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Evolution of confined quantum fields in curved spacetime: application to dynamical Casimir effect with screened scalar fields

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Abstract

In this thesis, we focus on the study of confined quantum fields within the framework of quantum field theory in curved spacetime. We develop a method to describe the evolution of a confined quantum scalar field affected by changes in the background geometry and/or the confining boundaries. To prove its validity, we implement the method in known examples, namely, cosmological particle creation and the dynamical Casimir effect. Moreover, we also apply it to the study of a confined quantum field perturbed by gravitational waves reproducing and extending the results of a gravitational wave detector proposed in 2014.

Finally, we employ this method to study the dynamical Casimir effect in the presence of a screened scalar field. Since screened scalar fields are dark energy/dark matter candidates, there are many experimental efforts to find or constrain them. In this work, we concentrate on one type of screened scalar fields, namely, the chameleon models. By showing that the particle production is reduced due to the presence of a chameleon field, we propose a new direction for constraining these models.

With all these applications, the method proves useful for studying relativistic effects on confined quantum systems. Furthermore, this method is able to address relevant effects beyond general relativity on quantum fields. We expect that our method will become a standard tool in the growing research area of quantum field theory in curved spacetimes for confined fields.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit eingesperrten Quantenfeldern im Rahmen der Quantenfeldtheorie in gekrümmten Raumzeiten. Wir entwickeln eine Methode zur Beschreibung der Evolution eines eingesperrten Quantenskalarfeldes beeinflusst durch Veränderungen der Hintergrundgeometrie und/oder der einsperrenden Grenzen. Um seine Gültigkeit zu beweisen, wenden wir die Methode auf bekannte Beispiele an, nämlich die kosmologische Erzeugung von Teilchen und den dynamischen Casimireffekt. Darüberhinaus verwenden wir sie für die Untersuchung eines eingesperrten Quantenfeldes, welches durch Gravitationswellen gestört wird. Dies reproduziert und erweitert die Ergebnisse eines Gravitationswellendetektors, der 2014 vorgeschlagen wurde.

Abschließend verwenden wir diese Methode, um den dynamischen Casimireffekt unter Präsenz eines abgeschirrmten Skalarfeldes zu untersuchen. Da abgeschirrmte Skalarfelder Kandidaten für dunkle Energie/dunkle Materie sind, gibt es enorme experimentelle Bemühungen, um sie zu finden oder ihre Modellparameter einzuschränken. In dieser Arbeit konzentrieren wir uns auf das Chamäleonmodell als einen bestimmten Typ abgeschirrmter Skalarfelder. Indem wir zeigen, dass die Erzeugung von Teilchen durch die Anwesenheit eines Chamäleonfeldes reduziert wird, schlagen wir eine neue Möglichkeit zur Einschränkung solcher Modelle vor.

Durch all diese Anwendungen erweist sich die Methode als nützlich für Untersuchungen relativistischer, eingesperrter Quantensysteme. Diese Methode ist außerdem dazu in der Lage, relevante Effekte auf Quantenfelder über die allgemeine Relativitätstheorie hinaus zu beschreiben. Wir erwarten, dass unsere Methode im wachsenden Forschungsbereich der Quantenfeldtheorie eingesperrter Felder in gekrümmten Raumzeiten ein Standardwerkzeug werden wird.

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Contents

It only takes two facing mirrors to build a labyrinth.

Jorge Luis Borges, Seven Nights

List of Publications

[1] Luis C. Barbado, Ana Lucía Báez-Camargo and Ivette Fuentes
Evolution of confined quantum scalar fields in curved spacetime. Part I: Spacetimes without boundaries or with static boundaries in a synchronous gauge
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Evolution of confined quantum scalar fields in curved spacetime. Part II: Spacetimes with moving boundaries in any synchronous gauge
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[3] Ana Lucía Báez-Camargo, Daniel Hartley, Christian Käding and Ivette Fuentes
Dynamical Casimir effect with screened scalar fields
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Part I

Preamble

1

Introduction

The last century saw the blossoming and development of two major physical theories, namely general relativity (GR) and quantum theory. These theories revolutionised the way we think about Nature. For instance, the idea of merging space and time into a single entity called spacetime is due to relativity while quantum theory opened our minds to a world of probabilities, denying the existence of objective reality. Unfortunately, unifying quantum theory and relativity is a problem that remains unsolved. Early attempts to find a solution to this problem sought to replace the Schrödinger equation with a [special] relativistic version of it. In particular, Klein [4] and Gordon [5] found an equation which describes spinless particles. The system characterised by this relativistic wave equation is not that of a single particle, but rather that of a field. This novel approach gave birth to quantum field theory in flat spacetime, which after decades of development, became the mathematical framework for the Standard Model of particle physics.

Similar to its flat counterpart, quantum field theory in curved spacetime (QFTCS) is the theory that studies the behaviour of quantum fields which propagate in a classical general relativistic background geometry. The main initial result of the theory is the pioneering work of Parker in which he predicts that the expansion of the Universe can create paricles out of the vacuum [6,7]. A few years later, Hawking concluded that there is particle creation as a result of the gravitational collapse of a body forming a black hole, such that black holes emit thermal radiation with temperature proportional to their surface gravity [8]. After these works, there was an extensive development of the theory with applications to different phenomena [9–11]. In order to perform such studies, different simplifications and techniques have been developed, adapted to the specific problems. For example, in the case of particle creation in cosmological models, approximations such as homogeneity or isotropy are often convenient to make any meaningful analytical progress.

Recently, there has been a great interest to verify QFTCS [12–15]. This is because of the rapid development of experimental techniques in quantum physics in order to create quantum technologies. These quantum experiments could also be used to explore relativistic effects on quantum systems. Thus, it is expected, in the near future, to experimentally verify theoretical predictions about quantum fields confined in cavities affected by relativistic features like changes in the background geometry or a non-inertial motion of the cavity [16]. As a consequence, new concrete systems are being studied within the framework of the theory, which need the development of new mathematical techniques.

For this reason, we consider confined quantum fields in this thesis. The main goal of this work is to present a general method for computing the evolution of a confined quantum scalar field in a globally hyperbolic spacetime by means of a time-dependent Bogoliubov transformation. This method can be used in spacetimes where the timelike boundaries remain static in some synchronous frame [1], or even when this is not the case [2]. In these works, we apply the method to well-known problems in QFTCS like cosmological particle creation and the dynamical Casimir effect (DCE) [17], which refers to the particle production due to the motion of boundaries [18]. Moreover, we apply this method to a recently developed scheme for dectecting gravitational waves [19].

In the last part of this thesis, we propose a new direction to find and constrain screened scalar fields, using the method presented in [3], by means of studying the impact of these fields on the dynamical Casimir effect. Screened scalar fields are a class of fifth force models with a *screening mechanism*, which allows the fifth force to vary depending on the environment [20]. Fifth force models arise from the need to explain phenomena that cannot be explained with the current theories of GR or the Standard Model (SM), which together describe the four fundamental interactions or forces known in Nature. Examples of phenomena that are still open problems are the observed accelerating expansion of the Universe or galaxies that do not behave as GR predicts, implying a new type of matter not included in the current SM. Thus, the idea of the existence of additional scalar fields comes from modifications of GR and from extensions of the SM. In this thesis, we focus on the modified gravity approach.

This thesis is organised as follows: In Chapter 2, a brief summary of concepts of differential geometry and GR is presented. In Chapter 3, we present an overview of QFTCS and, in particular, the technique that gave us the inspiration for the resulting model. Chapter 4 contains the introduction and basic useful results for screened scalar fields. The results of this thesis are presented in Part II. We conclude in Part III.

2

Differential Geometry and General Relativity

The mathematical framework of GR is differential geometry. Some excellent introductory texts for differential geometry and relativity are [21–23]. In this chapter, we will show some aspects of the geometrical part of the theory of relativity. In particular, we restrict our attention to the concepts needed for this thesis and give references to the reader for topics that lie beyond the scope of this work. This chapter will be more technical than the others because some mathematical concepts of GR are presented.

2.1 Spacetimes

Differential geometry studies *smooth manifolds*, which are infinitely differentiable manifolds. Essentially, a manifold of dimension n is a mathematical object that, when viewed in a small region, looks like $\mathbb{R}^n$. A particular type of manifold that is used in GR is a *Lorentzian manifold* [22, 23].

Definition 1. *A* Lorentzian manifold *is a smooth manifold M of dimension $n \geq 2$ equipped with a non-degenerate, symmetric, bilinear map $g : T_pM \times T_pM \to \mathbb{R}$ of signature $(-, +, \cdots, +)$, called a* Lorentzian metric[1].

At each point p of a Lorentzian manifold M, there exists a *tangent space* T_pM formed by *tangent vectors* at p. This real vector space contains all the directions

[1]Some authors prefer the convention $(+, -, \cdots, -)$ for the signature of the metric. The choice has no effect on any observable quantity when applied to physics.

in which one could tangentially pass through p. We are interested in classifying the vectors of this tangent space:

Definition 2. *A tangent vector $v \in T_pM$ is classified as*

- *timelike* *if* $g(v,v) < 0$,
- *null* *if* $g(v,v) = 0$,
- *causal if either timelike or null,*
- *spacelike* *if* $g(v,v) > 0$.

In each tangent space, the causal tangent vectors form a double cone called the *causal cone*. A *time-orientation* at p is a choice of an arrow of time which splits the causal cone into two convex cones in which the one that opens in the direction of the time's arrow is defined as the *future cone*, while the other is called the *past cone*. If it is possible to choose this splitting continuously on M, the manifold is said to be *time-orientable*. With these ingredients we can now define *spacetime* [24, 25].

Definition 3. *A* spacetime (M, g) *is a time-oriented connected Lorentzian manifold.*

The spacetime can be locally covered with coordinates $\{x^\mu | \mu = 0, 1, 2, \cdots, n-1\}$ in open subsets of $\mathbb{R}^n$. Thus, considering a patch of local coordinates, we can write $g(v,v) = g_{\mu\nu}v^\mu v^\nu = v^\mu v_\mu$, where $g_{\mu\nu}$ are the elements of the metric tensor and summation is implied over repeated indices. An infinitesimal distance between two neighbouring points of spacetime x^μ and $x^\mu + dx^\mu$ is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{2.1}$$

and is called *the line element*. In analogy to Def. 2, a parametrized curve $x^\mu(\lambda)$ is called *timelike/null/spacelike* if its tangent vector $v^\mu = \frac{dx^\mu(\lambda)}{d\lambda}$ is timelike/null/spacelike at every point [2]. A curve is said to be *causal* if it is not spacelike. Massive particles and observers follow timelike curves called *world-lines*, while massless particles follow null curves. Along any timelike curve C_t the time measured by a clock moving along the curve is called *proper time* and is defined as

$$\tau = \int_{C_t} \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{2.2}$$

[2] These definitions can be extended to higher dimensions, see [25].

Similarly, along a spacelike curve C_s, the *proper length* l is defined as

$$l = \int_{C_s} \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{2.3}$$

Note that in both the proper length and proper time the integrand is given by the line element (2.1).

2.2 On the causal structure

There are several causal relations between points in a spacetime (M, g). In this work we focus just on one. We recommend the reader to check [24] for a thorough review on the causal relations and structures of spacetimes.

Given two points or *events* in a spacetime $p, q \in M$, we say that the event p is *causally* related to the event q, denoted $p \leq q$, if there is a future-directed causal curve connecting p with q or if $p = q$. The relation becomes *strictly causal* if $p \neq q$.

Definition 4. *A spacetime (M, g) is called* causal *if it satisfies one of the following equivalent properties:*

- *No closed causal curve exists.*
- *Strict causal relation is* irreflexive, *i.e., for $p, q \in M$ such that $p < q \Rightarrow p \neq q$.*

In other words, in a causal spacetime, paradoxes derived from trips to the past (like the grandfather's paradox [26]) are avoided. Consequently, we can define the *causal past* J^- and *future* J^+ of an event.

Definition 5. *For a point $p \in M$, the* causal past (future) *of p is the set of all points q in M such that:*

$$\begin{aligned} J^-(p) &= \{q \in M : q \leq p\}, \\ J^+(p) &= \{q \in M : p \leq q\}. \end{aligned} \tag{2.4}$$

In this thesis, we will work with spacetimes that have the strongest type of causality known as *global hyperbolicity* which we define hereafter [24, 27]:

Definition 6. *A spacetime (M, g) is* globally hyperbolic *if:*

a) it is causal, and

b) the intersections $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$.

Consider p as the starting point of an observer, while q is the endpoint. Condition *b)* tells us that the *causal diamond* formed by the intersection of the future cone of event p with the past cone of event q is compact, i.e., closed and bounded [28]. Consequently, the observer will follow a world-line which cannot go beyond the causal diamond. Therefore, the compactness of the diamonds $J^+(p) \cap J^-(q)$ guarantees that there are no losses of information/energy in the spacetime. An important feature of globally hyperbolic spacetimes is the existence of spacelike surfaces that can define an "instant of time", and consequently, initial data can be completely specified. These type of hypersurfaces are defined as:

Definition 7. *A* Cauchy hypersurface *of a spacetime (M, g) is a submanifold Σ of codimension* 1[3] *intersected once and only once by each inextensible (with no ends) timelike curve.*

Any globally hyperbolic spacetime (M, g) admits a Cauchy temporal function t in the full spacetime, which provides a foliation in Cauchy hypersurfaces of constant time [29]. This result was later extended to the case of *globally hyperbolic spacetimes with a timelike-boundary* $(\bar{M} = M \cup \partial M, g)$ in [30]. Consequently, $\bar{M}$ splits smoothly as a product $\mathbb{R} \times \bar{\Sigma}$, where $\bar{\Sigma}$ is a Cauchy hypersurface with a boundary. In general, QFTCS studies fields propagating in globally hyperbolic spacetimes. The last result regarding spacetimes with a timelike boundary is relevant because we are interested in confined quantum fields in curved spacetimes.

2.3 On the spacetime's symmetries

In general, symmetries play a fundamental role in physics. In this context, these are symmetries of the spacetime metric, which are called *isometries*. An isometry is a diffeomorphism (a differentiable bijection with a differentiable inverse) that leaves the metric invariant. For example, the rotation of a sphere is a diffeomorphism.

Note that to detect a continuous symmetry in this way, a single discrete diffeomorphism is insufficient. We require a one-parameter group of diffeomorphisms and a derivative operator on the metric that characterises its rate of change under diffeomorphisms [21, 31, 32].

[3] The codimension of a submanifold $\Sigma \subset M$ is defined as $\mathrm{codim}(\Sigma) = \dim(M) - \dim(\Sigma)$.

Definition 8. *The Lie derivative for the metric tensor $g_{\mu\nu}$ with respect to a vector field V is defined as*

$$L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu. \tag{2.5}$$

Here ∇_μ is a covariant derivative, which is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{2.6}$$

where $\Gamma^\nu_{\mu\lambda}$ are known as *Christoffel symbols*[4] [33]. Thus, if a one-parameter group of isometries is generated by a vector field K, then it is known as a *Killing vector field*, which fulfills the condition

$$L_K g_{\mu\nu} = 0. \tag{2.7}$$

Equation (2.7) means that any observer following a Killing vector field can define quantities that do not change along the vector field, thus Killing vectors imply conserved quantities associated with the motion of free particles. If the Killing vector field is timelike, then these quantities are preserved in time.

2.4 The Einstein-Hilbert action

GR studies gravity and its relation with the spacetime. In words of John Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve" [34]. The theory is governed by the Einstein-Hilbert action [33]

$$S_{EH} = \int \mathrm{d}^4x \sqrt{-g} \frac{1}{2\kappa} \mathcal{R} + S_m \left[g_{\mu\nu}; \Phi_i \right], \tag{2.8}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{R}$ is the scalar curvature or Ricci scalar, κ is the Einstein gravitational constant, and S_m is the matter action that describes any matter fields Φ_i appearing in the theory. The Einstein gravitational constant is defined as $\kappa = 8\pi G c^{-4}$, where G is the Newton constant of gravitation and c is the speed of light in vacuum. The first term of Eq. (2.8) is usually referred to as the gravitational sector of the action. Varying the action (2.8) with respect to the

[4] The covariant derivative is given by the ordinary partial derivative plus a correction specified by the Christoffel symbols which depend on the metric and ordinary derivatives of the metric.

metric and applying the stationary-action principle yields the Einstein field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.9}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the energy-momentum tensor for matter[5]. The Einstein field equations (2.9) are a system of partial differential equations where the solution is given by the metric tensor of a spacetime. In this sense, GR relates the geometry of spacetime to the matter and energy content of such spacetime.

[5]It is possible to include a cosmological constant Λ to enlarge the action without introducing new degrees of freedom. In this case, the Einstein equations will have another term related to the cosmological constant.

3

Quantum Field Theory in Curved Spacetimes

In this chapter, we establish the basic concepts of QFTCS, where we consider confined quantum fields. For the interested reader we recommend [9–11]. This chapter is organised as follows. We recall some concepts of quantum mechanics that are used in QFTCS. Then, we introduce the Klein-Gordon field and its quantisation. Afterwards, we define the Bogoliubov transformation between two sets of solutions to a real scalar quantum field and, as an example, we explain a well-known toy model of cosmological particle creation. We finish with an overview of a previous technique for computing Bogoliubov coefficients for confined fields, which was the inspiration for the method presented in this thesis. Unless otherwise noted, natural units $\hbar = c = 1$ are used.

3.1 Quantum Mechanics

The basic mathematical structure in which quantum mechanics is supported is a *Hilbert space* which is defined as [35, 36]

Definition 9. *A Hilbert space $\mathcal{H}$ is a complex vector space equipped with an inner product $\langle . | . \rangle$, which is complete with respect to the norm induced by the inner product.*

A *quantum state* is represented by a vector (also known as *ket* or ray) $|\psi\rangle$ in $\mathcal{H}$. The complex conjugate transpose of a ket (also known as a *dual vector*) is represented by a *bra* $\langle\psi|$. For any quantum state, it is required that $\langle\psi|\psi\rangle = 1$, i.e., it is normalised to unity. Physical observables are represented by *linear operators* on such a Hilbert space. An operator O acts on a vector $|\psi\rangle \in \mathcal{H}$ by mapping it onto another vector $|\phi\rangle \in \mathcal{H}$, such that [37]

$$O\,|\psi\rangle = |\phi\rangle\,. \tag{3.1}$$

The operator O is linear if for any complex numbers a and b it satisfies

$$O\,[a\,|\psi\rangle + b\,|\chi\rangle] = a\;O\,|\psi\rangle + b\;O\,|\chi\rangle\,, \tag{3.2}$$

where $|\psi\rangle\,, |\chi\rangle \in \mathcal{H}$. For any linear operator O on $\mathcal{H}$, there exists its adjoint, denoted by $O^\dagger$, such that

$$\langle\psi|O\chi\rangle = \langle O^\dagger\psi|\chi\rangle\,. \tag{3.3}$$

In the case where the operator is *self-adjoint*, that is $O^\dagger = O$, the operator is called *Hermitian*. Furthermore, an operator U is called *unitary* if

$$UU^\dagger = U^\dagger U = I, \tag{3.4}$$

where I is the *identity operator* ($I\,|\psi\rangle = |\psi\rangle$ for all states). From (3.4), we see that $U^\dagger = U^{-1}$ and that unitary operators leave the inner product between vectors, $\langle U\psi|U\chi\rangle = \langle\psi|\chi\rangle$, invariant.

So far, we have presented concepts of a single Hilbert space, that is, of a single quantum system. For composite systems, we need to extend the Hilbert space. Let us consider a composite system of two quantum subsystems represented by the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimension d_A and d_B, respectively. Thus, the Hilbert space of the full quantum system is assembled using the *tensor product*, such that $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $d_{AB} = d_A d_B$. Given two vectors $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$, a combined vector belonging to $\mathcal{H}_{AB}$ is given by

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle\,. \tag{3.5}$$

In general, for a set of N quantum subsystems, the Hilbert space for the whole quantum system is defined as

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i. \tag{3.6}$$

3.2 The Klein-Gordon field

There are four basic ingredients in the construction of a quantum field theory, namely [38]

- The Lagrangian, or the equation of motion of the classical field theory;
- A quantisation procedure. In this thesis we will work with the canonical quantisation;
- The mathematical characterisation of quantum states;
- The physical interpretation of such states and of the observables.

In the next subsections we will discuss these ingredients.

3.2.1 The classical Lagrangian formulation

Consider a free real scalar field Φ propagating in a globally hyperbolic spacetime (M, g). Then the matter action S_m of (2.8) is given by

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}\left(\Phi, \nabla_\mu\Phi\right), \tag{3.7}$$

where $\mathrm{d}^4x\sqrt{-g}$ is the *covariant volume element* and $\mathcal{L}$ is the *Lagrangian density*, which is written as

$$\mathcal{L} = -\frac{1}{2}\left(g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + m^2\Phi^2\right). \tag{3.8}$$

Here $m \geq 0$ is the rest mass of the field[1]. Varying the action (3.7) with respect to the field, and requiring it to be stationary ($\delta S = 0$), yields the *Euler-Lagrange field equations*

$$\frac{\partial\sqrt{-g}\mathcal{L}}{\partial\Phi} - \partial_\mu\left(\frac{\partial\sqrt{-g}\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\right) = 0. \tag{3.9}$$

Substituting (3.8) into (3.9), we obtain the *Klein-Gordon equation*

$$\left(\Box - m^2\right)\Phi = 0, \tag{3.10}$$

[1] For simplicity, this Langrangian density describes a field that is not coupled to the gravitational field. However, the method presented in [1] and [2] takes into account cases where the field Φ is coupled to the scalar curvature $\mathcal{R}$.

where $\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Phi$. The main implication of global hyperbolicity is that the Klein-Gordon equation (3.10) is *well-posed*, i.e., there exists a unique solution that depends continuously on the data given in the problem [39]. Consequently, we can relate one-to-one initial conditions and solutions of the Klein-Gordon equation.

3.2.2 The canonical quantisation of the Klein-Gordon field

Given two classical solutions to Eq. (3.10), the *(pseudo) inner product*[2] is defined as

$$\langle\Phi_1, \Phi_2\rangle = -\mathrm{i}\int_\Sigma \mathrm{d}\Sigma^\mu[\Phi_1\partial_\mu\Phi_2^* - \Phi_2^*\partial_\mu\Phi_1], \tag{3.11}$$

where Σ is a spacelike Cauchy hypersurface and $\mathrm{d}\Sigma^\mu$ is the volume form. Using Gauss' theorem, one can show that the inner product is independent of the choice of the hypersurface [11]. This crucial property means that one can calculate the inner product at any point of the time coordinate and guarantees that it will be the same at any other point.

A complete set of orthonormal solutions with respect to (3.11) satisfies [9]

$$\begin{aligned}\langle\Phi_m, \Phi_n\rangle &= -\langle\Phi_m^*, \Phi_n^*\rangle = \delta_{mn},\\ \langle\Phi_m, \Phi_n^*\rangle &= 0.\end{aligned} \tag{3.12}$$

Notice that, in (3.12) the norm of the complex conjugates of the modes is negative, hence $\{\Phi_n^*\}$ is known as the *set of negative frequency solutions* and $\{\Phi_n\}$ is called the *set of positive frequency solutions*. In general, there is not a unique way of splitting the solutions into $\{\Phi_n\}$ and $\{\Phi_n^*\}$[3]. Since we are considering a complete basis of solutions $\{\Phi_n, \Phi_n^*\}$, the classical field can be expanded as

$$\Phi = \sum_n[a_n\Phi_n + a_n^*\Phi_n^*], \tag{3.13}$$

where a_n and a_n^* are the expansion coefficients[4]. *Canonical quantisation* is achieved by introducing the equal time canonical commutation relations (CCRs) converting

[2] It is usually called pseudo inner product because the norm of a vector admits negative values.
[3] This fact will be important for the physical characterisation of states.
[4] Since we are considering confined quantum fields, we have assumed a discrete set of solutions. However, the treatment can be reformulated for a continuous spectrum.

the coefficients a_n into operators

$$[\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn},$$
$$[\hat{a}_m, \hat{a}_n] = [\hat{a}_m^\dagger, \hat{a}_n^\dagger] = 0, \tag{3.14}$$

such that the Klein-Gordon field can be promoted to an operator of the form

$$\hat{\Phi} = \sum_n [\hat{a}_n \Phi_n + \hat{a}_n^\dagger \Phi_n^*]. \tag{3.15}$$

The $\hat{a}_n^\dagger$ and $\hat{a}_n$ are called *creation* and *annihilation* operators, respectively. Note that (3.14) are the same commutation relations as for the standard quantum harmonic oscillator, but since the norm generated by the pseudo inner product (3.11) is not positive-definite, linear combinations of solutions to the Klein-Gordon equation Φ_n and Φ_n^* cannot be interpreted as a single particle wave function. Thus, we can interpret the field as a collection of decoupled harmonic oscillators.

3.2.3 The mathematical characterisation of quantum states

The quantisation of the Klein-Gordon field is necessarily associated to a vector space in which the states are defined. As we have just discussed, the linear combinations of solutions to the Klein-Gordon equation cannot be interpreted as a single particle wave equation. Therefore, the single-particle Hilbert space is not appropriate to construct a quantum field theory. The right mathematical structure is given by a *Fock space*. This space is constructed as a direct sum of a zero-particle space, a one-particle Hilbert space, a two-particle Hilbert space and so on

$$\mathcal{F} \equiv (\mathbb{C}) \oplus (\underbrace{\mathcal{H}}_{1\,particle}) \oplus (\underbrace{\mathcal{H} \otimes \mathcal{H}}_{2\,particles}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots . \tag{3.16}$$

The vacuum state of a quantum field is defined as

$$\hat{a}_n |0\rangle = 0 \quad \forall\ n, \tag{3.17}$$

and is interpreted as the one which does not contain particles. Then, we can construct one-particle states by using creation operators acting on the vacuum, such that any of these states will be linear combinations of vectors of the form $|\phi_i\rangle = \hat{a}_i^\dagger |0\rangle$. That is,

$$|\phi^{(1)}\rangle = \sum_i b_i \, |\phi_i\rangle \,, \tag{3.18}$$

where $b_i \in \mathbb{C}$. All possible single-particle states are elements of the one-particle Hilbert space $\mathcal{H}$ in (3.16). A similar construction can be done for the two-particle Hilbert space. In this case, any two-particle state is given by linear combinations of vectors of the form

$$|\phi_m, \phi_n\rangle = \hat{a}_m^\dagger \hat{a}_n^\dagger \, |0\rangle \,. \tag{3.19}$$

We can construct Hilbert spaces with higher particle content in an analogous manner. Therefore, *mathematically*, we can define *particles* as the action of creation operators on the vacuum state.

3.2.4 The physical characterisation of quantum states

Note that the expansion (3.15) defines the vacuum state (3.17). In flat spacetime, the vacuum state is uniquely defined regardless of the Lorentz frame in which t is the time coordinate. Therefore, it is straightforward to physically characterise the vacuum state as the ground state of the field. However, in curved spacetime, the choice of $\{\Phi_n\}$ is not unique, and hence there is no unique notion of the vacuum state. Thus, the *physical* notion of particle becomes ambiguous.

If the spacetime is globally hyperbolic and there exists a timelike Killing vector field $K = \partial_\tau$ as in (2.7), then a choice of $\{\Phi_n\}$ can be determined. Let us choose the set of solutions for which their Lie derivative with respect to K satisfies

$$\begin{aligned} \mathrm{i}L_K\Phi_n &= \mathrm{i}K\Phi_n = \mathrm{i}\partial_\tau\Phi_n = +\omega_n\Phi_n, \\ \mathrm{i}L_K\Phi_n^* &= -\omega_n\Phi_n^*, \end{aligned} \tag{3.20}$$

where $\omega_n > 0$. In this sense, positive and negative energy solutions are defined according to the signs of their eigenvalues $\pm\omega_n$. Thus, Eq. (3.20) leads to the interpretation of *particles* associated with the positive frequency modes and *antiparticles* with negative frequency modes. Since a timelike Killing vector field defines conserved quantities, an observer flowing along K will have a constant notion of particles throughout spacetime. In addition, a vacuum state at one point in spacetime remains a vacuum state along a Killing vector field.

3.3 Bogoliubov transformations

Until now, we have presented the quantisation of a field for a particular set of solutions and we have also separated them into positive and negative frequency modes w.r.t. a timelike Killing vector field. However, a spacetime can admit different timelike Killing vector fields, each of which will have a different basis of mode functions. Given two bases of mode solutions $\{\Phi_n, \Phi_n^*\}$ and $\{\tilde{\Phi}_m, \tilde{\Phi}_m^*\}$, the field can be equivalently expanded as

$$\hat{\Phi} = \sum_n [\hat{a}_n \Phi_n + \hat{a}_n^\dagger \Phi_n^*] = \sum_m [\hat{b}_m \tilde{\Phi}_m + \hat{b}_m^\dagger \tilde{\Phi}_m^*]. \tag{3.21}$$

Consequently, the classification into positive and negative frequency solutions differ. It is natural to ask how these two sets of field solutions are related. The transformations that relate two bases of solutions are called *Bogoliubov transformations* and are defined as

$$\tilde{\Phi}_m = \sum_n [\alpha_{mn} \Phi_n + \beta_{mn} \Phi_n^*], \tag{3.22}$$

$$\hat{b}_m = \sum_n [\alpha_{mn}^* \hat{a}_n - \beta_{mn}^* \hat{a}_n^\dagger], \tag{3.23}$$

where

$$\alpha_{mn} = \langle \tilde{\Phi}_m, \Phi_n \rangle, \tag{3.24}$$

$$\beta_{mn} = -\langle \tilde{\Phi}_m, \Phi_n^* \rangle, \tag{3.25}$$

are called *Bogoliubov coefficients*. Notice that if the β-coefficients are zero, then the transformation (3.23) contains only annihilation operators affected by the α-coefficients. Thus, both bases define the same vacuum state as in (3.17). However, if the β-coefficients are non-zero, the two vacua do not coincide. Consequently, different Killing observers will in general observe a different number of particles. In summary, the role of the α-coefficients is to shift excitations between different modes, commonly known as *mode-mixing*, while the role of the β-coefficients is to account for the *particle creation* due to the transformation. For instance, starting with a vacuum state, the mean number of particles in mode m after a transformation is given by [9]

$$\mathcal{N}_m = \sum_n |\beta_{nm}|^2 . \tag{3.26}$$

The transformations given in Eqs. (3.22) and (3.23) can be written in a much more convenient way using matrices, such that in block notation

$$\begin{pmatrix} \tilde{\Phi}_m \\ \tilde{\Phi}^*_m \end{pmatrix} = U \begin{pmatrix} \Phi_n \\ \Phi^*_n \end{pmatrix}, \tag{3.27}$$

$$\begin{pmatrix} \hat{b}_m \\ \hat{b}^\dagger_m \end{pmatrix} = \left(U^{-1}\right)^T \begin{pmatrix} \hat{a}_n \\ \hat{a}^\dagger_n \end{pmatrix}, \tag{3.28}$$

where

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad \left(U^{-1}\right)^T = \begin{pmatrix} \alpha^* & -\beta^* \\ -\beta & \alpha \end{pmatrix}. \tag{3.29}$$

Here, α and β are matrices formed by the elements in (3.24) and (3.25), respectively. Since a Bogoliubov transformation preserves the inner product by construction, it must be unitary. Therefore, any Bogoliubov transformation must satisfy the following identity

$$UJU^\dagger = J, \tag{3.30}$$

where

$$U^\dagger = \begin{pmatrix} \alpha^\dagger & \beta^T \\ \beta^\dagger & \alpha^T \end{pmatrix}, \quad J := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{3.31}$$

Hence,

$$\begin{aligned} \alpha\alpha^\dagger - \beta\beta^\dagger &= \mathbb{1}, \\ \alpha\beta^T - \beta\alpha^T &= 0. \end{aligned} \tag{3.32}$$

All linear and unitary transformations on the Fock space can be represented by Bogoliubov transformations, which are the keystone of this work. To understand the usage of Bogoliubov transformations, it may be instructive to look at a specific example, namely a toy model of cosmological particle creation.

3.4 Cosmological particle creation

Parker predicted that the expansion of the Universe can create particles out of the vacuum [6]. In this section, we show a toy model of cosmological particle creation presented in [9], since it was reproduced in [1]. Consider a (1+1)-dimensional *Friedmann-*

Lemaître-Robertson-Walker spacetime [40] with line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2, \tag{3.33}$$

where $a(t)$ is known as *scale factor* since its role is to expand or contract the spatial sector of the metric. For convenience, the authors of [9] replaced the coordinate t by the *conformal time* η

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}, \tag{3.34}$$

such that the line element (3.33) may be rewritten as

$$\mathrm{d}s^2 = c(\eta)(-\mathrm{d}\eta^2 + \mathrm{d}x^2), \tag{3.35}$$

where $c(\eta) = a^2(\eta)$. Notice that the line element is *conformal* to Minkowski spacetime[5]. The conformal scale factor $c(\eta)$ specifies how the spacetime evolves. Consider that the spacetime is asymptotically flat in the past and future infinities, denoted as *in* and *out regions*, respectively. In both regions, the spacetime admits timelike Killing vector fields. Consequently, there will be two different field expansions as in (3.21), each one associated to one of these regions. A conformal scale factor that satisfies such a condition is given by

$$c(\eta) = A + B\tanh(\rho\eta), \tag{3.36}$$

where A, B and ρ are constants. Then, the in and out regions are found in the limits

$$\begin{aligned} \eta \to -\infty \Rightarrow \mathrm{d}s^2_{\text{in}} &= (A - B)\left(-\mathrm{d}\eta^2 + \mathrm{d}x^2\right), \\ \eta \to \infty \Rightarrow \mathrm{d}s^2_{\text{out}} &= (A + B)(-\mathrm{d}\eta^2 + \mathrm{d}x^2). \end{aligned} \tag{3.37}$$

We see that in the far past and future, the spacetime becomes flat, i.e., Minkowskian. Consider a massive scalar field in a spacetime with line element (3.35) and conformal scale factor (3.36). Thus, the Klein-Gordon equation (3.10) is given by

$$\left[-\partial_\eta^2 + \partial_x^2 - c(\eta)m^2\right]\Phi = 0. \tag{3.38}$$

[5] In the next chapter, we go deeper into the study of conformal frames.

The change to conformal time (3.34) was done in order to simplify the Klein-Gordon equation, such that the space and time variables are separated. Thus, the solutions to (3.38) are of the form

$$\Phi_k\left(\eta, x\right) = Ce^{\mathrm{ikx}}\xi_k(\eta), \tag{3.39}$$

where C is a constant. Substituting (3.39) in (3.38), we obtain a differential equation for $\xi_k(\eta)$

$$\partial_\eta^2 \xi_k\left(\eta\right) + \left(k^2 + c\left(\eta\right) m^2\right)\xi_k\left(\eta\right) = 0. \tag{3.40}$$

Eq. (3.40) is solved in terms of two hypergeometric functions, such that

$$\begin{aligned}
\xi_k^{(1)}\left(\eta\right) &= D\left(\eta\right)\, {}_2F_1\left[1 + \frac{\mathrm{i}\omega_-}{\rho}, \frac{\mathrm{i}\omega_-}{\rho}; 1 - \frac{\mathrm{i}\omega_{in}}{\rho}; \frac{1}{2}\left(1 + \tanh\left(\rho\eta\right)\right)\right], \\
\xi_k^{(2)}\left(\eta\right) &= D\left(\eta\right)\, {}_2F_1\left[1 + \frac{\mathrm{i}\omega_-}{\rho}, \frac{\mathrm{i}\omega_-}{\rho}; 1 + \frac{\mathrm{i}\omega_{in}}{\rho}; \frac{1}{2}\left(1 - \tanh\left(\rho\eta\right)\right)\right],
\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}
D\left(\eta\right) &= \left(\frac{e^{2\rho\eta}}{e^{2\rho\eta}+1}\right)^{-\mathrm{i}\frac{\omega_{in}}{2\rho}} \left(\frac{1}{e^{2\rho\eta}+1}\right)^{\mathrm{i}\frac{\omega_{out}}{2\rho}}, \\
\omega_{\mathrm{in}} &= \left(k^2 + \left(A - B\right)m^2\right)^{1/2}, \\
\omega_{\mathrm{out}} &= \left(k^2 + \left(A + B\right)m^2\right)^{1/2}, \\
\omega_\pm &= \frac{1}{2}\left(\omega_{\mathrm{out}} \pm \omega_{\mathrm{in}}\right).
\end{aligned} \tag{3.42}$$

Notice that in the limits

$$\begin{aligned}
\lim_{\eta\to-\infty} \xi_k^{(1)}\left(\eta\right) &= e^{-\mathrm{i}\omega_{\mathrm{in}}\eta}, \\
\lim_{\eta\to\infty} \xi_k^{(2)}\left(\eta\right) &= e^{-\mathrm{i}\omega_{\mathrm{out}}\eta}.
\end{aligned} \tag{3.43}$$

It is customary, for convenience, to impose periodic boundary conditions $\Phi\left(\eta, x = 0\right) = \Phi\left(\eta, x = L\right)$. For fields that propagate through all of spacetime, this is just a mathematical device, such that L is taken to infinity at the end of the computations[6] [11]. The normalised modes with respect to the Klein-Gordon inner product (3.11) associ-

[6]The toy model presented in [9] do not use this technique, but we consider it because we study confined fields. In the case of confined fields, L is not taken to infinity.

ated to the asymptotic flat regions are

$$\Phi_k^{\text{in}} \rightarrow \frac{1}{\sqrt{2L\omega_{\text{in}}}} e^{\mathrm{ikx}-\mathrm{i}\omega_{\text{in}}\eta},$$
$$\Phi_k^{\text{out}} \rightarrow \frac{1}{\sqrt{2L\omega_{\text{out}}}} e^{\mathrm{ikx}-\mathrm{i}\omega_{\text{out}}\eta}. \tag{3.44}$$

The normalisation constants given in (3.44) are different from the ones presented in [9], since we are considering a confined quantum field. We see that the in-modes are different from the out-modes. To know how much these bases of modes differ from one another, we need to compute the Bogoliubov coefficients. In order to do so, instead of computing the Bogoliubov coefficients using the inner product, we use the linear transformation properties of hypergeometric functions [41] to obtain the relation

$$\Phi_k^{\text{in}} = \alpha_k \Phi_k^{\text{out}}(\eta, x) + \beta_k \Phi_{-k}^{\text{out}*}(\eta, x), \tag{3.45}$$

where

$$\alpha_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2} \frac{\Gamma(1-\mathrm{i}\omega_{\text{in}}/\rho)\,\Gamma(-\mathrm{i}\omega_{\text{out}}/\rho)}{\Gamma(-\mathrm{i}\omega_{+}/\rho)\,\Gamma(1-\mathrm{i}\omega_{+}/\rho)},$$
$$\beta_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2} \frac{\Gamma(1-\mathrm{i}\omega_{\text{in}}/\rho)\,\Gamma(\mathrm{i}\omega_{\text{out}}/\rho)}{\Gamma(\mathrm{i}\omega_{-}/\rho)\,\Gamma(1+\mathrm{i}\omega_{-}/\rho)}. \tag{3.46}$$

Here Γ are gamma functions. From (3.22) and (3.45), we see that the Bogoliubov coefficients are given in terms of the quantities in (3.46) such that

$$\alpha_{kk'} = \alpha_k \delta_{kk'}, \qquad \beta_{kk'} = \beta_k \delta_{-kk'}. \tag{3.47}$$

Consider that in the in-region, the quantum field is in the vacuum state. Then from (3.26) and (3.46), we can compute the mean number of particles in mode k in the out-region

$$\mathcal{N}_k = |\beta_k|^2 = \frac{\sinh^2(\pi\omega_{-}/\rho)}{\sinh(\pi\omega_{\text{in}}/\rho)\sinh(\pi\omega_{\text{out}}/\rho)}. \tag{3.48}$$

From this toy model, we can get some insights:

- The only non-vanishing α-coefficients are the diagonal ones. Therefore, there is no mode-mixing. Moreover, the only non-vanishing β-coefficients are those of modes with the same frequency but opposite momentum.

- The vacuum state in the in-region is not the vacuum state in the out-region. Therefore, inertial observers in the in-region would see the vacuum state while unaccelerated observers in the far future would detect quanta in the same state.

- Substituting (3.42) in (3.48), we see that if $B = 0$ or $m = 0$, then the number of particles created will vanish. Therefore, particle creation can be interpreted as a consequence of the coupling of the spacetime expansion to the quantum field via the mass. The spacetime expansion changes the gravitational field and this change provides energy to the modes.

- It is convenient to change to the conformal time to solve the Klein-Gordon equation through separation of variables. Furthermore, to explicitly define the spacetime expansion helped to obtain the Bogoliubov coefficients between the asymptotic regions using some special properties of the hypergeometric functions.

In general, computing Bogoliubov coefficients is not an easy task. Other models of cosmological particle creation, as well as other problems in QFTCS, make use of different mathematical techniques and simplifications to make the computations manageable. Particularly, this thesis is focused in constructing mathematical techniques for confined quantum fields.

3.5 Non-uniform motion of cavities

In this section, we outline the mathematical model for computing Bogoliubov coefficients introduced in [42]. In this model, the authors considered a quantum field confined in a cavity in flat spacetime which undergoes non-uniform acceleration. The first approach was introduced in [43], where the cavity is initially inertial, then travels with uniform acceleration for some period and ends inertial.

Consider a massless scalar field in a $(1+1)$-dimensional rigid cavity of length L in Minkowski spacetime. The boundaries of the cavity are located at x_l (left) and x_r (right). We impose Dirichlet vanishing boundary conditions on the scalar field at the boundaries, so that when the cavity is inertial

$$\Phi\left(t, x_l\right) = \Phi\left(t, x_r\right) = 0. \tag{3.49}$$

In this case, the mode solutions are given by

$$\Phi_n\left(t,x\right)=\frac{1}{\sqrt{\omega_n L}}\sin\left(\frac{n\pi}{L}\left(x-x_l\right)\right)e^{\mathrm{i}\omega_n t}, \tag{3.50}$$

where

$$\omega_n=\sqrt{\left(\frac{n\pi}{L}\right)^2+m^2}. \tag{3.51}$$

The field can be expanded as in (3.15).

Let us now consider a uniformly accelerated cavity. In this case, the natural coordinates to use are the so-called *Rindler coordinates* (η,χ), see [44, 45], which are defined as

$$t=\tfrac{e^{A\chi}}{A}\sinh\left(A\eta\right), \qquad x=\tfrac{e^{A\chi}}{A}\cosh\left(A\eta\right), \tag{3.52}$$

where $0<\chi<\infty$, $-\infty<\eta<\infty$ and A is a transformation parameter with dimensions of acceleration. These coordinates are valid in the Minkowski region $0<x<\infty$, $-x<t<x$, which is called the *right Rindler wedge*. An observer who is stationary with respect to Rindler coordinates is uniformly accelerated with proper acceleration

$$a_p=Ae^{-A\chi}. \tag{3.53}$$

Inverting the relations in (3.52), we obtain

$$\eta=\tfrac{1}{A}\mathrm{arctanh}\left(t/x\right), \qquad \chi=\tfrac{1}{A}\ln\left(A\sqrt{x^2-t^2}\right). \tag{3.54}$$

Therefore, the line element in these coordinates is written as

$$\mathrm{d}s^2=e^{2A\chi}\left(-\mathrm{d}\eta^2+\mathrm{d}\chi^2\right). \tag{3.55}$$

Thus, the solutions to (3.10) are

$$\tilde{\Phi}_m\left(\eta,\chi\right)=\frac{1}{\sqrt{\Omega_m L'}}\sin\left(\frac{\pi m}{L'}\left(\chi-\chi_l\right)\right)e^{-\mathrm{i}\Omega_m\eta}, \tag{3.56}$$

where

$$\Omega_m=\sqrt{\left(\tfrac{m\pi}{L'}\right)^2+m^2}, \qquad L'=\chi_r-\chi_l=\tfrac{1}{A}\ln\left(\tfrac{x_r}{x_l}\right). \tag{3.57}$$

Let us consider that the cavity starts at rest, and at $t=0$ it suddenly accelerates.

