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Imagination and Formalism in (mathematical) Cognition

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Abstract: Imagination and Formalism in (mathematical) Cognition

While one could treat *numbers* in "classical" mathematics as *sensible* objects (at least the connection to "reality" cannot be ignored), in abstract (e.g. non-commutative) algebra this is quite difficult. But that would imply by Kant's "Critique" that there could not be any *cognition* (Erkenntnis) in fundamental parts of mathematics. Leading to a puzzling situation since mathematics is —not only for Kant— crucial for (not just natural) sciences. I want to show that this "paradox" can be avoided by clearly distinguishing between *cognition* —on the level of *reason* (Vernunft)— and *knowledge* —on that of *understanding* (Verstand)— and replace *sensibility* by *imagination* (Einbildungskraft) when we are *learning* autodidactic (to go beyond knowledge). In the classical setting, when we learn from someone with cognition (in some abstract area, not necessarily in mathematics), this imagination can be triggered via communication between our (human) bodies. In both cases *formalism* is *the* "language" of our understanding (Verstand), with the well known *decimal system* as prototypical example. In this way one can explain why we need not stumble when we leave intuition (Anschauung) completely, in particular when we can get orientation from *space* (or even "pure" language) within the theory of *free fractions.*

Zusammenfassung: Einbildungskraft und Formalismus in (mathematischer) Erkenntnis

Während man *Zahlen* in der “klassischen” Mathematik als *sinnliche* Objekte (zumindest kann die Verbindung zur “Wirklichkeit” nicht ignoriert werden) behandeln kann, ist das in der abstrakten (z.B. nicht-kommutativen) Algebra ziemlich schwierig. Aber das würde mit Kants “Kritik” implizieren, dass es keine *Erkenntnis* in fundamentalen Teilen der Mathematik geben könnte. Was zu einer rätselhaften Situation führen würde, ist doch die Mathematik —nicht nur für Kant— unabdingbar für die (nicht nur Natur-) Wissenschaften. Ich möchte zeigen, dass dieses “Paradoxon” durch die klare Trennung zwischen *Erkenntnis* —auf der Ebene der *Vernunft*— und *Wissen* —auf der des *Verstandes*— vermieden werden kann, wenn man die *Sinnlichkeit* durch die *Einbildungskraft* beim autodidaktischen *Lernen* ersetzt (um über Wissen hinauszugehen). In der klassischen Umgebung, wenn wir von jemandem mit Erkenntnis (in einem abstrakten Gebiet, nicht notwendigerweise Mathematik) lernen, kann diese Einbildungskraft durch die Kommunikation zwischen unseren (menschlichen) Körpern ausgelöst werden. In beiden Fällen ist der *Formalismus die* “Sprache” unseres Verstandes, mit dem allseits bekannten *Dezimalsystem* als prototypischem Beispiel. Auf diese Art kann man erklären warum man nicht stolpern muss, wenn man die Anschauung vollständig verlässt, insbesondere dann, wenn man Orientierung vom *Raum* (oder sogar der “reinen” Sprache) innerhalb der Theorie der *freien Brüche* bekommt.

Citations and Orthography

Except for some classical books like those from Plato and Kant, standardized page numberings are rarely available. To support orientation I included the section/chapter which should be invariant under translations. When a number system is available —like usually in mathematical references— I included that in the footnote.

The dash is used —just like before— such that it is easier to see what was inserted. Sometimes parenthesis could be used instead. But they are also used to "deemphasize" or for the creation of variants (of how to understand certain phrases). Or, like in the last sentence, for clarification or a recall of something earlier.

Quotation marks can have several meanings. That *projected* is often within them has to do with my mathematical background. The same holds for other technical terms used outside mathematics. In general they are used to increase awareness, sometimes it is necessary to hold in for a while and reflect. The standard uses are for citations and for terms consisting of several words (when this is not immediately clear). For the latter sometimes cursive/italics text is used.

The orthography in citations is not changed. Square brackets are used for insertions (usually from the original text before) that increase readability. Other changes (e.g. emphasis) are commented in the respective footnotes.

Prologue

Already talking about knowledge and/or cognition is quite challenging since it cannot be done without some (prior) "understanding of knowledge" and we need to use the language we have. Restricting to knowledge of mathematics would not suffice in general without assuming that there is some significant difference. But there is no indication for something special about "mathematical intelligence"[1].

To minimize troubles with this non-avoidable self-reference, we are going to build (almost literally) worlds around (the terms) *knowledge* and *cognition* such that we can clearly separate them when necessary (without becoming too technically and thus more unreadable) but still use them in a colloquial (individual coloured) way otherwise. That it will be non-trivial to overcome this difficulty should be clear immediately when we think about the use of these (and many other, sometimes only loosely connected) words in other languages (than English, the one used here).

To simplify things we are going to take up the parallel in Immanuel Kant's *Critique*[2] and mention (once in a while) his German "technical" terms. This should help to get acquainted faster because the context is (e.g. for translation) clear(er). However, there is no need to study Kant's work before. One can (start to) dig deeper along our journey with the starting point and central theme:

"It comes along with our nature that *intuition* (Anschauung) can never be other than *sensible*, i.e. that it contains only the way in which we are affected by objects.

[1] (Penrose, 1994, Introduction)
[2] Kant (1998b)

The faculty for *thinking* of objects of sensible intuition, on the contrary, is the *understanding* (Verstand). Neither of these properties is to be preferred to the other. Without sensibility no object would be given to us, and without understanding non would be thought. Thoughts without content are empty, intuitions without concepts are blind. It is thus just as necessary to make the mind's concepts sensible (...) as it is to make its intuitions understandable (...). Further, these two faculties or capacities cannot exchange their functions. The understanding is not capable of intuiting anything, and the senses are not capable of thinking anything. Only from their unification can cognition (Erkenntnis) arise."[3]

The main goal of this thesis is not to establish some "hard" knowledge about "imagination and formalism in (mathematical) cognition" but to increase our awareness for the fascinating *dynamics* "between" *imagination* and *formalism* that is the basis for learning mathematics, no matter on which level. A side-effect should be that we can find a similar —although maybe less sharply describable— interaction between our *senses* and our *understanding* also beyond mathematics ...

[3] (Kant, 1998a, B75)

Contents

List of Figures

List of Tables

Chapter 1

Introduction

What does it mean to *know* something, say in mathematics? Being able to reproduce a formula in a way one has (or should have) learned it? Being able to use it within a specific "usual" context? Knowing "enough" about its limitations, and/or how to solve it? Exact (if possible), or approximately? By using a specific program/software? If so, how much do we have to know about it to apply it rightly, to avoid traps?

Well, the answers —we are not primarily interested in— clearly depend heavily on the situation, and —in principle— we could ask someone with "more" knowledge (also on some meta level) to guide us. If it is concrete enough one could look up in some book. However, we succeed in finding (what we are looking for) only if that what bothers us (e.g. in some problem/calculation) is more or less "normal". Although the *internet* enables us to find also very "special" information (e.g. in discussion forums), it does not contain the answers to the "unasked" and there is a certain problem to get from *information* to *knowledge* (in particular also when we use tools like "artificial intelligence").

Cognition —as an (in principle) unlimited source of knowledge— will be the key. It is also the origin of *all* "objective knowledge", unfortunately the classification as "subjective" has lead to much less attention it would have deserved. We therefore start by reading Karl Popper in a *neutral* way: "I assert that we can, and that indeed we must, distinguish sharply between *knowledge in the subjective sense* and *knowledge in the objective sense*."[1]

Let us call the former *cognition*, the latter *knowledge*, and add *information* to Figure 1.1 (page 3). A first —but crucial— observation is that we can only have (more or less) direct access to some limited part of World 3: "From this determination of World 3, there are two remarkable consequences. The first concerns the *interaction between the worlds*, the second the *cognitivistical reduced interpretation of the third world*. Due to Popper the first and the second world are in direct exchange, such as the second and the third. The first and the third however interact only through World 2."[2]

A second observation is the important role of *information* as basis for (a priori "local") *knowledge* and *cognition*. However, and that is central to Kant's *Critique*, information needs interpretation guided by *concepts* (Begriffe) which themself need *content* (Inhalt) to become useful resp. to avoid shallowness. Now it should become clear that it was no coincidence that we talked about *information* (not knowledge) from the internet before. *Data structures*, i.e. knowledge about the interpretation of data (e.g. how numbers are stored depending on the hardware), should not only be relevant for computer scientists. Moreover, unless we know something about the reliability of the source (of information), we need to be careful.[3]

Last but not least, a third observation is the *dynamics* of cognition in sense of an individual process, being quite opposite to the *static* (here not understood completely timeless) knowledge (of World 3) we have in mind when we think of something like "objective knowledge" (in e.g. epistemology). This "cognitive" dynamics is stressed by the (graphical) presentation in Figure 1.1 (below).

[1] (Popper, 1978, Section X, Page 156)

[2] (Habermas, 2014, Chapter 3, Page 118), my translation

[3] But even that is nothing new. For serious books we usually do not ask if we could trust them (in general). Although in mathematics it is hard to imagine that someone writes (intentionally) "fake mathematics", there is always the small chance for typographical errors. Thus, processing information *always* needs some level of awareness, which could be indirect in e.g. plausibility checks if it is about huge data (processed with some machine).

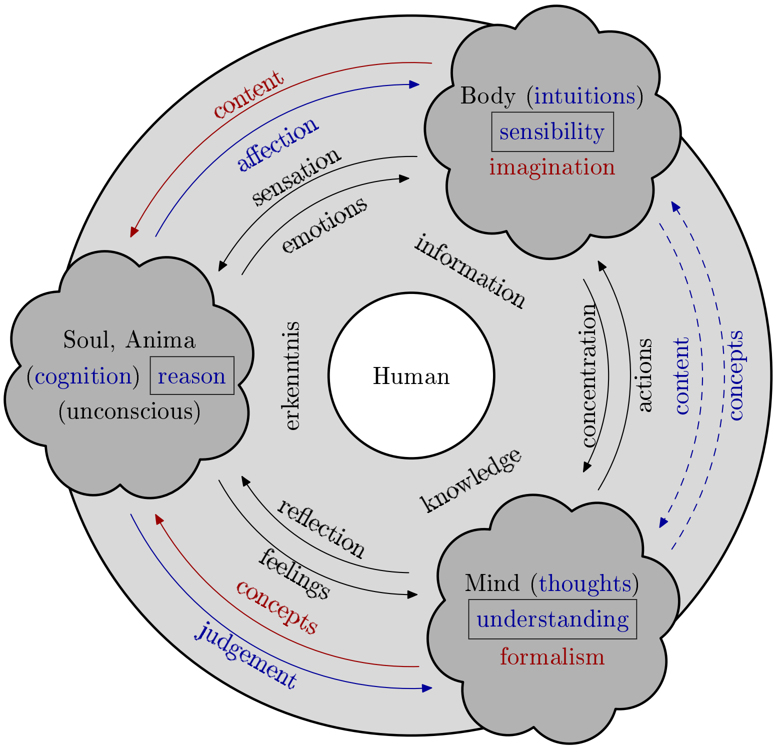

Figure 1.1: A non-trivial human model. What started as a very simple model (for an article in a university magazine in Graz in 2017) based on the distinction of *emotions* and *feelings* —as well as *sensation* (Körperempfindungen)— Damasio (2013) and *actions* in the context of "free will", e.g. from Bieri (2013), completed by the *soul* and the missing arrows "concentration" and "reflection", now contains an extra level with concepts very close to Kant (1998a). This trinity aligns almost naturally with the concept of *worlds* from Popper (1978), established in an essay about pluralism in mathematics, Section E.2. With respect to our main topic we can assign the three different dimensions *information*, *cognition* (Erkenntnis) and *knowledge* to World 1 (physical, sensibility), World 2 (actors, reason) resp. World 3 (concepts/theories/thoughts, understanding). Kant's *apperception* would relate to Popper's World 2. More details are available in the text starting with the introduction, Chapter 1. Notice that it is no miss-interpretation of Popper since the World 3 "depicted" here is only a *local* (individual) one. Actually, in a dynamical understanding, we need *cognition* to keep this "copy" (of a small part of the huge World 3) updated.

Since there is already some circularity in "Thoughts without content are empty, intuitions without concepts are blind."[4], we could add Kant's (power of) *judgement* (Urteilskraft) and *affection* to complete: Thoughts without judgement are meaningless (i.e. cannot be knowledge), intuitions without affection are valueless (i.e. cannot be information). And further: Only from the unification of *information* and *knowledge* can cognition arise.

Fortunately, taking the dynamics into account, the self-reference is only week in the sense that we need (resp. rely on) only "old" (or "partial") knowledge, not yet part of the whole World 3. Or, in other words, and that is in particular important for (the process of) *learning*: We can get knowledge not only from World 3 (by reading a book, asking someone, etc.) but also from thinking (here understood very broadly). In the latter case cognition is the key for the verification of knowledge to make it *consistent* (within "our" world). To avoid the term "subjective", one could call this type of (consistent) knowledge *locally*[5] *objective*.

To avoid misunderstanding we actually would need to specify which knowledge we use (e.g. about our language, arithmetic, etc.) and *how* we got some insight (and possible new knowledge). We need to find out what is relevant (to properly describe a theory for a specific group of people and maybe the relevance for another group) and what could be omitted. And there could be difficulties (e.g. in the application) that need special attention (and e.g. extra designed examples for illustration). Or one might draw a figure for visualization that reveals connections almost "invisible" in normal text or that depend on far too technical mathematical notation. *Where* does all that (meta) knowledge come from?

One mathematical concept, namely *continuous* versus *discrete*, fits as metaphor to Popper's view: "It is the objective thought content of a conjecture or theory on which the scientist's subjective thought process work. They are at work to improve

[4](Kant, 1998a, B75)

[5]The predicate "local" is frequently used in mathematics for properties on a connected set, i.e. not just on isolated elements.

the objective thought contents by way of *criticism*."[6]

Knowledge (also *local*, within a human) could be understood as *discrete* in the sense of a snapshot. Brought to the (conscious) mind it enables *reflection* (see Figure 1.1) and self-critique, brought to the audience of scientists (say at a conference) it helps to further develop a theory or identify gaps (in proofs). Cognition (as steady "flow") could be understood as *continuous* (not in a strict mathematical sense).

Summing up we get the basis for further discussions with a characterization of *knowledge* vs. *cognition* in Table 1.1 (below) . There would be quite a lot to say about *information* too because it is *only* via (the physical) World 1 we get access to World 3, e.g. via reading a book. What looks almost trivial becomes very subtle when one starts to ask, what symbols (e.g. in mathematics) mean. The essay in Section E.4 could be seen as a warm-up.

One can clearly discuss where to put the (conscious) mind[7] and how to integrate Kant's *outer* and *inner sense* (space resp. time)[8]. The former itself would be a topic on its own, to the latter we come back in Section 3.3. Our basic idea is to take the (physical) *body* as gateway to World 1 while the *mind* acts as gateway to (the global) World 3, coordinated by the *I* (the actor) in World 2. Including Damasio's understanding of *emotions* (body) versus *feelings* (mind) as "two sides of one coin" (see Figure 1.1) has the advantage of a more complete human model, in particular important for *communication*: While feelings might not (yet) be present —for whatever reason (e.g. a not fully developed concept)—, emotions could already be read by others, in particular those with a "lower" filter like *highly sensitive people*[9] *and* the necessary concept(s) for interpretation . . .

Remark. We will encounter Figure 1.1 (Page 3) in a variant later in Figure 3.1 (Page 31). These figures serve as a "workbench" for thinking about the topic. It is

[6](Popper, 1978, Section X, Page 156), my emphasis

[7]See e.g. (Popper and Eccles, 2018, Chapter E7) as a starting point, in particular Section 50 with the nice tabular "world" and the graphical "brain–mind interaction" overview.

[8](Kant, 1998a, B37)

[9]Aron and Aron (1997)

information (World 1)	cognition (World 2)	knowledge (World 3)
physical	subjective	objective
touchable	active	passive
measurable	dynamic	static
	individual	collective
	continuous	discrete
	thought process	thought content

Table 1.1: Characterization of *information*, *knowledge*, and *cognition* (Erkenntnis) in the context of the concept of *worlds* from Popper (1978). As mentioned in the text, it should not be understood too dogmatically. It's not just almost immediate to think of a local World 3 resp. *local knowledge* (see also Figure 1.1), it stresses that communication (between people) is mandatory for "alignment" to get something like (accepted) "global knowledge". The other way around, cognition could be much less "subjective" and attributed at least to research groups (having *paradigms* from Kuhn (2017) in mind), in particular with respect to dynamics and the importance of criticism.

a *graphical representation* of our considerations in the sense of Sybille Krämer's understanding of *diagrammatic*, a concept we will discuss —with the focus on mathematics— deeper in Section 3.1. It should be clear that reflections about mathematics —and thus in particular mathematical cognition (but not necessarily knowledge)— are at least partially "outside" mathematics. This is the reason why "mathematical" in the title is within parentheses. For completion some additional comments could be helpful: By *concentration* we understand the impact of the state of the body (tired, ill, stressed, etc.) on the mind. Although this seems trivial, it is not completely irrelevant with respect to concrete calculations since it could be a source of errors. By *reflection* we mean that —although we do not have direct access to the subconscious— we can change something, e.g. to reduce errors by exercising a lot. The latter is in particular relevant for mathematics: It does not help us to *know* (the rules of) the decimal system, we need to be able to apply it "like a machine"[10].

Chapter F contains a description of the basic theme of Figures 1.1 and 3.1 for blind people (or those who want to try how it is to build up a figure in mind). Reading the "clouds" *counterclockwise* (i.e. mathematically positive), *body*, *soul*, and *mind* correspond —more or less— to Popper's World 1, World 2, and World 3 respectively.

[10]The essay in Section E.7 provides an an introduction to the discussion "*mind* versus *machine*" in our context (of Popper's worlds).

