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Post-Lie algebra structures and decompositions of Lie algebras

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Mina Monadjem BSc MSc

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Abstract

In this thesis we study decompositions of Lie algebras, the existence of post-Lie algebra structures and their connection. Post-Lie algebra structures (PA-structures) appear in various areas of mathematics, such as crystallographic groups and affine structures on Lie groups. We consider the existence of post-Lie algebra structures depending on the algebraic properties of the Lie algebras. This will be done by examining decomposition questions. By a decomposition we mean a Lie algebra $\mathfrak{g}$ that can be written as a vector space sum of two of its subalgebras, i.e. $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$. We want to determine properties of $\mathfrak{g}$ depending on the algebraic properties of $\mathfrak{a}$ and $\mathfrak{b}$. We show that a Lie algebra that is a direct vector space sum of two semisimple subalgebras is semisimple. We will use this result to make statements about post-Lie algebra structures on semisimple Lie algebras. In particular, we show that for a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple, we have that $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic, so we have rigidity in this case. Moreover we want to generalize results about post-Lie algebra structures on semisimple Lie algebras to perfect Lie algebras. We show based on lower dimensional perfect Lie algebras, which pairs admit a post-Lie algebra structure. We see that there exist pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is reductive non-semisimple that admit a post-Lie algebra structure. In addition, we give an existence table, which shows for which pairs $(\mathfrak{g}, \mathfrak{n})$ there exists a post-Lie algebra structure. Furthermore, we investigate commutative PA-structures and generalize some existing results to new classes of PA-structures, namely λ-SPA-structures.

Zusammenfassung

In dieser Arbeit untersuchen wir Zerlegungen von Lie Algebren, die Existenz von Post-Lie Algebra Strukturen und deren Zusammenhang. Post-Lie Algebra Strukturen (PA-Strukturen) treten in verschiedenen Bereichen der Mathematik auf, wie etwa bei kristallographischen Gruppen und affinen Strukturen auf Lie Gruppen. Wir untersuchen die Existenz von Post-Lie Algebra Strukturen in Abhängigkeit von den algebraischen Eigenschaften der Lie Algebren. Dies wird anhand von Zerlegungsfragen beleuchtet. Mit einer Zerlegung einer Lie Algebra meinen wir eine Lie Algebra $\mathfrak{g}$ die wir als Vektorraumsumme von zwei Unteralgebren schreiben können, d.h. $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ wobei $\mathfrak{a}$ und $\mathfrak{b}$ Unteralgebren von $\mathfrak{g}$ sind. Dabei wollen wir Eigenschaften von $\mathfrak{g}$ in Abhängigkeit von den algebraischen Eigenschaften von $\mathfrak{a}$ und $\mathfrak{b}$ bestimmen. Wir zeigen, dass eine Lie Algebra, die die direkte Vektorraumsumme zweier halbeinfacher Unteralgebren ist, halbeinfach ist. Dieses Resultat verwenden wir um Aussagen über Post-Lie Algebra Strukturen auf halbeinfachen Lie Algebren zu treffen. Insbesondere, zeigen wir, dass für Post-Lie Algebra Strukturen auf Paaren $(\mathfrak{g}, \mathfrak{n})$ wo $\mathfrak{g}$ halbeinfach ist, gilt, dass $\mathfrak{g}$ und $\mathfrak{n}$ isomorph sind, also hat man Starrheit in diesem Fall. Weiters wollen wir Resultate über Post-Lie Algebra Strukturen auf halbeinfachen Lie Algebren für perfekte Lie Algebren verallgemeinern. Wir zeigen anhand niedrig dimensionaler Beispiele von perfekten Lie Algebren, welche Paare Post-Lie Algebra Strukturen zulassen. Wir sehen, dass es Paare $(\mathfrak{g}, \mathfrak{n})$, wo $\mathfrak{g}$ perfekt und $\mathfrak{n}$ reduktiv nicht-halbeinfach ist, gibt, die eine PA-Struktur zulassen. Weiters geben wir eine Existenztabelle an, die aufweist, für welche Paare $(\mathfrak{g}, \mathfrak{n})$ Post-Lie Algebra Strukturen existieren. Außerdem behandeln wir kommutative PA-Strukturen und verallgemeinern einige schon existierende Resultate auf eine neue Klasse an PA-Strukturen, nämlich λ-SPA-Strukturen.

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Introduction

Post-Lie algebra structures have appeared in different areas of mathematics. In this thesis we study the relationship between these structures and decompositions of Lie algebras, presenting results on the existence of post-Lie algebra structures.

Chapter 1 is a preliminary chapter recalling notions from algebraic Lie theory, such as representations, semisimple, reductive and solvable Lie algebras and Levi's Theorem. It is for the reader not familiar with certain definitions and theorems necessary for the subsequent chapters. Moreover, we fix notations that we use afterwards.

We then proceed with Section 1.2 and give an overview where post-Lie algebra structures arose and some motivation to study them. Since they appear in various areas, we restrict ourselves to their appearance in the setting of affine actions on Lie groups. We shortly discuss crystallographic groups and affine crystallographic groups before moving on to the algebraic study of post-Lie algebra structures.

In Section 1.3 we introduce post-Lie algebra structures and give some basic examples and results concerning these structures. We recall results on post-Lie algebra structures depending on various properties of the Lie algebras involved. Furthermore, we give a short survey on the connection between post-Lie algebra structures and Rota-Baxter operators.

We start with a survey on decompositions of Lie algebras in Chapter 2. By a decomposition of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra $\mathfrak{g}$ that can be written as a vector space sum of two subalgebras $\mathfrak{a}$ and $\mathfrak{b}$, i.e. $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. In case that $\mathfrak{a}$ and $\mathfrak{b}$ are semisimple subalgebras, we call $\mathfrak{g}$ disemisimple. If in addition, we have that the vector space sum is direct, we call $\mathfrak{g}$ strongly disemisimple. In this chapter, we focus in particular on the following question.

Question. Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ be a vector space sum of two subalgebras $\mathfrak{a}$ and $\mathfrak{b}$. What can we say about the algebraic properties of $\mathfrak{g}$ depending on the properties of $\mathfrak{a}$ and $\mathfrak{b}$?

We give a survey on disemisimple Lie algebras based on [42] and Onishchik's results on decompositions of reductive Lie groups, see [75] and [76]. These results we use to present a result on strongly disemisimple Lie algebras in Section 2.4.1, which is joint work with D. Burde and K. Dekimpe and has been presented in [28].

Theorem. Let $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ be a direct vector space sum of two semisimple subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Then $\mathfrak{g}$ is semisimple and $\mathfrak{g} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\oplus$ denotes a direct Lie algebra sum and $\dot{+}$ denotes a direct vector space sum.

The third chapter concerns the existence of post-Lie algebra structures and is based on [28] and [29]. More precisely, the sections are based on the following, which are both joint works with D. Burde and K. Dekimpe.

- Section 3.1 and 3.2: Burde, D; Dekimpe, K.; Monadjem M.: *Rigidity results for Lie algebras admitting a post-Lie algebra structure.* International Journal of Algebra and Computation, Vol. **32** (2022), Issue 08, 1495-1511.
- Section 3.3: Burde, D; Dekimpe, K.; Monadjem M.: *Post-Lie algebra structures for perfect Lie algebras.* To appear in Communications in Algebra (2024).

We first start by discussing the question of rigidity in Section 3.1.1. By rigidity we mean the following.

Question. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Under which assumptions do we necessarily have $\mathfrak{g} \cong \mathfrak{n}$?

We generalize the result for $\mathfrak{g}$ simple to $\mathfrak{g}$ semisimple using the decomposition theorem from above. We show the following.

Theorem. Suppose there exists a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is arbitrary. Then $\mathfrak{n}$ is semisimple too and $\mathfrak{g} \cong \mathfrak{n}$.

We then look at the case where $\mathfrak{g}$ is reductive with 1-dimensional center in Section 3.1.2. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ there exist pre-Lie algebra structures and they have been classified in [18]. The next natural question is what happens if $\mathfrak{n}$ is nilpotent. We first show that there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ is 2-step nilpotent. We then generalize this to the case where $\mathfrak{n}$ is nilpotent non-abelian.

Section 3.2 then looks at the existence of these structures in general given different assumptions on $\mathfrak{g}$ and $\mathfrak{n}$. In particular, we present new pairs of Lie algebras admitting a post-Lie algebra structure. This is gathered in a table using the assumptions abelian, nilpotent non-abelian, solvable non-nilpotent, simple, semisimple non-simple, reductive non-semisimple and complete non-semisimple. The checkmark means that there is some pair admitting a post-Lie algebra structure, "-" means there is no post-Lie algebra structure and the question mark is the case that remains open.

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ semi	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abe	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ nil	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ sol	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ sim	-	-	-	✓	-	-	-
$\mathfrak{g}$ semi	-	-	-	-	✓	-	-
$\mathfrak{g}$ red	✓	✓	✓	-	?	✓	✓
$\mathfrak{g}$ com	✓	✓	✓	✓	✓	✓	✓

This solves almost all open cases, the case where $\mathfrak{g}$ is reductive non-semisimple and $\mathfrak{n}$ is semisimple remains open.

In Section 3.3 we discuss post-Lie algebra structures for perfect Lie algebras. We want to generalize results obtained for semisimple Lie algebras to perfect Lie algebras. It turns out that here we get different results. We show that there exists a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is reductive non-semisimple. Moreover, there also exists a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is reductive non-semisimple and $\mathfrak{n}$ is perfect non-semisimple.

We discuss lower dimensional perfect Lie algebras and show when a post-Lie algebra structure exists and when not. Here we distinguish results depending on whether $\mathfrak{g}$ or $\mathfrak{n}$ is perfect non-semisimple. We include a column and a row in our existence table using the assumption perfect non-semisimple. However, some cases remain open, since the cohomology and structure of perfect Lie algebras is quite hard.

The fourth chapter discusses classes of post-Lie algebra structures. We begin with stating existing results on commutative post-Lie algebra structures and then generalize this to a new class of post-Lie algebra structures, called λ-SPA-structures. This is a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{,\})$ and $[x, y] = \lambda\{x, y\}$. Here we focus on perfect and complete Lie algebras. We look at the classification of these structures for lower dimensional solvable Lie algebras.

The last chapter is a chapter on open questions that have arisen in the study of post-Lie algebra structures and decompositions of Lie algebras. It gives an outlook on further work and we discuss possible ideas.

Contents

1 Preliminaries

In this chapter we recall some basic definitions and lemmas from the theory of Lie algebras that we use in the subsequent chapters. We then give some background and motivation to study post-Lie algebra structures. At the end of this chapter we introduce post-Lie algebra structures, give some examples and give an overview of what is known about the existence of these structures.

1.1 Some Lie theory

This section is merely a recollection of definitions and lemmas from the theory of Lie algebras. We omit proofs, they can be found in any introductory book on Lie algebras. We refer to [52], [58], [61] and [88] as our main reference for this section.

1.1.1 Basic definitions and examples

Definition 1.1. A *Lie algebra* $\mathfrak{g}$ over a field k is a k-vector space together with a k-bilinear map, called the Lie bracket,

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto [x, y]$$

such that for all $x, y, z \in \mathfrak{g}$, we have

(i) anti-symmetry: $[x, x] = 0$,

(ii) Jacobi-identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 1.2. We call a subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ a *subalgebra* if $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$. We call it an *ideal* if $[\mathfrak{g}, \mathfrak{a}] = [\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$.

Every Lie algebra has two trivial ideals 0 and itself. Note that the commutator $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$. Here are some examples of Lie algebras, that are important in various areas of mathematics.

Example 1.

- $\mathfrak{gl}_n(k)$ denotes the general linear Lie algebra of dimension n^2 and $\mathfrak{sl}_n(k)$, the set of all matrices in $\mathfrak{gl}_n(k)$ with trace zero, is a Lie subalgebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.
- The set of strictly upper triangular matrices forms a Lie algebra $\mathfrak{n}_n(k)$.

Definition 1.3. Let $\mathfrak{g}$ be a Lie algebra and $A \subseteq \mathfrak{g}$ be a subset. We call

$$Z_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} \mid [x, y] = 0 \,\forall y \in A\}$$

the *centralizer* of A in $\mathfrak{g}$. $Z_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$ is called the *center* of $\mathfrak{g}$. Note that the center is always an ideal.

1.1.2 Representations of Lie algebras

The representation theory of Lie algebras has a long history and is useful in both mathematics and physics. We start by defining what a derivation of a k-algebra is.

Definition 1.4. Let $(A, \cdot)$ be a k-algebra. A linear map $D \in \mathrm{End}(A)$ is called a *derivation* of A if for all $x, y \in A$

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y).$$

We denote by $\mathrm{Der}(A)$ the space of derivations of A.

Definition 1.5. For a Lie algebra $\mathfrak{g}$, an ideal that is preserved by derivations is called a *characteristic ideal*. We always have that $[\mathfrak{g}, \mathfrak{g}]$ and $Z(\mathfrak{g})$ are characteristic ideals.

Similar to groups, the representation theory of Lie algebras is crucial when one studies Lie algebras.

Definition 1.6. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A *Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ is a linear map such that $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$ for all $x, y \in \mathfrak{g}$.

Definition 1.7. A *representation* of a Lie algebra $\mathfrak{g}$ is a k-vector space V together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. The representation is called *faithful*, if ρ is injective.
A *subrepresentation* of a representation (ρ, V) is a representation (π, W) such that W is a vector subspace of V and $\rho|_W = \pi$.

Ado [2] and Iwasawa [60] showed that, every finite-dimensional Lie algebra over a field of characteristic zero or of prime characteristic has a faithful representation, meaning we can view every finite-dimensional Lie algebra explicitly.

Definition 1.8. We call a representation V of a Lie algebra $\mathfrak{g}$ *simple* or *irreducible*, if $V \neq 0$, and the zero representation is the only proper subrepresentation of V. We call a representation V of a Lie algebra $\mathfrak{g}$ *semisimple*, if for each subrepresentation there exists a complement.

Given a representation one also speaks of a module structure. These terms are used interchangeably in the literature.

Definition 1.9. Let (ρ, V) be a representation of a Lie algebra $\mathfrak{g}$. Then $x.v = \rho(x)(v)$ defines a k-bilinear operation

$$\begin{aligned} \mathfrak{g} \times V &\to V \\ (x, v) &\mapsto x.v \end{aligned}$$

such that for all $x, y \in \mathfrak{g}$ and $v \in V$, we get

$$[x, y].v = x.(y.v) - y.(x.v).$$

We call V together with this map a $\mathfrak{g}$-*module*.

Example 2. Any k-vector space V together with the trivial action $x.v = 0$ for all $x \in \mathfrak{g}$ and $v \in V$ defines a $\mathfrak{g}$-module structure. $V = k$ together with the trivial action is called the *trivial representation.* The zero vector space together with the trivial action is called the *zero representation.*

Definition 1.10. We call a linear map $\phi : V \to W$ between two $\mathfrak{g}$-modules a $\mathfrak{g}$-*module homomorphism* if it preserves the module structure, i.e.

$$\phi(x.v) = x.\phi(v) \quad \forall v \in V, x \in \mathfrak{g}.$$

One representation that often appears is the adjoint representation.

Example 3.

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto \mathrm{ad}(x),$$

where $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ with $\mathrm{ad}(x)(y) = [x, y]$ defines a representation, called the adjoint representation and $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. $\mathrm{ad}(\mathfrak{g})$ is also known as the space of *inner derivations*, denoted as $\mathrm{ad}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. Analogously to the group case of outer automorphisms, in the Lie algebra case we have the notion of *outer derivations* defined as $\mathrm{Out}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$.

Note that the ideals of a Lie algebra $\mathfrak{g}$ are exactly the subrepresentations of the adjoint representation.

Lemma 1.11. *Given $\phi : V \to W$ a homomorphism of representations of a Lie algebra $\mathfrak{g}$, we have that $\ker(\phi)$ is a subrepresentation of V and $\mathrm{im}(\phi)$ is a subrepresentation of W.*

Definition 1.12. Given a subrepresentation $U \subset V$, there is exactly one representation of $\mathfrak{g}$ on the quotient space V/U such that $\phi : V \to W, v \mapsto v + U$ is a homomorphism of representations. V/U is called the *quotient representation.*

Lemma 1.13. *Every subrepresentation and quotient representation of a semisimple representation is semisimple.*

Definition 1.14. Let V and W be two $\mathfrak{g}$-modules. Then we denote by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ the space of $\mathfrak{g}$-module homomorphisms. Define $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{g}}(V, V)$.

We end this section by stating Schur's Lemma for simple modules.

Proposition 1.15. *(Schur's Lemma) Let V and W be simple $\mathfrak{g}$-modules. Then we have:*

(i) *If $V \ncong W$, then* $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$.

(ii) $\mathrm{End}_{\mathfrak{g}}(V)$ *is a division algebra.*

(iii) *If V is a finite-dimensional k-vector space and k is algebraically closed, then* $\mathrm{End}_{\mathfrak{g}}(V) = k \cdot id$.

1.1.3 Semidirect products of Lie algebras

Given a family of Lie algebras $\mathfrak{g}_i, i \in I$, we can form the direct sum denoted by

$$\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i.$$

Let us denote by (x_i) the elements of $\mathfrak{g}$. Setting $[(x_j), (y_j)] = ([x_j, y_j])$ we get a Lie bracket on $\mathfrak{g}$. Then $\mathfrak{g}$ is called the *direct Lie algebra sum* of the $\mathfrak{g}_i$'s.

Remark 1. We always denote $+$ as a sum of vector spaces, $\dot{+}$ as a direct sum of vector spaces and $\oplus$ as a direct sum of Lie algebras. This notation is used throughout all the chapters.

Definition 1.16. Let $\mathfrak{a}$ be a subalgebra and $\mathfrak{b}$ be an ideal of a Lie algebra $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$ decomposes as a direct vector space sum. Then $\mathfrak{g}$ is called the *inner semidirect sum* or *inner semidirect product* of $\mathfrak{a}$ and $\mathfrak{b}$ denoted by $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$.

Definition 1.17. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras and $\phi : \mathfrak{a} \to \mathrm{Der}(\mathfrak{b})$ be a Lie algebra homomorphism. Then

$$[(x, a), (y, b)] = ([x, y], [a, b] + \phi(x)(b) - \phi(y)(a))$$

defines a Lie bracket on $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$. We denote this Lie algebra by $\mathfrak{g} = \mathfrak{a} \ltimes_\phi \mathfrak{b}$ and it is called the *outer semidirect product* of $\mathfrak{a}$ and $\mathfrak{b}$.

Note that a semidirect product $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ is direct if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$.

Proposition 1.18. *Every outer semidirect product is an inner semidirect product. Conversely, let $\mathfrak{g}$ be an inner semidirect sum of $\mathfrak{a}$ and $\mathfrak{b}$. Then $\mathfrak{a} \ltimes_\phi \mathfrak{b} \to \mathfrak{g}$,$(x, a) \mapsto x + a$ is a Lie algebra isomorphism where ϕ is as above.*

Given a semidirect product $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$, we get a corresponding split short exact sequence, i.e. there exists a split short exact sequence

$$0 \to \mathfrak{b} \to \mathfrak{g} \xrightarrow{\beta} \mathfrak{a} \to 0.$$

This means that there exists a Lie algebra homomorphism (section) $\tau : \mathfrak{a} \to \mathfrak{g}$ such that $\beta \circ \tau = \mathrm{id}$.

1.1.4 Semisimple and reductive Lie algebras

One class of Lie algebras that we are looking at more closely in the next chapters are simple, semisimple and reductive Lie algebras. All Lie algebras are finite-dimensional and over a field of characteristic zero, unless stated otherwise.

Definition 1.19. A Lie algebra $\mathfrak{g}$ is called *simple* if the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$ and the commutator ideal is not zero, that is $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We call a Lie algebra $\mathfrak{g}$ *semisimple*, if it is a direct sum of simple Lie algebras.

Definition 1.20. A Lie algebra $\mathfrak{g}$ is called *reductive* if for every ideal $\mathfrak{a}$ in $\mathfrak{g}$ we have a complementary ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \bigoplus \mathfrak{b}$. A subalgebra $\mathfrak{h}$ of a Lie algebra is called *reductive in* $\mathfrak{g}$ if the adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$ is semisimple.

Note that it follows immediately that a simple Lie algebra is always semisimple. Moreover we have that every semisimple Lie algebra is reductive. A semisimple Lie algebra is also always perfect and complete. Let us recall the definition.

Definition 1.21. A Lie algebra $\mathfrak{g}$ is called *complete*, if $Z(\mathfrak{g}) = 0$ and $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Definition 1.22. A Lie algebra $\mathfrak{g}$ that satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ is called *perfect*.

Lemma 1.23. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ is perfect, reductive and complete.*

The converse of this lemma does not hold. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is reductive but not semisimple. The Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ where $V(2)$ stands for the standard irreducible representation considered as an abelian Lie algebra is perfect but not semisimple. In dimension 2 we have a complete Lie algebra $\mathfrak{r}_2(\mathbb{C})$ which is not semisimple.

Lemma 1.24. *Let $\mathfrak{a} \ltimes \mathfrak{b}$ be the semidirect product of two semisimple Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$. Then $\mathfrak{a} \ltimes \mathfrak{b}$ is also semisimple and $\mathfrak{a} \ltimes \mathfrak{b} \cong \mathfrak{a} \oplus \mathfrak{b}$.*

Theorem 1.25. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then we have the following:*

(i) Every ideal $\mathfrak{a}$ in $\mathfrak{g}$ is reductive and $\mathfrak{g}/\mathfrak{a}$ is reductive.

(ii) We can write $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \bigoplus Z(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

(iii) $\mathfrak{g}$ is semisimple if and only if $Z(\mathfrak{g}) = 0$.

We end this section with Weyl's Theorem.

Theorem 1.26. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra $\mathfrak{g}$. Then ρ is semisimple.*

Remark 2. The converse also holds. We also have that if every finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is semisimple.

1.1.5 Solvable Lie algebras

Another class that we closely examine are nilpotent and solvable Lie algebras. Therefore we recall some results concerning solvable and nilpotent Lie algebras. We assume all Lie algebras to be finite-dimensional and over a field of characteristic zero, unless stated otherwise.

Definition 1.27. A Lie algebra $\mathfrak{g}$ is called *abelian*, if $[\mathfrak{g}, \mathfrak{g}] = 0$.

In analogy to the group case, we define the lower central series and derived series for a Lie algebra $\mathfrak{g}$.

Definition 1.28. Let $\mathfrak{g}$ be a Lie algebra. We define

(i) the *lower central series* $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$;

(ii) the *derived series* $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

Definition 1.29. A Lie algebra $\mathfrak{g} \neq 0$ is called *k-step nilpotent*, if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. A Lie algebra $\mathfrak{g} \neq 0$ is called *k-step solvable*, if $\mathfrak{g}^{(k)} = 0$ and $\mathfrak{g}^{(k-1)} \neq 0$. Note that $\mathfrak{g}^i$ and $\mathfrak{g}^{(j)}$ are characteristic ideals for all i, j.

Example 4. The Heisenberg Lie algebra $\mathfrak{n}_3(\mathbb{C})$ with basis (e_1, e_2, e_3) and Lie bracket $[e_1, e_2] = e_3$ is 2-step nilpotent. The 2-dimensional Lie algebra $\mathfrak{r}_2(\mathbb{C})$ with basis (e_1, e_2) and Lie bracket $[e_1, e_2] = e_2$ is solvable of derived length 2.

Note that every abelian Lie algebra is nilpotent of class 1. We have that $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i$, thus every nilpotent Lie algebra is solvable. The nilpotency class $c(\mathfrak{g})$ of a nilpotent Lie algebra is the length of the lower central series and $d(\mathfrak{g})$ is called the derived length of a solvable Lie algebra, i.e. the length of the derived series.

For nilpotent Lie algebras, we get the following:

Theorem 1.30. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then the following holds:*

(i) For any ideal $\mathfrak{a} \neq 0$ in $\mathfrak{g}$, we have $\mathfrak{a} \cap Z(\mathfrak{g}) \neq 0$. In particular, it holds that $Z(\mathfrak{g}) \neq 0$.

(ii) Every Lie subalgebra and homomorphic image of $\mathfrak{g}$ is nilpotent.

(iii) Let

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

be a short exact sequence where $\mathfrak{a}$ and $\mathfrak{g}$ are nilpotent, and suppose $\mathfrak{a} \subseteq Z(\mathfrak{h})$, then $\mathfrak{h}$ is nilpotent too.

Nilpotency is not an extension property in general. One needs the assumption $\mathfrak{a} \subseteq Z(\mathfrak{h})$, a counterexample is the Lie algebra $\mathfrak{r}_2(\mathbb{C}) = \mathbb{C} \ltimes \mathbb{C}$.

Definition 1.31. We call the maximal nilpotent ideal in a Lie algebra $\mathfrak{g}$ the *nilradical* of $\mathfrak{g}$. We denote it by $\mathrm{nil}(\mathfrak{g})$. Similarly, the largest solvable ideal in $\mathfrak{g}$ is called the *solvable radical of* $\mathfrak{g}$ and we denote it by $\mathrm{rad}(\mathfrak{g})$.

In analogy to the nilpotent case, we can formulate the following for solvable Lie algebras. In this case however we have that solvability is an extension property.

Theorem 1.32. *(i) Every subalgebra and homomorphic image of a solvable Lie algebra $\mathfrak{g}$ is solvable.*

(ii) Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$, where $\mathfrak{a}$ is solvable. Suppose $\mathfrak{g}/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable too.

(iii) Let $\mathfrak{a}$ and $\mathfrak{b}$ be solvable ideals in $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ is also solvable.

For semisimple Lie algebras, every ideal is semisimple, which leads to the following for semisimple Lie algebras.

Lemma 1.33. *Suppose $\mathfrak{g}$ is a semisimple Lie algebra, then* $\mathrm{rad}(\mathfrak{g}) = 0$*.*

Moreover, we have the following.

Corollary 1.34. *Let $\mathfrak{g}$ be a Lie algebra. Then $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ is a nilpotent ideal of $\mathfrak{g}$. We have $D(\mathrm{rad}(\mathfrak{g})) \subseteq \mathrm{nil}(\mathfrak{g})$ for all derivations D of $\mathfrak{g}$.*

1.1.6 Levi's Theorem

In this section we want to mention Levi's Theorem and the theorem of Levi-Malcev. Also here we have that all Lie algebras are finite-dimensional and over a field of characteristic zero. Note that we include zero in this section in the definition of semisimplicity. Levi's Theorem states the following:

Theorem 1.35. *Every short exact sequence of Lie algebras*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{s} \to 0$$

with a semisimple Lie algebra $\mathfrak{s}$ splits.

Moreover, we want to state that every Lie algebra admits a Levi complement. So let us start by defining a Levi complement.

Definition 1.36. Let $\mathfrak{g}$ be a Lie algebra. A *Levi complement* is a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ with $\mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$.

Corollary 1.37. *For every Lie algebra $\mathfrak{g}$ there exists a Levi complement in $\mathfrak{g}$.*

A consequence of Levi's Theorem is the following.

Corollary 1.38. *Let $\mathfrak{s}$ be a Levi complement in $\mathfrak{g}$. Then*

$$[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{s} \ltimes [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})].$$

Lemma 1.39. *Let $\mathfrak{g}$ be a Lie algebra and D a nilpotent derivation. Then e^D is an automorphism on $\mathfrak{g}$.*

Now we state what is known as Levi-Malcev Theorem.

Theorem 1.40. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ two Levi subalgebras of $\mathfrak{g}$. Then there exists an automorphism $\phi = \exp(\mathrm{ad}(z))$ with $z \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathrm{nil}(\mathfrak{g})$ such that $\phi(\mathfrak{s}_1) = \mathfrak{s}_2$.*

Consequently, we obtain the following results.

Corollary 1.41. *Every semisimple subalgebra $\mathfrak{h}$ is contained in a Levi subalgebra of $\mathfrak{g}$.*

Corollary 1.42. *The Levi complements of* $\mathfrak{g}$ *are exactly the maximal semisimple subalgebras of* $\mathfrak{g}$.

Corollary 1.43. *Let* $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$ *and let* $\mathfrak{g} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$ *be a Levi decomposition of* $\mathfrak{g}$. *Then*

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes (\mathfrak{a} \cap \operatorname{rad}(\mathfrak{g}))$$

is a Levi decomposition of $\mathfrak{a}$.

We end this chapter by stating a decomposition result for the radical of a semidirect product, which we have not found explicitly in any introductory book or literature.

