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Area and Volume Comparison in Lorentzian Geometry

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Abstract

In recent decades, various approaches have been explored to transfer theorems which compare volumes of sets in manifolds, from Riemannian geometry, where such theorems have been studied throughout the twentieth century, to Lorentzian geometry. This work deals with two such approaches and their applications. In particular, these results pave the way for alternative proofs of the Hawking singularity theorem and for an alternative definition of spacetime singularities.

In an introductory chapter, we present fundamental concepts and results from semi-Riemannian geometry. In particular, we show a formula that allows us to express the change in measure of submanifolds by using their curvature, as well as a comparison result for solutions of Riccati equations. These theorems enable us to derive inequalities for volumes from inequalities of curvature quantities.

In the first application of these results, we consider certain spacelike hypersurfaces transported along timelike geodesics. By assuming bounds on the timelike Ricci curvature and the mean curvature of a hypersurface, we obtain bounds on the area of the transported hypersurface. By integrating over all these hypersurfaces, we also obtain bounds on volumes of subsets of the spacetime. Using similar methods, we can also derive bounds on the areas of spacelike slices of the light cone. Finally, the volume bounds are used as motivation for a new way to describe spacetime singularities, where a manifold is called incomplete if it possesses points whose future is of finite volume.

Abriss

In den letzten Jahrzehnten wurden diverse Möglichkeiten untersucht Theoreme, in denen Volumina von Mengen in Mannigfaltigkeiten verglichen werden, aus der Riemann Geometrie, in der solche Sätze bereits über das zwanzigste Jahrhundert hinweg erforscht wurden, auf die Lorentzgeometrie zu übertragen. Diese Arbeit beschäftigt sich mit zwei derartigen Zugängen sowie deren Anwendungen. Insbesondere ebnen diese Resultate den Weg zu alternativen Beweisen für das Singularitäten Theorem von Hawking, sowie zu einer alternativen Definition von Raumzeit Singularitäten.

In einem einleitenden Kapitel werden zunächst grundlegende Resultate aus der semi-Riemannschen Geometrie bewiesen. Insbesondere zeigen wir eine Formel, mit welcher wir die Veränderung von Maßen von Untermannigfaltigkeiten mittels deren Krümmung ausdrücken können, sowie ein Vergleichs Resultat für Lösungen von Riccati Gleichungen. Diese Sätze erlauben es uns aus Ungleichungen von Krümmungsgrößen, Ungleichungen für Volumina zu gewinnen.

In der ersten Anwendung dieses Wissens, betrachten wir bestimmte raumartige Hyperflächen, welche entlang zeitartiger Geodäten transportiert werden. Durch Schranken für zeitartige Ricci Krümmung und die mittlere Krümmung einer Hyperfläche, erlangen wir Schranken für die Oberflächen der transportierten Hyperfläche. Mittels Integration über all diese Hyperflächen bekommen wir darüber hinaus auch Schranken für Volumina von Teilmengen der Raumzeit. Mit ähnlichen Methoden können wir ebenfalls Schranken für die Oberflächen von raumartigen Schnitten des Lichtkegels gewinnen. Zu guter Letzt werden die Schranken für Volumina als Motivation für eine neuartige Beschreibung von Raumzeit Singularitäten verwendet, in welcher eine Mannigfaltigkeit unvollständig genannt wird, falls sie Punkte besitzt, deren Zukunft ein endliches Volumen haben.

Contents

1 Lorentzian Geometry Background

While throughout this thesis we will assume basic knowledge in semi-Riemannian- and in particular, Lorentzian geometry, we will start by giving some essential results, which are not covered in many books on the topic. We will roughly use the contents of [11] as a standard for what knowledge will be assumed. Before starting, also quickly note that throughout the thesis we will use the notation $\langle ., . \rangle$ for the metric g of a semi-Riemannian manifold and we will write $\|X\| := \sqrt{|\langle X, X \rangle|}$. For the Riemann curvature tensor we will use the convention $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$.

1.1 Integration on Semi-Riemannian Manifolds

Fittingly for a thesis on volumes and areas we will start by recalling some basics about integration on semi-Riemannian manifolds as well as proving a few important results needed later. We will mostly follow chapter 16 of [10] and chapter 5 of [13]

Recall that at a much more basic level, namely in $\mathbb{R}^n$, we can measure the volume of the parallelepiped spanned by n vectors by computing the determinant of the linear map that maps the standard basis onto these vectors. Taking this as a motivation we say that on a general finite dimensional vector space a function describes a volume if it satisfies a certain equality involving the determinant.

Definition 1.1.1. Let V be an n-dimensional vector space. We call a map $\mu : V^n \to \mathbb{R}$ a density if for all $v_1, \dots, v_n \in V$ and for all $A \in \mathrm{End}(V)$ we have:

$$\mu(Av_1, \dots, Av_n) = |\det(A)| \cdot \mu(v_1, \dots, v_n) \tag{1.1}$$

We write $|\Lambda^n|V^*$ for the set of all densities on V.

As this notation suggests there is a strong connection between densities and n-forms on a vector space. However we will mostly stick to densities with the exception of noting that if ω is an n-form then $|\omega|$ is a density since n-forms satisfy (1.1) without the absolute value.

Remark 1.1.2. (i) It is easy to see that if $\mu \in |\Lambda^n|V^*$ and $v_1, \dots, v_n$ are linearly dependent then $\mu(v_1, \dots, v_n) = 0$.

To see this, choose a maximal subset of $\{v_1, \dots, v_n\}$ that is linearly independent (without loss of generality that set is $\{v_1, \dots, v_k\}$), add linearly independent $w_{k+1}, \dots, w_n$ to make it a basis and express $v_{k+1}, \dots, v_n$ in this basis. The linear map A mapping the first k basis vectors to itself and w_i to v_i for $i \geq k$ has determinant 0. Inserting this into (1.1) gives the result.

(ii) Densities are not multilinear, but certain expressions still behave similarly. For example:
$$\mu\left(v_1,\dots,v_{n-1},\sum_{i=1}^n v_i\right)=\mu(v_1,\dots,v_{n-1},v_n) \tag{1.2}$$
If $v_1,\dots,v_n$ are linearly dependent this holds since by (i) both sides become 0. So let $v_1,\dots,v_n$ be linearly independent vectors of the n-dimensional vector space V and hence a basis. We consider the linear map, mapping:
$$v_1\mapsto v_1,\dots,v_{n-1}\mapsto v_{n-1},v_n\mapsto\sum_{i=1}^n v_i$$
By writing this map as a matrix with respect to the basis $v_1,\dots,v_n$ we clearly see that it has determinant 1 so (1.1) immediately gives (1.2).

(iii) An elementary calculation shows that $|\Lambda^n|V^*$ is a one-dimensional vector space. Clearly the function $|\det|$ itself is a density, so every density on V is just a multiple of the absolute value of the determinant. This provides us with a different approach to see (i) and (ii)

(iv) If $A:W\to V$ is a linear map then we can define the pullback of a density $\mu\in|\Lambda^n|V^*$ via A as
$$(A^*\mu)(w_1,\dots,w_n)=\mu(Aw_1,\dots,Aw_n) \tag{1.3}$$
If A is an isomorphism, $A^*\mu$ is a density on W.

So far a density is only an object that makes sense on an individual tangent space of a manifold. We want to combine all densities on all tangent spaces into a vector bundle. So we define the set
$$|\Lambda^n|T^*M:=\bigsqcup_{p\in M}|\Lambda^n|T_pM^*$$
Together with the obvious bundle projection $|\Lambda^n|T^*M\to M$ this can clearly be turned into a vector bundle, called the density bundle. Sections of the density bundle are once again called densities.

If we have two charts (x,U) and (y,U) we can use the defining equation (1.1) pointwise to obtain
$$\mu\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right)=\left|\det\left(\left(\frac{\partial y^i}{\partial x^j}\right)_{ij}\right)\right|\cdot\mu\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n}\right), \tag{1.4}$$
where by $\det\left(\left(\frac{\partial y^i}{\partial x^j}\right)_{ij}\right)$ we mean the determinant of the tangent map of the chart transition $y\circ x^{-1}$.

If we are given a density on a manifold M we can now define an integral. We start off as simple as possible and only consider continuous functions f with compact support, i.e. $f \in \mathcal{C}_c(M)$, such that their support is contained in the chart neighborhood U of the chart (U, x). In this case the integral is defined as follows:

$$\int_U f \, d\mu = \int_{x(U)} \left(f \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x^{-1} d\lambda^n$$

With the help of a partition of unity we can extend this definition to all $f \in \mathcal{C}_c(M)$.

Definition 1.1.3. Let μ be a density on M, let $f \in \mathcal{C}_c(M)$ and let $\{(U_i, x_i)\}_{i \geq 1}$ a countable, locally finite atlas of M. If $\{\chi_i\}_{i \geq 1}$ is a subordinate partition of unity we define:

$$\int_M f \, d\mu := \sum_{i \geq 1} \int_{x_i(U_i)} \left(f \cdot \chi_i \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x_i^{-1} d\lambda^n \tag{1.5}$$

Proposition 1.1.4. *Expression* (1.5) *is finite and independent of the choices of atlas and partition of unity. Hence the integral is well-defined.*

Proof. To show finiteness first note that since $\{\operatorname{supp} \chi_i\}_{i \geq 1}$ is locally finite, only finitely many of them intersect the compact set $\operatorname{supp} f$. Hence the sum in the definition is a finite one. Also, each of the integrands is a continuous function with compact support in $\mathbb{R}^n$ hence each of the integrals is finite. Altogether the integral is finite. To show independence of the atlas and partition of unity choose another atlas $\{(V_j, y_j)\}_{j \geq 1}$ and a subordinate partition of unity $\{\eta_j\}_{j \geq 1}$. Then by a simple calculation:

$$\begin{aligned}
&\sum_{i \geq 1} \int_{x_i(U_i)} \left(f \cdot \chi_i \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x_i^{-1} d\lambda^n = \\
&= \sum_{i,j \geq 1} \int_{x_i(U_i)} \left(f \cdot \chi_i \cdot \eta_j \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x_i^{-1} d\lambda^n = \\
&= \sum_{i,j \geq 1} \int_{x_i(U_i \cap V_j)} \left(f \cdot \chi_i \cdot \eta_j \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x_i^{-1} d\lambda^n = \\
&= \sum_{i,j \geq 1} \int_{x_i(U_i \cap V_j)} \left(\left| \det \left(\left(\frac{\partial y^\alpha}{\partial x^\beta} \right)_{\alpha\beta} \right) \right| \cdot f \cdot \chi_i \cdot \eta_j \cdot \mu \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right) \right) \circ x_i^{-1} d\lambda^n = \\
&= \sum_{i,j \geq 1} \int_{y_j(U_i \cap V_j)} \left(f \cdot \chi_i \cdot \eta_j \cdot \mu \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right) \right) \circ y_j^{-1} d\lambda^n = \\
&= \sum_{j \geq 1} \int_{y_j(V_j)} \left(f \cdot \eta_j \cdot \mu \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right) \right) \circ y_j^{-1} d\lambda^n
\end{aligned}$$

This proves the Proposition. □

Now if $\Phi : N \to M$ is a diffeomorphism and we are given a density μ on M, then we can pull back μ along Φ. Since the tangent map of a diffeomorphism at any point is a linear isomorphism, analogous to Remark 1.1.2(iv), this pullback gives a density on N. Moreover there is a nice formula to calculate the integral with respect to the pullback of a density.

Proposition 1.1.5. *Let μ be a density on M, let $\Phi : M \to N$ be a diffeomorphism and let $f \in \mathcal{C}_c(N)$. Then*

$$\int_N f \, d\Phi^* \mu = \int_M (f \circ \Phi^{-1}) d\mu \tag{1.6}$$

Proof. Without loss of generality assume that the support of f lies in the chart domain of (x, U). Note that since Φ is a diffeomorphism, $(y, V) = (x \circ \Phi^{-1}, \Phi(U))$ is a chart of M. Hence:

$$\begin{aligned}
\int_N f \, d\Phi^* \mu &= \int_{x(U)} \left(f \cdot \Phi^* \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x^{-1} d\lambda^n = \\
&= \int_{x(U)} \left(f \cdot \mu \left(T\Phi \left(\frac{\partial}{\partial x^1} \right), \ldots, T\Phi \left(\frac{\partial}{\partial x^n} \right) \right) \right) \circ x^{-1} d\lambda^n = \\
&= \int_{x(U)} \left(|\det T\Phi| \cdot f \cdot \mu \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \right) \circ x^{-1} d\lambda^n = \\
&= \int_{y(V)} \left(f \circ \Phi^{-1} \cdot \mu \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right) \right) \circ y^{-1} d\lambda^n = \int_M \left(f \circ \Phi^{-1} \right) d\mu
\end{aligned}$$

□

While in this way we were able to define an integral for certain functions, so far we have not yet made any measure theoretic considerations. To go from our integral for continuous functions with compact support to an integral in the measure theoretic sense we will need the help of the following central theorem from measure theory:

Theorem 1.1.6 (Riesz)**.** *Let X be a locally compact Hausdorff space and $I : \mathcal{C}_c(X) \to \mathbb{R}$ a positive linear functional.*

Then there exists a unique Radon measure on the Borel σ-algebra $\mathcal{B}(X)$, such that:

$$I(f) = \int_X f \, d\mu \qquad (f \in \mathcal{C}_c(X))$$

For a proof of this theorem see [3, chapter 8, §2].

We note that if μ is a positive density, i.e. a density only taking positive values, then it defines a positive linear functional on the locally compact Hausdorff space M. So we can immediately apply Theorem 1.1.6 and see the following.

Theorem 1.1.7. *Let μ a positive density on M. Then there exists a unique Radon measure on $\mathcal{B}(M)$, which we will again denote by μ, such that for $f \in \mathcal{C}_c(M)$ the integral is given by expression* (1.5)

Note that for any function in $L^1(M, d\mu)$ its integral can still be expressed with the defining formula (1.5). As a consequence the pullback formula (1.6) also still holds for integrable functions $f \in L^1(N, d\Phi^*\mu)$.

Now that we are on a measure theoretically sound footing, we turn our attention to semi-Riemannian manifolds. With the help of the metric we want to find a density that captures what we think of when we talk of volumes or areas of manifolds. Recall from elementary analysis that if $M \in \mathbb{R}^n$ is an $(n-1)$-dimensional hypersurface with global parametrization $\phi : \mathbb{R}^{n-1} \supset U \to \mathbb{R}^n$, we calculate the surface area with the formula

$$\int_M d\phi := \int_U \phi \cdot \sqrt{G(D\phi)}\, d\lambda^{n-1},$$

where $G(D\phi) = \det(\langle \partial_i \phi, \partial_j \phi \rangle_{ij})$ is the Gram Determinant.

Definition 1.1.8. Let (M, g) be a semi-Riemannian manifold and let $\{(x_i, U_i)\}_{i \in I}$ be an atlas. Then on each chart neighborhood we define the semi-Riemannian density:

$$\mu_g\big|_{U_i} := \sqrt{|\det(g_{ij})|} \cdot |dx^1 \wedge \ldots \wedge dx^n| \tag{1.7}$$

By a similar calculation as before we can see that this definition is independent of the atlas and the object hence is well defined. Also, as we have mentioned before, the absolute value of an n-form is a density, so this definition really does give a density. So finally by Theorem 1.1.7 this gives us a Radon measure on M, which we will denote by μ_g and call the semi-Riemannian measure of (M, g)

Remark 1.1.9. The semi-Riemannian density maps any orthonormal basis to 1. To see this at some point $p \in M$ choose Riemannian Normal Coordinates with respect to the orthonormal basis. Then $g_{ij}\big|_p = \delta_{ij}$ and of course $|dx^1 \wedge \ldots \wedge dx^n| \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = 1$.

As the final result of this section we want to prove a semi-Riemannian Fubini Theorem. The idea will be as follows:

If a function $f : (M, g) \to (N, h)$ is a surjective semi-Riemannian submersion, then the preimage of each $q \in N$ is a semi-Riemannian submanifold of M. Moreover, we have $M = \bigcup_{q \in N} f^{-1}(q)$ and so, recalling the classical Fubini theorem from measure theory, one might hope that we can calculate an integral over M by first integrating over each $f^{-1}(q)$ and then integrating the results over N.

Before proving this we first introduce some notation and make some observations.

First recall that a semi-Riemannian submersion is a submersion f such that each fiber $f^{-1}(q)$ (which we will denote as N_q from now on) is a semi-Riemannian submanifold and the map $T_p f\big|_{T_p N_q^\perp} : T_p N_q^\perp \to T_{f(p)} N$ is an isometry. We will write μ_q for the semi-Riemannian measure on N_q

We now set $\mathcal{V} := \bigsqcup_{q\in N} TN_q \subset TM$ and due to nondegeneracy of N_q can define $\mathcal{H} := \mathcal{V}^\perp$. Then $TM = \mathcal{V} \oplus \mathcal{H}$ and due to the isometry property of semi-Riemannian submersions $Tf\big|_{\mathcal{H}} : \mathcal{H} \to TN$ is an isometry. Restricting the metric g to either $\mathcal{V}$ or $\mathcal{H}$ gives us another "metric" in the sense that at each point the metric is still a nondegenerate scalar product. We can hence define induced semi-Riemannian densities $\mu_\mathcal{V}$ and $\mu_\mathcal{H}$ on $\mathcal{V}$ and $\mathcal{H}$. If $X_1, \dots, X_n$ are smooth sections of $\mathcal{V}$ and $X_{n+1}, \dots, X_m$ are smooth sections of $\mathcal{H}$, then we clearly get the formula:

$$\mu_g(X_1, \dots, X_m) = \mu_\mathcal{V}(X_1, \dots, X_n) \cdot \mu_\mathcal{H}(X_{n+1}, \dots, X_m)$$

We now have the necessary tools to prove the following:

Theorem 1.1.10 (Fubini)**.** *Let $f : (M, g) \to (N, h)$ be a surjective semi-Riemannian submersion and $\varphi \in L^1(M, d\mu_g)$. Then $\varphi\big|_{N_q} \in L^1(N_q, d\mu_q)$ for almost every $q \in N$, $\int_{N_q} \varphi\big|_{N_q} d\mu_q$ as a function of q is in $L^1(N, d\mu_h)$ and*

$$\int_M \varphi \, d\mu_g = \int_N \int_{N_q} \varphi\big|_{N_q} d\mu_q \, d\mu_h$$

Proof. Let $\dim M = m$, $\dim N = n$ and $p \in M$. Since f is a submersion, by the rank theorem there exist charts (x, U) and (y, V) around p and $f(p)$ respectively, such that: $x(U) = (-\varepsilon, \varepsilon)^m$, $y(V) = (-\varepsilon, \varepsilon)^n$, $f(U) = V$ and $y \circ f \circ x^{-1} = pr_m$. We get

$$Tf\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \qquad\qquad (i \leq n) \tag{1.8}$$

and

$$Tf\left(\frac{\partial}{\partial x^i}\right) = 0. \qquad\qquad (i \geq n+1) \tag{1.9}$$

Equation (1.9) tells us that $\frac{\partial}{\partial x^{n+1}}, \dots, \frac{\partial}{\partial x^m}$ span $\ker Tf = \mathcal{V} \cap TU$. On the other hand $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ span TV. Since Tf is an isometry between $\mathcal{H}$ and TN there are vector fields $\xi_i \in \mathfrak{X}(U)$, obtained as the preimage of $\frac{\partial}{\partial y^i}$ under $Tf\big|_{\mathcal{H}}$, which span $\mathcal{H} \cap TV$. By equation (1.8) these vector fields must be of the form:

$$\xi_i = \frac{\partial}{\partial x^i} + \sum_{j=n+1}^{m} \alpha_i^j \frac{\partial}{\partial x^j}$$

The transition matrix for the change of basis from $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}$ to $\xi_1, \dots, \xi_n, \frac{\partial}{\partial x^{n+1}}, \dots, \frac{\partial}{\partial x^m}$ is:

$$A = \left(\begin{array}{c|c} I_n & \mathbf{0} \\ \hline (\alpha_i^j)_{ji} & I_m \end{array}\right) \tag{1.10}$$

Now we can calculate:

$$\mu_g\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}\right) = \overbrace{|\det(A)|}^{1}\cdot\mu_g\left(\xi_1,\dots,\xi_n,\frac{\partial}{\partial x^{n+1}},\dots,\frac{\partial}{\partial x^m}\right) =$$
$$= \mu_{\mathcal{H}}(\xi_1,\dots,\xi_n)\cdot\mu_{\mathcal{V}}\left(\frac{\partial}{\partial x^{n+1}},\dots,\frac{\partial}{\partial x^m}\right) =$$
$$= \left(\mu_h\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n}\right)\circ f\right)\cdot\mu_{\mathcal{V}}\left(\frac{\partial}{\partial x^{n+1}},\dots,\frac{\partial}{\partial x^m}\right)$$

Finally, we can put this into our integral formula.

$$\int_U \varphi\, d\mu_g = \int_{x(U)}\left(\varphi\cdot\mu_g\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}\right)\right)\circ x^{-1} d\lambda^m =$$
$$= \int_{x(U)}\left(\varphi\cdot\left(\mu_h\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n}\right)\circ f\right)\cdot\mu_{\mathcal{V}}\left(\frac{\partial}{\partial x^{n+1}},\dots,\frac{\partial}{\partial x^m}\right)\right)\circ x^{-1} d\lambda^m$$

Now all results follow from the classical Fubini Theorem for $\mathbb{R}^m$ $\square$

1.2 The First Variation of Area

In this section we will examine how the area of a semi-Riemannian submanifold $N \subset M$ changes when it is transported along a vector field $X \in \mathfrak{X}(M)$. The idea is that for small t the image $\mathrm{Fl}_t^X(N)$ under the flow of X should still be a semi-Riemannian submanifold with new semi-Riemannian density μ_t. So, after interchanging differentiation and integration, computing the change of area of N should come down to computing a time-derivative of μ_t.

Looking at how each μ_t is defined (as in equation (1.7)), to compute such a derivative we first need some technical results about time-derivatives of determinants and the metric. We follow [2].

Lemma 1.2.1. *Let $A : (-\varepsilon,\varepsilon) \to \mathbb{R}^{n\times n}$ be a smooth matrix-valued curve such that $A(0) = I_n$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \operatorname{tr} A'(0)$$

Proof. First note that around $t = 0$ we can write $A(t) = A(0) + tA'(0) + \mathcal{O}(t^2)$. So we can easily see that $\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \left.\frac{d}{dt}\right|_{t=0} \det(A(0) + tA'(0))$. Also, a simple calculation with the characteristic polynomial and eigenvalues $\lambda_1, \dots, \lambda_n$ of $A'(0)$ yields

$$\begin{aligned}\det(A(0) + tA'(0)) &= \det(I_n + tA'(0)) = \\ &= (-t)^n \det\left(-\frac{1}{t}I_n - A'(0)\right) = \\ &= (-t)^n \left(-\frac{1}{t} - \lambda_1\right) \cdots \left(-\frac{1}{t} - \lambda_n\right) = \\ &= 1 + t(\underbrace{\lambda_1 + \cdots + \lambda_n}_{\operatorname{tr}(A'(0))}) + c_2 t^2 + \cdots + c_n t^n\end{aligned}$$

Combining these two facts we get

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA'(0)) = \operatorname{tr}(A'(0))$$

□

Lemma 1.2.2. *Let A, B and X be three vector fields on (M, g). Then the Lie derivative of the metric g satisfies the following formula:*

$$(L_X g)(A, B) = \langle \nabla_A X, B\rangle + \langle \nabla_B X, A\rangle$$

Proof. The Lie derivative is a tensor derivative which satisfies a product rule. So we can compute

$$\begin{aligned}(L_X g)(A, B) &= X\langle A, B\rangle - \langle L_X A, B\rangle - \langle L_X B, A\rangle = \\ &= \langle \nabla_X A, B\rangle + \langle \nabla_X B, A\rangle - \langle [X, A], B\rangle - \langle [X, B], A\rangle = \\ &= \langle \nabla_X A - \underbrace{[X, A]}_{\nabla_X A - \nabla_A X}, B\rangle + \langle \nabla_X B - \underbrace{[X, B]}_{\nabla_X B - \nabla_B X}, A\rangle = \\ &= \langle \nabla_A X, B\rangle + \langle \nabla_B X, A\rangle\end{aligned}$$

□

Before using the last two results to compute a time-derivative of μ_s, quickly recall that for any n-dimensional semi-Riemannian submanifold $N \subset M$ and vector field $X \in \mathfrak{X}(M)$, we have an induced divergence locally given via any orthonormal frame $(E_i)_{i=1}^n$ of N as

$$\operatorname{div}_N(X) = \sum_{i=1}^n \epsilon_i \langle \nabla_{E_i} X, E_i\rangle, \tag{1.11}$$

where $(\epsilon_i)_{i=1}^n$ is the signature of $(E_i)_{i=1}^n$. Going forward we will write ϵ for $\epsilon_1 \cdots \epsilon_n$. Note that of course both $\operatorname{div}_N(X)$ and ϵ are independent of the choice of orthonormal basis.

Proposition 1.2.3. *Let $N \subset M$ be a semi-Riemannian submanifold and let $X \in \mathfrak{X}(M)$ be a vector field such that $\mathrm{Fl}_t^X(N) =: N_t$ is a semi-Riemannian submanifold with metric g^t and semi-Riemannian density μ_t for $t \in (-\varepsilon, \varepsilon)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} (\mathrm{Fl}_t^X)^* \mu_t = \epsilon \operatorname{div}_N(X) \cdot \mu_0 \tag{1.12}$$

Proof. Denote by g^t the metric on N_t and by μ_t the corresponding semi-Riemannian density. For ease of notation we write $\tilde{\mu_t}$ for $(\mathrm{Fl}_t^X)^*\mu_t$ and $\tilde{g}^t$ for $(\mathrm{Fl}_x^X)^*g^t$. First note that the equality (1.12) is pointwise, so we only need to prove this for an arbitrary $p \in N$. Choose a chart (x, U) of N around p such that $\mathrm{Fl}_t^X(p) \in U$ and let $X_1, \ldots, X_n \in T_pN$. Then:

$$\begin{aligned}
\tilde{\mu_t}(X_1, \ldots X_n) &= \mu_t(T_p \mathrm{Fl}_t^X X_1, \ldots, T_p \mathrm{Fl}_t^X X_n) = \\
&= \sqrt{|\det(g_{ij}^t)|} \cdot |dx^1 \wedge \ldots \wedge dx^n|(T_p \mathrm{Fl}_t^X X_1, \ldots, T_p \mathrm{Fl}_t^X X_n) = \\
&= |\det(T_p \mathrm{Fl}_t^X)| \cdot \sqrt{|\det(g_{ij}^t)|} \cdot |dx^1 \wedge \ldots \wedge dx^n|(X_1, \ldots, X_n) = \\
&= \sqrt{|\det(\tilde{g}_{ij}^t)|} \cdot |dx^1 \wedge \ldots \wedge dx^n|(X_1, \ldots, X_n),
\end{aligned}$$

where to see the final equality one simply has to calculate $\tilde{g}_{ij}^t$. Altogether we see that

$$\tilde{\mu_t} = \sqrt{|\det(\tilde{g}_{ij}^t)|} \cdot |dx^1 \wedge \ldots \wedge dx^n|$$

Without loss of generality let $(x^1, \ldots, x^n)$ be normal coordinates of N. Then we get $\tilde{g}_{ij}^0 = \epsilon_i \delta_{ij}$ and hence $\det(\tilde{g}_{ij}^0) = \epsilon_1 \cdots \epsilon_n = \epsilon$. Note that by continuity and nondegeneracy of $\tilde{g}_{ij}^t$ we have $\mathrm{sgn}(\det(\tilde{g}_{ij}^t)) = \epsilon$. We use all of this to calculate:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \tilde{\mu_t} &= \overbrace{\frac{1}{2\sqrt{|\det(\tilde{g}_{ij}^0)|}}}^{\frac{1}{2}} \cdot \overbrace{\mathrm{sgn}(\det(\tilde{g}_{ij}^t))}^{\epsilon} \cdot \overbrace{\frac{d}{dt}\Big|_{t=0} \det(\tilde{g}_{ij}^t)}^{\mathrm{tr}\left(\frac{d}{dt}\big|_{t=0}\tilde{g}_{ij}^t\right)} \cdot \overbrace{|dx^1 \wedge \ldots \wedge dx^n|}^{\tilde{\mu_0}} = \\
&= \frac{\epsilon}{2} \cdot \mathrm{tr}\,\left(\frac{d}{dt}\Big|_{t=0} \tilde{g}_{ij}^t\right) \cdot \tilde{\mu_0}
\end{aligned} \tag{1.13}$$

Here we used Lemma 1.2.1. Next note that $\frac{d}{dt}\big|_{t=0}\tilde{g}^t = L_X g\big|_N$, so after choosing an orthonormal frame $(E_i)_{i=1}^n$ we see:

$$\begin{aligned}
\mathrm{tr}\left(\frac{d}{dt}\Big|_{t=0} \tilde{g}_{ij}^t\right) = \mathrm{tr}(L_X g\big|_N) &= \sum_{i=1}^n \epsilon_i (L_X g)(E_i, E_i) = \\
&= 2\sum_{i=1}^n \epsilon_i \langle \nabla_{E_i} X, E_i \rangle = 2\ \mathrm{div}_N X
\end{aligned} \tag{1.14}$$

Here the penultimate equality follows from Lemma 1.2.2. Combining (1.13) and (1.14) we finally get:

$$\frac{d}{dt}\Big|_{t=0} \tilde{\mu_t} = \epsilon \, \mathrm{div}_N X \cdot \mu_0$$

□

Before proving the First Variation of Area formula, quickly recall that by area K, where K is a compact subset of the semi-Riemannian submanifold $N \subset M$, we mean the integral $\int_K d\mu_N$. We will sometimes also write vol K instead of area K when the dimension of N is equal to that of M.