We introduce the parameter

$$h = a_c L, \tag{3.58}$$

where a_c is the proper acceleration (3.53) at the center of the cavity and L is the length of the cavity in its rest frame. Expressing (3.53) in (t, x) coordinates, we have that $a_p = \frac{1}{x}$, such that the proper acceleration at the center of the cavity is given by

$$a_c = \frac{1}{x_l + \frac{L}{2}}. \tag{3.59}$$

Thus, the cavity boundaries are

$$x_l = \left(\frac{1}{h} - \frac{L}{2}\right) L, \qquad x_r = \left(\frac{1}{a_c} + \frac{L}{2}\right) L. \tag{3.60}$$

Substituting (3.60) in (3.57), we see that $0 < h < 2$, in order that the proper acceleration at both ends of the cavity is finite. Expressing the Rindler mode functions (3.56) in terms of Minkowski coordinates using (3.54), we can compute the Bogoliubov coefficients ((3.24)-(3.25)) for the transformation from the Minkowski to the Rindler frame and encode them in matrix form as in (3.29), such that

$${}_0K = \begin{pmatrix} {}_0\alpha & {}_0\beta \\ {}_0\beta^* & {}_0\alpha^* \end{pmatrix}. \tag{3.61}$$

For the computation of these Bogoliubov coefficients, it is convenient to consider small accelerations, such that it is possible to expand the integrands of Eqs. (3.24) and (3.25) in a power series around $h = 0$.

So far, we have the method for computing the Bogoliubov coefficients for a uniformly accelerating cavity, but we are interested in a non-uniform accelerated motion of the cavity. The key to construct more general cavity trajectories lies in the *basic building block* (BBB), which consists of the transformation from an inertial cavity, back to an inertial cavity, with an intermediate period of uniform acceleration. In terms of Bogoliubov transformations, the BBB is a composition of matrices such that

$${}_B\mathcal{U} = {}_0K^{-1}\tilde{\mathcal{G}}\left(\eta_1\right)\, {}_0K, \tag{3.62}$$

where $\tilde{\mathcal{G}}\left(\eta_1\right) = \mathcal{G}\left(\eta_1\right) \oplus \mathcal{G}^*\left(\eta_1\right)$ and $\mathcal{G} = \mathrm{diag}\left\{e^{\mathrm{i}\Omega_1\eta}, e^{\mathrm{i}\Omega_2\eta}, \cdots\right\}$. Here the transformation from the inertial region to the uniformly accelerated one is given by (3.61). The intermediate region of uniform acceleration, which goes from $t = 0$ to Rindler coordinate time $\eta = \eta_1$, is characterised by phase factors and the transformation back to an inertial region at $\eta = \eta_1$ is given by the inverse transformation of (3.61). In this sense, one can construct more general travel scenarios by composing BBBs, i.e., by pasting together segments of inertial evolution and uniform acceleration. It is important to notice that two BBBs can be connected by an intermediate inertial period of proper time τ. The inertial period is represented by a matrix $G(\tau)$, which is composed as $\tilde{\mathcal{G}}$ in (3.62). Therefore, the Bogoliubov transformation for a generic travel scenario that connects two inertial regions with n BBBs is written as

$$U = {}_B\mathcal{U}_n G(\tau_{n-1})\, {}_B\mathcal{U}_{n-1} \cdots G(\tau_2)\, {}_B\mathcal{U}_2 G(\tau_1)\, {}_B\mathcal{U}_1. \tag{3.63}$$

Until here, the method has achieved the computation of Bogoliubov transformations for a finite and sharp number of jumps between inertial and accelerated motions of the cavity field modes. However, in [42] the authors achieved a method to study the effects of smoothly varying accelerations. This was attained by constructing the "continuous" Bogoliubov transformation as a limit of (3.63). Since the acceleration is now arbitrary, it is possible to remove the inertial segments, such that

$$U(\tau, \tau_0) = \lim_{\substack{N\to\infty \\ \tilde{\tau}_k \to 0}} {}_B\mathcal{U}(\tilde{\tau}_N, h_N)\, {}_B\mathcal{U}(\tilde{\tau}_{N-1}, h_{N-1}) \cdots {}_B\mathcal{U}(\tilde{\tau}_1, h_1), \tag{3.64}$$

where h is the parameter defined in (3.58) and the total time is given by

$$\sum_{k=1}^{N} \tilde{\tau}_k = \tau - \tau_0. \tag{3.65}$$

Here τ and τ_0 are proper times in the inertial frames, while $\tilde{\tau}_k$ are proper times in uniformly accelerated frames with acceleration a_{p_k}. An infinitesimal increase in time for the Bogoliubov transformation in (3.64) can be written as

$$U(\tau + \mathrm{d}\tau, \tau_0) = {}_B\mathcal{U}(\mathrm{d}\tau, h)\, U(\tau, \tau_0), \tag{3.66}$$

where

$${}_B\mathcal{U}(\mathrm{d}\tau, h) = {}_0K^{-1}(h)\, \tilde{G}(\mathrm{d}\tau)\, {}_0K(h) = {}_0K^{-1}(h) \left(\mathbb{1} + \mathrm{i}\tilde{\Omega}(h)\, \mathrm{d}\tau\right) {}_0K(h). \tag{3.67}$$

Here, we have neglected terms of $\mathcal{O}\left(\mathrm{d}\tau^2\right)$ and we define

$$\tilde{\Omega}\left(h\right) = \begin{pmatrix} \Omega\left(h\right) & 0 \\ 0 & -\Omega\left(h\right) \end{pmatrix}, \tag{3.68}$$

with $\Omega\left(h\right) = \mathrm{diag}\left\{\Omega_1\left(h\right), \Omega_2\left(h\right), \cdots\right\}$. Deriving the transformation (3.64) and using Eqs. 3.66 and (3.67), we obtain

$$\begin{aligned}\frac{\partial}{\partial\tau}U\left(\tau,\tau_0\right) &= \lim_{\mathrm{d}\tau\to 0}\frac{U\left(\tau+\mathrm{d}\tau,\tau_0\right)-U\left(\tau,\tau_0\right)}{\mathrm{d}\tau} \\ &= \mathrm{i}\,{}_0K^{-1}\left(h\left(\tau\right)\right)\tilde{\Omega}\left(h\left(\tau\right)\right)\,{}_0K\left(h\left(\tau\right)\right)U\left(\tau,\tau_0\right).\end{aligned} \tag{3.69}$$

The solution to (3.69) is

$$U\left(\tau_f,\tau_0\right) = T\exp\left[\mathrm{i}\int_{\tau_0}^{\tau_f}{}_0K^{-1}\left(h\left(\tau\right)\right)\tilde{\Omega}\left(h\left(\tau\right)\right)\,{}_0K\left(h\left(\tau\right)\right)\mathrm{d}\tau\right], \tag{3.70}$$

where τ_f is the moment at which the acceleration finishes and T denotes the *time-ordered exponential* (see [46]). Eq. (3.70) is the Bogoliubov transformation between two inertial segments, one that ends at τ_0 and another one that starts at τ_f. Some insights should be highlighted:

- The Bogoliubov transformation (3.70) is general in the sense that it allows $h\left(\tau\right)$ to vary arbitrarily during the acceleration period within the rigidity constraint. Moreover, no approximation has been made regarding acceleration.

- For any time interval where h is constant, the Bogoliubov transformation evolves by phases. Therefore, any effects of particle creation or mode-mixing are a consequence of changes in the acceleration of the cavity, not by the acceleration.

- This technique can be applied only in Minkowski spacetime. However, there are extensions to other metrics [19, 47–50], considering their conformal invariance.

The method introduced in [1, 2] can be regarded as a generalisation of the technique presented in this section.

4

Modified Gravity

It has been more than hundred years since Einstein published his theory of GR [51]. Throughout this time, many experiments have been executed proving the success of his gravitational theory, from gravitational lensing [52] to the recent detection of gravitational waves [53] (see [54] for a review). However, GR has some well known limitations, such as the cosmological constant problem, the breakdown of the equivalence principle at the singularities in black holes, the galactic observations suggesting the existence of dark matter or the accelerating expansion of the Universe implying dark energy. These issues indicate that GR is not the end of the story and we might need to go beyond it. As a consequence, many different modifications to GR have been proposed [55, 56]. Amongst these modified theories of gravity, scalar-tensor theories are some of the most studied. The reason for this is that a scalar-tensor theory is one of the simplest ways to modify gravity. Another motivation to study additional scalar fields in Nature arises from extensions to the Standard Model (SM) of particle physics. For example, the Higgs-portal theories [57, 58], where the scalar field couples to the Higgs field [59, 60]. Moreover, many theories that go beyond the SM predict the existence of scalar fields [61–63].

This chapter is organised as follows. In Sec. 4.1 we introduce some aspects of scalar-tensor theories of gravity. Subsequently, in Sec. 4.2, we explain screening mechanisms and we outline the chameleon field model.

4.1 Scalar-tensor theories of gravity

Scalar-tensor theories of gravitation study the modifications of GR due to an additional scalar field φ which is coupled to a metric tensor of gravity. In this sense, the Einstein-Hilbert action (2.8) is modified schematically as in the following example[1]

$$\tilde{S} = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\frac{M_{Pl}^2}{2}\frac{\tilde{\mathcal{R}}}{A^2(\varphi)} + \tilde{S}_m\left[\tilde{g}_{\mu\nu};\Phi_i\right] + \tilde{S}_\varphi, \tag{4.1}$$

where $M_{Pl} = [8\pi G/c\hbar]^{-1/2}$ is the reduced Planck mass[2], $A^2(\varphi)$ is a function depending on the scalar field, $\tilde{S}_m$ is the matter action and $\tilde{S}_\varphi$ comprises the kinetic and potential terms of the scalar field action. Note that the scalar field φ only couples to the gravitational sector through the function A, but there is no direct coupling to the matter fields, i.e., in the form of a term proportional to $\varphi\Phi$.

4.1.1 Conformal transformations

The coupling between the scalar field φ and the metric tensor is usually performed through a *conformal transformation*, which is defined as

$$\tilde{g}_{\mu\nu} = A^2(\varphi(\mathbf{x}))\, g_{\mu\nu}, \tag{4.2}$$

where A^2 is some arbitrary function of the spacetime coordinate $\mathbf{x}$ known as the *conformal factor*. It is important to mention that conformal transformations are different from general coordinate transformations. The transformation applied to the line element (2.1) transforms as $\mathrm{d}\tilde{s}^2 = A^2\mathrm{d}s^2$ under a conformal transformation whereas it is invariant under general coordinate transformations. In other words, a conformal transformation changes the line element at the same rate. In the case of scalar-tensor theories, the conformal factor depends on the scalar field φ. Thus, scalar-tensor theories of gravity are defined up to a conformal transformation leading from one *conformal frame* to another[3].

Since conformal frames are simply different mathematical formulations, the theo-

[1] This is not the most general example of a scalar-tensor action. For a more general form, refer to [64].

[2] Recall that we use natural units $\hbar = c = 1$.

[3] There are more general ways to couple the scalar field and the metric tensor. For example, the so-called disformal coupling [65].

retical prediction for an observable quantity is identical in different conformal frames. However, the interpretation of the physics can be different. In order to explain the differences between conformal frames, we will consider two popular conformal frames, namely, the Jordan and Einstein frame.

4.1.2 Einstein and Jordan frames

Following [66], a conformal transformation from the Jordan to the Einstein frame is

$$\tilde{g}_{\mu\nu} = A^2(\varphi)\, g_{\mu\nu}, \tag{4.3}$$

and

$$\sqrt{-\tilde{g}} = A^4(\varphi)\sqrt{-g}. \tag{4.4}$$

Therefore, the action (4.1) is transformed as

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{Pl}^2}{2}\mathcal{R} + S_m\left[A^2(\varphi)\, g_{\mu\nu}; \Phi_i\right] + S_\varphi. \tag{4.5}$$

In this case, note that the scalar field φ does not couple to the gravitational section but there is a direct coupling to the matter fields. Thus, the Jordan frame action in (4.1) and the Einstein frame action in (4.5) lead to different physical interpretations. For example, test particles in the Jordan frame follow geodesics of $\tilde{g}$ which are different from those predicted in GR, since the scalar field couples to the gravitational sector. However, test particles in the Einstein frame would follow GR's geodesics (since the gravitational section is not coupled to φ) but are also subject to a *fifth force* of Nature carried by the additional scalar field and, consequently, deviating the particles from the geodesics. In other words, in the Jordan frame gravity is modified, while in the Einstein frame gravity is unchanged but a new interaction is introduced. It is important to recall that even though the physical interpretation is different, in both cases the physical measured trajectory of a test particle will be the same, since conformal frames are nothing else than different mathematical formulations[4].

[4] It is still a topic of research whether this is also true at the quantum level, see [67–70] and references therein.

4.2 Screened scalar fields models

Conformally coupled scalar-tensor theories of gravity introduce a fifth force of Nature mediated by the additional scalar field in the Einstein frame. The problem is that we have not yet seen any evidence of this extra fifth force and such forces are tightly constrained in our Solar System [71, 72]. In order to produce interesting effects on cosmological scales while fulfilling the known constraints in our Solar System, these scalar fields should be hidden in our local environment. For this purpose, so-called *screening mechanisms* have been proposed. There are different types of screening mechanisms, but their common ground is the fact that the local matter density is many orders of magnitude larger than the mean density in cosmological scales. It is possible to distribute the most studied models in the following categories [73]

- Weak coupling: the coupling to matter depends on the environment, so that the scalar couples with gravitational strength in low-density regions and turns weaker the denser the region becomes. In this category we find the symmetron [74–76] and environment-dependent dilaton [77–81] models.

- Large inertia: the kinetic term in the action S_φ of (4.5) depends on the environment, so that the kinetics of the field is suppressed in dense regions. This is known as the *Vainshtein screening* [82]. Examples are the galileon models [83, 84].

- Large mass: the effective mass of the scalar field depends on the environment, so that it increases in regions of high density causing the fifth force to be short-ranged. This is known as the *chameleon mechanism* [85, 86] and is the main focus of [3].

A screening mechanism would allow additional scalar fields to contribute to dark energy circumventing current experimental constraints on fifth forces [87–89]. Furthermore, some screened scalar field fifth forces could be potential alternatives for particle dark matter [90–93]. All these works treat screened scalar fields as classical fields, including [3]. However, there have been initial discussions of screened scalar fields as quantum fields [94–97].

4.2.1 Chameleon field models

As its name suggests, the chameleon field adapts depending on the local environment and becomes almost impossible to detect in regions of high matter density. The action in the Einstein frame (4.5) of the chameleon theories is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{Pl}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) + S_m\left[A^2(\varphi)\, g_{\mu\nu};\Phi_i\right], \tag{4.6}$$

where $V(\varphi)$ is the chameleon potential. Varying the action with respect to the scalar field, we obtain the equation of motion

$$\Box\varphi = V_{,\varphi} - \frac{A_{,\varphi}}{A(\varphi)}g^{\mu\nu}T_{\mu\nu}, \tag{4.7}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{4.8}$$

is the stress-energy tensor. For non-relativistic matter, one has $g^{\mu\nu}T_{\mu\nu} = -\rho \approx -\tilde{\rho} \approx \tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu}$, where $\tilde{}$ refers to quantities in the Jordan frame, while the other quantities are from the Einstein frame [20]. Thus, the equation of motion is

$$\Box\varphi = V_{,\varphi} - \frac{\mathrm{d}\ln A}{\mathrm{d}\varphi}\rho. \tag{4.9}$$

From the right hand side of (4.9), we see that the dynamics of the chameleon field φ are not solely governed by $V(\varphi)$, but rather by an *effective potential*

$$V_{\text{eff}}(\varphi) = V(\varphi) + \ln A(\varphi)\rho. \tag{4.10}$$

To obtain a chameleon field model, the potential $V(\varphi)$ and coupling factor $A^2(\varphi)$ in (4.6), should be chosen so that the effective potential can develop a minimum at some finite value φ_{min}. This condition may be satisfied by the following choices [98, 99]

$$V(\varphi) = \frac{\Lambda^{4+N}}{\varphi^N}, \tag{4.11}$$

$$A^2(\varphi) = e^{2\varphi/M}, \tag{4.12}$$

where $N \in \mathbb{Z}^+ \cup 2\mathbb{Z}^- \backslash \{-2\}$ distinguishes between different chameleon models, the parameter Λ determines the strength of the self-interaction and M is a mass scale determining the strength of the chameleon-matter coupling. The model $N = -2$ is not valid since it has only a constant mass $m_\varphi \sim \Lambda$, and the cases where $N \in 2\mathbb{Z}^- - 1$ are not valid since they lead to imaginary vacuum expectation values. Thus, in an environment of homogeneous density ρ, the effective potential (4.10) has a minimum at

$$\varphi_{\rm min} = \left(N \Lambda^{4+N} \frac{M}{\rho} \right)^{\frac{1}{N+1}}, \tag{4.13}$$

and consequently a chameleon effective mass

$$m^2_{\rm eff}(\varphi) = \frac{\mathrm{d}^2 V_{\rm eff}}{\mathrm{d}\varphi^2} = \frac{N(N+1)\Lambda^{4+N}}{\varphi^{N+2}_{\rm min}}. \tag{4.14}$$

Note that due to $\varphi_{\rm min}$, the chameleon mass depends on the environmental density ρ. Moreover, the effective mass (4.14) increases with ρ, regardless of the chameleon model. For the chameleon model with $N = -4$, the potential (4.11) takes the form

$$V(\varphi) = -\frac{\lambda}{4!}\varphi^4, \tag{4.15}$$

where $\lambda = (\Lambda/\Lambda_{DE})^4$ and $\Lambda_{DE} = 2.4$meV. Hence, the minimum of the effective potential (4.10) is

$$\varphi_{\rm min} = -\left(\frac{6\rho}{\lambda M} \right)^{1/3}. \tag{4.16}$$

A fifth force denoted as *chameleon force* on a test particle of mass m is given by [100]

$$\mathbf{F}_\varphi = -m \nabla \ln A(\varphi) = -m \frac{\nabla \varphi}{M}. \tag{4.17}$$

The chameleon force usually has a Yukawa-like suppression, as will be seen in the following example. Thus, its range is shorter the larger the chameleon mass (4.14) is. Consequently, in dense environments, the chameleon force is screened. In addition, notice that this force scales with M.

4.2.1.1 Profile of a spherical object

Let us illustrate the chameleon model with an example. Consider a static spherically symmetric source of radius R and density ρ_{obj} immersed in a homogeneous medium of density ρ_{bg}. From (4.9), the equation of motion is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) = -n\frac{\Lambda^{4+N}}{\varphi^{N+1}} + \frac{\rho}{M}. \tag{4.18}$$

The field profile outside of the source is [85]

$$\varphi = \varphi_{\mathrm{bg}} - \frac{R}{r}(\varphi_{\mathrm{bg}} - \varphi_{\mathrm{obj}})\,\mathrm{e}^{-m_{\mathrm{bg}}r}, \tag{4.19}$$

where φ_{bg} is the potential minimum value of the chameleon field outside the source, φ_{obj} is the potential minimum value of the chameleon field inside the source and m_{bg} is the chameleon mass (4.14) outside the source. The chameleon fifth force (4.17) on a test particle is

$$F_{\varphi} \sim \partial_r\varphi(r) \sim \exp(-m_{\mathrm{bg}}r)/r. \tag{4.20}$$

Note that the force is of shorter range the heavier the chameleon is, which turns it into a weaker force, i.e. screened. If the source is screened, then φ_{obj} is the minimum of the chameleon field within the source except for a thin shell near the surface. This is due to the short range of the fifth force inside a large object, which allows only a thin layer of matter on the surface of this object to source the chameleon force and is known as the *thin-shell effect*.

In [3] (reprinted below), we used an approximate solution of Eq. (4.18), namely, the consideration that the field profile is still within an ambient Compton wavelength $r < m_{\mathrm{bg}}^{-1}$. Thus the solution is the same as in (4.19), without the exponential function.

Part II

Publications

5

Evolution of confined quantum scalar fields in curved spacetime. Part I

Own contribution: Luis C. Barbado and I were looking for a general form to describe the evolution of quantum fields using the framework of QFTCS in a separate manner, both supervised by Ivette Fuentes. The original idea for the construction of the method was of Luis C. Barbado. However, we equally contributed to the development of the method. The results of both applications of the method were derived by me with input of Luis C. Barbado, David Bruschi and Ivette Fuentes. The publication draft was written by me and Luis C. Barbado. All authors contributed to correcting the draft and responding to review comments.

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Evolution of confined quantum scalar fields in curved spacetime. Part I

Spacetimes without boundaries or with static boundaries in a synchronous gauge

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Abstract We develop a method for computing the Bogoliubov transformation experienced by a confined quantum scalar field in a globally hyperbolic spacetime, due to the changes in the geometry and/or the confining boundaries. The method constructs a basis of modes of the field associated to each Cauchy hypersurface, by means of an eigenvalue problem posed in the hypersurface. The Bogoliubov transformation between bases associated to different times can be computed through a differential equation, which coefficients have simple expressions in terms of the solutions to the eigenvalue problem. This transformation can be interpreted physically when it connects two regions of the spacetime where the metric is static. Conceptually, the method is a generalisation of Parker's early work on cosmological particle creation. It proves especially useful in the regime of small perturbations, where it allows one to easily make quantitative predictions on the amplitude of the resonances of the field, providing an important tool in the growing research area of confined quantum fields in table-top experiments. We give examples within the perturbative regime (gravitational waves) and the non-perturbative regime (cosmological particle creation). This is the first of two articles introducing the method, dedicated to spacetimes without boundaries or which boundaries remain static in some synchronous gauge.

1 Introduction

Quantum field theory in curved spacetime is the theory that studies the evolution of quantum fields which propagate in a classical general relativistic background geometry. To this date, the theory has been able to yield many quantitative theoretical predictions for physically relevant problems within its scope, such as cosmological particle creation [1,2], the Unruh effect [3] or Hawking radiation [4]. The success in the study of those concrete well-known problems can be related to the use of different mathematical techniques and simplifications, adapted to the specific problem and which yield the computations tractable. For instance, in the study of phenomena such as Hawking radiation or the Unruh effect, the presence of horizons (or "would-be horizons" [5,6]) allows to simplify the computations leading to a thermal spectrum for the quantum field, at least within most of the usual approaches to these phenomena. Also, the presence of symmetries like homogeneity or isotropy is convenient to obtain quantitative results on particle creation in cosmological models.

In the recent years, the interest to verify the theory experimentally has increased significantly [7–13]. Thus far, an analogous of Hawking radiation in Bose-Einstein condensates has already been experimentally demonstrated [14] (although the result is not free of controversy, see for example [15]). In particular, and due to the great improvement of the precision of quantum measurements in table-top experiments, it is expected that the theoretical predictions on quantum fields confined in cavities, and under the effect of small changes in the background geometry or the non-inertial motion of the cavity, will be tested experimentally in the near future [16–19]. Consequently, new open problems within the framework of the theory are arising, related to the new concrete systems under study, and therefore new adapted mathematical techniques will be necessary to approach them.

In this work, we introduce a general method for computing the evolution of a confined quantum scalar field in a globally hyperbolic spacetime, by means of a time-dependent Bogoliubov transformation. As it will be clearly shown along the

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article, the method proves to be especially useful to address the new kind of problems that we just mentioned, related to quantum fields confined in cavities and undergoing small perturbations, since it provides very simple recipes for computing the resonance spectrum and sensibility of the field to a given perturbation of the background metric or the boundary conditions. However, we wish to emphasise that the method can be applied to any confined quantum scalar field, both real or complex, evolving as a result of any kind of regular changes in the background metric and/or the boundary conditions, just under some minor assumptions. This generality will also be illustrated through a concrete example of an application of the method outside the scope of small perturbations (in cosmological particle creation) that will be considered in the article.

The method is based on the foliation of the spacetime in spacelike Cauchy hypersurfaces using a time coordinate. Since the method is intended for confined fields, these hypersurfaces will be compact, although not necessarily with boundaries. The core idea is to construct a basis of modes naturally associated to each hypersurface, and then provide a differential equation in time for the Bogoliubov transformation between the modes associated to two hypersurfaces at different times. This way, the evolution of the field in time is not obtained by solving the Klein-Gordon equation, but rather by solving a differential equation for the time-dependent Bogoliubov transformation.

The core idea of transferring the dynamics from the mode functions to the Bogoliubov coefficients appears for the first time in the pioneer work by Parker [2]. In this sense, our work is a conceptual extension of Parker's work to the general case of any confined quantum scalar field (with minor assumptions). However, the key mathematical construction has been developed as a generalisation of the technique introduced in [20]. That previous technique, although with the strong limitation of being applicable only in Minkowski spacetime (the influence in the evolution of the field coming from the motion of the cavity), has already proven to be of special practical use in the study of quantum fields confined in cavities. Benefiting from conformal invariance, extensions to a few other metrics were also considered [17,21–24]. Yet, the scope of that method was notoriously reduced, something which the present work overcomes.

We want to emphasise that our way of proceeding makes the method completely different to the Hamiltonian diagonalisation approach introduced for the problem of cosmological particle creation [25,26]. In such approach, the construction of bases of modes associated to each time is used to diagonalise the Hamiltonian of the quantum field by a time-dependent Bogoliubov transformation in the Fock representation, such that the Hamiltonian is diagonal in terms of particle operators at every moment; and then this diagonalised Hamiltonian is used to compute the evolution of a state in time directly in the Fock representation. Therefore, this approach introduces a different notion of particles *at each instant of time.* Problems with this approach appear because the Fock representations for different times can be unitarily inequivalent [27–29]; that is, because the particle interpretation in this framework is in general wrong. Conversely, the method introduced in this article does not intend to quantise the field with a Fock representation at all times during its evolution. The construction of the modes at each time will be just an intermediate mathematical tool used to compute a time-dependent Bogoliubov transformation. Such Bogoliubov transformation has the usual physical interpretation in terms of effects on quantum particles (mode-mixing and particle creation) only when it relates regions of spacetime where the Fock quantisation has a valid physical interpretation in terms of particles, in particular due to the presence of a timelike Killing field. The Bogoliubov transformation then unambiguously describes the evolution of the field between the two regions, since it allows to compute the scattering matrix between the bases of the two different Fock quantisations [29–31]. We provide strong physical arguments to explain why the Fock representations that we consider are unitary equivalent and physically meaningful.

Following the above discussion, we take the opportunity to stress that, for the quantisation of the field, we simply rely on canonical Fock quantisation in the regions where it is clearly physically meaningful. In other words, the method is of general *practical* applicability to compute the scattering matrix between two physically valid Fock representations of the field, and it is indeed an extremely powerful tool for such purpose; while, on the other hand, we have no intention of providing a fully consistent field quantisation in any region of a general spacetime (see the approaches to such problem in e.g. [32–36]).

This article constitutes the first of two articles introducing the method. In this first part we consider spacetimes without boundaries, or with timelike boundaries that remain static in some synchronous gauge, as explained in the next section. Spacetimes with timelike boundaries which do not remain static in any synchronous gauge require an specific treatment, which due to its extension we leave for a second forthcoming article. The present article is organised as follows. In Sect. 2 we state the general physical problem for which we will construct the method, introducing the background metric, the field theory and the different assumptions that we consider; and also define two important mathematical objects that we will use. The nuclear part of the article is Sect. 3. In this section we construct the basis of modes associated to each hypersurface of the foliation of the spacetime, and formally compute the time-dependent Bogoliubov transformation between the modes of two different hypersurfaces, for which we give a differential equation and a formal solution. We also discuss the physical meaning of both the modes and the transformations. In Sect. 4 we consider the particularly important case of small perturbations and resonances, obtaining especially simple recipes for its solution. As an example of application, we consider the problem of a confined quantum field in the presence of a gravitational wave. In Sect. 5 we apply the method to an example of cosmological particle creation. Finally, in Sect. 6 we present the summary and conclusions. In addition, in Appendix A we discuss in detail the different conditions that we require for the method to work. In Appendix B we describe how the method handles the problems in which modes with exponential time dependence or zero-frequency modes are present. In Appendix C we prove that the bases of solutions to the Klein-Gordon equation on which we build our method are complete and orthonormal. In Appendix D we provide the detailed computation of the differential equation provided in Sect. 3, and prove that the Bogoliubov transformations obtained satisfy the Bogoliubov identities. In Appendix E we further support and extend the results obtained on small perturbations. In Appendix F we provide for convenience a summary of the useful formulae for the application of the method.

2 Preliminaries

2.1 Statement of the problem

We consider a globally hyperbolic spacetime (M, g) of dimension $N+1$, possibly with timelike boundary ∂M [37]. In this spacetime we define a Klein-Gordon scalar field Φ satisfying the equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - m^2\Phi - \xi R\Phi = 0; \qquad (1)$$

where $m \geq 0$ is the rest mass of the field, $g^{\mu\nu}$ is the spacetime metric, R its scalar curvature and $\xi \in \mathbb{R}$ is a coupling constant (we use natural units $\hbar = c = 1$). In the case of the existence of boundaries, we impose one of the following four possible boundary conditions to the field:

(a) Dirichlet vanishing boundary conditions

$$\Phi(x) = 0, \quad x \in \partial M. \qquad (2)$$

(b) Neumann vanishing boundary conditions

$$n^\mu\nabla_\mu\Phi(x) = 0, \quad x \in \partial M; \qquad (3)$$

where $n^\mu(x)$ is the normal vector to ∂M.

(c) Robin vanishing boundary conditions

$$n^\mu\nabla_\mu\Phi(x) + \gamma\Phi(x) = 0, \quad x \in \partial M; \qquad (4)$$

where $\gamma \in \mathbb{R} \setminus \{0\}$ is the boundary parameter. Later on, we will introduce the possibility that this parameter is replaced by a function.

(d) Mixed vanishing boundary conditions

In this case, Dirichlet, Neumann and Robin vanishing boundary conditions apply each to complementary regions of the boundary. Being just a combination of the other cases, we mention them here as a possibility, but for simplicity we will not consider them explicitly when constructing the method.

The types of boundary conditions considered are by far the most common ones in physical problems. We do not discard that the method could also apply to other less used conditions, but we simply do not approach such possibility within this work.

Thanks to the global hyperbolicity, it is always possible to construct a Cauchy temporal function t in the full spacetime [37], which provides a foliation in Cauchy hypersurfaces Σ_t of constant time. We introduce the Klein–Gordon inner product between two solutions of (1), given by

$$\langle\Phi', \Phi\rangle := -\mathrm{i}\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left[\Phi'(\tilde{t})\, \partial_t\Phi(t)^*\big|_{t=\tilde{t}} - \Phi(\tilde{t})^*\, \partial_t\Phi'(t)\big|_{t=\tilde{t}}\right]; \qquad (5)$$

which, for convenience, we already evaluated at a given Cauchy hypersurface $\Sigma_{\tilde{t}}$, with $\mathrm{d}V_{\tilde{t}}$ being its volume element. Both in the absence of boundaries, or under the boundary conditions (2–4), this inner product is independent of $\Sigma_{\tilde{t}}$.

Finally, we introduce the three conditions on the Cauchy hypersurfaces and the temporal function that we need to ensure the applicability of the method. These conditions are:

(A) The Cauchy hypersurfaces Σ_t must be compact.

(B) For any Cauchy hypersurface Σ_t, the Cauchy problem for the Klein–Gordon equation (1) must be well-posed; that is, given as initial conditions the value of the field and of its first derivative with respect to t at Σ_t, there exists an unique solution to the Klein–Gordon equation in the whole spacetime satisfying these conditions.[1]

(C) Using the temporal function as a coordinate, the metric should be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + h_{ij}(t, \mathbf{x})\mathrm{d}x^i\mathrm{d}x^j, \qquad (6)$$

where $h_{ij}(t)$ is a regular Riemannian metric.[2] This is called a synchronous gauge.

[1] Under the presence of boundaries, these initial conditions must be compatible with the boundary conditions at the intersection $\partial\Sigma_t = \Sigma_t \cap \partial M$.

[2] For simplicity in the notation, from here on we will omit the dependence of $h_{ij}(t)$ on the spatial coordinates, but we are considering the most general case in which this metric can have any (regular)

The necessity of each condition will become clear when constructing the method. A detailed discussion on their physical meaning and the limitations they introduce can be found in Appendix A. In this first article introducing the method, we are going to consider the problems for which M either has no boundary, or its boundary remains parallel to the gradient of the temporal function t, that is, it remains static in some synchronous gauge. Problems for which there exists no adequate temporal function (satisfying the required conditions) such that the boundary remains parallel to its gradient will be considered in the second article.

2.2 Space of functions over Σ_t and self-adjoint operator

Let us introduce two mathematical objects that are pivotal for the method. First, we define Γ_t as a subspace of the space of square integrable smooth functions over a Cauchy hypersurface, $\Gamma_t \subseteq C^\infty(\Sigma_t) \cap L^2(\Sigma_t)$. If M (and therefore Σ_t) has no boundary, then $\Gamma_t = C^\infty(\Sigma_t) \cap L^2(\Sigma_t)$. If there are boundaries, then Γ_t is the restriction of $C^\infty(\Sigma_t) \cap L^2(\Sigma_t)$ to functions satisfying the boundary conditions in $\partial\Sigma_t$ compatible with the boundary conditions in ∂M; which, since the boundaries are parallel to ∂_t, read

$$\Phi(\mathbf{x}) = 0 \qquad \text{(Dirichlet)}, \tag{7}$$

$$\mathbf{n} \cdot \nabla_{h(t)} \Phi(\mathbf{x}) = 0 \qquad \text{(Neumann)}, \tag{8}$$

$$\mathbf{n} \cdot \nabla_{h(t)} \Phi(\mathbf{x}) + \gamma \Phi(\mathbf{x}) = 0 \qquad \text{(Robin)}; \tag{9}$$

for $\mathbf{x} \in \partial\Sigma_t$, where $\mathbf{n}(\mathbf{x})$ is the normal vector to the boundary $\partial\Sigma_t$ and $\nabla_{h(t)}$ is the covariant derivative corresponding to the spatial metric $h_{ij}(t)$. At this point, we mention that the boundary parameter γ could be also considered a function $\gamma(\mathbf{x})$, but it must be independent of t.[3]

Second, we define the operator $\hat{\mathcal{O}}(t)$ on Γ_t as

$$\hat{\mathcal{O}}(t) := -\nabla^2_{h(t)} + \xi R^h(t) + m^2 + F(t); \tag{10}$$

where $\nabla^2_{h(t)}$ is the Laplace–Beltrami differential operator, $R^h(t)$ the scalar curvature corresponding to the spatial metric $h_{ij}(t)$ and $F(t) \geq 0$ a time-dependent non-negative quantity. The quantity $F(t)$ is given by the condition that the operator $\hat{\mathcal{O}}(t)$ at each time must be positive definite in the corresponding space Γ_t (any values of $F(t)$ that meet this requirement would be valid). Notice that, thanks to condition A and eventually to the boundary conditions, this operator is self-adjoint [with respect to the usual scalar product in $L^2(\Sigma_t)$] and has a discrete spectrum with no accumulation points.[4] Moreover, if we fixed $F(t) = 0$ the operator would always have a minimum eigenvalue. Consequently, a finite value for $F(t)$ such that the operator is positive definite can always be found. We refer to Appendix B for a discussion on the role of the quantity $F(t)$, related to the presence of exponential time dependence of the solutions and of zero-frequency modes.[5] In many problems (such as the examples provided in this article) it is possible to fix $F(t) = 0$ while keeping $\hat{\mathcal{O}}(t)$ positive definite, in which case the quantity can simply be ignored.

Once we have introduced the operator $\hat{\mathcal{O}}(t)$, and making use of condition C, we shall simplify the Klein-Gordon equation (1) taking into account the form of the metric in (6), obtaining

$$\partial_t^2 \Phi = -\hat{\mathcal{O}}(t)\Phi - q(t)\partial_t \Phi - \xi \bar{R}(t)\Phi + F(t)\Phi; \tag{11}$$

where

$$q(t) := \partial_t \log \sqrt{h(t)}, \tag{12}$$

with $h(t)$ being the determinant of the spatial metric $h_{ij}(t)$ (it is a factor which depends on the change of the metric of the spacelike hypersurfaces with time), and $\bar{R}(t) := R(t) - R^h(t)$ is the part of the full scalar curvature of $g_{\mu\nu}$ which depends on time derivatives, given by

$$\bar{R}(t) = 2\partial_t q(t) + q(t)^2 - \frac{1}{4}[\partial_t h^{ij}(t)][\partial_t h_{ij}(t)]. \tag{13}$$

The key role of condition C has been to yield Eq. (11), in which all the spatial derivatives present are those in the Laplace–Beltrami operator contained in $\hat{\mathcal{O}}(t)$.

3 Construction of the method

3.1 Construction of the bases of modes

For each Cauchy hypersurface $\Sigma_{\tilde{t}}$ we will construct a set of modes $\{\Phi_n^{[\tilde{t}]}(t)\}$ fulfilling the following two properties:

Footnote 2 continued
dependence on the coordinates. We will not write the explicit dependence on the spatial coordinates for other quantities either, except when necessary. Also, in (6) for simplicity we are assuming that for each hypersurface Σ_t there is one coordinate chart $(x^1, \ldots, x^N)$ that completely covers it. This might not be the case, but considering several coordinate charts would be straightforward and would not affect the construction of the method, as far as for all of them the metric can be written as in (6).

[3] The justification for this apparently arbitrary limitation can be found in Appendix C.

[4] Assertions about the self-adjoint nature of the operator and about the properties of its spectrum should strictly be done over the extension of the operator defined on Γ_t to the full Hilbert space $L^2(\Sigma_t)$ (of which Γ_t is a dense subspace). This is a well-known mathematical procedure in the analysis of partial differential equations with elliptic operators and boundary value problems, which is the context in which we will make use of the operator. Therefore, we shall not get into the details of it. Another possibility is to directly consider the operator as acting on a Sobolev space over the Riemannian manifold. See for example [38].

[5] The discussion in Appendix B cannot however be fully understood until reading Sect. 3.

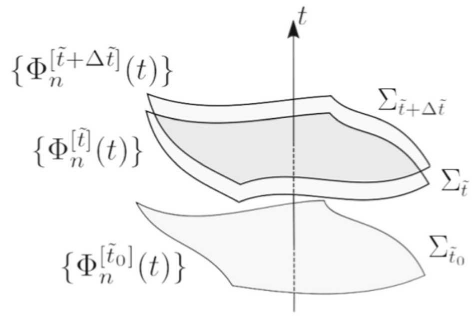

Fig. 1 Association of bases of modes $\{\Phi_n^{[\tilde t]}(t)\}$ to Cauchy hypersurfaces $\Sigma_{\tilde t}$

(I) They will form, together with their complex conjugates $\{\Phi_n^{[\tilde t]}(t)^*\}$, a complete orthonormal basis of the space of global solutions to the Klein–Gordon Eq. (1) with respect to the inner product (5). We stress that each mode of the basis is defined in the whole spacetime, the label $[\tilde t]$ meaning only that we *associate* it to the corresponding hypersurface. Specifically, it is in this hypersurface that we will set its initial conditions.

(II) In the regions where ∂_t behaves like a Killing field around the hypersurface $\Sigma_{\tilde t}$ [that is, $h_{ij}(t)$ remains constant for a long enough period around $t=\tilde t$], and where the function $F(t)$ introduced in (10) can be made to vanish, the modes will be those with positive frequency with respect to t in that region (the corresponding complex conjugates will be those with negative frequency).[6] When this is not the case, the way we choose the modes can be thought just as an unambiguous recipe to associate an orthonormal basis of solutions to each hypersurface, even if no notion of positive frequency modes exists.

In Fig. 1 we provide a graphical depiction of this association of bases of modes to Cauchy hypersurfaces, which shall be helpful when following the construction of the method. Since we impose that the modes are solutions to the Klein–Gordon equation, then because of condition B the only quantities left to fully determine them are the initial conditions; which, as mentioned before, we are going to fix at $\Sigma_{\tilde t}$. That is, we need to fix the quantities $\Phi_n^{[\tilde t]}(\tilde t)$ and $\partial_t\Phi_n^{[\tilde t]}(t)|_{t=\tilde t}$ for each mode. The evaluations of the modes at $t=\tilde t$, which will be functions in $\Gamma_{\tilde t}$, are going to be given by the eigenvectors of the operator $\hat{\mathcal{O}}(\tilde t)$ in (10):

$$\hat{\mathcal{O}}(\tilde t)\Phi_n^{[\tilde t]}(\tilde t)=(\omega_n^{[\tilde t]})^2\Phi_n^{[\tilde t]}(\tilde t). \tag{14}$$

Notice that $\tilde t$ in (14) is just a parameter: this is a partial differential equation in the spatial coordinates of the given $\Sigma_{\tilde t}$.[7] Since $\hat{\mathcal{O}}(\tilde t)$ is positive definite, we can define the real and positive quantities $\omega_n^{[\tilde t]}:=+\sqrt{(\omega_n^{[\tilde t]})^2}$ out of the eigenvalues of Eq. (14). We also impose the following orthogonality and normalisation condition to the solutions:

$$\int_{\Sigma_{\tilde t}}\mathrm{d}V_{\tilde t}\,\Phi_n^{[\tilde t]}(\tilde t)\Phi_m^{[\tilde t]}(\tilde t)^*=\frac{\delta_{nm}}{2\omega_n^{[\tilde t]}}. \tag{15}$$

Finally, the first time derivative of the modes evaluated at $t=\tilde t$ will be given by

$$\partial_t\Phi_n^{[\tilde t]}(t)\Big|_{t=\tilde t}=-\mathrm{i}\omega_n^{[\tilde t]}\Phi_n^{[\tilde t]}(\tilde t). \tag{16}$$

Equations (14–16) fully determine the initial conditions, and therefore their time evolution determines the modes $\{\Phi_n^{[\tilde t]},(\Phi_n^{[\tilde t]})^*\}$. In Appendix C we prove that this set of modes constitutes a complete orthonormal basis [in the Klein–Gordon inner product (5)] of the space of solutions to the Klein–Gordon equation (satisfying the possible boundary conditions).

We have then proven that the modes fulfil property I. Let us now consider property II. One can easily realise that, when ∂_t behaves like a Killing field in some spacetime region S around $\Sigma_{\tilde t}$, then $q(t)=\bar R(t)=0$ in S. Also, because of the time symmetry and the requisite that $F(t)=0$ the eigenvalue problem (14) will have the same solutions for all the hypersurfaces Σ_t within S. Taking into account these facts, it is easy to check that the modes of the form

$$\Phi_n^{[\tilde t]}(t)=\Phi_n^{[\tilde t]}(\tilde t)\mathrm{e}^{-\mathrm{i}\omega_n^{[\tilde t]}(t-\tilde t)} \tag{17}$$

satisfy both the initial conditions (14–16) in $\Sigma_{\tilde t}$ and the Klein–Gordon equation (11) in the whole region S. Therefore, these modes correspond to the modes that we are actually assigning to the hypersurface $\Sigma_{\tilde t}$. Now, if the interval of time that S embraces is large enough so as to explore the minimum frequency in the spectrum (now we can really call $\omega_n^{[\tilde t]}$ a frequency), then it is clear that we can talk about the modes (17) as forming an orthonormal basis of modes with well-defined positive frequency with respect to t. Property II is therefore also fulfilled.

3.2 Time-dependent Bogoliubov transformation

Let us write down the Bogoliubov coefficients between any two orthonormal bases of modes, associated to the hypersur-

[6] Regions in which $F(t)$ cannot be made to vanish, but can be made arbitrarily small, may also be considered. We refer to Appendix B for further details on this condition.

[7] As we commented previously for the operator $\hat{\mathcal{O}}(\tilde t)$, this equation should also be posed in the full Hilbert space $L^2(\Sigma_t)$. It is known, however, that the functions representing the eigenvectors can be taken to belong to the dense subspace $\Gamma_{\tilde t}$ [38].

faces $\Sigma_{\tilde{t}_0}$ and $\Sigma_{\tilde{t}}$:

$$\alpha_{nm}(\tilde{t},\tilde{t}_0) := \langle \Phi_n^{[\tilde{t}]}, \Phi_m^{[\tilde{t}_0]}\rangle, \tag{18}$$

$$\beta_{nm}(\tilde{t},\tilde{t}_0) := -\langle \Phi_n^{[\tilde{t}]}, (\Phi_m^{[\tilde{t}_0]})^*\rangle. \tag{19}$$

Since, according to condition B, all the modes are well defined globally, these coefficients exist and are unique as functions of time for all times. The objective of the method is to provide a differential equation in time that allows for their computation, without having to explicitly compute the time dependence of the modes.