Lemma 1.44. *Let* $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ *be a semidirect product of a Lie algebra* $\mathfrak{a}$ *and a solvable Lie algebra* $\mathfrak{b}$. *Let* $\mathfrak{a} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{a})$ *be a Levi decomposition of* $\mathfrak{a}$. *Then we have*

$$\operatorname{rad}(\mathfrak{g}) = \operatorname{rad}(\mathfrak{a}) \ltimes \mathfrak{b}.$$

Proof. We have that $\mathfrak{b}$ is an $\mathfrak{a}$-module by construction, so it is a $\operatorname{rad}(\mathfrak{a})$-module, hence we have $\operatorname{rad}(\mathfrak{a}) \ltimes \mathfrak{b}$. Let us write $\mathfrak{h} = \operatorname{rad}(\mathfrak{a}) \ltimes \mathfrak{b}$. We have that $\mathfrak{h}$ is solvable, because the semidirect product of solvable Lie algebras is solvable by Theorem 1.32. We claim that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Indeed, we have

$$[\mathfrak{g}, \mathfrak{h}] = [\mathfrak{a} + \mathfrak{b}, \operatorname{rad}(\mathfrak{a}) + \mathfrak{b}] \subseteq [\mathfrak{a}, \operatorname{rad}(\mathfrak{a})] + [\mathfrak{a}, \mathfrak{b}] + [\mathfrak{b}, \operatorname{rad}(\mathfrak{a}) + \mathfrak{b}] \subseteq \operatorname{rad}(\mathfrak{a}) + \mathfrak{b} \subseteq \mathfrak{h}.$$

Hence $\mathfrak{h}$ is a solvable ideal of $\mathfrak{g}$, so that $\mathfrak{h} \subseteq \operatorname{rad}(\mathfrak{g})$. Let $r \in \operatorname{rad}(\mathfrak{g})$, then there exists a unique $s \in \mathfrak{s}$ and a unique $h \in \mathfrak{h}$ such that $r = s + h$. This means that $r - h = s \in \mathfrak{s} \cap \operatorname{rad}(\mathfrak{g}) = 0$ so that $r = h$, which means that $\operatorname{rad}(\mathfrak{g}) \subseteq \mathfrak{h}$. Therefore we get

$$\operatorname{rad}(\mathfrak{g}) = \mathfrak{h} = \operatorname{rad}(\mathfrak{a}) \ltimes \mathfrak{b}.$$

□

1.2 Motivation

We start this chapter first by giving some motivation on post-Lie algebra structures and where they arise. As they appear in various areas, we hope that the following chapter gives an overview and encourages the reader to delve more deeply into the topic.

The first time post-Lie algebra structures, or more precisely post-Lie algebras, appeared was by Valette [92] regarding homology of partition posets and Koszul operads. Loday [69] has also studied them in connection with generalized bialgebras. They are also related to Rota-Baxter operators and the classical Yang-Baxter equation, see [37] and [8] and the references therein. In [31] it was found that they appear in a different setting as well, which we will give a short survey for, based on [22] and the references therein. Here they appear in the context of affine actions on Lie groups or nil-affine crystallographic actions, as generalizations of pre-Lie algebra structures.

1.2.1 Crystallographic groups

Let us first start by introducing some matrix groups.

Definition 1.45. Let $E(n)$ be the *isometry group* of $\mathbb{R}^n$, which is given by

$$E(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in O_n(\mathbb{R}), v \in \mathbb{R}^n \right\}.$$

We have that the group of translations form a normal subgroup of $E(n)$

$$T(n) = \left\{ \begin{pmatrix} E_n & v \\ 0 & 1 \end{pmatrix} \;\middle|\; v \in \mathbb{R}^n \right\}$$

and $E(n) \cong O_n(\mathbb{R}) \ltimes T(n) \cong O_n(\mathbb{R}) \ltimes \mathbb{R}^n$. Moreover, we have that $E(n)$ acts on the Euclidean space $\mathbb{R}^n$ by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av + b \\ 1 \end{pmatrix}.$$

One can generalize this notion to introduce the affine group of the Euclidean space $\mathbb{R}^n$.

Definition 1.46. The group

$$A(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in GL_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$$

is called the *affine group* of the Euclidean space $\mathbb{R}^n$. We have $A(n) \cong GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$.

Definition 1.47. A subgroup Γ of $E(n)$ is called a *Euclidean crystallographic group* (ECG) if Γ is discrete and cocompact, i.e. $\mathbb{R}^n/\Gamma$ is compact.

These crystallographic groups are known as wallpaper groups in the two-dimensional Euclidean space and as symmetry groups of crystals in the three dimensional Euclidean space, hence the name crystallographic groups. There are exactly up to isomorphism 17 wallpaper groups, see [50]. Moreover, these wallpaper groups appear in arts and architecture. It is conjectured, that all of the 17 wallpaper groups can be found in the Alhambra in Granada in Spain [48]. Federov and Schönflies proved that there are 219 three dimensional crystallographic groups, see [50].

Hilbert 18 [57] raises the question of whether there are only finitely many different crystallographic groups in any dimension. The answer to that is positive and became known as Bieberbach's theorem, see [15], [16].

Theorem 1.48 ((Bieberbach 1))**.** *Let Γ be a Euclidean crystallographic group. Then the translation subgroup $\mathbb{Z}^n$ is a normal subgroup of Γ where $\Gamma/\mathbb{Z}^n$ is finite.*

Theorem 1.49 ((Bieberbach 2))**.** *Two ECGs are isomorphic if and only if they are conjugated in $A(n)$.*

Theorem 1.50 ((Bieberbach 3))**.** *There are only finitely many ECGs in each dimension.*

The classification of these crystallographic groups in lower dimension has been studied. Up to dimension 6 one knows how many crystallographic groups exist.

In dimension 4 one has the following classification, see [17].

Proposition 1.51. *There are exactly* 4783 *different Euclidean crystallographic groups in dimension* 4*.*

For dimension 5 and dimension 6 one has the following due to Plesken and Schulz [82].

Proposition 1.52. *There are exactly* 222018 *different Euclidean crystallographic groups in dimension* 5 *and* 28927922 *different Euclidean crystallographic groups in dimension* 6*.*

1.2.2 Affine crystallographic groups

Euclidean crystallographic groups have then been generalized to hyperbolic and spherical crystallographic groups, see [22]. Another generalization are affine and nil-affine crystallographic groups, the main motivation for the study of post-Lie algebra structures, see also [22]. We start with a locally compact topological Hausdorff space X and the group of homeomorphisms of X, which we denote by G.

Definition 1.53. A subgroup Γ of G is called *crystallographic* if G acts properly discontinuously and cocompactly on X. We call a continuous action of a group Γ on X a *crystallographic action* if the action is properly discontinuous and cocompact.

Definition 1.54. Let $X = \mathbb{A}^n$ be the affine space and $G = A(n)$, then a crystallographic group Γ is called an *affine crystallographic group* (ACG).

The question of whether the Bieberbach Theorems also hold for ACGs was then studied. It was shown that the Bieberbach Theorems do not hold for ACGs. There is a counterexample in dimension 3, see Example 4.2 in [22].

Auslander [5] studied the question of whether every ACG is virtually polycyclic, i.e. has a polycyclic subgroup of finite index, giving a generalization for Bieberbach's First Theorem. He stated, leaving out the compactness of the manifold, that the fundamental group of every complete affine manifold is virtually polycyclic, which is a more general statement. The proof however has a mistake. Also Milnor, see [73], studied this question. He showed that every torsion-free virtually polycyclic group is the fundamental group of some complete flat affine manifold. Milnor conjectured, as Auslander did, that the fundamental group of every complete affine manifold is virtually polycyclic. This however is wrong and a counterexample in dimension 3 was given by Margulis [71].

Proposition 1.55. *There exist non-compact complete affine manifolds in dimension* 3 *with a free non-abelian fundamental group of rank* 2*.*

This lead to the following conjecture, including compactness, which is known as Auslander's conjecture.

Conjecture (Auslander)**.** Every ACG is virtually polycyclic.

This question remains open. There are results in lower dimension. In dimension $n \leq 3$ this was proved by Fried and Goldman, see [51]. For $n \leq 5$ the conjecture also holds and was proved by Tomanov, see [89]. Abels, Margulis and Soifer also studied this question for dimension less than 7, see [1].

At the heart of the study of post-Lie algebra structures or more precisely pre-Lie algebra structures, was the following question posed by Milnor in [73], which used the term left-invariant affine structures on Lie groups.

Question 1 ((Milnor 1977))**.** Does every virtually polycyclic group admit an affine crystallographic action? Does every solvable n-dimensional Lie group G admit a complete left-invariant affine structure, or equivalently, does the universal covering group $\tilde{G}$ operate simply transitively by affine transformations on $\mathbb{R}^n$?

The answer is negative and counterexamples were given by Benoist in [14] and Burde in [19]. There are also counterexamples by Burde and Grunewald in [36]. To provide a negative answer to this question pre-Lie algebra structures were used. Because this does not hold, a natural question was then to generalize this to larger groups, say $\mathrm{Aff}(N) = \mathrm{Aut}(N) \ltimes N$, where N is a connected and simply-connected nilpotent Lie group. Here we have the following for a crystallographic action of $\mathrm{Aut}(N)$.

Definition 1.56. Let $\rho : \Gamma \to \mathrm{Aff}(N)$ be a representation for some connected and simply connected nilpotent Lie group N, where ρ acts properly discontinuously and cocompactly on N. Then ρ is called a *nil-affine crystallographic action* and $\rho(\Gamma)$ is called a *nil-affine crystallographic group.*

One has the following generalization, see [47].

Proposition 1.57. *Every virtually polycyclic group* Γ *admits a nil-affine crystallographic action* $\rho : \Gamma \to \mathrm{Aff}(N)$*.*

Coming back to Milnor's question, although the answer is negative, it is known that for any such G there exists a nilpotent Lie group N and $\rho : G \to \mathrm{Aff}(N)$, letting G act simply transitively on N, see [46], [10]. Studying these pairs of groups (G, N) and translating this to the Lie algebra level gives rise to the notion of a post-Lie algebra structure, which is the subject of the next chapter.

1.3 Post-Lie algebra structures

We already mentioned, where post-Lie algebra structures arise and one motivation to study them. This chapter is dedicated to results regarding the algebraic study of post-Lie algebra structures. We give an overview of what is already known and give some basic tools to study post-Lie algebra structures.

1.3.1 Elementary results and definitions

Let us start by giving the definition of a post-Lie algebra first, see [92].

Definition 1.58. A *post-Lie algebra* $(V, \cdot, \{,\})$ is a vector space V over a field k together with two k-bilinear operations $x \cdot y$ and $\{x, y\}$ such that $(V, \{,\})$ is a Lie algebra and

(1) $\{x, y\} \cdot z = (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z)$

(2) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$

for all $x, y, z \in V$.

If the Lie algebra is abelian, i.e. $\{x, y\} = 0$, then a post-Lie algebra is called a *pre-Lie algebra*. Given a post-Lie algebra one gets another Lie bracket that satisfies certain properties, see [31]. We mention these two results and give a proof here as well.

Proposition 1.59. *Let $(V, \cdot, \{,\})$ be a post-Lie algebra. Then*

$$[x, y] = x \cdot y - y \cdot x + \{x, y\}$$

defines another Lie bracket.

Proof. Let us check anti-symmetry and the Jacobi identity. Let $x, y, z \in V$, then we have

$$[x, y] = x \cdot y - y \cdot x + \{x, y\} = -(y \cdot x - x \cdot y + \{y, x\}) = -[y, x]$$

Now let us check the Jacobi-identity. We have

$$\begin{aligned} [x, [y, z]] &= x \cdot [y, z] - [y, z] \cdot x + \{x, [y, z]\} \\ &= x \cdot (y \cdot z - z \cdot y + \{y, z\}) \\ &\quad - (y \cdot z - z \cdot y + \{y, z\}) \cdot x \\ &\quad + \{x, y \cdot z - z \cdot y + \{y, z\}\} \\ &= x \cdot (y \cdot z) - x \cdot (z \cdot y) + x \cdot \{y, z\} \\ &\quad - (y \cdot z) \cdot x + (z \cdot y) \cdot x - \{y, z\} \cdot x \\ &\quad + \{x, y \cdot z\} - \{x, z \cdot y\} + \{x, \{y, z\}\}. \end{aligned}$$

So we get

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= x \cdot (y \cdot z) - x \cdot (z \cdot y) + x \cdot \{y, z\} \\ &\quad - (y \cdot z) \cdot x + (z \cdot y) \cdot x - \{y, z\} \cdot x \\ &\quad + \{x, y \cdot z\} - \{x, z \cdot y\} + \{x, \{y, z\}\} \\ &\quad + y \cdot (z \cdot x) - y \cdot (x \cdot z) + y \cdot \{z, x\} \\ &\quad - (z \cdot x) \cdot y + (x \cdot z) \cdot y - \{z, x\} \cdot y \\ &\quad + \{y, z \cdot x\} - \{y, x \cdot z\} + \{y, \{z, x\}\} \\ &\quad + z \cdot (x \cdot y) - z \cdot (y \cdot x) + z \cdot \{x, y\} \\ &\quad - (x \cdot y) \cdot z + (y \cdot x) \cdot z - \{x, y\} \cdot z \\ &\quad + \{z, x \cdot y\} - \{z, y \cdot x\} + \{z, \{x, y\}\} \\ &= 0. \end{aligned}$$

$\square$

Proposition 1.60. *Let $(V, \cdot, \{,\})$ be a post-Lie algebra. The associated Lie bracket $[,]$ satisfies the following condition*

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

Proof. Let $x, y, z \in V$, then we have by the first condition of a post-Lie algebra and Proposition 1.59

$$\begin{aligned}
[x, y] \cdot z &= (x \cdot y - y \cdot x + \{x, y\}) \cdot z \\
&= (x \cdot y) \cdot z - (y \cdot x) \cdot z + \{x, y\} \cdot z \\
&= (x \cdot y) \cdot z - (y \cdot x) \cdot z + (y \cdot x) \cdot z \\
&\quad - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z) \\
&= x \cdot (y \cdot z) - y \cdot (x \cdot z).
\end{aligned}$$

$\square$

We see that in fact we have two Lie algebras. Reformulating this gives rise to the notion of a post-Lie algebra structure on pairs of Lie algebras, see [31].

Definition 1.61. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{,\})$ be two Lie algebras having the same underlying vector space. A *post-Lie algebra structure* or short *PA-structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a k-bilinear product $x \cdot y$ on V satisfying the following

(1) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$

(2) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$

(3) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$

for all $x, y, z \in V$.

Remark 3. Let us denote by $L(x)$ the left multiplication operators, i.e. $L(x)(y) = x \cdot y$. The second condition implies that $L : \mathfrak{g} \to \mathrm{End}(V), x \mapsto L(x)$ defines a representation of $\mathfrak{g}$ and the third condition implies that all $L(x)$ are derivations of $\mathfrak{n}$. Thus we can think of a post-Lie algebra structure as a homomorphism from $\mathfrak{g}$ into the derivation algebra of $\mathfrak{n}$, i.e.

$$L : \mathfrak{g} \to \mathrm{Der}(\mathfrak{n}).$$

Remark 4.
- If $\mathfrak{n}$ is abelian, i.e. $\{x, y\} = 0$, then $x \cdot y$ is called a *pre-Lie algebra structure* on $\mathfrak{g}$.
- If $\mathfrak{g}$ is abelian, i.e. $[x, y] = 0$, then $-x \cdot y$ is called an *LR-structure* on $\mathfrak{n}$.

Pre-Lie algebra structures have been widely discussed in the literature, see [20], [65] and the references therein. Also LR-structures have been studied, see [25], [26], [30].

Let us recall Lemma 2.5 from [31].

Lemma 1.62. *Let $x \cdot y$ be a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$. Then the following identities hold:*

$$\begin{aligned}
\{x, y\} \cdot z &= (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z), \\
z \cdot [x, y] &= z \cdot (x \cdot y) - z \cdot (y \cdot x) + z \cdot \{x, y\}, \\
[x \cdot y, z] + [y, x \cdot z] - x \cdot [y, z] &= (x \cdot y) \cdot z - (x \cdot z) \cdot y + y \cdot (x \cdot z) \\
&\quad - x \cdot (y \cdot z) + x \cdot (z \cdot y) - z \cdot (x \cdot y), \\
x \cdot \{y, z\} + y \cdot \{z, x\} + z \cdot \{x, y\} &= \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\}, \\
\{x, y\} \cdot z + \{y, z\} \cdot x + \{z, x\} \cdot y &= \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\} \\
&\quad + [\{x, y\}, z] + [\{y, z\}, x] + [\{z, x\}, y],
\end{aligned}$$

for all $x, y, z \in V$.

Example 5. Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where the Lie brackets of $\mathfrak{g}$ and $\mathfrak{n}$ coincide, i.e. $[x, y] = \{x, y\}$ for all $x, y \in V$. Then $x \cdot y = 0$ defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$ with $[x, y] = \{x, y\}$ for all $x, y \in V$ are called *commutative post-Lie algebra structures*, see [24], and give rise to a certain class of post-Lie algebra structures, which we discuss in Chapter 4.

Throughout the next chapters we give more examples of non-trivial post-Lie algebra structures. In this chapter, we would like to draw more attention to lemmas and propositions already known with regards to post-Lie algebra structures. Throughout this chapter all Lie algebras are finite-dimensional and over a field of characteristic zero, unless stated otherwise.

We recall the definition of an inner post-Lie algebra structure, see [37]. These play an important part in the study of post-Lie algebra structures on pairs of semisimple Lie algebras.

Definition 1.63. Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$. We call the PA-structure *inner* if there exists a $\phi \in \operatorname{End}(V)$ such that

$$x \cdot y = \{\phi(x), y\}$$

for all $x, y \in V$.

For a complete Lie algebra $\mathfrak{n}$, any post-Lie algebra structure is inner, see Lemma 2.9 and Proposition 2.10 in [31].

Lemma 1.64. *Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is complete. Then there is a unique $\phi \in \operatorname{End}(V)$ such that $x \cdot y = \{\phi(x), y\}$, i.e. the PA-structure is inner.*

Proposition 1.65. *Let $x \cdot y$ be a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $Z(\mathfrak{n}) = 0$. Let $\phi \in \operatorname{End}(V)$. Then $x \cdot y = \{\phi(x), y\}$ is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if*

$$\begin{aligned}
\{\phi(x), y\} + \{x, \phi(y)\} &= [x, y] - \{x, y\} \\
\phi([x, y]) &= \{\phi(x), \phi(y)\}
\end{aligned}$$

for all $x, y \in V$.

The next results will be used later on when looking at the existence of post-Lie algebra structures. We already know that we can think of a post-Lie algebra structure as a homomorphism from $\mathfrak{g}$ into the derivation algebra of $\mathfrak{n}$. More generally we get an embedding of $\mathfrak{g}$ into the holomorph of $\mathfrak{n}$, i.e. into $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$. Note that the Lie bracket of $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is given by

$$[(x, D), (x', D')] = (\{x, x'\} + D(x') - D'(x), [D, D']).$$

In [31] the following result was shown. This embedding will be often used in the next chapters.

Proposition 1.66. *Suppose that* $x \cdot y$ *is a PA-structure on a pair* $(\mathfrak{g}, \mathfrak{n})$. *Then*

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

is an injective Lie algebra homomorphism. Conversely, any homomorphism of this form with the identity map on the first component gives a PA-structure on $(\mathfrak{g}, \mathfrak{n})$.

Consequently, one obtains the following, see Proposition 2.11 in [31].

Proposition 1.67. *Post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$ *are in one-to-one correspondence with subalgebras* $\mathfrak{h} \leq \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ *for which the projection* $p_1 : \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \to \mathfrak{n}$ *onto the first factor induces a Lie algebra isomorphism of* $\mathfrak{h}$ *onto* $\mathfrak{g}$.

For a semisimple Lie algebra $\mathfrak{n}$, we have that $\mathfrak{n} \cong \mathrm{Der}(\mathfrak{n}) \cong \mathrm{ad}(\mathfrak{n})$. So we get $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \cong \mathfrak{n} \rtimes \mathfrak{n}$. Moreover, we have that $\mathfrak{n} \rtimes \mathfrak{n} \cong \mathfrak{n} \oplus \mathfrak{n}$ by Lemma 1.24. Therefore the correspondence becomes the following, see [31].

Proposition 1.68. *Let* $\mathfrak{n}$ *be a semisimple Lie algebra. A post-Lie algebra structure on a pair* $(\mathfrak{g}, \mathfrak{n})$ *is in one-to-one correspondence with subalgebras* $\mathfrak{h} \leq \mathfrak{n} \oplus \mathfrak{n}$ *for which* $p_1 - p_2 : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}, (x, y) \mapsto x - y$ *induces an isomorphism of* $\mathfrak{h}$ *onto* $\mathfrak{g}$. *Here* p_i *stands for the projection onto the* $i - th$ *component.*

1.3.2 Examples of post-Lie algebra structures

We recall here some examples in lower dimension and special classes of post-Lie algebra structures from [31]. There we have the following. All Lie algebras are finite-dimensional and over a field of characteristic zero, unless stated otherwise.

Proposition 1.69. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras and* $\lambda \notin \{0, 1\}$. *Then* $x \cdot y = \lambda[x, y]$ *defines a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if* $\{x, y\} = (1 - 2\lambda)[x, y]$ *and both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are nilpotent of class at most* 2.

Analogously, one gets a similar result using $x \cdot y = \mu\{x, y\}$, see [31]. Of course here $\lambda = \frac{1}{2}$ gives a pre-Lie algebra structure on $\mathfrak{g}$. One immediately sees here special classes of post-Lie algebra structures arising, which we discuss in Chapter 4, namely those of commutative post-Lie algebra structures and post-Lie algebra structures where one Lie bracket is given by a scalar of the other.

Classifying post-Lie algebra structures for lower dimensional Lie algebras is quite hard. In dimension two we have the following classification result up to isomorphism, see [31].

Proposition 1.70. *Let $(V, \cdot, \{,\})$ be a complex two dimensional post-Lie algebra. Then V is isomorphic to one of the following algebras.*

V	*Products*	$[,]$	$\{,\}$
V_1	-	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_2	$e_1 \cdot e_1 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_3	$e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_4	$e_1 \cdot e_2 = e_1, e_2 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_5	$e_2 \cdot e_2 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = 0$
V_6	$e_1 \cdot e_1 = e_1, e_2 \cdot e_1 = -e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$
V_7	$e_1 \cdot e_2 = e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$
V_8	$e_2 \cdot e_1 = -e_1$	$[e_1, e_2] = 0$	$\{e_1, e_2\} = -e_1$
$V_9(\alpha)$	$e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = \alpha e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
$V_{10}(\beta), \beta \neq 0$	$e_1 \cdot e_2 = \beta e_1, e_2 \cdot e_1 = (\beta - 1)e_1, e_2 \cdot e_2 = \beta e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{11}	$e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = e_1 - e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{12}	$e_1 \cdot e_1 = e_2, e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = -2e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
V_{13}	$e_1 \cdot e_2 = e_1, e_2 \cdot e_2 = e_1 + e_2$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = 0$
$V_{14,\alpha_1}, \alpha_1 \neq 0$	$e_2 \cdot e_1 = (1 - \alpha_1)e_1$	$[e_1, e_2] = \alpha_1 e_1$	$\{e_1, e_2\} = e_1$
V_{15}	$e_2 \cdot e_2 = e_1$	$[e_1, e_2] = e_1$	$\{e_1, e_2\} = e_1$
$V_{16,\alpha_1}, \alpha_1 \neq 0$	$e_1 \cdot e_2 = -e_1, e_2 \cdot e_1 = -\alpha_1 e_1$	$[e_1, e_2] = \alpha_1 e_1$	$\{e_1, e_2\} = e_1$
V_{17}	$e_1 \cdot e_2 = -e_1, e_2 \cdot e_1 = e_1, e_2 \cdot e_2 = e_1$	$[e_1, e_2] = -e_1$	$\{e_1, e_2\} = e_1$

Moreover, we have the following see Proposition 4.7 in [31].

Example 6. Suppose that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$ or to one of the Lie algebras $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ for $\lambda \neq -1$. Moreover, all of these possibilities occur. Indeed, let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ with Lie bracket $\{e_1, e_2\} = e_3, \{e_1, e_3\} = -2e_1, \{e_2, e_3\} = 2e_2$. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then we have the following structures

- $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $x \cdot y = 0$ or $x \cdot y = [x, y]$.
- $\mathfrak{g} \cong \mathfrak{r}_{3,\lambda}(\mathbb{C})$ and for $\alpha \neq \beta$ two complex parameters, $\beta \neq 0$, $\lambda = \frac{-\alpha}{\beta} \neq -1$

$$
\begin{aligned}
e_2 \cdot e_1 &= -\alpha e_1 + e_3, & e_3 \cdot e_1 &= \frac{2\beta}{\alpha - \beta} e_1 \\
e_2 \cdot e_2 &= \alpha e_2 + \frac{\alpha^2 - \beta^2}{4} e_3, & e_3 \cdot e_2 &= -\frac{2\beta}{\alpha - \beta} e_2 - \beta e_3, \\
e_2 \cdot e_3 &= \frac{\beta^2 - \alpha^2}{2} e_1 - 2e_2, & e_3 \cdot e_3 &= 2\beta e_1
\end{aligned}
$$

and the Lie bracket of $\mathfrak{g}$ is given by

$$\begin{aligned}
[e_1, e_2] &= \alpha e_1, \\
[e_1, e_3] &= -\frac{2\alpha}{\alpha - \beta} e_1, \\
[e_2, e_3] &= \frac{\beta^2 - \alpha^2}{2} e_1 + \frac{2\beta}{\alpha - \beta} e_2 + \beta e_3.
\end{aligned}$$

For semisimple Lie algebras $\mathfrak{n}$, we always get these two examples, see [23].

Example 7. Let $\mathfrak{n}$ be a semisimple Lie algebra. Then $x \cdot y = \{\phi(x), y\}$ with $\phi = 0$ defines a post-Lie algebra structure where $[x, y] = \{x, y\}$ and $x \cdot y = \{\phi(x), y\}$ with $\phi = -\operatorname{id}$ defines a post-Lie algebra structure where $[x, y] = -\{x, y\}$.

1.3.3 Structure results for post-Lie algebra structures

Existence questions for post-Lie algebra structures depend on the algebraic properties of $\mathfrak{g}$ and $\mathfrak{n}$. The results here vary with respect to $\mathfrak{g}$ and $\mathfrak{n}$. We recall some results where we use different assumptions on $\mathfrak{g}$ and $\mathfrak{n}$.

For pre-Lie algebra structures, we have the following see [56].

Proposition 1.71. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ does not admit a pre-Lie algebra structure.*

For the other special case, namely those of LR-structures we have the following, see [26].

Proposition 1.72. *Let $\mathfrak{n}$ be a Lie algebra admitting an LR-structure. Then $\mathfrak{n}$ is solvable of class at most 2.*

In [31] we find a result that gives an embedding of the holomorph of a 2-step nilpotent Lie algebra into the holomorph of an abelian Lie algebra.

Lemma 1.73. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra and $\mathfrak{m}$ be the abelian Lie algebra with the same underlying vector space as $\mathfrak{n}$. Then we have an injective Lie algebra homomorphism*

$$\begin{aligned}
\psi : \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) &\to \mathfrak{m} \rtimes \operatorname{Der}(\mathfrak{m}) \\
(x, D) &\mapsto (x, \frac{1}{2} \operatorname{ad}(x) + D).
\end{aligned}$$

This already leads to the following, see Proposition 4.2 in [31].

Proposition 1.74. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is 2-step nilpotent. Then $\mathfrak{g}$ admits a pre-Lie algebra structure. In particular, $\mathfrak{g}$ is not semisimple.*

If $\mathfrak{g}$ is nilpotent, then $\mathfrak{n}$ has to already be solvable, see [31].

Proposition 1.75. *Assume* $(\mathfrak{g}, \mathfrak{n})$ *admits a PA-structure, where* $\mathfrak{g}$ *is nilpotent. Then* $\mathfrak{n}$ *is solvable.*

The proof here uses a result by Goto, see [53], that the sum of two nilpotent Lie algebras is solvable. Decomposition results play an important part in the study of post-Lie algebra structures, which we discuss in Chapter 2 and Chapter 3 in more detail.

For simple Lie algebras, we only obtain the following post-Lie algebra structures, see [31].

Proposition 1.76. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras where both* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are simple. Suppose* $(\mathfrak{g}, \mathfrak{n})$ *admits a PA-structure. Then either* $x \cdot y = 0$ *for all* x *and* y *and* $[x, y] = \{x, y\}$ *or* $x \cdot y = [x, y] = -\{x, y\}$*.*

We can always obtain a post-Lie algebra structure by decomposing $\mathfrak{n}$ into the direct vector space sum of two subalgebras, see Proposition 2.13 in [23].

Proposition 1.77. *Let* $\mathfrak{n}$ *be a Lie algebra which is a direct vector space sum* $\mathfrak{n} = \mathfrak{a} \dot{+} \mathfrak{b}$ *of two subalgebras* $\mathfrak{a}$ *and* $\mathfrak{b}$*. Let* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ *where the Lie bracket is given by*

$$[a_1 + b_1, a_2 + b_2] = \{a_1, a_2\} - \{b_1, b_2\}.$$

Then we get a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ *where the product is given by*

$$(a_1 + b_1) \cdot (a_2 + b_2) = -\{b_1, a_2 + b_2\}.$$

For a semisimple Lie algebra $\mathfrak{n}$, we can construct a post-Lie algebra structure using triangular decompositions, see [23].