Theorem 1.2.4 (First Variation of Area). *Let $N \subset M$ be a semi-Riemannian submanifold and let $X \in \mathfrak{X}(M)$ be a vector field such that $\mathrm{Fl}_t^X(N) =: N_t$ is a semi-Riemannian submanifold for $t \in (-\varepsilon, \varepsilon)$. Let $K \in N$ be compact and set $K_t := \mathrm{Fl}_t^X(K)$. Then*

$$\frac{d}{dt}\bigg|_{t=0} \operatorname{area} K_t = \int_K \epsilon \operatorname{div}_N X d\mu_0$$

Proof. First note that $\mathrm{Fl}_t^X\big|_N : N \to N_t$ is bijective with inverse function $\mathrm{Fl}_{-t}^X\big|_{N_t}$. Since for fixed t the flow is smooth, the restriction to a submanifold is smooth, so both the aforementioned functions are smooth and hence diffeomorphisms. This means we will be able to use formula (1.6) for the pullback of a density:

$$\operatorname{area} K_t = \int_{K_t} d\mu_t = \int_K d(\mathrm{Fl}_t^X)^* \mu_t$$

We pick a locally finite open cover $\{(U_i, x_i)\}_{i \in I}$ of N and a subordinate partition of unity $\{\chi_i\}_{i \in I}$. Then we can rewrite the integral:

$$\operatorname{area} K_t = \sum_{i \in I} \int_{x_i(U_i \cap K)} \left(\chi_i \cdot ((\mathrm{Fl}_t^X)^* \mu_t) \left(\frac{\partial}{\partial x_i^1}, \dots, \frac{\partial}{\partial x_i^n} \right) \right) \circ x_i^{-1} d\lambda^n$$

We need to differentiate this term with respect to t. Clearly if we can interchange the order of differentiation and integration we can apply Proposition 1.2.3 and we are done.

Let us call the function in the integrand $f(t, .)$ defined on $(-\varepsilon, \varepsilon) \times U$. Choose any sequence $h_n \to 0$, then we define $f_n(.) := \frac{f(h_n,.) - f(0,.)}{h_n}$. Clearly $f_n(.)$ converges to $\frac{d}{dt}\big|_{t=0} f(t, .)$ pointwise, so if we can find a dominating function for f_n we can use the dominated convergence Theorem to interchange the order of differentiation and integration. This would finish our proof.

By the mean value theorem we can find a function $\theta_n(x)$ such that $|\theta_n| \leq |h_n|$ and

$$f_n(x) = \frac{d}{dt}\big|_{t=0} f(x, t + \theta_n(x))$$

Hence $f_n(x) \leq \sup_{t \in (-\varepsilon, \varepsilon)} \frac{d}{dt} f(x, t)$. This inequality still holds if we restrict t to a small enough compact subinterval $[-\varepsilon_1, \varepsilon_1]$ (we can just assume that all h_n are smaller than ε_1). Now it is easy to see that $\sup_{t \in [-\varepsilon_1, \varepsilon_1]} \frac{d}{dt} f(., t)$ is bounded and in particular integrable meaning it can serve as our desired dominating function. $\square$

1.3 Timelike Distance Functions

In this section (M, g) will always denote a Lorentzian manifold with time orientation, i.e. a spacetime.

We will examine functions describing how much time has passed between points and sets in a spacetime. We interpret their level sets as "time slices" evolving along the gradient field of the function. This will be the first setting to which we apply the results from the previous two sections. We follow [15].

Definition 1.3.1. A function $\tau \in \mathcal{C}^\infty(M)$ is called a timelike distance function if it satisfies:

$$\langle \operatorname{grad} \tau, \operatorname{grad} \tau \rangle = -1 \tag{1.15}$$

Without loss of generality we will always assume that the gradient of a timelike distance function τ is past directed. If it wasn't then the gradient of the timelike distance function $-\tau$ would be past directed.

We will now prove the first simple observations:

Proposition 1.3.2. *Let $\tau \in \mathcal{C}^\infty(M)$ be a timelike distance function. Then:*

$$\nabla_{\operatorname{grad} \tau} \operatorname{grad} \tau = 0 \tag{1.16}$$

In particular, integral curves of $\operatorname{grad} \tau$ *are past directed, timelike geodesics.*

Proof. Let $X \in \mathfrak{X}(M)$. If we can show $\langle \nabla_{\operatorname{grad} \tau} \operatorname{grad} \tau, X \rangle = 0$ then we are done by the nondegeneracy of the metric. So we calculate:

$$\begin{aligned}\langle \nabla_{\operatorname{grad} \tau} \operatorname{grad} \tau, X \rangle &= \operatorname{Hess}(\tau)(\operatorname{grad} \tau, X) = \operatorname{Hess}(\tau)(X, \operatorname{grad} \tau) = \\ &= \langle \nabla_X \operatorname{grad} \tau, \operatorname{grad} \tau \rangle = \frac{1}{2} X \langle \operatorname{grad} \tau, \operatorname{grad} \tau \rangle = 0,\end{aligned}$$

where in the second equality we used the symmetry of the Hessian and in the final equality we used that the tangential map of a constant function like $\langle \operatorname{grad} \tau, \operatorname{grad} \tau \rangle \equiv -1$ is always 0.

The second claim immediately follows from the first. □

Proposition 1.3.3. *Let $\tau \in \mathcal{C}^\infty(M)$ be a timelike distance function.*

(i) The level sets $\mathcal{S}_t := \tau^{-1}(t)$ are spacelike hypersurfaces with unit normal $\mathrm{n} := -\operatorname{grad} \tau$*.*

(ii) τ is a semi-Riemannian submersion.

Proof. By equation (1.15) the tangent map of the timelike distance function τ is nonzero, hence it is a submersion into the 1-dimensional manifold $\mathbb{R}$. As a result the level sets $\mathcal{S}_t$ are hypersurfaces of M.

Next note that for any $p \in \mathcal{S}_t$ and vector $v \in T_p\mathcal{S}_t$ we have:

$$\langle v, \mathrm{n}\big|_p\rangle = \langle v, \operatorname{grad}\tau\big|_p\rangle = d\tau\big|_p(v) = 0$$

so indeed n is normal to the level sets $\mathcal{S}_t$ and since by (1.15) $\|\mathrm{n}\| = 1$, it is in fact a unit normal. This means that $\mathcal{S}_t$ is normal to a timelike vector field and hence spacelike. This proves (i).

To prove (ii) the only thing left to do is showing that $T_p\tau\big|_{T_p\mathcal{S}_t^\perp} : T_p\mathcal{S}_t^\perp \to \mathbb{R}$ is an isometry (Here $\mathbb{R}$ is equipped with the metric $g_0(a,b) = -(a \cdot b)$). This is true since

$$g_0(d\tau(\mathrm{n}), d\tau(\mathrm{n})) = -d\tau(\mathrm{n})^2 = -\langle -\mathrm{n}, \mathrm{n}\rangle^2 = -1 = \langle \mathrm{n}, \mathrm{n}\rangle.$$

Because the spaces are one dimensional, proving this equality for n already finishes the proof. □

Remark 1.3.4. Since we assumed without loss of generality that $\operatorname{grad}\tau$ is always past directed we now have that $\mathrm{n} = -\operatorname{grad}\tau$ is always future directed.

Proposition 1.3.5. *Let $\tau \in \mathcal{C}^\infty(M)$ be a timelike distance function and consider the flow $\mathrm{Fl}^{\mathrm{n}} : \mathcal{U} \subset \mathbb{R} \times M \to M$. Let $p \in \mathcal{S}_t$ and $s \in \mathbb{R}$ such that $(s,p) \in \mathcal{U}$. Then we have*

$$\mathrm{Fl}^{\mathrm{n}}_s(p) \in \mathcal{S}_{t+s}$$

Proof. Making a simple calculation using the chain rule we see:

$$\frac{d}{ds}\tau(\mathrm{Fl}^{\mathrm{n}}_s(p)) = d_{\mathrm{Fl}^{\mathrm{n}}_s(p)}\tau(\mathrm{n}\big|_{\mathrm{Fl}^{\mathrm{n}}_s(p)}) = \langle -\mathrm{n}, \mathrm{n}\rangle\big|_{\mathrm{Fl}^{\mathrm{n}}_s(p)} = 1$$

It follows that $\tau(\mathrm{Fl}^{\mathrm{n}}_s(p)) = s + t$ and hence $\mathrm{Fl}^{\mathrm{n}}_s(p) \in \mathcal{S}_{t+s}$ □

As mentioned at the beginning of this section our goal will be to transport sets $K \subset \mathcal{S}_t$ along the flow of a vector field "into the future" and compare how the area of these sets changes in the process. Now on the one hand we think of the level set $\mathcal{S}_{t+s}$ as the future of $\mathcal{S}_t$ after time s, but on the other hand the obvious choice of vector field along which to transport K is n. Proposition 1.3.5 makes sure these two pictures fit together. If we flow along the geodesics normal to $\mathcal{S}_t$ for time s then $\mathcal{S}_{t+s}$ is really where we end up.

Definition 1.3.6. Let $\mathcal{S}_t$ be a level set of a timelike distance function with unit normal n. Then we define the shape operator $S_t \in \mathcal{T}^1_1(\mathcal{S}_t)$ as follows:

$$S_t(v) := \nabla_v \mathrm{n} \qquad\qquad (v \in T\mathcal{S}_t)$$

We also define the mean curvature $H_t \in \mathcal{C}^\infty(N)$ as $H_t := \operatorname{tr}(S_t)$

Remark 1.3.7. (i) The shape operator can be defined in different ways. The way we did it is the most convenient for us, but from this definition it isn't immediately obvious that the shape operator at $p \in N$ is really a map from $T_p\mathcal{S}_t$ to itself. To see this let $v \in T_p\mathcal{S}_t$, then:

$$\langle S_t(v), \mathrm{n}\rangle = \langle \nabla_v \mathrm{n}, \mathrm{n}\rangle = \overbrace{v\langle \mathrm{n}, \mathrm{n}\rangle}^{0} - \langle \mathrm{n}, \nabla_v \mathrm{n}\rangle = -\langle S_t(v), \mathrm{n}\rangle$$

$$\Longrightarrow \langle S_t(v), \mathrm{n}\rangle = 0$$

$T_p\mathcal{S}_t$ is a nondegenerate codimension 1 subspace normal to n, hence $S_t(v) \in T_p\mathcal{S}_t$.

(ii) The shape operator is self adjoint. Indeed let $X, Y \in \mathfrak{X}(\mathcal{S}_t)$, then

$$\begin{aligned}
\langle S_t(X), Y\rangle &= \langle \nabla_X \mathrm{n}, Y\rangle = \\
&= X\langle \mathrm{n}, Y, \rangle - \langle \mathrm{n}, \nabla_X Y\rangle = \\
&= -\langle \mathrm{n}, [X, Y]\rangle + \langle \mathrm{n}, \nabla_Y X\rangle = \\
&= \cdots = \langle S_t(Y), X\rangle
\end{aligned}$$

Note that $\nabla_X Y$ does not describe the covariant derivative of the Riemannian manifold $\mathcal{S}_t$, but the induced connection as described in [11, Corollary 4.2]. Also note that $\langle \mathrm{n}, [X, Y]\rangle = 0$ because $[X, Y] \in \mathfrak{X}(\mathcal{S}_t)$ for $X, Y \in \mathfrak{X}(\mathcal{S}_t)$.

A crucial observation for us is that if we pick an orthonormal frame $(E_i)_{i=1}^{m-1}$ around some point of $\mathcal{S}_t$ and write down the mean curvature explicitly, then it is clearly just the induced divergence introduced in (1.11):

$$H_t = \mathrm{tr}(S_t) = \sum_{i=1}^{m-1} \langle S_t(E_i), E_i\rangle = \sum_{i=1}^{m-1} \langle \nabla_{E_i} \mathrm{n}, E_i\rangle = \mathrm{div}_{\mathcal{S}_t} \mathrm{n} \tag{1.17}$$

This allows us to formulate the first variation of area formula from Proposition 1.2.4 in this more specific setting. The semi-Riemannian submanifolds N_t become the level sets $\mathcal{S}_t$, the vector field X becomes the vector field n and instead of the induced divergence we integrate the mean curvature. Altogether we get:

Corollary 1.3.8. *Let τ be a timelike distance function, $t \in \mathrm{im}(\tau)$ and $K \subset \mathcal{S}_t$ compact such that Fl^n is defined on $[-\varepsilon, \varepsilon] \times K$ for some $\varepsilon > 0$. Calling $\mathrm{Fl}^n_s(K) =: K_s$ we get:*

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{area}\, K_s = \int_K H_t \, d\mu_t$$

Remark 1.3.9. Equation (1.17) can be extended a little further. At any point p of $\mathcal{S}_t$ an orthonormal basis of $T_p\mathcal{S}_t$ can be completed to a basis of T_pM by adding the vector $\mathrm{n} = -\operatorname{grad}\tau$ as the m-th basis vector E_m. Hence:

$$\begin{aligned}
H_t &= -\sum_{i=1}^{m-1} \langle \nabla_{E_i} \operatorname{grad}\tau, E_i\rangle = \\
&= -\sum_{i=1}^{m-1} \langle \nabla_{E_i} \operatorname{grad}\tau, E_i\rangle + \langle \underbrace{\nabla_{\operatorname{grad}\tau} \operatorname{grad}\tau}_{0}, \operatorname{grad}\tau\rangle = \\
&= -\sum_{i=1}^{m} \epsilon_i \langle \nabla_{E_i} \operatorname{grad}\tau, E_i\rangle = \\
&= -\operatorname{div}(\operatorname{grad}\tau) = -\Box\tau
\end{aligned}$$

Going back to Corollary 1.3.8 we clearly see that the area of a set in the future at time t_1 is determined by the curvature of the hypersurfaces $\mathcal{S}_s$ for $s < t_1$. So in order to gain an understanding about areas we need to gain an understanding of S_t and as a result H_t. As a first step in that direction we turn the family of objects S_t into one global object.

Definition 1.3.10. For $\mathrm{n} = -\operatorname{grad}\tau$, where τ is a timelike distance function, we define $S \in \mathcal{T}^1_1(M)$ via:

$$S(v) := \nabla_v \mathrm{n} \qquad\qquad (v \in TM)$$

Clearly the restriction of S to the set $T\mathcal{S}_t$ is just the shape operator S_t.

The crucial property of this globally defined endomorphism field S is that it satisfies a certain differential equation. Understanding solutions of this equation will tell us something about how S_t changes with respect to t.

Theorem 1.3.11. *Let S be defined as above, denote by S^2 the pointwise composition $S \circ S$ and let $R_\mathrm{n} \in \mathcal{T}^1_1(M)$ be the operator $R_\mathrm{n}(v) = R(v,\mathrm{n})\,\mathrm{n}$. Then:*

$$\nabla_\mathrm{n} S + S^2 + R_\mathrm{n} = 0 \tag{1.18}$$

Proof. It is enough to prove that the left hand side applied to any $X \in \mathfrak{X}(M)$ equals zero. So we calculate:

$$\begin{aligned}
(\nabla_\mathrm{n} S)(X) &= \nabla_\mathrm{n}(S(X)) - S(\nabla_\mathrm{n} X) = \\
&= \nabla_\mathrm{n}(\nabla_X \mathrm{n}) - S(\nabla_n X) = \\
&\overset{*}{=} \nabla_X(\nabla_\mathrm{n} \mathrm{n}) + R(\mathrm{n}, X)\,\mathrm{n} + \nabla_{[\mathrm{n},X]}\,\mathrm{n} - S(\nabla_\mathrm{n} X) = \\
&\overset{**}{=} -R(X,\mathrm{n})\,\mathrm{n} + S([\mathrm{n},X]) - S(\nabla_\mathrm{n} X) = \\
&= -R_\mathrm{n}(X) + S(\nabla_n X - \nabla_X \mathrm{n}) - S(\nabla_\mathrm{n} X) = \\
&= -R_\mathrm{n}(X) - S(\nabla_X \mathrm{n}) = \\
&= -R_\mathrm{n}(X) - S^2(X)
\end{aligned}$$

Note that $*$ follows from writing out the Riemann tensor explicitly and $**$ follows from $\nabla_{\mathrm{n}}\,\mathrm{n} = \nabla_{\operatorname{grad}\tau}\operatorname{grad}\tau = 0$ $\square$

Nonlinear first order differential equations of the form of (1.18) are called Riccati equations. There is a rich theory about these equations, and Theorem 1.3.11 allows us to apply that theory to our setting. Doing so will be the aim of our next section. Before that there is one more result we want to prove. We want to apply the semi-Riemannian Fubini Theorem 1.1.10 to integrate functions on M by first integrating on $\mathcal{S}_t$ and then integrating with respect to t.

Corollary 1.3.12 (Coarea formula)**.** *Let τ be a timelike distance function with level sets $\mathcal{S}_t$ and let $\varphi \in L^1(M, d\mu_g)$. Then $\varphi\big|_{\mathcal{S}_t} \in L^1(\mathcal{S}_t, d\mu_t)$ for almost all $t \in \operatorname{im}(\tau)$ and:*

$$\int_M \varphi \, d\mu_g = \int_{\mathbb{R}} \int_{\mathcal{S}_t} \varphi\big|_{\mathcal{S}_t} d\mu_t \, dt$$

Proof. This follows immediately from Theorem 1.1.10 applied to the function $\tau : M \to \mathbb{R}$, which is indeed a semi-Riemannian submersion by Proposition 1.3.3. $\square$

1.4 Riccati Comparison Theory

We return to equation (1.18), changing the viewpoint a bit. Each point p in M lies on an integral curve of the vector field n, so instead of looking at the entire operator S as defined in Definition 1.3.10 we look at one integral curve at a time and interpret S as a collection of tensor fields along curves. The covariant derivative ∇_{n} then simply becomes an induced covariant derivative on the integral curves γ_{n}. Informally we can reinterpret this again as a function $\mathbb{R} \supset I \to \operatorname{End}(E)$ (with E being an m-dimensional vector space), where instead of taking the induced covariant derivative we simply derive with respect to the scalar variable.

All of this will be made precise when the results from this section are actually applied. As of now this should simply be taken as motivation for examining differential equations of the form:

$$S' + S^2 + R = 0$$

with $S, R : I \to \operatorname{End}(E)$.

In this section E will always be an n-dimensional vector space with positive definite inner product $\langle .\,,.\rangle$. Denote by $\operatorname{Sym}(E) \subset \operatorname{End}(E)$ the linear maps from E to itself which are self-adjoint with respect to $\langle .\,,.\rangle$. Recall that $\operatorname{Sym}(E)$ is partially ordered by the relation:

$$A \geq B :\Leftrightarrow \langle (A-B)v, v\rangle \geq 0 \qquad \forall v \in E$$

For curves $A, B : I \to \operatorname{Sym}(E)$ by $A \geq B$ we mean that $A(t) \geq B(t)$ for all $t \in I$. By saying that such a curve A is bounded from below we mean that there exists a $c \in \mathbb{R}$ such that $A \geq c \cdot \mathrm{id}_E$.

The following central Theorem and its proof are taken from [4].

Theorem 1.4.1. *Let* $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}(E)$ *be smooth with* $R_1 \geq R_2$*. For* $i = 1, 2$ *assume that* $S_i : (0, t_i) \to \mathrm{Sym}(E)$ *are solutions of*

$$S_i' + S_i^2 + R_i = 0 \tag{1.19}$$

such that S_2 *cannot be extended beyond* $t_2 \in (0, \infty]$*. Let* $U := S_2 - S_1$*. If* U *can be continuously extended to* 0 *such that* $U(0) \geq 0$ *then the following hold:*

(i) $t_1 \leq t_2$ *and* $S_1 \leq S_2$ *on* $(0, t_1)$

(ii) The function $t \mapsto \dim \ker U(t)$ *is monotonously decreasing on* $(0, t_1)$

(iii) If there exists a $t \in (0, t_1)$ *such that* $S_1(t) = S_2(t)$ *then* $S_1 = S_2$ *and* $R_1 = R_2$ *on all of* $(0, t]$.

Proof. Let $t_0 = \min\{t_1, t_2\}$. Differentiating U and using the Riccati equation (1.19) we see:

$$U' = S_2' - S_1' = -S_2^2 - R_2 + S_1^2 + R_1 = XU + UX + Y,$$

where $X := -\frac{1}{2}(S_2 + S_1)$ and $Y := R_1 - R_2 \geq 0$. We can interpret this as U being a solution of the differential equation

$$U' = X \cdot U + U \cdot X + Y \tag{1.20}$$

Again by equation (1.19) we see that $-S_i' = S_i^2 + R_i \geq R_i$, where we used that any self-adjoint map squared is positive semidefinite. So if we pick any $s_0 \in (0, t_0)$ then for $t \in (0, s_0]$ we can write:

$$S_i(t) = S_i(s_0) - \int_0^{s_0 - t} S_i'(s_0 - s)\, ds \geq S_i(s_0) + \int_0^{s_0 - t} R_i(s_0 - s)\, ds$$

R_i is continuous and defined on all of $\mathbb{R}$, so it is bounded on $(0, s_0]$, meaning that S_i is bounded from below on the same interval. Hence by its definition X is bounded from above on some interval $(0, \varepsilon]$ meaning there exists a $c \in \mathbb{R}$ such that:

$$X(t) \leq c \cdot \mathrm{id}_E \qquad (t \in (0, \varepsilon]) \tag{1.21}$$

Next we examine nonsingular solutions $G : (0, t_0) \to \mathrm{End}(E)$ of the differential equation:

$$G' = X \cdot G \tag{1.22}$$

We quickly show that if a solution of (1.22) is nonsingular at any $s_0 \in (0, t_0)$ then it is nonsingular on all of $(0, t_0)$. Let G be a solution of (1.22) such that $G(s_0)$ is invertible. Consider the initial value problem:

$$\tilde{G}' = -\tilde{G} \cdot X \qquad \tilde{G}(s_0) = G(s_0)^{-1}$$

Now $(G\tilde{G})' = G'\tilde{G} + G\tilde{G}' = XG\tilde{G} - G\tilde{G}X$. So $G\tilde{G}$ is the solution of the initial value problem

$$A' = X \cdot A - A \cdot X \qquad\qquad A(s_0) = \mathrm{id}_E$$

But id_E is clearly also a solution so $G\tilde{G} = \mathrm{id}_E$, implying G is nonsingular on all of $(0, t_0)$, as desired. So let $G : (0, t_0) \to \mathrm{End}(E)$ be a solution of (1.22), which we can assume to be nonsingular. Then we can consider yet another initial value problem:

$$V' = G^{-1} \cdot Y \cdot (G^{-1})^T \qquad\qquad V(s_0) = G^{-1}(s_0) \cdot U(s_0) \cdot G(s_0)$$

Note that Y is self-adjoint, so V' is self-adjoint as well. We want to show that $U = GVG^T$. Differentiating yields:

$$\begin{aligned}(GVG^T)' &= G'VG^T + GV'G^T + GV(G^T)' = \\ &= XGVG^T + GG^{-1}Y(G^{-1})^T G^T + GVG^T X^T = \\ &= X(GVG^T) + Y + (GVG^T)X\end{aligned}$$

In the second equality we used that $(G^T)' = (G')^T$ and in the final one we used that X is self-adjoint. Altogether we see that GVG^T satisfies equation (1.20) and since the initial values of V were chosen specifically such that $U(s_0) = G(s_0) \cdot V(s_0) \cdot G(s_0)^T$ we indeed get $U = GVG^T$.

This implies two important things. Firstly, self-adjointness of U implies that V must be self-adjoint too. Secondly, we see that $U \geq 0$ if and only if $V \geq 0$. Since the former is part of the claim in (i) it will be our next aim to prove $V \geq 0$.

From $Y \geq 0$ it follows that $V' \geq 0$ since for all $v \in E$:

$$\langle V'v, v\rangle = \langle G^{-1}Y(G^{-1})^T v, v\rangle = \langle Y(G^{-1})^T v, (G^{-1})^T v\rangle \geq 0$$

Now we fix $v \in E$. We want to show that $\langle Vv, v\rangle$ can be continuously extended to 0 with this limit being nonnegative. We define $H := (G^{-1})^T$, then:

$$\begin{aligned}&\langle Vv, v\rangle = \langle G^{-1}U(G^{-1})^T v, v\rangle = \langle UHv, Hv\rangle \\ &\implies |\langle Vv, v\rangle| \leq \|U\| \cdot \|Hv\|^2\end{aligned}$$

We define $f := \|Hv\|^2$ and want to estimate its derivative. To this end we calculate:

$$\begin{aligned}&(G^{-1})' = -G^{-1}G'G^{-1} = -G^{-1}XGG^{-1} = -G^{-1}X \\ &\implies H' = ((G^{-1})^T)' = (-G^{-1}X)^T = -X^T(G^{-1})^T = -XH\end{aligned}$$

Hence on $(0, \varepsilon]$:

$$f' = 2\langle H'v, Hv\rangle = -2\langle XHv, Hv\rangle \geq -2c\,\langle Hv, Hv\rangle = -2cf,$$

where the inequality follows from equation (1.21). For the function $\tilde{f} := f(\varepsilon - .)$ we get the inequality

$$\tilde{f}'(t) = -f'(\varepsilon - t) \leq 2cf(\varepsilon - t) = 2c\tilde{f}(t),$$

for all $t \in [0, \varepsilon)$. By the Grönwall inequality [8, Chapter III Theorem 1.1] we get $\tilde{f} \leq g$, where g is the solution of the initial value problem

$$g' = 2cg \qquad\qquad g(0) = \tilde{f}(0),$$

which is bounded on $[0, \varepsilon)$. As a result f is bounded from above on $(0, \varepsilon]$ and by its definition also from below. This together with the previously stated fact that $\langle Vv, v\rangle$ is monotonous implies that $\lim_{t\to 0}\langle V(t)v, v\rangle$ exists. Moreover, due to the boundedness of Hv we can pick a sequence $s_k \to 0$ such that $y_k := H(s_k)v$ converge to some y. Then:

$$\begin{aligned}\lim_{t\to 0}\langle V(t)v, v\rangle &= \lim_{t\to 0}\langle UH(t)v, H(t)v\rangle = \\ &= \lim_{k\to\infty}\langle UH(s_k)v, H(s_k)v\rangle = \\ &= \lim_{k\to\infty}\langle U(s_k)y_k, y_k\rangle = \langle U(0)y, y\rangle \geq 0\end{aligned}$$

Now we can use $V' \geq 0$ to see that $\langle Vv, v\rangle \geq 0$ on all of $(0, t_0)$, meaning $V \geq 0$ and finally $S_2 - S_1 = U \geq 0$. To finish the proof of (i) it remains to show that $t_1 \leq t_2$. If $t_2 = \infty$ this is trivial so assume $t_2 < \infty$. Recall from earlier that by the Riccati equation S_2' is bounded from above on $(0, t_2)$. This implies that if we find a $v \in E$ such that $|\langle S_2(t)v, v\rangle| \to \infty$ for $t \to t_2$, then we must have $\langle S_2(t)v, v\rangle \to -\infty$. Suppose no such v exists then without loss of generality $S_2 \geq 0$ (otherwise $S_2 + k \cdot \mathrm{id}_E \geq 0$ for some $k > 0$). Now note that the norm of the solution of an ODE which cannot be extended beyond some finite time must go to infinity, so $\|S_2(t)\| \to \infty$ for $t \to t_2$. S_2 is positive semidefinite and self-adjoint so it has a self-adjoint square root. The norm of this square root also goes to infinity:

$$\|\sqrt{S_2(t)}\|^2 \geq \|\sqrt{S_2(t)}^2\| = \|S_2(t)\| \to \infty \qquad (t \to t_2)$$

In particular there exists a $v \in E$ such that $\|\sqrt{S_2(t)}v\|$ goes to infinity which implies:

$$\langle S_2(t)v, v\rangle = \langle\sqrt{S_2(t)}v, \sqrt{S_2(t)}v\rangle = \|\sqrt{S_2(t)}v\|^2 \to \infty \qquad (t \to t_2)$$

But, as previously mentioned, this contradicts the boundedness of S_2'. So there must exist a $v \in E$ such that for $t \to t_2$ we have $\langle S_2(t)v, v\rangle \to -\infty$. But then we already know $S_1 \leq S_2$, so clearly there exists a $\bar{t} \leq t_2$ such that $\langle S_1(t)v, v\rangle \to -\infty$ for $t \to \bar{t}$. This means $t_1 \leq t_2$.

To prove (ii) again use that the self-adjoint positive semidefinite map V has a square root $\sqrt{V}$ which is again self-adjoint and positive semidefinite and satisfies $\ker\sqrt{V} \subset \ker V$. Now pick $s_0 \in (0, t_1)$ and let $v \in \ker U(s_0)$. We need to show that $v \in \ker U(t) = \ker V(t)$ for $t \in (0, s_0)$. Clearly $\langle V(s_0)v, v\rangle = 0$ so by monotonicity of V have:

$$0 = \langle V(t)v, v\rangle = \langle\sqrt{V(t)}v, \sqrt{V(t)}v\rangle \qquad (t \in (0, s_0))$$

But then $v \in \ker \sqrt{V(t)} \subset \ker V(t) = \ker U(t)$ and we are done.

Finally to see (iii) assume that $U(s_0) = 0$ for some $s_0 \in (0, t_1)$. Then $\dim\ker U(s_0) = n$ and by (ii) $\dim\ker U(t) = n$ for all $t \in (0, s_0]$. This immediately gives $U(t) = 0$ for all $t \in (0, s_0]$, so indeed $S_1 = S_2$. Inserting this into equation (1.19) gives $R_1 = R_2$ and we have finished our proof. □

Remark 1.4.2. Recall that we motivated our examination of the Riccati equation with the tensor field S defined in Definition 1.3.10. Note that in Theorem 1.4.1 we assumed our endomorphism valued curves S_i to be self-adjoint. In contrast to this, S is not self-adjoint. However, the shape operator S_t of the hypersurfaces $\mathcal{S}_t$ from Definition 1.3.6, which one obtains by restricting S to the tangent space of $\mathcal{S}_t$, is self-adjoint. So the correct model of the situation from Theorem 1.4.1 is not S, but S restricted to the sub-bundle $\bigsqcup_{t\in \mathrm{im}(\tau)} T\mathcal{S}_t \subset TM$.

Also recall that in the previous section we considered the mean curvature H_t of hypersurfaces $\mathcal{S}_t$, which we obtained by taking the trace of the shape operator S_t. This leads us to our next definition. In the following I will always denote an interval in $\mathbb{R}$. We follow [15].

