to reconstruction. The results proved in this section are included in the work [7].

We start by showing a simple result for compactly supported windows g.

Theorem 3.7. *Let $g \in \mathcal{C}(\mathbb{R})$, real-valued, and even with $\operatorname{supp}(g) \subseteq [-m,m]$ that generates an orthonormal Wilson basis. Let $\Lambda \subseteq \mathbb{R}$ be a set of sampling for $PW_b^2(g,\mathbb{R})$ defined in (3.0.3). Then, $\Lambda \cap (\alpha,\beta)$ contains at least $\lceil \beta - \alpha - 2m \rceil + \sum_{n/2 \in [\alpha+m,\beta-m]} b(n)$ points.*

To show the result we use an easy argument that involves counting dimensions as in [5].

Proof. Let $I = (\alpha,\beta)$ such that $|I| \geq 2m$ and

$$\begin{aligned} PW_b^2(g,I) &= \operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \subseteq I\} \\ &= \operatorname{span}\{\psi_{n,0} : n \in [\alpha+m,\beta-m]\} \\ &\quad \cup \operatorname{span}\{\psi_{n,l} : n/2 \in [\alpha+m,\beta-m],\ l = 1,...,b(n)\}. \end{aligned}$$

We make the following three observations:

(i) $PW_b^2(g,I)$ is a subspace of $PW_b^2(g,\mathbb{R})$ and thus the sampling inequalities hold for $PW_b^2(g,I)$.

(ii) Since the Wilson basis functions are compactly supported, then every $f \in PW_b^2(g, I)$ has support in (α, β).

(iii) $PW_b^2(g, I)$ is finite-dimensional, and

$$\dim(PW_b^2(g, I)) = \#([\alpha + m, \beta - m] \cap \mathbb{Z}) + \sum_{n/2 \in [\alpha+m, \beta-m]} b(n). \tag{3.3.1}$$

Since for $f \in PW_b^2(g, I)$, we have

$$\sum_{\lambda \in \Lambda \cap (\alpha,\beta)} |f(\lambda)|^2 = \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \geq A \|f\|_2^2,$$

the map $f \in PW_b^2(g, I) \to \{f(\lambda)\}_{\lambda \in \Lambda \cap (\alpha,\beta)}$ is one-to-one. It follows that

$$\dim(PW_b^2(g, I)) \leq \lceil \beta - \alpha - 2m \rceil + \sum_{n/2 \in [\alpha+m, \beta-m]} b(n) \leq \#(\Lambda \cap (\alpha, \beta)).$$

□

To study necessary conditions for sampling in $PW_b^2(g, \mathbb{R})$ for a more general class of windows g with a decay of the type $|g(x)| \leq C(1 + |x|)^{-1-\epsilon}$ for $C > 0$, $\epsilon > 0$, we apply the results on sampling in general reproducing kernel Hilbert spaces from [34] in the form of Theorem 1.19. We only need to verify the conditions on the reproducing kernel.

Theorem 3.8. *Let $g \in \mathcal{C}(\mathbb{R})$ be a real-valued, even function, such that $|g(x)| \leq C(1+|x|)^{-1-\epsilon}$ for some $C > 0$ and $\epsilon > 0$ and assume that g generates an orthonormal Wilson basis. Let $\Lambda \subseteq \mathbb{R}$ be a set of sampling for $PW_b^2(g, \mathbb{R})$ defined in* (3.0.3) *with $b(n) \geq 1$ for every $n \in \mathbb{Z}$. Then*

$$\liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(\Lambda \cap [x - r, x + r])}{2r} \geq 1 + \overline{b}$$

where $\overline{b} = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2r} \sum_{\frac{n}{2} \in [x-r,x+r]} b(n)$.

Proof. By Lemma 3.5, $PW_b^2(g, \mathbb{R})$ is a reproducing kernel Hilbert space. Firstly, we need to show that the reproducing kernel (3.1.2) satisfies conditions (i) and (ii) of Theorem 1.19 and, finally we compute the averaged trace of the kernel.

Step 1. We prove that the diagonal $k(x, x)$ of the reproducing kernel is bounded below (condition (i) of Theorem 1.19).
Let $x \in \mathbb{R}$ and substitute the definition of a Wilson basis in (3.1.2), then

$$k(x, x) = \sum_{n \in \mathbb{Z}} \sum_{l=0}^{b(n)} \left|\psi_{n,l}(x)\right| = \sum_{n \in \mathbb{Z}} \left|g(x - n)\right|^2 + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{l=1}^{b(n)} \left|e^{2\pi i l x} + (-1)^{l+n} e^{-2\pi i l x}\right|^2 \left|g\left(x - \frac{n}{2}\right)\right|^2.$$

Since $k(x, x)$ is continuous and periodic with period 1, then $k(x, x) > 0$ on $[0, 1)$ implies that $\inf_{x \in \mathbb{R}} k(x, x) > 0$.

Consequently, it suffices to show that $k(x,x) > 0$ on $[0,1)$. Recalling that $b(n) \geq 1$ for every $n \in \mathbb{Z}$, we obtain

$$k(x,x) \geq \sum_{n\in\mathbb{Z}} \left|g(x-n)\right|^2 + \frac{1}{2}\sum_{n\in\mathbb{Z}} \left|e^{2\pi i x} + (-1)^{1+n}e^{-2\pi i x}\right|^2 \left|g\left(x - \frac{n}{2}\right)\right|^2.$$

By dividing the second sum into odd and even parts and applying (iii) of Theorem 3.3, we get

$$\begin{aligned}
k(x,x) &\geq \sum_{n\in\mathbb{Z}} \left|g(x-n)\right|^2 + 2\sum_{n\in\mathbb{Z}} |\sin 2\pi x|^2 \left|g(x-n)\right|^2 + 2\sum_{n\in\mathbb{Z}} |\cos 2\pi x|^2 \left|g\Big(x - n - \frac{1}{2}\Big)\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|g(x-n)\right|^2 (1 + 2|\sin 2\pi x|^2) + 2\sum_{n\in\mathbb{Z}} |\cos 2\pi x|^2 \left|g\Big(x - n - \frac{1}{2}\Big)\right|^2 \qquad (3.3.2) \\
&\geq \min\{1 + 2|\sin 2\pi x|^2, 2|\cos 2\pi x|^2\} \sum_{n\in\mathbb{Z}} \left|g\Big(x - \frac{n}{2}\Big)\right|^2 \\
&= 2\min\{1 + 2|\sin 2\pi x|^2, 2|\cos 2\pi x|^2\}.
\end{aligned}$$

The only possible zeros of $k(x,x)$ for $x \in [0,1)$ are for $\cos 2\pi x = 0$, i.e., at $x = \frac{1}{4}$ or $x = \frac{3}{4}$. Since g is even, $k(x,x)$ is even, and thus periodicity yields $k(\frac{1}{4},\frac{1}{4}) = k(-\frac{1}{4},-\frac{1}{4}) = k(\frac{3}{4},\frac{3}{4})$ and $k(\frac{1}{4},\frac{1}{4}) + k(\frac{3}{4},\frac{3}{4}) = k(\frac{1}{4},\frac{1}{4}) + k(-\frac{1}{4},-\frac{1}{4})$. Applying (iii) of Theorem 3.3 and from the inequality (3.3.2), we obtain

$$\begin{aligned}
k\Big(\frac{1}{4},\frac{1}{4}\Big) + k\Big(-\frac{1}{4},-\frac{1}{4}\Big) &\geq 3\Big(\sum_{n\in\mathbb{Z}} \left|g\Big(\frac{1}{4} - n\Big)\right|^2 + \sum_{n\in\mathbb{Z}} \left|g\Big(-\frac{1}{4} - n\Big)\right|^2\Big) \\
&= 3\Big(\sum_{n\in\mathbb{Z}} \left|g\Big(\frac{1}{4} - n\Big)\right|^2 + \sum_{n\in\mathbb{Z}} \left|g\Big(\frac{1}{4} - n - \frac{1}{2}\Big)\right|^2\Big) = 6.
\end{aligned}$$

By symmetry $k(\frac{1}{4},\frac{1}{4}) = k(\frac{3}{4},\frac{3}{4}) = 3 \neq 0$.
It follows that $k(x,x) > 0$ for all $x \in [0,1)$ and thus $\inf_{x\in\mathbb{R}} k(x,x) = \min_{x\in\mathbb{R}} k(x,x) > 0$.

Step 2. We verify the required decay condition of the kernel (condition (ii) of Theorem 1.19). Let $x, y \in \mathbb{R}$. Recall that $b(n) \leq B$ for every $n \in \mathbb{Z}$ and $|\psi_{n,l}(x)| \leq \sqrt{2}g(x - n/2)$ for $l \neq 0$, then we have

$$\begin{aligned}
|k(x,y)| &= \Big|\sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} \overline{\psi_{n,l}(y)}\psi_{n,l}(x)\Big| \\
&= \Big|\sum_{n\in\mathbb{Z}} \overline{\psi_{n,0}(y)}\psi_{n,0}(x) + \sum_{n\in\mathbb{Z}} \sum_{l=1}^{b(n)} \overline{\psi_{n,l}(y)}\psi_{n,l}(x)\Big| \\
&\leq \sum_{n\in\mathbb{Z}} \left\{\left|g(y-n)g(x-n)\right| + 2b(n)\left|g(y-n/2)g(x-n/2)\right|\right\} \\
&\leq \sum_{n\in\mathbb{Z}} \left\{\left|g(y-n)g(x-n)\right| + 2B\left|g(y-n/2)g(x-n/2)\right|\right\}. \qquad (3.3.3)
\end{aligned}$$

Set $w(x) = (1 + |x|)^{-1-\epsilon}$, then w is even, and by [39, Chapter 11] there exist constants $C_0 > 0$ and $C_1 > 0$ such that

$$\frac{1}{C_0} w(x) \leq \min_{|y|\leq 1} w(x+y) \leq \max_{|y|\leq 1} w(x+y) \leq C_0 w(x) \tag{3.3.4}$$

$$\text{and} \;\; w|_{\mathbb{Z}} * w|_{\mathbb{Z}} \leq C_1 \, w|_{\mathbb{Z}} \, . \tag{3.3.5}$$

w is called **subconvolutive**. Consider the first term of (3.3.3), apply the assumption on the decay of g and the properties (3.3.4) and (3.3.5) of w, then

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |g(y-n)g(x-n)| &\leq C^2 \sum_{n\in\mathbb{Z}} w(y-n)w(x-n) \\
&\leq C^2 C_0^2 \sum_{n\in\mathbb{Z}} w(\lfloor y \rfloor - n) w(\lfloor x \rfloor - n) \\
&= C^2 C_0^2 \sum_{n\in\mathbb{Z}} w(n) w(\lfloor x \rfloor - \lfloor y \rfloor - n) \\
&\leq C^2 C_0^2 C_1 w(\lfloor x \rfloor - \lfloor y \rfloor) \\
&\leq C^2 C_0^3 C_1 w(x-y).
\end{aligned}$$

Similarly, the second term in (3.3.3) is bounded by

$$\sum_{n\in\mathbb{Z}} \big| g(y - n/2) g(x - n/2) \big| \leq \widetilde{C} w(x-y)$$

and we obtain the desired off-diagonal decay of the kernel

$$|k(x,y)| \leq N(1 + |x-y|)^{-1-\epsilon} \quad \text{ for all } x, y \in \mathbb{R} \text{ and for } \epsilon > 0.$$

Conditions (i) and (ii) of Theorem 1.19 are satisfied.

Step 3. It remains to compute the averaged trace $\frac{1}{2r}\int_{x-r}^{x+r} k(y,y)dy$ and let r tend to ∞. We fix $x \in \mathbb{R}$ and we write $B_r(x) = [x-r, x+r]$ in the following. Fix $\varepsilon > 0$ and choose $\rho > 0$, such that $\int_{B_\rho(0)} |\psi_{0,l}(x)|^2 dx \geq 1 - \varepsilon$ for $l = 0, \dots, B$. Then

$$\begin{aligned}
&\int_{B_\rho(n/2)} |\psi_{n,l}(x)|^2 dx \geq 1 - \varepsilon \quad \text{for } l = 1, \dots, B \quad \text{ and} \\
&\int_{B_\rho(n)} |\psi_{n,0}(x)|^2 dx \geq 1 - \varepsilon.
\end{aligned}$$

By (iii) of Theorem 3.3, we have

$$\sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} \big|\psi_{n,l}(x)\big|^2 = \sum_{n\in\mathbb{Z}} |g(x-n)|^2 + \sum_{n\in\mathbb{Z}} \sum_{l=1}^{b(n)} \big|\psi_{n,l}(x)\big|^2 \leq 2 + 4B.$$

Therefore, we apply dominated convergence to interchange summation and integration, to obtain

$$\frac{1}{2r}\int_{B_r(x)} k(y,y)dy = \frac{1}{2r}\int_{B_r(x)} \sum_{n\in\mathbb{Z}}\sum_{l=0}^{b(n)} |\psi_{n,l}(y)|^2 dy = \frac{1}{2r}\sum_{n\in\mathbb{Z}}\sum_{l=0}^{b(n)}\int_{B_r(x)} |\psi_{n,l}(y)|^2 dy.$$

If $\frac{n}{2} \in B_{r-\rho}(x)$, then $B_\rho(\frac{n}{2}) \subseteq B_r(x)$. The estimate of the average trace is

$$\begin{aligned}\frac{1}{2r}\int_{B_r(x)} k(y,y)dy &\geq \frac{1}{2r}\sum_{n\in B_{r-\rho}(x)}\int_{B_r(x)} |\psi_{n,0}(y)|^2 dy + \frac{1}{2r}\sum_{\frac{n}{2}\in B_{r-\rho}(x)}\sum_{l=1}^{b(n)}\int_{B_r(x)} |\psi_{n,l}(y)|^2 dy \\ &\geq \frac{1}{2r}(1-\varepsilon)\Big[\#(B_{r-\rho}(x)\cap\mathbb{Z}) + \sum_{\frac{n}{2}\in B_{r-\rho}(x)} b(n)\Big] \\ &\geq \frac{1}{2r}(1-\varepsilon)\Big[2(r-\rho) + \sum_{\frac{n}{2}\in B_{r-\rho}(x)} b(n)\Big] \\ &= (1-\varepsilon)\frac{r-\rho}{r} + \frac{1}{2r}(1-\varepsilon)\sum_{\frac{n}{2}\in B_{r-\rho}(x)} b(n).\end{aligned}$$

Letting $r \to \infty$, we get

$$\liminf_{r\to\infty}\inf_{x\in\mathbb{R}}\frac{1}{2r}\int_{B_r(x)} k(y,y)dy \geq 1-\varepsilon + (1-\varepsilon)\liminf_{r\to\infty}\inf_{x\in\mathbb{R}}\frac{1}{2r}\sum_{\frac{n}{2}\in B_r(x)} b(n).$$

Define $\bar{b} := \liminf_{r\to\infty}\inf_{x\in\mathbb{R}}\frac{1}{2r}\sum_{\frac{n}{2}\in B_r(x)} b(n)$ as the average bandwidth in $B_r(x)$. Since $\varepsilon > 0$ is arbitrary, Theorem 1.19 yields

$$D^-(\Lambda) \geq 1 + \bar{b}.$$

□

In this result, $\bar{b}$ represents a sort of average bandwidth. The theorem shows that whenever the window function g has some mild decay properties, the lower Beurling density $D^-(\Lambda)$ indicates that we have at least $1+\bar{b}$ samples of Λ contained in a ball of radius r.

The properties of Wilson bases play a crucial role in satisfying the conditions of the theorem. In particular, the combination of property (iii) of the characterization Theorem 3.3 and the symmetry of the window function g serves as a key ingredient in showing that the trace of the reproducing kernel has a bounded infimum.

In the following corollary we point out that for compactly supported windows g, Theorem 3.7 implies directly Theorem 3.8.

Corollary 3.9. *Let $g \in \mathcal{C}(\mathbb{R})$, real-valued, and even with $\operatorname{supp}(g) \subseteq [-m,m]$ that generates an orthonormal Wilson basis and assume that $\Lambda \subseteq \mathbb{R}$ is a set of sampling for $PW_b^2(g,\mathbb{R})$. Then*

$$D^-(\Lambda) := \liminf_{r\to\infty}\inf_{x\in\mathbb{R}}\frac{\#(\Lambda\cap[x-r,x+r])}{2r} \geq 1+\bar{b}$$

where $\overline{b} = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2r} \sum_{\frac{n}{2}\in[x-r,x+r]} b(n)$.

Proof. Fix $x \in \mathbb{R}$ and $r > 0$ and take $(\alpha,\beta) = (x-r, x+r) = B_r(x)$. Applying Theorem 3.7 and dividing by $2r$, we obtain

$$\frac{\#(\Lambda \cap B_r(x))}{2r} \geq \frac{\lceil 2r - 2m \rceil}{2r} + \frac{1}{2r} \sum_{\frac{n}{2}\in B_{r-m}(x)} b(n).$$

Letting $r \to \infty$, we get

$$D^-(\Lambda) \geq 1 + \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2r} \sum_{\frac{n}{2}\in B_r(x)} b(n).$$

□

3.4 Sufficient conditions for sampling

This section investigates the question under which conditions on a sampling set Λ a function in $PW_b^2(g,\mathbb{R})$ can be reconstructed completely from its evaluation on a discrete set of points. We provide a sufficient condition for sampling for $PW_b^2(g,\mathbb{R})$ that confirms the intuition that the parameter $b(k)$ measures the local bandwidth of a function $f \in PW_b(g,\mathbb{R})$. For the proof, we resort once again to the adaptive weights method described in [32, 38] and we construct the space $PW_b^2(g,\mathbb{R})$ using a Wilson basis with compactly supported window g. As seen in the proofs of Theorem 2.3 and 2.4, the condition that g is compactly supported is essential.

The sufficient condition for sampling for $PW_b^2(g,\mathbb{R})$ relates the maximal gap between consecutive points of the discrete set Λ in an interval of $\mathbb{R}$ of the type $[k/2, k/2+1/2)$ to the maximal bandwidth over the given interval.

With this constructive method, we produce quantitative results that are applied as a base for numerical experiments in Chapter 5. The result proved in the following is included in the work [7].

As done before, we start by indexing the set of points Λ as follows:

$$\Lambda = \left\{ x_{\frac{k}{2},j} \in \mathbb{R} : x_{\frac{k}{2},j} = \frac{k}{2} + \eta_{\frac{k}{2},j},\ k\in\mathbb{Z}, j = 1,\dots,j_{\max}(k/2),\ \text{and } \eta_{\frac{k}{2},j} \in [0,1/2) \right\}, \tag{3.4.1}$$

where the sampling points in Λ are ordered by magnitude,

$$\cdots < x_{\frac{k}{2},j} < x_{\frac{k}{2},j+1} < \cdots < x_{\frac{k+1}{2},j} < x_{\frac{k+1}{2},j+1} < \dots \quad .$$

The $x_{k/2,j}$ are consecutive points in the interval $[k/2, k/2+1/2)$,

$$\delta_{\frac{k}{2}} = \max_{j=1,\dots j_{\max}(\frac{k}{2})-1} (x_{\frac{k}{2},j+1} - x_{\frac{k}{2},j})$$

is the maximum gap between the samples, and $j_{\max}(\frac{k}{2})$ depends on the specified interval. We define $y_{\frac{k}{2},j} = \frac{1}{2}(x_{\frac{k}{2},j} + x_{\frac{k}{2},j+1})$ to be the midpoints between the samples, $y_{\frac{k}{2},0} = \frac{k}{2}$, and $y_{\frac{k}{2},j_{\max}(\frac{k}{2})} = \frac{k}{2} + \frac{1}{2}$, and set $\chi_{\frac{k}{2},j} = \chi_{[y_{\frac{k}{2},j-1}, y_{\frac{k}{2},j})}$.

Theorem 3.10 (Sufficient conditions for sampling for Wilson basis)**.** *Let $g \in \mathcal{C}^1(\mathbb{R})$ be a real-valued, even function with* $\operatorname{supp}(g) \subseteq [-m, m]$ *that generates an orthonormal Wilson basis. Let $PW^2_b(g, \mathbb{R})$ be the space of variable bandwidth defined in* (3.0.3) *and let $\Lambda \subseteq \mathbb{R}$ be as in* (3.4.1) *with $x_{\frac{k}{2},1} - \frac{k}{2} \leq \frac{1}{2}\delta_{\frac{k}{2}}$ and $\frac{k+1}{2} - x_{\frac{k}{2},j_{\max}(\frac{k}{2})} \leq \frac{1}{2}\delta_{\frac{k}{2}}$. Define*

$$D \coloneqq (4m) \cdot \max\{2\pi\|g\|_\infty, \|g'\|_\infty\}$$

and

$$\mu_{\frac{k}{2}} \coloneqq \max_{n \in (k-2m, k+2m+1) \cap \mathbb{Z}} \Big(b(n) + 1\Big). \tag{3.4.2}$$

If

$$\sup_{k \in \mathbb{Z}} \mu_{\frac{k}{2}} \delta_{\frac{k}{2}} < \frac{\pi}{D}, \tag{3.4.3}$$

then Λ is a sampling set for $PW^2_b(g, \mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \leq \sum_{k \in \mathbb{Z}} \sum_{j=1}^{j_{\max}(\frac{k}{2})} |f(x_{\frac{k}{2},j})|^2 w_{\frac{k}{2},j} \leq B\|f\|_2^2 \qquad \text{for all } f \in PW^2_b(g, \mathbb{R}), \tag{3.4.4}$$

where $w_{\frac{k}{2},j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in PW^2_b(g, \mathbb{R})$ can be reconstructed completely from the samples in Λ.

Proof. Let $f \in PW^2_b(g, \mathbb{R})$. We recall from Section 2.1 the idea of the proof, which consists of creating an approximation operator A that is built using only the evaluation of the function at the sampling points $f\big(x_{\frac{k}{2},j}\big)$ and show that $\|f - Af\|_2 \leq \gamma\|f\|_2$ with $\gamma < 1$. We apply Lemma 1.17 with $\mathcal{K} = PW^2_b(g, \mathbb{R})$, $[\alpha_k, \alpha_{k+1}) = [k/2, k/2 + 1/2)$ for every $k \in \mathbb{Z}$, Λ as in (3.4.1), and we take the usual approximation operator formed by a piecewise constant function

$$Af = P\Big(\sum_{k \in \mathbb{Z}} \sum_{j=1}^{j_{\max}(\frac{k}{2})} f\big(x_{\frac{k}{2},j}\big)\chi_{\frac{k}{2},j}\Big), \tag{3.4.5}$$

where P is the orthogonal projection from $L^2(\mathbb{R})$ onto $PW^2_b(g, \mathbb{R})$. We obtain

$$\|f - Af\|_2^2 \leq \sum_{k \in \mathbb{Z}} \frac{1}{\pi^2}\delta^2_{\frac{k}{2}} \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx.$$

Step 1. Estimate of local smoothness $\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx$.

