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Entanglement-Based Producibility Criteria using Graph States in Quantum Networks

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Abstract

Recent years have seen the advancement of long-range quantum information technology. For the purpose of applications such as quantum cryptography, this progress necessitates the rigorous analysis of the limitations of quantum networks and their characterization in general. A crucial question in this topic is the question of producibility. There, the task lies in determining whether some quantum state can be produced in a given network topology. This question is particularly relevant for photonic applications: Preparing multipartite entangled states for photons is quite challenging and typically relies on bipartite sources. Previous work in this direction has provided necessary, but not sufficient criteria to determine producibility in quantum network states. A primary focus of this research is the so-called triangle network, where three sources distribute photon pairs to three parties, such that every party receives two photons prepared by different sources. However, the quest for general and practically applicable necessary and sufficient criteria is still ongoing. This thesis seeks to further this line of research, by complementing previous insights into producibility in quantum networks with the theory of graph states, to allow for a more rigorous analysis of producible and non-producible correlations in a given network topology. Namely, the chosen method of this thesis is to fully characterize the set of producible and non-producible six-qubit graph states in an independent triangle network. This line of inquiry reveals a set of eight equivalence classes, four of which are producible by distributing a given number of Bell pairs and separable states, with the other four being non-producible. Based on these findings, we conjecture a criterion that exploits the presence of GHZ-like correlations in three of the four non-producible equivalence classes. As a framework this criterion makes use of generalized Bloch decompositions, in particular the correlation tensor and its norm. We provide a number of approaches and use them in an attempt to prove the aforementioned conjecture, but these have not been successful. Lastly, we present a potential outlook on other methods that might be more likely to aid in proving the conjecture, that the scope of this thesis did not permit to explore. Graph states permit one to characterize correlations in quantum states in a manner that allows one to work with a finite set of states, rather than an uncountably infinite one. In particular for quantum networks this approach has been shown to be fruitful, and could be further extended to weighted graph states, which could lead to a more complete characterization of correlations in network states. This characterization of correlations using graph states has revealed a potential criterion that is based on existing methods of entanglement theory and is accessible to local measurements. Given the correctness of the conjecture, there could be a potential class of criteria, based on similar approaches, that could serve to advance the field in the question of producibility in quantum networks.

Kurzfassung

In den letzten Jahren wurden große Fortschritte auf dem Gebiet der Quanteninformationstechnologien gemacht. Für Zwecke von Anwendungen wie Quantenkryptographie wird die rigorose Analyse von Einschränkungen und Charakterisierung im Allgemeinen von Quantennetzwerken benötigt. In diesem Kontext ist die Frage der Produzierbarkeit bedeutend. Hier liegt die Aufgabe darin, zu entscheiden ob ein gegebener Quantenzustand produziert werden kann in einer gegebenen Quantennetzwerktopologie. Diese Frage ist besonders relevant für photonische Anwendungen, denn das präparieren von multipartiten verschränkten Zuständen für Photonen ist herausfordernd und daher beschränkt man sich oftmals auf bipartite Photonenquellen. Frühere Arbeiten in diese Richtung haben notwendige, aber nicht ausreichende Kriterien geliefert um die Produzierbarkeit von Quantennetzwerkzuständen zu bestimmen. Ein primärer Fokus dieser Forschung ist das sogenannte Dreiecks-Netzwerk, wo drei Quellen je ein Photonenpaar generieren und an drei Benutzer im Netzwerk verteilen, sodass am Ende jeder Benutzer zwei Photonen von zwei verschiedenen Quellen erhält. Jedoch ist die Suche nach praktisch anwendbaren, notwending und ausreichenden Kriterien immernoch andauernd. Diese Arbeit strebt danach frühere Erkenntnisse von Produzierbarkeit in Quantennetzwerken zu ergänzen mit der Theorie der Graphzustände, um eine umfassendere Analyse von produzierbaren und nicht produzierbaren Korrelationen zu erlauben. Die gewählte Methode dieser Arbeit ist es alle Sechs-Qubit Graphzustände in einem unabhängigem Dreiecksnetzwerk in produzierbar und nicht produzierbar einzuteilen. Dieser Zugang hat acht Äquivalenzklassen offenbart, von denen vier produzierbar sind, indem zwischen null und drei Bell-Paaren verteilt werden, und vier nicht produzierbar sind. Auf diesen Ergebnissen basierend schlagen wir ein mögliches Kriterion vor, dass die Präsenz von GHZ-ähnlichen Korrelationen in drei der vier nicht produzierbaren Äquivalenzklassen ausnutzt. Dieses Kriterion basiert auf generalisierten Bloch-Zerlegungen, im Speziellen dem Korrelationstensor, sowie dessen Norm. Letztlich präsentieren wir einen Ausblick auf Ansätze mit welchen dieses Kriterion bewiesen werden könnte, sowie jene welche nicht erfolgreich waren. Graphzustände ermöglichen einfachere Charakterisierung von Korrelationen in Quantenzuständen indem eine endliche Menge an Zuständen in Betracht gezogen wird, anstatt der vollen Menge der Quantenzustände, welche unzählbar unendlich ist. Dieser Zugang könnte potenziell auf gewichtete Graphzustände angewandt werden, welche eine allgemeinere Charakterisierung erlauben könnten. Unser Ansatz hat ein potenzielles Kriterion offenbart, welches auf bestehenden Methoden der Verschränkungstheorie, sowie lokalen Messungen, basiert. Falls unsere Konjektur korrekt ist, könnte es eine ganze Klasse an Kriterien geben, die auf ähnlichen Prinzipien basieren, welche substanziellen Fortschritt in der Frage von Produzierbarkeit darstellen würden.

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List of Acronyms

QKD	quantum key distribution
LOSR	local operations and shared randomness
SDP	semidefinite programming
GHZ	Greenberger–Horne–Zeilinger (state)
GME	genuine multipartite entanglement
NP	nondeterministic polynomial time
LOCC	local operations and classical communication
GGM	generalized Gell-Mann (matrices)
SU(d)	d-dimensional special unitary group
POB	polarization operator basis
CPTP	completely postive and trace preserving (maps)
LO	local operations
SLOCC	stochastic local operations and classical communication
SEP	separable transformations
LC	local complementation
CZ	controlled Z (gate)
QMP	quantum marginal problem
QMA	quantum Merlin Arthur
SPDC	spontanteous parametric down conversion
ITN	independent triangle network
BTN	basic triangle network
IUTN	independent unitary triangle network
ICTN	independent channel triangle network
NSI	no-signalling and independence
CTN	correlated triangle network
CCTN	correlated channel triangle network
SDPs	semidefinite programs
NPA	Navascués, Pironio, and Acín (hierarchy)
CUTN	correlated unitary triangle network
SO(d)	d-dimensional special orthognal group

1 Introduction

In the last few decades significant progress was made in the development, both theoretical and experimental, of quantum technologies, see [1]. Quantum communication constitutes one of the primary pillars of this research under the umbrella of quantum technologies. One of the primary use cases for quantum communication is quantum cryptography. Recent years have seen the rapid development of such long-range quantum communication technology for the purpose of quantum cryptography, over distances beyond 100 km as well as satellite-based experiments, see [2] and [3], respectively.

However, these applications are oftentimes limited to cryptographic protocols with two users [4], and under consideration of a potential future quantum internet, these developments must necessarily be scaled up to accommodate multi-user protocols. One of the difficulties of multi-user quantum cryptography is the practical realization of general transformations involving two or more photons, because photons do not interact with each other. For this reason such multi-user quantum cryptographic protocols cannot rely on arbitrary multipartite quantum states being generated by an N-photon source that, e.g., supplies the qubits necessary for a quantum key distribution (QKD) protocol. Therefore, a conventional limitation for the purpose of multi-user QKD is the reliance on bipartite photon sources and at most two-photon interactions at the sites of the parties, see e.g. [5]. Intuitively, such a *quantum network* has limitations in the sense that there may be quantum states that might not be producible by such a setup consisting of a number of bipartite sources and transformations on pairs of photons. This is where a primary obstacle in the context of quantum networks comes into focus, namely the question of *producibility*.

There are numerous publications dedicated to deriving criteria for producibility in quantum networks. Oftentimes they presume a limitation in accessible operations, e.g. unitary transformations versus general quantum channels, and whether the sources are independent or allow shared randomness, which is subsumed in the paradigm of local operations and shared randomness (LOSR) [6]. Some of these criteria, rather than testing the producibility of quantum states in a given network topology, are designed to be applicable for arbitrary network structures, to allow simultaneous probing of desired network configurations; These are often referred to as tests of network topology. In any case, up until this point no necessary and sufficient criteria have been found, even in the most simple quantum networks. Some of the methods used to derive producibility criteria are symmetry conditions of entangled states [7], bounding correlations via semidefinite programming (SDP) [8], fidelity estimation via local measurements [9], coherence theory [10, 11], no-signaling and independence conditions of the sources [12], lower-bounding

fidelities to Greenberger–Horne–Zeilinger (GHZ) states and W states via quantum inflation and semidefinite programming [13], Bell-like inequalities using quantum inflation and semidefinite programming [14], producibility of graph states using quantum inflation [15, 16], properties of density matrices with independent sources as well as convex optimization for correlated sources [17], coherence theory in conjunction with semidefinite programming [18], and lastly symmetries of inflated network graphs [19].

The goal of this thesis is to gain a better understanding of the entanglement in states that are producible in quantum networks. In particular, we examine the independent triangle network; A network with three bipartite sources and three parties that receive two qubits each. Previous results have shown that states producible in the independent triangle network, and generic networks for that matter, cannot prepare states that are arbitrarily strongly entangled. For this reason, we expect an inquiry into the entanglement of network states to reveal fundamental limitations of the set of producible states in the independent network case, which can then be extended to a correlated network, where the sources share a classical randomness resource.

The results acquired in this thesis rely on the theory of graph states, see [20], as well as previous results in the theory of quantum networks, as outlined in the paragraph above. Furthermore, the conjecture put forward in Section 5 is based on entanglement theory, in particular the theory of generalized Bloch decompositions [21], as well as the norms thereof [22].

The structure of this thesis is as follows; First off, Section 2 will serve as an introduction to some basic and some advanced concepts in quantum information theory, as well as introduce the necessary tools for the subsequent sections. Secondly, Section 3 will further motivate the field of quantum networks and will elaborate on the difficulties in this field, as well as function as an extensive review of the literature of producibility in quantum networks and the criteria derived therein. Subsequently, in Section 4 we will outline how specifically the tools introduced in Sections 2.3 and 3.1.2 allow one to extend the theory of graph states to network scenarios, as well as put forward the results that we obtained. Section 5 will serve as a discussion of the results presented in the preceding chapter, where we will outline possible implications for potential entanglement-based producibility criteria and propose a conjecture for such a criterion, which we have not been able to prove within the scope of this thesis. Lastly, in Section 6 we will summarize the results of this project, what the answer to the research question of this thesis is, as well as reflect on the issues of the approaches used to arrive at the results of this thesis, what can be done to improve upon them, as well as list potential avenues that can be taken to further build upon the findings of this thesis, and what the potential impact of such inquiries might be.

2 Fundamentals of Quantum Information

This chapter serves as a brief introduction to the basics of quantum information theory and the methods that are used in subsequent chapters of this thesis. However, some prior knowledge in linear algebra and quantum mechanics is assumed. We refer the interested reader to [23] for an introduction to linear algebra, [24], as well as Section I of [25], for an introduction to the basic principles of quantum mechanics, and [26], as well as Parts II and III of [25] for a more complete introduction to quantum information theory.

2.1 Quantum States

A state in physics is a mathematical object, that has all available information about a system encoded in it. The instructions of how a state changes under physical transformations and what predictions can be made, are specified by the according physical theory. The underlying theory that is employed in quantum information is naturally quantum theory. The following sections provide a short introduction to the mathematical formalism of quantum states, more specifically discrete-variable quantum states, as these will be exclusively considered in this thesis.

2.1.1 Pure States

Pure quantum states, referred to as pure states in the following, are an idealized form of quantum states. In the case of discrete-variable systems a pure state is associated with a complex-valued *state vector*,

$$|\psi\rangle = \sum_{i=0}^{d-1} c_i |i\rangle, \tag{2.1}$$

where $c_i \in \mathbb{C}$, satisfying $\sum_i |c_i|^2 = 1$, are the components of the vector with respect to an orthonormal basis $\{|i\rangle\}_{i=0}^{d-1}$ of a Hilbert space $\mathcal{H} = \mathbb{C}^d$ with finite dimension d. Each vector $|\psi\rangle$ has a corresponding adjoint element in the dual space $\langle\psi| = |\psi\rangle^\dagger$. We denote the scalar product of a state vector $|\psi\rangle \in \mathcal{H}$ with an element of the dual Hilbert space $\langle\phi| \in \mathcal{H}^*$ as $\langle\phi|\psi\rangle$. State vectors have to be normalized, i.e. $\langle\psi|\psi\rangle = 1$, as the elements of a state vector represent measurement statistics. Namely, the probability of finding a system in the state $|\phi\rangle$ that has been prepared in the state $|\psi\rangle$ is specified by the *Born rule*,

$$P(|\phi\rangle, |\psi\rangle) \coloneqq |\langle\phi|\psi\rangle|^2. \tag{2.2}$$

Measurements in quantum theory are associated with Hermitian operators, or matrices in the case of discrete-variable quantum systems. The mean value of measurement outcomes of a system prepared in the state $|\psi\rangle$ for an operator A, is determined by "sandwiching" the operator, i.e.

$$\langle A\rangle \coloneqq \langle\psi|A|\psi\rangle. \tag{2.3}$$

Unitary transformations or *unitary matrices*, called just *unitaries* for short, fulfill $UU^\dagger = \mathbb{1}$, where $\mathbb{1}$ is the identity matrix. Applying a unitary to a pure state results in a pure state, e.g. $|\tilde{\psi}\rangle = U|\psi\rangle$, and can therefore be considered to conserve information. Pure states contain the full information about a quantum system and can therefore not account for effects such as decoherence in an experimental setup. They are nevertheless useful, as calculations with pure states are typically simpler than in the more general case of so-called mixed states.

2.1.2 Mixed States

Mixed quantum states, referred to as mixed states in the following, encompass experimental setups where less than the full possible information about a system is available. They can be thought of as statistical ensembles of systems, prepared in pure states, that one samples from. They are represented as *density matrices* or *density operators*,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2.4}$$

where $p_i \in \mathbb{R}$, $\sum_i p_i = 1$ and $|\psi_i\rangle \in \mathcal{H}$. The states $|\psi_i\rangle$ are eigenstates of the density matrix, provided they are orthogonal to each other, with corresponding eigenvalues p_i. These density matrices are elements of a Hilbert-Schmidt space $\mathcal{B}_{HS}(\mathcal{H})$, that is isometrically isomorphic to $\mathcal{H}\otimes\mathcal{H}^*$. For brevity's sake, the term Hilbert space will be used to both refer to the state vector space of pure states as well as the Hilbert-Schmidt space of density matrices. Density matrices have a few properties, namely that they are Hermitian, normalized and positive semidefinite:

$$\text{Hermitian:} \qquad \rho^\dagger = \rho, \tag{2.5}$$

$$\text{Positive semidefinite:} \qquad \rho \geq 0, \tag{2.6}$$

$$\text{Normalization:} \qquad \mathrm{Tr}\,(\rho) = 1, \tag{2.7}$$

where $\mathrm{Tr}\,(\rho)$ is the trace of the state ρ, which we will introduce shortly. Unitaries are applied to density matrices as $\tilde{\rho} = U\rho U^\dagger$. In general, a density matrix has infinitely many decompositions of the form $\rho = \sum_i p_i\rho_i$, where ρ_i are normalized elements of the Hilbert-Schmidt space. However, the decomposition in Eq. (2.4) is special, as it corresponds to the representation in which the density matrix is diagonal, provided $\{|\psi_i\rangle\}_i$ is an orthonormal basis and $p_i \neq p_j$ for $i \neq j$, i.e. all of the eigenvalues are unique. Pure states $|\psi\rangle$ can be

expressed as density matrices, when setting $p_i = 1, \forall j \neq i : p_j = 0$, resulting in a *rank*-1 *projector*,

$$\rho_{\text{pure}} = |\psi\rangle\langle\psi|. \tag{2.8}$$

For this reason pure states only have a single pure state decomposition, as they only "inhabit" the space spanned by themselves. Furthermore, pure-state density operators are equal to their own square $\rho_{\text{pure}}^2 = \rho_{\text{pure}}$. Operators fulfilling this property are called projectors, which are further distinguished as rank-N projectors. Pure-state density operators $|\psi\rangle\langle\psi|$, as mentioned above, are rank-1 projectors and, when applied to a state ρ, project the state onto the subspace spanned by $|\psi\rangle$. This property can be used to determine the mixedness, or conversely the purity, of a quantum state. For this purpose we will now introduce the trace of a density matrix.

Trace

To determine the measurement statistics of density matrices, one can define the trace

$$\text{Tr}\,(\rho) \coloneqq \sum_{n=0}^{d-1} \langle n|\rho|n\rangle, \tag{2.9}$$

where $\{|n\rangle\}_{n=0}^{d-1}$ is an orthonormal base. The quantity in Eq. (2.9) is equal to one, due the normalization condition of density matrices,

$$\begin{aligned} \text{Tr}\,(\rho) &= \sum_{n=0}^{d-1} \langle n|\rho|n\rangle = \sum_{n=0}^{d-1} \langle n| \sum_i p_i |\psi_i\rangle\langle\psi_i|n\rangle = \sum_{n=0}^{d-1} \sum_i p_i \langle n|\psi_i\rangle\langle\psi_i|n\rangle \\ &= \sum_i p_i \langle\psi_i| \underbrace{\sum_{n=0}^{d-1} |n\rangle\langle n|}_{\mathbb{1}} |\psi_i\rangle = \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1. \end{aligned} \tag{2.10}$$

This property can also be understood as the sum of the diagonal elements of the matrix summing to one. Furthermore, one can use the trace to generalize the "sandwiching" of a measurement operator in Eq. (2.3), namely

$$\text{Tr}\,(A\rho) = \sum_{n=0}^{d-1} \langle n|A\rho|n\rangle = \sum_{n=0}^{d-1} \sum_i p_i \langle\psi_i|A|\psi_i\rangle\langle\psi_i|n\rangle = \sum_i p_i \langle\psi_i|A|\psi_i\rangle, \tag{2.11}$$

where the transformation in the third step is possible due to the scalar products being freely permutable with each other, as well as using the linearity of the sum. This is in line with the intuition that the mean value of a measurement on an ensemble of states, should be equal to the average of the mean values of the microstates. One can check that in the case of pure states, i.e. $\text{Tr}(|\psi\rangle\langle\psi||\phi\rangle\langle\phi|)$ this simplifies to the Born rule in Eq. (2.2).