Definition 1.4.3. Let $S : I \to \mathrm{End}(E)$. We define the expansion $\theta \in \mathcal{C}^\infty(I)$, the vorticity $\omega : I \to \mathrm{End}(E)$ and the shear $\sigma : I \to \mathrm{End}(E)$ as follows:

- $\theta(t) := \operatorname{tr} S(t)$
- $\omega(t) := \frac{1}{2}\left(S(t) - S(t)^T\right)$
- $\sigma(t) := \frac{1}{2}\left(S(t) + S(t)^T\right) - \frac{\theta(t)}{n}\,\mathrm{id}_E$

Lemma 1.4.4 (Raychaudhuri Equation)**.** *Let $S : I \to \mathrm{End}(E)$ be a solution of $S' + S^2 + R = 0$. Then the expansion, the vorticity and the shear satisfy:*

$$\theta' + \frac{\theta^2}{n} + \operatorname{tr}(\omega^2) + \operatorname{tr}(\sigma^2) + \operatorname{tr}(R) = 0 \tag{1.23}$$

Moreover, if S is self-adjoint then:

$$\theta' + \frac{\theta^2}{n} + \operatorname{tr}(\sigma^2) + \operatorname{tr}(R) = 0 \tag{1.24}$$

Proof. The proof is a straightforward calculation. We start by simply taking the trace of the Riccati equation:

$$0 = \operatorname{tr}(S' + S^2 + R) = \underbrace{\operatorname{tr}(S')}_{(*)} + \underbrace{\operatorname{tr}(S^2)}_{(**)} + \operatorname{tr}(R) \tag{1.25}$$

Now we calculate each of the summands. Clearly $(*) = \operatorname{tr}(S') = \operatorname{tr}(S)' = \theta'$.

Next we note that $S = \omega + \sigma + \frac{\theta}{n}\operatorname{id}_E$, so we can calculate:

$$(**) = \operatorname{tr}\left((\omega + \sigma + \frac{\theta}{n}\operatorname{id}_E)^2\right) =$$
$$= \operatorname{tr}(\omega^2) + \operatorname{tr}(\sigma^2) + \left(\frac{\theta}{n}\right)^2 \operatorname{tr}(\operatorname{id}_E^2) + \operatorname{tr}(\omega\sigma) + \operatorname{tr}(\sigma\omega) + \frac{2\theta}{n}\operatorname{tr}(\omega) + \frac{2\theta}{n}\operatorname{tr}(\sigma)$$

Again calculating the individual terms we see:

- $\operatorname{tr}(\omega) = \frac{1}{2}\left(\operatorname{tr}(S) - \operatorname{tr}(S^T)\right) = 0 \qquad$ (because $\operatorname{tr}(S) = \operatorname{tr}(S^T)$)
- $\operatorname{tr}(\sigma) = \frac{1}{2}\left(\operatorname{tr}(S) + \operatorname{tr}(S^T)\right) - \frac{\theta}{n}\operatorname{tr}(\operatorname{id}_E) = \frac{1}{2}\left(\operatorname{tr}(S) + \operatorname{tr}(S)\right) - \frac{\operatorname{tr}(S)}{n} n = 0$
- $\operatorname{tr}(\omega\sigma) = \operatorname{tr}\left((\omega\sigma)^T\right) = \operatorname{tr}\left(\sigma^T\omega^T\right) = -\operatorname{tr}(\sigma\omega)$

In the last equality we used that $\omega^T = -\omega$ and $\sigma^T = \sigma$. Inserting this back into $(**)$ gives:

$$(**) = \operatorname{tr}(\omega^2) + \operatorname{tr}(\sigma^2) + \left(\frac{\theta}{n}\right)^2 \operatorname{tr}(\operatorname{id}_E^2) = \operatorname{tr}(\omega^2) + \operatorname{tr}(\sigma^2) + \frac{\theta}{n}$$

So inserting all of this back into equation (1.25) gives:

$$0 = \theta' + \operatorname{tr}(\omega^2) + \operatorname{tr}(\sigma^2) + \frac{\theta^2}{n} + \operatorname{tr}(R) = 0$$

This is precisely (1.23). To see (1.24) just note that the vorticity ω becomes 0 for S self-adjoint. $\square$

Combining Lemma 1.4.4 with Theorem 1.4.1 gives the following result:

Theorem 1.4.5. *Let $R : \mathbb{R} \to \operatorname{Sym}(E)$ be smooth with $\operatorname{tr}(R) \geq n \cdot \kappa$ for some $\kappa \in \mathbb{R}$. Let $S : (0, b) \to \operatorname{Sym}(E)$ be a solution of $S' + S^2 + R = 0$ and let $s_\kappa : (0, b_\kappa) \to \mathbb{R}$ be a solution of $s_\kappa' + s_\kappa^2 + \kappa = 0$, which cannot be extended beyond b_κ. If $\lim_{t\searrow 0}\left(s_\kappa(t) - \frac{\operatorname{tr} S(t)}{n}\right)$ exists and is nonnegative then $b \leq b_\kappa$ and*

$$\operatorname{tr} S(t) \leq n \cdot s_\kappa(t) \qquad\qquad (t \in (0, b))$$

If there exists some t_0 in $(0, b)$ such that $\operatorname{tr} S(t_0) = n \cdot s_\kappa(t_0)$, then equality holds on all of $(0, t_0]$. Moreover on $(0, t_0]$, S and R are given by $S(t) = s_\kappa(t)\operatorname{id}_E$ and $R(t) = \kappa \operatorname{id}_E$.

Proof. Define $r := \frac{1}{n}\left(\operatorname{tr}(\sigma^2) + \operatorname{tr}(R)\right)$. Then by Lemma 1.4.4 $\frac{\operatorname{tr}(S)}{n}$ satisfies the scalar Riccati equation

$$\left(\frac{\operatorname{tr}(S)}{n}\right)' + \left(\frac{\operatorname{tr}(S)}{n}\right)^2 + r = 0$$

Recall that the self-adjoint σ has a nonnegative trace since for any basis $(v_i)_{i=1}^n$ of E:

$$\operatorname{tr}(\sigma^2) = \sum_{i=1}^n \langle \sigma^2 v_i, v_i\rangle = \sum_{i=1}^n \langle \sigma v_i, \sigma v_i\rangle = \sum_{i=1}^n \|\sigma v_i\| \geq 0$$

So by assumption

$$r = \frac{\operatorname{tr}(\sigma^2) + \operatorname{tr}(R)}{n} \geq \frac{\operatorname{tr}(R)}{n} \geq \kappa \tag{1.26}$$

We have found two solutions s_κ and $\frac{\operatorname{tr}(S)}{n}$ of a scalar Riccati equation equation where the inhomogeneous terms satisfy inequality (1.26) and the limit for $t \searrow 0$ of the difference $s_\kappa - \frac{\operatorname{tr}(S)}{n}$ exists and is nonnegative by assumption. Altogether we can apply Theorem 1.4.1 to get $b \leq b_\kappa$ and $\frac{\operatorname{tr}(S)}{n} \leq s_\kappa$. This proves the first claim of our Theorem.

If $\frac{\operatorname{tr} S(t_0)}{n} = s_\kappa(t_0)$ for some $t_0 \in (0, b)$, then Theorem 1.4.1 immediately gives us equality on all of $(0, t_0]$. It remains to validate that S and R are indeed of the claimed form. To this end note that Theorem 1.4.1 also implies equality in (1.26). In particular this means that $\operatorname{tr}(\sigma^2) = 0$ and $\operatorname{tr}(R) = n \cdot \kappa$ on $(0, t_0]$. Using the definition of σ we get:

$$\begin{aligned}
0 = \operatorname{tr}(\sigma^2) &= \operatorname{tr}\left((S - \frac{\operatorname{tr}(S)}{n} \operatorname{id}_E)^2\right) = \\
&= \operatorname{tr}(S^2) + \operatorname{tr}\left(2\frac{\operatorname{tr}(S)}{n} S \operatorname{id}_E\right) + \left(\frac{\operatorname{tr}(S)}{n}\right)^2 n = \\
&= \operatorname{tr}(S^2) - 2\frac{\operatorname{tr}(S)^2}{n} + \frac{\operatorname{tr}(S)^2}{n} = \\
&= \operatorname{tr}(S^2) - \frac{1}{n} \operatorname{tr}(S)^2
\end{aligned}$$

Hence $\frac{1}{n} \operatorname{tr}(S)^2 = \operatorname{tr}(S^2)$. Noting that $\langle\langle A, B \rangle\rangle := \operatorname{tr}(AB)$ defines a scalar product on $\operatorname{End}(E)$, the Cauchy-Schwarz inequality implies that this equality can only hold if S and id_E are linearly dependent. So in order for $\frac{\operatorname{tr}(S)}{n} = s_\kappa$ to hold we must have $S = s_\kappa \cdot \operatorname{id}_E$ on $(0, t_0]$. Inserting this into the Riccati equation for S gives:

$$s_\kappa' \cdot \operatorname{id}_E + s_\kappa^2 \cdot \operatorname{id}_E + R = 0$$

So on $(0, t_0]$ R must be a multiple of id_E too and by the scalar Riccati equation it must in fact be $R = \kappa \cdot \operatorname{id}_E$. $\square$

2 Comparison Theory for Spacelike Hypersurfaces

We will now develop the setting in which we apply the results about timelike distance functions and the Riccati equation. We start by introducing the central notions of this chapter.

2.1 The Signed Time Separation and Causally Complete Sets

We follow the beginning of [15].

We start by recalling that the Lorentzian arc-length of a piecewise smooth curve $\gamma : [a, b] \to M$ in a Lorentzian manifold is simply defined as the integral

$$L(\gamma) := \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{|\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle|} dt,$$

where $t_1 = a, t_n = b$ and for $1 < i < n$ the t_i are the breakpoints, where γ is only continuous and not smooth. Using this we can define the time separation on a spacetime M.

Definition 2.1.1. Let $U(p, q)$ be the set of all piecewise smooth curves γ, which are future directed causal from p to q. The time separation $\tau : M \times M \to [0, \infty]$ is defined as:

$$\tau(p, q) := \begin{cases} \sup_{\gamma \in U} \{L(\gamma)\} & p < q \\ 0 & \text{else} \end{cases}$$

As with the Riemannian distance on Riemannian manifolds one could wonder under which conditions this "distance" is attained by an admissible curve. In Riemannian Geometry this question is answered by the Hopf-Rinow Theorem, stating that this is the case for complete Riemannian manifolds. In Lorentzian Geometry we need the manifold to be globally hyperbolic, meaning it is strongly causal and the causal diamonds $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$. Then we get the Avez-Seifert Theorem:

Theorem 2.1.2. *Let M be globally hyperbolic and $p < q$ in M. Then there exists a causal geodesic γ from p to q such that:*

$$L(\gamma) = \tau(p, q)$$

γ is without conjugate points before q. Moreover, τ is finite-valued and continuous.

Proofs of all parts of this statement can be found in chapter 14 of [11].

Analogous to the distance in metric spaces we can define the time separation between points and sets of a spacetime.

Definition 2.1.3. Let $A \subset M$ be acausal, meaning that for all $p, q \in A$ we have $p \not< q$. Then we define the signed time separation $\tau_A : M \to [-\infty, \infty]$ as:

$$\tau_A(q) := \begin{cases} \sup_{p \in A} \tau(p, q) & q \in I^+(A) \\ -\sup_{p \in A} \tau(q, p) & q \in I^-(A) \\ 0 & \text{else} \end{cases}$$

Note that the assumption of A being acausal was necessary in order to exclude the case where q is both in $I^+(A)$ and in $I^-(A)$. Once again, we would like to have a condition that guarantees the supremum in the definition to be attained. However it is easy to see that global hyperbolicity of M is no longer enough. Take for example the Minkowski space $\mathbb{R}^2_1$ as the manifold M and $\{0\} \times (0, 1]$ as the acausal set A. Then clearly for the point $q = (1, 0)$ we have $\tau_A(q) = 1$ but there is no causal curve of Lorentzian arclength 1 connecting A to q since $(0, 0) \notin A$.

In order for the supremum to be attained we will need to impose an additional condition on A. The following definition will provide precisely this.

Definition 2.1.4. A set $A \subset M$ is called future causally complete (FCC) if for all $q \in I^+(A)$ the set $\overline{J^-(q) \cap A}$ is compact in A. Past causal completeness (PCC) is defined analogously. A is called causally complete if it is both future causally complete and past causally complete.

Example 2.1.5. (i) Using our previous example of the Minkowski space with the subset $A = \{0\} \times (0, 1]$ and $q = (1, 0)$ we indeed see that A is not FCC since $\overline{J^-(q) \cap A} = A$ and hence not compact.

(ii) Let $M = \mathbb{R}^n_1$ and A the set of all past directed vectors v such that $\langle v, v \rangle = -1$, meaning A is one half of a two-sheeted hyperboloid. Then $A \subset M$ is not FCC since for example for $q = (0, 0)$ we have $\overline{J^-(q) \cap A} = A$ and A is not compact. However if we consider $I^-(0)$ instead of M, A is FCC as a subset of $I^-(0)$.

(iii) It can be shown that any Cauchy hypersurface is causally complete (for example this is an immediate consequence of [11, Lemma 14.40]. This fits well together with (ii) since the set A is not a Cauchy hypersurface of the manifold M but it is a Cauchy hypersurface of the manifold $I^-(0)$.

In the following all results stated for FCC sets have an analogous version for PCC sets. The proofs always work the same way for both cases.

Theorem 2.1.6. *Let M be globally hyperbolic and let $A \subset M$ be a FCC set. Then for every $q \in J^+(A)$ there exists a $p \in A$ such that*

$$\tau_A(q) = \tau(p, q)$$

Moreover, if A is a spacelike, acausal submanifold then all maximizing curves from A to q are timelike, unbroken geodesics, normal to A and without focal points before q.

Proof. Let $q \in J^+(A)$. We first try to find a $p \in A$ such that $\tau_A(q) = \tau(p, q)$. By Theorem 2.1.2 the function $\tau(., q) : M \to \mathbb{R}$ is continuous. Hence, after restricting $\tau(., q)$ to the compact set $K := \overline{J^-(q) \cap A}$, there exists a point $p \in K$ where it attains its maximum. So we get:

$$\tau(p, q) = \sup_{p' \in K} \tau(p', q) = \sup_{p' \in A} \tau(p', q) = \tau_A(q)$$

where in the second equality we used that $\operatorname{supp} \tau(., q) \subset J^-(q)$.

The fact that this p and q are connected via a maximizing geodesic is just Theorem 2.1.2 again. The remaining facts about maximizing curves are standard results about the first and second variation of arclength formulas as they can for example be found in chapter 10 of [11]. □

Proposition 2.1.7. *Let M be globally hyperbolic and let $A \subset M$ be a FCC set. Then τ_A is finite-valued and continuous on $J^+(A)$.*

Proof. Finiteness is clear by the finiteness of τ together with Theorem 2.1.6. It remains to prove continuity.

Let $q \in J^+(A)$, $p \in A$ such that $\tau_A(q) = \tau(p, q)$ and let $(q_i)_{i \in \mathbb{N}}$ be a sequence in $J^+(A)$ such that $q_i \to q$. Suppose $\tau_A(q_i) \nrightarrow \tau_A(q) = \tau(p, q)$. Then there exists an $\varepsilon > 0$ such that $|\tau_A(q_i) - \tau(p, q)| > \varepsilon$ for infinitely many $i \in \mathbb{N}$. Note that

$$\tau_A(q_i) \geq \tau(p, q_i) \to \tau(p, q)$$

so without loss of generality we can assume $\tau_A(q_i) > \tau(p, q) + \varepsilon$ holds for all $i \in \mathbb{N}$. By Theorem 2.1.6 there exists a sequence $(p_i)_{i \in \mathbb{N}}$ in A such that $\tau_A(q_i) = \tau(p_i, q_i)$ for all $i \in \mathbb{N}$. We can pick a $q^+ \in I^+(q)$ and note that $I^-(q^+)$ is an open neighbourhood of q. This means almost all q_i and as a consequence almost all p_i lie in $J^-(q^+)$. So without loss of generality we assume that all p_i are contained in the compact set $\overline{J^-(q^+) \cap A}$. Therefore they have a convergent subsequence, for simplicity again called $(p_i)_{i \in \mathbb{N}}$, converging to some $p' \in A$. But then using continuity of τ:

$$\tau_A(q) + \varepsilon < \tau_A(q_i) = \tau(p_i, q_i) \to \tau(p', q) \leq \tau_A(q)$$

This is a contradiction. □

Let us take a moment to analyse why the last two results are important for us. As we have mentioned at the very beginning of this chapter, we want to apply our theory on timelike distance functions to a concrete function. Let Σ be a smooth, spacelike, acausal, FCC hypersurface, then τ_Σ is a natural candidate for the role of such a timelike distance function. Recall that in section 1.3 we used the integral curves of τ to connect a point in our manifold to a "time slice". We then saw in Proposition 1.3.5 that taking the difference of the τ values of set and point and measuring the length of the curve connecting the two, amounted to the same thing. In other words, the level set and the point were connected by something similar to a maximizing geodesic. In this sense τ and τ_Σ behave similarly.

However now we run into a problem. The function τ from section 1.3 did not just connect a point to a "time slice" with a "maximizing geodesic". It did so in a unique way. This seems reasonable. If we transport a set A along geodesics and think about the resulting set as the future of A, then we would like to be able to reverse the procedure. This does not work for τ_Σ. Each point in our manifold can be connected to Σ with a maximizing geodesic, but it is easy to construct an example where there is more than one such geodesic:

Example 2.1.8. Let $M = \mathbb{R}^2_1$ and Σ the graph of the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{\sqrt{2}}(1+x) - \sqrt{2} & x < -1 \\ -\sqrt{1+x^2} & x \in [-1,1] \\ \frac{1}{\sqrt{2}}(1-x) - \sqrt{2} & x > 1 \end{cases}$$

This looks a bit confusing, but think of it as the set of past directed vectors v satisfying $\langle v, v\rangle = -1$, straightened out far enough away from 0 to make it a smooth, spacelike, acausal, FCC hypersurface. The point 0 lies in $J^+(\Sigma)$ and it is easy to see that $\tau_\Sigma(0) = 1$. But there are many curves from Σ to 0 of exactly that length. For example, the straight line from $(-1, 0)$ to 0 and the straight line from $(-\sqrt{2}, 1)$ to 0 both have Lorentzian arclength 1. We also note that if we were to extend theses straight lines beyond 0 then they would stop being maximizing.

This problem cannot be fixed, meaning that in general we will not be able to turn τ_Σ into a timelike distance function on M. However, we note that if we simply remove the set where τ_Σ is making trouble, we might get a new manifold where everything works out fine. We will do this in two steps. First, we will show that close enough to Σ every point can be connected to Σ by a maximizing geodesic in a unique way. Then we will try to find the maximal set on which this holds by excluding the points where the geodesics stop being maximizing.

As we have seen in Theorem 2.1.6, maximizing geodesics are always perpendicular to the hypersurface they are emanating from. So to study these geodesics, we introduce the normal bundle and the normal exponential map.

Definition 2.1.9. Let M be a semi-Riemannian manifold and let $P \subset M$ be a semi-Riemannian submanifold. Then the set

$$NP := \bigsqcup_{p \in P} T_p P^{\perp},$$

is called the normal bundle of P.

With the obvious bundle projection $NP \to P$ the normal bundle becomes a vector bundle. We define the set $\mathcal{D}$ of all vectors v in NP such that the geodesic with starting vector v is defined on all of $[0,1]$. Then, analogously to the exponential map for a point, we can define the normal exponential map:

Definition 2.1.10. The normal exponential map $\exp^{\perp} : \mathcal{D} \to M$ is defined as:

$$\exp^{\perp}(v) = \gamma_v(1),$$

where γ_v is the geodesic such that $\dot{\gamma_v}(0) = v$.

One can show that $\exp^{\perp}$ is a smooth map on an open neighbourhood of the set $Z := \{0_p \,|\, p \in P\}$, similarly to how it is done for the usual exponential map. We sum up some important properties of $\exp^{\perp}$ in the following proposition:

Proposition 2.1.11. *Let M be a semi-Riemannian manifold and let $P \subset M$ be a semi-Riemannian submanifold.*

(i) There exist open neighbourhoods $U \subset NP$ of Z and $V \subset M$ of P such that $\exp^{\perp}\big|_U : U \to V$ is a diffeomorphism.

(ii) For $v \in \mathcal{D}$, $\gamma_v(1)$ is a focal point of P along γ_v if and only if v is a singular point of $\exp^{\perp}$

The proofs of these statements can be found in chapter 7 and chapter 10 of [11]. Next, we will show some technical results about causally complete sets.

Lemma 2.1.12. *Let M be globally hyperbolic and let $A \subset M$ be FCC. Then:*

(i) A is closed

(ii) $J^+(A)$ is closed

Proof. (i) Let $(p_i)_{i \in \mathbb{N}}$ be a sequence in A such that $p_i \to p$ for some $p \in M$. We show $p \in A$. Let $q \in I^+(p) \subset I^+(A)$. Then $I^-(q)$ is an open neighbourhood of p, so without loss of generality we can assume that all p_i lie in $I^-(q) \subset J^-(q)$. But then in particular all p_i must lie in the compact set $J^-(q) \cap A$ so they must have a subsequence converging to some $p' \in A$. But then clearly $p' = p$, which proves the first part.

(ii) Now let $(p_i)_{i \in \mathbb{N}}$ be a sequence in $J^+(A)$ such that $p_i \to p$ for some $p \in M$. We show $p \in J^+(A)$. Again, let $q \in I^+(p)$ then (like in (i)) we can assume that all p_i lie in $J^-(q)$. For each p_i there exists a $p_i^- \in A$ such that $p_i^- \leq p_i$. Then all p_i^- must lie in the compact set $\overline{J^-(q) \cap A}$ and have a subsequence converging to some $p^- \in A$. In globally hyperbolic manifolds $\leq$ is a closed relation (see for example [11, Lemma 14.22]) so $p_i^- \leq p_i$ implies $p^- \leq p$, meaning that $p \in J^+(p^-) \subset J^+(A)$. $\square$

Proposition 2.1.13. *Let M be globally hyperbolic, let $A \subset M$ be FCC and let $\gamma_i : [0, b_i] \to M$ be a sequence of future directed causal curves starting in A. If there exists a $q \in M$ such that $\gamma_i(b_i) \to q$ then one of two cases holds:*

- *$q \in A$ and $(\gamma_i)_{i\in\mathbb{N}}$ collapses to q.*
- *There exist a subsequence $(\gamma_{i_j})_{j\in\mathbb{N}}$, a point $p \in A$ and a future directed causal curve γ from p to q such that $\gamma_{i_j}(0) \to p$ and $L(\gamma) \geq \lim_{j\to\infty} L(\gamma_{i_j})$.*

Before we can prove this, recall the following technical result from [11, Lemma 14.14]:

Lemma 2.1.14. *Let M be globally hyperbolic and let $K \subset M$ be compact. Let $(\gamma_i)_{i\in\mathbb{N}}$ be a sequence of future directed causal curves, entirely contained in K, such that $\gamma_i(0) \to p$ and $\gamma_i(b_i) \to q \neq p$. Then there exists a subsequence $(\gamma_{i_j})_{j\in\mathbb{N}}$ and a future directed causal broken geodesic γ from p to q such that $L(\gamma) \geq \lim_{j\to\infty} L(\gamma_{i_j})$.*

Proof of Proposition 2.1.13. By Lemma 2.1.12 (ii) $J^+(A)$ is closed, hence $q \in J^+(A)$. Let $q^+ \in I^+(q)$, then without loss of generality all $\gamma_i(b_i)$ lie in $J^-(q^+)$. But then all γ_i and in particular all points $p_i := \gamma_i(0)$ must be entirely contained in $J^-(q^+)$, which means that there exists a $p \in A$ such that a subsequence of $(p_i)_{i\in\mathbb{N}}$ (which we will again call $(p_i)_{i\in\mathbb{N}}$) converges to p. Now there are two cases:

- $p = q$ and $(\gamma_i)_{i\in\mathbb{N}}$ collapses to p.
- $p \neq q$

If the first case occurs we are done. If the second case occurs it remains to prove that p and q can be connected by a future directed causal curve γ such that $L(\gamma) \geq \lim_{j\to\infty} L(\gamma_{i_j})$. If we could apply Lemma 2.1.14 then we would be done, but to do so we still need to show that all curves γ_i are contained in a compact set. But this can easily be done by picking some $p^- \in I^-(p)$ and noting that now, without loss of generality, all $p_i \in J^+(p^-)$. Altogether all curves γ_i are entirely contained in $J^+(p^-) \cap J^-(q^+)$, which is compact by global hyperbolicity. $\square$

All of this technical work was necessary for the next proposition:

Proposition 2.1.15. *Let M be globally hyperbolic and let $P \subset M$ be a spacelike, acausal, FCC submanifold. Let $\gamma : [0, b)$ be a future directed causal geodesic starting in $p \in P$ such that $v := \dot\gamma(0) \in T_pP^\perp$. Then there exists a $\delta > 0$ such that $L(\gamma\big|_{[0,\delta]}) = \tau_P(\gamma(\delta))$ i.e. $\gamma\big|_{[0,\delta]}$ is maximizing.*

Proof. Suppose $\gamma\big|_{[0,\delta]}$ is not maximizing for any $\delta > 0$. For all $i \in \mathbb{N}$ such that $\frac{1}{i} < b$ we set $q_i := \gamma(\frac{1}{i})$. By Theorem 2.1.6 there exist maximizing geodesics $\gamma_i : [0, a_i] \to M$ from P to q_i such that $v_i := \dot{\gamma}_i(0) \in NP$. Note that we can assume that v_i and v are not multiples of each other since this would contradict the assumption of $\gamma\big|_{[0,\delta]}$ not being maximizing for any δ. $\gamma_i(a_i) = q_i$ converges to $p = \gamma(0)$ so we are in the setting of Proposition 2.1.13. This means either all curves collapse to the point p or there exists a future directed causal curve from P to p. Clearly the latter cannot happen by acausality of P. It remains to exclude the first case.

Suppose all curves collapse to p. Without loss of generality assume that all vectors v_i are of length 1. Then we would necessarily have $a_i \to 0$ and hence $a_i v_i \to 0_p \in T_pM$. Now note that $a_i v_i \neq \frac{1}{i} v$ for all i and $\exp^\perp(a_i v_i) = \exp^\perp(\frac{1}{i} v)$. But since $\frac{1}{i} v \to 0_p$ this means that $\exp^\perp$ is not injective in any neighbourhood of 0_p. But by Proposition 2.1.11 (i) this is clearly impossible and we have a contradiction. □

Remark 2.1.16. We have shown that a geodesic γ starting perpendicular to P maximizes the signed time separation for small parameters. In addition for $t > 0$ small enough $\gamma\big|_{[0,t]}$ is the unique maximizing geodesic from $\gamma(0)$ to $\gamma(t)$. To see this let $\delta > 0$ such that $\gamma\big|_{[0,\delta]}$ is maximizing. But then for $t \in (0, \delta)$ the curve $\gamma\big|_{[0,t]}$ is also maximizing and if there was another curve from P to $\gamma(t)$ of equal length we could concatenate these two curves to obtain a new maximizing curve connecting P and $\gamma(\delta)$. By construction this curve would be a broken geodesic. However by Theorem 2.1.6 such maximizing curves can never be broken geodesics, which is a contradiction.

By this argument we see that a maximizing geodesic can only stop being the unique maximizing geodesic at a point where it stops to maximize the signed time separation altogether. This leads us to our next important concept, the future cut locus.

2.2 The Future Cut Locus

In the following M is always a globally hyperbolic spacetime and $\Sigma \subset M$ is always a smooth, spacelike, acausal, FCC hypersurface. As in the previous section one could just as well work with PCC hypersurfaces, define the past cut locus and arrive at completely analogous results on $J^-(\Sigma)$. We will follow section 3.2 of [14].

Definition 2.2.1. We define the future normal bundle as the set

$$S^+N\Sigma := \{v \in N\Sigma \mid v \text{ future directed and } \langle v, v\rangle = -1\}.$$

For each $v \in S^+N\Sigma$ denote by $\gamma_v : I_v \to M$ the geodesic with starting vector $\dot{\gamma_v}(0) = v$. Then we can define the Σ-future cut function $s^+_\Sigma : S^+N\Sigma \to (0, \infty]$ by:

$$s^+_\Sigma(v) := \sup\{t \in I_v \mid \tau_\Sigma(\gamma_v(t)) = L(\gamma_v\big|_{[0,t]})\}$$

Note that we know that s^+_Σ is larger than 0 for all $v \in S^+N\Sigma$ due to Proposition 2.1.15. Now $t_0 = s^+_\Sigma(v)$ can be one of two things. It could be the point where the geodesic stops to be maximizing or, in the case of a geodesically incomplete manifold M, it could be the maximal time of existence of the geodesic γ_v. Right now we primarily care about the former.

Definition 2.2.2. If $s^+_\Sigma(v) \in I_v$ we call the point $\gamma_v(s^+_\Sigma(v))$ a Σ-cut point of γ_v. The collection of all such points is

$$\mathrm{Cut}^+(\Sigma) := \{\exp^\perp(s^+_\Sigma(v)v) \,|\, v \in S^+N\Sigma \text{ and } s^+_\Sigma(v) \in I_v\} \subset M$$

and we call it the future cut locus of Σ.

In order to understand the structure of the future cut locus we first need some regularity for the Σ-future cut function.

Proposition 2.2.3. *The Σ-future cut function $s^+_\Sigma : S^+N\Sigma \to (0, \infty]$ is lower semicontinuous.*

Remark 2.2.4. Before beginning the proof, note that because Σ is a hypersurface, the set $T_pM \cap S^+N\Sigma$ for some $p \in \Sigma$ consists of only one vector v_p. Locally Σ can be written as the zero set of some submersion h, which means that locally we can write v_p as $\frac{\operatorname{grad} h(p)}{\|\operatorname{grad} h(p)\|}$. Hence v_p depends smoothly on the point p. Consequently if a sequence p_i converges to p in Σ, the corresponding vectors v_{p_i} in $S^+N\Sigma$ will converge to v_p. This will be used multiple times

Proof. Let $(v_i)_{i\in\mathbb{N}}$ and v be a sequence and a vector in $S^+N\Sigma$ such that $v_i \to v$. We have to show that $s^+_\Sigma(v) \leq s_0 := \liminf_{i\to\infty} s^+_\Sigma(v_i)$. We can go over to a subsequence and assume $s_0 = \lim_{i\to\infty} s^+_\Sigma(v_i)$. If $s_0 = \infty$ the assertion is trivially satisfied so we assume $s_0 < \infty$. Suppose $s^+_\Sigma(v) > s_0$ then we try to obtain a contradiction.