We consider the expansion of f in terms of a trigonometric polynomial as (3.1.1)

$$f(x) = \sum_{n \in \mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l}\Psi_{n,l}(x) = \sum_{n \in \mathbb{Z}} P_n(x) g(x - n/2),$$

and $\sum_{n\in\mathbb{Z}}\|P_n\|_2^2 = \|f\|_2^2$. The derivative f' is then

$$f'(x) = \sum_{n\in\mathbb{Z}} \Big(P_n'(x)T_{\frac{n}{2}}g(x) + P_n(x)T_{\frac{n}{2}}g'(x)\Big).$$

Since $\operatorname{supp}(g) \subseteq [-m, m]$, the sum is locally finite. Assuming $x \in \big[\frac{k}{2}, \frac{k}{2} + \frac{1}{2}\big)$ requires $x - \frac{n}{2} \in (-m, m)$. This restricts the summation indices $n \in (k - 2m, k + 1 + 2m) \cap \mathbb{Z}$. We abbreviate this interval by

$$F_k \coloneqq (k - 2m, k + 1 + 2m) \cap \mathbb{Z}.$$

Then

$$\begin{aligned}
&\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx \\
&\quad= \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big|\sum_{n\in F_k} \Big(P_n'(x)T_{\frac{n}{2}}g(x) + P_n(x)T_{\frac{n}{2}}g'(x)\Big)\Big|^2 dx \\
&\quad= \sum_{n,i\in F_k} \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big(P_n'(x)T_{\frac{n}{2}}g(x) + P_n(x)T_{\frac{n}{2}}g'(x)\Big)\overline{\Big(P_i'(x)T_{\frac{i}{2}}g(x) + P_i(x)T_{\frac{i}{2}}g'(x)\Big)} dx \\
&\quad= \sum_{n,i\in F_k} \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big(P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i'(x)T_{\frac{i}{2}}g(x)} \\
&\qquad\qquad + 2\operatorname{Re}\Big\{P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i(x)T_{\frac{i}{2}}g'(x)}\Big\} \\
&\qquad\qquad + P_n(x)T_{\frac{n}{2}}g'(x)\overline{P_i(x)T_{\frac{i}{2}}g'(x)}\Big) dx.
\end{aligned} \tag{3.4.6}$$

To bound the first term, we apply Bernstein's inequality (1.3.2) to the polynomial P_n of degree $b(n)$ with $|I| = 1/2$, $\beta = 1$. Then $\lceil\beta|I|\rceil = 1$ and we use the estimate $\|T_{\frac{n}{2}}gT_{\frac{i}{2}}\overline{g}\|_\infty \leq \|g\|_\infty^2$. We obtain

$$\begin{aligned}
\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big(P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i'(x)T_{\frac{i}{2}}g(x)}\Big) dx &\leq \|T_{\frac{n}{2}}gT_{\frac{i}{2}}\overline{g}\|_\infty\|P_n'\|_2\|P_i'\|_2 \\
&\leq \|g\|_\infty^2 4\pi^2 b(n)b(i)\|P_n\|_2\|P_i\|_2.
\end{aligned}$$

Summing over n and i, we get

$$\begin{aligned}
&\sum_{n,i\in F_k} \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big(P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i'(x)T_{\frac{i}{2}}g(x)}\Big) dx \\
&\quad\leq 4\pi^2\|g\|_\infty^2 \sum_{n,i\in F_k} b(n)b(i)\|P_n\|_2\|P_i\|_2 \\
&\quad= 4\pi^2\|g\|_\infty^2 \Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)^2.
\end{aligned} \tag{3.4.7}$$

For the middle terms of (3.4.6) we obtain in a similar way that

$$\begin{aligned}
&2\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \operatorname{Re}\left\{P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i(x)T_{\frac{i}{2}}g'(x)}\right\}dx \\
&\leq 2\|T_{\frac{n}{2}}gT_{\frac{i}{2}}\overline{g'}\|_\infty \|P_n'\|_2\|P_i\|_2 \\
&\leq 4\pi\|g\|_\infty\|g'\|_\infty b(n)\|P_n\|_2\|P_i\|_2.
\end{aligned}$$

After summing over n and i, we have

$$\begin{aligned}
&\sum_{n,i\in F_k}\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} 2\operatorname{Re}\Big(P_n'(x)T_{\frac{n}{2}}g(x)\overline{P_i(x)T_{\frac{i}{2}}g'(x)}\Big)dx \\
&\qquad\leq 4\pi\|g\|_\infty\|g'\|_\infty\Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|P_n\|_2\Big).
\end{aligned} \tag{3.4.8}$$

The last term of (3.4.6) is bounded by

$$\begin{aligned}
&\sum_{n,i\in F_k}\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} \Big(P_n(x)T_{\frac{n}{2}}g'(x)\overline{P_i(x)T_{\frac{i}{2}}g'(x)}\Big)dx \\
&\qquad\leq \sum_{n,i\in F_k}\|T_{\frac{n}{2}}g'T_{\frac{i}{2}}\overline{g'}\|_\infty\|P_n\|_2\|P_i\|_2 \\
&\qquad\leq \|g'\|_\infty^2\Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2.
\end{aligned} \tag{3.4.9}$$

Adding all terms, the local estimate for the derivative is

$$\begin{aligned}
\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2dx &\leq 4\pi^2\|g\|_\infty^2\Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)^2 + \|g'\|_\infty^2\Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2 \\
&\qquad + 4\pi\|g\|_\infty\|g'\|_\infty\Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|P_n\|_2\Big) \\
&= \Big(2\pi\|g\|_\infty\sum_{n\in F_k} b(n)\|P_n\|_2 + \|g'\|_\infty\sum_{n\in F_k}\|P_n\|_2\Big)^2 \\
&\leq \max\{2\pi\|g\|_\infty, \|g'\|_\infty\}^2\Big(\sum_{n\in F_k}\|P_n\|_2\big(b(n)+1\big)\Big)^2.
\end{aligned}$$

Therefore, by taking the sum over all $k\in\mathbb{Z}$, we get

$$\begin{aligned}
\|f-Af\|_2^2 &\leq \sum_{k\in\mathbb{Z}}\frac{1}{\pi^2}\delta_{\frac{k}{2}}^2\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}}|f'(x)|^2dx \\
&\leq \frac{1}{\pi^2}\max\{2\pi\|g\|_\infty, \|g'\|_\infty\}^2\sum_{k\in\mathbb{Z}}\delta_{\frac{k}{2}}^2\Big(\sum_{n\in F_k}\|P_n\|_2\big(b(n)+1\big)\Big)^2.
\end{aligned}$$

We define
$$\mu_{\frac{k}{2}} := \max_{n\in(k-2m,k+2m+1)\cap\mathbb{Z}} \Big(b(n)+1\Big)$$
to be the maximal bandwidth over the interval $[k/2, k/2+1/2)$. Then we write
$$\|f-Af\|_2^2 \le \frac{1}{\pi^2}\max\{2\pi\|g\|_\infty, \|g'\|_\infty\}^2 \sum_{k\in\mathbb{Z}} \delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2 \Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2.$$

Step 2. Final estimate for $\|f-Af\|_2$.
We interpret $\sum_{n\in F_k}\|P_n\|_2$ as a convolution. Let $q:\mathbb{Z}\to\mathbb{R}$ be the sequence $q(n)=\|P_n\|_2$ and $\chi_{(-2m-1,2m)}$ be the characteristic function of $(-2m-1,2m)$. Then
$$\begin{aligned}\Big(q*\chi_{(-2m-1,2m)}\Big)(k) &= \sum_{n\in\mathbb{Z}} q(n)\chi_{(-2m-1,2m)}(k-n) = \sum_{n\in\mathbb{Z}} q(n)\chi_{(k-2m,k+2m+1)}(n)\\ &= \sum_{n\in F_k} q(n) = \sum_{n\in F_k}\|P_n\|_2.\end{aligned}$$
We can rewrite the error as
$$\|f-Af\|_2^2 \le \frac{1}{\pi^2}\max\{2\pi\|g\|_\infty, \|g'\|_\infty\}^2 \sum_{k\in\mathbb{Z}} \delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2 \Big(q*\chi_{(-2m-1,2m)}(k)\Big)^2.$$
The right-hand side can readily be estimated with Young's inequality as
$$\begin{aligned}\sum_{k\in\mathbb{Z}} \delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2 \Big(q*\chi_{(-2m-1,2m)}(k)\Big)^2 &\le \sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)\|q*\chi_{(-2m-1,2m)}\|_2^2\\ &\le \sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)\|\chi_{(-2m-1,2m)}\|_1^2\|q\|_2^2\\ &\le \sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)(4m)^2\sum_{k\in\mathbb{Z}}\|P_k\|_2^2\end{aligned}$$
since $\|\chi_{(-2m-1,2m)}\|_1 = 4m$. Defining $D := 4m\max\{2\pi\|g\|_\infty, \|g'\|_\infty\}$, we get
$$\|f-Af\|_2^2 \le \frac{1}{\pi^2}D^2\sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)\sum_{k\in\mathbb{Z}}\|P_k\|_2^2.$$
Therefore, the error constant $\gamma^2 := \frac{1}{\pi^2}D^2\sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)$ satisfies $\gamma<1$ by (3.4.3). Hence,
$$\|f-Af\|_2^2 \le \gamma^2\sum_{k\in\mathbb{Z}}\|P_k\|_2^2 = \gamma^2\sum_{k\in\mathbb{Z}}\sum_{l=0}^{b(k)}|\langle f,\psi_{k,l}\rangle|^2 = \gamma^2\|f\|_2^2 \tag{3.4.10}$$
and the operator A is invertible on $PW_b^2(g,\mathbb{R})$. By Lemma 1.18, the weighted sampling inequality (3.4.4) holds with $A = \Big(1-\frac{D}{\pi}\sqrt{\sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)}\Big)^2$ and $B = \Big(1+\frac{D}{\pi}\sqrt{\sup_{k\in\mathbb{Z}}\big(\delta_{\frac{k}{2}}^2 \mu_{\frac{k}{2}}^2\big)}\Big)^2$.
□

Remark. An alternative sufficient condition for sampling which uses the maximal bandwidth over an interval can be obtained by applying the convolution with $q(n) = \|P_n\|_2$ and $\chi_{(-2m-1,2m)}$ already at (3.4.7), (3.4.8) and (3.4.9). Let $\nu_{k/2} \coloneqq \max_{n\in(k-2m,k+2m+1)\cap\mathbb{Z}} b(n)$, then

$$\sum_{k\in\mathbb{Z}} (3.4.7) = \sum_{k\in\mathbb{Z}} 4\pi^2\|g\|_\infty^2\Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)^2 = \sum_{k\in\mathbb{Z}} 4\pi^2\|g\|_\infty^2\nu_{\frac{k}{2}}^2\big(q*\chi_{(-2m-1,2m)}(k)\big)^2.$$

$$\begin{aligned}\sum_{k\in\mathbb{Z}} (3.4.8) &= \sum_{k\in\mathbb{Z}} 4\pi\|g\|_\infty\|g'\|_\infty\Big(\sum_{n\in F_k} b(n)\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|P_n\|_2\Big) \\ &= \sum_{k\in\mathbb{Z}} 4\pi\|g\|_\infty\|g'\|_\infty\nu_{\frac{k}{2}}\big(q*\chi_{(-2m-1,2m)}(k)\big)^2.\end{aligned}$$

$$\sum_{k\in\mathbb{Z}} (3.4.9) = \sum_{k\in\mathbb{Z}}\|g'\|_\infty^2\Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2 = \sum_{k\in\mathbb{Z}}\|g'\|_\infty^2\big(q*\chi_{(-2m-1,2m)}(k)\big)^2.$$

We apply Young's inequality to obtain

$$\begin{aligned}\|f-Af\|_2^2 &\le \frac{1}{\pi^2}\sum_{k\in\mathbb{Z}}\delta_{\frac{k}{2}}^2(2\pi\|g\|_\infty\nu_{\frac{k}{2}}+\|g'\|_\infty)^2\big(q*\chi_{(-2m-1,2m)}(k)\big)^2 \\ &\le \frac{1}{\pi^2}\sup_{k\in\mathbb{Z}}\big\{\delta_{\frac{k}{2}}^2(2\pi\|g\|_\infty\nu_{\frac{k}{2}}+\|g'\|_\infty)^2\big\}\|q*\chi_{(-2m-1,2m)}\|_2^2 \\ &\le \frac{1}{\pi^2}(4m)^2\sup_{k\in\mathbb{Z}}\big\{\delta_{\frac{k}{2}}^2(2\pi\|g\|_\infty\nu_{\frac{k}{2}}+\|g'\|_\infty)^2\big\}\sum_{k\in\mathbb{Z}}\|P_k\|_2^2.\end{aligned}$$

By setting

$$\sup_{k\in\mathbb{Z}}\delta_{\frac{k}{2}}(2\pi\|g\|_\infty\nu_{\frac{k}{2}}+\|g'\|_\infty) < \frac{\pi}{4m},$$

we have a different maximal gap condition than the one of Theorem 3.10.

Example B. Consider the window function $g(x) = \sqrt{2}\cos(\pi x)\chi_{[-1/2,1/2]}(x)$, illustrated in Figure 5.1. Applying the characterization (iii) from Theorem 3.3, it is evident that the corresponding Wilson system $\{\psi_{n,l}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Despite the lack of differentiability of g at points $\{-1/2, 1/2\}$, Theorem 3.10 remains applicable. Its proof merely necessitates the differentiability of g on each interval $(k/2, k/2+1/2)$ with one-sided derivatives at $k/2$, with $k\in\mathbb{Z}$.

This Wilson basis underlines a critical aspect of Theorem 3.10, demonstrating that the local quantity $\mu_{k/2} = \max_{n\in(k-2m,k+2m+1)\cap\mathbb{Z}}\big(b(n)+1\big)$ must be addressed in the sufficient condition. Given that $\operatorname{supp}(g) = (-1/2, 1/2)$, only two translations of g overlap. Consequently, when a general function $f = \sum_n P_n(x)g(x-n/2)$ is restricted to $[0,1/2]$, the expression simplifies to:

$$
\begin{aligned}
f\Big|_{[0,\frac{1}{2}]} &= P_0(x)g(x) + P_1(x)g\Big(x-\frac{1}{2}\Big) \\
&= P_0(x)\cos\pi x + P_1(x)\sin\pi x \\
&= \frac{e^{-i\pi x}}{2}\Big(P_0(x)(e^{2\pi i x}+1) - iP_1(x)(e^{2\pi i x}-1)\Big).
\end{aligned}
$$

If $\deg P_0 = b(0)$ and $\deg P_1 = b(1)$, then $e^{-i\pi x} f\Big|_{[0,\frac{1}{2}]}$ represents a trigonometric polynomial of degree $\max\{b(0), b(1)\} + 1$, in alignment with definition (3.4.2). The formulation of Theorem 3.10 extends this insight, emphasizing the significance of $\mu_{k/2}$ as a more precise measure of local bandwidth in comparison to $b(k)$.

3.5 Sufficient conditions for sampling - Wilson Riesz basis

The same methodology can be employed when substituting a Wilson orthonormal basis with a Wilson Riesz basis. Interestingly, the use of $\{\psi_{n,l}\}_{n\in\mathbb{Z},l=0,\dots,b(n)}$ as an orthonormal basis is only crucial in the final equality of (3.4.10). However, if we assume $\{\psi_{n,l}\}_{n\in\mathbb{Z},l=0,\dots,b(n)}$ to be a Wilson Riesz basis, the sufficient condition needs to be adjusted to take care of the lower Riesz bound. Moreover, choosing a Wilson Riesz basis allows us to have more freedom in the choice of the window function g since the orthogonality of the basis elements is not required. The decision to use a Wilson Riesz basis for the construction of a variable bandwidth space was made to be able to present a sufficient condition that highlights the influence of the neighboring windows in the reconstruction in the style of Theorem 2.3. We will be able to make a direct comparison with the result in Section 2.1 and we will see that also for the space of variable bandwidth constructed with a Wilson Riesz basis, the bandwidths of the two central windows have a stronger impact on the density of the points that make Λ a sampling set.

The next result is a sufficient condition for sampling theorem for the space of variable bandwidth $\widetilde{PW}^2_b(g,\mathbb{R})$ constructed using a Wilson Riesz basis. The following result is included in [6].

Theorem 3.11 (Sufficient conditions for sampling for Wilson Riesz basis)**.** *Let $g \in \mathcal{C}^1(\mathbb{R})$ with* $\operatorname{supp}(g) \subseteq [-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$ *and $0 < \varepsilon < 1/2$, and such that g generates a Wilson Riesz basis. Assume g satisfies the following conditions:*

(i) $g(x) = g(-x)$, *for every $x \in \mathbb{R}$.*

(ii) $g(x) = 1$, *for $x \in [-\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon]$.*

(iii) $g(1-x) = 1 - g(x)$, *for $x \in [\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$.*

(iv) g is decreasing in $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$.

Let $\widetilde{PW}_b^2(g,\mathbb{R})$ be the space of variable bandwidth defined in (3.0.3) *constructed with a Wilson Riesz basis with lower bound A' and let $\Lambda \subseteq \mathbb{R}$ be as in* (3.4.1) *with $x_{\frac{k}{2},1} - \frac{k}{2} \leq \frac{1}{2}\delta_{\frac{k}{2}}$ and $\frac{k+1}{2} - x_{\frac{k}{2},j_{\max}(\frac{k}{2})} \leq \frac{1}{2}\delta_{\frac{k}{2}}$. If*

$$\frac{8}{A'}\sup_{k\in\mathbb{Z}}\left\{\left(\delta^2_{\frac{k}{2}-1} + 4\delta^2_{\frac{k}{2}-\frac{1}{2}} + 4\delta^2_{\frac{k}{2}} + \delta^2_{\frac{k}{2}+\frac{1}{2}}\right)\left(b^2(k) + \frac{\|g'\|^2_\infty}{\pi^2}\right)\right\} < 1 \tag{3.5.1}$$

then Λ is a sampling set for $\widetilde{PW}_b^2(g,\mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}}\sum_{j=1}^{j_{\max}(\frac{k}{2})} |f(x_{\frac{k}{2},j})|^2 w_{\frac{k}{2},j} \leq B\|f\|_2^2 \qquad \textit{for all } f \in \widetilde{PW}_b^2(g,\mathbb{R})\,,$$

where $w_{\frac{k}{2},j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in \widetilde{PW}_b^2(g,\mathbb{R})$ can be reconstructed completely from the samples in Λ.

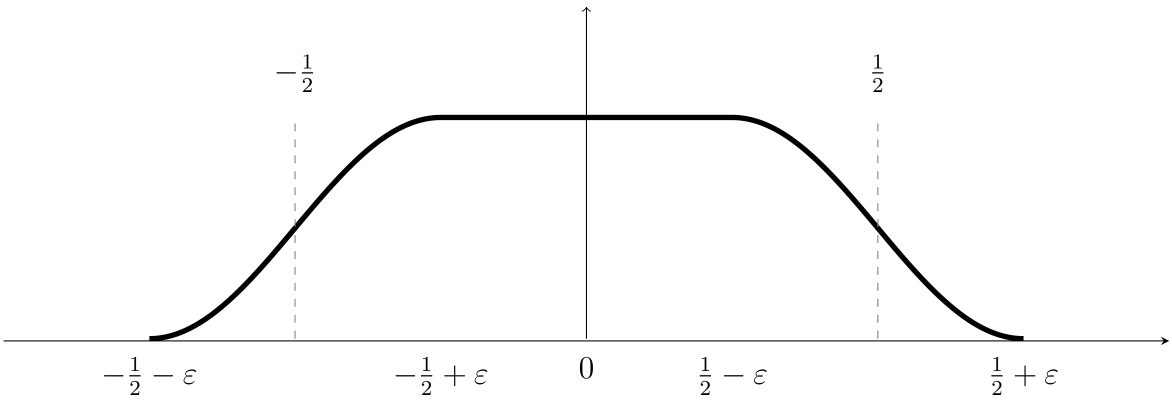


Figure 3.2: Example of a window function g which satisfies (i), (ii), (iii), and (iv).

The proof follows the same core concept as Theorem 3.10, with certain technical adjustments.

Proof. Let $f \in \widetilde{PW}_b^2(g,\mathbb{R})$. Applying Lemma 1.17 with $\mathcal{K} = \widetilde{PW}_b^2(g,\mathbb{R})$, $[\alpha_k,\alpha_{k+1}) = [k/2, k/2+1/2)$ for every $k \in \mathbb{Z}$, and defining an approximation operator Af as presented in (3.4.5), we derive the following inequality:

$$\|f - Af\|_2^2 \leq \sum_{k\in\mathbb{Z}} \frac{1}{\pi^2}\delta^2_{\frac{k}{2}} \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx.$$

Step 1. Estimate of local smoothness $\int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx$.

Consider $x \in \left[\frac{k}{2}, \frac{k}{2}+\frac{1}{2}\right)$, $x = \frac{k}{2}+\eta$ where $\eta \in [0,1/2)$. Then

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(k/2+\eta)g(k/2+\eta-n/2).$$

Since g is compactly supported, the only terms that remain are $n = k$, $n = k \pm 1$ and $n = k + 2$. Hence,

$$\begin{aligned} f(x) = P_k(\eta + k/2)g(\eta) + P_{k-1}(\eta + k/2)g(\eta + 1/2) \\ + P_{k+1}(\eta + k/2)g(\eta - 1/2) + P_{k+2}(\eta + k/2)g(\eta - 1). \end{aligned}$$

Since $\operatorname{supp}(g) \subseteq [-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ and $g(\eta) = 1$ for $\eta \in [-\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon]$, we have:

- $g(\eta + \frac{1}{2}) \neq 0$, only if $\eta \in [0, \varepsilon]$.
- $g(\eta - 1) \neq 0$, only if $\eta \in [\frac{1}{2} - \varepsilon, \frac{1}{2})$.
- $g'(\eta) \neq 0$, only if $\eta \in [\frac{1}{2} - \varepsilon, \frac{1}{2})$.
- $g'(\eta - \frac{1}{2}) \neq 0$, only if $\eta \in [0, \varepsilon]$.

Moreover, by properties (ii) and (iv) we get that $\|g\|_\infty = 1$. By taking the derivative of $f(x)$, we get

$$\begin{aligned} f'(x) = P'_k(\eta + k/2)g(\eta) + P'_{k-1}(\eta + k/2)g(\eta + 1/2)\chi_{[0,\varepsilon]}(\eta) \\ + P'_{k+1}(\eta + k/2)g(\eta - 1/2) + P'_{k+2}(\eta + k/2)g(\eta - 1)\chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2})}(\eta) \\ + P_k(\eta + k/2)g'(\eta)\chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2})}(\eta) + P_{k-1}(\eta + k/2)g'(\eta + 1/2)\chi_{[0,\varepsilon]}(\eta) \\ + P_{k+1}(\eta + k/2)g'(\eta - 1/2)\chi_{[0,\varepsilon]}(\eta) + P_{k+2}(\eta + k/2)g'(\eta - 1)\chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2})}(\eta). \end{aligned}$$

We bound the L^2-norm of each term and we start by computing a bound for the terms containing the derivative of the trigonometric polynomial using Bernstein's inequality (1.3.2) with $\beta = 1$ and $\lceil \beta |I| \rceil = 1$.

- $\|P'_k g\|_2^2 \leq \Big(\sup_{x \in [0, \frac{1}{2}]} |g(x)| \Big)^2 \|P'_k\|_2^2 \leq (2\pi b(k))^2 \|P_k\|_2^2.$
- $\|P'_{k+1} g(\cdot - 1/2)\|_2^2 \leq \Big(\sup_{x \in [0, \frac{1}{2}]} |g(x - 1/2)| \Big)^2 \|P'_{k+1}\|_2^2 \leq (2\pi b(k+1))^2 \|P_{k+1}\|_2^2.$

Note that $g(\cdot - 1)$ is increasing in $[\frac{1}{2} - \varepsilon, \frac{1}{2}]$, therefore the maximum is achieved at $x = 1/2$. By symmetry and property (iii), we have that $g(1/2 - 1) = g(1 - 1/2) = 1 - g(1/2)$. Hence, $g(1/2) = 1/2$. In a similar way we can bound the $\sup_{x \in [0, \varepsilon]} |g(x + 1/2)|$ to obtain

- $\|P'_{k-1} g(\cdot + 1/2)\chi_{[0,\varepsilon]}\|_2^2 \leq \Big(\sup_{x \in [0, \varepsilon]} |g(x + 1/2)| \Big)^2 \|P'_{k-1}\|_2^2 \leq \dfrac{1}{4}(2\pi b(k-1))^2 \|P_{k-1}\|_2^2.$
- $\|P'_{k+2} g(\cdot - 1)\chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2}]}\|_2^2 \leq \Big(\sup_{x \in [\frac{1}{2}-\varepsilon, \frac{1}{2}]} |g(x - 1)| \Big)^2 \|P'_{k+2}\|_2^2 \leq \dfrac{1}{4}(2\pi b(k+2))^2 \|P_{k+2}\|_2^2.$

We bound now the terms that contain the derivative of g.