Using the trace one can define the purity and mixedness, also called linear entropy, of quantum states, namely

$$\Pi(\rho) \coloneqq \mathrm{Tr}(\rho^2), \tag{2.12}$$

$$S_L(\rho) \coloneqq 1 - \Pi(\rho) = 1 - \mathrm{Tr}(\rho^2). \tag{2.13}$$

Using these quantities we can define a *maximally mixed* state,

$$\rho = \frac{1}{d}\sum_{i=0}^{d-1} |i\rangle\langle i| = \frac{1}{d}\mathbb{1}. \tag{2.14}$$

The maximally mixed state has uniformly distributed measurement statistics independent of the choice of $\{|i\rangle\}_{i=0}^{d-1}$. If a physical system is prepared in the maximally mixed state, it can be said to contain no information, as all measurement outcomes are equally likely. In terms of thermodynamics, it can be considered to be a state of maximum *entropy*.

Moreover, utilizing the trace operation allows us to define a scalar product and subsequently a norm on the Hilbert-Schmidt space. The Hilbert-Schmidt inner product of two operators A and B is denoted as $\langle A, B\rangle_{\mathrm{HS}} \coloneqq \mathrm{Tr}(A^\dagger B)$. If instead of arbitrary operators A and B we consider pure quantum states, i.e. projectors, we can see that the purity arises as a special case of this scalar product. Additionally, we can introduce the trace norm defined as

$$||A||_{\mathrm{Tr}} \coloneqq \mathrm{Tr}(\sqrt{A^\dagger A}). \tag{2.15}$$

Extending the concepts of scalar products and norms to mixed states becomes crucial when analysing quantum operations that may either reduce, for example, the trace norm of quantum states or leave it invariant.

2.1.3 Composite Systems

Until this point we have considered states that describe single systems. However, quantum networks require the distribution of multiple quantum systems between spatially separated parties. For this purpose we introduce here the tensor product that is essential for the construction of states that describe composite systems, i.e. that consist of multiple systems. It has the following properties

$$\text{Distributivity:} \quad v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w, \tag{2.16}$$

$$v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2), \tag{2.17}$$

$$\text{Scalar Multiplication:} \quad c(v \otimes w) = c \cdot v \otimes w = v \otimes c \cdot w, \tag{2.18}$$

$$\forall v, v_1, v_2 \in \mathcal{H}_A, w, w_1, w_2 \in \mathcal{H}_B, c \in \mathbb{C}.$$

Applying the tensor product to some pure states $|\psi\rangle_A \in \mathcal{H}_A$, $|\phi\rangle_B \in \mathcal{H}_B$, results in a state $|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$ that "inhabits" the product space of the state spaces of the

original states, i.e. $|\Psi\rangle_{AB} \in \mathcal{H}_{AB} \coloneqq \mathcal{H}_A \otimes \mathcal{H}_B$. Composite mixed systems are constructed in complete analogy to pure states. However, not all states that inhabit a product space $\mathcal{H}_{AB}$ as in our example have the form of the state $|\Psi\rangle_{AB}$, which are refered to as *product states*. States that do not have this form are called *entangled* states, or are said to have *entanglement*, which is discussed in Section 2.1.4.

In the context of composite systems, one does not need to take the trace over all subsystems. A trace that does not act on the whole space of a state is referred to as a *partial trace*. Additionally, we adapt our notation to specify the system to be traced out as

$$\mathrm{Tr}_B(\rho_{AB}) = \sum_{n=0}^{d_B-1} \langle n|_B \rho |n\rangle_B = \rho_A, \tag{2.19}$$

where $\{|n\rangle_B\}_{n=0}^{d_B-1}$ is a basis of $\mathcal{H}_B$. Physically this represents the act of preparing two systems in a composite state ρ_{AB} and then discarding system B. Given a state on, e.g., two subsystems ρ_{AB}, we refer to the the reduced state of subsystem A, i.e. ρ_A as its *marginal*.

2.1.4 Entanglement

A natural question that comes up in the study of composite systems, be they quantum or other, is that of correlations. Correlations in classical systems can also be exhibited by quantum systems. However, there are quantum correlations that are impossible to achieve in classical systems. This feature of genuine quantum correlations is called *entanglement*. Entanglement is at the heart of many phenomena in quantum theory. Exploiting entanglement is a cornerstone of many quantum technologies, be it in quantum cryptography [27, 28], quantum teleportation [29, 30] and quantum networks, among many others. For this purpose we give a brief introduction to the topic of entanglement. For further reading on entanglement, we refer the interested reader to Chapters 15 through 18 of [25].

Entanglement is defined by its negation, *separability*. We define a pure biseparable state, with respect to a tensor-factorization $\mathcal{H}_{A_i} \otimes \mathcal{H}_{\bar{A}_i}$, as

$$|\psi_{\text{bisep}}\rangle_A = |\psi\rangle_{A_i} \otimes |\phi\rangle_{\bar{A}_i}, \tag{2.20}$$

where $A_i \subset A$, and $\bar{A}_i$ is the complement of A_i in A. It is common practice to identify a tensor-factorization $\mathcal{H}_{A_i} \otimes \mathcal{H}_{\bar{A}_i}$ instead by a *partition* of subsystems, i.e. $A_i|\bar{A}_i$. In particular, the aforementioned partition is often referred to as a *bipartition*, as there are only two tensor factors. Therefore, a state that is not biseparable is entangled, i.e. $|\psi_{\text{ent}}\rangle \neq |\psi\rangle_{A_i} \otimes |\phi\rangle_{\bar{A}_i}$. However, we have to make a distinction between the case where A is a bipartite system and the case where it contains more than two subsystems, conventionally referred to as a multipartite system. If A is a bipartite system then $|\psi_{ent}\rangle$ has bipartite entanglement, and if it is a multipartite system, and there is no choice of A_i, such that the state can be expressed in the form in Eq. (2.20), then we refer to this state

as having *genuine multipartite entanglement*, or GME for short. In analogy we define states to be k-separable that can be factorized as

$$|\psi_{\mathrm{k-sep}}\rangle_A = \bigotimes_i^k |\psi_i\rangle_{A_i}, \tag{2.21}$$

where we assume that $\bigcup_i^k A_i = A$. The marginals $|\psi_i\rangle$ may themselves be composite systems and contain entanglement. This reveals one of the hurdles in characterizing entanglement, namely that multipartite systems states can be entangled in many different ways and when referring to a state being entangled, this implicitly contains the specification which subsystems or collections of subsystems are entangled.

When considering mixed states this problem becomes further exacerbated, as mixed states in general have infinitely many decompositions into pure states. Then the definition of a biseparable mixed state is

$$\rho_{bisep} = \sum_i p_i \rho_{A_i} \otimes \rho_{\bar{A}_i}, \tag{2.22}$$

where we sum over the subsystem index A_i, because biseparability in mixed states does not refer to biseparability with respect to one bipartition, but rather to biseparability with respect to any combination of bipartitions. This is the big problem in the characterization of entanglement in mixed states, as in general it is not immediately clear whether a mixed state has a biseparable decomposition and determining if it has one is an NP (non-deterministic polynomial-time) hard problem, which means that there is no polynomial time algorithm to solve it. Furthermore, if we assume ρ_{bisep} is a tripartite system, the state is by construction biseparable, so it is not GME, but it can still contain bipartite entanglement between any pair of subsystems.

Over recent decades a number of entanglement measures has been developed, see [31], that seek to be computable for some states or that can be checked via measurements. The discussions of types of entanglement, how they relate to each other, how they can be detected and how they could be used in applications would each by themselves be an extensive topic for a thesis. For this reason we will leave the discussion of these topics to the literature referred to above and instead list some final pertinent examples of entangled states, that will be used in later chapters and briefly lay out some of their properties.

GHZ State The Greenberger-Horne-Zeilinger state, or GHZ state for short, is a N-partite quantum state that is GME. It is defined as

$$|\mathrm{GHZ}^{(N)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \tag{2.23}$$

A notable feature of GHZ-type states is that if one of the subsystems is traced out the state is no longer GME, but rather only classically correlated, as the resulting state is of the form $\frac{1}{2}(|0...0\rangle\langle 0...0| + |1...1\rangle\langle 1...1|)$.

W State A different type of GME state is the W state, which is defined as

$$|\mathrm{W}^{(N)}\rangle = \frac{1}{\sqrt{N}}(|10\ldots0\rangle + |01\ldots0\rangle + \ldots + |00\ldots1\rangle). \tag{2.24}$$

While the GHZ state loses its entanglement when discarding a subsystem, the W state still remains entangled in this case. Furthermore, the GHZ and W states belong to distinct classes of GME states that cannot be transformed into each other via local operations assisted by classical communication (LOCC), which will be explained shortly. This is another feature of multipartite entanglement, as in bipartite entanglement there is only a single class of entangled states.

2.1.5 Entropy

The notion of entropy used in classical thermodynamics can straightforwardly be generalized into the quantum domain, via the von Neumann entropy,

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho). \tag{2.25}$$

It is possible, via the spectral decomposition theorem, to simplify the above expression into a function of the eigenvalues λ_i of the density matrix ρ, as

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i. \tag{2.26}$$

The von Neumann entropy fulfills the following properties,

Invariance under Unitaries:	$S(\rho) = S(U\rho U^\dagger)$,	(2.27)
Range:	$0 \leq S(\rho) \leq \ln d$,	(2.28)
Subadditivity:	$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$,	(2.29)
Strong Subadditivity:	$S(\rho_{AC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$,	(2.30)
Concavity:	$\sum_i p_i S(\rho_i) \leq S(\sum_i p_i \rho_i)$.	(2.31)

Similarly to the linear entropy and purity of a quantum state, it is possible to use the von Neumann entropy to quantify the mixedness of a quantum state. Specifically, only pure states minimize the von Neumann entropy to 0, which can be straightforwardly calculated using the rule of l'Hôpital, whereas the maximally mixed state maximizes the von Neumann entropy. The invariance under unitaries of the entropy can be easily seen, when considering that unitary operations only transform the eigenvectors of a matrix, but leave the eigenvalues unchanged. The concavity of the entropy is essential as it demonstrates the intuition that mixing quantum states, i.e. increasing the number of microstates, must increase entropy.

2.1.6 Bloch Decomposition

Rather than representing quantum states as convex sums of density matrices, it is possible to represent states through the *Bloch decomposition*. In this approach, a quantum state is expressed using a basis of matrices from the Hilbert-Schmidt space. These basis matrices are chosen to satisfy several useful conditions, including tracelessness, hermiticity, and unitarity. However, beyond a matrix dimension of two, it is not possible find a basis that simultaneously fulfills all of these properties. Therefore, the choice of basis and which properties it fulfills depends on the specific application. The Bloch decomposition of a single system is

$$\rho = \frac{1}{d}\left(\mathbb{1} + \sum_{i=1}^{d^2-1} a_i \lambda_i\right), \tag{2.32}$$

where a_i are the Bloch coefficients, and the vector made up of these coefficients is called Bloch vector. The Bloch decomposition serves as a common tool for examining composite quantum systems and their entanglement content, as discussed in [22]. In this context, we introduce a Bloch tensor, whose rank equals the number of systems, rather than a Bloch vector as in the single-system case. Then the Bloch decomposition becomes

$$\rho = \frac{1}{d^N}\left(\sum_{i_1,\ldots,i_N=0}^{d^2-1} T_{i_1\ldots i_N} \lambda_{i_1} \otimes \ldots \otimes \lambda_{i_N}\right), \tag{2.33}$$

where $T_{i_1,\ldots i_N}$ are the elements of the aforementioned Bloch tensor or *correlation tensor* T, as it captures classical and quantum correlations in measurements, and $T_{0,\ldots 0} = 1$ because $\lambda_0 = \mathbb{1}$. As can be seen in Eq. (2.32), it is necessary to add an identity term that ensures normalization, as the matrices that are conventionally chosen as a basis are traceless. Conventionally, we refer to a *n-body* correlation tensor, which is the tensor that is constructed by choosing n indices in $T_{i_1,\ldots i_N}$ to be non-zero, where each combination constitutes one of the tensor elements. The significance of tracelessness becomes evident when considering the normalization condition for density matrices, which is easily fulfilled by traceless generators. Additionally, the hermiticity of the density matrix naturally arises if the chosen generators are Hermitian. Moreover, employing hermitian generators enables the identification of the elements of the Bloch vector as the mean outcomes of measurements. This aligns with the intuition that a quantum state should be characterizable using its measurement statistics. Furthermore, in order for these matrices to be considered a basis they must be orthogonal, i.e. $\mathrm{Tr}(\lambda_i \lambda_j) = \delta_{ij}$, up to a dimension-dependent normalization factor. However, this factor can be omitted, as any basis can be normalized, by absorbing the normalization factor into the definition of the basis elements.

We will now lay out some commonly used choices of generators. For in-depth explanations, as well as examples for certain dimensions, we refer to [21].

Pauli Matrices

For a 2-dimensional quantum system, the natural choice of generators for the Bloch decomposition is the set of *Pauli matrices*. These matrices are unique in that they are the only generators that satisfy all three conditions mentioned previously: tracelessness, hermiticity, and unitarity. Furthermore, it is not possible to fulfill all three of these conditions simultaneously except when $d = 2$. The Pauli matrices are denoted as

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.34}$$

While the Pauli matrices themselves are straightforward, they possess a multitude of useful properties that warrant detailed exploration. Therefore, for brevity's sake we refer the interested reader to Chapters 2 and 17 of [32] for an in-depth discussion.

Generalized Gell-Mann Matrices

The generalized Gell-Mann (GGM) matrices are a higher-dimensional generalization of the Pauli matrices and the Gell-Mann matrices in 3 dimensions. They are ordinarily used as the generators of the $SU(d)$ group. Although they cannot satisfy all the properties mentioned earlier due to their higher dimension compared to the Pauli matrices, they do possess the important characteristics of being Hermitian and traceless. There are three distinct sets of GGM matrices, that pertain to their symmetry:

1. $\frac{d(d-1)}{2}$ Symmetric GGM Matrices:

$$\lambda_s^{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \ 1 \leq j < k \leq d, \tag{2.35}$$

2. $\frac{d(d-1)}{2}$ Anti-symmetric GGM Matrices:

$$\lambda_a^{jk} = -i|j\rangle\langle k| + i|k\rangle\langle j|, \ 1 \leq j < k \leq d, \tag{2.36}$$

3. $(d-1)$ Diagonal GGM Matrices:

$$\lambda_d^l = \sqrt{\frac{2}{l(l+1)}} \left(\sum_{j=0}^{l-1} |j\rangle\langle j| - l|l\rangle\langle l| \right), \ 1 \leq l \leq d-1. \tag{2.37}$$

Due to the hermiticity of these operators we can identify the matrices with measurements, the mean outcomes of which give the Bloch vector components, i.e. $a_i = \mathrm{Tr}(\rho\lambda_i)$. Furthermore, these do not include the identity, meaning that it must be added for the first term in the generalized Bloch decomposition in Eq. (2.33)

Weyl Matrices

The Weyl matrices are a basis consisting of d^2 matrices that are traceless and unitary, except for the identity λ_{00}, which is only unitary. They can be straightforwardly defined as,

$$\lambda_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}kn} |k\rangle\langle(k+m) \bmod d|, \;\; n, m = 0, 1, \ldots, d-1, \tag{2.38}$$

where λ_{00} corresponds to the identity. It is straightforward to show through explicit calculation that in the case of $d = 2$ the Weyl matrices correspond to the Pauli matrices. The Weyl matrices being unitary is useful for calculations as the product of unitary matrices remains unitary, i.e. if one applies unitaries to the state $U\rho U^\dagger$, then the decomposition contains operators of the form $\tilde{\lambda}_{nm} = U\lambda_{nm}U^\dagger$, which are still unitary.

Polarization Operator Basis (POB)

The polarization operator basis (POB) represents a set of generators that differs from the aforementioned examples. Unlike the generalized Gell-Mann matrices or the Weyl matrices, they are neither hermitian nor unitary; rather, they solely satisfy the property of tracelessness. They are defined as the following $d \times d$ matrices,

$$\lambda_{LM} = \sqrt{\frac{2L+1}{2s+1}} \sum_{k,l=1}^{d} C^{sm_k}_{sm_l,LM} |k\rangle\langle l|, \tag{2.39}$$

where $s = \frac{d-1}{2}$, $L = 0, 1, \ldots, 2s$, $M = -L, -L+1, \ldots, L-1, L$, and $m_1 = s$, $m_2 = s-1, \ldots, m_d = -s$. The coefficients $C^{sm_k}_{sm_l,LM}$ are the Clebsch-Gordan coefficients, that have been introduced for the addition of angular momenta. Furthermore, it is important to note here that λ_{00} represents the identity matrix. Additionally, since the Clebsch-Gordan coefficients are real numbers, the Bloch vector must contain complex-valued elements.

Tensor Norm

For the purpose of entanglement certification, it can be beneficial to define a tensor norm of the correlation tensor, as it encapsulates information about the measurement correlations exhibited by a quantum state in different bases. This tensor norm serves as a generalization of the standard vector norm, $||a||^2 = \sum_i a_i^2$, where we sum over all indices of the correlation tensor:

$$||T||_2^2 \coloneqq \sum_{i_1,\ldots,i_N} T^2_{i_1,\ldots,i_N} = \sum_{i_1,\ldots,i_N} \mathrm{Tr}(\rho\lambda_{i_1} \otimes \ldots \otimes \lambda_{i_N})^2. \tag{2.40}$$

Here, we have the flexibility to choose which indices to sum over, allowing us to analyse the correlations in a given number of systems of a quantum state. For bounds on the correlation tensor norm and criteria for separability, we refer the interested reader to [22, 33].

2.1.7 Fidelity

While the scalar product of our Hilbert space provides a natural choice of metric for both pure and mixed states, these distance measures often lack operational meaning. In contrast, the *fidelity* serves as a meaningful measure of the closeness of quantum states. For two quantum states ρ and σ, the fidelity is defined as

$$\mathcal{F}(\rho, \sigma) \coloneqq \left(\mathrm{Tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right)^2 . \tag{2.41}$$

If one of the states is a pure state, the expression simplifies to

$$\mathcal{F}(\rho, |\psi\rangle) = \langle\psi|\rho|\psi\rangle, \tag{2.42}$$

which is identical to the expression of the probability of measuring the state ρ and obtaining the outcome $|\psi\rangle$. In realistic scenarios, it is often not possible to prepare a system in a pure quantum state. Therefore, it becomes essential to model the noise that a system exhibits, in particular in relation to an ideal *target state*. The fidelity can serve as a measure that gives an operational meaning to the closeness of an experimental state ρ and a pure state $|\psi\rangle$, that is desired for, e.g., some quantum communication protocol. We will later see, that probabilistic state preparation is an often-considered scenario in quantum networks, where long distances and operations with sub-unit success rate limit the fidelity that is possible for any given protocol.

2.2 Quantum Operations

Up until this point, we have focused on describing quantum systems using static quantum states. However, it is crucial to consider dynamical transformations that act on these quantum states, as manipulating systems is fundamental in any physical application. To this end, we introduce different kinds of quantum operations: *unitary transformations* and *quantum channels*. Unitary transformations are reversible and preserve the purity of quantum states, while non-unitary quantum channels are irreversible transformations that may change the purity of quantum states. For a review on quantum channels we refer the interest reader to Chapter 21 of [25].