There exists a $\delta > 0$ such that $s^+_\Sigma(v) > s_0 + \delta$. Hence we can define the point $q := \gamma_v(s_0 + \delta)$ where γ_v is the geodesic with starting vector v, which by assumption is the maximizing geodesic between Σ and q. Next we define $b_i := s^+_\Sigma(v_i) + \delta$ then $b_i \to s_0 + \delta < s^+_\Sigma(v)$ meaning, after going to another subsequence, we can assume $b_i < s^+_\Sigma(v)$ for all $i \in \mathbb{N}$. Further denote by γ_i the geodesics such that $\dot\gamma_i(0) = v_i$. Now we note that

$$b_i v_i \to (s_0 + \delta)v \in \mathcal{D} \subset N\Sigma$$

$\mathcal{D}$ is open, so again after going to a subsequence we can assume $b_i v_i \in \mathcal{D}$ for all $i \in \mathbb{N}$. This allows us to define the sequence $q_i := \gamma_i(b_i v_i)$, which converges to q by continuity of $\exp^\perp$. $b_i > s^+_\Sigma(v_i)$ so γ_i is not maximizing between Σ and q_i, but by Theorem 2.1.6 we know that such a maximizing geodesic exists. We denote it by $\alpha_i : [0, a_i] \to M$ and call its starting vector $\dot\alpha_i(0) =: w_i \in S^+N\Sigma$. Now $\alpha_i(a_i) = q_i \to q \notin \Sigma$, which means we can apply Proposition 2.1.13:

There exists a subsequence, still denoted by $(\alpha_i)_{i\in\mathbb{N}}$, a point $p \in \Sigma$ and a future directed causal curve $\alpha : [0, a] \to M$ from p to q such that $\alpha_i(0) \to p$ and $L(\alpha) \geq \lim_{i\to\infty} L(\alpha_i)$.

This curve α needs to be maximizing since, by maximality of α_i, an even longer curve between Σ and q would violate continuity of the signed time separation τ_Σ. Moreover we get:

$$a = L(\alpha) = \tau_\Sigma(q) = \lim_{i\to\infty} \tau_\Sigma(q_i) = \lim_{i\to\infty} L(\alpha_i) = \lim_{i\to\infty} a_i$$

This together with Remark 2.2.4 implies that $a_i w_i \to aw$, where $w := \dot\alpha(0)$. Now both α and γ_v are maximizing between Σ and q, but γ_v remains maximizing beyond q (because $s^+_\Sigma(v) > s_0+\delta$). If $\alpha \neq \gamma_v\big|_{[0,s_0+\delta]}$ this allows us to construct a maximizing broken geodesic as in Remark 2.1.16, which stands in contradiction to Theorem 2.1.6. So there is only one maximizing geodesics between Σ and q, meaning $\alpha = \gamma_v\big|_{[0,s_0+\delta]}$ and in particular $w = v$. Altogether we get:

$$a_i w_i \to aw = (s_0 + \delta)v \leftarrow b_i v_i$$

But at the same time we have $\exp^\perp(a_i w_i) = q_i = \exp^\perp(b_i v_i)$, meaning $\exp^\perp$ is not injective in any neighbourhood of $(s_0 + \delta)v$. This means $(s_0 + \delta)v$ is a singular point of $\exp^\perp$ and by Proposition 2.1.11 (ii) q is a focal point of Σ along γ_v. But again according to Theorem 2.1.6 maximizing geodesics do not encounter focal points and since γ_v is maximizing beyond q we have derived a contradiction. $\square$

Next we define the set $\mathcal{J}^+_T(\Sigma) := \{tv \,|\, v \in S^+\Sigma \text{ and } t \in [0, s^+_\Sigma(v))\}$ and its interior $\mathcal{I}^+_T(\Sigma) := \mathcal{J}^+_T(\Sigma)^\circ$. The idea is that mapping these vectors to M via $\exp^\perp$ gives us precisely the points in M at which geodesics normal to Σ have not yet encountered the future cut locus of Σ. This leads us to the following central Theorem:

Theorem 2.2.5. *The set $\mathcal{I}^+(\Sigma) := \exp^\perp(\mathcal{I}^+_T(\Sigma)) \subset M$ satisfies the following properties:*

(i) $\mathcal{I}^+(\Sigma)$ is open and diffeomorphic to $\mathcal{I}^+_T(\Sigma)$ via $\exp^\perp$.

(ii) $\mathcal{I}^+(\Sigma) = I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$

(iii) Every point in $\mathcal{I}^+(\Sigma)$ can be connected to Σ with a unique maximizing geodesic.

(iv) $\mathrm{Cut}^+(\Sigma) \subset M$ is closed.

Proof. (i) First note that $\exp^\perp$ is injective (and hence bijective as a map to $\mathcal{I}^+(\Sigma)$) on $\mathcal{I}^+_T(\Sigma)$ because two maximizing geodesics cannot remain maximizing once they intersect according to Theorem 2.1.6. By the same Theorem geodesics also cannot remain maximizing after encountering a focal point, which are precisely the singular points of $\exp^\perp$ by Proposition 2.1.11(ii). So, by the inverse function theorem locally $\exp^\perp$ can be inverted in a smooth way, which together with bijectivity already means that $\exp^\perp$ is a diffeomorphism onto its image, which must be open.

(ii) Let $q \in I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$. If there were multiple maximizing geodesics connecting Σ to q then neither would remain maximizing after q and q would be in $\mathrm{Cut}^+(\Sigma)$. Hence there exists exactly one such geodesic γ_v with $v \in S^+N\Sigma$. So $q = \exp^\perp(tv)$ for some

$t < s^+_\Sigma(v)$, implying $tv \in \mathcal{J}^+_T(\Sigma)$. It remains to show $tv \in \mathcal{I}^+_T(\Sigma)$. Let $\delta > 0$ such that $t < s^+_\Sigma(v) - \delta$. But then by lower semicontinuity of s^+_Σ there exists a neighbourhood $W \subset S^+N\Sigma$ of v such that:

$$t < s^+_\Sigma(v) - \delta < s^+_\Sigma(w) \qquad \forall w \in W$$

So there exists a neighbourhood $I \subset \mathbb{R}$ of t such that for all $w \in W$ we have $I \subset [0, s^+_\Sigma(w))$. But then $I \cdot W \subset N\Sigma$ is a neighbourhood of tv which by definition is entirely contained in $\mathcal{J}^+_T(\Sigma)$. So altogether we have $tv \in \mathcal{J}^+_T(\Sigma)^\circ = \mathcal{I}^+_T(\Sigma)$.

The converse direction is clear by the definition of the sets.

(iii) Existence of such a geodesic already followed from Theorem 2.1.6. Then at the beginning of (ii) we already showed that for $q \in I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$ this geodesic must be unique and finally by (ii) again $\mathcal{I}^+(\Sigma)$ is precisely the set $I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$.

(iv) Let $(q_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Cut}^+(\Sigma)$ such that there exists a $q \in M$ satisfying $q_i \to q$. We need to show $q \in \mathrm{Cut}^+(\Sigma)$. By Theorem 2.1.6 there exist maximizing geodesics $\gamma_i : [0, b_i] \to M$ from Σ to q_i. Since $q_i \in \mathrm{Cut}^+(\Sigma)$ we have $b_i = s^+_\Sigma(v_i)$. Now by Proposition 2.1.13 one of two possibilities must occur:

- The curves $(\gamma_i)_{i\in\mathbb{N}}$ collapse to $q \in \Sigma$.
- There exist a subsequence, again denoted by $(\gamma_i)_{i\in\mathbb{N}}$, a point $p \in \Sigma$ and a future directed causal curve $\gamma : [0, b] \to M$ from p to q such that $\gamma_i(0) \to p$ and $L(\gamma) \geq \lim_{i\to\infty} L(\gamma_i)$.

In the first case we call the unique vector in $T_qM \cap S^+N\Sigma$ v and see that $v_i \to v$ by Remark 2.2.4. By lower semi continuity of s^+_Σ we get $s^+_\Sigma(v) \leq \liminf_{i\to\infty} s^+_\Sigma(v_i) = 0$ by assumption. But s^+_Σ is a strictly positive function so this is a contradiction. Hence the second case must hold.

In the second case continuity of the signed time separation once again implies that γ must be a maximizing geodesic with some starting vector $v = \dot\gamma(0)$, so $s^+_\Sigma(v) \geq b$. But at the same time, again by Remark 2.2.4, we have $v = \lim_{i\to\infty} v_i$, so by lower semicontinuity of s^+_Σ:

$$s^+_\Sigma(v) \leq \liminf_{i\to\infty} s^+_\Sigma(v_i) = b,$$

where the equality at the end again holds by continuity of the signed time separation and because $L(\gamma_i) = b_i$. But so altogether we have $b = s^+_\Sigma(v)$ and $q = \exp^\perp(s^+_\Sigma(v)v) \in \mathrm{Cut}^+(\Sigma)$. □

One key takeaway from this Theorem is that because $\mathcal{I}^+(\Sigma)$ is open, it is again a manifold and by (iii) it is in fact a manifold where every point can be connected to Σ by a unique geodesic maximizing the signed time separation. However, what we have not proven is that it really is the maximal such set. This does in fact hold but is not proven in this thesis. For a proof see [14, Proposition 3.2.30]. However, we will still prove that $\mathcal{I}^+(\Sigma)$ is maximal up to a set of measure zero, because the future cut locus is itself a zero set.

Proposition 2.2.6. $\mathrm{Cut}^+(\Sigma)$ *has measure zero with respect to the Lorentzian measure* μ_g*. In particular* $\mathrm{Cut}^+(\Sigma)$ *has no interior points.*

Remark 2.2.7. While we claim that $\mathrm{Cut}^+(\Sigma)$ has measure 0 with respect to μ_g, note that a set of measure 0 is already determined by the differentiable structure of the manifold. More precisely, if measures μ_1, μ_2 on a manifold are defined via smooth, nowhere vanishing densities, also called μ_1 and μ_2, then for a measurable set A we have $\mu_1(A) = 0$ if and only if $\mu_2(A) = 0$.

To see this assume without loss of generality that $A \subset U$, where U is the domain of the chart (x, U). Now:

$$\mu_i(A) = \int_{x(A)} \mu_i \left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^n} \right) \circ x^{-1} \, d\lambda^n$$

By assumption the integrand is nowhere vanishing, so we have $\mu_1(A) = 0$ if and only if $\lambda^n(x(A)) = 0$ if and only if $\mu_2(A) = 0$. Due to this we will not need to specify the measures in the ensuing proof.

Proof of Proposition 2.2.6. For $n \in \mathbb{N}$ we define the set $U_n := \{v \in S^+N\Sigma \mid s^+_\Sigma(v) < n\}$ and the function $s_n := s^+_\Sigma|_{U_n} : U_n \to (0, n)$. Denote the set of future directed vectors in $N\Sigma$ by $N^+\Sigma$, then

$$\begin{aligned} \Phi : S^+N\Sigma \times (0, \infty) &\to N^+\Sigma \\ (v, t) &\mapsto t \cdot v \end{aligned}$$

is clearly a diffeomorphism hence we can identify the two sets. Define $\mathrm{Cut}^+_T(\Sigma) := (\exp^\perp)^{-1}(\mathrm{Cut}^+(\Sigma))$ and denote by $\Gamma(s_n) \subset S^+N\Sigma \times (0, \infty)$ the graph of s_n, then:

$$\mathrm{Cut}^+_T(\Sigma) \subset \bigcup_{n \in \mathbb{N}} \Gamma(s_n)$$

The functions s_n are lower semicontinuous and in particular Borel-measurable. Note that the graph of a measurable function must have measure 0 by Fubini's Theorem. As a result the measure of $\Gamma(s_n)$ is 0 for all $n \in \mathbb{N}$. This implies that the measure of $\mathrm{Cut}^+_T(\Sigma)$ is 0 and consequently its image under a smooth map $\exp^\perp(\mathrm{Cut}^+_T(\Sigma)) = \mathrm{Cut}^+(\Sigma)$ is of measure 0 too (see [10, Proposition 6.5]). $\square$

Our next step is to restrict the signed time separation τ_Σ to $\mathcal{I}^+(\Sigma) \subset M$. We want to show that on this set τ_Σ is a timelike distance function.

Proposition 2.2.8. *The function* $\tau_\Sigma : M \to \mathbb{R}$ *satisfies the following properties:*

(i) τ_Σ *restricted to* $\mathcal{I}^+(\Sigma)$ *is smooth.*

(ii) For each $q \in \mathcal{I}^+(\Sigma)$ *we have* $\mathrm{grad}\, \tau_\Sigma|_q = -\dot\gamma(\tau_\Sigma(q))$*, where* $\gamma : [0, \tau_\Sigma(q)] \to M$ *is the unique maximizing geodesic from* Σ *to* q*.*

(iii) On $\mathcal{I}^+(\Sigma)$ $\mathrm{grad}\, \tau_\Sigma$ *is a past directed timelike vector field satisfying* $\|\mathrm{grad}\, \tau_\Sigma\| \equiv 1$*.*

Proof. (i) Recall that the length of a maximizing geodesic $\gamma_v = \exp(. \cdot v)$ is given by $\|tv\|$. So using that by Theorem 2.2.5(i) $\exp^{\perp}$ is a diffeomorphism between $\mathcal{I}_T^+(\Sigma)$ and $\mathcal{I}^+(\Sigma)$ we can express the signed time separation for $q \in \mathcal{I}^+(\Sigma)$ as:

$$\tau_\Sigma(q) = \|(\exp^{\perp})^{-1}(q)\|$$

$(\exp^{\perp})^{-1}$ is smooth and so is $\|.\|$ away from zero. Since τ_Σ is strictly positive on $\mathcal{I}^+(\Sigma)$ we have shown smoothness.

(ii) Let $q \in \mathcal{I}^+(\Sigma)$, let $b := \tau_\Sigma(q)$ and let $\gamma : [0, b] \to M$ be the unit speed maximizing geodesic from Σ to q. Pick any $v \in T_qM$ and choose another curve $\alpha : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = q$ and $\dot\alpha(0) = v$. $\mathcal{I}^+(\Sigma)$ is open, so for ε small enough $\mathrm{im}(\alpha) \subset \mathcal{I}^+(\Sigma)$. This allows us to define the following variation σ of the curve γ:

$$\begin{aligned} \sigma &: (-\varepsilon, \varepsilon) \times [0, b] \to M \\ \sigma_s(t) &:= \exp^{\perp}\left(\frac{t}{b} \cdot (\exp^{\perp})^{-1}(\alpha(s))\right) \end{aligned}$$

For each $s \in (-\varepsilon, \varepsilon)$ the curve $\sigma_s : [0, b] \to M$ is a maximizing geodesic from Σ to $\alpha(s)$ meaning $L(\sigma_s) = \tau_\Sigma(\alpha(s))$. Recall that the derivative of $L(\sigma_s)$ with respect to s can be calculated with the formula

$$L'(\sigma_s)\Big|_{s=0} = \int_0^b \langle \ddot\gamma(t), V(t)\rangle \, dt - \langle \dot\gamma(t), V(t)\rangle \Big|_0^b,$$

where $V(t)$ is the variation vector field $\frac{\partial \sigma_s(t)}{\partial s}\big|_{s=0}$. A proof of this formula can be found in [11, Proposition 10.2]. We can now apply this to calculate the change of the signed time separation along α:

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} \tau_\Sigma(\alpha(s)) &= \int_0^b \langle 0, V(t)\rangle \, dt - \langle \dot\gamma(b), V(b)\rangle + \langle \dot\gamma(0), V(0)\rangle = \\ &= -\langle \dot\gamma(b), \dot\alpha(0)\rangle = -\langle \dot\gamma(b), v\rangle, \end{aligned} \tag{2.1}$$

where we used that $V(b) = \dot\alpha(0)$ and that $\sigma_s(0) \in \Sigma$ for all $s \in (-\varepsilon, \varepsilon)$, implying $V(0) \in T\Sigma$ and is hence orthogonal to $\dot\gamma(0)$. The left hand side of (2.1) can also be calculated by using the chain rule:

$$\frac{d}{ds}\Big|_{s=0} \tau_\Sigma(\alpha(s)) = d\tau_\Sigma\big|_{\alpha(s)}(\dot\alpha(s)) = \langle \operatorname{grad} \tau_\Sigma\big|_q, v\rangle$$

Since $v \in T_qM$ was arbitrary this implies that $\operatorname{grad} \tau_\Sigma\big|_q = -\dot\gamma(\tau_\Sigma(q))$, which is what we wanted to prove.

(iii) By (ii) the integral curves of the vector field $-\operatorname{grad} \tau_\Sigma$ are maximizing, future directed timelike, unit speed geodesics. This already proves (iii). □

Recall that by Definition 1.15 a timelike distance function is a smooth function τ satisfying $\langle \operatorname{grad} \tau, \operatorname{grad} \tau \rangle = -1$. Hence Proposition 2.2.8(iii) says that τ_Σ is a timelike distance function on $\mathcal{I}^+(\Sigma)$.

If one wants to find the maximal set on which τ_Σ behaves this nicely one might not be satisfied yet. Of course, the signed time separation is defined on $I^-(\Sigma)$ too, so for a past causally complete hypersurface Σ we could define the past cut locus and the set $\mathcal{I}^-(\Sigma)$, which has precisely the same properties as $\mathcal{I}^+(\Sigma)$. Then we can combine the sets $\mathcal{I}^+(\Sigma)$, $\mathcal{I}^-(\Sigma)$ and Σ itself, to form one large manifold on which the signed time separation is a timelike distance function.

Proposition 2.2.9. *Let M be a globally hyperbolic spacetime and let Σ be a smooth, spacelike, acausal, causally complete hypersurface. Then $\tau_\Sigma : M \to \mathbb{R}$ satisfies:*

(i) For each $q \in \mathcal{I}^\pm(\Sigma)$ we have $\operatorname{grad} \tau_\Sigma\big|_q = \mp\dot\gamma(\pm\tau_\Sigma(q))$, where $\gamma : [0, \pm\tau_\Sigma(q)] \to M$ is the unique maximizing geodesic from Σ to q.

(ii) τ_Σ is smooth on $\mathcal{I}^-(\Sigma) \cup \Sigma \cup \mathcal{I}^+(\Sigma)$ and the restriction of $\operatorname{grad} \tau_\Sigma$ to Σ is a past directed unit normal.

Proof. (i) follows immediately from the previous Proposition.

For (ii) we only need to check smoothness for points in Σ. So let $p \in \Sigma$ and choose neighbourhoods $U \subset M$ of p and $V \subset N\Sigma$ of 0_p such that $\exp^\perp\big|_V : V \to U$ is a diffeomorphism. Such neighbourhoods exist by Proposition 2.1.11(i). Now Let n be the future directed unit normal of $\Sigma \cap U$, which is a section of $N\Sigma\big|_{\Sigma\cap U}$. Then we can construct the function $f : V \to \mathbb{R}$

$$f(v) := -\langle v, \mathrm{n}_{\pi_{N\Sigma}(v)} \rangle$$

and the function $h : U \to \mathbb{R}$

$$h := f \circ (\exp^\perp\big|_V)^{-1}.$$

Now let $q \in \mathcal{I}^\pm(\Sigma) \cap U$, then there exists a unique $p \in \Sigma$ such that $q = \exp^\perp(\tau_\Sigma(q)\, \mathrm{n}_p)$. So we have:

$$h(q) = f(\tau_\Sigma(q)\, \mathrm{n}_p) = -\tau_\Sigma(q)\langle \mathrm{n}_p, \mathrm{n}_p \rangle = \tau_\Sigma(q)$$

Moreover, for $q \in \Sigma \cap U$ we have $h(q) = \tau_\Sigma(q) = 0$, hence $h = \tau_\Sigma$ on $\mathcal{I}^\pm(\Sigma) \cap U$. By the construction of h this is smooth. Now $\operatorname{grad} \tau_\Sigma$ is a past directed, timelike unit vector field on $\mathcal{I}^\pm(\Sigma)$, so these properties must extend to Σ by continuity. Moreover, the restriction of $\operatorname{grad} \tau_\Sigma$ to Σ must give a normal vector field because its integral curves intersect Σ normally.

□

Although the result of this Proposition is quite satisfying, we will not actually use it going forward. For our purposes it will usually be enough to view τ_Σ as a function on $\mathcal{I}^+(\Sigma)$. This is a little inconvenient because it means we cannot interpret Σ as a level set of τ_Σ (because Σ is not a subset of $\mathcal{I}^+(\Sigma)$). However, we can still continuously extend $\operatorname{grad}_\Sigma$ to Σ to obtain a past directed unit normal.

The reason we don't use smoothness on $\mathcal{I}^-(\Sigma) \cup \Sigma \cup \mathcal{I}^+(\Sigma)$ is because to do so we would need to assume that Σ is not just future causally complete but also past causally complete. So, by avoiding $\mathcal{I}^-(\Sigma)$ we can put fewer assumptions on Σ.

2.3 Comparison Geometries

We now follow section 4 of [15]. For the rest of this chapter M is an $(n+1)$-dimensional globally hyperbolic spacetime and Σ a smooth, spacelike, acausal, FCC hypersurface with signed time separation $\tau_\Sigma : M \to \mathbb{R}$. We first introduce some new notation:

Definition 2.3.1. Let $t > 0$, then we define the future sphere and the restricted future sphere around Σ to be the sets:

$$S^+_\Sigma(t) := \tau_\Sigma^{-1}(t) \quad \text{and} \quad \mathcal{S}^+_\Sigma(t) := S^+_\Sigma(t) \cap \mathcal{I}^+(\Sigma)$$

We also define the future ball and restricted future ball around Σ as the sets:

$$B^+_\Sigma(t) := \bigcup_{s \in (0,t)} S^+_\Sigma(s) \quad \text{and} \quad \mathcal{B}^+_\Sigma(t) := B^+_\Sigma(t) \cap \mathcal{I}^+(\Sigma)$$

Next, for $A \subset \Sigma$ we define the truncated spheres:

$$S^+_A(t) := \{q \in S^+_\Sigma(t) \,|\, \exists p \in A : \tau_\Sigma(q) = \tau(p,q)\}$$

The sets $\mathcal{S}^+_A(t)$, $B^+_A(t)$ and $\mathcal{B}^+_A(t)$ are defined analogously.

By Proposition 2.2.8, τ_Σ is a timelike distance function on $\mathcal{I}^+(\Sigma)$ so we quickly recall what some of the results in section 1.3 imply for our current setting.

We again define $\mathrm{n} := -\operatorname{grad} \tau_\Sigma$, then by Proposition 1.3.3 the sets $\mathcal{S}^+_\Sigma(t)$ are spacelike hypersurfaces of $\mathcal{I}^+(\Sigma)$ with future directed unit normal n. If S_t is the shape operator and H_t is the mean curvature of the hypersurface $\mathcal{S}^+_\Sigma(t)$, then for every $q \in \mathcal{S}^+_\Sigma(t)$:

$$H_t(q) = \operatorname{tr} S_t\big|_q = -\Box \tau_\Sigma(q)$$

Further, if we define the global shape operator S as in Definition 1.3.10, then it satisfies the Riccati equation

$$\nabla_{\mathrm{n}} S + S^2 + R_{\mathrm{n}} = 0.$$

This leads us back to Riccati Comparison Theory from section 1.4. Recall that all our comparison results there worked similarly. If we had some solution S of a Riccati equation with inhomogeneous term R, then a global bound on R together with a bound on the initial value of S gave us a global bound on S. In particular in Theorem 1.4.5 S and R were curves of operators and we assumed bounds on their traces. In our current setting the bound on the trace of S at time 0 will turn into a bound on the mean curvature of Σ. Meanwhile the global bound on the trace of R will turn into a bound on the Ricci curvature since with a local frame $(E_i)_{i=1}^{n+1}$ of M we can compute:

$$\operatorname{tr}(R_{\mathrm{n}}) = \sum_{i=1}^{n+1} \epsilon_i \langle R(E_i, \mathrm{n})\,\mathrm{n}, E_i\rangle = \operatorname{Ric}(\mathrm{n}, \mathrm{n})$$

We now use this information to define the kind of manifolds we will subsequently examine.

Definition 2.3.2. Let $\kappa, \beta \in \mathbb{R}$, then we say the pair (M, Σ) satisfies the cosmological comparison condition $\mathrm{CCC}(\kappa, \beta)$ if the following two conditions hold:

- For all timelike vectors $v \in TM$ the Ricci tensor of M satisfies $\operatorname{Ric}(v, v) \geq n\kappa$.
- The mean curvature H of Σ, with respect to n, satisfies $H \leq \beta$.

Now the idea is as follows. Given parameters κ and β, we want to find a manifold and a hypersurface $(M_{\kappa,\beta}, \Sigma_{\kappa,\beta})$ such that this pair satisfies the $\mathrm{CCC}(\kappa, \beta)$ with equality. Then after applying our Riccati comparison theorems the mean curvature of the future of Σ is bounded by the mean curvature of the future of $\Sigma_{\kappa,\beta}$. For the sake of finding the correct comparison geometries we will examine warped products of the following form:

Let $(a, b) \subset \mathbb{R}$ be an open interval, let $f : (a, b) \to (0, \infty)$ be a smooth function and let (N, h) be a complete, n-dimensional Riemannian manifold. Then we will look at Lorentzian manifolds (M, g) such that:

$$M = (a, b) \times N \qquad \text{and} \qquad \mathrm{g} = -dt^2 + f(t)^2\,\mathrm{h} \tag{2.2}$$

Now the question is how precisely to choose the parts of the above equations in order to get the right comparison manifold. First note that the vector field ∂_t i.e. the dual vector to the 1-form dt, is by definition timelike. We will always assume that the time orientation is chosen to make ∂_t future directed. Now step by step we will show that with the right function f and the right manifold (N, h) these warped products give suitable comparison spaces.

Proposition 2.3.3. *The warped product (M, g) as defined above is globally hyperbolic.*

Proof. In order to show this we follow [1, Lemma A.5.14]. First note that by [11, Corollary 14.39] if a spacetime contains a Cauchy Hypersurface then it is globally hyperbolic. By its definition a Cauchy hypersurface is characterized by being met by any inextendible timelike curve precisely once. So we will take any $t_0 \in (a, b)$ and try to show that the set $N_{t_0} := \{t_0\} \times N$ satisfies this condition.

We first show that N_{t_0} is even acausal. To this end let $\gamma(s) = (t(s), \sigma(s))$ be a causal curve, then we must have:

$$0 \geq \mathrm{g}(\dot\gamma(s), \dot\gamma(s)) = -(t'(s))^2 + f(t(s))^2 \cdot \mathrm{h}(\dot\sigma(s), \dot\sigma(s)) \tag{2.3}$$

In particular we have $|t'| \geq f \cdot \|\dot\sigma\|_N$. This means that if $t'(s) = 0$ then also $\dot\sigma(s) = 0$ and hence $\dot\gamma(s) = 0$, which is a contradiction to γ being causal. Therefore $t' > 0$ or $t' < 0$ on all of (a, b), implying (M, g) is acausal because $t(s) = t_0$ holds for at most one s.

Now we want to prove that any inextendible causal curve $\gamma = (t, \sigma)$ meets N_{t_0}. Without loss of generality let γ be future directed, which by the previous argument means $t' > 0$. Consequently we can assume that after reparametrization γ has the form $\gamma(t) = (t, \sigma(t))$. We claim that this curve is inextendible, which would imply that it is defined on all of (a, b). Suppose this is not the case i.e. γ is only defined on some proper subinterval $(\tilde{a}, \tilde{b})$. Without loss of generality let $\tilde{a} \in (a, b)$, then there exists an $\varepsilon > 0$ such that $[\tilde{a} - \varepsilon, \tilde{a} + \varepsilon] \subset (a, b)$. There exist constants $K_2 > K_1 > 0$ such that

$$K_1 \leq f(t) \leq K_2. \qquad t \in [\tilde{a} - \varepsilon, \tilde{a} + \varepsilon]$$

Note that (2.3) becomes $0 \geq -1 + f(t)^2 \cdot \|\dot{\sigma}(t)\|_N^2$, which implies

$$\|\dot{\sigma}(t)\|_N \leq \frac{1}{f(t)} \leq \frac{1}{K_1}. \qquad t \in (\tilde{a} - \varepsilon, \tilde{a} + \varepsilon) \tag{2.4}$$

Let $(t_i)_{i \in \mathbb{N}}$ be a sequence such that $t_i \searrow \tilde{a}$, which without loss of generality lies entirely in $(\tilde{a}, \tilde{a} + \varepsilon)$. By (2.4) the length of the curve $\sigma\big|_{[t_i, t_j]}$ is bounded above by $|t_j - t_i| \frac{1}{K_1}$, so we get the following inequality for the Riemannian distance:

$$d(\sigma(t_i), \sigma(t_j)) \leq \frac{|t_j - t_i|}{K_1}$$

In particular $(\sigma(t_i))_{i \in \mathbb{N}}$ is a Cauchy sequence and since the manifold N is complete by assumption, this sequence has a limit $p \in N$. Clearly this limit is independent of the choice of sequence because a different sequence $(s_i)_{i \in \mathbb{N}}$ could be combined with $(t_i)_{i \in \mathbb{N}}$ to give a new Cauchy sequence which must again converge to p. This means σ has a continuous extension to $[\tilde{a}, \tilde{b})$ such that $\sigma(\tilde{a}) = p$. Now we further extend σ in any piecewise smooth way such that

$$\|\dot{\sigma}(t)\|_N \leq \frac{1}{K_2}. \qquad t \in (\tilde{a} - \varepsilon, \tilde{a})$$

Then for $t \in (\tilde{a} - \varepsilon, \tilde{a})$ the new curve $\gamma = (t, \sigma)$ satisfies:

$$\mathrm{g}(\dot{\gamma}(t), \dot{\gamma}(t)) = -1 + f(t)^2 \cdot \|\dot{\sigma}(t)\|_N^2 \leq -1 + \frac{f(t)^2}{K_2} \leq 0$$

Thus we have found a causal extension of γ beyond $\tilde{a}$, which is a contradiction. □

Aside from proving that (M, g) is globally hyperbolic we also just showed that the sets $N_t := \{t\} \times N$ are smooth, acausal Cauchy hypersurfaces. They are clearly spacelike because the timelike vector field ∂_t is normal to the sets N_t. Moreover by Example 2.1.5(iii) any Cauchy hypersurface is FCC. These were all prerequisites for the pairs (M, N_t) to satisfy the cosmological comparison condition for some κ and β. Now we will first try to satisfy the Ricci curvature bound with equality i.e. with given κ we will try to choose f and (N, h) such that:

$$\mathrm{Ric}(v, v) = n\kappa \qquad (v \in TM \text{ such that } \mathrm{g}(v, v) = -1) \tag{2.5}$$

First note that for an arbitrary timelike vector v this means

$$n\kappa = \mathrm{Ric}\left(\frac{v}{\|v\|}, \frac{v}{\|v\|}\right) = \frac{1}{\|v\|^2}\,\mathrm{Ric}(v,v)$$

and hence (2.5) is equivalent to $\mathrm{Ric}(v,v) = -n\kappa\,\mathrm{g}(v,v)$ for all v timelike. This condition is weaker than assuming this equality holds for all $v \in TM$ but we will use it as motivation to require just that. Such manifolds, where the Ricci tensor is a scalar multiple of the metric tensor, are called Einstein. Of course a priori it is not at all clear that we can just assume our warped product to be Einstein. We will need some work to do so.