Proposition 1.78. *Let* $\mathfrak{n}$ *be a semisimple Lie algebra. Then there exists a solvable, non-nilpotent Lie algebra* $\mathfrak{g}$ *such that there is a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

However if $\mathfrak{g}$ is a solvable unimodular Lie algebra, then there exists no post-Lie algebra structure, see Theorem 3.3 in [23].

Theorem 1.79. *Let* $\mathfrak{n}$ *be a semisimple Lie algebra, and* $\mathfrak{g}$ *be a solvable and unimodular Lie algebra. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

Moreover, in [23] we also find a structural result on post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is solvable. The proof uses a lemma, which we later on need. We state it here, see Lemma 4.1 in [23].

Lemma 1.80. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* M *be a* $\mathfrak{g}$*-module with* $\dim(M) \leq \dim(\mathfrak{g})$*. Then* $\mathrm{Ann}(m) \neq 0$ *for all* $m \in M$*.*

The proof uses that there exists no pre-Lie algebra structure on a semisimple Lie algebra. Lemma 1.80 also holds for perfect Lie algebras, since there are no pre-Lie algebra structures on perfect Lie algebras, see [85]. This result was then used to show the following, see Theorem 4.2 in [23].

Theorem 1.81. *There exists no post-Lie algebra structure on a pair* $(\mathfrak{g}, \mathfrak{n})$ *where* $\mathfrak{g}$ *is semisimple and* $\mathfrak{n}$ *is solvable.*

1.3.4 Rota-Baxter operators and PA-structures

Post-Lie algebra structures are closely linked to Rota-Baxter operators. These operators were first introduced by G. Baxter in [11] in connection to an analytic problem from probability theory. They are also related to the classical and modified Yang Baxter equation, see [86], [96], and other areas of mathematics. When studying post-Lie algebra structures for semisimple Lie algebras, they can be used to determine the existence of these structures, see [37], [38].

We refer to [37] and the references therein as our main reference point for this section. Let A be a non-associative algebra over a field k, see Schafer [84]. Let k be a field of characteristic zero unless stated otherwise.

Definition 1.82. A linear operator $R : A \to A$ is called a *Rota-Baxter operator on A of weight λ* where $\lambda \in k$ if it satisfies the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy),$$

for all $x, y \in A$. We use RB-operators for Rota-Baxter operators.

Example 8. The trivial operators $R = 0$ and $R = \lambda \operatorname{id}$ are Rota-Baxter operators.

In [54] we find the following three results that connect Rota-Baxter operators to decompositions of algebras.

Lemma 1.83. *Let R be a Rota-Baxter operator of weight λ on A. Then also $-(R+\lambda \operatorname{id})$ is an RB-operator of weight λ on A and $\lambda^{-1}R$ is an RB-operator of weight 1 on A for all $\lambda \neq 0$.*

Lemma 1.84. *Let B be a countable direct sum of an algebra A. Then*

$$R((a_1, a_2, ..., a_n, ...)) = (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, ...)$$

is an RB-operator of weight 1 on B.

Proposition 1.85. *Let $A = A_1 \dotplus A_2$ be the direct vector space sum of two subalgebras. Then*

$$R(a_1 + a_2) = -\lambda a_2,$$

where $a_1 \in A_1, a_2 \in A_2$ defines an RB-operator of weight λ on A.

Remark 5. RB-operators as in Proposition 1.85 are called *split* operators and are used to study post-Lie algebra structures, see [37].

Moreover, we have the following two results from [13].

Proposition 1.86. *Let R be an RB-operator of weight λ on A and ϕ be an automorphism of A. Then also $\phi^{-1}R\phi$ is an RB-operator of weight λ on A.*

Lemma 1.87. *Let R be an RB-operator of weight $\lambda \neq 0$ on A. Then R is split if and only if $R(R + \lambda \operatorname{id}) = 0$.*

For Rota-Baxter operators on direct sums we have the following, see [37].

Proposition 1.88. *Let $B = A \oplus A$ and $\phi \in \mathrm{Aut}(A)$. Then R defined by*

$$R((a_1, a_2)) = (0, \phi(a_1))$$

defines a Rota-Baxter operator of weight 1 on B. Moreover,

$$R((a_1, a_2)) = (-a_1, -\phi(a_1))$$

defines a Rota-Baxter operator of weight 1 on B.

We can define a Rota-Baxter operator on a direct sum pointwise, see [68].

Proposition 1.89. *Let $A = A_1 \oplus A_2$, R_1 a Rota-Baxter operator of weight λ on A_1 and R_2 a Rota-Baxter operator of weight λ on A_2. Then*

$$R((a_1, a_2)) = (R_1(a_1), R_2(a_2))$$

defines a Rota-Baxter operator of weight λ on A.

Let us look more closely at the relationship between Rota-Baxter operators and post-Lie algebra structures, see [8].

Proposition 1.90. *Let $\mathfrak{n}$ be a Lie algebra and R be a Rota-Baxter operator of weight one on $\mathfrak{n}$, so we have*

$$\{R(x), R(y)\} = R(\{R(x), y\} + \{x, R(y)\} + \{x, y\}),$$

for all $x, y \in V$. Then

$$x \cdot y = \{R(x), y\}$$

defines an inner PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where

$$[x, y] = x \cdot y - y \cdot x + \{x, y\}$$

is the Lie bracket for $\mathfrak{g}$.

Relating this to semisimple Lie algebras, or more generally to Lie algebras with trivial center, we get the following, see [37].

Proposition 1.91. *Let $\mathfrak{n}$ be a Lie algebra with $Z(\mathfrak{n}) = 0$. Then any inner post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is of the form $x \cdot y = \{R(x), y\}$ where R is a RB-operator of weight 1 on $\mathfrak{n}$. If in addition we have that every derivation is inner, i.e. $\mathfrak{n}$ is complete, then every PA-structure on $(\mathfrak{g}, \mathfrak{n})$ is inner.*

This leads to the description of post-Lie algebra structures for complete Lie algebras through Rota-Baxter operators, see [37].

Corollary 1.92. *Let $\mathfrak{n}$ be a complete Lie algebra. Then there is a one-to-one correspondence between post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and Rota-Baxter operators of weight 1 on $\mathfrak{n}$.*

This allows us to study post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is semisimple, since semisimple Lie algebras are complete. This result is something we later on use more often.

Moreover, we want to recollect these three results from [37].

Proposition 1.93. *Let $x \cdot y = \{R(x), y\}$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where R is a RB-operator of weight 1 on $\mathfrak{n}$. Then R is also an RB-operator of weight 1 on $\mathfrak{g}$, so*

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + [x, y]),$$

for all $x, y \in V$.

Corollary 1.94. *Let $x \cdot y$ be an inner PA-structure arising from an RB-operator of weight 1 on $\mathfrak{n}$. Let us denote by $\mathfrak{g}_i$ the Lie algebras defined by*

$$\begin{aligned} [x, y]_0 &= \{x, y\} \\ [x, y]_{i+1} &= [R(x), y]_i - [R(y), x]_i + [x, y]_i \end{aligned}$$

for all $i \geq 1$. Then we get a PA-structure on each pair $(\mathfrak{g}_{i+1}, \mathfrak{g}_i)$.

Proposition 1.95. *Let $x \cdot y = \{R(x), y\}$ be an inner post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where R is an RB-operator of weight 1 on $\mathfrak{n}$ and let us denote by $\mathfrak{g}^{(i)}$ and $\mathfrak{n}^{(i)}$ the derived ideals of $\mathfrak{g}$ and $\mathfrak{n}$. Then we have $\dim(\mathfrak{g}^{(i)}) \leq \dim(\mathfrak{n}^{(i)})$ for all $i \geq 1$.*

We can say more about the existence of post-Lie algebra structures with respect to $\mathfrak{n}$ being complete, see [37].

Corollary 1.96. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is complete, i.e. $x \cdot y = \{R(x), y\}$ where R is an RB-operator of weight 1 on $\mathfrak{n}$. If $\mathfrak{n}$ is solvable, so is $\mathfrak{g}$. If $\mathfrak{g}$ is perfect, then so is $\mathfrak{n}$.*

Given two non-isomorphic Lie algebras, one can say something about the kernel of a Rota-Baxter operator, see [37].

Proposition 1.97. *Let $x \cdot y = \{R(x), y\}$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ arising by a Rota-Baxter operator of weight 1 on $\mathfrak{n}$. Then we have the following:*

(i) If $\mathfrak{g} \not\cong \mathfrak{n}$, then both $\ker(R)$ and $\ker(R + id)$ are non-trivial.

(ii) If either $\mathfrak{g}$ or $\mathfrak{n}$ is not solvable, then at least one of the kernels of the operators R and $R + id$ is non-trivial.

Let us conclude this chapter by giving a method to construct post-Lie algebra structures given a decomposition of a Lie algebra. We start with a direct vector space sum decomposition of $\mathfrak{n}$ and use split Rota-Baxter operators to construct examples of post-Lie algebra structures.

Proposition 1.98. *Let* $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2$ *be a direct vector space decomposition of a Lie algebra* $\mathfrak{n}$ *into two subalgebras* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$*. Let* R *be a split Rota-Baxter operator of weight* 1 *on* $\mathfrak{n}$*. Then* $x \cdot y = \{R(x), y\}$ *defines a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *where* $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$*.*

Proof. Let R be a split RB-operator of weight 1 on $\mathfrak{n}$, i.e. $R(n_1 + n_2) = -n_2$. We know that $x \cdot y = \{R(x), y\}$ defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ by Proposition 1.90. Now let $n_1 \in \mathfrak{n}_1$ and $n_2 \in \mathfrak{n}_2$, then we have

$$[n_1, n_2] = \{R(n_1), n_2\} + \{n_1, R(n_2)\} + \{n_1, n_2\} = 0.$$

Moreover we have that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are subalgebras of $\mathfrak{g}$, since

$$[\mathfrak{n}_1, \mathfrak{n}_1] = \{\mathfrak{n}_1, \mathfrak{n}_1\} \subseteq \mathfrak{n}_1$$

and

$$[\mathfrak{n}_2, \mathfrak{n}_2] = -\{\mathfrak{n}_2, \mathfrak{n}_2\} \subseteq \mathfrak{n}_2.$$

Altogether we obtain $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. $\square$

2 Decompositions of Lie algebras

This chapter is dedicated to decompositions of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra that decomposes into the sum of two of its subalgebras $\mathfrak{a}$ and $\mathfrak{b}$, i.e. $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ as a vector space sum. What can we say about $\mathfrak{g}$ depending on the assumptions on $\mathfrak{a}$ and $\mathfrak{b}$? This question first arose for groups. We start this chapter with a short survey on factorizations of groups based on [21] and the references therein. Motivated by this, similar questions on a Lie algebra level arose and we devote the subsequent sections to dinilpotent Lie algebras and disemisimple Lie algebras. We give a survey on dinilpotent Lie algebras, also based on [21]. For disemisimple Lie algebras, we refer to [42] as our main reference point. We then proceed with reductive decompositions in Section 2.4 based on Onishchik's work [75], [76]. In Section 2.4.1 we present a result on strongly disemisimple Lie algebras, which is joint work with D. Burde and K. Dekimpe and has been presented in [28].

2.1 Factorizations of groups

As already mentioned, the question for decompositions of Lie algebras was historically studied first for groups. Let us first define what we mean by a decomposition of a group. Here the notion factorization of a group is more frequently used, see [4].

Definition 2.1. Let G be a group and let A and B be two subgroups. By a *factorization* of a group G we mean that $G = AB$ is a product of two of its subgroups.

Remark 6. Let $G = AB$ be a factorization of G. Then AB is a subgroup of G if and only if $AB = BA$.

These groups have a long history of being studied, see [4] and [70] and the references therein. It is quite natural to ask oneself what one can say about a group G that is the product of two subgroups having certain algebraic properties. In particular, what can we say about such a group being the product of nilpotent or abelian groups. These are called dinilpotent, see [70].

Definition 2.2. Let $G = AB$ be a group that is the product of two nilpotent subgroups A and B. Then G is called *dinilpotent* or *nildecomposable.*

One result about such groups is known as the Wielandt-Kegel Theorem, see [64].

Theorem 2.3. *Let G be a finite group and dinilpotent, i.e. $G = AB$ where A and B are nilpotent subgroups. Then G is solvable.*

Moreover, Itô, see [59], prior to that already showed the following.

Theorem 2.4. *Let $G = AB$, where A and B are abelian. Then G is metabelian.*

This lead to the study of these solvable groups and their derived length. In particular, how is the derived length of such a group related to the nilpotency class of the subgroups.

Here the first conjecture was to bound the derived length of such a solvable group by the sum of its nilpotency classes, i.e.

$$d(G) \leq c(A) + c(B).$$

By Theorem 2.4, this holds for A and B abelian. Moreover this is true for A and B having coprime orders, see [55]. According to [78], the study of these groups reduces to the study of p-groups. However, already in this case, the conjecture is false, a counterexample is provided in [44]. There is already a counterexample of a 4-step solvable group being the product of an abelian and 2-step solvable subgroup. Since the conjecture is not true, it was then natural to ask oneself what happens if the bound is a linear function of the nilpotency classes. For p-groups, we have the following result by L. Kazarin, see [63].

Proposition 2.5. *Let $G = AB$, where G is a finite p-group and A and B are nilpotent subgroups with $|A| = p^m$ and $|B| = p^n$. Then*

$$d(G) \leq 2m + 2n + 2.$$

This bound was then improved by Morigi, see [74].

Proposition 2.6. *Let G, A, and B be as in Proposition* 2.5*. Then*

$$d(G) \leq 2m + n + 2.$$

In addition, we have the following bound for a dinilpotent group, where A is abelian and B is 2-step solvable, see [45].

Proposition 2.7. *Let $G = AB$ where G is finite and A abelian and B nilpotent and* 2*-step solvable. Then G is solvable and*

$$d(G) \leq 4.$$

2.2 Dinilpotent Lie algebras

In analogy to the group case, dinilpotent Lie algebras where studied, see [21] and the references therein. First let us define what we mean by a decomposition of a Lie algebra.

Definition 2.8. A triple $(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ of Lie algebras is called a *decomposition* if $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Here the sum is a vector space sum, which is not necessarily direct.

Definition 2.9. A Lie algebra $\mathfrak{g}$ is called *dinilpotent* or *nildecomposable* if $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are nilpotent subalgebras.

Remark 7. Of course every direct Lie algebra sum of nilpotent Lie algebras is dinilpotent. However, it is obvious, that a dinilpotent Lie algebra need not be nilpotent itself. Already in dimension two we have

$$\mathfrak{r}_2(\mathbb{C}) = < e_1 > + < e_2 >$$

However for finite-dimensional Lie algebras over a field of characteristic zero one has, analogously to the Wielandt-Kegel theorem, the following result by Goto, see [53].

Theorem 2.10. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero and $\mathfrak{g}$ be nildecomposable, i.e. $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are nilpotent subalgebras. Then $\mathfrak{g}$ is solvable.*

This result holds also for dinilpotent Lie algebras over a field of characteristic $p > 2$, see [77], [97].

This again lead to the study of the derived length of such a Lie algebra. For finite-dimensional Lie algebras over fields of characteristic zero we do not know whether the analogous conjecture holds, i.e.

$$d(\mathfrak{g}) \leq c(\mathfrak{a}) + c(\mathfrak{b}).$$

The conjecture does not hold for infinite-dimensional Lie algebras, see [7]. For $\mathfrak{a}$ and $\mathfrak{b}$ abelian, one can apply Itô's result to the Lie algebra case and the conjecture is true, see [79]. If one of the summands is an ideal, this is also true, see [21].

Since the conjecture holds for $\mathfrak{a}$ and $\mathfrak{b}$ abelian, the next natural question would be to ask what happens if one of the subalgebras is 2-step nilpotent. Here one has the following bound, see [80].

Proposition 2.11. *Let $\mathfrak{g} = \mathfrak{a}+\mathfrak{b}$ be a dinilpotent Lie algebra over a field of characteristic different from two, where $\mathfrak{a}$ is abelian and $\mathfrak{b}$ is 2-step nilpotent. Then*

$$d(\mathfrak{g}) \leq 10.$$

2.3 Disemisimple Lie algebras

Analogously to dinilpotent Lie algebras, disemisimple Lie algebras have been recently studied. We give a short survey based on [42].

Definition 2.12. A Lie algebra $\mathfrak{g}$ is called *disemisimple* if $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are semisimple subalgebras of $\mathfrak{g}$.

Also here there is a study for the group case, i.e. a finite group that is the product of two simple groups. We refer to [94] and the references therein.

There is a classification of all finite dimensional associative algebras over a field that are sums of two simple subalgebras, which can be found in [6]. In there we have the following result:

Theorem 2.13. *Let A be a finite dimensional associative algebra over an algebraically closed field k. Let $A = S_1 + S_2$ be a vector space sum of two simple subalgebras S_1 and S_2. Suppose $\dim(S_1) \geq \dim(S_2)$. Then we have one of the cases:*

(i) A is simple and $A = S_1$.

(ii) $A = A_1 \oplus A_2$ *is an algebra sum of two proper simple ideals* A_1 *and* A_2*, with* $\dim(A_1) \geq \dim(A_2)$*,* $A_1 = S_1$ *and either*

   (a) $A_2 = S_2$ *or*

   (b) there exists a subalgebra $A_2' \subseteq A_1$ *with* $A_2' \cong A_2$ *and* B *is the diagonal subalgebra.*

(iii) $A = S_1 \oplus N$*, where* N *is a nilpotent ideal of* A *which is cyclic as an* S_1*-module on one side and a trivial module on the other. Moreover for an* $x \in N$*, we have that* $S_2 = (1+x)S_1$ *or* $S_2 = S_1(1+x)$*, depending on the sides.*

As for Lie algebras over fields of characteristic zero, we have the following result, see also [42]. We give a proof here as well, to see that we can generalize this result to perfect Lie algebras.

Lemma 2.14. *Let* $\mathfrak{g}$ *be a disemisimple Lie algebra. Then* $\mathfrak{g}$ *is perfect and the radical of* $\mathfrak{g}$ *is nilpotent.*

Proof. Suppose $\mathfrak{g}$ is disemisimple, i.e. there exist semisimple subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ such that $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$. Then we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{s}_1 + \mathfrak{s}_2 \\ &\subseteq [\mathfrak{s}_1, \mathfrak{s}_1] + [\mathfrak{s}_2, \mathfrak{s}_2] \\ &\subseteq [\mathfrak{g}, \mathfrak{g}] + [\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \end{aligned}$$

This is because a semisimple Lie algebra is perfect by Lemma 1.23, and hence we get $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Moreover the radical of a perfect Lie algebra is nilpotent. Indeed, we have

$$\operatorname{rad}(\mathfrak{g}) = \mathfrak{g} \cap \operatorname{rad}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \cap \operatorname{rad}(\mathfrak{g}) = [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})].$$

For the last equality, we obviously have $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] \subseteq [\mathfrak{g}, \mathfrak{g}] \cap \operatorname{rad}(\mathfrak{g})$. For the other direction, let $x \in [\mathfrak{g}, \mathfrak{g}] \cap \operatorname{rad}(\mathfrak{g})$, then we have that there exists $g_1, g_2 \in \mathfrak{g}$ such that $x = [g_1, g_2]$. Moreover, we have that there exists $s_1, s_2 \in \mathfrak{s}$ for a Levi subalgebra $\mathfrak{s}$ and $r_1, r_2 \in \operatorname{rad}(\mathfrak{g})$ such that $g_1 = s_1 + r_1$ and $g_2 = s_2 + r_2$, so that

$$x = [s_1 + r_1, s_2 + r_2] = [s_1, s_2] + [s_1, r_2] + [r_1, s_2] + [r_1, r_2].$$

But this means that $x - [s_1, r_2] - [r_1, s_2] - [r_1, r_2] = [s_1, s_2] \in \mathfrak{s} \cap \operatorname{rad}(\mathfrak{g}) = 0$ and therefore we get

$$x = [s_1, r_2] + [r_1, s_2] + [r_1, r_2] \in [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})],$$

so we have shown the last equality.

We know that $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$ is nilpotent by Corollary 1.34 and hence we get that $\operatorname{rad}(\mathfrak{g}) \subseteq \operatorname{nil}(\mathfrak{g})$ and is therefore nilpotent. $\square$

Remark 8. This holds more generally for Lie algebras decomposing into the sum of perfect subalgebras. Note that we only used the perfectness of semisimple Lie algebras. Therefore we have that a Lie algebra that decomposes into the sum of two perfect subalgebras is perfect.

In [42] one sees that disemisimple Lie algebras lead to the study of prehomogeneous modules, which are also related to pre-Lie algebra structures, see [35]. We recall these results in the next section.

2.3.1 Prehomogeneous modules

We first recall the definition of a prehomogeneous module for a group G, see [35], [66].

Definition 2.15. Let G be a connected complex algebraic group. A rational G-module V is called *prehomogeneous* if V has a Zariski-open G-orbit $\mathcal{O}$. If in addition we have $\dim(\mathcal{O}) = \dim(G)$, then the module is called *étale*.

For Lie algebras the definition of a prehomogeneous module is the following, see [93].

Definition 2.16. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module V is called *prehomogeneous* if there exists a $v \in V$ such that $\mathfrak{g}.v = V$. If in addition we have $\dim(\mathfrak{g}) = \dim(V)$, then the module is called *étale*.

Remark 9. For a prehomogeneous $\mathfrak{g}$-module V we always have $\dim(V) \leq \dim(\mathfrak{g})$. This is because for every $x \in V$ there exists a $g \in \mathfrak{g}$ and $v \in V$, such that $x = g.v$.

If the Lie algebra $\mathfrak{g}$ is semisimple, we have the following result, see [35].

Proposition 2.17. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then there exists no étale $\mathfrak{g}$-module.*

This is due to the fact that a semisimple Lie algebra admits no pre-Lie algebra structure, see [56].

Therefore we get the following result, see [42].

Corollary 2.18. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero and suppose $\mathfrak{g}$ admits a non-trivial prehomogeneous module V. Then $2 \leq \dim(V) \leq \dim(\mathfrak{g}) - 1$.*

Theorem 2.7 in [42] allows us to translate the question of disemisimplicity to that of prehomogeneous modules. We state this theorem and repeat the proof. From now on in this section all Lie algebras are finite-dimensional and over a field of characteristic zero, unless stated otherwise.

Theorem 2.19. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$ be a Levi decomposition of $\mathfrak{g}$. Then the following are equivalent:*

(a) *$\mathfrak{g}$ is disemisimple*

(b) $\operatorname{rad}(\mathfrak{g}) = \operatorname{nil}(\mathfrak{g})$ *is a prehomogeneous $\mathfrak{s}$-module, i.e. there exists an $x \in \operatorname{nil}(\mathfrak{g})$ such that* $\operatorname{nil}(\mathfrak{g}) = [x, \mathfrak{s}]$.

Proof. First assume that $\mathfrak{g}$ is disemisimple, i.e. $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are semisimple subalgebras. We have that $\mathfrak{s}_1 \subseteq \mathfrak{s}$ and $\mathfrak{s}_2 \subseteq \tilde{\mathfrak{s}}$ where $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are two Levi subalgebras by Corollary 1.41. Hence we get

$$\mathfrak{g} \subseteq \mathfrak{s} + \tilde{\mathfrak{s}} \subseteq \mathfrak{g}.$$

Therefore we get a decomposition of $\mathfrak{g} = \mathfrak{s} + \tilde{\mathfrak{s}}$. By the Levi-Malcev theorem, there exists a $z \in \mathrm{rad}(\mathfrak{g})$ such that $\tilde{\mathfrak{s}} = \tau(\mathfrak{s})$, where $\tau = \exp(\mathrm{ad}(z))$. We have that

$$\mathfrak{g} = \mathfrak{s} \dotplus \mathrm{rad}(\mathfrak{g}) = \mathfrak{s} + \tau(\mathfrak{s}) = \mathfrak{s} + (\tau - \mathrm{id})(\mathfrak{s}).$$

Note that $(\tau - \mathrm{id})(\mathfrak{s}) \subseteq \mathrm{rad}(\mathfrak{g})$, and we get that $\mathfrak{s} \cap (\tau - \mathrm{id})(\mathfrak{s}) = 0$. So we get that $\mathfrak{s} \dotplus \mathrm{rad}(\mathfrak{g}) = \mathfrak{s} \dotplus (\tau - \mathrm{id})(\mathfrak{s})$. This means that $\mathrm{rad}(\mathfrak{g}) = (\tau - \mathrm{id})(\mathfrak{s})$.

Now defining $\psi : \mathrm{rad}(\mathfrak{g}) \to \mathrm{rad}(\mathfrak{g})$ by

$$\psi = \sum_{i=1}^{\infty} \frac{1}{i!} \mathrm{ad}(z)^{i-1},$$

we see that ψ is a vector space automorphism. This is because $\mathrm{ad}(z)$ is a nilpotent endomorphism, since $\mathfrak{g}$ is disemisimple and hence perfect, so that $\mathrm{rad}(\mathfrak{g}) = \mathrm{nil}(\mathfrak{g})$. We get that

$$\mathrm{rad}(\mathfrak{g}) = \mathrm{nil}(\mathfrak{g}) = (\tau - \mathrm{id})(\mathfrak{s}) = \psi([\mathfrak{s}, -z]).$$

This implies that $\mathrm{nil}(\mathfrak{g}) = \psi^{-1}(\mathrm{nil}(\mathfrak{g})) = [\mathfrak{s}, -z]$. So that $\mathrm{nil}(\mathfrak{g})$ is a prehomogeneous $\mathfrak{s}$-module.

Conversely, let $\mathrm{rad}(\mathfrak{g}) = \mathrm{nil}(\mathfrak{g})$ be a prehomogeneous $\mathfrak{s}$-module, i.e. there exists a $z \in \mathrm{nil}(\mathfrak{g})$ such that $\mathrm{nil}(\mathfrak{g}) = [z, \mathfrak{s}]$. Consider the automorphism $\tau = \exp(\mathrm{ad}(z))$, then $\tau(\mathfrak{s})$ is also a Levi subalgebra which we will denote by $\tilde{\mathfrak{s}}$. Then we get

$$\begin{aligned} \mathfrak{s} + \tilde{\mathfrak{s}} &= \mathfrak{s} + \tau(\mathfrak{s}) \\ &= \mathfrak{s} + (\tau - \mathrm{id})(\mathfrak{s}) \\ &= \mathfrak{s} + \psi([z, \mathfrak{s}]) \\ &= \mathfrak{s} + \psi(\mathrm{nil}(\mathfrak{g})) \\ &= \mathfrak{s} + \mathrm{nil}(\mathfrak{g}) = \mathfrak{g} \end{aligned}$$

Therefore we get that $\mathfrak{g}$ is disemisimple. □

In [93] we find a classification of all, irreducible and reducible, prehomogeneous modules for complex simple Lie algebras. First let us mention the irreducible ones, where we denote by $L(\omega_i)$ the irreducible highest weight module with fundamental weight ω_i.

Theorem 2.20. *Let $\mathfrak{g}$ be a complex simple Lie algebra and V be a non-trivial irreducible $\mathfrak{g}$-module. Then V is a prehomogeneous $\mathfrak{g}$-module if and only if V is contained in the following list:*

$\mathfrak{g}$	$\dim(\mathfrak{g})$	V	$\dim(V)$
$A_l, l \geq 1$	$l(l+2)$	$L(\omega_1)$,$L(\omega_l)$	$l+1$
$A_{2l}, l \geq 2$	$4l(l+1)$	$L(\omega_2), L(\omega_{2l-1})$	$l(2l+1)$
$C_l, l \geq 2$	$l(2l+1)$	$L(\omega_1)$	$2l$
D_5	45	$L(\omega_4), L(\omega_5)$	16

For the reducible ones, this is done in Theorem 8 in [93].

Theorem 2.21. *Let $\mathfrak{g}$ be a complex simple Lie algebra and V be a non-trivial reducible $\mathfrak{g}$-module. Then V is a prehomogeneous $\mathfrak{g}$-module if and only if V is contained in the following list:*

$\mathfrak{g}$	$\dim(\mathfrak{g})$	V	$\dim(V)$
$A_l, l \geq 2$	$l(l+2)$	$mL(\omega_1)$,$mL(\omega_l)$, $2 \leq m \leq l$	$m(l+1)$
$A_{2l}, l \geq 2$	$4l(l+1)$	$L(\omega_1) \oplus L(\omega_{2l-1}), L(\omega_2) \oplus L(\omega_{2l})$	$(l+1)(2l+1)$
	$4l(l+1)$	$L(\omega_2) \oplus L(\omega_2),\ L(\omega_{2l-1}) \oplus L(\omega_{2l-1})$	$2l(2l+1)$

In the case where the Levi subalgebra is semisimple but not simple, there is not an explicit classification. However, Sato and Kimura [66] have classified prehomogeneous modules up to strong equivalence and castling equivalence, see definition 4 and 10 in [66].