Chapter 2

Basics and Concepts

This chapter serves three main purposes: (1) It provides an ultra-compact background (and some hints on further literature) for those who are not that familiar with *philosophy of mathematics* in general (Section 2.1) and (mathematical) *formalism* in particular (Section 2.2). (2) It summarizes the key ideas of Karl R. Poppers concept of *three worlds* (Section 2.3). And (3), it should —if read before— prepare for diverse(r) perspectives and/or —if read later (after Chapter 3 and/or Chapter 4)— be useful to sharpen the readers *own* thoughts on *mathematical knowledge* (Section 2.4) and *imagination in mathematics* (Section 2.5). These two last sections should therefore not be understood as "complete" or "closed", not even partially; rather as "open" and slightly provoking; in any case fragmentary . . .

2.1 Philosophy of Mathematics

Mathematics is a "philosophical challenge"[1] because it "seems to be practiced by means of reflection and proof alone, without any reliance on sense experience or experimentation; and it seems to deliver [a priori] knowledge of necessary truths that

[1] (Linnebo, 2017, Chapter 1)

are concerned with abstract things such as numbers, sets, and functions."[2] And nevertheless it is so successful in other (more empirical) sciences. Or because of that?

Mathematics is often much more than just a tool: "I assert, however, that in any special doctrine of nature there can be only as much *proper* science as there is *mathematics* therein."[3]

One can question (Kant's) *apriorism* (see Section 2.4). For nominalists there are no *abstract objects*[4]. For platonists "Mathematical objects are at least as real as ordinary physical objects."[5] Frege's logicism ended in a disaster.[6] We will have a closer look on *formalism* in (the following) Section 2.2. Hilbert's instrumentalism can be seen "between" realism and nominalism.[7] "*Structuralism* is a philosophical view that emphasizes mathematics' concern with abstract structures, as opposed to particular systems of objects and relations that realize these structures."[8] Gödel's incompleteness theorem had a deep impact on Hilbert's program; for Brouwer's intuitionism it was important to develop the "contentful, finitistic part of mathematics."[9]

The *necessity* is out of question. But we recall in Chapter 3 that the *context* makes $7 + 5 = 12$ true (or false). For numbers —we typically write using the *decimal system*— this is admittedly somewhat pedantic. However, we would like to get a step further and "calculate with words" later in Chapter 4. It *highly* matters which symbols are part of the basic alphabet (to form non-commutative polynomials) and which are used "freely" (as non-commuting variables) to denote expressions.

Later we will see that also *apriorism* versus *empiricism* and *realism* (platonism) versus *nominalism* very much depend on the context resp. the perspective. It is

[2]Ibid. Page 5

[3](Kant, 2002, Preface, Page 185, 4:470)

[4](Linnebo, 2017, Chapter 7). One prominent defender of nominalism is Field (2016).

[5](Linnebo, 2017, Section 1.4, Page 11) "An object is said to be *abstract*, we recall, if it lacks spatiotemporal location and is causally inefficacious; otherwise it is said to be *concrete*." (Ibid. Page 9)

[6]Ibid. Chapter 2. A reminder to *Russell's paradox* will appear in a remark at the end of Section 4.3.

[7](Detlefsen, 1986, Chapter 1, Page 2)

[8](Linnebo, 2017, Chapter 11, Page 154)

[9](Linnebo, 2017, Chapter 5, Page 73)

difficult to sharply set the limits e.g. what one still considers as *number* (Section 3.3 but also Section 2.3). After deeper reflections about the essence of —not just some "ancient"— mathematics it will turn out anyway that we don't come that far with any dogmatism. Mainly because we are not interested in a static view but in the dynamical of *cognition*. Not just primarily to gain (new) knowledge but to be able to *explain* mathematics (to someone else). Therefore it is important to be able to shift the focus: From the concrete to the structure, from the formal to the sensual, etc. (and back again). In this sense of *diversity*, mathematics —from ancient over "modern" to "cutting edge" results— can be understood as timeless . . .

After all, mathematics can be seen as *language*, including the *application* and drawing the parallel to *action*: "Our speech is at the same time also an action."[10] Rephrased in the essay in Section E.4 as "Our calculation is at the same time also an application."

$$\begin{tikzcd}
& \text{mathematical system} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{calculation} \arrow[rr, leftrightarrow] & & \text{application}
\end{tikzcd}$$

Philosophy of mathematics[11] is —in this sense— in particular philosophy of language.

2.2 Mathematical Formalism

When there are problems with the reference to "objects" and "meaning" of (mathematical) expressions, wouldn't it be better to concentrate on the *syntax* alone? We recall: "*Syntax* is the study of linguistic expressions abstracted from any meaning they may have. The expressions are understood as mere strings of characters. *Semantics*, by contrast, has meaning as its primary concern. It is the study of the truth,

[10](Krämer, 2017, Section 13.1, Page 241) on Judith Butler's "performativity of language"; John L. Austin initiated the discussion of joining speech and action. My translation.

[11]For a focus on "doing" (mathematics) and many more "philosophical thoughts about proofs, applications, and other mathematical activities" see Hacking (2014).

reference, and linguistic meaning that can attach to linguistic expressions."[12]

There is a whole zoo of formalisms, e.g. game formalism, term formalism, deductivism, Tractarian formalism, nominalist formalism, empirico-semantic formalism, instrumental formalism, etc.[13] Here we focus on David Hilbert's understanding of formalism because it can also serve as a good starting point for studying variants.

Hilbert distinguishes between *real* (reale) and *ideal* (ideale) entities. (One could use *contentful* resp. *non-contentful*[14] instead.) He starts with *finitistic* propositions and adjoins the *ideal* propositions. In this way one can introduce $i = \sqrt{-1}$ to get $i^2 = -1$.[15] To get the basic idea its worth to "listen" to himself:

"The method of the algebraic 'calculation with letters' (Buchstabenrechnung) isn't included in the contentful-vivid (inhaltlich-anschaulich) number theory, as we understood it until now. In its context, formulae were used solely for communication; letters stand for 'numerical words' (Zahlzeichen), and via an equation the correspondence between two signs (Zeichen) were communicated. Contrary, in algebra we regard 'expressions of letters' (Buchstabenausdrücke) as such as independent things and the contentful theorems of number theory are formalized by them. Instead of propositions about numerical words, their place is taken by formulae which are now concrete objects of a vivid (anschaulich) observation; and the place of a contentful number theoretic proof is taken by the derivation of a formula from another formula by specific rules."[16]

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[12](Linnebo, 2017, Section 3.1, Page 38)

[13]For the first three see e.g. (Linnebo, 2017, Sections 3.2, 3.3, and 3.4), for the Tractarian and the nominalist formalism (Weir, 2022, Section 3 resp. Section 5), and for the last two Detlefsen (2021). For more background on formalism in general and Hilbert's formalism in particular see Detlefsen (2005).

[14](Detlefsen, 2021, Introduction, Page 530)

[15](Hilbert, 1926, Page 174)

[16](Hilbert, 1926, Page 174), my translation. I tried to keep some very subtle details (from the German original). In formal language theory, a *number* is just a word over an alphabet whose letters we call digits. A *number* is much more than a *numerical word* (Zahlzeichen). This should become clear later in Chapter 4: Using the decimal system for *adding* two *numbers* actually is nothing more than to "add" two *numerical words*, i.e. to apply purely *formal* rules.

Hilbert's approach combines *term* formalism for *finitary* (real, contentful) mathematics and *game* formalism for *infinitary* (ideal, non-contentful) mathematics. Since it is not clear whether finitary mathematics can provide realizations of infinite structures he reduces the semantic problem (of model existence) to *formal consistency* where a derivation with *finitely* many steps would be sufficient to show some contradiction, e.g. $0 = 1$.[17] However, as Kurt Gödel showed with his *incompleteness theorems* (1930), "Finitary mathematics cannot prove its own inconsistency."[18]

Unfortunately we needed to make a huge step and ignored a lot of details. One might consider *Hilbert's program*[19] as "dead" (after Gödel's results[20]), but his important contributions to mathematical *instrumentalism* are lasting. And —turning back to our "cognitivistic" focus— one could pick up the *dynamics* of Hilbert's program:

"What is it that shapes mathematical arguments into proofs that are intelligible to us, and what is it that allows us to find proofs efficiently? — This is the informal question I intend to address by investigating, on the one hand, the abstract ways of the axiomatic method in modern mathematics and, on the other hand, the concrete ways of proof construction suggested by modern proof theory. (. . .) The subtle interaction between understanding and reasoning, i.e. between *introducing concepts* and *proving theorems*, is crucial. It suggests principles for structuring proofs conceptually and brings out the dynamic role of *leading ideas*. Hilbert's work provides a perspective that allows us to weave these strands into a fascinating intellectual fabric and to connect, in novel and surprising ways, classical themes with deep contemporary problems. The connections reach from proof theory through computer science and cognitive psychology to philosophy of mathematics and all the way back."[21]

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[17](Linnebo, 2017, Section 4.1, Page 56/57)
[18]Ibid. Section 4.6, Page 71
[19]For in depth discussions see Detlefsen (1986) and/or Sieg (2013).
[20]For a comprehensive presentation see Smith (2007).
[21](Sieg, 2013, Chapter III.3, Page 377)

2.3 Three Worlds

Karl R. Popper proposes *three* interacting worlds (or "sub-universes"): (The physical) World 1 consists of "physical bodies: of stones and of stars; of plants and of animals; but also of radiation, and of other forms of physical energy."[22] (The mental or psychological) World 2 consists of "feelings of pain and of pleasure, of our thoughts, of our decisions, of our perceptions and our observations; (...) Human suffering belongs to world 2."[23] World 3 consists of "the products of the human mind, such as languages; tales and stories and religious myths; scientific conjectures or theories, and mathematical constructions; songs and symphonies; paintings and sculptures. But also aeroplanes and airports and other feasts of engineering."[24] Often objects from World 3 are also part of World 1, a typical example is that of a *book*, which could have many different *physical realizations*.[25]

What existence means for (abstract) objects (in World 3) is another story. But it is obvious that we do distinguish between "Where do I find Kant's *Critique*?" (in a library), "How do you find Kant's *Critique*?", and "Do I find something about cognition in Kant's *Critique*?" (neither a concrete book nor the language it is written in is relevant here).

Popper then explains in detail how the arguments (against his three worlds) would look from a materialist or physicalist monist insisting on *one* world only, namely World 1.[26] Instead of his example of Beethoven's Fifth Symphony, one could actually take also some *number*: We use the language as if it were a real (physical) object.

Dualists, as Popper continues, accept also World 2 because of "the great *experience* of listening to Beethoven's Fifth Symphony."[27] In particular interesting here is the emphasis of the *action*: If there is something we like, we can spend enormous effort

[22] (Popper, 1978, Section I, Page 143)
[23] Ibid.
[24] Ibid. Page 144
[25] Ibid. Page 145
[26] (Popper, 1978, Section III, Page 146)
[27] (Popper, 1978, Section IV, Page 147)

to be able to get (or listen to) it (live in a concert hall).

For arguing against a pluralist, he uses again the (example of a) symphony and different *performances*.[28] There is actually a very interesting parallel to Ferdinand de Saussure, who splits *language* in his "Cours de linguistique générale"[29] into *language system* (as subject in linguistics) and (individual) *speech*. One of his key observations is: "A performance so judged [marvellous, few appreciated it] is a world 3 object in my terminology — of course, one that is embodied, or physically realized — and it *can* be judged as a world 3 object."[30]

How "real" are objects in World 3? "[T]hey may be real in that they may *have a causal effect* upon us, upon our world 2 experiences, and further upon our world 1 brains, and thus upon material bodies."[31] That would also apply to mathematical objects and (partially) contradict "An object is said to be *abstract*, we recall, if it lacks spatiotemporal location and is causally inefficacious; otherwise it is said to be *concrete*."[32] We leave that open but just mention that we connect the *conscious* mind directly to (a local version of) World 3 (Figure 1.1). One could understand "causality" via *actions* triggered by something called *free will*.

After some thoughts about art[33], he comes to his central question: "Are world 3 objects, such as Newton's or Einstein's theories of gravitation, *real* objects?"[34] For him they are real, "real in a sense very much like the sense in which the physicalist would call physical forces, and fields of forces, real, or really existing."[35] This is then argued step by step, starting from physical objects we can easily handle to tiny and huge objects and further magnetic forces, etc., since they all "have [directly or indirectly] a *a causal effect* upon physical things, (. . .)"[36]

[28](Popper, 1978, Section V, Page 148)
[29]Wunderli (2013)
[30](Popper, 1978, Section V, Page 149)
[31](Popper, 1978, Section VI, Page 150)
[32](Linnebo, 2017, Section 1.4, Page 9)
[33](Popper, 1978, Section VII, Page 151)
[34](Popper, 1978, Section VIII, Page 152)
[35]Ibid.
[36]Ibid. Page 153

His key argument is then our use of science, “by using world 3 *conjectures or theories* as instruments of change.”[37] After explaining what monists and dualists would say, he turns to *subjective* (World 2) and *objective* (World 3) knowledge[38]. This is what we took up in the introduction in (the previous) Chapter 1.[39] With Einstein’s “Special Theory of Relativity” he makes clear that it is *not* about an “abstraction from a concrete thought process, but an *object* very much like other [abstract] objects.”[40] His point is, that consequences of such theories can go far beyond of what the “inventor” had thought. This is due to possible logical relationships.[41] At the end, the *language* is needed: “Thought contents are, we may conjecture, products of human language; and human languages, in their turn, are the most important and basic of world 3 objects. But languages have, of course, also a physical aspect, while the content of what has been thought or said is something abstract.”[42]

Important is “the transition from a non-linguistic thought to a linguistically formulated thought”[43]. Or, with our focus (and bias), to a *mathematically* formulated (thought) that can be *manipulated* (first of all *locally*). What starts in Section 3.1 will be further developed in Chapter 4.

Although there is no doubt that the (ring of) *integers* are part of World 3, his example is problematic because *understanding* the infinite is maybe the most difficult

[37](Popper, 1978, Section IX, Page 154)

[38](Popper, 1978, Section X, Page 156)

[39]We did not focus however on the question how real mathematical objects are. It is indeed quite tricky as we mentioned just above: Physics take much use of mathematics. But the theories used as “tools” are not genuine mathematical. On the other hand, the “origin” of mathematics is a *geometrical* one, very much tied to practical problems (e.g. measuring the area of land). So by a similar argument (like Popper used starting from “handling simple objects”) *all* of mathematics could have —at least indirectly— a causal effect. For a serious discussion one needs to distinguish between a “pure” formula and one with the necessary background/context. Our main goal here is to build the fundament for dealing with such tricky questions ...

[40](Popper, 1978, Section X, Page 158)

[41]Ibid.

[42]Ibid. Page 159

[43](Popper, 1978, Section XI, Page 159)

part in studying mathematics[44]: "Children learn to count. This is a skill, a man-made invention. We learn so to count that we can construct to *any* given number its successor number, without end. So we come to understand the infinite sequence of natural numbers."[45] This is the reason we mentioned also Jürgen Habermas in the introduction: We cannot really work directly with World 3 objects, we might use simplifications[46] (possibly without knowing).

"Geographical maps, a plan of a building, (. . .) are based upon theories; they are, precisely like books, embodiments of world 3 objects. Their causal efficacy is very obvious. (. . .) But like books, they are valueless for those who cannot read them."[47] Together with the example of *computer programs* Popper stresses the possibility for improvement by criticism, and cooperation.[48]

The last sections[49] could be briefly summarized by "mind versus machine" (the essay in Section E.7 is related to that topic), "thinking and consciousness", "causally open world 2", "(human) culture", and "summary" respectively.

In the introduction (Chapter 1) we have explained why we place the *conscious* mind in (or better: related to) a local World 3: It acts as *the* gateway to (the global) World 3. More precise with Popper's arguments: The (conscious) thought itself is understood in World 2, the *content* however is in (our) World 3. The latter is the relevant view (for us) because it enables *manipulation*. We dig deeper to find more about the stunning interaction between *body* (imagination) and *mind* (formalism) to

[44] There are actually many infinities, one distinguishes in particular the *countable infinite* (e.g. integers) from the *uncountable infinite* (e.g. reals or powerset of the integers). To grasp that even partially a lot of concepts are necessary, notions of *limit processes* lead the way. While —at least for mathematicians— there is no real difference between a two, a three, and an n-dimensional space, from linear algebra (with matrices as linear operators between *finite*-dimensional vectors spaces) to *functional analysis* (with operators on potentially infinite vectors spaces) it is a *huge* step since highly non-trivial phenomena can appear.

[45] (Popper, 1978, Section XII, Page 161)

[46] (Habermas, 2014, Chapter 3, Page 118)

[47] (Popper, 1978, Section XIII, Page 162)

[48] Ibid.

[49] (Popper, 1978, Sections XIV to XVIII)

gain *cognition* (in World 2) in Section 3.3 and later in Chapter 4.

2.4 Mathematical Knowledge

Philip Kitcher challenges *mathematical apriorism* (e.g. from Kant) by developing an alternative theory, namely that of *mathematical empiricism*.[50] He takes the historical development into account and does not ignore the fact, that we have to learn mathematics.[51]

Interestingly, mathematics is somehow special —compared to (natural) sciences— insofar as it steadily generalizes *without* loosing its "roots" as special cases. These (the latter) are important for learning as we are going to stress several times later, e.g. in Chapter 4. In the beginning of Chapter 3 we mention the dispute about *synthetic* (a priori) versus *analytical* (mathematical propositions). Now, having Anna Sfard's "processes and objects"[52] in mind (something we discuss in more detail in Section 3.2), one might understand also (synthetic) *a priori* versus *empirical* in a similar way: *object level* versus *process level*. However, this yields to a problematic confusion because then *synthetic a priori* would be simultaneously process and object level. Another difficulty would be with the term *process* which we (primarily) understand as that of an *actor*. To avoid that, one could try to use Saussure and distinguish between *synchronic* (static) and *diachronic* (evolutive) linguistics[53] resp. mathematics[54].