Proposition 2.3.4. *Let* (M,g) *be the warped product defined as in* (2.2) *and let* $\kappa \in \mathbb{R}$. *Then the following are equivalent:*

(i) $\mathrm{Ric} = -n\kappa\,\mathrm{g}$.

(ii) There exists a constant $\kappa_N \in \mathbb{R}$ *such that* $\mathrm{Ric}_N = (n-1)\kappa_N\,\mathrm{h}$ *and the function* f *satisfies the following system of equations:*

$$\begin{cases} f'' = -\kappa \cdot f \\ (f')^2 + \kappa_N = f \cdot f'' \end{cases} \tag{2.6}$$

Before beginning the proof, quickly recall the following characterization of the Ricci tensor in warped products from [11, Corollary 7.43]. Let $X, Y \in \mathfrak{X}(N_t)$, then:

- $\mathrm{Ric}(\partial_t, \partial_t) = -n\frac{f''}{f}$
- $\mathrm{Ric}(\partial_t, X) = 0$
- $\mathrm{Ric}(X,Y) = \mathrm{Ric}_N(X,Y) + \left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)\mathrm{g}(X,Y)$

In the final equation we used that the tangent space of N_t can be identified with the tangent space of N.

Proof. Let us first assume that $\mathrm{Ric}_N = (n-1)\kappa_N\,\mathrm{h}$ and f satisfies the system (2.6). We can split any $v \in TM$ into some vector $X \in TN_t$ and a multiple of ∂_t. So to verify $\mathrm{Ric} = -n\kappa\,\mathrm{g}$ it suffices to show $\mathrm{Ric}(\partial_t, \partial_t) = n\kappa$ and $\mathrm{Ric}(X,Y) = -n\kappa\,\mathrm{g}(X,Y)$, where $X, Y \in T_pN_t$ for some $t \in (a,b)$ and $p \in N_t$. The former follows immediately from the above characterization of the Ricci tensor and the first part of (2.6):

$$\mathrm{Ric}(\partial_t, \partial_t) = -n\frac{f''}{f} = n\kappa$$

For the latter we again use the characterization of the Ricci tensor, both equations in (2.6) and the assumption on the Ricci tensor of N:

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \mathrm{Ric}_N(X,Y) + \left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)\mathrm{g}(X,Y) = \\
&= (n-1)\kappa_N\,\mathrm{h}(X,Y) + \left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2)}\right)\mathrm{g}(X,Y) = \\
&= \left(\frac{n-1}{f^2}\big(\kappa_N + (f')^2\big) + \frac{f''}{f}\right)\mathrm{g}(X,Y) = \\
&= \left(\frac{n-1}{f^2}ff'' - \kappa\right)\mathrm{g}(X,Y) = \\
&= \big(-(n-1)\kappa - \kappa\big)\,\mathrm{g}(X,Y) = -n\kappa\,\mathrm{g}(X,Y)
\end{aligned}$$

So we have proven the first part of our claim.

Conversely assume that $\mathrm{Ric} = -n\kappa\,\mathrm{g}$. Then for $X, Y \in T_pN_t$ as before we can calculate:

$$\begin{aligned}
\mathrm{Ric}_N(X,Y) &= \mathrm{Ric}(X,Y) - \left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)\mathrm{g}(X,Y) = \\
&= \left(-n\kappa - \frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)f^2\,\mathrm{h}(X,Y)
\end{aligned} \tag{2.7}$$

This formula must hold for all $t \in (a,b)$ and is independent of the point $p \in N_t$. Hence the factor in front of $\mathrm{h}(X,Y)$ must be a constant. We define κ_N such that this constant is $(n-1)\kappa_N$ and it remains to show that f satisfies the system (2.6). The first equation holds because by the characterization of the Ricci tensor we still have

$$-n\frac{f''}{f} = \mathrm{Ric}(\partial_t, \partial_t) = -n\kappa\,\mathrm{g}(\partial_t, \partial_t) = n\kappa,$$

implying $f'' = -\kappa f$. For the second equation in (2.6) we insert the first one into our formula (2.7) for κ_N:

$$\begin{aligned}
\kappa_N &= \frac{1}{n-1}\big(-n\kappa f^2 - f''f + (n-1)(f')^2\big) = \\
&= \frac{1}{n-1}\big(nf''f - f''f + (n-1)(f')^2\big) = f''f - (f')^2
\end{aligned}$$

So we have proven the second part of our claim. □

We can now see that constructing a warped product satisfying $\mathrm{Ric} = -n\kappa\,\mathrm{g}$ comes down to solving the differential equations (2.6) and finding a complete Riemannian manifold (N, h) that is Einstein with $\mathrm{Ric}_N = (n-1)\kappa_N\,\mathrm{h}$. We first focus on the former.

With given initial conditions both equations have maximal solutions. We have to choose the initial conditions in a way to make these solutions coincide. This is not

Table 2.1: Solutions for f and H

$\kappa < 0$	$\kappa_N > 0$	$f(t) = \pm\sqrt{\kappa_N/\lvert\kappa\rvert}\cosh\left(\sqrt{\lvert\kappa\rvert}t + b\right)$	$H_t = n\sqrt{\lvert\kappa\rvert}\tanh\left(\sqrt{\lvert\kappa\rvert}t + b\right)$
$\kappa < 0$	$\kappa_N = 0$	$f(t) = e^{\pm\sqrt{\lvert\kappa\rvert}t}$	$H_t = \pm n\sqrt{\lvert\kappa\rvert}$
$\kappa < 0$	$\kappa_N < 0$	$f(t) = \pm\sqrt{\lvert\kappa_N\rvert/\lvert\kappa\rvert}\sinh\left(\sqrt{\lvert\kappa\rvert}t + b\right)$	$H_t = n\sqrt{\lvert\kappa\rvert}\coth\left(\sqrt{\lvert\kappa\rvert}t + b\right)$
$\kappa = 0$	$\kappa_N = 0$	$f(t) = e^b$	$H_t = 0$
$\kappa = 0$	$\kappa_N < 0$	$f(t) = \pm\sqrt{\lvert\kappa_N\rvert}t + b$	$H_t = n/(t \pm b/\sqrt{\lvert\kappa_N\rvert})$
$\kappa > 0$	$\kappa_N < 0$	$f(t) = \sqrt{\lvert\kappa_N\rvert/\kappa}\sin\left(\sqrt{\kappa}t + b\right)$	$H_t = n\sqrt{\kappa}\cot\left(\sqrt{\kappa}t + b\right)$

always possible, for example for $\kappa > 0$ and $\kappa_N = 0$ a solution of the first equation is of the form

$$f(t) = c_1 \cos\sqrt{\kappa}t + c_2 \sin\sqrt{\kappa}t,$$

while a solution of the second equation is of the form

$$f(t) = e^{k_1 t + k_2}$$

Clearly these solutions cannot coincide for any choice of constants c_1, c_2, k_1 and k_2. Similarly the cases $\kappa > 0$, $\kappa_N > 0$ and $\kappa = 0$, $\kappa_N > 0$ cannot be matched. However, for all remaining cases there exist solutions, which are summarized in table 2.1.

Recall that in the beginning we were only given a fixed value for κ and not for κ_N, so we can simply assume $\kappa_N \in \{-1, 0, 1\}$. Then we claim that the hyperbolic space $\mathbb{H}^n$, the n-dimensional plane $\mathbb{R}^n$ and the sphere $\mathbb{S}^n$ satisfy $\mathrm{Ric}_N = (n-1)\kappa_N\,\mathrm{h}$ with the respective constant κ_N. This is true because $\mathbb{H}^n$, $\mathbb{R}^n$ and $\mathbb{S}^n$ have constant sectional curvature -1, 0 and 1 respectively (see [11, Proposition 4.29]) and a Riemannian manifold (N, h) with constant sectional curvature κ_N satisfies

$$R(X,Y)Z = \kappa_N(\mathrm{h}(Z,Y)X - \mathrm{h}(Z,X)Y),$$

(see [11, Corollary 3.43]), which we can use to calculate:

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} \mathrm{h}(R(E_i,X)Y, E_i) = \\
&= \kappa_N \sum_{i=1}^{n} \mathrm{h}(Y,X)\overbrace{\mathrm{h}(E_i,E_i)}^{1} - \mathrm{h}(Y,E_i)\,\mathrm{h}(X,E_i) = \\
&= \kappa_N(n\,\mathrm{h}(X,Y) - \mathrm{h}(X,Y)) = \kappa_N(n-1)\,\mathrm{h}(X,Y)
\end{aligned}$$

Here $X, Y \in T_pN$ for some $p \in N$ and $(E_i)_{i=1}^n$ is an orthonormal basis of T_pN.

To summarize, by choosing N to be one of $\mathbb{H}^n$, $\mathbb{R}^n$ or $\mathbb{S}^n$ and solving the differential equations (2.6) we have found a warped product with the desired Ricci tensor. In order to see that (N, g) satisfies the $\mathrm{CCC}(\kappa, \beta)$ with equality it remains to show that the time

Table 2.2: Solutions for $f_{\kappa,\beta}$ and $N_{\kappa,\beta}$

$\kappa < 0$	$\lvert\beta\rvert < n\sqrt{\lvert\kappa\rvert}$	$N_{\kappa,\beta} = \mathbb{S}^n$	$f_{\kappa,\beta}(t) = \pm\sqrt{\lvert\kappa\rvert^{-1}}\cosh(\sqrt{\lvert\kappa\rvert}t + \operatorname{artanh}(\beta/(n\sqrt{\lvert\kappa\rvert})))$
$\kappa < 0$	$\lvert\beta\rvert = n\sqrt{\lvert\kappa\rvert}$	$N_{\kappa,\beta} = \mathbb{R}^n$	$f_{\kappa,\beta}(t) = e^{\pm\sqrt{\lvert\kappa\rvert}t}$
$\kappa < 0$	$\lvert\beta\rvert > n\sqrt{\lvert\kappa\rvert}$	$N_{\kappa,\beta} = \mathbb{H}^n$	$f_{\kappa,\beta}(t) = \pm\sqrt{\lvert\kappa\rvert^{-1}}\sinh(\sqrt{\lvert\kappa\rvert}t + \operatorname{arcoth}(\beta/(n\sqrt{\lvert\kappa\rvert})))$
$\kappa = 0$	$\beta = 0$	$N_{\kappa,\beta} = \mathbb{R}^n$	$f_{\kappa,\beta}(t) = \text{const}$
$\kappa = 0$	$\beta \neq 0$	$N_{\kappa,\beta} = \mathbb{H}^n$	$f_{\kappa,\beta}(t) = \pm t \pm n/\beta$
$\kappa > 0$	$\beta \in \mathbb{R}$	$N_{\kappa,\beta} = \mathbb{H}^n$	$f_{\kappa,\beta}(t) = \sqrt{\kappa^{-1}}\sin(\sqrt{\kappa}t + \operatorname{arccot}(\beta/(n\sqrt{\kappa})))$

slice $N_0 = \{0\} \times N$ has mean curvature equal to β.

We first try to compute the shape operator of N_t. Pick a point $p \in N_t$ and vectors $X, Y \in T_pN_t$ and extend them to local vector fields of N_t such that their Lie-brackets vanish. This can be achieved by letting them have constant components with respect to some coordinate vector field. Then we can use the Koszul formula (see for example [11, Theorem 3.11]) to compute the shape operator S_t:

$$\begin{aligned} \mathrm{g}(S_t(X), Y) = \mathrm{g}(\nabla_X \partial_t, Y) &= \\ &= \frac{1}{2}\big(X\,\mathrm{g}(\partial_t, Y) + \partial_t\,\mathrm{g}(X, Y) - Y\,\mathrm{g}(X, \partial_t)\big) = \\ &= \frac{1}{2}\partial_t(f(t)^2\,\mathrm{h}(X, Y)) = f'(t)f(t)\,\mathrm{h}(X, Y) = \frac{f'(t)}{f(t)}\,\mathrm{g}(X, Y) \end{aligned}$$

So the shape operator of N_t is given by $S_t = \frac{f'(t)}{f(t)}\,\mathrm{id}_{TN_t}$. As a result we immediately get

$$H_t = \operatorname{tr} S_t = n \cdot \frac{f'(t)}{f(t)}. \tag{2.8}$$

Because this is independent of the concrete point in N_t, the mean curvature of each time slice is constant. So, in order to satisfy $H_0 = \beta$ we simply need to choose the initial values for the function f such that $n \cdot \frac{f'(0)}{f(0)} = \beta$. Comparing this to table 2.1 we see that for any numbers $\kappa, \beta \in \mathbb{R}$ there is precisely one way to choose $\kappa_N \in \{-1, 0, 1\}$ and f, a solution of (2.6), such that both $\mathrm{Ric} = -n\kappa\,\mathrm{g}$ and $H_0 = \beta$. We summarize the results for all possible choices of κ and β in table 2.2. This allows us to finally define our comparison geometries:

Definition 2.3.5. Let $\kappa, \beta \in \mathbb{R}$, then we define $(M_{\kappa,\beta}, \mathrm{g}_{\kappa,\beta})$ to be the Lorentzian manifold

$$M_{\kappa,\beta} := (a_{\kappa,\beta}, b_{\kappa,\beta}) \times N_{\kappa,\beta} \qquad \mathrm{g}_{\kappa,\beta} = -dt^2 + f_{\kappa,\beta}(t)^2\,\mathrm{h}_{\kappa,\beta},$$

with $N_{\kappa,\beta}$ and $f_{\kappa,\beta}$ as in table 2.2. $(a_{\kappa,\beta}, b_{\kappa,\beta})$ is chosen to be the maximal interval containing 0 such that $f_{\kappa,\beta} > 0$ on $(a_{\kappa,\beta}, b_{\kappa,\beta})$.

Now we define $\Sigma_{\kappa,\beta} := \{0\} \times N_{\kappa,\beta}$ and see that this set is a smooth, spacelike, acausal, causally complete hypersurface. In particular the pair $(M_{\kappa,\beta}, \Sigma_{\kappa,\beta})$ satisfies the CCC(κ, β) with equality, as desired.

Remark 2.3.6. (i) We can define the signed time separation with respect to the hypersurface $\Sigma_{\kappa,\beta}$ and for simplicity call it $\tau_{\kappa,\beta} : M_{\kappa,\beta} \to [-\infty, \infty]$. A geodesic maximizing the signed time separation must emanate perpendicularly from $\Sigma_{\kappa,\beta}$ at some point $(0,p)$, so its tangent vector at that point must be a multiple of $\partial_t\big|_{(0,p)}$. But then the geodesic γ starting at $(0,p)$ such that $\dot\gamma(0) = c \cdot \partial_t$ is given by $\gamma(t) = (c \cdot t, p)$ (see [11, Corollary 7.36]). So clearly the signed time separation is coinciding with the function $t := \mathrm{pr}_1 : M_{\kappa,\beta} \to (a_{\kappa,\beta}, b_{\kappa,\beta})$. Because each point in $M_{\kappa,\beta}$ is met by exactly one such geodesic they never stop maximizing, implying $\mathrm{Cut}^{\pm}(\Sigma_{\kappa,\beta}) = \emptyset$. Moreover, $\tau_{\kappa,\beta} = t$ implies that the future spheres are given by

$$S^+_{\kappa,\beta}(t) = \mathcal{S}^+_{\kappa,\beta}(t) = \{t\} \times N_{\kappa,\beta}.$$

(ii) We will continue to sometimes adapt the notation to make a clear distinction between our comparison manifolds and general manifolds. If an expression has indices κ, β then that expression will always describe the corresponding object belonging to $(M_{\kappa,\beta}, \mathrm{g}_{\kappa,\beta})$. For example we will write $H_{\kappa,\beta}(t)$ instead of H_t.

(iii) The shape operator of the hypersurfaces $\{t\} \times N_{\kappa,\beta}$ and its trace, the mean curvature, once again satisfy Riccati equations. This can be seen either by applying Theorem 1.3.11 or directly, using the differential equations (2.6). For example, for the mean curvature we get:

$$\frac{1}{n}H'_{\kappa,\beta}(t) + \frac{1}{n}H^2_{\kappa,\beta} + \kappa = \left(\frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)}\right)' + \left(\frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)}\right)^2 + \kappa = \frac{f''_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)} + \kappa =$$
$$= -\kappa + \kappa = 0$$

(iv) The fact that we have an expression for the mean curvature $H_{\kappa,\beta}(t)$ allows us to calculate the areas of future spheres explicitly. Let $B \subset \Sigma_{\kappa,\beta}$ be compact such that $\mathrm{area}_{\kappa,\beta}\, B > 0$, then also $S^+_B(t) = \{t\} \times B$ is compact and $\mathrm{area}_{\kappa,\beta}\, S^+_B(t) > 0$, so by the first variation of area formula from Corollary 1.3.8 we get:

$$\begin{aligned}
\frac{d}{dt}\log\big(\mathrm{area}_{\kappa,\beta}\, S^+_B(t)\big) &= \frac{1}{\mathrm{area}_{\kappa,\beta}\, S^+_B(t)} \cdot \int_{S^+_B(t)} H_{\kappa,\beta}(t)\, d\mu_{\kappa,\beta,t} = \\
&= \frac{1}{\mathrm{area}_{\kappa,\beta}\, S^+_B(t)} \cdot \int_{S^+_B(t)} n \cdot \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)}\, d\mu_{\kappa,\beta,t} = \\
&= n \cdot \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)} \cdot \frac{\mathrm{area}_{\kappa,\beta}\, S^+_B(t)}{\mathrm{area}_{\kappa,\beta}\, S^+_B(t)} = \\
&= n \cdot \frac{d}{dt}\log(f_{\kappa,\beta}(t)) = \frac{d}{dt}\log(f_{\kappa,\beta}(t)^n)
\end{aligned}$$

Integrating this equation with respect to t gives

$$\log\left(\frac{\operatorname{area}_{\kappa,\beta} S_B^+(t)}{\operatorname{area}_{\kappa,\beta} B}\right) = \log\left(\frac{f_{\kappa,\beta}(t)^n}{f_{\kappa,\beta}(0)^n}\right),$$

and hence

$$\operatorname{area}_{\kappa,\beta} S_B^+(t) = \operatorname{area}_{\kappa,\beta} B \cdot \frac{f_{\kappa,\beta}(t)^n}{f_{\kappa,\beta}(0)^n}.$$

These expressions will serve as explicit bounds in our comparison Theorems.

2.4 Comparison Theorems

We will now finally prove some comparison theorems. We keep following [15].

Theorem 2.4.1 (d'Alembertian Comparison)**.** *Let $\kappa, \beta \in \mathbb{R}$ and let M and Σ satisfy the CCC(κ,β). Then each $q \in \mathcal{I}^+(\Sigma)$ satisfies $\tau_\Sigma(q) < b_{\kappa,\beta}$ and*

$$-\Box\tau_\Sigma(q) = H_{\tau_\Sigma(q)}(q) \leq H_{\kappa,\beta}(\tau_\Sigma(q)) = -\Box_{\kappa,\beta}\tau_{\kappa,\beta}\big|_{S_{\kappa,\beta}(\tau_\Sigma(q))} \tag{2.9}$$

Proof. The two equalities in (2.9) are true by Remark 1.3.9, so it remains to prove the inequality.

Recall that the "global shape operator" S and $R_{\mathrm{n}} = R(.,\mathrm{n})\,\mathrm{n}$ satisfy the Riccati equation

$$\nabla_{\mathrm{n}} S + S^2 + R_{\mathrm{n}} = 0 \tag{2.10}$$

Now fix $q \in \mathcal{I}^+(\Sigma)$ and let $\gamma : [0, \tau_\Sigma(q)] \to M$ be the unique maximizing geodesic from some $p \in \Sigma$ to q. We choose an orthonormal basis $E_1, \dots, E_n$ of $T_p\Sigma$ and transport it parallelly along γ. For any $t \in [0, \tau_\Sigma(q)]$ the vectors $E_1(t), \dots, E_n(t)$ form an orthonormal basis of $\dot\gamma(t)^\perp = T_{\gamma(t)}\mathcal{S}_\Sigma^+(t)$. Note that the images of S and R_{n} restricted to $\dot\gamma(t)^\perp$ still lie in $\dot\gamma(t)^\perp$ so we can express $S\big|_{\dot\gamma(t)^\perp}$ and $R_{\mathrm{n}}\big|_{\dot\gamma(t)^\perp}$ in the basis $E_1, \dots, E_n$. This gives rise to the smooth functions $\mathcal{S}_i^j, \mathcal{R}_i^j : [0, \tau_\Sigma(q)] \to \mathbb{R}$ such that $S(E_i) = \mathcal{S}_i^j E_j$ and $R_{\mathrm{n}}(E_i) = \mathcal{R}_i^j E_j$ $(1 \leq i, j \leq n)$. Now note that by equation (2.10) we have:

$$(\nabla_{\mathrm{n}} S + S^2 + R_{\mathrm{n}})(E_i) = 0 \qquad (1 \leq i \leq n)$$

The covariant derivative of S can be expressed by the derivative of the functions $\mathcal{S}_j^i$ since:

$$\begin{aligned}(\nabla_{\mathrm{n}} S)(E_i) &= \nabla_{\mathrm{n}}(S(E_i)) - S(\nabla_{\mathrm{n}} E_i) = \\ &= \nabla_{\mathrm{n}}(\mathcal{S}_i^j E_j) = \mathcal{S}_j^{i\prime} E_j + \mathcal{S}_j^i \nabla_{\mathrm{n}} E_j = \mathcal{S}_j^{i\prime} E_j\end{aligned}$$

So we obtain a new matrix Riccati equation

$$\mathcal{S}' + \mathcal{S}^2 + \mathcal{R} = 0, \tag{2.11}$$

where $\mathcal{S}$ and $\mathcal{R}$ are the matrix valued curves with components $\mathcal{S}_j^i$ and $\mathcal{R}_j^i$.

Next note that S restricted to $\dot\gamma(t)^\perp$ is just the self-adjoint shape operator of the hypersurface $\mathcal{S}^+_\Sigma(t)$ at the respective point. R_n restricted to $\dot\gamma(t)^\perp$ is also self-adjoint by a symmetry of the Riemann tensor. This means that the matrix representation with an orthonormal basis of these maps is self-adjoint with respect to the Euclidean scalar product. Finally, because (Σ, M) satisfies the CCC(κ,β) we get

$$\operatorname{tr}\mathcal{R}(t) = \operatorname{tr} R_\mathrm{n}(t) = \operatorname{Ric}(\dot\gamma(t),\dot\gamma(t)) \geq n\kappa$$

and

$$\operatorname{tr}\mathcal{S}(0) = \operatorname{tr} S\big|_{\dot\gamma(0)^\perp} = H_0(p) \leq \beta$$

We are hence positioned to apply the scalar Riccati comparison Theorem 1.4.5. Defining $s_{\kappa,\beta} := \frac{1}{n}H_{\kappa,\beta} : (0, b_{\kappa,\beta}) \to \mathbb{R}$ we note that by Remark 2.3.6(iii) it satisfies the scalar Riccati equation $s'_{\kappa,\beta} + s^2_{\kappa,\beta} + \kappa = 0$. Our comparison geometry was specifically constructed such that $s_{\kappa,\beta}(0) = \frac{1}{n}\beta$, meaning:

$$\lim_{t\searrow 0}\left(s_{\kappa,\beta}(t) - \frac{\operatorname{tr}\mathcal{S}(t)}{n}\right) \geq 0$$

Moreover $s_{\kappa,\beta}$ cannot be extended beyond $b_{\kappa,\beta}$. So all conditions of Theorem 1.4.5 are satisfied and we get $\tau_\Sigma(q) < b_{\kappa,\beta}$ and $H_t(\gamma(t)) = \operatorname{tr}\mathcal{S}(t) \leq H_{\kappa,\beta}(t)$. Finally, $t = \tau_\Sigma(q)$ gives the inequality in equation (2.9). □

We now want to use this Theorem to prove area and volume comparison Theorems. Recall that by the first variation of area formula in Corollary 1.3.8 the rate of change of the area of future spheres is given by integrals of the mean curvature. We already used this in Remark 2.3.6(iv) to get an explicit formula for $\operatorname{area}_{\kappa,\beta} S^+_B(t)$. Now we also know that if (Σ, M) satisfies the CCC(κ,β) then the future mean curvatures of M are bounded by the future mean curvatures of $M_{\kappa,\beta}$. So we would expect areas of future spheres in $M_{\kappa,\beta}$ to grow faster than in M. Making this precise will be the content of the next Theorem:

Theorem 2.4.2 (Area Comparison)**.** *Let $\kappa,\beta\in\mathbb{R}$ and let M and Σ satisfy the CCC(κ,β). Let $B\subset\Sigma_{\kappa,\beta}$ be a compact set satisfying $\operatorname{area}_{\kappa,\beta} B > 0$ and let $A\subset\Sigma$. Then*

$$t \mapsto \frac{\operatorname{area}\mathcal{S}^+_A(t)}{\operatorname{area}_{\kappa,\beta} S^+_B(t)} \qquad (t\in[0,b_{\kappa,\beta})), \tag{2.12}$$

is nonincreasing. Moreover we get

$$\lim_{t\searrow 0}\frac{\operatorname{area}\mathcal{S}^+_A(t)}{\operatorname{area}_{\kappa,\beta} S^+_B(t)} = \frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B}, \tag{2.13}$$

and

$$\operatorname{area}\mathcal{S}^+_A(t) \leq \frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B}\cdot\operatorname{area}_{\kappa,\beta} S^+_B(t) \tag{2.14}$$

for all $t\in[0,b_{\kappa,\beta})$.

Proof. First note that, as in the proof of Proposition 1.2.3, for any $t \in [0, b_{\kappa,\beta})$ the semi-Riemannian density of $\mathcal{S}_\Sigma^+(t)$ can be expressed with the pullback of the semi-Riemannian density of Σ. This means that if $\operatorname{area} A = 0$ then $\operatorname{area} \mathcal{S}_A^+(t) = 0$, too. So in the case of $\operatorname{area} A = 0$ all assertions follow immediately and we can focus on the case $\operatorname{area} A > 0$.

Let $0 < t_1 < t_2 < b_{\kappa,\beta}$ and let $(K_i)_{i\in\mathbb{N}}$ be a sequence of compact sets in $\mathcal{S}_A^+(t)$ such that $\operatorname{area} K_i \nearrow \operatorname{area} \mathcal{S}_A^+(t_2)$. Note that such a sequence exists because by Theorem 1.1.7 semi-Riemannian measures are Radon measures, which have precisely this property. Also note that $\mathcal{S}_A^+(t_2)$ is a well defined set even if $t_2 > \sup \tau_\Sigma$ because then, by its definition, $\mathcal{S}_A^+$ is simply the empty set. Every point in $\mathcal{S}_A^+(t_2)$ can be connected to Σ via a unique maximizing geodesic which is an integral curve of the vector field $\mathrm{n} = -\operatorname{grad} \tau_\Sigma$. So for $t \in [0, t_2]$ we define:

$$K_i(t) := \mathrm{Fl}^{\mathrm{n}}_{t-t_2}(K_i) \subset \mathcal{S}_A^+(t)$$

Pick any $i \in \mathbb{N}$, then without loss of generality $\operatorname{area} K_i(t) > 0$. As the image of a compact set under a continuous function the set $K_i(t)$ is still compact, so we can apply the first variation of area formula from Corollary 1.3.8 to compute the rate of change of its area:

$$\begin{aligned}
\frac{d}{dt}\log(\operatorname{area} K_i(t)) &= \frac{1}{\operatorname{area} K_i(t)} \int_{K_i(t)} H_t \, d\mu_t \leq \\
&\leq \frac{1}{\operatorname{area} K_i(t)} \int_{K_i(t)} H_{\kappa,\beta}(t) \, d\mu_t = \\
&= H_{\kappa,\beta}(t) = \\
&= \frac{1}{\operatorname{area}_{\kappa,\beta} S_B^+(t)} \int_{S_B^+(t)} H_{\kappa,\beta}(t) \, d\mu_{\kappa,\beta,t} = \frac{d}{dt}\log(\operatorname{area}_{\kappa,\beta} S_B^+(t))
\end{aligned}$$

Note that the inequality follows from Theorem 2.4.1. So we see that $t \mapsto \frac{\operatorname{area} K_i(t)}{\operatorname{area}_{\kappa,\beta} S_B^+(t)}$ is non increasing on $[0, t_2]$. Consequently we get the inequality

$$\frac{\operatorname{area} K_i(t_2)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_2)} \leq \frac{\operatorname{area} K_i(t_1)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_1)} \leq \frac{\operatorname{area} \mathcal{S}_A^+(t_1)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_1)},$$

and taking the limit

$$\frac{\operatorname{area} \mathcal{S}_A^+(t_2)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_2)} = \lim_{i\to\infty} \frac{\operatorname{area} K_i(t_2)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_2)} \leq \frac{\operatorname{area} \mathcal{S}_A^+(t_1)}{\operatorname{area}_{\kappa,\beta} S_B^+(t_1)}.$$

To finish the proof, equation (2.13) clearly holds because of the continuous dependence of the areas on t and equation (2.14) follows immediately by combining (2.12) and (2.13). □

One immediate consequence of this is the following Corollary:

Corollary 2.4.3. *Let $\kappa, \beta \in \mathbb{R}$ and let M and Σ satisfy the CCC(κ, β). Let $B \subset \Sigma_{\kappa,\beta}$ be compact and let $A \subset \Sigma$ such that* $\operatorname{area}_{\kappa,\beta} B = \operatorname{area} A > 0$. *Then:*

$$\operatorname{area} \mathcal{S}_A^+(t) \leq \operatorname{area}_{\kappa,\beta} S_B^+(t) \qquad (t \in [0, b_{\kappa,\beta}))$$

Proof. This is simply equation (2.14) with $\frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B} = 1$. □

Next up we would like to get a similar result for the volume of $\mathcal{B}_A^+(t)$. Due to the coarea formula in Corollary 1.3.12 we would expect the area comparison result from Theorem 2.4.2 to directly translate to a volume comparison result. However, we need an additional result about the monotonicity of fractions of integrals first.