For convenience, we write the Bogoliubov transformation in the matrix notation as

$$U(\tilde{t},\tilde{t}_0) := \begin{pmatrix} \alpha(\tilde{t},\tilde{t}_0) & \beta(\tilde{t},\tilde{t}_0) \\ \beta(\tilde{t},\tilde{t}_0)^* & \alpha(\tilde{t},\tilde{t}_0)^* \end{pmatrix}. \tag{20}$$

In Appendix D.1 we prove that this time-dependent Bogoliubov transformation satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} U(\tilde{t},\tilde{t}_0) = \left[\mathrm{i}\Omega(\tilde{t}) + K(\tilde{t})\right] U(\tilde{t},\tilde{t}_0); \tag{21}$$

where we define the quantities

$$\Omega(\tilde{t}) := \mathrm{diag}(\omega_1^{[\tilde{t}]}, \omega_2^{[\tilde{t}]}, \ldots, -\omega_1^{[\tilde{t}]}, -\omega_2^{[\tilde{t}]}, \ldots),$$
$$K(\tilde{t}) := \begin{pmatrix} \hat{\alpha}(\tilde{t}) & \hat{\beta}(\tilde{t}) \\ \hat{\beta}(\tilde{t})^* & \hat{\alpha}(\tilde{t})^* \end{pmatrix}; \tag{22}$$

with

$$\begin{aligned}\hat{\alpha}_{nm}(\tilde{t}) :=\ & (\omega_m^{[\tilde{t}]} + \omega_n^{[\tilde{t}]}) \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \Phi_n^{[\tilde{t}]}(\tilde{t})\right] \Phi_m^{[\tilde{t}]}(\tilde{t})^* \\ & + \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t}) \left[\omega_n^{[\tilde{t}]} q(\tilde{t}) + \mathrm{i}\xi \bar{R}(\tilde{t})\right] \Phi_m^{[\tilde{t}]}(\tilde{t})^* \\ & + \frac{\delta_{nm}}{2\omega_n^{[\tilde{t}]}} \left[\frac{\mathrm{d}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} - \mathrm{i}F(\tilde{t})\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{\beta}_{nm}(\tilde{t}) :=\ & (\omega_m^{[\tilde{t}]} - \omega_n^{[\tilde{t}]}) \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \Phi_n^{[\tilde{t}]}(\tilde{t})\right] \Phi_m^{[\tilde{t}]}(\tilde{t}) \\ & - \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t}) \left[\omega_n^{[\tilde{t}]} q(\tilde{t}) + \mathrm{i}\xi \bar{R}(\tilde{t})\right] \Phi_m^{[\tilde{t}]}(\tilde{t}) \\ & - \left[\frac{\mathrm{d}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} - \mathrm{i}F(\tilde{t})\right] \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t}) \Phi_m^{[\tilde{t}]}(\tilde{t}).\end{aligned} \tag{24}$$

With the initial condition $U(\tilde{t}_0,\tilde{t}_0) = I$, Eq. (21) has the formal solution

$$U(\tilde{t}_\mathrm{f},\tilde{t}_0) = \mathcal{T} \exp\left\{\int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t} \left[\mathrm{i}\Omega(\tilde{t}) + K(\tilde{t})\right]\right\}, \tag{25}$$

where $\mathcal{T}$ denotes time ordering.

Equations (21) and (25) are the main result of this work: They switch from the time evolution of the modes to the time evolution of the transformation between bases. We also stress that one of the strengths of the method is that all the quantities appearing in (22–24), which are the coefficients of our differential equation, are known just by constructing the orthonormal basis of modes solutions to the eigenvalue Eq. (14), for which the time $\tilde{t}$ is just a parameter.

We can get rid of the trivial phase in (21) and write the evolution in a more compact form. Apart from the phase introduced by the term $\mathrm{i}\Omega(\tilde{t})$, one can check that the diagonal elements $\hat{\alpha}_{nn}(\tilde{t})$ are purely imaginary [see relation (D.23) in Appendix D.2]. Thus, we can get rid of the full trivial phase by defining the diagonal matrix

$$\Theta(\tilde{t}) := \exp\left\{\int^{\tilde{t}} \mathrm{d}\tilde{t}'[\mathrm{i}\Omega(\tilde{t}') + A(\tilde{t}')]\right\}, \tag{26}$$
$$A(\tilde{t}) := \mathrm{diag}(\hat{\alpha}_{11}, \hat{\alpha}_{22}, \ldots, -\hat{\alpha}_{11}, -\hat{\alpha}_{22}, \ldots);$$

and writing the evolution in terms of a new operator $Q(\tilde{t},\tilde{t}_0)$ defined by

$$Q(\tilde{t},\tilde{t}_0) := \Theta(\tilde{t})^* U(\tilde{t},\tilde{t}_0). \tag{27}$$

Replacing (27) in (21), we get the differential equation [notice that $\Theta(\tilde{t})^{-1} = \Theta(\tilde{t})^*$, since $A(\tilde{t})^* = -A(\tilde{t})$]

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} Q(\tilde{t},\tilde{t}_0) = \Theta(\tilde{t})^* \bar{K}(\tilde{t}) \Theta(\tilde{t}) Q(\tilde{t},\tilde{t}_0), \tag{28}$$
$$\bar{K}(\tilde{t}) := K(\tilde{t}) - A(\tilde{t});$$

which, again with the initial condition $Q(\tilde{t}_0,\tilde{t}_0) = I$, has the formal solution

$$Q(\tilde{t}_\mathrm{f},\tilde{t}_0) = \mathcal{T} \exp\left[\int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\ \Theta(\tilde{t})^* \bar{K}(\tilde{t}) \Theta(\tilde{t})\right]. \tag{29}$$

In Appendix D.2 we check that the Bogoliubov transformations (25) and (29) obtained with this method satisfy the Bogoliubov identities.

3.3 Physical interpretation

The time-dependent Bogoliubov transformation that the method provides contains all the information necessary to compute the evolution of the field in time. However, as we advanced in the Introduction, we do not pretend to give a quantisation for each and every basis of modes that we have constructed at each time. It is only in those regions S with time symmetry, that we described at the end of the Sect. 3.1, where we can proceed to the usual Fock quantisation of the field;[8] that is, defining the corresponding Fock space with its vacuum state and creation and annihilation operators (ones the adjoints of the others in the case of a real field) associated to the mode decomposition given by the method, which in such region would be a decomposition in modes with well-defined frequency (for brevity, we do not expose this whole construction explicitly, see e.g. [29]). We rely on the fact that, under the existence of a timelike Killing vector field, Fock representation gives the correct physical description of a field in terms of particles perceived by the family of observers following the integral trajectories of the timelike Killing vector field. The "time-dependent Bogoliubov coefficients" that we constructed in the previous Subsection can be interpreted as true Bogoliubov coefficients relating the annihilation and creation operators of different mode decompositions only when they connect two regions in time where this valid quantisation procedure can be done. When this is not the case, within the scope of this article they should be considered just as an intermediate computational tool. Whether there is still some physical interpretation of the intermediate times between two static regions will be studied in future works. The Bogoliubov coefficients at those intermediate times may well be related to field observables (such as the stress-energy tensor) instead of particle observables, since those remain physically meaningful in the absence of timelike Killing fields.

A concern that can still be raised is that, even if two Fock quantisations done in two different regions S_1 and S_2, in which ∂_t is a Killing field, are both valid and physically correct (which we know to be the case), they could eventually still be unitarily inequivalent. Unfortunately, we do not have a rigorous mathematical demonstration against such possibility occurring, but we can provide a solid physical argument for the cases considered with our method, taking into account that we only consider regular metrics. The condition for unitary inequivalence in the case of two quantisations related by a Bogoliubov transformation is that $\sum_{n,m} |\beta_{nm}|^2 \to \infty$. This implies a divergent number of particles in one of the quantisations for any state of the field with a finite number of particles in the other quantisation. If the energies of the particles in each quantisation were not limited from below, a divergent number of particles could happen without a divergent amount of energy (the so-called *infrared catastrophe* [39]). But thanks to condition A and the positive definiteness of $\hat{\mathcal{O}}(t)$ a minimum non-zero energy always exists. Therefore, for the problems considered with this method unitarily inequivalent quantisations would imply that, for states with finite energy in one of the quantisations, the energy diverges according to the other quantisation. However, for physically valid quantisations this possibility is also discarded by the regularity of the metric and the compactness of the spatial hypersurfaces, since these two facts imply that only finite amounts of energy can be interchanged with the field in finite times. Therefore, for the problems considered with this method, physically valid quantisations must be also unitarily equivalent.

4 Small perturbations and resonances

Let us consider now the case in which the spatial metric $h_{ij}(t)$ only changes in time by a small perturbation around some constant metric h^0_{ij}; that is,

$$h_{ij}(t) = h^0_{ij} + \varepsilon \Delta h_{ij}(t), \tag{30}$$

where $\varepsilon \ll 1$. Since, if boundaries exist, they remain parallel to ∂_t, all the spacelike hypersurfaces will be identical, and for convenience we will call them $\Sigma^0 := \Sigma_{\tilde{t}}$. We additionally require that $F(t)$ remains $O(\varepsilon)$, so that the solutions to the problem for $\varepsilon = 0$ are modes with well-defined frequency. Later on, we will check that such value of $F(t)$ actually does not contribute at all to the physically relevant result of resonances, and thus the function $F(t)$ can be fully ignored in the small perturbations regime. Staying to first order in ε, we can write the relevant quantities for computing the Bogoliubov coefficients as

$$\begin{aligned} \Phi^{[\tilde{t}]}_n(\tilde{t}) &\approx \Phi^0_n + \varepsilon \Delta \Phi^{[\tilde{t}]}_n(\tilde{t}), & q(\tilde{t}) &\approx \varepsilon \Delta q(\tilde{t}), \\ \omega^{[\tilde{t}]}_n &\approx \omega^0_n + \varepsilon \Delta \omega^{[\tilde{t}]}_n(\tilde{t}), & \bar{R}(\tilde{t}) &\approx \varepsilon \Delta \bar{R}(\tilde{t}), \\ & & F(\tilde{t}) &\approx \varepsilon \Delta F(\tilde{t}); \end{aligned} \tag{31}$$

where Φ^0_n and ω^0_n are the modes and frequencies solution to (14) with h^0_{ij} as the metric. It is immediate that $\hat{\alpha}_{nm}(\tilde{t})$ and $\hat{\beta}_{nm}(\tilde{t})$ in (23) and (24) are $O(\varepsilon)$, since the quantities $\mathrm{d}\Phi^{[\tilde{t}]}_n(\tilde{t})/\mathrm{d}\tilde{t}$, $\mathrm{d}\omega^{[\tilde{t}]}_n/\mathrm{d}\tilde{t}$, $q(\tilde{t})$, $\bar{R}(\tilde{t})$ and $F(\tilde{t})$ are all $O(\varepsilon)$. We can then write $\hat{\alpha}_{nm}(\tilde{t}) \approx \varepsilon \Delta \hat{\alpha}_{nm}(\tilde{t})$ and $\hat{\beta}_{nm}(\tilde{t}) \approx \varepsilon \Delta \hat{\beta}_{nm}(\tilde{t})$, with

$$\begin{aligned} \Delta \hat{\alpha}_{nm}(\tilde{t}) := (\omega^0_m + \omega^0_n) \int_{\Sigma^0} \mathrm{d}V^0 \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \Delta \Phi^{[\tilde{t}]}_n(\tilde{t}) \right] (\Phi^0_m)^* \\ + \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi^0_n \left[\omega^0_n \Delta q(\tilde{t}) + \mathrm{i}\xi \Delta \bar{R}(\tilde{t}) \right] (\Phi^0_m)^* \\ + \frac{\delta_{nm}}{2\omega^0_n} \left[\frac{\mathrm{d}\Delta\omega^{[\tilde{t}]}_n}{\mathrm{d}\tilde{t}} - \mathrm{i}\Delta F(\tilde{t}) \right], \end{aligned} \tag{32}$$

$$\begin{aligned} \Delta \hat{\beta}_{nm}(\tilde{t}) := (\omega^0_m - \omega^0_n) \int_{\Sigma^0} \mathrm{d}V^0 \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \Delta \Phi^{[\tilde{t}]}_n(\tilde{t}) \right] \Phi^0_m \\ - \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi^0_n \left[\omega^0_n \Delta q(\tilde{t}) + \mathrm{i}\xi \Delta \bar{R}(\tilde{t}) \right] \Phi^0_m \\ - \left[\frac{\mathrm{d}\Delta\omega^{[\tilde{t}]}_n}{\mathrm{d}\tilde{t}} - \mathrm{i}\Delta F(\tilde{t}) \right] \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi^0_n \Phi^0_m; \end{aligned} \tag{33}$$

[8] There is also the requisite that $F(t) = 0$ to avoid negative eigenvalues (see Appendix B). For simplicity, we omit to mention this requisite explicitly along the rest of the subsection.

where $\mathrm{d}V^0$ is the volume element of the metric h^0_{ij}. Later on, we will further simplify these expressions for practical purposes.

Let us obtain the Bogoliubov coefficients to order ε in terms of these expressions. In the case of small perturbations, it is more convenient to consider the evolution as expressed by the Bogoliubov transformation $Q(\tilde{t}_\mathrm{f}, \tilde{t}_0)$ in (29). Noticing that $\bar{K}(\tilde{t})$ is $O(\varepsilon)$, we only need to consider the zeroth order in the matrix $\Theta(\tilde{t})$. To first order in ε, the transformation reads

$$Q(\tilde{t}_\mathrm{f}, \tilde{t}_0) \approx I + \varepsilon \int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\; \Theta^0(\tilde{t})^* \Delta\bar{K}(\tilde{t}) \Theta^0(\tilde{t}); \tag{34}$$

where

$$\begin{aligned}
\Theta^0(\tilde{t}) &:= \mathrm{diag}(\mathrm{e}^{\mathrm{i}\omega_1^0 \tilde{t}}, \mathrm{e}^{\mathrm{i}\omega_2^0 \tilde{t}}, \ldots, \mathrm{e}^{-\mathrm{i}\omega_1^0 \tilde{t}}, \mathrm{e}^{-\mathrm{i}\omega_2^0 \tilde{t}}, \ldots),\\
\Delta\bar{K}(\tilde{t}) &:= \begin{pmatrix} \Delta\hat{\alpha}(\tilde{t}) & \Delta\hat{\beta}(\tilde{t}) \\ \Delta\hat{\beta}(\tilde{t})^* & \Delta\hat{\alpha}(\tilde{t})^* \end{pmatrix}\\
&\quad - \mathrm{diag}(\Delta\hat{\alpha}_{11}, \Delta\hat{\alpha}_{22}, \ldots, -\Delta\hat{\alpha}_{11}, -\Delta\hat{\alpha}_{22}, \ldots).
\end{aligned} \tag{35}$$

We can show the resonance behaviour of the field in a clear way if we write explicitly the expressions for the Bogoliubov coefficients:

$$\begin{aligned}
&\alpha_{nn}(\tilde{t}_\mathrm{f}, \tilde{t}_0) \approx 1;\\
&\alpha_{nm}(\tilde{t}_\mathrm{f}, \tilde{t}_0) \approx \varepsilon \int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\; \mathrm{e}^{-\mathrm{i}(\omega_n^0 - \omega_m^0)\tilde{t}} \Delta\hat{\alpha}_{nm}(\tilde{t}), \qquad (36)\\
&\qquad n \neq m;\\
&\beta_{nm}(\tilde{t}_\mathrm{f}, \tilde{t}_0) \approx \varepsilon \int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\; \mathrm{e}^{-\mathrm{i}(\omega_n^0 + \omega_m^0)\tilde{t}} \Delta\hat{\beta}_{nm}(\tilde{t}). \qquad (37)
\end{aligned}$$

One can see that, in general, the Bogoliubov transformation will differ from the identity just by terms of first order in ε, except for the cases where there are resonances. That is, if the perturbation of the metric contains some characteristic frequency ω_p, then the same frequency will usually be also present in the quantities $\Delta\hat{\alpha}_{nm}(\tilde{t})$ and $\Delta\hat{\beta}_{nm}(\tilde{t})$; and if this frequency coincides with some difference between the frequencies of two modes, $\omega_\mathrm{p} = \omega_n^0 - \omega_m^0$ (it is in resonance), then the corresponding coefficient $\alpha_{nm}(\tilde{t}_\mathrm{f}, \tilde{t}_0)$ will grow linearly with the time difference $\tilde{t}_\mathrm{f} - \tilde{t}_0$, and after enough time it will overcome the $O(\varepsilon)$. Respectively, if the characteristic frequency coincides with some sum between the frequencies of two modes, $\omega_\mathrm{p} = \omega_n^0 + \omega_m^0$, then the corresponding coefficient $\beta_{nm}(\tilde{t}_\mathrm{f}, \tilde{t}_0)$ will grow linearly in time and eventually overcome the $O(\varepsilon)$. For completeness, in Appendix E.1 we show that the resonances remain stable under small deviations of the frequency of the perturbation from the exact resonant frequency.

If the Fourier transform $\mathcal{F}$ of $\Delta\hat{\alpha}_{nm}(\tilde{t})$ [respectively $\Delta\hat{\beta}_{nm}(\tilde{t})$] exists as a well-defined function, which necessarily implies that the perturbation vanishes fast enough in the asymptotic past and future, another way to consider the resonances is by taking the limits $\tilde{t}_0 \to -\infty$ and $\tilde{t}_\mathrm{f} \to \infty$ in (36) and (37) and writing

$$\begin{aligned}
&\alpha_{nn}(-\infty, \infty) \approx 1;\\
&\alpha_{nm}(-\infty, \infty) \approx \varepsilon\sqrt{2\pi}\; \mathcal{F}[\Delta\hat{\alpha}_{nm}](\omega_n^0 - \omega_m^0), \qquad (38)\\
&\qquad n \neq m;\\
&\beta_{nm}(-\infty, \infty) \approx \varepsilon\sqrt{2\pi}\; \mathcal{F}[\Delta\hat{\beta}_{nm}](\omega_n^0 + \omega_m^0). \qquad (39)
\end{aligned}$$

That is, the Bogoliubov coefficients between the asymptotic past and future are proportional to the Fourier transforms evaluated at the corresponding subtraction (respectively addition) of frequencies. Evidently, resonances will occur if the frequency spectrum is peaked around one or more of these values.

It is important to remark that, since there is no strict time symmetry [neither $F(t)$ vanishes in general], strictly speaking there would be no notion of particles associated to modes with well-defined frequency, except if the perturbation really vanishes in certain regions. However, even while the perturbation is ongoing, the deviation from the time symmetry is of order ε. Thus, in the resonance regime, when one or more coefficients grow beyond this order, this lack of symmetry can be neglected with a solid physical argument: The deviation from time symmetry is small, and in particular much smaller than the effects measured. Therefore, in the case of resonance one can safely say that the physical effects are taking place for particles associated to modes with well-defined frequency.

In Appendix E.2, we further support the interpretation in the previous paragraph by proving that the results on resonances do not depend on the particular choice of the basis of modes to order ε that we have used to compute the evolution. In order to give such proof, in Appendix E.2 we also prove a result that is useful at this point to further simplify the expressions in (32) and (33): Any contribution of the form

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} - \mathrm{i}(\omega_n^0 - \omega_m^0)X(\tilde{t}), \tag{40}$$

respectively

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} - \mathrm{i}(\omega_n^0 + \omega_m^0)X(\tilde{t}); \tag{41}$$

where $X(\tilde{t})$ can be any function of time, appearing in the expression of $\Delta\hat{\alpha}_{nm}(\tilde{t})$, respectively $\Delta\hat{\beta}_{nm}(\tilde{t})$, will not affect the resonance behaviour of the corresponding modes. In simple words, this happens because, if $X(\tilde{t})$ contains the correct resonant frequency in its Fourier expansion, the contribution of the corresponding term of the expansion to (40) [resp. (41)]

vanishes. We refer to Appendix E.2 for the details. Taking into account this fact, the expressions in (32) and (33) can be simplified to

$$\Delta\hat{\alpha}_{nm}(\tilde{t}) \equiv \mathrm{i}[(\omega_n^0)^2 - (\omega_m^0)^2] \int_{\Sigma^0} \mathrm{d}V^0 \, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})(\Phi_m^0)^* + \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi_n^0 \left[\omega_n^0 \Delta q(\tilde{t}) + \mathrm{i}\xi \Delta\bar{R}(\tilde{t})\right](\Phi_m^0)^*, \tag{42}$$

$$\Delta\hat{\beta}_{nm}(\tilde{t}) \equiv -\mathrm{i}[(\omega_n^0)^2 - (\omega_m^0)^2] \int_{\Sigma^0} \mathrm{d}V^0 \, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})\Phi_m^0 - \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi_n^0 \left[\omega_n^0 \Delta q(\tilde{t}) + \mathrm{i}\xi \Delta\bar{R}(\tilde{t})\right]\Phi_m^0 - [2\mathrm{i}\omega_n^0 \Delta\omega_n^{[\tilde{t}]} - \mathrm{i}\Delta F(\tilde{t})] \int_{\Sigma^0} \mathrm{d}V^0 \, \Phi_n^0 \Phi_m^0; \tag{43}$$

where the symbol '$\equiv$' denotes the equivalence relation "gives the same resonances as". Since resonances are the only physically meaningful result to be obtained from this computation, given a concrete problem one can always use these simplified expressions to compute the sensibility of the field to each resonance, by identifying the corresponding term of the Fourier expansion or value of the Fourier transform.

It is useful to notice that, although in principle we have assumed the orthonormality of the modes $\Phi_n^{[\tilde{t}]}(t)$, for using the expressions in (42) and (43) only the non-perturbed modes Φ_n^0 need to be strictly orthogonal [with respect to the usual inner product in $L^2(\Sigma^0)$] and with the correct normalisation given by (15) to zeroth order. The modes $\Phi_n^{[\tilde{t}]}(\tilde{t})$ need only to satisfy the unavoidable orthogonality already guaranteed by being correct solutions (to order ε) of the eigenvalue problem (14); that is, they will necessarily be orthogonal (to order ε) unless they have the same eigenvalue $(\omega_n^{[\tilde{t}]})^2$. It is easy to check that any change of order ε in the normalisation of the modes (or in the inner product between modes with the same eigenvalue) would not contribute to (42) or (43), because the corresponding factor $(\omega_n^0)^2 - (\omega_m^0)^2$ would make it to vanish. Therefore, when solving a specific problem, one only needs to explicitly ensure orthogonality and properly normalise the modes to zeroth order.

Finally, we give other alternative simplified formulas for the computation of $\Delta\hat{\alpha}_{nm}(\tilde{t})$ and $\Delta\hat{\beta}_{nm}(\tilde{t})$, which justification is given in Appendix E.3. These formulas have the great advantage that they only imply the modes Φ_n^0 and eigenvalues ω_n^0 of the static problem:

$$\Delta\hat{\alpha}_{nm}(\tilde{t}) \equiv \int_{\Sigma^0} \mathrm{d}V^0 \, [\hat{\Delta}(\tilde{t})\Phi_n^0](\Phi_m^0)^*, \tag{44}$$

$$\Delta\hat{\beta}_{nm}(\tilde{t}) \equiv -\int_{\Sigma^0} \mathrm{d}V^0 \, [\hat{\Delta}(\tilde{t})\Phi_n^0]\Phi_m^0; \tag{45}$$

where $\hat{\Delta}(\tilde{t})$ is a linear operator defined by its action on the basis $\{\Phi_n^0\}$ as

$$\hat{\Delta}(\tilde{t})\Phi_n^0 := \left[\mathrm{i}\Delta\hat{\mathcal{O}}(\tilde{t}) + \omega_n^0 \Delta q(\tilde{t}) + \mathrm{i}\xi \Delta\bar{R}(\tilde{t}) - \mathrm{i}\Delta F(\tilde{t})\right]\Phi_n^0, \tag{46}$$

with $\varepsilon\Delta\hat{\mathcal{O}}(\tilde{t})$ being the first order term in ε of the operator $\hat{\mathcal{O}}(\tilde{t})$. For problems in which it is easy to compute the eigenvalues and eigenfunctions to first order in ε, it might be easier to apply (42) and (43); while for other problems the expressions in (44–46) are definitely more convenient. Finally, as we already advanced, we can check in these last expressions that the quantity $\Delta F(\tilde{t})$ never contributes to the resonances, since in (46) its direct contribution to $\hat{\Delta}(\tilde{t})$ cancels with its contribution through $\Delta\hat{\mathcal{O}}(\tilde{t})$ [recall the definition in (10)].

The resonance can be consistently described in the regime of duration of the perturbation $\Delta\tilde{t}$ such that $1 \ll \omega_{\mathrm{p}}\Delta\tilde{t} \ll 1/\varepsilon$; since one needs the period of time to be reasonably larger than the inverse of the frequency being described, but on the other hand, one should keep the second order term in ε that we dropped in (34) significantly smaller than the first order term that we kept, so that the first order expansion of the exponential in (29) remains valid. Notice that, because of these limitations, the net effect after integrating in time under resonance might be of order greater than ε, but still should be small as compared to unity. However, even the effect of such small Bogoliubov transformation could in principle be measured using the adequate probe states as the initial states of the field and quantum metrology techniques (see [17,21, 40–42]). Thus, even in these situations, in which the total energy involved was too small to produce a pair of particles or to promote an existing particle to a higher energy state, this would not mean that there are no measurable effects at all. Moreover, in this work we restrict ourselves to an idealised system in which the evolution of the field is perfectly unitary. More realistic systems may also present decoherence any time quanta is created or promoted, something that can then lead to cumulative effects.

4.1 A concrete example

In order to show the method in action, we will consider an example of application within the perturbative regime. In particular, we consider a field trapped in a cavity which is perturbed by a gravitational wave. This is a very interesting problem in the growing research field of gravitational wave detection with Bose-Einstein condensates. In [21] the authors solve a similar problem by using the restricted version of the method in [20]. They introduce a simplified toy model in one spatial dimension, taking advantage then of the conformal invariance in order to "translate" the problem into an equivalent one in Minkowski spacetime. With the general method presented here, it is possible to go beyond that scheme and solve the problem in three dimensions. The three-dimensional problem was also considered in [42] with a very different approach. We will reproduce and extend the result in that work with very simple and straightforward calculations.

As in [21] and [42], we consider our field to be the phonon field of a trapped Bose-Einstein condensate, therefore a real scalar massless quantum field. Following the work in [43], one can see that, in the case that the condensate remains stationary, the phonon field experiences an effective metric (with minimal coupling) which is the gravitational wave metric with the speed of light appearing in the component g_{00} replaced by the speed of sound in the condensate c_s. This speed of sound is the one that we normalise for this problem, $c_\mathrm{s}=1$. We work in the TT-gauge and consider a wave with amplitude ε and frequency Ω propagating in the z-direction and with polarisation in the xy-directions. This yields the metric

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}t^2+[1+\varepsilon\sin(\Omega t)]\mathrm{d}x^2\\ &\quad+[1-\varepsilon\sin(\Omega t)]\mathrm{d}y^2+\mathrm{d}z^2,\end{aligned} \tag{47}$$

where we have simplified $\sin[\Omega(t-z/c)]\to\sin(\Omega t)$, as we have that $c\gg\Omega L_z$ (being L_z the size of the condensate in the z-direction), because of the orders of magnitude between the speed of light and the speed of sound. For simplicity, we consider that the condensate is trapped in a rectangular prism of lengths L_x, L_y and L_z aligned with the directions of propagation and polarisation of the wave. The boundaries are free-falling, and we impose Dirichlet vanishing boundary conditions.

For the metric in (47) and for $m=\xi=F(t)=0$ the eigenvalue Eq. (14) reads[9]

$$\begin{aligned}&\Bigg[-\frac{1}{1+\varepsilon\sin(\Omega t)}\,\partial_x^2-\frac{1}{1-\varepsilon\sin(\Omega t)}\,\partial_y^2\\ &\quad-\partial_z^2\Bigg]\Phi^{[t]}_{nm\ell}(t)=(\omega^{[t]}_{nm\ell})^2\Phi^{[t]}_{nm\ell}(t),\end{aligned} \tag{48}$$

where it is clear that we will need three quantum numbers. Since the boundaries are free-falling, the boundary conditions read $\Phi(x=0)=\Phi(x=L_x)=0$, and equivalently for the other dimensions. The solutions to order ε, with the normalisation (15) to zeroth order, are

$$\begin{aligned}\Phi^{[t]}_{nm\ell}(t) &\approx \frac{2}{\sqrt{L_xL_yL_z\omega^0_{nm\ell}}}\sin(k^x_n x)\ \sin(k^y_m y)\ \sin(k^z_\ell z)\\ &=\Phi^0_{nm\ell},\quad n,m,\ell\in\mathbb{N}^*;\end{aligned} \tag{49}$$

where $k^x_n:=\pi n/L_x$, and equivalently for the other dimensions; with the eigenvalues to order ε given by

$$\begin{aligned}\omega^{[t]}_{nm\ell} &\approx \omega^0_{nm\ell}+\varepsilon\,\frac{(k^y_m)^2-(k^x_n)^2}{2\omega^0_{nm\ell}}\sin(\Omega t),\\ \omega^0_{nm\ell} &= \sqrt{(k^x_n)^2+(k^y_m)^2+(k^z_\ell)^2}.\end{aligned} \tag{50}$$

Notice that, for this particular problem, the free-falling boundaries make the eigenvalues to depend on time while the modes are time-independent to order ε. We plug the solutions into (42) and (43), together with $\Delta q(t)=0$ computed from (12), obtaining

$$\begin{aligned}&\Delta\hat{\alpha}^{n'm'\ell'}_{nm\ell}(t)\equiv\Delta\hat{\beta}^{n'm'\ell'}_{nm\ell}(t)\equiv 0,\\ &\qquad (n',m',\ell')\neq(n,m,\ell);\end{aligned} \tag{51}$$

$$\Delta\hat{\beta}^{nm\ell}_{nm\ell}(t)\equiv \mathrm{i}\,\frac{(k^x_n)^2-(k^y_m)^2}{2\omega^0_{nm\ell}}\sin(\Omega t). \tag{52}$$

By looking at these expressions, and considering the resonance behaviour found in (37), it is straightforward to obtain the conclusions: When $\Omega=2\omega^0_{nm\ell}$, the corresponding diagonal β-coefficient will grow linearly in time due to resonance as

$$\beta^{nm\ell}_{nm\ell}(t_\mathrm{f},t_0)\approx\varepsilon\,\frac{(k^x_n)^2-(k^y_m)^2}{4\omega^0_{nm\ell}}(t_\mathrm{f}-t_0), \tag{53}$$

while the rest of the coefficients will remain trivial in any case.

In [42], the authors consider the more realistic case in which the gravitational wave lasts some approximate time τ and vanishes asymptotically, by replacing $\sin(\Omega t)\to \mathrm{e}^{-t^2/\tau^2}\sin(\Omega t)$ in the metric (47). We can easily handle this perturbation with our method by using the expression with the Fourier transform (39), obtaining (for the only non trivial coefficients)

$$\begin{aligned}\beta^{nm\ell}_{nm\ell}(-\infty,\infty) &\approx \varepsilon\sqrt{\pi}\,\frac{(k^x_n)^2-(k^y_m)^2}{4\omega^0_{nm\ell}}\,\tau\\ &\quad\times\left[\mathrm{e}^{-(\Omega-2\omega^0_{nm\ell})^2\tau^2/4}-\mathrm{e}^{-(\Omega+2\omega^0_{nm\ell})^2\tau^2/4}\right].\end{aligned} \tag{54}$$

Equation (54) exactly reproduces, through a very simple calculation (just a trivial eigenvalue problem and a trivial Fourier transform), the relation found in [42] (equation 2.21).

[9] When constructing the integration method, we have been using the different notations t and $\tilde{t}$ for time, in order to distinguish between the evolution of modes in time and the choice of a concrete basis of modes associated to a Cauchy hypersurface $\Sigma_{\tilde{t}}$. This was indeed necessary for constructing the method. But for its application to a concrete problem we will not need to explicitly consider the evolution in time t of a basis of modes anymore. Thus, we can relax the notation and replace $\tilde{t}\to t$.

When $\Omega \simeq 2\omega^0_{nm\ell}$, it reproduces the resonance behaviour for the range of durations $1/\Omega \ll \tau \ll 1/|\Omega - 2\omega^0_{nm\ell}|$, as discussed in Appendix E.1.

We can obtain some further conclusions on this simple model. For example, it is straightforward to check that imposing Neumann vanishing boundary conditions leads to the same results. One would need to temporarily introduce a small rest mass term Δm in the field equation, in order to dodge the presence of a zero-frequency mode, as described in Appendix B; but it is easy to check that taking the limit $\Delta m \to 0$ after solving the problem is in this case trivially well defined. Also, we can discuss what happens if we include the *cross-polarisation* in the metric. This polarisation would add a term proportional to $\partial_x \partial_y$ in the operator in (48). By using the expressions (44–46), it is easy to check that this does not contribute to any Bogoliubov coefficient. Therefore, for waves coming perpendicular to any of the faces, the field is sensitive only to the polarisation which is aligned with the other faces. Again by using those expressions, it would not be difficult to even consider a wave propagating in an arbitrary direction, by applying the corresponding rotation matrices with the Euler angles to the metric, but for brevity we will not approach this problem here. In future publications [44], the authors will apply the method introduced in this article to study in detail the resonance properties of this and other models of trapping cavities, in what will be a fruitful application of the method to the study of gravitational wave detection with Bose-Einstein condensates.

5 Application to cosmological particle creation

In this section, we give an example of the application of the method beyond the perturbative regime. In particular, we use it to analyse a massive, minimally coupled Klein–Gordon field in a one-dimensional FLRW flat spacetime. The line element of this spacetime is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}x^2. \tag{55}$$

For the metric in (55), where $h_{xx}(t) = a(t)^2$, and for $\xi = F(t) = 0$, the eigenvalue Eq. (14) takes the form[10]

$$\left[-\frac{1}{a(t)^2}\partial_x^2 + m^2\right] \Phi_n^{[t]}(t) = (\omega_n^{[t]})^2 \Phi_n^{[t]}(t). \tag{56}$$

As it is customary [2,29], we consider the spatial dimension to be a 1-torus, for which purpose we introduce periodic boundary conditions in space, so that $\Phi(x=0) = \Phi(x=L)$, with L being the length of the torus. The solutions to (56), with the normalisation in (15), are given by

$$\Phi_n^{[t]}(t) = \frac{1}{\sqrt{2\omega_n^{[t]}|a(t)|L}}\, \mathrm{e}^{\mathrm{i}k_n x}, \quad n \in \mathbb{Z}; \tag{57}$$

where $\omega_n^{[t]} = \sqrt{\frac{k_n^2}{a(t)^2} + m^2}$ and $k_n := \frac{2\pi n}{L}$. From (12) we get $q(t) = \dot{a}(t)/a(t)$. Computing (23) and (24) with the field modes (57) we obtain

$$\begin{aligned} &\hat{\alpha}_{nm}(t) = 0; \quad \hat{\beta}_{nm}(t) = 0, \quad m \neq -n; \\ &\hat{\beta}_{n(-n)}(t) = -\frac{\dot{a}(t) m^2}{2a(t)(\omega_n^{[t]})^2}. \end{aligned} \tag{58}$$

Observe that if the scale factor is constant, then we obtain $\hat{\beta}_{n(-n)}(t) = 0$ and the transformation is trivial. The transformation is also trivial when the scalar field is massless, which agrees with the fact that particles are created in $(1+1)$-dimensions only if a non-zero mass breaks the conformal invariance [29].

With the result obtained in (58), the differential equation (28) written directly for the Bogoliubov coefficients reads

$$\begin{aligned} \dot{\alpha}_{nm}(t, t_0) &= \mathrm{e}^{-2\mathrm{i}\int^t \mathrm{d}t' \omega_n^{[t']}} \hat{\beta}_{n(-n)}(t) \beta_{(-n)m}(t, t_0)^*, \\ \dot{\beta}_{nm}(t, t_0) &= \mathrm{e}^{-2\mathrm{i}\int^t \mathrm{d}t' \omega_n^{[t']}} \hat{\beta}_{n(-n)}(t) \alpha_{(-n)m}(t, t_0)^*. \end{aligned} \tag{59}$$

By replacing $n \to -n$ in the second equation, we realise that the system of equations can be decoupled in independent systems of two differential equations for each pair $\{\alpha_{nm}, \beta_{(-n)m}\}$. Moreover, since the equations are homogeneous and the initial conditions are $\alpha_{nm}(t_0, t_0) = \delta_{nm}$ and $\beta_{nm}(t_0, t_0) = 0$, only the systems for which $m = n$ will have a non-trivial evolution, while the rest of the coefficients will all stay equal to zero. For the non-trivial systems, we can write down the two first order differential equations more conveniently as the following second order differential equation for $\alpha_{nn}(t, t_0)$:

$$\begin{aligned} \ddot{\alpha}_{nn}(t, t_0) + \left[2\mathrm{i}\,\omega_n^{[t]} - \frac{\dot{\hat{\beta}}_{n(-n)}(t)}{\hat{\beta}_{n(-n)}(t)}\right] \dot{\alpha}_{nn}(t, t_0) \\ - \hat{\beta}_{n(-n)}(t)^2 \alpha_{nn}(t, t_0) = 0, \end{aligned} \tag{60}$$

with the initial conditions given by $\alpha_{nn}(t_0, t_0) = 1$ and $\dot{\alpha}_{nn}(t_0, t_0) \propto \beta_{(-n)n}(t_0, t_0)^* = 0$. Once computed the $\alpha_{nn}(t, t_0)$ coefficients, computing another integral would give the $\beta_{(-n)n}(t, t_0)$ coefficients. However, it is much easier to compute them (up to an unimportant phase) out of the Bogoliubov identity $\sum_m (|\alpha_{nm}|^2 - |\beta_{nm}|^2) = 1$, which in this case simplifies to

$$|\beta_{(-n)n}(t, t_0)|^2 = |\alpha_{nn}(t, t_0)|^2 - 1. \tag{61}$$

[10] As for the previous example in Sect. 4.1, we replace $\tilde{t} \to t$ in the notation.

The mathematical results up to here are completely general, there is no assumption made on the form of the scale factor. As it was already discussed, the Bogoliubov coefficients computed will only have a clear physical interpretation when they connect regions in which ∂_t is a Killing field. We already reproduce the known result [2,29] that in the problem considered there is never mode-mixing (the only non-zero α's are the diagonal ones), but only particle creation in pairs of fully entangled particles with the same frequency and equal but opposite momentum. Finally, we point out that we could easily take the limit of an unconfined quantum field ($L \to \infty$) by simply replacing the role of the quantum number n by the wave number now raised to a continuum quantity, $k_n \to k \in \mathbb{R}$; and the discrete frequencies by the corresponding dispersion relation, $\omega_n^{[t]} \to \omega^{[t]}(k)$.

5.1 A concrete example

To illustrate the previous general result, let us use it to compute numerically the Bogoliubov coefficients for a well-known toy model of cosmological expansion; in particular, the toy model appearing in [29, pp. 59–62], and which was first proposed in [45]. In this model, the expansion is explicitly written in terms of the conformal time η, defined by $\mathrm{d}\eta/\mathrm{d}t = 1/a(t)$. In this time, the scale factor is given by

$$a(\eta) = \sqrt{A + B\tanh(\rho\eta)}, \tag{62}$$

where A, B and ρ are constants. One can see that, in the asymptotic past and future ($\eta, t \to \pm\infty$), the spacetime approaches the Minkowskian flat spacetime. In [29] the Bogoliubov coefficients between the asymptotic regions $\alpha_{nn}(\infty, -\infty)$ and $\beta_{(-n)n}(\infty, -\infty)$ are obtained analytically by first solving the Klein-Gordon equation in conformal time through separation of variables.

Unfortunately, for the given expression of $a(\eta)$ we cannot obtain an explicit analytic expression for $a(t)$ (one could expect this to happen, since the toy model is prepared *ad hoc* to be solved in conformal time). Nonetheless, we can solve this problem numerically using the differential equation (60) that we have obtained. We first compute the functions appearing in the coefficients of the equation, and finally solve the differential equation itself. The asymptotic initial conditions are $\alpha_{nn}(-\infty, -\infty) = 1$ and $\dot{\alpha}_{nn}(-\infty, -\infty) = 0$. Being trivially related by (61), it is equivalent to follow the evolution of the α's or the β's. For convenience, we consider the second ones, since they directly give the interesting result of particle creation for the quantum field. In Fig. 2, we plot a graphic with an example of numerical results for $|\beta_{(-n)n}(t, -\infty)|^2$ for a concrete choice of the parameters in the problem, comparing them to the asymptotic values obtained analytically in [29].

One can clearly see that the asymptotic values in [29] are correctly reproduced. We also claim the attention about the

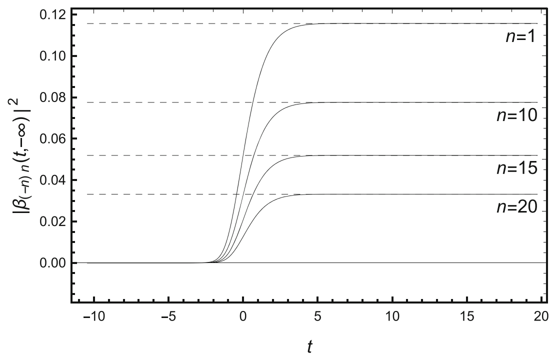

Fig. 2 Time evolution of $|\beta_{(-n)n}(t, -\infty)|^2$ as obtained from numerical computation (solid lines), and asymptotic values $|\beta_{(-n)n}(\infty, -\infty)|^2$ as given in [29] (dashed lines). The mass of the field is $m = 0.1$, periodic boundary conditions are imposed for a length of $L = 1000$, and the parameters of $a(\eta)$ are $A = 2.5$, $B = 1.5$ and $\rho = 1$

fact that, even if the intermediate values of the coefficients cannot be assigned an immediate interpretation in terms of particle creation, the evolution of the coefficients in time is perfectly smooth and monotonically increasing, with a time dependence qualitatively similar to that of the scale factor itself. This "well-behaved" result is remarkable, and makes us to suspect that those intermediate values may have some physical meaning in terms of some sort of approximation, although the discussion of this idea lies beyond the scope of this article.

6 Summary and conclusions

We have developed a method for computing the evolution of a confined quantum scalar field under general changes of the spacetime metric, under the assumptions given in Sect. 2.1. In order to keep track of the evolution of the field, instead of solving the Klein-Gordon differential equation, the method computes a time-dependent Bogoliubov transformation between the bases of modes associated to different spacelike hypersurfaces. It provides the way to obtain the initial conditions of the basis of modes associated to each spacelike hypersurface as an eigenvalue problem in that hypersurface [Eq. (14)]. Once obtained these initial conditions, the method provides a system of homogeneous first order differential equations in time [Eq. (21) or (28)] for the Bogoliubov transformation between the bases of modes associated to different times. The coefficients of such system of equations are computed out of the initial conditions of the modes themselves [Eqs. (22–24)]. Finally, we can also arrive to the formal generic solution of the system of equations [Eq. (25) or (29)]. In this article we have considered the geometries without boundaries and those for which the boundaries remain static in some synchronous gauge, leaving the case of "moving boundaries" for a second article.

The time-dependent Bogoliubov transformation obtained with this method cannot, in general, be given a direct physical interpretation in terms of mode-mixing and particle creation phenomena, simply because the modes constructed for each spacelike hypersurface do not correspond (in general) to modes with well-defined frequency with respect to some time, and therefore they do not yield a physically unambiguous quantisation in terms of particles. However, if there is a region of spacetime (long enough in time) in which the time translation along the chosen time coordinate is a symmetry,[11] then automatically the modes as constructed with the method correspond to modes with well-defined frequency in that region. In such cases, the quantisation in these modes has the adequate interpretation in terms of particles, and the Bogoliubov transformation between two such regions has also its usual physical meaning in terms of mode-mixing and particle creation. In the regions where this is not the case, the modes and Bogoliubov transformations constructed can be considered just as an "integration tool" for computing the physically meaningful cases, although their potential meaning as some sort of approximation will be analysed in future works.

We stress that the method is of general applicability, just under the considered assumptions. However, as much as we stress this generality, we should also emphasise that the method proves to be of special practical use in the case of small perturbations of the metric (or the boundary conditions, within the next forthcoming article on the method) around a static situation. For clarity, we number here the reasons of this utility:

- The perturbations of the modes can be obtained out of the eigenvalue Eq. (14) just with perturbation theory to first order. In the same way, for computing the quantities $\Delta\hat{\alpha}(\tilde{t})$ and $\Delta\hat{\beta}(\tilde{t})$ one should only keep the first order in the perturbation, taking advantage of the equivalence relation with respect to resonances described in Appendix E.2 to further simplify the expressions [Eqs. (42) and (43)]. Moreover, one can alternatively use the expressions (44–46) to obtain the results directly from the solutions to the corresponding static problem.
- There is no need to explicitly solve the differential equation (28) or to compute the exact transformation (29). In (36) and (37) we can already see that, to first order, there is just a resonance behaviour. Thus, writing down the Fourier expansion of the perturbations $\Delta\hat{\alpha}(\tilde{t})$ and $\Delta\hat{\beta}(\tilde{t})$ automatically gives which Bogoliubov coefficients will stay to order ε (the order of the perturbation) and which (if any) will grow linearly in time due to resonance, and at which rate. Alternatively, when it is possible one can also take the Fourier transform to find the Bogoliubov coefficients between the asymptotic regions using (38) and (39).
- Since the problem consist of a small perturbation of order ε around a static situation, in which there would be an exact notion of particles with well-defined frequency, the indefiniteness in such a notion will stay to order ε. Thus, for the modes which are in resonance (the interesting ones), for which the Bogoliubov coefficients will grow beyond the order ε, the physical interpretation in terms of mode-mixing and particle creation is perfectly valid.
- In Appendix E we prove that the result on resonances is stable under small deviations from the resonant frequency. We also verify that the resonances obtained are the same regardless of which bases of modes to order ε one uses to integrate the evolution, which further supports that they are the physically meaningful result.