Furthermore, in the context of quantum networks, as well as other applications, understanding the set of operations available to spatially separated parties is vital for characterizing the network. Different sets of operations may be considered based on the physical restrictions imposed by an application. This differentiation leads to the distinction between *global operations* and *local operations*. Physically, modeling and controlling interactions between three or more systems is challenging, leading to a preference for restricting operations to those involving at most two subsystems, such as nearest neighbor interactions. Additionally, global operations allow for the transformation of separable states into entangled states. Therefore, in frameworks where entanglement is considered a limited resource, it is natural to restrict operations to local ones. Local operations can be

further classified into different regimes, which may involve parties sharing classical random resources, communicating using classical channels, and more. We refer the interested reader to [34, 35], wherein the former article provides rigorous arguments for why the LOSR class is a particularly useful choice of allowed operations in Bell scenarios.

In the following sections, we will provide an overview of unitary transformations and quantum channels, and then delve into various classes of local operations that are pertinent to entanglement theory and quantum networks.

2.2.1 Unitary Transformations

Unitary transformations arise naturally out of the Schrödinger Equation and make a natural choice of evolution for quantum states for this reason. They are characterized by unitary matrices, i.e. matrices U that fulfill $UU^\dagger = \mathbb{1}$. Unitary transformations are reversible, because U and $U^\dagger$ are both valid transformations to apply to quantum states, and they are inverse to each other. Furthermore, we can employ the spectral theorem, which allows us to express a quantum state in terms of its eigenvalues and eigenvectors as $\rho = \sum_i \lambda_i |i\rangle\langle i|$, to realize that unitary operations leave the eigenvalues of quantum states invariant and only change the eigenvectors,

$$U\rho U^\dagger = \sum_i \lambda_i U|i\rangle\langle i|U^\dagger = \sum_i \lambda_i |\tilde{i}\rangle\langle\tilde{i}|. \tag{2.43}$$

This implies that unitary transformations leave the purity of a quantum state invariant and as mentioned before, do not increase or decrease entropy, see Eq. (2.27).

Due to the conservation of probabilities under unitary transformations, as explained in Section 2.1.1, it is not possible to model decoherence or open quantum systems exclusively via unitary transformations. However, the interaction of multiple quantum systems, as modeled by the Schrödinger equation, is still mediated via unitary operations. Such interacting particles become entangled with each other and if one were to dismiss one of the involved particles, one would witness a loss of coherence, i.e. increase of entropy, as highlighted in Section 2.1.4. Indeed, this provides the basis for modeling quantum channels, i.e. irreversible transformations that can increase and locally decrease entropy.

2.2.2 Quantum Channels

Quantum channels provide a generalization of transformation from unitary transformations. In general a transformation Λ, that transforms a state ρ into another state $\Lambda[\rho]$, must be some kind of map from one Hilbert-Schmidt space to the same or another Hilbert-Schmidt space,

$$\begin{aligned} \Lambda : \mathcal{B}_{HS}(\mathcal{H}_A) &\longrightarrow \mathcal{B}_{HS}(\mathcal{H}_B), \\ \rho &\longmapsto \Lambda[\rho]. \end{aligned} \tag{2.44}$$

While a quantum channel may in principal map a state from one Hilbert-Schmidt space to any other, we will only consider maps that map a Hilbert-Schmidt space to itself in the following, i.e. $A = B$, and drop the subscript. It is important to determine what properties such a transformation must have. Intuitively, any physical transformation of a quantum state must yield a quantum state in return. This already strongly restricts the types of transformations that can be considered, as $\Lambda[\rho]$ needs to be a valid density matrix. Therefore, we can impose the following conditions on a state $\Lambda[\rho] \in \mathcal{B}_{HS}(\mathcal{H})$, that is related to a quantum state $\rho \in \mathcal{B}_{HS}(\mathcal{H})$ by a quantum channel Λ:

$$\begin{aligned} &\text{Trace Preserving :} && \mathrm{Tr}(\Lambda[\rho]) = \mathrm{Tr}(\rho) = 1 && (2.45)\\ &\text{Positive Semidefinite :} && \Lambda[\rho] \geq 0 && (2.46)\\ &\text{Complete Positivity :} && (\Lambda_A \otimes \mathbb{1}_B)[\rho_{AB}] \geq 0 && (2.47)\\ &\text{Linearity :} && \Lambda[\textstyle\sum_i p_i \rho_i] = \sum_i p_i \Lambda[\rho_i]. && (2.48) \end{aligned}$$

The trace being preserved is necessary to preserve probabilities in the quantum state. Similarly, linearity makes sure that a mixture of quantum states being transformed, is the same as the individual states being transformed and then mixed. Quantum states being necessarily positive semidefinite, requires the same being the case for the transformed state. Complete positivity is a slightly subtler condition, as it is necessary to make sure that even if a transformation acts on a subsystem, rather than a single system on its own, that the resulting global state is still a valid quantum state. In particular, maps that are positive but not completely positive are useful to verify entanglement. Conventionally quantum channels are also referred to as *completely positive and trace preserving* (CPTP) maps signifying the properties laid out above. However, this does not specify the mathematical construction of a CPTP map in any way.

Within quantum mechanics, evolution of quantum states is on a fundamental level solely governed via unitary transformations. Therefore, it is desirable to base CPTP maps on unitary transformations. One way to achieve this is by considering unitaries on a larger Hilbert space, i.e.

$$\Lambda[\rho_A] = \mathrm{Tr}_B(U \rho_A \otimes |\omega\rangle\langle\omega|_B U^\dagger), \tag{2.49}$$

where the unitary U acts on the subsystems A, B and C. This transformation can be physically realized by having the subsystems prepared in their respective state, having the transformation governed by a Hamiltonian that induces the unitary transformation U, and afterwards discarding systems A and C, making it physically well-motivated. Given a CPTP map of the form in Eq. (2.49), it is possible to express the map in terms of the Kraus decomposition, as

$$\Lambda[\rho_A] = \sum_i K_i \rho_A K_i^\dagger, \tag{2.50}$$

where K_i are the Kraut operators, which are defined as $K_i = \langle i|_B U |\omega\rangle_B$ and fulfill $\sum_i K_i^\dagger K_i = \mathbb{1}_A$. It is important to note that these Kraus operators are not unitary and

furthermore act as maps from the space of subsystem A to the space of subsystem B:

$$\begin{aligned} \Lambda : \mathcal{B}_{HS}(\mathcal{H}) &\longrightarrow \mathcal{B}_{HS}(\mathcal{H}), \\ \rho &\longmapsto \Lambda[\rho_A] = \sum_i K_i \rho_A K_i^\dagger. \end{aligned} \tag{2.51}$$

Importantly, the three formulations of quantum channels as CPTP maps, unitaries on a larger Hilbert space and the Kraus decomposition are all equivalent. Quantum channels find use in scenarios where coherence is lost in a system, e.g. through coupling with an environment or a subsystem in an entangled pair being lost, which explains how this view of discarding the primary system might be useful.

2.2.3 Global vs. Local Operations

The physical implementation of an arbitrary unitary transformation, or even quantum channel, on a system made up of a large number of subsystems is difficult. Such a process would require an extremely well controlled interaction involving all of the subsystems, i.e. a lot of particles, and is therefore not possible in a setting that involves spatially separated parties that only have access to their respective subsystems. While global unitaries and quantum channels find use when trying to answer fundamental questions, any implementation, e.g. in quantum networks or quantum computation, operations are typically expressed by decomposing such transformations into a set of single-system and two-system transformations that are performed in sequence. However, for this purpose it is necessary to specify which resources, e.g. common classical randomness, the parties have access to. Resources in this context refers to resources in terms of resource theories, where we assume some properties or transformations to have a cost and others to be free, i.e. there are *free operations* and anything that cannot be created by free operations is a *resource*. We refer the interested reader to [36], which introduces the notion of resource theories in a general manner. In the following, we will lay out a number of interesting sets of local operations that one may restrict to in a realistic setting. Each of the presented sets is a true subset of the set of operations that follows it.

Local Operations

The set of local operations (LO) is made up of all transformations that are either local unitary or quantum channels, that only act on single subsystems at a time. Local unitaries act on individual subsystems, rather than on the entire Hilbert space, i.e. $U = U_{A_1} \otimes \ldots \otimes U_{A_N}$. Consequently, states that are equivalent under local unitary transformations differ only in the choice of basis states. Furthermore, properties of states such as entropy, purity, norms, and entanglement measures remain invariant under local unitary transformations. Similarly, local quantum channels act on single subsystems of a composite system, i.e. $\Lambda_{A_1} \otimes \ldots \otimes \Lambda_{A_N}[\rho_{A_1,\ldots,A_N}]$. In any case local operations cannot increase or create entanglement, and in the case of local quantum channels, can even

reduce entanglement. Furthermore, local operations cannot create classical correlations, as the operations lack even a common classical resource.

Local Operations and Shared Randomness

The set of operations local operations and shared randomness (LOSR) is characterized as the set of all convex combinations of local operations. This means that given a quantum state ρ and a set operations $\{\Lambda_i\}_i$, with $\Lambda_i \in \mathrm{LO}$, a general LOSR operation $\Lambda_{\mathrm{LOSR}} \in \mathrm{LOSR}$ is defined as

$$\Lambda_{\mathrm{LOSR}} = \sum_i p_i \Lambda_i[\rho], \tag{2.52}$$

where $\sum_i p_i = 1$. By construction the LOSR set of operations is the convex hull of the set of LO operations. The titular shared randomness of the LOSR set refers to the random variable p_i, that is shared between all parties, as it permits them to simultaneously perform a set of local operations based on the outcome of this local variable. This could for example mean that a coin is flipped, the outcome of which is broadcast to each of the parties and based on the outcome all parties apply either a σ_X or leave the system as it is. In this case a shared classical resource can be used to prepare classically correlated states, e.g. $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) = \frac{1}{2}(\rho + \sigma_X \otimes \sigma_X \rho \sigma_X \otimes \sigma_X)$, where $\rho = |00\rangle\langle 00|$. LOSR operations are particularly pertinent to quantum networks as they correspond to the server-client structure that is common in classical networks, as both the quantum states and the classical resource may be broadcast by a server or source, to which the clients or parties have access.

Local Operations and Classical Communication

In LOSR the parties share a classical random variable, but are not allowed to communicate using classical channels. The set of local operations and classical communication (LOCC), is the set of operations that allow exactly that. As a classical channel between the parties can in principle be used to broadcast some shared randomness instead, it can be intuitively understood that $\mathrm{LOSR} \subset \mathrm{LOCC}$. The set of LOCC operations is the most common paradigm that is used in entanglement theory, as from a practical standpoint, there is no reason not to allow classical communication to take place between spatially distant parties, save for quantum cryptography, which conventionally does not allow for classical communication between the parties during, e.g., state preparation as these can compromise the security of the protocol. However, despite this set being well-motivated from a practical standpoint, it comes at the significant cost of there not being a construction for a general LOCC operation. This makes the examination of LOCC difficult from a theoretical point of view. LOCC operations are particularly often used in the question of state convertibility, where one tries to determine, if and with how many operations it is possible to convert one multipartite state into another.

Furthermore, it is possible to define a class of operations called SLOCC, or stochastic local operations and classical communication. This set is characterized by not requiring unit probability of success when attempting to transform a given state into another, but rather having a finite probability of success. Naturally, this set is larger than LOCC, as any LOCC transformation can be seen as an SLOCC operation that happens to have a probability of success of 1. On average SLOCC operations cannot increase entanglement, i.e. averaged over a representative number of attempts. However, given post-selection on successful runs, these protocols can increase entanglement, as one tries to find methods of discarding runs in which the protocol was unsuccessful. The SLOCC set inherits the same issues as LOCC, as there is no general construction for its elements.

Separable Transformations

The set of separable transformations (SEP), is the set of all operations that cannot increase or create entanglement. They are a particularly useful tool for calculations, as they permit an explicit construction, see [37]. That is, a general transformation $\Lambda_{\text{SEP}} \in \text{SEP}$ has the form

$$\Lambda_{\text{SEP}}[\rho] = \sum_i K_i \rho K_i^\dagger, \tag{2.53}$$

where $K_i = \bigotimes_j^N K_{i,j}$ are separable Kraus operators. It holds that LOCC $\subset$ SEP, therefore any LOCC transformation can be decomposed into the form in Eq. (2.53), i.e. one can use the construction in Eq. (2.53) to characterize LOCC transformations as well. While this makes SEP quite useful, it comes with its own caveats, namely that there exist transformations of the form Eq. (2.53), that are not physically implementable. This can be seen by considering, as suggested in [35], that any operation in SEP, can be approximated with a finite chance of success via SLOCC. However, this does not mean that the transformation itself can be implemented, rather that there is a chance to achieve a given SEP operation via post selection. Therefore, the usefulness of SEP in theoretical applications is undeniable, but quite limited if one considers experimental implementations, e.g. quantum networks.

2.3 Graph States

A major part of the results in this thesis is based on the notion of so-called *graph states*. Graph states are pure states with a particular construction, and they are relevant because many states, that have applications in quantum information and quantum computing, are graph states. Some examples include GHZ states and 1-dimensional as well as 2-dimensional cluster states. For an in-depth discussion of examples of graph states and their application in quantum information, see Section 4 of [20]. Due to this, it sometimes suffices to consider graph states in theoretical analysis, as they can reveal structures that might be otherwise obfuscated in a more all-encompassing analysis. In particular,

because in a given scenario there is always a set of only countably many graph states, this makes it easier to analyse than the entire state space, while still providing a family of states that covers many interesting features, e.g. entanglement structures. There are two equivalent pictures of interpretation for graph states, one of which interprets them as states resulting from bipartite interactions, see Section 2.3.1, the other interprets them as being defined by their stabilizers, see Section 2.3.2. We refer the interested reader to [20], which presents an extensive introduction to the theory of graph states.

In either picture, any graph state can be visualized as a set of edges, that connect a number of vertices in a graph. We define a graph to be set of vertices $V = \{A_1, A_2, \ldots, A_N\}$, in complete analogy to our labelling of subsystems in multipartite quantum states. A vertex A_k represents a single quantum system, in particular a qubit-system, where the vertices in totality can be viewed as a network, where the edges denote interactions between systems. Edges are defined as a set with two elements, that specify the vertices that are connected by such an edge, e.g. $E_{k,k'} = \{A_k, A_{k'}\}$ connects the vertices A_k and $A_{k'}$. Now we define the set of all edges as E. Any graph is defined as a pair $G = (V, E)$ that specifies the number and labelling of vertices in a graph and the edges that connect the vertices. It is important to note that the choice of edges can be arbitrary, and depends on the system that is considered, and a valid graph must naturally only contain edges between vertices that are part of the graph. Figure 2.1 shows an example of a graph, with five vertices and three edges connecting the vertices. Naturally it is important to

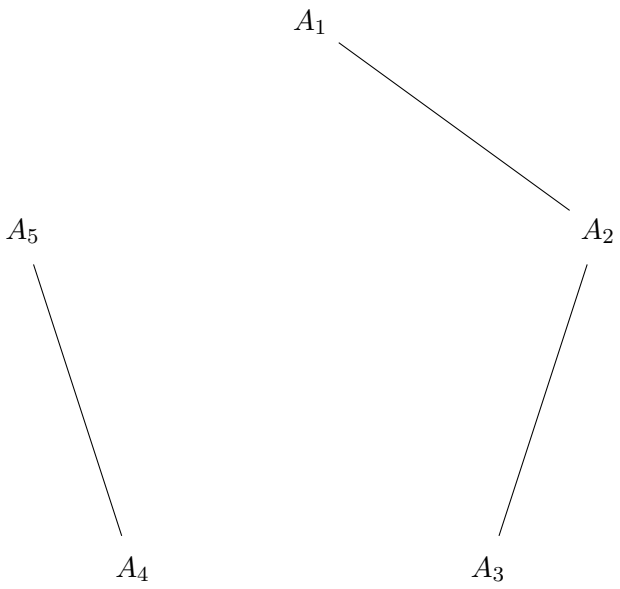


Figure 2.1: A graph with five vertices and three edges.

introduce a definition that captures whether a given vertex has any edges, and if so, how many there are, which is the so-called neighbourhood of a vertex. A neighbourhood of a

vertex A_k is defined as the set of all nodes that are directly connected to A_k via an edge, i.e. $N(A_k) = \{A_i | \{A_k, A_i\} \in E\}$. Furthermore, we call the number of edges connected to a vertex $|N(A_k)|$ the *connectivity*. Through simple combinatorics one can determine that the number of possible edges in a graph of N vertices is

$$\binom{N}{2} = \frac{N(N-1)}{2}, \tag{2.54}$$

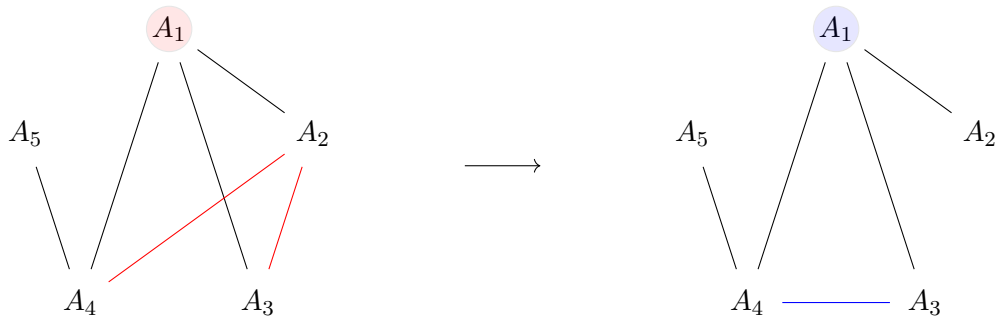
e.g. there are ten possible positions for edges in our example graph in Figure 2.1. As each edge can either be present or absent, that leaves us with a total of $2^{\binom{N}{2}}$ distinct graphs, given N vertices. In order to define the addition and removal of edges from a graph, we define $G + F \coloneqq (V, E + F)$, where

$$E + F = (E \cup F) \setminus (E \cap F), \tag{2.55}$$

i.e. $E + F$ is the symmetric difference of the sets E and F. This essentially acts as an exclusive OR on the edges contained in a graph; If an edge is only contained in E or F it will be an element in $E + F$, if it is contained in both or neither, it will not be an element in $E + F$. This definition is essential to define a rule called *local complementation* (LC), or local complementation rule. Given a vertex $A_i \in N$, in a graph $G = (V, E)$, we define the local complement of G with respect to the neighbourhood of A_i as

$$\tau_{A_i} : G \mapsto \tau_{A_i}(G) \coloneqq G + N_{A_i}. \tag{2.56}$$

In essence this means that any edge that is connected to A_i is left unchanged, and any edge that would connect two vertices in the neighbourhood of A_i is removed, if it is in the graph, and added, if it is not in the graph. Another useful tool for the examination of



(a) (b)

Figure 2.2: A graph with six edges, where LC is performed with respect to A_1 and edges that are removed and added are highlighted in red and blue, respectively.

graph states is the *adjacency matrix*. This adjacency matrix $\Gamma_G = \Gamma$, of a given graph G, is a $N \times N$-matrix and shows the number of edges connected to a vertex. It is defined as

$$\Gamma_{ij} = \begin{cases} 1, & \text{if } \{A_i, A_j\} \in E, \\ 0, & \text{else.} \end{cases} \tag{2.57}$$

In Section 4 we will make repeated use of a modified adjacency matrix, that further specifies the connectivity of every vertex. This makes it easier to make sure that the adjacency matrix was correctly constructed and makes it easier to sort graph states by number of edges. This modified adjacency matrix is defined as

$$\Gamma'_{ij} = \begin{cases} |N_{A_i}|, & \text{if } i = j, \\ -1, & \text{if } \{A_i, A_j\} \in E, \\ 0, & \text{else.} \end{cases} \tag{2.58}$$

This definition makes sure that the modified adjacency matrix has the following properties, $\sum_i \Gamma'_{ij} = 0$ and $\sum_j \Gamma'_{ij} = 0$, rather than $\sum_i \Gamma'_{ij} = |N_{A_i}|$ and $\sum_j \Gamma'_{ij} = |N_{A_j}|$, respectively, in the case of the conventional adjacency matrix.