Lemma 2.4.4. *Let $f, g : [a,b] \to [0,\infty)$ be locally integrable and nonzero on $[a,b]$. Assume $\frac{f}{g}$ is nonincreasing on $[a,b]$, then the functions $F, G : [a,b] \to (0,\infty)$,*

$$F(x) = \int_a^x f(y)\,dy, \qquad G(x) = \int_a^x g(y)\,dy,$$

are continuous and $\frac{F}{G}$ is nonincreasing on $(a,b]$.

Proof. First note that F and G are well defined due to local integrability of f and g. Continuity can easily be shown using the dominated convergence Theorem (see for example [3, Theorem 4.5.6]) so it remains to prove the monotonicity assertion.

Choose $r \leq R$ in $[a,b]$ then

$$\begin{aligned}
\int_a^r f(x)\,dx \cdot \int_r^R g(x)\,dx &= \int_a^r \frac{f(x)}{g(x)} g(x)\,dx \cdot \int_r^R \frac{g(x)}{f(x)} f(x) \geq \\
&\geq \int_a^r \frac{f(r)}{g(r)} g(x)\,dx \cdot \int_r^R \frac{g(r)}{f(r)} f(x)\,dx = \\
&= \int_a^r g(x)\,dx \cdot \int_r^R f(x)\,dx
\end{aligned}$$

We use this to get the following inequality:

$$\begin{aligned}
F(r) \cdot G(R) &= \int_a^r f(x)\,dx \cdot \int_a^R g(x)\,dx = \\
&= \int_a^r f(x)\,dx \cdot \int_a^r g(x)\,dx + \int_a^r f(x)\,dx \cdot \int_r^R g(x)\,dx \geq \\
&\geq \int_a^r f(x)\,dx \cdot \int_a^r g(x)\,dx + \int_a^r g(x)\,dx \cdot \int_r^R f(x)\,dx = \\
&= \int_a^R f(x)\,dx \cdot \int_a^r g(x)\,dx = F(R) \cdot G(r)
\end{aligned}$$

Altogether we have:

$$\frac{F(r)}{G(r)} \geq \frac{F(R)}{G(R)}$$

□

Theorem 2.4.5 (Volume Comparison)**.** *Let $\kappa, \beta \in \mathbb{R}$ and let M and Σ satisfy the CCC(κ, β). Let $B \subset \Sigma_{\kappa,\beta}$ be a compact set satisfying* $\operatorname{area}_{\kappa,\beta}(B) > 0$ *and let $A \subset \Sigma$. Then*

$$t \mapsto \frac{\operatorname{vol} \mathcal{B}_A^+(t)}{\operatorname{vol}_{\kappa,\beta} B_B^+(t)} \qquad (t \in [0, b_{\kappa,\beta})) \tag{2.15}$$

is nonincreasing. Moreover we get

$$\lim_{t \searrow 0} \frac{\operatorname{vol} \mathcal{B}_A^+(t)}{\operatorname{vol}_{\kappa,\beta} B_B^+(t)} = \frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B}, \tag{2.16}$$

and

$$\operatorname{vol} \mathcal{B}_A^+(t) \leq \frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B} \cdot \operatorname{vol}_{\kappa,\beta} B_B^+(t) \tag{2.17}$$

for all $t \in [0, b_{\kappa,\beta})$.

Proof. By Corollary 1.3.12 for $t \in [0, b_{\kappa,\beta})$ we have:

$$\operatorname{vol} \mathcal{B}_A^+(t) = \int_0^t \operatorname{area} \mathcal{S}_A^+(s)\, ds \tag{2.18}$$

Let $0 < t_1 < t_2 < b_{\kappa,\beta}$, then we make a case distinction:

The case where $\operatorname{vol} \mathcal{B}_A^+(t_2) = 0$ is obvious, so we first assume that $\operatorname{vol} \mathcal{B}_A^+(t_2) = \infty$. Then by formula (2.18) there must exist some $s_0 \in (0, t_2]$ such that $\operatorname{area} \mathcal{S}_A^+(s_0) = \infty$. By Theorem 2.4.2 it follows that $\operatorname{area} \mathcal{S}_A^+(s) = \infty$ for all $s \in (0, s_0)$ and consequently $\operatorname{vol} \mathcal{B}_A^+(t_1) = \infty$.

Now let $\operatorname{vol} \mathcal{B}_A^+(t_2) < \infty$. We can assume that $\operatorname{area} \mathcal{S}_A^+(t_2) > 0$ because combining area monotonicity with equation (2.18) tells us that the numerator of the quotient in (2.15) stays constant after the first zero of $\operatorname{area} \mathcal{S}_A^+(t)$. We examine the functions

$$s \mapsto \operatorname{area} \mathcal{S}_A^+(s) \qquad \text{and} \qquad s \mapsto \operatorname{area}_{\kappa,\beta} S_B^+(s).$$

By assumption both functions are nonzero and again by assumption and by equation (2.18) they are locally integrable on $[0, t_2]$. Moreover

$$t \mapsto \frac{\operatorname{area} \mathcal{S}_A^+(t)}{\operatorname{area}_{\kappa,\beta} S_B^+(t)}$$

is nonincreasing on $[0, t_2]$, so we can apply Lemma 2.4.4, which proves monotonicity of (2.15).

To prove equation (2.16), rewrite it with (2.18), note that both numerator and denominator go to 0 for $t \searrow 0$ and use L'Hôpital's rule:

$$\lim_{t \searrow 0} \frac{\operatorname{vol} \mathcal{B}_A^+(t)}{\operatorname{vol}_{\kappa,\beta} B_B^+(t)} = \lim_{t \searrow 0} \frac{\operatorname{area} \mathcal{S}_A^+(t)}{\operatorname{area}_{\kappa,\beta} S_B^+(t)} = \frac{\operatorname{area} A}{\operatorname{area}_{\kappa,\beta} B},$$

where we the final equality follows from equation (2.13). Finally, the last equation (2.17) is an immediate consequence of the first two. $\square$

Just like in the "area case" we immediately get the following Corollary:

Corollary 2.4.6. *Let $\kappa, \beta \in \mathbb{R}$ and let M and Σ satisfy the CCC(κ, β). Let $B \subset \Sigma_{\kappa,\beta}$ be compact and let $A \subset \Sigma$ such that* $\operatorname{area}_{\kappa,\beta} B = \operatorname{area} A > 0$. *Then:*

$$\operatorname{vol} \mathcal{B}_A^+(t) \leq \operatorname{vol}_{\kappa,\beta} B_B^+(t) \qquad (t \in [0, b_{\kappa,\beta}))$$

These comparison theorems allow us to find alternative proofs for the Hawking Singularity Theorem as it can for example be found in [11, Theorem 14.55]. We will prove this in two ways. First, directly with the d'Alembertian Comparison Theorem 2.4.1. Then using the Area Comparison Theorem 2.4.2.

Theorem 2.4.7 (Hawking)**.** *Let M be globally hyperbolic, let $\Sigma \subset M$ be a smooth, spacelike, acausal, FCC hypersurface and let (M, Σ) satisfy the CCC(κ, β) with $\kappa = 0$ and $\beta < 0$. Then τ_Σ is bounded above by $\frac{n}{|\beta|}$ and in particular M is timelike geodesically incomplete.*

Proof via d'Alembertian Comparison. Let $q \in J^+(\Sigma)$, then by Theorem 2.1.6 there exists a maximizing unit speed geodesic $\gamma : [0, b] \to M$ which is future directed timelike and starts perpendicular to Σ. We need to show that $b \leq \frac{n}{|\beta|}$. For $t \in (0, b)$ the point $\gamma(t)$ lies in $\mathcal{I}^+(\Sigma)$, so we can use the d'Alembertian Comparison Theorem to see:

$$H_t(\gamma(t)) \leq H_{0,\beta}(t)$$

We can explicitly compute the right hand side of this equation by using Table 2.2 and equation (2.8):

$$H_{0,\beta}(t) = n \cdot \frac{f'_{0,\beta}(t)}{f_{0,\beta}(t)} = \frac{n}{t + \frac{n}{\beta}} = \frac{n}{t - \frac{n}{|\beta|}}$$

For $t \to \frac{n}{|\beta|}$ this converges to $-\infty$. Meanwhile by smoothness of τ_Σ the value of $H_t(\gamma(t)) = -\Box \tau_\Sigma(\gamma(t))$ cannot go to $-\infty$ before b. In particular $b \leq \frac{n}{|\beta|}$ as desired. The fact that M is timelike geodesically incomplete is now obvious. $\square$

Proof via Area Comparison. We claim that $S_\Sigma^+(\frac{n}{|\beta|}) \subset \operatorname{Cut}^+(\Sigma)$. If we can show this then no curve starting in Σ can remain maximizing after reaching Lorentzian arclength $\frac{n}{|\beta|}$. This would immediately give us the bound $\tau_\Sigma \leq \frac{n}{|\beta|}$. It remains to prove the claim.

Assume $q \in S_\Sigma^+(\frac{n}{|\beta|}) \setminus \operatorname{Cut}^+(\Sigma) = \mathcal{S}_\Sigma^+(\frac{n}{|\beta|})$. The cut locus is closed, so there would exist an open neighbourhood $U \subset \mathcal{S}_\Sigma^+(\frac{n}{|\beta|})$ of q which must satisfy $\operatorname{area} U > 0$. Define $A := \mathrm{Fl}^{\mathrm{n}}_{-\frac{n}{|\beta|}}(U) \subset \Sigma$ and choose any compact set $B \subset \Sigma_{0,\beta}$ such that $\operatorname{area}_{0,\beta} > 0$. By Theorem 2.4.2 we have:

$$\begin{aligned}\operatorname{area} \mathcal{S}_A^+(t) &\leq \frac{\operatorname{area} A}{\operatorname{area}_{0,\beta} B} \cdot \operatorname{area}_{0,\beta} S_B^+(t) = \\ &= \frac{\operatorname{area} A}{\operatorname{area}_{0,\beta} B} \cdot \operatorname{area}_{0,\beta} B \cdot \frac{f_{0,\beta}(t)^n}{f_{0,\beta}(0)^n} = \\ &= \operatorname{area} A \cdot \left(\frac{(-t - \frac{n}{\beta})}{-\frac{n}{\beta}} \right)^n\end{aligned}$$

for all $t \in [0, \frac{n}{|\beta|})$. Note that the penultimate equality followed from Remark 2.3.6(iv) while the final one followed from Table 2.2. For $t \to \frac{n}{|\beta|}$ the right hand side goes to 0 so in particular we must have $\operatorname{area} \mathcal{S}_A^+(t_0) = 0$ for some $t_0 \le \frac{n}{|\beta|}$. But this would imply $\operatorname{area} U \le \operatorname{area} \mathcal{S}_A^+(\frac{n}{|\beta|}) = 0$, which is a contradiction. □

Remark 2.4.8. Note that the only thing the above proofs use about $f_{0,\beta}$, is that the function becomes 0 at the finite time $b_{0,\beta} = \frac{n}{|\beta|}$. Something similar can also happen for different pairs (κ, β). For $\kappa < 0$ and $\beta < -n\sqrt{|\kappa|}$ or for $\kappa > 0$ and any $\beta \in \mathbb{R}$, $f_{\kappa,\beta}$ also has a zero at $b_{\kappa,\beta} < \infty$ by Table 2.2. As a consequence, under the same assumptions as before but with (M, Σ) satisfying the $\mathrm{CCC}(\kappa, \beta)$ for these other pairs (κ, β), the spacetime is still timelike geodesically incomplete.

3 Area and Volume Comparison on Null Cones

Our next objective is to prove area comparison theorems, similar to those in the previous chapter, but in a different setting. Instead of observing how the areas of spacelike hypersurfaces change along timelike geodesics, we will observe how areas of sets change as they are transported along null geodesics. Throughout this chapter (M, g) will always be an $(n+1)$-dimensional spacetime. We will follow [6].

3.1 Null Cones

In this section we will develop our new setting for Comparison Theorems. We will examine null cones, which, loosely speaking, are sets consisting of all null geodesics emanating from a point. We give the precise definition as in [7].

Let $p \in M$ be fixed. Denote by $\mathcal{D}$ the maximal domain of $\exp_p$ and denote by N^+ the set of all future directed null vectors in T_pM. Then we define the future null cone:

$$\mathcal{N}^+ := \exp_p(\mathcal{D} \cap N^+)$$

We would like to work on $\mathcal{N}^+$ but quickly note that it might not be a smooth manifold. The simplest example is perhaps a cylinder $\mathbb{R} \times S^1$, embedded in the Minkowski space $\mathbb{R}^3_1$. If we pick $p = (0, 1, 0)$ then two null geodesics going around the cylinder in different directions will meet again in the point $(\pi, -1, 0)$. Clearly the set $\mathcal{N}^+$ is not a smooth manifold in a neighbourhood of this point. In order to avoid this we need to restrict $\exp_p$ to a smaller set.

As a first step towards this, we must normalize null vectors. To this end let $T \in T_pM$ be a unit length, future directed timelike vector. Then we define

$$S_t^+ := \{v \in N^+ \mid \langle v, T\rangle = -t\}$$

and

$$N_t^+ := \bigcup_{s \in (0,t)} S_s^+. \tag{3.1}$$

Now always assuming $t > 0$ is chosen small enough so that all vectors in S_t^+ and N_t^+ are still contained in $\mathcal{D}$, we define

$$\mathcal{S}_t^+ := \exp_p(S_t^+)$$

and

$$\mathcal{N}_t^+ := \exp_p(N_t^+).$$

Clearly S_t^+ and N_t^+ are $(n-1)$, respectively n-dimensional submanifolds of T_pM. We define the null injectivity radius as the supremum over all $t > 0$ such that $\exp_p : N_t^+ \to \mathcal{N}_t^+$ is both bijective and a local diffeomorphism. For the rest of this chapter we fix $t > 0$ as some parameter smaller than the null injectivity radius. With this choice of t we now clearly see that $\mathcal{S}_t^+$ and $\mathcal{N}_t^+$ are submanifolds of M. We will usually use the letter s to describe some parameter $0 < s < t$ such that $\mathcal{S}_s^+ \subset \mathcal{N}_t^+$.

Remark 3.1.1. Note that the sets $S_s^+, N_s^+, \mathcal{S}_s^+$ and $\mathcal{N}_s^+$ as well as the null injectivity radius all depend on the choice of vector T. The results in this chapter should hence be read as holding for all choices of T and all subsequent choices of t smaller than the null injectivity radius depending on T.

Next up we note that the sets $\mathcal{S}_s^+$ are not just smooth submanifolds of M, they are in fact Riemannian submanifolds. To see this we need to show that all tangent vectors in $T_q\mathcal{S}_s^+$ are spacelike. Indeed, by the Gauss Lemma, such vectors v are orthogonal to $\dot\gamma(s)$, where γ is the normalized (with respect to T) null geodesic starting at p and going through q. As a result v cannot be timelike. If it was null, it would have to be a multiple of $\dot\gamma(s)$. This cannot be the case since $T_q \exp_p^{-1}(v)$ and $T_q \exp_p^{-1}(\dot\gamma(s))$ are not multiples of each other either and $\exp_p$ is a local diffeomorphism. Hence v is spacelike and the metric restricted to $\mathcal{S}_s^+$ is Riemannian.

We denote by g_s the metric on $\mathcal{S}_s^+$ and by μ_s the respective Riemannian measure. In order to stick to previous notation, we will write

$$\operatorname{area} \mathcal{S}_s^+ := \mu_s(\mathcal{S}_s^+)$$

In contrast to $\mathcal{S}_s^+$, the manifold $\mathcal{N}_t^+$ is not a semi-Riemannian submanifold of M, since the restriction of the metric is degenerate. To see this we first define the vector field $\ell \in \mathfrak{X}(\mathcal{N}_t^+)$ by assigning to each point $q \in \mathcal{N}_t^+$ the tangent vector of the normalized, future directed null geodesic starting in p and going through q. Then at each such q, $\ell(q)$ is a null vector, which is also orthogonal to all vectors in $T_q\mathcal{S}_{s(q)}^+$ (again by the Gauss Lemma). Here $s(q)$ is the unique parameter such that $q \in \mathcal{S}_{s(q)}^+$. Since $T_q\mathcal{N}_t^+ = T_q\mathcal{S}_{s(q)}^+ \oplus \langle \ell(q) \rangle$ this implies degeneracy.

We will nonetheless try to define certain operations from semi-Riemannian geometry on $\mathcal{N}_t^+$. Firstly, if we have some tensor field $\theta \in \mathcal{T}(\mathcal{N}_t^+)$, we can still take the induced covariant derivative along the generating null geodesics:

$$\theta' = \nabla_{\dot\gamma_\ell}\theta = \nabla_\ell \theta$$

Locally we can smoothly expand the vector field ℓ to a vector field in an open neighbourhood in M. This allows us to also define covariant derivatives of the form $\nabla_v \ell$, for some vector $v \in T\mathcal{N}_t^+$. This operation is independent of the choice of extension (see [11, Lemma 4.1]) and allows us to define an object similar to the shape operator.

Definition 3.1.2. We define the map $S_s : \mathfrak{X}(\mathcal{S}_s^+) \to \mathfrak{X}(\mathcal{S}_s^+)$ by the relation

$$\langle S_s(v), u\rangle := \langle \nabla_v \ell, u\rangle \qquad (u \in T\mathcal{S}_s^+)$$

and call it the null shape operator of $\mathcal{S}_s^+$ with respect to ℓ. We also define the null mean curvature $H_s \in \mathcal{C}^\infty(\mathcal{S}_s^+)$ as $H_s := \operatorname{tr}(S_s)$.

In other words, $S_s(v)$ is the orthogonal projection of $\nabla_v \ell$ onto $T\mathcal{S}_s^+$. This definition should of course be compared to the definition of the shape operator and the mean curvature as in Definition 1.3.6. The main difference between the two definitions is that the shape operator in the usual sense, is defined on a hypersurface, which, up to sign, already fixes a unit normal. For codimension 2 semi-Riemannian submanifolds, such as $\mathcal{S}_s^+$, this is no longer the case. However, with the additional condition that the "unit normal" must also lie in $T\mathcal{N}_t^+$, we come to the conclusion that ℓ is the appropriate vector field with respect to which we define the shape operator. Also note that in Definition 1.3.6 no orthogonal projection was necessary to make it a map from $T\mathcal{S}_s$ to itself, as could be seen in the subsequent remark. The calculation we did there no longer works in the null-case, so using the projection in the definition is really necessary.

We can now use this to obtain a new variant of the first variation of area formula.

Corollary 3.1.3. *The null-mean curvature satisfies:*

$$\frac{d}{ds} \operatorname{area} \mathcal{S}_s^+ = \int_{\mathcal{S}_s^+} H_s \, d\mu_s \tag{3.2}$$

Proof. We want to apply Theorem 1.2.4. To this end first note that the sets $\mathcal{S}_s^+$ are compact, because they are the image of the compact set S_s^+ under the continuous map $\exp_p$. Also note that $\mathcal{S}_{t+\varepsilon}^+ = \operatorname{Fl}_\varepsilon^\ell(\mathcal{S}_s^+)$. So the first variation of area formula gives

$$\frac{d}{dt} \operatorname{area} \mathcal{S}_s^+ = \int_{\mathcal{S}_s^+} \epsilon \operatorname{div}_{\mathcal{S}_s^+} \ell \, d\mu_s,$$

where $\epsilon = 1$ because $\mathcal{S}_s^+$ is Riemannian. Using (1.11) we can calculate the divergence by locally choosing an orthonormal frame $(E_i)_{i=1}^{n-1}$ of $\mathcal{S}_s^+$:

$$\operatorname{div}_{\mathcal{S}_s^+} \ell = \sum_{i=1}^{n-1} \langle \nabla_{E_i} \ell, E_i\rangle = \sum_{i=1}^{n-1} \langle S_s(E_i), E_i\rangle = \operatorname{tr} S_s = H_s$$

This proves the claim. □

Example 3.1.4. Consider the Minkowski space $\mathbb{R}_1^{n+1}$ with $p = 0$ and timelike vector $T = (1, 0, \ldots, 0)$. We write points $q \in \mathbb{R}_1^{n+1}$ as $(t, x_1, \ldots, x_n)$ and write $\|x\|$ for the euclidean norm of the vector $(x_1, \dots, x_n)$. Then the set $\mathcal{S}_s^+ = S_s^+$ consists of all points satisfying

$$s = t = \|x\|,$$

i.e. the $(n-1)$-dimensional sphere of radius s, with center $(t, 0, \dots, 0)$, rotating around the t-axis. We want to calculate the null-shape operator and the null-mean curvature. To do this we use that we can write $\ell = \frac{1}{s}P$, where $P = x^i \frac{\partial}{\partial x^i}$ is the position vector field. Then for $v \in TS_s^+$:

$$\begin{aligned} \nabla_v \ell &= \frac{1}{s}\nabla_v P = \\ &= \frac{1}{s}\nabla_v \left(x^i \frac{\partial}{\partial x^i} \right) = \\ &= \frac{1}{s}\Big(\underbrace{v(x^i)\frac{\partial}{\partial x^i}}_{v} + \underbrace{x^i \nabla_v \frac{\partial}{\partial x^i}}_{0} \Big) = \frac{1}{s} v \end{aligned}$$

Consequently we have

$$S_s = \frac{1}{s}\,\mathrm{id}_{TS_s^+} \qquad \text{and} \qquad H_s = \frac{n-1}{s}.$$

Inserting this into equation (3.2) we obtain the differential equation

$$\frac{d}{ds}\,\mathrm{area}\, S_s^+ = \frac{n-1}{s}\,\mathrm{area}\, S_s^+.$$

Calling the area of S_1^+ ω_{n-1}, which is the same as the area of the $(n-1)$-unit sphere in $\mathbb{R}^n$, this equation has the solution

$$\mathrm{area}\, S_s^+ = \omega_{n-1} s^{n-1},$$

which is of course just the well known formula for the area of the $(n-1)$-sphere of radius s. Next we will show that in a more general setting, these expressions are still the limiting forms of the null shape operator and the null mean curvature as $s \searrow 0$.

Proposition 3.1.5. *The null shape operator and the null mean curvature satisfy*

$$S_s = \frac{1}{s}\,\mathrm{id}_{T\mathcal{S}_s^+} + O(s) \qquad \textit{and} \qquad H_s = \frac{n-1}{s} + O(s),$$

as $s \searrow 0$. *Moreover the area of* $\mathcal{S}_s^+$ *satisfies*

$$\lim_{s \searrow 0} \frac{\mathrm{area}\, \mathcal{S}_s^+}{\mathrm{area}\, S_s^+} = 1, \tag{3.3}$$

where S_s^+ *denotes the corresponding future sphere in Minkowski space, as in Example 3.1.4*

Proof. We choose normal coordinates in a neighbourhood of p. Note that if $v = v^i \frac{\partial}{\partial x^i}\big|_p \in T_pM$, then by the definition of the coordinates we have:

$$x^i(\exp_p(sv)) \frac{\partial}{\partial x^i}\bigg|_p = sv^i \frac{\partial}{\partial x^i}\bigg|_p \tag{3.4}$$

In particular $x^i(\exp_p(sv)) = sv^i$. Due to this the tangent vectors of the geodesic $\exp_p(sv)$ are given by $\exp_p(sv)' = v^i \frac{\partial}{\partial x^i} = \frac{1}{s} x^i \frac{\partial}{\partial x^i}$, where the final equality follows from (3.4). Hence, as in the previous remark, we can write the vector field ℓ as:

$$\ell = \frac{1}{s}\left(x^i \frac{\partial}{\partial x^i}\right)\bigg|_{\mathcal{N}_t^+}$$

We will drop the restriction to $\mathcal{N}_t^+$ and just write ℓ for the vector field of tangent vectors of the geodesics emanating from p. Then:

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} \ell = \frac{1}{s} \nabla_{\frac{\partial}{\partial x^i}} \left(x^j \frac{\partial}{\partial x^j}\right) &= \\ = \frac{1}{s}\left(\frac{\partial x^j}{\partial x^i}\frac{\partial}{\partial x^j} + x^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) &= \\ = \frac{1}{s}\left(\frac{\partial}{\partial x^i} + x^j \Gamma_{ij}^m \frac{\partial}{\partial x^m}\right)&\end{aligned}$$

Now note that both $\Gamma_{ij}^m(\exp_p(sv))$ and $x^j(\exp_p(sv))$ are smooth functions of s on some open interval including 0 and they go to 0 as $s \searrow 0$. In particular they are both $O(s)$. If we restrict everything to $\mathcal{N}_t^+$ this shows $S_s = \frac{1}{s} \operatorname{id}_{T\mathcal{S}_s^+} + O(s)$. The second claim follows immediately by taking the trace of S_s.

Finally we calculate the limit of the area. We choose a chart $(U, (y^1, \ldots, y^{n-1}))$ for the Riemannian manifold $(\mathcal{S}_{s_0}^+, h^{s_0})$, such that U is still contained in our original normal neighbourhood. We can hence also interpret this as a chart of $(S_{s_0}^+, \overline{h^{s_0}})$, noting that the coordinates $h_{ij}^{s_0}$ and $\overline{h_{ij}^{s_0}}$ are different. Moreover for $s \in (0, s_0)$ we obtain charts (U_s, y_s) of the manifolds $(\mathcal{S}_s^+, h^s)$, where $y_s = y \circ \mathrm{Fl}_{s_0 - s}^{\ell}$ is a chart since the flow of ℓ gives a diffeomorphism between $\mathcal{S}_s^+$ and $\mathcal{S}_{s_0}^+$. Again we can also interpret (U_s, y_s) as a chart of S_s^+. Since we are still in our original normal neighbourhood we can express the metric h^s as follows:

$$h_{ij}^s = \left\langle \frac{\partial}{\partial y_s^i}, \frac{\partial}{\partial y_s^j} \right\rangle = \left\langle \frac{\partial x^k}{\partial y_s^i}\frac{\partial}{\partial x^k}, \frac{\partial x^m}{\partial y_s^j}\frac{\partial}{\partial x^m} \right\rangle = \frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j} g_{km}$$

In particular we can write the Riemannian density of $\mathcal{S}_s^+$ on U_s as

$$\mu_{h^s} = \sqrt{\left|\det\left(\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j} g_{km}\right)\right|} \cdot |dy_s^1 \wedge \ldots \wedge dy_s^{n-1}|.$$

Next we claim that $g_{km}(\exp_p(s\ell)) = \varepsilon_k \delta_{km} + O(s^2)$, where ε_k is -1 if $k = 1$ and 1 otherwise. Indeed taking the Taylor expansion of g_{km} around p gives us

$$g_{km}(x) = \underbrace{g_{km}(p)}_{\varepsilon_k\delta_{km}} + \underbrace{\left.\frac{\partial g_{km}}{\partial x^i}\right|_p}_{0} x^i + \frac{1}{2}\left.\frac{\partial^2 g_{km}}{\partial x^i \partial x^j}\right|_p x^i x^j + O(|x|^3),$$

which implies:

$$g_{km}(\exp_p(s\ell)) = \varepsilon_k \delta_{km} + \frac{1}{2}\left.\frac{\partial^2 g_{km}}{\partial x^i \partial x^j}\right|_p s^2 \ell^i \ell^j + O(s^3)$$

In particular we can write the components of the metric g_{km} as $\varepsilon_k\delta_{km} + s^2 b_{km}$, where b_{km} is some bounded function defined around p. So, using differentiability of the determinant we can write:

$$\begin{aligned} &\det\left(\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j}(\varepsilon_k\delta_{km} + s^2 b_{km})\right) = \\ &= \det\left(\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j}\varepsilon_k\delta_{km} + s^2\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j} b_{km}\right) = \\ &= \det\left(\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j}\varepsilon_k\delta_{km}\right) + O(s^2) \end{aligned}$$

In particular, since the square root satisfies $\sqrt{|a+b|} \leq \sqrt{|a|} + \sqrt{|b|}$, we can write the density μ_{h^s} as follows:

$$\mu_{h^s} = \mu_{\overline{h^s}} + O(s) \cdot |dy_s^1 \wedge \ldots \wedge dy_s^{n-1}|,$$

where

$$\mu_{\overline{h^s}} = \sqrt{\left|\det\left(\frac{\partial x^k}{\partial y_s^i}\frac{\partial x^m}{\partial y_s^j}\varepsilon_k\delta_{km}\right)\right|} \cdot |dy_s^1 \wedge \ldots \wedge dy_s^{n-1}|$$

is the coordinate expression of the Riemannian density of S_s^+. Writing area $\mathcal{S}_s^+$ and area S_s^+ as integrals of these densities we can see that their fraction goes to 1 as $s \searrow 0$. $\square$

As in previous chapters the idea going forward will be to apply the first variation of area formula to inequalities of the shape operator, to prove inequalities for areas and volumes. Of course in order to do this we first need inequalities for the null-shape operator. To get those we will once again use Riccati techniques.