Proposition 2.22. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $V \neq 0$ be an irreducible prehomogeneous $\mathfrak{g}$-module. Then V is strongly equivalent to one of the following castling-reduced prehomogeneous triples:*

$\mathfrak{g}$	$\dim(\mathfrak{g})$	V	$\dim(V)$	*conditions*
$\mathfrak{s}_1 \oplus A_m$	$s + m(m+2)$	$L(\lambda) \otimes L(\omega_1)$	$(n+1)(m+1)$	$m > n > 2$
$A_n \oplus A_m$	$n(n+2) + m(m+2)$	$L(\omega_1) \otimes L(\omega_1)$	$(n+1)(m+1)$	$m \geq 2n+1 \geq 1$
A_{2m}	$4m(m+1)$	$L(\omega_2)$	$m(2m+1)$	$m \geq 1$
$A_1 \oplus A_{2m}$	$4m(m+1)+3$	$L(\omega_1) \otimes L(\omega_2)$	$2m(2m+1)$	$m \geq 1$
$C_n \oplus A_m$	$n(2n+1) + 4m(m+1)$	$L(\omega_1) \otimes L(\omega_1)$	$2n(2m+1)$	$n \geq m+1 \geq 1$
D_5	45	$L(\omega_4)$	16	-

In [42], we find some examples of disemisimple Lie algebras. The list of prehomogeneous modules is useful to decide which Lie algebras are disemisimple in low dimension, see Corollary 2.12 in [42].

Corollary 2.23. $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ *and* $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ *are the only complex disemisimple Lie algebras of dimension $n \leq 6$.*

In general we have the following, see Example 4.10 in [38].

Example 9. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$ where $V(n)$ is the natural $\mathfrak{sl}_n(\mathbb{C})$-module considered as an abelian Lie algebra and $n \geq 2$. Then $\mathfrak{g}$ is disemisimple and we have the following decomposition

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) + \mathfrak{sl}_n(\mathbb{C}).$$

This shows that a disemisimple Lie algebra need not be semisimple. However, we have that every semisimple Lie algebra is disemisimple, since $\mathfrak{s} = \mathfrak{s} + \mathfrak{s}$ for a semisimple

Lie algebra $\mathfrak{s}$. Moreover, Onishchik has classified semisimple decompositions of simple Lie algebras [76]. We give a short survey in the next section.

Looking at the examples in lower dimension, one sees that often the radical of a disemisimple Lie algebra is abelian. We do not know whether this is true in general. However, under certain conditions this is true, see Theorem 3.7 in [42].

Theorem 2.24. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$ be a disemisimple Lie algebra and $\mathfrak{s}$ be a simple Levi subalgebra. Then $\mathrm{rad}(\mathfrak{g})$ is abelian.*

This is due to Vinberg's classification and prehomogeneous modules of a given type, see [42], which we recall here.

Definition 2.25. Let $\mathfrak{s}$ be a semisimple Lie algebra and V be an $\mathfrak{s}$-module. The module V is said to be of *type* 1, if $V = A \oplus B$ where A and B are two non-trivial irreducible $\mathfrak{s}$-submodules such that B can be embedded in the wedge product $\Lambda^2(A) = A \wedge A$.
We say it is of *type* 2, if $V = A \oplus B \oplus C$ where A, B, C are non-trivial irreducible $\mathfrak{s}$-submodules of V such that C can be embedded in the tensor product $A \otimes B$.

The study of disemisimple Lie algebras where the radical is non-abelian, if they exist, reduces to that of prehomogeneous modules of type 1 or 2, see Theorem 3.2 in [42].

Theorem 2.26. *Let $\mathfrak{s}$ be a semisimple Lie algebra. Then there exists a disemisimple Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$ with non-abelian radical if and only if there is a prehomogeneous $\mathfrak{s}$-module of type 1 or of type 2.*

It turns out that simple Lie algebras do not admit prehomogeneous modules of type 1 or 2, see Lemma 3.6 in [42].

Lemma 2.27. *Let $\mathfrak{s}$ be a simple Lie algebra. Then $\mathfrak{s}$ does not admit prehomogeneous modules of type 1 or 2.*

Therefore, we see that Theorem 2.24 holds. We do not know whether this holds more generally for all Levi subalgebras. However one has for specific Levi subalgebras that this is the case.

Definition 2.28. A disemisimple Lie algebra $\mathfrak{g}$ is called *A-free*, if it has no simple quotient of type A, i.e. not isomorphic to $\mathfrak{sl}_n(\mathbb{C})$ with $n \geq 2$.

We have the following result, see Theorem 4.3 in [42].

Theorem 2.29. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$ be a disemisimple, A-free Lie algebra. Then $\mathrm{rad}(\mathfrak{g})$ is abelian.*

We believe that this holds more generally for any disemisimple Lie algebra.

Conjecture 1. Let $\mathfrak{g}$ be a disemisimple Lie algebra. Then the solvable radical of $\mathfrak{g}$ is abelian.

2.4 Reductive decompositions

Onishchik has studied decompositions of reductive Lie groups in [75] and [76]. He first studied decompositions of compact Lie groups, in particular compact Lie algebras, and then looked at the non-compact case involving compact real forms. We recall some results from these two papers, that are relevant for the subsequent chapters.

Definition 2.30. A decomposition $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ is called a *reductive decomposition*, if $\mathfrak{g}$ is reductive and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are reductive in $\mathfrak{g}$. It is called a *semisimple decomposition* if $\mathfrak{g}$,$\mathfrak{g}_1$, $\mathfrak{g}_2$ are semisimple.

In [76], Theorem 3.2 we find the following result.

Theorem 2.31. *Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{R}$, and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be reductive in $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{s} \oplus Z(\mathfrak{g})$, $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus Z(\mathfrak{g}_1)$ and $\mathfrak{g}_2 = \mathfrak{s}_2 \oplus Z(\mathfrak{g}_2)$. Moreover, let $\tilde{Z}_1$ and $\tilde{Z}_2$ be the projections of the algebras $Z(\mathfrak{g}_1)$ and $Z(\mathfrak{g}_2)$ onto $Z(\mathfrak{g})$. Then $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ if and only if $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2$ and $Z(\mathfrak{g}) = \tilde{Z}_1 + \tilde{Z}_2$.*

The study of reductive decompositions was first done for compact Lie algebras in [75] and later on generalized to the non-compact case, see Theorem 3.3 in [76]. We have the following classification result, see [75].

Theorem 2.32. *Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two subalgebras of $\mathfrak{g}$, which are necessarily reductive. Then all proper decompositions, meaning that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are proper subalgebras are given as follows:*

$\mathfrak{g}$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_1 \cap \mathfrak{g}_2$
$A_{2n-1}, n > 1$	C_n	$A_{2n-2}, A_{2n-2} \oplus T$	$C_{n-1}, C_{n-1} \oplus T$
$D_{n+1}, n > 2$	B_n	$A_n, A_n \oplus T$	$A_{n-1}, A_{n-1} \oplus T$
$D_{2n}, n > 1$	B_{2n-1}	$C_n, C_n \oplus T,\ C_n \oplus A_1$	C_{n-1},$C_{n-1} \oplus T,\ C_{n-1} \oplus A_1$
B_3	G_2	$B_2, B_2 \oplus T, D_3$	$A_1, A_1 \oplus T, A_2$
D_4	B_3	$B_2, B_2 \oplus T, B_2 \oplus A_1,$ $D_3, D_3 \oplus T, B_3$	$A_1, A_1 \oplus T, A_1 \oplus A_1,$ $A_2, A_2 \oplus T, G_2$
D_8	B_7	B_4	B_3

We have the following result, see Theorem 3.1 in [76].

Theorem 2.33. *Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be reductive in $\mathfrak{g}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let G_1 and G_2 be its connected Lie subgroups corresponding to the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then $G = G_1 G_2$ if and only if $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$.*

Moreover, Corollary 3.1 in [76] shows the following.

Corollary 2.34. *Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a complex reductive algebraic decomposition. Then there exists a decomposition $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ of the compact real form $\mathfrak{l}$ of $\mathfrak{g}$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ map into $\mathfrak{l}_1^{\mathbb{C}}$ and $\mathfrak{l}_2^{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}$.*

Onishchik then, right after Corollary 3.1 in [76], further concludes that this means that the complex reductive algebraic decompositions and the semisimple decompositions are in one-to-one correspondence with the compact decompositions that can be found in [75].

Moreover, the proof of Corollary 2 in [75], mentions the following, which is not explicitly stated as a result, but mentioned in the proof.

Lemma 2.35. *Let $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ be a compact semisimple decomposition. Then $\mathfrak{l}_1 \cap \mathfrak{l}_2$ is zero or semisimple.*

For the Lie group case and the relationship between decompositions of compact connected Lie groups and reductive algebraic decompositions we have the following, see Theorem 2.1 in [76].

Theorem 2.36. *Let K be a compact connected Lie group, K_1 and K_2 closed connected subgroups with $K = K_1K_2$. Then $K^{\mathbb{C}} = K_1^{\mathbb{C}}K_2^{\mathbb{C}}$. Conversely, let G be a complex connected reductive algebraic group and let $G = G_1G_2$, where G_1, G_2 are connected complex Lie subgroups and $G_1 \cap G_2$ has a finite number of connected components. Let K, K_1, K_2 be maximal compact subgroups in G, G_1, G_2 with $K_1, K_2 \subset K$. Then $K = K_1K_2$. Suppose G_{1_r} and G_{2_r} are maximal reductive algebraic subgroups in G_1 and G_2, then $G = G_{1_r}G_{2_r}$.*

Moreover, Onishchik has classified all semisimple decompositions of non-compact simple Lie algebras, see Theorem 4.1 in [76].

Theorem 2.37. *Let $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ be a semisimple decomposition of a non-compact simple Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^{\mathbb{C}}$ is simple. Then all decompositions up to conjugacy are given as follows:*

$\mathfrak{g}$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_1 \cap \mathfrak{g}_2$
$\mathfrak{su}(2p, 2(n-p))$	$\mathfrak{sp}(p, n-p)$	$\mathfrak{su}(2p, 2(n-p)-1)$	$\mathfrak{sp}(p, n-p-1)$
$\mathfrak{su}(2p, 2(n-p))$	$\mathfrak{sp}(p, n-p)$	$\mathfrak{su}(2p-1, 2(n-p))$	$\mathfrak{sp}(p-1, n-p)$
$\mathfrak{su}(n, n)$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{su}(n-1, n)$	$\mathfrak{sp}(n-1, \mathbb{R})$
$\mathfrak{sl}(2n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(2n-1, \mathbb{R})$	$\mathfrak{sp}(n-1, \mathbb{R})$
$\mathfrak{so}(3, 4)$	G_2	$\mathfrak{so}(2, 4)$	$\mathfrak{su}(1, 2)$
$\mathfrak{so}(3, 4)$	G_2	$\mathfrak{so}(3, 3)$	$\mathfrak{sl}(3, \mathbb{R})$
$\mathfrak{so}(3, 4)$	G_2	$\mathfrak{so}(1, 4)$	$\mathfrak{su}(2)$
$\mathfrak{so}(3, 4)$	G_2	$\mathfrak{so}(2, 3)$	$\mathfrak{su}(1, 1)$
$\mathfrak{so}(2p, 2(n-p))$	$\mathfrak{so}(2p, 2(n-p)-1)$	$\mathfrak{su}(p, n-p)$	$\mathfrak{su}(p, n-p-1)$
$\mathfrak{so}(2p, 2(n-p))$	$\mathfrak{so}(2p-1, 2(n-p))$	$\mathfrak{su}(p, n-p)$	$\mathfrak{su}(p-1, n-p)$
$\mathfrak{so}(n, n)$	$\mathfrak{so}(n-1, n)$	$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{sl}(n-1, \mathbb{R})$
$\mathfrak{so}(4p, 4(n-p))$	$\mathfrak{so}(4p, 4(n-p)-1)$	$\mathfrak{sp}(p, n-p)$	$\mathfrak{sp}(p, n-p-1)$
$\mathfrak{so}(4p, 4(n-p))$	$\mathfrak{so}(4p-1, 4(n-p))$	$\mathfrak{sp}(p, n-p)$	$\mathfrak{sp}(p-1, n-p)$
$\mathfrak{so}(4p, 4(n-p))$	$\mathfrak{so}(4p, 4(n-p)-1)$	$\mathfrak{sp}(p, n-p) \oplus \mathfrak{sp}(1)$	$\mathfrak{sp}(p, n-p-1) \oplus \mathfrak{sp}(1)$
$\mathfrak{so}(4p, 4(n-p))$	$\mathfrak{so}(4p-1, 4(n-p))$	$\mathfrak{sp}(p, n-p) \oplus \mathfrak{sp}(1)$	$\mathfrak{sp}(p-1, n-p) \oplus \mathfrak{sp}(1)$
$\mathfrak{so}(2n, 2n)$	$\mathfrak{so}(2n-1, 2n)$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n-1, \mathbb{R})$
$\mathfrak{so}(2n, 2n)$	$\mathfrak{so}(2n-1, 2n)$	$\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$	$\mathfrak{sp}(n-1, \mathbb{R}) \oplus \mathfrak{sp}(1, \mathbb{R})$
$\mathfrak{so}(8, 8)$	$\mathfrak{so}(7, 8)$	$\mathfrak{so}(1, 8)$	$\mathfrak{so}(7)$
$\mathfrak{so}(8, 8)$	$\mathfrak{so}(7, 8)$	$\mathfrak{so}(4, 5)$	$\mathfrak{so}(3, 4)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(3, 4)$	G_2
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(2, 4)$	$\mathfrak{su}(1, 2)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(3, 3)$	$\mathfrak{sl}(3, \mathbb{R})$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(1, 4)$	$\mathfrak{so}(3)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(2, 3)$	$\mathfrak{so}(1, 2)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(1, 4) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(3, 4)$	$\mathfrak{so}(2, 3) \oplus \mathfrak{so}(1, 2)$	$\mathfrak{so}(1, 2) \oplus \mathfrak{so}(1, 2)$

Onishchik's results in [75] and [76] use methods from topology. In [7] Proposition 1 and Theorem 3.1, there is an algebraic description of decompositions. Here we have the following result.

Theorem 2.38. *Let $\mathfrak{g} = \mathfrak{sl}(V)$. Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are proper simple subalgebras of $\mathfrak{g}$, which act reducibly on V. Then $\mathfrak{g} \neq \mathfrak{g}_1 + \mathfrak{g}_2$.*

2.4.1 Strongly disemisimple Lie algebras

In this section, we prove a result on strongly disemisimple Lie algebras. The results in this section are joint work with D. Burde and K. Dekimpe and have been presented in [28]. All Lie algebras are complex and finite-dimensional, unless stated otherwise.

Definition 2.39. Let $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ be a disemisimple Lie algebra. The Lie algebra $\mathfrak{g}$ is called strongly disemisimple if $\mathfrak{s}_1 \cap \mathfrak{s}_2 = 0$, i.e. $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ as a direct vector space sum.

Koszul, see [67], has shown that a semisimple Lie algebra, which is a direct vector space sum of semisimple subalgebras is isomorphic to the direct Lie algebra sum. More generally, he has shown it for direct reductive decompositions. This result can also be found in Onishchik's work [75]. Onishchik uses topological methods, while Koszul uses Lie algebra cohomology.

Theorem 2.40. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{a}$ and $\mathfrak{b}$ be subalgebras that are reductive in $\mathfrak{g}$. Suppose $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$. Then $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$.*

It turns out that a Lie algebra that decomposes into the direct vector space sum of semisimple subalgebras is necessarily semisimple. Note that the assumption that the vector space sum is direct is crucial, as can be seen in Example 9, where we have a disemisimple Lie algebra that is itself not semisimple.

Theorem 2.41. *Suppose $\mathfrak{g}$ is strongly disemisimple, i.e. $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are semisimple subalgebras. Then $\mathfrak{g}$ is semisimple too and $\mathfrak{g} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$.*

The proof of this result is organized into three parts. One is based on the non-existence of pre-Lie algebra structures for semisimple Lie algebras, see [23]. The second is using Levi-Malcev theorem to get a semisimple decomposition of a Levi subalgebra. The third ingredient is a consequence of Onishchik's work on reductive decompositions, see [75] and [76].

We show a result which gives a restriction on the semisimple subalgebras for strongly disemisimple Lie algebras.

Lemma 2.42. *Let $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ be a strongly disemisimple Lie algebra. Then neither $\mathfrak{s}_1$ nor $\mathfrak{s}_2$ can be a Levi subalgebra of $\mathfrak{g}$.*

Proof. Suppose, without loss of generality, that $\mathfrak{s}_1$ is a Levi subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \operatorname{rad}(\mathfrak{g})$. Then we have

$$\dim(\operatorname{rad}(\mathfrak{g})) = \dim(\mathfrak{g}) - \dim(\mathfrak{s}_1) = \dim(\mathfrak{s}_2).$$

Since $\mathfrak{s}_2$ is a semisimple subalgebra it lies in a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. By the Levi-Malcev theorem, we have that there exists an element $z \in \operatorname{rad}(\mathfrak{g})$ such that $\phi(\mathfrak{s}) = \mathfrak{s}_1$ with $\phi = \exp(\operatorname{ad}(z))$. Since ϕ is an automorphism we have that $\phi(\mathfrak{s}_2)$ is a semisimple subalgebra of $\mathfrak{s}_1$. We get that $\operatorname{rad}(\mathfrak{g})$ is a $\mathfrak{s}$-module, so in particular it is a $\mathfrak{s}_2$-module with $\dim(\operatorname{rad}(\mathfrak{g})) = \dim(\mathfrak{s}_2)$. Hence by Lemma 1.80, there exists a non-zero $x \in \mathfrak{s}_2$ such that $x.z = 0$, where . denotes the action in the semidirect product. Therefore we get $[x, z] = 0$, and hence $x \in \mathfrak{s}_2$ is a fixed point of ϕ, i.e. $\phi(x) = x \in \mathfrak{s}_1$, so that $x \in \mathfrak{s}_2 \cap \mathfrak{s}_1 = 0$, which is a contradiction to x being non-zero. $\square$

Here we used Lemma 1.80, where the main ingredient of the proof is that there exists no pre-Lie algebra structure on a semisimple Lie algebra. This was the first part, let us proceed with the second, namely that we get a semisimple decomposition of a Levi subalgebra.

Lemma 2.43. *Let* $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ *be a strongly disemisimple Lie algebra and* $\mathfrak{s}$ *be a Levi subalgebra with* $\mathfrak{s}_1 \subset \mathfrak{s}$*. Then there exists a semisimple subalgebra* $\mathfrak{s}_3$ *such that* $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_3$ *is disemisimple and* $\dim(\mathfrak{s}_1 \cap \mathfrak{s}_3) = \dim(\operatorname{rad}(\mathfrak{g}))$.

Proof. Again by the Levi-Malcev theorem we have that there exists a $z \in \operatorname{rad}(\mathfrak{g})$ such that $\mathfrak{s}_2 = \phi(\mathfrak{s}_3)$ where $\phi = \exp(\operatorname{ad}(z))$ for some semisimple subalgebra $\mathfrak{s}_3 \subset \mathfrak{s}$. This means that $\mathfrak{s}_3 = \phi^{-1}(\mathfrak{s}_2)$, where $\phi^{-1} = \exp(\operatorname{ad}(-z))$. Now let $x \in \mathfrak{g}$. Then there exist unique $s_1 \in \mathfrak{s}_1$ and $s_2 \in \mathfrak{s}_2$ such that $x = s_1 + s_2$. So we get

$$\begin{aligned}
x = s_1 + s_2 &= s_1 + \phi^{-1}(s_2) + s_2 - \phi^{-1}(s_2) \\
&= s_1 + \phi^{-1}(s_2) + (s_2 - s_2 - \sum_{k=1}^{\infty} \frac{(\operatorname{ad}(-z))^k}{k!}(y)) \\
&= s_1 + \phi^{-1}(s_2) - \sum_{k=1}^{\infty} \frac{(\operatorname{ad}(-z))^k}{k!}(y) \\
&\in \mathfrak{s}_1 + \mathfrak{s}_3 + \operatorname{rad}(\mathfrak{g})
\end{aligned}$$

Hence, we get

$$\mathfrak{g} \subseteq \mathfrak{s}_1 + \mathfrak{s}_3 + \operatorname{rad}(\mathfrak{g}) \subseteq \mathfrak{g},$$

so

$$\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_3 + \operatorname{rad}(\mathfrak{g}). \tag{1}$$

Now let $x \in \mathfrak{s}$. Then because of (1), we have that there exists $s_1 \in \mathfrak{s}_1$, $s_3 \in \mathfrak{s}_3$ and $r \in \operatorname{rad}(\mathfrak{g})$ such that

$$x = s_1 + s_3 + r.$$

So we have $r = x - s_1 - s_3 \in \mathfrak{s}$, therefore $r \in \mathfrak{s} \cap \operatorname{rad}(\mathfrak{g}) = 0$. Altogether we get

$$\mathfrak{s} \subseteq \mathfrak{s}_1 + \mathfrak{s}_3 \subseteq \mathfrak{s},$$

so $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_3$. This implies

$$\dim(\mathfrak{s}) = \dim(\mathfrak{s}_1) + \dim(\mathfrak{s}_3) - \dim(\mathfrak{s}_1 \cap \mathfrak{s}_3).$$

Since $\dim(\mathfrak{g}) = \dim(\mathfrak{s}_1) + \dim(\mathfrak{s}_2) = \dim(\mathfrak{s}) + \dim(\operatorname{rad}(\mathfrak{g}))$ and $\dim(\mathfrak{s}_2) = \dim(\mathfrak{s}_3)$ we get

$$\begin{aligned}
\dim(\mathfrak{s}) &= \dim(\mathfrak{g}) - \dim(\operatorname{rad}(\mathfrak{g})) \\
&= \dim(\mathfrak{s}_1) + \dim(\mathfrak{s}_2) - \dim(\operatorname{rad}(\mathfrak{g})) \\
&= \dim(\mathfrak{s}_1) + \dim(\mathfrak{s}_3) - \dim(\operatorname{rad}(\mathfrak{g}))
\end{aligned}$$

Therefore we get $\dim(\operatorname{rad}(\mathfrak{g})) = \dim(\mathfrak{s}_1 \cap \mathfrak{s}_3)$. $\square$

This concludes the second part. As mentioned before, the third part is based on Onishchik's work [75], [76]. This is not explicitly stated, however one can deduce it from it. Already looking at the classification of semisimple decompositions of simple Lie algebras gives the hint.

Lemma 2.44. *Let $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2$ be a semisimple decomposition, i.e. $\mathfrak{s}, \mathfrak{s}_1, \mathfrak{s}_2$ are semisimple Lie algebras. Then $\mathfrak{s}_1 \cap \mathfrak{s}_2$ is semisimple or zero.*

Proof. Let $G_{\mathfrak{s}}, G_{\mathfrak{s}_1}, G_{\mathfrak{s}_2}$ be the connected algebraic groups corresponding to $\mathfrak{s}, \mathfrak{s}_1$ and $\mathfrak{s}_2$. Then by the proof of Theorem 3.1 in [76] we get $G_{\mathfrak{s}} = G_{\mathfrak{s}_1} G_{\mathfrak{s}_2}$. Moreover, we have by Corollary 2.34 that there exists a real compact form $\mathfrak{l}$ of $\mathfrak{s}$ such that $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are the real compact forms of the Lie algebras $\mathrm{Ad}(x)(\mathfrak{s}_1)$ and $\mathrm{Ad}(y)(\mathfrak{s}_2)$ corresponding to some conjugates $xG_{\mathfrak{s}_1}x^{-1}$ and $yG_{\mathfrak{s}_2}y^{-1}$ of $G_{\mathfrak{s}_1}$ and $G_{\mathfrak{s}_2}$. Then we have by Lemma 2.35 that $\mathfrak{l}_1 \cap \mathfrak{l}_2$ is zero or semisimple. So that also the complexification $(\mathfrak{l}_1 \cap \mathfrak{l}_2)^{\mathbb{C}} = \mathfrak{l}_1^{\mathbb{C}} \cap \mathfrak{l}_2^{\mathbb{C}} = \mathrm{Ad}(x)(\mathfrak{s}_1) \cap \mathrm{Ad}(y)(\mathfrak{s}_2)$ is zero or semisimple. Now the claim is that

$$\mathfrak{s}_1 \cap \mathfrak{s}_2 \cong \mathrm{Ad}(x)(\mathfrak{s}_1) \cap \mathrm{Ad}(y)(\mathfrak{s}_2).$$

Let $x = ba$, where $b \in G_{\mathfrak{s}_2}$ and $a \in G_{\mathfrak{s}_1}$. Then we have

$$\begin{aligned}
\mathrm{Ad}(x)(\mathfrak{s}_1) \cap \mathrm{Ad}(y)(\mathfrak{s}_2) &= \mathrm{Ad}(b)(\mathrm{Ad}(a)(\mathfrak{s}_1) \cap \mathrm{Ad}(b^{-1}y)(\mathfrak{s}_2)) \\
&= \mathrm{Ad}(b)(\mathfrak{s}_1 \cap \mathrm{Ad}(b^{-1}y)(\mathfrak{s}_2)) \\
&\cong \mathfrak{s}_1 \cap \mathrm{Ad}(b^{-1}y)(\mathfrak{s}_2).
\end{aligned}$$

Writing $b^{-1}y = a'b'$ with $a' \in G_{\mathfrak{s}_1}$ and $b' \in G_{\mathfrak{s}_2}$, we get

$$\begin{aligned}
\mathfrak{s}_1 \cap \mathrm{Ad}(b^{-1}y)(\mathfrak{s}_2) &= \mathfrak{s}_1 \cap \mathrm{Ad}(a')\,\mathrm{Ad}(b')(\mathfrak{s}_2) \\
&= \mathfrak{s}_1 \cap \mathrm{Ad}(a')(\mathfrak{s}_2) \\
&= \mathrm{Ad}(a')(\mathfrak{s}_1 \cap \mathfrak{s}_2) \\
&\cong \mathfrak{s}_1 \cap \mathfrak{s}_2
\end{aligned}$$

Altogether we obtain $\mathfrak{s}_1 \cap \mathfrak{s}_2 \cong \mathrm{Ad}(x)(\mathfrak{s}_1) \cap \mathrm{Ad}(y)(\mathfrak{s}_2)$. So we get that $\mathfrak{s}_1 \cap \mathfrak{s}_2$ is semisimple or zero. □

This concludes the third part, so now we can go back to Theorem 2.41.

Proof of Theorem 2.41. Suppose $\mathfrak{g}$ is not semisimple. Then we have that $\mathrm{rad}(\mathfrak{g}) \neq 0$, so that $\dim(\mathrm{rad}(\mathfrak{g})) \geq 1$. Let $\mathfrak{s}$ be a Levi subalgebra with $\mathfrak{s}_1 \subset \mathfrak{s}$. There exists a semisimple subalgebra $\mathfrak{s}_3$ such that $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_3$ by Lemma 2.43. Moreover, we have $\dim(\mathfrak{s}_1 \cap \mathfrak{s}_3) = \dim(\mathrm{rad}(\mathfrak{g})) \geq 1$. By Lemma 2.44 we get that $\mathfrak{s}_1 \cap \mathfrak{s}_3 \subset \mathfrak{s}$ is semisimple. Therefore $\mathfrak{s}_1 \cap \mathfrak{s}_3$ is a Levi subalgebra of $(\mathfrak{s}_1 \cap \mathfrak{s}_3) \ltimes \mathrm{rad}(\mathfrak{g})$. Now we claim that

$$(\mathfrak{s}_1 \cap \mathfrak{s}_3) \ltimes \mathrm{rad}(\mathfrak{g}) = (\mathfrak{s}_1 \cap \mathfrak{s}_3) \dot{+} \phi(\mathfrak{s}_1 \cap \mathfrak{s}_3),$$

where $\phi = \exp(\mathrm{ad}(z))$ is the special automorphism with $\phi(\mathfrak{s}_3) = \mathfrak{s}_2$ from Lemma 2.43.