2.3.1 Gate Picture

This picture for physically interpreting graph states is motivated by interactions between subsystems, where we interpret graphs as rules for applying specific gates to a fixed initial state. Here we assume all systems to be initialized in the state

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{2.59}$$

where $|+\rangle$ is the $+1$ eigenstate of the first Pauli matrix, i.e. $\sigma_X|+\rangle = +|+\rangle$. Therefore, the global state before applying gates, which are specified by the edges, is $|+\rangle^{\otimes N}$. If two vertices, or qubits in this case, are connected by an edge, e.g. A_k and $A_{k'}$, then this is interpreted as a CZ-gate (controlled Z) that acts on these two subsystems. The CZ-gate is defined as the following unitary two-qubit operation

$$CZ = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_Z. \tag{2.60}$$

For example the state in Figure 2.1 would be

$$|\psi_{2.1}\rangle = CZ_{A_1A_2} CZ_{A_2A_3} CZ_{A_4A_5} |+\rangle^{\otimes 5}, \tag{2.61}$$

where we omit the identity operator where it is unnecessary, i.e. $CZ_{A_1A_2} = CZ_{A_1A_2} \otimes \mathbb{1}_{A_3} \otimes \mathbb{1}_{A_4} \otimes \mathbb{1}_{A_5}$, as is conventional. The order in which the gates corresponding to different edges are applied is irrelevant, because CZ gates commute with each other. In the gate-picture, the LC rule can be interpreted as the equivalence of graph states under local unitary transformations. This is related to the Bloch decomposition of quantum states, as two quantum states sharing the same, but permuted, correlation tensor elements, can be transformed into each other using local unitaries.

Two vertices connected by an edge are, up to local unitaries, equivalent to a Bell state, whereas three vertices connected by two edges are equivalent to the three-qubit GHZ state. Indeed, a fully connected graph of N vertices is equivalent to a N-partite GHZ state,

as local complementation allows to convert a fully connected graph into a graph where one vertex is connected to all others, while being the only vertex in the neighbourhood of all other vertices. Figure 2.3 shows a graph state with five nodes that represents the five-qubit GHZ state.

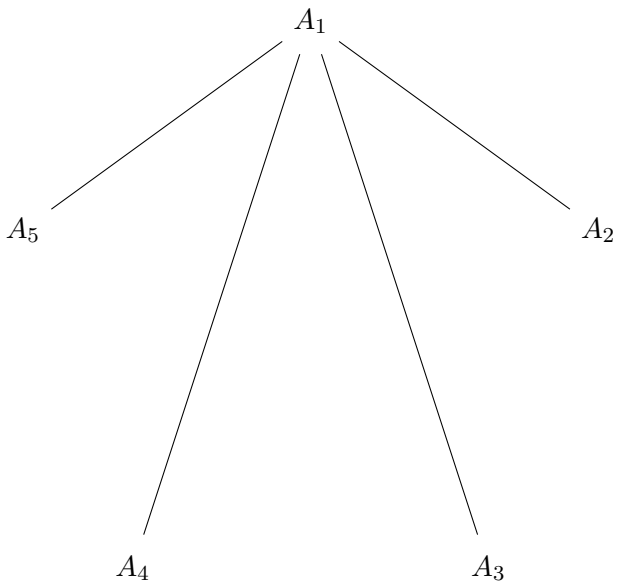


Figure 2.3: A graph with five nodes that represents the five-qubit GHZ state.

2.3.2 Stabilizer Picture

A different way of viewing graph states is through the stabilizer formalism. The stabilizers s of a quantum state $|\psi\rangle$ are the operators that, when applied to this quantum state, leave it invariant, i.e. $s|\psi\rangle = |\psi\rangle$. Differently put, states are the $+1$ eigenstates of their stabilizers. Naturally, given a set of stabilizers $\{s_i\}_i$ of a state $|\psi\rangle$, the product of any combination of stabilizers is again a stabilizer. This allows us to define a stabilizer group, with respect to a state as

$$S_\psi = \{s \in SU(d) | s|\psi\rangle = |\psi\rangle\}, \tag{2.62}$$

where $SU(d)$ is the d-dimensional special unitary group. This stabilizer group of a quantum state is unique, meaning that it is possible to define some stabilizers and with it identify a state, rather than defining a state and then identifying its stabilizers. Typical choices of stabilizers include Pauli matrices and tensor products thereof, e.g. $\sigma_x|+\rangle = |+\rangle$. Indeed, for graph states it is possible to directly infer the stabilizers from their graph structure. A graph state $|G\rangle$ is the unique $+1$ eigenstate of the set of stabilizers defined

as

$$K_{A_i} = \sigma_X^{A_i} \sigma_Z^{N(A_i)} \coloneqq \sigma_X^{A_i} \prod_{A_k \in N(A_i)} \sigma_Z^{A_k}, \tag{2.63}$$

where $A_i \in N$. While this uniquely determines the state $|G\rangle$, these are not all stabilizers of the state. However, the missing stabilizers can be calculated explicitly as the products of all combinations of K_{A_i}. Disjoint graphs, as the one in Figure 2.1, allow us to determine the stabilizers of the subgraphs separately. The set of stabilizers of the joint graph is then constructed via the tensor products of all stabilizers of the subgraphs. Due to the fact that the number of stabilizers increases exponentially with the number of vertices, namely as $2^N - 1$, we leave out an explicit calculation of all stabilizers of graphs as shown above, for the sake of brevity.

The attentive reader may realize that stabilizers are strongly connected to the generalized Bloch decomposition, introduced in Section 2.1.6. Namely, the correlation-tensor elements $T_{i_1 \dots i_N}$ of a graph states are $+1$ for the measurement operators where $\sigma_{i_1} \otimes \dots \otimes \sigma_{i_N}$ coincide with a stabilizer. Thereby one can also interpret a stabilizer state as the unique state that exhibits a mean measurement value of $+1$ for all its stabilizers, and 0 otherwise. As one can characterize the correlations of a state via its correlation tensor, this further cements the usefulness of graph states as representative of larger classes of states that exhibit similar correlations, albeit in different measurements bases.

2.4 Coherence Theory

Coherence theory is a quantum resource theory that regards coherence in quantum states as a resource and considers those operations to be free that do not generate coherence. It is important to note, that while it is a resource theory, the practical significance of the restriction to such operations is subject of ongoing debate. Creating coherence in a quantum state is, for separable states, only a question of applying local unitary transformations, i.e. performing a basis change, which constitutes a free operation in the resource theory of entanglement. However, entanglement is a coherence between orthogonal states of composite systems, and therefore can be considered a physical resource in that respect. In any case, coherence theory has been shown to be useful in applications of optics and quantum networks as a mathematical framework, which is why we will give a brief introduction into the methods that have been used to derive earlier results. For an in-depth review on coherence theory, we refer the interested reader to [38, 39].

One of the tools that we make use of in the context of coherence theories is the *covariance matrix*. The covariance matrix of a quantum state is another measure of the correlations of observables, in particular observables where spatially separated parties measure their respective subsystems. Previous works include results that are pertinent to the subject of this thesis, namely quantum networks, see [10, 11], but the covariance matrix has also been used to characterize the entanglement content of states, see [40, 41]. The matrix

elements of a covariance matrix, given a quantum state ρ, are defined as

$$[\Gamma(\{M_i\},\rho)]_{mn} = \langle M_m M_n\rangle_\rho - \langle M_m\rangle_\rho\langle M_n\rangle_\rho, \tag{2.64}$$

where $\langle A\rangle_\rho \coloneqq \mathrm{Tr}(A\rho)$ and $\{M_i\}$ is a set of observables. If we define a set of tripartite measurement operators $M_{ijk} \coloneqq A_i \otimes B_j \otimes C_k$, that each act on the subspace of one of three parties, we can find a block-matrix of a tripartite system as

$$\Gamma(\{A_i, B_j, C_k\},\rho) = \begin{pmatrix} \Gamma_A & \gamma_E & \gamma_F \\ \gamma_E^T & \Gamma_B & \gamma_G \\ \gamma_F^T & \gamma_G^T & \Gamma_C \end{pmatrix}, \tag{2.65}$$

where $\Gamma_A \coloneqq \Gamma(\{A_i\},\rho)$ and $\gamma_E \coloneqq \gamma(\{A_i \otimes B_j\},\rho)$. This means that Γ_X are the covariance matrices of the subsystems given some measurement operators and γ_X define covariances, which encode the statistical correlations between observables between different subsystems.

2.5 Quantum Inflation

As discussed in Section 1, quantum inflation is an important tool for addressing questions of producibility and has been used to derive many previous results. For this reason, we will briefly introduce the basic ideas related to quantum inflation and refer the interested reader to [42] for an extensive introduction to it.

The inflation method was originally developed for questions related to the marginal problem, see [43]. The marginal problem is the question of whether a set of marginal random variables permit a common global probability distribution. Similarly, the *quantum marginal problem* (QMP) is the question of whether a set of marginal density matrices are compatible with a common global quantum state. Solving this problem is quite hard in general, as it is a so-called quantum Merlin-Arthur (QMA) complete problem, see [44], but it is possible to solve this problem in specific scenarios. Furthermore, one can sometimes use quantum inflation to rule out the existence of a common global quantum state and likewise this technique can be used to determine whether a global quantum state can exist that fulfills the restrictions imposed by a quantum network. Quantum inflation is based on the idea, that in some cases it should be possible to construct the global quantum state, e.g. of N qubits, from a quantum state that has even more subsystems, e.g. $2N$ qubits. Such an *inflated* global quantum state may impose additional constraints on the specific QMP in question and may rule out whether a global quantum state can exist in some instances. A similar line of reasoning can be employed to determine whether some states can be produced in a quantum network. In the following section, we will introduce many of the methods for determining producibility derived in previous publications, including quantum inflation, as well as give examples of states, and discuss where they can be useful and what their restrictions are.

3 Quantum Networks

Research into modern quantum technologies branches into various fields, including, but not limited to:

1. Quantum computation
2. Quantum communication
3. Quantum simulation
4. Quantum metrology.

Quantum communication requires the preparation, manipulation and distribution of quantum states between parties that are spatially separated, even over distances of over a hundred kilometers [2, 3]. This necessity of spatial separation, sometimes referred to as *distant-labs* paradigm, poses an additional challenge for real-world implementations of quantum communication, in contrast to other quantum technologies. An example of this would be an implementation of the Ekert-91 protocol [27], where Alice and Bob want to perform *quantum key distribution* (QKD) between their laboratories, that are situated, e.g., in Vienna and on the Kahlenberg; For this they require the preparation and distribution of a certain number N of Bell pairs between them, which they can realize either via optical fibre or free-space transmission.

It is important to note, that in the case of two parties, it is still comparatively easy to prepare arbitrary bipartite quantum states. This task becomes significantly more challenging in a multi-party setting, where we want to prepare, e.g., a tripartite GHZ state. The reason for this is that photons do not interact with each other, and currently the main method for preparing entangled photonic quantum states is to use non-linear crystals, pumped by lasers. The effect by which the generation of entangled states is then accomplished is *spontaneous parametric down conversion* (SPDC). Here, photons of particular wavelengths can be converted into pairs of photons of larger wavelengths. Due to conservation of momentum and angular momentum, output photon pairs can be entangled in these degrees of freedom, see e.g. [45]. This is possible, because SPDC is a coherent process, allowing for the generation of superpositions in composite quantum states. However, the generation of such pairs is probabilistic in nature and the preparation of multipartite states has a low probability. Instead of relying on higher-order SPDC processes it is also possible to rely on successive SPDC processes, see [46].

A different way of preparing multipartite photonic quantum states is to employ several sources to supply photons to multiple parties. Such a structure of sources that distribute

quantum systems among spatially separated parties is called a *quantum network*. The simplest network one might imagine is again the network of one source that transmits two qubits, one to Alice and one to Bob, respectively. Due to the issues discussed above, a practical restriction that one might impose on a quantum network, is to only use bipartite sources that supply the parties. However, in such a scenario, the question which states can be prepared has not yet been fully answered.. This question can be further categorized into the cases where the sources are fully independent—only local operations (LO) are permitted—or share a source of randomness—local operations and shared randomness (LOSR) is permitted—and the cases of different operations that the parties may use on their respective subsystems, i.e. unitary operations or general quantum channels. On a related note, a different approach to dealing with the difficulties of quantum networks, in the sense of attempting to perform multi-user quantum cryptography, is to establish a different kind of quantum network, that instead seeks to permit a number of parties to freely carry out two-user QKD protocols, thereby allowing all parties to have access to secure keys, which they can use to establish, e.g., a common key distributed between all parties using classical cryptography. For an example of such an approach, see [47].

In this thesis, the so-called *independent triangle network* will be mostly considered, which is the case where the sources are fully independent and there are only three parties, as well as sources. This setup imposes the simplest mathematical structure and provides a baseline to tackle more complex versions of this problem. However, an introduction to other network paradigms will be provided, as well a review of existing producibility criteria. Additionally, we refer the interested reader to [48] for an extensive review of the literature and existing producibility criteria. In the following we shall lay out the different mathematical formulations of this question, as well as the difficulties in answering it, for different cases of network constraints. We will first consider the case where the sources are independent, beginning with a restriction to unitary operations and then generalizing to quantum channels, before moving on to the case where the sources share a classical source of randomness.

3.1 The Independent Triangle Network

The simplest non-trivial quantum network that we can imagine is a *triangle network*, were we require the sources to independently prepare bipartite quantum states. This so-called *independent triangle network* (ITN) imposes the following structure: We have three separate parties, Alice, Bob, and Charlie, and three bipartite sources that prepare two qudits each. Each of these sources distributes its qudits between two different parties in such a way that every party ends up with two qudits. Finally, each party may locally apply a two-qudit quantum channel. Figure 3.1 shows this structure.

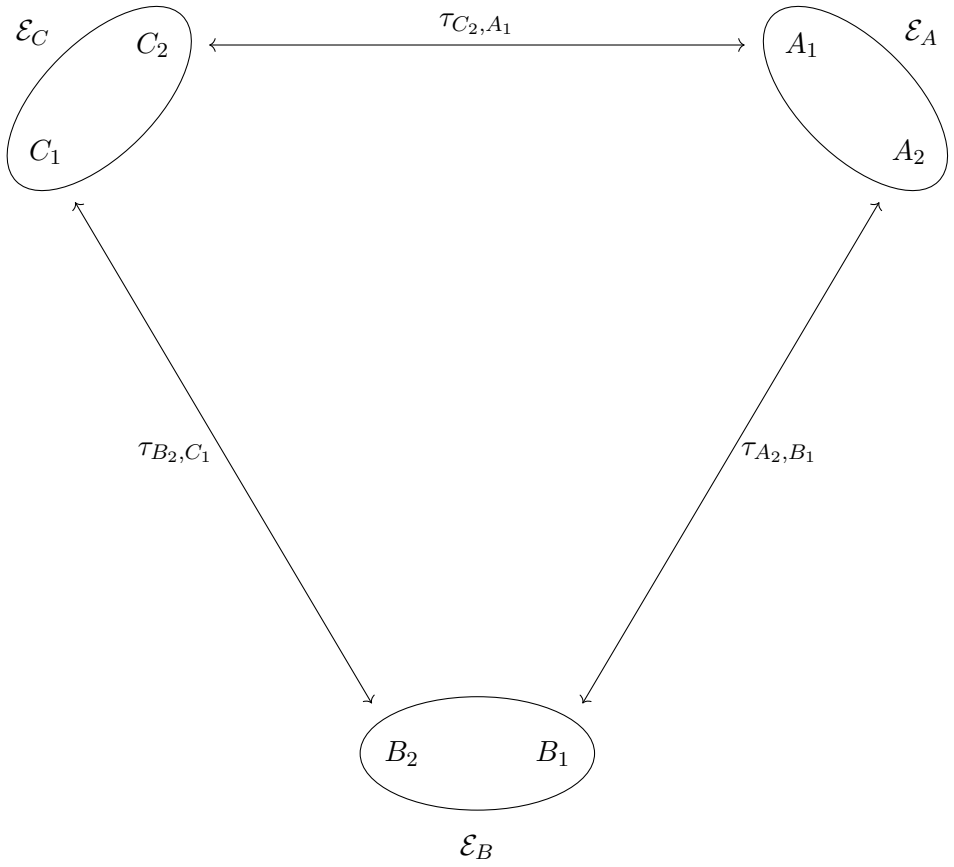


Figure 3.1: Three Sources distribute pairs of quantum states between the parties Alice, Bob, Charlie. Each party may apply a channel to their respective systems. The state τ_{A_2,B_1} is a bipartite qudit quantum state, where the subsystems A_2 and B_1 get distributed to Alice and Bob, respectively, in complete analogy for τ_{B_2,C_1} and $\tau_{C_2A_1}$ and $\mathcal{E}_A$ is a quantum channel, locally applied by Alice to her respective subsystems, in complete analogy for $\mathcal{E}_B$ and $\mathcal{E}_C$

3.1.1 The Basic Triangle Network

For the sake of completeness let us briefly mention the *basic triangle network* (BTN). In the BTN the parties are not allowed to apply any operations on their respective systems, effectively meaning that they cannot entangle their respective subsystems X_1 and X_2, where $X_i \in \{A_i, B_i, C_i\}$. Therefore, states that are producible in such a network are of the form,

$$\rho = \tau_{A_2,B_1} \otimes \tau_{B_2,C_1} \otimes \tau_{C_2,A_1}, \tag{3.1}$$

where, τ_{A_2,B_1} is a two-qudit quantum state, likewise for B_2C_2 and C_2A_1. It has to be noted that due to the tensor-product structure of such a state, known methods of detecting bipartite entanglement, e.g. checking the bipartite mutual information across the imposed partitions, can be used to discern whether a given state admits such a decomposition. Furthermore, it is possible, via local measurements, to determine whether or not a state can arise in a BTN. This criterion is based on the covariance matrix, see Eq. (2.65), of the quantum state. Because of the relative simplicity of checking producibility in the BTN and the complexity of the expressions of this criterion, we refer to [11], which initially introduced criteria of this type. Even though the structure of states producible in the BTN is very simple, it is nevertheless instructive to consider it initially, as it reveals the difficulty of attempting to do the same when permitting operations for the parties.