As an example of the application of the method, we have considered the perturbation of a confined field by gravitational waves in a realistic $3+1$ dimensional background. Through very simple calculations, we obtain the Bogoliubov coefficients, which agree with those given in [42] for the same problem, and further discuss the role of the boundary conditions and the polarisation. The perturbative method could also prove its utility in other problems which are now under study, in which quantum systems are perturbed by small gravitational effects [16,46,47].

Finally, we have included an example of an application of the method to cosmological particle creation. We have obtained a differential equation that allows to compute the Bogoliubov coefficients for any one-dimensional FLRW flat spacetime, and applied this differential equation to an already known toy model of cosmological expansion, reproducing the correct results for the Bogoliubov coefficients between the asymptotic regions. In this way, we checked that the method can also be useful in problems beyond the perturbative regime. Out of the results obtained for the toy model, we highlight the smooth and monotonic behaviour in time of the Bogoliubov coefficients computed, even when they do not necessarily have an immediate physical interpretation. As we already mentioned, we leave for future work the analysis of the possible meaning of those values as some sort of approximation. We also leave for future work potential new applications of the integration method introduced in this article to other cosmological scenarios, as well as further applications in other problems in quantum field theory in curved spacetime.

[11] The requisite that $F(t) = 0$ to avoid negative eigenvalues must be also satisfied. See Appendix B.

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Data availability This manuscript has no associated data or the data will not be deposited. [Authors' comment: The only data generated for this article corresponds to the numerical solutions to the differential equation (60) plotted in Fig. 2. This data can be very easily reproduced following the explanations in the article and using standard software and basic numerical analysis skills. Therefore, we do not find it useful to deposit such data with the article.]

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Appendix A Discussion on the conditions for the method

In this Appendix we provide some further discussion on the conditions A, B and C introduced in Sect. 2.1.

Condition A reflects the fact that the method applies only for confined fields. It plays its role in ensuring that the operator $\hat{\mathcal{O}}(t)$ in (10) has a discrete spectrum, something crucial for the matrix formalism that we employ. In some situations (such as the example we give in Sect. 5), it might be possible to obtain results for unconfined fields by taking the limit of some typical length of the confinement region going to infinity after solving the problem [2,29]. However, in most of the situations this limit may not be well defined, or may not commute with a limit previously taken when solving the problem, such as a late time limit.

Condition B is introduced to ensure that the Cauchy problem for the scalar field is well posed, so that we can relate one-to-one initial conditions and solutions. Notice that this condition amounts to the predictability of the behaviour of the field in the given geometry. The aim of the method presented in this article is not to prove such predictability, but rather to compute the evolution of the field, for which it is reasonable to assume the predictability as given. If M has no boundaries, then condition B is automatically ensured by the well-known theorems on existence and uniqueness of global solutions to partial differential equations. Unfortunately, there is to our knowledge no extension of these theorems in the presence of timelike boundaries. For Dirichlet boundary conditions, a proof of local well-posedness of the Cauchy problem for the Klein-Gordon equation is given in [48] (where the authors consider the uniform Lopatinski boundary condition, which includes Dirichlet as a specific case). Global well-posedness has been proven for the Dirac equation under MIT-boundary conditions, which are the analogous to Neumann boundary conditions for a scalar field [49]. In the case of Robin boundary conditions, global well-posedness is proven in [50] for the Klein–Gordon equation in static spacetimes of bounded geometry (for a wider class of boundary conditions including the Robin case). It is probable that global well-posedness of the Cauchy problem for the Klein–Gordon equation can be proven for the boundary conditions considered in this article as an extension of the results cited, but approaching such task is not in the aim of the present work, which goals are mainly computational. Therefore, lacking (so far) definitive theorems, we prefer to explicitly impose global well-posedness as a requisite. Nonetheless, we strongly believe that it will be satisfied, in the worst case, except for too stilted geometries, most probably without any physical interest.

Condition C, which may seem unmotivated, is crucial for being able to write the Klein–Gordon equation as in (11), with all the spatial derivatives collected in the self-adjoint operator $\hat{\mathcal{O}}(t)$. Without this, the construction of the bases of modes and the time-dependent Bogoliubov transformation would not be possible, at least in the form we provide it. The existence of a gauge in which the crossed time-space terms in the metric cancel is guaranteed by the global hyperbolicity, with or without boundaries [37]. In most of the cases of physical interest, it should be also possible to make the lapse function $-g_{tt}$ equal to one globally. In any case, this is always possible locally, so even if no global synchronous gauge exists, one may still apply the method "by segments". However, we cannot provide a general recipe on how to approach the joint of the results of the different segments.

Appendix B Function $F(t)$, exponential time dependence and zero-frequency modes

The quantity $F(t)$ has been introduced in the operator $\hat{\mathcal{O}}(t)$ in (10) for a purely technical reason; namely, in order to have a positive-definite operator, so that all the eigenvalues obtained from the Eq. (14) are positive, something necessary for the construction of bases of modes and the computation of Bogoliubov coefficients to hold as it has been exposed. Since, while there is no timelike Killing field, the construction of bases and Bogoliubov transformations is purely instrumental, there is no need to further discuss the role of $F(t)$ in such case. However, when stating the properties of the bases constructed in Sect. 3.1, in property II, together with ∂_t being a timelike Killing field in a region of spacetime, we require also that $F(t) = 0$ in such region so that the modes constructed are modes with well-defined frequency (and we prove that this is the case). We can better understand this requisite, and in general the role of $F(t)$, if we analyse what happens when ∂_t is a timelike Killing field but $F(t) > 0$. There are two possibilities:

- $F(t)$ must take some finite positive value, so that the operator $\hat{\mathcal{O}}(t)$ is positive definite as required. If we insisted in setting $F(t) = 0$, we would obtain an operator with some negative eigenvalues. This would imply the existence of imaginary values for the quantities $\omega_n^{[\tilde{t}]}$ obtained from the eigenvalue Eq. (14). It is not difficult to check that in such case there exist fundamental solutions of the Klein-Gordon equation that have an exponential dependence on t, by considering equation (17) with imaginary $\omega_n^{[\tilde{t}]}$.[12] Under this circumstance, it is not possible to proceed to the Fock quantisation of the field, even under the existence of a timelike Killing field. However, the method can still keep track of the evolution of the field in that region, if not for quantising it. By introducing the positive function $F(t)$, we build bases of solutions associated to each hypersurface, although none of these solutions correspond to the fundamental solutions with exponential behaviour. They are just bases of solutions as instrumental as in the cases in which there is no timelike Killing field. But, precisely because they are not fundamental solutions, the Bogoliubov transformations between these different basis, obtained from the dynamics of the field, are not trivial, since $F(t)$ is present in (23) and (24). In other words, the exponential behaviour that is "artificially dropped" from the modes by the introduction of $F(t)$ is picked back and reproduced by the Bogoliubov transformation between them.
- $F(t)$ could be made arbitrarily small, but not zero, in order for the operator $\hat{\mathcal{O}}(t)$ to be positive definite. In this case, setting $F(t) = 0$ would yield a positive semidefinite operator, which implies the existence of a fundamental zero-frequency mode. If one is not interested in the quantisation of the field in the region, one can proceed by giving $F(t)$ some fixed value and compute the corresponding Bogoliubov coefficients. However, if one wishes to attempt quantisation in such region, one could instead add a small increment to the rest mass Δm, so that the operator $\hat{\mathcal{O}}(t)$ is positive definite even with $F(t) = 0$. Yet, after solving the problem one should take the limit $\Delta m \to 0$ in order to recover the original field. This limit might not be trivial, and the method cannot give an *a priori* prescription on how to take it. Even so, in Sect. 4.1 we give an example where this procedure proves to be successful in a trivial way. Notice however, that this difficulty with a zero-frequency mode is intrinsic to the field properties and not peculiar to the method.

We can identify some circumstances in which one can always set $F(t) = 0$ and forget about the quantity. In particular, let us consider the important case of a minimally coupled field ($\xi = 0$). In such case, non-zero rest mass and/or Dirichlet boundary conditions or Robin boundary conditions with $\gamma > 0$ [52] automatically make the operator $\hat{\mathcal{O}}(t)$ to be positive definite; while for massless fields without boundaries or with Neumann boundary conditions the operator is positive semidefinite and could be handled with the procedure described in the second point above.

Appendix C Completeness and orthonormality of the bases of modes

Let us prove that the set $\{\Phi_n^{[\tilde{t}]}, (\Phi_n^{[\tilde{t}]})^*\}$ constructed in Sect. 3.1 is a complete basis of the space of solutions to the Klein-Gordon equation which satisfy the possible boundary conditions. Because of condition B and the linearity of the Klein–Gordon equation, it is equivalent to prove that the set formed by the initial conditions for each mode in that set at $\Sigma_{\tilde{t}}$ is itself a basis of the space of initial conditions at $\Sigma_{\tilde{t}}$. If we recall (16), this set of initial conditions of the modes is given by the following set of pairs of values of the field and of its first time derivative at $\Sigma_{\tilde{t}}$, $(\Phi(\tilde{t}), \partial_t\Phi(t)|_{t=\tilde{t}})$:

$$\{(\Phi_n^{[\tilde{t}]}(\tilde{t}), -\mathrm{i}\omega_n^{[\tilde{t}]}\Phi_n^{[\tilde{t}]}(\tilde{t})), (\Phi_n^{[\tilde{t}]}(\tilde{t})^*, \mathrm{i}\omega_n^{[\tilde{t}]}\Phi_n^{[\tilde{t}]}(\tilde{t})^*)\}. \quad (C.1)$$

It is clear that the space of initial conditions at $\Sigma_{\tilde{t}}$ [pairs of functions $(\Phi(\tilde{t}), \partial_t\Phi(t)|_{t=\tilde{t}})$] is $L^2(\Sigma_{\tilde{t}}) \oplus L^2(\Sigma_{\tilde{t}})$. We will prove in short that the set in (C.1) is indeed a basis of such space. However, we also need to ensure that the elements in the basis satisfy the possible boundary conditions at $\partial\Sigma_{\tilde{t}}$

[12] This situation should not be confused with the necessary presence of superradiance. See the Appendix in [28], or [51].

compatible with the boundary conditions in ∂M, when they exist. For the first component of the elements in (C.1) (the evaluation of the mode at $\Sigma_{\tilde{t}}$), this compatibility means being in $\Gamma_{\tilde{t}}$, since this is the definition of such subspace [together with the smoothness condition that simply picks up a representative function of the vector in the Hilbert space $L^2(\Sigma_{\tilde{t}})$]. Such condition is clearly satisfied. Let us consider the second component of the elements in (C.1) (the time derivative of the mode at $\Sigma_{\tilde{t}}$). If we take the total time derivative of any of the boundary conditions in (2–4) along the boundary, since the boundary itself remains static, we will easily find that the same boundary condition that applies to $\Phi(t)$ must apply also to $\partial_t\Phi(t)$. Therefore, we have that the second component of the elements in (C.1) must also belong to $\Gamma_{\tilde{t}}$, something which is also clearly satisfied.

Observe that the fact that both the evaluation of the mode and of its first time derivative at some Cauchy hypersurface $\Sigma_{\tilde{t}}$ satisfy the same boundary conditions is what allows for the construction of the modes as it has been done in this article [that is, imposing equation (16)]. This construction was therefore feasible thanks to the condition that the boundaries shall remain static in the synchronous gauge considered. If this is not the case and the boundaries are not parallel to the time flow ∂_t, the boundary conditions at ∂M as given in (2–4) (or their derivatives along the boundary itself) mix spatial and time derivatives of the mode in the same equation, and the construction of initial conditions as presented in this article cannot be done any more. This is the situation that we will consider in the second forthcoming article on the method.[13]

Finally, let us prove that the set (C.1) is a basis of $L^2(\Sigma_{\tilde{t}})\oplus L^2(\Sigma_{\tilde{t}})$. For convenience, we temporarily simplify the notation for the functions and eigenvalues by replacing $\Phi_n^{[\tilde{t}]}(\tilde{t}) \to \Phi_n$ and $\omega_n^{[\tilde{t}]} \to \omega_n$. We notice first that clearly both sets of eigenvectors $\{\Phi_n\}$ and $\{\Phi_n^*\}$ are complete bases of $L^2(\Sigma_{\tilde{t}})$.[14] This means that the modes of the second basis can be expanded in the first basis with unique coefficients:

$$\Phi_n^* = \sum_m p_{nm}\Phi_m; \tag{C.2}$$

where

$$p_{nm} \neq 0 \Leftrightarrow \omega_n = \omega_m \tag{C.3}$$

since the subspaces generated by eigenfunctions of the same operator with different eigenvalues are orthogonal, and p_{nm} is clearly invertible. Also, any two arbitrary functions Φ_A and Φ_B in $L^2(\Sigma_{\tilde{t}})$ can be expanded in the basis $\{\Phi_n\}$ with unique coefficients:

$$\Phi_A = \sum_n a_n\Phi_n, \quad \Phi_B = \sum_n b_n\Phi_n. \tag{C.4}$$

Using the expansions in (C.2) and (C.4) we will prove that the generic pair $(\Phi_A, \Phi_B) \in L^2(\Sigma_{\tilde{t}})\oplus L^2(\Sigma_{\tilde{t}})$, that is, a generic initial condition, can be expanded in the set of pairs in (C.1) with unique coefficients, which proves that such set is a basis of the space of initial conditions. If we write

$$(\Phi_A, \Phi_B) = \sum_n \left[c_n(\Phi_n, -\mathrm{i}\omega_n\Phi_n) + d_n(\Phi_n^*, \mathrm{i}\omega_n\Phi_n^*)\right], \tag{C.5}$$

comparing each component with (C.4) and using (C.2) we obtain the following set of equations:

$$c_m + \sum_n p_{nm}d_n = a_m, \tag{C.6}$$

$$\omega_m c_m - \sum_n p_{nm}\omega_n d_n = \mathrm{i}b_m. \tag{C.7}$$

Multiplying (C.6) by ω_m and subtracting both equations, we arrive at

$$\sum_n p_{nm}(\omega_n+\omega_m)d_n = \omega_m a_m - \mathrm{i}b_m$$
$$\Leftrightarrow \sum_n p_{nm}d_n = \frac{1}{2}\left(a_m - \mathrm{i}\frac{b_m}{\omega_m}\right), \tag{C.8}$$

where for obtaining the last equation we have used (C.3). Since p_{nm} is invertible, the coefficients d_n exist and are unique. Using (C.6) we also obtain unique coefficients c_n. This completes the proof.

With respect to the orthonormal character of the bases, using the initial conditions and the normalisation given in (15) one can easily compute the Klein-Gordon inner product (5) between any two modes of the basis, obtaining:

$$\langle\Phi_n^{[\tilde{t}]}, \Phi_m^{[\tilde{t}]}\rangle = (\omega_n^{[\tilde{t}]} + \omega_m^{[\tilde{t}]})\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t})\Phi_m^{[\tilde{t}]}(\tilde{t})^*$$
$$= \delta_{nm}, \tag{C.9}$$

$$\langle\Phi_n^{[\tilde{t}]}, (\Phi_m^{[\tilde{t}]})^*\rangle = (\omega_n^{[\tilde{t}]} - \omega_m^{[\tilde{t}]})\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t})\Phi_m^{[\tilde{t}]}(\tilde{t})$$
$$= 0, \tag{C.10}$$

$$\langle(\Phi_n^{[\tilde{t}]})^*, (\Phi_m^{[\tilde{t}]})^*\rangle = -(\omega_n^{[\tilde{t}]} + \omega_m^{[\tilde{t}]})\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t})^*\Phi_m^{[\tilde{t}]}(\tilde{t})$$
$$= -\delta_{nm}; \tag{C.11}$$

where in (C.10) we have used that the integral is non-zero only when the eigenvalues coincide. Therefore, we have

[13] We can see at this point why, if we promote the boundary parameter γ of Robin boundary conditions to a function, a possibility given below equation (9), such function must be time independent. Otherwise, the boundary conditions applying to the field and to its time derivative would also be different.

[14] Here $L^2(\Sigma_{\tilde{t}})$ must be understood of course as a Hilbert space, that is, the space of square integrable functions modulo differences in a set of measure 0.

proven the completeness and orthonormality of the bases of modes that we are constructing for each hypersurface.

Appendix D Calculations on Bogoliubov transformations

Appendix D.1 Detailed calculation of the main result

We consider the Bogoliubov transformation between the modes associated with $\Sigma_{\tilde{t}_0}$ and with $\Sigma_{\tilde{t}+\Delta\tilde{t}}$. By composition of Bogoliubov transformations, we have that

$$U(\tilde{t}+\Delta\tilde{t},\tilde{t}_0)=U(\tilde{t}+\Delta\tilde{t},\tilde{t})U(\tilde{t},\tilde{t}_0). \tag{D.12}$$

With the help of this relation, we will find the differential equation in time (21) for $U(\tilde{t},\tilde{t}_0)$. If we differentiate, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}U(\tilde{t},\tilde{t}_0)&=\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}U(\tilde{t}+\Delta\tilde{t},\tilde{t}_0)\bigg|_{\Delta\tilde{t}=0}\\&=\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}U(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0}U(\tilde{t},\tilde{t}_0).\end{aligned} \tag{D.13}$$

Now we will compute the first factor on the r.h.s. Let us write explicitly the matrix elements of this factor:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\alpha_{nm}(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0}=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\,\Big[\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t})\;\partial_t\Phi_m^{[\tilde{t}]}(t)^*\Big|_{t=\tilde{t}}\\&-\Phi_m^{[\tilde{t}]}(\tilde{t})^*\;\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}}\Big]\bigg|_{\Delta\tilde{t}=0},\end{aligned} \tag{D.14}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\beta_{nm}(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0}=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\,\Big[\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t})\;\partial_t\Phi_m^{[\tilde{t}]}(t)\Big|_{t=\tilde{t}}\\&-\Phi_m^{[\tilde{t}]}(\tilde{t})\;\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}}\Big]\bigg|_{\Delta\tilde{t}=0}.\end{aligned} \tag{D.15}$$

We need to compute the derivative with respect to $\Delta\tilde{t}$ of the quantities inside the integrals. For that, we use the *local* evolution in time around $\tilde{t}+\Delta\tilde{t}$ (and to first order in $\Delta\tilde{t}$) of $\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)$, which is given by (16); and of $\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)$, which we will obtain through the Klein-Gordon equation. Notice that conditions (14) and (16), when replaced in the Klein-Gordon equation as written in (11) and evaluated at $t=\tilde{t}$, imply that

$$\begin{aligned}\partial_t^2\Phi_n^{[\tilde{t}]}(t)\Big|_{t=\tilde{t}}&=\mathrm{i}\omega_n^{[\tilde{t}]}\left[\mathrm{i}\omega_n^{[\tilde{t}]}+q(\tilde{t})\right]\Phi_n^{[\tilde{t}]}(\tilde{t})\\&\quad-[\xi\bar{R}(\tilde{t})-F(\tilde{t})]\Phi_n^{[\tilde{t}]}(\tilde{t}).\end{aligned} \tag{D.16}$$

That is, out of the initial conditions, the Klein-Gordon equation provides the value of the second time derivative of $\Phi_n^{[\tilde{t}]}(t)$ at $t=\tilde{t}$ [and *only* at $t=\tilde{t}$, Eq. (D.16) is evidently *not* a differential equation in time].

Because of relations (16) and (D.16) (replacing $\tilde{t}\to\tilde{t}+\Delta\tilde{t}$), we have that

$$\begin{aligned}\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t})&=\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}+\Delta\tilde{t})-\Delta\tilde{t}\;\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}+\Delta\tilde{t}}\\&\quad+O(\Delta\tilde{t})^2=\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}+\Delta\tilde{t})\\&\quad\left(1+\mathrm{i}\omega_n^{[\tilde{t}]}\Delta\tilde{t}\right)+O(\Delta\tilde{t})^2,\end{aligned} \tag{D.17}$$

$$\begin{aligned}\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}}&=\partial_t\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}+\Delta\tilde{t}}\\&\quad-\Delta\tilde{t}\;\partial_t^2\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}+\Delta\tilde{t}}+O(\Delta\tilde{t})^2\\&=-\mathrm{i}\omega_n^{[\tilde{t}+\Delta\tilde{t}]}\Phi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}+\Delta\tilde{t})\left\{1+\left[\mathrm{i}\omega_n^{[\tilde{t}]}+q(\tilde{t})\right]\Delta\tilde{t}\right\}\\&\quad+\Phi_n^{[\tilde{t}]}(\tilde{t})[\xi\bar{R}(\tilde{t})-F(\tilde{t})]\Delta\tilde{t}+O(\Delta\tilde{t})^2.\end{aligned} \tag{D.18}$$

This is the local evolution of the needed quantities to first order in $\Delta\tilde{t}$, and therefore we are ready to compute the derivatives in (D.14) and (D.15). If we plug (D.17) and (D.18) into (D.14) and (D.15), we get, with some manipulation,

$$\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\alpha_{nm}(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0}=\mathrm{i}\omega_n^{[\tilde{t}]}\delta_{nm}+\hat{\alpha}_{nm}(\tilde{t}), \tag{D.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}\beta_{nm}(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0}=\hat{\beta}_{nm}(\tilde{t}); \tag{D.20}$$

where the quantities are those defined in (23) and (24). This completes the proof of equation (21).

Appendix D.2 Bogoliubov identity checking

Any Bogoliubov transformation B [written in matrix notation as in (20)] must satisfy the following identity:

$$B^{-1}=JB^{\dagger}J,\quad J:=\begin{pmatrix}I&\\&-I\end{pmatrix}. \tag{D.21}$$

The Bogoliubov transformations that we have found in (25) and (29) both have the form $B=\mathrm{e}^{D}$ (the temporal ordering only affects the limits of the integrals and clearly does not play any role). In such a case, noticing that J is invertible, the condition (D.21) implies the following condition for D:

$$\mathrm{e}^{-D}=J\mathrm{e}^{D^{\dagger}}J=\mathrm{e}^{JD^{\dagger}J}\Leftrightarrow D=-JD^{\dagger}J. \tag{D.22}$$

Thus, noticing that $\Omega(\tilde{t})$, $\Theta(\tilde{t})$ and $A(\tilde{t})$ are diagonal, it is clear that both $U(\tilde{t},\tilde{t}_0)$ and $Q(\tilde{t},\tilde{t}_0)$ will satisfy the Bogoliubov identity (D.21) if and only if identity (D.22) is satisfied for $D=K(\tilde{t})$. This condition is equivalent to the relations

$$\hat{\alpha}_{nm}(\tilde{t})=-\hat{\alpha}_{mn}(\tilde{t})^*,\quad\hat{\beta}_{nm}(\tilde{t})=\hat{\beta}_{mn}(\tilde{t}). \tag{D.23}$$

Let us prove these relations by taking the derivative with respect to $\tilde{t}$ of the inner products given in (C.9) and (C.10).

Since these inner products are constant, with some manipulation we have that

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\langle \Phi_n^{[\tilde{t}]}, \Phi_m^{[\tilde{t}]}\rangle = \frac{\delta_{nm}}{\omega_n^{[\tilde{t}]}}\frac{\mathrm{d}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} \\
&+ (\omega_m^{[\tilde{t}]} + \omega_n^{[\tilde{t}]}) \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Phi_n^{[\tilde{t}]}(\tilde{t})\right]\Phi_m^{[\tilde{t}]}(\tilde{t})^* \right. \\
&\left. + \Phi_n^{[\tilde{t}]}(\tilde{t})\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Phi_m^{[\tilde{t}]}(\tilde{t})^*\right] + \Phi_n^{[\tilde{t}]}(\tilde{t})q(\tilde{t})\Phi_m^{[\tilde{t}]}(\tilde{t})^* \right\}, \qquad \text{(D.24)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\langle \Phi_n^{[\tilde{t}]}, (\Phi_m^{[\tilde{t}]})^*\rangle = \left(\frac{\mathrm{d}\omega_m^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} - \frac{\mathrm{d}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}}\right)\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, \Phi_n^{[\tilde{t}]}(\tilde{t})\Phi_m^{[\tilde{t}]}(\tilde{t}) \\
&+ (\omega_m^{[\tilde{t}]} - \omega_n^{[\tilde{t}]}) \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Phi_n^{[\tilde{t}]}(\tilde{t})\right]\Phi_m^{[\tilde{t}]}(\tilde{t}) \right. \\
&\left. + \Phi_n^{[\tilde{t}]}(\tilde{t})\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Phi_m^{[\tilde{t}]}(\tilde{t})\right] + \Phi_n^{[\tilde{t}]}(\tilde{t})q(\tilde{t})\Phi_m^{[\tilde{t}]}(\tilde{t}) \right\}. \qquad \text{(D.25)}
\end{aligned}
$$

The last term in each expression appears due to the derivative of $\mathrm{d}V_{\tilde{t}} = \sqrt{h(\tilde{t})}\prod_i \mathrm{d}x^i$ [and using (12)]. Comparing the identities (D.24) and (D.25) with the expressions in (23) and (24) it is straightforward to check that the relations in (D.23) hold, and thus the transformations satisfy the Bogoliubov identity (D.21).

Appendix E Robustness of the results on resonances

Appendix E.1 Stability under small deviations in the frequency

Let us check that resonances still occur when the perturbation frequency slightly deviates from the exact resonant frequency. Consider that a perturbation contains a frequency ω_p, so that for some pair of modes we get $\Delta\hat{\alpha}_{nm}(\tilde{t}) = C\mathrm{e}^{\mathrm{i}\omega_\mathrm{p}\tilde{t}}$, with C constant. Consider now that this frequency is very close to the resonant frequency $\omega_n^0 - \omega_m^0$ for the α-coefficient between this pair of modes, so that $\omega_\mathrm{p} = \omega_n^0 - \omega_m^0 + \Delta\omega$, with $|\Delta\omega| \lll |\omega_n^0 - \omega_m^0|$. From (36) we have that

$$
\begin{aligned}
\alpha_{nm}(\tilde{t}_\mathrm{f}, \tilde{t}_0) &\approx \varepsilon C \int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\, \mathrm{e}^{\mathrm{i}\Delta\omega\tilde{t}} \\
&= \varepsilon C \frac{2}{\Delta\omega}\sin\left[\frac{\Delta\omega}{2}(\tilde{t}_\mathrm{f} - \tilde{t}_0)\right]\mathrm{e}^{\mathrm{i}\varphi} \approx \varepsilon C(\tilde{t}_\mathrm{f} - \tilde{t}_0)\mathrm{e}^{\mathrm{i}\varphi}, \qquad \text{(E.26)}
\end{aligned}
$$

where $\varphi := \Delta\omega(\tilde{t}_\mathrm{f} + \tilde{t}_0)/2$ is an irrelevant phase, and we have also assumed that $\tilde{t}_\mathrm{f} - \tilde{t}_0 \ll 1/|\Delta\omega|$. Since we also need that $\tilde{t}_\mathrm{f} - \tilde{t}_0 \gg 1/|\omega_n^0 - \omega_m^0|$ for the resonance to take place, this means that, as far as the frequency is sufficiently close to the resonant one, there is a wide window of time between $1/|\omega_n^0 - \omega_m^0|$ and $1/|\Delta\omega|$ in which resonance occurs. Analogous arguments apply to the resonance for the β-coefficients.

Appendix E.2 Invariance under small changes in the basis of modes

In the perturbative regime, the modes with physical interpretation, those that can be used for quantisation, are the resonance modes of the static configuration; that is, the static solutions Φ_n^0. The only physically meaningful effect that can be described are the resonances between them, since anything else remains as tiny as the perturbation, and therefore as tiny as the degree of arbitrariness in the choice of the solutions to order ε that we use to keep track of the evolution. Here, we prove that the results on resonances do not depend on the concrete perturbed bases used to integrate the evolution of the field (as far as they are bases of solutions correctly computed to order ε).

Consider that, instead of the bases $\{\Phi_n^{[\tilde{t}]}(t), \Phi_n^{[\tilde{t}]}(t)^*\}$, we used some other bases of solutions $\{{}^1\chi_n^{[\tilde{t}]}(t), {}^2\chi_n^{[\tilde{t}]}(t)\}$ which differ just by order ε of the original ones (notice that we do not even require that "half" of the modes are the complex conjugate of the others). Since both are complete bases of solutions, they have to be related by

$$
\begin{pmatrix} {}^1\chi_1^{[\tilde{t}]}(t) \\ \cdots \\ {}^2\chi_1^{[\tilde{t}]}(t) \\ \cdots \end{pmatrix} = [I + \varepsilon T(\tilde{t})] \begin{pmatrix} \Phi_1^{[\tilde{t}]}(t) \\ \cdots \\ \Phi_1^{[\tilde{t}]}(t)^* \\ \cdots \end{pmatrix}, \qquad \text{(E.27)}
$$

where $T(\tilde{t})$ can be any time-dependent matrix that remains $O(1)$ in ε. If we call $V(\tilde{t}, \tilde{t}_0)$ the transformation between the new bases at different times, it is clear that it will be related to the $U(\tilde{t}, \tilde{t}_0)$, which transforms the original bases, by

$$
V(\tilde{t}, \tilde{t}_0) = [I + \varepsilon T(\tilde{t})]U(\tilde{t}, \tilde{t}_0)[I - \varepsilon T(\tilde{t}_0)]. \qquad \text{(E.28)}
$$

Consider now the differential equation (21). Taking into account that $K(\tilde{t})$ has no zeroth order terms in ε, the corresponding differential equation for $V(\tilde{t}, \tilde{t}_0)$ correct to order ε will be

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}V(\tilde{t}, \tilde{t}_0) &= \Big\{\mathrm{i}\Omega(\tilde{t}) + K(\tilde{t}) \\
&+ \varepsilon\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}T(\tilde{t}) + \mathrm{i}\Big(T(\tilde{t})\Omega^0 - \Omega^0 T(\tilde{t})\Big)\right]\Big\}V(\tilde{t}, \tilde{t}_0), \qquad \text{(E.29)}
\end{aligned}
$$

with $\Omega^0 := \Omega(\tilde{t})|_{\varepsilon=0} = \mathrm{diag}(\omega_1^0, \omega_2^0, \ldots, -\omega_1^0, -\omega_2^0, \ldots)$. One can see that, in principle, there is a non-zero contribution to the differential equation due to the change of bases. Let us now write down the matrix $T(\tilde{t})$ in the block form

$$
T(\tilde{t}) := \begin{pmatrix} X(\tilde{t}) & Y(\tilde{t}) \\ Z(\tilde{t}) & W(\tilde{t}) \end{pmatrix}. \qquad \text{(E.30)}
$$

This is just for notational purposes, there is no need that the coefficients satisfy any relation. We consider for example the coefficient $X_{nm}(\tilde{t})$, which would contribute to the quantity $\Delta\hat{\alpha}_{nm}(\tilde{t})$ in the new transformation with the amount

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}X_{nm}(\tilde{t}) - \mathrm{i}(\omega_n^0 - \omega_m^0)X_{nm}(\tilde{t}), \tag{E.31}$$

as one can read out of the new contribution appearing in (E.29). Consider now that the coefficient $X_{nm}(\tilde{t})$ contains some frequency ω_T, so that $X_{nm}(\tilde{t}) = C\mathrm{e}^{\mathrm{i}\omega_T\tilde{t}}$, with C constant. Then the contribution reads

$$\mathrm{i}C(\omega_T - \omega_n^0 + \omega_m^0)\mathrm{e}^{\mathrm{i}\omega_T\tilde{t}}. \tag{E.32}$$

Finally, when computing the corresponding Bogoliubov coefficient between two different times $\alpha_{nm}(\tilde{t},\tilde{t}_0)$ using (36), the extra contribution due to the change of bases will be

$$\begin{aligned}&\mathrm{i}\varepsilon C(\omega_T - \omega_n^0 + \omega_m^0)\int_{\tilde{t}_0}^{\tilde{t}_\mathrm{f}} \mathrm{d}\tilde{t}\, \mathrm{e}^{\mathrm{i}(\omega_T - \omega_n^0 + \omega_m^0)\tilde{t}}\\ &\quad = \varepsilon C\left[\mathrm{e}^{\mathrm{i}(\omega_T - \omega_n^0 + \omega_m^0)\tilde{t}_\mathrm{f}} - \mathrm{e}^{\mathrm{i}(\omega_T - \omega_n^0 + \omega_m^0)\tilde{t}_0}\right],\end{aligned} \tag{E.33}$$

which remains $O(\varepsilon)$ at any time, and therefore does not contribute to the resonances. If we had considered the Bogoliubov coefficient as given by the Fourier transform in (38), the conclusion is even more obvious, since the Fourier transform of (E.31) identically vanishes when evaluated at $\omega_n^0 - \omega_m^0$. Analogous arguments apply to the other entries of the transformation.

Notice that, if the transformation $I + \varepsilon T(\tilde{t})$ is not a Bogoliubov transformation to order ε, then the Bogoliubov coefficients obtained will not fulfil the Bogoliubov identities to order ε. This happens because the bases of perturbed solutions used in this case are not orthonormal to order ε. However, this is completely irrelevant, since these bases are just an intermediate mathematical tool to correctly compute the evolution. The modes in the bases need to be correct solutions to order ε and coincide with the static solutions to zeroth order, nothing else. The final Bogoliubov transformation should not be interpreted physically to order ε, but only beyond it; that is, only for the resonances, which as we have proven remain the same, no matter which transformation (to order ε) is done to the bases.

It is straightforward to connect the computation done in this Appendix to the claim done in Sect. 4, which states that the quantities of the form in (40) appearing in $\Delta\hat{\alpha}_{nm}(\tilde{t})$ [respectively of the form in (41) in $\Delta\hat{\beta}_{nm}(\tilde{t})$] will not contribute to the resonances: As one can see from (E.31), adding or subtracting these quantities is equivalent to an irrelevant change in the choice of bases to order ε.

Appendix E.3 Further simplification of the expressions

Let us obtain expressions (44–46) from (38) and (39). First, we write the operator $\hat{\mathcal{O}}(\tilde{t})$ to first order in ε as

$$\hat{\mathcal{O}}(\tilde{t}) \approx \hat{\mathcal{O}}^0 + \varepsilon\Delta\hat{\mathcal{O}}(\tilde{t}). \tag{E.34}$$

Zeroth and first orders in ε of equation (14) then read, respectively,

$$\hat{\mathcal{O}}^0\Phi_n^0 = (\omega_n^0)^2\Phi_n^0, \tag{E.35}$$

$$\hat{\mathcal{O}}^0\Delta\Phi_n^{[\tilde{t}]}(\tilde{t}) + \Delta\hat{\mathcal{O}}(\tilde{t})\Phi_n^0 = (\omega_n^0)^2\Delta\Phi_n^{[\tilde{t}]}(\tilde{t}) + 2\omega_n^0\Delta\omega_n^{[\tilde{t}]}\Phi_n^0. \tag{E.36}$$

Using these two expressions and the fact that $\hat{\mathcal{O}}^0$ is self-adjoint, we can write

$$\begin{aligned}&(\omega_m^0)^2\int_{\Sigma^0}\mathrm{d}V^0\, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})(\Phi_m^0)^*\\ &\quad = \int_{\Sigma^0}\mathrm{d}V^0\, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})[\hat{\mathcal{O}}^0(\Phi_m^0)^*]\\ &\quad = \int_{\Sigma^0}\mathrm{d}V^0\, [\hat{\mathcal{O}}^0\Delta\Phi_n^{[\tilde{t}]}(\tilde{t})](\Phi_m^0)^*\\ &\quad = -\int_{\Sigma^0}\mathrm{d}V^0\, [\Delta\hat{\mathcal{O}}(\tilde{t})\Phi_n^0](\Phi_m^0)^*\\ &\qquad + (\omega_n^0)^2\int_{\Sigma^0}\mathrm{d}V^0\, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})(\Phi_m^0)^* + \Delta\omega_n^{[\tilde{t}]}\delta_{nm};\end{aligned} \tag{E.37}$$

which implies

$$\begin{aligned}&\mathrm{i}[(\omega_n^0)^2 - (\omega_m^0)^2]\int_{\Sigma^0}\mathrm{d}V^0\, \Delta\Phi_n^{[\tilde{t}]}(\tilde{t})(\Phi_m^0)^* + \mathrm{i}\Delta\omega_n^{[\tilde{t}]}\delta_{nm}\\ &\quad = \mathrm{i}\int_{\Sigma^0}\mathrm{d}V^0\, [\Delta\hat{\mathcal{O}}(\tilde{t})\Phi_n^0](\Phi_m^0)^*.\end{aligned} \tag{E.38}$$

Finally, we notice that, because of the equivalence relation with respect to resonances, any diagonal term in $\Delta\hat{\alpha}_{nm}(\tilde{t})$ is always meaningless. We can thus ignore the second term in the l.h.s., obtaining (44) and (46) for $\Delta\hat{\alpha}_{nm}(\tilde{t})$. An identical derivation with the replacement $(\Phi_m^0)^* \to \Phi_m^0$ yields (45) for $\Delta\hat{\beta}_{nm}(\tilde{t})$, in which case the diagonal terms are relevant.

Appendix F Summary of formulae for the application of the method

In this Appendix, we provide Tables 1 and 2 with all the formulae necessary in order to apply the method to a concrete problem. As we indicated in the examples in Sects. 4.1 and 5, once the expressions for the method have been found, there is no need to consider the explicit evolution in time of the modes constructed any more. Therefore, the notation for the "time label" of the different modes can be simplified replacing $\tilde{t} \to t$. In the tables we use this simplification.

Table 1 Summary of formulae for the application of the method (part 1)

Computation of modes and eigenvalues	
Eigenvalue equation (14)	$\hat{\mathcal{O}}(t)\Phi_n^{[t]}(t) = (\omega_n^{[t]})^2 \Phi_n^{[t]}(t);$
with the operator (10)	$\hat{\mathcal{O}}(t) = -\nabla^2_{h(t)} + \xi R^h(t) + m^2 + F(t),$ with $F(t)$ such that $\hat{\mathcal{O}}(t)$ is positive definite;
(possibly) boundary conditions (7–9)	$\Phi_n^{[t]}(t, \mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Sigma_t \quad \text{(Dirichlet)},$ $\mathbf{n} \cdot \nabla_{h(t)} \Phi_n^{[t]}(t, \mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Sigma_t \quad \text{(Neumann)},$ $\mathbf{n} \cdot \nabla_{h(t)} \Phi_n^{[t]}(t, \mathbf{x}) + \gamma \Phi_n^{[t]}(t, \mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Sigma_t \quad \text{(Robin)};$
and orthonormalisation condition (15)	$\int_{\Sigma_t} \mathrm{d}V_t\, \Phi_n^{[t]}(t)\Phi_m^{[t]}(t)^* = \frac{\delta_{nm}}{2\omega_n^{[t]}}.$
Non-perturbative regime	
Quantities $\hat{\alpha}$ and $\hat{\beta}$ (23, 24)	$\hat{\alpha}_{nm}(t) = (\omega_m^{[t]} + \omega_n^{[t]}) \int_{\Sigma_t} \mathrm{d}V_t \left[\frac{\mathrm{d}}{\mathrm{d}t}\Phi_n^{[t]}(t)\right] \Phi_m^{[t]}(t)^*$ $+ \int_{\Sigma_t} \mathrm{d}V_t\, \Phi_n^{[t]}(t) \left[\omega_n^{[t]} q(t) + \mathrm{i}\xi \bar{R}(t)\right] \Phi_m^{[t]}(t)^*$ $+ \frac{\delta_{nm}}{2\omega_n^{[t]}} \left[\frac{\mathrm{d}\omega_n^{[t]}}{\mathrm{d}t} - \mathrm{i}F(t)\right],$ $\hat{\beta}_{nm}(t) = (\omega_m^{[t]} - \omega_n^{[t]}) \int_{\Sigma_t} \mathrm{d}V_t \left[\frac{\mathrm{d}}{\mathrm{d}t}\Phi_n^{[t]}(t)\right] \Phi_m^{[t]}(t)$ $- \int_{\Sigma_t} \mathrm{d}V_t\, \Phi_n^{[t]}(t) \left[\omega_n^{[t]} q(t) + \mathrm{i}\xi \bar{R}(t)\right] \Phi_m^{[t]}(t)$ $- \left[\frac{\mathrm{d}\omega_n^{[t]}}{\mathrm{d}t} - \mathrm{i}F(t)\right] \int_{\Sigma_t} \mathrm{d}V_t\, \Phi_n^{[t]}(t)\Phi_m^{[t]}(t);$
with (12, 13)	$q(t) = \partial_t \log\sqrt{h(t)},$ $\bar{R}(t) = 2\partial_t q(t) + q(t)^2 - [\partial_t h^{ij}(t)][\partial_t h_{ij}(t)]/4.$
Bogoliubov coefficients (with the trivial phase)	
Differential equation (21)	$\frac{\mathrm{d}}{\mathrm{d}t} U(t, t_0) = [\mathrm{i}\Omega(t) + K(t)]\, U(t, t_0);$
with (22)	$\Omega(t) = \mathrm{diag}(\omega_1^{[t]}, \omega_2^{[t]}, \ldots, -\omega_1^{[t]}, -\omega_2^{[t]}, \ldots),$ $K(t) = \begin{pmatrix} \hat{\alpha}(t) & \hat{\beta}(t) \\ \hat{\beta}(t)^* & \hat{\alpha}(t)^* \end{pmatrix}.$
Formal solution (25)	$U(t_\mathrm{f}, t_0) = \mathcal{T} \exp\left\{\int_{t_0}^{t_\mathrm{f}} \mathrm{d}t\, [\mathrm{i}\Omega(t) + K(t)]\right\}.$
Bogoliubov coefficients (without the trivial phase)	
Differential equation (28)	$\frac{\mathrm{d}}{\mathrm{d}t} Q(t, t_0) = \Theta(t)^* \bar{K}(t)\Theta(t) Q(t, t_0);$
with (26)	$\Theta(t) = \exp\left\{\int^t \mathrm{d}t' [\mathrm{i}\Omega(t') + A(t')]\right\},$ $\bar{K}(t) = K(t) - A(t),$ $A(t) = \mathrm{diag}(\hat{\alpha}_{11}, \hat{\alpha}_{22}, \ldots, -\hat{\alpha}_{11}, -\hat{\alpha}_{22}, \ldots).$
Formal solution (29)	$Q(t_\mathrm{f}, t_0) = \mathcal{T} \exp\left[\int_{t_0}^{t_\mathrm{f}} \mathrm{d}t\, \Theta(t)^* \bar{K}(t)\Theta(t)\right].$

Table 2 Summary of formulae for the application of the method (part 2)

Quantities $\Delta\hat{\alpha}$ and $\Delta\hat{\beta}$ (Perturbative regime)	
Using the modes computed to first order in ε (42, 43)	$\begin{aligned} \Delta\hat{\alpha}_{nm}(t) \equiv{}& \mathrm{i}[(\omega_n^0)^2-(\omega_m^0)^2]\int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Phi_n^{[t]}(t)(\Phi_m^0)^* \\ &+\int_{\Sigma^0}\mathrm{d}V^0\ \Phi_n^0\left[\omega_n^0\Delta q(t)+\mathrm{i}\xi\Delta\bar{R}(t)\right](\Phi_m^0)^*, \\ \Delta\hat{\beta}_{nm}(t) \equiv{}& -\mathrm{i}[(\omega_n^0)^2-(\omega_m^0)^2]\int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Phi_n^{[t]}(t)\Phi_m^0 \\ &-\int_{\Sigma^0}\mathrm{d}V^0\ \Phi_n^0\left[\omega_n^0\Delta q(t)+\mathrm{i}\xi\Delta\bar{R}(t)\right]\Phi_m^0 \\ &-[2\mathrm{i}\omega_n^0\Delta\omega_n^{[t]}-\mathrm{i}\Delta F(t)]\int_{\Sigma^0}\mathrm{d}V^0\ \Phi_n^0\Phi_m^0. \end{aligned}$
Using the modes solution to the static problem (44, 45)	$\begin{aligned} \Delta\hat{\alpha}_{nm}(t) &\equiv \int_{\Sigma^0}\mathrm{d}V^0\ [\hat{\Delta}(t)\Phi_n^0](\Phi_m^0)^*, \\ \Delta\hat{\beta}_{nm}(t) &\equiv -\int_{\Sigma^0}\mathrm{d}V^0\ [\hat{\Delta}(t)\Phi_n^0]\Phi_m^0; \end{aligned}$
with (46, E.34)	$\begin{aligned} \hat{\Delta}(t)\Phi_n^0 &= \left[\mathrm{i}\Delta\hat{\mathcal{O}}(t)+\omega_n^0\Delta q(t)+\mathrm{i}\xi\Delta\bar{R}(t)-\mathrm{i}\Delta F(t)\right]\Phi_n^0, \\ \Delta\hat{\mathcal{O}}(t) &= \partial_\varepsilon\hat{\mathcal{O}}(t)\vert_{\varepsilon=0}. \end{aligned}$
Bogoliubov coefficients (perturbative regime)	
Explicit time evolution (resonances) (36, 37)	$\begin{aligned} \alpha_{nn}(t_\mathrm{f},t_0) &\approx 1; \\ \alpha_{nm}(t_\mathrm{f},t_0) &\approx \varepsilon\int_{t_0}^{t_\mathrm{f}}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(\omega_n^0-\omega_m^0)t}\,\Delta\hat{\alpha}_{nm}(t), \quad n\neq m; \\ \beta_{nm}(t_\mathrm{f},t_0) &\approx \varepsilon\int_{t_0}^{t_\mathrm{f}}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(\omega_n^0+\omega_m^0)t}\,\Delta\hat{\beta}_{nm}(t). \end{aligned}$
Asymptotic values using Fourier transforms (38, 39)	$\begin{aligned} \alpha_{nn}(-\infty,\infty) &\approx 1; \\ \alpha_{nm}(-\infty,\infty) &\approx \varepsilon\sqrt{2\pi}\ \mathcal{F}[\Delta\hat{\alpha}_{nm}](\omega_n^0-\omega_m^0), \quad n\neq m; \\ \beta_{nm}(-\infty,\infty) &\approx \varepsilon\sqrt{2\pi}\ \mathcal{F}[\Delta\hat{\beta}_{nm}](\omega_n^0+\omega_m^0). \end{aligned}$

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6

Evolution of confined quantum scalar fields in curved spacetime. Part II

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Evolution of confined quantum scalar fields in curved spacetime. Part II

Spacetimes with moving boundaries in any synchronous gauge

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Abstract We develop a method for computing the Bogoliubov transformation experienced by a confined quantum scalar field in a globally hyperbolic spacetime, due to the changes in the geometry and/or the confining boundaries. The method constructs a basis of solutions to the Klein–Gordon equation associated to each compact Cauchy hypersurface of constant time. It then provides a differential equation for the linear transformation between bases at different times. The transformation can be interpreted physically as a Bogoliubov transformation when it connects two regions in which a time symmetry allows for a Fock quantisation. This second article on the method is dedicated to spacetimes with timelike boundaries that do not remain static in any synchronous gauge. The method proves especially useful in the regime of small perturbations, where it allows one to easily make quantitative predictions on the amplitude of the resonances of the field. Therefore, it provides a crucial tool in the growing research area of confined quantum fields in table-top experiments. We prove this utility by addressing two problems in the perturbative regime: Dynamical Casimir Effect and gravitational wave resonance. We reproduce many previous results on these phenomena and find novel results in an unified way. Possible extensions of the method are indicated. We expect that our method will become standard in quantum field theory for confined fields.