For $q \in \mathcal{N}_t^+$ we henceforth denote by P the orthogonal projection from T_qM to $T_q\mathcal{S}_{s(q)}^+$.

Definition 3.1.6. Let $q \in \mathcal{N}_t^+$ then we define:

$$R_\ell : T_q\mathcal{S}_{s(q)}^+ \to T_q\mathcal{S}_{s(q)}^+, \qquad R_\ell(v) := P(R(v,\ell)\ell)$$

We only defined this operator at a single point in $\mathcal{N}_t^+$ but it can also be seen as a tensor field on all of $\mathcal{N}_t^+$. Defining $T\mathcal{S} := \bigcup_{0<s<t} T\mathcal{S}_s^+$, which is a subbundle of $T\mathcal{N}_t^+$, we similarly collect all null-shape operators of sets $\mathcal{S}_s^+$ into one global object.

$$S : TM\big|_{\mathcal{N}_t^+} \to T\mathcal{S}, \qquad S(v) := P(\nabla_v \ell)$$

We see that S restricted to $T\mathcal{S}$ coincides with the shape operator as it was defined in Definition 3.1.2. Before proving that this global null-shape operator once again satisfies a Riccati equation we need to introduce one more technical tool. One of the inconveniences of working on $\mathcal{N}_t^+$ is that we can no longer choose local, orthonormal frames. Instead we will work with a different kind of basis.

Definition 3.1.7. A null basis at $q \in M$ is a basis $(\ell, \mathrm{n}, e_1, \dots, e_{n-1})$ satisfying the following relations:

$$\langle \ell, \mathrm{n} \rangle = -2, \qquad \langle e_i, e_j \rangle = \delta_{ij}, \qquad \langle \ell, \ell \rangle = \langle \mathrm{n}, \mathrm{n} \rangle = \langle e_i, \ell \rangle = \langle e_i, \mathrm{n} \rangle = 0, \tag{3.5}$$

for all $1 \leq i, j \leq n-1$.

Lemma 3.1.8. *Let $q \in \mathcal{N}_t^+$, then there exists a neighbourhood U of q (in $\mathcal{N}_t^+$) and vector fields $\ell, \mathrm{n}, e_1, \dots, e_{n-1} \in \bar{\mathfrak{X}}(U)$ such that for all $q' \in U$ they form a null basis. In addition they satisfy the relations*

$$\nabla_\ell \ell = 0, \qquad \nabla_\ell e_i = \alpha_i \ell, \qquad \nabla_\ell \mathrm{n} = 2 \sum_{i=1}^{n-1} \alpha_i e_i \tag{3.6}$$

for some smooth functions $\alpha_i \in \mathcal{C}^\infty(U)$.

Note that by $\bar{\mathfrak{X}}(U)$ we mean vector fields along the inclusion map $U \hookrightarrow M$.

Proof. Let ℓ be the vector field of null vectors as defined before Definition 3.1.2. By definition, the integral curves of ℓ are null geodesics, so we immediately get $\nabla_\ell \ell = 0$. Now consider the vector bundle $\ell^\perp \subset TM\big|_{\mathcal{N}_t^+}$. The fiber of $\ell^\perp$ at $q \in \mathcal{N}_t^+$ can be written as $\langle \ell \rangle \oplus T_q\mathcal{S}_{s(q)}^+$ so we pick an orthonormal frame $(\tilde{e}_i)_{i=1}^{n-1}$ in a neighbourhood of q in $\mathcal{S}_{s(q)}^+$. At q all relations in (3.5) only involving ℓ and $\tilde{e}_i$ are satisfied. We now need to extend this to an entire neighbourhood in $\mathcal{N}_t^+$. In order to achieve this we define the parallel transport

$$\Gamma : (-\varepsilon, \varepsilon) \times T\mathcal{S}_{s(q)}^+ \to TM,$$

along ℓ. Transporting the orthonormal basis we get vectors $\Gamma_a(\tilde{e}_i) \in T_{\gamma_\ell(s(q)+a)}M$. By definition of the parallel transport, their scalar products remain unchanged, meaning together with ℓ they still form a basis of $T_{\gamma_\ell(s(q)+a)}\mathcal{N}_t^+$. However, $\Gamma_a(\tilde{e}_i)$ might no longer be in $T\mathcal{S}$. To change this we need to take the orthogonal projection onto $T\mathcal{S}_{s(q)+a}^+$. The resulting basis might no longer be orthogonal, which we fix by applying the Gram-Schmidt process. This gives us a frame, for simplicity again called $(\tilde{e}_i)_{i=1}^{n-1}$, which is smooth, because all steps (i.e. parallel transport, orthogonal projection and Gram-Schmidt) can be applied smoothly. Note that all relations in (3.5) only involving ℓ and $\tilde{e_i}$ are already satisfied, but the relations in (3.6) might still fail. Next we calculate

$$\langle \nabla_\ell \tilde{e_i}, \ell\rangle = \ell\langle \tilde{e_i}, \ell\rangle - \langle \tilde{e_i}, \nabla_\ell \ell\rangle = 0 \qquad (1 \leq i \leq n-1),$$

so $\nabla_\ell \tilde{e_i}$ is still in $\ell^\perp$. Consequently we can write

$$\nabla_\ell \tilde{e_i} = \tilde{\alpha}_i \ell + \tilde{\beta}_i^j \tilde{e_j},$$

for some smooth functions $\tilde{\alpha}_i, \tilde{\beta}_i^j$. Calculating the $\tilde{\beta}_i^j$ we see:

$$\tilde{\beta}_i^j = \langle \tilde{\beta}_i^k \tilde{e_k}, \tilde{e_j}\rangle = \langle \nabla_\ell \tilde{e_i}, \tilde{e_j}\rangle = -\langle \nabla_\ell \tilde{e_j}, \tilde{e_i}\rangle = -\langle \tilde{\beta}_j^k \tilde{e_k}, \tilde{e_i}\rangle = -\tilde{\beta}_j^i \tag{3.7}$$

Now let $\Lambda : (-\varepsilon, \varepsilon) \to \mathrm{SO}(n-1)$ be a smooth matrix valued curve to be specified later. We define the vectors $e_i := \Lambda_i^j \tilde{e_j}$, which still form an orthonormal basis. By the same calculation as before we can still express the covariant derivative in direction ℓ as

$$\nabla_\ell e_i = \alpha_i \ell + \beta_i^j \tilde{e_j},$$

for some smooth functions α_i, β_i^j. Note that the left hand side can also be calculated as

$$\begin{aligned}\nabla_\ell e_i = \nabla_\ell(\Lambda_i^j \tilde{e_j}) &= \\ &= (\Lambda_i^j)' \tilde{e_j} + \Lambda_i^j \nabla_\ell \tilde{e_j} = \\ &= (\Lambda_i^j)' \tilde{e_j} + \Lambda_i^j \tilde{\alpha}_i \ell + \Lambda_i^j \tilde{\beta}_j^k \tilde{e_k},\end{aligned}$$

so altogether we have $\beta = \Lambda' + \Lambda\tilde{\beta}$. Now we finally choose Λ to be the solution of the ordinary differential equation:

$$\Lambda'(s) + \Lambda(s)\tilde{\beta}(s) = 0, \qquad \Lambda(s) \to I_{n-1} \quad (\text{as } s \to -\varepsilon)$$

First note that the solution of this equation is indeed in $\mathrm{SO}(n-1)$. Indeed we have

$$\begin{aligned}(\Lambda\Lambda^T)' &= \Lambda'\Lambda^T + \Lambda(\Lambda^T)' = \\ &= -\Lambda\tilde{\beta}\Lambda^T + \Lambda\tilde{\beta}\Lambda^T = 0,\end{aligned}$$

where we used that $\tilde{\beta}^T = -\tilde{\beta}$ according to equation (3.7). So since $\Lambda(s)\Lambda^T(s) \to I_{n-1}$ as $s \to -\varepsilon$, we must have $\Lambda(s)\Lambda^T(s) = I_{n-1}$ for all $s \in (-\varepsilon, \varepsilon)$. So indeed $\Lambda \in \mathrm{SO}(n-1)$. This means that with this choice of Λ we get $\beta = 0$ and hence

$$\nabla_\ell e_i = \alpha_i \ell$$

Finally the vector field n is already uniquely determined by (3.5). It remains to show that $\nabla_\ell \mathrm{n} = 2\sum_{i=1}^{n-1} \alpha_i e_i$. Indeed

$$0 = \ell\langle \mathrm{n}, e_i\rangle = \langle \nabla_\ell \mathrm{n}, e_i\rangle + \langle \mathrm{n}, \nabla_\ell e_i\rangle = \langle \nabla_\ell \mathrm{n}, e_i\rangle + \langle \mathrm{n}, \alpha_i \ell\rangle = \langle \nabla_\ell \mathrm{n}, e_i\rangle - 2\alpha_i,$$

and similarly

$$\langle \nabla_\ell \mathrm{n}, \mathrm{n}\rangle = 0 = \langle \nabla_\ell \mathrm{n}, \ell\rangle,$$

which altogether implies $\nabla_\ell \mathrm{n} = 2\sum_{i=1}^{n-1} \alpha_i e_i$ as desired. □

Let $v \in T_qM$ and let $(\ell, \mathrm{n}, e_1, \ldots, e_{n-1})$ be a null basis at q. Then we can write

$$v = -\frac{1}{2}\langle v, \mathrm{n}\rangle \ell - \frac{1}{2}\langle v, \ell\rangle \,\mathrm{n} + \sum_{i=1}^{n-1} \langle v, e_i\rangle e_i \tag{3.8}$$

To see this we call the right hand side $\pi(v)$ and calculate

$$\begin{aligned}
\langle v - \pi(v), \ell\rangle &= \langle v + \frac{1}{2}\langle v, \ell\rangle \,\mathrm{n}, \ell\rangle = \langle v, \ell\rangle - \langle v, \ell\rangle = 0 \\
\langle v - \pi(v), \mathrm{n}\rangle &= \langle v + \frac{1}{2}\langle v, \mathrm{n}\rangle \ell, \mathrm{n}\rangle = \langle v, \mathrm{n}\rangle - \langle v, \mathrm{n}\rangle = 0 \\
\langle v - \pi(v), e_i\rangle &= \langle v - \langle v, e_i\rangle e_i, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0
\end{aligned}$$

Altogether this gives (3.8) by nondegeneracy of the metric. We will use that vectors can be expressed like this in the proof of the following Theorem.

Theorem 3.1.9. *The null-shape operator along the integral curves of ℓ satisfies the following Riccati equation:*

$$P(\nabla_\ell S + S^2 + R_\ell)\big|_{T\mathcal{S}} = 0 \tag{3.9}$$

For simplicity, in the ensuing proof we will not be as strict with Einstein summation convention and sum over all k, if e_k appears in both sides of some product, even if both k are lower indices.

Proof. We choose a null basis $(\ell, \mathrm{n}, e_1, \ldots, e_{n-1})$ in a neighbourhood of $\mathcal{N}_t^+$, as it was constructed in Lemma 3.1.8. Since in equation (3.9) we are only looking at the restriction to $T\mathcal{S}$, it suffices to prove $\langle e_i, (\nabla_\ell S + S^2 + R_\ell)(e_j)\rangle = 0$ for $1 \le i, j \le n-1$.

$$\begin{aligned}
\langle e_i, (\nabla_\ell S)(e_j)\rangle &= \langle e_i, \nabla_\ell(S(e_j)) - \overbrace{S(\nabla_\ell e_j)}^{S(\alpha_j \ell) = \alpha_j \nabla_\ell \ell = 0} \rangle = \\
&= \langle e_i, \nabla_\ell \nabla_{e_j} \ell\rangle = \\
&= \langle e_i, R(\ell, e_j)\ell + \nabla_{[\ell, e_j]}\ell + \nabla_{e_j}\nabla_\ell \ell\rangle = \\
&= -\langle e_i, R_\ell(e_j) - \nabla_{[e_j, \ell]}\ell\rangle
\end{aligned} \tag{3.10}$$

Now we calculate the Lie bracket:

$$\begin{aligned}
[e_j, \ell] &= \nabla_{e_j}\ell - \nabla_\ell e_j = \\
&= \nabla_{e_j}\ell - \alpha_j \ell = \\
&= \langle \nabla_{e_j}\ell, e_k\rangle e_k - \frac{1}{2}\langle \nabla_{e_j}\ell, \mathrm{n}\rangle \ell - \frac{1}{2}\langle \nabla_{e_j}\ell, \ell\rangle \,\mathrm{n} - \alpha_j \ell = \\
&= \underbrace{\Big(-\frac{1}{2}\langle \nabla_{e_j}\ell, \mathrm{n}\rangle - \alpha_j \Big)}_{=:f} \ell + \langle \nabla_{e_j}\ell, e_k\rangle e_k
\end{aligned}$$

Table 3.1: Comparison functions

$\kappa > 0$	$\mathrm{sn}_\kappa(s) = \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} s)$
$\kappa = 0$	$\mathrm{sn}_\kappa(s) = s$
$\kappa < 0$	$\mathrm{sn}_\kappa(s) = \frac{1}{\sqrt{\lvert\kappa\rvert}} \sinh(\sqrt{\lvert\kappa\rvert} s)$

In the penultimate equality we used (3.8) and in the final one that $\langle \nabla_{e_j}\ell, \ell\rangle = -\langle \nabla_{e_j}\ell, \ell\rangle$, which implies that the expression is 0. Finally we see:

$$\begin{aligned}\langle e_i, \nabla_{[e_j,\ell]}\ell\rangle &= \langle e_i, f \overbrace{\nabla_\ell \ell}^{0}\rangle + \langle e_i, \langle \nabla_{e_j}\ell, e_k\rangle \nabla_{e_k}\ell\rangle = \\ &= \langle e_k, \nabla_{e_j}\ell\rangle\langle e_i, \nabla_{e_k}\ell\rangle = \\ &= \langle e_k, S(e_j)\rangle\langle e_i, S(e_k)\rangle = \\ &= \langle e_i, S(\underbrace{\langle S(e_j), e_k\rangle e_k}_{S(e_j)})\rangle = \langle e_i, S^2(e_j)\rangle\end{aligned}$$

Combining this with (3.10) finishes the proof. □

3.2 Comparison Results

We now have the tools to first prove a comparison result for the null-shape operator, which we can subsequently use for area and volume comparison theorems on the null cone.

Theorem 3.2.1. *Let γ_ℓ be the null geodesic starting in p with normalized starting vector $\ell \in S_1^+$, let $\kappa \in \mathbb{R}$ and let $\tilde{t}$ be the minimum of t and the first positive zero of* sn_κ.

(i) Suppose $\mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \geq \kappa(n-1)$ *along γ_ℓ, then*

$$H_s(\gamma_\ell(s)) \leq (n-1)\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)} \qquad 0 < s < \tilde{t} \tag{3.11}$$

(ii) Suppose $R_\ell(\gamma_\ell(s)) \leq \kappa\, \mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$ along γ_ℓ, then

$$S_s(\gamma_\ell(s)) \geq \frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}\, \mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+} \qquad 0 < s < \tilde{t} \tag{3.12}$$

In particular:

$$H_s(\gamma_\ell(s)) \geq (n-1)\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)} \qquad 0 < s < \tilde{t} \tag{3.13}$$

Proof. We choose a null basis $(\ell, \mathrm{n}, e_1, \ldots, e_{n-1})$ in a neighbourhood of the curve γ_ℓ in $\mathcal{N}_t^+$. Note that this can be done because in the proof of Lemma 3.1.8 we could simply parallelly transport vectors in $T_q\mathcal{S}^+_{s(q)}$ along the entire geodesic γ_ℓ. Let $\mathcal{S}_i^j, \mathcal{R}_i^j$ be the smooth functions along γ_ℓ such that:

$$S(e_i) = \mathcal{S}_i^j e_j, \qquad R_\ell(e_i) = \mathcal{R}_i^j e_j$$

The corresponding matrices $\mathcal{S}$ and $\mathcal{R}$ are symmetric, because the operators S and R_ℓ restricted to $T\mathcal{S}$ are self-adjoint (for S this can be proven just like it was done in Remark 1.3.7(ii)). We want to show that these matrices once again satisfy a Riccati equation. To see this, first note that

$$\begin{aligned} P(\nabla_\ell S)(e_i) &= P(\nabla_\ell(S(e_i)) - S(\nabla_\ell e_i)) = \\ &= P(\nabla_\ell(\mathcal{S}_i^j e_j) - S(\alpha_i \ell)) = \\ &= P((\mathcal{S}_i^j)' e_j + \mathcal{S}_i^j(\nabla_\ell e_j) - \alpha_i \nabla_\ell \ell) = \\ &= P((\mathcal{S}_i^j)' e_j + \mathcal{S}_i^j \alpha_j \ell) = (\mathcal{S}_i^j)' e_j. \end{aligned}$$

This together with Theorem 3.1.9 immediately gives:

$$\mathcal{S}' + \mathcal{S}^2 + \mathcal{R} = 0$$

Now we prove the two claims:

(i) Assume $\mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \geq \kappa(n-1)$, then we have

$$\begin{aligned} \mathrm{tr}(\mathcal{R}) &= \sum_{i=1}^{n-1} \langle e_i, R_\ell(e_i)\rangle = \sum_{i=1}^{n-1} \langle e_i, R(e_i, \dot\gamma_\ell)\dot\gamma_\ell\rangle = \\ &= \mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) + \frac{1}{2}\langle \dot\gamma_\ell, R(\mathrm{n}, \dot\gamma_\ell)\dot\gamma_\ell\rangle + \frac{1}{2}\langle \mathrm{n}, R(\dot\gamma_\ell, \dot\gamma_\ell)\dot\gamma_\ell\rangle = \\ &= \mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \geq \kappa(n-1). \end{aligned} \tag{3.14}$$

We define the function $s_\kappa := \frac{\mathrm{sn}_\kappa'}{\mathrm{sn}_\kappa}$, which satisfies a scalar Riccati equation:

$$s_\kappa'(s) = \left(\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}\right)' = \frac{\mathrm{sn}_\kappa''(s)\,\mathrm{sn}_\kappa(s) - \mathrm{sn}_\kappa'(s)^2}{\mathrm{sn}_\kappa(s)^2} = -|\kappa| - \left(\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}\right)^2 \tag{3.15}$$

By Proposition 3.1.5 we know that as $s \searrow 0$, the mean curvature H_s behaves like $\frac{n-1}{s}$, which implies that $\lim_{s\searrow 0}\left(\frac{\mathrm{tr}\,\mathcal{S}(s)}{n-1} - s_\kappa(s)\right) = 0$. Consequently all conditions to apply Theorem 1.4.5 are met and we get

$$H_s(\gamma_\ell(s)) = \mathrm{tr}\,\mathcal{S}(s) \leq (n-1)\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}.$$

(ii) Now assume $R_\ell(\gamma_\ell(s)) \leq \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$. We define the operators $A(s) := \frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$ and $B(s) := \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$, which satisfy

$$\nabla_\ell A + A^2 + B = 0,$$

due to (3.15). In particular the respective matrix representations $\mathcal{A}$ and $\mathcal{B}$ satisfy

$$\mathcal{A}' + \mathcal{A}^2 + \mathcal{B} = 0.$$

Again by Proposition 3.1.5 we know that as $s \searrow 0$ the operator $S_s(\gamma_\ell(s))$ behaves like $\frac{1}{s}\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$, which implies that $\lim_{s\searrow 0}(\mathcal{S}(s) - \mathcal{A}(s)) = 0$. Now we can apply Theorem 1.4.1 to get $\mathcal{S} \geq \mathcal{A}$ for $0 < s < t$. Switching back to our operators S and A this reads:

$$S_s(\gamma_\ell(s)) \geq \frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$$

Finally, taking the trace gives:

$$H_s(\gamma_\ell(s)) \geq (n-1)\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}$$

□

Remark 3.2.2. (i) Note that in the above Theorem, if $\kappa \leq 0$, then $\tilde{t} = t$ and hence the inequalities for the null-shape operator and the null-mean curvature hold on all of $\mathcal{N}_t^+$. However, for $\kappa > 0$ we need to be more careful since $\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)}$ goes to $-\infty$ as $s \nearrow \frac{\pi}{\sqrt{\kappa}}$.

If we are in the case of Ricci curvature bounded from below, Theorem 1.4.5 tells us that $t < \frac{\pi}{\sqrt{\kappa}}$, implying our results still hold on all of $\mathcal{N}_t^+$. In particular, the null injectivity radius is also bounded from above by $\frac{\pi}{\sqrt{\kappa}}$. If we are in the case of curvature bounded from above, it might happen that $\frac{\pi}{\sqrt{\kappa}} < t$. In this case we no longer have a bound for the null-shape operator or the null-mean curvature on the interval $\left[\frac{\pi}{\sqrt{\kappa}}, t\right)$.

(ii) The curvature assumption in Theorem 3.2.1(ii) is strictly stronger than a Ricci curvature assumption as in (i). Indeed if we had $R_\ell(\gamma_\ell(s)) \geq \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$, then making essentially the same calculation as in (3.14) yields $\mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \geq \kappa(n-1)$. With this stronger assumption we could have gotten a full operator inequality analogous to (3.12), but with the reverse inequality.

Conversely, if in (ii) we had only assumed $\mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \leq \kappa(n-1)$, we could not have recovered any of the results, so making the stronger assumption was indeed necessary. The reason a Ricci bound from above is not a sufficiently strong assumption boils down to the fact that taking the trace of a matrix Riccati equation does not give a scalar Riccati equation, since in general $\mathrm{tr}(S^2) \neq \mathrm{tr}(S)^2$.

Theorem 3.2.3. *Let $\kappa \in \mathbb{R}$ and $\mathrm{Ric}(\dot\gamma_\ell, \dot\gamma_\ell) \geq \kappa(n-1)$ for all null geodesics γ_ℓ, generating $\mathcal{N}_t^+$. Then the function*

$$s \mapsto \frac{\mathrm{area}\,\mathcal{S}_s^+}{\omega_{n-1}\,\mathrm{sn}_\kappa(s)^{n-1}}, \qquad (0 < s < t) \tag{3.16}$$

is nonincreasing. Moreover this quotient goes to 1 *as* $s \searrow 0$ *and*

$$\operatorname{area} \mathcal{S}_s^+ \leq \omega_{n-1} \operatorname{sn}_\kappa(s)^{n-1}. \qquad (0 < s < t) \tag{3.17}$$

Recall that in Example 3.1.4 we already showed that in Minkowski space we have $\operatorname{area} \mathcal{S}_s^+ = \omega_{n-1} s^{n-1}$. In particular this means that for $\kappa = 0$, inequalities (3.16) and (3.17) can be read as comparing areas of future spheres with areas of future spheres in Minkowski space, i.e.

$$s \mapsto \frac{\operatorname{area} \mathcal{S}_s^+}{\operatorname{area}_0 S_s^+}$$

is nonincreasing and we have

$$\operatorname{area} \mathcal{S}_s^+ \leq \operatorname{area}_0 S_s^+.$$

Similar considerations can be made for all subsequent comparison theorems.

Proof. The proof works similar to that of Theorem 2.4.2. Applying Corollary 3.1.3 and Theorem 3.2.1(i) we get:

$$\begin{aligned}
\frac{d}{ds} \log(\operatorname{area} \mathcal{S}_s^+) &= \frac{1}{\operatorname{area} \mathcal{S}_s^+} \int_{\mathcal{S}_s^+} H_s \, d\mu_s \leq \\
&\leq (n-1) \frac{\operatorname{sn}_\kappa'(s)}{\operatorname{sn}_\kappa(s)} = \\
&= (n-1) \frac{d}{ds} \log(\operatorname{sn}_\kappa(s)) = \frac{d}{ds} \log(\operatorname{sn}_\kappa(s)^{n-1})
\end{aligned}$$

In particular $\frac{d}{ds} \log \left(\frac{\operatorname{area} \mathcal{S}_s^+}{\operatorname{sn}_\kappa(s)^{n-1}} \right) \leq 0$ so the term is nonincreasing. Composition with a monotonous function and multiplication by the positive constant ω_{n-1} does not change monotonicity so we get (3.16). Convergence to 1 as $s \searrow 0$ is once again a result of Proposition 3.1.5:

$$\lim_{s \to 0} \frac{\operatorname{area} \mathcal{S}_s^+}{\omega_{n-1} \operatorname{sn}_\kappa(s)^{n-1}} = \lim_{s \to 0} \frac{\operatorname{area} \mathcal{S}_s^+}{\operatorname{area}_0 S_s^+} = 1$$

Note that the use of the area in Minkowski space is warranted because all functions $\operatorname{sn}_\kappa(s)^{n-1}$ are asymptotically equivalent as $s \searrow 0$, independently of κ. Finally the convergence to 1 together with (3.16) immediately gives (3.17). □

As in section 2.4 we would like to use bounds on areas of $\mathcal{S}_s^+$ to derive bounds on the volume of $\mathcal{N}_t^+$. However, we run into the problem that it is not at all clear what the "volume" of $\mathcal{N}_t^+$ even is, since $\mathcal{N}_t^+$ is not a semi-Riemannian manifold and hence does not posses a canonical semi-Riemannian measure. To work around this issue, recall that previously, the way to go from area comparison to volume comparison, was to use the coarea formula from Corollary 1.3.12. We use this as motivation to define the volume of the null cone as

$$\operatorname{vol} \mathcal{N}_s^+ := \int_0^s \operatorname{area} \mathcal{S}_u^+ \, du.$$

Analogously we integrate the comparison function $\omega_{n-1}\operatorname{sn}_\kappa^{n-1}$ to define:

$$\operatorname{vol}_\kappa(s) := \omega_{n-1}\int_0^s \operatorname{sn}_\kappa(u)^{n-1}\,du$$

Theorem 3.2.4. *Let $\kappa \in \mathbb{R}$ and $\operatorname{Ric}(\dot\gamma_\ell,\dot\gamma_\ell) \geq \kappa(n-1)$ for all null geodesics γ_ℓ, generating $\mathcal{N}_t^+$. Then the function*

$$s \mapsto \frac{\operatorname{vol}\mathcal{N}_s^+}{\operatorname{vol}_\kappa(s)}, \qquad (0 < s < t) \tag{3.18}$$

is nonincreasing. Moreover this quotient goes to 1 as $s \searrow 0$ and

$$\operatorname{vol}\mathcal{N}_s^+ \leq \operatorname{vol}_\kappa(s). \qquad (0 < s < t) \tag{3.19}$$

Proof. We define the functions $f(s) := \operatorname{area}\mathcal{S}_s^+$ and $g(s) := \omega_{n-1}\operatorname{sn}_\kappa^{n-1}$, then by Theorem 3.2.3 the function $\frac{f}{g}$ is nonincreasing. By Lemma 2.4.4 we get that

$$\frac{\int_0^s f(u)\,du}{\int_0^s g(u)\,du} = \frac{\operatorname{vol}\mathcal{N}_s^+}{\operatorname{vol}_\kappa(s)},$$

is nonincreasing, which proves (3.19). The behaviour for $s \searrow 0$ follows from L'Hôpital's rule together with the behaviour as $s \searrow 0$ in the area case, as it was shown in Theorem 3.2.3. Finally (3.19) once again follows by combining the first two claims. □

Next we consider the case, where we have a curvature bound from above. Assuming a bound of the form $R_\ell(\gamma_\ell(s)) \leq \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$, we get an area comparison theorem, similar to that of Theorem 3.2.3.

Theorem 3.2.5. *Let $\kappa \in \mathbb{R}$ and $R_\ell(\gamma_\ell(s)) \leq \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$ for all null geodesics γ_ℓ, generating $\mathcal{N}_t^+$. Then the function*

$$s \mapsto \frac{\operatorname{area}\mathcal{S}_s^+}{\omega_{n-1}\operatorname{sn}_\kappa(s)^{n-1}}, \qquad (0 < s < \tilde t) \tag{3.20}$$

is nondecreasing. Moreover this quotient goes to 1 as $s \searrow 0$ and

$$\operatorname{area}\mathcal{S}_s^+ \geq \omega_{n-1}\operatorname{sn}_\kappa(s)^{n-1}. \qquad (0 < s < \tilde t) \tag{3.21}$$

Here $\tilde t$ is defined as the minimum of t and the first positive zero of sn_κ.

Proof. The proof works precisely as the one of Theorem 3.2.3, using (3.13) instead of (3.11). □

At first one might look at this result and immediately try to use it for a volume comparison, just how we used Theorem 3.2.3 to prove Theorem 3.2.4. However, this does not work because no analogue to Lemma 2.4.4 exists. Indeed it is easy to find locally integrable, positive functions f and g such that their quotient is nondecreasing, but the quotient of their integrals is not. Consider $f(x) := e^x$ and $g(x) = 1$, then $\frac{f(x)}{g(x)} = e^x$ is clearly nondecreasing. However

$$\frac{\int_0^x f(u)\,du}{\int_0^x g(u)\,du} = \frac{e^x}{x},$$

is decreasing on $(0, 1)$. As a result we will not be able to recover a full analogue of Theorem 3.2.4 for curvature bounded from above. However, part of the statement can be recovered.

Theorem 3.2.6. *Let $\kappa \in \mathbb{R}$ and $R_\ell(\gamma_\ell(s)) \le \kappa\,\mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$ for all null geodesics γ_ℓ, generating $\mathcal{N}_t^+$. Then*

$$\operatorname{vol}\mathcal{N}_s^+ \ge \operatorname{vol}_\kappa(s), \qquad (0 < s < \tilde{t}) \tag{3.22}$$

where $\tilde{t}$ is defined as the minimum of t and the first positive zero of sn_κ.

Proof. Using (3.21) we calculate:

$$\frac{d}{ds}\operatorname{vol}\mathcal{N}_s^+ = \operatorname{area}\mathcal{S}_s^+ \ge \omega_{n-1}\,\mathrm{sn}_\kappa(s)^{n-1} = \frac{d}{ds}\operatorname{vol}_\kappa(s)$$

Since the quotient of $\operatorname{vol}\mathcal{N}_s^+$ and $\operatorname{vol}_\kappa(s)$ converges to 1 as $s \searrow 0$ we get (3.22). □

3.3 Application

A simple application of the results of the previous section is that they allow to measure how much certain quantities along the null cone deviate from those of the null cone in Minkowski space. Throughout this section we will restrict ourselves to 4-dimensional spacetimes satisfying $\mathrm{Ric} = 0$.