What we understand by "mathematics" will raise our attention later (not just once). Kitcher addresses an important point when he writes "I compare mathematical change with scientific change, attempting to show that the growth of mathematical knowledge is far more similar to the growth of scientific knowledge than is usually ap-

[50](Kitcher, 1985, Introduction, Page 3)
[51]Ibid. Page 7
[52]Sfard (1991)
[53](Wunderli, 2013, Section I.3.1, Page 189)
[54]Or "mathematistics"? Taking the parallel to languages is the main theme in the essay in Section E.4.

preciated (. . .).”[55] Not just Kant’s understanding of mathematics[56] was far away from our “modern” mathematics. One cannot directly transfer his thoughts to our time. And, grasping current (new) mathematical theories for *deep* philosophical discussions is everything else than simple. This thesis wants to close this gap by building the base for the former and providing a complete new view on mathematics by illustrating *free fractions*[57] as alternative to “classical” (non-commutative) rational expressions. In many cases it is not so much a new mathematical theory that matters (itself) but the *perspective* it can contribute to understand what we are (more) familiar with.[58]

Knowledge is not just there. What one really knows if one reads a theorem is difficult to say. Even if it is formulated in a language one understands (with not too much special symbolism). Kitcher mentions the role of *proofs*[59]. Implicitly, one can see the important role of the *context*. And, what the understanding of cognition as *process* means: Reading a proof —even if it is one’s (maybe “sketchy”) own— step by step leads to reflections about what one (already) knows and what one does not understand. If one (eventually) understands the proof, one gets automatically deeper trust in the (new) knowledge. (If one “knows” the proof already, one can start to see where it is not yet clear enough.) At this point it is not just a copy from World 3 anymore.

What happens if one focuses too much on *information* is another story.[60] Mathematics wouldn’t be mathematics if it would allow (known) *inconsistencies*. Yes, everyone takes another way (to learn mathematics), some are more laborious than others. But at the end, one can “forget” this (personal) road. And that would jus-

[55] (Kitcher, 1985, Introduction, Page 8)

[56] For a detailed discussion we refer to Koriako (1999). It’s no coincidence that Koriako (re-)appears in the beginning of Chapter 3.

[57] Schrempf (2018b) with an extended summary in English in Chapter E

[58] Just to mention another example: Very often we talk about probability. But what does that mean from the perspective of (non-commutative) *free probability*? A starting point to the latter could be Rota (2001).

[59] (Kitcher, 1985, Introduction, Page 10)

[60] (Kitcher, 1985, Introduction, Page 11); He doesn’t use “information” (in our sense) but it seems that this is what he has in mind.

tify to speak of "objective" mathematical knowledge. That is —simplified— Kant's position[61]. Viewing mathematical propositions from the *object* perspective relies on learning the necessary concepts (e.g. that of a *ring*). In this case it is not that easy to (completely) ignore the *empirical* character ...

Comparing scientific change and mathematical change is central to Kitcher: "Our first task will be to uncover the picture of scientific change which underlies the complaint that, unlike the natural sciences, mathematics does not grow by responding to observation and experiment."[62] Now one should not be too pedantic here with regard to the difference in terminology: In mathematics one could say "observation and examples". We emphasize the important role of examples several times, e.g. in Sections 3.2 and 3.3. In the latter we refer to Kant: "This is also the sole and great utility of examples: that they sharpen the power of judgement."[63] What differs between mathematics and (not just natural) sciences is, that —unfortunately— most of the process towards a *mathematical* theory is "hidden" (or forgotten once a "watertight" proof is available) while non-mathematical theories should actually be called *conjectures* (from a mathematical perspective). But this does not contradict Kitcher, nor supports his "opponents" (he calls *apriorists*). There would be quite a lot more to say, in the beginning of Chapter 3 we mention an article of van der Waerden about "the sources of [his] book *modern algebra*"[64] (he will appear a little bit later again with another fascinating article). The main point is that —to reach "deep" *cognition* (and not just "pure" knowledge)— we need much more than "pure" mathematics ("projected" into World 3).[65]

One "pattern" of mathematical change is "the extension of mathematical language

[61] For a detailed discussion see (Kitcher, 1985, Section 1.III, Page 21).

[62] (Kitcher, 1985, Section 7.I, Page 150)

[63] (Kant, 1998a, B173)

[64] van der Waerden (1975)

[65] Reconstructing (a simplification of) the history can however be quite difficult. But it is important insofar as it helps to discover open questions and "possible mathematics". (The essay in Section E.3 contains thoughts about that.) And, it is certainly interesting what the great mathematicians —on one builds up theories— were looking for. Sometimes this can be even *crucial* to avoid getting lost.

by *generalization*".[66] This is something we prominently take use of here: Since we know how to calculate with (rational) *numbers* (using fractions), we get some basic orientation for what it would mean to "calculate with words"[67].

Abstraction clearly has its price: One needs heavy symbolism to be able to write statements compactly (enough to work with). Although the "abbreviation of expressions"[68] is (hopefully) clearly stated in publications, it can take a lot of time to have them *immediately* at hand while following a proof (or reading a book, e.g. Cohn's "Free Ideal Rings"[69]). This is —as Kitcher argues— not completely irrelevant when one wants to understand "mathematical knowledge" . . .

2.5 Imagination in Mathematics

Gottlob Frege opposes[70] Kant's "Without sensibility no object would be given to us, and without understanding none would be thought."[71] and brings an example of a huge number, namely 1000^{1000}, we cannot "imagine". But what does that mean? Aren't we using here *space* and a special "formal" representation?

Section 4.4 is dedicated to how one could understand *imagination* (in the context of mathematics). That it is hard to separate it (completely) from *formalism* will become already clear(er) in Section 3.1. In Section 4.3 we sketch a definition of (mathematical) languages based on the (spatial) dimension. Coming back to Frege: Why should we *implicitly* assume only the *zero*-dimensional (the position does not matter) to imagine (say) one hundred (distinguishable) dots with random positions

[66] (Kitcher, 1985, Section 9.IV, Page 207)

[67] Such an orientation is also important while developing a (mathematical) theory: It is one thing to have several results: Schrempf (2018a) from 2017/01, Schrempf (2019) from 2017/06, Schrempf (2020a) from 2017/12, and Schrempf (2020b) from 2018/03. But another to see their —non-trivial— interplay and being able to describe it: Schrempf (2018b) from 2018/07. For an explanation of "lexetic" see the end of Section 3.3, just before the Interlude.

[68] (Kitcher, 1985, Section 7.V, Page 170)

[69] Cohn (2006)

[70] (Frege, 1987, §89)

[71] (Kant, 1998a, B75)

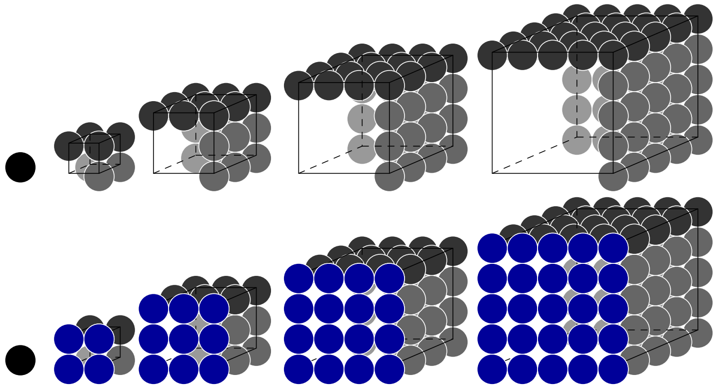
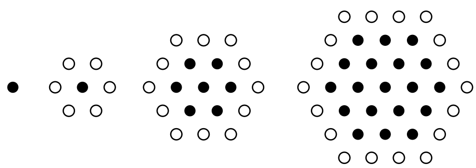


Figure 2.1: The partial sums of the hexagonal numbers 1, 7, 19, 37, 61, 91, 127, ... are cubical (Penrose, 1994, Page 120): $1 = 1^3$, $1+7 = 8 = 2^3$, $1+7+19 = 27 = 3^3$, $1+7+19+37 = 64 = 4^3$, $1+7+19+37+61 = 125 = 5^3$, ... For $1 = 1^3$ it is clear. To get the next "cube", we add a layer to the top, the right and the back (top). Drawing resp. imagining the back layer in the front yields directly the (here somewhat distorted) *hexagonal* form of the numbers (bottom).

in mind (or "on paper")? Do we get any insight unless we group them "as square" and find out that there are indeed 10^2 of them?[72]

And are we allowed to use "external" tools like paper and pen, e.g. to *prove* that the partial sums of the *hexagonal numbers* 1, 7, 19, 37, 61, 91, 127, ... are *cubical* by *geometrical insight*?[73] See Figure 2.1.



A (partial) characterization of *imagination* based on the underlying "formalism" (or "language") might be possible. There would be at least *three* aspects: (a) Which *world* is dominating? World 1 (the "form") or World 3 (the "pure" formal)? (b)

[72] Here the position of the digits 1 and 0 matters. Calling that *one*-dimensional is justified because we can (and typically write) `10^2` (or `10e2`) for interaction with a computer.

[73] (Penrose, 1994, Page 119): $1 = 1^3$, $1+7 = 8 = 2^3$, $1+7+19 = 27 = 3^3$, ...

Where is the *focus*? Purely mathematical or in applications (to solve problems)? And (c), what is —based on the *spatial dimension*— the main *language*? With a look on *diagrammatic* (in Section 3.1) it should be clear that there are plenty of combinations possible.

The *dimension* is also relevant for the "static" (flat) surface: "It is not that hard to transfer the flat image on the page to a 'solid' three-dimensional structure."[74] Penrose uses a "rotating" dodecahedron and a cube for visualization. In this way one can (partially) "visualize" the time dimension. And in applied mathematics one can use *coloring* as an additional "half" dimension for the visualization of complex three-dimensional "real-world" simulations[75] (Figure 2.2).

Visualization is undoubtedly important for *learning* mathematics (which itself is essential for understanding many practical problems). One however needs to state the type more precisely: "We showed that visual–spatial representations can be reliably classified into one of these types [*schematic* and *pictorial* representions] and that the types are differentially related to problem-solving success. Use of schematic representations is positively related to success in mathematical problem solving, whereas use of pictorial representations is negatively related to success in mathematical problem solving [because it takes the problem solver's attention from the main problem]."[76]

Our vision system clearly dominates, but "seeing the unseen"[77] (e.g. in data) goes

[74] (Penrose, 1994, Page 110)

[75] There is also a "hidden" but extremely important aspect here, namely that these simulations give a *feedback* to the programmers of the respective software (or modules). Knowing the fine details on how grids (for volumes or curved surfaces) are implemented adds a lot more *structure* that "leads" imagination, e.g. to optimize the coupling of methods, for speedup, to improve the visualization, etc. In reality this is an iterative process. The "formal" source code is just another side of a "multi-dimensional" coin and in particular important for bringing the —in the simulation *invisible*— details to the foreground. On the other hand, "reading" (and understanding) even essential parts of such programs is hard to impossible *without* visualizations. We discuss that —from a different perspective— in more detail in Section 3.1.

[76] (Hegarty and Kozhevnikov, 1999, Discussion, Page 688): 33 students (mean age of 12 years) participated.

[77] (Arcavi, 2003, Page 216)

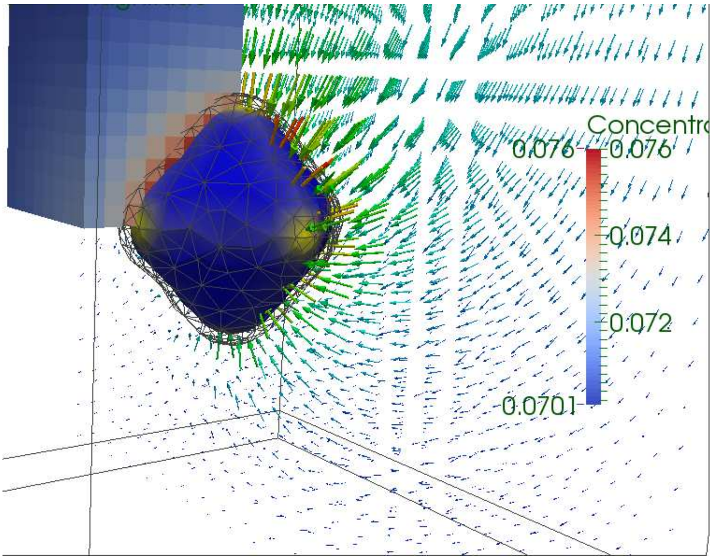


Figure 2.2: Simulation of a dendritic solidification of an aluminium copper alloy with explicit front tracking in 3D: Shrinkage induces a convection (field of arrows). The crystallographic orientations of a —in the beginning spherical— nucleus become visible after a few iterations since the solidification is faster in these (here six) directions. In the small upper left block a (small) concentration gradient becomes visible. The rest of the surrounding (liquid) metal is hidden to enable a look "inside". The solidified part (in the middle) has an artificial (transparent) hull that shows the time-discretization. (...) There would be more to say, a lot of additional data (curvature, temperature, velocity, etc.) can be shown, even a look into the highly complex dendritic solidified structure is possible. Understanding the full "picture" is only possible after studying several different movies (to see the dynamics) and investigating the data (from the simulation) interactively in specialized software like `https://paraview.org` (June 20, 2024) *and* having the mathematical description of the problem (in particular with the main governing *partial differential equations*) at hand (and being at least roughly able to read it). Imagination has many facets here: That for (computer) *programming*, that for *mathematics* and that for the actual (here physical) *problem.* For "deep" cognition —one could say that— not only *imagination* and *formalism* must work together, but all relevant imaginations and formalisms need to interact in harmony. (Figure from unpublished results, May 2017.)

beyond “pure” vision. Furthermore, one can even visualize “impossible objects”[78]. For our focus on *imagination* it is helpful to listen to a blind mathematician: “Bagett says he has never been very good in geometry and cannot easily visualize complicated topological objects. But this is not because he is blind; in visualizing, say, a four-dimensional sphere, he said, ‘I don’t know why being able to see makes it any easier.’”[79]

One important aspect of visualization is the support for *reflection*: “But seeing the thing [e.g. a 4,000 times aplification of a blood cell] itself, with the aid of technology which overcomes the limitation of our sight, provides not only a fulfillment of our desire to ‘see’ and the subsequent enjoyment, but it may also sharpen our understanding, or serve as a springboard for questions which we were not able to formulate before.”[80]

With *technology* we come back to mathematics and the formalism that can indirectly support our imagination via *compressive sensing*[81], e.g. in magnetic resonance imaging, radar, and sampling theory[82]. This is one aspect, namely the “active” for discovery. Another is the “passive” with “imagination as faculty of representation [of the visual]”[83]. But even such a characterization is relative. With the focus on *cognition* (rather than “pure” knowledge[84]), understanding proofs or explanations becomes important. And here it turns out that “Visual thinking can occur as a non-superfluous part of thinking through a proof and it can at the same time be irreplaceable, in the sense that one could not think through the same proof by a process of thought in

[78] (Krämer, 2016, Section 2.7)

[79] (Jackson, 2002, Page 1249)

[80] (Arcavi, 2003, Page 216)

[81] (Foucart and Rauhut, 2013, Chapter 1)

[82] “Reconstructing a continuous-time signal from a discrete set of samples is an important task in many technological and scientific applications. Examples include image processing, sensor technology in general, and analog-to-digital conversion appearing, for instance, in audio entertainment systems or mobile communication devices.” (Foucart and Rauhut, 2013, Section 1.2, Page 15)

[83] (Arana, 2016, Page 463)

[84] How and whether one can obtain “knowledge through imagination” is another fascination question. The book from Kind and Kung (2016) is dedicated to that.

which the visual thinking is replaced by some thinking of a different kind."[85]

Last but not least, also as appetizer for Chapter 4: "Finally, visual thinking is useful in calculation. What I have in mind is the rearrangement of symbol arrays in imagination or on paper. (...) Symbol manipulation also plays a significant role in algebra, though contrary to popular belief there is more to algebra than that. Matrix algebra provides a significant example of visual operations on symbol displays, and the importance of matrices in applications again attests to the utility of symbol manipulation in calculation."[86]

[85] (Giaquinto, 2008, Section 1.5, Page 39)

[86] (Giaquinto, 2008, Section 1.4, Page 39). Notice however that we are formalizing the *symbolic* (rational) "work with matrices", which is "just" non-commutative algebra. The use of "classical" matrix algebra is not sufficient any more, a generalized form of *linear representations* is crucial: Cohn and Reutenauer (1994), Cohn and Reutenauer (1999).

Chapter 3

Mathematical Cognition: Motivation

To be able to talk about "mathematical cognition" at all, we need to become aware what we mean by *mathematics*. Taking a closer look, e.g. from the pluralistic perspective in Section E.2, something that seems so familiar like mathematics is not that easy to grasp. In particular when we refer to Kant (more than two centuries ago) and seek for a (more or less) "timeless" understanding.

Therefore it is no surprise that Darius Koriako warns already in the beginning of his introduction to "Kant's philosophy of mathematics": "But nobody seems to ask whether we are allowed to project these modern concepts into Kant's theory. Couldn't it be that especially this attitude stands in our way to comprehend the specific in Kant's concept of mathematics? And couldn't it be that this inability, to grasp older conceptions of mathematics in its peculiarity, is also responsible for not seeing the radical changes that distances us from these conceptions?"[1]

Skipping a lot of fascinating details, one significant change towards "modern mathematics" could be pinned down with van der Waerden's book "Modern Algebra"[2].