To show the claim, let $y \in \mathrm{rad}(\mathfrak{g})$, then we have that there exist unique $s_1 \in \mathfrak{s}_1, s_2 \in \mathfrak{s}_2$, such that

$$y = s_1 + s_2.$$

For this s_2, we have that there exists a unique $s_3 \in \mathfrak{s}_3$ such that $s_2 = \phi(s_3)$. Therefore we get

$$y = s_1 + s_2 = s_1 + \phi(s_3) = s_1 + s_3 + (\phi(s_3) - s_3).$$

We have that $(\phi(s_3) - s_3) = \sum_{k=1}^{\infty} \frac{(\mathrm{ad}(z))^k}{k!}(s_3) \in \mathrm{rad}(\mathfrak{g})$ and $s_1 + s_3 \in \mathfrak{s}$, so that

$$y - (\phi(s_3) - s_3) = s_1 + s_3 \in \mathfrak{s} \cap \mathrm{rad}(\mathfrak{g}) = 0.$$

Hence $s_3 = -s_1 \in \mathfrak{s}_1 \cap \mathfrak{s}_3$ and $y = \phi(s_3) - s_3$. Let $x \in \mathfrak{s}_1 \cap \mathfrak{s}_3$ and $y \in \mathrm{rad}(\mathfrak{g})$ as before. Then we have

$$x + y = x + \phi(s_3) - s_3 = (x - s_3) + \phi(s_3) \in (\mathfrak{s}_1 \cap \mathfrak{s}_3) \dot{+} \phi(\mathfrak{s}_1 \cap \mathfrak{s}_3).$$

Conversely let $y, z \in \mathfrak{s}_1 \cap \mathfrak{s}_3$, then

$$y + \phi(z) = (y + z) + (\phi - \mathrm{id})(z) \in (\mathfrak{s}_1 \cap \mathfrak{s}_3) \ltimes \mathrm{rad}(\mathfrak{g}).$$

Thus we have shown the claim. But this means we get a Lie algebra, namely $(\mathfrak{s}_1 \cap \mathfrak{s}_3) \ltimes \mathrm{rad}(\mathfrak{g})$, that decomposes into the direct vector space sum of two Levi subalgebras, which is a contradiction to Lemma 2.42. Hence we get that $\mathfrak{g}$ is semisimple. Then we get that $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ is a reductive decomposition, since semisimple subalgebras are reductive in $\mathfrak{g}$. By Theorem 2.40, we get that

$$\mathfrak{g} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2.$$

□

Remark 10. This does not hold for reductive Lie algebras. The direct sum of two reductive Lie algebras need not be reductive. We already get

- $\mathfrak{r}_2(\mathbb{C}) = \mathfrak{a} \dot{+} \mathfrak{b}$ is the sum of two 1-dimensional abelian subalgebras.
- $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2) = \mathfrak{sl}_2(\mathbb{C}) \dot{+} V(2)$, which is the sum of a semisimple and an abelian Lie subalgebra.

However if we assume that we have a Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \dot{+} \mathfrak{g}_2$ that decomposes as a direct vector space sum of two subalgebras that are reductive in $\mathfrak{g}$, then we do not know. We believe the following in this case.

Conjecture 2. Let $\mathfrak{g} = \mathfrak{g}_1 \dot{+} \mathfrak{g}_2$, where $\mathfrak{g}_1, \mathfrak{g}_2$ are reductive in $\mathfrak{g}$. Then $\mathfrak{g}$ is reductive and $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

3 Existence of post-Lie algebra structures

In this chapter we want to study the existence of post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ given assumptions on $\mathfrak{g}$ or $\mathfrak{n}$. First, we recall some results already known and give some examples. We then proceed with the study of post-Lie algebra structures on pairs of reductive Lie algebras, in particular semisimple Lie algebras. In the subsequent chapter we want to generalize results obtained for semisimple Lie algebras to perfect Lie algebras. The results in Section 3.1 and 3.2 are joint work with D. Burde and K. Dekimpe and have been published in [28], with the exception of Lemma 3.16 and Theorem 3.17, which have not been presented yet. The results in Section 3.3 are also joint work with D. Burde and K. Dekimpe and have been presented in [29].

3.1 Existence results for pairs of Lie algebras where one is reductive

This part is dedicated to the study of post-Lie algebra structures on pairs of Lie algebras where one of the Lie algebras is simple, semisimple or reductive. We recall some results already known on the existence of these structures and give new results concerning rigidity and the existence of post-Lie algebra structures. We start by studying the notion of rigidity for post-Lie algebra structures and then present some examples of post-Lie algebra structures. We have already discussed what we know about the existence of post-Lie algebra structures on semisimple Lie algebras in Section 1.3. We are able to generalize these results and show stronger statements regarding rigidity. All Lie algebras are finite dimensional and complex, unless stated otherwise.

3.1.1 Rigidity results

We have seen examples of pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ that admit post-Lie algebra structures where $\mathfrak{g}$ and $\mathfrak{n}$ are not isomorphic. We ask ourselves under which assumptions do we have rigidity.

Question 2. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a PA-structure. For which algebraic properties of $\mathfrak{g}$ and $\mathfrak{n}$ can we conclude that $\mathfrak{g} \cong \mathfrak{n}$?

We look at the properties simple, semisimple and reductive. For solvable Lie algebras already in lower dimension we find examples where $\mathfrak{g}$ and $\mathfrak{n}$ are not isomorphic.

Example 10. Let $\mathfrak{g} = \mathfrak{r}_{3,1}(\mathbb{C})$ with basis (e_1, e_2, e_3) and Lie brackets $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$ and let $\mathfrak{n} = \mathfrak{r}_3(\mathbb{C})$ given by $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$. Then there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ given by $e_1 \cdot e_3 = -e_2$ and all other products are zero.

If $\mathfrak{g}$ and $\mathfrak{n}$ are simple, then we have rigidity, see [31], which we already mentioned, see Proposition 1.76, but not as a rigidity statement.

Proposition 3.1. *Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ and $\mathfrak{n}$ are simple. Then $x \cdot y = 0$, so $[x, y] = \{x, y\}$ for all $x, y \in \mathfrak{g}$ or $x \cdot y = [x, y]$ and $[x, y] = -\{x, y\}$ for all $x, y \in \mathfrak{g}$. So we have that $\mathfrak{g} \cong \mathfrak{n}$.*

For $\mathfrak{g}$ simple, we even have strong rigidity, see [24].

Theorem 3.2. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras admitting a PA-structure where $\mathfrak{g}$ is simple and $\mathfrak{n}$ is arbitrary. Then $\mathfrak{n}$ is also simple and $\mathfrak{g} \cong \mathfrak{n}$.*

In [37] we find these three rigidity results for semisimple Lie algebras using the theory of Rota-Baxter operators which we mention here.

Theorem 3.3. *Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is simple. Then $\mathfrak{g}$ is also simple and $\mathfrak{g} \cong \mathfrak{n}$.*

Proposition 3.4. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is the direct sum of simple ideals and $\mathfrak{n}$ is semisimple. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then $\mathfrak{g} \cong \mathfrak{n}$.*

Proposition 3.5. *All PA-structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$ where $\mathfrak{s}_i$ are simple ideals for $i = 1, 2$ arise by the trivial Rota-Baxter operators or by one of the following Rota-Baxter operators on $\mathfrak{n}$ (up to permuting the factors and application of $\psi(R) = -(R + \mathrm{id})$)*

$$\begin{aligned} R((s_1, s_2)) &= (-s_1, -\phi(s_1)) \\ R((s_1, s_2)) &= (0, \phi(s_1)) \\ R((s_1, s_2)) &= (-s_1, 0) \end{aligned}$$

where $\phi \in \mathrm{Aut}(\mathfrak{n})$.

Moreover in [38] we can find the following two results.

Theorem 3.6. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ arising from an RB-operator of weight 1 on $\mathfrak{n}$. Suppose that $\mathfrak{g}$ is semisimple. Then $\mathfrak{n}$ is also semisimple.*

Corollary 3.7. *Let $x \cdot y$ be a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is complete. Then $\mathfrak{n}$ is also semisimple.*

We are able to generalize all these results and show strong rigidity also in the case where $\mathfrak{g}$ is semisimple using Theorem 2.41. Let us first show a Lemma, which shows that the study of post-Lie algebra structures is closely connected to the study of decompositions of Lie algebras.

Lemma 3.8. *Suppose that $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then we get a decomposition $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}$ where $\mathfrak{h} = L(\mathfrak{g})$ and ϕ is the embedding from Proposition 1.66.*

Proof. Recall that we get an embedding by Proposition 1.66

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)). \end{aligned}$$

Let $\mathfrak{h} = L(\mathfrak{g})$. We claim now that $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}$. Indeed, let $(x, L(y)) \in \mathfrak{n} \rtimes \mathfrak{h}$, then we can write

$$(x, L(y)) = (x, L(x)) - (0, L(x) - L(y)) \in \phi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

Conversely, for $(x, L(x)) \in \phi(\mathfrak{g})$ and $(0, L(y)) \in \mathfrak{h}$, we have

$$(x, L(x)) + (0, L(y)) = (x, L(x + y)) \in \mathfrak{n} \rtimes \mathfrak{h}.$$

So that we get $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}$. ☐

Theorem 3.9. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is arbitrary. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then $\mathfrak{n}$ is also semisimple and $\mathfrak{g} \cong \mathfrak{n}$.*

Proof. We know that there is an injective homomorphism

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

by Proposition 1.66 and writing $\mathfrak{h} = L(\mathfrak{g})$, we get by Lemma 3.8 a decomposition

$$\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

We have that both $\phi(\mathfrak{g})$ and $\mathfrak{h}$ are semisimple because they are the homomorphic image of a semisimple Lie algebra. Therefore $\mathfrak{n} \rtimes \mathfrak{h}$ is strongly disemisimple, i.e. a direct vector space sum of two semisimple subalgebras $\phi(\mathfrak{g})$ and $\mathfrak{h}$. By Theorem 2.41 we get that $\mathfrak{n} \rtimes \mathfrak{h}$ is semisimple and so also $\mathfrak{n}$, because it is an ideal in $\mathfrak{n} \rtimes \mathfrak{h}$. By Lemma 1.24 we get $\mathfrak{n} \rtimes \mathfrak{h} \cong \mathfrak{n} \oplus \mathfrak{h}$ and by Theorem 2.41 we also have $\mathfrak{n} \rtimes \mathfrak{h} \cong \phi(\mathfrak{g}) \oplus \mathfrak{h}$. We can write

$$\begin{aligned} \phi(\mathfrak{g}) &\cong \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_i \\ \mathfrak{n} &\cong \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_j \\ \mathfrak{h} &\cong \mathfrak{c}_1 \oplus \ldots \oplus \mathfrak{c}_k \end{aligned}$$

as a direct sum of simple ideals. Since $\phi(\mathfrak{g}) \oplus \mathfrak{h} \cong \mathfrak{n} \oplus \mathfrak{h}$, we have

$$(\mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_i) \oplus (\mathfrak{c}_1 \oplus \ldots \oplus \mathfrak{c}_k) \cong (\mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_j) \oplus (\mathfrak{c}_1 \oplus \ldots \oplus \mathfrak{c}_k).$$

The decomposition of a semisimple Lie algebra into simple ideals is unique up to permutation, therefore we obtain $i = j$ and $\phi(\mathfrak{g}) \cong \mathfrak{n}$, so $\mathfrak{g} \cong \mathfrak{n}$. ☐

Remark 11. This does not hold for $\mathfrak{n}$ semisimple, see Proposition 1.78. Also if $\mathfrak{g}$ and $\mathfrak{n}$ are reductive we do not have rigidity. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n} = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}^{n^2}$ then $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure by taking a post-Lie algebra structure on $(\mathfrak{gl}_n(\mathbb{C}), \mathbb{C}^{n^2})$ that comes from a pre-Lie algebra structure and the trivial structure on $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}))$.

For $\mathfrak{g}$ reductive and $\mathfrak{n}$ simple, we have the following rigidity result, see [38].

Theorem 3.10. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras where* $\mathfrak{g}$ *is reductive and* $\mathfrak{n}$ *is simple. Suppose* $(\mathfrak{g}, \mathfrak{n})$ *admits a post-Lie algebra structure. Then* $\mathfrak{g}$ *is also simple and* $\mathfrak{g} \cong \mathfrak{n}$*.*

We believe that this can be generalized to $\mathfrak{n}$ semisimple.

Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is reductive and $\mathfrak{n}$ is semisimple. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then $\mathfrak{g}$ is also semisimple and $\mathfrak{g} \cong \mathfrak{n}$.

3.1.2 Post-Lie algebra structures where $\mathfrak{g}$ is reductive with one-dimensional center

We have seen that for post-Lie algebra structures where $\mathfrak{g}$ is semisimple, we have rigidity. This does not hold for reductive Lie algebras. However, it is natural to ask what happens if the center is one-dimensional. For $\mathfrak{gl}_n(\mathbb{C})$ there exist pre-Lie algebra structures and they have been classified in [18]. We show results for $\mathfrak{g}$ reductive with 1-dimensional center and $\mathfrak{n}$ solvable non-nilpotent or 2-step nilpotent. For the more general case where $\mathfrak{g}$ is reductive with 1-dimensional center and $\mathfrak{n}$ is nilpotent non-abelian, we give a proof at the end where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

For the question of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is reductive with 1-dimensional center and $\mathfrak{n}$ is solvable non-nilpotent we have the following result.

Theorem 3.11. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras where* $\mathfrak{g}$ *is reductive with one-dimensional center and* $\mathfrak{n}$ *is solvable non-nilpotent. Then there exists no PA-structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

Proof. Suppose there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{,\})$ and $\dim(V) = n$. By the third condition of a post-Lie algebra structure, we have that all $L(x)$ are derivations of $\mathfrak{n}$. By Corollary 1.34, we have $D(\mathrm{rad}(\mathfrak{n})) \subseteq \mathrm{nil}(\mathfrak{n})$ for all $D \in \mathrm{Der}(\mathfrak{n})$. Since $\mathfrak{n}$ is solvable we get $D(\mathfrak{n}) \subseteq \mathrm{nil}(\mathfrak{n})$. This means we get

$$\mathfrak{n} \cdot \mathfrak{n} = L(\mathfrak{n})(\mathfrak{n}) \subseteq \mathrm{nil}(\mathfrak{n}).$$

Because $\mathfrak{n}$ is solvable, we have that $\{\mathfrak{n}, \mathfrak{n}\}$ is a nilpotent ideal by Corollary 1.34, so that $\{\mathfrak{n}, \mathfrak{n}\} \subseteq \mathrm{nil}(\mathfrak{n})$. So for $x, y \in V$, we get

$$[x, y] = x \cdot y - y \cdot x + \{x, y\} \in \mathrm{nil}(\mathfrak{n}).$$

We get $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{nil}(\mathfrak{n})$. We have that $\dim([\mathfrak{g}, \mathfrak{g}]) = n - 1$, because $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$ and the center is one-dimensional. This means that $\dim(\mathrm{nil}(\mathfrak{n})) \geq n - 1$. Since $\mathfrak{n}$ is solvable non-nilpotent, we obtain $\dim(\mathrm{nil}(\mathfrak{n})) \leq n - 1$. Together we have $\dim(\mathrm{nil}(\mathfrak{n})) = n - 1$ and hence $[\mathfrak{g}, \mathfrak{g}] = \mathrm{nil}(\mathfrak{n})$ as vector spaces. However, this means that we get a post-Lie algebra structure on $([\mathfrak{g}, \mathfrak{g}], \mathrm{nil}(\mathfrak{n}))$ which is a contradiction to Theorem 3.9, because $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. □

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ solvable non-nilpotent, we therefore do not get a post-Lie algebra structure. However, if $\mathfrak{g}$ is reductive with $\dim(Z(\mathfrak{g})) \geq 2$, then we have the following result.

Proposition 3.12. *For any $n \geq 2$ there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \oplus \mathbb{C}$ with basis $(y_1, \ldots y_{n^2}, x)$ where $\mathbb{C} = \operatorname{span}\{x\}$ and $\mathfrak{n}$ is the 2-step solvable Lie algebra with $\{x, y_i\} = y_i$ for $1 \leq i \leq n^2$.*

Proof. Let $\mathfrak{a} = \operatorname{span}\{y_1, y_2, \ldots y_{n^2}\}$, then we have $\mathfrak{n} = < x > \ltimes \mathfrak{a}$, where x acts on $\mathfrak{a}$ by the derivation $\mathrm{id}_{\mathfrak{a}}$, so $\operatorname{ad}(x)|_{\mathfrak{a}} = \mathrm{id}_{\mathfrak{a}}$. By definition, we have that $\mathfrak{a}$ is an abelian ideal in $\mathfrak{n}$. Now we can choose a pre-Lie algebra structure $y_i \cdot y_j$ on $\mathfrak{gl}_n(\mathbb{C})$, which exists, see [18]. Then we get a post-Lie algebra structure on $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{a})$. Now we can extend this to $(\mathfrak{gl}_n(\mathbb{C}) \oplus \mathbb{C}, \mathfrak{n})$ by

$$\begin{aligned} x \cdot x &= 0 \\ y_i \cdot x &= 0 \\ x \cdot y_i &= -y_i = -\{x, y_i\}, \end{aligned}$$

for all $1 \leq i \leq n^2$. We have to check that this is in fact a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Let us first look at the first condition, i.e.

$$u \cdot v - v \cdot u = [u, v] - \{u, v\}.$$

For $u = y_i$ and $v = y_j$, this holds because we have a pre-Lie algebra structure on $\mathfrak{gl}_n(\mathbb{C})$ and $\{y_i, y_j\} = 0$. For $u = x$ and $v = x$, this is trivially satisfied. So the remaining case is where one of them is x and the other y_i for $1 \leq i \leq n^2$. By symmetry we may assume that $u = x$ and $v = y_i$, then we have

$$x \cdot y_i - y_i \cdot x = -y_i = [x, y_i] - \{x, y_i\}.$$

For the third identity, we need to check that all $L(v)$ are derivations of $\mathfrak{n}$ for all v. In both $L(x)$ and $L(y_i)$ the last row and last column are zero. For $v = y_i$ the remaining matrix of size n^2 is a derivation of $\mathfrak{a}$, since we have a post-Lie algebra structure on $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{a})$. For $v = x$, the remaining matrix is $-I$, and therefore a derivation, so we get $L(v) \in \operatorname{Der}(\mathfrak{n})$ for all v.

For the second identity, we have to show that $L : v \mapsto L(v)$ is a representation of $\mathfrak{g}$. Since also here we have that $L(y_i)$ and $L(x)$ have the last row and column zero and for the remaining matrix we obtain the result for $L(y_i)$ from the induced post-Lie algebra structure on $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{a})$ and for $L(x)$ because the remaining matrix is $-I$. In particular, $L(x)$ commutes with all operators $L(y_i)$. □

Now let us look again at the case of $\mathfrak{gl}_n(\mathbb{C})$. Let us first consider it for $\mathfrak{gl}_2(\mathbb{C})$.

Proposition 3.13. *Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$ and $\mathfrak{n}$ be nilpotent non-abelian. Then there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Up to isomorphism, we have two non-abelian nilpotent Lie algebras of dimension 4. One is $\mathfrak{n} = \mathfrak{n}_4(\mathbb{C})$ and the other $\mathfrak{n} = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$. Let us first look at the case $\mathfrak{n} = \mathfrak{n}_4(\mathbb{C})$. We claim that any derivation of $\mathfrak{n}_4(\mathbb{C})$ is a lower triangular matrix. Indeed, let

$D \in \mathrm{Der}(\mathfrak{n})$ and let $< e_1, e_2, e_3, e_4 >$ be a basis of $\mathfrak{n}$ with Lie bracket $\{e_1, e_2\} = e_3$ and $\{e_1, e_3\} = e_4$. Let $D = (a_{ij})$ with $i, j = 1, 2, 3, 4$. Then we have that D is of the form

$$D = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} & 0 \\ a_{41} & a_{42} & a_{32} & 2a_{11} + a_{22} \end{pmatrix}$$

This means that $\mathrm{Der}(\mathfrak{n})$ is solvable. But then also $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is solvable. Now let us assume that there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then we get an injective homomorphism $\phi : \mathfrak{g} \to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ by Proposition 1.66, but $\mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ is solvable and therefore also $\phi(\mathfrak{g}) \cong \mathfrak{g}$ is solvable, which is a contradiction. Now assume $\mathfrak{n} = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$. Suppose there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Let $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{s}$ acts on $\mathfrak{n}$ via $L : \mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$. We claim that $\mathfrak{s}$ acts trivially on $Z(\mathfrak{n})$. Let $\{\mathfrak{n}, \mathfrak{n}\} =< z >\subset Z(\mathfrak{n}) =< z, t >$. By Weyl's Theorem, there exists a $\mathfrak{s}$-submodule W of $\{\mathfrak{n}, \mathfrak{n}\}$ in $Z(\mathfrak{n})$ such that $Z(\mathfrak{n}) = W \oplus \{\mathfrak{n}, \mathfrak{n}\}$ and $\dim(\{\mathfrak{n}, \mathfrak{n}\}) = 1$. Since $\mathfrak{s}$ is simple, it acts trivially on both W and $\{\mathfrak{n}, \mathfrak{n}\}$, and hence $\mathfrak{s} \cdot Z(\mathfrak{n}) = 0$. We have that $Z(\mathfrak{n})$ is a $\mathfrak{s}$-submodule of $\mathfrak{n}$ and again by Weyl's Theorem, there exists a $\mathfrak{s}$-submodule U of $\mathfrak{n}$ such that

$$\mathfrak{n} = Z(\mathfrak{n}) \oplus U$$

with $U =< u, v >$. Suppose $\{U, U\} = 0$, then $\{\mathfrak{n}, \mathfrak{n}\} = \{U \oplus Z(\mathfrak{n}), U \oplus Z(\mathfrak{n})\} = 0$, which is a contradiction. Therefore $\{u, v\} = w$ is nonzero and $\mathfrak{h} =< u, v, w >$ is a subalgebra of $\mathfrak{n}$ isomorphic to $\mathfrak{n}_3(\mathbb{C})$. We have $\mathfrak{s} \cdot \mathfrak{h} \subseteq \mathfrak{h}$ by construction and hence

$$[x, y] = x \cdot y - y \cdot x + \{x, y\} \in \mathfrak{h}$$

for all $x, y \in \mathfrak{g}$. We get $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{h}$ and since $\dim(\mathfrak{h}) = \dim(\mathfrak{s})$, we get $\mathfrak{s} = \mathfrak{h}$ as vector spaces. But this means we get a post-Lie algebra structure on $(\mathfrak{s}, \mathfrak{h})$ where $\mathfrak{s}$ is semisimple and $\mathfrak{h}$ is nilpotent, which is a contradiction to Theorem 3.9. ☐

Now let us show the more general case $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{n})$ where $\mathfrak{n}$ is 2-step nilpotent.

Lemma 3.14. *Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is 2-step nilpotent with Lie bracket $\{x, y\}$. Then*

$$x \circ y = \frac{1}{2}\{x, y\} + x \cdot y$$

defines a pre-Lie algebra structure on $\mathfrak{g}$.

Proof. This follows from Lemma 1.73. One can also compute it directly. Indeed, let $x, y, z \in \mathfrak{g}$. Then

$$x \circ y - y \circ x = x \cdot y - y \cdot x + \{x, y\} = [x, y]$$

and

$$\begin{aligned}
[x,y]\circ z &= \frac{1}{2}\{[x,y],z\} + [x,y]\cdot z \\
&= \frac{1}{2}\{[x,y],z\} + (x\cdot(y\cdot z) - y\cdot(x\cdot z)) \\
&= \frac{1}{2}\{x\cdot y - y\cdot x + \{x,y\},z\} + (x\cdot(y\cdot z) - y\cdot(x\cdot z)) \\
&= \frac{1}{2}\{x\cdot y,z\} - \frac{1}{2}\{y\cdot x,z\} + x\cdot(y\cdot z) - y\cdot(x\cdot z).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
x\circ(y\circ z) - y\circ(x\circ z) &= \frac{1}{2}\{x,y\cdot z\} + \frac{1}{2}x\cdot\{y,z\} + x\cdot(y\cdot z) \\
&\quad - \frac{1}{2}\{y,x\cdot z\} - \frac{1}{2}y\cdot\{x,z\} - y\cdot(x\cdot z) \\
&= \frac{1}{2}\{x\cdot y,z\} + x\cdot(y\cdot z) - \frac{1}{2}\{y\cdot x,z\} - y\cdot(x\cdot z)
\end{aligned}$$

by the third condition of a post-Lie algebra structure. Therefore we get

$$[x,y]\circ z = x\circ(y\circ z) - y\circ(x\circ z)$$

$\square$

Let us define the linear operators $l(x), L(x), \mathrm{ad}(x)$ by $L(x)(y) = x\cdot y$, $l(x)(y) = x\circ y$ and $\mathrm{ad}(x)(y) = \{x,y\}$. Then we can write the pre-Lie algebra structure as

$$l(x) = \frac{1}{2}\,\mathrm{ad}(x) + L(x).$$

Proposition 3.15. *Let $(\mathfrak{g},\mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ is 2-step nilpotent and non-abelian. Then there exists no post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$.*

Proof. We can assume $n \geq 3$, because of Proposition 3.13. Suppose there exists a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$ and let $L : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{Der}(\mathfrak{n})$ be the representation given by $x \mapsto L(x)$. Then we obtain a $\mathfrak{s} = \mathfrak{sl}_n(\mathbb{C})$ module M_L via L. By Lemma 3.14 we also obtain a $\mathfrak{g}$-module via l, so also a $\mathfrak{s}$-module which we denote as M_l and that defines a pre-Lie algebra structure on $\mathfrak{g}$. Theorem 4.5 in [9] implies that the $\mathfrak{s}$-module M_l is *special*, see [9] for the definition, and equivalent to $M_n(\mathbb{C})$. The module action is given by left multiplication of matrices and $M_n(\mathbb{C})$ is equivalent to $L(\omega_1)^{\oplus n}$ and to $(L(\omega_1)^*)^{\oplus n}$ where $L(\omega_1)$ denotes the n-dimensional natural $\mathfrak{s}$-module and $L(\omega_1)^*$ its dual. We can therefore assume that the irreducible $\mathfrak{s}$-submodules of M_l are of the form $L(\omega_1)$. Given the formula in Lemma 3.14 we see that M_l and M_L have the same irreducible $\mathfrak{s}$-submodules, and therefore also M_L has only irreducible $\mathfrak{s}$-submodules of type $L(\omega_1)$. We have that $\{\mathfrak{n},\mathfrak{n}\}$ is a characteristic ideal of $\mathfrak{n}$, so it is also an $\mathfrak{s}$-submodule of $\mathfrak{n}$ and hence of M_L. By

Weyl's Theorem there exists a $\mathfrak{s}$-submodule $V = V_1 \oplus ... \oplus V_k$ of $\mathfrak{n}$ where the V_i's are irreducible such that

$$\mathfrak{n} = V \oplus \{\mathfrak{n}, \mathfrak{n}\}.$$

This implies that V is a generating space for $\mathfrak{n}$. Because $\mathfrak{n}$ is 2-step nilpotent non-abelian we have that the $\mathfrak{s}$-submodule $\{\mathfrak{n}, \mathfrak{n}\}$ is an $\mathfrak{s}$-submodule of

$$V \wedge V = (V_1 \oplus ... \oplus V_k) \wedge (V_1 \oplus ... \oplus V_k).$$

We get by Lemma 3.4 in [42]

$$V_i \wedge V_j = \begin{cases} L(\omega_1) \otimes L(\omega_1) \cong L(\omega_2) \oplus L(2\omega_1) & \text{if} \quad i \neq j \\ \Lambda^2(V_i) \cong L(\omega_2) & \text{if} \quad i = j. \end{cases}$$

This means that $V \wedge V$ contains an irreducible $\mathfrak{s}$-submodule which is is not of type $L(\omega_1)$. This implies that $\{\mathfrak{n}, \mathfrak{n}\} = 0$, which is a contradiction to $\mathfrak{n}$ being 2-step nilpotent or that $\{\mathfrak{n}, \mathfrak{n}\}$ does not contain an irreducible $\mathfrak{s}$-submodule of type $L(\omega_1)$, which is a contradiction to M_L containing only irreducible submodules of type $L(\omega_1)$. □

This can be generalized also to $\mathfrak{g}$ being reductive with 1-dimensional center and simple Levi subalgebras using the same method. In [28] we posed the following question.

Question 3. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is reductive with 1-dimensional center and $\mathfrak{n}$ is nilpotent non-abelian. Is it true that there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$?

We now give a proof which is a generalization of the above Theorem for the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ nilpotent non-abelian, which has not been presented yet. First let us start with a lemma.