- $\|P_k g' \chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2}]}\|_2^2 \leq \Big(\sup_{x\in[\frac{1}{2}-\varepsilon,\frac{1}{2}]} |g'(x)| \Big)^2 \|P_k\|_2^2 \leq \|g'\|_\infty^2 \|P_k\|_2^2.$
- $\|P_{k+1} g'(\cdot - 1/2) \chi_{[0,\varepsilon]}\|_2^2 \leq \Big(\sup_{x\in[0,\varepsilon]} |g'(x-1/2)| \Big)^2 \|P_{k+1}\|_2^2 \leq \|g'\|_\infty^2 \|P_{k+1}\|_2^2.$
- $\|P_{k-1} g'(\cdot + 1/2) \chi_{[0,\varepsilon]}\|_2^2 \leq \Big(\sup_{x\in[0,\varepsilon]} |g'(x+1/2)| \Big)^2 \|P_{k-1}\|_2^2 \leq \|g'\|_\infty^2 \|P_{k-1}\|_2^2.$
- $\|P_{k+2} g'(\cdot - 1) \chi_{[\frac{1}{2}-\varepsilon,\frac{1}{2}]}\|_2^2 \leq \Big(\sup_{x\in[\frac{1}{2}-\varepsilon,\frac{1}{2}]} |g'(x-1)| \Big)^2 \|P_{k+2}\|_2^2 \leq \|g'\|_\infty^2 \|P_{k+2}\|_2^2.$

Step 2. Final estimate for $\|f - Af\|_2$.
We recall that $|\sum_{i\in F} f_i|^2 \leq \#F \sum_{i\in F} |f_i|^2$ and we add the bounds. We get

$$
\begin{aligned}
\frac{1}{\pi^2}\delta_{\frac{k}{2}}^2 \int_{\frac{k}{2}}^{\frac{k}{2}+\frac{1}{2}} |f'(x)|^2 dx \leq{}& \frac{8}{\pi^2}\delta_{\frac{k}{2}}^2 \Big\{ \|P_k\|_2^2 \Big[(2\pi b(k))^2 + \|g'\|_\infty^2 \Big] \\
&+ \|P_{k+1}\|_2^2 \Big[(2\pi b(k+1))^2 + \|g'\|_\infty^2 \Big] \\
&+ \|P_{k-1}\|_2^2 \Big[(\pi b(k-1))^2 + \|g'\|_\infty^2 \Big] \\
&+ \|P_{k+2}\|_2^2 \Big[(\pi b(k+2))^2 + \|g'\|_\infty^2 \Big] \Big\} \\
\leq{}& \frac{8}{\pi^2}\delta_{\frac{k}{2}}^2 \Big\{ 4\|P_k\|_2^2 \Big[(\pi b(k))^2 + \|g'\|_\infty^2 \Big] \\
&+ 4\|P_{k+1}\|_2^2 \Big[(\pi b(k+1))^2 + \|g'\|_\infty^2 \Big] \\
&+ \|P_{k-1}\|_2^2 \Big[(\pi b(k-1))^2 + \|g'\|_\infty^2 \Big] \\
&+ \|P_{k+2}\|_2^2 \Big[(\pi b(k+2))^2 + \|g'\|_\infty^2 \Big] \Big\}.
\end{aligned}
$$

Summing all the terms, we obtain

$$
\|f - Af\|_2^2 \leq 8 \sum_{k\in\mathbb{Z}} \Big\{ \|P_k\|_2^2 \Big(\delta_{\frac{k}{2}-1}^2 + 4\delta_{\frac{k}{2}-\frac{1}{2}}^2 + 4\delta_{\frac{k}{2}}^2 + \delta_{\frac{k}{2}+\frac{1}{2}}^2 \Big) \Big(b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2} \Big) \Big\}.
$$

We define

$$
D := 8 \sup_{k\in\mathbb{Z}} \Big\{ \Big(\delta_{\frac{k}{2}-1}^2 + 4\delta_{\frac{k}{2}-\frac{1}{2}}^2 + 4\delta_{\frac{k}{2}}^2 + \delta_{\frac{k}{2}+\frac{1}{2}}^2 \Big) \Big(b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2} \Big) \Big\}.
$$

Applying Theorem 1.3 and setting $\{\tilde{\psi}_{k,l}\}_{k\in\mathbb{Z},l=0,\dots,b(k)}$ to be the dual Riesz basis of $\{\psi_{k,l}\}_{k\in\mathbb{Z},l=0,\dots,b(k)}$ with upper bound $1/A'$ we have

$$
\|f - Af\|_2^2 \leq D \sum_{k\in\mathbb{Z}} \|P_k\|_2^2 = D \sum_{k\in\mathbb{Z}} \sum_{l=0}^{b(k)} |c_{k,l}|^2 = D \sum_{k\in\mathbb{Z}} \sum_{l=0}^{b(k)} |\langle f, \tilde{\psi}_{k,l} \rangle|^2 \leq \frac{D}{A'} \|f\|_2^2.
$$

By condition (3.5.1), we have $D < A'$, therefore $\gamma^2 := D/A' < 1$ and

$$\|f - Af\|_2^2 \leq \gamma^2 \|f\|_2^2.$$

Again, the sampling inequality follows from Lemma 1.18 with $A = \big(1 - \sqrt{\frac{D}{A'}}\big)^2$ and $B = \big(1 + \sqrt{\frac{D}{A'}}\big)^2$, which concludes the proof. $\square$

3.6 Paley-Wiener space embeddings

To better understand the spaces of variable bandwidth it is important to explore their relation with the Paley-Wiener spaces. There is no Wilson basis with compactly supported window g, such that the variable bandwidth space $PW_b^2(g, \mathbb{R})$ is contained in a Paley-Wiener space because the basis elements must then consist of bandlimited functions. However, with a bandlimited atom, one can compare spaces of variable bandwidth and Paley-Wiener spaces.

Example C. Consider the bandlimited function $g(x) = \frac{\sin \pi x}{\pi x}$ with Fourier transform $\hat{g} = \chi_{[-1/2,1/2]}$. Using Theorem 3.4 it can be proved that g generates a orthonormal Wilson basis. By taking $b(n) = 0$ for all n, we obtain that $PW_b^2(g, \mathbb{R}) = \operatorname{span}\{T_n g : n \in \mathbb{Z}\}$ coincides with the classical Paley-Wiener space $PW_{1/2} = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-1/2, 1/2]\}$. In this sense, the spaces of variable bandwidth are a generalization of the Paley-Wiener space. However, if we consider a frequency truncation sequence b satisfying $1 \leq \Omega_0 \leq b(n) \leq \Omega_1$, then

$$PW_{\Omega_0 - 1/2} \subseteq PW_b^2(g, \mathbb{R}) \subseteq PW_{\Omega_1 + 1/2}\,,$$

and both inclusions are strict.

With small modifications, such strict embeddings also hold for more general bandlimited generators with $\operatorname{supp}(\hat{g}) \subseteq [-1, 1]$.

Proposition 3.12. *Let $\hat{g}$ be a real-valued function with $\operatorname{supp}(\hat{g}) \subseteq [-1, 1]$ such that g generates a orthonormal Wilson basis and let $1 < \Omega_0 \leq b(n) \leq \Omega_1$ for every $n \in \mathbb{Z}$. Let $PW_b^2(g, \mathbb{R})$ be the space of variable bandwidth defined in* (3.0.3) *and PW_Ω be the Paley-Wiener space defined in* (0.0.1). *Then*

$$PW_{\Omega_0 - 1} \subseteq PW_b^2(g, \mathbb{R}) \subseteq PW_{\Omega_1 + 1}.$$

Proof. **Step 1. $\boldsymbol{PW_{\Omega_0-1} \subseteq PW_b^2(g, \mathbb{R})}$.**
Let $f \in PW_{\Omega_0 - 1}$, then $f \in L^2(\mathbb{R})$ and therefore f can be expand in terms of an orthonormal Wilson basis as

$$f = \sum_{n \in \mathbb{Z}} \sum_{l=0}^{\infty} c_{n,l} \psi_{n,l} = \sum_{n \in \mathbb{Z}} \sum_{l=0}^{\infty} \langle f, \psi_{n,l} \rangle \psi_{n,l}.$$

Claim: If $c_{n,l} = \langle f, \psi_{n,l}\rangle = 0$ for every $l \geq \Omega_0$ and for every $n \in \mathbb{Z}$, then $f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lceil\Omega_0\rceil} c_{n,l}\psi_{n,l}$ and $f \in PW_b^2(g,\mathbb{R})$ with $b(n) \geq \Omega_0$ for every $n \in \mathbb{Z}$.
Note that by applying the Fourier transform to the Wilson basis functions (3.0.1) we get

$$\hat{\psi}_{n,l}(\omega) = \begin{cases} M_{-n}\hat{g}(\omega), & l = 0 \\ \frac{1}{\sqrt{2}}e^{-\pi i l n}M_{-\frac{n}{2}}\left[\hat{g}(\omega - l) + (-1)^{l+n}\hat{g}(\omega + l)\right], & l \neq 0. \end{cases} \tag{3.6.1}$$

for $n, l \in \mathbb{Z}$ and $l \geq 0$.

Fix $l = \lceil\Omega_0\rceil$. Then by Parseval's formula, we have

$$\langle f, \psi_{n,\lceil\Omega_0\rceil}\rangle = \langle \hat{f}, \hat{\psi}_{n,\lceil\Omega_0\rceil}\rangle = \Big\langle \hat{f}, \frac{1}{\sqrt{2}}e^{-\pi i\lceil\Omega_0\rceil n}M_{-\frac{n}{2}}\left[T_{\lceil\Omega_0\rceil}\hat{g} + (-1)^{\lceil\Omega_0\rceil+n}T_{-\lceil\Omega_0\rceil}\hat{g}\right]\Big\rangle.$$

Since $f \in PW_{\Omega_0-1}$, then $\operatorname{supp}(\hat{f}) \subseteq [-\Omega_0+1, \Omega_0-1]$. Therefore $\operatorname{supp}(\hat{f}) \cap \operatorname{supp}(T_{\lceil\Omega_0\rceil}\hat{g}) = \emptyset$ and $\operatorname{supp}(\hat{f}) \cap \operatorname{supp}(T_{-\lceil\Omega_0\rceil}\hat{g}) = \emptyset$. We have that $c_{n,l} = 0$ for every $l \geq \Omega_0$ and for every $n \in \mathbb{Z}$ and $f \in PW_b^2(g,\mathbb{R})$ with $b(n) \geq \Omega_0$ for all $n \in \mathbb{Z}$.

Step 2. $\boldsymbol{PW_b^2(g,\mathbb{R}) \subseteq PW_{\Omega_1+1}}$.
Let $f \in PW_b^2(g,\mathbb{R})$ with $b(n) \leq \Omega_1$ for every $n \in \mathbb{Z}$. Since $b(n) \in \mathbb{N}$, for every $n \in \mathbb{Z}$, we have $b(n) \leq \lfloor\Omega_1\rfloor$ for every $n \in \mathbb{Z}$ then $f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lfloor\Omega_1\rfloor} c_{n,l}\psi_{n,l}$. We consider the Fourier transform of f and we take the Fourier transform of the Wilson basis functions (3.6.1), to get

$$\hat{f}(\omega) = \sum_{n\in\mathbb{Z}}\sum_{l=0}^{\lfloor\Omega_1\rfloor} c_{n,l}\hat{\psi}_{n,l} = \sum_{n\in\mathbb{Z}}\Big[c_{n,0}M_{-n}\hat{g}(\omega) + \sum_{l=1}^{\lfloor\Omega_1\rfloor}\frac{c_{n,l}}{\sqrt{2}}e^{-\pi i l n}M_{-\frac{n}{2}}\big(\hat{g}(\omega-l)+(-1)^{l+n}\hat{g}(\omega+l)\big)\Big].$$

Since $\operatorname{supp}(\hat{g}) \subseteq [-1,1]$, then $\operatorname{supp}(\hat{f}) \subseteq [-\lfloor\Omega_1\rfloor - 1, \lfloor\Omega_1\rfloor + 1] \subseteq [-\Omega_1 - 1, \Omega_1 + 1]$ and $f \in PW_{\Omega_1+1}$.

□

The proposition shows that even if the classical Paley-Wiener spaces and the spaces of variable bandwidth are comparable and share similar properties, e.g., sampling theorems, $PW_b^2(g,\mathbb{R})$ is not a straight generalization of the classical definition.
The next lemma provides an example of a function g that satisfies Proposition 3.12.

Lemma 3.13. *There exists a window function g that generates a orthonormal Wilson basis such that $\hat{g} \in C^\infty(\mathbb{R})$ is a real-valued function with* $\operatorname{supp}(\hat{g}) \subseteq [-1,1]$.

Proof. We want to find g such that $\hat{g}$ satisfies condition (ii) of Theorem 3.4. Fix $l = 0$ and since $\operatorname{supp}(\hat{g}) \subseteq [-1,1]$, we have $\hat{g}(\omega)\hat{g}(\omega - 2j) = 0$ for every $\omega \in \mathbb{R}$. We need to show that $\sum_{l\in\mathbb{Z}} \hat{g}(\omega - l)^2 = 1$. Since the sum is periodic in ω with period 1, we need to check the condition for $\omega \in [0,1]$. $\hat{g}$ has support in $[-1,1]$ then it suffices to find $\hat{g}(\omega)^2 + \hat{g}(\omega - 1)^2 = 1$ for $\omega \in [0,1]$. Take $\phi \in C^\infty(\mathbb{R})$ such that

$$\phi(\omega) = \begin{cases} 0, & \omega \leq 0 \\ 1, & \omega \geq 1. \end{cases}$$

and $0 \leq \phi(\omega) \leq 1$ for every $\omega \in \mathbb{R}$.

Define

$$\hat{g}(\omega) = \begin{cases} \sin\left(\frac{\pi}{2}\phi(\omega+1)\right), & \omega \le 0 \\ \cos\left(\frac{\pi}{2}\phi(\omega)\right), & \omega \ge 0. \end{cases}$$

As $\phi \in C^\infty(\mathbb{R})$, it implies that $\hat{g} \in C^\infty(\mathbb{R})$. Moreover, $\text{supp}(\hat{g}) \subseteq [-1, 1]$ and $\hat{g}$ satisfies (ii) of Theorem 3.4 and g generates a orthonormal Wilson basis. □

3.7 Comparison and conclusions

(i) The two Theorems 3.10 and 3.11 present distinct sufficient conditions for sampling involving the maximal gap between consecutive points of the sampling set Λ. In the previous chapter, when we studied the space of variable bandwidth constructed with a Gabor Riesz sequence, these two conditions illuminate different aspects of sampling. Theorems 3.10 and 3.11 do the same for $PW_b^2(g, \mathbb{R})$.

Given our model space $PW_b^2(g, \mathbb{R})$ and the local bandwidth on $[k/2, k/2 + 1/2)$ being approximately $b(k)$ (or potentially $b(k+l)$ for $|l| \le m$ when considering the influence of neighboring intervals), we expected that the maximum gap $\delta_{k/2}$ on $[k/2, k/2 + 1/2)$ would be correlated with $b(k)$. All formulations in the theorems of this section validate our initial intuition that the parameter $b(k)$ measures the local bandwidth of $f \in PW_b^2(g, \mathbb{R})$. However, we have also come to understand that accounting for the influence of neighboring intervals is crucial. In Theorem 3.10, the constant $\mu_{k/2}$ (3.4.2) emerges as a precise measure for the local bandwidth and represents the maximal bandwidth that has an influence on the reconstruction in the interval $[k/2, k/2 + 1/2)$. For an illustrative example, we refer to the discussion of Chapter 5. The sufficient condition (3.4.3) of Theorem 3.10

$$\sup_{k \in \mathbb{Z}} \mu_{\frac{k}{2}} \delta_{\frac{k}{2}} < \frac{\pi}{D}$$

can be compared to the inequality (2.2.2) in Theorem 2.4

$$\sup_{k \in \mathbb{Z}} \mu_k \delta_k < \frac{\sqrt{A'\beta}\pi}{D}.$$

These two inequalities are very similar except for the presence in the condition for $V_b^2(g, \mathbb{R})$ of the lower bound A' of the Riesz sequence and the parameter β. Allowing $\alpha = 1/2$, $\beta = 1$, $A' = 1$, and $\text{supp}(g) = [0, 2m]$, we would have two equivalent conditions. On the other hand, the density theorem 1.5 tells us that $1/2 = \alpha\beta \le 1$ implies that the system $\{M_l T_{n/2} g\}_{n,l \in \mathbb{Z}}$ can not be a Riesz sequence for $L^2(\mathbb{R})$.

(ii) A space of variable bandwidth constructed via a Wilson Riesz basis allows us to give up the orthogonality condition that an orthonormal basis requires. Therefore, it gives more flexibility in the choice of the window function g, which can be taken to be similar to the one of Theorem 2.3. Results where the window function g is specifically chosen, allow to underline the role of the neighboring windows in the reconstruction. Indeed, the bandwidths of the two central windows yield a more substantial impact on the sampling rate, as reflected by the presence of the constant 4 in (3.5.1). In contrast, the windows at the boundaries offer a milder influence in determining the sampling rate. Consequently, despite incurring a cost in terms of the lower Riesz constant A', the condition (3.5.1) presents another point of view in understanding the variable bandwidth spaces and provides valuable information regarding the role played by the overlap of different translations of the window function g.

We can make a direct comparison between condition (2.1.2) of Theorem 2.3

$$\sup_{k\in\mathbb{Z}} \left\{ \frac{\lceil\beta\rceil}{\beta} \frac{6}{A'} (\delta_{k-1}^2 + 4\delta_k^2 + \delta_{k+1}^2) \Big(\beta^2 b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2} \Big) \right\} < 1$$

and inequality (3.5.1) of Theorem 3.11

$$\sup_{k\in\mathbb{Z}} \left\{ \frac{8}{A'} (\delta_{\frac{k}{2}-1}^2 + 4\delta_{\frac{k}{2}-\frac{1}{2}}^2 + 4\delta_{\frac{k}{2}}^2 + \delta_{\frac{k}{2}+\frac{1}{2}}^2) \Big(b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2} \Big) \right\} < 1.$$

By taking $\beta = 1$, the two conditions look pretty similar; what changes is the choice of the partition. The structure of a Wilson basis, i.e., the translation parameter $n/2$ of the window function g imposes a natural partition of $\mathbb{R}$ in intervals of length $1/2$. To the contrary, the translation parameter $\alpha = 1$ in Theorem 2.3 suggests a partition in intervals of length 1. Therefore, we obtain that 3 densities play a role in the reconstruction for $V_b^2(g,\mathbb{R})$ and 4 densities for the reconstruction in the Wilson Riesz basis scenario.

(iii) Except for very special cases, as in [8, 38, 71], the adaptive weights method does not produce optimal results and fails to come close to the necessary sampling density. To stick to the essentials, we have therefore not aimed to optimize (and complicate) the proof at all steps. The estimates can be slightly sharpened in several places, but this does not seem relevant in the numerical simulations.

(iv) As elaborated in (ii) of Section 2.3, it is important to note that the proof does not require the separation of points. In practice, any local fluctuations in density can be effectively handled using the adaptive weights method. Nonetheless, if the points are indeed separated, it results in a lower bound for the weights.

4 Variable bandwidth via local Fourier bases

In this chapter, we study whether there exists a sufficient condition for sampling in the space of variable bandwidth generated by a local Fourier basis.

In signal processing, there is often a need to analyze the local properties of a signal f, particularly within a bounded interval. For an interval I there are orthonormal bases for $L^2(I)$ formed by trigonometric functions:

(i) $\sin\left(\frac{2l+1}{2}\frac{\pi}{|I|}x\right)\chi_I(x),\ \ l = 0, 1, 2, \ldots;$

(ii) $\sin\left(l\frac{\pi}{|I|}x\right)\chi_I(x),\ \ l = 1, 2, 3, \ldots;$

(iii) $\cos\left(\frac{2l+1}{2}\frac{\pi}{|I|}x\right)\chi_I(x),\ \ l = 0, 1, 2, \ldots;$

(iv) $\cos\left(l\frac{\pi}{|I|}x\right)\chi_I(x),\ \ l = 0, 1, 2, \ldots.$

Given a partition $\{\alpha_n\}_{n\in\mathbb{Z}}$ of $\mathbb{R}$, and the fact that $L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} L^2[\alpha_n, \alpha_{n+1}]$, these bases can be combined to form orthonormal bases for $L^2(\mathbb{R})$. Despite being well-localized in the time domain, the characteristic functions introduce abrupt transitions, which prevents good frequency localization for these bases.

The concept of local Fourier bases involves replacing the characteristic function with a smooth function, often referred to as a **bell function**, while still preserving the orthogonality of the basis elements. A family of functions of the form

$$\phi_{n,l}(x) = \sqrt{\frac{2}{\ell_n}}g_n(x)\cos\Big(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x-\alpha_n)\Big),\ \text{for } n\in\mathbb{Z},\ l\in\mathbb{N}\cup\{0\},$$

where the cosine can be replaced by the trigonometric functions in (i), (ii), and (iv) are called local Fourier bases. $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, $\ell_n = \alpha_{n+1} - \alpha_n$, and g_n is a smooth and compactly supported function which replaces the characteristic function.

The bell functions g_n are supported in the interval $[\alpha_n - \varepsilon_n, \alpha_{n+1} + \varepsilon_{n+1}]$, where $\varepsilon_n > 0$ for every $n \in \mathbb{Z}$. Note that each basis element admits a frequency within a band near $\frac{2l+1}{4\ell_n}$. By letting $b : \mathbb{Z} \to [0, \infty)$ being a bounded and positive sequence, we define the space of variable bandwidth constructed with a local Fourier basis as

$$LF_b^2(\mathbb{R}) := \Big\{f\in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}}\sum_{l=0}^{\lceil 2b(n)\ell_n - \frac{1}{2}\rceil} c_{n,l}\phi_{n,l},\ c\in\ell^2(\mathbb{Z}\times\mathbb{N})\Big\}.$$

By the construction of a local Fourier basis, g_n are compactly supported, then $b(n)$ is a measure of the local frequency on the interval $[\alpha_n, \alpha_{n+1}]$ of the partition.

The chapter is dedicated to the study of sufficient conditions for sampling for $LF_b^2(\mathbb{R})$.

Looking at the structure of the local Fourier bases we can already anticipate that the sufficient conditions are a bit different from those in the previous chapters. In fact, unlike the case of variable bandwidth spaces constructed with Gabor Riesz sequences (Chapter 2) and Wilson bases (Chapter 3), where there was only one translated window function g spanning the space, in the case of $LF_b^2(\mathbb{R})$, there exists a collection of bell functions $\{g_n\}_{n\in\mathbb{Z}}$ with varying supports and decay rates at the boundaries, allowing for greater adaptability. Therefore, the properties of g_n, coming from the use of the adaptive weight method, can not be treated as a constant anymore but their influence in the reconstruction may vary for different intervals.

Following [10], we start Section 4.1 with the construction by Auscher, Weiss, and Wickerhauser of the local Fourier bases and we show that, under special assumptions, Wilson bases are a particular case of local Fourier bases. Section 4.2 introduces the main object of the chapter: the space of variable bandwidth constructed via a local Fourier basis $LF_b^2(\mathbb{R})$ and the proof that $LF_b^2(\mathbb{R})$ is a reproducing kernel Hilbert space. Sections 4.3 and 4.4 are dedicated to the proofs of the first and second versions of the sufficient condition for sampling theorem, respectively. In Section 4.5 we present an analysis and comparison of the results of the chapter.

4.1 Construction of local Fourier bases

In this section, we briefly present the construction of the so-called **local Fourier bases** or **Malvar bases**. These bases were first introduced by Malvar [59] and emerged in the context of subband coding theory: he wanted to eliminate the aliasing effect of subband coding when two blocks overlap in the block-by-block discrete cosine transform. His solution consists of a modulated lapped transform which cancels the aliasing effects and allows a perfect reconstruction. These bases were independently constructed in a generalized form by Coifman and Meyer [25]. The work [10] by Auscher, Weiss and Wickerhauser shows that these two apparently different constructions are actually particular cases of a family called **local Fourier bases**.

The idea of Auscher and his collaborators is to find projections over intervals that have an arbitrarily smooth cut-off to avoid the undesirable effects produced by multiplication by the characteristic function in the trigonometric orthonormal bases (i),(ii),(iii), and (iv). In this section we present the construction of the Malvar basis by Auscher, Weiss and Wickerhauser in [10] and by Auscher in [9].

Let us start by introducing the bell function g_n.