[1] (Koriako, 1999, Page 1), my translation

[2] van der Waerden (1940); for the genesis we recommend van der Waerden (1975)

Another one —connected to Abel and Galois— lies in the beginning of the nineteenth century with the question whether polynomials can be solved by radicals.[3]

Kant distinguishes clearly between *philosophy* and *mathematics*: "*Philosophical* cognition is *rational cognition* from *concepts*, mathematical cognition that from the *construction* of concepts."[4]

For basic mathematics such constructions are *geometrical* and *intuitive*, e.g. triangle, circle, etc. There is a subtle interplay between space, orientation and intuition in the context of *diagrammatic*.[5] (We discuss that in detail in the following Section 3.1.) Our question will be how *formalism* (and imagination) can be used to provide constructions for highly abstract concepts.

There is another notion of "construction", namely *in* mathematics. As it turned out only some decades later, "constructive mathematics" does have fundamental limitations: "But mathematics does not merely construct magnitudes (quanta), as in geometry, but also mere magnitude (quantitatem), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, *extraction of roots*, etc., and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition;"[6]

Computing roots can be quite delicate.[7] What "computation" means is another topic because except from special cases, we can achieve only approximations. And —as mentioned before— using exact methods to solve polynomial equations is very limited. The possibility for (exact) reformulations of problems is however important for dealing with the problem of rounding (errors) and thus highly relevant for the

[3](Alten et al., 2014, Section 6.4)
[4](Kant, 1998a, B741)
[5](Krämer, 2016, Section 8.3)
[6](Kant, 1998a, B745), my emphasis
[7]A possible starting point could be Schleicher and Stoll (2017).

practical use of mathematics (e.g. in engineering).[8] Reflecting (more) about the limit(s) of formalism would be another story ...

Although we cannot comment here (in detail) on the "never-ending" dispute whether mathematical propositions are *synthetic a priori* or *analytical* judgements —this is the focus of the essay in Section E.1— it is important to stress that there are (now, with "modern algebra") two perspectives: Kant's prototypical example "$7 + 5 = 12$"[9] could be read "sensual" (having the *process* of counting in mind), or "formal" (having *objects* in the ring of integers in mind).

If we focus on *knowledge* only, it does not matter how we gained $7 + 5 = 12$. But since we are interested in *cognition*, we need to have a look on the *interplay* between our senses (delivering content) and our understanding (delivering concepts). Independent of how we treat (abstract) mathematical objects, one cannot ignore that *counting* (in particular using our fingers) is something highly sensual. In such a "trivial" (mathematical) proposition, cognition could mean e.g. to realize that the sum of two odd numbers is even (and eventually being able to find a way to gain a more general insight), or by being able to explain the context ($7 + 5 = 12$ would be wrong using the octal or hexadecimal system).

The other way around: Can we obtain more than (pure) knowledge (of single trivial mathematical propositions) if we only know how to count? (Having a child in mind that starts to learn the decimal system.) And what if there is only the formalism (of the decimal system)? (Having a computer in mind.)

At this point it is worth to recall Figure 1.1 (Page 3) in the slightly modified version of Figure 3.1 (below) with an attempt to use *diagrammatic* itself to "visualize" parts of

[8] An —even superficial— discussion of the "applicability problem" is out of scope here. An appetizer could be Wilholt (2006), who starts by citing Hilbert: "We are confronted with the peculiar fact that matter seems to comply well and truly to the formalism of mathematics. There arises an unforeseen unison of being and thinking, which for the present we have to accept like a miracle." (Wilholt, 2006, Page 69) To get a feeling for subtleties around "floating-point arithmetic" I highly recommend the "introduction" Rump (2010).

[9] (Kant, 1998a, B15)

the main ideas[10]. However, our (two-dimensional) theme serves also the *multi-dimensional* understanding of concepts: Although we put "formalism" in the realm of our (conscious) mind (in our local World 3), it has a tangent to our body (in World 1) in terms of the concrete (e.g. programming) language we use (for manipulation). By the way: There are (now mostly unknown) human artifacts, which in particular support the communication between World 1 and World 3, namely the *slide rule*[11] (Rechenschieber). In some sense it is —compared to the abacus— "semi-sensible", and —compared to (modern) algebra— "semi-formalistic". After getting acquainted with its usage, it supports reflecting about mathematical cognition and thus serves a little bit as philosophical "tool".

To shade a light on this interplay of *intuitions* (in the faculty of sensibility) and *thoughts* (in the faculty of understanding) we now take a closer look on how formalistic mathematics is (Section 3.1), what processes and objects are "as two sides of the same coin"[12] (Section 3.2), and finally how we can overcome the incompatibility of intuitions and thoughts by Kant's concept of *schematism*[13] (Section 3.3).

3.1 How formalistic is mathematics?

Already the abstract of the essay in Section E.5 shows that such a question is difficult, the slide rule —mentioned before— indicates just that. But it get's even trickier when Sybille Krämer asks, if there is a "constitutive nonsensuality of the formal": "There is an assumption that is widespread not only in everyday understanding but also in science's understanding of itself: the combination of numerical computability and the formalization of semiotic operations results in a process of the highest level of abstraction: All figurative or sensual attributes are eliminated in formal procedures;

[10](Krämer, 2014, Section 5.2)

[11]For a starting point with respect to the discussion whether one needs symbolic systems for calculating we refer to (Maddy, 2018, Page 18).

[12]Sfard (1991)

[13](Kant, 1998a, B176–187)

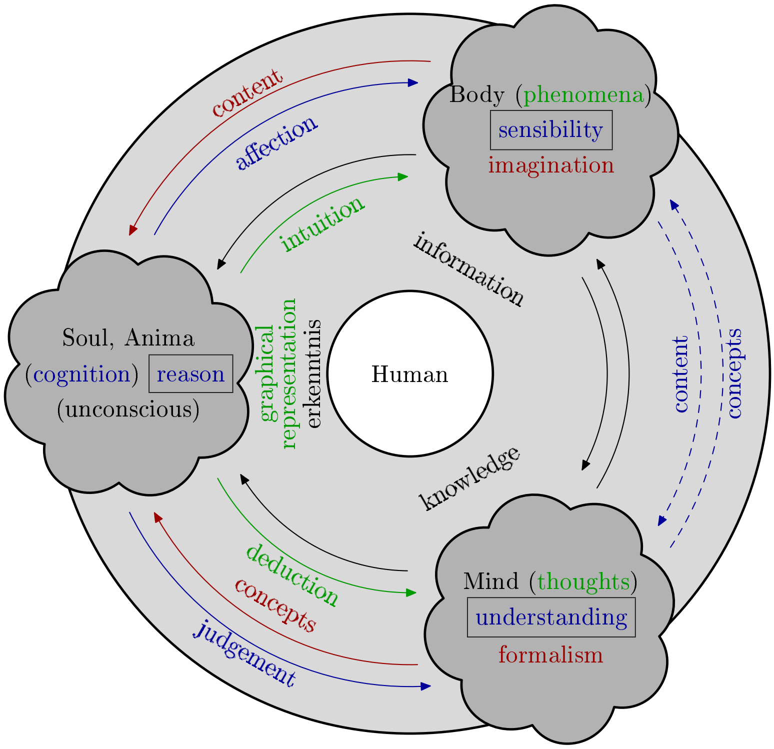


Figure 3.1: This variant of Figure 1.1 (Page 3) joins the main essences of the sections in this chapter using *diagrammatic* (Section 3.1) to bring "invisible" to the foreground. Although this is just a snapshot, it should be understood as a *process* or a "dynamical tool". Recalling (Kant, 1998a, B75) from the prologue, one would initially put the arrows "content" and "concepts" to the right (directly between body and mind). Digging deeper into *schematism*, one realizes that these arrows need to be (in) *dashed* (style) because there cannot be any direct communication. Being sceptical is always a good idea: "For the presented objects one shall neither ask how others assessed them nor what we suspect, but what we clearly and evidently can recognize intuitively or reliably deduce; only in this way one attains knowledge." (Descartes, 2018, Rule III; my translation) Notice that the focus here is not knowledge (itself) but the "process" of gaining knowledge, i.e. cognition in our sense. One interpretation could be that *intuition* and *deduction* controls the body resp. the mind to obtain *content* resp. *concepts* for joining them to cognition. One can distinguish between *concepts* (in the *global* World 3) and *conceptions* (in the *local* version) like (Sfard, 1991, Page 3).

what remains is a schematic of the manipulation of arbitrary signs."[14]

Our goal is not to agree or disagree on this assumption —she wants to revise— but sharpen our understanding of (not just) formalism in mathematics. Firstly we need some definition of *diagrammatic*: "This technique [of performative spatialization for the development of cognition] consists of using two-dimensional spatial relations to visually represent and operate with nonspatial, invisible objects of knowledge. The realm of this schematic visualization is broad: it stretches from notations to lists, tables, diagrams, graphs, and even maps. We want to call this realm of different kinds of inscriptions: the diagrammatic."[15]

Now, if we think of how we use the decimal system "on paper" —resp. how we learned it at school—, we find exactly that, namely the use of *two* dimensions although mathematics is —as language— essentially *one*-dimensional: Numbers are just *words* over an alphabet whose letters we call *digits*. For calculations (say addition or multiplication) it does not matter how these digits look like, as long as one can distinguish them clearly. However, working with orderings other than

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9, \tag{3.1}$$

or non-standard multiplication tables makes it very difficult and time consuming for humans (even if we ignore the problem of introducing errors). One should be aware of the fact that learning the decimal system —the very basics of arithmetic— takes roughly ten years[16]! This aspect adds *automation* and eventually *time* to our "problem" of formalization (in mathematics). Without proper training one cannot concentrate on the calculation (within some context) because the mind is occupied with the manipulation of signs. This "overflow of the working memory" is a central topic in *dyscalculia*[17].

[14](Krämer, 2014, Section 1, Page 346)

[15](Krämer, 2014, Section 1, Page 346)

[16]Two years for "familiarization" with digits (and small numbers) before school, four years for "basics" at primary school, and another four years for "fractions and some practice" (ISCED, lower secondary education).

[17]For a deeper discussion about the role of the *working memory* see e.g. (Landerl et al., 2013,

The way how Krämer characterizes *calculus* by the three aspects "visuality—spatiality", "symbolism—technique", "operation—interpretation"[18] stresses the human perspective and usage; *formal languages* (alone) are not suitable (for "human" formalization).

Here however, its worth to have a closer look on "practical" *programming languages* which are somehow the World 1 counterparts of formal languages. Although both are *one*-dimensional (the former explicit, the latter implicit), the interpretation in *time* (compilation of code as serial process) resp. *space* (free[19] non-commutativity of letters) gives us the possibility to view them as different sides of cognition. (We will come back to that in the following sections.)

At a first glance, one would immediately disagree, even without thinking of *assembler* or even *machine code* (in hexadecimal notation). From an almost ancient book on computer programming in *C*: It said something about how bizarre it might look compared to (the programming language) *Pascal* and warned that "real" *C* programs are usually much more cryptic.[20]

But even without sophisticated programming environments, which actually produce a lot of *graphical* visualization, we typically use a lot of —for the machine irrelevant— structure (indentation[21]), "readable" names as identifiers, and (colored) syntax highlighting (just to mention a few). Therefore we can also apply Krämer's characterization from above (to programming languages) because it can be used to distinguish "good" (readable) from "bad" (unreadable) code. The latter is problematic with respect to possible *hidden* (conceptual) errors. In other words: This extra dimension —by not arbitrarily cutting a long line into chunks to fit a paper (or a

Section 1.5.2).

[18](Krämer, 2014, Section 2)

[19]Here "free" means *without* (multiplicative) *relations* like $2 \cdot 3 = 3 \cdot 2$ in the commutative. The (natural) multiplication in "normal" languages is the concatenation.

[20]Turbo C 2.0, original documentation in several volumes in German. My own wording from fragments I kept in memory. But it's easy to demonstrate, with *C* it's not difficult to write —for humans— almost unreadable programs without even the use of "goto".

[21]There are a view exceptions. But even in these cases one needs to distinguish between the way the text is encoded and how it is displayed.

screen)— can bring the invisible to our attention!

Using "graphical" tools (partially similar to flow charts) for programming or —more general— tools to generate (structured and readable) code can have advantages (in particular for learning the first steps). To turn that upside down might look strange because when we consider learning "classical" programming as difficult, producing sophisticated high quality figures for *visualization* (in books) using e.g. *MetaPost*[22] can be considered as even harder because one needs additionally a highly trained power of *imagination* in three-dimensional space. However, and that is —recalling Krämer's definition of *diagrammatic* from the beginning of this section— a crucial point, namely transforming a "static" figure (in the two-dimensional space) into the *procedual* "performative" language suitable for a computer to do the drawing for us, can bring the "invisible in the visible" to the foreground!

Usually we are used to draw a *concrete* circle and see the *abstract* mathematical object: "A concrete shape is replaced by an abstract structure. Both the textualization of arithmetic by the invention of written reckoning and the algebraization of geometry by the invention of analytical geometry seem to show that progress in methods of calculation can be interpreted as the elimination of imagination and intuition."[23]

Reading that upside down (as we have illustrated just before): We write an abstract structure (formal program) and get —using a printer— arbitrary (almost perfect looking) instances of concrete shapes. Even more: We can explore different perspectives and detect possible inconsistencies. Not everything that looks like a circle is indeed a circle. Something like that is highly relevant e.g. in *computer aided design* (CAD).

At the end we do not have to go that far to come to the conclusion that it is not that simple to get rid of *imagination* in the formal. Clearly, seeing problems in complicated three-dimensional structures is sooner than later difficult to impossible. But we certainly do not need a computer for such a performative act: It is e.g. inherent

[22] https://tug.org/metapost.html (May 21, 2024)

[23] (Krämer, 2014, Section 1, Page 346)

in Plato's "Menon"[24] and a leading theme in Lakatos' "Proofs and Refutations"[25].

To explain the "power of the line" Krämer goes back to Descartes. Simplified illustrated in Figure 3.1 in (above): "The specification of cognition as intuition and deduction places perceptivity at the center of Descartes' theory of cognition. But what are the propositions that can be either immediately and intuitively perceived or organized like the links of a deductive chain? These propositions are neither phenomena nor linguistic sentences; rather they are cognitive objects that are visually represented in a graphical reference system. And this transformation of phenomena and thoughts into graphically embodied scientific objects is performed precisely by the power of imagination."[26]

What we get if we strip off the sensual from the formal (understood in the sense of diagrammatic but not necessarily to graphical representation of objects) is "pure" knowledge (instead of cognition), or using Descartes' words/example: "We do not become mathematician by memorizing all proofs of the others but by having the power of mind to be able to solve every particular problem;"[27]

At the bottom line there are different ways of understanding formalism, in a "pure" sense (restricted/"projected" to World 3), and in a "diagrammatic" sense: "Formalism is therefore not the opponent of perceptivity, but rather it presupposes and generates a radical form of perceptivity by referring to the surface of simultaneous visibility."[28]

And as we have seen, the latter could be reinterpreted somewhat more general by using (two-dimensional) visualization also for abstract languages. Something we actually use in particular in mathematics anyway, e.g. for high quality mathematical typesetting. The perspective (from World 2) makes the difference. Later, in Chapter 4, we will dig a little deeper in highly abstract (non-commutative) algebra and

[24] (Ebert, 2018, 82a–85b)
[25] Lakatos (2015)
[26] (Krämer, 2014, Section 5.2, Page 353)
[27] (Descartes, 2018, Comments to Rule III), my translation; For how difficult the problems could be one should recall Rule II.
[28] (Krämer, 2014, Page 355)

will see that there is actually a purely *two*-dimensional (mathematical) "language" carrying a natural diagrammatic.

Instead of asking (directly) about the "formal in the imagination" —since we have to join them later anyway— we try to approach that via firstly deepening out knowledge of the sensual versus the abstract and secondly bringing the difficulty of *concepts* versus *content* to our awareness in the now following sections.

3.2 Processes and Objects

To put the soon following citation of Anna Sfard into the right context, some comments could be helpful. Mathematicians[29] tend to say that there is nothing special about "mathematical intelligence": "I feel certain that there is no fundamental difference between mathematical and other kinds of thinking."[30]

Non-mathematicians tend to say that mathematics is difficult, at least one can agree that learning takes time. And it is usually a long way until one is able publish a new theory (without the support of much more senior colleagues): Additionally to the mentioned ten years for the very basics, there are roughly 15 years more necessary for "higher" mathematics (at school), university degree, PhD, and some post-doctoral experience. How that differs fundamentally from say music and what is necessary (talent, techniques, experience, etc.) to be able to compose a symphony we need to leave open ...

"It seems that to put our finger on the source of its ostensibly surprising difficulty [of learning mathematics], we must ask ourselves the most basic epistemological

[29] In this sense I do have certainly also a bias. But this shouldn't be a problem here since people who are fluent in several languages (or using instruments) tend to say that learning languages (resp. playing music) is easy. As always it depends: We are not aware of learning our *first* tongue and we do not count the thousands of hours we practice (with *all* our senses) until we are able to read a book (fast enough). Learning a new language —even if it's close to our mother tongue— takes time. Learning a complete new alphabet for another language takes time. But the more one learns, the easier it becomes. With respect to that, there is nothing special with mathematics.

[30] (Penrose, 1994, Page 107)

question regarding *the nature of mathematical knowledge*. Indeed, since in its inaccessibility mathematics seems to surpass all the other scientific disciplines, there must be something really special and unique in the kind of thinking involved in constructing a mathematical universe. (...) The real question which should be asked here is qualitative rather than quantitative: How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develops, in its functions and applications?"[31]

As in the section before, it is not about agreeing or disagreeing because one needs a lot more context and opens —already with "teaching" (not only at school)— a complete new chapter, we would like to focus on the basic principles that allow us to learn —actually not only— mathematics *step by step*, starting from *concrete* objects (in World 1).[32]

Mathematics is not just there (in World 3), we need to get access to it by learning the very basics. Basics we need also to be able to use a computer (to solve "mathematical" problems). And learning certainly takes time because *everyone* needs to start from (almost) zero: Digits, the correct order, their correspondence to (a small number of) *real* physical objects, rules for addition, multiplication table(s), etc. At this point we are still far away from *fractions* ...