1 Introduction

Quantum field theory in curved spacetime studies the evolution of quantum fields which propagate in a classical general relativistic background geometry. Beyond its core mathematical construction (see e.g. [1–4]), the theory has been successful in approaching different concrete problems, such as Hawking and Unruh radiations [5,6] or cosmological particle creation [4,7,8]. This has required the development of different mathematical techniques and simplifications adapted to each specific problem, which allow for quantitative theoretical predictions. A family of problems of especial interest are quantum fields confined in cavities and under the effect of small changes in the background geometry or the non-inertial motion of the cavity boundaries. The theoretical predictions on these problems may be tested experimentally in the near future [9–12], thanks to the great improvement of the precision of quantum measurements in table-top experiments. Consequently, new mathematical techniques are necessary to address these problems and make quantitative predictions, which can then be contrasted with the experimental results.

In the preceding article [13], which we shall call “Part I”, we constructed a method for computing the evolution of a confined quantum scalar field in a globally hyperbolic spacetime, by means of a time-dependent Bogoliubov transformation. The method proved especially useful for addressing the kind of problems just mentioned, related to confined quantum fields undergoing small perturbations, although it is of general applicability (under some minor assumptions). However, the mathematical construction of Part I only allowed to approach spacetimes without boundaries or with static boundaries in some synchronous gauge.

In this second article we extend the method to spacetimes with timelike boundaries which do not remain static in any

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synchronous gauge. The essence of the procedure remains the same, but we require a more involved mathematical construction than the one undertaken in Part I. In particular, we need a specific treatment for each different boundary condition that we may impose to the field. The method is based on the foliation of the spacetime in compact spacelike Cauchy hypersurfaces using a time coordinate. The core idea is to construct a basis of modes naturally associated to each hypersurface, and then provide a differential equation in time for the linear transformation between the modes associated to two hypersurfaces at different times. This way, the evolution of the field in time is not obtained by solving the Klein–Gordon equation, but rather by solving a differential equation for a time-dependent linear transformation between the bases. Such linear transformation can be interpreted physically as a Bogoliubov transformation when it relates regions in which the time symmetry allows for a Fock quantisation in terms of particles associated to the corresponding bases of modes.

The conception of transferring the time evolution from the mode functions to the Bogoliubov transformation appears for the first time in the pioneer work by Parker [8]. This idea of a time-dependent Bogoliubov transformation has since then been developed specifically for other concrete problems (see e.g. [14–19]). Our work is therefore a generalisation (for confined fields) of the previous specific results. Operationally, it is mostly inspired by the construction in [20] for periodically accelerated cavities.

As in Part I, the method is of general applicability (with minor assumptions), but proves especially useful in the regime of small perturbations, since it provides very simple recipes for computing the resonance spectrum and sensibility of the field to a given perturbation of the background metric or the boundary conditions. We show with concrete examples that, in the small perturbations regime, with this unique method it is possible to easily solve different problems, each of which has so far required its own specific (and way more involved) treatment. Moreover, we easily handle a so far unsolved problem, namely that of a quantum field inside a three-dimensional rigid cavity and perturbed by a gravitational wave. We manage to explain it physically as a combination of the direct effect of the gravitational wave on the field plus a Dynamical Casimir Effect.

The contribution of the method to the understanding of quantum fields in curved spacetime is threefold. First, as we just mentioned, its direct application to concrete problems within its range allows to easily solve many important problems of physical interest. Second, the general structure of the method is very likely to be extensible (with the necessary adaptations) to other scenarios, such as other quantum fields, boundary conditions or metric gauges [21–26]. And third, the mathematical time-dependent linear transformation obtained may be given a physical interpretation beyond the one in terms of particle quantisation considered here; for example, in relation to adiabatic expansions [2,27,28] or to approaches to quantum field theory in curved spacetime based on field-related quantities [1,3,29].

The article is organised as follows. In Sect. 2 we state the general physical problem for which we construct the method, introducing the background metric, the field theory and the different assumptions that we consider; and also define three important mathematical objects that we use. The nuclear part of the article is Sect. 3. In this section we construct the basis of modes associated to each hypersurface of the foliation of the spacetime, and formally compute the time-dependent linear transformation between the modes of two different hypersurfaces. We give a differential equation and a formal solution for it, which constitute one of the two main results of the work. We also discuss the physical meaning of both the modes and the transformation. In Sect. 4 we consider the particularly important case of small perturbations and resonances, obtaining especially simple recipes for its solution, which constitute the other main result of the work. We apply the recipes to the Dynamical Casimir Effect and the gravitational wave perturbation problems. Finally, in Sect. 5 we present the summary and conclusions. In addition, in “Appendix A” we explain why a different treatment as that of Part I is needed in the case of “moving” boundaries. In “Appendix B” we prove that the properties we assign to the sets of modes that we build are fulfilled. In “Appendix C” we provide the detailed computation of the differential equation provided in Sect. 3. In “Appendix D” we derive the expressions given in Sect. 4. In “Appendices E and F” we provide the derivation of auxiliary expressions used in “Appendix D”. “Appendix G” is dedicated to the case of Dirichlet vanishing boundary conditions. In “Appendix H” we prove a necessary proposition about certain sets of eigenvalues. In “Appendix I” we provide for convenience a summary of the useful formulae for the application of the method.[1]

2 Preliminaries

2.1 Statement of the problem

We consider a globally hyperbolic spacetime (M, g) of dimension $N+1$ with timelike boundary ∂M [30]. In this geometry we introduce a scalar field Φ satisfying the Klein–Gordon equation

[1] As mentioned, this article introduces a more involved mathematical construction, as compared to Part I. Therefore, although the article is self-contained, it is mostly devoted to the development of such construction. The underlying ideas and the physical picture behind the method, which are analogous in both parts, are thus explained in more detail in Part I.

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - m^2\Phi - \xi R\Phi = 0; \tag{1}$$

where $m \geq 0$ is the rest mass of the field, $g^{\mu\nu}$ is the spacetime metric, R its scalar curvature and $\xi \in \mathbb{R}$ is a coupling constant (we use natural units $\hbar = c = 1$).

We impose one of the following two boundary conditions to the field:

(a) Dirichlet vanishing boundary conditions

$$\Phi(t, \mathbf{x}) = 0, \quad (t, \mathbf{x}) \in \partial M. \tag{2}$$

(b) Neumann vanishing boundary conditions

$$n^\mu\nabla_\mu\Phi(t, \mathbf{x}) = 0, \quad (t, \mathbf{x}) \in \partial M; \tag{3}$$

where $n^\mu(t, \mathbf{x})$ is the normal vector to ∂M.

We treat explicitly Dirichlet and Neumann vanishing boundary conditions since they are arguably the most common ones in physical problems. However, we do not discard that a specific treatment for other boundary conditions is also possible. The treatment of Dirichlet boundary conditions (2) requires a subtle reformulation of the boundary conditions themselves, which nonetheless does not modify the physical problem being addressed. Due to the need of a specific discussion, we leave Dirichlet boundary conditions for "Appendix G". Therefore, from here on we consider only Neumann boundary conditions (3) (except for "Appendix G" or unless otherwise stated).

Thanks to the global hyperbolicity, it is always possible to construct a Cauchy temporal function t in the full spacetime [30]. This provides a foliation in Cauchy hypersurfaces Σ_t of constant time. We introduce the Klein-Gordon inner product between two solutions of (1), given by

$$\begin{aligned}\langle\Phi', \Phi\rangle := \; -\mathrm{i}\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \; \Big[\Phi'(\tilde{t})\; \partial_t\Phi(t)^*\big|_{t=\tilde{t}} \\ -\Phi(\tilde{t})^*\; \partial_t\Phi'(t)\big|_{t=\tilde{t}}\Big];\end{aligned} \tag{4}$$

which, for convenience, we already evaluated at a given Cauchy hypersurface $\Sigma_{\tilde{t}}$, with $\mathrm{d}V_{\tilde{t}}$ being its volume element. Under the boundary conditions (2) or (3) this inner product is independent of $\Sigma_{\tilde{t}}$.

Finally, we introduce the three conditions on the Cauchy hypersurfaces and the temporal function that we need to ensure the applicability of the method. These conditions are:

(A) The Cauchy hypersurfaces Σ_t must be compact.

(B) For any Cauchy hypersurface Σ_t, the Cauchy problem for the Klein–Gordon equation (1) must be well-posed; that is, given as initial conditions the value of the field and of its first derivative with respect to t at Σ_t (compatible with the boundary conditions at the intersection $\partial\Sigma_t = \Sigma_t \cap \partial M$), there exists an unique solution to the Klein–Gordon equation in the whole spacetime satisfying these conditions.

(C) Using the temporal function as a coordinate, the metric should be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + h_{ij}(t, \mathbf{x})\mathrm{d}x^i\mathrm{d}x^j, \tag{5}$$

where $h_{ij}(t)$ is a regular Riemannian metric.[2] This is called a *synchronous gauge.*

The necessity of each condition will become clear when constructing the method. A detailed discussion on their physical meaning and the limitations they introduce can be found in "Appendix A" of Part I. In this second article, we consider the cases in which at least some parts of the boundary ∂M have non-zero velocity in the coordinates chosen. We shall mention that, if the boundaries are not static, an alternative solution could be to change to a new coordinate system in which the boundaries remain static, and then use the method as exposed in Part I. However, in general, in this new coordinate system the metric may not look like (5) and (since Condition C is also a requirement in Part I) the integration method will not apply.

2.2 Space of initial conditions at Σ_t, inner product and self-adjoint operator

Let us introduce three mathematical objects that are pivotal for the method. First, we define Γ_t as a subspace of the space of *pairs of* square integrable smooth functions over a Cauchy hypersurface, representing possible initial conditions $(\Phi, \partial_t\Phi)|_{\Sigma_t}$. That is, $\Gamma_t \subset [C^\infty(\Sigma_t) \cap L^2(\Sigma_t)]^{\oplus 2}$. Specifically, Γ_t is the restriction of the full space of pairs of functions to initial conditions $(\Phi, \partial_t\Phi)|_{\Sigma_t}$ satisfying Neumann vanishing boundary conditions (3) at $\partial\Sigma_t$. This can be rewritten in terms of the initial conditions as

$$\begin{cases}\mathbf{n}\cdot\nabla_{h(t)}\Phi(t, \mathbf{x}) = -v_{\mathrm{B}}(t, \mathbf{x})\partial_t\Phi(t, \mathbf{x}), \\ \text{and } \mathbf{n}\cdot\nabla_{h(t)}\partial_t\Phi(t, \mathbf{x}) = 0 \text{ if } v_{\mathrm{B}}(t, \mathbf{x}) = 0;\end{cases} \tag{6}$$

where $\mathbf{x} \in \partial\Sigma_t$; $\mathbf{n}(t, \mathbf{x})$ is the normal vector to $\partial\Sigma_t$ and pointing outwards from Σ_t; $v_{\mathrm{B}}(t, \mathbf{x}) := (\mathbf{n}\cdot\mathbf{v}_{\mathrm{B}})(t, \mathbf{x})$, where $\mathbf{v}_{\mathrm{B}}(t, \mathbf{x})$ is the velocity vector of the boundary, and therefore $v_{\mathrm{B}}(t, \mathbf{x})$ is its normal component; and $\nabla_{h(t)}$ is the covariant derivative corresponding to the spatial metric $h_{ij}(t)$. The

[2] From here on we omit the explicit dependence of $h_{ij}(t)$ and other quantities on the spatial coordinates. Also, in (5) for simplicity we are assuming that for each hypersurface Σ_t there is one coordinate chart $(x^1, \ldots, x^N)$ that completely covers it. This might not be the case, but considering several coordinate charts would be straightforward and would not affect the construction of the method.

first line of (6) is just the reformulation of (3) separating the spatial and temporal partial derivatives. The second line corresponds to the total time derivative of (3) along the boundary when $v_{\mathrm{B}} = 0$. Clearly, that time derivative must also vanish, and this shall be taken into account when considering the compatibility of the initial conditions with the boundary conditions. However, in the regions where the boundary is moving in the chosen coordinates ($v_{\mathrm{B}} \neq 0$), the total time derivative of (3) along the boundary involves second order partial time derivatives of the field. In such case, the fulfilment of the first time derivative of the boundary conditions at $\partial\Sigma_t$ already depends on the dynamical evolution given by the Klein–Gordon equation (1), and not just on the initial conditions at Σ_t. Therefore, its fulfilment is guaranteed by Condition B. On the other hand, in the regions where the boundary remains parallel to ∂_t ($v_{\mathrm{B}} = 0$), the first time derivative of the boundary condition reads as in the second line of (6), involving only up to the first partial time derivative of the field, and thus depending only on the initial conditions at Σ_t. Therefore, in such regions it imposes the corresponding constraint on these initial conditions.

The second mathematical object that we introduce is the following inner product in the Hilbert space of pairs of functions $L^2(\Sigma_t) \oplus L^2(\Sigma_t) \supset \Gamma_t$:

$$\left\langle \begin{pmatrix} \Phi' \\ \partial_t \Phi' \end{pmatrix}, \begin{pmatrix} \Phi \\ \partial_t \Phi \end{pmatrix} \right\rangle_{\Sigma_t} := \int_{\Sigma_t} \mathrm{d}V_t \, [\xi R^h(t) + m^2 + F(t)]\Phi' \Phi^* + \int_{\Sigma_t} \mathrm{d}V_t \, \partial_t \Phi' \partial_t \Phi^* + \int_{\Sigma_t} \mathrm{d}V_t \, (\nabla_{h(t)} \Phi') \cdot (\nabla_{h(t)} \Phi^*); \tag{7}$$

where $R^h(t)$ is the scalar curvature corresponding to the spatial metric $h_{ij}(t)$ and $F(t) \geq 0$ a time-dependent non-negative quantity given by

$$F(t) := \begin{cases} 0 & \text{if } \xi R^h(t, \mathbf{x}) + m^2 > 0 \text{ a.e. in} \Sigma_t, \\ -\text{ess inf}\{\xi R^h(t, \mathbf{x}) + m^2, \mathbf{x} \in \Sigma_t\} + \epsilon & \text{i.o.c.}; \end{cases} \tag{8}$$

where "a.e." stands for *almost everywhere,* "ess inf" stands for *essential infimum,* "i.o.c." stands for *in other case* and $\epsilon > 0$ is an arbitrarily small positive quantity. The quantity $F(t)$ ensures that the inner product is positive-definite.[3]

Finally, we introduce the following operator in Γ_t:

$$\hat{\mathscr{M}}(t) := \begin{pmatrix} 0 & 1 \\ \hat{\mathscr{O}}(t) & 0 \end{pmatrix}; \tag{9}$$

with

$$\hat{\mathscr{O}}(t) := -\nabla^2_{h(t)} + \xi R^h(t) + m^2 + F(t), \tag{10}$$

where $\nabla^2_{h(t)}$ is the Laplace–Beltrami differential operator and $F(t)$ has been defined in (8). With the inner product in (7), the boundary condition (6), and using Green's first identity, one can easily check that the operator $\hat{\mathscr{M}}(t)$ is self-adjoint.[4]

Before finishing this section, let us mention that using the operator $\hat{\mathscr{O}}(t)$ and Condition C we shall simplify the Klein–Gordon equation (1) taking into account the form of the metric in (5), obtaining

$$\partial_t^2 \Phi = -\hat{\mathscr{O}}(t)\Phi - q(t)\partial_t \Phi - \xi \bar{R}(t)\Phi + F(t)\Phi; \tag{11}$$

where

$$q(t) := \partial_t \log \sqrt{h(t)} \tag{12}$$

is a factor which depends on the change of the metric of the spacelike hypersurfaces with time, with $h(t)$ the determinant of the spatial metric $h_{ij}(t)$, and $\bar{R}(t) := R(t) - R^h(t)$ is the part of the full scalar curvature of $g_{\mu\nu}$ which depends on time derivatives, given by

$$\bar{R}(t) = 2\partial_t q(t) + q(t)^2 - \frac{1}{4}[\partial_t h^{ij}(t)][\partial_t h_{ij}(t)]. \tag{13}$$

The key role of Condition C has been to yield Eq. (11), in which all the spatial derivatives present are those in the Laplace–Beltrami operator contained in $\hat{\mathscr{O}}(t)$.

3 Construction of the method

3.1 Construction of the bases of modes

For each spacelike hypersurface $\Sigma_{\tilde{t}}$ we construct a set of modes $\{{}^{\pm}\Phi_n^{[\tilde{t}]}(t)\}$ fulfilling the following two Properties:

(I) They form a complete basis of the space of solutions to the Klein–Gordon equation (1). We stress that each mode of the basis is defined in the whole spacetime, the label $[\tilde{t}]$ meaning only that we *associate* it to the corresponding hypersurface. Specifically, it is in this hypersurface that we set its initial conditions.

[3] The quantity $F(t)$ plays here an analogous role to that of the function with the same name introduced in Part I, which meaning is discussed in "Appendix B" there. In many problems of interest, such as a massive field with minimal coupling ($\xi = 0$), the quantity $F(t)$ vanishes and can be ignored.

[4] Assertions about the self-adjoint nature of the operator should strictly be done over the extension of the operator defined on Γ_t to the full Hilbert space $L^2(\Sigma_t) \oplus L^2(\Sigma_t)$ (of which Γ_t is a dense subspace). This is a well-known mathematical procedure in the analysis of partial differential equations with elliptic operators and boundary value problems. Since this is the context in which we make use of the operator, we shall not get into the details of it. See for example [31].

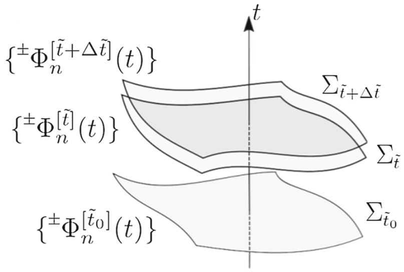

Fig. 1 Association of bases of modes $\{^{\pm}\Phi_n^{[\tilde{t}]}(t)\}$ to Cauchy hypersurfaces $\Sigma_{\tilde{t}}$

(II) If there exists a region S of the spacetime around $\Sigma_{\tilde{t}}$ where

- ∂_t behaves like a Killing field [$h_{ij}(t)$ is constant],
- the boundaries remain parallel to ∂_t ("static"),
- and the function $F(t)$ in (8) vanishes;

the modes form an orthonormal basis with respect to the Klein-Gordon inner product (4) of positive frequency modes (modes $^{+}\Phi_n^{[\tilde{t}]}$) and negative frequency modes (modes $^{-}\Phi_n^{[\tilde{t}]}$) with respect to t.[5] The region S needs to fully embrace the spatial hypersurfaces Σ_t along an interval of time t around $t = \tilde{t}$ which is long enough so as to explore the minimum frequency in the spectrum given by the modes.

In Fig. 1 we provide a graphical depiction of this association of bases of modes to Cauchy hypersurfaces, which shall be helpful when following the construction of the method. In order to construct the bases $\{^{\pm}\Phi_n^{[\tilde{t}]}\}$, we first construct auxiliary bases of modes $\{\Psi_n^{[\tilde{t}]}\}$, also associated to each hypersurface $\Sigma_{\tilde{t}}$, and then introduce a linear transformation to the bases $\{^{\pm}\Phi_n^{[\tilde{t}]}\}$. Since the modes $\Psi_n^{[\tilde{t}]}(t)$ are also solutions to the Klein–Gordon equation, because of Condition B the only quantities left to fully determine them are the initial conditions, which we are going to fix at $\Sigma_{\tilde{t}}$. That is, we need to fix the pair $(\Psi_n^{[\tilde{t}]}(\tilde{t}), \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}}) \in \Gamma_{\tilde{t}}$ for each mode. These pairs are going to be given by the eigenvectors of the operator $\hat{\mathscr{M}}(\tilde{t})$ in (9):

$$\hat{\mathscr{M}}(\tilde{t})\begin{pmatrix}\Psi_n^{[\tilde{t}]}(\tilde{t})\\ \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}}\end{pmatrix} = \omega_n^{[\tilde{t}]}\begin{pmatrix}\Psi_n^{[\tilde{t}]}(\tilde{t})\\ \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}}\end{pmatrix}. \tag{14}$$

Notice that $\tilde{t}$ in (14) is just a parameter.[6] Since $\hat{\mathscr{M}}(\tilde{t})$ is self-adjoint with respect to the inner product (7), the eigenvalues $\omega_n^{[\tilde{t}]}$ are real. Therefore, we can also impose that the pairs of functions are of real functions and that they satisfy the following orthogonality and normalisation condition:

$$\left\langle \begin{pmatrix}\Psi_n^{[\tilde{t}]}(\tilde{t})\\ \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}}\end{pmatrix}, \begin{pmatrix}\Psi_m^{[\tilde{t}]}(\tilde{t})\\ \partial_t\Psi_m^{[\tilde{t}]}(t)|_{t=\tilde{t}}\end{pmatrix}\right\rangle_{\Sigma_{\tilde{t}}} = |\omega_n^{[\tilde{t}]}|\delta_{nm}. \tag{15}$$

In "Appendix H" we prove that there are no zero eigenvalues $\omega_n^{[\tilde{t}]}$, so the normalisation criterion is valid. Finally, notice that thanks to Condition A we can be sure that the spectrum is discrete, as we had implicitly assumed with the notation.

In order to operationally find the quantities $\Psi_n^{[\tilde{t}]}(\tilde{t})$, we summarise (6) and (14) in the two equations

$$\hat{\mathscr{O}}(\tilde{t})\Psi_n^{[\tilde{t}]}(\tilde{t}) = (\omega_n^{[\tilde{t}]})^2\Psi_n^{[\tilde{t}]}(\tilde{t}), \tag{16}$$

$$\mathbf{n}\cdot\nabla_{h(\tilde{t})}\Psi_n^{[\tilde{t}]}(\tilde{t},\mathbf{x}) = -\omega_n^{[\tilde{t}]}v_{\mathrm{B}}(\tilde{t},\mathbf{x})\Psi_n^{[\tilde{t}]}(\tilde{t},\mathbf{x}), \quad \mathbf{x}\in\partial\Sigma_{\tilde{t}}; \tag{17}$$

and the equation for the partial time derivative

$$\partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}} = \omega_n^{[\tilde{t}]}\Psi_n^{[\tilde{t}]}(\tilde{t}). \tag{18}$$

This last equation plays a role in the construction of the method, but in order to apply the method to a concrete problem only the first two are necessary. Because of this last equation, when $v_{\mathrm{B}}(\tilde{t},\mathbf{x}) = 0$ Eq. (17) is also imposing the second condition in (6). Notice that Eq. (16) cannot be taken as an eigenvalue problem posed directly for $\Psi_n^{[\tilde{t}]}(\tilde{t})$, since the boundary conditions (17) of such problem would not be fixed (they would depend on the eigenvalue).

By Condition B, for each pair $(\Psi_n^{[\tilde{t}]}(\tilde{t}), \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}}) \in \Gamma_{\tilde{t}}$ that is solution to (14) we have an unique mode $\Psi_n^{[\tilde{t}]}(t)$. Let us now relabel the modes in the (infinite countable) set $\{\Psi_n^{[\tilde{t}]}\}$ and group them into two (also infinite countable) subsets, $\{^{+}\Psi_n^{[\tilde{t}]}\}$ and $\{^{-}\Psi_n^{[\tilde{t}]}\}$, with their respective sets of eigenvalues $\{^{+}\omega_n^{[\tilde{t}]}\}$ and $\{^{-}\omega_n^{[\tilde{t}]}\}$. In "Appendix B" we prove that, at least for the problems in which we can give a physical interpretation to the results of the method, there is an infinite number of both positive and negative eigenvalues $\omega_n^{[\tilde{t}]}$. We organise the solutions as follows:

$$\cdots \le {}^{-}\omega_2^{[\tilde{t}]} \le {}^{-}\omega_1^{[\tilde{t}]} < 0 < {}^{+}\omega_1^{[\tilde{t}]} \le {}^{+}\omega_2^{[\tilde{t}]} \le \cdots. \tag{19}$$

We notice that, in general, the solutions $^{\pm}\Psi_n^{[\tilde{t}]}$ and eigenvalues $^{\pm}\omega_n^{[\tilde{t}]}$ in each subset are independent.

Because of Condition B and the linearity of the Klein–Gordon equation, we can trivially consider the inner product (7), defined for initial conditions at each hypersurface $\Sigma_{\tilde{t}}$, as an inner product in the space of solutions: For any two solutions, their inner product is that of their corresponding

[5] Regions in which $F(t)$ does not vanish, but takes an arbitrarily small value ϵ, may also be considered. We refer to "Appendix B" of Part I for further details on this condition.

[6] As we commented previously for the operator $\hat{\mathscr{M}}(\tilde{t})$, this equation should also be posed in the full Hilbert space $L^2(\Sigma_t)\oplus L^2(\Sigma_t)$. It is known, however, that the functions representing the eigenvectors can be taken to belong to the dense subspace $\Gamma_{\tilde{t}}$ [31].

initial conditions at the given hypersurface. Using this definition of inner product in the space of solutions, and carefully taking into account the relabelling $\Psi_n^{[\tilde{t}]} \to {}^{\pm}\Psi_n^{[\tilde{t}]}$, we can write the orthonormalisation condition (15) in an equivalent way, but directly for the modes ${}^{\pm}\Psi_n^{[\tilde{t}]}$, as[7]

$$\langle {}^{\pm}\Psi_n^{[\tilde{t}]}, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]} \rangle_{\Sigma_{\tilde{t}}} = |{}^{\pm}\omega_n^{[\tilde{t}]}| \delta_{nm} \delta_{\pm\hat{\pm}}, \tag{20}$$

where the quantity $\delta_{\pm\hat{\pm}}$ equals 1 if the signs coincide and 0 otherwise. We also stress that this orthonormalisation is only correct in the inner product of the hypersurface $\Sigma_{\tilde{t}}$ to which the modes are associated.

Finally, we build the set of modes $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$ by taking a linear transformation from the set of modes $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$. This linear transformation reads

$$\begin{pmatrix} {}^{+}\Phi_1^{[\tilde{t}]} \\ \dots \\ {}^{-}\Phi_1^{[\tilde{t}]} \\ \dots \end{pmatrix} = M \begin{pmatrix} {}^{+}\Psi_1^{[\tilde{t}]} \\ \dots \\ {}^{-}\Psi_1^{[\tilde{t}]} \\ \dots \end{pmatrix}, \tag{21}$$

where, in obvious block notation,

$$M := \frac{1}{2} \begin{pmatrix} (1-\mathrm{i})I & (1+\mathrm{i})I \\ (1+\mathrm{i})I & (1-\mathrm{i})I \end{pmatrix}. \tag{22}$$

The set of modes $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$, constructed this way for each $\tilde{t}$, is a basis of modes satisfying Properties I and II. We prove this in "Appendix B". In particular, in the regions S described in Property II, where it is possible to construct modes with well-defined frequency with respect to t, we have that ${}^{-}\Psi_n^{[\tilde{t}]}(\tilde{t}) = {}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t})$ and ${}^{-}\omega_n^{[\tilde{t}]} = -({}^{+}\omega_n^{[\tilde{t}]})$, and the modes in the basis are

$${}^{\pm}\Phi_n^{[\tilde{t}]}(t) = {}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t}) \mathrm{e}^{\mp \mathrm{i}({}^{+}\omega_n^{[\tilde{t}]})(t-\tilde{t})}; \tag{23}$$

that is, ${}^{+}\Phi_n^{[\tilde{t}]}(t)$ are the modes with positive frequencies ${}^{+}\omega_n^{[\tilde{t}]} > 0$, and ${}^{-}\Phi_n^{[\tilde{t}]}(t) = {}^{+}\Phi_n^{[\tilde{t}]}(t)^*$ the corresponding negative frequency modes.

The construction of the bases of modes done here has been significantly different to that in Part I. We discuss in "Appendix A" why the construction done in Part I does not work here.

3.2 Time-dependent linear transformation

Let us write down formally the linear transformation $U(\tilde{t}, \tilde{t}_0)$ between any two bases of modes, associated to the hypersurfaces $\Sigma_{\tilde{t}_0}$ and $\Sigma_{\tilde{t}}$:

$$\begin{pmatrix} {}^{+}\Phi_1^{[\tilde{t}]} \\ \dots \\ {}^{-}\Phi_1^{[\tilde{t}]} \\ \dots \end{pmatrix} = U(\tilde{t}, \tilde{t}_0) \begin{pmatrix} {}^{+}\Phi_1^{[\tilde{t}_0]} \\ \dots \\ {}^{-}\Phi_1^{[\tilde{t}_0]} \\ \dots \end{pmatrix}. \tag{24}$$

In "Appendix C" we prove that this time-dependent linear transformation between bases satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} U(\tilde{t}, \tilde{t}_0) = M \hat{V}(\tilde{t}) M^* U(\tilde{t}, \tilde{t}_0); \tag{25}$$

where

$$\hat{V}(\tilde{t}) = \begin{pmatrix} \hat{V}^{++} & \hat{V}^{+-} \\ \hat{V}^{-+} & \hat{V}^{--} \end{pmatrix}, \tag{26}$$

with

$$\begin{aligned} \hat{V}_{nm}^{\pm\hat{\pm}} = & -({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}) \delta_{nm} \delta_{\pm\hat{\pm}} \\ & \hat{\pm} \Bigg\{ \Big[({}^{\pm}\omega_n^{[\tilde{t}]}) + ({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}) \Big] \\ & \times \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \right] {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ & + \left[2({}^{\pm}\omega_n^{[\tilde{t}]})^2 + \frac{\mathrm{d}^{\pm}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} - F(\tilde{t}) \right] \\ & \times \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ & + \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \left[{}^{\pm}\omega_n^{[\tilde{t}]} q(\tilde{t}) + \xi \bar{R}(\tilde{t}) \right] {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ & - \int_{\partial\Sigma_{\tilde{t}}} \mathrm{d}S_{\tilde{t}}\, v_{\mathrm{B}}(\tilde{t}) \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \right] {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \Bigg\}, \end{aligned} \tag{27}$$

where $\mathrm{d}S_{\tilde{t}}$ is the surface element of $\partial\Sigma_{\tilde{t}}$. With the initial condition $U(\tilde{t}_0, \tilde{t}_0) = I$, Eq. (25) has the formal solution

$$U(\tilde{t}_{\mathrm{f}}, \tilde{t}_0) = \mathscr{T} \exp \left[\int_{\tilde{t}_0}^{\tilde{t}_{\mathrm{f}}} \mathrm{d}\tilde{t}\, M \hat{V}(\tilde{t}) M^* \right], \tag{28}$$

where $\mathscr{T}$ denotes time ordering.

Equations (25–28) are one of the two main results of this work: They switch from the time evolution of the modes to the time evolution of the transformation between bases. The time-dependent linear transformation obtained relates the bases $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$, which are those satisfying Properties I and II. However, one of the strengths of the method is that all the quantities appearing in (27), which are the coefficients of our differential equation, are known just by computing the initial conditions of the auxiliary bases $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$, which are the solutions to the Eqs. (16) and (17), for which the time $\tilde{t}$ is just a parameter.

[7] Any time that two '±' signs are involved in an equation, we use a *hat* '$\hat{\pm}$' to distinguish one of them.

3.3 Physical interpretation

The time-dependent linear transformation obtained contains all the information necessary to compute the evolution of the field in time. However, as we advanced in the Introduction, we do not pretend to give a quantisation for each and every basis of modes that we have constructed at each time. It is only in those regions S described within Property II of Sect. 3.1, where we can proceed to the usual Fock quantisation of the field. That is, defining the corresponding Fock space with its vacuum state and creation and annihilation operators (ones the adjoints of the others in the case of a real field) associated to the mode decomposition given by the method. This is the case because, in such regions, the decomposition is done in modes with well-defined frequency with respect to a timelike Killing field [see Eq. (23)]. We rely on the fact that, in such situation, Fock representation gives the correct physical description of a field in terms of particles associated to those modes.[8] When connecting two regions where this Fock quantisation procedure can be done, the time-dependent linear transformation that we constructed in the previous section really becomes a Bogoliubov transformation, taking the well-known form

$$U(\tilde{t},\tilde{t}_0)=\begin{pmatrix}\alpha(\tilde{t},\tilde{t}_0) & \beta(\tilde{t},\tilde{t}_0)\\ \beta(\tilde{t},\tilde{t}_0)^* & \alpha(\tilde{t},\tilde{t}_0)^*\end{pmatrix}. \tag{29}$$

These Bogoliubov coefficients also relate in the usual way the annihilation and creation operators of the mode decompositions associated to the different regions.

We refer to Sect. 3.3 of Part I for additional discussions on different aspects of the physical interpretation, which also apply here.

4 Small perturbations and resonances

Let us consider the case in which the spatial metric $h_{ij}(t)$ only changes in time by a small perturbation around some constant metric h^0_{ij}; that is,

$$h_{ij}(t)=h^0_{ij}+\varepsilon\Delta h_{ij}(t), \tag{30}$$

where $\varepsilon \ll 1$. Also, the boundaries may experience small displacements of order ε, meaning that we allow the hypersurfaces Σ_t to slightly change around some fixed hypersurface Σ^0. We call $\varepsilon\Delta x(t,\mathbf{x})$ the proper distance between the boundary $\partial\Sigma_t$ and the fixed boundary $\partial\Sigma^0$ at the point $(t,\mathbf{x})$ along the direction normal and outwards to $\partial\Sigma^0$. Therefore, we have that

$$v_{\mathrm{B}}(t,\mathbf{x})\approx\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\Delta x(t,\mathbf{x}). \tag{31}$$

[8] For brevity, we do not expose the details of the quantisation procedure explicitly, see e.g. [2].

Finally, we require that $F(t)$ remains $O(\varepsilon)$, so that the solutions to the problem for $\varepsilon=0$ are modes with well-defined frequency. In "Appendix D" we prove that such value of $F(t)$ actually does not contribute at all to the physically relevant result of resonances. Therefore, once we have required that $F(t)$ remains $O(\varepsilon)$, without loss of generality we consider that $F(t)=0$.

We write down the quantities in (10) and (13) to first order in ε:

$$\begin{aligned}&\hat{\mathscr{O}}(\tilde{t})\approx\hat{\mathscr{O}}^0+\varepsilon\Delta\hat{\mathscr{O}}(\tilde{t}),\\&\bar{R}(\tilde{t})\approx\varepsilon\Delta\bar{R}(\tilde{t});\end{aligned} \tag{32}$$

where $\bar{R}(\tilde{t})$ vanishes when there is no perturbation because it only depends on time derivatives. The perturbation of the quantity $q(\tilde{t})$ in (12) does not directly appear in the perturbative regime, but rather its primitive with respect to $\tilde{t}$, given by

$$\Delta r(\tilde{t}):=\frac{1}{2}\left.\frac{\partial}{\partial\varepsilon}\log h(\tilde{t})\right|_{\varepsilon=0}. \tag{33}$$

As we prove in "Appendix D", we manage to write down the Bogoliubov coefficients without explicitly computing the solutions ${}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})$ and ${}^{\pm}\omega_n^{[\tilde{t}]}$ to first order in ε, by using the perturbation of the operator $\Delta\hat{\mathscr{O}}(\tilde{t})$ in (32). Thus, we only need the solutions to (16) and (17) for the static problem (for $\varepsilon=0$). We denote them as Ψ_n^0 and $\omega_n^0>0$ for the ${}^{+}\Psi_n^{[\tilde{t}]}$ modes. Therefore, those corresponding to the ${}^{-}\Psi_n^{[\tilde{t}]}$ modes are Ψ_n^0 and $-\omega_n^0$ (see "Appendix B"). These solutions satisfy

$$\hat{\mathscr{O}}^0\Psi_n^0=(\omega_n^0)^2\Psi_n^0, \tag{34}$$

$$\mathbf{n}\cdot\nabla_{h^0}\Psi_n^0(\mathbf{x})=0,\quad \mathbf{x}\in\partial\Sigma^0. \tag{35}$$

Being solutions to a static problem, they must also fulfil the orthonormalisation condition given in (B.7), namely,

$$\int_{\Sigma^0}\mathrm{d}V^0\,\Psi_n^0\Psi_m^0=\frac{\delta_{nm}}{2\omega_n^0}, \tag{36}$$

where $\mathrm{d}V^0$ is the volume element of Σ^0.

We want to solve the differential equation (25) in the perturbative regime. Let us first compute the coefficient $M\hat{V}(\tilde{t})M^*$ explicitly to zeroth order in ε. Using the solutions to zeroth order and (36) in (27) it is easy to check that

$$\hat{V}_{nm}^{\pm\hat{\pm}}=\hat{\pm}\omega_n^0\delta_{nm}\delta_{\mp\hat{\pm}}+O(\varepsilon). \tag{37}$$

Using now (22) and (26), we can write

$$M\hat{V}(\tilde{t})M^*\approx\mathrm{i}\Omega^0+\varepsilon\Delta K(\tilde{t}); \tag{38}$$

with

$$\Omega^0 := \mathrm{diag}(\omega_1^0, \omega_2^0, \ldots, -\omega_1^0, -\omega_2^0, \ldots),$$
$$\Delta K(\tilde t) := \begin{pmatrix} \Delta\hat\alpha(\tilde t) & \Delta\hat\beta(\tilde t) \\ \Delta\hat\beta(\tilde t)^* & \Delta\hat\alpha(\tilde t)^* \end{pmatrix}, \tag{39}$$

where the entries of $\Delta K(\tilde t)$ are the contributions to first order in ε, that depend on the perturbation and which explicit expressions we provide later on. If we introduce the result (38) in the differential equation (25), we clearly see that for $\varepsilon = 0$ the modes evolve just with a trivial phase with constant frequency $\pm\omega_n^0$, as one should expect for a static metric.

In order to properly compute the evolution to first order in ε, we shall first absorb any phase evolution, given by the diagonal terms. This is done by writing the evolution in terms of a new linear transformation $Q(\tilde t, \tilde t_0)$ defined by

$$Q(\tilde t, \tilde t_0) := \Theta(\tilde t)^* U(\tilde t, \tilde t_0); \tag{40}$$

where

$$\Theta(\tilde t) := \exp\left\{\int^{\tilde t} \mathrm{d}\tilde t' [\mathrm{i}\Omega^0 + \varepsilon \Delta A(\tilde t')]\right\},$$
$$\Delta A(\tilde t) := \mathrm{diag}(\Delta\hat\alpha_{11}, \Delta\hat\alpha_{22}, \ldots, -\Delta\hat\alpha_{11}, -\Delta\hat\alpha_{22}, \ldots). \tag{41}$$

Replacing (40) in (25), we get the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tilde t} Q(\tilde t, \tilde t_0) = \varepsilon \Theta^0(\tilde t)^* \Delta\bar K(\tilde t) \Theta^0(\tilde t) Q(\tilde t, \tilde t_0),$$
$$\Delta\bar K(\tilde t) := \Delta K(\tilde t) - \Delta A(\tilde t),$$
$$\Theta^0(\tilde t) := \mathrm{e}^{\mathrm{i}\Omega^0 \tilde t}; \tag{42}$$

where we dropped the terms to first order in ε from $\Theta(\tilde t)$ because of the overall factor ε appearing on the r.h.s. With the initial condition $Q(\tilde t_0, \tilde t_0) = I$, to first order in ε the transformation reads

$$Q(\tilde t_\mathrm{f}, \tilde t_0) \approx I + \varepsilon \int_{\tilde t_0}^{\tilde t_\mathrm{f}} \mathrm{d}\tilde t\; \Theta^0(\tilde t)^* \Delta\bar K(\tilde t) \Theta^0(\tilde t). \tag{43}$$

We can show the resonance behaviour of the field in a clear way if we write explicitly the expressions for the Bogoliubov coefficients:

$$\alpha_{nn}(\tilde t_\mathrm{f}, \tilde t_0) \approx 1;$$
$$\alpha_{nm}(\tilde t_\mathrm{f}, \tilde t_0) \approx \varepsilon \int_{\tilde t_0}^{\tilde t_\mathrm{f}} \mathrm{d}\tilde t\; \mathrm{e}^{-\mathrm{i}(\omega_n^0 - \omega_m^0)\tilde t} \Delta\hat\alpha_{nm}(\tilde t), \tag{44}$$
$$n \neq m;$$
$$\beta_{nm}(\tilde t_\mathrm{f}, \tilde t_0) \approx \varepsilon \int_{\tilde t_0}^{\tilde t_\mathrm{f}} \mathrm{d}\tilde t\; \mathrm{e}^{-\mathrm{i}(\omega_n^0 + \omega_m^0)\tilde t} \Delta\hat\beta_{nm}(\tilde t). \tag{45}$$

In general, the Bogoliubov transformation differs from the identity just by terms of first order in ε, except for the cases where there are resonances. That is, if the perturbation considered contains some characteristic frequency ω_p, then the same frequency is usually also present in the quantities $\Delta\hat\alpha_{nm}(\tilde t)$ and $\Delta\hat\beta_{nm}(\tilde t)$. If such frequency coincides with some difference between the frequencies of two modes, $\omega_\mathrm{p} = \omega_n^0 - \omega_m^0$ (it is in resonance), then the corresponding coefficient $\alpha_{nm}(\tilde t_\mathrm{f}, \tilde t_0)$ grows linearly with the time difference $\tilde t_\mathrm{f} - \tilde t_0$, and after enough time it will overcome the $O(\varepsilon)$. Respectively, if the characteristic frequency coincides with some sum between the frequencies of two modes, $\omega_\mathrm{p} = \omega_n^0 + \omega_m^0$, then the corresponding coefficient $\beta_{nm}(\tilde t_\mathrm{f}, \tilde t_0)$ grows linearly in time and eventually overcomes the $O(\varepsilon)$.[9]

If the Fourier transform $\mathscr{F}$ of $\Delta\hat\alpha_{nm}(\tilde t)$ [respectively $\Delta\hat\beta_{nm}(\tilde t)$] exists as a well-defined function, which necessarily implies that the perturbation vanishes fast enough in the asymptotic past and future, then another way to consider the resonances is by taking the limits $\tilde t_0 \to -\infty$ and $\tilde t_\mathrm{f} \to \infty$ in (44) and (45) and writing

$$\alpha_{nn}(-\infty, \infty) \approx 1;$$
$$\alpha_{nm}(-\infty, \infty) \approx \varepsilon\sqrt{2\pi}\; \mathscr{F}[\Delta\hat\alpha_{nm}](\omega_n^0 - \omega_m^0), \tag{46}$$
$$n \neq m;$$
$$\beta_{nm}(-\infty, \infty) \approx \varepsilon\sqrt{2\pi}\; \mathscr{F}[\Delta\hat\beta_{nm}](\omega_n^0 + \omega_m^0). \tag{47}$$

That is, the Bogoliubov coefficients between the asymptotic past and future are proportional to the Fourier transforms evaluated at the corresponding substraction (respectively addition) of frequencies. Evidently, resonances occur if the frequency spectrum is peaked around one or more of these values.