Definition 4.1. Let $I_n = [\alpha_n, \alpha_{n+1}]$, with $-\infty < \alpha_n < \alpha_{n+1} < +\infty$ such that $\alpha_n + \varepsilon_n \leq \alpha_{n+1} - \varepsilon_{n+1}$ with $\varepsilon_n, \varepsilon_{n+1} > 0$. Let $\theta_{\varepsilon_n}(x)$ be an odd and increasing function such that

$\theta_{\varepsilon_n}(x) \geq \pi/4$ for $x \geq \varepsilon_n$ for all $n \in \mathbb{Z}$. Define

$$s_{\varepsilon_n}(x) := \sin\left(\theta_{\varepsilon_n}(x) + \frac{\pi}{4}\right) \quad \text{and} \quad c_{\varepsilon_n}(x) := \cos\left(\theta_{\varepsilon_n}(x) + \frac{\pi}{4}\right).$$

A **bell function over I_n** is defined by

$$g_n(x) := s_{\varepsilon_n}(x - \alpha_n) c_{\varepsilon_{n+1}}(x - \alpha_{n+1}), \quad \text{for all } x \in \mathbb{R}.$$

We have the following basic properties of g_n.

Proposition 4.2 (Properties of g_n)**.** *The function g_n introduced in Definition 4.1 satisfies*

(i) $\operatorname{supp}(g_n) \subseteq [\alpha_n - \varepsilon_n, \alpha_{n+1} + \varepsilon_{n+1}]$.

(ii) $g_n(x) = 1$ *when* $x \in [\alpha_n + \varepsilon_n, \alpha_{n+1} - \varepsilon_{n+1}]$.

(iii) $g_n^2(x) + g_n^2(2\alpha_n - x) + g_n^2(2\alpha_{n+1} - x) = 1$ *on* $\operatorname{supp}(g_n)$.

Properties on the interval $[\alpha_n - \varepsilon_n, \alpha_n + \varepsilon_n]$*:*

(iv) $g_n(x) = s_{\varepsilon_n}(x - \alpha_n)$;

(v) $g_n(2\alpha_n - x) = s_{\varepsilon_n}(\alpha_n - x) = c_{\varepsilon_n}(x - \alpha_n)$;

(vi) $g_n^2(x) + g_n^2(2\alpha_n - x) = s_{\varepsilon_n}^2(x - \alpha_n) + c_{\varepsilon_n}^2(x - \alpha_n) = 1$.

(vii) $\operatorname{supp}(g_n(x) g_n(2\alpha_n - x)) \subseteq [\alpha_n - \varepsilon_n, \alpha_n + \varepsilon_n]$.

Properties on the interval $[\alpha_{n+1} - \varepsilon_{n+1}, \alpha_{n+1} + \varepsilon_{n+1}]$*:*

(viii) $g_n(x) = c_{\varepsilon_{n+1}}(x - \alpha_{n+1})$;

(ix) $g_n(2\alpha_{n+1} - x) = c_{\varepsilon_{n+1}}(\alpha_{n+1} - x) = s_{\varepsilon_{n+1}}(x - \alpha_{n+1})$;

(x) $g_n^2(x) + g_n^2(2\alpha_{n+1} - x) = 1$.

(xi) $\operatorname{supp}(g_n(x) g_n(2\alpha_{n+1} - x)) \subseteq [\alpha_{n+1} - \varepsilon_{n+1}, \alpha_{n+1} + \varepsilon_{n+1}]$.

We consider the Definition 4.1 of the bell function and the properties in Proposition 4.2, then the operators P_n defined by

$$(P_n f)(x) = g_n^2(x) f(x) \pm g_n(x) g_n(2\alpha_n - x) f(2\alpha_n - x) \pm g_n(x) g_n(2\alpha_{n+1} - x) f(2\alpha_{n+1} - x)$$

are orthogonal projections and they represent a smooth version of the operator of pointwise multiplication by the characteristic function of I_n.

We have four choices for the sign associated with the endpoints α_n and α_{n+1}. The choice of $\pm$ associated with α_n is referred to as the **polarity of P_n at α_n**, and the choice of $\pm$ associated with α_{n+1} is the **polarity of P_n at α_{n+1}**. Polarities are very important when studying the properties of P_{n-1} and P_n for two adjacent intervals I_{n-1} and I_n. Given two adjacent intervals $I_{n-1} = [\alpha_{n-1}, \alpha_n]$ and $I_n = [\alpha_n, \alpha_{n+1}]$ and $\varepsilon_{n-1}, \varepsilon_n, \varepsilon_{n+1} > 0$ we require

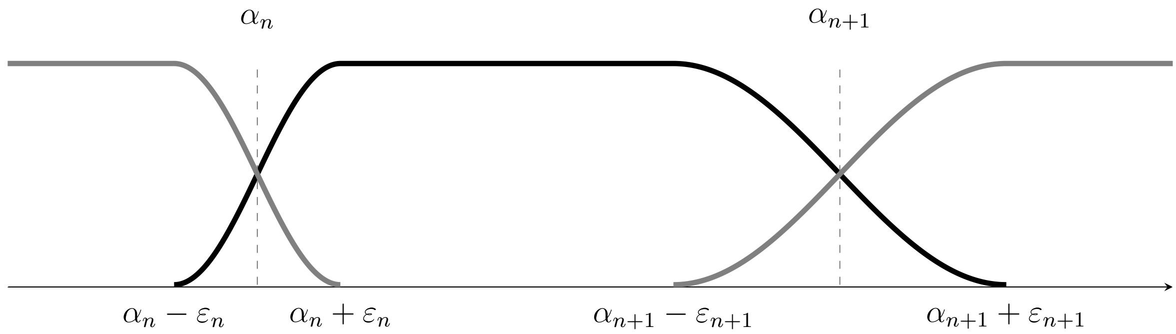


Figure 4.1: The bell function g_n with $\mathrm{supp}(g_n) \subseteq [\alpha_n - \varepsilon_n, \alpha_{n+1} + \varepsilon_{n+1}]$.

(a) $\alpha_{n-1} + \varepsilon_{n-1} \leq \alpha_n - \varepsilon_n < \alpha_n + \varepsilon_n \leq \alpha_{n+1} - \varepsilon_{n+1}$ and,

(b) $g_{n-1}(x) = s_{\varepsilon_{n-1}}(x - \alpha_{n-1})c_{\varepsilon_n}(x - \alpha_n), \quad g_n(x) = s_{\varepsilon_n}(x - \alpha_n)c_{\varepsilon_{n+1}}(x - \alpha_{n+1})$.

Two intervals satisfying (a) and (b) are said to have **compatible bell functions $\boldsymbol{g_{n-1}}$ and $\boldsymbol{g_n}$**.

If P_{n-1} and P_n have opposite polarities at α_n and the intervals I_{n-1} and I_n have compatible bell functions, then P_{n-1} and P_n are orthogonal to each other and $P_{n-1} + P_n = P_{I_{n-1} \cup I_n}$. Orthogonal projections with these properties provide a way to decompose $L^2(\mathbb{R})$ as a direct sum of orthogonal subspaces.

Theorem 4.3. *Let $\{\alpha_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence such that $\lim_{n\to\pm\infty} \alpha_n = \pm\infty$. Let $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be such that $\varepsilon_n > 0$ for every $n \in \mathbb{Z}$. Let $I_{n-1} = [\alpha_{n-1}, \alpha_n]$ and $I_n = [\alpha_n, \alpha_{n+1}]$ be two adjacent intervals with compatible bell functions and P_{n-1} and P_n have opposite polarity at α_n for every $n \in \mathbb{Z}$. Then*

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} P_n(L^2(\mathbb{R})).$$

Thanks to Theorem 4.3, we can state the following theorem which provides the existence of the so-called **local Fourier bases** for $L^2(\mathbb{R})$.

Theorem 4.4 (Local Fourier bases)**.** *Let $\{\alpha_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence such that $\lim_{n\to\pm\infty} \alpha_n = \pm\infty$ and define $\ell_n = \alpha_{n+1} - \alpha_n$ for all $n \in \mathbb{Z}$. Let $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be such that $\varepsilon_n > 0$ for every $n \in \mathbb{Z}$. Let $I_{n-1} = [\alpha_{n-1}, \alpha_n]$ and $I_n = [\alpha_n, \alpha_{n+1}]$ be two adjacent intervals with compatible bell functions for every $n \in \mathbb{Z}$. If we choose the polarities $(-,+)$ for each smooth projection P_n we obtain that the system*

(i) $\left\{\sqrt{\frac{2}{\ell_n}} g_n(x) \sin\left(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x - \alpha_n)\right),\ n \in \mathbb{Z},\ l \in \mathbb{N} \cup \{0\}\right\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Moreover, if $f \in L^2(\mathbb{R})$, then the series

$$\sum_{n\in\mathbb{Z}} \sum_{l=0}^{\infty} \sqrt{\frac{2}{\ell_n}} c_{n,l} g_n(x) \sin\left(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x-\alpha_n)\right)$$

converges to $f(x)$ almost everywhere.

If we choose the polarities $(-,-)$, $(+,-)$ and $(+,+)$ at (α_n, α_{n+1}), the same result is true if we use, respectively, the systems

(ii) $\left\{\sqrt{\frac{2}{\ell_n}} g_n(x) \sin\left(l\frac{\pi}{\ell_n}(x-\alpha_n)\right),\ n\in\mathbb{Z},\ l\in\mathbb{N}\right\}$;

(iii) $\left\{\sqrt{\frac{2}{\ell_n}} g_n(x) \cos\left(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x-\alpha_n)\right),\ n\in\mathbb{Z},\ l\in\mathbb{N}\cup\{0\}\right\}$;

(iv) $\left\{\sqrt{\frac{1}{\ell_n}} g_n(x), \sqrt{\frac{2}{\ell_n}} g_n(x) \cos\left(l\frac{\pi}{\ell_n}(x-\alpha_n)\right),\ n\in\mathbb{Z},\ l\in\mathbb{N}\right\}$.

We can categorize local Fourier bases into two types: the first type occurs when all intervals have different polarity at their endpoints, either $(-,+)$ for all or $(+,-)$ for all, i.e., (i) and (iii).

The second type contains bases with the same polarity at the endpoints, necessitating an alternating sequence of pairs of polarity, such as $(-,-)$ and $(+,+)$, i.e., (ii) and (iv).

Remark. It is important to emphasize that the local Fourier basis of the type (ii) and (iv) can be viewed as a refinement of those in (i) and (iii). Specifically, by splitting a general interval I with polarity $(-,+)$ at the endpoints into two parts, we obtain polarity $(-,-)$ for the left part and polarity $(+,+)$ for the right part. Similarly, if we start with an interval having polarity $(-,+)$, we can achieve the same result. This implies that local Fourier bases of the type (i) and (iii) are, in a sense, generic for this entire collection of bases.

In the next subsection, we present a result by Auscher [9] which relates local Fourier bases and Wilson bases.

4.1.1 Wilson bases as a special case of local Fourier bases

Auscher shows that in the special case of a uniform partition and by considering a suitable polarity, Wilson bases are a particular case of local Fourier bases of the second type [9].

Consider a partition of $\mathbb{R}$ in intervals of the type $I_n = [\frac{n}{2} - \frac{1}{4}, \frac{n}{2} + \frac{1}{4}]$ for $n \in \mathbb{Z}$. If n is even, we assign the polarity of I_n to be $(+,+)$, and if n is odd, we set I_n to have the polarity $(-,-)$. We get a uniform covering of $\mathbb{R}$ with compatible adjacent intervals. Let $g_n(x) = \frac{1}{\sqrt{2}} g\left(x - \frac{n}{2}\right)$ be the bell over I_n, defined as the $n/2$ translation of a fixed window function $g \in L^2(\mathbb{R})$.

Assume n is even and $n = 2n'$. By taking the local Fourier basis (iv) in Theorem 4.4, we get

$$\begin{aligned}\tilde{\Psi}_{2n',l}(x) &= \sqrt{2}g(x-n')\cos\Big(2\pi l\Big(x-\Big(-\frac{1}{4}+n'\Big)\Big)\Big)\\ &= \sqrt{2}g(x-n')\cos\Big(2\pi l\Big(x+\frac{1}{4}\Big)\Big)\\ &= \sqrt{2}g(x-n')\begin{cases}(-1)^{[l/2]}\cos(2\pi lx), & \text{if } l \text{ is even},\\ (-1)^{[l/2]+1}\sin(2\pi lx), & \text{if } l \text{ is odd},\end{cases}\end{aligned} \tag{4.1.1}$$

where $[l/2]$ is the integer part of $l/2$.

Let $n = 2n'+1$ and consider the basis (ii) in Theorem 4.4. Then

$$\begin{aligned}\tilde{\Psi}_{2n'+1,l}(x) &= \sqrt{2}g\Big(x-\frac{2n'+1}{2}\Big)\sin\Big(2\pi l\Big(x-\Big(-\frac{1}{4}+\frac{2n'+1}{2}\Big)\Big)\Big)\\ &= \sqrt{2}g\Big(x-\frac{2n'+1}{2}\Big)\sin\Big(2\pi l\Big(x-\frac{1}{4}\Big)\Big)\\ &= \sqrt{2}g\Big(x-\frac{2n'+1}{2}\Big)\begin{cases}(-1)^{[l/2]}\sin(2\pi lx), & \text{if } l \text{ is even},\\ (-1)^{[l/2]+1}\cos(2\pi lx), & \text{if } l \text{ is odd}.\end{cases}\end{aligned} \tag{4.1.2}$$

Therefore, the basis $\tilde{\Psi}_{n,l}$ generated by equations (4.1.1) and (4.1.2) is equivalent to $\psi_{n,l}$ in (3.0.2), up to unimodular multiplicative constants.

4.2 Space of variable bandwidth constructed with local Fourier bases

The local Fourier bases of Theorem 4.4 are the time-frequency analysis tools to construct the last variation of spaces of variable bandwidth.

We consider the local Fourier basis (iii) of Theorem 4.4

$$\phi_{n,l}(x) = \sqrt{\frac{2}{\ell_n}}g_n(x)\cos\Big(\frac{2l+1}{2}\frac{\pi}{\ell_n}(x-\alpha_n)\Big), \quad \text{for } n\in\mathbb{Z} \text{ and } l\in\mathbb{N}\cup\{0\}. \tag{4.2.1}$$

We recall that $g_n(x) = s_{\varepsilon_n}(x-\alpha_n)c_{\varepsilon_{n+1}}(x-\alpha_{n+1})$ with $\operatorname{supp}(g_n)\subseteq[\alpha_n-\varepsilon_n, \alpha_{n+1}+\varepsilon_{n+1}]$ and $\ell_n = \alpha_{n+1}-\alpha_n$. Using the translation and modulation operators (1.2.1), we can rewrite the elements of the basis (4.2.1) as

$$\phi_{n,l} = \frac{1}{2}\sqrt{\frac{2}{\ell_n}}T_{\alpha_n}\Big[M_{\frac{2l+1}{4\ell_n}} + M_{-\frac{2l+1}{4\ell_n}}\Big]T_{-\alpha_n}g_n.$$

On every interval $[\alpha_n, \alpha_{n+1}]$, the local frequency of $\phi_{n,l}$ is approximately $\frac{2l+1}{4\ell_n}$. Let $b:\mathbb{Z}\to[0,\infty)$ be a bounded and positive sequence that represents the local frequency truncation, then the index $l = 0,\dots,\lceil 2b(n)\ell_n-\frac{1}{2}\rceil$ for every $n\in\mathbb{Z}$.

We define a different variation of the space of variable bandwidth by using the local Fourier basis (4.2.1).

Definition 4.5. Let $b : \mathbb{Z} \to [0, \infty)$ be a bounded positive sequence such that $b(n) \leq B < \infty$ for every $n \in \mathbb{Z}$ and let $\{\phi_{n,l}\}_{n\in\mathbb{Z},l\in\mathbb{N}\cup\{0\}}$ be the local Fourier basis (4.2.1). We define the **space of variable bandwidth generated by a local Fourier basis** as

$$LF_b^2(\mathbb{R}) \coloneqq \Big\{ f \in L^2(\mathbb{R}) : f = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lceil 2b(n)\ell_n - \frac{1}{2}\rceil} c_{n,l}\phi_{n,l},\ c \in \ell^2(\mathbb{Z}\times\mathbb{N}) \Big\}. \tag{4.2.2}$$

Since the bell functions g_n have compact support with $\operatorname{supp}(g_n) \subseteq [\alpha_n - \varepsilon_n, \alpha_{n+1} + \varepsilon_{n+1}]$, the sequence b represents the local bandwidth of f within each interval $[\alpha_n, \alpha_{n+1}]$.

As for the other spaces of variable bandwidth $V_b^2(g, \mathbb{R})$ and $PW_b^2(g, \mathbb{R})$, the space $LF_b^2(\mathbb{R})$ defined by (4.2.2) is a reproducing kernel Hilbert space.

Lemma 4.6. *Let $LF_b^2(\mathbb{R})$ be the space of variable bandwidth defined in* (4.2.2) *generated by the local Fourier basis* (4.2.1)*. Then $LF_b^2(\mathbb{R})$ is a reproducing kernel Hilbert space with reproducing kernel*

$$k(x, y) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lceil 2b(n)\ell_n - \frac{1}{2}\rceil} \overline{\phi_{n,l}(y)}\phi_{n,l}(x).$$

Proof. For all $n \in \mathbb{Z}$, we set

$$a(n) \coloneqq \lceil 2b(n)\ell_n - \tfrac{1}{2}\rceil.$$

To apply Lemma 1.15, we show that $\sum_{n\in\mathbb{Z}} \sum_{l=0}^{a(n)} \big|\phi_{n,l}(x)\big|^2 < \infty$ for every $x \in \mathbb{R}$.

Fix $x \in [\alpha_k, \alpha_{k+1}]$. Since g_n is compactly supported for every $n \in \mathbb{Z}$, we have that $\operatorname{supp}(g_n) \cap [\alpha_k, \alpha_{k+1}] \neq \emptyset$ only for $n = \{k-1, k, k+1\}$. Hence,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \sum_{l=0}^{a(n)} \big|\phi_{n,l}(x)\big|^2 &= \sum_{l=0}^{a(k-1)} \big|\phi_{k-1,l}(x)\big|^2 + \sum_{l=0}^{a(k)} \big|\phi_{k,l}(x)\big|^2 + \sum_{l=0}^{a(k+1)} \big|\phi_{k+1,l}(x)\big|^2 \\
&\leq 2\Big\{ \frac{1}{\ell_{k-1}} \sum_{l=0}^{a(k-1)} \big|g_{k-1}(x)\big|^2 + \frac{1}{\ell_k} \sum_{l=0}^{a(k)} \big|g_k(x)\big|^2 + \frac{1}{\ell_{k+1}} \sum_{l=0}^{a(k+1)} \big|g_{k+1}(x)\big|^2 \Big\} \\
&\leq 2 \max_{n\in\{k-1,k,k+1\}} \frac{a(n)+1}{\ell_n} \sum_{n=k-1}^{k+1} \big|g_n(x)\big|^2.
\end{aligned}$$

By property (iii) of Proposition 4.2 we have that $\sum_{n\in\mathbb{Z}} \big|g_n(x)\big|^2 = 1$ and since $b(n) \leq B < \infty$ for every $n \in \mathbb{Z}$, we obtain

$$\sum_{n\in\mathbb{Z}} \sum_{l=0}^{a(n)} \big|\phi_{n,l}(x)\big|^2 \leq 2 \max_{n\in\{k-1,k,k+1\}} \frac{\lceil 2B\ell_n - \frac{1}{2}\rceil + 1}{\ell_n}.$$

Applying Lemma 1.15 with $\{e_n\}_{n\in\mathbb{Z}} = \{\phi_{n,l}\}_{n\in\mathbb{Z},l=0,\dots,a(n)}$ we have that $LF_b^2(\mathbb{R})$ is a reproducing kernel Hilbert space and since $\{\phi_{n,l}\}_{n\in\mathbb{Z},l=0,\dots,a(n)}$ is an orthonormal basis for $LF_b^2(\mathbb{R})$, the reproducing kernel is given by

$$k(x,y) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\lceil 2b(n)\ell_n - \frac{1}{2}\rceil} \overline{\phi_{n,l}(y)}\phi_{n,l}(x).$$

□

Similarly to how we addressed the Gabor Riesz sequence case in Chapter 2 and the Wilson basis case in Chapter 3, in the following sections we present two sufficient conditions for sampling for the space $LF_b^2(\mathbb{R})$. The proofs of these theorems follow the adaptive weights method.

4.3 Sufficient conditions for sampling - first version

In the first version of the sufficient conditions for sampling theorem, we prove a result in the same style as Theorem 2.3 and Theorem 3.11. The theorem explicitly expresses the dependence of the sufficient condition on the neighboring window functions and assigns a specific weight to each of them.

The more convenient partition for $\mathbb{R}$ for the space $LF_b^2(\mathbb{R})$ is again inherited from the chosen time-frequency atom, i.e., the local Fourier basis itself. The partition in intervals $[\alpha_k, \alpha_{k+1})$ given from Theorem 4.4 ensures that the expansion of a function at each sample $f(x_{k,j})$ in the space $LF_b^2(\mathbb{R})$ involves only three bell functions: g_{k-1}, g_k, and g_{k+1}. We show that as it was for the spaces $V_b^2(g,\mathbb{R})$ and $PW_b^2(g,\mathbb{R})$, the bandwidth of the central window has a stronger impact on the sampling rate and those at the boundaries exercise a more modest influence.

As for the previous results, we conveniently label the set of points Λ.

$$\Lambda := \Big\{x_{k,j} \in \mathbb{R} : x_{k,j} \in [\alpha_k, \alpha_{k+1}),\ k \in \mathbb{Z} \text{ and } j = 1, \dots, j_{\max(k)}\Big\}, \tag{4.3.1}$$

where the sampling points in Λ are ordered by magnitude.

$$\delta_k = \max_{j=1,\dots,j_{\max(k)-1}} (x_{k,j+1} - x_{k,j}),$$

is the maximum gap between the samples and $j_{\max(k)}$ depends on the specific interval. We define $y_{k,j} = \frac{1}{2}(x_{k,j} + x_{k,j+1})$ for $j = 1, \dots, j_{\max(k)} - 1$ to be the midpoints between the samples and $y_{k,0} = \alpha_k$ and $y_{k,j_{\max(k)}} = \alpha_{k+1}$. We set $\chi_{k,j} = \chi_{[y_{k,j-1}, y_{k,j})}$ and $\chi_k = \chi_{[\alpha_k,\alpha_{k+1})}$.

First, we prove a modification of Bernstein's inequality of Theorem 1.13 for a trigonometric polynomial of the type

$$P_n = \sum_{l=0}^{a(n)} c_{n,l} \frac{1}{2} T_{\alpha_n} \sqrt{\frac{2}{\ell_n}} \Big[M_{\frac{2l+1}{4\ell_n}} + M_{-\frac{2l+1}{4\ell_n}}\Big]$$

of degree $a(n)$ and period $4\ell_n$.

Corollary 4.7 (Modification of Bernstein's inequality). *Let $I \subset \mathbb{R}$ be an interval and*

$$P_n = \sum_{l=0}^{a(n)} c_{n,l} \frac{1}{2} T_{\alpha_n} \sqrt{\frac{2}{\ell_n}} \Big[M_{\frac{2l+1}{4\ell_n}} + M_{-\frac{2l+1}{4\ell_n}} \Big]$$

be a trigonometric polynomial of degree $a(n)$ and period $4\ell_n$. Then for every $n \in \mathbb{Z}$, we have

$$\|P_n' \chi_I\|_2 \leq \pi \sqrt{\Big\lceil \tfrac{|I|}{4\ell_n} \Big\rceil} \frac{2a(n)+1}{2\ell_n} \|P_n\|_2, \tag{4.3.2}$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$.