Definition 3.3.1. The area radius of the future sphere $\mathcal{S}_s^+$ is

$$r(s) := \sqrt{\frac{\operatorname{area}\mathcal{S}_s^+}{4\pi}}.$$

This definition is of course motivated by the formula for the area of a 4-dimensional sphere of radius r, which is $4\pi r^2$. In our current setting it is no longer obvious what the radius is, while the area of the set $\mathcal{S}_s^+$ is defined by the metric. So we define the radius by its relation with the area instead of the other way around.

Note that in Minkowski space $\mathbb{R}_1^4$, with point $p = 0$ and vector $T = (1, 0, \dots, 0)$, as in Example 3.1.4, we can now express the null-mean curvature with the area radius:

$$H_s = \frac{n-1}{s} = \frac{2}{r(s)}$$

We are thus interested in estimating $H_s - \frac{2}{r(s)}$. Estimates of such terms are used in the investigation of Cauchy Problems of the Einstein equation as in [9].

Proposition 3.3.2. *Let M be a 4-dimensional Ricci flat spacetime, let $\kappa \geq 0$ and let $R_\ell(\gamma_\ell(s)) \leq \kappa\, \mathrm{id}_{T_{\gamma_\ell(s)}\mathcal{S}_s^+}$ along all null geodesics γ_ℓ generating $\mathcal{N}_t^+$. Then we have:*

$$H_s - \frac{2}{r(s)} \leq 0 \qquad (0 < s < t)$$

and

$$H_s - \frac{2}{r(s)} \geq -2\sqrt{\kappa}\, \tan\left(\frac{\sqrt{\kappa}}{2}s\right) \qquad \left(0 < s < \frac{\pi}{\sqrt{\kappa}}\right)$$

Remark 3.3.3. Note that the right hand side of the final inequality is a continuous, bijective function $[0, \frac{\pi}{\sqrt{\kappa}}) \to [0, -\infty)$. In particular this means that for any $\varepsilon > 0$ we can find some $s_0 > 0$ such that for $0 < s < s_0$:

$$-\varepsilon \leq H_s - \frac{2}{r(s)} \leq 0$$

In other words, we can assume that $H_s - \frac{2}{r(s)}$ is smaller or equal to 0 but arbitrarily close to 0 for s in a small enough time interval

Proof. We can use both parts of Theorem 3.2.1 to obtain:

$$2\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)} \leq H_s \leq \frac{2}{s} \tag{3.23}$$

The first inequality is (3.13) and the second one (3.11). Moreover our area comparison theorems imply

$$4\pi\, \mathrm{sn}_\kappa(s)^2 \leq \mathrm{area}\, \mathcal{S}_s^+ \leq 4\pi s^2,$$

where the first inequality is (3.21) and the second one (3.17). Inserting this into the formula for the area radius gives

$$\mathrm{sn}_\kappa(s) \leq r(s) \leq s. \tag{3.24}$$

Combining the second inequalities in both (3.23) and (3.24) gives

$$H_s \leq \frac{2}{s} \leq \frac{2}{r(s)},$$

which already proves the first claim. For the second claim we use the first inequalities in both (3.23) and (3.24):

$$\begin{aligned} H_s - \frac{2}{r(s)} &\geq 2\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)} - \frac{2}{r(s)} \geq \\ &\geq 2\frac{\mathrm{sn}_\kappa'(s)}{\mathrm{sn}_\kappa(s)} - \frac{2}{\mathrm{sn}_\kappa(s)} = \\ &= s\sqrt{\kappa}\frac{\cos(\sqrt{\kappa}s) - 1}{\sin(\sqrt{\kappa}s)} = -2\sqrt{\kappa}\tan\left(\frac{\sqrt{\kappa}}{2}s\right), \end{aligned}$$

where the final equality is just a trigonometric identity. □

Remark 3.3.4. Note that we can show a similar result by working with the volume radius $r_V(s) := \left(\frac{3\,\mathrm{vol}\,\mathcal{N}_s^+}{4\pi}\right)^{\frac{1}{3}}$ instead. Indeed, with the same bounds on the curvature we again get:

$$H_s - \frac{2}{r_V(s)} \leq 0 \qquad (0 < s < t)$$

and

$$H_s - \frac{2}{r_V(s)} \geq -2\sqrt{\kappa}\,\tan\left(\frac{\sqrt{\kappa}}{2}s\right) \qquad \left(0 < s < \frac{\pi}{\sqrt{\kappa}}\right)$$

The proof works essentially like the proof of Proposition 3.3.2, using volume comparison instead of area comparison. For more details see [6, Proposition 5.3].

4 Volume Singularities

4.1 Definition and basic facts

The final chapter of this thesis will be concerned with a notion of singularities of spacetimes that was proposed by García-Heveling in [5]. Instead of using geodesic incompleteness as a criterion for calling a spacetime singular, as we already did in Theorem 2.4.7, we now use the volume of sets $I^+(p)$ instead. We follow [5].

Definition 4.1.1. A spacetime (M, g) is future (respectively past) volume incomplete if there exists a point $p \in M$ such that $\mathrm{vol}(I^\pm(p)) < \infty$.

We start with the following central result.

Theorem 4.1.2. *Let (M, g) be a chronological spacetime. Then the following are equivalent:*

(i) There exists a $p \in M$ such that $\mathrm{vol}(I^\pm(p)) < \infty$.

(ii) For all $\varepsilon > 0$ there exists a $p_\varepsilon \in M$ such that $\mathrm{vol}(I^\pm(p_\varepsilon)) < \varepsilon$.

Proof. The implication (ii) $\Longrightarrow$ (i) is trivial, so we assume that there exists a $p \in M$ such that $\mathrm{vol}(I^+(p)) < \infty$. The case of $\mathrm{vol}(I^-(p)) < \infty$ works analogously. We define the functions:

$$\begin{aligned} v(q) &:= \mathrm{vol}(I^+(q)) \\ f(q) &:= \inf\{v(x) \mid x \in I^+(q)\} \end{aligned}$$

First note that for $q_1, q_2 \in I^+(p)$ we have $v(q_i) < \infty$ $(i = 1, 2)$, by monotonicity of measures. In fact it is easy to see that for $p \ll q_1 \ll q_2$ we have

$$v(p) > v(q_1) > v(q_2). \tag{4.1}$$

To see this, note that $U := I^+(q_1) \cap I^-(q_2)$ is a nonempty open set, which must hence have positive volume. By achronality of M we have $U \cap I^+(q_2) = \emptyset$, so $v(q_1) \geq \mathrm{vol}(U) + v(q_2) > v(q_2)$. On the other hand with the same p, q_1, q_2 we clearly have,

$$f(p) \leq f(q_1) \leq f(q_2) \tag{4.2}$$

because the infimum over a set is always smaller or equal than the infimum over a subset. Finally we also have

$$v(q) > f(q) \tag{4.3}$$

for all $q \in M$ since we can always find a $\tilde{q} \gg q$ to which we can apply (4.1). If we could show $f(p) = 0$, we would be done, so we assume $f(p) > 0$ and try to obtain a contradiction. We choose a sequence $(q_i)_{i\in\mathbb{N}}$ such that $q_i \ll q_{i+1}$ and:

$$v(q_{i+1}) \leq \frac{v(q_i) + f(q_i)}{2} \tag{4.4}$$

We can find such a sequence because by definition of f for every $C > f(q_i)$ there exists a $q_{i+1} \gg q_i$ such that $f(q_i) \leq v(q_{i+1}) \leq C$. In particular such a q_{i+1} exists for $C = \frac{v(q_i)+f(q_i)}{2}$. $(v(q_i))_{i\in\mathbb{N}}$ and $(f(q_i))_{i\in\mathbb{N}}$ are monotonous sequences, which are bounded from below by $f(p)$ and from above by $v(p)$ respectively, so they have limits $v_\infty > 0$ and f_∞. Next we note that by dominated convergence

$$\mathrm{vol}\left(\bigcap_{i\in\mathbb{N}} I^+(q_i)\right) = \lim_{j\to\infty} \mathrm{vol}\left(\bigcap_{1\leq i\leq j} I^+(q_i)\right) = \lim_{j\to\infty} v(q_j) = v_\infty > 0,$$

so in particular

$$A := \bigcap_{i\in\mathbb{N}} I^+(q_i) \neq \emptyset$$

We choose a point $q_\infty \in A$ and immediately get:

$$f_\infty \overset{(4.2)}{\leq} f(q_\infty) \overset{(4.3)}{<} v(q_\infty) \overset{(4.1)}{\leq} v_\infty \tag{4.5}$$

However if we take the limit in (4.4) we get

$$v_\infty \leq \frac{v_\infty + f_\infty}{2},$$

which implies $v_\infty \leq f_\infty$. This contradicts (4.5) and we have finished our proof. $\square$

Theorem 4.1.2 tells us that if our spacetime is future volume incomplete, and satisfies the mild condition of being chronological, then it must contain points with a future of arbitrarily small volume. Intuitively we think of these points as being close to a singularity, although this singularity is of course not an actual point in our spacetime and hence not a well defined notion. We will hence loosely use the term “volume singularity” while always talking about volume incompleteness when making precise mathematical statements.

Next up we try to answer how the concept of volume incompleteness could potentially be an improvement over geodesic incompleteness. To this end we define the following set:

$$\mathfrak{S} := \{p \in M \mid \mathrm{vol}(I^+(p)) < \infty\}$$

We get the following simple result:

Proposition 4.1.3. *Let M be a chronological spacetime, then the set $\mathfrak{S} \subset M$ is a future set i.e. $I^+(\mathfrak{S}) \subset \mathfrak{S}$.*

Proof. Let $q \in I^+(\mathfrak{S})$, then there exists a $p \in \mathfrak{S}$ such that $q \in I^+(p)$. This implies $I^+(q) \subset I^+(p)$ and by monotonicity of measures we have

$$\mathrm{vol}(I^+(q)) \leq \mathrm{vol}(I^+(p)) < \infty.$$

□

A basic result in Lorentzian Geometry now tells us that the boundary of a future set is a closed, achronal, topological hypersurface (see [11, Corollary 14.27]). In particular the boundary of $\mathfrak{S}$ is an achronal, topological hypersurface. We can hence interpret $\partial\mathfrak{S}$ as a horizon, that can only be crossed one way by observers. This can be strengthened further if we assume our spacetime to be globally hyperbolic:

Proposition 4.1.4. *Let M be globally hyperbolic and let $\gamma : [0, b) \to M$ be an inextendible curve such that either*

(i) γ is future directed timelike and $\gamma(0) \in \mathfrak{S}$ or

(ii) γ is future directed causal and $\gamma(0) \in \mathfrak{S}^\circ$.

Then $\lim_{t\to b} \mathrm{vol}\left(I^+(\gamma(t))\right) = 0$

Proof. (i) By dominated convergence we have:

$$\mathrm{vol}\left(\bigcap_{t\in[0,b)} I^+(\gamma(t))\right) = \lim_{t\to b} \mathrm{vol}\left(I^+(\gamma(t))\right) =: c \tag{4.6}$$

Note that this limit indeed exists because $\mathrm{vol}\left(I^+(\gamma(t))\right)$ is monotonously decreasing and bounded from below by 0. Assume $c > 0$ then the set $A := \bigcap_{t\in[0,b)} I^+(\gamma(t))$ is nonempty, hence we can choose some $q \in A$. The entire curve γ must be contained in $J^+(\gamma(0)) \cap J^-(q)$, which is compact by global hyperbolicity. But future inextendible causal curves in strongly causal spacetimes must leave any compact set [11, Lemma 14.13], which is a contradiction. As a consequence $c = 0$, which proves (i).

(ii) We claim that:

$$\mathrm{vol}\left(\bigcap_{t\in[0,b)} J^+(\gamma(t))\right) = \lim_{t\to b} \mathrm{vol}\left(J^+(\gamma(t))\right)$$

If this was true then by proceeding as in (i) we could show that $\lim_{t\to b} \mathrm{vol}\left(J^+(\gamma(t))\right) = 0$, which would immediately imply (ii) since $I^+(\gamma(t)) \subset J^+(\gamma(t))$. Hence it remains to show the claim. By taking a normal neighbourhood of $\gamma(0)$, which is still contained in $\mathfrak{S}$, it is easy to find a point $p \in I^-(\gamma(0)) \cap \mathfrak{S}$. By the push-up principle we have $J^+(\gamma(0)) \subset I^+(p)$, which implies that $\mathrm{vol}\left(J^+(\gamma(0))\right) < \infty$. As a result we can use dominated convergence to prove the claim. □

Roughly speaking, Proposition 4.1.4 allows us to interpret the set $\mathfrak{S}$ as the union of the regions of M where the spacetime becomes volume singular. Further it strengthens the interpretation of $\partial\mathfrak{S}$ as an event horizon, since it tells us that any causal curve starting in $\mathfrak{S}$ must eventually meet the singularity. This is a clear distinction to geodesic incompleteness, where singularities do not automatically come with this kind of horizon.

Remark 4.1.5. The investigation of singularities in Lorentzian Geometry is motivated by the singularity theorems of Penrose and Hawking, both of which describe geodesically incomplete spacetimes. Nonetheless the geodesic incompleteness has a different interpretation in both settings, due to Hawking's Theorem describing timelike geodesic incompleteness, and Penrose's Theorem describing null geodesic incompleteness. While the former can be interpreted as a resting observer reaching the "end" of the spacetime after some finite time, the latter has no such interpretation. The Penrose singularity theorem is often associated with singularities in black holes, but it does not for example show that the singularity is shielded by an event horizon, which is the weak censorship hypothesis that was formulated by Penrose in [12]. If one was to interpret the set $\mathfrak{S}$ as the interior of black holes, as it is done in [5], a theorem similar to that of Penrose but for volume singularities, would solve this problem due to Proposition 4.1.3. To make this more precise, we would like to have a result similar to the following:

Let M be a spacetime satisfying:

- $\mathrm{Ric}(v, v) \geq 0$ for all null vectors v.
- There exists a Cauchy hypersurface $\Sigma \subset M$ that is not compact
- There exists a compact, achronal, spacelike codimension 2 future trapped surface $S \subset M$ (i.e. the metric contraction of its second fundamental form is past directed timelike).

Then $\mathrm{vol}(I^+(S)) < \infty$.

If this was proven then the boundary of the connected component of $\mathfrak{S}$ which contains S, could be interpreted as the event horizon of the singularity. Since no such result exists for singularities in the geodesic sense, this would be an improvement. The caveat is of course that the statement above is unproven. The hope for a volume-Penrose singularity theorem is therefore mostly grounded in the fact that such a volume version exists for the Hawking singularity theorem. Proving this will be our next objective.

Theorem 4.1.6. *Let M be an $(n+1)$-dimensional globally hyperbolic spacetime and let $\Sigma \subset M$ be a Cauchy hypersurface such that* $\mathrm{area}\,\Sigma < \infty$. *Assume that (M, Σ) satisfy the CCC(κ, β) either for $\kappa < 0$ and $\beta \leq -n\sqrt{|\kappa|}$ or for $\kappa = 0$ and $\beta < 0$ or for $\kappa > 0$ and any $\beta \in \mathbb{R}$. Then M is future volume incomplete.*

Recall that the CCC(κ, β) (cosmological comparison condition) was defined in Definition 2.3.2 and says that $\mathrm{Ric}(v, v) \geq n\kappa$ for all timelike vectors $v \in TM$ and that the mean curvature H of the hypersurface Σ satisfies $H \leq \beta$. In the ensuing proof we will frequently use notions from chapter 2, in particular quantities of comparison geometries as they were defined in Table 2.1 and Table 2.2. Finally also note that the assumption $\operatorname{area}\Sigma < \infty$ is in particular satisfied if Σ is compact.

Proof. For all the permitted choices of κ and β, except $\kappa < 0$ and $\beta = -n\sqrt{|\kappa|}$, the function $f_{\kappa,\beta}$ becomes 0 at $b_{\kappa,\beta}$. Therefore, in these cases, by Remark 2.4.8 the length of timelike geodesics emanating from Σ is bounded from above by $b_{\kappa,\beta} < \infty$. Hence we have

$$I^+(\Sigma) = B^+_\Sigma(b_{\kappa,\beta}) \tag{4.7}$$

Now let A be any set in $\Sigma_{\kappa,\beta} = f^{-1}_{\kappa,\beta}(0) \subset M_{\kappa,\beta}$, which is of positive area. Recall from Proposition 2.2.6 that the future cut locus has volume 0, so Theorem 2.4.5 implies

$$\operatorname{vol} B^+_\Sigma(t) = \operatorname{vol} \mathcal{B}^+_\Sigma(t) \leq \frac{\operatorname{area}\Sigma}{\operatorname{area}_{\kappa,\beta} A} \operatorname{vol}_{\kappa,\beta} B^+_A(t) = \frac{\operatorname{area}\Sigma}{f_{\kappa,\beta}(0)^n} \int_0^t f_{\kappa,\beta}(s)^n \, ds, \tag{4.8}$$

where the final equality follows from Remark 2.3.6(iv). The limit $t \nearrow b_{\kappa,\beta}$ on the left hand side is $\operatorname{vol}(I^+(\Sigma))$ by equation (4.7) and because $B_\Sigma(t) \subset I^+(\Sigma)$ for all $t < b_{\kappa,\beta}$. The limit $t \nearrow b_{\kappa,\beta}$ on the right hand side remains bounded because $b_{\kappa,\beta} < \infty$ and the function $f_{\kappa,\beta}$ is bounded on $[0, b_{\kappa,\beta}]$. So we have shown $\operatorname{vol}(I^+(\Sigma)) < \infty$, which proves that M is future volume incomplete.

It remains to check the case $\kappa < 0$ and $\beta = -n\sqrt{|\kappa|}$. Here we have:

$$f_{\kappa,\beta}(t) := e^{-\sqrt{|\kappa|}t}$$

In contrast to the other cases $f_{\kappa,\beta}$ is defined on all of $[0,\infty)$, so our previous strategy no longer works. Instead, again choosing some $A \subset \Sigma_{\kappa,\beta}$ of positive area, we directly compute the volume:

$$\operatorname{vol}_{\kappa,\beta} B^+_A(t) = \frac{\operatorname{area}_{\kappa,\beta} A}{f_{\kappa,\beta}(0)^n} \int_0^t e^{-\sqrt{|\kappa|}ns} \, ds = \frac{\operatorname{area}_{\kappa,\beta} A}{\sqrt{|\kappa|}n} \cdot \left(1 - e^{-\sqrt{|\kappa|}nt}\right) \leq \frac{\operatorname{area}_{\kappa,\beta} A}{\sqrt{|\kappa|}n}$$

Therefore we have

$$\operatorname{vol} B^+_\Sigma(t) = \operatorname{vol} \mathcal{B}^+_\Sigma(t) \leq \frac{\operatorname{area}\Sigma}{\operatorname{area}_{\kappa,\beta} A} \operatorname{vol}_{\kappa,\beta} B^+_A(t) \leq \frac{\operatorname{area}\Sigma}{\sqrt{|\kappa|}n}. \tag{4.9}$$

We can now finish the proof as in the first case, using (4.9) instead of (4.8). □

Note that in all cases except for $\beta = -n\sqrt{|\kappa|}$ we used that the Lorentzian arclengths of timelike curves emanating from Σ are uniformly bounded by $b_{\kappa,\beta}$. In other words, we used the classical Hawking singularity theorem to prove the volume version. In this sense the classical version is stronger. In fact we even saw that in the case $\beta = -n\sqrt{|\kappa|}$ Theorem 4.1.6 still holds while in general Theorem 2.4.7 fails. Indeed the manifold $M_{\kappa,\beta}$ serves as an example of a timelike geodesically complete spacetime satisfying all conditions from the Hawking singularity theorem, which is future volume incomplete.

This raises the question whether geodesic incompleteness is stronger than volume incompleteness in general. It is easy to find an example showing this does not hold in general.

Example 4.1.7. Let $M = \mathbb{R}^2 \setminus \{(t,x) | t \geq x\}$ be equipped with the usual Minkowski metric restricted to M. The geodesics in this spacetime are just straight lines, so M is clearly both timelike and null geodesically incomplete. However, for each point $p \in M$ the set $J^+(p)$ is an infinitely long rectangle bounded by two null geodesics emanating from p as well as the boundary of M. This set has infinite volume so the manifold is future volume complete. In fact, just like $M_{\kappa,\beta}$ from the previous Theorem, this manifold is globally hyperbolic. So, geodesic incompleteness and volume incompleteness are even independent in the class of globally hyperbolic spacetimes.

4.2 The Singularity of the Schwarzschild Spacetime

Finally, we will examine the singularity of the Schwarzschild metric, a solution of the Einstein field equations modelling a static black hole without electric charge. Throughout this section we fix $m > 0$, which should be interpreted as the mass of the black hole. The construction of the Schwarzschild spacetime follows [11, chapter 13].

We start by constructing the Schwarzschild half-plane. We take the set $P = \mathbb{R} \times (0, \infty)$ and try to endow it with the metric

$$\bar{g} := -h(r)dt^2 + \frac{1}{h(r)}dr^2,$$

where $h(r) = \left(1 - \frac{2m}{r}\right)$. We write “try” because the factor $\frac{1}{h(r)}$ goes to $\pm\infty$ as $r \to 2m$ meaning this metric is not well defined at points $(t, 2m)$. In this sense we have really defined two manifolds $P_1 := \mathbb{R} \times (0, 2m)$ and $P_2 := \mathbb{R} \times (2m, \infty)$. It can be shown that this singularity is removable by using different coordinates (see [11, chapter 13]) but for our purposes it will be enough to look at P_1 and P_2 as two separate spacetimes. Note that h also becomes singular as $r \to 0$. This singularity cannot be removed and is commonly interpreted as a singularity at the center of a black hole. It can be shown that causal curves cannot escape P_1 and that it is indeed both timelike and null geodesically incomplete (see [11, Proposition 13.30]), while causal geodesics starting in P_2 can exist forever. We want to test whether the notion of volume incompleteness behaves similarly, i.e. we want to check whether the manifold P_1 is future volume incomplete while P_2 is not.

To get a feeling for the geometry of this half-plane, we try to find the causal vectors at a point $p \in P_i$. Such a vector $v = a\partial_t + b\partial_r$ must satisfy:

$$0 \geq \bar{g}(v, v) = -a^2 h(r) + \frac{b^2}{h(r)},$$

which is equivalent to

$$\begin{cases} |a| \cdot |h(r)| \geq |b| & \text{if } h(r) > 0 \\ |a| \cdot |h(r)| \leq |b| & \text{if } h(r) < 0 \end{cases} \tag{4.10}$$

We see that $h(r) \to 1$ as $r \to \infty$, so, since the set of vectors $a\partial_t + b\partial_r$ such that $|a| \geq |b|$ is just the set of causal vectors in the Minkowski metric, we get the intuition that far away from the boundary of the half plane, P_2 behaves like Minkowski space. If $r \searrow 2m$ we get $h(r) \searrow 0$ so (4.10) implies that $b \to 0$ and the radial component of causal vectors vanishes as we approach the boundary between P_1 and P_2. Conversely, if $r \nearrow 2m$ we get $h(r) \nearrow 0$. (4.10) now implies that for null vectors $a\partial_t + b\partial_r$, $\frac{|b|}{|a|}$ goes to 0 as $r \nearrow 0$, meaning the radial component again becomes negligible. However, the vectors "above" the null vectors are now spacelike and no longer timelike. In this sense the cone of causal vectors has flipped by 90 degrees and causal vectors now point towards the singularity at $r = 0$. This behaviour precisely describes that causal curves beyond an event horizon can no longer escape a black hole.

To give this model a bit more physical relevance we want to turn the 2-dimensional Schwarzschild plane into a 4-dimensional manifold, modeling the geometry of a black hole in a 4-dimensional spacetime. We do this by taking a warped product of P_i with the unit sphere S^2.

Definition 4.2.1. We define the Schwarzschild interior spacetime and the Schwarzschild exterior spacetime to be the sets:

$$M_i := P_i \times S^2 \qquad (i = 1, 2)$$

with the metric

$$g = \bar{g} + r^2 d\sigma,$$

where σ is the Riemannian metric of the unit sphere S^2.

We choose the time orientation such that the future directed causal vectors are those with negative radial component in M_1 and those with positive time component in M_2. We now show that the Schwarzschild interior is future volume incomplete, while the Schwarzschild exterior is future volume complete, which confirms our intuition for singularities in the Schwarzschild metric.

Proposition 4.2.2. *(i) The Schwarzschild interior spacetime is future volume incomplete, with* $\operatorname{vol}(I^+(p)) < \infty$ *for any* $p \in M_1$.

(ii) The Schwarzschild exterior spacetime is future volume complete.

Proof. (i) Let $\gamma(s) = (t(s), r(s), \tilde{\gamma}(s))$ be a future directed timelike curve starting at $p_0 = (t_0, r_0, \tilde{p}_0) \in M_1$. We have:

$$0 > g(\dot{\gamma}, \dot{\gamma}) = -h(r)\dot{t}^2 + \frac{1}{h(r)}\dot{r}^2 + r^2\sigma(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) \geq -h(r)\dot{t}^2 + \frac{1}{h(r)}\dot{r}^2$$

Because h is negative for $r < 2m$ this is equivalent to:

$$h(r)^2\dot{t}^2 < \dot{r}^2 \tag{4.11}$$

By our consideration of the cone of causal vectors in P_1 it is clear that $r(s)$ is decreasing. Combining this with (4.11) gives

$$h(r_0)^2\dot{t}^2 \leq h(r)^2\dot{t}^2 < \dot{r}^2$$

and in particular

$$|\dot{t}| < |h(r_0)|^{-1} \cdot |\dot{r}|.$$

Since $r \in (0, r_0]$, we must hence have $t \in (\underbrace{t_0 - |h(r_0)|^{-1}r_0}_{=:t_-}, \underbrace{t_0 + |h(r_0)|^{-1}r_0}_{=:t_+})$. We want to use this to compute the volume of $I^+(p_0)$. To do this we will need the semi-Riemannian Fubini Theorem 1.1.10. The function $r : M_1 \to (0, 2m)$ is a semi-Riemannian submersion and on the level sets N_r, the function $t : N_r \to \mathbb{R}$ is again a semi-Riemannian submersion. We can hence compute volumes by computing triple integrals over $(0, 2m)$, $\mathbb{R}$ and S^2. Note however that the manifolds $(0, 2m)$ and $\mathbb{R}$ are equipped with the semi-Riemannian metrics $\frac{1}{h(r)}dr^2$, respectively $-h(r)dt^2$, hence we integrate with respect to the densities $\sqrt{|\frac{1}{h(r)}|}|dr|$, respectively $\sqrt{|h(r)|}|dt|$. Conveniently the factors in front of $|dr|$ and $|dt|$ cancel each other out in the integral, so we can compute:

$$\mathrm{vol}(I^+(p_0)) \leq \int_0^{r_0}\int_{t_-}^{t_+}\int_{S^2} d\mu_\sigma\, dt\, dr = \int_0^{r_0}\int_{t_-}^{t_+} 4\pi r^2\, dt\, dr = \frac{8}{3}\pi|h(r_0)|^{-1}r_0^4 < \infty$$

This finishes the first part.

(ii) Now let $p_0 = (t_0, r_0, \tilde{p}_0) \in M_2$. We try to construct a set of infinite volume, lying in $I^+(p_0)$. As a first step let $\tilde{q}$ be any point in S^2, then there exists a σ-geodesic $\tilde{\gamma}$ in S^2 connecting $\tilde{p}_0$ to $\tilde{q}$. In particular, if the initial speed of $\tilde{\gamma}$ is chosen such that $\sigma(\dot{\tilde{\gamma}}(0), \dot{\tilde{\gamma}}(0)) = \frac{1}{r_0^2}h(r_0)$, we can assume that $\tilde{\gamma}(s)$ meets $\tilde{q}$ for $s \leq \frac{1}{h(r_0)}r_0^2\pi =: s_0$. This is true because the geodesics in S^2 are just parametrizations of great cycles, which connect two opposite poles of the sphere with a curve of length π. Now note that the curve $\gamma(s) = (t_0 + s, r_0, \tilde{\gamma}(s))$ connects p_0 to $(t_0 + s_0, r_0, \tilde{q})$ and satisfies

$$g(\dot{\gamma}, \dot{\gamma}) = -h(r_0) + r_0^2 \cdot \frac{1}{r_0^2}h(r_0) = 0,$$

meaning it is a null curve. Since $\tilde{q} \in S^2$ was arbitrary this means that the set $A_0 := (t_0 + s_0, r_0) \times S^2$ is contained in $J^+(p_0)$. Starting at any point $(t_1, r_0, \tilde{q}_1)$ in A_0 we see that the curve $(t_1 + s, r_0 + Cs, \tilde{q}_1)$ is timelike as long as $h(r_0 + Cs) > \frac{C^2}{h(r_0+Cs)}$. Fixing some $s_1 > 0$, we can find a $C_1 > 0$ such that for all $|C| < C_1$, the curve $(t_1 + s, r_0 + Cs, \tilde{q}_1)$ is timelike on $(0, s_1]$. Therefore the set

$$A_1 := \{(t_1 + s_1, r_0 + Cs_1, \tilde{q}) \,|\, |C| < C_1,\, \tilde{q} \in S^2\}$$

is contained in $J^+(p_0)$. Finally for each point $(t_2, r_2, \tilde{q}_2) \in A_1$, the curve $(t_2 + s, r_2, \tilde{q}_2)$, is timelike, so the set

$$A_2 := \{(t, r, \tilde{q}) \,\big|\, t > t_2,\ r_0 - C_1 s_1 < r < r_0 + C_1 s_1,\ \tilde{q} \in S^2\}$$

is contained in $I^+(p_0)$, by the push-up principle. Now we can compute the volume of A_2 via Fubinis Theorem as in (i) to obtain:

$$\mathrm{vol}(I^+(p_0)) \geq \mathrm{vol}(A_2) = \int_{t_2}^{\infty} \int_{r_0 - C_1 s_1}^{r_0 + C_1 s_1} 4\pi r^2 \, dr \, dt = \infty$$

Since $p_0 \in M_2$ was arbitrary, this finishes the proof. $\square$

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