Let us remark that, in the presence of resonances, the physically meaningful modes for which the effects take place can be taken as the stationary modes given by $\{\Psi_n^0 \exp(\mp\mathrm{i}\omega_n^0 t)\}$. This is because the exact modes $\{{}^{\pm}\Phi_n^{[\tilde t]}(t)\}$ differ from them just to order ε (see "Appendix B"), which is the degree of indefiniteness of the well-defined frequency modes due to the perturbation. Only when the neat effect overcomes this order (and thus the degree of indefiniteness), the effect can be interpreted physically. We shall also mention that the resonance can be consistently described in the regime of duration of the perturbation $\Delta\tilde t$ such that $1 \ll \omega_\mathrm{p}\Delta\tilde t \ll 1/\varepsilon$. The reason is that one needs the period of time to be reasonably larger than the inverse of the frequency being described, but on the other hand, one should keep the second order term in ε that we dropped in (43) significantly smaller than the first order term that we kept. We refer to Sect. 4 and "Appendix E.2"

[9] In "Appendix E.1" of Part I we show that these resonances remain stable under small deviations of the frequency of the perturbation from the exact resonant frequency.

in Part I for additional discussion on the interpretation of the modes and on the regime of validity of the perturbative computation.

Finally, we provide the explicit expressions for the entries of $\Delta K(\tilde{t})$. They are computed in detail in "Appendix D". In order to find resonances, the following expressions can be used

$$\begin{aligned}\Delta\hat{\alpha}_{nm}(\tilde{t}) \equiv\, & \mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{-}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & + \mathrm{i}\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x(\tilde{t})\Big[(\nabla_{h^0}\Psi^0_n)\cdot(\nabla_{h^0}\Psi^0_m) \\ & + (\xi R^{h^0} + m^2 - \omega^0_n\omega^0_m)\Psi^0_n\Psi^0_m\Big], \qquad (48)\\ \Delta\hat{\beta}_{nm}(\tilde{t}) \equiv & -\mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{+}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & - \mathrm{i}\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x(\tilde{t})\Big[(\nabla_{h^0}\Psi^0_n)\cdot(\nabla_{h^0}\Psi^0_m) \\ & + (\xi R^{h^0} + m^2 + \omega^0_n\omega^0_m)\Psi^0_n\Psi^0_m\Big]; \qquad (49)\end{aligned}$$

where $\mathrm{d}S^0$ is the surface element of $\partial\Sigma^0$, ∇_{h^0} is the connection associated to the static metric h^0_{ij}, R^{h^0} its scalar curvature, and ${}^{\pm}_{m}\hat{\Delta}(\tilde{t})$ are linear operators defined by their action on the basis $\{\Psi^0_n\}$ as

$$\begin{aligned}&{}^{\pm}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n \\ &:= \big[\Delta\hat{\mathscr{O}}(\tilde{t}) + \omega^0_n(\omega^0_n \pm \omega^0_m)\Delta r(\tilde{t}) + \xi\Delta\bar{R}(\tilde{t})\big]\Psi^0_n.\end{aligned} \qquad (50)$$

The expressions in (44–50) are the second main result of this work. As we will show with concrete examples, they provide a very simple recipe for computing the resonance frequencies and amplitudes of a trapped quantum field in the perturbative regime. We highlight again that they do not even require the computation of the modes to first order in the perturbations, but only the solutions of the static problem in (34–36).

The symbol '$\equiv$' in (48) and (49) denotes the equivalence relation "gives the same resonances as". This means that the expressions in (48) and (49) have been simplified by dropping terms that are non-zero, but that nonetheless never contribute to the resonances when replaced in (44) and (45) [or in (46) and (47)]. Since resonances are the only physically meaningful result to be obtained from this computation, one can always use these expressions to compute the sensibility of the field to each resonance. We refer to "Appendix D", and again to Sect. 4 and "Appendix E.2" in Part I, for more details on the interpretation of the resonances and on the meaning of the "equivalence for resonances" relation given by '$\equiv$'.[10]

In the expressions (48) and (49) we can see a clear separation between the contributions due to the change of the metric (the volume integrals) and due to the motion of the boundaries (the surface integrals). As it must be the case, the first contributions are equivalent to those found in Part I [Eqs. (44) and (45)].

4.1 Example: dynamical Casimir effect

In order to provide an illustrative example, let us apply the method to arguably the simplest problem with moving boundary conditions, which is the Dynamical Casimir Effect for a minimally coupled ($\xi = 0$) massive scalar field in 1+1-dimensional Minkowski spacetime. The spacetime metric is simply

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2. \qquad (51)$$

The field is trapped inside a cavity of average proper length L, with the boundaries placed at x_- (left) and x_+ (right). The boundaries oscillate with frequency Ω and amplitude $\varepsilon L/2 \ll L$. We consider three different configurations for such oscillations:

$$\begin{aligned}&x_- = -L/2,\quad x_+ = L[1+\varepsilon\sin(\Omega t)]/2 && \text{(i)};\\ &x_\pm = \pm L[1+\varepsilon\sin(\Omega t)]/2 && \text{(ii)};\\ &x_\pm = \pm L[1\pm\varepsilon\sin(\Omega t)]/2 && \text{(iii)}.\end{aligned} \qquad (52)$$

In (i) only the right boundary oscillates, in (ii) the boundaries oscillate in opposite directions (the cavity expands and contracts) and in (iii) the boundaries oscillate in the same direction (the cavity shakes).

For the problem under consideration, it is straightforward to obtain the quantities needed to compute (48) and (49). In particular, we have that

$$\begin{aligned}&\Delta\hat{\mathscr{O}} = \Delta r = \Delta\bar{R} = {}^{\pm}_{m}\hat{\Delta} = 0;\\ &\Delta x(-L/2) = 0,\ \ \Delta x(L/2) = L\sin(\Omega t)/2 && \text{(i)};\\ &\Delta x(\pm L/2) = L\sin(\Omega t)/2 && \text{(ii)};\\ &\Delta x(\pm L/2) = \pm L\sin(\Omega t)/2 && \text{(iii)}.\end{aligned} \qquad (53)$$

We solve the problem both for Neumann and Dirichlet boundary conditions (see "Appendix G" for the expressions in this latter case). The eigenvalue Eq. (34) and the boundary conditions (35) [respectively (G.57)] read

$$\begin{aligned}&\big(-\partial_x^2 + m^2\big)\Psi^0_n = (\omega^0_n)^2\Psi^0_n;\\ &\pm\partial_x\Psi^0_n|_{x=\pm L/2} = 0 && \text{(Neumann)},\\ &\qquad\Psi^0_n(\pm L/2) = 0 && \text{(Dirichlet)};\end{aligned} \qquad (54)$$

[10] This equivalence relation, which concrete expressions are given in "Appendix D" [Eqs. (D.27) and (D.42)], can also be used to further simplify concrete expressions for the coefficients found in a specific problem.

and the solutions to these problems are

$$\Psi_n^0 = \frac{1}{\sqrt{L\omega_n^0}} \cos\left[k_n\left(x+\frac{L}{2}\right)\right] \qquad \text{(Neumann)},$$
$$\Psi_n^0 = \frac{1}{\sqrt{L\omega_n^0}} \sin\left[k_n\left(x+\frac{L}{2}\right)\right] \qquad \text{(Dirichlet)};$$
$$\omega_n^0 = \sqrt{k_n^2+m^2}; \qquad n \in \mathbb{N}^{(*)}; \tag{55}$$

where $k_n := \pi n/L$ and the mode with $n = 0$ is excluded for Dirichlet boundary conditions.

From (53) it is immediate that the first integral of the quantities (48) and (49) [respectively of (G.58) and (G.59)] vanishes. Since we are considering one spatial dimension, the "surface integral" is simply the evaluation of the integrand at the two boundaries. Plugging the corresponding quantities into (48) and (49), we easily obtain the solutions for Neumann boundary conditions:[11]

$$\Delta\hat{\alpha}_{nm}(t) \equiv -\Delta\hat{\beta}_{nm}(t) \equiv \frac{\mathrm{i}C_{nm}(\Omega^2-k_n^2-k_m^2)}{4\sqrt{\omega_n^0\omega_m^0}}\sin(\Omega t). \tag{56}$$

The factor C_{nm} depends on the oscillation configuration, and is given by

$$C_{nm} = (-1)^{n+m} \qquad \text{(i)},$$
$$C_{nm} = (-1)^{n+m}+1 \qquad \text{(ii)},$$
$$C_{nm} = (-1)^{n+m}-1 \qquad \text{(iii)}. \tag{57}$$

In order to obtain the expression in (56) we replaced $|\omega_n^0 - \omega_m^0| \to \Omega$ in the computation of $\Delta\hat{\alpha}_{nm}(t)$ and $\omega_n^0+\omega_m^0 \to \Omega$ in the computation of $\Delta\hat{\beta}_{nm}(t)$. This is legitimate within the equivalence relation with respect to resonances, since the only (positive) frequency present in the perturbation is Ω. Respectively, the solutions for Dirichlet boundary conditions are obtained by plugging the corresponding quantities into (G.58) and (G.59):

$$\Delta\hat{\alpha}_{nm}(t) \equiv -\Delta\hat{\beta}_{nm}(t) \equiv -\frac{\mathrm{i}C_{nm}k_nk_m}{2\sqrt{\omega_n^0\omega_m^0}}\sin(\Omega t). \tag{58}$$

In general, we find mode mixing and/or particle production due to the moving boundaries, which reproduces the Dynamical Casimir Effect. For example, if the frequency Ω coincides with the difference between the frequencies $|\omega_n^0 - \omega_m^0|$, plugging (56) into (44) we find that

$$\alpha_{nm}(t_\mathrm{f}, t_0) \approx \pm\varepsilon\frac{C_{nm}(\Omega^2-k_n^2-k_m^2)}{8\sqrt{\omega_n^0\omega_m^0}}(t_\mathrm{f}-t_0). \tag{59}$$

For long enough times, this quantity can overcome the order ε and become significant to zeroth order; that is, to the resonant modes of the cavity, and therefore significant mode mixing takes place between the corresponding modes. Analogous arguments apply for the β-coefficients and the corresponding particle creation, for any boundary conditions and configuration.

We notice that, out of the final results, we can take the limit of a massless field in a straightforward well-defined way.[12] The results with Dirichlet boundary conditions for configuration (i) in the massless case exactly reproduce the results obtained in [32] and independently in [33], while for configuration (iii) they reproduce the results obtained in [20] both for the massive and the massless case.

4.2 Example: gravitational wave resonance

Confined quantum fields undergo Bogoliubov transformations when perturbed by gravitational waves. This was shown in [33] considering a scalar field in a one-dimensional rigid trap. The authors proposed to exploit this effect in order to detect gravitational waves using phonons in a Bose-Einstein condensate. In Part I, we extended this work by computing the field transformations in the three-dimensional case considering free-falling boundary conditions (and thus static in the synchronous gauge). Free-falling boundary conditions were also studied in [34] using a different technique. Considering free-falling boundary conditions is interesting from a mathematical point of view. However, in practice, phononic gravitational wave detectors require inter-atomic interactions and thus, rigid or semi-rigid boundary conditions. The method introduced in this article enables the study of the phonon field transformations in a three-dimensional rigid or semi-rigid cavity. Therefore, the method will be useful in extending [33] to improve the detection of gravitational waves by using three-dimensional trapped Bose-Einstein condensates.

In particular, in this section we explicitly compute the Bogoliubov transformations for the phonon field when trapped in a fully rigid three-dimensional cavity. The phonon field can be described by a real scalar massless quantum field. In the case that the condensate remains stationary, the quantum field obeys a Klein–Gordon equation in an effective metric (with minimal coupling) which corresponds to the gravitational wave metric with the speed of sound in the condensate c_s replacing the speed of light in the g_{00} component [35–38]. We work in the TT-gauge and normalise the speed of sound $c_\mathrm{s} = 1$. We consider a wave with amplitude ε and

[11] Since we do not need to explicitly consider the evolution in time t of a basis of modes anymore, we can relax the notation and replace $\tilde{t} \to t$.

[12] The existing "zero-frequency mode" in the massless limit (in the case of Neumann boundary conditions) would not be normalisable, but the perturbation does not introduce any effect for it (all the coefficients would vanish for $n = 0$ or $m = 0$). Thus such mode can simply be ignored.

frequency Ω propagating in the z-direction and with polarisation in the xy-directions. Therefore, the metric is given by

$$
\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}t^2 + [1+\varepsilon\sin(\Omega t)]\mathrm{d}x^2 \\
&+[1-\varepsilon\sin(\Omega t)]\mathrm{d}y^2 + \mathrm{d}z^2,
\end{aligned}
\tag{60}
$$

where we have simplified $\sin[\Omega(t-z/c)] \to \sin(\Omega t)$, as we have that $c \gg \Omega L_z$ (being L_z the size of the condensate in the z-direction), because of the orders of magnitude between the speed of light and the speed of sound.

For simplicity, we consider that the field is trapped in a rectangular prism of proper lengths L_x, L_y and L_z aligned with the directions of propagation and polarisation of the wave. Since the cavity is rigid, these proper lengths must stay constant at all times. We consider that the centre of mass of the cavity (which by symmetry coincides with its geometrical centre) is in free-fall, and we fix it at the origin of the coordinate system. Therefore, the boundaries of the prism are placed at (in obvious notation):

$$
\begin{aligned}
x_\pm &= \pm\frac{L_x}{2\sqrt{1+\varepsilon\sin(\Omega t)}}, \\
y_\pm &= \pm\frac{L_y}{2\sqrt{1-\varepsilon\sin(\Omega t)}}, \\
z_\pm &= \pm\frac{L_z}{2}.
\end{aligned}
\tag{61}
$$

Although we are considering the physical problem of a massless field, we can take advantage of the versatility of our method and address the more general mathematical problem of a massive field with equal ease. The physical problem is then recovered by taking the massless limit. Therefore, from here on we consider $m \geq 0$. Thus, the eigenvalue equation (34) reads

$$
\left(-\partial_x^2 - \partial_y^2 - \partial_z^2 + m^2\right)\Psi^0_{nm\ell} = (\omega^0_{nm\ell})^2\Psi^0_{nm\ell};
\tag{62}
$$

where n, m and ℓ are quantum numbers. We first consider Dirichlet boundary conditions. The boundary conditions imposed to the static modes (G.57) are $\Psi^0_{nm\ell} = 0$ at the boundaries given in (61) for $\varepsilon = 0$. The solutions to this problem with the orthonormalisation in (36) are

$$
\begin{aligned}
\Psi^0_{nm\ell} &= \frac{2}{\sqrt{L_xL_yL_z\omega^0_{nm\ell}}}\sin\left[k^x_n\left(x+\frac{L_x}{2}\right)\right] \\
&\quad\times\sin\left[k^y_n\left(y+\frac{L_y}{2}\right)\right]\sin\left[k^z_n\left(z+\frac{L_z}{2}\right)\right], \\
\omega^0_{nm\ell} &= \sqrt{(k^x_n)^2+(k^y_m)^2+(k^z_\ell)^2+m^2}, \quad n,m,\ell\in\mathbb{N}^*;
\end{aligned}
\tag{63}
$$

where $k^x_n := \pi n/L_x$, and equivalently for the other dimensions. The remaining quantities needed to compute (G.58) and (G.59) are

$$
\begin{aligned}
&\Delta r = \Delta\bar{R} = 0, \\
&{}^{\pm}_{m}\hat{\Delta} = \Delta\hat{\mathscr{O}} = \sin(\Omega t)(\partial_x^2 - \partial_y^2); \\
&\Delta x(x = \pm L_x/2) = -L_x\sin(\Omega t)/4, \\
&\Delta x(y = \pm L_y/2) = L_y\sin(\Omega t)/4, \\
&\Delta x(z = \pm L_z/2) = 0.
\end{aligned}
\tag{64}
$$

Plugging all the quantities into (G.58) and (G.59), we obtain

$$
\begin{aligned}
\Delta\hat{\alpha}^{n'm'\ell'}_{nm\ell}(t) \equiv{}& \frac{\mathrm{i}\sin(\Omega t)}{4\sqrt{\omega^0_{nm\ell}\omega^0_{n'm'\ell'}}}\delta^{\ell'}_{\ell} \\
&\times\Big\{[(-1)^{n+n'}+1]k^x_nk^x_{n'}\delta^{m'}_m \\
&-[(-1)^{m+m'}+1]k^y_mk^y_{m'}\delta^{n'}_n\Big\},
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\Delta\hat{\beta}^{n'm'\ell'}_{nm\ell}(t) \equiv{}& \frac{\mathrm{i}\sin(\Omega t)}{2\omega^0_{nm\ell}}\left[(k^x_n)^2-(k^y_m)^2\right]\delta^{n'm'\ell'}_{nm\ell} \\
&-\Delta\hat{\alpha}^{n'm'\ell'}_{nm\ell}(t).
\end{aligned}
\tag{66}
$$

An equivalent procedure for Neumann boundary conditions yields the following results:

$$
\begin{aligned}
\Delta\hat{\alpha}^{n'm'\ell'}_{nm\ell}(t) \equiv{}& \frac{\mathrm{i}\sin(\Omega t)}{8\sqrt{\omega^0_{nm\ell}\omega^0_{n'm'\ell'}}}\delta^{\ell'}_{\ell} \\
&\times\Big\{[(-1)^{m+m'}+1][\Omega^2-(k^y_m)^2-(k^y_{m'})^2]\delta^{n'}_n \\
&-[(-1)^{n+n'}+1][\Omega^2-(k^x_n)^2-(k^x_{n'})^2]\delta^{m'}_m\Big\},
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\Delta\hat{\beta}^{n'm'\ell'}_{nm\ell}(t) \equiv{}& \frac{\mathrm{i}\sin(\Omega t)}{2\omega^0_{nm\ell}}\left[(k^x_n)^2-(k^y_m)^2\right]\delta^{n'm'\ell'}_{nm\ell} \\
&-\Delta\hat{\alpha}^{n'm'\ell'}_{nm\ell}(t);
\end{aligned}
\tag{68}
$$

where in this case the quantum numbers can take zero values.[13] Just as we did in the previous example in Sect. 4.1, we can use (44) and (45) to compute the linear growing in time of the Bogoliubov coefficients when resonances are present.

Let us give some physical interpretation to the results. The first term in (66) and (68) corresponds to the contribution of the perturbation of the metric, and coincides with the result in Sect. 4.1 of Part I for free-falling boundaries. The remaining contributions are due to the rigidity of the cavity, and therefore the motion of its boundaries in the TT-gauge. A direct comparison with the results in Sect. 4.1 clearly shows that these contributions correspond to a superposition of two

[13] As for the previous example, we have replaced $\tilde{t} \to t$ in the notation, and $|\omega^0_n \pm \omega^0_m| \to \Omega$ according to the equivalence relation with respect to resonances.

"anti-synchronised" Dynamical Casimir Effects in the two transversal directions, with configuration (ii) in the notation of Sect. 4.1 (with the cavity expanding and contracting). This is exactly the effect that one would expect from a gravitational wave on a rigid cavity, considering the concrete shape of the cavity and its interaction with the wave.

Thanks to the contributions due to the rigidity of the cavity, both mode-mixing and particle creation between different modes are present. We notice that this could not physically happen in the case of free-falling boundaries. The reason is that any mode-mixing or particle creation between *different* modes (non-diagonal Bogoliubov coefficients) implies local exchange of momentum with the field (in the basis of stationary modes that we are considering). However, a gravitational wave can provide momentum locally to a free field or to a free-falling boundary only in the direction of its propagation, something which in this case is negligible due to the orders of magnitude between the speed of light and the speed of sound. Hence, the direct effect due to the perturbation of the metric is pure cosmological particle creation, which does not exchange momentum and therefore can only affect the diagonal coefficients. On the contrary, the forces keeping the rigidity of the cavity redistribute the momentum so that it is locally non-zero (the boundaries move), and then transmit this momentum locally to the field (although of course the total momentum still vanishes).

Finally, and more interestingly, one can check that for a rigid cavity it is the diagonal quantities $\Delta\hat{\beta}^{nm\ell}_{nm\ell}(t)$ (the only non-zero quantities for free-falling boundaries) that vanish, for both boundary conditions. This means that the contribution due to the rigidity of the cavity exactly cancels the direct contribution from the change in the metric: Somehow the fact that the cavity keeps its own proper lengths shields the sensibility of the field to any length contractions and expansions from the metric. This is a novel and physically very plausible result. Nonetheless, we think that it is also a non-trivial result, which would be worthy exploring beyond the perturbative regime.

5 Summary and conclusions

In this second article we have extended the method developed in Part I for computing the evolution of a confined quantum scalar field in a globally hyperbolic spacetime, to the cases in which the timelike boundaries of the spacetime do not remain static in any synchronous gauge. Despite the more sophisticated technical construction required, we have shown that the core ideas of the method can still be used in such situation. Namely, we could construct bases of modes associated to different Cauchy hypersurfaces, a time-dependent linear transformation between them, and a first-order differential equation in time for such transformation. In this case, the coefficients of the transformation depend on the initial conditions of some auxiliary bases, that are solutions to an eigenvalue problem for which the time is just a parameter. If the time-dependent linear transformation connects two regions in which (thanks to a time symmetry) a valid Fock quantisation in terms of the bases of modes associated to each region is possible, then the linear transformation is actually a Bogoliubov transformation, and can be interpreted physically as such in terms of mode-mixing and particle creation between the different modes.

The extension of the method presented here is still of general applicability (as in Part I), just under some minor assumptions introduced in Sect. 2.1. However, we shall stress again that it proves to be especially useful to compute quantitative results on resonances in the perturbative regime (of the metric and the motion of the boundaries). Such usefulness stands out from the simple and practical expressions obtained in Sect. 4 (and at the end of "Appendix G"). We have also illustrated this fact with two examples within the perturbative regime which we could easily solve, namely the Dynamical Casimir Effect (where we reproduced and extended known results) and the perturbation of a field in a rigid cavity by a gravitational wave (which is a completely novel computation). We highlight how the simple expressions obtained [results (56–58) and (65–68), respectively] embrace several physical configurations in an unified way. The perturbative method could also prove its utility in other problems which are now under study, in which quantum systems are perturbed by small gravitational effects [9,39,40].

The main aim of this work, both of Part I and Part II, is to provide an useful method to compute the evolution of confined quantum fields in concrete physical situations. By applying the method to many different concrete physical problems, mainly (but not only) in the perturbative regime, we have provided plenty of evidence that the method is truly successful in this practical purpose. Specifically, with the examples provided in Parts I and II we reproduce previous results in [2,20,32–34,41]. Those results were found using very different approaches and techniques in each work, which implied longer and way more involved calculations. The method presented here manages to reproduce all of the results in an unified way and with a concise calculation for each case. Moreover, the method also extends some of those previous results, easily handling generalisations and variations of them; in particular, some non-trivial variations such as the rigid cavity under a gravitational wave perturbation considered in Sect. 4.2. Finally, the results obtained always had consistent physical interpretations. The method will surely prove fruitful in addressing many other relevant problems, and we expect it to become standard in the toolbox of Quantum Field Theory in Curved Spacetime for confined fields, especially in the perturbative regime.

Together with the promising practical applications of the method, there are also future directions of research on the theoretical side. In particular, these include the possible physical interpretations of the time-dependent linear transformations obtained, when they cannot be interpreted directly as Bogoliubov transformations between different Fock quantisations; and their possible connection to field-related (instead of particle-related) quantities. Further extensions of the method for different boundary conditions, quantum fields and/or metric gauges may be also approached.

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Appendix A: Why the procedure considered in Part I does not work here?

In Part I, we constructed the modes $\Phi_n^{[\tilde{t}]}(t)$ by imposing the following initial conditions:

$$\hat{\mathcal{O}}(\tilde{t})\Phi_n^{[\tilde{t}]}(\tilde{t}) = (\omega_n^{[\tilde{t}]})^2\Phi_n^{[\tilde{t}]}(\tilde{t}), \tag{A.1}$$

$$\partial_t\Phi_n^{[\tilde{t}]}(t)\Big|_{t=\tilde{t}} = -\mathrm{i}\omega_n^{[\tilde{t}]}\Phi_n^{[\tilde{t}]}(\tilde{t}). \tag{A.2}$$

In that case, we could solve the eigenvalue problem (A.1) because the boundary conditions could be written *separately* for the initial condition $\Phi(\tilde{t})$ and the partial time derivative $\partial_t\Phi(t)|_{t=\tilde{t}}$. Moreover, these boundary conditions were homogeneous for both quantities, which allowed us to impose Eq. (A.2) consistently. On the contrary, in the case where the boundary conditions are in the form of (6) (evaluated at $t = \tilde{t}$), they involve in the same equation *both* the gradient of $\Phi(\tilde{t})$ and the partial time derivative $\partial_t\Phi(t)|_{t=\tilde{t}}$ (except for the regions of the boundary which remain static). As a consequence, we cannot pose a valid eigenvalue problem just for $\Phi_n^{[\tilde{t}]}(\tilde{t})$, as we did in Part I. Therefore, we need to construct the basis of modes associated to each hypersurface $\Sigma_{\tilde{t}}$ as the solutions to an eigenvalue problem posed directly on the full space of initial conditions $(\Phi(\tilde{t}), \partial_t\Phi(t)|_{t=\tilde{t}})$, so that the boundary conditions for the problem can be properly imposed.

A way to summarise Eqs. (A.1) and (A.2) as an eigenvalue problem in the space of initial conditions would be to use the vector eigenvalue equation

$$\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i}\hat{\mathcal{O}}(\tilde{t}) & 0 \end{pmatrix}\begin{pmatrix} \Phi_n^{[\tilde{t}]}(\tilde{t}) \\ \partial_t\Phi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}} \end{pmatrix} = \omega_n^{[\tilde{t}]}\begin{pmatrix} \Phi_n^{[\tilde{t}]}(\tilde{t}) \\ \partial_t\Phi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}} \end{pmatrix}. \tag{A.3}$$

However, for moving boundary conditions in general the operator in this eigenvalue equation is not self-adjoint. In fact, it is straightforward to find simple problems with moving boundary conditions for which (A.3) has no solutions with real $\omega_n^{[\tilde{t}]}$. In conclusion: In general, modes with "locally well-defined frequency" with respect to time t [which is how we may call the modes satisfying (A.2)] cannot be solutions to the problem because they do not even fulfil the boundary conditions when the boundaries are moving. This is to be expected, since modes satisfying (A.2) would be "locally stationary" oscillations in phase with respect to t around $t = \tilde{t}$, and therefore could not readjust to any motion of the boundary to first order in t.

In order to break this impasse we have considered a Wick rotation in the coordinate time. This rotation has transformed Eq. (A.3) into the eigenvalue equation (14), which always provides valid bases of initial conditions, although they do not correspond to modes with "locally well-defined frequency".[14] This Wick rotation is then reversed by the linear transformation (21) from the modes $\Psi_n^{[\tilde{t}]}$ to the modes ${}^{\pm}\Phi_n^{[\tilde{t}]}$. Neither these modes can be directly interpreted as modes with "locally well-defined frequency" in general. However, this is completely irrelevant, since in the regions S described in Property II they do behave as modes with well-defined frequency; and this is all we really need, since those are the only regions where the modes can be used for quantisation. Analogous arguments apply for the case of Dirichlet vanishing boundary conditions as developed in "Appendix G".

Finally, let us notice that, since we are working with different bases of modes as compared to Part I, the time-dependent linear transformation (28) does not even formally satisfy any Bogoliubov identities. This is because, in general, the bases related are not orthonormal in the Klein-Gordon scalar product. However, the cases in which the transformation can be interpreted as a Bogoliubov transformation are those in which (according to Property II) the bases related are indeed orthonormal. Therefore, in such cases the coefficients of the transformation do satisfy the Bogoliubov identities.

Appendix B: Proof of the fulfilment of Properties I and II

Let us check first Property I, namely that the set $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$ is a basis of the space of solutions to the Klein–Gordon equation. We first realise that it is equivalent to check that the set $\{\Psi_n^{[\tilde{t}]}\} = \{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$ is also a basis of the space of solutions, since the set $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$ is obtained by the linear invertible transformation (21) of the elements in $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$. Because of Condition B and the linearity of the Klein–Gordon equation, this is the case if and only if the set of initial conditions $\{(\Psi_n^{[\tilde{t}]}(\tilde{t}), \partial_t\Psi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}})\}$ is a basis of the space of initial conditions at $\Sigma_{\tilde{t}}$, which is clearly the space $L^2(\Sigma_{\tilde{t}}) \oplus L^2(\Sigma_{\tilde{t}})$. This is definitely fulfilled, since this set of initial conditions is constructed with the eigenvectors of the self-adjoint operator $\hat{\mathscr{M}}(\tilde{t})$, that are solutions to the eigenvalue problem (14).

In order to check Property II, let us consider a spacetime region S in which the conditions listed in Property II hold. Then, it is easy to check that, for each mode ${}^{+}\Psi_n^{[\tilde{t}]}(t)$ with positive eigenvalue ${}^{+}\omega_n^{[\tilde{t}]} > 0$ there is a corresponding mode ${}^{-}\Psi_n^{[\tilde{t}]}(t)$ with negative eigenvalue ${}^{-}\omega_n^{[\tilde{t}]} = -({}^{+}\omega_n^{[\tilde{t}]})$, and with the same initial condition ${}^{-}\Psi_n^{[\tilde{t}]}(\tilde{t}) = {}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t})$. If we put each mode of the pair in one of the subsets $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$ and we use the same index n for them, then the modes ${}^{\pm}\Phi_n^{[\tilde{t}]}$ satisfy the following initial conditions in $\Sigma_{\tilde{t}}$:

$$\begin{aligned} {}^{\pm}\Phi_n^{[\tilde{t}]}(\tilde{t}) &= {}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t}), \\ \partial_t {}^{\pm}\Phi_n^{[\tilde{t}]}(t)|_{t=\tilde{t}} &= \mp \mathrm{i}({}^{+}\omega_n^{[\tilde{t}]})\,{}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t}). \end{aligned} \tag{B.4}$$

Moreover, the eigenvalue problem (14) has the same solutions for all the hypersurfaces Σ_t within S. Taking into account these facts, and also that we are considering only real spatial functions ${}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})$, the modes of the form (23) satisfy both the initial conditions (B.4) in $\Sigma_{\tilde{t}}$ and the Klein–Gordon equation (11) in the whole region S. Therefore, these modes correspond to the modes that we are actually assigning to the hypersurface $\Sigma_{\tilde{t}}$. Now, if the interval of time that S embraces is large enough so as to explore the minimum frequency in the spectrum, then we can talk about the modes (23) as modes with well-defined positive and negative frequency with respect to t.

In Sect. 3.1 we claimed that, for the problems in which we can give a physical interpretation to the results (that is, the problems where Fock quantisation is possible at least in some regions), there is an infinite number of both positive and negative eigenvalues $\omega_n^{[\tilde{t}]}$ for every $\tilde{t}$. Now we can justify this claim. The reason is that, for regular metrics, the eigenvalues obtained at the different hypersurfaces $\Sigma_{\tilde{t}}$ should be a continuous function of $\tilde{t}$. Since, at least in the regions where Fock quantisation is possible, these eigenvalues are indeed divided into infinitely many positive and negative, and since in no case $\omega_n^{[\tilde{t}]} = 0$ (see "Appendix H"), then they must stay divided in such way even in the regions where Fock quantisation is not possible.

It remains to be proven that, under the conditions given in Property II for the region S, the basis $\{{}^{\pm}\Phi_n^{[\tilde{t}]}\}$ is orthonormal. First, let us write down the Klein-Gordon inner product (4) between the modes ${}^{\pm}\Psi_n^{[\tilde{t}]}$. Using (18) we get

$$\begin{aligned} &\left\langle {}^{\pm}\Psi_n^{[\tilde{t}]}, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]} \right\rangle \\ &= \mathrm{i}\left[\left({}^{\pm}\omega_n^{[\tilde{t}]}\right) - \left({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}\right)\right] \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})\, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}). \end{aligned} \tag{B.5}$$

We need to evaluate the integral in (B.5). In order to do so, we evaluate the inner product (7) between the initial conditions of these two modes and compare this evaluation with the orthonormalisation condition that we imposed in (20). Using (14) and Green's first identity, we obtain

$$\begin{aligned} &{}^{\pm}\omega_n^{[\tilde{t}]}\left[\left({}^{\pm}\omega_n^{[\tilde{t}]}\right) + \left({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}\right)\right] \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})\, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ &+ \int_{\partial\Sigma_{\tilde{t}}} \mathrm{d}S_{\tilde{t}}\, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t})\, \mathbf{n}\cdot\nabla_{h(\tilde{t})}\, {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) = |{}^{\pm}\omega_n^{[\tilde{t}]}|\delta_{nm}\delta_{\pm\hat{\pm}}. \end{aligned} \tag{B.6}$$

[14] The fact that we are immersed in the Wick rotation while manipulating the $\Psi_n^{[\tilde{t}]}$ modes, is what forces us to use real eigenvectors for their initial conditions.

Equation (B.6) is general, that is, we have not used yet the specific conditions holding in region S. Let us now impose those conditions. In particular, imposing $v_{\mathrm{B}}(t)=0$ and using (17) we have that the surface integral vanishes. If we pick now the '++' sign prescripts, noticing that ${}^{+}\omega_n^{[\tilde{t}]}>0$ we obtain

$$\int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}}\, {}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t})\, {}^{+}\Psi_m^{[\tilde{t}]}(\tilde{t}) = \frac{\delta_{nm}}{2\left({}^{+}\omega_n^{[\tilde{t}]}\right)}. \tag{B.7}$$

Moreover, we also have that ${}^{+}\Psi_n^{[\tilde{t}]}(\tilde{t}) = {}^{-}\Psi_m^{[\tilde{t}]}(\tilde{t})$, so the previous equation holds for any sign prescripts of the spatial functions. Using this fact and ${}^{-}\omega_n^{[\tilde{t}]} = -({}^{+}\omega_n^{[\tilde{t}]})$, we can finally compute the Klein-Gordon inner product in (B.5), obtaining

$$\langle {}^{\pm}\Psi_n^{[\tilde{t}]}, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]} \rangle = \pm \mathrm{i}\delta_{nm}\delta_{\mp\hat{\pm}}. \tag{B.8}$$

Finally, using the linear transformation (21), we get

$$\left\langle {}^{\pm}\Phi_n^{[\tilde{t}]}, {}^{\hat{\pm}}\Phi_m^{[\tilde{t}]} \right\rangle = \pm\delta_{nm}\delta_{\pm\hat{\pm}}, \tag{B.9}$$

which proves that the basis is orthonormal.

Appendix C: Proof of the differential equation for the transformation

Let us call $V(\tilde{t},\tilde{t}_0)$ the linear transformation between the basis $\{{}^{\pm}\Psi_n^{[\tilde{t}_0]}\}$ and the basis $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$. By composition of linear transformations, it is clear that

$$V(\tilde{t}+\Delta\tilde{t},\tilde{t}_0) = V(\tilde{t}+\Delta\tilde{t},\tilde{t})V(\tilde{t},\tilde{t}_0). \tag{C.10}$$

We can use this relation to obtain the following differential equation for $V(\tilde{t},\tilde{t}_0)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}V(\tilde{t},\tilde{t}_0) &= \frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}V(\tilde{t}+\Delta\tilde{t},\tilde{t}_0)\bigg|_{\Delta\tilde{t}=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}V(\tilde{t}+\Delta\tilde{t},\tilde{t})\bigg|_{\Delta\tilde{t}=0} V(\tilde{t},\tilde{t}_0).\end{aligned} \tag{C.11}$$

Notice now that, because of the relation (21) between bases, we have that the linear transformation $U(\tilde{t},\tilde{t}_0)$ is related to the linear transformation $V(\tilde{t},\tilde{t}_0)$ by

$$U(\tilde{t},\tilde{t}_0) = MV(\tilde{t},\tilde{t}_0)M^*, \tag{C.12}$$

since $M^{-1}=M^*$. Then, if we call $\hat{V}(\tilde{t})$ the first factor on the r.h.s. of (C.11), replacing (C.12) we get the differential equation (25). Therefore, what remains to be obtained are the expressions for $\hat{V}(\tilde{t})$ given by (26) and (27) out of the definition

$$\hat{V}(\tilde{t}) := \frac{\mathrm{d}}{\mathrm{d}(\Delta\tilde{t})}V(\tilde{t}+\Delta\tilde{t},\tilde{t}_0)\bigg|_{\Delta\tilde{t}=0}, \tag{C.13}$$

where $V(\tilde{t}+\Delta\tilde{t},\tilde{t})$ is so far only implicitly defined by being the linear transformation between the $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$ bases at different times.

Since the basis $\{{}^{\pm}\Psi_n^{[\tilde{t}]}\}$ has the corresponding orthonormalisation given by (20), it is clear that we can compute the elements of $V(\tilde{t}+\Delta\tilde{t},\tilde{t})$ as

$$V_{nm}^{\pm\hat{\pm}}(\tilde{t}+\Delta\tilde{t},\tilde{t}) = \frac{1}{|{}^{\hat{\pm}}\omega_m^{[\tilde{t}]}|}\left\langle {}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}, {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}\right\rangle_{\Sigma_{\tilde{t}}}. \tag{C.14}$$

According to the definition of the inner product between modes associated to the hypersurface $\Sigma_{\tilde{t}}$, which is given by the inner product in (7) with the values of the corresponding modes and their first time derivatives at $\Sigma_{\tilde{t}}$ as entries, we have that

$$\begin{aligned}&\hat{V}_{nm}^{\pm\hat{\pm}}(\tilde{t})\\ &= \frac{1}{|{}^{\hat{\pm}}\omega_m^{[\tilde{t}]}|}\frac{\mathrm{d}}{\mathrm{d}\Delta\tilde{t}}V_{nm}^{\pm\hat{\pm}}(\tilde{t}+\Delta\tilde{t},\tilde{t})\\ &= \frac{1}{|{}^{\hat{\pm}}\omega_m^{[\tilde{t}]}|}\frac{\mathrm{d}}{\mathrm{d}\Delta\tilde{t}}\left\{\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\;\partial_t{}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}}\;\partial_t{}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(t)\Big|_{t=\tilde{t}}\right.\\ &\quad+\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\left[\xi R^h(\tilde{t})+m^2+F(\tilde{t})\right]{}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}){}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t})\\ &\quad\left.+\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\left[\nabla_{h(\tilde{t})}{}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t})\right]\cdot\left[\nabla_{h(\tilde{t})}{}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t})\right]\right\}.\end{aligned} \tag{C.15}$$

We need to compute the derivative with respect to $\Delta\tilde{t}$ of the quantities inside the integrals. For that, we use the *local* evolution in time around $\tilde{t}+\Delta\tilde{t}$ (and to first order in $\Delta\tilde{t}$) of ${}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)$, which is given by (18); and of $\partial_t{}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)$, which we obtain through the Klein–Gordon equation. Notice that conditions (16) and (18), when replaced in the Klein–Gordon equation as written in (11) and evaluated at $t=\tilde{t}$, imply that

$$\begin{aligned}\partial_t^2{}^{\pm}\Psi_n^{[\tilde{t}]}(t)\Big|_{t=\tilde{t}} =& -({}^{\pm}\omega_n^{[\tilde{t}]})\left[{}^{\pm}\omega_n^{[\tilde{t}]}+q(\tilde{t})\right]{}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})\\ &-\left[\xi\bar{R}(\tilde{t})-F(\tilde{t})\right]{}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}).\end{aligned} \tag{C.16}$$

That is, out of the initial conditions, the Klein–Gordon equation provides the value of the second time derivative of ${}^{\pm}\Psi_n^{[\tilde{t}]}(t)$ at $t=\tilde{t}$ [and *only* at $t=\tilde{t}$, Eq. (C.16) is evidently *not* a differential equation in time].

Because of relations (18) and (C.16) (replacing $\tilde{t} \to \tilde{t} + \Delta\tilde{t}$), we have that

$$\begin{aligned} &{}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}) \\ &\quad = {}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}+\Delta\tilde{t})\left[1 - ({}^{\pm}\omega_n^{[\tilde{t}]})\Delta\tilde{t}\right] + \mathcal{O}\left(\Delta\tilde{t}\right)^2, \end{aligned} \tag{C.17}$$

$$\begin{aligned} &\partial_t {}^{\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(t)\Big|_{t=\tilde{t}} \\ &\quad = {}^{\pm}\omega_n^{[\tilde{t}+\Delta\tilde{t}]\pm}\Psi_n^{[\tilde{t}+\Delta\tilde{t}]}(\tilde{t}+\Delta\tilde{t})\left\{1 + \left[{}^{\pm}\omega_n^{[\tilde{t}]} + q(\tilde{t})\right]\Delta\tilde{t}\right\} \\ &\qquad + \left[\xi\bar{R}(\tilde{t}) - F(\tilde{t})\right]{}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t})\Delta\tilde{t} + \mathcal{O}\left(\Delta\tilde{t}\right)^2. \end{aligned} \tag{C.18}$$

This is the local evolution of the needed quantities to first order in $\Delta\tilde{t}$, and therefore we are ready to compute the derivative in (C.15). Plugging (C.17) and (C.18) into (C.15), and using Green's first identity, the orthonormalisation condition in (20) and the boundary condition (17), after a tedious but straightforward calculation we obtain the expression for the elements in (27). This completes the proof of equation (25).