Proof. By the same computation as in Lemma 1.12 with $\beta = \frac{1}{4\ell_n}$, we have

$$\|P_n' \chi_I\|_2 \leq \sqrt{\Big\lceil \tfrac{|I|}{4\ell_n} \Big\rceil} \|P_n'\|_2.$$

Note that the derivative of the basis (iii) of Theorem 4.4 is the basis (i) multiplied by $\frac{2l+1}{2\ell_n}\pi$, therefore

$$\begin{aligned} \|P_n'\|_2^2 &= \int_0^{4\ell_n} \Big| \sum_{l=0}^{a(n)} c_{n,l} \frac{2l+1}{2\ell_n} \pi \sqrt{\frac{2}{\ell_n}} \sin\Big(\frac{2l+1}{2} \frac{\pi}{\ell_n} (x - \alpha_n) \Big) \Big|^2 dx \\ &\leq \pi^2 \Big(\frac{2a(n)+1}{2\ell_n} \Big)^2 \|P_n\|_2^2. \end{aligned}$$

□

We are ready to prove the first version of the sufficient condition for sampling theorem for $LF_b^2(\mathbb{R})$.

Theorem 4.8 (Sufficient conditions for sampling for local Fourier basis - first version). *Let $g_n \in \mathcal{C}^1(\mathbb{R})$ for every $n \in \mathbb{Z}$ and let $LF_b^2(\mathbb{R})$ be the space of variable bandwidth defined in* (4.2.2) *generated by the local Fourier basis* (4.2.1). *Let Λ be as in* (4.3.1) *with $x_{k,1} - \alpha_k \leq \delta_k/2$ and $\alpha_{k+1} - x_{k,j_{\max(k)}} \leq \delta_k/2$, and $\chi_k = \chi_{[\alpha_k, \alpha_{k+1})}$. If*

$$\sup_{k \in \mathbb{Z}} \Big\{ 12\ell_k \big(\delta_{k-1}^2 + 2\delta_k^2 + \delta_{k+1}^2 \big) \Big(\Big(\frac{2\lceil 2b(k)\ell_k - \frac{1}{2} \rceil + 1}{2\ell_k} \Big)^2 + 2 \frac{\|g_k' \chi_k\|_\infty^2}{\pi^2} \Big) \Big\} < 1, \tag{4.3.3}$$

then Λ is a sampling set for $LF_b^2(\mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \leq \sum_{k \in \mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} |f(x_{k,j})|^2 w_{k,j} \leq B\|f\|_2^2 \qquad \text{for all } f \in LF_b^2(\mathbb{R}),$$

where $w_{k,j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in LF_b^2(\mathbb{R})$ can be reconstructed completely from the samples in Λ.

Proof. Let $f \in LF^2_b(\mathbb{R})$. As done in the previous sampling theorems, we apply the adaptive weights method. Therefore, we construct an approximation operator Af of f which contains the evaluation of the function at the sampling points $f(x_{k,j})$ and we show that $\|f - Af\|_2 \leq \gamma \|f\|_2$ with $\gamma < 1$. We apply Lemma 1.17 with $\mathcal{K} = LF^2_b(\mathbb{R})$, Λ as in (4.3.1), and we consider the approximation operator

$$Af = P\Big(\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} f(x_{k,j})\chi_{k,j}\Big). \tag{4.3.4}$$

Then

$$\|f - Af\|_2^2 \leq \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx.$$

Let $a(n) = \lceil 2b(n)\ell_n - 1/2 \rceil$, then

$$P_n = \sum_{l=0}^{a(n)} c_{n,l} \sqrt{\frac{2}{\ell_n}} \cos\Big(\frac{2l+1}{2\ell_n}\pi(x - \alpha_n)\Big) = \sum_{l=0}^{a(n)} c_{n,l} \frac{1}{2} T_{\alpha_n} \sqrt{\frac{2}{\ell_n}} \Big[M_{\frac{2l+1}{4\ell_n}} + M_{-\frac{2l+1}{4\ell_n}}\Big]$$

is a trigonometric polynomial of degree $a(n)$ and periond $4\ell_n$ for every $n \in \mathbb{Z}$. Therefore,

$$\|P_n\|_2^2 = 4\ell_n \sum_{l=0}^{a(n)} |c_{n,l}|^2. \tag{4.3.5}$$

and we can write every function $f \in LF^2_b(\mathbb{R})$ as

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(x) g_n(x).$$

Step 1. Estimate of local smoothness $\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx$.
Fix $x \in [\alpha_k, \alpha_{k+1})$ and let $\chi_k = \chi_{[\alpha_k,\alpha_{k+1})}$. By construction of the local Fourier basis in Section 4.1, only the bell functions g_{k-1}, g_k and g_{k+1} play a role in the expansion of f on $[\alpha_k, \alpha_{k+1})$. Therefore,

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(x) g_n(x) = \sum_{n=k-1}^{k+1} P_n(x) g_n(x). \tag{4.3.6}$$

We recall that $|\sum_{i\in F} f_i|^2 \leq \#F \sum_{i\in F} |f_i|^2$, therefore

$$\begin{aligned}\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx &= \int_{\alpha_k}^{\alpha_{k+1}} \Big| \sum_{n=k-1}^{k+1} P_n'(x) g_n(x) + P_n(x) g_n'(x) \Big|^2 dx \\ &\leq 6 \sum_{n=k-1}^{k+1} \int_{\alpha_k}^{\alpha_{k+1}} \Big(\big|P_n'(x) g_n(x)\big|^2 + \big|P_n(x) g_n'(x)\big|^2 \Big) dx. \end{aligned} \tag{4.3.7}$$

By the properties of g_k in Proposition 4.2, we have that

(a) $\operatorname{supp}(g_{k-1}) \subseteq [\alpha_{k-1} - \varepsilon_{k-1}, \alpha_k + \varepsilon_k]$,

(b) $\operatorname{supp}(g_k) \subseteq [\alpha_k - \varepsilon_k, \alpha_{k+1} + \varepsilon_{k+1}]$,

(c) $\operatorname{supp}(g_{k+1}) \subseteq [\alpha_{k+1} - \varepsilon_{k+1}, \alpha_{k+2} + \varepsilon_{k+2}]$,

with $\alpha_{k-1} + \varepsilon_{k-1} \leq \alpha_k - \varepsilon_k < \alpha_k + \varepsilon_k \leq \alpha_{k+1} - \varepsilon_{k+1}$.

We expand the sum in (4.3.7) and we get

$$\begin{aligned}\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx \leq 6\Big[&\int_{\alpha_k}^{\alpha_k+\varepsilon_k} \big|P'_{k-1}(x) g_{k-1}(x)\big|^2 dx + \int_{\alpha_k}^{\alpha_k+\varepsilon_k} \big|P_{k-1}(x) g'_{k-1}(x)\big|^2 dx \\ &+ \int_{\alpha_k}^{\alpha_{k+1}} \big|P'_k(x) g_k(x)\big|^2 dx + \int_{\alpha_k}^{\alpha_{k+1}} \big|P_k(x) g'_k(x)\big|^2 dx \\ &+ \int_{\alpha_{k+1}-\varepsilon_{k+1}}^{\alpha_{k+1}} \big|P'_{k+1}(x) g_{k+1}(x)\big|^2 dx + \int_{\alpha_{k+1}-\varepsilon_{k+1}}^{\alpha_{k+1}} \big|P_{k+1}(x) g'_{k+1}(x)\big|^2 dx\Big].\end{aligned}$$

We bound each term of the inequality.

$$\begin{aligned}\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx \leq 6\Big[&\|g_{k-1}\chi_{[\alpha_k,\alpha_k+\varepsilon_k]}\|_\infty^2 \|P'_{k-1}\chi_{[\alpha_k,\alpha_k+\varepsilon_k]}\|_2^2 \\ &+ \|g'_{k-1}\chi_{[\alpha_k,\alpha_k+\varepsilon_k]}\|_\infty^2 \|P_{k-1}\chi_{[\alpha_k,\alpha_k+\varepsilon_k]}\|_2^2 \\ &+ \|g_k\chi_k\|_\infty^2 \|P'_k\chi_k\|_2^2 + \|g'_k\chi_k\|_\infty^2 \|P_k\chi_k\|_2^2 \\ &+ \|g_{k+1}\chi_{[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]}\|_\infty^2 \|P'_{k+1}\chi_{[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]}\|_2^2 \\ &+ \|g'_{k+1}\chi_{[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]}\|_\infty^2 \|P_{k+1}\chi_{[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]}\|_2^2\Big].\end{aligned}$$

By construction of the bell functions, we have

$$\|g_k\chi_k\|_\infty = \sup_{x\in[\alpha_k,\alpha_{k+1}]} |g_k(x)| = 1,$$

$$\|g_{k-1}\chi_{[\alpha_k,\alpha_k+\varepsilon_k]}\|_\infty = \sup_{x\in[\alpha_k,\alpha_k+\varepsilon_k]} |g_{k-1}(x)| = \frac{\sqrt{2}}{2},$$

$$\|g_{k+1}\chi_{[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]}\|_\infty = \sup_{x\in[\alpha_{k+1}-\varepsilon_{k+1},\alpha_{k+1}]} |g_{k+1}(x)| = \frac{\sqrt{2}}{2}.$$

Recall that the trigonometric polynomials P_n have period $4\ell_n$ for every $n \in \mathbb{Z}$. By construction of the local Fourier bases $\varepsilon_k < \ell_{k-1}$ and $\varepsilon_k < \ell_k$ for every $k \in \mathbb{Z}$, then $\left\lceil \frac{\varepsilon_k}{4\ell_{k-1}} \right\rceil = 1$ and $\left\lceil \frac{\varepsilon_{k+1}}{4\ell_{k+1}} \right\rceil = 1$. We apply the modification of Bernstein's inequality (4.3.2) to each term, then

$$\begin{aligned}\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx \leq 6\Big[&\frac{1}{2}\pi^2\Big(\frac{2a(k-1)+1}{2\ell_{k-1}}\Big)^2 \|P_{k-1}\|_2^2 + \|g'_{k-1}\chi_k\|_\infty^2 \|P_{k-1}\|_2^2 \\ &+ \pi^2\Big(\frac{2a(k)+1}{2\ell_k}\Big)^2 \|P_k\|_2^2 + \|g'_k\chi_k\|_\infty^2 \|P_k\|_2^2 \\ &+ \frac{1}{2}\pi^2\Big(\frac{2a(k+1)+1}{2\ell_{k+1}}\Big)^2 \|P_{k+1}\|_2^2 + \|g'_{k+1}\chi_k\|_\infty^2 \|P_{k+1}\|_2^2\Big].\end{aligned}$$

Step 2. Final estimate for $\|f - Af\|_2$.
Consider the sum over all $k \in \mathbb{Z}$, then

$$\begin{aligned}
\|f - Af\|_2^2 \leq \frac{6}{\pi^2} \sum_{k\in\mathbb{Z}} \delta_k^2 \Big\{ &\|P_{k-1}\|_2^2 \Big[\frac{1}{2}\pi^2\Big(\frac{2a(k-1)+1}{2\ell_{k-1}}\Big)^2 + \|g'_{k-1}\chi_k\|_\infty^2\Big] \\
&+ \|P_k\|_2^2 \Big[\pi^2\Big(\frac{2a(k)+1}{2\ell_k}\Big)^2 + \|g'_k\chi_k\|_\infty^2\Big] \\
&+ \|P_{k+1}\|_2^2 \Big[\frac{1}{2}\pi^2\Big(\frac{2a(k+1)+1}{2\ell_{k+1}}\Big)^2 + \|g'_{k+1}\chi_k\|_\infty^2\Big]\Big\} \\
\leq \frac{6}{\pi^2} \sum_{k\in\mathbb{Z}} \delta_k^2 \Big\{ &\frac{1}{2}\|P_{k-1}\|_2^2 \Big[\pi^2\Big(\frac{2a(k-1)+1}{2\ell_{k-1}}\Big)^2 + 2\|g'_{k-1}\chi_k\|_\infty^2\Big] \\
&+ \|P_k\|_2^2 \Big[\pi^2\Big(\frac{2a(k)+1}{2\ell_k}\Big)^2 + 2\|g'_k\chi_k\|_\infty^2\Big] \\
&+ \frac{1}{2}\|P_{k+1}\|_2^2 \Big[\pi^2\Big(\frac{2a(k+1)+1}{2\ell_{k+1}}\Big)^2 + 2\|g'_{k+1}\chi_k\|_\infty^2\Big]\Big\}.
\end{aligned}$$

By rewriting, we get

$$\|f - Af\|_2^2 \leq 3 \sum_{k\in\mathbb{Z}} \big(\delta_{k-1}^2 + 2\delta_k^2 + \delta_{k+1}^2\big) \Big[\Big(\frac{2a(k)+1}{2\ell_k}\Big)^2 + 2\frac{\|g'_k\chi_k\|_\infty^2}{\pi^2}\Big] \|P_k\|_2^2.$$

We recall that $\|P_k\|_2^2 = 4\ell_k \sum_{l=0}^{a(k)} |c_{k,l}|^2$ and we define

$$\gamma^2 := \sup_{k\in\mathbb{Z}} \Big\{ 12\ell_k \big(\delta_{k-1}^2 + 2\delta_k^2 + \delta_{k+1}^2\big) \Big[\Big(\frac{2a(k)+1}{2\ell_k}\Big)^2 + 2\frac{\|g'_k\chi_k\|_\infty^2}{\pi^2}\Big]\Big\}.$$

Therefore, by inequality (4.3.3), $\gamma < 1$ and

$$\|f - Af\|_2^2 \leq \gamma^2 \sum_{k\in\mathbb{Z}} \sum_{l=0}^{a(k)} |c_{k,l}|^2 = \gamma^2 \|f\|_2^2.$$

Hence, the operator A is invertible on $LF_b^2(\mathbb{R})$ and f is completely determined by $f(x_{k,j})$. The weighted sampling inequality follows from Lemma 1.18 with $A = (1-\gamma)^2$ and $B = (1+\gamma)^2$. □

4.4 Sufficient conditions for sampling - second version

As done in the previous two chapters, we study a different sufficient condition for sampling for $LF_b^2(\mathbb{R})$. Once again, we show that we can construct a quantitative solution that depends on the maximal bandwidth over only one interval $[\alpha_k, \alpha_{k+1})$ of the partition. However, compared to the spaces $V_b^2(g, \mathbb{R})$ and $PW_b^2(g, \mathbb{R})$, for the space $LF_b^2(\mathbb{R})$ the number of bandwidth among which we have to take the maximum is always 3. In fact, by the construction of a local Fourier basis there are only three bell functions that play a role in the reconstruction of $f(x_{k,j})$ in each interval of the partition.

Theorem 4.9 (Sufficient conditions for sampling for local Fourier basis - second version)**.** *Let $g_n \in \mathcal{C}^1(\mathbb{R})$ for every $n \in \mathbb{Z}$ and let $LF^2_b(\mathbb{R})$ be the space of variable bandwidth defined in* (4.2.2) *generated by the local Fourier basis* (4.2.1)*. Let Λ be as in* (4.3.1) *with $x_{k,1}-\alpha_k \leq \delta_k/2$ and $\alpha_{k+1} - x_{k,j_{\max(k)}} \leq \delta_k/2$, and $\chi_k = \chi_{[\alpha_k,\alpha_{k+1})}$. Define*

$$\zeta_k := \max_{n\in\{k-1,k,k+1\}} \sqrt{\left\lceil \frac{\ell_k}{4\ell_n} \right\rceil} \Big(\pi \frac{2\lceil 2b(n)\ell_n - \frac{1}{2} \rceil + 1}{2\ell_n} + \|g_n'\chi_k\|_\infty \Big). \tag{4.4.1}$$

If

$$\sup_{k\in\mathbb{Z}} \delta_k \zeta_k \sqrt{\ell_k} < \frac{\pi}{6}, \tag{4.4.2}$$

then Λ is a sampling set for $LF^2_b(\mathbb{R})$ and satisfies a weighted sampling inequality

$$A\|f\|_2^2 \leq \sum_{n\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k)}} |f(x_{k,j})|^2 w_{k,j} \leq B\|f\|_2^2 \qquad \text{for all } f \in LF^2_b(\mathbb{R})\,,$$

where $w_{k,j} > 0$ is a suitable sequence of weights. As a consequence, every $f \in LF^2_b(\mathbb{R})$ can be reconstructed completely from the samples in Λ.

Proof. Let $f \in LF^2_b(\mathbb{R})$. We use the adaptive weights method and we show that $\|f - Af\|_2 \leq \gamma\|f\|_2$ with $\gamma < 1$. By applying Lemma 1.17 with $\mathcal{K} = LF^2_b(\mathbb{R})$, the sequence $\{\alpha_k\}_{k\in\mathbb{Z}}$ as a partition of $\mathbb{R}$, Λ as in (4.3.1), and the approximation operator (4.3.4), we get

$$\|f - Af\|_2^2 \leq \sum_{k\in\mathbb{Z}} \frac{\delta_k^2}{\pi^2} \int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx.$$

Step 1. Estimate of local smoothness $\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|^2 dx$.
Let $x \in [\alpha_k, \alpha_{k+1})$ and consider again the expansion of f in terms of the trigonometric polynomial as in (4.3.6):

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(x) g_n(x) = \sum_{n=k-1}^{k+1} P_n(x) g_n(x).$$

Define $F_k = \{k-1, k, k+1\}$, then

$$\begin{aligned}
&\int_{\alpha_k}^{\alpha_{k+1}} |f'(x)|dx \\
&\quad = \int_{\alpha_k}^{\alpha_{k+1}} \Big| \sum_{n=k-1}^{k+1} P_n'(x)g_n(x) + P_n(x)g_n'(x)\Big|^2 dx \\
&\quad = \sum_{i,n\in F_k} \int_{\alpha_k}^{\alpha_{k+1}} \Big(P_i'(x)g_i(x) + P_i(x)g_i'(x)\Big)\overline{\Big(P_n'(x)g_n(x) + P_n(x)g_n'(x)\Big)}dx \\
&\quad = \sum_{i,n\in F_k} \int_{\alpha_k}^{\alpha_{k+1}} \Big(P_i'(x)g_i(x)\overline{P_n'(x)g_n(x)} \\
&\qquad\qquad + 2\,\mathrm{Re}\Big\{P_i'(x)g_i(x)\overline{P_n(x)g_n'(x)}\Big\} \\
&\qquad\qquad + P_i(x)g_i'(x)\overline{P_n(x)g_n'(x)}\Big)dx. \qquad (4.4.3)
\end{aligned}$$

To bound the first term, we apply the modification of Bernstein's inequality of Corollary 4.7 to the polynomials P_i and P_n of degrees $a(i)$ and $a(n)$ and periods $4\ell_i$ and $4\ell_n$, respectively. $|I| = \ell_k$ and we use the estimate $\|g_i\overline{g_n}\chi_k\|_\infty \le \|g_k\chi_k\|_\infty^2 = 1$. We obtain

$$\begin{aligned}
&\int_{\alpha_k}^{\alpha_{k+1}} \Big(P_i'(x)g_i(x)\overline{P_n'(x)g_n(x)}\Big)dx \\
&\quad \le \|g_i\overline{g_n}\chi_k\|_\infty \|P_i'\chi_k\|_2 \|P_n'\chi_k\|_2 \\
&\quad \le \|g_k\chi_k\|_\infty^2 \|P_i'\chi_k\|_2 \|P_n'\chi_k\|_2 \\
&\quad \le \pi^2 \sqrt{\Big\lceil \frac{\ell_k}{4\ell_i}\Big\rceil} \frac{2a(i)+1}{2\ell_i}\|P_i\|_2 \sqrt{\Big\lceil \frac{\ell_k}{4\ell_n}\Big\rceil} \frac{2a(n)+1}{2\ell_n}\|P_n\|_2.
\end{aligned}$$

Summing over i and n, we get

$$\begin{aligned}
&\sum_{i,n\in F_k} \int_{\alpha_k}^{\alpha_{k+1}} \Big(P_i'(x)g_i(x)\overline{P_n'(x)g_n(x)}\Big)dx \\
&\quad \le \pi^2 \sum_{i,n\in F_k} \sqrt{\Big\lceil \frac{\ell_k}{4\ell_i}\Big\rceil} \frac{2a(i)+1}{2\ell_i}\|P_i\|_2 \sqrt{\Big\lceil \frac{\ell_k}{4\ell_n}\Big\rceil} \frac{2a(n)+1}{2\ell_n}\|P_n\|_2 \\
&\quad = \pi^2 \Big(\sum_{n\in F_k} \sqrt{\Big\lceil \frac{\ell_k}{4\ell_n}\Big\rceil} \frac{2a(n)+1}{2\ell_n}\|P_n\|_2\Big)^2.
\end{aligned}$$

For the middle terms of (4.4.3) we obtain in a similar way that

$$\begin{aligned}
2\int_{\alpha_k}^{\alpha_{k+1}} \mathrm{Re}\Big\{P_i'(x)g_i(x)\overline{P_n(x)g_n'(x)}\Big\}dx &\le 2\|g_i\overline{g_n'}\chi_k\|_\infty \|P_i'\chi_k\|_2 \|P_n\chi_k\|_2 \\
&\le 2\|g_i\chi_k\|_\infty \|g_n'\chi_k\|_\infty \|P_i'\chi_k\|_2 \|P_n\chi_k\|_2 \\
&\le 2\pi \|g_n'\chi_k\|_\infty \sqrt{\Big\lceil \frac{\ell_k}{4\ell_i}\Big\rceil} \frac{2a(i)+1}{2\ell_i}\|P_i\|_2 \sqrt{\Big\lceil \frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2.
\end{aligned}$$

Therefore,

$$\sum_{i,n\in F_k}\int_{\alpha_k}^{\alpha_{k+1}} 2\,\mathrm{Re}\Big\{P_i'(x)g_i(x)\overline{P_n(x)g_n'(x)}\Big\}dx$$
$$\leq 2\Big(\sum_{n\in F_k}\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\pi\frac{2a(n)+1}{2\ell_n}\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|g_n'\chi_k\|_\infty\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2\Big).$$

The bound for the last term of (4.4.3) is

$$\sum_{i,n\in F_k}\int_{\alpha_n}^{\alpha_{n+1}}\Big(P_i(x)g_i'(x)\overline{P_n(x)g_n'(x)}\Big)dx \leq \sum_{i,n\in F_k}\|g_i'\overline{g_n'}\chi_k\|_\infty\|P_i\chi_k\|_2\|P_n\chi_k\|_2$$
$$\leq \Big(\sum_{n\in F_k}\|g_n'\chi_k\|_\infty\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2\Big)^2.$$

Adding all terms, the local estimate for the derivative is

$$\begin{aligned}\int_{\alpha_k}^{\alpha_{k+1}}|f'(x)|dx &\leq \Big(\sum_{n\in F_k}\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\pi\frac{2a(n)+1}{2\ell_n}\|P_n\|_2\Big)^2+\Big(\sum_{n\in F_k}\|g_n'\chi_k\|_\infty\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2\Big)^2\\
&\quad+2\Big(\sum_{n\in F_k}\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\pi\frac{2a(n)+1}{2\ell_n}\|P_n\|_2\Big)\Big(\sum_{n\in F_k}\|g_n'\chi_k\|_\infty\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2\Big)\\
&=\Big(\sum_{n\in F_k}\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\pi\frac{2a(n)+1}{2\ell_n}\|P_n\|_2+\sum_{n\in F_k}\|g_n'\chi_k\|_\infty\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\|P_n\|_2\Big)^2\\
&=\Big(\sum_{n\in F_k}\|P_n\|_2\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\Big[\pi\frac{2a(n)+1}{2\ell_n}+\|g_n'\chi_k\|_\infty\Big]\Big)^2.\end{aligned}$$

We take the sum over all $k\in\mathbb{Z}$, then

$$\begin{aligned}\|f-Af\|_2^2 &\leq \sum_{k\in\mathbb{Z}}\frac{\delta_k^2}{\pi^2}\int_{\alpha_k}^{\alpha_{k+1}}|f'(x)|^2dx\\
&\leq \sum_{k\in\mathbb{Z}}\frac{\delta_k^2}{\pi^2}\Big(\sum_{n\in F_k}\|P_n\|_2\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\Big[\pi\frac{2a(n)+1}{2\ell_n}+\|g_n'\chi_k\|_\infty\Big]\Big)^2.\end{aligned}\tag{4.4.4}$$

We define

$$\zeta_k:=\max_{n\in\{k-1,k,k+1\}}\sqrt{\Big\lceil\frac{\ell_k}{4\ell_n}\Big\rceil}\Big(\pi\frac{2a(n)+1}{2\ell_n}+\|g_n'\chi_k\|_\infty\Big)$$

and we rewrite (4.4.4) as

$$\|f-Af\|_2^2\leq\frac{1}{\pi^2}\sum_{k\in\mathbb{Z}}\delta_k^2\zeta_k^2\Big(\sum_{n\in F_k}\|P_n\|_2\Big)^2.$$

Step 2. Final estimate for $\|f - Af\|_2$.
We interpret $\sum_{n\in F_k}\|P_n\|_2$ as a convolution. Let the sequence $q:\mathbb{Z}\to\mathbb{R}$ be $q(n)=\|P_n\|_2$ and $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1,1]$. Following the usual computations, we have

$$\begin{aligned}\big(q*\chi_{[-1,1]}\big)(k) &= \sum_{n\in\mathbb{Z}} q(n)\chi_{[-1,1]}(k-n) = \sum_{n\in\mathbb{Z}} q(n)\chi_{[k-1,k+1]}(n)\\ &= \sum_{n\in F_k} q(n) = \sum_{n\in F_k}\|P_n\|_2.\end{aligned}$$

By rewriting the error, we get

$$\|f-Af\|_2^2 \le \frac{1}{\pi^2}\sum_{k\in\mathbb{Z}}\delta_k^2\zeta_k^2\big(q*\chi_{[-1,1]}(k)\big)^2.$$

We use Young's inequality and the fact that $\|\chi_{[-1,1]}\|_1=3$, to obtain

$$\begin{aligned}\|f-Af\|_2^2 &\le \frac{1}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\big)\big\|q*\chi_{[-1,1]}\big\|_2^2\\ &\le \frac{1}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\big)\big\|\chi_{[-1,1]}\big\|_1^2\|q\|_2^2\\ &\le \frac{9}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\big)\sum_{k\in\mathbb{Z}}\|P_k\|_2^2.\end{aligned}$$

Applying equality (4.3.5), we have

$$\begin{aligned}\|f-Af\|_2^2 &\le \frac{9}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\big)\sum_{k\in\mathbb{Z}}\|P_k\|_2^2\\ &= \frac{9}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\big)\sum_{k\in\mathbb{Z}}4\ell_k\sum_{l=0}^{a(k)}\big|c_{k,l}\big|^2\\ &\le \frac{36}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\ell_k\big)\|f\|_2^2.\end{aligned}$$

We define the error constant $\gamma^2 := \frac{36}{\pi^2}\sup_{k\in\mathbb{Z}}\big(\delta_k^2\zeta_k^2\ell_k\big)$. By condition (4.4.2), we have that $\gamma<1$, and by Lemma 1.18, Λ is a sampling set for $LF_b^2(\mathbb{R})$ with constants $A=(1-\gamma)^2$ and $B=(1+\gamma)^2$. □

4.5 Comparison and conclusions

(i) In the same spirit as the previous chapters, in Sections 4.3 and 4.4 we showed two sufficient conditions for sampling. Unlike the case of variable bandwidth spaces constructed with a Gabor Riesz sequence (Chapter 2) and a Wilson basis (Chapter

3), there is an important feature of local Fourier basis which makes Theorems 4.8 and 4.9 different from those in the previous chapters. The functions in $V_b^2(g,\mathbb{R})$ and $PW_b^2(g,\mathbb{R})$ are sums of only one translated and modulated window function g. Therefore, $\|g'\|_\infty$ could be treated as a constant independent of the partition element. In the case of $LF_b^2(\mathbb{R})$, there exists a collection of bell functions $\{g_n\}_{n\in\mathbb{Z}}$ with different supports and decay rates at the boundaries that expand each function in $LF_b^2(\mathbb{R})$.