3.1.2 The Unitary Triangle Network

Similarly to the BTN, in the *independent unitary triangle network* (IUTN) each party receives two qudits. However, as the name suggests, each party is permitted to apply any two-qudit unitary on their respective systems. Based on this structure, any state that is producible in the IUTN must be of the form

$$\rho = U_A \otimes U_B \otimes U_C \; \tau_{A_2,B_1} \otimes \tau_{B_2,C_1} \otimes \tau_{C_2,A_1} \; U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger, \tag{3.2}$$

where, τ_{A_2,B_1} is a two-qudit quantum state, likewise for B_2C_2 and C_2A_1, and U_X is a two-qudit unitary, with $X \in \{A, B, C\}$, acting on systems X_1 and X_2, i.e. $U_X \coloneqq U_{X_1,X_2}$, where the label of the corresponding party, rather than the subsystems, will be used for brevity. We define the set of all states that are producible in the IUTN as Δ_{IUTN}, i.e. all quantum states that are of the form of Eq. (3.2).

In the following we will lay out some known criteria that can be used to determine producibility in the IUTN. For a more detailed discussion of these criteria we refer the interested reader to [11, 17], where we will note in each of the following sections in which publication each criterion was originally published.

Tripartite Mutual Information

The tripartite mutual information provides a necessary, but not sufficient, criterion to determine whether an given state can be produced in the IUTN, see Observation 1 in

[17]. The tripartite mutual information is defined as

$$I_3(\rho) \coloneqq S(\rho_{ABC}) + S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{AB}) - S(\rho_{AC}) - S(\rho_{BC}), \tag{3.3}$$

with $S(\rho_X)$ being the von-Neumann entropy of the marginal of systems X and ρ_{ABC} is the global tripartite quantum state. The value of the mutual information is zero for producible states, because unitaries leave the entropy invariant and the mutual information is additive on tensor products. Removing the unitaries in Eq. (3.2) leaves a state that factorizes with respect to the tensor product, as it is separable across the different sources, leading to the different terms cancelling out. Therefore, one can define a criterion for producibility.

Criterion 1 (C1): Let $I_3(\rho_{\text{IUTN}})$ be the tripartite mutual information of a state $\rho_{\text{IUTN}} \in \Delta_{\text{IUTN}}$. Then

$$I_3(\rho_{\text{IUTN}}) = 0. \tag{C1}$$

While (C1) can be straightforwardly calculated, it has one major caveat, namely that it trivially reduces to zero for all pure states. To see this one may note that the first term in Eq. (3.3) disappears, as all pure states have a von-Neumann entropy of zero. Furthermore, one can match each bipartite entropy with a matching single system entropy, such that they make up two marginals of the total network, e.g. $S(A)$ and $S(BC)$. These terms then cancel pairwise, as complementary marginals of a pure composite quantum state always have the same entropy. This means that (C1) can only be used to check the producibility of mixed states. Using this criterion one can show that many states are not producible in the IUTN, e.g. by considering a single-parameter family of states, such as the GHZ state mixed with white noise.

Tripartite Mutual Information of the Noisy Six-Qubit GHZ and W States: Mixtures of pure states and white noise, sometimes referred to as *noisy states*, lend themselves to examine producibility as they only depend one parameter and can be considered to be lines through the Hilbert space. Furthermore, examining states that may exhibit entanglement, up to a certain threshold, permit us to make statements about the amount of entanglement that can be distributed in a network. For this purpose we examine noisy six-qubit GHZ and W states, which are defined as,

$$\rho^{(6)}_{\text{GHZ}}(p) = p\,|\text{GHZ}^{(6)}\rangle\langle\text{GHZ}^{(6)}| + (1-p)\frac{1}{2^6}\mathbb{I}_{2^6}, \tag{3.4}$$

$$\rho^{(6)}_{\text{W}}(p) = p\,|\text{W}^{(6)}\rangle\langle\text{W}^{(6)}| + (1-p)\frac{1}{2^6}\mathbb{I}_{2^6}, \tag{3.5}$$

with $|\text{GHZ}^{(6)}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes 6} + |1\rangle^{\otimes 6})$ and $|\text{W}^{(6)}\rangle = \frac{1}{\sqrt{6}}\sum_{i=1}^{6}|1\rangle_{A_i} \otimes |0\rangle^{\otimes 5}_{\bar{A}_i}$, where $A_i \in A \coloneqq \{A_i\}_i^6$ and $\bar{A}_i$ is the complement of A_i, i.e. $A_i \cup \bar{A}_i = A$. The states in Eq. (3.4) are valid quantum states for $p \in [-1/(2^6-1), 1]$. The tripartite mutual information of these noisy states is shown in Figure 3.2, which demonstrates that the noisy GHZ as well as the noisy

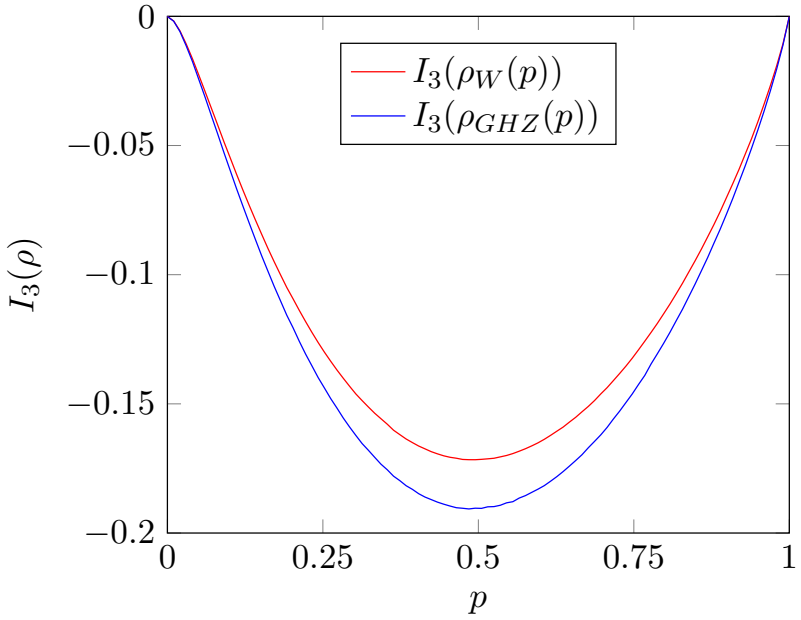


Figure 3.2: Tripartite Mutual Information I_3 plotted against mixing ratio p, for noisy six-qubit GHZ and W states.

W state fulfill (C1) for the maximally mixed state at $p = 0$ and the pure states $p = 1$, but for no other parameter values in between. Figure 3.2 suggests that the set of states producible in the IUTN is small, as there is no other producible state along the entire line through the Hilbert space of the noisy GHZ and W states, and in our two one-parameter families of states even states arbitrarily close to the maximally mixed state cannot be produced in the IUTN.

Additionally, as can be seen in Eq. (3.3), the tripartite mutual information is a sum of entropy functions. The entropy is a sum of functions of the form $-x \log x$, which is a continuous function. It can be shown that the composition of continuous functions, in this case the composition of the sum and the function $x \log x$, is itself continuous, see e.g. Proposition 2.13 in [49]. This, combined with the fact that noisy GHZ and W states that are arbitrarily close to the identity are not producible, suggests that there are no non-zero volumes of IUTN-producible states along the lines of our families of states.

This makes sense as Eq. (3.2) imposes that IUTN-producible states may only depend on a limited number of parameters,, i.e. $d^2 - 1$ per state and unitary, resulting in a total of $6(d^2 - 1)$ parameters. This of course is much smaller than the $d^{12} - 1$ parameters for a general state of six qubits. Combining these facts, this suggests that the set of producible states does not just take up a smaller volume of the Hilbert space than non-producible states, However, as mentioned before, pure states trivially fulfill (C1), therefore one cannot make a statement on the producibility of the pure GHZ and W states, without further criteria.

Marginal Ranks

Another necessary, but not sufficient, criterion that one can derive is based on the ranks of producible states and their marginals, see [17]. Even though the parties may choose to apply some unitaries to their respective subsystems, these unitaries cannot change the rank of the total state. Furthermore, the state and its marginals must fulfill a set of conditions, based on the ranks of the marginals.

Criterion 2 (C2): $\rho \in \Delta_{\text{IUTN}} \implies \exists$ integers $r_\alpha, r_\beta, r_\gamma \in [1, d^2]$ and $r_\alpha^B, r_\alpha^C, r_\beta^A, r_\beta^C, r_\gamma^A, r_\gamma^B \in [1, d]$, such that

$$\begin{aligned} \operatorname{rank}(\rho) &= r_\alpha r_\beta r_\gamma, \\ \operatorname{rank}\big(\operatorname{Tr}_A(\rho)\big) = r_\alpha r_\beta^C r_\gamma^B, &\quad \operatorname{rank}\big(\operatorname{Tr}_{BC}(\rho)\big) = r_\beta^A r_\gamma^A, \\ \operatorname{rank}\big(\operatorname{Tr}_B(\rho)\big) = r_\alpha^C r_\beta r_\gamma^A, &\quad \operatorname{rank}\big(\operatorname{Tr}_{AC}(\rho)\big) = r_\alpha^B r_\gamma^B, \\ \operatorname{rank}\big(\operatorname{Tr}_C(\rho)\big) = r_\alpha^B r_\beta^A r_\gamma, &\quad \operatorname{rank}\big(\operatorname{Tr}_{AB}(\rho)\big) = r_\alpha^C r_\beta^C. \end{aligned} \tag{C2}$$

One can understand (C2) qualitatively in the following way. Each source ξ in the network contributes a state of rank r_ξ. Therefore, the total rank of the state must be equal to the product of the individual ranks. If one traces out one party, then the state from one source remains unaltered, while the contributions from the other two sources only have the ranks of their respective marginals remaining. However, tracing out two of the parties, e.g. A and B, leaves only the ranks of the two respective marginals that are contained in the remaining party C, i.e. the remaining ranks belong to the marginals of the two sources that supplied the remaining party, in this case r_α^C and r_β^C.

A major disadvantage of (C2) is that it is trivially fulfilled for states with full rank and can therefore not be used to determine producibility in noisy states, as demonstrated in Figure 3.2 for (C1). However, this criterion has the advantage that it still holds true for pure states, in contrast to (C1), and even has a simplified form for pure states. Indeed, (C1) and (C2) were both initially published in [17] and are intended to be applicable in opposite scenarios; That is (C1) is particularly useful for mixed states of full ranks, which trivially fulfill (C2), while (C2) can detect pure states, which are not detected by (C1). However, there are still states that are not detected as non-producible by either criterion. Therefore we continue our examination of the GHZ and W states, while making use of (C2), to determine their respective producibility.

Rank Criterion for the 6-Qubit GHZ and W States: Checking whether or not the GHZ and W states fulfill (C2) consists of determining the ranks of the marginals, as any pure state has rank 1, and then trying to find a solution to the remaining system of equations. In this case, the rank of the total state is 1, therefore three of the coefficients necessarily have to be equal to 1 as well. Inserting this into (C2) leaves us with six equations of six coefficients.

Let us start with the W state, where each two-party marginal has a rank of 5 and each single-party marginal has a rank of 3. Therefore, the set of equations reads:

$$\begin{aligned}
\operatorname{rank}\big(\operatorname{Tr}_A(\rho)\big) &= r^C_\beta r^B_\gamma = 5, && \text{(3.6a)}\\
\operatorname{rank}\big(\operatorname{Tr}_B(\rho)\big) &= r^C_\alpha r^A_\gamma = 5, && \text{(3.6b)}\\
\operatorname{rank}\big(\operatorname{Tr}_C(\rho)\big) &= r^B_\alpha r^A_\beta = 5, && \text{(3.6c)}\\
\operatorname{rank}\big(\operatorname{Tr}_{BC}(\rho)\big) &= r^A_\beta r^A_\gamma = 3, && \text{(3.6d)}\\
\operatorname{rank}\big(\operatorname{Tr}_{AC}(\rho)\big) &= r^B_\alpha r^B_\gamma = 3, && \text{(3.6e)}\\
\operatorname{rank}\big(\operatorname{Tr}_{AB}(\rho)\big) &= r^C_\alpha r^C_\beta = 3. && \text{(3.6f)}
\end{aligned}$$

This set of equations does not have a solution, as Eq. (3.6a), Eq. (3.6b), and Eq. (3.6c) impose the restriction of each pair of coefficients being equal to 5 and 1, or vice versa, whereas Eq. (3.6d), Eq. (3.6e), and Eq. (3.6f) require one of them being equal to 3 and the other being equal to 1. This is a contradiction, therefore the six-qubit W state is not producible in the IUTN.

Doing the same for the GHZ state we are left with the following set of equations

$$\begin{aligned}
\operatorname{rank}\big(\operatorname{Tr}_A(\rho)\big) &= r^C_\beta r^B_\gamma = 2, && \text{(3.7a)}\\
\operatorname{rank}\big(\operatorname{Tr}_B(\rho)\big) &= r^C_\alpha r^A_\gamma = 2, && \text{(3.7b)}\\
\operatorname{rank}\big(\operatorname{Tr}_C(\rho)\big) &= r^B_\alpha r^A_\beta = 2, && \text{(3.7c)}\\
\operatorname{rank}\big(\operatorname{Tr}_{BC}(\rho)\big) &= r^A_\beta r^A_\gamma = 2, && \text{(3.7d)}\\
\operatorname{rank}\big(\operatorname{Tr}_{AC}(\rho)\big) &= r^B_\alpha r^B_\gamma = 2, && \text{(3.7e)}\\
\operatorname{rank}\big(\operatorname{Tr}_{AB}(\rho)\big) &= r^C_\alpha r^C_\beta = 2. && \text{(3.7f)}
\end{aligned}$$

Unfortunately, in this case the set of equations has two solutions, where one half of the parameters is equal to 2 and the other to 1 and vice versa for the second solution. This means we can neither rule out the GHZ state as being non-producible, nor declare it producible, because (C2) is only a necessary criterion for determining producibility.

Covariance Matrix

Another approach for determining producibility is to again make use of *covariance matrices*. One can check, whether or not a given state fulfills the following condition that was introduced in [11].

Criterion 3 (C3): Let $\Gamma(\rho)$ be the covariance matrix of quantum state ρ, and O be an orthogonal matrix. Then

$$\rho_{\text{IUTN}} \in \Delta_{\text{IUTN}} \implies \Gamma(\rho_{\text{IUTN}}) = O^T \Gamma(\rho_{BTN}) O. \qquad \text{(C3)}$$

In combination with the criterion for BTN states, (C3) can be used to make another statement on the structure of the covariance matrix. Namely, that there must be a block decomposition of the covariance matrix of ρ_{IUTN}, after applying the inverse orthogonal transformation, as

$$\Gamma(\rho_{UTN}) = \underbrace{\begin{pmatrix} \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{O^T T_C O} + \underbrace{\begin{pmatrix} \blacksquare & 0 & \blacksquare \\ 0 & 0 & 0 \\ \blacksquare & 0 & \blacksquare \end{pmatrix}}_{O^T T_B O} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \end{pmatrix}}_{O^T T_A O} + \underbrace{\begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{pmatrix}}_{O^T R O}, \tag{3.8}$$

where O are orthogonal matrices, that depend on the unitary matrices applied by the parties, T_A are positive semidefinite block matrices, whose entries correspond to correlations between subsystems B and C, and in complete analogy for T_B and T_C, and R is a positive semidefinite block-diagonal matrix.

A problem arises when trying to apply (C3) to determine the producibility of a state, as in general the transformations O are not known, as it would require both the knowledge on the applied unitaries and a BTN-producible state that can be transformed into ρ_{IUTN}. However, these are not known in general, therefore making it difficult in practice to check if a decomposition of the form of Eq. (3.8) exists. Fortunately, (C3) can be used to determine a more general criterion that is fulfilled even for networks with arbitrary quantum channels.

3.1.3 The Channel Triangle Network

In analogy to the IUTN we can define the *independent channel triangle network* (ICTN). Here, instead of just permitting the use of arbitrary unitaries acting on the respective subsystems of each party, the parties can apply any quantum channel on their qudits. Therefore, a general ICTN state can be described as

$$\rho_{\mathrm{ICTN}} = \mathcal{E}_A \otimes \mathcal{E}_B \otimes \mathcal{E}_C[\rho_{A_2,B_1} \otimes \rho_{B_2,C_1} \otimes \rho_{C_2,A_1}], \tag{3.9}$$

where $\mathcal{E}_A$, is a channel acting on party A, similarly for B and C. Analogously to before, it is possible to derive criteria for a state fulfilling Eq. (3.9), by characterizing quantum channels, e.g., by their Kraus operators.

Covariance Matrix

Given a state $\rho \in \Delta_{\mathrm{ICTN}}$, a criterion can be derived that is based on the off-diagonal blocks of the covariance matrix, see Proposition 6 of [11].

Criterion 4 (C4): Let Γ be the covariance matrix of a quantum state $\rho \in \Delta_{\text{ICTN}}$ with local observables $\{A_i, B_j, C_k\}$, and let γ_X be the off-diagonal blocks of the coherence matrix, see Eq. (2.65), and $||\cdot||_{\text{Tr}}$ is the trace norm, see Eq. (2.15). Then

$$\text{Tr}(\Gamma) \geq 2(||\gamma_E||_{\text{Tr}} + ||\gamma_F|_{\text{Tr}} + ||\gamma_G||_{\text{Tr}}). \tag{C4}$$

It is possible to directly compute (C4), given the local operators $\{A_i, B_j, C_k\}$. It is important to note that whether a given non-producible state violates the equality is dependent on the choice of $\{A_i, B_j, C_k\}$. However, given that these amount to basic matrix computations, one can quickly check different sets of operators. An instructive example to consider (C4) for, is the noisy three-qubit GHZ state, given by

$$\rho^{(3)}_{\text{GHZ}}(p) = p|\text{GHZ}^{(3)}\rangle\langle\text{GHZ}^{(3)}| + \frac{1-p}{2^3}\mathbb{1}_{2^3}, \tag{3.10}$$

where we set the local observables as $\{\sigma_Z\mathbb{1}_2\mathbb{1}_2, \mathbb{1}_2\sigma_Z\mathbb{1}_2, \mathbb{1}_2\mathbb{1}_2\sigma_Z\}$, i.e. locally measuring each qubit along Z, while leaving the remaining qubits untouched. A simple computation of the covariance matrix and (C4) reveals that $\rho_{\text{GHZ}}(p)$ is not producible for $p > \frac{1}{2}$. Interestingly, the three-qubit GHZ state can be converted to a six-qubit GHZ state via local operations applied by the parties. This can be expressed in terms of equivalence classes of graph states, which will be laid out in Section 4.1. For this reason, we can now conclusively answer the question of producibility with respect to the GHZ state in the negative; The six-qubit GHZ state cannot be produced in an independent quantum network with quantum channels as allowed operations. Note that the set of states producible in the IUTN is a subset of the states producible in the ICTN, i.e. $\Delta_{\text{IUTN}} \subset \Delta_{\text{ICTN}}$, as the set of quantum channels contains the set of unitary operations. This means that the six-qubit GHZ state is not producible when one only has access to unitary operations. Therefore, (C4) can also be used as a criterion for producibility in the IUTN.