Lemma 3.16. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ is nilpotent. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then $\mathfrak{n}$ becomes a $\mathfrak{sl}_n(\mathbb{C})$-module via L and $\mathfrak{n}$ cannot contain a 1-dimensional $\mathfrak{sl}_n(\mathbb{C})$-submodule.*

Proof. Since we assume that a post-Lie algebra structure exists, we get an embedding by Proposition 1.66

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

and a decomposition $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dotplus \mathfrak{h}$ by Lemma 3.8, where $\mathfrak{h} = L(\mathfrak{sl}_n(\mathbb{C})) \oplus L(Z(\mathfrak{g}))$. Moreover we have that $L(\mathfrak{sl}_n(\mathbb{C})) \neq 0$ cause otherwise we would obtain $\mathfrak{sl}_n(\mathbb{C})$ as a subalgebra of $\mathfrak{n}$ and we get a contradiction. Now let us write $L(\mathfrak{sl}_n(\mathbb{C})) = \mathfrak{s} \cong \mathfrak{sl}_n(\mathbb{C})$. Then $\mathfrak{n}$ is a $\mathfrak{s}$-module. So the first part is done. Now let us assume that $\mathfrak{n}$ contains a 1-dimensional $\mathfrak{s}$-submodule, i.e. there exists a $V \subseteq \mathfrak{n}$ with $\dim(V) = 1$. By Weyl's Theorem, there exists a $\mathfrak{s}$-submodule W such that

$$\mathfrak{n} = V \oplus W.$$

We have $\dim(W) = \dim(\mathfrak{s})$, because $\dim(\mathfrak{n}) = \dim(\mathfrak{g}) = \dim(\mathfrak{s}) + 1$. Let us denote $\phi(\mathfrak{g}) = \tilde{\mathfrak{s}} \oplus \tilde{\mathfrak{a}}$. We have that both $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are Levi subalgebras of $\mathfrak{h} \ltimes \mathfrak{n}$ by Lemma 1.44. Note that they are also Levi subalgebras of the Lie algebra $\mathfrak{s}\ltimes\mathfrak{n}$, because $\tilde{\mathfrak{s}} \subseteq \mathfrak{s}\ltimes\mathfrak{n}$. By the Levi-Malcev theorem, they are conjugated by a special automorphism $\tau = \exp(\mathrm{ad}(z))$ with $z \in \mathfrak{n}$, i.e. $\tau(\mathfrak{s}) = \tilde{\mathfrak{s}}$. For this $z \in \mathfrak{n}$, we have that there exists a unique $z_1 \in V$ and unique $z_2 \in W$ such that $z = z_1 + z_2$. By Lemma 1.80 we have that $\mathrm{Ann}(z_2) \neq 0$, so that there exists a non-zero $x \in \mathfrak{s}$ such that $x.z_2 = 0$, where the dot denotes the action in the semidirect product $\mathfrak{s} \ltimes \mathfrak{n}$. Moreover we have that $[s, z_1] = 0$ for all $s \in \mathfrak{s}$ since V is a 1-dimensional $\mathfrak{s}$-module. So we get that there exists a non-zero $x \in \mathfrak{s}$ such that $[x, z] = [x, z_1 + z_2] = 0$. But this means that $\tau(x) = x$, so that $x \in \mathfrak{s} \cap \tilde{\mathfrak{s}} = 0$, and we get a contradiction. $\square$

Theorem 3.17. *Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ be nilpotent non-abelian. Then there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then by Proposition 1.66 we have an embedding

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

and we again obtain a decomposition by Lemma 3.8

$$\mathfrak{h} \ltimes \mathfrak{n} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h},$$

where $\mathfrak{h} = L(\mathfrak{g}) = L(\mathfrak{sl}_n(\mathbb{C})) \oplus L(Z(\mathfrak{g})) =: \mathfrak{s} \oplus \mathfrak{a}$. Let us denote $\phi(\mathfrak{g}) = \phi(\mathfrak{sl}_n(\mathbb{C})) \oplus \phi(Z(\mathfrak{g})) = \tilde{\mathfrak{s}} \oplus \tilde{\mathfrak{a}}$. We start with a case distinction.

Case 1. $\mathfrak{a} = 0$. Then $L(Z(\mathfrak{g})) = 0$, so that $\tilde{\mathfrak{a}} \subseteq \mathfrak{n}$. Note that $\mathfrak{n}$ is also a $\tilde{\mathfrak{s}}$-module, since both $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are Levi subalgebras of $\mathfrak{s} \ltimes \mathfrak{n}$. But this means that $\tilde{\mathfrak{a}}$ is a 1-dimensional submodule of $\mathfrak{n}$, which is a contradiction to Lemma 3.16.

Case 2. $\mathfrak{a} \neq 0$. By the Levi-Malcev theorem, we have that $\tau(\tilde{\mathfrak{s}}) = \mathfrak{s}$, where $\tau = \exp(\mathrm{ad}(z))$ with $z \in \mathfrak{n}$. Then $\tau(\tilde{\mathfrak{a}})$ is also an $\mathfrak{s}$-module, so that we have

$$\tau(\tilde{\mathfrak{a}}) \cap \mathfrak{a} = \begin{cases} 0 \\ \mathfrak{a} \end{cases}$$

because $\dim(\mathfrak{a}) = 1$. If $\tau(\tilde{\mathfrak{a}}) \cap \mathfrak{a} = 0$, then we have that $\tau(\tilde{\mathfrak{a}}) \dot{+} \mathfrak{a}$ is an $\mathfrak{s}$-submodule of $\mathrm{rad}(\mathfrak{h} \ltimes \mathfrak{n})$ and $\mathrm{rad}(\mathfrak{h} \ltimes \mathfrak{n}) = \mathfrak{a} \dot{+} \mathfrak{n}$ by Lemma 1.44, so there exists an $\mathfrak{s}$-module V such that $\mathrm{rad}(\mathfrak{h} \ltimes \mathfrak{n}) = \tau(\tilde{\mathfrak{a}}) \dot{+} \mathfrak{a} \dot{+} V \cong \mathfrak{a} \dot{+} \mathfrak{n}$, as $\mathfrak{s}$-modules. But this means that $\mathfrak{n} \cong \tau(\tilde{\mathfrak{a}}) \dot{+} V$ as $\mathfrak{s}$-module, so that $\mathfrak{n}$ contains a 1-dimensional $\mathfrak{s}$-submodule, which is a contradiction to Lemma 3.16.
Therefore we get $\tau(\tilde{\mathfrak{a}}) = \mathfrak{a}$ and hence $\tau(\phi(\mathfrak{g})) = \mathfrak{h}$. Now let us look at the linear map

$$\begin{aligned} \psi : \mathfrak{h} &\to \mathfrak{n} \\ x &\mapsto x.z. \end{aligned}$$

Then ψ is bijective, because otherwise there exists a non-zero $x \in \mathfrak{h}$ such that $\tau(x) = x$ and hence $x \in \mathfrak{h} \cap \phi(\mathfrak{g}) = 0$ which is a contradiction. Let ρ be the representation associated to $\mathfrak{n}$. Then consider the $\mathfrak{h}$-module M given by $\psi^{-1}\rho\psi$. We get that $M \cong \mathfrak{n}$ as $\mathfrak{h}$-modules and that M induces a pre-Lie algebra structure on $\mathfrak{h}$, see Proposition 3.3 in [20]. Then we follow the line of argument as in Proposition 3.15 and we get that M and $\mathfrak{n}$ have the same irreducible $\mathfrak{s}$-modules, which are of the form $L(\omega_1)$, see [9]. We have that $\{\mathfrak{n}, \mathfrak{n}\}$ is a characteristic ideal, so it is also a $\mathfrak{s}$-submodule of $\mathfrak{n}$. Again by Weyl's Theorem, there exists a $\mathfrak{s}$-submodule V of $\mathfrak{n}$ such that

$$\mathfrak{n} = V \oplus \{\mathfrak{n}, \mathfrak{n}\}.$$

We have that $\{\mathfrak{n}, \{\mathfrak{n}, \mathfrak{n}\}\}$ is a characteristic ideal, and hence a $\mathfrak{s}$-submodule of $\{\mathfrak{n}, \mathfrak{n}\}$. Therefore by Weyl's Theorem there exists a $\mathfrak{s}$-submodule W of $\{\mathfrak{n}, \mathfrak{n}\}$, such that

$$\mathfrak{n} = V \oplus W \oplus \{\mathfrak{n}, \{\mathfrak{n}, \mathfrak{n}\}\}.$$

We have that W can be embedded into $V \wedge V$. Writing $V = V_1 \oplus ... \oplus V_k$ as the sum of irreducible $\mathfrak{s}$-submodules V_i we get that

$$V \wedge V = (V_1 \oplus ... \oplus V_k) \wedge (V_1 \oplus ... \oplus V_k).$$

Again by Lemma 3.4 in [42], we have

$$V_i \wedge V_j = \begin{cases} L(\omega_1) \otimes L(\omega_1) \cong L(\omega_2) \oplus L(2\omega_1) & \text{if} \quad i \neq j \\ \Lambda^2(V_i) \cong L(\omega_2) & \text{if} \quad i = j. \end{cases}$$

This means that $V \wedge V$ contains an irreducible $\mathfrak{s}$-submodule which is is not of type $L(\omega_1)$. This implies that $W = 0$, which is a contradiction to $\mathfrak{n}$ being nilpotent non-abelian or that W does not contain an irreducible $\mathfrak{s}$-submodule of type $L(\omega_1)$, which is a contradiction to $\mathfrak{n}$ containing only irreducible submodules of type $L(\omega_1)$.

□

3.2 The existence table

In [22] a table was introduced showing what is known about the existence of post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$ given different algebraic properties of $\mathfrak{g}$ and $\mathfrak{n}$. The table was given as follows.

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ semi	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abe	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ nil	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ sol	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ sim	-	-	-	✓	-	-	-
$\mathfrak{g}$ semi	-	-	-	-	✓	?	-
$\mathfrak{g}$ red	✓	?	?	-	?	✓	✓
$\mathfrak{g}$ com	✓	✓	✓	?	?	✓	✓

The properties here are abelian, nilpotent, solvable, simple, semisimple, reductive and complete. To avoid overlaps, we assume nilpotent non-abelian, solvable non-nilpotent, semisimple non-simple, reductive non-semisimple and complete non-semisimple. A complete Lie algebra here can be solvable. A checkmark means that there is some pair admitting a post-Lie algebra structure, not that all pairs admit a PA-structure. "-" means that these pairs do not admit a post-Lie algebra structure and the question mark represents the cases that are still open.

We are able to solve all open cases except one and we get the following table.

Theorem 3.18. *The existence table for post-Lie algebra structures on pairs* $(\mathfrak{g}, \mathfrak{n})$ *is given as follows.*

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ *abe*	$\mathfrak{n}$ *nil*	$\mathfrak{n}$ *sol*	$\mathfrak{n}$ *sim*	$\mathfrak{n}$ *semi*	$\mathfrak{n}$ *red*	$\mathfrak{n}$ *com*
$\mathfrak{g}$ *abe*	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ *nil*	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ *sol*	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ *sim*	-	-	-	✓	-	-	-
$\mathfrak{g}$ *semi*	-	-	-	-	✓	-	-
$\mathfrak{g}$ *red*	✓	✓	✓	-	?	✓	✓
$\mathfrak{g}$ *com*	✓	✓	✓	✓	✓	✓	✓

We look at the cases one by one, which concludes the proof for Theorem 3.18. Let us first look at the case where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is reductive non-semisimple.

Proposition 3.19. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras where* $\mathfrak{g}$ *is semisimple and* $\mathfrak{n}$ *is reductive non-semisimple. Then there exists no PA-structure on* $(\mathfrak{g}, \mathfrak{n})$.

Proof. Suppose there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then since $\mathfrak{g}$ is semisimple, we have by Theorem 3.9 strong rigidity, i.e. $\mathfrak{g} \cong \mathfrak{n}$. This is a contradiction since $\mathfrak{n}$ is reductive non-semisimple. □

Now let us go the next row, i.e. where $\mathfrak{g}$ is reductive non-semisimple. Here we have the following for the open cases.

Proposition 3.20. *There exists a post-Lie algebra structure on the pairs*

$$(\mathfrak{g}, \mathfrak{n}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4, \mathbb{C}^4 \oplus \mathfrak{n}_3(\mathbb{C}))$$
$$(\mathfrak{g}, \mathfrak{n}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^3, \mathbb{C}^4 \oplus \mathfrak{r}_2(\mathbb{C}))$$

Proof. Let $x \cdot y$ be a pre-Lie algebra structure on $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, which exists and these structures have been classified in [18], Theorem 3. This defines a post-Lie algebra structure on the pair $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \mathbb{C}^4)$. Let $x \circ y$ be an LR-structure on $\mathfrak{n}_3(\mathbb{C})$, which also exists see [26]. This defines a post-Lie algebra structure on $(\mathbb{C}^3, \mathfrak{n}_3(\mathbb{C}))$. Then we can define a post-Lie algebra structure on $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4, \mathbb{C}^4 \oplus \mathfrak{n}_3(\mathbb{C}))$ by

$$(x_1, y_1) \bullet (x_2, y_2) = (x_1 \cdot x_2, y_1 \circ y_2)$$

induced by a pre-Lie algebra structure on $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$ and an LR-structure on $\mathfrak{n}_3(\mathbb{C})$. For the second post-Lie algebra structure, the construction works analogously. Choose an LR-structure $x \circ y$ on $\mathfrak{r}_2(\mathbb{C})$, which is a post-Lie algebra structure on $(\mathbb{C}^2, \mathfrak{r}_2(\mathbb{C}))$. Again we take a pre-Lie algebra structure $x \cdot y$ on $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, which is a post-Lie algebra structure on $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \mathbb{C}^4)$. Thus again we get a post-Lie algebra structure

$$(x_1, y_1) \bullet (x_2, y_2) = (x_1 \cdot x_2, y_1 \circ y_2)$$

induced by a pre-Lie algebra structure and an LR-structure. $\square$

This solves the two open cases in the row of $\mathfrak{g}$ reductive non-semisimple. The case of pairs $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is reductive non-semisimple and $\mathfrak{n}$ is semisimple remains open. However, we believe rigidity in this case, which implies for the existence of post-Lie algebra structures the following.

Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is reductive non-semisimple and $\mathfrak{n}$ is semisimple. Then there exists no PA-structure on the pair $(\mathfrak{g}, \mathfrak{n})$.

Let E_{ij} be the matrix with entry 1 at position (i, j) and all other entries are equal to zero. Let $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$ with basis

$$\begin{aligned} &e_1 = E_{12}, \;\; e_2 = E_{13}, \;\; e_3 = E_{21}, \;\; e_4 = E_{23}, \;\; e_5 = E_{31}, \\ &e_6 = E_{32}, \;\; e_7 = E_{11} - E_{22}, \;\; e_8 = E_{22} - E_{33}. \end{aligned}$$

Then the Lie brackets of $\mathfrak{n}$ are defined by

$$\begin{aligned}
&\{e_1, e_3\} = e_7, && \{e_2, e_6\} = e_1, && \{e_4, e_6\} = e_8, \\
&\{e_1, e_4\} = e_2, && \{e_2, e_7\} = -e_2, && \{e_4, e_7\} = e_4, \\
&\{e_1, e_5\} = -e_6, && \{e_2, e_8\} = -e_2, && \{e_4, e_8\} = -2e_4, \\
&\{e_1, e_7\} = -2e_1, && \{e_3, e_6\} = -e_5, && \{e_5, e_7\} = e_5, \\
&\{e_1, e_8\} = e_1, && \{e_3, e_7\} = 2e_3, && \{e_5, e_8\} = e_5, \\
&\{e_2, e_3\} = -e_4, && \{e_3, e_8\} = -e_3, && \{e_6, e_7\} = -e_6, \\
&\{e_2, e_5\} = e_7 + e_8, && \{e_4, e_5\} = e_3, && \{e_6, e_8\} = 2e_6.
\end{aligned}$$

Let $\mathfrak{aff}_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ be the affine Lie algebra of dimension $n^2 + n$, which is a simply complete Lie algebra. Take a basis $(f_1, ..., f_8)$ for $\mathfrak{aff}_2(\mathbb{C}) \oplus \mathfrak{aff}_1(\mathbb{C})$ with Lie brackets

$$\begin{aligned}
&[f_1, f_2] = f_2, && [f_2, f_4] = f_2, && [f_4, f_6] = f_6, \\
&[f_1, f_3] = -f_3, && [f_2, f_6] = f_5, && [f_7, f_8] = f_7, \\
&[f_1, f_5] = f_5, && [f_3, f_4] = -f_3, && \\
&[f_2, f_3] = f_1 - f_4, && [f_3, f_5] = f_6, &&
\end{aligned}$$

where $(f_1, ... f_6) = (E_{11}, E_{12}, E_{21}, E_{22}, E_{13}, E_{23})$ is a basis for $\mathfrak{aff}_2(\mathbb{C})$ and (f_7, f_8) a basis of $\mathfrak{aff}_1(\mathbb{C}) = \mathfrak{r}_2(\mathbb{C})$.

Now let us look at the last row, so the case where $\mathfrak{g}$ is complete non-semisimple. Then we have the following.

Proposition 3.21. *Let $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Then there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} \cong \mathfrak{aff}_2(\mathbb{C}) \oplus \mathfrak{aff}_1(\mathbb{C})$ given by $x \cdot y = \{R(x), y\}$ with*

$$\begin{aligned} R(e_3) &= -e_3 - e_7, \\ R(e_4) &= -e_4 - e_5, \\ R(e_i) &= 0 \ \text{ for } \ i = 1, 2, 5, 6, 7, 8 \end{aligned}$$

and the Lie brackets for $\mathfrak{g}$ are given by

$$\begin{aligned} [e_1, e_3] &= 2e_1, & [e_2, e_5] &= e_7 + e_8, & [e_4, e_7] &= -e_5, \\ [e_1, e_4] &= e_6, & [e_2, e_6] &= e_1, & [e_4, e_8] &= -e_5, \\ [e_1, e_5] &= -e_6, & [e_2, e_7] &= -e_2, & [e_5, e_7] &= e_5, \\ [e_1, e_7] &= -2e_1, & [e_2, e_8] &= -e_2, & [e_5, e_8] &= e_5, \\ [e_1, e_8] &= e_1, & [e_3, e_4] &= e_4, & [e_6, e_7] &= -e_6, \\ [e_2, e_3] &= e_2, & [e_3, e_5] &= e_5, & [e_6, e_8] &= 2e_6, \\ [e_2, e_4] &= -e_7 - e_8, & [e_3, e_6] &= -e_6. \end{aligned}$$

More precisely, the post-Lie algebra structure is given by

$$\begin{aligned} e_3 \cdot e_1 &= -2e_1 + e_7, & e_3 \cdot e_6 &= e_5 - e_6, & e_4 \cdot e_4 &= e_3, \\ e_3 \cdot e_2 &= -e_2 - e_4, & e_3 \cdot e_7 &= -2e_3, & e_4 \cdot e_5 &= -e_3, \\ e_3 \cdot e_3 &= 2e_3, & e_3 \cdot e_8 &= e_3, & e_4 \cdot e_6 &= -e_8, \\ e_3 \cdot e_4 &= e_4, & e_4 \cdot e_1 &= e_2 - e_6, & e_4 \cdot e_7 &= -e_4 - e_5, \\ e_3 \cdot e_5 &= e_5, & e_4 \cdot e_2 &= e_7 + e_8, & e_4 \cdot e_8 &= 2e_4 - e_5. \end{aligned}$$

Proof. Since $\mathfrak{n}$ is complete, we have by Corollary 1.92 that post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ are in bijection with Rota-Baxter operators of weight 1 on $\mathfrak{n}$. Moreover $R(e_3) = -e_3 - e_7$, $R(e_4) = -e_4 - e_5$, $R(e_i) = 0$ for $i = 1, 2, 5, 6, 7, 8$ defines a Rota-Baxter operator of weight 1 on $\mathfrak{n}$. We have a Lie algebra isomorphism

$$f : \mathfrak{g} \to \mathfrak{aff}_2(\mathbb{C}) \oplus \mathfrak{aff}_1(\mathbb{C})$$

given by

$$f = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\det(f) = -3$. One can also use that $R(\mathfrak{n}) \cong \mathfrak{r}_2(\mathbb{C})$, so that $\ker(R)$ is a 6-dimensional complete subalgebra of $\mathfrak{n}$. The only 6-dimensional subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ is the parabolic one, and therefore it is isomorphic to $\mathfrak{aff}_2(\mathbb{C})$. $\square$

Remark 12. One can also use a decomposition of $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2 \cong \mathfrak{aff}_1(\mathbb{C}) \dot{+} \mathfrak{aff}_2(\mathbb{C})$ and a split Rota-Baxter operator of weight 1 on $\mathfrak{n}$, just as in Proposition 1.98.

For semisimple non-simple Lie algebras we have the following result, see also [37]. Let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ with basis

$$e_1 = E_{12}, e_2 = E_{21}, e_3 = E_{11} - E_{22}, e_4 = E_{45}, e_5 = E_{54}, e_6 = E_{44} - E_{55}.$$

Proposition 3.22. *Let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Then there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ given by $x \cdot y = \{R(x), y\}$ with*

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$.

Proof. Since $\mathfrak{n}$ is complete, we have that the post-Lie algebra structure is in bijection with a Rota-Baxter operator of weight 1 on $\mathfrak{n}$, i.e. $x \cdot y = \{R(x), y\}$ by Corollary 1.92. Now let us decompose $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2$ where $\mathfrak{n}_1 = \mathrm{span}\{e_1, e_3, e_4, e_6\}$ and $\mathfrak{n}_2 = \mathrm{span}\{e_2, e_3 + e_5\}$. So we have $\mathfrak{n}_1 \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ and $\mathfrak{n}_2 \cong \mathfrak{r}_2(\mathbb{C})$. By Proposition 1.85, $R(n_1 + n_2) = -n_2$ for all $n_1 \in \mathfrak{n}_1, n_2 \in \mathfrak{n}_2$ defines a Rota-Baxter operator of weight 1 on $\mathfrak{n}$. This means, we have $R(e_1) = R(e_3) = R(e_4) = R(e_6) = 0$ and $R(e_2) = -e_2, R(e_3 + e_5) = -e_3 - e_5$ giving the above matrix. Then by Proposition 1.98 we get

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}).$$

$\square$

This completes the proof of Theorem 3.18.

3.3 Post-Lie algebra structures for perfect Lie algebras

In this section, we want to generalize results for post-Lie algebra structures on semisimple Lie algebras to perfect Lie algebras. There is some literature on perfect non-semisimple Lie algebras, see [12], [81]. However cohomological questions for perfect non-semisimple Lie algebras are quite difficult, which makes it harder to generalize results for post-Lie algebra structures on perfect Lie algebras. We show the existence of post-Lie algebra structures on pairs of Lie algebras where one of the Lie algebras is perfect. We first look at the case, where $\mathfrak{g}$ is perfect, then where $\mathfrak{n}$ is perfect. Note that existence results vary with respect to $\mathfrak{g}$ and $\mathfrak{n}$. All Lie algebras are finite-dimensional and complex, unless stated otherwise.

3.3.1 Existence results where $\mathfrak{g}$ is perfect

Since historically first pre-Lie algebra structures were introduced, we start with the following result due to Helmstetter, see [56].

Theorem 3.23. *Let $\mathfrak{g}$ be a perfect Lie algebra. Then $\mathfrak{g}$ does not admit a pre-Lie algebra structure. This means that there exists no post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is abelian.*

The next question is whether it is possible to generalize this result to $\mathfrak{g}$ perfect and $\mathfrak{n}$ solvable. Here we have the following results, see Proposition 4.4 in [31]. The proof uses that derivations send the radical of a Lie algebra to its nilradical. We provide a different proof using decompositions of Lie algebras.

Proposition 3.24. *Let $\mathfrak{g}$ be a perfect Lie algebra and $\mathfrak{n}$ be a solvable non-nilpotent Lie algebra. Then there exists no PA-structure on the pair $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Suppose there exists a PA-structure on the pair $(\mathfrak{g}, \mathfrak{n})$. Then by Lemma 3.8 we get a decomposition

$$\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h},$$

where ϕ is the embedding from Proposition 1.66. We have that $\phi(\mathfrak{g})$ and $\mathfrak{h} = L(\mathfrak{g})$ are perfect because $\mathfrak{g}$ is perfect. We have therefore that $\mathfrak{n} \rtimes \mathfrak{h}$ is perfect by Remark 8. Since $\mathfrak{n}$ is a solvable ideal of $\mathfrak{h} \ltimes \mathfrak{n}$, we have that $\mathfrak{n} \subseteq \operatorname{rad}(\mathfrak{h} \ltimes \mathfrak{n})$. But $\mathfrak{h} \ltimes \mathfrak{n}$ is perfect, so that the radical is nilpotent and hence $\mathfrak{n}$ is nilpotent, which is a contradiction to our assumption. $\square$

Lemma 3.25. *There exists no PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is 2-step nilpotent.*

Proof. This an immediate consequence of Proposition 1.74. $\square$

For a 3-step nilpotent Lie algebra $\mathfrak{n}$, we do not know whether or not a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect exists. Observing the results in lower dimension and the results that have already been known, we formulate the following conjecture.

Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is nilpotent. Then there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

In [90] there is a classification of Lie algebras with non-trivial Levi decomposition over $\mathbb{R}$ up to dimension 8, including explicit Lie brackets. From this, one can deduce the classification for complex perfect Lie algebras of dimension $n \leq 8$. There is one Lie algebra missing in Turkowski's list, namely $\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus \mathfrak{n}_3(\mathbb{C}))$. In [3] there is another classification over an algebraically closed field of characteristic zero, which includes this Lie algebra. Let us state the classification. Since we are only interested in perfect Lie algebras, we mention the perfect ones here.

Proposition 3.26. *Every complex perfect non-semisimple Lie algebra of dimension $n \leq 8$ is isomorphic to one of the following Lie algebras.*

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$\dim(Z(\mathfrak{g}))$
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$	5	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$	6	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$	6	1
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(4)$	7	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(2))$	7	0
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$	8	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(5)$	8	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(3))$	8	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus \mathfrak{n}_3(\mathbb{C}))$	8	1
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{f}_{2,3}(\mathbb{C})$	8	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_5(\mathbb{C})$	8	1
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_5(\mathbb{C})$	8	1

For Lie algebras of dimension 9 there is also a classification by Turkowsi, see [91]. We mention the perfect ones here.

Proposition 3.27. *Every complex perfect non-semisimple Lie algebra of dimension 9 is isomorphic to one of the following Lie algebras:*

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$\dim(Z(\mathfrak{g}))$
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3))$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C}))$	9	1
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(6)$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(4))$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(3) \oplus V(3))$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \oplus V(2) \oplus V(2))$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(3) \oplus \mathfrak{n}_3(\mathbb{C}))$	9	1
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (\mathfrak{n}_3(\mathbb{C}) \oplus \mathfrak{n}_3(\mathbb{C}))$	9	2
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{f}_{3,2}(\mathbb{C})$	9	0
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathcal{A}_{6,4}(\mathbb{C})$	9	2

Moreover, the nilradical of such an algebra always has nilpotency class $c \leq 2$.

Now let us look at results for post-Lie algebra structures on lower-dimensional perfect non-semisimple Lie algebras.

For dimension 5, we have the following, see [24].

Proposition 3.28. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ and $\mathfrak{n}$ be nilpotent. Then there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

The result uses the classification of nilpotent Lie algebras of dimension 5. It is not easy to see how this can be generalized to $\mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$, though it might be possible to generalize this result using that a perfect Lie algebra does not admit a pre-Lie algebra structure.

For dimension 6, we have the following result.

Proposition 3.29. *Let $\mathfrak{g}$ be a perfect non-semisimple Lie algebra of dimension 6 and $\mathfrak{n}$ be nilpotent. Then there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. By Proposition 3.26, we have that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{a}$ where $\mathfrak{a}$ is either $V(3)$ or $\mathfrak{n}_3(\mathbb{C})$. Suppose there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then we get an embedding by Proposition 1.66

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) \\ x &\mapsto (x, L(x)). \end{aligned}$$

We either have $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = 0$ or $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$, since $\mathfrak{sl}_2(\mathbb{C})$ is simple and $\ker(L)$ is an ideal of $\mathfrak{g}$. Suppose $\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$, then we have $L(x) = 0$ for all $x \in \mathfrak{sl}_2(\mathbb{C})$. Hence by the first condition of a post-Lie algebra structure we get

$$[x, y] - \{x, y\} = x \cdot y - y \cdot x = 0$$

for all $x, y \in \mathfrak{sl}_2(\mathbb{C})$. But this means that $\mathfrak{sl}_2(\mathbb{C})$ is a subalgebra of $\mathfrak{n}$, which is a contradiction to $\mathfrak{n}$ being nilpotent. Therefore we get

$$\ker(L) \cap \mathfrak{sl}_2(\mathbb{C}) = 0.$$

Let us denote $\mathfrak{h} = L(\mathfrak{g}) = L(\mathfrak{sl}_2(\mathbb{C})) \ltimes L(\mathfrak{a}) = \mathfrak{s} \ltimes \mathfrak{r}$. Then we know by Lemma 3.8 that we get a vector space decomposition

$$\mathfrak{h} \ltimes \mathfrak{n} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

$\phi(\mathfrak{g})$ and $\mathfrak{h}$ are perfect subalgebras, because they are the homomorphic image of a perfect Lie algebra and therefore also $\mathfrak{h} \ltimes \mathfrak{n} = (\mathfrak{s} \ltimes \mathfrak{r}) \ltimes \mathfrak{n}$ is perfect with nilradical $\mathfrak{m} = \mathfrak{r} \dot{+} \mathfrak{n}$ by Lemma 1.44. Since $\mathfrak{h} \ltimes \mathfrak{n}$ is perfect, we have that $\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}]$ cannot contain a 1-dimensional $\mathfrak{s}$-module. But this means that also $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{m}/[\mathfrak{m}, \mathfrak{m}]$ cannot contain a 1-dimensional $\mathfrak{s}$-module. Thus the Lie algebra $\mathfrak{s} \ltimes \mathfrak{n}$ is perfect and $\dim(\mathfrak{s} \ltimes \mathfrak{n}) = \dim(\mathfrak{s}) + \dim(\mathfrak{n}) = 9$. By Proposition 3.27, we have for the nilpotency class of $\mathfrak{n}$ that $c(\mathfrak{n}) \leq 2$. But this means that there exists a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is 2-step nilpotent, which is a contradiction to Lemma 3.25. $\square$

We have looked at the case of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is nilpotent. Now let us look at the case where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is semisimple in lower dimensions. In dimension 3, there is no pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is semisimple, since the only perfect Lie algebra in dimension 3 is $\mathfrak{sl}_2(\mathbb{C})$. The next semisimple Lie algebra occurs in dimension 6 namely $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Post-Lie algebra structures for a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ have been classified in [37]. Let us recall this result.