What really would help are concepts like a *ring* with its prototypical example of the *integers* $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ (with "normal" addition and multiplication). But than we would need to understand what axioms mean, what the implications of a *commutative* multiplication (to use it for "working with numbers") are, etc. Children cannot learn the grammar of a language they do not speak yet.[33]

[31] (Sfard, 1991, Intoduction, Page 2)

[32] In (Sfard, 1991, Figure 4, Page 22) this looks like a staircase: "Processes on concrete objects" can lead via *interiorization*, *condensation*, and *reification* to (abstract) "objects" which themselves appear in other (meta) processes, ...

[33] And learning (the first time) the grammar of a language one speaks fluently is everything else than trivial. I would compare that with learning mathematics. However, while one can speak (not necessarily write) fluently *without* knowing much about the grammar (and all the exceptions), in mathematics this is not that clear. The funny thing however is, that —at the end—, when we (should)

We are already deep in Anna Sfard's topic of "processes and objects" and the difficult question of how (mathematical) cognition arises. In other words (and with reference to the full title of her work): Knowing the integers (counting, process level) and knowing rings (abstract concepts, object level) does *not* imply that we know of the possible interpretation that they (integers and rings[34]) are "different sides of the same coin"[35]. (We will discuss the incompatibility of *concepts* and *content* in the next section about *schematism* in greater detail.)

Only after checking that the *integers* indeed fulfill the axioms of a *ring* we know that the former can serve as an example. What looks trivial in such a familiar case can be difficult (and take a lot of time) even when we know the (abstract) concept already. Looking for a *new* concept is usually extremely challenging:

"The solution [of the delicate problem of solving algebraic equations by radicals (by Galois)] does not consist of a comprehensible condition on the coefficients of the equations under consideration, say in terms of a formula. On the contrary, even to be formulated it requires a new language, more precisely, a new way of thinking in combination with new concepts, that could only be established in a long process of trial and error and of studying examples."[36]

The main point is that "the ability of seeing a function or a number both as a

have learned how to do basic arithmetic *fluently*, we can "forget" the (mathematical) "grammar" (resp. formalism) since it's "stored" in the subconsciousness. It seems somewhat paradox, that actually mathematics should be much easier because there are *no* exceptions.

[34] To be precise, one would need to compare "integers" (process view) with "elements in some ring" (object view). Mathematicians are not pedantic when they speak about the "ring of integers" (or the "field of rational numbers", etc.) because there are many possible structures, e.g. the "additive group of the integers" or the "multiplicative monoid of the integers". But in these cases, we have already left the *process level* (and counting) and are deep in the world of objects, in particular if we consider other operations (instead of the "classical" addition and/or multiplication). Later, in Chapter 4, when we encounter *free fractions* it becomes even "messier" because there are *simultaneously* several levels, structures, and operations. Indeed, it would become almost unreadable. Therefore one prefers simplicity and writes a lot "between the lines".

[35] Sfard (1991)

[36] (Bosch, 2018, Introduction, Page 5)

process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of 'understanding' is."[37] For her the *operational* view (as processes) and the *structural* view (as objects) are complementary (for a mathematical notion).

The naturally arising question how to "mediate" between them was already (indirectly) a topic in the previous section. One difficulty with mathematics is certainly that we need bridges: "As such, geometry is a natural and possibly irreplaceable intermediary between *ordinary language* and mathematical formalism, where each object is reduced to a symbol and the group of equivalences is reduced to the identity of the written symbol with itself."[38]

With a look at Figure 3.1 and the only possibility for abstract *objects* to be placed in the (conscious) *mind* (representing —not only— our partical/incomplete local "copy" of World 3), the *process*(es) would occour ("in time") in (our) World 1, triggering —at a first glance— a contradiction:

"Unlike material objects, however, advanced mathematical constructs are totally inaccessible to our senses — they can only be seen with our mind's eyes. Indeed, even when we draw [the graph of] a function or write down a number, we are very careful to emphasize that the sign on the paper is but one among many possible *representations* of some abstract entity, which by itself can be neither seen nor touched."[39]

To overcome this "contradiction" we need to recall the *performative* (or procedual, algorithmic, operative) aspect of Krämer's *diagrammatic* from the last section and think about the limits of our senses in the (physical) World 1: We can climb a "small" mountain but can we really "perceive" something big like the Mount Everest? (At this point it is worth to re-read the mathematician René Thom from the previous page.)

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[37](Sfard, 1991, Introduction, Page 5)

[38](Thom, 1971, Page 698), my emphasis

[39](Sfard, 1991, Introduction, Page 3). Adding "the graph of" wouldn't be necessary in colloquial mathematical language where one deals with (at least piecewise) *smooth* functions. In general however, there are functions impossible to sketch even in "small" parts.

Unless one denies that mathematical cognition is possible (which would imply that there cannot be much mathematics in World 3) one needs to get rid of this "contradiction". Although we can in principle count (using our fingers) in World 1, we can *simulate* that in our mind: "Degrees of abstraction and of integration are but quantitative characteristics, whereas the crucial, qualitative, difference between the two modes of thinking lies in the basic, usually implicit, beliefs about the nature of mathematical entities. In other words, *there is a deep ontological gap between operational and structural conceptions.*"[40]

How to close this gap —if it is possible at all— will be the focus of the next section which is —like the one before and this one— just another way of looking at a *duality* which itself is constructed by concepts and hardly —if not impossible— to grasp if it is not viewed in the wider context of the *trinity* we have in mind here. Maybe this gap could be understood positively taking Heisenberg's *uncertainty principle* as metaphor? In other words: What is the *content* (example) and what the *concept*? What would we gain if we think if a "continuum" between the integers and a ring? Using (again) concepts from mathematics: The *limit* of abstraction would than be a *pure* "formal" concept (in World 3) while the *limit* of specialization would be a *pure* "sensual" content/example (in World 1, which does not exclude the principal possibility to count[41] without any limit because we can count objects arbitrarily often).

In any case, this gap opens the "space" for the dynamics of cognition we already mentioned in the introduction (Chapter 1) steming from the *operational* (we can place in World 1 due to Krämer): "Thus, whereas the structural conception is static (. . .), instantaneous, and integrative, the operational is dynamic, sequential, and detailed."[42]

[40]Ibid. Page 4

[41]Notice the "formal in the sensitive": We can count only because of a "machinery" for producing *systematic* names for (sufficiently big) numbers.

[42](Sfard, 1991, Intoduction, Page 4)

3.3 Schematism

Due to Kant's "Analytic of Principles"[43], understanding (Verstand) deals with *concepts*, the power of judgement (Urteilskraft) with *judgements*, and the reason (Vernunft) with *inferences* (Schlüsse). How *perception* can enter here is quite delicate, ignoring it would leave our thoughts "empty". Furthermore, since we do have mathematical cognition in mind, there are "abstract" *examples* which we cannot directly perceive with our senses and nevertheless we need to see in a bigger context for "deeper" understanding (i.e. cognition). Fortunately, Kant seems to assume such a capacity working also for abstract examples in philosophy:

"A physician therefore, a judge, or a statesman, can have many fine pathological, juridical, or political rules in his head, of which he can even be a thorough teacher, and yet can easily stumble in their application, either because he is lacking in natural power of judgement (though not in understanding), and to be sure understands the universal *in abstracto* but cannot distinguish whether a case *in concrete* belongs under it, or also because he has not received adequate training for this judgement through examples and actual business. This is also the sole and great utility of examples: that they sharpen the power of judgment."[44]

Here one should make a pause because we have several possibilities to read Kant: In a *static* way as (pure) text in (Popper's) World 3 (ignoring that there is a language —"living" in World 1— involved), or in a *dynamical* way[45] in *our* (human) World 2.

[43] (Kant, 1998a, B169)

[44] (Kant, 1998a, B173)

[45] The latter is actually what I assume because it distinguishes texts (and mathematics) which are just written from what *had* to be written. Rephrasing "say something" (Halmos, 1970, Section 2): Kant wanted to tell us something. It includes a lot of what one could call *passion*. Halmos writes: "It might seem unnecessary to insist that in order to say something well you must have something to say, but it's no joke. Much bad writing, mathematical and otherwise, is caused by a violation of that principle." (Page 124); and continues after some examples: "To have something to say is by far the most important ingredient of good exposition—so much so that if the idea is important enough, the work has the chance to be immortal even if it is confusingly misorganized and awkwardly expressed." (Page 125). During (the process of) *learning*, language (used to describe/explain) can be "confuse";

However "difficult" or "obscure" Kant's section about schematism might be[46], it gives us some insight into his understanding of the (human) soul[47]:

"This schematism of our understanding with regard to appearances and their mere form is a hidden art in the depths of the human soul, whose true operations we can divine from nature and lay unveiled before our eyes only with difficulty. We can say only this much: the *image* is a product of the empirical faculty of productive imagination, the *schema* of sensible concepts (such as figures in space) is a product and as it were a monogram of pure *a priori* imagination, through which and in accordance with which the images first become possible, but which must be connected with the concept, to which they are in themselves never fully congruent, always only by means of the schema that they designate."[48]

Anyway, it is definitely tricky. Therefore we recall that (a) *time* is the form of the *inner sense*, (b) *space* is the form of the *outer sense*, and (c) the *soul* and the *body* are —mathematically speaking— the "projections" of the *I* in the sense of "I think." into the "dimension" of the inner resp. the outer sense.[49] For simplicity —and the connection to mathematics— we concentrate on *numbers*:

"The pure image of all magnitudes (quantorum) for outer sense is space; for all

for some examples in mathematics see e.g. (Meyer and Tiedemann, 2017, Section 1.1).

[46] For an overview about (or as a starting point for) the ongoing discussion see e.g. (Krämer, 2016, Section 8.2). Instead of contributing another interpretation or judgement (of Kant's schematism) I read the invitation between his lines to take up the dynamics and spirit and try ourself to understand this interplay of *senses* and *mind*. This is —with a bias on abstract mathematics— what I want to to do here (in this thesis). I leave the judgement —about how successful this attempt is— to others.

[47] I need to stress here that *only* by writing this thesis I came to this conclusion I want to share. Although reading some parts of Kant's *Critique* long before I had a first draft of the "triangle" *body—soul—mind* in mind (for a small genesis see the beginning of the description in Figure 1.1), even after studying Kant in greater detail from 2019 onwards (in the beginning without mathematical motivation) I was not aware of the many parallels. Only roughly half a year before submission of this thesis —after seminars on (mathematical) formalism and the role of imagination for knowledge— I rediscovered my *diagram* and tried to align it with Kant. My essay about *pluralism* (Section E.2) contains only an idea that indicates that one should possibly dig deeper.

[48] (Kant, 1998a, B181)

[49] (Kant, 1998a, B49, B42, B399 resp. B400)

objects of the senses in general, it is time. The *schema of magnitude* (quantitatis), however, as a concept of the understanding, is *number*, which is a representation that summarizes the successive addition of one (homogeneous) unit to another."[50]

This immediately reminds us of Sfard's "processes and objects"[51] we discussed in (the previous) Section 3.2. How far we can go with a generalization of "sensible" numbers is not that clear. In the sense of "measureability" in the context of electricity (resp. electromagnetic fields[52]) one might go to the (field of) complex numbers. What the implications would be, if we include even the ("slightly" non-commutative field of) *Hamiltonian quaternions*[53], is not that easy to say since they —being a *four*-dimensional algebra over the reals— enable the description of movements (of bodies) in (three-dimensional) space ...

If we take the *body* as object in the (physical) World 1 for the representation of *time* and the *mind* as "working in time" —in the sense of a stream of pictures forming a movie— for the representation of *space* we would have a "pre-entanglement" that can be joined to *cognition* by our soul. Taking Einstein's *relativity theory* into account, thinking in *spacetime* is a prerequisite for any cognition (and thus knowledge) about our physical world. For something like "Imagination and Mathematics in (physical) Cognition" it is worth to listen to Einstein himself in Max Wertheimers "Productive Thinking"[54].

This is the way the Figures 1.1 and 3.1 should be read. There *cannot* be any formalism without space! Because any symbolism —in the "purest" way one could even imagine— relies inherently on *relations*. And it is the power of "pure" non-commutativity (as mathematical concept in World 3) that abstracts that of *language* (directly) and *life* (indirectly via DNA, RNA resp. Proteins[55]). Whether one can

[50](Kant, 1998a, B182)

[51]Sfard (1991); notice that she does not refer to Kant at all.

[52]In German: *Feld* (singular). Not to be confused with the mathematical *field* (Körper).

[53]For an exposition with focus on the discovery I recommend van der Waerden (1973).

[54](Wertheimer, 1964, Chapter VII)

[55]That this is a very, very simplified view can be seen in the section "The atomic structure of proteins" in (Alberts, 2022, Chapter 3). There would be much more to say, in particular about

"touch" mathematical (abstract) non-commutativity one can argue. Via the "classical" multiplication[56] of numbers it is not possible. But once we have the concept (in mind) we can *perceive* it everywhere: In daily communication when the word order changes (e.g. to distinguish between a statement and a question), in mathematics when we mix up digits in numbers, in music when notes are at the wrong place[57], etc. The other way around: Unless we "see" these examples (maybe indirectly via the immediate result) the corresponding concept is —using Kant's word— *empty*. But once we realize this *connection* between some content and its concept, we can *imagine* other situations with the help of *formalism* which provides some kind of orientation.

When we started to ask about how formal mathematics is (in Section 3.1) we realized —with the help of Krämer's *diagrammatic*— that it is not straight forward to ignore the *sensual* in the formal. Now the other way around: The formal (decimal system) provides a "machinery" to produce *names* (not necessarily *concepts*) for reference in particular *numbers*. Although one can hardly perceive more than very small numbers *directly* (without microscopes and telescopes there is a lot we cannot see at all), formal concepts (or World 3) can help to "extend" our senses. The essay in Section E.6 was a first attempt to grasp that.

The decimal system is *the* language for *arithmetic*, i.e. for working with *numbers*. Later, in Chapter 4, we will have a very brief look at *free fractions*[58], the "decimal system for *words*", as a new language for *lexetic*[59].

folding (also of DNA/RNA) which takes place in *three*-dimensional space, adding an extreme level of complexity to the (pure) mathematical notion of (free) non-commutativity we will meet later.

[56] The multiplication of "pure" letters (in the free monoid) is concatenation. If we take *digits*, we get numbers, we usually think as e.g. $512 = 5 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0$.

[57] The situation here is more tricky due to the inherent use of a second dimension (in at least writing down "classical" notes). For a starting point to "Spatialization in Music" see (Krämer, 2014, Section 5.1).

[58] Schrempf (2018)

[59] From the non-existing Greek word λεξητικος (from λεξις for *word*) as analogon to αριθμητικος (from αριθμος for *number*) Schrempf (2021).

Interlude

Instead of answering the —in the beginning of the previous chapter (indirectly) asked— question what *mathematics* is, we opened another, namely what *formalism* is. Since they are heavily intertwined this should not be any surprise. What might be less expected is the *sensual* in the formal (Section 3.1), but also the *formal* in the sensual, *in particular* in mathematics. What looks like a contradiction, causing some struggles in Section 3.2[1], should get clearer by shading a light on the *two* dimensions of the formal, namely the "abstract" in (our local) World 3, and and the "real" in (our part of) World 1. It is the latter, where *imagination* can take place and *guide* us to find the right steps in the "game"[2] of formal systems which are so dominant in mathematics.

Before we can continue we need a side step (again) to *diagrammatic*: "What about the connection between 'narrative diagrammatics' and real 'graphical diagrams'? To find the link between diagrammatics and diagrams, we have to interpret both as phenomena of surfaces. To be able to reflect on diagrams and their operative potential, what matters is not only their visibility, but the two-dimensional spatiality. Texts and diagrams use artificial flatness as a medium of depiction and inscription."[3]

[1] The way the corresponding parts are written —one could call that "inner" form— should reflect this uncertainty. By restricting this thesis to "pure" text (whatever the result would be, say by "projecting" it onto World 3) something would be lost.

[2] For an introduction to *game* versus *term* formalism see (Linnebo, 2017, Section 3.2) resp. (Section 3.3).

[3] (Krämer, 2014, Abstract, Page 11)

Now, when we have the *operative potential* in mind, e.g. to lose oneself in a poem from Emily Dickinson[4] like

Wonder – is not precisely knowing
And not precisely knowing not –
A beautiful but bleak condition
He has not lived who has not felt –

Suspense – is his maturer Sister –
Whether Adult Delight is Pain
Or of itself a new misgiving –
This is the Gnat that mangels man –

or for learning some new mathematics (say derivation at school, as preparation for integration to be able to compute "fancier" volumes), can we really ignore the *second* dimension? The *form* of a text that leads our eyes to something relevant, to hold in for a while? The *structure* that guides us while reading, introducing marks like (3.1) (Page 32) for reference?[5]

One can argue that in "pure" diagrams we do have more freedom in using the two-dimensional space, but certainly *meaningful* —for humans readable— text (e.g. to get access to abstract concepts in World 3) needs more than one dimension[6]. But even

[4](Dickinson, 2015, Nr. 1347, Page 1046) from 1874

[5]There is a lot more we usually do not see unless we have a look into the "source code" (which includes information about formatting, e.g. in HTML, XML, or TeX/LaTeX). Even when there are not two versions of line spacings (a larger is demanded for the official thesis), it takes a lot of fine tuning to ensure that blocks (like a poem) are not unnecessarily split (to avoid disturbances for the reader). In mathematics this can be a substantial part of a publication in particular if it brings several results together into a bigger *coherent* theory, e.g. Schrempf (2018b). (We come back to that later, in the following chapter. To get an idea of the "pure" form I recommend to browse fast through my PhD thesis page by page in a scaling such that even the (bold) section headings are almost unreadable.) Already finding a *consistent* notation (in particular for the use of *symbols*) can be rather challenging and time consuming. Usually this is —and should be— invisible. It can become visible when we stumble or are distracted, something which is highly problematic while learning.