Constraints on Non-Locality via No-Signaling and Independence

Another method of certifying producibility in a quantum network relies on analysing measurement outcomes, while requiring *no-signaling and independence* (NSI) of the sources. No-signaling is the principle that instantaneous action at a distance is not possible, as it would lead to causality issues within the theory of relativity. Therefore, no-signaling is a condition that any physical theory must obey to not be acausal. If one arranges for the sources in a quantum network to independently prepare quantum states, i.e. with the prepared states not being correlated, it is reasonable to assume that the correlations in a quantum network must be limited in the sense of having to obey this no-signaling principle. In particular, in such an approach one would exclusively find restrictions on the quantum correlations of the network, rather than the classical ones, as these are not existent in this setup.

In [12], NSI is applied to derive constraints on the expectation values of binary measurements in a triangle network. This approach makes use of the probability distributions associated with these binary measurements, rather than investigating the full density matrix of a quantum state. Furthermore, [12] employs quantum inflation to examine how an inflated hexagon quantum network would be indistinguishable from the triangle network, from the point of view of two of the three parties. With these tools a number of inequalities of the expectation values in the network can be derived.

Criterion 5 (C5): Let E_X, E_{XY}, where $X \in \{A, B, C\}$ and $\{X, Y\} \subset \{A, B, C\}$, be the expectation values of a measurement with binary outcomes. Then

$$\begin{aligned} &(1 + |E_A| + |E_B| + E_{AB})^2 \\ +&(1 + |E_A| + |E_C| + E_{AC})^2 \\ +&(1 + |E_B| + |E_C| + E_{BC})^2 \\ \leq&6(1 + |E_A|)(1 + |E_B|)(1 + |E_C|). \end{aligned} \tag{C5}$$

There are two simplifications to (C5) that can be considered. Firstly, by considering the case where the single-party expectation values are equal to zero, i.e. $E_A = E_B = E_C = 0$, and secondly the symmetric case where the expectation values are the same for the single-party expectation values and two-party expectation values, i.e. $E_1 := E_A = E_B = E_C$ and $E_2 := E_{AB} = E_{AC} = E_{BC}$. In the former case (C5) simplifies to $(1+E_{AB})^2+(1+E_{AC})^2+(1+E_{BC})^2 \leq 6$, and in the latter to $(1+2|E_1|+E_2)^2 \leq 2(1+|E_1|^3)$.

3.2 The Correlated Triangle Network

Examining states that can be prepared in the ITN, e.g. Eq. (3.2), one can verify that classically correlated states are not preparable in an independent network scenario using (C1). To illustrate this, consider a classically correlated state, e.g.

$$\rho = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|), \tag{3.11}$$

where a measurement in the computational basis yields perfectly correlated results. This state ist not producible, as can be seen from (C1). However, extending the ITN to a *correlated triangle network* (CTN), like,

$$\rho \;=\; \sum_{\lambda} \mathcal{E}_A^{\lambda} \otimes \mathcal{E}_B^{\lambda} \otimes \mathcal{E}_C^{\lambda} \left[\tau_{A_2,B_1} \otimes \tau_{B_2,C_1} \otimes \tau_{C_2,A_1}\right], \tag{3.12}$$

states such as Eq. (3.11) can be prepared. Interestingly, as noted in [19], if one permits the dimension of the states produced by the sources, i.e. τ_{A_2,B_1}, τ_{B_2,C_1} and τ_{C_2,A_1}, to be unbounded it is not necessary to have them depend on the shared random variable λ. This is because we can encode any potential mixture of source states in a state with larger

dimension. On the other hand, the converse is also possible, namely that we can choose to apply only a single map $\mathcal{E}$ on the source states, rather than having them be dependent on a shared classical variable, by moving the dependency on λ onto the source states. This flexibility is useful, because some degrees of freedom of photons are in principal infinite dimensional, e.g. orbital angular momentum and photon number. For this reason it may be desirable to instead consider a higher-dimensional Hilbert space, rather than considering a mixture of source states.

Furthermore, we have not specified, as of yet, the nature of the maps $\mathcal{E}$, because in principle it is possible to restrict considerations to only local unitaries or to more general quantum channels. In the correlated scenario, the comparatively simple structure imposed by the ITN is lost. Consequently, one could argue that restricting operations to unitary transformations is unnecessary. This is because one must contend with the typical issue of dealing with infinitely many decompositions of mixed states and a larger state space. Additionally, we will observe that the majority of criteria derived in correlated networks already take into account quantum channels, which is in contrast to many methods developed for independent network scenarios.

The criteria previously discussed only considered a deterministic state-preparation scenario, i.e. it is assumed that the sources are set up to produce a specific quantum state that is desired, e.g. for some given protocol. In contrast to this approach, it is also possible to consider a set of sources and operations, that, rather than exactly producing the desired quantum state, produce a state that is as *close* to some desired state as possible, which is referred to as *stochastic state preparation*. This notion of closeness between quantum states is often quantified using the fidelity, as introduced in Section 2.1.7.

In such a scenario it is not possible to achieve a higher fidelity to a pure target state by creating a mixed state. The reason for this is that the fidelity of a mixed state ρ to a pure state $|\psi\rangle$ is equivalent to the average fidelity of the states ρ_i of a decomposition of the original mixed state, i.e.

$$F(\rho, |\psi\rangle) = F(\sum_i p_i \rho_i, |\psi\rangle) = \sum_i p_i F(\rho_i, |\psi\rangle). \tag{3.13}$$

Naturally this average is upper bounded by the state in the decomposition ρ_i that maximizes this fidelity, i.e.

$$\sum_i p_i F(\rho_i, |\psi\rangle) \leq \max_i (F(\rho_i, |\psi\rangle)). \tag{3.14}$$

Therefore, stochastic state preparation cannot be used to produce states with a higher fidelity with pure target states, which are useful in, e.g., QKD protocols. However, it is possible to produce states in such a scenario that would otherwise not be producible in the ITN and can furthermore be used to create classical correlations. These differences between correlated and independent networks come into play in different tasks, e.g. when one seeks to determine the structure of a network in terms of structure of the sources and number of parties, etc. This provides the motivation to examine correlated networks, even if they are not more useful in terms of applications of quantum networks.

3.2.1 The Correlated Channel Triangle Network

Similar to the case of the ITN, we can again consider the scenario, where the parties can apply arbitrary quantum channels to their respective systems, which we refer to as the *correlated channel triangle network* (CCTN). As mentioned before, the resulting quantum state would be of the form of Eq. (3.12), where $\mathcal{E}$ may be general quantum channels, rather than just unitary operations. Attempts to fully characterize the effect of these quantum channels are of course even harder than in the case where only unitaries or even independent sources are considered.

Inflation Techniques, Semidefinite Programming and Fidelity to GHZ-State and W-State

In [13], bounds on the achievable fidelities to tripartite qudit GHZ and W states were established by analysing inflations of the triangle network and deriving entropy inequalities for these inflated networks. These entropy inequalities were derived using semidefinite programming (SDP).

SDP is a type of linear optimization that seeks to optimize an objective function over the cone of positive semidefinite matrices. This approach allows us to make certain statements concerning the convergence of final solutions which has the benefit of finding optimal solutions in applications suitable for semidefinite programs (SDPs). SDPs have been used in many applications in quantum information [50], particularly in quantum networks, see [8, 14, 18, 51]. The following criterion is one such application, and some of the subsequent criteria also rely on SDP, which we will note accordingly.

Criterion 6 (C6): Let $F(\rho, |\psi_d\rangle) \coloneqq \langle\psi_d|\rho|\psi_d\rangle$ be the fidelity of $\rho \in \Delta_{CCTN}$ with a quantum state $|\psi_d\rangle$. Then

$$\begin{aligned} F(\rho, |\mathrm{GHZ}_d\rangle) &\leq \frac{2d(3d + \sqrt{2d-1})}{1 - 2d + 9d^2}, \\ F(\rho, |\mathrm{GHZ}_2\rangle) &< \frac{1+\sqrt{3}}{4}, \\ F(\rho, |\mathrm{W}_2\rangle) &< 0.7602. \end{aligned} \tag{C6}$$

Naturally, (C6) is not tight, as not all states that are producible in the CCTN necessarily have a high fidelity with the GHZ and W states. For this consider for example a GHZ state where the relative phase has its sign changed

$$|\mathrm{GHZ}_2^-\rangle \coloneqq \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle). \tag{3.15}$$

The state in Eq. (3.15) is orthogonal to both the three-qubit GHZ and W states, but it exhibits the similar correlations as the GHZ state, up to local unitaries. For this reason it is not producible, yet (C6) is not able to detect this. Of course it is possible to extend the

fidelity bounds of (C6) to states that are equivalent up to local unitaries. Additionally, in [11] a stronger bound on the fidelity of a GHZ state was derived, further showing that (C6) is not tight.

However, it is useful to identify volumes of states around the GHZ and W states as non-producible, with little effort. Unfortunately this comes at the cost of the criterion not being easily generalizable. Indeed, the fidelity bounds given in the first and second line of (C6) do not coincide, as the first bound was derived with more general entropy inequalies, that are applicable to any dimension d, and using an SDP a stronger inequality was found, but only for qubits. Furthermore, these bounds have different values, as the weaker, more general bound would equate to $F(\rho, |\mathrm{GHZ}_{d=2}\rangle) \leq \frac{4}{33}(6+\sqrt{3}) \approx 0.9372$ and the stronger bound equates to $F(\rho, |\mathrm{GHZ}_2\rangle) < \frac{1+\sqrt{3}}{4} \approx 0.6803$. This suggests that similar, stronger bounds can be found for higher dimension, but cannot immediately be generalized to any dimension, at least not with the methods used in [13], due to the reliance on SDP.

Inflation Techniques and Symmetries

It is possible to derive different criteria via inflation techniques, see Section 2.5, by applying symmetry conditions on the network. In [19], different types of symmetries are considered, namely symmetries with respect to stabilizers of the quantum state, e.g. Pauli operators and tensor products of Pauli operators, and permutational symmetries, with projectors onto symmetric subspaces of the quantum state. The way quantum inflation is used in [19], is to consider inflated networks in which the original and inflated network share some marginals. By considering observables which (anti)commute and only have eigenvalues of ± 1, one can put bounds on the sum of their expectation values. In particular, if one has a set of observables $\{M_i\}_i$, which pairwise anticommute, then these fulfill the inequality $\sum_i \langle M_i \rangle^2 \leq 1$. Some of the results of [19] using such techniques are only expressed in terms of observations, for which we refer the interested reader to Observations 1 and 2 in [19], as these cannot easily be expressed in terms of mathematical formulas or criteria. However, one of the results of [19] is the following inequality that can be used to determine producibility.

Criterion 7 (C7): Let σ_i^X be a Pauli matrix acting on subsystem X, with $i = 1, 2, 3$, and $X, Y, Z = A, B, C$ and $X \neq Y \neq Z$, where $\langle M \rangle^2 \coloneqq \mathrm{Tr}(M \rho_{\mathrm{CCTN}})^2$ is the squared expectation value of an operator with a state $\rho_{\mathrm{CCTN}} \in \Delta_{\mathrm{CCTN}}$. Then

$$\langle \sigma_i^X \sigma_i^Y \sigma_i^Z \rangle^2 + \left(\langle \sigma_j^X \sigma_j^Y \rangle + \langle \sigma_j^Y \sigma_j^Z \rangle - 1 \right)^2 \leq 1. \tag{C7}$$

Using (C7) it possible to put an upper bound on the fidelity a producible state has with the GHZ state, namely $F(\rho_{\mathrm{CCTN}}, |\mathrm{GHZ}^{(3)}\rangle) \leq \frac{1}{\sqrt{2}}$, see Supplementary Note 2 in [19]. This bound is worse than the one obtained via convex optimization methods, e.g. (C6), but is an improvement on previous methods relying solely on analytical tools. The same

methods used to arrive at the inequality in (C7) can also be used to derive bounds for a square network, which can be used to determine that cluster states and other graph states cannot be produced with a fidelity of $F(\rho, |C_4\rangle) > 0.7377$. We refer the interested reader to [19] for how to arrive at this result, as it is not related to producibility in a triangle network. Producibility of graph states was also considered in [15, 16].

As [15, 16, 19] show, graph states are a useful tool to analyse producibility in quantum networks. In [15] it has been proven that no states can be produced in a network that have a given minimum fidelity with a graph state, where the N nodes of the graph correspond to N parties of a network with bipartite sources. In [16] similar methods have been used as the ones used to derive (C7), but instead of observables, the stabilizers of graph states have been considered, which also allow to put bounds on the achievable fidelity with graph states of a state in a network with bipartite sources. Interestingly, these approaches deal with the nodes of a graph state as the parties in a network, rather than, e.g., considering the subsystems prepared by the sources as the nodes of the graph. This approach will be explored in Section 4.

Covariance Matrix

Similarly to the way that criteria that are based on the covariance matrix can be derived in the independent network scenario, it is possible to adjust these criteria to the correlated case. In [11] an argument is made that any state from a correlated network is necessarily a convex combination of states that are preparable in the independent network. Therefore, one can make the argument, that given any network state in the independent network that maximizes the fidelity with a given target state $|\psi\rangle$, one cannot construct a network state in a correlated network that would increase this fidelity, i.e. $\max_{\Delta_C} \langle\psi|\rho_{\Delta_{\mathrm{ICTN}}}|\psi\rangle = \max_{\Delta_{\mathrm{CCTN}}} \langle\psi|\rho_{\Delta_{CCTN}}|\psi\rangle$. A consequence of this is that using (C4) one can find optimal states for stochastic state preparation, rather than deterministic preparation. The covariance matrices of states produced in particular quantum networks obey additional restrictions due to the network structure [10], but these are not directly relevant for determining producibility and we therefore do not further discuss these here.

Symmetry Conditions of Entangled States

The structure of a quantum network imposes a number of restrictions on quantum states. In particular, for symmetric entangled states one can find bounds on the fidelity with a producible state. The reason for this is that entangled states, that are symmetric under permutation of subsystems, can be transformed to have contributions from states which are known to be non-producible, e.g. the state $\frac{1}{2}\sum_i^4 |iii\rangle$ can be transformed into a product of three-qubit GHZ states, which can also be extended to larger classes of states, see Theorem 1 in [7]. Using this approach, some conditions that must be fulfilled by producible states, were found in [7]. For the sake of brevity we refer the interested reader to Theorem 3 of [7], which puts upper bounds on the fidelity a producible state

can have with the GHZ state, with a Dicke state, superpositions of Dicke states, and with an arbitrary pure entangled state that is producible. These fidelity upper bounds can be used similarly to (C6) and have therefore some use in upper-bounding entanglement of producible states.

Criterion 8 (C8): For any N-partite state ρ, it holds that $\rho \in \Delta_{\mathrm{CCTN}}$, where $F(\rho, |\psi\rangle) \coloneqq \langle\psi|\rho|\psi\rangle$ is the fidelity, if

$$\begin{aligned} F(\rho, |\mathrm{GHZ}_d\rangle) &\leq \max\{a_0^2, \ldots, a_{d-1}^2\}, \\ F(\rho, |D_{k,n}\rangle) &\leq \frac{n-1}{n+k-1}, \quad \text{for } k = 1, \ldots, \lfloor \tfrac{nd-n-1}{2} \rfloor, \\ F(\rho, |\Phi\rangle) &\leq \max_{A \subset \{A_1, \ldots, A_N\}} \max\{\sigma(\rho_A)\}, \end{aligned} \tag{C8}$$

where $|\mathrm{GHZ}\rangle$ is a generalized GHZ state with coefficients $\{a_0, \ldots a_{d-1}\}$, $|D_{k,n}\rangle$ is a Dicke state and $|\Phi\rangle$ is a general non-producible state, where $\sigma(\rho_A)$ are the eigenvalues of the marginal ρ_A.

The criteria (C8) have been derived analytically, and can therefore more easily be modified to be applicable to cases of higher dimensions, as well as to larger numbers of parties. This makes the approach used to derive these results useful in trying to find more general results, in contrast to (C6) which, while qualitatively similar, does not permit an extension to different cases without reexamining the SDP used to arrive at the results.

3.3 General Network methods

In this section we will outline further methods that have been developed in the field of quantum networks, that are not necessarily related to the question of producibility. The following paragraphs are dedicated to criteria that are used to determine whether the correlations in a network originate from quantum states, whether the measurement statistics in a network can arise from bipartite sources, and what the general structure of the network is. As these are only indirectly related to the research question of this thesis, we will forego explicit calculations and applications of these, as they do not give further insight into the question of producibility, that we are interested in here.

Furthermore, machine learning has the potential to be useful for analysing and classifying network structures based on measurement data. While computational approaches have been used less than analytical methods up until this point, apart from SPDs, they have the potential to prove useful numerical bounds on fidelities and other quantities that could be used for producibility, tests for network topology and determining the origin of correlations. It remains an open questions, whether such approaches will become more common in the future.

Bounding Correlations via Semidefinite Programming

As noted in Section 3.2.1, there have been a number of different approaches in quantum networks that in some capacity utilize SDPs. Here we make note of one such use case, discussed in [8], where bounds on the classical and quantum correlations present in quantum network are derived. Namely, using the Navascués, Pironio, and Acín (NPA) hierarchy [52, 53], a criterion was derived that allows one to determine whether the correlations present in a network have a quantum origin. The NPA hierarchy is a hierarchy of conditions that distinguishes whether the correlations in measurements outcomes are of quantum or of non-quantum origin.

Criterion 9 (C9): Let $p(a,b,\ldots|x,y,\ldots) = \mathrm{Tr}(\rho\Pi^A_{a|x}\otimes\Pi^B_{b|y}\otimes\ldots)$ be a multipartite probability distribution with a quantum realization, with measurement operators $\{\Pi^q_{o|i}\}_{o,i}$ for $q = A,B,\ldots$. Then for every set of products of measurement operators, $S = \{S_1,\ldots,S_n\}$, the moment matrix Γ is positive semidefinite, which is defined as

$$\Gamma_{ij} = \mathrm{Tr}(\rho S_i^\dagger S_j), \tag{C9}$$

This criterion is related to the methods in [12], i.e. (C5), which was itself not intended as a criterion for producibility originally, but nevertheless can be used as one. However, here we cannot determine producibility using this method, as it only determines if the measurement statistics in a network are non-local, rather than determining if they can arise from a triangle-network-producible states. For this reason this criterion is more useful in applications of non-locality, which have uses in device-independent (DI) QKD, see [28, 54]. DI QKD is a paradigm, where one seeks to ensure the security of a QKD protocol while not requiring knowledge about the inner workings of a measurement apparatus, provided it reports the measurement outcomes that it actually measures. This is often achieved by relying on the violation of Bell inequalities, see [31].