Proposition 3.30. *Suppose there exists a PA-structure on a pair* $(\mathfrak{g}, \mathfrak{n})$ *where* $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$*, Then* $\mathfrak{g}$ *is isomorphic to one of the following Lie algebras and all admit a post-Lie algebra structure.*

- $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$,
- $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\lambda \neq -1$,
- $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathfrak{r}_{3,\mu}(\mathbb{C})$, $(\lambda, \mu) \neq (-1, -1)$,
- $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$,
- $\mathfrak{r}_2(\mathbb{C}) \oplus (\mathbb{C}^3 \ltimes \mathbb{C}) =< x_1, ..., x_6 >$ *and for* $\alpha \neq 0, \beta \neq 0, -1$ *the following Lie bracket*

$$[x_1, x_2] = x_1, [x_3, x_6] = x_3, [x_4, x_6] = \alpha x_4, [x_5, x_6] = \beta x_5,$$

- $\mathbb{C} \oplus ((\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}) \ltimes \mathbb{C}) =< x_1, ..., x_6 >$ *and for* $\lambda \neq 0, \alpha \neq 0, -1$ *the following Lie bracket*

$$[x_2, x_4] = x_2, [x_3, x_4] = \lambda x_3, [x_3, x_6] = x_3, [x_5, x_6] = \alpha x_5,$$

- $(\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}^2) \ltimes \mathbb{C} =< x_1, ..., x_6 >$ *and for* $\lambda \neq 0, \alpha_1, \alpha_2 \neq 0$ *and* $(\lambda, \alpha_1, \alpha_2) \neq (-1, \alpha_1, -\alpha_1 - 1)$,

$$[x_1, x_3] = x_1, [x_2, x_3] = \lambda x_2, [x_2, x_6] = \alpha_1 x_2, [x_4, x_6] = x_4, [x_5, x_6] = \alpha_2 x_5,$$

- $(\mathbb{C}^2 \oplus \mathbb{C}^2) \ltimes \mathbb{C}^2 =< x_1, ..., x_6 >$ *and Lie brackets*

$$\begin{aligned} &[x_1, x_5] = x_1, \quad [x_2, x_5] = \alpha_2 x_2, [x_3, x_5] = \alpha_4 x_3, [x_4, x_5] = \alpha_6 x_4 \\ &[x_1, x_6] = \alpha_1 x_1, [x_2, x_6] = \alpha_3 x_2, [x_3, x_6] = \alpha_5 x_3, [x_4, x_6] = \alpha_7 x_4 \end{aligned}$$

  with one of the following:

  (i) $\alpha_3 = 1$, $\alpha_5 = \alpha_1 \alpha_7$, $\alpha_6 = \alpha_2 \alpha_4$, $\alpha_1 \alpha_2 \neq 1$, $\alpha_4, \alpha_7 \neq 0, -1$,

  (ii) $\alpha_4 = \alpha_1 - 1$, $\alpha_5 = -\alpha_1$, $\alpha_6 = \alpha_2(\alpha_1 - 1)$, $\alpha_7 = \alpha_1 \alpha_3 - \alpha_1^2 \alpha_2 - \alpha_3$, $\alpha_3 - \alpha_1 \alpha_2 \neq 0$, $\alpha_1 \neq 0, 1$.

None of the Lie algebras is perfect and non-semisimple. Therefore we have the following.

Proposition 3.31. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras where* $\mathfrak{g}$ *is perfect non-semisimple and* $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$*. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

Now let us consider the case where $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. For this we have to start with the following.

Lemma 3.32. *Let* $i : \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2) \to \mathfrak{sl}_3(\mathbb{C})$ *be an injective homomorphism. By conjugating with a matrix in* $\mathfrak{gl}_3(\mathbb{C})$ *we can assume that the image is of the following form*

$$\mathrm{im}(i) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & -a_1 & a_5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\} \quad or \quad \mathrm{im}(i) = \left\{ \begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & -b_1 & 0 \\ b_4 & b_5 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}.$$

Proof. We get that $\mathbb{C}^3$ becomes an $\mathfrak{sl}_2(\mathbb{C})$-module via i restricted to $\mathfrak{sl}_2(\mathbb{C})$. So that either $\mathbb{C}^3 \cong V(2) \oplus V(1)$ or $\mathbb{C}^3 \cong V(3)$ as $\mathfrak{sl}_2(\mathbb{C})$-module. For the first case using the natural representation for $\mathfrak{sl}_2(\mathbb{C})$ we get

$$i(\mathfrak{sl}_2(\mathbb{C})) = \left\{ \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\},$$

where $i(e_1) = E_{12}, i(e_2) = E_{21}, i(e_3) = E_{11} - E_{22}$ and (e_1, e_2, e_3) is a basis of $\mathfrak{sl}_2(\mathbb{C})$. Extending this to $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ we get an additional $V(2)$-block and obtain one of the two forms listed above. For the second case we get $i(e_1) = E_{12}+2E_{23}, i(e_2) = E_{21}+2E_{32}$ and $i(e_3) = 2E_{11} - 2E_{33}$. This cannot be extended to $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$. $\square$

We moreover need the following.

Lemma 3.33. *Let $j : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_3(\mathbb{C})$ be an injective Lie algebra homomorphism. Let us denote by $c_i : \mathfrak{gl}_3(\mathbb{C}) \to \mathbb{C}^3$ the projection of a matrix in $\mathfrak{gl}_3(\mathbb{C})$ to its i-th column and $r_i : \mathfrak{gl}_3(\mathbb{C}) \to \mathbb{C}^3$ the projection to its i-th row. Then none of the linear maps $c_i \circ j$, $r_i \circ j : \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}^3$ for $i = 1, 2, 3$ is bijective.*

Proof. It suffices to show that this holds for the columns. For the rows this can be done by using the isomorphism $\mathfrak{sl}_3(\mathbb{C}) \to \mathfrak{sl}_3(\mathbb{C})$, $X \mapsto -X^T$. Let us look at $c_3 \circ j$. We have that $\mathbb{C}^3$ is an $\mathfrak{sl}_2(\mathbb{C})$-module via j. We get that the map $c_3 \circ j$ is

$$\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}^3, x \mapsto x. \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\dim(\mathfrak{sl}_2(\mathbb{C})) = \dim(\mathbb{C}^3)$, we have by Lemma 1.80 that there exists a non-zero $x \in \mathfrak{sl}_2(\mathbb{C})$ such that $x. \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$. Hence we get that $c_3 \circ j$ is not injective. Similarly we get that the map $c_2 \circ j$ is not injective and also $c_1 \circ j$ is not injective. $\square$

Lemma 3.34. *Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be Lie algebras isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Then the ideals of the Lie algebra $\mathfrak{g} = \mathfrak{s}_1 \oplus (\mathfrak{s}_2 \ltimes V(2))$ are given by*

$$0, \mathfrak{s}_1, \mathfrak{s}_2 \ltimes V(2), \mathfrak{s}_1 \oplus V(2), V(2), \mathfrak{g}.$$

Proof. It is obvious that these are ideals. Conversely, let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Let us make a case distinction. Suppose first, that $\mathfrak{a} \cap \mathfrak{s}_1 \neq 0$, then $\mathfrak{s}_1 \subseteq \mathfrak{a}$ and $\mathfrak{a}/\mathfrak{s}_1$ is an ideal of $\mathfrak{g}/\mathfrak{s}_1 \cong \mathfrak{s}_2 \ltimes V(2)$. The only ideals of $\mathfrak{s}_2 \ltimes V(2)$ are $0, V(2)$ and $\mathfrak{s}_2 \ltimes V(2)$. Therefore we have that if $\mathfrak{a} \cap \mathfrak{s}_1 \neq 0$, then all ideals are given by $\mathfrak{s}_1, \mathfrak{s}_1 \oplus V(2)$ and $\mathfrak{g}$.

Suppose now that $\mathfrak{a} \cap \mathfrak{s}_1 = 0$. We claim that $\mathfrak{a} \subseteq \mathfrak{s}_2 \ltimes V(2)$. Suppose there exists a $x \in \mathfrak{a} \backslash (\mathfrak{s}_2 \ltimes V(2))$. Then there exist unique $x_1 \in \mathfrak{s}_1$ and $x_2 \in \mathfrak{s}_2 \ltimes V(2)$ such that $x = x_1 + x_2$ where $x_1 \neq 0$. We have that there exists a $y \in \mathfrak{s}_1$ such that $[y, x_1] \neq 0$. Therefore we get that $0 \neq [y, x] = [y, x_1 + x_2] = [y, x_1] \in \mathfrak{s}_1 \cap \mathfrak{a}$, which is a contradiction. Hence $\mathfrak{a} \subseteq \mathfrak{s}_2 \ltimes V(2)$, so that we get the ideals $0, V(2)$ and $\mathfrak{s}_2 \ltimes V(2)$. □

Lemma 3.35. *There is no direct vector space decomposition* $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \dotplus \mathfrak{b}$ *where* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are subalgebras of* $\mathfrak{sl}_3(\mathbb{C})$ *with* $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ *and* $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{C})$.

Proof. Suppose there exists a decomposition $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \dotplus \mathfrak{b}$ with $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ and $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{C})$. Then by Lemma 3.32 we can assume that

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & -a_1 & a_5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\} \quad \text{or} \quad \mathfrak{a} = \left\{ \begin{pmatrix} b_1 & b_2 & 0 \\ b_3 & -b_1 & 0 \\ b_4 & b_5 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}) \right\}.$$

Since $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \dotplus \mathfrak{b}$ we get that the row projection map $r_3 : \mathfrak{b} \to \mathbb{C}^3$ is bijective in the first case, and the column projection map $c_3 : \mathfrak{b} \to \mathbb{C}^3$ is bijective in the second case, which is impossible by Lemma 3.33. □

We can apply these results now to the case of post-Lie algebra structures on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{g}$ has a Levi subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Proposition 3.36. *Let* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ *and* $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. *Then there exists no post-Lie algebra structure on the pair* $(\mathfrak{g}, \mathfrak{n})$.

Proof. Suppose first that there exists a PA-structure where $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{s}_1 \oplus (\mathfrak{s}_2 \ltimes V(2))$. By Proposition 1.68 there exists an injective Lie algebra homomorphism

$$j : \mathfrak{g} \to \mathfrak{n} \oplus \mathfrak{n}, x \mapsto (j_1(x), j_2(x))$$

such that $j_1 - j_2$ is a bijective linear map. Since $\ker(j_1)$ is an ideal in $\mathfrak{g}$, we get the following six possibilities.

- Case 1. $\ker(j_1) = 0$. Then $j_1 : \mathfrak{g} \to \mathfrak{n}$ is an isomorphism. But $\mathfrak{g}$ is not semisimple, so we get a contradiction.
- Case 2. $\ker(j_1) = \mathfrak{g}$. Then j_1 is the zero map. But since $j_1 - j_2$ is bijective we get that $j_2 : \mathfrak{g} \to \mathfrak{n}$ is an isomorphism, which is a contradiction.
- Case 3. $\ker(j_1) = V(2)$. Then the representation

  $$j_1|_{\mathfrak{s}_1 \oplus \mathfrak{s}_2} : \mathfrak{s}_1 \oplus \mathfrak{s}_2 \to \mathfrak{sl}_3(\mathbb{C})$$

  is faithful. But the smallest dimension of a faithful representation of $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ is equal to 4, see [39]. This is impossible.

- Case 4. $\ker(j_1) = \mathfrak{s}_1 \oplus V(2)$. Then
$$j_2|_{\mathfrak{s}_1 \oplus V(2)} = (j_2 - j_1)|_{s_1 \oplus V(2)} : \mathfrak{s}_1 \oplus V(2) \to \mathfrak{sl}_3(\mathbb{C})$$
has to be injective. However, $\mathfrak{s}_1 \oplus V(2)$ has a 3-dimensional abelian subalgebra, but $\mathfrak{sl}_3(\mathbb{C})$ does not, so this is not possible.

- Case 5. $\ker(j_1) = \mathfrak{s}_2 \ltimes V(2)$. Then $(\mathfrak{s}_2 \ltimes V(2)) \cap \ker(j_2) = 0$. Because $\ker(j_2)$ is an ideal, which cannot be zero, see case 1, we get $\ker(j_2) = \mathfrak{s}_1$. For every $x \in \mathfrak{g}$ there exists a unique $x_1 \in \mathfrak{s}_1$ and a unique $x_2 \in \mathfrak{s}_2 \ltimes V(2)$ such that $x = x_1 + x_2$. Then we have $(j_1 - j_2)(x) = j_1(x_1) - j_2(x_2)$. Moreover, $j_1 - j_2$ is injective if and only if $\operatorname{im}(j_1) \cap \operatorname{im}(j_2) = 0$. This means that
$$\mathfrak{sl}_3(\mathbb{C}) = \operatorname{im}(j_1) \dot{+} \operatorname{im}(j_2)$$
with $\operatorname{im}(j_1) \cong \mathfrak{s}_1$ and $\operatorname{im}(j_2) \cong \mathfrak{s}_2 \ltimes V(2)$, which is not possible, see Lemma 3.35.

- Case 6. $\ker(j_1) = \mathfrak{s}_1$. Then $\mathfrak{s}_1$ cannot be contained in $\ker(j_2)$. So for $\ker(j_2)$ we either have $0, V(2)$ or $\mathfrak{s}_2 \ltimes V(2)$. $\ker(j_2)$ cannot be zero, so we either have $\ker(j_2) = V(2)$ or $\ker(j_2) = \mathfrak{s}_2 \ltimes V(2)$. If $\ker(j_2) = V(2)$, then we get a contradiction as in case 3. If $\ker(j_2) = \mathfrak{s}_2 \ltimes V(2)$, then we get a contradiction as in case 5.

□

More generally, we get the following.

Theorem 3.37. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$. Then by Proposition 3.26 we have $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})$. The first case implies $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2))$ by Proposition 3.26. Then $(\mathfrak{g}, \mathfrak{n})$ does not admit a PA-structure by Proposition 3.36. In the second case, we have $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{r}$ where $\mathfrak{r}$ is isomorphic to one of the following Lie algebras
$$V(5), V(2) \oplus V(3), V(2) \oplus \mathfrak{n}_3(\mathbb{C}), \mathfrak{n}_5(\mathbb{C}), \mathfrak{f}_{2,3}(\mathbb{C}).$$
We look again at the maps
$$j_i : \mathfrak{g} \to \mathfrak{n}$$
and assume that $j_1 - j_2$ is a bijective linear map. In the first two cases, either j_1 or j_2 must be injective on the factor $V(3)$ respectively $V(5)$. But $\mathfrak{sl}_3(\mathbb{C})$ does not contain an abelian subalgebra of dimension $n \geq 3$. Assume that now $\mathfrak{r} \cong V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. If $\ker(j_1)$ is non-solvable, then $\ker(j_1) = \mathfrak{g}$, so that $j_2 : \mathfrak{g} \to \mathfrak{n}$ is an isomorphism, which is a contradiction. Therefore $\ker(j_1)$ is solvable, so $\ker(j_1) \subseteq V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. $\ker(j_1)$ is a subalgebra and a submodule of $V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. As modules we have $V(2) \oplus \mathfrak{n}_3(\mathbb{C}) \cong V(2) \oplus V(2) \oplus V(1)$. We have that $V(2) \oplus V(2)$ is not an ideal, since there is no subalgebra corresponding to it. So we get the following cases for $\ker(j_1)$ as a $\mathfrak{sl}_2(\mathbb{C})$-submodule:

- Case 1. $\ker(j_1) = 0$. Then j_1 is an isomorphism, which is a contradiction.
- Case 2. $\ker(j_1) \cong V(2)$. Then $j_1|_{\mathfrak{sl}_2(\mathbb{C})\ltimes\mathfrak{n}_3(\mathbb{C})}$ is injective. However $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$ has a 3-dimensional subalgebra, while $\mathfrak{sl}_3(\mathbb{C})$ does not. This is a contradiction.
- Case 3. $\ker(j_1) \cong \mathfrak{n}_3(\mathbb{C})$. Then j_2 induces an injection $\mathfrak{sl}_2(\mathbb{C}) \ltimes \ker(j_1) \to \mathfrak{sl}_3(\mathbb{C})$, which is impossible as in case 2.
- Case 4. $\ker(j_1) \cong V(2) \oplus V(1)$ with $V(1) \cong Z(\mathfrak{n}_3(\mathbb{C}))$. Then j_2 is injective on $\ker(j_1)$, and $V(2) \oplus V(1)$ is an abelian subalgebra of dimension 3, which is impossible.
- Case 5. $\ker(j_1) \cong V(2) \oplus \mathfrak{n}_3(\mathbb{C})$. Then $j_2|_{V(2)\oplus\mathfrak{n}_3(\mathbb{C})}$ is injective and we again obtain a three dimensional abelian subalgebra, which is impossible because $\mathfrak{sl}_3(\mathbb{C})$ does not have a 3-dimensional abelian subalgebra.

Now assume $\mathfrak{r} \cong \mathfrak{n}_5(\mathbb{C})$. We claim that either j_1 or j_2 is injective on $\mathfrak{n}_5(\mathbb{C})$. Otherwise $j_1(Z(\mathfrak{n}_5(\mathbb{C})) = j_2(Z(\mathfrak{n}_5(\mathbb{C}))) = 0$, so that $(j_1 - j_2)|_{Z(\mathfrak{n}_5(\mathbb{C}))} = 0$, which is a contradiction to $j_1 - j_2$ being bijective. So j_1 or j_2 are injective on $\mathfrak{n}_5(\mathbb{C})$, which is impossible because $\mathfrak{n}_5(\mathbb{C})$ contains a three dimensional abelian subalgebra.

Assume now that $\mathfrak{r} \cong \mathfrak{f}_{2,3}(\mathbb{C})$. Then either j_1 or j_2 must be injective on $\mathfrak{r}$. If j_1 is not injective on $\mathfrak{r}$, then $Z(\mathfrak{r}) \cap \ker(j_1) \neq 0$. We claim that $Z(\mathfrak{r}) \subseteq \ker(j_1)$. This is because every ideal $\mathfrak{a}$ of $\mathfrak{g}$ satisfying $\mathfrak{a} \cap Z(\mathfrak{r}) \neq 0$ also satisfies $Z(\mathfrak{r}) \subseteq \mathfrak{a}$. Indeed, note that $Z(\mathfrak{r}) = [\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]]$ and the action of $\mathfrak{sl}_2(\mathbb{C})$ on $[\mathfrak{r}, \mathfrak{r}]/[\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]]$ is trivial, since the quotient is 1-dimensional. Hence the action of $\mathfrak{sl}_2(\mathbb{C})$ on $Z(\mathfrak{r})$ coincides with the action on $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$. We have that it does not contain a 1-dimensional submodule, since $\mathfrak{g}$ is perfect. Hence also $Z(\mathfrak{r})$ has no 1-dimensional submodule, so that $\dim(\mathfrak{a} \cap Z(\mathfrak{r})) \geq 2$. Because $\dim(Z(\mathfrak{r})) = 2$, we have $Z(\mathfrak{r}) \subseteq \mathfrak{a}$. So if both j_1 and j_2 are not injective on $\mathfrak{r}$, then $Z(\mathfrak{r})$ is contained in $\ker(j_1)$ and $\ker(j_2)$, which means that $(j_1 - j_2)|_{Z(\mathfrak{r})} = 0$, which is a contradiction to $j_1 - j_2$ being bijective. Hence $j_i : \mathfrak{f}_{2,3}(\mathbb{C}) \to \mathfrak{sl}_3(\mathbb{C})$ is injective for some i, which is impossible since $\mathfrak{f}_{2,3}(\mathbb{C})$ has a 3-dimensional abelian subalgebra. □

For a perfect Lie algebra with a simple Levi subalgebra $\mathfrak{s}$ and an irreducible $\mathfrak{s}$-module, we have the following result.

Proposition 3.38. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is semisimple. Suppose we have a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes V$, where $\mathfrak{s}$ is a simple Levi subalgebra and V is an irreducible $\mathfrak{s}$-module, considered as an abelian Lie algebra. Then $(\mathfrak{g}, \mathfrak{n})$ admits no PA-structure.*

Proof. Suppose there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then by Corollary 1.92 we have that $x \cdot y = \{R(x), y\}$ for a Rota-Baxter operator R of weight 1 on $\mathfrak{n}$. Because $\mathfrak{g}$ and $\mathfrak{n}$ are not isomorphic, we have that $\ker(R) \neq 0$ and $\ker(R + \mathrm{id}) \neq 0$ are nonzero ideals of $\mathfrak{g}$ by Proposition 1.97. Moreover, we have

$$\ker(R) \cap \ker(R + \mathrm{id}) = 0.$$

Indeed, suppose $x \in \ker(R) \cap \ker(R + \mathrm{id})$. Then $0 = (R + \mathrm{id})(x) = R(x) + x = x$. We show that the only ideals of $\mathfrak{g} = \mathfrak{s} \ltimes V$ are $0, V$ and $\mathfrak{g}$. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$, then by Corollary 1.43 we have

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes (\mathfrak{a} \cap V).$$

Because $\mathfrak{a} \cap \mathfrak{s}$ is an ideal in $\mathfrak{s}$ and $\mathfrak{s}$ is simple, we either have $\mathfrak{a} \cap \mathfrak{s} = 0$ or $\mathfrak{a} \cap \mathfrak{s} = \mathfrak{s}$. Since $\mathfrak{a} \cap V$ is an $\mathfrak{s}$-submodule of V and V is irreducible, we either have $\mathfrak{a} \cap V = 0$ or $\mathfrak{a} \cap V = V$.

- Case 1. $\mathfrak{a} \cap V = 0$. Then $\mathfrak{a} \cap \mathfrak{s}$ is either zero or $\mathfrak{s}$. Hence $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{s}$. But $\mathfrak{s}$ is not an ideal of $\mathfrak{g}$, hence $\mathfrak{a} = 0$.

- Case 2. $\mathfrak{a} \cap V = V$. Then $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{s}) \ltimes V$, which means either $\mathfrak{a} = V$ or $\mathfrak{a} = \mathfrak{g}$.

Since $\ker(R) \neq 0$ and $\ker(R + \mathrm{id}) \neq 0$, we get that $V \subseteq \ker(R)$ and $V \subseteq \ker(R + \mathrm{id})$. But this means that $V \subseteq \ker(R) \cap \ker(R + \mathrm{id}) = 0$, which is a contradiction. $\square$

If $\mathfrak{n}$ is reductive non-semisimple, then we obtain a post-Lie algebra structure. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ with basis $(e_1, ..., e_5)$ and Lie brackets

$$\begin{aligned}
&[e_1, e_2] = e_3, && [e_2, e_3] = 2e_2, && [e_3, e_4] = e_4 \\
&[e_1, e_3] = -2e_1, && [e_2, e_4] = e_5, && [e_3, e_5] = -e_5, \\
&[e_1, e_5] = e_4.
\end{aligned}$$

Example 11. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$. Then $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure defined by

$$\begin{aligned}
e_1 \cdot e_5 &= e_4, & e_2 \cdot e_4 &= e_5, \\
e_3 \cdot e_4 &= e_4, & e_3 \cdot e_5 &= -e_5.
\end{aligned}$$

Note that $\mathfrak{n}$ has a 2-dimensional center. This is not possible when $\mathfrak{n}$ has a 1-dimensional center.

Proposition 3.39. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is reductive with 1-dimensional center. Then $(\mathfrak{g}, \mathfrak{n})$ does not admit a PA-structure.*

Proof. Suppose there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ and denote by $L(x)(y) = x \cdot y$ the left multiplication operator. Then by Proposition 1.66 we have an embedding

$$\begin{aligned}
\phi : \mathfrak{g} &\to \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n}) \\
x &\mapsto (x, L(x)).
\end{aligned}$$

Writing $\mathfrak{h} = L(\mathfrak{g})$ we again obtain a direct vector space sum by Lemma 3.8

$$\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

We have that $\phi(\mathfrak{g})$ and $\mathfrak{h}$ are perfect subalgebras of $\mathfrak{n} \rtimes \mathfrak{h}$, because they are homomorphic images of a perfect Lie algebra. Therefore $\mathfrak{n} \rtimes \mathfrak{h}$ is also perfect by Remark 8. We have that $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus Z(\mathfrak{n})$ where $\mathfrak{s} = [\mathfrak{n}, \mathfrak{n}]$ is semisimple and $\dim(Z(\mathfrak{n})) = 1$. Since both are characteristic ideals of $\mathfrak{n}$ and $\mathfrak{n}$ is an ideal in $\mathfrak{n} \rtimes \mathfrak{h}$, we have that both $[\mathfrak{n}, \mathfrak{n}]$ and $Z(\mathfrak{n})$ are ideals in $\mathfrak{n} \rtimes \mathfrak{h}$. We claim that $[Z(\mathfrak{n}), \mathfrak{h}] = 0$ for the Lie bracket in $\mathfrak{n} \rtimes \mathfrak{h}$. We have that $Z(\mathfrak{n})$ is an ideal of $\mathfrak{n} \rtimes \mathfrak{h}$, so that $[\mathfrak{h}, Z(\mathfrak{n})] \subseteq Z(\mathfrak{n})$, and therefore $Z(\mathfrak{n})$ is a 1-dimensional $\mathfrak{h}$-module and is trivial. The proof works analogously to the semisimple case, see [58]. Hence $[Z(\mathfrak{n}), \mathfrak{h}] = 0$. It follows that

$$[\mathfrak{n}, \mathfrak{h}] = [\mathfrak{s} + Z(\mathfrak{n}), \mathfrak{h}] = [\mathfrak{s}, \mathfrak{h}] \subseteq \mathfrak{s},$$

because $\mathfrak{s}$ is an ideal of $\mathfrak{n} \rtimes \mathfrak{h}$. Since $\mathfrak{n} \rtimes \mathfrak{h}$ is perfect, we have

$$\begin{aligned} \mathfrak{n} + \mathfrak{h} &= [\mathfrak{n} + \mathfrak{h}, \mathfrak{n} + \mathfrak{h}] \\ &= [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{n}, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\ &= \mathfrak{s} + \mathfrak{h}. \end{aligned}$$

This together with $\mathfrak{n} \cap \mathfrak{h} = 0$ and $\mathfrak{s} \cap \mathfrak{h} = 0$, gives us $1 + \dim(\mathfrak{s}) + \dim(\mathfrak{h}) = \dim(\mathfrak{n}) + \dim(\mathfrak{h}) = \dim(\mathfrak{s}) + \dim(\mathfrak{h})$, which is a contradiction. □

If $\mathfrak{n}$ is complete non-perfect, we have the following.

Proposition 3.40. *Let $\mathfrak{g}$ be a perfect Lie algebra and $\mathfrak{n}$ be a complete non-perfect Lie algebra. Then $(\mathfrak{g}, \mathfrak{n})$ admits no post-Lie algebra structure.*

Proof. Suppose there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Because $\mathfrak{n}$ is complete, the post-Lie algebra structure arises by a Rota-Baxter operator of weight 1 on $\mathfrak{n}$, i.e. $x \cdot y = \{R(x), y\}$. Because $\mathfrak{g}$ is perfect, we have that also $\mathfrak{n}$ is perfect by Corollary 1.96 which is a contradiction to our assumption. □

3.3.2 Existence results where $\mathfrak{n}$ is perfect

Let us again look at the special case of LR-structures.

Proposition 3.41. *There is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is abelian and $\mathfrak{n}$ is perfect.*

Proof. A PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is abelian corresponds to an LR-structure on $\mathfrak{n}$. By Proposition 2.1 in [26], we have that $\mathfrak{n}$ is 2-step solvable. Therefore there exists no PA-structure on $(\mathfrak{g}, \mathfrak{n})$. □

More generally, we have the following result for $\mathfrak{g}$ nilpotent.

Proposition 3.42. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{n}$ be a perfect Lie algebra. Then there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Suppose there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is nilpotent. Then $\mathfrak{n}$ is solvable by Proposition 1.75. Therefore there cannot exist a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. □

For $\mathfrak{g}$ solvable non-nilpotent, we have the following example of a post-Lie algebra structure.