[6]As already mentioned earlier, non-commutativity, i.e. in general non-commuting letters/symbols, needs one (spatial) dimension.

that shouldn't surprise: Spoken "serial" language is one-dimensional (in time) but contains a lot of meta-information (intonation, etc.) due to *acoustics* in the (physical) *three*-dimensional space. (Ignoring non-verbal communication via our body in World 1.)

Coming back to our starting point we can (try to) sum up: Undisputedly the *formal* (here with focus on World 3) in mathematics dominates, but the *form* (here with focus on World 1) cannot be ignored. Especially not when we have "living" (active) —compared to "dead"— mathematics in mind. Since usually we are not (consciously) aware of the form with respect to perception (of concepts), we tend to summarize all "perfect" (axiomatizable) formal under the "umbrella" of *mathematics*. At least today. Other possible viewpoints are the *diagrammatic*[7] or the *geometric* (which does have some limitations in "modern" abstract mathematics). It is also worth having a look in Ancient Greek[8]: μαθησις (matesis) means (1) learning, cognition, (2) topics of teaching, and (3) knowledge, science; μαθηματικος (matematikos) means (1) part of learning, and (2) part of mathematics.

The (hidden historical) aspect of *learning* is interesting for our purpose, both because *cognition* (Erkenntnis) is crucial for learning (not only mathematics) and it indeed characterizes mathematics as that what we have to learn[9]. There is a lot we should learn and a lot we must. Unfortunately mathematics can be sometimes characterized as that what we are forced to learn. This however is not the fault of mathematics itself but those who have a too simplified understanding of it, criticized e.g. by the (outstanding) mathematician René Thom: "Everything considered, the excessive optimism bred by the use of set theory symbols has its roots in a philosophical error. It was believed that by teaching the use of the symbols $\in$, $\subset$, $\cup$ and $\cap$ it was possible to make explicit the mechanisms underlying all reasoning and deduction. (...) Modern protagonists of set theory should realize that this theory is insufficient

[7] For a brief historical overview see e.g. (Krämer, 2014, Section 1).

[8] Gemoll (2012), my translation from the German

[9] Compared to our —once we become aware of our self— "built-in" mother tongue.

to account for even the most elementary deductive steps of ordinary thought."[10]

Since there is no danger that the formal in "modern" mathematics is lost, this thesis could be read as pleading for its *sensual* side. For the importance of *imagination* and the time it takes learning to find the right rule(s) in a highly formal "game" (and the necessary "translations" within the context of practical problems). Therefore "learning" will be a central theme in the following chapter where we get in touch with highly abstract algebra by (very briefly) discussing a (relatively) new "language" admitting the "calculation with words". A language which is not only highly sensual —in the diagrammatic sense— but simultaneously provides an *object level* to (simplified) "classical" language (process level) as source of (new) knowledge.

[10](Thom, 1971, Page 698)

Chapter 4

Cognition as Source of Knowledge

How difficult basic mathematical problems are depends very much on how they are stated. Mathematics is not without cultural context. Although it seems that we do have some "mathematical seed" when we are born, it is hard work to grow something useful we are sufficiently familiar with:

"The challenge of learning [mathematics] at school is that average talented students must learn contents [resp. concepts] in a few years, at whose development highly talented scientists worked for several centuries."[1]

Actually, if it were only "content", it wouldn't be that difficult. But we start from pure *counting*, i.e. Hilbert strokes[2], and go further and further into a more and more

[1](Stern, 2013, Page 143), my translation. One needs to be a little careful since here *content* (Inhalte) is meant in a colloquial way. Therefore I added *concepts* (Begriffe), which we also need to learn as "content of the lectures". The challenge is thus that we get in touch with plenty of *concepts*, a lot of new *content* (now in our normal understanding in the sense of Kant), and "uncountable" ways of combing them —not only— for the application to (often idealized) practical problems.

[2](Hilbert, 1926, Page 171)

abstract world. It is not sufficient to learn a “formula”, already Pythagoras’

$$a^2 + b^2 = c^2$$

opens a small universe because it combines symbolism with other abstract geometrical objects which do have significant practical relevance in our (physical) world. By *calculating*, i.e. working with a lot of similar examples, we hope to get a fraction of a big concept (in World 3) *heavily* connected with many other concepts.

For those who consider “Pythagoras” as trivial, one can recommend “operator-valued” versions[3]. When linear algebra is too boring, one should consider *numerical* linear algebra[4], or —since usually an algebraically closed (commutative) field is assumed— linear algebra over *non-commutative* (skew) fields[5].

There are at least three aspects here to consider: (1) Cognition is *relative*. What is only knowledge on a very abstract level can be “deep” cognition on a lower level and a huge source of knowledge when one is able to see a concept as a special case of a more general one. In this case one can understand the role of certain axioms, not just stating them. (2) One needs to start with the basics, i.e. with linear operators on *finite dimensional* vector spaces (over commutative fields), based on a sufficient understanding of vectors in “our” (at least locally) Euclidean three-dimensional world, after . . . ; there are hardly any shortcuts because one needs the special cases as “intuitive” examples later. And (3) —maybe less obvious— one has to *un*-learn almost hard-coded[6] “facts” like $(a+b)^2 = a^2 + 2ab + b^2$ that do not hold for *non-commuting* variables (e.g. for matrices). The difficulty here is that one needs to become aware about what is “stored” in our memory and how *reliable* it is in a certain context.

[3] Kadison (2002a,b)

[4] For example Demmel (1997)

[5] In this case one cannot talk anymore about *the* algebraic closure since there are several different definitions (Cohn, 1995, Section 8.1). Not that much is known even in very special cases.

[6] This is actually a serious topic in computer algebra systems since a lot of special cases are indeed hard-coded.

Writing[7]

$$(a+b)^2 = (a+b)(a+b) = a^2 + ba + ab + b^2$$

looks trivial but is actually the first step because it makes explicit the order of the multiplicands. Those who wonder if this is only a problem for people less literate with (abstract) mathematics could have a closer look on the famous[8] *eigenvalue equation*[9]

$$Ax = \lambda x.$$

That there is something wrong with this definition of *left eigenvalues*[10] should already raised ones attention when they write "In general, a quaternionic matrix similarity is meaningless for the left eigenvalues."[11] in the very beginning.

Wrong? Well, in mathematics one is free to define whatever one wants as long as it is *consistent* (within some broader context). From the *knowledge*-perspective (World 3), a formula is just a formula and thus neither right nor wrong.[12] From the *cognition*-perspective, each formula has a *history*. In this case it is about the *diagonalization*[13] of a (square) matrix A with dimension n, namely

$$AX = XP$$

with $P = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$. Picking a *column* (eigen-)vector x and the corresponding *right* eigenvalue ρ yields $Ax = x\rho$. In some cases —in particular when working

[7] Notice that it is a long road from explicitly writing the multiplication in say $2 \times 3 = 6$ to the implicit between variables, expressions in parentheses, and/or coefficients.

[8] Omnipresent (explicit and implicit) at university in mathematics, engineering, informatics (e.g. for extracting relevant information of huge sparse networks like "searching the internet"), etc.

[9] (Strang, 2003, Section 6.1); Notice that mathematicians rarely write $A\mathbf{x} = \lambda\mathbf{x}$ with vectors typed in boldface. That x here is a vector is clear from the context. At least if the context is clear ...

[10] (Ahmad et al., 2021, Definition 2.1, Page 130)

[11] Ibid. Page 128

[12] Even that is very idealistic since if World 3 contains too much (for an application in some science) completely irrelevant, how could we find the "right" concepts?

[13] There are a lot of interesting details we need to skip here. In general one needs to ask about the structure (e.g. some extension field) where this is possible or specify the exceptions (e.g. block diagonalization over the *real* instead of the *complex* numbers).

over a *commutative* field— one can write $Ax = \rho' \cdot x$ where ρ' is a *conjugate* of ρ and the multiplication is changed to that of a scalar in the vector space. From $YA = \Lambda Y$ one can easily derive the equation $yA = \lambda y$ for *left* eigenvalues with their corresponding *row* vectors y and (possible) conjugates λ' in $yA = y \cdot \lambda'$.[14]

Since for Kant already *schematism* (Section 3.3) is "obscure", one could easily adapt his words to "*cognition* (...) is a hidden art in the depths of the human soul"[15]. Nevertheless, it (cognition) is highly relevant to (a) find the right context, (b) detect interconnections, and —finally— (c) generate *new* knowledge. This is possible only because —rephrasing the three aspects from before— (1) cognition —as we understand it here— is a *process* that (2) builds (almost naturally) hierarchies and (3) can —at least partially— reference to itself. The latter is clearly limited since self-awareness is: "The mind is in particular in its origin *subconscious* mind."[16]

Knowledge about the context —in particular about the assumptions— is important to avoid errors. Even if we have a perfect "machine" (a formal system in World 3 with some terminal in World 1), a problem needs to be stated/formulated correctly. To find (new) connections it is sufficient to have a *vague* understanding of the "embedding" of a special case into a more general concept (which might lead to a full theory later). This is just an interpretation of Hans Reichenbach famous "context of discovery" versus "context of justification"[17].

[14]That we are used to write scalars (and coefficients) usually on the left, is just convention. The same holds for functions we typically write as $f(x)$. But sometimes a more delicate notation is necessary, e.g. $(x)f$ or x^f (with the same meaning). This can make reading (e.g. of Paul M. Cohn's "advanced" books) quite difficult because it takes time to get acquainted and an extremely high level of concentration is necessary.

[15](Kant, 1998a, B181), my change and emphasis

[16](Frankl, 2009, Page 65), my translation and emphasis

[17](Reichenbach, 1938, Section I.1, Page 7); see also (Reichenbach, 1938, Section V.42) for a discussion about Einstein's discoveries. In principle there is nothing special with mathematics. But since there cannot be something like "absolute" knowledge outside (e.g. in physics), it is much trickier to deal with knowledge. Therefore Reichenbach uses the concept of probability. Taking the *mathematical* concept of probability would lead to interesting questions concerning the limitations (due to the problem of how we can measure the reliability of theories). One also needs to be careful what *theory*

In Section 3.1 we have seen how (Krämer's) *diagrammatic* can facilitate understanding by joining *imagination* and *formalism* within some *graphical* (two-dimensional) representation (visualized in Figure 3.1). The hidden principle deserves a closer look: Starting from an essentially *one*-dimensional language, we introduce another dimension or —mathematically speaking— *degree of freedom*, making everything more complex (at a first glance). However, we do have additional "rules". Rules that are *not* part of the decimal system itself. With their help we gain an *orientation* (in World 1) which helps us to apply something purely *formal* (in World 3). Usually we do not separate them. And this justifies to see the "sensual in the formal" (as Sybille Krämer does in her understanding of diagrammatic). For the addition (of positive integers), say by writing $17 + 302 + 5 + 94$ as

$$\begin{array}{r}17\\302\\5\\94\end{array}, \qquad \text{and } \textit{not} \text{ as} \qquad \begin{array}{l}17\\302\\5\\94\end{array} \qquad \text{or even} \qquad \begin{array}{l}17\\302\\5\\94\end{array},$$

this might look trivial but if we consider computing the square root of an arbitrary positive integer by hand, it indeed shows a highly sophisticated system we have to learn additionally to the (rather simple "purely" formal) rules (of the decimal system.)

Now one could argue that these "form rules" (and multiplication *tables*) are just for humans while computers use *serial* memory[18]. And to the following comment of René Thom one could reply that computers —contrary to humans— do not need any meaning:

"But if these procedures [in natural languages] are systematized into a series of rules which are then pursued blindly to their local conclusion, the resulting sentences soon become so long and complex that they lose all meaning. I see no reason why a similar phenomenon could not happen in mathematics; in extrapolating a formal mechanism to the limit of its generative capacities, it does not take long to assemble

means. Being (mathematically) pedantic one would need to say "Einsteins relativity conjecture".

[18] To be more precise: address memory in a serial way.

formulas that are so long and complex that all possibility of intuitive interpretation disappears."[19]

As always, by taking a closer look, things turn out to be less simple, e.g. by starting with *rational identities*[20] like $y^{-1}x^{-1} = (xy)^{-1}$, or Hua's identity[21]

$$x - \left(x^{-1} + (y^{-1} - x)^{-1}\right)^{-1} = xyx. \tag{4.1}$$

Proving the latter is already highly non-trivial. But such identities could become arbitrarily complex, and, this is the main point, are almost completely inaccessible to computers! At least so far. This is where *free fractions*[22] —as a *genuine* two-dimensional (mathematical) language— come into place. With *free fractions* we can "calculate" with words almost like with numbers (using "classical" fractions). But before we can go a little bit deeper, we need to sharpen our eyes for the *graphical* (two-dimensional) patterns they admit.

4.1 Patterns in Free Fractions

In a typical introductory course to *linear algebra* one learns a "rule" similar to (in the following we omit the dots for the multiplication unless they increase readability)

$$(X \cdot Y \cdot Z)^{-1} = Z^{-1} \cdot Y^{-1} \cdot X^{-1} \tag{4.2}$$

for *invertible* (square) matrices X, Y, Z, usually over some *commutative* field (e.g. the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$). Notice that here the letters X, Y, and Z are just symbols where we (learn to) plug in matrices (of appropriate size), just like we are used to "classical" variables for numbers. Both sides of (4.2) are just two different rational expressions for the *same* element (we

[19] (Thom, 1971, Page 697)

[20] See Amitsur (1966), and —for a bigger (historical) context— (Cohn, 2006, Notes and comments on Chapter 7, Page 516).

[21] For details and how to prove it *systematically* using free fractions see (Schrempf, 2018b, Section 2.6).

[22] Schrempf (2018b) resp. (Schrempf, 2018b, Chapter E) for an English summary

denote by) f^{-1} in the *universal field of fractions* $\mathbb{F}$ of a *free associative algebra* (aka "non-commutative polynomials"). The latter is the (abstract) algebraic view based on a deep theory[23] of Paul M. Cohn (from 1970) that ensures that we can invert each *non-zero* expression, in particular symbols/letters. In a (not necessarily commutative) field we expect $f\,f^{-1} = f^{-1}f = 1$ for $0 \neq f \in \mathbb{F}$. And this is exactly what the rule (4.2) ensures:

$$XYZ \cdot (XYZ)^{-1} = XYZ \cdot Z^{-1}Y^{-1}X^{-1} = XY \cdot Y^{-1}X^{-1} = X \cdot X^{-1} = 1.$$

In general however it is not that simple because rational expressions can soon become far too complex to deal with them directly. But there is a way to solve such problems "graphically" using *admissible linear systems*[24] (ALSs), the technical side of the (more colloquial) term *free fractions*. An ALS[25] for XYZ is (lower dots denote non-emphasized zeros)

$$\begin{bmatrix} 1 & -X & 0 & 0 \\ . & 1 & -Y & 0 \\ . & . & 1 & -Z \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}. \tag{4.3}$$

Notice that "classical" fractions like $s = 1/3$ can be written in a similar form as equation $3s = 1$. In our more general case we need *systems of equations* (over non-commutative rings). Using the *minimal inverse*[26] one gets the corresponding

[23]For the mathematical theory see (Cohn, 2006, Section 7.5), for an overview about the development the end of (Cohn, 2006, Chapter 7), and for a brief overview about the construction (Schrempf, 2018b, Chapter C, Page 129).

[24](Schrempf, 2018b, Definition 2.1.10, Page 16) or (Schrempf, 2018a, Definition 1.10, Page 1212)

[25]There are many more, but the *minimal* ones, i.e. with the smallest possible dimension, can be easily transformed into each other. And if we are dealing with *words* like XYZ here, an ALS can always be transformed into such an upper-triangular form (with ones in the diagonal). For details see (Schrempf, 2018b, Proposition 2.2.1, Page 18) or (Schrempf, 2018a, Proposition 4.1, Page 1222).

[26](Schrempf, 2018b, Theorem 2.5.13, Page 31) or (Schrempf, 2018a, Theorem 4.13, Page 1229)

equations for $(XYZ)^{-1}$, namely

$$\begin{bmatrix} +Z & -1 & . \\ 0 & +Y & -1 \\ 0 & 0 & +X \end{bmatrix} s = \begin{bmatrix} . \\ . \\ 1 \end{bmatrix}.$$

We get the system matrix for the inverse by reversing the rows and columns (and changing the sign) of the upper right 3×3 block in the system matrix of (4.3). To read off the *solution* s_1 (in the first component of the *unique* solution vector s) as *rational expression* (we are usually more used to), one starts with the last row in

$$\begin{bmatrix} Z & -1 & 0 \\ . & Y & -1 \\ . & . & X \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} . \\ . \\ 1 \end{bmatrix},$$

that is, $Xs_3 = 1$, thus $s_3 = X^{-1}$, $Ys_2 - s_3 = 0$, thus $Ys_2 = X^{-1}$ resp. (after "dividing" by Y from the left) $s_2 = Y^{-1}X^{-1}$, and finally $Zs_1 - s_2 = 0$, thus $s_1 = Z^{-1}Y^{-1}X^{-1}$. The latter is just the right hand side of (4.2).

The *patterns* of the upper right zeros in the system matrix of (4.3) reveal the *factorization*[27] of XYZ into e.g. $X \cdot YZ$ (block of zeros of size 1×2) or $XY \cdot Z$ (block of zeros of size 2×1). In the *equivalent* ALS (also representing the word XYZ)

$$\begin{bmatrix} 1 & -X & -2 & 2Z \\ . & 1 & -Y & 0 \\ . & . & 1 & -Z \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}$$

one can look for row (from the bottom to the top) and column (from left to right, the first column must not be used) operations to find such zero blocks (here by adding 2-times row 3 to row 1).

[27] For the complete theory (for polynomials) see (Schrempf, 2019, Section 2.3), for a generalized factorization theory (Schrempf, 2020a, Section 4), and for the exposition (in German) within a bigger context (Schrempf, 2018b, Chapter 3).

With some experience one can read off much more information. Finding "hidden" blocks of zeros *systematically* is not always that easy, but programming that in some *computer algebra system*[28] (CAS) is actually guided by the two-dimensional representation.