Appendix D: Derivation of the expressions in the perturbative regime

The entries $\Delta\hat{\alpha}_{nm}(\tilde{t})$ and $\Delta\hat{\beta}_{nm}(\tilde{t})$ in $\Delta K(\tilde{t})$ (and their respective complex conjugates) correspond to the first order contributions in ε to the factor $M\hat{V}(\tilde{t})M^*$ in (25). In order to compute them, we temporarily introduce the expressions for the solutions to first order in ε:[15]

$$\begin{aligned} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) &\approx \Psi_n^0 + \varepsilon {}^{\pm}\Delta\Psi_n, \\ {}^{\pm}\omega_n^{[\tilde{t}]} &\approx \pm(\omega_n^0 + \varepsilon {}^{\pm}\Delta\omega_n). \end{aligned} \tag{D.19}$$

Just as the solutions to the static problem satisfied the zeroth order in ε of (16) and (17) [given by (34) and (35)], the perturbations appearing in (D.19) must satisfy the first order in ε of those equations, which reads

$$\hat{\mathcal{O}}^0({}^{\pm}\Delta\Psi_n) + \Delta\hat{\mathcal{O}}\Psi_n^0 = (\omega_n^0)^2({}^{\pm}\Delta\Psi_n) + 2\omega_n^0({}^{\pm}\Delta\omega_n)\Psi_n^0, \tag{D.20}$$

$$\mathbf{n}\cdot\nabla_{h^0}{}^{\pm}\Delta\Psi_n + \Delta x\,\mathbf{n}\cdot\nabla_{h^0}\left(\mathbf{n}\cdot\nabla_{h^0}\Psi_n^0\right) = \mp\omega_n^0\Psi_n^0\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x; \tag{D.21}$$

where all the quantities in the second equation are evaluated at the boundary $\partial\Sigma^0$. The second term of the second equation takes into account the displacement of the point where the boundary condition is imposed. We also temporarily introduce the perturbation of $F(\tilde{t})$ to first order in ε (since we need to prove here that indeed it does not contribute to the resonances):

$$F \approx \varepsilon\Delta F. \tag{D.22}$$

Let us calculate the first order in ε of the quantity $\hat{V}_{nm}^{\pm\hat{\pm}}$ in (27), which in terms of the perturbed quantities in (32), (D.19) and (D.22) reads

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\hat{V}_{nm}^{\pm\hat{\pm}}\Big|_{\varepsilon=0} = \hat{\pm}\Bigg\{ &-({}^{\pm}\Delta\omega_n)\delta_{nm}\delta_{\pm\hat{\pm}} \\ &+(\pm\omega_n^0\hat{\pm}\omega_m^0)\int_{\Sigma^0}\mathrm{d}V^0\left(\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}{}^{\pm}\Delta\Psi_n\right)\Psi_m^0 \\ &+2({}^{\pm}\Delta\omega_n)\delta_{nm} + \left(\pm\frac{\mathrm{d}^{\pm}\Delta\omega_n}{\mathrm{d}\tilde{t}} - \Delta F\right)\frac{\delta_{nm}}{2\omega_n^0} \\ &+2(\omega_n^0)^2\left[\int_{\Sigma^0}\mathrm{d}V^0({}^{\pm}\Delta\Psi_n)\Psi_m^0 + \Delta J_{nm}^{\hat{\pm}}\right] \\ &+\int_{\Sigma^0}\mathrm{d}V^0\,\Psi_n^0\left(\pm\omega_n^0\Delta q + \xi\Delta\bar{R}\right)\Psi_m^0\Bigg\}, \end{aligned} \tag{D.23}$$

where $\Delta q := \partial_\varepsilon q|_{\varepsilon=0}$ and $\Delta J_{nm}^{\pm}$ is defined as

$$\Delta J_{nm}^{\pm} := \frac{\partial}{\partial\varepsilon}\int_{\Sigma_{\tilde{t}}}\mathrm{d}V_{\tilde{t}}\,\Psi_n^{0\pm}\Psi_m^{[\tilde{t}]}(\tilde{t})\Big|_{\varepsilon=0}. \tag{D.24}$$

This quantity contains contributions from ${}^{\pm}\Psi_m^{[\tilde{t}]}(\tilde{t})$, but also from $\mathrm{d}V_{\tilde{t}}$ and from the change in the domain $\Sigma_{\tilde{t}}$, since these two objects depend on ε. In "Appendix E" we prove that this quantity takes the value

$$\begin{aligned} \Delta J_{nm}^{\pm} = -\frac{1}{2(\omega_n^0)^2}\Bigg\{&\frac{1}{2}\left[({}^{+}\Delta\omega_n) + ({}^{-}\Delta\omega_n)\right]\delta_{nm} \\ &+\omega_n^0(\omega_n^0 \pm \omega_m^0)\int_{\Sigma^0}\mathrm{d}V^0\,{}^{+}\Delta\Psi_n\Psi_m^0 \\ &+\omega_n^0(\omega_n^0 \mp \omega_m^0)\int_{\Sigma^0}\mathrm{d}V^0\,{}^{-}\Delta\Psi_n\Psi_m^0\Bigg\}. \end{aligned} \tag{D.25}$$

We provide the explicit computation of $\Delta\hat{\beta}_{nm}(\tilde{t})$. The computations of the other entries in $\Delta K(\tilde{t})$ follow an analogous procedure. Using (22) and (26) we have that

$$\Delta\hat{\beta}_{nm}(\tilde{t}) = \frac{1}{2}\frac{\partial}{\partial\varepsilon}\left[\hat{V}_{nm}^{+-} + \hat{V}_{nm}^{-+} + \mathrm{i}(\hat{V}_{nm}^{--} - \hat{V}_{nm}^{++})\right]_{\varepsilon=0}. \tag{D.26}$$

The first part of the computation consists of plugging the expressions (D.23) and (D.25) into (D.26). We also use at this point the equivalence relation with respect to resonances for the terms appearing in $\Delta\hat{\beta}_{nm}(\tilde{t})$, which reads

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} \equiv \mathrm{i}(\omega_n^0 + \omega_m^0)X(\tilde{t}), \tag{D.27}$$

[15] From here on, we omit the explicit dependence on $\tilde{t}$ for most of the quantities.

where $X(\tilde{t})$ can be any function of $\tilde{t}$. As we advanced, this equivalence means that, although the two quantities are not the same, when $X(\tilde{t})$ contains a term with the correct resonant frequency $\omega_n^0 + \omega_m^0$, the contribution of this term to the Bogoliubov coefficient (45) or (47) is the same for the two quantities in (D.27). We refer to Appendix E.2 in Part I for a careful mathematical derivation of this equivalence. We use the equivalence in order to simplify the derivatives with respect to $\tilde{t}$ appearing in (D.23). After a tedious but mechanical calculation, we obtain

$$\begin{aligned}\Delta\hat{\beta}_{nm}(\tilde{t}) \equiv & -\mathrm{i}\left[(\omega_n^0)^2 - (\omega_m^0)^2\right]\int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Psi_n\Psi_m^0 \\ & -\mathrm{i}\int_{\Sigma^0}\mathrm{d}V^0\ \Psi_n^0\left[\omega_n^0(\omega_n^0+\omega_m^0)\Delta r + \xi\Delta\bar{R}\right]\Psi_m^0 \\ & -\mathrm{i}\left(\Delta\omega_n - \frac{\Delta F}{2\omega_n^0}\right)\delta_{nm}; \qquad \text{(D.28)}\end{aligned}$$

where Δr is given by (33), and we have defined the quantities

$$\Delta\Psi_n := \frac{1}{2}\left[(1-\mathrm{i})^+\Delta\Psi_n + (1+\mathrm{i})^-\Delta\Psi_n\right], \qquad \text{(D.29)}$$

$$\Delta\omega_n := \frac{1}{2}\left[(1-\mathrm{i})^+\Delta\omega_n + (1+\mathrm{i})^-\Delta\omega_n\right]. \qquad \text{(D.30)}$$

The second part of the computation consists of a first attempt for getting rid of the perturbations of the solutions $\Delta\Psi_n$ and $\Delta\omega_n$ (usually harder to compute for a given problem) by introducing the perturbation of the operator $\Delta\hat{\mathscr{O}}$ in (32) (which is in principle trivial to compute). By considering a linear combination of (D.20) for the positive and negative prescripts, we can write that equation for the quantities defined in (D.29) and (D.30):

$$\hat{\mathscr{O}}^0\Delta\Psi_n + \Delta\hat{\mathscr{O}}\Psi_n^0 = (\omega_n^0)^2\Delta\Psi_n + 2\omega_n^0\Delta\omega_n\Psi_n^0. \qquad \text{(D.31)}$$

Using (34) and (D.31), Green's second identity and the known properties of the solutions to zeroth order, we can do the following calculation:

$$\begin{aligned}(\omega_m^0)^2\int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Psi_n\Psi_m^0 &= \int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Psi_n(\hat{\mathscr{O}}^0\Psi_m^0) \\ &= \int_{\Sigma^0}\mathrm{d}V^0(\hat{\mathscr{O}}^0\Delta\Psi_n)\Psi_m^0 + \int_{\partial\Sigma^0}\mathrm{d}S^0\ \Psi_m^0\mathbf{n}\cdot\nabla_{h^0}\Delta\Psi_n \\ &= -\int_{\Sigma^0}\mathrm{d}V^0(\Delta\hat{\mathscr{O}}\Psi_n^0)\Psi_m^0 + (\omega_n^0)^2\int_{\Sigma^0}\mathrm{d}V^0\ \Delta\Psi_n\Psi_m^0 \\ &\quad + \Delta\omega_n\delta_{nm} + \int_{\partial\Sigma^0}\mathrm{d}S^0\ \Psi_m^0\mathbf{n}\cdot\nabla_{h^0}\Delta\Psi_n. \qquad \text{(D.32)}\end{aligned}$$

Rearranging terms between the first and the last line we can use this computation to simplify (D.28). In particular, the direct contribution of ΔF in (D.28) cancels out with its contribution through $\Delta\hat{\mathscr{O}}$. Also, after the simplification the only remaining quantity that could depend on ΔF is $\mathbf{n}\cdot\nabla_{h^0}\Delta\Psi_n$ in the surface integral in (D.32). But from (D.21) and (D.29) one can see that such quantity evaluated at the surface is fully determined by other quantities which do not depend on ΔF. Therefore, we have already proven that ΔF does not contribute to the resonances, and from here on we can consider again $F = 0$. Consequently, (D.28) becomes

$$\begin{aligned}\Delta\hat{\beta}_{nm}(\tilde{t}) \equiv & -\mathrm{i}\int_{\Sigma^0}\mathrm{d}V^0\ ({}^+_m\hat{\Delta}\Psi_n^0)\Psi_m^0 \\ & +\mathrm{i}\int_{\partial\Sigma^0}\mathrm{d}S^0\ \Psi_m^0\mathbf{n}\cdot\nabla_{h^0}\Delta\Psi_n, \qquad \text{(D.33)}\end{aligned}$$

where ${}^+_m\hat{\Delta}(\tilde{t})$ has been defined in (50).

We can see that the perturbation of the modes still appears in the remaining surface integral. The third and last part of the computation consists of further manipulating this surface integral in order to fully get rid of the perturbation of the modes. In this last part of the computation for convenience we consider the notation simplifications $h^0 \to h$ and $\nabla_{h^0} \to \nabla$. We notice that the boundary condition (D.21) for the quantities in (D.29) and (D.30) reads

$$\mathbf{n}\cdot\nabla\Delta\Psi_n + \Delta x\mathbf{n}\cdot\nabla\left(\mathbf{n}\cdot\nabla\Psi_n^0\right) = \mathrm{i}\omega_n^0\Psi_n^0\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x. \qquad \text{(D.34)}$$

We can use this boundary condition and again the equivalence (D.27) to write

$$\begin{aligned}\mathbf{n}\cdot\nabla\Delta\Psi_n \equiv & -\Delta x\ \mathbf{n}\cdot\nabla\left(\mathbf{n}\cdot\nabla\Psi_n^0\right) \\ & -\omega_n^0(\omega_n^0+\omega_m^0)\Psi_n^0\Delta x. \qquad \text{(D.35)}\end{aligned}$$

We split the first term on the r.h.s. into two terms using index notation:

$$\mathbf{n}\cdot\nabla\left(\mathbf{n}\cdot\nabla\Psi_n^0\right) = (n^i\nabla_i n^j)\nabla_j\Psi_n^0 + n^i n^j\nabla_i\nabla_j\Psi_n^0. \qquad \text{(D.36)}$$

In "Appendix F" we prove the following relation [42]:

$$n^i\nabla_i n^j = \frac{1}{\Delta x}(n^i n^j - h^{ij})\nabla_i\Delta x. \qquad \text{(D.37)}$$

We also notice that

$$n^i n^j\nabla_i\nabla_j = (h^{ij} - {}^\partial h^{ij})\nabla_i\nabla_j = \nabla^2 - D^2, \qquad \text{(D.38)}$$

where ${}^\partial h^{ij}$ is the induced metric on the boundary $\partial\Sigma^0$ and D its associated connection. If we replace the results (D.35–D.38) in the surface integral in (D.33), we obtain

$$
\begin{aligned}
&\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Psi_m^0 \mathbf{n}\cdot\nabla\Delta\Psi_n \\
&\equiv \int_{\partial\Sigma^0} \mathrm{d}S^0 (\nabla^i \Delta x)(\nabla_i \Psi_n^0)\Psi_m^0 \\
&\quad - \int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x (\nabla^2 \Psi_n^0)\Psi_m^0 \\
&\quad + \int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x (D^2 \Psi_n^0)\Psi_m^0 \\
&\quad -\omega_n^0(\omega_n^0+\omega_m^0)\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x\, \Psi_n^0\Psi_m^0. \qquad \text{(D.39)}
\end{aligned}
$$

Let us work out the first integral on the r.h.s. We can use the boundary condition (35) to exchange the full connection ∇ and the induced connection D any time the normal component vanishes. With this in mind, we have:

$$
\begin{aligned}
&\int_{\partial\Sigma^0} \mathrm{d}S^0 (\nabla^i \Delta x)(\nabla_i \Psi_n^0)\Psi_m^0 \\
&= \int_{\partial\Sigma^0} \mathrm{d}S^0 (D^i \Delta x)(D_i \Psi_n^0)\Psi_m^0 \\
&= -\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x (D^2 \Psi_n^0)\Psi_m^0 \\
&\quad - \int_{\partial\Sigma^0} \mathrm{d}S^0 \Delta x (D_i \Psi_n^0)(D^i \Psi_m^0) \\
&= -\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x (D^2 \Psi_n^0)\Psi_m^0 \\
&\quad - \int_{\partial\Sigma^0} \mathrm{d}S^0 \Delta x (\nabla_i \Psi_n^0)(\nabla^i \Psi_m^0), \qquad \text{(D.40)}
\end{aligned}
$$

where in the second step we have used the divergence theorem and the fact that $\partial\Sigma^0$ has no boundary.

The second integral on the r.h.s. of (D.39) can be simplified with the definition of the operator $\hat{\mathcal{O}}^0$ and Eq. (34), obtaining

$$
\begin{aligned}
&-\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x (\nabla^2 \Psi_n^0)\Psi_m^0 \\
&= \int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x \left[\left(\omega_n^0\right)^2 - \xi R^{h^0} - m^2\right] \Psi_n^0\Psi_m^0. \qquad \text{(D.41)}
\end{aligned}
$$

Replacing (D.39–D.41) in (D.33) we obtain (49), completing the proof.

The other entries of $\Delta K(\tilde{t})$ are computed in a similar way. The only relevant difference is that for $\Delta\hat{\alpha}_{nm}(\tilde{t})$ the equivalence relation for resonances reads

$$
\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} \equiv \mathrm{i}(\omega_n^0 - \omega_m^0) X(\tilde{t}). \qquad \text{(D.42)}
$$

Notice in particular that this equivalence relation implies that any term in $\Delta\hat{\alpha}_{nn}(\tilde{t})$ is equivalent to zero, and thus that these diagonal elements [but not the $\Delta\hat{\beta}_{nn}(\tilde{t})$] are irrelevant.

Appendix E: Derivation of the expression for $\Delta J^{\pm}_{nm}$

We obtain (D.25) indirectly by using the first order in ε of Eq. (B.6). If we pick the ++ sign prescripts in that equation, using (35) and (36) the first order in ε reads

$$
\begin{aligned}
&\omega_n^0(\omega_n^0+\omega_m^0)\left[\Delta J^{+}_{nm} + \int_{\Sigma^0} \mathrm{d}V^0\, {}^{+}\Delta\Psi_n\Psi_m^0\right] + ({}^{+}\Delta\omega_n)\delta_{nm} \\
&\quad + \int_{\partial\Sigma^0} \mathrm{d}S^0\, \Psi_m^0 \Big[\Delta x\, \mathbf{n}\cdot\nabla_{h^0}\left(\mathbf{n}\cdot\nabla_{h^0}\Psi_n^0\right) \\
&\quad + \mathbf{n}\cdot\nabla_{h^0}{}^{+}\Delta\Psi_n\Big] = 0, \qquad \text{(E.43)}
\end{aligned}
$$

where the first term of the surface integral comes from the contribution due to the displacement of the surface $\partial\Sigma_{\tilde{t}}$ with ε. We can simplify the integrand of the surface integral using (D.21). Doing the same procedure with all of the sign prescripts, we obtain

$$
\begin{aligned}
&\omega_n^0(\omega_n^0+\omega_m^0)\left[\Delta J^{+}_{nm} + \int_{\Sigma^0} \mathrm{d}V^0\, {}^{+}\Delta\Psi_n\Psi_m^0\right] \\
&\quad + ({}^{+}\Delta\omega_n)\delta_{nm} - \omega_n^0 \int_{\partial\Sigma^0} \mathrm{d}S^0 \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x\right)\Psi_n^0\Psi_m^0 = 0, \qquad \text{(E.44)} \\
&\omega_n^0(\omega_n^0-\omega_m^0)\left[\Delta J^{-}_{nm} + \int_{\Sigma^0} \mathrm{d}V^0\, {}^{+}\Delta\Psi_n\Psi_m^0\right] \\
&\quad + \frac{1}{2}\left[({}^{+}\Delta\omega_n) - ({}^{-}\Delta\omega_n)\right]\delta_{nm} \\
&\quad - \omega_n^0 \int_{\partial\Sigma^0} \mathrm{d}S^0 \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x\right)\Psi_n^0\Psi_m^0 = 0, \qquad \text{(E.45)} \\
&\omega_n^0(\omega_n^0-\omega_m^0)\left[\Delta J^{+}_{nm} + \int_{\Sigma^0} \mathrm{d}V^0\, {}^{-}\Delta\Psi_n\Psi_m^0\right] \\
&\quad + \frac{1}{2}\left[({}^{-}\Delta\omega_n) - ({}^{+}\Delta\omega_n)\right]\delta_{nm} \\
&\quad + \omega_n^0 \int_{\partial\Sigma^0} \mathrm{d}S^0 \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x\right)\Psi_n^0\Psi_m^0 = 0, \qquad \text{(E.46)} \\
&\omega_n^0(\omega_n^0+\omega_m^0)\left[\Delta J^{-}_{nm} + \int_{\Sigma^0} \mathrm{d}V^0\, {}^{-}\Delta\Psi_n\Psi_m^0\right] \\
&\quad + ({}^{-}\Delta\omega_n)\delta_{nm} + \omega_n^0 \int_{\partial\Sigma^0} \mathrm{d}S^0 \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\Delta x\right)\Psi_n^0\Psi_m^0 = 0. \qquad \text{(E.47)}
\end{aligned}
$$

Adding (E.44) and (E.46) on the one side, and (E.45) and (E.47) on the other side, and solving for $\Delta J^{\pm}_{nm}$ respectively, one obtains (D.25).

Appendix F: Derivation of the differential geometry relation

In this Appendix we prove the relation (D.37) [42]. Let us consider the family of surfaces $\partial\Sigma_{\tilde{t}}(\varepsilon)$ around $\partial\Sigma^0$ given by different values of ε, and n^i the vector field normal to the family of surfaces. In an arbitrarily close neighbourhood of $\partial\Sigma^0$ we can use the coordinate chart

$$(x^1 = \varepsilon, x^2, \ldots, x^N), \tag{F.48}$$

where the vector fields ∂_i for $i > 1$ are tangent to the surfaces. Notice that in this coordinate chart the metric is such that $h^{1i} = 0$ for $i > 1$. By definition, the quantity Δx is the proper length displacement per unit ε. It is therefore given by

$$\Delta x = \| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \begin{pmatrix} x^1 \\ x^2 \\ \ldots \\ x^N \end{pmatrix} \| = \| \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \| = \sqrt{h_{11}}, \tag{F.49}$$

where $\| \cdot \|$ is the norm given by the metric. The vector n^i is pointing in the direction ∂_1 and has unit norm. Therefore, $n^i = \delta^i_1/\Delta x$. Using this last expression we can easily compute

$$\begin{aligned} n^i \nabla_i n^j &= \frac{1}{\Delta x}\nabla_1 n^j = \frac{1}{\Delta x}(\partial_1 n^j + \Gamma^j_{1i} n^i) \\ &= \frac{1}{\Delta x^2}\left(\Gamma^j_{11} - \delta^j_1 \frac{\partial_1 \Delta x}{\Delta x}\right). \end{aligned} \tag{F.50}$$

We have to compute the Christoffel symbols appearing in (F.50). We do so by using $h^{1i} = 0$ for $i > 1$, and $h_{11} = \Delta x^2$ from (F.49), finding

$$\begin{aligned} \Gamma^j_{11} &= \frac{1}{2} h^{ij}(2\partial_1 h_{1i} - \partial_i h_{11}) \\ &= \frac{1}{2}\left(h^{1j}\partial_1 h_{11} - \sum_{i>1} h^{ij}\partial_i h_{11}\right) \\ &= \delta^j_1 \frac{\partial_1 \Delta x}{\Delta x} - \Delta x \sum_{i>1} h^{ij}\partial_i \Delta x. \end{aligned} \tag{F.51}$$

By replacing (F.51) in (F.50) we find that

$$\begin{aligned} n^i \nabla_i n^j &= -\frac{1}{\Delta x}\sum_{i>1} h^{ij}\partial_i \Delta x \\ &= \frac{1}{\Delta x}(n^i n^j - h^{ij})\nabla_i \Delta x. \end{aligned} \tag{F.52}$$

The last expression can be obtained from the previous one by checking independently that they coincide both for $j = 1$ and for $j > 1$. This expression is manifestly covariant and therefore valid in any coordinate chart.

Appendix G: Dirichlet vanishing boundary conditions

The construction of the method for Dirichlet vanishing boundary conditions is analogous to the one done for Neumann vanishing boundary conditions. That is, we also construct bases of modes associated to Cauchy hypersurfaces satisfying Properties I and II, and then find a differential equation in time for the linear transformation between these bases, which can be interpreted as a Bogoliubov transformation when the conditions given in Property II are met.

However, the imposition of Dirichlet boundary conditions with moving boundaries in a way that makes them compatible with the application of the method is more intricate than what it may seem on a first stage. One may think that condition (2) does not relate the field with its time derivative, as Neumann condition does, and therefore that a construction like the one done for the Neumann condition in this second article is not necessary, and one can proceed as in Part I. However, when (2) is considered as a boundary condition constraining the possible initial conditions on a Cauchy hypersurface Σ_t, it should be taken into account also how the time derivative of this global boundary condition may constrain the first partial time derivative of a mode at the boundary of the hypersurface. If one takes the total derivative with respect to t along the boundary of (2), it is easy to obtain

$$\partial_t \Phi(t, \mathbf{x}) = -v_{\mathrm{B}}(t, \mathbf{x})\mathbf{n} \cdot \nabla_{h(t)}\Phi(t, \mathbf{x}); \tag{G.53}$$

which is an expression identical to the first line of (6), but replacing $v_{\mathrm{B}} \to 1/v_{\mathrm{B}}$. It is clear then that we should impose both (2) and (G.53) to the possible initial conditions. However, imposing these two boundary conditions raises a technical difficulty for computing an useful auxiliary basis $\{\Psi_n^{[\tilde{t}]}\}$ of initial conditions. By useful, we mean that it should be possible to linearly transform it into a basis of "locally well-defined frequency" modes (in the regions where those can be defined), as we did in the case of Neumann boundary conditions with the transformation (21). In order for this transformation to provide "locally well-defined frequency" modes, one crucial ingredient is the proportionality relation in (18). But this proportionality relation, together with conditions (2) and (G.53), would imply that both $\Psi_n^{[\tilde{t}]}(\tilde{t})$ and $\mathbf{n}\cdot\nabla_{h(\tilde{t})}\Psi_n^{[\tilde{t}]}(\tilde{t})$ vanish at the boundary (except when $v_{\mathrm{B}} = 0$). If we wished now to find the set of initial conditions as solutions to an eigenvalue problem in $\Sigma_{\tilde{t}}$ with an elliptic operator, in general we would not find non-zero solutions.

How do we get out of this blind alley? When considering how the boundary conditions constrain the initial conditions, we renounce to require (2) and keep only condition (G.53), *except* for the regions of the boundary which remain static ($v_{\mathrm{B}} = 0$), in which case we impose both conditions. This decision might look not legitimated, since after all we will construct modes that, in general, do not satisfy Dirichlet vanishing boundary conditions. However, a careful discussion shows that the construction obtained is valid when the method is used to compute the evolution between regions for which a physically valid Fock quantisation can be done, as described in Sect. 3.3. In particular, notice that in those regions the boundaries must remain static (although in the time between the regions they may of course not). Therefore, when constructing the initial conditions of the modes associated to those regions, we *do* also impose their initial conditions to vanish at the boundary.

With that in mind, let us justify why we can leave the condition (2) only for the static regions of the boundary. We notice that condition (G.53) (which is satisfied by all modes) was obtained by taking the total time derivative of (2) along the boundary. Therefore, the fulfilment of this condition guarantees that the value at the boundary is *preserved* in time. That is, the bases of solutions that we obtain expand the space of solutions to the Klein–Gordon equation for which the evaluation of the field at the boundary remains constant (along the direction of the projection of ∂_t over the boundary). If we then use the method to compute the evolution of a mode which vanishes at the intersection of some Cauchy hypersurface with the boundary, we can be sure that this zero value is preserved along the whole spacetime boundary. This means that, even if for expanding this mode at a different time we use a basis of modes which, individually, do not vanish at the boundary, their linear combination given by the expansion necessarily vanishes.

In summary: We are, strictly speaking, not constructing the method *directly* for Dirichlet vanishing boundary conditions. Rather, we are constructing the method for Dirichlet "time-preserved and vanishing-when-static" boundary conditions. But, at the same time, we are making sure to use the method *only* with bases of modes that vanish at the boundary at *some* time, and therefore at *every* time, thus fulfilling Dirichlet vanishing boundary conditions, as desired.

Once the discussion above has been done, the remaining development of the method follows in a completely analogous way to the one done for Neumann boundary conditions. Therefore, we do not reproduce all the calculations and discussions in detail. We rather list all the objects and formulas in the article, excluding the examples and the Appendices, that change when considering Dirichlet boundary conditions. Any formula that is not listed below can be used with both boundary conditions.

In Sect. 2.2 the subspace of initial conditions $\Gamma_t \subset [C^\infty(\Sigma_t) \cap L^2(\Sigma_t)]^{\oplus 2}$ is the restriction of the full space of pairs of functions to initial conditions $(\Phi, \partial_t \Phi)|_{\Sigma_t}$ satisfying

$$\begin{cases} \partial_t \Phi(t, \mathbf{x}) = -v_{\mathrm{B}}(t, \mathbf{x}) \mathbf{n} \cdot \nabla_{h(t)} \Phi(t, \mathbf{x}), \\ \text{and } \Phi(t, \mathbf{x}) = 0 \text{ if } v_{\mathrm{B}}(t, \mathbf{x}) = 0; \end{cases} \tag{G.54}$$

where $\mathbf{x} \in \partial \Sigma_t$ [instead of (6)]. That is, pairs of initial conditions satisfying (G.53) always and (2) only when the boundary is static, as we advanced.

In Sect. 3.1 Eq. (17) must be replaced by

$$\omega_n^{[\tilde{t}]} \Psi_n^{[\tilde{t}]}(\tilde{t}, \mathbf{x}) = -v_{\mathrm{B}}(\tilde{t}, \mathbf{x}) \mathbf{n} \cdot \nabla_{h(\tilde{t})} \Psi_n^{[\tilde{t}]}(\tilde{t}, \mathbf{x}), \quad \mathbf{x} \in \partial \Sigma_{\tilde{t}}. \tag{G.55}$$

Notice that, since $\omega_n^{[\tilde{t}]} \neq 0$ (see "Appendix H"), when $v_{\mathrm{B}} = 0$ this equation is also imposing the second condition in (G.54).

In Sect. 3.2 the expression for $\hat{V}_{nm}^{\pm\hat{\pm}}$ in (27) is slightly changed (only the surface integral changes[16]):

$$\begin{aligned} &\hat{V}_{nm}^{\pm\hat{\pm}} \\ &\quad = -({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}) \delta_{nm} \delta_{\pm\hat{\pm}} \\ &\quad\quad \hat{\pm} \left\{ \left[({}^{\pm}\omega_n^{[\tilde{t}]}) + ({}^{\hat{\pm}}\omega_m^{[\tilde{t}]}) \right] \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \right] {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \right. \\ &\quad\quad + \left[2({}^{\pm}\omega_n^{[\tilde{t}]})^2 + \frac{\mathrm{d}^{\pm}\omega_n^{[\tilde{t}]}}{\mathrm{d}\tilde{t}} - F(\tilde{t}) \right] \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ &\quad\quad + \int_{\Sigma_{\tilde{t}}} \mathrm{d}V_{\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \left[{}^{\pm}\omega_n^{[\tilde{t}]} q(\tilde{t}) + \xi \bar{R}(\tilde{t}) \right] {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \\ &\quad\quad \left. + \frac{1}{{}^{\hat{\pm}}\omega_m^{[\tilde{t}]}} \int_{\partial\Sigma_{\tilde{t}}} \mathrm{d}S_{\tilde{t}} \left[\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} {}^{\pm}\Psi_n^{[\tilde{t}]}(\tilde{t}) \right] \mathbf{n} \cdot \nabla_{h(\tilde{t})} {}^{\hat{\pm}}\Psi_m^{[\tilde{t}]}(\tilde{t}) \right\}. \end{aligned} \tag{G.56}$$

Finally, in Sect. 4 Eq. (35) becomes

$$\Psi_n^0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial \Sigma^0; \tag{G.57}$$

and the entries of the matrix $\Delta K(\tilde{t})$ in (39), given by (48) and (49), must be replaced by

$$\begin{aligned} \Delta\hat{\alpha}_{nm}(\tilde{t}) \equiv\ & \mathrm{i} \int_{\Sigma^0} \mathrm{d}V^0\, [{}^{-}_{m}\hat{\Delta}(\tilde{t}) \Psi_n^0] \Psi_m^0 \\ & - \mathrm{i} \int_{\partial\Sigma_0} \mathrm{d}S^0\, \Delta x(\tilde{t}) \left(\mathbf{n} \cdot \nabla_{h^0} \Psi_n^0 \right) \left(\mathbf{n} \cdot \nabla_{h^0} \Psi_m^0 \right), \end{aligned} \tag{G.58}$$

$$\begin{aligned} \Delta\hat{\beta}_{nm}(\tilde{t}) \equiv\ & -\mathrm{i} \int_{\Sigma^0} \mathrm{d}V^0\, [{}^{+}_{m}\hat{\Delta}(\tilde{t}) \Psi_n^0] \Psi_m^0 \\ & + \mathrm{i} \int_{\partial\Sigma_0} \mathrm{d}S^0\, \Delta x(\tilde{t}) \left(\mathbf{n} \cdot \nabla_{h^0} \Psi_n^0 \right) \left(\mathbf{n} \cdot \nabla_{h^0} \Psi_m^0 \right). \end{aligned} \tag{G.59}$$

Appendix H: Proof that $\omega_n^{[\tilde{t}]} \neq 0$

Let us consider a solution of (14) for which $\omega_n^{[\tilde{t}]} = 0$. Then there exists a function $\Psi_n^{[\tilde{t}]}(\tilde{t})$ satisfying (16) and (17) with $\omega_n^{[\tilde{t}]} = 0$. But (17) with $\omega_n^{[\tilde{t}]} = 0$ are simply Neumann vanishing boundary conditions for the *spatial* function $\Psi_n^{[\tilde{t}]}(\tilde{t})$, and the operator $\hat{\mathcal{O}}(\tilde{t})$ is clearly positive defi-

[16] The surface integral in (27) had been simplified using (17). The expression given in (G.56) is actually valid for both boundary conditions.

nite for functions satisfying such boundary conditions. Since, according to (16), $\Psi_n^{[\tilde{t}]}(\tilde{t})$ is an eigenfunction of this operator with eigenvalue $(\omega_n^{[\tilde{t}]})^2$, we must have $(\omega_n^{[\tilde{t}]})^2 > 0$, which is a contradiction. Therefore $\omega_n^{[\tilde{t}]} \neq 0$.

In the case of Dirichlet boundary conditions, the proof that $\omega_n^{[\tilde{t}]} \neq 0$ is slightly more subtle. We also start by considering a solution of (14) for which $\omega_n^{[\tilde{t}]} = 0$. Then there exists a function $\Psi_n^{[\tilde{t}]}(\tilde{t})$ satisfying (16) and (G.55) with $\omega_n^{[\tilde{t}]} = 0$. But (G.55) with $\omega_n^{[\tilde{t}]} = 0$ implies that the spatial function $\Psi_n^{[\tilde{t}]}(\tilde{t})$ satisfies Neumann vanishing boundary conditions at least in the regions of the boundary where $v_{\mathrm{B}} \neq 0$. But in the regions of the boundary where $v_{\mathrm{B}} = 0$ we know that the second line of (G.54) must be imposed, and therefore the function $\Psi_n^{[\tilde{t}]}(\tilde{t})$ must satisfy Dirichlet vanishing boundary conditions there.[17] Therefore, the function $\Psi_n^{[\tilde{t}]}(\tilde{t})$ satisfies mixed vanishing boundary conditions. But the operator $\hat{\mathscr{O}}(\tilde{t})$ is clearly positive definite for functions satisfying such boundary conditions. Then, again according to (16) we must have $(\omega_n^{[\tilde{t}]})^2 > 0$, which is a contradiction. Therefore $\omega_n^{[\tilde{t}]} \neq 0$.

Appendix I: Summary of formulae for the application of the method

In this Appendix, we provide Tables 1 and 2 with all the formulae necessary in order to apply the method to a concrete

[17] Notice that (G.55) implies the second line of (G.54) only *after* having already found that $\omega_n^{[\tilde{t}]} \neq 0$. Since here we are assuming the opposite, we must impose the second line of (G.54) explicitly.

Table 1 Summary of formulae for the application of the method (part 1)

Computation of the auxiliary modes and eigenvalues	
(Pseudo-)eigenvalue equation (16)	$\hat{\mathscr{O}}(t)\Psi_n^{[t]}(t) = (\omega_n^{[t]})^2\Psi_n^{[t]}(t), \quad \Psi_n^{[t]}(t)$ real;
with the operator (10, 8)	$\hat{\mathscr{O}}(t) = -\nabla^2_{h(t)} + \xi R^h(t) + m^2 + F(t),$ $F(t) = \begin{cases} 0 & \text{if } \xi R^h(t,\mathbf{x}) + m^2 > 0 \text{ a.e. in } \Sigma_t, \\ -\text{ess inf}\{\xi R^h(t,\mathbf{x}) + m^2, (t,\mathbf{x}) \in \Sigma_t\} + \epsilon & \text{i.o.c.}; \end{cases}$
boundary conditions (for $\mathbf{x} \in \partial\Sigma_t$) (17, G.55)	$\mathbf{n}\cdot\nabla_{h(t)}\Psi_n^{[t]}(t) = -\omega_n^{[t]} v_{\mathrm{B}}(t)\Psi_n^{[t]}(t) \quad$ (Neumann), $\omega_n^{[t]}\Psi_n^{[t]}(t) = -v_{\mathrm{B}}(t)\mathbf{n}\cdot\nabla_{h(t)}\Psi_n^{[t]}(t) \quad$ (Dirichlet);
and orthonormalisation condition (15, 18, 7)	$\int_{\Sigma_t} \mathrm{d}V_t\, [\xi R^h(t) + m^2 + F(t) + \omega_n^{[t]}\omega_m^{[t]}]\Psi_n^{[t]}(t)\Psi_m^{[t]}(t) + \int_{\Sigma_t} \mathrm{d}V_t\, [\nabla_{h(t)}\Psi_n^{[t]}(t)]\cdot[\nabla_{h(t)}\Psi_m^{[t]}(t)] = \lvert\omega_n^{[t]}\rvert\delta_{nm}.$
Time-dependent linear transformation	
Differential equation (25)	$\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = M\hat{V}(t)M^* U(t,t_0);$
with (22, 26, G.56, 27, 12, 13)	$M = \frac{1}{2}\begin{pmatrix} (1-\mathrm{i})I & (1+\mathrm{i})I \\ (1+\mathrm{i})I & (1-\mathrm{i})I \end{pmatrix}, \quad \hat{V}(t) = \begin{pmatrix} \hat{V}^{++} & \hat{V}^{+-} \\ \hat{V}^{-+} & \hat{V}^{--} \end{pmatrix},$ $\hat{V}_{nm}^{\pm\hat{\pm}} = -(^{\hat{\pm}}\omega_m^{[t]})\delta_{nm}\delta_{\pm\hat{\pm}} \hat{\pm}\Big\{\Big[(^{\pm}\omega_n^{[t]}) + (^{\hat{\pm}}\omega_m^{[t]})\Big]\int_{\Sigma_t}\mathrm{d}V_t\Big[\frac{\mathrm{d}}{\mathrm{d}t}{}^{\pm}\Psi_n^{[t]}(t)\Big]{}^{\hat{\pm}}\Psi_m^{[t]}(t) + \Big[2(^{\pm}\omega_n^{[t]})^2 + \frac{\mathrm{d}^{\pm}\omega_n^{[t]}}{\mathrm{d}t} - F(t)\Big]\int_{\Sigma_t}\mathrm{d}V_t\,{}^{\pm}\Psi_n^{[t]}(t){}^{\hat{\pm}}\Psi_m^{[t]}(t) + \int_{\Sigma_t}\mathrm{d}V_t\,{}^{\pm}\Psi_n^{[t]}(t)\Big[{}^{\pm}\omega_n^{[t]}q(t) + \xi\bar{R}(t)\Big]{}^{\hat{\pm}}\Psi_m^{[t]}(t) + (\mathrm{SI})\Big\};$ $(\mathrm{SI}) := \begin{cases} (^{\hat{\pm}}\omega_m^{[t]})^{-1}\int_{\partial\Sigma_t}\mathrm{d}S_t\,\Big[\frac{\mathrm{d}}{\mathrm{d}t}{}^{\pm}\Psi_n^{[t]}(t)\Big]\mathbf{n}\cdot\nabla_{h(t)}{}^{\hat{\pm}}\Psi_m^{[t]}(t) \\ \text{(Dirichlet-Neumann)}, \\ -\int_{\partial\Sigma_t}\mathrm{d}S_t\, v_{\mathrm{B}}(t)\Big[\frac{\mathrm{d}}{\mathrm{d}t}{}^{\pm}\Psi_n^{[t]}(t)\Big]{}^{\hat{\pm}}\Psi_m^{[t]}(t) \\ \text{(Neumann)}; \end{cases}$ $q(t) = \partial_t \log\sqrt{h(t)},$ $\bar{R}(t) = 2\partial_t q(t) + q(t)^2 - [\partial_t h^{ij}(t)][\partial_t h_{ij}(t)]/4.$
Formal solution (28)	$U(t_{\mathrm{f}},t_0) = \mathscr{T}\exp\Big[\int_{t_0}^{t_{\mathrm{f}}}\mathrm{d}t\; M\hat{V}(t)M^*\Big].$

Table 2 Summary of formulae for the application of the method (part 2)

Perturbative regime	
Quantities $\Delta\hat{\alpha}$ and $\Delta\hat{\beta}$	
Neumann boundary conditions (48, 49)	$$\begin{aligned}\Delta\hat{\alpha}_{nm}(t) \equiv{} & \mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{-}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & + \mathrm{i}\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x(t)\Big[(\nabla_{h^0}\Psi^0_n)\cdot(\nabla_{h^0}\Psi^0_m) \\ & + (\xi R^{h^0} + m^2 - \omega^0_n\omega^0_m)\Psi^0_n\Psi^0_m\Big], \\ \Delta\hat{\beta}_{nm}(t) \equiv{} & -\mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{+}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & - \mathrm{i}\int_{\partial\Sigma^0} \mathrm{d}S^0\, \Delta x(t)\Big[(\nabla_{h^0}\Psi^0_n)\cdot(\nabla_{h^0}\Psi^0_m) \\ & + (\xi R^{h^0} + m^2 + \omega^0_n\omega^0_m)\Psi^0_n\Psi^0_m\Big];\end{aligned}$$
Dirichlet boundary conditions (G.58, G.59)	$$\begin{aligned}\Delta\hat{\alpha}_{nm}(t) \equiv{} & \mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{-}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & - \mathrm{i}\int_{\partial\Sigma_0} \mathrm{d}S^0\, \Delta x(t)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi^0_n\right)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi^0_m\right), \\ \Delta\hat{\beta}_{nm}(t) \equiv{} & -\mathrm{i}\int_{\Sigma^0} \mathrm{d}V^0\, [{}^{+}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n]\Psi^0_m \\ & + \mathrm{i}\int_{\partial\Sigma_0} \mathrm{d}S^0\, \Delta x(t)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi^0_n\right)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi^0_m\right);\end{aligned}$$
with the static modes (34)	$\hat{\mathcal{O}}^0\Psi^0_n = (\omega^0_n)^2\Psi^0_n$, Ψ^0_n real;
with boundary conditions (for $\mathbf{x}\in\partial\Sigma^0$) (35, G.57)	$\mathbf{n}\cdot\nabla_{h^0}\Psi^0_n(\mathbf{x}) = 0$ (Neumann), $\Psi^0_n(\mathbf{x}) = 0$ (Dirichlet);
and orthonormalisation condition (36)	$\int_{\Sigma^0} \mathrm{d}V^0\, \Psi^0_n\Psi^0_m = \frac{\delta_{nm}}{2\omega^0_n}$;
and with the operators (50)	${}^{\pm}_{m}\hat{\Delta}(\tilde{t})\Psi^0_n = \left[\Delta\hat{\mathcal{O}}(t) + \omega^0_n(\omega^0_n \pm \omega^0_m)\Delta r(t) + \xi\Delta\bar{R}(t)\right]\Psi^0_n$,
with (33)	$\Delta r(t) = \partial_\varepsilon \log h(t)\vert_{\varepsilon=0}/2$,
and	$\Delta F(t) = 0$.
Bogoliubov coefficients	
Explicit time evolution (resonances) (44, 45)	$$\begin{aligned}&\alpha_{nn}(t_{\mathrm{f}}, t_0) \approx 1; \\ &\alpha_{nm}(t_{\mathrm{f}}, t_0) \approx \varepsilon\int_{t_0}^{t_{\mathrm{f}}} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(\omega^0_n-\omega^0_m)t}\,\Delta\hat{\alpha}_{nm}(t), \quad n\neq m; \\ &\beta_{nm}(t_{\mathrm{f}}, t_0) \approx \varepsilon\int_{t_0}^{t_{\mathrm{f}}} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(\omega^0_n+\omega^0_m)t}\,\Delta\hat{\beta}_{nm}(t).\end{aligned}$$
Asymptotic values using Fourier transforms (46, 47)	$$\begin{aligned}&\alpha_{nn}(-\infty,\infty) \approx 1; \\ &\alpha_{nm}(-\infty,\infty) \approx \varepsilon\sqrt{2\pi}\,\mathscr{F}[\Delta\hat{\alpha}_{nm}](\omega^0_n-\omega^0_m), \quad n\neq m; \\ &\beta_{nm}(-\infty,\infty) \approx \varepsilon\sqrt{2\pi}\,\mathscr{F}[\Delta\hat{\beta}_{nm}](\omega^0_n+\omega^0_m).\end{aligned}$$

problem. As we indicated in the examples in Sect. 4.1 and 4.2, once the expressions for the method have been found, there is no need to consider the explicit evolution in time of the modes constructed any more. Therefore, the notation for the "time label" of the different modes can be simplified replacing $\tilde{t} \to t$. In the Tables we use this simplification.

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Dynamical Casimir effect with screened scalar fields

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Dynamical Casimir effect with screened scalar fields

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Understanding the nature of dark energy and dark matter is one of modern physics' greatest open problems. Scalar-tensor theories with screened scalar fields like the chameleon model are among the most popular proposed solutions. In this article, we present the first analysis of the impact of a chameleon field on the dynamical Casimir effect, whose main feature is the particle production associated with a resonant condition of boundary periodic motion in cavities. For this, we employ a recently developed method to compute the evolution of confined quantum scalar fields in a globally hyperbolic spacetime by means of time-dependent Bogoliubov transformations. As a result, we show that particle production is reduced due to the presence of the chameleon field. In addition, our results for the Bogoliubov coefficients and the mean number of created particles agree with known results in the absence of a chameleon field. Our results initiate the discussion of the evolution of quantum fields on screened scalar field backgrounds.

Keywords: Chameleons, Dynamical Casimir effect, Quantum field theory in curved spacetime

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DCE with screened scalar fields

I. INTRODUCTION

Quantum field theory in curved spacetime (QFTCS) studies the behaviour of quantum fields propagating in a classical relativistic background geometry[1–3]. This theory has predicted many physical phenomena, such as cosmological particle creation[3–5], Hawking radiation[6] and the Unruh effect[7], as well as the dynamical Casimir effect (DCE)[8], which refers to the generation of particles due to the motion of boundaries (see Refs.[9–11] for reviews). The first computations of the DCE were done in flat spacetime[8,12]. Over the past five decades numerous developments have appeared including distinct geometries of the cavities[13–15], entanglement generation[16–19], and extensions to a few other metrics[20,21]. Performing a mathematically rigorous study of the DCE is challenging due to the complexity of studying quantum field theory with dynamical boundary conditions[22,23]. For this reason, within the framework of QFTCS, in Refs.[24,25], some of the authors of the present work introduced a general method to compute the evolution of a confined quantum scalar field in a globally hyperbolic spacetime by means of a time-dependent Bogoliubov transformation. Part I[24] considers spacetimes without boundaries or with timelike boundaries that remain static in some synchronous frame, while Part II[25] considers spacetimes with timelike boundaries that do not remain static in any synchronous frame.

QFTCS builds on the framework of general relativity (GR), which has proven to be a remarkably successful theory of gravity and cosmology[26,27]. Many physical predictions of GR have been experimentally validated over the last century, the most recent being the detection of gravitational waves[28]. However, GR has some well known limitations, such as the breakdown of the equivalence principle at singularities, or the accelerating expansion of the Universe and the mystery of dark energy (responsible for this accelerated expansion). Therefore, many different modifications to GR have been proposed. Amongst these modified theories of gravity, scalar-tensor theories[29] are some of the most studied. There are two major reasons to study such theories; firstly, it is one of the simplest ways to modify GR, and secondly, some extensions of the Standard Model of particle physics predict the existence of scalar fields[30,31]. This is further motivated by the experimentally confirmed existence of one scalar field in Nature, namely the Higgs field[32–34]. Moreover, there are several proposed explanations for the nature of dark energy based on scalar-tensor theories[35,36]. Some of these models predict a fifth force, which has not yet been detected on Earth or in the Solar

System[37–39].