Inflation Techniques and Symmetries

Based on (C7) another criterion can be derived, making use of the stabilizers of a producible state and permutational symmetries with respect to the parties as they are stated in [19].

Criterion 10 (C10): Let σ_i^X be the i-th Pauli matrix acting on subsystem X, with $i = 1,2,3$, and $X,Y,Z = A,B,C$ and $X \neq Y \neq Z$ and P_{R_i} are observables acting on disjoints subsets of different systems, i.e. $R_i \cap R_j = \{\}$. Then

$$\langle\sigma_i^X\sigma_i^Y P_{R_1}\rangle^2 + \langle\sigma_j^X\sigma_j^Y P_{R_2}\rangle^2 + \langle\sigma_k^X\sigma_k^Y P_{R_3}\rangle^2 \leq 1, \tag{C10}$$

A violation of (C10) implies that a bipartite source supplying X and Y must exist and is distributing entanglement. Furthermore, this criterion is experimentally testable as it only depends on the expectation values of local measurements. This criterion can also be understood as a tool for testing network topology, as it can determine whether a network has, e.g., a bipartite source distributing single-qubit states between A and B.

3.3.1 Network Topology Tests

As noted in Section 1, some previous works[9, 14, 18], rather than deriving criteria for a given network topology, e.g. a triangle network, seek to find tests that can determine the structure of a quantum network. A frequent approach is to analyse the measurement outcomes of a state in a network. We refer to these approaches as tests of network topology here, which are also sometimes called witnesses of network incompatibility, and will give a brief list of publications that have contributed to this field. It is important to note that tests of network topology and producibility criteria are closely related, and sometimes can be used interchangeably. This is intuitive, as determining that a state ρ can be produced in a triangle network immediately shows that the structure of the network that produced ρ could be a triangle network.

In [9] a potential method for hypothesis testing has been proposed, which considers the fidelity with N-partite GHZ states in networks with bipartite sources. Another approach is to analyse the correlations that can be present in a network, as has been done in e.g. [11, 13, 17]. By deriving witnesses that can detect incompatible probability distributions, [14] presents an experimental implementation of a device-independent analysis of quantum network structure. These witnesses are derived by checking the compatibility of a measurement outcome distribution with an inflated quantum state, which can be achieved using SPDs.

Similar to methods used for producibility, a covariance-matrix based approach has also been used in [18] to derive witnesses of network topology. This allows testing the compatibility of a network topology with the measured covariances in a network. The method by which this was achieved was to was to impose so-called *semidefinite decompositions*, where the covariance matrix of a state produced in a given network topology has to admit a certain decomposition into positive semidefinite matrices. This semidefinite decomposition can be recast into SDPs and therefore be efficiently optimized to arrive at *semidefinite tests* for network topology compatibility.

4 Results

In this chapter the main results of this thesis will be laid out, using the methods introduced in Sections 2 and 3.

4.1 Graph States in the ITN

As discussed in the preceding chapter, the large parameter space of the independent unitary triangle network (IUTN) poses challenges for direct characterization. Consequently, a possible approach would be to constrain the state space under consideration to enhance our understanding of the producible states in the IUTN. Graph states, as demonstrated earlier, offer simplicity in construction and serve as proxies for broader categories of states sharing similar correlations as their corresponding graphs. Hence, our initial focus will be on comprehensively characterizing the set of graph states in a triangle network and their producibility, if the parties have access to unitary operations.

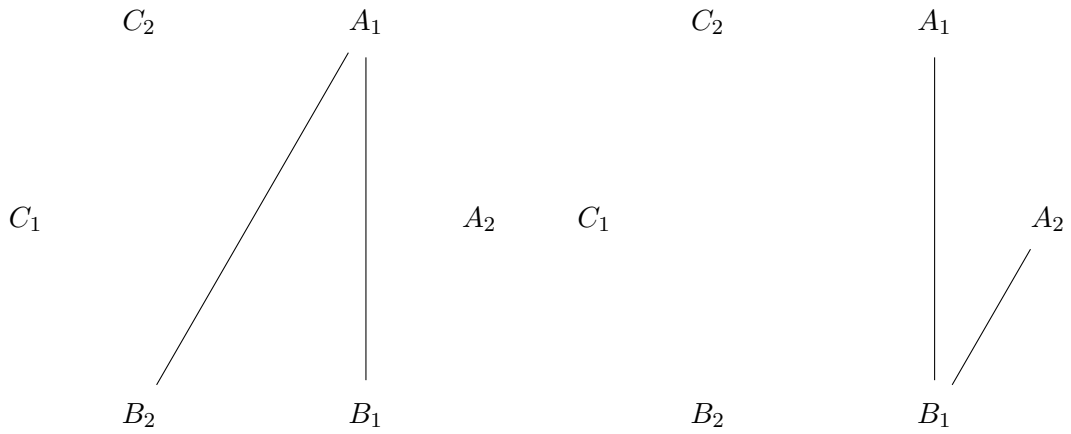
The IUTN can be represented as a graph with six vertices, indicating interactions between subsystems. With 15 potential edges in this graph, there are 2^{15} distinct graphs to examine. From now on, we will refer to distinct graph states, when the graphs themselves are different, and inequivalent graph states, when the graphs are different but there is no transformation from one into the other. This large number poses challenges in understanding the properties of states within Δ_{IUTN}. To address this, it is necessary to consider classes of states that are equivalent, up to local unitaries, to narrow down the set of states for consideration. Instead of considering all 2^{15} distinct graph states independently, we identify equivalence classes of states that can be transformed into each other by the following operations:

- Party SWAP
- Local SWAP
- Local CZ
- Local Complementation,

where we will elaborate on each of these operations in detail in the following paragraphs. Furthermore, we will from now on refer to graph states that are equivalent or inequivalent under these transformations as *network-(in)equivalent* states, to signify that these do not necessarily correspond to the same equivalence classes as for generic graph states. In particular, we will occasionally refer to *reducing* the number of network-inequivalent states,

where we mean, that adding one of the transformations above into account, reduces the number of inequivalent states, in comparison to the case where one would neglect it. This is important, as the number of network-inequivalent states, under the transformations above, is of course constant and cannot be reduced, but rather the number of states we know are network-inequivalent is reduced, where we refer to the latter, when a reduction of the number of network-inequivalent states is mentioned. Here, we will briefly discuss the intuitive basis for these transformations and provide a rough estimate of their impact on reducing the number of network-inequivalent graph states.

Party SWAP In the triangle network there are three parties, each of which receives two qubits, that originate from two different sources. As the name triangle network suggests, this network is symmetric under the permutation of any of the three parties, be it cyclical or acyclical. Therefore, we can identify any graph state that is identical to another under the exchange of two parties by one of the equivalent states. Figure 4.1 shows two graph states that are equivalent under this transformation. Naively, one could assume that the number of network-inequivalent graph states is halved for the permutation of each of the three parties, reducing the number of network-inequivalent states, when allowing arbitrary party permutation, by a factor of about $\frac{1}{8}$. However, one can imagine that there are states that are invariant under the permutation of parties, meaning that this estimate is only a lower bound.

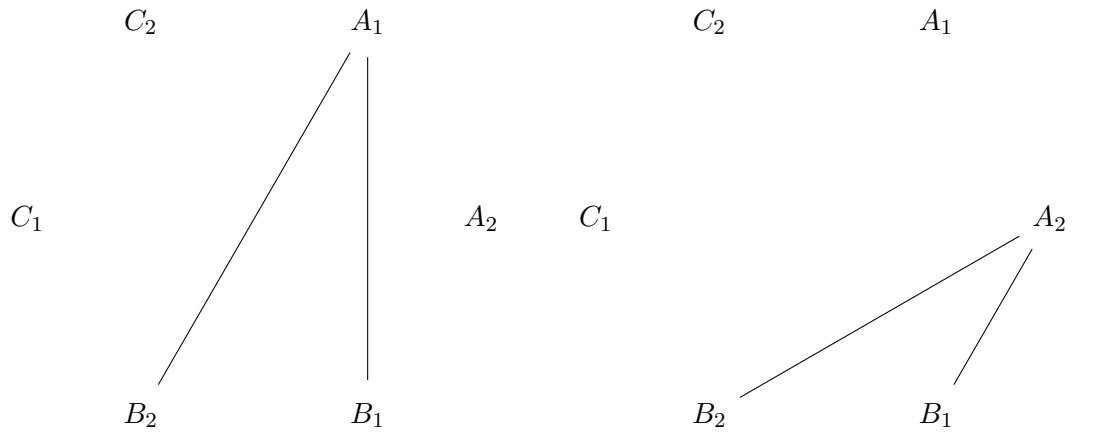


(a) An example of a graph state.

(b) A distinct graph state, that is equivalent to the graph on the left up to party permutation.

Figure 4.1: Example of two graphs that are equivalent up to party permutation.

Local SWAP Similarly to the equivalence of graph states under exchange of parties, we can assume two graphs to be equivalent, if they are the same under permutation of local subsystems. As each party is permitted to apply arbitrary bipartite unitary



(a) An example of a graph state.

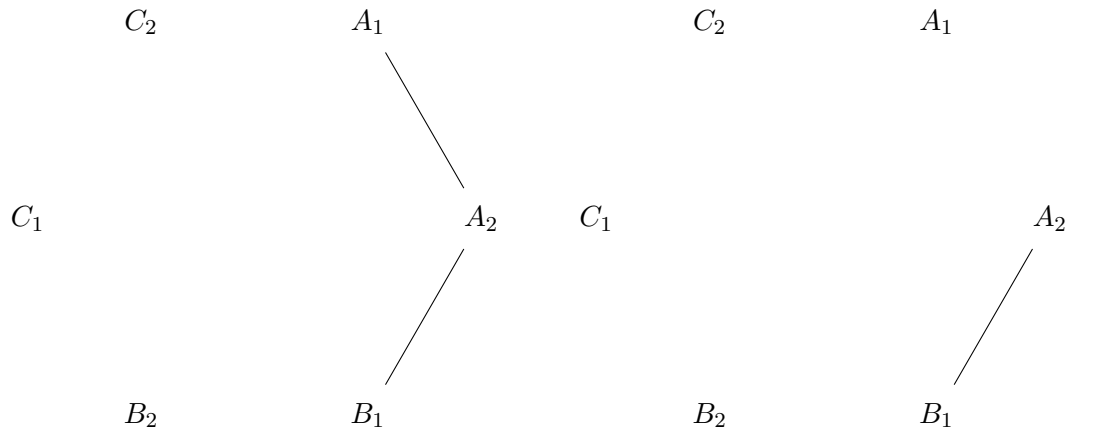
(b) A distinct graph state, that is equivalent to the graph on the left up to a SWAP gate.

Figure 4.2: Example of two graphs that are equivalent up to SWAPs of the two subsystems of any fixed party.

transformations on their respective subsystems, we can declare states to be equivalent if they are related by a SWAP operation applied to the two subsystems of any fixed party. Figure 4.2 shows two graphs that are network-equivalent under this transformations. Similarly to the party SWAP, we can find some exceptions of graph states that are invariant under a SWAP operation, therefore we can also roughly assume the number of network-inequivalent graph states to be reduced, at most, by a factor of $\frac{1}{8}$.

Local CZ The parties can freely choose to apply bipartite unitary transformations. But instead of applying SWAP operations, the parties can apply a local CZ gate on their respective subsystems. If there is an edge present in a graph state, between e.g. A_1 and A_2, we can remove this edge by applying $CZ_{A_1A_2}$. Figure 4.3 shows two graph states that are network-equivalent under the symmetry of A applying a local CZ gate. Here the estimation of the reduction of network-inequivalent graph states is more straightforward, as it effectively reduces the number of pertinent edges to 12, leaving 2^{12} network-inequivalent states.

Local Complementation As has been laid out in Section 2.3, graph states form sets of states that are equivalent under local unitary transformations. These symmetries can be found using the local complementation (LC) rule. However, in contrast to usual graph-state applications, we can use the LC rule in conjunction with the local CZ rule, and thereby drastically increase the number of graph states equivalent under local unitaries. An estimation of the number of graph states that are equivalent when using the LC rule has been done in [20]. However, here we have a reduced permutation symmetry, as usually



(a) An example of a graph state.

(b) A distinct graph state, that is equivalent to the graph on the left, if we assume local CZ symmetry.

Figure 4.3: Example of two graphs that are equivalent up to local CZ gates.

graph states are equivalent under any permutation of the subsystems, rather than the parties and the local subsystems. For this reason we cannot employ the polynomial time algorithm that has been developed to determine the LC equivalence of generic graph states, see Section 2.2.4 of [20]. This makes it difficult, using previous results, to estimate how many graph states are network-inequivalent when applying the LC rule.

4.1.1 Graph States as Binary Strings and Network-Equivalent States

Using these rules it is possible to drastically reduce the initial number of 2^{15} distinct graph states that need to be considered. However, this requires a method of encoding each distinct graph state and then determining the network-equivalent graph states by checking which graph states are equivalent using the operations laid out in the preceding sections. In the following paragraphs, we will lay out the algorithm and methods used to determine network-equivalent graph states and the representation chosen for each graph.

The method we have employed to achieve this, is to encode a graph state as a binary string, where each digit represents whether a certain edge in the graph is present. The convention we have chosen is to represent the six systems A_1, A_2, B_1, B_2, C_1, C_2, instead as numbers 1 through 6, respectively. Each edge can then be represented as a set $\{n, m\}$, where $\{n, m\} \subset \{1, 2, 3, 4, 5, 6\}$. Furthermore, we chose to represent each edge by having the leading digit be the smaller one $n < m$, e.g. $\{1, 3\}$ and further ordering edges in ascending order, prioritizing the first digit. This means that every graph can be represented as a binary string, where the first digit signifies the presence or absence of the edge $\{1, 2\}$, the second digit corresponds to $\{1, 3\}$, and so on, with the last digit corresponding to $\{5, 6\}$.

However, due to the equivalence under local CZ transformations, we can choose to skip the digits that correspond to the edges that connect the subsystems of each party, i.e. $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$. This leaves us with a 12-bit string, that uniquely identifies every distinct graph state, when already including the local CZ symmetry. As before we define E as the set of the edges that are present in a graph, e.g. $\{\{1,3\},\{2,3\},\{4,5\}\}$. In order to identify network-equivalent states, we construct the modified adjacency matrix Γ', see Eq. (2.58), from each binary string and bring each matrix into a canonical form, checking whether previous graphs have the same canonical form, allowing us to identify network-equivalent states.

The canonical form we have chosen is to permute the subsystems in such a way that the vertex with the larger connectivity is in the top left of the block, e.g. $\Gamma'_{11} \leq \Gamma'_{22}$ is permuted to $\Gamma'_{11} \geq \Gamma'_{22}$, of course respecting which edges belong to which vertex. This represents an implementation of the equivalence under local SWAPs. Here the property of the modified adjacency matrix that each row and column sums to 0 is useful, as it can be used to make sure that the permuted matrix is still valid. In the next step we permute the parties in such a way that $\Gamma'_{11} \geq \Gamma'_{33} \geq \Gamma'_{55}$ and $\Gamma'_{22} \geq \Gamma'_{44} \geq \Gamma'_{66}$. Similarly, this can be seen as a check of the equivalence under party SWAPs, discarding graphs that turn out to be equivalent to previously evaluated graph states. Implementing the equivalence classes under the first three types of transformations in this way reveals that the number of network-inequivalent states is 146.

Lastly, it would be necessary to find an algorithm that applies the LC symmetry and further reduces this number. However, the known algorithm for determining LC-equivalence in generic graph states is not applicable in our scenario. The biggest problem in adapting this algorithm for a quantum network, is that we have access to local CZ gates, i.e. we can arbitrarily add or remove the edges that belong to nodes of the same party, which is not the case in general graph states, where we cannot remove or add edges, except via the LC rule itself. Currently there is no such extension to the LC-equivalence algorithm. For this reason, we have instead checked the LC symmetry on the remaining 146 states by hand.

Doing so reveals that there are only eight network-inequivalent states when taking advantage of all symmetries that were laid out before. These are shown in Figure. 4.4. The first row of states is equivalent to the scenario where the three independent sources distribute zero, one, two and three Bell pairs, respectively. The second row of states correspond to $|\mathrm{GHZ}^{(3)}\rangle \otimes |+\rangle^{\otimes 3}$, $|\mathrm{GHZ}^{(3)}\rangle \otimes |\mathrm{GHZ}^{(3)}\rangle$, $|\Phi^+\rangle \otimes |\Phi^+\rangle \otimes |+\rangle^{\otimes 2}$ and $|\mathrm{GHZ}^{(3)}\rangle \otimes |\Phi^+\rangle \otimes |+\rangle$, respectively, where $|\mathrm{GHZ}^{(3)}\rangle$ is the three-qubit GHZ state, see Section 2.1.4, and $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a Bell state.

4.1.2 Producibility of Network-Inequivalent States

The next step is to ascertain which of the states depicted in Figures 4.4a through 4.4h can be generated within the IUTN and which cannot. Initially, it is evident that the states in

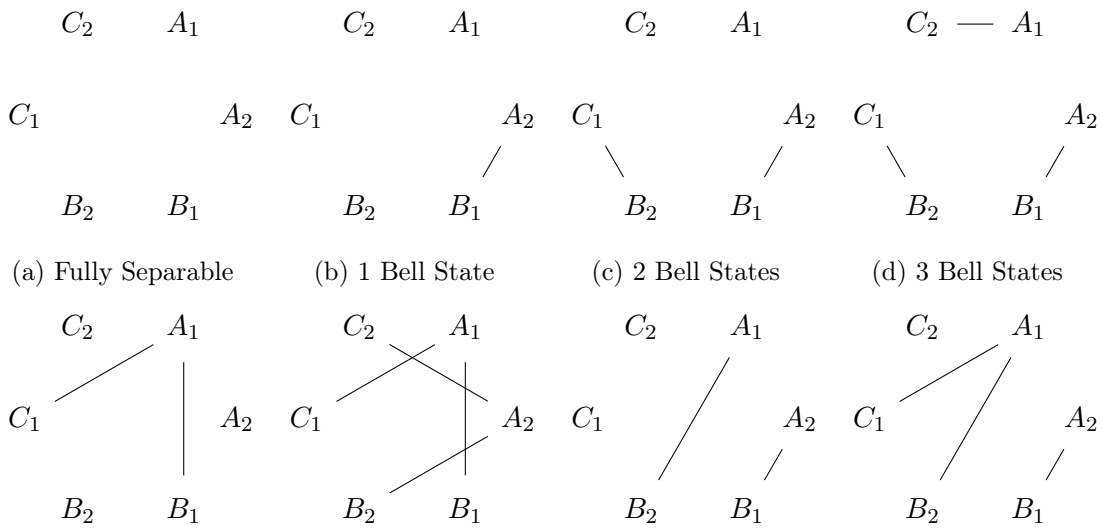


Figure 4.4: Network-Inequivalent States in the IUTN.

the first row of Figure 4.4 can be produced by distributing zero to three Bell pairs among the parties, respectively, while the remaining sources distribute separable states.