For this reason, we could not treat $\|g_n'\chi_k\|_\infty$ as a constant since it might vary for each interval of the partition and we have to include it in the definition of ζ (4.4.1).

Additionally, the structure of the local Fourier basis is reflected in the presence of the length of the partition $\ell_k = \alpha_{k+1} - \alpha_k$ in the condition of Theorem 4.8

$$\sup_{k\in\mathbb{Z}} \left\{ 12\ell_k \left(\delta_{k-1}^2 + 2\delta_k^2 + \delta_{k+1}^2\right)\left(\left(\frac{2\lceil 2b(k)\ell_k - \frac{1}{2}\rceil + 1}{2\ell_k}\right)^2 + 2\frac{\|g_k'\chi_k\|_\infty^2}{\pi^2}\right)\right\} < 1, \quad (4.5.1)$$

and in the sufficient condition of Theorem 4.9

$$\sup_{k\in\mathbb{Z}} \delta_k \zeta_k \sqrt{\ell_k} < \frac{\pi}{6}. \quad (4.5.2)$$

(ii) As done for Theorem 2.3 in the context of Gabor Riesz sequences and Theorem 3.11 for the space constructed with a Wilson Riesz basis, the Theorem 4.8 illustrates the dependence of the sufficient condition (4.5.1) on the properties of the bell function and how neighboring bell windows impact the reconstruction in every interval. The chosen partition, organized in intervals of the form $[\alpha_k, \alpha_{k+1})$, ensures that the expansion of a function $f(x_{k,j})$ in the space $LF_b^2(\mathbb{R})$ involves three bell functions: g_{k-1}, g_k, and g_{k+1}.

This implies that only 3 densities δ_{k-1}, δ_k and δ_{k+1}, are contained in condition (4.5.1). As it was proved for $V_b^2(g,\mathbb{R})$ and $PW_b^2(g,\mathbb{R})$, the bandwidth of the central window has a stronger impact on the sampling rate, indicated by the constant 2 in (4.5.1), whereas the windows at the boundaries exercise a more modest influence on the density of the space.

Condition (4.5.1) can be compared to condition (2.1.2) of Theorem 2.3 for $V_b^2(g,\mathbb{R})$

$$\sup_{k\in\mathbb{Z}} \left\{ \frac{\lceil\beta\rceil}{\beta}\frac{6}{A'}\left(\delta_{k-1}^2 + 4\delta_k^2 + \delta_{k+1}^2\right)\left(\beta^2 b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\right)\right\} < 1$$

and (3.5.1) of Theorem 3.11 for $\widetilde{PW}_b^2(g,\mathbb{R})$

$$\sup_{k\in\mathbb{Z}} \left\{ \frac{8}{A'}\left(\delta_{\frac{k}{2}-1}^2 + 4\delta_{\frac{k}{2}-\frac{1}{2}}^2 + 4\delta_{\frac{k}{2}}^2 + \delta_{\frac{k}{2}+\frac{1}{2}}^2\right)\left(b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\right)\right\} < 1.$$

The three results are very similar. In particular, when we compare (4.5.1) with the sufficient condition (2.1.2) for $V_b^2(g,\mathbb{R})$, we have that $\beta b(n)$ is the highest frequency in $[k,k+1)$, while $\frac{2\lceil 2b(k)\ell_k-\frac{1}{2}\rceil+1}{2\ell_k}$ is the highest frequency in $[\alpha_k,\alpha_{k+1})$ for $LF_b^2(\mathbb{R})$. Moreover, both the inequalities present the dependence on the period of their respective trigonometric polynomials, i.e., $1/\beta$ for (2.1.2) and $4\ell_k$ (hidden in $12\ell_k$) for $LF_b^2(\mathbb{R})$.

(iii) Theorem 4.9 isolates the reconstruction in every interval by presenting a sufficient condition (4.5.2) that solely depends on the δ_k. This result is comparable to Theorem 2.4 for $V_b^2(g,\mathbb{R})$ and Theorem 3.10 for $PW_b^2(g,\mathbb{R})$. However, for the space $LF_b^2(\mathbb{R})$ the number of bandwidth among which we have to take the maximum is always 3. In fact, even if we can choose the support of every bell function, by the construction of local Fourier bases there are only three bell functions that play a role in the reconstruction of $f(x_{k,j})$ in each interval of the partition.

(iv) Another interesting aspect is that the other bases of Theorem 4.4 can be used to construct similar spaces of variable bandwidth with local Fourier bases. Sufficient conditions for sampling theorems could be proved in an entirely similar way.

(v) As discussed in detail in (ii) of Section 2.3 and (iv) in Section 3.7, it is crucial to emphasize that the proof does not necessitate the separation of points.

5 Numerical simulations

In this chapter, we present various sampling scenarios and the results of numerical reconstructions. The main point is the numerical and visual illustration of our notion of variable bandwidth. We hope to convince the reader that the spaces constructed using Wilson bases $PW_b^2(g,\mathbb{R})$ are meaningful and useful notions to capture variable bandwidth.

In principle the proof of Theorem 3.10 provides a reconstruction procedure as follows: from the samples $\{f(x_{\frac{k}{2},j})\}$ we form the approximation operator Af and then invert A to obtain the reconstruction $f = A^{-1}(Af)$. Since this reconstruction requires the orthogonal projection P onto $PW_b^2(g,\mathbb{R})$ and the inversion of A, this method is not practical.

A second reconstruction method can be based on the sampling inequality (3.4.4). Since $PW_b^2(g,\mathbb{R})$ is a reproducing kernel Hilbert space, (3.4.4) can be read as the frame inequality for the reproducing kernels associated to the sampling points. Then any off-the-shelf frame algorithm [23] yields a reconstruction.

In both cases, one still has to deal with a numerical discretization from the infinite-dimensional space $PW_b^2(g,\mathbb{R})$ (requiring infinitely many data) to some finite-dimensional problem that uses only finite information as input. One of the advantages of our model of variable bandwidth is that such a discretization is not necessary! Precisely, if the window g generating the Wilson basis has compact support and if the samples are taken from an interval I, then we may try to find a reconstruction or approximation in the finite-dimensional space $PW_b^2(g,I)$. In this case, a reconstruction amounts to the solution of a finite-dimensional linear system and can be treated with numerical linear algebra.

The chapter is organized as follows: Section 5.1 presents the matrix formulation of the reconstruction problem. In Section 5.2 we define the variables of the space $PW_b^2(g,I)$ and we reconstruct a compactly supported function that belongs to the given space. Section 5.3 provides a reconstruction for a function in $PW_b^2(g,\mathbb{R})$ and finally, in Section 5.4, we explore the problem of reconstructing a chirp, which does not belong to our space of variable bandwidth. The simulations in Sections 5.1, 5.2, 5.3, and 5.4 are reported verbatim from [7]. Section 5.5 contains the reconstructions for a space of variable bandwidth constructed via Gabor frames.

5.1 Matrix formulation of the reconstruction

Assume that g has compact support in $[-m,m]$. Let $I=[\alpha,\beta]$ and recall that

$$PW_b^2(g,I) = \operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \subseteq I\}$$

is the local version of the variable bandwidth space $PW_b^2(g,\mathbb{R})$. According to the proof of Theorem 3.7, $PW_b^2(g,I)$ is a finite-dimensional subspace of $PW_b^2(g,\mathbb{R})$ consisting of those functions $f \in PW_b^2(g,\mathbb{R})$ with $\mathrm{supp}(f) \subseteq I$.

Let $\Lambda = \{x_j : j = 1,\dots,L\}$ be a sampling set contained in I.

To reconstruct a function $f \in PW_b^2(g,I)$ from its samples $f(x_j)$, we need to solve the linear system of equations

$$\sum_{n,l\in\mathbb{Z}:\mathrm{supp}(\psi_{n,l})\subseteq I} c_{n,l}\psi_{n,l}(x_j) = \sum_{n\in[\alpha+m,\beta-m]\cap\mathbb{Z}} c_{n,0}\psi_{n,0}(x_j) + \sum_{n/2\in[\alpha+m,\beta-m]\cap\mathbb{Z}} \sum_{l=1}^{b(n)} c_{n,l}\psi_{n,l}(x_j) = f(x_j)\,,$$

$j = 1,\dots,L$, for the coefficients $c_{n,l}$.

This linear system can be written in matrix notation as

$$Uc = y,$$

where y is the input sequence $\{f(x_j)\}$, and the matrix U has the entries

$$U_{j,(n,l)} = \psi_{n,l}(x_j). \tag{5.1.1}$$

In general the samples are not exact and may contain measurement errors or noise, so that the input vector y is given by entries $y_j = f(x_j) + \epsilon_j$. It may also happen that the sampled function is not contained in $PW_b^2(g,I)$. In both cases we seek a function $\tilde{f} \in PW_b^2(g,I)$ that approximates the data vector y optimally. In the absence of additional information, such as sparsity, we use a least squares approach and seek $\tilde{f}$ as the solution of

$$\begin{aligned}\tilde{f} &= \mathrm{argmin}_{f\in PW_b^2(g,I)} \sum_j |f(x_j) - y_j|^2 \\ &= \mathrm{argmin}_{c_{n,l}} \sum_j \Big| \sum_{n,l\in\mathbb{Z}:\mathrm{supp}(\psi_{n,l})\subseteq I} c_{n,l}\psi_{n,l}(x_j) - y_j\Big|^2 \\ &= \mathrm{argmin}_{c_{n,l}} \|Uc - y\|_2^2\,. \end{aligned} \tag{5.1.2}$$

The solution to the matrix least squares problem uses the Moore-Penrose pseudo-inverse matrix $U^\dagger$ of U and is given by

$$c = U^\dagger y. \tag{5.1.3}$$

The approximating function $\tilde{f} \in PW_b^2(g,I)$ is then

$$\tilde{f} = \sum_{n,l\in\mathbb{Z}:\mathrm{supp}(\psi_{n,l})\subseteq I} c_{n,l}\psi_{n,l} \tag{5.1.4}$$

and solves (5.1.2). If the data are exact, $y_j = f(x_j)$ for $f \in PW_b^2(g,I)$, then $\tilde{f} = f$ and the reconstruction is exact.

For a detailed exposition of the least squares approximation method, see [16, 37, 54].

5.2 Numerical set-up

For the numerical simulations we adopt (5.1.3) and (5.1.4) as our starting point. Next we explain the numerics and the selection of parameters in the experiments.

5.2.1 Window function

For the numerical simulations we use the window function $g(x) = \sqrt{2}\cos(\pi x)\chi_{[-1/2,1/2]}(x)$ from Example B and Figure 5.1. Then g generates an orthonormal Wilson basis. In this case the constants of Theorem 3.10 are readily computed and are listed in Table 5.1.

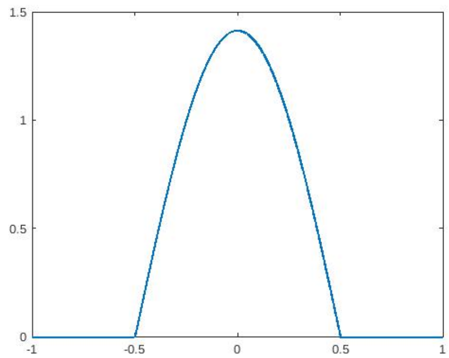


Figure 5.1: Window $\sqrt{2}\cos(\pi x)\chi_{[-1/2,1/2]}(x)$ used to generate $PW_b^2(g,\mathbb{R})$.

Window	$\sqrt{2}\cos(\pi x)\chi_{[-1/2,1/2]}(x)$
m	0.5
$\|g\|_\infty$	$\sqrt{2}$
$\|g'\|_\infty$	$\sqrt{2}\pi$

Table 5.1: Constants of the window needed in Theorem 3.10.

5.2.2 Numerical error

In the numerical simulations we start with a known function $f \in PW_b^2(g, I)$ and then compute a reconstruction $\tilde{f}$ from its samples. To assess the quality of the reconstruction, we evaluate the discretized error on a uniform grid $\{\gamma n : n \in \mathbb{Z}\}$ on the interval of reconstruction I. We use the relative errors in the ℓ^2-norm and in the ℓ^∞-norm defined by

$$E(f,2) := \frac{\left\{\sum_{\gamma n \in I} |\tilde{f}(\gamma n) - f(\gamma n)|^2\right\}^{1/2}}{\left\{\sum_{\gamma n \in I} |f(\gamma n)|^2\right\}^{1/2}},$$

$$E(f,\infty) := \frac{\max_{\gamma n \in I} |\tilde{f}(\gamma n) - f(\gamma n)|}{\max_{\gamma n \in I} |f(\gamma n)|},$$

where the step-size γ is chosen to be of the order $\mathcal{O}(10^{-4})$.

5.2.3 Least squares approximation

The numerical approximation is carried out with MATLAB, and the solution to (5.1.2) is obtained with the MATLAB function `pinv`. Since our aim is to validate the notion of variable bandwidth with Wilson basis, we have not tried to optimize the numerical part. To set up the sampling matrix (5.1.1) we choose a suitable enumeration of the indices (n, l) satisfying $\mathrm{supp}(\psi_{n,l}) \subseteq I$, namely,

$$i(n,l) = \sum_{\substack{r<n: \\ \mathrm{supp}(\psi_{r,l}) \subseteq I}} b(r) + \#\{r \leq n : \mathrm{supp}(\psi_{r,0}) \subseteq I\} + l\,.$$

The entries of (5.1.1) are identified by

$$U_{j,(n,l)} = U_{j,i}.$$

As always, the generation of U is the most expensive part of the computation.

5.2.4 Test signal

We sample a signal on the interval $I = [0, 6]$, and we choose a suitable test signal $f_1 \in PW_b^2(g, I)$ by generating its coefficients. To begin, we generate a bandwidth sequence $b : \mathbb{Z} \to \mathbb{N}$, where

$$b(n) \sim \mathcal{U}_{[20,500]} \text{ for } \{n \in \mathbb{Z} : \mathrm{supp}(\psi_{n,l}) \subseteq I\},$$

with $\mathcal{U}$ denoting the uniform distribution of integers in the range $[20, 500]$. The resulting bandwidth sequence is $b = (102, 35, 499, 444, 341, 111, 197, 241, 492, 95, 431)$, illustrated by the center of the yellow bands in Figure 5.2b. For clarity, we assume that only the maximum frequency on each interval $[k/2, k/2 + 1/2]$ is active, so that the test signal is of the form [1]

$$f_1 = \sum_{n=1}^{11} c_{n,b(n)} \psi_{n,b(n)} \in PW_b^2(g, I)\,.$$

We randomize the coefficients by setting for $\{n \in \mathbb{Z} : \mathrm{supp}(\psi_{n,l}) \subseteq I\}$

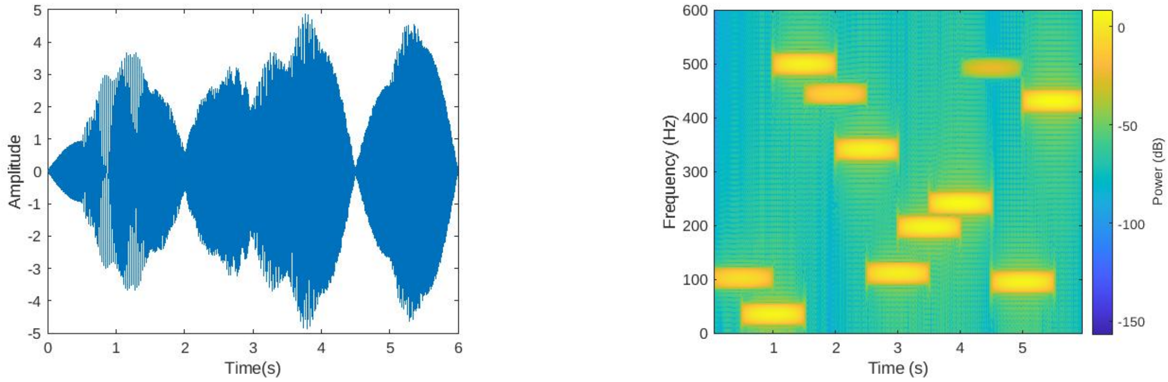
$$c_{n,b(n)} \sim \mathcal{U}_{[-200,200]} \cdot 10^{-2} \text{ and all the other coefficients are set to be zero.} \tag{5.2.1}$$

By (3.3.1), the dimension of $PW_b^2(g, I)$ is given by

$$\begin{aligned} \dim(PW_b^2(g, I)) &= \dim(\mathrm{span}\{\psi_{n,l} : \mathrm{supp}(\psi_{n,l}) \subseteq I\}) \\ &= \#([0.5, 5.5] \cap \mathbb{Z}) + \sum_{n/2 \in [0.5, 5.5]} b(n) = 5 + \sum_{n=1}^{11} b(n) = 2993. \end{aligned}$$

Figure 5.2 represents the generated signal $f_1 \in PW_b^2(g, I)$ that needs to be reconstructed. The local frequency spectrum is visualized by means of the spectrogram of f_1.

[1]Note that this is a sparse signal that could be reconstructed with compressed sensing methods. Our choice f_1 is motivated by the need to obtain a reasonable and interpretable plot (Figure 5.2a). General signals in $PW_b^2(g, I)$ are *not* sparse, and the least square solution is the only option.



(a) Function $f_1 \in PW_b^2(g, I)$.

(b) Spectrogram of f_1.

Figure 5.2: Test signal $f_1 \in PW_b^2(g, I)$ generated by the bandwidth sequence $b = (102, 35, 499, 444, 341, 111, 197, 241, 492, 95, 431)$, and coefficients sequence of the form (5.2.1).

5.2.5 Sampling set

For every signal we use two types of sampling sets. One set is chosen to satisfy the sufficient conditions of Theorem 3.10, the other type is not covered by the theory.

Since in our example the signal space $PW_b^2(g, I)$ is assumed to be known, we can determine the local maximal gaps $\delta_{k/2}$ on I in advance by means of (3.4.3). We define

$$\mathcal{M} := \left\{ k \in \mathbb{Z} : I \cap \left[\frac{k}{2}, \frac{k+1}{2}\right) \neq \emptyset \right\}$$

to be the set of indices $k \in \mathbb{Z}$, for which we have to compute $\delta_{k/2}$. With $\mathcal{M} = \{0, \dots, 11\}$, we choose the sampling set

$$\Lambda_1 := \left\{ x_{k/2,j} = \frac{k}{2} + \delta_{k/2} j : \ k \in \mathcal{M}, \ j = 0, \dots, \left\lfloor \frac{1}{2\delta_{k/2}} \right\rfloor \right\}. \tag{5.2.2}$$

The sampling points are equispaced according to the local bandwidth within the interval $[\frac{k}{2}, \frac{k}{2} + \frac{1}{2}]$ for every $k \in \mathcal{M}$, and by construction Λ_1 satisfies the conditions of Theorem 3.10. We then evaluate the Wilson basis elements on Λ_1 and assemble them in the matrix U according to (5.1.1), and then solve (5.1.3) and (5.1.4) by means of the MATLAB function `pinv`.

The reconstruction $\tilde{f}_1$ is accurate up to machine precision. Table 5.2 reports the relative errors in the ℓ^2-norm and ℓ^∞-norm. The pointwise difference between the original function f_1 and its reconstruction $\tilde{f}_1$ is plotted in Figure 5.3.

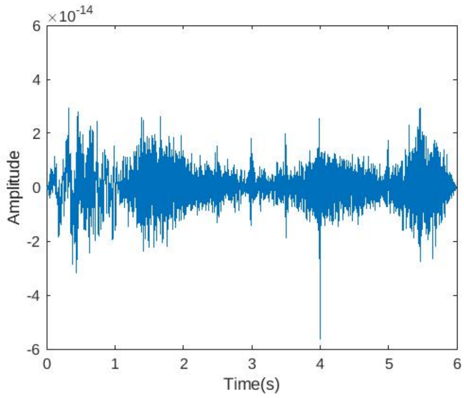


Figure 5.3: Difference between the original signal f_1 and the reconstruction $\tilde{f}_1$. Maximal gap condition (3.4.3) is satisfied.

Relative errors	
$E(f,2)$	3.776×10^{-15}
$E(f,\infty)$	6.016×10^{-15}

Table 5.2: Relative errors in ℓ^2-norm and ℓ^∞-norm.

However, the conditions of Theorem 3.10 are much too pessimistic. According to Theorem 3.7 we need at least $\dim(PW_b^2(g,I)) = 2993$ samples, whereas $\#\Lambda_1 = 12121$. Consequently, the redundancy, or oversampling factor, is given by

$$q_1 = \frac{\#\Lambda_1}{\dim(PW_b^2(g,I))} = 4.05.$$

It is therefore not surprising that with such a high oversampling the reconstruction works perfectly.