When one looks for the left (or right) *greatest common divisor* of two non-commutative polynomials one can use free fractions (resp. admissible linear systems) to find (upper right) blocks of zeros in "L-form"[29]. What sometimes looks quite trivial (from a graphical point of view) can lead to a highly technical implementation. For those with some experience in "graphical programming" (as mentioned in Section 3.1) this is maybe not so surprising because it just shows the "power of visualization".

By adding one dimension (degree of freedom) to the "normal" (mathematical) language (of rational expressions) we are able to write a "simple" *word* like XYZ in many different ways, even if we use *minimal* admissible linear systems. However, only this extra freedom makes such *patterns* (or the *form* in World 1) possible![30] In other words: Free fractions combine the "pure" *formal* (in World 3) and the *sensual* (in World 1) in a natural way.

In some sense one can say that the *form* (the sensual) can be formalized because otherwise the underlying theory (establishing a correspondence between blocks of zeros and purely algebraic properties) could not be proved (rigorously) and in particular not implemented (for a "machine"). But such a "formalization" is always *secondary* (to the form) because —with some experience— it can be done (almost) ad hoc. The main ingredient is —and this is the main point— *imagination*:

One can get easily from a "simple" example to a general *block form* by interpreting a *letter* as a symbol for a matrix. What is part of the (basic) alphabet is clear from

[28] For example FriCAS (FriCAS team, 2019, Packages `FDALG` resp. `LINPEN`). Some general comments on implementations of free fractions can be found in (Schrempf, 2018b, Section B.5).

[29] (Schrempf, 2018b, Section A.1) or (Schrempf, 2020b, Example 5.4, Page 1383)

[30] There are actually other, more restricted two-dimensional representations like *proper linear systems* (Salomaa and Soittola, 1978, Section II.1) where such patterns are not always possible. And, these systems are not general enough for the inversion of words (like XYZ in our example).

the context. The many more other symbols (in particular with indices) are “free” for the use as (generalized) variables. If one needs to formalize that even partially, the notation can become quite involved[31]. The difficulty here is, that one needs a meta-level which “formalizes” *how* and *what* we substitute. And this brings as back to the question:

4.2 What is mathematics?

What do we see in the formula (4.2), $(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}$, in the beginning of the previous section? Usually, even for graduate students in mathematics, it is a “not completely formalized” *expression* which gets its meaning by plugging in (appropriate) matrices or —possibly infinite dimensional— operators. Not that much context is needed here. For “pure” formalism we wound need to ask *where* these elements (of the formula) “live” to be able to settle the right framework for manipulations (based on the underlying axioms). But already with *invalid* expressions —based on Hua’s identity (4.1)— like

$$\Big(x - \big(x^{-1} + (y^{-1} - x)^{-1}\big)^{-1} - xyx\Big)^{-1}$$

we would get into big troubles[32]. Seeing (valid) non-commutative rational expressions as elements in *free fields* (universal fields of fractions of their respective free associative algebra) would be the natural choice, but this is a huge step into highly abstract

[31] See e.g. (Schrempf, 2021, Section 3).

[32] There is hardly any computer algebra system that can deal with such expressions “natively” (if they have basic non-commutative capabilities at all). One package (for MATHEMATICA) which in principle should be able to deal with such expressions (or check their validity) is `https://mathweb.ucsd.edu/~ncalg/` (June 9, 2024). But neither is the usage straight forward nor can assumptions (order of the variables such that the underlying non-commutative Gröbner basis algorithm terminates) be completely avoided. In numerical programs like OCTAVE `https://www.octave.org` (June 9, 2024) there isn’t any warning if you invert the —to Hua’s identity corresponding— *zero matrix* (by plugging in even moderate sized random matrices of dimension 500). Admittedly, this is —due to rounding errors and the way expressions are evaluated— difficult do handle. At the end it’s in the responsibility of the programmers to avoid such nonsense ...

algebra (far beyond what one usually learns in mathematics at university and partially beyond when one specializes in algebra).

A step we would need to go anyway if we consider mathematics as the "pure" formal only. A step we actually should go because it would give us the *object* level —discussed in Section 3.2— to our *process* level of *words* (in our languages), just like the decimal system "adds" that to our *intuitive* counting. What seems to make this step so difficult is the focus on the formal (in mathematics):

"Workers in commutative ring theory, and its partners algebraic geometry and number theory, far outnumber those in non-commutative rings, and tend to regard the latter area as excessively 'far out'. ('If we don't know our rings to be commutative, how can we *trust* anything we *know*?') Among non-commutative ring theorists there is a tension between the tendencies to hug close to the border with the more popular commutative theory, judging results as 'good' to the extend that they look like standard results from the commutative case, and to venture far from the commutative and discover what results are natural to the rings and finds there. Paul was one who strode into the wilds of the non-commutative, and uncovered great beauty." [33]

Now one should add that the theory of *free fractions* is only a very special case of Cohn's theory (of *free ideal rings*[34]). And his formalism is admittedly highly demanding, even if one is acquainted with the prototypical case of *free associative algebras* (aka "algebra of non-commutative polynomials"). But who learns about *rings*[35] (at university) without being familiar with their prototypical (commutative) case of the *integers*? (One could actually call non-commutative polynomials —with their special case of "normal" *words*— "free integers".)

Unless something is "purely" formalized we need "normal" language to describe

[33] (Bergman and Stuart, 2014, Section 8), my emphasis. A *ring* is more general then a *field* like the rational, real or complex numbers in which one can divide by each element except zero.

[34] FIRs, Cohn (2006)

[35] To be precise: The integers are *the* prototypical case of *principal ideal domains* (PID), a very special class of rings. *Free ideal rings* (FIRs) generalize PIDs in a very natural way.

the context (and thus assumptions, restrictions/limitations, etc.) Mathematics seems so clear because we are conditioned to see "numbers" and "formulas" in a unique way. Until one studies *mathematics* which consists of analysis, algebra, topology, logic, numerics, measure theory, functional analysis, graph theory, etc. Except from basic mathematics, the *context* matters (because it is less clear). In applied mathematics, say in engineering and economics, the (complete) formalization of the context is difficult due to the use in the "real" world. Already that of the pure mathematical part is challenging.[36] In "pure" mathematics "total" formalization (possibly understood as "machine-readable") is —in principle— possible. But that has —as we have discussed with that of free fractions— its price: From the computer code —even if it is written highly readable— it is almost impossible to get the *graphical representation* (in two-dimensional space) back. With dramatic consequences:

(1) We lose a great *source of knowledge* because cognition —in the sense of Kant— is difficult to impossible. As we have sketched in Chapter 3 from different perspectives: *Diagrammatic* (resp. graphical representation) has the great potential to "mediate" between our *senses* (in World 1) and our *understanding* (in World 3) by combining *imagination* and *formalism*!

(2) If mathematics is only partially understood as "that what we have to learn"

[36] As an example one can take acoustic-structure interactions in Schrempf (2008), or simulations of dendritic solidification with explicit front tracking in 3D to solve the *Stefan problem*. In the latter one has to deal with *non-linearities* in systems of partial differential equations, combined with non-trivial boundary conditions. Already the basic formulation take use of heavy symbolism. Then one needs to chose some kind of *discretization* (in space and time). This is where numerics comes into place, with different concepts like *finite* and *boundary element methods*, *finite difference methods*, and *finite volume methods*, just to mention some. In practical problems however, one usually needs to combine them. This is everything else than straight forward because we are usually interested in a way that *practically* works. Interestingly, there is a close connection to free fractions via the —in numerics— ubiquitous *Schur complement*, a rule for block decomposition of matrices one learns. From an abstract algebraic point of view this is just the use of *Gaussian elimination* over the *free field* (universal field of fractions), with row operations "from the left" and column operations "from the right". Here another wheel turns full cycle: Most of all that is done manually "on paper" and hard-coded in supplementary functions in huge (often very expensive) simulation systems.

—as we tried to argue in the *Interlude* (before this chapter) with reference to Ancient Greek—, then it gets "isolated" (in World 3) and only very special gifted people can take some (limited) advantage. But isn't that (yet) another characterization of mathematics, namely that *everyone* learns it? (At least the basics.) That it became a "lingua franca"?[37]

At the end it is up to us how we understand mathematics: More *sensual* (and applied) in World 1, more *formal* (and abstract) in World 3, or *alive* —as source of (not only other mathematical) knowledge— in World 2.

4.3 Processes and Objects

As long as we focus on (single) words only, it is not clear why *free fractions* should provide an object level to an even unclearer process level in terms of words. But similar to numbers, we would not gain anything with the decimal system as long as we deal with a hand full of (Hilbert) strokes only:

$$IIIII\;II + IIIII = IIIII\;IIIII\;II.$$

However, *counting* understood as "the successive addition of one (homogeneous) unit to another"[38] takes place in *time*. This is the idea of (Kant's) schematism briefly discussed in Section 3.3. Since time is (in principle) unbounded, we do have the *potential* infinite"[39] *integers*. The decimal system provides us with an object level (to the *actual* infinite ring of integers) insofar as it enables to address the elements *directly*. For the addition (or the multiplication) we pick two arbitrary numbers and the result is again an element (in the ring of integers).

[37] In the essay about pluralism of mathematics in Section E.2 I try to grasp the differences between basic mathematics (for world-wide communication), advanced applied mathematics (in particular in engineering) and abstract mathematics in terms of the "level" of pluralism. One additionally needs to distinguish between the "mathematical system" (more monistic) and "calculations" (more pluralistic) in the sense of the essay in Section E.4.

[38] (Kant, 1998a, B182)

[39] For an overview about the "potential" versus the "actual" infinite we refer to (Linnebo, 2017, Section 4.2).

In the non-commutative we are dealing with *rational power series*[40] as *formal* sums of words. If these sums are *finite*, we call them *polynomials* (understandable as “multi-words”). But even very simple (non-commutative) rational expressions can result in *infinite* sums (1 is called *empty word*, it has zero length):

$$(1 - xyz)^{-1} = 1 + xyz + \underbrace{xyzxyz}_{=(xyz)^2} + \underbrace{xyzxyzxyz}_{=(xyz)^3} + \underbrace{xyzxyzxyzxyz}_{=(xyz)^4} + \ldots$$

(Before we used capital letters which are typically used for denoting matrices. Here we do not intend to plug in matrices and have the “pure” letters in mind.) This “addition” (of words) could be understood as *process*. Working with such infinite series is possible, at least theoretically, e.g. to show that the sum and the product (but not necessarily the inverse) are again such (rational) series. Using free fractions we can represent $1 - xyz$ via the system

$$\begin{bmatrix} 1 & x & . & 1 \\ . & 1 & -y & . \\ . & . & 1 & -z \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}$$

and get immediately —recalling to take the “reversed” upper right block with changed signs of the system matrix of the *admissible linear system* (ALS)— the inverse

$$\begin{bmatrix} z & -1 & . \\ . & y & -1 \\ -1 & . & -x \end{bmatrix} s = \begin{bmatrix} . \\ . \\ 1 \end{bmatrix}.$$

Notice that both have *finite* dimension! This is not a coincidence but a characterization of —even infinite— *rational series*. Furthermore, as we have seen before, one can use free fractions to represent even the inverse of “pure” words (which do not have any representation in terms of formal power series anymore).

Recalling the comment from René Thom about the complexity just before Hua’s identity (4.1) one can ask how “readable” infinite series are. Fortunately rational

[40] For an introduction see e.g. Berstel and Reutenauer (2011).

series do have some "predictable" pattern comparable to "normal" fractions like

$$1/7 = 0.142857\,142857\,1428571\,\ldots$$

Using the expression $(1-xyz)^{-1}$ for our example from before does not really help because it is —except for trivial cases— difficult to impossible to use for calculations. The following —de facto unreadable[41]— "one-dimensional" (non-commutative) rational expression

$$\begin{aligned} g &= x_1(y_{1,1} - y_{1,2}y_{2,2}^{-1}y_{2,1})^{-1}z_1 + x_1(y_{2,1} - y_{2,2}y_{1,2}^{-1}y_{1,1})^{-1}z_2 \\ &\quad + x_2(y_{1,2} - y_{1,1}y_{2,1}^{-1}y_{2,2})^{-1}z_1 + x_2(y_{2,2} - y_{2,1}y_{1,1}^{-1}y_{1,2})^{-1}z_2 \end{aligned}$$

can be written simply as $g = XY^{-1}Z$[42] using the *matrix factorization*[43], resp. —written as ("two-dimensional") linear system, a star "$*$" denotes an arbitrary entry—

$$\begin{bmatrix} 1 & -x_1 & -x_2 & . \\ . & y_{1,1} & y_{1,2} & -z_1 \\ . & y_{2,1} & y_{2,2} & -z_2 \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}, \quad s = \begin{bmatrix} g \\ * \\ * \\ 1 \end{bmatrix}.$$

Before we prove that $(1-xyz)(1-xyz)^{-1} = 1$ using free fractions, it is worth to recapture the main points: Although we are used to group strokes (or dots) to increase readability, their position actually does not matter! One could call that "language" *zero*-dimensional. This is the *process* level. Adding one dimension, we introduce *non-commutativity* on the digits in our *numbers* (in the decimal system). Now the position matters, the *zero*[44] becomes important. $405 \neq 504$ is immediate by comparing the "words". (Comparing 405 strokes to 504 strokes is already challenging if we group them.)

[41] Notice that there are many more, even "uglier" ways for writing this element in the *free field* (Cohn, 2006, Section 7.5), the universal field of fractions of a free associative algebra.

[42] Here the upper case letters stand for (block) matrices of symbols and are not part of the alphabet. Such as f, g (here), which we use to denote (general) rational elements

[43] (Schrempf, 2019, Section 3)

[44] For a historical overview see e.g. (Krämer, 2014, Section 1).

Now rational expressions and formal (power) series are essentially *one*-dimensional. And become soon —as we have seen— unreadable (or infinite). If we add again a dimension, working with such elements becomes feasible, in particular using the computer! (Which however needs a "pure" formalization.) Again we switched from the *process* to the *object* level.

Operations[45] (addition, multiplication, and inversion) involving free fractions take place in two-dimensional space. The product of $1 - xyz$ and $(1 - xyz)^{-1}$ can be represented by the *non-minimal* admissible linear system (ALS)

$$\begin{bmatrix} 1 & x & . & 1 & 0 & . & . \\ . & 1 & -y & . & 0 & . & . \\ . & . & 1 & -z & 0 & . & . \\ . & . & . & 1 & -1 & . & . \\ & & & & z & -1 & . \\ & & & & . & y & -1 \\ & & & & -1 & . & -x \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix}.$$

Notice the block structure in the system matrix: The upper left corresponds to $1-xyz$, the lower right to $(1 - xyz)^{-1}$, and the left column of the upper right to the negative right hand side of the ALS for $1 - xyz$.

What we do now, namely *minimizing*[46] an ALS, can be compared to *cancellation* in calculations with "normal" fractions, e.g. from $2/4$ to $1/2$. Firstly, we add row 5 to row 3, row 6 to row 2, and row 7 to row 1 (notice that row operations apply also

[45] (Schrempf, 2018b, Proposition 2.3.1, Page 21)
[46] For details see (Schrempf, 2018b, Chapter 4) or (Schrempf, 2020b, Section 4).

to the right hand side):

$$\begin{bmatrix} 1 & x & . & 1 & -1 & . & -x \\ . & 1 & -y & . & . & y & -1 \\ . & . & 1 & -z & z & -1 & . \\ . & . & . & 1 & -1 & . & . \\ & & & & z & -1 & . \\ & & & & . & y & -1 \\ & & & & -1 & . & -x \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix}.$$

Now we add column 2 to column 7, column 3 to column 6, and column 4 to column 5:

$$\begin{bmatrix} 1 & x & . & 1 & 0 & 0 & 0 \\ . & 1 & -y & . & 0 & 0 & 0 \\ . & . & 1 & -z & 0 & 0 & 0 \\ . & . & . & 1 & 0 & 0 & 0 \\ & & & & z & -1 & . \\ & & & & . & y & -1 \\ & & & & -1 & . & -x \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix}.$$

Recall that we are interested only in the first component s_1 of the solution vector s. Because of the upper right block of zeros we know that the components s_5, s_6 and s_7 do not contribute. Therefore we can concentrate on the smaller system

$$\begin{bmatrix} 1 & x & . & 1 \\ . & 1 & -y & . \\ . & . & 1 & -z \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ . \\ . \end{bmatrix},$$

in which it is immediate that $s_4 = 0$, which implies that $s_3 = 0$, and further $s_2 = 0$. Leaving the ALS $[1]s = [1]$, representing $1 = (1 - xyz)(1 - xyz)^{-1}$.

Remark. Those who know the little story of "Flatland"[47] might ask what happens if we go a step further, into the *third* dimension. This is something one can describe

[47] Abbott et al. (2009)

partially, a section in the essay in Section E.1 is dedicated to that: One might come from *free fractions* to *free roots* (a theory that does not yet exist) generalizing *algebraic* formal (power) series. Going further is pure speculation in which one might end in something one could call "Kant's Space-Time-Singularity". If a *four*-dimensional language (whatever that is) is powerful enough[48] to represent the "whole" mathematics in a single element within a huge algebra[49] which is itself a part of mathematics, we would almost certainly run into a problem similar to the well known *Russell's paradox*[50].

4.4 Imagination and Formalism

Recalling Figure 3.1 (Page 31), *imagination* should give content to our (mathematical) *thoughts* (when our "normal" senses could not help), whereas *formalism* should provide concepts to our *intuitions* (when our "normal" understanding get's in trouble). This can only work when there is something where one can start and something one can trust. As we have seen before, "pure" formalism can (soon) become far too complex to be of any use. This is well known in (computer) programming, we briefly discussed the role of additional structure (e.g. two-dimensional representation of code, flow diagrams, etc.) in Section 3.1.