One way to mitigate this tension between theory and observation is by introducing a "screening mechanism"[40], which allows the effects of the additional scalar fields to vary depending on the environment. Therefore, a screening mechanism would enable additional scalar fields to contribute to dark energy or dark matter while evading current experimental constraints on fifth forces. There are several models for such screened scalar fields with different types of screening mechanisms, such as chameleons[41,42]; symmetrons[43–50], whose fifth forces have been suggested as alternatives to particle dark matter[51–54]; galileons[55–57]; and environment-dependent dilatons[45,58–63]. Most of these models have been or are proposed to be tested in a zoo of different experiments and observations, for example, Refs.[40,64–90]. Furthermore, in recent years, there have been initial attempts to study screened scalars as quantum fields[91–94], and it was proposed to study screened scalar-tensor theories in analogue gravity simulations[95].

Additional proposals in the particular case of chameleon fields suggest that experiments which measure Casimir forces may also be used to constrain chameleon theories[9697–103]. From a theoretical point of view, a natural extension of the previous proposals then arises: if the chameleon field can be constrained by the static Casimir effect, then it might also be constrained by the dynamical Casimir effect. Thus, the aim of the present work is to study the DCE in the presence of a chameleon field, and to explore the relationship between the particle production and the chameleon field parameters. As a first step towards estimating the feasibility of constraining chameleon fields with the DCE, we consider a toy model with only the effect of the chameleon field and no gravity. Since the problem we want to solve in this work is that of a confined quantum field with moving boundaries, we will use the techniques developed in Ref.[25].

The article is organized as follows. In Sec. II A, we introduce screened scalar fields using the example of the chameleon mechanism; and, in Sec. II B, we describe some relevant aspects of QFTCS applied to the DCE and, in particular, the method developed in Ref.[25]. Sec. III is the nuclear part of the article, where we obtain the main result, and analyze it both analytically and with a numerical example. We conclude in Sec. IV. In addition, in Appendix A, we show the derivation of the normalization constant of the cavity modes. We use natural units $\hbar = c = 1$ throughout the article.

II. BACKGROUND

In this section, we give an overview of scalar-tensor theories of gravitation, in particular screened scalar fields and the chameleon model. In addition, we show schematically the usual approach to studying the DCE within the framework of QFTCS, and we then outline the techniques that are used in the present work.

A. Screened scalar fields

The aim of scalar-tensor theories of gravitation is to study the modifications of GR due to an additional scalar field which is coupled to the metric tensor. A common way of performing such a coupling between a scalar field φ and the metric tensor $g_{\mu\nu}$ is through a conformal factor $A^2(\varphi)$, such that

$$\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}. \tag{1}$$

In this sense, scalar-tensor theories of gravity are defined up to a conformal transformation leading from one so-called conformal frame to another[104]. These conformal frames are merely different mathematical formulations. Hence, the theoretical prediction for an observable quantity cannot be altered due to a change of conformal frame. The advantage is that some calculations might be easier to perform in one frame than in another. Two popular conformal frames are the Jordan frame - with a metric we denote $\tilde{g}_{\mu\nu}$ - and the Einstein frame denoted as $g_{\mu\nu}$.

Even though the physical measurement cannot be changed, the physical interpretation can actually differ from one frame to another. For example, in the Jordan frame formulation, Einstein's theory of gravity is modified in such a way that test particles follow different geodesics from those predicted in GR, while in the Einstein frame formulation, test particles still follow GR's geodesics but are also subject to a gravity-like fifth force of Nature carried by the additional scalar field φ. The problem with such a prediction is that fifth forces are tightly constrained in our Solar System. An interesting way to solve this issue is given by so-called screening mechanisms. Such a mechanism allows the fifth force to be weak within our Solar System but cosmologically significant on intergalactic scales. As we describe in the Introduction, Sec. I, there are several models for such screened scalar fields with different types of screening mechanisms such as the chameleon model, which will be presented in more detail in Sec. II A 2.

1. Einstein-frame action

In this article, we consider a universe containing a free scalar test particle Φ, which we denote as the "matter", with mass m_Φ; and an additional scalar field φ conformally coupling to the metric tensor. In the Einstein frame, this universe's action is schematically given by

$$S_{Universe} = S_{gravity} + S_m + S_\varphi, \tag{2}$$

where $S_{gravity}$ is the usual Einstein-Hilbert gravitational action, S_m the matter action, and S_φ the action of the scalar field φ. Following Ref.[92], the conformal coupling to the metric tensor induces an interaction between Φ and φ, which in turn leads to a rescaling of the free field's mass by the conformal factor. Consequently, the Lagrangian matter density associated with the action S_m is given by

$$\mathcal{L}_m = -\frac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}A^2(\varphi)\, m_\Phi^2\Phi^2. \tag{3}$$

Subsequently, from the Euler-Lagrange equations, we obtain the equation of motion for the probe field Φ:

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi - A^2(\varphi)\, m_\Phi^2\Phi = 0. \tag{4}$$

Later, in Sec. III, we will ignore gravity for simplicity and consequently set $g_{\mu\nu} = \eta_{\mu\nu}$.

2. Chameleon model

A chameleon scalar field model has the defining property of coupling to matter in such a way that its effective mass increases with increasing local matter density. As its name suggests, the chameleon field adapts to its environment and becomes almost impossible to detect in regions of high matter density like our Solar System. The conformal coupling factor in Eq. (1) of a chameleon is given by

$$A^2(\varphi) = \mathrm{e}^{2\varphi/M}, \tag{5}$$

where M is a mass scale which determines the strength of the chameleon-matter coupling. As is common practice when dealing with chameleons, we assume that $\varphi/M \ll 1$. The

Lagrangian density describing the chameleon field and associated to the action S_φ in Eq. (2) is

$$\mathcal{L}_\varphi = -\frac{1}{2}\left(\partial_\varphi\right)^2 - \frac{\Lambda^{4+N}}{\varphi^N} - \frac{\varphi}{M}\rho \ , \tag{6}$$

where $N \in \mathbb{Z}^+ \cup 2\mathbb{Z}^- \backslash \{-2\}$ distinguishes between different chameleon models; the parameter Λ determines the strength of the self-interaction; and ρ is the density of non-relativistic matter that the chameleon is interacting with. The sum of the last two terms in Eq. (6) results in an effective potential with a local minimum, and consequently a non-vanishing chameleon mass which increases with the matter density. Since the chameleon fifth force usually has a Yukawa-like suppression[41], its range is the shorter the larger the chameleon's mass. Consequently, in environments of sufficiently high density, the chameleon fifth force is effectively quite feeble, i.e., screened.

Consider a static spherically symmetric source of radius R and homogeneous density ρ_{obj} immersed in a homogeneous medium of density ρ_{bg}. The field profile outside of this source, but still within an ambient Compton wavelength ($r < m_{bg}^{-1}$) with m_{bg} being the chameleon's mass in the medium of density ρ_{bg}, is approximately given by[72]

$$\varphi \simeq \varphi_{bg} - \frac{R}{r}\left(\varphi_{bg} - \varphi_{obj}\right), \tag{7}$$

where φ_{obj} is the value of the chameleon field inside the source and φ_{bg} is the value of the chameleon field outside the source or the so-called background value. In the case of a large density contrast $\rho_{obj} \gg \rho_{bg}$, we can consider $\varphi_{bg} \gg \varphi_{obj}$. If the source is screened, then φ_{obj} is actually the minimum of the chameleon within the source apart from a thin shell near the surface. Only the matter in this thin shell sources the chameleon fifth force in the exterior while the interior is not contributing. This is due to the short range of the fifth force in case of a large effective chameleon mass, and is known as the *thin-shell effect*. In order to know if the chameleon field is screened or not, we define the shell thickness

$$\Delta R = \frac{M\varphi_{bg}}{\rho_{obj} R}. \tag{8}$$

The object is said to be screened if $\Delta R \ll R$ or

$$\frac{M\varphi_{bg}}{\rho_{obj} R^2} \ll 1. \tag{9}$$

B. Dynamical Casimir effect, quantum field theory and particle content

The usual approach to studying the DCE is to consider a free scalar field Φ in a one-dimensional[105] cavity with perfectly reflecting boundaries satisfying the Klein-Gordon equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - m_\Phi^2\Phi - \xi\mathcal{R}\Phi = 0, \tag{10}$$

where $m_\Phi \geq 0$ is the rest mass of the field, $g^{\mu\nu}$ is the spacetime metric, $\mathcal{R}$ its scalar curvature and $\xi \in \mathbb{R}$ is a coupling constant. Let us consider flat spacetime in inertial coordinates (t, x). The boundaries of the cavity are moved during the time $t_0 < t < t_f$. Since we are considering ideally reflecting boundaries, we impose Dirichlet vanishing boundary conditions

$$\Phi\left(t, x = x_l\left(t\right)\right) = \Phi\left(t, x = x_r\left(t\right)\right) = 0, \tag{11}$$

where the functions $x_l(t)$ and $x_r(t)$ determine the positions of the left and right boundaries for $t_0 < t < t_f$, respectively. Before the boundaries move ($t < t_0$), we assume that the walls are static. For such initial conditions, the quantized field operator is decomposed as follows[1]:

$$\hat{\Phi}\left(t, x\right) = \sum_n \left[\hat{a}_n\phi_n\left(t, x\right) + \hat{a}_n^\dagger\phi_n^*\left(t, x\right)\right], \tag{12}$$

where the mode functions $\phi_n(t, x)$ are solutions to the Klein-Gordon equation (10). In addition, $\hat{a}_n$ and $\hat{a}_n^\dagger$ are the bosonic annihilation and creation operators, respectively. Hence, the Fock space and vacuum state are defined in the canonical way. Two sets of mode solutions are related by a Bogoliubov transformation. In this way, the effects of the moving boundaries on the quantum field can be computed using a Bogoliubov transformation[1], such that

$$\begin{aligned} \tilde{\phi}_m &= \sum_n \left[\alpha_{mn}\phi_n + \beta_{mn}\phi_n^*\right], \\ \tilde{a}_m &= \sum_n \left[\alpha_{mn}^*\hat{a}_n - \beta_{mn}^*\hat{a}_n^\dagger\right], \end{aligned} \tag{13}$$

where α_{mn} and β_{mn} are called Bogoliubov coefficients. Note that if $\beta_{mn} \neq 0$, then the transformation of the annihilation operator of Eq. (13) contains creation operators. Therefore, the two vacua do not coincide. Hence, the β-coefficients quantify particle creation due to the transformation. Starting with a vacuum state, the average number of particles in mode m after a Bogoliubov transformation is given by

$$\mathcal{N}_m = \sum_n \left|\beta_{nm}\right|^2. \tag{14}$$

In general, the computation of the Bogoliubov coefficients is difficult. Thus, mathematical techniques and simplifications adapted to a specific problem make the computations manageable. For instance, the presence of symmetries like homogeneity or isotropy is convenient to obtain results on particle creation in cosmological models. In Refs.[24,25], the authors developed a method to compute the Bogoliubov transformation experienced by a confined quantum scalar field in a globally hyperbolic spacetime due to the changes in the geometry and/or the confining boundaries. The second part[25] extends the method to cases in which the timelike boundaries of the spacetime do not remain static in any synchronous frame. This method is especially useful in the presence of resonances of the field modes due to small perturbations of the metric and/or the motion of the cavity boundaries. This is because in these cases, the Bogoliubov coefficients take the following simple expressions

$$\alpha_{nn}(t_{\rm f}, t_0) \approx 1;$$

$$\alpha_{nm}(t_{\rm f}, t_0) \approx \varepsilon \int_{t_0}^{t_{\rm f}} {\rm d}t\, {\rm e}^{-{\rm i}(\omega_n^0-\omega_m^0)t} \Delta\hat{\alpha}_{nm}(t), \quad n \neq m; \tag{15}$$

$$\beta_{nm}(t_{\rm f}, t_0) \approx \varepsilon \int_{t_0}^{t_{\rm f}} {\rm d}t\, {\rm e}^{-{\rm i}(\omega_n^0+\omega_m^0)t} \Delta\hat{\beta}_{nm}(t); \tag{16}$$

where $\varepsilon \ll 1$ is a small parameter that characterizes the perturbation of the confined field (e.g. oscillation amplitude), and ω_n^0 are the mode frequencies for the static problem ($\varepsilon = 0$). For Dirichlet boundary conditions,

$$\Delta\hat{\alpha}_{nm}(t) \equiv \ {\rm i}\int_{\Sigma^0} {\rm d}V^0 \ [{}^{-}_{m}\hat{\Delta}(t)\Psi_n^0]\Psi_m^0 - {\rm i}\int_{\partial\Sigma_0} {\rm d}S^0 \ \Delta x(t) \left(\mathbf{n}\cdot\nabla_{h^0}\Psi_n^0\right)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi_m^0\right), \tag{17}$$

$$\Delta\hat{\beta}_{nm}(t) \equiv \ -{\rm i}\int_{\Sigma^0} {\rm d}V^0 \ [{}^{+}_{m}\hat{\Delta}(t)\Psi_n^0]\Psi_m^0 + {\rm i}\int_{\partial\Sigma_0} {\rm d}S^0 \ \Delta x(t) \left(\mathbf{n}\cdot\nabla_{h^0}\Psi_n^0\right)\left(\mathbf{n}\cdot\nabla_{h^0}\Psi_m^0\right). \tag{18}$$

Here, Σ^0 is a fixed spatial hypersurface, around which the perturbation occurs, with volume element dV^0; boundary $\partial\Sigma^0$; boundary surface element dS^0; proper distance $\varepsilon\Delta x(t)$ between the boundary $\partial\Sigma_t$ and the fixed boundary $\partial\Sigma^0$; and connection ∇_{h^0} associated to the static metric h_{ij}^0. ${}^{\pm}_{m}\hat{\Delta}(t)$ are linear operators determined by their actions on the mode basis $\{\Psi_n^0\}$ as defined in Eq. (50) of Ref.[25].

The Bogoliubov transformation differs maximally from the identity just by terms of first order in ε, except for the cases where there are resonances. If the perturbation considered contains some characteristic frequency $\omega_{\rm p}$, such that it coincides with some difference between the frequencies of two modes, $\omega_{\rm p} = \omega_n^0 - \omega_m^0$, then the corresponding coefficient $\alpha_{nm}(t_{\rm f}, t_0)$ grows linearly with the time difference $t_{\rm f} - t_0$ and eventually grows to be a non-perturbative correction. Respectively, if the characteristic frequency coincides with some sum between the frequencies of two modes, $\omega_{\rm p} = \omega_n^0 + \omega_m^0$, then the coefficient $\beta_{nm}(t_{\rm f}, t_0)$ grows linearly in time. The duration of the perturbation Δt should be such that $1 \ll \omega_{\rm p}\Delta t \ll 1/\varepsilon$. This is because the period of time should be reasonably larger than the inverse of the frequency being described, but on the other hand, one should keep higher order terms in ε significantly smaller than the first order term to ensure the validity of the perturbative computation.

III. DYNAMICAL CASIMIR EFFECT IN A SPACETIME WITH A SCREENED SCALAR FIELD

In this section, we study the toy model of a DCE for a minimally coupled massive quantum scalar field in a spacetime affected by a chameleon field $\varphi = \varphi(\mathbf{x})$. Let us consider the spacetime metric to be the Minkowski metric $\eta_{\mu\nu}$ and a quantum field trapped inside an effectively one-dimensional cavity[106] of average proper length L. The cavity is placed at a distance d to a sphere of radius R, which acts as a source for the chameleon force. We consider coordinates centered on the sphere, where x is the radial distance to the center of the sphere. The boundaries of the cavity are placed at x_l (left) and x_r (right), and the right boundary oscillates with frequency Ω and amplitude $\varepsilon L \ll L$, such that the cavity is oscillating as:

$$x_l = s, \quad x_r(t) = s + L\left[1 + \varepsilon \sin(\Omega t)\right], \tag{19}$$

where $s = R + d$. We impose Dirichlet boundary conditions on the scalar field at the boundaries and ignore the gravitational field of the chameleon source mass. This toy model will help us to understand the qualitative behaviour of a confined quantum field with moving boundaries in a spacetime with a screened scalar field.

Since the Minkowski metric $\eta_{\mu\nu}$ is a synchronous frame, we can use the method displayed in Ref.[25] right away. Focusing on the small perturbations regime for the problem under consideration, the quantities needed to compute Eqs. (17) and (18) are

$$\mathop{\hat{\Delta}}\limits^{\pm}_{m} = 0; \tag{20}$$

$$\Delta x(s) = 0, \quad \Delta x(s+L) = L\sin(\Omega t). \tag{21}$$

Using Eq. (4), the static spatial eigenvalue equation (34) of Ref.[25] and the boundary conditions read

$$\begin{aligned}
&\partial_x^2 \Psi_n^0 + \left[\left(\omega_n^0\right)^2 - A^2\left(\varphi\right) m_\Phi^2\right] \Psi_n^0 = 0, \\
&\Psi_n^0\left(x_l = s\right) = \Psi_n^0\left(x_r(0) = s + L\right) = 0.
\end{aligned} \tag{22}$$

A. Solution to the static eigenvalue equation

Since we have assumed $\varphi/M \ll 1$, see Sec. II A 2, the chameleon coupling function in Eq. (5) can be approximated by

$$A^2\left(\varphi\right) = 1 + 2\frac{\varphi\left(x\right)}{M} + \mathcal{O}\left(\frac{\varphi^2}{M^2}\right). \tag{23}$$

The chameleon profile given in Eq. (7) is a function of $1/r$ with $r > 0$. Thus, assuming that the cavity is sufficiently far from the source mass center, i.e. $L \ll s$, it is possible to linearize the field profile within the cavity $x \in [x_l, x_r]$, such that

$$\varphi\left(x\right) \approx \varphi_{bg} - 2\varphi_{bg}\frac{R}{s} + \varphi_{bg}\frac{R}{s^2}x. \tag{24}$$

Substituting Eq. (24) in the eigenvalue equation (22), we have

$$\partial_x^2 \Psi_n^0 - 2\frac{\varphi_{bg}}{M}\frac{R}{s^2} x m_\Phi^2 \Psi_n^0 + \left[\left(\omega_n^0\right)^2 - \left[1 + 2\frac{\varphi_{bg}}{M} - 4\frac{\varphi_{bg}}{M}\frac{R}{s}\right] m_\Phi^2\right] \Psi_n^0 = 0. \tag{25}$$

Following the technique presented in Ref.[107], we define the quantities

$$a := 2\frac{\varphi_{bg}}{M}\frac{R}{s^2} m_\Phi^2, \tag{26}$$

$$b := 1 + 2\frac{\varphi_{bg}}{M} - 4\frac{\varphi_{bg}}{M}\frac{R}{s}, \tag{27}$$

$$\lambda_n^2 := \left(\omega_n^0\right)^2 - b m_\Phi^2. \tag{28}$$

Furthermore, we introduce a new variable

$$u := u_n\left(x\right) = \left(\frac{\lambda_n^2}{a} - x\right)\left(a\right)^{1/3}. \tag{29}$$

From here on, we omit the explicit dependence of u on n for simplicity in the notation. Then the eigenvalue equation (25) can be rewritten as

$$\partial_u^2 \Psi_n^0 + u \Psi_n^0 = 0, \tag{30}$$

which is an Airy differential equation. The solution of Eq. (30) is given by means of Bessel functions:

$$\Psi_n^0(u) = \sqrt{u}\left[c_1 J_{1/3}\left(\frac{2}{3}u^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}u^{3/2}\right)\right]. \tag{31}$$

To progress further in this derivation, we must assume that $u \gg 1$. Therefore, we can apply the asymptotic form of Eq. (31) as seen in Ref.[107], such that

$$\Psi_n^0(u) = \mathcal{A}_n u^{-1/4}\sin\left(\frac{2}{3}u^{3/2} + \phi\right), \tag{32}$$

with $\mathcal{A}_n$ and ϕ being constants. While $u \gg 1$ is not globally true, we show in Sec. III C that the region of the parameter space of the considered chameleon models, for which this assumption can be applied, is largely unconstrained by experiments.

To guarantee that the field satisfies the Dirichlet boundary conditions, we require that

$$\frac{2}{3}\left[u^{3/2}(x_r) - u^{3/2}(x_l)\right] = n\pi, \qquad n \in \mathbb{N}. \tag{33}$$

The next approximation we make is that the variation of u within the cavity is small. This lets us linearize $[u(x)]^{3/2}$ at the point s,

$$[u(x)]^{3/2} \approx [u(s)]^{1/2}\left[u(s) + \frac{3}{2}\left[u'(s)\right](x-s)\right]. \tag{34}$$

Then Eq. (33) becomes

$$[u(s)]^{1/2}\, u'(s)\, L = n\pi. \tag{35}$$

Substituting Eqs. (29) and (28) in Eq. (35), we obtain

$$\left(\omega_n^0\right)^2 = k_n^2 + m_\Phi^2\left(1 + 2\frac{\varphi_{bg}}{M}\left[1 - \frac{R}{s}\right]\right), \tag{36}$$

where $k_n = \frac{n\pi}{L}$. Note that if the chameleon field is turned off, we recover the usual frequencies of the static problem in flat spacetime for a massive field[1]

$$\omega_{n_\mathrm{f}}^0 = \sqrt{k_n^2 + m_\Phi^2}. \tag{37}$$

From Eq. (32) and the boundary condition $\Psi_n^0(x_l) = 0$, we see that

$$\phi = -\frac{2}{3}u^{3/2}(x_l). \tag{38}$$

Applying the normalization condition given in Eq. (36) of Ref.[25], we obtain

$$\mathcal{A}_n = \left[2\omega_n^0\int_{x_l}^{x_r} \mathrm{d}x\, u^{-1/2}(x)\sin^2\left(\frac{2}{3}u^{3/2}(x) - \frac{2}{3}u^{3/2}(x_l)\right)\right]^{-1/2}. \tag{39}$$

DCE with screened scalar fields

Using Eq. (34) and doing another linearization of $u^{-1/2}(x)$ at the point s, we obtain after some algebra

$$\mathcal{A}_n = \frac{2\left(k_n^2\right)^{3/4}}{\left(\omega_n^0 L\right)^{1/2}(a)^{1/6}\left(4k_n^2 + aL\right)^{1/2}}. \tag{40}$$

The full derivation of this normalization constant can be found in the Appendix A.

B. Bogoliubov coefficients

In order to compute the Bogoliubov coefficients in Eqs. (15) and (16), we first need to compute the quantities in Eqs. (17) and (18). Note that the first integrals of Eqs. (17) and (18) vanish since the operator $\underset{m}{\pm}\hat{\Delta}$ is zero (as seen in Eq. (20)). For the second integrals, we substitute Eq. (21). Since we are considering one spatial dimension, the "surface integral" is simply the evaluation of the integrand at the two static boundaries, such that

$$\begin{aligned}\Delta\hat{\alpha}_{nl} &= -\mathrm{i}L\sin(\Omega t)(-1)^{n+l}\mathcal{A}_n\mathcal{A}_l(a)^{1/3}\left[\left(\lambda_n^2 - ax_r\right)\left(\lambda_l^2 - ax_r\right)\right]^{1/4} \\ &= -\Delta\hat{\beta}_{nl}.\end{aligned} \tag{41}$$

Substituting Eq. (40) in Eq. (41) and considering that the oscillation frequency of the boundary coincides with the difference of the mode frequencies, that is $\Omega = |\omega_n^0 - \omega_l^0|$, then we see that the α-coefficients from Eq. (15) are given by

$$\alpha_{nl}(t_f, t_0) \approx -\varepsilon\frac{2(-1)^{n+l}(t_f - t_0)}{\left(\omega_n^0\omega_l^0\right)^{1/2}}\frac{\left[k_n^2k_l^2\right]^{3/4}\left[k_n^2k_l^2 - aL\left(k_n^2 + k_l^2\right) + a^2L^2\right]^{1/4}}{\left[16k_n^2k_l^2 + aL\left(4k_n^2 + 4k_l^2\right) + a^2L^2\right]^{1/2}}. \tag{42}$$

If, instead, the oscillation frequency of the boundary coincides with the sum of the mode frequencies, that is $\Omega = \omega_n^0 + \omega_l^0$, then Eq. (16) is

$$\beta_{nl}(t_f, t_0) \approx \varepsilon\frac{2(-1)^{n+l}(t_f - t_0)}{\left(\omega_n^0\omega_l^0\right)^{1/2}}\frac{\left[k_n^2k_l^2\right]^{3/4}\left[k_n^2k_l^2 - aL\left(k_n^2 + k_l^2\right) + a^2L^2\right]^{1/4}}{\left[16k_n^2k_l^2 + aL\left(4k_n^2 + 4k_l^2\right) + a^2L^2\right]^{1/2}}. \tag{43}$$

Eqs. (42) and (43) are the general results of this work. Recall that these coefficients are obtained in the presence of resonances where the corresponding coefficient grows linearly with time. Hence, after enough time the effect becomes significant and non-perturbative.

C. Analysis

If we expand Eq. (43) around the small parameter $\frac{\varphi_{bg}}{M}$ up to first order (since the second order is negligible, see Eq. (23)), we obtain

$$\beta_{nl}\left(t_f, t_0\right) \approx \frac{\left(-1\right)^{n+l} \varepsilon k_n k_l \left(t_f - t_0\right)}{2\left(\omega^0_{n_\mathrm{f}} \omega^0_{l_\mathrm{f}}\right)^{1/2}} \left\{1 + B_{nl} + \mathcal{O}\left(\frac{\varphi^2_{bg}}{M^2}\right)\right\}. \tag{44}$$

We see that the zeroth order approximation, when the chameleon field is turned off, gives the usual coefficients of the DCE in Minkowski spacetime[25,108,109]. Thus, the zeroth order approximation exhibits the familiar resonance behaviour in the β-coefficients. The first order approximation is given by

$$B_{nl} = \frac{m^2_\Phi}{2} \frac{\varphi_{bg}}{M} \left[-\left(\frac{1}{\left[\omega^0_{n_\mathrm{f}}\right]^2} + \frac{1}{\left[\omega^0_{l_\mathrm{f}}\right]^2}\right) + \frac{R}{s}\left(\frac{1}{\left[\omega^0_{n_\mathrm{f}}\right]^2} + \frac{1}{\left[\omega^0_{l_\mathrm{f}}\right]^2}\right) - \frac{3}{2}\frac{RL}{s^2}\left(\frac{k^2_n + k^2_l}{k^2_n k^2_l}\right)\right]. \tag{45}$$

Eq. (45) gives a novel contribution due to the chameleon field, where the first term is a constant independent of the geometry. The second contribution depends on the position of the cavity in relation to the chameleon source, while the third term depends on the length of the cavity and tells us about the strength of the chameleon gradient between the two ends of the cavity, reminiscent of the structure of a linearized Newtonian gravitational potential. Since we are already in the resonance regime $\Omega = \omega^0_n + \omega^0_l$, the sum in the number of particles in Eq. (14) disappears, such that the average particle number is given by

$$\left|\beta_{nl}\left(t_f, t_0\right)\right|^2 \approx \frac{\varepsilon^2 k^2_n k^2_l \left(t_f - t_0\right)^2}{4\left(\omega^0_{n_\mathrm{f}} \omega^0_{l_\mathrm{f}}\right)} \left\{1 + 2B_{nl}\right\}. \tag{46}$$

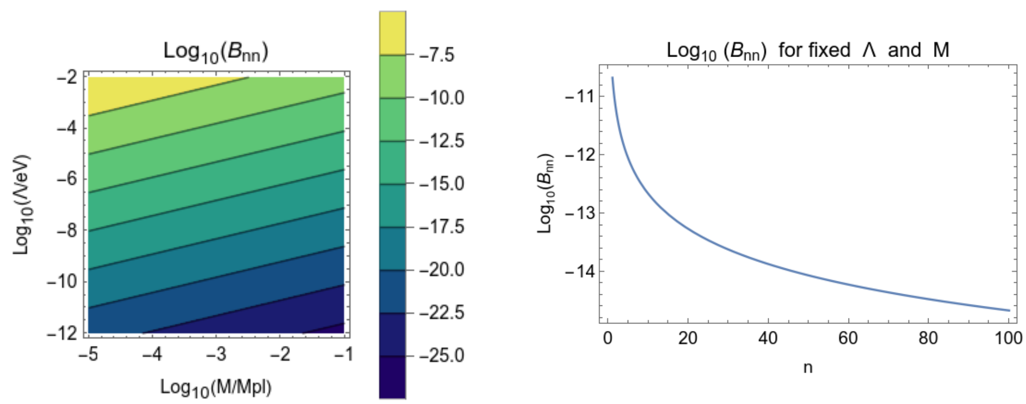
To see how the chameleon contribution affects the β-coefficients and thus the particle number, let us consider that the cavity and the chameleon source are inside a vacuum chamber of radius R_{vac}. We plot the contours for B_{nl} using the parameters shown in Tab. I. Note that we consider a kaon K^0 as the massive quantum scalar field in our toy model. In order to obtain the chameleon background value, we use the relation given in Refs.[69,72]

$$\varphi_{bg}\left(\Lambda\right) = \xi\left(N\left(N+1\right)\Lambda^{4+N} R^2_{vac}\right)^{1/N+2}, \tag{47}$$

where ξ is a "fudge factor" largely insensitive to N, Λ and M, as well as to the assumed chamber geometry. Here, we assume the conservative value of $\xi = 0.55$[72]. We consider the chameleon models $N = 1$ and $N = -4$, which are the most studied ones[40]. For the

TABLE I: Parameters used to compute the chameleon contribution to the particle content of the confined quantum field.

Parameter	**Symbol**	**Value**
Cavity length	L	$50\ \text{eV}^{-1}$ ($\sim 10^{-5}$ m)
Cavity field mass	m_Φ	497×10^6 eV (mass of kaon K^0)
Source mass radius	R	$1000\ \text{eV}^{-1}$ (2×10^{-4} m)
Distance from center of source mass to cavity	s	$5000\ \text{eV}^{-1}$ ($\sim 10^{-3}$ m)
Vacuum chamber radius	R_{vac}	$2.5 \times 10^5\ \text{eV}^{-1}$ (5×10^{-2} m)



(a) (b)

FIG. 1: Plots of the logarithmic values of the chameleon contribution to the β-coefficients and the particle number for the chameleon model $N = 1$. Figure (a) shows the contour plot of such a contribution as a function of Λ and M for fixed the quantum number $n = 1$. For this part of the parameter space, $u \gg 1$ is true. Plot (b) shows the contribution as a function of the quantum number n for $\Lambda = 1 \times 10^{-3}$ eV and $M = 1 \times 10^{-1} M_{Pl}$.

chameleon model with $N = -4$, the Lagrangian in Eq. (6) changes to

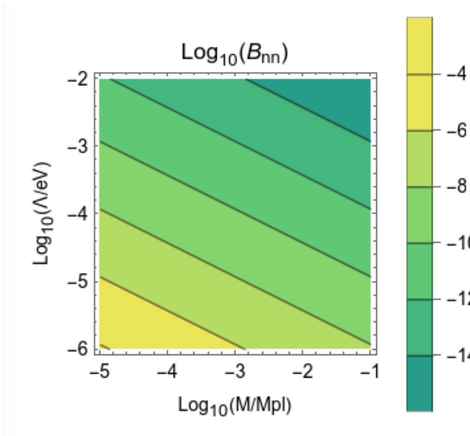
$$\mathcal{L} = -\frac{1}{2}\left(\partial\varphi\right)^2 - \frac{\lambda}{4!}\varphi^4 - \frac{\varphi}{\mathcal{M}}\rho. \tag{48}$$

Hence, in this case, Eq. (47) is given by

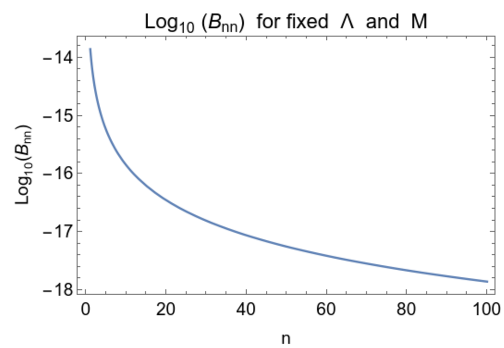
$$\varphi_{bg} = \xi\sqrt{\frac{2}{\lambda}}\frac{1}{R}, \tag{49}$$

where $\lambda = \left(\Lambda/\Lambda_{DE}\right)^4$ and $\Lambda_{DE} = 2.4$ meV is the dark energy scale[40].

Fig. 1 shows the chameleon contribution to the Bogoliubov coefficients and consequently to the particle number for the chameleon model $N = 1$, where, more precisely, Fig. 1a depicts



(a)



(b)

FIG. 2: Plots of the logarithmic values of the chameleon contribution to the β-coefficients and the particle number for the chameleon model $N = -4$. Plot (a) shows the contribution as a function of Λ and M for the fixed quantum number $n = 1$. For this part of the parameter space, $u \gg 1$ is true. Plot (b) shows the contribution as a function of the quantum number n for $\Lambda = 1 \times 10^{-3}$ eV and $M = 1 \times 10^{-1} M_{Pl}$.

the chameleon contribution as a function of Λ and M. The parameter M is essentially unconstrained but probably below the reduced Planck mass $M_{Pl} \approx 2.4 \times 10^{18}$ GeV[69]. Here, we use a subset of the parameter spaces shown in Refs.[82,86], where the assumption made in Eq. (29) is fulfilled, namely $u \gg 1$[110]. Fig. 1b shows the chameleon contribution as a function of the cavity mode number n. Note that the chameleon contribution is stronger for the upper left corner in Fig. 1a. In addition, also note that, for fixed Λ and M, the chameleon contribution is the strongest for the quantum number $n = 1$, but decays with increasing n.

Furthermore, in Fig. 2, we present the chameleon contribution to the Bogoliubov coefficients and consequently to the particle number for the chameleon model $N = -4$. Fig. 2a shows the chameleon contribution as a function of Λ and M, and Fig. 2b depicts it as a function of the cavity mode number n. Note that, in contrast to Fig. 1a, the chameleon contribution is stronger in the lower left corner in Fig. 2a, while, for fixed Λ and M, it behaves in the same way as it did for the chameleon model $N = 1$.

This difference in behaviour can be understood by examining the chameleon-induced force. The acceleration experienced by a test particle due to a chameleon field near a spherical source mass is given by Ref.[111]

$$\mathbf{a} = -\nabla \ln A(\varphi) = -\frac{\nabla\varphi}{M} = -\frac{R}{Mr^2}\left(\varphi_{bg} - \varphi_{obj}\right)\mathbf{r}, \tag{50}$$

where A is the conformal coupling factor defined in Eqs. (1) and (5), and we have used the chameleon field profile in Eq. (7). It can immediately be seen that a smaller M results in a stronger force. When comparing Eqs. (47) and (49), we see that the chameleon field scales oppositely with Λ for $N = 1$ and $N = -4$ models, i.e., φ_{bg} increases with increasing Λ for $N = 1$ but decreases for $N = -4$.

Therefore, we come to the natural conclusion that the chameleon field effect on the cavity particle production is the strongest where the chameleon-induced force is also the strongest. In both considered cases for N, we find a reduction in the particle number due to the presence of the chameleon scalar field.

IV. CONCLUSIONS

In this paper, we have shown the effect of a chameleon field on the number of particles created in a massive quantum scalar field by the DCE. We have considered an effectively one-dimensional cavity, with one of its boundaries allowed to move, placed near a chameleon source mass. Since the chameleon field is coupled to the mass of the quantum field, the Klein-Gordon equation and, in particular, the Lagrange-Beltrami operator acting on the quantum field in a spatial hypersurface, are affected by the chameleon field. We then computed the Bogoliubov coefficients in the presence of parametric resonances using the techniques developed in Ref.[25]. For this computation, we have linearized the chameleon field profile, in analogy to studies of linearizations of the Newtonian potential[20]. As expected, when the chameleon is turned off, the Bogoliubov coefficients are those of the DCE in Minkowski spacetime. Finally, we have analyzed how the particle content is affected by the presence of the chameleon field. We showed that the mean number of created particles is diminished by the presence of the chameleon field, and we gave representative numerical estimates for how the particle content is affected depending on the choice of the chameleon model, the parameters of the model, and the mode number of the quantum field.

This work can also be seen as an extension of the method presented in Refs.[24,25] since, for the first time, we were effectively considering a spatially dependent mass, which we can define as $\tilde{m}_\Phi(x) := A^2(\varphi(x))\, m_\Phi$.

DCE with screened scalar fields

To our knowledge, this article is the first work on the effect of screened scalar fields on particle creation. In the future, it will be interesting to create a more realistic study, also taking into account the gravitational field, not linearizing the chameleon field, and treating the chameleon as a quantum field as in Refs.[91–94]. The last would lead to loop corrections that could potentially be significant for the DCE, cp. Refs.[112–116]. However, such a fully quantum treatment is beyond the scope of the present article. In addition, other screened scalar field models could also be studied in the same way, and we leave for future work the study of entanglement between modes, their relation to the chameleon parameters, and the implementation of quantum metrology to estimate and constrain chameleon parameters.

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CONFLICT OF INTEREST

The authors have no conflicts to disclose

AUTHOR CONTRIBUTIONS

Ana Lucía Báez-Camargo: Conceptualization (equal); Data curation (supporting); Formal analysis (lead); Funding aquisition (equal); Investigation (lead); Methodology (equal); Software (equal); Visualization (equal); Writing - original draft (lead); Writing - review & editing (supporting). **Daniel Hartley:** Conceptualization (supporting); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (supporting); Software (lead); Validation (equal); Visualization (equal); Writing - original draft (equal); Writing - review & editing (equal). **Christian Käding:** Conceptualization (supporting); Data curation (lead); Formal analysis (equal); Funding aquisition (equal); Investigation (equal); Methodology (supporting); Software (supporting); Validation (equal); Visualization (equal); Writing - original draft (equal); Writing - review & editing (equal). **Ivette Fuentes-Guridi:** Conceptualization (equal); Formal analysis (supporting); Funding aquisition (equal); Investigation (supporting); Methodology (equal); Supervision (lead); Validation (equal); Visualization (equal); Writing - review & editing (lead).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

Appendix A: Derivation of the normalization constant

In order to compute the normalization constant in Eq. (39), we use Eq. (34) and another linearization of $u^{-1/2}(x)$ at the point s. Thus, the integral of the constant in Eq. (39) is

$$\begin{aligned} I &= \int_{x_l}^{x_r} \mathrm{d}x\, u^{-1/2}(x) \sin^2\left(\frac{2}{3}u^{3/2}(x) - \frac{2}{3}u^{3/2}(x_l)\right) \\ &= \int_{x_l}^{x_r} \mathrm{d}x \left([u(s)]^{-1/2}\left(1 - \frac{1}{2}[u(s)]^{-1} u'(s)(x-s)\right)\right) \\ &\quad \cdot \sin^2\left(\frac{2}{3}[u(s)]^{1/2}\left[\frac{3}{2}u'(s)\,x - \frac{3}{2}u'(s)\,x_l\right]\right). \end{aligned} \tag{A1}$$

Hence,

$$\begin{aligned} I &= [u(s)]^{-1/2}\left\{\frac{(x-s)}{2}\left(1 + \frac{1}{2}su^{-1}(s)\,u'(s)\right) - \frac{1}{2}u^{-1}(s)\,u'(s)\left(\frac{(x^2-s^2)}{4}\right)\right\}_{x_l}^{x_r} \\ &= [u(s)]^{-1/2}\frac{L}{2}\left\{1 + \left(\frac{s}{2} - \frac{2s+L}{4}\right)u^{-1}(s)\,u'(s)\right\}. \end{aligned} \tag{A2}$$

Substituting Eq. (29) in Eq. (A2), we obtain

$$\begin{aligned} I &= \left[\left(\frac{\lambda_n^2}{a} - s\right)(a)^{1/3}\right]^{-1/2}\frac{L}{2}\left\{1 + \left(\frac{s}{2} - \frac{2s+L}{4}\right)[u(s)]^{-1}u'(s)\right\} \\ &= \frac{(a)^{1/2}}{[\lambda_n^2 - as]^{1/2}(a)^{1/6}}\frac{L}{2}\left\{1 + \frac{L}{4}\left(\frac{a}{(\lambda_n^2 - as)}\right)\right\}. \end{aligned} \tag{A3}$$

Replacing Eq. (A3) in Eq. (39), we have that

$$\mathcal{A}_n = \left[2\omega_n^0 \frac{(a)^{1/3}}{[\lambda_n^2 - as]^{1/2}}\frac{L}{2}\left\{1 + \frac{L}{4}\left(\frac{a}{(\lambda_n^2 - as)}\right)\right\}\right]^{-1/2}. \tag{A4}$$

Plugging Eq. (28) into Eq. (A4), we finally obtain Eq. (40).

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[105]There are some authors that compute the DCE in three dimensions[20,120,121]. We consider the one-dimensional case because the transverse momentum does not appear, which simplifies the calculations. Moreover, experimental investigations of the DCE primarily work in one dimension[122–124].

[106]The reduction from a three spatial dimensional problem is done by assuming that two cavity dimensions are much smaller than the third one, such that the system can be effectively treated in one spatial dimension. This is due to the direction of the chameleon field gradient being radial, and therefore no significant effects occurring in the transverse

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Part III

Conclusion

8

Conclusion

This thesis presented a general method for computing the evolution of a confined quantum scalar field in a globally hyperbolic spacetime under general changes of the metric and/or the boundary conditions. We took into account cases in which the timelike boundaries remain static in a synchronous frame in [1] and when they do not in [2]. The core idea of the method is to construct a time-dependent Bogoliubov transformation between the bases of modes associated to different spacelike hypersurfaces. Such a time-dependent transformation can be interpreted physically in terms of mode-mixing and particle creation when it relates regions of spacetime in which the time symmetry allows for a Fock quantisation in terms of particles. In this sense, the method can be regarded as an "integration tool" to compute the evolution of a system between spacetime regions where there is a well defined particle interpretation.

The method is of general applicability and in order to prove so, we used this method in [1] to study the known prediction of cosmological particle creation, where we have obtained a differential equation to compute the Bogoliubov coefficients for *any* one-dimensional spatially flat FLRW spacetime. In addition, we applied this differential equation to a known toy model that predicts particle creation, reproducing the same results for the Bogoliubov coefficients between the asymptotic flat regions as in [9]. Moreover, we obtained a smooth and monotonic behaviour of the time-dependent Bogoliubov coefficients, even in regions where the physical interpretation is not well defined.

This method is especially useful in the case of small perturbations of the metric (or boundary conditions) around a static situation, since it provides simple and practical formulas for computing the Bogoliubov coefficients in the presence of resonances. In [1], we considered the case of a quantum field trapped in a cavity in free fall which is perturbed by gravitational waves in a $(3+1)$-dimensional spacetime. This example reproduces and extends the result in Ref. [101] with much simpler calculations.

To verify and illustrate the effectiveness of the method in the case of moving boundary conditions presented in [2], we considered the dynamical Casimir effect for a massive scalar field in (1+1)-dimensional Minkowski spacetime with three different periodic movements of the boundaries. Again, we reproduced and extended known results. Furthermore, we studied for the first time the case of a quantum field trapped in a rigid cavity which is perturbed by gravitational waves in a $(3+1)$-dimensional spacetime. Thus, the method will be useful to improve the detection of gravitational waves by using trapped Bose-Einstein condensates as proposed in Ref. [19].

In [3], we used the method to study dark energy and dark matter candidates. In particular, we analysed the impact of a chameleon field on the dynamical Casimir effect. In order to do so, we computed the Bogoliubov coefficients in the presence of resonances using the method developed in [2] and we showed that the number of particles created is reduced by the presence of the chameleon field. Furthermore, we gave representative numerical estimates for the change in the particle number depending on the chameleon model, the parameters of the model and the mode number of the quantum field.

We have provided plenty of evidence that the method presented here is a useful tool to compute the evolution of confined quantum fields in many different concrete physical situations. Using this method, we reproduced (and easily extended) results which used different techniques and had previously required more involved calculations. Moreover, we proposed a new way to find and constrain chameleon fields. Therefore, we expect this method to become a standard tool in QFTCS for confined fields.

There are future directions of research using this method. For example, it would be interesting to study quantum fields in other cosmological or astrophysical scenarios like collapsing stars or the study of other dark energy models. The theoretical side of the method leaves open the study of physical interpretations of the time-dependent transformations in regions that lack a time symmetry. In addition, further extensions of the method for different boundary conditions or other quantum fields may be constructed.

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