For the sake of completeness, we conducted the same experiment with a function f_2 that shares the same bandwidth sequence as before but has a full set of non-zero coefficients as follows:

$$c_{n,l} \sim \mathcal{U}_{[-200,200]} \cdot 10^{-2}, \text{ for } \{n, l : \operatorname{supp}(\psi_{n,l}) \subseteq I\}.$$

Consequently, the test signal can be expressed as

$$f_2 = \sum_{n=1}^{5} c_{n,0}\psi_{n,0} + \sum_{n=1}^{11}\sum_{l=1}^{b(n)} c_{n,l}\psi_{n,l} \in PW_b^2(g,I).$$

We use the same sampling set Λ_1 as before, so that the redundancy is again $q_2 = q_1 = 4.05$. Then the sufficient condition of Theorem 3.10 provides again a good reconstruction, with relative errors in the ℓ^2-norm and ℓ^∞-norm of the order $\mathcal{O}(10^{-15})$. The pointwise difference between the function f_2 and its reconstruction $\tilde{f}_2$ is plotted in Figure 5.4. Table 5.3 reports the relative errors in the ℓ^2-norm and ℓ^∞-norm.

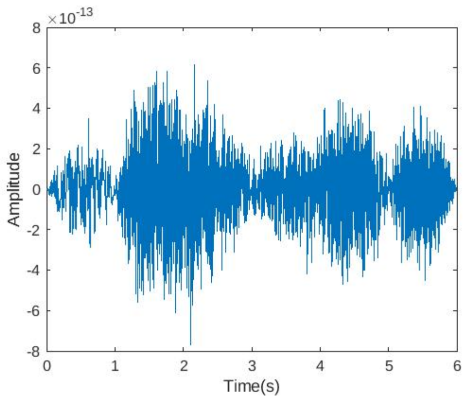


Figure 5.4: Difference between the original signal f_2 and the reconstruction $\tilde{f}_2$. Maximal gap condition (3.4.3) is satisfied.

Relative errors	
$E(f,2)$	5.218×10^{-15}
$E(f,\infty)$	4.956×10^{-15}

Table 5.3: Relative errors in ℓ^2-norm and ℓ^∞-norm.

In the following experiment, we test sampling sets that are not covered by the theoretical conditions. Recall that $\mu_{k/2} = \max_{n\in(k-2m,k+2m+1)\cap\mathbb{Z}}\big(b(n)+1\big)$ is the active local bandwidth. We now choose a sampling set of the form

$$\Gamma_1^\rho \coloneqq \Big\{x_{k/2,j} = \frac{k}{2} + \frac{1}{2}\frac{1}{\rho\mu_{k/2}}j:\ k\in\mathcal{M},\ j=0,\dots,\lfloor\rho\mu_{k/2}\rfloor\Big\}. \tag{5.2.3}$$

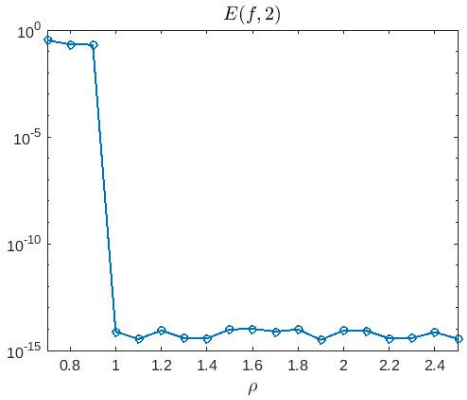
with a parameter $\rho>0$. If ρ is sufficiently large, precisely $\rho\ge D/(2\pi)=2.83$, then Γ_1^ρ is covered by Theorem 3.10. For $\rho<D/(2\pi)$ we do not have any theoretical guarantees for the quality of the reconstruction. The errors of the least square approximation are illustrated in Figure 5.5 for different sampling sets Γ_1^ρ with parameters $\rho=0.7,0.8,\dots,2.4,2.5$. In all cases $\rho\ge 1$, machine precision is reached. The case $\rho=1$ corresponds to an oversampling parameter of 1.43 and the starting point $\rho=0.7$ represents an oversampling parameter of 1.004.

This simulation suggests that the numerical reconstruction still works near the critical density, although there is currently no theoretical performance guarantee in this regime.

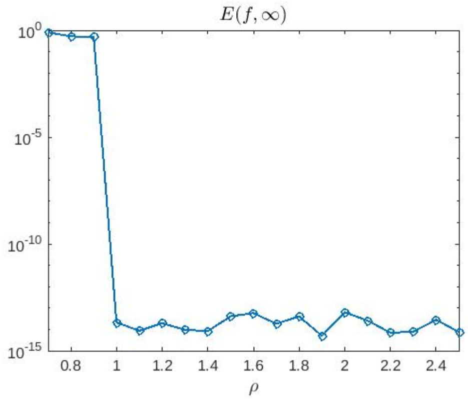
5.3 Local reconstruction of a function of $PW_b^2(g,\mathbb{R})$

We now modify the scenario and assume that a function $f_3\in PW_b^2(g,\mathbb{R})$ is given. We try to reconstruct or approximate f_3 on an interval I from samples in I. This situation is different, because a sample $x\in I$ sees all basis functions $\psi_{n,l}$ with $x\in\operatorname{supp}(\psi_{n,l})$. Consequently, we have to solve the linear equations

$$\sum_{(n,l):\operatorname{supp}(\psi_{n,l})\cap I\neq\emptyset} c_{n,l}\psi_{n,l}(x_j) = y_j \qquad j=1,\dots,L\,.$$



(a) Relative error in the ℓ^2-norm.



(b) Relative error in the ℓ^∞-norm.

Figure 5.5: Behavior of the reconstruction of $f_1 \in PW_b^2(g, I)$ with increasing parameter $\rho = 0.7, 0.8, \ldots, 2.4, 2.5$.

For $I = [0, 6]$ and the window g from Example B with $\text{supp}(g) \subseteq [-1/2, 1/2]$, the resulting equations are

$$\sum_{n=0}^{6} c_{n,0}\psi_{n,0}(x_j) + \sum_{n=0}^{12}\sum_{l=1}^{b(n)} c_{n,l}\psi_{n,l}(x_j) = y_j\,, \qquad j = 1, \ldots, L\,.$$

The dimension of this problem (the number of variables) is now

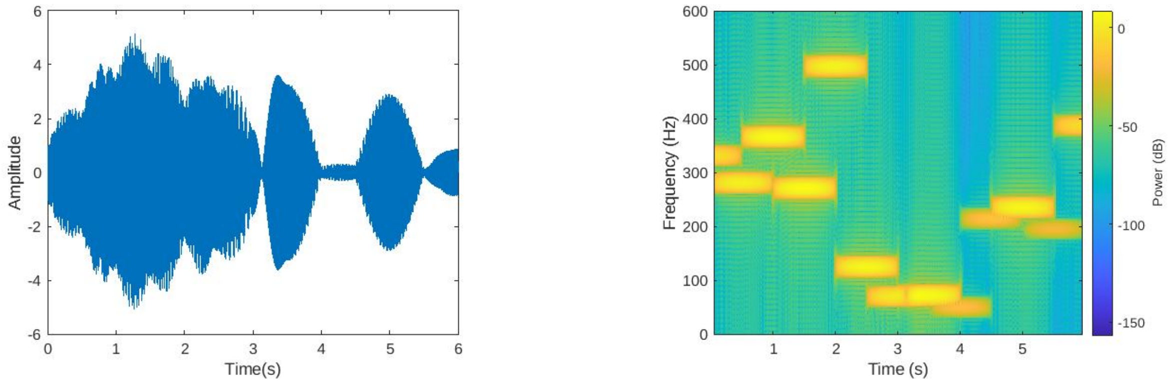
$$\begin{aligned}\dim(\text{span}\{\psi_{n,l} : \text{supp}(\psi_{n,l}) \cap I \neq \emptyset\}) &= \#\{n \in \mathbb{Z} : \text{supp}(\psi_{n,0}) \cap I\} + \sum_{\substack{n\in\mathbb{Z}:\\ \text{supp}(\psi_{n,l})\cap I}} b(n) \\ &= 7 + \sum_{n=0}^{12} b(n) = 3101.\end{aligned}$$

Figure 5.6 illustrates the function $f_3\chi_I$ that we aim to reconstruct.

Again we use two types of sampling sets. The first set Λ_3 is chosen as in (5.5.3) so that it formally satisfies the sufficient conditions of Theorem 3.10. The redundancy is $q_3 = 3.11$.

The pointwise error between $f_3\chi_I$ and its approximation $\tilde{f}_3 \in \text{span}\{\psi_{n,l} : \text{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$ is plotted in Figure 5.7, and the relative errors are reported in Table 5.4.

We see that the reconstruction is highly accurate of the order $\mathcal{O}(10^{-14})$ in the central part of the signal, but oscillates with a pointwise error of a significantly larger order $\mathcal{O}(10^{-5})$ at the boundary of I. This is not surprising: the samples near the boundary of I, in our case $\Lambda_3 \cap ([0, 0.5] \cup [5.5, 6])$ are needed to recover the coefficients of all terms $\psi_{n,l}$ with $\text{supp}(\psi_{n,l}) \cap I \neq \emptyset$ and $\text{supp}(\psi_{n,l}) \not\subseteq I$. Even if the sufficient conditions of Theorem 3.10 are satisfied, there may not be enough samples near the boundary, and the sampling matrix has some rank deficiency.



(a) Function $f_3\chi_I$.

(b) Spectrogram of $f_3\chi_I$.

Figure 5.6: Signal $f_3\chi_I$ generated by the bandwidth sequence $b = (331, 281, 366, 271, 497, 125, 70, 72, 50, 214, 235, 195, 387)$ and coefficients sequence of the form (5.2.1) with the modified index set.

To mitigate this effect, we add sampling points outside the interval I and generate a sampling set that satisfies the sufficient conditions (5.5.3) on the enhanced interval $\tilde{I} = I \cup [-0.5, 0] \cup [6, 6.5]$. The resulting oversampling parameter is 3.77.

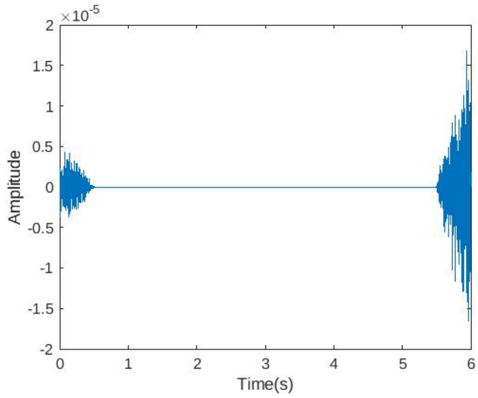


Figure 5.7: Difference between $f_3\chi_I$ and $\tilde{f}_3 \in \operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$.

Relative errors	
$E(f,2)$	7.264×10^{-7}
$E(f,\infty)$	3.273×10^{-6}

Table 5.4: Relative errors in ℓ^2-norm and ℓ^∞-norm.

As shown in Figure 5.8 and Table 5.5, the numerical reconstruction on I (not $\tilde{I}$) is now again accurate to machine precision. The theoretical explanation is simple: by sampling on the extended interval $\tilde{I}$, we replace the given signal $f_3 \in PW_b^2(g,\mathbb{R})$ by a function in the local variable bandwidth space $f_4 \in PW_b^2(g,\tilde{I})$ such that $f_3\big|_I = f_4\big|_I$. We are now in the situation of the first scenario from Section 5.2 and try to recover $f_4 \in PW_b^2(g,\tilde{I})$ from samples in $\tilde{I}$.

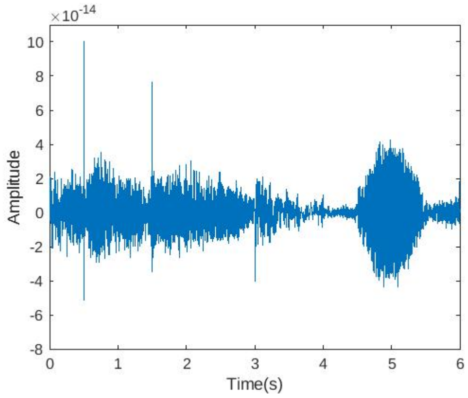
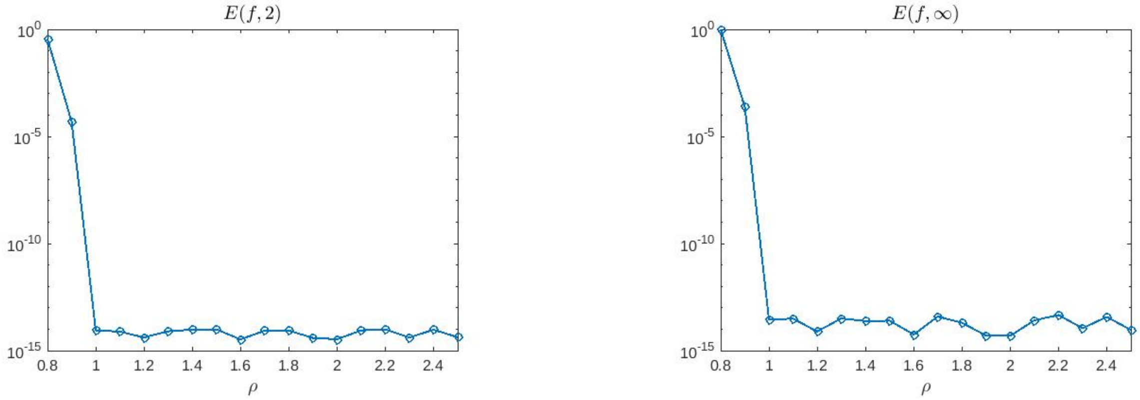


Figure 5.8: Difference between $f_3\chi_I$ and its reconstruction by sampling in $[-0.5, 6.5)$.

Relative errors	
$E(f,2)$	6.235×10^{-15}
$E(f,\infty)$	1.953×10^{-14}

Table 5.5: Relative errors in ℓ^2-norm and ℓ^∞-norm.

Additionally, we investigate the reconstruction from sampling sets Γ_3^ρ as in (5.2.3) for $\rho > 0$. Figure 5.9 displays the relative errors in the ℓ^2-norm and ℓ^∞-norms for $\rho = 0.8, 0.9, \ldots, 2.4, 2.5$.



(a) Relative error in the ℓ^2-norm of the reconstruction by sampling in $[-0.5, 6.5)$.

(b) Relative error in the ∞-norm by sampling in $[-0.5, 6.5)$.

Figure 5.9: Behavior of the reconstruction in $\operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$ by sampling in $[-0.5, 6.5)$, with increasing parameter $\rho = 0.8, 0.9, \ldots, 2.4, 2.5$.

The results indicate that the sufficient conditions of Theorem 3.10 remain pessimistic. For $\rho = 1$, corresponding to an oversampling parameter of 1.33, the reconstruction already achieves a relative error in the ℓ^2-norm of the order $\mathcal{O}(10^{-15})$.

5.4 Reconstruction of a chirp

In the final example, we aim to reconstruct a linear chirp in the interval $I = [0, 6]$, described by the equation

$$p(x) = \sin\Big(\phi(0) + \pi\frac{\omega(T) - \omega(0)}{T}x^2 + 2\pi\omega(0)x\Big),$$

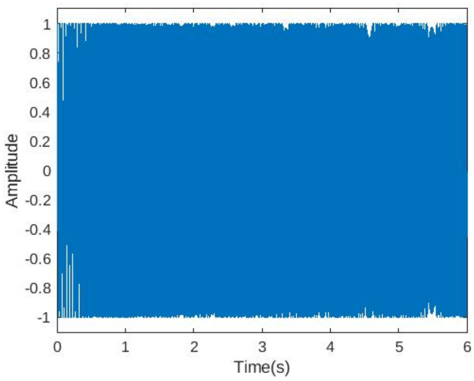
where $\phi(0)$ is the initial phase, $\omega(0)$ is the instantaneous frequency at time 0, and $\omega(T)$ is the instantaneous frequency at time T. For a linear chirp, the instantaneous frequency function ω is given by the formula

$$\omega(t) = \frac{\omega(T) - \omega(0)}{T} t + \omega(0).$$

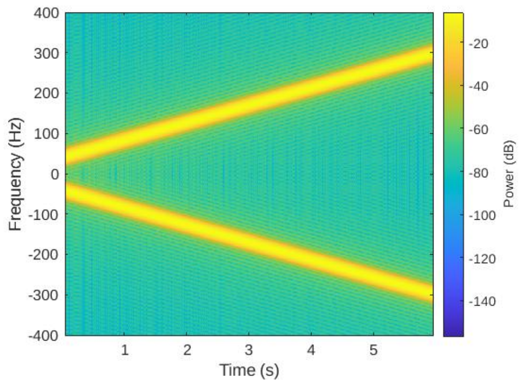
In our simulations, we take the parameters $T = 6$, $\phi(0) = 0$, $\omega(0) = 40$, and $\omega(T) = 300$ to get the chirp

$$p(x) = \sin\left(\frac{130}{3}\pi x^2 + 80\pi x\right),$$

shown in Figure 5.10.



(a) Chirp p.



(b) Spectrogram of p.

Figure 5.10: Chirp p with $\phi(0) = 0$, $\omega(0) = 40$, and $\omega(6) = 300$.

In contrast to the previous examples, a general chirp p does not belong to a space $PW_b^2(g, \mathbb{R})$ of variable bandwidth, but it can be modeled and approximated by a function in $PW_b^2(g, \mathbb{R})$ with a suitable choice of the local bandwidth $b(n)$.

Since we know the structure of the chirp in advance, we model the bandwidth sequence $b : \mathbb{Z} \to \mathbb{N}$ by a linear function

$$b(n) := \omega\Big(\frac{n}{2} + \frac{1}{2}\Big) + 30 = \frac{65}{3}(n+1) + 70, \quad \text{for } n = 0, \dots, 12.$$

The number 30 is a safety margin to ensure that the chosen bandwidths contain all the relevant frequencies. We choose $\operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$ as the reconstruction space and the dimension of the problem is then

$$\dim(\operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}) = 7 + \sum_{n=0}^{12} b(n) = 2893.$$

To mitigate boundary effects, we sample p on the extended interval $\tilde{I} = [-0.5, 6.5]$ and we consider sampling set Λ_5 satisfying Theorem 3.10. The oversampling parameter is

$q_5 = \#\Lambda_5 / \dim(PW_b^2(g, \tilde{I})) = 3.18$. The reconstruction algorithm (5.1.3) and (5.1.4) yields an approximation $\tilde{p} \in PW_b^2(g, \tilde{I})$ to the original chirp p.

Figure 5.11 shows the pointwise error between $p\chi_I$ and $\tilde{p}\chi_I$. Although the error of order $\mathcal{O}(10^{-3})$ seems much larger than in the previous simulations, the result is nearly optimal. The chirp p does not belong to $PW_b^2(g, \mathbb{R})$ and the best approximation by a function in $PW_b^2(g, \mathbb{R})$ we can hope for is the orthogonal projection Pp. In Figure 5.12 we therefore plot the pointwise difference between the chirp p and its orthogonal projection onto $\operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$, where the coefficients $\langle p, \psi_{n,l} \rangle$ arising in Pp are obtained by numerical integration with the trapezoidal method (we used the MATLAB function `trapz`). In Tables 5.6 and 5.7 the respective relative errors in the ℓ^2-norm and ℓ^∞-norm are presented.

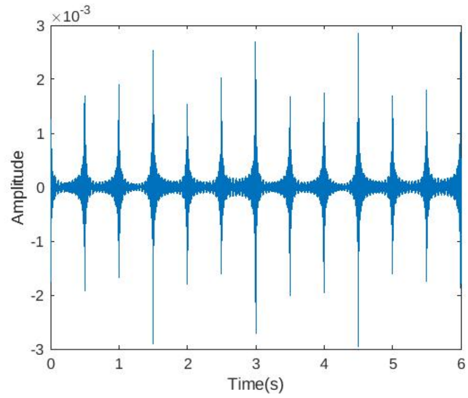


Figure 5.11: Difference between the original chirp p and the reconstruction $\tilde{p}$ using Theorem 3.10 by sampling in $[-0.5, 6.5)$.

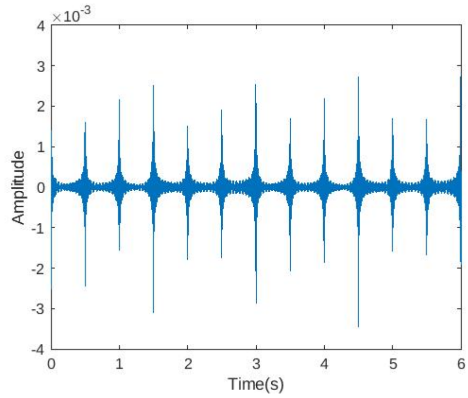


Figure 5.12: Difference between the chirp p and its orthogonal projection Pp onto $\operatorname{span}\{\psi_{n,l} : \operatorname{supp}(\psi_{n,l}) \cap I \neq \emptyset\}$.

Relative errors	
$E(f,2)$	3.926×10^{-4}
$E(f,\infty)$	2.867×10^{-3}

Table 5.6: Relative errors for $\tilde{p}$.

Relative errors	
$E(f,2)$	3.946×10^{-4}
$E(f,\infty)$	2.731×10^{-3}

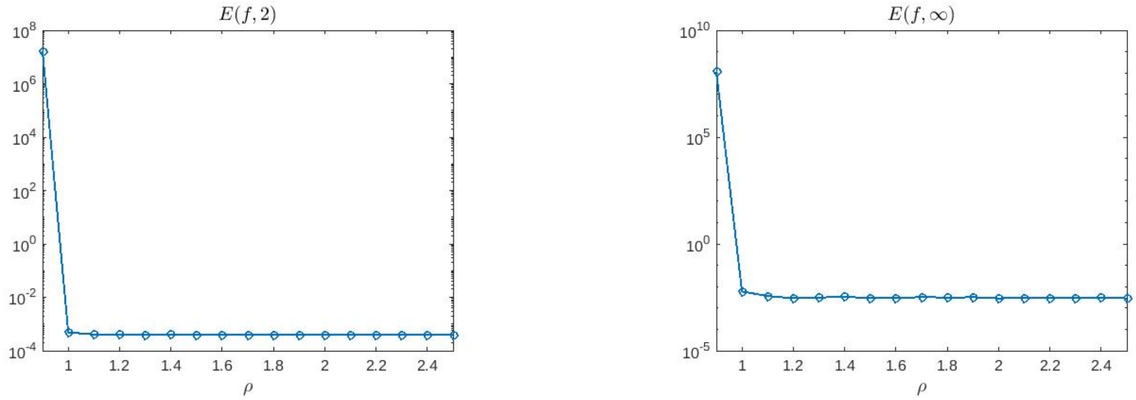
Table 5.7: Relative errors for Pp.

A comparison of the two error plots shows that there is hardly a difference between $p - \tilde{p}$ and $p - Pp$. We conclude that the approximation of p from samples is (almost) the orthogonal projection. It is interesting to note that the largest errors occur at the half integers $k/2$ for $k \in \mathbb{Z}$. This is to be expected because the chirp is smooth everywhere, whereas the window g from Figure 5.1 and hence every $f \in PW_b^2(g, \mathbb{R})$ is continuous, but not differentiable at $\frac{1}{2}\mathbb{Z}$.

Finally, we use the sampling sets Γ_5^ρ from (5.2.3) and approximate the chirp p from its samples on $\tilde{I}$ for $\rho = 0.9, 1, \dots, 2.4, 2.5$.

The relative errors in the ℓ^2-norm and the ℓ^∞-norm are displayed in Figure 5.13.

The relative errors reach the errors of the orthogonal projection at $\rho = 1$, which corresponds



(a) Relative errors in the ℓ^2-norm of $\tilde{p}$.

(b) Relative errors in the ℓ^∞-norm of $\tilde{p}$.

Figure 5.13: Behavior of the reconstruction $\tilde{p}$ of the chirp p with increasing parameter $\rho = 0.9, 1, \ldots, 2.4, 2.5$ and by sampling in $[-0.5, 6.5)$.

to an oversampling parameter of 1.12. Again the relative errors are almost equal to the difference $\|(p - Pp)\big|_I\|_2$, and we obtain an almost perfect reconstruction of Pp rather than p itself.