Now, for concepts like "sheep"[51] or small and/or often used numbers like e.g. "one", "two", "hundred", "thousand", or "million", one can do as if they are just there (once we have learned them). But in general, mathematical concepts (Begriffe) need *construction*. One could say that the "pure" *symbol* "712" gets its content by naming it "seven hundred and twelve". What seems rather trivial is actually quite subtle: Understanding a simple text with numbers is difficult without being able to give them a

[48] Algebraic (or context-free) languages might not be sufficient. For an overview with linguistic motivation see Chomsky and Schützenberger (1963), for more algebraic Cohn (1975).

[49] An algebra is a ring with special additional structure.

[50] (Linnebo, 2017, Section 2.7)

[51] With a (more) concrete meaning depending on the context.

name resp. "reading" it.[52] One might be still able to "copy" it (in the sense of taking it as "pure" knowledge). And, as we have seen with the "simple" equation $Ax = \lambda x$ in the beginning of this chapter, the *context* contributes a lot. The symbol π is not always the number $3.14159\ldots$ for computing the (approximative) circumference of a (perfect) circle or volume of a (perfect) sphere. In functional analysis it is often used for naming projections. And although one would be free to use symbols arbitrarily, in an exposition it matters how "intuitively" the notation is chosen![53]

For Kant, "*Imagination* (Einbildungskraft) is the faculty for representing an object even *without its presence* in intuition."[54] And further: "Now since all of our intuition is sensible, the imagination, on account of the subjective condition under which alone it can give a corresponding intuition to the concepts of understanding, belongs to *sensibility*; but insofar as its synthesis is still an exercise of spontaneity, which is determining and not, like sense, merely determinable, and can thus determine the form of sense a priori in accordance with the unity of apperception, the imagination is to this extent a faculty for determining the sensibility a priori, (...)"[55]

He distinguishes two types of imagination: "Now insofar as the imagination is spontaneity, I also occasionally call it the *productive* imagination, and thereby distinguish it from the *reproductive* imagination, whose synthesis is subject solely to empirical laws, namely those of association, and that therefore contributes nothing to the explanation of the possibility of cognition a priori, and on that account belongs not in transcendental philosophy but in psychology."[56]

[52] This is what makes in particular abstract mathematics difficult when one is not sufficiently acquainted with the notation. It does not really help if one can lookup that "symbol by symbol" like trying to read something in an alphabet one is not familiar with.

[53] The topic with *left* and *right* eigenvalues is difficult enough, even specialized mathematicians can struggle. Using ρ for *right* and λ for *left* eigenvalues makes it much easier to follow. There was a temptation to use λ with X and ρ with Y because then the alphabetical order would be also consistent. Often this is very useful. Here it would have been too confusing since the "normal" eigenvalue equation (typically written with *row* vector x) is actually that for *right* eigenvalues.

[54] (Kant, 1998a, B151)

[55] (Kant, 1998a, B151/152)

[56] (Kant, 1998a, B152)

Kant's understanding of *productive imagination* is also what J. Michael Young stresses: "He holds instead [of the mental imaging of absent objects] that it is the capacity to construe or interpret the sensibly present as something other, or something more, than what it immediately presents itself as being. It is the capacity, therefore, for representing in intuition something which, strictly speaking, is not present there, at least not fully."[57]

It is the *construction*, where (productive) *imagination* and *formalism* intertwine. In particular in mathematics, where constructions follow precise rules. However, as we have pointed out several times, there are ways to introduce degrees of freedom to enable dynamics in the formal. Most prominently *diagrammatic* (in the sense of Sybille Krämer, discussed in Section 3.1), which can be understood more broadly also for *one*-dimensional text and (mathematical) formulas (in World 3) that get a special *form* (with carefully chosen "intuitive" symbols) by using the *two*-dimensional (sub-)space of the (idealized) physical world.

Another way to open dynamics, is the view of special cases within a more general (in the beginning usually less developed) concept (that later might become a theory on its own). When we are familiar with the (ring of) integers, we can ask what makes them special, e.g. *unique factorization*[58]: $2 \cdot 3 = 6 = 3 \cdot 2$. (The factorization into *irreducible elements* or *atoms* is unique up to the order of the factors[59].) A *commutative* FIR (free ideal ring) is a PID (principal ideal domain)[60]. The prototypical example of a FIR is the *free associative algebra* (say over a *commutative* field and a *finite* alphabet). Therefore it admits a *similarity unique factorization*[61]: $x(1-yx) = x - xyx = (1-xy)x$. The number of factors is unique and that of different factorizations are related by similarity. The abstract definition of the latter is far from

[57] (Young, 1992, Section III, Page 170)

[58] For an overview see e.g. (Cohn, 2006, Section 0.9).

[59] For the integers this corresponds to the factorization into prime numbers. However, in general, these concepts are different.

[60] (Cohn, 2006, Proposition 2.2.2, Page 111)

[61] For an overview (in German) and further references see (Schrempf, 2018b, Section 3.1).

intuitive. In terms of *free factions* however, there is a *graphical* interpretation[62]:

$$\begin{bmatrix} 1 & -x & 0 & 0 \\ . & 1 & y & -1 \\ . & . & 1 & -x \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix} \quad \text{versus} \quad \begin{bmatrix} 1 & -x & -1 & 0 \\ . & 1 & y & 0 \\ . & . & 1 & -x \\ . & . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}.$$

We do not want to discuss, how "real" *abstract*[63] (mathematical) objects are. The special with free fractions is, that we can *deliberately* manipulate them (e.g. for factorization), *without* changing the represented element, e.g. to reveal (non-trivial) *knowledge*. What one could call *inner* degree of freedom here, is more an *outer* in the use of the decimal system ("on the paper"). Both cases facilitate *cognition* which —when "deep" enough— can bring forward *new* knowledge that becomes "formally" part of (the global) World 3 by the publication of a well-formulated and -explained (hopefully also illustrated using simple but non-trivial examples) article that is much more than the "pure" formal.

[62] For the construction see (Schrempf, 2018b, Example 3.2.1, Page 42), for the factorization of (non-commutative) polynomials (Schrempf, 2018b, Section 3.3) or Schrempf (2019).

[63] "An object is said to be *abstract*, we recall, if it lacks spatiotemporal location and is causally inefficacious; otherwise it is said to be *concrete*." (Linnebo, 2017, Section 1.4)

Epilogue

Whatever *mathematical* objects[1] are, they definitely (can) affect us, even if they are not accessible directly using our senses. This is where imagination takes over: "*Imagination* (Einbildungskraft) is the faculty for representing an object even *without its presence* in intuition. Now since all of our intuition is sensible, the imagination (...) belongs to *sensibility*;"[2]

For *cognition* we need to unify *mind* (for concepts) and *senses* (for content).[3] The trivial side of thinking —say about the ring of integers— is that we need to know (at least) the concepts (e.g. that of a ring). In "modern" mathematics this is very much about the governing axioms. Understanding them means to know how to *manipulate* (mathematical) expressions. (At least in principle.) This is —sloppy speaking— what *formalism* is about. It shifts the focus in mathematics from the passive to the active, it can be seen as a tool for extrapolation of the abstract.

But what would be *imagination* without *formalism*? (In particular in mathematics.) And how could we transform even simple —say non-commutative— expressions without imagination? To be more precise: If we know the rules, we can apply them one by one. (If the problem is small enough and the solution is known[4] one can even

[1]One might call them *thin*: "Thus, if mathematical objects are thin [in the sense that their existence does not make a substantial demand on the world], this will explain the striking fact that mathematics operates with an ontology that is far more abundant than that of any other science. The idea of thin objects is elusive, however." (Linnebo, 2018, Preface, Page xi)

[2](Kant, 1998a, B151)

[3](Kant, 1998a, B75)

[4]Hua's identity (4.1) on page 54 could be used as a test case.

find a proof.) But to find complete new paths, *formalism* and *imagination* must go hand in hand. This is what we tried to show.

Sketching (very briefly) David Hilbert's contributions to formalism in Section 2.2 has also[5] to do with his struggling as mathematician: "Hilbert's Program was founded on a concern for the phenomenon of paradox in mathematics. To Hilbert, the paradoxes, which are at once both absurd and irresistible, revealed a deep philosophical truth: namely, that there is a discrepancy between the laws according to which the mind of *homo mathematicus* works, and the laws governing objective mathematical fact. Mathematical epistemology is, therefore, to be seen as a struggle between a mind that naturally works in one way and a reality that works in another. Knowledge occurs when the two cooperate."[6]

Some might have recognized the inconsistent use of "unconscious" and "subconscious" (here). For our purpose it should not be any problem. And for "the unconscious and discovery"[7] in our context of imagination and formalism another thesis would be necessary. For a focus on *memory*[8] one could claim something similar.

Our focus was on cognition and its two aspects: The (active) process involved in *learning*, and the (passive) "result" as source of knowledge. Even if a theory is (formally) published, there might be much more "hidden" knowledge. Learning is also tied to *communication*, something one could understand quite broadly also in the autodidactic case. (See Chapter 1 for the role of Popper's World 3 for improvement

[5]That one could go step by step (to the abstract) by giving *content* to the —in the beginning— ideal elements should be clear by now. A *free field* (universal field of fraction of a free associative algebra) —with an alphabet of at least two letters— is *infinite-dimensional* over its center. Therefore it's difficult to work with its elements. With "finite" free fractions it becomes much easier. By the way: Paul M. Cohn used more general *admissible systems* which do not reveal that much patterns. (Schrempf, 2018b, Section C.2, Page 130).

[6](Detlefsen, 1986, Preface, Page ix)

[7](Hadamard, 2020, Section III); published 1945.

[8]Starting points could be Robert E. Briscoe's "Superimposed Mental Imagery", in particular (Macpherson and Dorsch, 2018, Section 8.2.2) on diagrammatic reasoning, or Felipe De Brigard's "Memory and Imagination" (Bernecker and Kourken, 2017, Chapter 10).

and criticism.) One should not forget that (typical) mathematical publications are for very specific groups with the necessary background. Rewriting it for other (maybe non-mathematical) audience is difficult to impossible without *cognition*. And what would “pure” knowledge be worth if it cannot be written in different “languages”, if it cannot be explained to others (not even to those with a degree in mathematics)?

Appendix E

Essays

Some of the main ideas of this master thesis go back to essays where I developed thoughts around several aspects which themself arose during the finalization of my (mathematical) PhD thesis[1] about *free fractions*. Here only the abstracts are included.

E.1 Immanuel Kant and the Logical Empiricism as a source of mathematical cognition

Abstract of an essay (in German) within a seminar around the philosophy of science from September 2020 (University of Vienna):

For Kant, mathematical judgements are *synthetic* (a priori). For supporters of the Logical Empiricism (e.g. from the Vienna Circle) they are *analytic* because they could be deduced with logic. And Quine called this distinction (between analytic and synthetic) as one of the dogmas in empiricism. Indeed however, one could understand this "dual nature" of mathematics *dialectical* and as actual source for mathematical cognition, whose two sides are the *sensual* and that of the *understanding*.

[1] Schrempf (2018b)

E.2 Pluralism in mathematics

Abstract of an essay (in German) within a seminar around the philosophy of science from March 2022 (University of Vienna):

How pluralistic is mathematics? Since already this question depends on the viewpoint (resp. perspective) —e.g. related to mathematics at school, the application in other (natural) sciences, or the very *heterogeneous* research topic mathematics—, we will attempt to make the terms precise. Because when a discipline appears —like many others— very divers under a closer look, it does not mean necessarily "pluralism". On the other hand, within —from the outside *monistic* appearing— mathematics there is an important *duality*, we want to stress here, namely that of the "sensual" *counting* and that of the "abstract" (resp. mechanical) *calculating* (using the decimal system). Because without these "two sides of the same coin" we cannot learn mathematics (at school). Moreover there are more —even among mathematicians almost unknown— dualities, e.g. probability theories or the representation of non-commutative rational functions, we want to sketch here to point out how important "not too much" pluralism can be for cognition.

E.3 Possible, new, and innovative mathematics?

Abstract of an essay within a seminar around "modelling possibilities" from August 2022 (University of Vienna):

Usually *mathematics* is treated as if is just there (ignoring the ongoing development), "fallen out of time", and just applied to *science* and *technology.* (Actually, since mathematics is applied almost everywhere, one can understand science quite broad here, not just restricted to natural sciences.) However, mathematics is also applied to mathematics itself, e.g. abstract algebra to develop algorithms for applied mathematics, *even* if it is itself under change. This —outside of only a few research groups completely invisible— transformation from the *possible* to the *new* is of particular interest since it enables a closer look on *how* mathematics evolve by a combination of "tools" (or technology) and (human) "intuition" (or biology). The "space of the possible" (in mathematics) is not just an abstract concept, it can be grasped by open concrete (important) books in any mathematical library, say that of the University of Vienna. Unfortunately, this is neither obvious nor trivial ...

E.4 Mathematics and calculating: About the "language(s)" in/of mathematics

Abstract of an essay (in German) within a seminar around the philosophy of mathematics from April 2023 (University of Vienna):

Ferdinand de Saussure splits *language* in his "Cours de linguistique générale" into *language system* (as subject in linguistics) and (individual) *speech*. To which extend is mathematics a language and could it be split into *mathematics* (mathematical system) —as subject in some "mathematistics"— and the (individual) *calculation* analogously? Where do language and mathematics differ (fundamentally)? Languages themself play an important role *in* mathematics. The other way around, (simple) mathematics is inseparably interwoven with language, to measure or arrange something is —*neither* in speech *nor* in the language system— unthinkable without numbers. This essay is about the investigation of these complex interactions.

E.5 How formalistic is mathematics?

Abstract of an essay (in German) within a seminar around formalism from September 2023 (University of Vienna):

Looking at "modern" mathematics in general and algebra (in particular from the twentieth century) in particular, one discovers *formalism* especially during the use of computers for the "processing" of algorithms. Because only the detachment from *interpretation* (resp. semantics) enables the "mechanization". But already a closer look at the "old" *decimal system* shows that it is not much more than the *systematic* manipulation of (differentiable) symbols. And nevertheless it —mathematics— is so useful (not just) in science. Or because of that? Following Hilbert('s program) we trace this fundamental puzzle in *philosophy of mathematics*, at the "border" between mathematics and its *very useful* application in the "real" world.

E.6 Augmented imagination in (mathematical) knowledge acquisition

Abstract of an essay within a seminar around "knowledge and imagination" from February 2023 (University of Vienna):

Reading literature about *visualization* and *imagination* (both meant very general) in mathematics one gets the impression that (1) it is only *secondary*, i.e. only for learning and support (and "real" mathematicians would not need it because of "better" abstract formalism), (2) it is *direct* in the sense that knowledge is immediate (at least when one can see the relevant pattern), and (3) that it is *complete* and one only needs cute figures that either fit on a page or one can do that easily in mind, i.e. imagine it. Unfortunately, getting a full picture about *visualization* and *imagination* (in mathematics) is impossible, since psychological investigations can only focus on rather primitive mathematics (in particular in school) to get statistically significant results, and even philosophy of mathematics tend to focus on quite old "common" mathematics. Therefore I propose "augmented imagination" (erweiterte Einbildungskraft[2]) as a faculty that is able to gain fundamental new mathematical insight and knowledge and show how it is hidden in the theory (and the development) of *free fractions*. Interestingly the latter brings together what is usually completely separated: *language* and *mathematics*.

[2] In the essay I used "Anschauungskraft". But this was on a very early stage where I still was developing the terminology.

E.7 Gödel's incompleteness theorems and the mind–machine undecidability problem

Abstract of an essay within a seminar around Gödel's incompleteness theorems and the relevance for philosophy from March 2023 (University of Vienna):

Although the question whether humans (or animals) are (just) machines —or cannot be— might be as old as philosophy, Gödel's *incompleteness theorems* (1931) mark a turning point for new fuel (to avoid "arguments" here) in discussions, in particular because of the *mathematical* rigor involved. Unfortunately, the debate sometimes became quite polemic and seemed to miss what one could call the *mind–machine*[3] *undecidability problem.* To resolve most of the confusion we are going to use Karl Popper's concept of *worlds* and show that (human) mind and machine are basically *incompatible.*

[3]Notice that the two terms are *not* separated by a hyphen.

Appendix F

Figures

The basic theme of Figures 1.1 and 3.1 consists of a circle with the label "Human" in the center. There are three cloudy shapes (with diameter roughly one third of the big circle) at angles 60, 180 and 300 degrees (counterclockwise) depicting the *body* (intuitions, sensibility), the *soul/anima* (cognition, reason, unconscious), and the *mind* (thoughts, understanding). The "clouds" mean that it's hard to draw an exact boundary (which doesn't exist anyway). The (upper right) body corresponds to *information*, the (left) soul to *erkenntnis*[1] (cognition), and the (lower right) mind to *knowledge*.

There are three inner connection pairs (depending on the concrete figure) and three outer: *content* from the body to the soul and *affection* the other way; *judgement* from the soul to the mind and *concepts* the other way; *concepts* from the mind to the body and *content* the other way (the arrows of the latter two are dashed).

The "clouds" *body* and *mind* contain our central concepts *imagination* and *for-*

[1]In the figures I kept the German term which I thought could be useful as technical term "the erkenntnis" in English. Later I found that *cognition* is used in Kant (1998a), the translation of Kant I used as reference. The glossary in the appendix with terms in German (with English translation, starting page 757) and terms in English (with German translation, starting page 766) was very helpful.

malism respectively.

F.1 Description Figure 1

The figure in the introduction (Chapter 1) has the following three inner connection pairs: *sensation* from the body to the soul and *emotions* the other way; *reflection* from the mind to the soul and *feelings* the other way; *actions* from the mind to the body and *concentration* the other way.

F.2 Description Figure 2

The figure in Section 3.1 emphasises only two of the inner connections, namely *intuition* from the soul to body, and *deduction* from the soul to the (conscious) mind. The *graphical representation* corresponds —additionally to erkenntnis (cognition)— to the soul (anima).

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