The states in the second row necessitate the application of criteria to determine their producibility. Assessing (C2) for each state reveals that the states in Figures 4.4g and 4.4h are not producible. However, as noted in Section 3.1.2 the GHZ state is not detected as non-producible by (C2), even though intuition would suggest that this is the case. Hence, the states in Figures 4.4e and 4.4f cannot be conclusively ruled out as non-producible using (C2). As detailed in Section 3.1.3, it is evident that the GHZ state is not producible according to (C4). Indeed, even the noisy GHZ state cannot be generated beyond a mixing parameter of $p > \frac{1}{2}$. Consequently, the state in Figure 4.4e is not producible, as it is equivalent to a GHZ state under local unitaries. Namely this choice of local unitaries is to first apply the local unitaries $\mathbb{1} \otimes H \otimes H$, where the unitaries are applied to the systems $A_1B_1C_1$ and H is the Hadamard gate $H = \frac{1}{2}(\sigma_X + \sigma_Z)$, which rotates the local basis to change from the Z to the X basis and vice versa. Afterwards one similarly has to rotate the local basis another time in all systems by changing from Z to Y and vice versa, which corresponds to the unitary $H' = \frac{1}{2}(\sigma_Y + \sigma_Z)$, resulting in a total unitary $H' \otimes H' \otimes H'$. Note that it would have also been possible to identify this state as non-producible using other criteria that identify the three-qubit GHZ state $|\mathrm{GHZ}^{(3)}\rangle$ as non-producible, such as (C6).

Thus, we have narrowed down the set of potentially producible states, in addition to the trivially producible states, to just the state in Figure 4.4f. Using the unitary transformation laid out above we can transform this state into $|\mathrm{GHZ}\rangle_{A_1B_1C_1} \otimes |\mathrm{GHZ}\rangle_{A_2B_2C_2}$. Expressed

in the computational basis this is equal to the state

$$|\mathrm{GHZ}\rangle_{A_1B_1C_1} \otimes |\mathrm{GHZ}\rangle_{A_2B_2C_2} = \frac{1}{2}(|000000\rangle + |010101\rangle + |101010\rangle + |111111\rangle), \quad (4.1)$$

where on the right-hand side we have permuted the subsystems into the order $A_1A_2B_1B_2C_1C_2$, because this allows us to express our six-qubit state as a three-ququart state instead. Here we choose our computational ququart basis as

$$|0\rangle \coloneqq |00\rangle, |1\rangle \coloneqq |01\rangle, |2\rangle \coloneqq |10\rangle, |3\rangle \coloneqq |11\rangle. \quad (4.2)$$

With this definition at hand we can express the state in Figure 4.4f as

$$|\mathrm{GHZ}_{d=4}\rangle_{ABC} = \frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |333\rangle), \quad (4.3)$$

where $A \coloneqq A_1A_2$, and in complete analogy for B and C. Expressed in this form, we can reexamine the producibility of the state, and straightforwardly identify it as non-producible using (C6), which puts an upper bound on the fidelity a state may have with a d-dimensional GHZ state. Trivially, the state in Eq. (4.3) has a unit fidelity with itself and is definitively non-producible for this reason.

Therefore, we have now fully characterized the set of all six-qubit graph states as either producible or non-producible in the IUTN, as each six-qubit graph state is network-equivalent to the eight graph states in Figure 4.4, under the application of party SWAPs, local SWAP gates, local CZ gates and the LC rule.

5 Discussion

We will now briefly summarize the findings of this thesis from the previous chapter, as well as analyse and discuss these results. In particular we will conjecture a potential criterion for producibility in correlated unitary triangle networks (CUTNs), independent of whether the sources have access to shared randomness.

The parameter space that has to be considered in the independent unitary triangle network (IUTN) is too large for direct characterization. For this reason graph states make a useful substitute, as they represent equivalence classes of states related by local unitaries. States within any such class hence display entanglement across the same cuts as the representative graph state. Furthermore, by identifying equivalence classes of states produced in the IUTN, we can identify network-equivalent states. Conversely, we can determine a fundamental set of network-inequivalent states, listed in Figure 4.4, that characterize all six-qubit graph states in a triangle network. By employing (C2) and (C6), we can identify four of the states in this set as being non-producible. The remaining states can be shown to be producible, as the states can be obtained by distributing a given number of Bell states at the sources, up to local unitary transformations.

This reveals that all producible states must be equivalent, up to local two-qubit unitaries applied by each party to their respective qubits, to bipartite entangled or separable states distributed by the sources. This conclusion aligns with the construction of states in the IUTN, as illustrated in Eq. (3.2). However, this revelation offers limited insight into the structure of producible states in the IUTN, consistent with previous findings that have only provided necessary but not sufficient criteria for producibility.

In this thesis, the focus was put solely on graph states, where each edge denotes the application of a CZ gate. However, there exist extensions known as *weighted* graph states, where the strength of entanglement between two vertices is variable. This extension is implemented physically by choosing a single-parameter family of entangling gates, that includes the CZ gate, via a suitable interaction Hamiltonian, and varying the interaction time. Expanding our considerations from graph states to *weighted* graph states could potentially reveal a more intricate structure, closer to a comprehensive description of the IUTN. Unfortunately, such an endeavor would exceed the scope of this thesis, as it would require substantial modifications to the algorithm used for determining network-inequivalent states. Particularly, the LC rule does not allow for the removal and addition of edges in the same manner in weighted graph states as in non-weighted ones.

Nevertheless, non-producible states in the set of network-inequivalent states display more intricate features than their producible counterparts. Interestingly, one of the

non-producible states is equivalent to two Bell pairs that are distributed between two of the parties. This can be seen as representative of the fact that when Alice and Bob receive a Bell state and receive two additional single-qubit states, they cannot create a second Bell state from the resources that they start out with, no matter what operations they have access to locally. Note that this is not the case, if they have access to high-dimensional entanglement from the sources and can access quantum channels, which would allow them to prepare such a state. Although this could be seen as a useful tool for determining producibility, the task of determining producibility in the triangle network seeks to distinguish between states that must have originated from a triangle network and states that can only be prepared by a tripartite source. Therefore, verifying states that only display entanglement between two parties is not of great use. Fortunately, the remaining non-producible states display tripartite entanglement.

Namely, three of the non-producible states contain contributions that are equivalent to a GHZ state up to local unitaries. This may seem unremarkable initially, as we have detailed in Section 3, GHZ states cannot be produced in a triangle network structure, which has been established in previous works. However, the fact that a majority of the non-producible states in Figure 4.4 display GHZ-like correlations indicates that this is possibly a fundamental feature of non-producible states in a triangle network. This opens the avenue of attempting to employ entanglement monotones, or other entanglement witnesses, for GME that could identify strongly entangled states as non-producible. Having said that, it has been pointed out in [17] that even strongly entangled GME states like the six-qubit ring cluster state can be produced in the triangle network, namely by distributing a Bell pair at each source and applying a CNOT gate at the site of each party. This is a hurdle in attempting to directly put an upper limit on the GME of network states. Furthermore, the fact that there exist inequivalent classes of GME in multipartite systems, makes the effort of quantifying entanglement using entanglement measures, beyond bipartite systems, tedious at best and impossible at worst.

Another possible approach for a producibility criterion based on GME would be to employ the correlation-tensor norm, introduced in Section 2.1.6. Namely, by checking the three-body correlation-tensor norm of network states, where the three systems are distributed between all three parties, i.e.

$$||T^{A_1B_1C_1}||_2^2 = \sum_{i,j,k=1}^{3} \mathrm{Tr}(\rho_{\Delta_{IUTN}} \sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes \sigma_k^{C1})^2, \tag{5.1}$$

where one can equivalently choose A_2, B_2 or C_2 instead of the respective subsystem of the same party, due to equivalence under local SWAP gates. Checking the correlation-tensor norm for the set of network-inequivalent graph states, it becomes apparent that all producible states fulfill

$$||T^{A_1B_1C_1}|| \leq 3, \tag{5.2}$$

where a Bell state exhibits a correlation-tensor norm of 3 and single-qubit states give a value of 1. In the case of product states, the correlation-tensor norm is the product of

the correlation-tensor norms of the marginals. As all producible graph states are product states of some number of Bell states and separable single qubit states, the norm must be upper bounded by 3 for producible graph states.

Furthermore, GHZ-like states exhibit a norm of 4. This means that three of the network-inequivalent graph states, that represent the equivalence classes that are not producible, do not fulfill the bound in Eq. (5.2). In addition to that, the graph states that represent the producible equivalence classes do fulfill the bound, in addition to some examples of producible states that we have checked by hand. Based on this observation, we can conjecture an upper bound on the three-body correlation-tensor norm for general producible states.

Conjecture: Let $\rho_{\Delta_{\mathrm{CUTN}}}$ be the density matrix of a quantum state that was prepared in a correlated unitary triangle network. Then the three-body correlation-tensor norm of this state must fulfill

$$||T^{A_{i_1}B_{i_2}C_{i_3}}|| \leq 3, \tag{Conj}$$

for all $i_1, i_2, i_3 = 1, 2$

Notably, while (Conj) is a necessary criterion for producibility in the set of network-inequivalent graph states, it is not sufficient, as one of the non-producible states saturates the bound. This is not a problem in itself, as no necessary and sufficient criteria have been found up to this point, and we must be content with necessary, but not sufficient, criteria in any case.

The above conjecture has been expressed in terms of a criterion for the correlated triangle network, rather than a criterion for independent networks. The reason for this is that the set of states producible in a correlated network is the convex hull of the set of states producible in an independent network and the correlation-tensor norm is a convex measure, i.e.

$$T[\sum_i p_i \rho_i] \leq \sum_i p_i T[\rho_i], \tag{5.3}$$

where we have introduced the notation $T[\rho]$, to signify the correlation-tensor norm of a state ρ, for the sake of brevity. Therefore, if all IUTN states fulfill (Conj), so must all states in Δ_{CUTN}. The results of this thesis were found using criteria that restrict the operations that the parties may apply to unitary transformations. For this reason, it is not possible to make any statement about the correlation-tensor norm in networks with, e.g., general quantum channels.

However, we have not yet found a proof for (Conj), despite different analytical methods having been tried. At this point it is unclear which tools might yield a proof for this conjecture. The scope of this thesis has not allowed further attempts at proving this conjecture. For this reason, we will leave this conjecture as an open question, that will hopefully spark further inquiry. Due to the convexity of the correlation-tensor norm, it

suffices to prove the conjecture for states that are producible in the IUTN. Despite this simplification this scenario still poses a considerable challenge, as it requires to characterize the action of the unitaries acting on arbitrary two-qubit states. Therefore, initial attempts at proving the conjecture sought to characterize the action of these two-qubit unitary operations.

SU(4) Elements

On a fundamental level, 4-by-4 unitary matrices can be expressed as elements of the 4-dimensional special unitary group, SU(4). Such elements can be parameterized using an exponential map of the generators of SU(4). In order to have access to commutation relations of Pauli matrices, we have chosen generators of the form

$$T_{\alpha\beta} \coloneqq \sigma_\alpha \otimes \sigma_\beta, \tag{5.4}$$

where $\sigma_\alpha, \sigma_\beta$ are the identity and the Pauli matrices, for $\alpha, \beta = 0$ and $\alpha, \beta = 1, 2, 3$, respectively. This permits us to express a general two-qubit unitary operation U via the exponential map

$$U = e^{i\theta a^{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta}, \tag{5.5}$$

where θ is a rotation angle and $a^{\alpha\beta}$ specify the direction of the rotation. Ways this parameterization has been used include using the Baker-Campbell-Hausdorff Lemma to simplify $U\sigma_i \otimes \mathbb{1} U^\dagger$, combining the Bloch decomposition of network states and the Taylor expansion of the exponential map to evaluate terms of the form

$$\mathrm{Tr}(\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_{i,0}), \tag{5.6}$$

where Greek letters indicate that variables may take values $0, 1, 2, 3$ and Latin letters can only take values $1, 2, 3$. Another approach was to try to find a third-order polynomial expression of the exponential map, similar to methods used in [55]. Additionally, we have attempted to use the first-order Taylor expansion of the exponential map, to determine whether the optimal strategies we have found up until this point, which saturate the bound, are local maxima. However, the commutation and trace relations of Pauli matrices are not enough to simplify expressions of the form of Eq. (5.6) in a way that yielded conclusive results.

Pauli Decomposition

Similar to the Bloch decomposition for quantum states, where we know that the Bloch coefficients need to fulfill certain requirements to yield a valid quantum state, we have attempted to decompose unitary matrices in such a way, namely

$$U = u_{\alpha\beta} \sigma^\alpha \otimes \sigma^\beta, \tag{5.7}$$

where the elements $u_{\alpha\beta}$ cannot be arbitrary in order for U to be unitary. However, we were not able to characterize the restrictions on these elements, and can therefore not make any assumptions about what the action of a transformation of this form would be, including whether or not it is unitary, for some choice of $u_{\alpha\beta}$.

SO(8) Transformations

Rather than characterizing the action of a two-qubit unitary, we have instead attempted to directly determine the action on the Bloch coefficients of the states prepared by the sources. This is possible because the transformation of the Pauli matrices may instead be understood as a transformation of the Bloch coefficients, where in the single-qubit case one can express the SU(2) transformation instead as an SO(3) transformation. Similarly, we have attempted to express SO(8) transformations via the action of SU(4). However, here we ran into similar problems, namely that the expression in Eq. (5.6) did not permit a simplification that was useful because it required to characterize the action of the unitary transformations on the Pauli matrices.

Pure States

As the correlation-tensor norm is a convex measure it should suffice to consider pure states rather than mixed states, as mixed states are convex mixtures of pure states. Even though this simplifies the structure of the problem, it does not allow to discard the action of the two-qubit transformations. Therefore, whether or not proving the conjecture for pure states is sufficient to prove it for mixed states still remains an open question.

Factorization of Unitary Transformations

In [56] the authors introduced a recursive factorization, based on the Haar measure, that allows one to express an SU(d) transformation as an SU($d-1$), SU(2) and SU($d-1$) transformation performed in succession, where the transformations act on different blocks of the full d-by-d matrix. This factorization allows us to express a transformation in SU(4) using 15 parameters. Up until this point, we have used this approach to perform Monte-Carlo searches of the 90-parameter space to find violations of the bound in (Conj), but these have not been successful. However, this tool provides the ground work for more directed searches, using deterministic search algorithms, rather than randomly probing the parameter space. Primary contenders for optimization algorithms in this context are *stochastic gradient-descent* search algorithms and *semidefinite programs*. We aim to employ such approaches in the future in an attempt to check our conjecture. It is important to note that while numerical searches may provide necessary insight into the problem, it would be preferential to have an analytical proof, as methods used in such a proof may be applicable in attempts to find similar criteria for non-triangle networks.

6 Conclusion

As laid out in the preceding chapters, we have, using equivalence classes of states in the unitary triangle network and graph states, determined an irreducible set of eight network-inequivalent states in the unitary triangle network. Each of the 2^{15} possible graph states in a graph with six vertices is network-equivalent to a state in this set of network-inequivalent states and thereby represented by it. Additionally, we have determined, that four of these states are producible in a IUTN, with the other four being non-producible. Each state represented by one of these eight network-inequivalent states shares the same property of being either producible or non-producible, as its representative.

The first of these network-inequivalent states can be prepared if each of the sources prepares a pure, fully separable two-qubit state, and the parties apply no further transformations on their respective pair of qubits. Similarly, the second, third and fourth state of this set can be prepared, when one, two, or three sources prepare a Bell state, respectively.

Naturally, as the remaining four states are not producible, there is no way to prepare them in the IUTN. The fifth state exhibits two Bell pairs being distributed between two parties, e.g. A and B, where C only has access to single-qubit states that are not entangled to any other qubits. The sixth, seventh and eight state all share the property of a GHZ state being distributed among the three parties, where in the sixth state, the second qubit of each party is again separable, in the seventh state two parties share an additional Bell state, and the eighth state has an additional GHZ state shared between all three parties. Note that these descriptions are accurate up to local unitary transformations, which have been omitted for simplicity's sake.

Based on this classification, it is apparent that in particular GHZ-like correlations are indicative of a state being non-producible. For this reason we have conjectured a criterion (Conj) that is based on the norm of the correlation tensor. At this point we have not found a proof that conclusively shows this conjecture to be either true or false. However, based on numerical searches and pertinent examples being checked analytically we believe that this conjecture is true.

This thesis sought to further develop methods and criteria to be used in the question of producibility in triangle networks, which have the potential to be generalised to other network topologies. The method by which this was attempted, was to characterize the set of graph states in a graph with six vertices with respect to their producibility in order to reveal properties of producible states. This approach has proven useful, and indicates that a feature of non-producible states may be the presence of GHZ-like entanglement distributed between the parties.

Despite the relative success of the approach chosen to classify correlations in network graph states, there are some limitations to this methodology. First off, our considerations have been limited to generic graph states, in contrast to so-called *weighted graph states*. Weighted graph states are set apart in that they specify the strength of the correlation in an edge. While, a priori there is no reason to not extend considerations to weighted graph states, there is a major difficulty that would arise in such a scenario. Namely, the equivalence classes of graph states in the triangle network rely on the edges in a graph state being equivalent. Otherwise, the different correlations between different pairs of subsystems, being inequivalent, would not allow to, e.g., classify the states in Figure 4.2 as network-equivalent, unless they happen to have identical weights in their edges. Therefore, such an investigation would require different methods to determine states to be network-equivalent.

It is reasonable to assume that, if the conjecture is true, this feature would be present in different network topologies and could serve as the basis for various criteria. Such a criterion, while still only necessary and not sufficient, would nevertheless prove useful in that it is a convex measure, which reflects the set of producible states which is itself convex. As it stands, a complete characterization of network-producible states seems to be out of reach. Therefore, a reasonable compromise is the development of criteria that are useful in different scenarios and in conjunction with each other may rule out larger classes of states. This conjecture, if proven, would constitute such a criterion and would for this reason benefit the field of quantum networks, and consequently quantum cryptography, in particular for devising novel multipartite cryptographic protocols, by elucidating which quantum states can be prepared in a given network topology.

However, as of yet, we have not proven the conjecture which would give rise to a necessary criterion. In conclusion, it can be said that the goal of this thesis was achieved in part, as we have been able to characterize the set of six-qubit graph states in a triangle network, but we have not proven our conjecture. This current state of knowledge on this problems suggests that further research is warranted, in particular further work that goes into investigating the conjecture that was put forward in this thesis, as well as potential criteria that exploit the properties of the non-GHZ-like network-inequivalent state. Additionally, a potential follow-up on the results obtained in this thesis could be to extend to weighted graph states, in order to model entanglement between subsystems more intricately, as well as applying the results to so-called *hypergraph states*, which generalize the notion of graph states. Such an inquiry could reveal further properties of the class of non-producible states and embed the findings of this thesis into a larger context of non-producibility. Furthermore, the methods used in this thesis straightforwardly apply to any network with bipartite sources, that is set up analogously as the triangle network. However, this would require a generalization of (C2) for such networks.

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