In conclusion, although chirps, in particular gravitational waves, are not in a variable bandwidth space, these spaces are convenient parametric models. In this particular application, the structure of the signal is known in advance, therefore the use of $PW_b^2(g, \mathbb{R})$ and of the associated sampling sets as in (5.2.3) appears to be a promising approach for their reconstructions.

5.5 The Gabor frame case

In the first part of this chapter, we performed experiments of reconstruction in a variable bandwidth space built with a Wilson basis $PW_b^2(g, \mathbb{R})$. In this last section, we turn our attention to analogous simulations within the realm of variable bandwidth spaces constructed using a Gabor frame.

Let g be compactly supported in $[0, m]$, $I = [a, b]$ and define

$$\widetilde{V}_b^2(g, I) := \operatorname{span}\{M_{\beta l} T_{\alpha n} g : \operatorname{supp}(M_{\beta l} T_{\alpha n} g) \subseteq I,\ |l| \leq b(n)\}.$$

From frame theory, we have that every finite sequence is a frame for the span of the elements. Hence, $\{M_{\beta l} T_{\alpha n} g : \operatorname{supp}(M_{\beta l} T_{\alpha n} g) \subseteq I,\ |l| \leq b(n)\}$ is a frame for $\widetilde{V}_b^2(g, I)$. Moreover, we recall that Linnell [55] demonstrated a fundamental result: if the function g is non-zero, then any finite collection of functions $\{M_{\beta l} T_{\alpha n} g\}_{(n,l) \in \mathcal{F}}$, where $\mathcal{F} \subset \mathbb{Z}^2$, is linear independent. We have that the Gabor system $\{M_{\beta l} T_{\alpha n} g : \operatorname{supp}(M_{\beta l} T_{\alpha n} g) \subseteq I,\ |l| \leq b(n)\}$ is a linearly independent frame for $\widetilde{V}_b^2(g, I)$ and therefore it is a Riesz basis for $\widetilde{V}_b^2(g, I)$.

Given these considerations, we test the sufficient condition for sampling of Theorem 2.4 on a space constructed with a Gabor frame.

The initial expectation was similar to the case of $PW_b^2(g,\mathbb{R})$. At first glance, the reconstruction problem in $\widetilde{V}_b^2(g,I)$ remains, at least for the infinite-dimensional case, essentially the same. Indeed, in Theorem 2.4 we obtain a sufficient condition for sampling which can easily be applied for numerical simulations. The use of a frame instead of a basis for the reconstruction has the difference that the lower frame bound A' and the modulation parameter β are included in the condition (2.2.2).

In the following experiments we discuss which obstacles could arise when working with the truncation of frames.

5.5.1 Matrix formulation and least squares approximation

The matrix formulation of the problem is similar to the one in Section 5.1 but follows some technical modifications to be adapted to the space $\widetilde{V}_b^2(g,I)$.

The linear system representing the reconstruction problem is

$$\sum_{\substack{n\in\mathbb{Z}:\mathrm{supp}(M_{\beta l}T_{\alpha n}g)\subseteq I,\\ |l|\leq b(n)}} c_{n,l}M_{\beta l}T_{\alpha n}g(x_j) = \sum_{\alpha n\in[a,b-m]\cap\mathbb{Z}}\ \sum_{|l|\leq b(n)} c_{n,l}M_{\beta l}T_{\alpha n}g(x_j) = f(x_j)\,,$$

for $j=1,\dots,L$, where $\Lambda=\{x_j\in I : j=1,\dots,L\}$ is the sampling set. The matrix formulation of the linear system is $Uc=y$, where y is the input sequence $\{f(x_j)\}$, and the matrix

$$U_{j,(n,l)} = M_{\beta l}T_{\alpha n}g(x_j). \tag{5.5.1}$$

In the same spirit as in (5.1.2) we search the solution $\tilde{f}\in\widetilde{V}_b^2(g,I)$ of the least squares problem by using the Moore-Penrose pseudo-inverse matrix $U^\dagger$ computed with the MATLAB function `pinv`. The approximating function $\tilde{f}\in\widetilde{V}_b^2(g,I)$ is then

$$\tilde{f} = \sum_{\substack{n\in\mathbb{Z}:\mathrm{supp}(M_{\beta l}T_{\alpha n}g)\subseteq I,\\ |l|\leq b(n)}} c_{n,l}M_{\beta l}T_{\alpha n}g.$$

To build the sampling matrix (5.5.1), we conveniently label the indices in the following way

$$i(n,l) = l + b(n) + 1 + \sum_{\substack{r<n:\\ \mathrm{supp}(M_{\beta l}T_{\alpha r}g)\subseteq I}} (2b(r)+1)\,.$$

The entries of (5.5.1) are $U_{j,(n,l)} = U_{j,i}$.

5.5.2 Window function

For the numerical simulations we use the same window function as in Example B. Since g generates a Wilson orthonormal basis for $L^2(\mathbb{R})$, by Theorem 3.3 the system $\{M_lT_{n/2}g\}_{n,l\in\mathbb{Z}}$ generates a tight frame for $L^2(\mathbb{R})$ of redundancy 2. For simplicity we consider a shifted version by 1/2 of g, i.e., $g_1(x)=\sqrt{2}\cos(\pi(x-1/2))\chi_{[0,1]}(x)$. The constants of Theorem 2.4 are listed in Table 5.8.

Window	$\sqrt{2}\cos(\pi(x-1/2))\chi_{[0,1]}(x)$
α	$1/2$
β	1
m	1
$\|g_1\|_\infty$	$\sqrt{2}$
$\|g_1'\|_\infty$	$\sqrt{2}\pi$
A'	2

Table 5.8: Constants of the window needed in Theorem 2.4.

5.5.3 Test signal

As in Subsection 5.2.4, we collect data from a signal over the interval $I = [0, 6]$ and select an appropriate test signal $f_1 \in \widetilde{V}_b^2(g_1, I)$ by constructing its coefficients. We create a bandwidth sequence $b : \mathbb{Z} \to \mathbb{N}$, where $b(n) \sim \mathcal{U}_{[20,200]}$ for $\{n \in \mathbb{Z} : \operatorname{supp}(M_l T_{n/2} g) \subseteq I\}$. The resulting bandwidth sequence is
$b = (167, 183, 42, 185, 134, 37, 70, 118, 193, 194, 48)$, again depicted by the central part of the yellow bands in Figure 5.14b. We assume that only the maximum frequency within each interval $[k/2, k/2 + 1/2]$ is active, thus the trial signal conforms to the format:

$$f_1 = \sum_{n=1}^{11} c_{n,-b(n)} M_{-b(n)} T_{n/2} g + \sum_{n=1}^{11} c_{n,b(n)} M_{b(n)} T_{n/2} g \in \widetilde{V}_b^2(g_1, I).$$

We assign coefficients as follows:

$$c_{n,b(n)} = c_{n,-b(n)} \sim \mathcal{U}_{[-200,200]} \cdot 10^{-2} \text{ while all other coefficients are set to zero,} \tag{5.5.2}$$

for $\{n \in \mathbb{Z} : \operatorname{supp}(M_l T_{n/2} g) \subseteq I\}$. Due to the symmetry $c_{n,b(n)} = c_{n,-b(n)}$, the signal f_1 is real valued.

The dimension of $\widetilde{V}_b^2(g_1, I)$ is given by

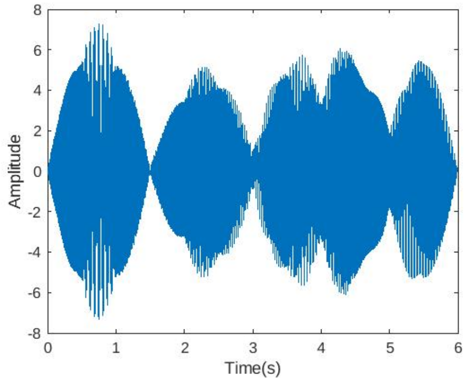
$$\begin{aligned}\dim(\widetilde{V}_b^2(g_1, I)) &= \dim\big(\operatorname{span}\{M_l T_{n/2} g_1 : \operatorname{supp}(M_l T_{n/2} g_1) \subseteq I\}\big)\\ &= \sum_{n/2 \in [0,5]} \big(2b(n) + 1\big) = \sum_{n=0}^{10} \big(2b(n) + 1\big) = 2753.\end{aligned}$$

Figure 5.14 represents the generated signal $f_1 \in \widetilde{V}_b^2(g_1, I)$ that needs to be reconstructed.

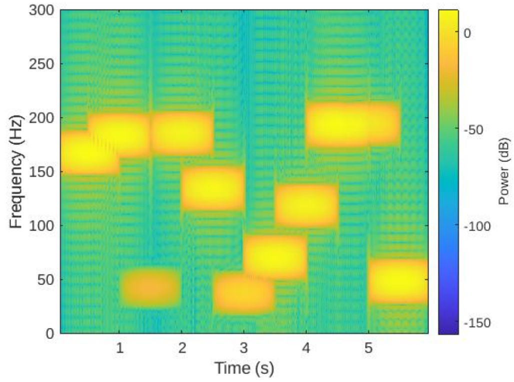
5.5.4 Sampling set

Since $\alpha = 1/2$ and $\beta = 1$ are the chosen parameters to generate the space $\widetilde{V}_b^2(g_1, I)$, the real line is partitioned in the same way as it was for the case of Wilson bases. Indeed,

$$\mathcal{M}_1 = \Big\{k \in \mathbb{Z} : I \cap \Big[\frac{k}{2}, \frac{k+1}{2}\Big) \neq \emptyset\Big\} = \{0, \dots, 11\}$$



(a) Function $f_1 \in \widetilde{V}_b^2(g_1, I)$.



(b) Spectrogram of f_1.

Figure 5.14: Test signal $f_1 \in \widetilde{V}_b^2(g_1, I)$ generated by the bandwidth sequence $b = (167, 183, 42, 185, 134, 37, 70, 118, 193, 194, 48)$, and coefficients sequence of the form (5.5.2).

is the set of indices $k \in \mathbb{Z}$, for which we compute δ_k. We calculate the local maximum gaps δ_k on I using inequality (2.2.2), with $D = 4\sqrt{2}\pi$ and $\delta_k < 1/4\mu_k$ for every $k \in \mathcal{M}_1$. Then we take as sampling set

$$\Lambda_1 := \Big\{ x_{k,j} = \frac{k}{2} + \delta_k j : \; k \in \mathcal{M}_1, \; j = 0, \dots, \Big\lfloor \frac{1}{2\delta_k} \Big\rfloor \Big\}. \tag{5.5.3}$$

The sampling points in Λ_1 are therefore structured such that they fulfill the maximal gap condition of Theorem 2.4. We evaluate the Gabor frames elements at Λ_1 and assemble them in the matrix U according to (5.5.1). We solve the least squares problem using the MATLAB function `pinv` with a tolerance parameter of 10^{-6}.

The reconstruction $\tilde{f}_1$ is not as accurate as it was for the Wilson basis case of Section 5.2. Table 5.9 reports the relative errors in the ℓ^2-norm and ℓ^∞-norm. The pointwise difference between the original function f_1 and its reconstruction $\tilde{f}_1$ is plotted in Figure 5.15.

The oversampling parameter for this experiment is given by

$$q_1 = \frac{\#\Lambda_1}{\dim(\widetilde{V}_b^2(g_1, I))} = \frac{4173}{2753} = 1.52.$$

It is interesting to point out that for this experiment we do not obtain point precision in the reconstruction, but an error of the order $\mathcal{O}(10^{-10})$ for the ℓ^2-norm. This is due to a large rank deficiency. For this example, the rank of the matrix U, computed with the MATLAB function `rank`, is $\mathrm{rank}(U) = 2043$, and the rank deficiency is given by

$$\text{rank deficiency} = 1 - \frac{\mathrm{rank}(U)}{\dim(\widetilde{V}_b^2(g_1, I))} = 1 - \frac{2043}{2753} = 0.26.$$

It is remarkable that we achieved a precision of $\mathcal{O}(10^{-10})$ if we consider that a significant portion, approximately 26%, of the columns of U lacks (numerical) linear independence.

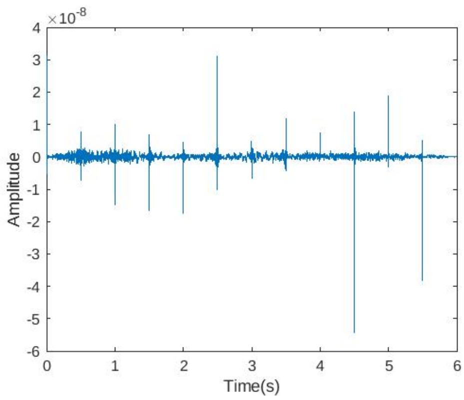


Figure 5.15: Difference between the original signal f_1 and the reconstruction $\tilde{f}_1$. Maximal gap condition (2.2.2) is satisfied.

Relative errors	
$E(f,2)$	5.499×10^{-10}
$E(f,\infty)$	7.461×10^{-9}

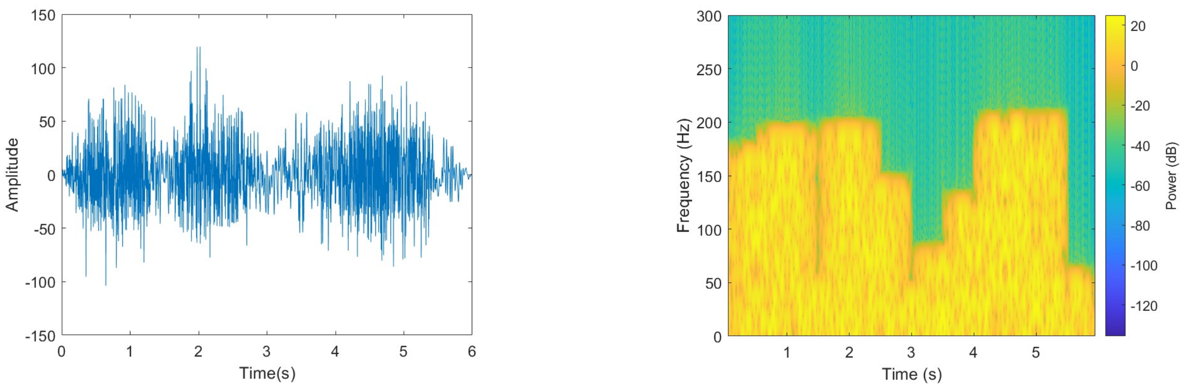
Table 5.9: Relative errors in ℓ^2-norm and ℓ^∞-norm.

This experiment sheds light on a crucial aspect of utilizing frames in practical numerical approximations.

Even if the truncated tight frame $\{M_l T_{n/2} g_1\}_{n=0,\ldots,10,|l|\leq b(n)}$ is a frame for its span and thus satisfies a frame condition, its frame bounds $\tilde{A}$ and $\tilde{B}$ may exhibit erratic behavior, even when the infinite frame bounds A' and B' are moderate. In this case $A' = B' = 2$.

Therefore, the rank deficiency of the matrix U results from the fact that the truncated frame has lower bound that tends to 0 even if the tight frame $\{M_l T_{n/2} g\}_{n,l\in\mathbb{Z}}$ has frame constant 2. Therefore, ill-conditioning correlates with poorly behaved frame bounds of the truncated frame.

For comparison, we repeated the experiment with a function f_2 possessing an identical bandwidth sequence as previously, yet featuring a complete set of non-zero coefficients of the type $c_{n,l} = c_{n,-l} \sim \mathcal{U}_{[-200,200]} \cdot 10^{-2}$, for $\{n, l : \operatorname{supp}(M_{l\beta} T_{n\alpha} g) \subseteq I \text{ and } |l| \leq b(n)\}$. The sampling set Λ_1 is the same as before, and so is the oversampling parameter.



(a) Function $f_2 \in \widetilde{V}_b^2(g_1, I)$.

(b) Spectrogram of f_2.

Figure 5.16: Test signal $f_2 \in \widetilde{V}_b^2(g_1, I)$ generated by the bandwidth sequence $b = (167, 183, 42, 185, 134, 37, 70, 118, 193, 194, 48)$, and a full set of non-zero coefficients.

In Figure 5.16 is plotted the function f_2 to be reconstructed. The pointwise difference between f_2 and its reconstruction $\tilde{f}_2$ is depicted in Figure 5.17. Table 5.10 reports the relative errors in the ℓ^2-norm and ℓ^∞-norm.

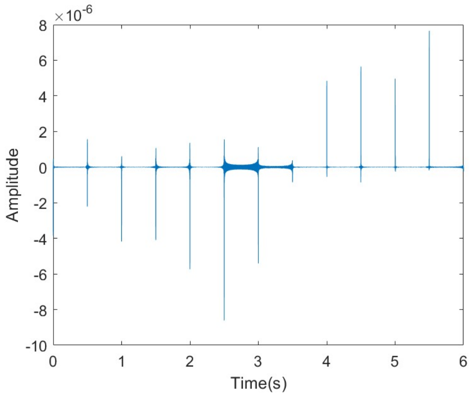


Figure 5.17: Difference between the original signal f_2 and the reconstruction $\tilde{f}_2$. Maximal gap condition (2.2.2) is satisfied.

Relative errors	
$E(f,2)$	8.698×10^{-9}
$E(f,\infty)$	7.193×10^{-8}

Table 5.10: Relative errors in ℓ^2-norm and ℓ^∞-norm.

5.5.5 Reconstruction with a Gabor frame built with a smooth window

We perform the same experiment by starting with a Gabor frame for $L^2(\mathbb{R})$ instead of a Gabor tight frame. We construct a space $\widetilde{V}_b^2(g_2, I)$ with a smooth and compactly supported window g_2. Let

$$g_2(x) = \frac{1}{2}(-\cos(2\pi x) + 1)\chi_{[0,1/2)}(x) + 1\chi_{[1/2,1+1/2)}(x) + \frac{1}{2}(-\cos(2\pi x) + 1)\chi_{[1+1/2,2)}(x),$$

as plotted in Figure 5.18.

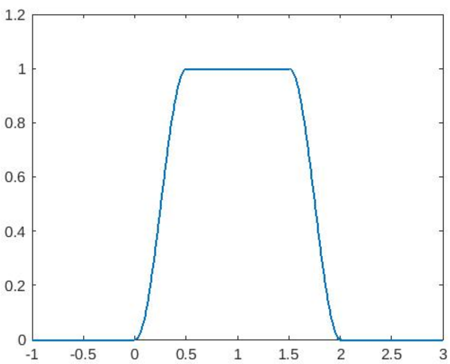


Figure 5.18: Window g_2.

Window	g_2
α	1
β	$1/(m+0.1)$
m	2
$\|g_2\|_\infty$	1
$\|g_2'\|_\infty$	π
A'	$1/\beta$

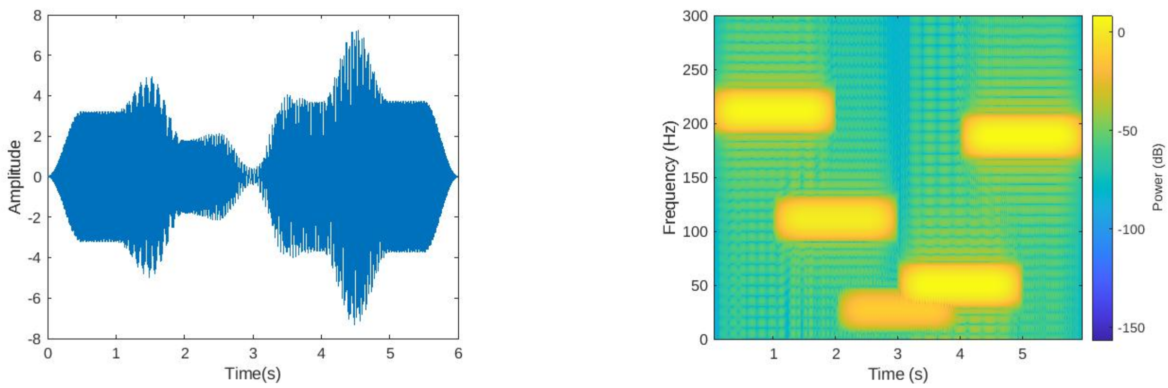
Table 5.11: Constants of $\widetilde{V}_b^2(g_2, I)$.

Then $m = |\text{supp}(g_2)| = 2$. We take $\alpha = 1$ and $\beta = 1/(m+0.1) = 0.476$, then by Proposition 1.8, the system $\{M_{\beta l} T_n g_2\}_{n,l\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower bound $A' = 1/\beta$. We consider $I = [0, 6]$ and the bandwidth sequence $b = (211, 111, 27, 50, 188)$. We want to reconstruct a function of the space

$$\widetilde{V}_b^2(g_2, I) \coloneqq \operatorname{span}\{M_{\beta l} T_n g_2 : \operatorname{supp}(M_{\beta l} T_n g_2) \subseteq I,\ |l| \le b(n)\}.$$

The Table 5.11 presents the constants needed to apply Theorem 2.4.

We generate the coefficients as in (5.5.2) and Figure 5.19 represents the function $f_3 \in \widetilde{V}_b^2(g_2, I)$ that we aim to reconstruct.



(a) Function $f_3 \in \widetilde{V}_b^2(g_2, I)$.

(b) Spectrogram of f_3.

Figure 5.19: Test signal $f_3 \in \widetilde{V}_b^2(g_2, I)$ generated by the bandwidth sequence $b = (211, 111, 27, 50, 188)$, and coefficients sequence of the form (5.5.2).

The dimension of the space is

$$\dim(\widetilde{V}_b^2(g_2, I)) = \sum_{n \in [0,4]} \big(2b(n) + 1\big) = 2475.$$

Since $\alpha = 1$, the partition has changed with respect to the previous experiment and is of the form $[k, k+1)$. Therefore the set $\mathcal{M}_2$ of indices for which we compute δ_k is

$$\mathcal{M}_2 = \Big\{k \in \mathbb{Z} : I \cap [k, k+1) \neq \emptyset\Big\} = \{0, \dots, 5\}.$$

We construct the sampling set

$$\Lambda_2 \coloneqq \left\{x_{k,j} = k + \delta_k j :\ k \in \mathcal{M}_2,\ j = 0, \dots, \left\lfloor \frac{1}{\delta_k} \right\rfloor\right\}$$

satisfying Theorem 2.4 and we get an oversampling parameter $q_3 = 1.64$. The difference between the original signal and its reconstruction is plotted in Figure 5.20 and Table 5.12 presents the relative errors in the ℓ^2-norm and ℓ^∞-norm.

The errors are comparable to those obtained by reconstructing in the space $\widetilde{V}_b^2(g_1, I)$. The $\operatorname{rank}(U) = 2056$ and we get a rank deficiency of 17%. Therefore, also with the window g_2, the lower bound of the truncated frame tends to 0, creating instability for the reconstruction.

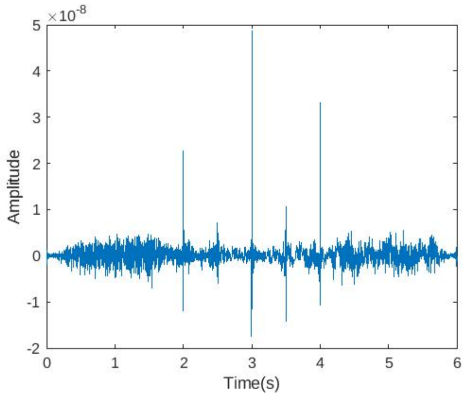


Figure 5.20: Difference between the original signal f_3 and the reconstruction $\widetilde{f}_3$. Maximal gap condition is satisfied.

Relative errors	
$E(f,2)$	1.138×10^{-9}
$E(f,\infty)$	6.652×10^{-9}

Table 5.12: Relative errors in ℓ^2-norm and ℓ^∞-norm.

We can conclude this section with two important observations. Firstly, we anticipated that the reconstruction would work similarly for functions in $\widetilde{V}_b^2(g,I)$ as it does for $PW_b^2(g,I)$. However, we discovered that using frames instead of bases introduces an issue of rank deficiency. Despite this considerable rank deficiency, we were still able to achieve a good numerical approximation of the functions.

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