Example 12. Let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ with Lie brackets as in Example 11. Then we can decompose $\mathfrak{n} = \mathfrak{n}_1 \dot{+} \mathfrak{n}_2$, where $\mathfrak{n}_1 = \mathrm{span}\{e_1, e_4, e_5\}$ and $\mathfrak{n}_2 = \mathrm{span}\{e_2, e_3\}$, so $\mathfrak{n}_1 \cong \mathfrak{n}_3(\mathbb{C})$ and $\mathfrak{n}_2 \cong \mathfrak{r}_2(\mathbb{C})$ as Lie algebras. Then by Proposition 1.85 we have that $R(x_1 + x_2) = -x_2$ for all $x_1 \in \mathfrak{n}_1$ $x_2 \in \mathfrak{n}_2$ defines a Rota-Baxter operator of weight 1 on $\mathfrak{n}$ and $x \cdot y = \{R(x), y\}$ with

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ and the Lie brackets of $\mathfrak{g}$ are given by $[e_1, e_5] = e_4$ and $[e_2, e_3] = -2e_2$.

We also have another example of a post-Lie algebra structure for this pair.

Example 13. The pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ admits a post-Lie algebra structure given by

$$\begin{aligned} &e_2 \cdot e_1 = e_3, && e_3 \cdot e_1 = -2e_1, && e_3 \cdot e_4 = -e_4, \\ &e_2 \cdot e_3 = -2e_2, && e_3 \cdot e_2 = 2e_2, && e_3 \cdot e_5 = e_5, \\ &e_2 \cdot e_4 = -e_5. \end{aligned}$$

For PA-structures where $\mathfrak{g}$ is semisimple, we have the following result.

Proposition 3.43. *There is no PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is perfect non-semisimple.*

Proof. Suppose we have a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple, then $\mathfrak{g} \cong \mathfrak{n}$ by Theorem 3.9 which is a contradiction to our assumption. $\square$

However when $\mathfrak{g}$ is reductive non-semisimple, there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is perfect non-semisimple.

Example 14. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$ and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$. Then $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure given by

$$\begin{aligned} &e_4 \cdot e_2 = e_5, && e_5 \cdot e_1 = e_4 \\ &e_4 \cdot e_3 = e_4, && e_5 \cdot e_3 = -e_5, \end{aligned}$$

where we use the Lie brackets for $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ as in Example 11. This post-Lie algebra structure corresponds to a split Rota-Baxter operator R of weight 1 on $\mathfrak{n}$ given by

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

which we get from the decomposition $\mathfrak{n} = \mathfrak{n}_1 \dotplus \mathfrak{n}_2$ where $\mathfrak{n}_1 = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{n}_2 = V(2)$.

Also for the case where $\mathfrak{g}$ is complete non-perfect, we obtain a post-Lie algebra structure.

Example 15. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ with the standard Lie brackets for $\mathfrak{sl}_2(\mathbb{C}) = \operatorname{span}\{e_1, e_2, e_3\}$ and $[e_4, e_5] = e_4$ for $\mathfrak{r}_2(\mathbb{C})$ and let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$. Then there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ given by

$$\begin{aligned} e_4 \cdot e_2 &= e_5 & e_5 \cdot e_1 &= e_4, & e_5 \cdot e_4 &= -e_4, \\ e_4 \cdot e_3 &= e_4, & e_5 \cdot e_3 &= -e_5, & e_5 \cdot e_5 &= -e_5. \end{aligned}$$

We end this chapter with the existence table, now adding perfect Lie algebras.

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ semi	$\mathfrak{n}$ red	$\mathfrak{n}$ com	$\mathfrak{n}$ perf
$\mathfrak{g}$ abe	✓	✓	✓	-	-	-	✓	-
$\mathfrak{g}$ nil	✓	✓	✓	-	-	-	✓	-
$\mathfrak{g}$ sol	✓	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ sim	-	-	-	✓	-	-	-	-
$\mathfrak{g}$ semi	-	-	-	-	✓	-	-	-
$\mathfrak{g}$ red	✓	✓	✓	-	?	✓	✓	✓
$\mathfrak{g}$ com	✓	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ perf	-	?	-	?	?	✓	-	✓

For the last question marks, we already mentioned that we believe that there exists no post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is nilpotent.
In the case where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is semisimple, we believe that rigidity holds.

Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is semisimple. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Then $\mathfrak{g}$ is also semisimple and $\mathfrak{g} \cong \mathfrak{n}$.

4 Classes of post-Lie algebra structures

This section is dedicated to certain classes of post-Lie algebra structures. We first recall some results about commutative post-Lie algebra structures and then generalize them to λ-scalar-post-Lie algebra structures. Throughout this chapter we assume all Lie algebras to be finite-dimensional and complex, unless stated otherwise.

4.1 Commutative post-Lie algebra structures

Let us start by giving a definition of a commutative post-Lie algebra structure, i.e. one where the product is commutative, so $x \cdot y = y \cdot x$. These were introduced in [24].

Definition 4.1. A *commutative post-Lie algebra structure* or *CPA-structure* on a Lie algebra $\mathfrak{g}$ is a k-bilinear product $x \cdot y$ satisfying

$$x \cdot y = y \cdot x \tag{2}$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \tag{3}$$

$$x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z] \tag{4}$$

for all $x, y, z \in V$.

This means a commutative post-Lie algebra structure is a post-Lie algebra structure where $[x, y] = \{x, y\}$.

Example 16. The trivial product, i.e. $x \cdot y = 0$ is a commutative post-Lie algebra structure on $\mathfrak{g}$.

For an abelian Lie algebra $\mathfrak{g}$ we get a commutative, associative algebra, see Example 5.2 in [24].

Example 17. A commutative post-Lie algebra structure on an abelian Lie algebra $\mathfrak{g}$ corresponds to a commutative, associative algebra.

For semisimple Lie algebras we have the following result, see Proposition 5.4 in [24].

Proposition 4.2. *Any commutative post-Lie algebra structure on a semisimple Lie algebra $\mathfrak{g}$ is trivial.*

Moreover we have the following two results, see [24].

Corollary 4.3. *Let $x \cdot y$ be a CPA-structure on $\mathfrak{g}$. Then $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{rad}(\mathfrak{g})$.*

Corollary 4.4. *Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ be a direct Lie algebra sum of a semisimple Lie algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$. Then for any CPA-structure on $\mathfrak{g}$ we have $\mathfrak{s} \cdot \mathfrak{g} = \mathfrak{g} \cdot \mathfrak{s} = 0$.*

In lower dimension we have a classification for abelian Lie algebras, see [32] [83]. For the two-dimensional non-abelian case, we have the following, see [24].

Proposition 4.5. *Let $\mathfrak{g} = \mathfrak{r}_2(\mathbb{C})$ be the 2-dimensional non-abelian Lie algebra with $[e_1, e_2] = e_1$. Then every CPA-structure on $\mathfrak{g}$ is isomorphic to one of the following:*

(1) $L(e_1) = 0$, $L(e_2) = 0$,

(2) $L(e_1) = 0$, $L(e_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(3) $L(e_1) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $L(e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

Note that the second corresponds to an LR-algebra and a pre-Lie algebra.

In dimension 3, it becomes more complicated and the classification is long. In [24] we find the following classification for two solvable Lie algebras.

Proposition 4.6. *Let $\mathfrak{g} = \mathfrak{r}_{3,1}(\mathbb{C})$ be the 3-dimensional solvable Lie algebra with $[e_1, e_2] = e_2$ and $[e_1, e_3] = e_2 + e_3$. Every CPA-structure on $\mathfrak{g}$ is isomorphic to one of the following:*

- $L(e_1) = 0$, $L(e_2) = 0$, $L(e_3) = 0$,
- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $L(e_2) = 0$, $L(e_3) = 0$,
- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $L(e_2) = 0$, $L(e_3) = 0$,
- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Proposition 4.7. *Let $\mathfrak{g} = \mathfrak{n}_3(\mathbb{C})$ be the 3-dimensional Heisenberg Lie algebra with $[e_1, e_2] = e_3$. Every CPA-structure on $\mathfrak{g}$ is isomorphic to one of the following structures:*

- $L(e_1) = 0$, $L(e_2) = 0$, $L(e_3) = 0$,
- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}$, $L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}$, $L(e_3) = 0$,
- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $L(e_2) = 0$, $L(e_3) = 0$

- $L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $L(e_3) = 0$.

We now mention some definitions and results for CPA-structures from [40] that we will generalize in the subsequent chapter to other classes of post-Lie algebra structures.

Definition 4.8. A CPA-structure $(A, \cdot)$ on $\mathfrak{g}$ is called *non-degenerate* if the annihilator

$$\mathrm{Ann}_A = \ker(L) = \{x \in \mathfrak{g} \mid L(x) = 0\}$$

is trivial.

Proposition 4.9. *Let $(A, \cdot)$ be a CPA-structure on $\mathfrak{g}$. Then there exists an ideal I_∞ such that*

(i) $I_\infty^{[k]} \subseteq \mathrm{Ann}_A \subseteq I_\infty$ *for all k large enough,*

(ii) the CPA-structure on $\mathfrak{g}/I_\infty$ is non-degenerate,

where for an ideal I we have $I^{[n]} := [I, [I, [I, ...]]]...]$.

Definition 4.10. We call a CPA-structure on $\mathfrak{g}$ *weakly inner* if there is a $\phi \in \mathrm{End}(V)$ such that

$$x \cdot y = [\phi(x), y].$$

It is called *inner* if $\phi \in \mathrm{End}(\mathfrak{g})$, so ϕ is a Lie algebra homomorphism.

For Lie algebras with trivial center, we have the following, see [40].

Lemma 4.11. *Let $\mathfrak{g}$ be a Lie algebra with $Z(\mathfrak{g}) = 0$. Then any weakly inner CPA-structure on $\mathfrak{g}$ is inner.*

For complete Lie algebras we even have more, see [40].

Corollary 4.12. *Let $\mathfrak{g}$ be a complete Lie algebra. Then any CPA-structure is inner.*

Moreover, we have the following results, see [40].

Lemma 4.13. *Let $(A, \cdot)$ be an inner CPA-structure on $\mathfrak{g}$. Then the ascending chain of ideals I_n defined by $I_0 = 0$ and $I_n = \{x \in A \mid x \cdot A \subseteq I_{n-1}\}$ are invariant under ϕ and all Lie algebra ideals of $\mathfrak{g}$ are ideals of A. Conversely, if $(A, \cdot)$ is non-degenerate, then all ideals of A are Lie algebra ideals.*

Lemma 4.14. *Let $x \cdot y = [\phi(x), y]$ be an inner CPA-structure on $\mathfrak{g}$ and let $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ϕ. Then we have the following*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha\beta}$$
$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0 \implies \alpha + \beta = 0$$

Moreover for Lie algebras admitting an inner CPA-structure we have a decomposition result, see [40]. For this let us recall the notion of nil-inner CPA-structures.

Definition 4.15. We call a CPA-structure on $\mathfrak{g}$ *nil-inner* if it can be written as $x \cdot y = [\phi(x), y]$ with a nilpotent Lie algebra homomorphism $\phi \in \operatorname{End}(\mathfrak{g})$.

Theorem 4.16. *Let $\mathfrak{g}$ be a Lie algebra admitting an inner CPA-structure, so $x \cdot y = [\phi(x), y]$ with $\phi \in \operatorname{End}(\mathfrak{g})$. Then $\mathfrak{g}$ decomposes into the sum of ϕ-invariant ideals*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$$

with

(i) $\phi|_{\mathfrak{n}}$ is a nilpotent endomorphism of $\mathfrak{n}$ such that the CPA-structure on $\mathfrak{n}$ is nil-inner.

(ii) $\phi|_{\mathfrak{h}}$ is an automorphism of $\mathfrak{h}$ and $\mathfrak{h}$ is metabelian.

This implies the following, see [40].

Corollary 4.17. *Let $\mathfrak{g}$ be a Lie algebra admitting a non-degenerate inner CPA-structure. Then $\mathfrak{g}$ is metabelian.*

For semisimple and more generally, perfect Lie algebras, any CPA-structure is trivial, see Proposition 3.1 and Theorem 3.3 in [40].

Theorem 4.18. *Let $\mathfrak{g}$ be a perfect Lie algebra. Then any CPA-structure on $\mathfrak{g}$ is trivial.*

For a Lie algebra admitting a non-degenerate CPA-structure, we have the following, see [40].

Corollary 4.19. *Suppose that $\mathfrak{g}$ admits a non-degenerate CPA-structure. Then $\mathfrak{g}$ is solvable.*

For solvable Lie algebras one can always obtain a non-trivial CPA-structure, see Corollary 3.9 in [40].

Corollary 4.20. *Let $\mathfrak{g}$ be a non-trivial solvable Lie algebra. Then $\mathfrak{g}$ admits a non-trivial commutative post-Lie algebra structure.*

4.1.1 CPA-structures for complete Lie algebras

There is a study on complete Lie algebras by Meng, see [72]. He shows that a complete Lie algebra decomposes into the direct sum of simply-complete ideals. Let us recall the definition and proposition from [72].

Definition 4.21. Let $\mathfrak{g}$ be a complete Lie algebra. It is called *simply-complete* if no non-trivial ideal in $\mathfrak{g}$ is complete.

Proposition 4.22. *(1) Let $\mathfrak{g}_1, ..., \mathfrak{g}_n$ be simply-complete Lie algebras. Then the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ is also complete.*

(2) Let $\mathfrak{g}$ be a complete Lie algebra. Then there exist simply-complete ideals $\mathfrak{g}_1, ..., \mathfrak{g}_n$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$.

(3) Such a decomposition is unique up to permutation of the simply-complete ideals.

For CPA-structures on complete Lie algebras we also have that they are reduced to the study of simply-complete Lie algebras, see Proposition 4.3 in [40].

Proposition 4.23. *Let $\mathfrak{g}_1, ..., \mathfrak{g}_n$ be simply-complete Lie algebras, each with a CPA-structure. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ admits a CPA-structure, which is given componentwise:*

$$(x_1, ..., x_n) \cdot (y_1, ..., y_n) = (x_1 \cdot y_1, ..., x_n \cdot y_n).$$

Conversely we have that for any complete Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ with simply-complete ideals $\mathfrak{g}_i$, any CPA-structure on $\mathfrak{g}$ is given as above.

In [40] the study of CPA-structures on complete Lie algebras is done in two cases, where either the Lie algebra is metabelian or it is not metabelian. For the metabelian case, we have the following result, see Proposition 4.4 in [40] which is a consequence of [87].

Proposition 4.24. *The only simply-complete metabelian Lie algebra is isomorphic to $\mathfrak{r}_2(\mathbb{C})$.*

We have already given the classification of CPA-structures on $\mathfrak{r}_2(\mathbb{C})$ in Proposition 4.5. For the non-metabelian case, we have the following theorem, see [40].

Theorem 4.25. *Let $\mathfrak{g}$ be a simply-complete non-metabelian Lie algebra. Suppose that $\mathfrak{g}$ satisfies $\mathrm{nil}(\mathfrak{g}) = [\mathfrak{g}, \mathrm{nil}(\mathfrak{g})]$. Then there is a bijective correspondence between CPA-structures on $\mathfrak{g}$ and elements $z \in Z([\mathfrak{g}, \mathfrak{g}])$ given by*

$$x \cdot y = [[z, x], y].$$

4.1.2 CPA-structures for nilpotent Lie algebras

In this section we want to recall some results for commutative post-Lie algebra structures on nilpotent Lie algebras. As can be seen in Proposition 4.7 the left multiplication operators are all nilpotent. The question under which assumptions the left multiplication operators are nilpotent was studied, see [24], [27]. For non-abelian nilpotent Lie algebras which are generated by two elements this is the case, see [24].

Proposition 4.26. *Let $\mathfrak{g}$ be a non-abelian nilpotent Lie algebra which is generated by two elements. Then for any CPA-structure on $\mathfrak{g}$ the left multiplication operators $L(x)$ are nilpotent.*

The same holds for lower dimensional nilpotent Lie algebras without abelian factor, see [24].

Corollary 4.27. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension $n \le 4$ without abelian factor. Then for any CPA-structure on $\mathfrak{g}$ the left multiplication operators $L(x)$ are nilpotent.*

This does not hold for nilpotent Lie algebras with abelian factor, see Example 7.3 in [24].

Example 18. Let $\mathfrak{g} = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$. Let (e_1, e_2, e_3, e_4) be a basis of $\mathfrak{g}$ with $[e_1, e_2] = e_3$. Then

$$L(e_1) = L(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, L(e_2) = 0, L(e_3) = 0$$

defines a CPA-structure on $\mathfrak{g}$ where not all maps $L(x)$ are nilpotent.

However the result might be true for all nilpotent Lie algebras without abelian factor, see Question 7.4 in [24].

Question 4. Is is true that all left multiplication maps are nilpotent for every CPA-structure on a nilpotent Lie algebra $\mathfrak{g}$ without abelian factor?

Moreover in [27] stem nilpotent Lie algebras were studied and it was shown that for stem nilpotent Lie algebras all left multiplication operators are nilpotent. Let us start by defining what stem Lie algebras and complete CPA-structures are, see [27].

Definition 4.28. A Lie algebra $\mathfrak{g}$ is called a *stem* Lie algebra, if $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$.

Definition 4.29. We call a CPA-structure on a Lie algebra $\mathfrak{g}$ *complete*, if all left multiplication operators $L(x)$ and therefore also all right multiplication operators $R(x)$ are nilpotent.

The main theorem, Theorem 3.6 in [27] shows that CPA-structures on stem nilpotent Lie algebras are complete.

Theorem 4.30. *Let $\mathfrak{g}$ be a stem nilpotent Lie algebra. Then every commutative post-Lie algebra structure on $\mathfrak{g}$ is complete.*

For stem nilpotent Lie algebras we even have more, see [27].

Definition 4.31. Let $\mathfrak{g}$ be a nilpotent Lie algebra. A CPA-structure on $\mathfrak{g}$ is called *central* if $\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g})$.

Lemma 4.32. *Let $\mathfrak{g}$ be a stem nilpotent Lie algebra. For a central CPA-structure on $\mathfrak{g}$ we have*

$$\mathfrak{g} \cdot Z(\mathfrak{g}) = \mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0.$$

In [27] we also find the following two examples.

Example 19. All CPA-structures on the free nilpotent Lie algebra $F_{2,3}$ with two generators and nilpotency class 3 are central and given by

$$\begin{aligned} e_1 \cdot e_1 &= \alpha e_4 + \beta e_5, \\ e_1 \cdot e_2 &= \gamma e_4 + \delta e_5, \\ e_2 \cdot e_2 &= \epsilon e_4 + \kappa e_5, \end{aligned}$$

for $\alpha, \beta, \gamma, \delta, \epsilon, \kappa \in \mathbb{C}$.

Example 20. Let $F_{3,2}$ be the free nilpotent Lie algebra on 3 generators of nilpotency class 2, then

$$\begin{aligned} e_1 \cdot e_1 &= e_2, \\ e_1 \cdot e_2 &= -e_5, \\ e_1 \cdot e_5 &= e_6, \\ e_2 \cdot e_3 &= -2e_6 \end{aligned}$$

defines a CPA-structure with $\mathfrak{g} \cdot Z(\mathfrak{g}) \neq 0$.

It turns out that for free nilpotent Lie algebras on 3 generators and nilpotency class $c \geq 3$ all CPA-structures are central, see Theorem 4.3 in [27].

Theorem 4.33. *All commutative post-Lie algebra structures on $F_{3,c}$ are central, where $c \geq 3$.*

Moreover, we find the following conjecture in [27].

Conjecture. All CPA-structures on $F_{g,c}$ with $c \geq 3$ and $g \geq 2$ are central.

There is a study on commutative post-Lie algebra structures on nilpotent Lie algebras relating them to Poisson structures and Poisson algebras, see [33]. Furthermore, in [43] we find results for CPA-structures on infinite-dimensional Lie algebras and CPA-structures on Kac-Moody algebras. For a study on commutative post-Lie algebra structures on Lie algebras over fields of characteristic $p > 0$ we refer to [41], relating these structures to the Zassenhaus conjecture, see [95].

4.2 λ-scalar-post-Lie algebra structures

In this chapter we introduce a new class of post-Lie algebra structures, namely λ-scalar post-Lie algebra structures or λ-SPA-structures. We want to generalize some of the results for commutative post-Lie algebra structures to λ-scalar-post Lie algebra structures. These structures have already been studied in Chapter 3.3 and 4.3 in [49]. We formally introduce the notion of a λ-SPA-structure and that of an inner λ-SPA-structure. The study of these inner λ-SPA-structures and looking at these structures on complete Lie algebras allows us to give a different approach to Chapter 4.3 in [49]. Moreover, we give a different proof to λ-SPA-structures on semisimple Lie algebras in Chapter 3.3 in [49]. Let us start with the definition of a λ-scalar-post-Lie algebra structure. Note that we already mentioned in the beginning of Chapter 4 that we restrict ourselves to finite-dimensional and complex Lie algebras. Moreover, we assume $\lambda \in \mathbb{C}$ and non-zero.

Definition 4.34. A λ-*scalar-post-Lie algebra structure* or short λ-*SPA-structure* on a Lie algebra $\mathfrak{g}$ is a bilinear product $x \cdot y$ satisfying

(1) $x \cdot y - y \cdot x = (1-\lambda)[x, y]$

(2) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$

(3) $x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z]$

for all $x, y, z \in V$.

Remark 13. A λ-SPA-structure is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where the Lie bracket of $\mathfrak{n}$ is given by $\lambda[x, y] = \{x, y\}$. Note that we have that $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic.

Definition 4.35. We call $(A, \cdot)$ where $\cdot$ is the product coming from a λ-SPA-structure a λ-*scalar-post-Lie algebra.* A subspace $I \subseteq A$ is called a *left algebra ideal* if $I \cdot A \subseteq I$. It is called a *right algebra ideal* if $A \cdot I \subseteq I$. It is called an *ideal* if it is both a left and a right ideal.

4.2.1 λ-SPA-structures for perfect Lie algebras

In this section we show that for perfect Lie algebras, only the trivial structures can occur. Let us start with semisimple Lie algebras.

Proposition 4.36. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \cdot y$ a λ-SPA-structure on $\mathfrak{g}$. Then $x \cdot y = 0$ or $x \cdot y = [x, y]$.*

Proof. Since $\mathfrak{g}$ is semisimple, we have $x \cdot y = \lambda[R(x), y]$ where R is an RB-operator of weight 1 on $\mathfrak{g}$ by Corollary 1.92. Now we know that

$$\begin{aligned} [R(x), y] + [x, R(y)] &= \frac{1-\lambda}{\lambda}[x, y] \\ R([x, y]) &= \lambda[R(x), R(y)]. \end{aligned}$$

Write $R = \frac{1-\lambda}{\lambda}\mu$ for some endomorphism μ, then we get

$$[\mu(x), y] + [x, \mu(y)] = [x, y].$$

We can rewrite $\mu = d + \frac{1}{2}\,\mathrm{id}$ for some linear map d. Then we get

$$[d(x), y] + [x, d(y)] = 0$$

and by Proposition 5.10 in [23], we get $d = 0$. We therefore obtain two cases. For the one case we have $R = 0$ and hence $\lambda = 1$, which is then a CPA-structure and by Proposition 5.1 in [24], is trivial.
The other case is $R = \frac{1-\lambda}{2\lambda}\,\mathrm{id}$ and $\lambda \neq 1$, then by the second axiom we get

$$\left(\frac{1-\lambda}{2\lambda}\right)([x, y]) = \lambda\left[\left(\frac{1-\lambda}{2\lambda}\right)(x), \left(\frac{1-\lambda}{2\lambda}\right)(y)\right]$$

and therefore $\lambda = -1$, and $x \cdot y = [x, y]$. $\square$

The next two results also appear in [49], Lemma 3.46 and Corollary 3.47. Since we used a different method for Proposition 4.37 we recall the results here and give the proof as well.

Lemma 4.37. *Let $x \cdot y$ be a λ-SPA-structure on a Lie algebra $\mathfrak{g}$ and $\mathfrak{g} \cong \mathfrak{s} \ltimes \operatorname{rad}(\mathfrak{g})$ with $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$. Then we have $\mathfrak{g} \cdot \mathfrak{g} \subseteq \operatorname{rad}(\mathfrak{g})$ or $\lambda = -1$.*

Proof. Let $\mathfrak{g} \cong \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition with Levi subalgebra $\mathfrak{s}$ and $\mathfrak{r} = \operatorname{rad}(\mathfrak{g})$. We have by definition that the left multiplication operators $L(x)$ are derivations of $\mathfrak{g}$ for all $x \in \mathfrak{g}$. Since the radical is a characteristic ideal it is invariant under derivations and we get $L(x)(\mathfrak{r}) \subseteq \mathfrak{r}$ and therefore $\mathfrak{g} \cdot \mathfrak{r} \subseteq \mathfrak{r}$. We also have

$$\mathfrak{r} \cdot \mathfrak{g} \subseteq (1-\lambda)[\mathfrak{r}, \mathfrak{g}] + \mathfrak{g} \cdot \mathfrak{r} \subseteq \mathfrak{r} + \mathfrak{r} = \mathfrak{r}.$$

We can define a bilinear map $(x+\mathfrak{r}) \circ (y+\mathfrak{r})$ on $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}$, by

$$\begin{aligned} \circ : \mathfrak{s} \times \mathfrak{s} &\to \mathfrak{s} \\ (x+\mathfrak{r}, y+\mathfrak{r}) &\mapsto x \cdot y + \mathfrak{r} \end{aligned}$$

and the Lie bracket on $\mathfrak{s}$ is given by $[x+\mathfrak{r}, y+\mathfrak{r}] = [x,y] + \mathfrak{r}$. Note that the second Lie bracket is given by $\lambda[x+\mathfrak{r}, y+\mathfrak{r}] = \{x+\mathfrak{r}, y+\mathfrak{r}\}$. This is a λ-SPA-structure on $\mathfrak{s}$. Indeed, we have

$$\begin{aligned}
&(1) \quad (x+\mathfrak{r}) \circ (y+\mathfrak{r}) - (y+\mathfrak{r}) \circ (x+\mathfrak{r}) \\
&= x \cdot y + \mathfrak{r} - y \cdot x + \mathfrak{r} \\
&= x \cdot y - y \cdot x + \mathfrak{r} = (1-\lambda)[x,y] + \mathfrak{r} \\
&= (1-\lambda)[x+\mathfrak{r}, y+\mathfrak{r}]
\end{aligned}$$

$$\begin{aligned}
&(2) \quad [x+\mathfrak{r}, y+\mathfrak{r}] \circ (z+\mathfrak{r}) \\
&= ([x,y]+\mathfrak{r}) \circ (z+\mathfrak{r}) \\
&= [x,y] \cdot z + \mathfrak{r} = x \cdot (y \cdot z) - y \cdot (x \cdot z) + \mathfrak{r} \\
&= x \cdot (y \cdot z) + \mathfrak{r} - y \cdot (x \cdot z) + \mathfrak{r} \\
&= (x+\mathfrak{r}) \circ ((y+\mathfrak{r}) \circ (z+\mathfrak{r})) - (y+\mathfrak{r}) \circ ((x+\mathfrak{r}) \circ (z+\mathfrak{r}))
\end{aligned}$$

$$\begin{aligned}
&(3) \quad (x+\mathfrak{r}) \circ [y+\mathfrak{r}, z+\mathfrak{r}] \\
&= (x+\mathfrak{r}) \circ ([y,z]+\mathfrak{r}) \\
&= x \cdot [y,z] + \mathfrak{r} = [x \cdot y, z] + [y, x \cdot z] + \mathfrak{r} \\
&= [x \cdot y, z] + \mathfrak{r} + [y, x \cdot z] + \mathfrak{r} \\
&= [x \cdot y + \mathfrak{r}, z+\mathfrak{r}] + [y+\mathfrak{r}, x \cdot z + \mathfrak{r}] \\
&= [(x+\mathfrak{r}) \circ (y+\mathfrak{r}), z+\mathfrak{r}] + [y+\mathfrak{r}, (x+\mathfrak{r}) \circ (z+\mathfrak{r})]
\end{aligned}$$

Since $\mathfrak{s}$ is semisimple, we get two cases by Proposition 4.36.

- $(x+\mathfrak{r})\circ(y+\mathfrak{r})=0$. This implies $x\cdot y\subseteq\mathfrak{r}$, therefore $\mathfrak{g}\cdot\mathfrak{g}\subseteq\operatorname{rad}(\mathfrak{g})$.
- $(x+\mathfrak{r})\circ(y+\mathfrak{r})=[x+\mathfrak{r},y+\mathfrak{r}]$. And therefore $\lambda=-1$.

□

This leads to the following result, which shows that for certain λ we have that the Lie algebra $\mathfrak{g}$ admitting a λ-SPA-structure has to be solvable. Recall again, that $\lambda\neq 0$ as mentioned in the beginning of Section 4.2.

Corollary 4.38. *Let $x\cdot y$ be a λ-SPA-structure on $\mathfrak{g}$ and $\lambda\neq 1,-1$. Then $\mathfrak{g}$ is solvable.*

Proof. Suppose $\mathfrak{g}$ is not solvable and $\mathfrak{g}\cong\mathfrak{s}\ltimes\operatorname{rad}(\mathfrak{g})$. Let $s_1,s_2\in\mathfrak{s}$, we have

$$s_1\cdot s_2-s_2\cdot s_1=(1-\lambda)[s_1,s_2].$$

The right-hand side is obviously contained in $\mathfrak{s}$ and therefore $s_1\cdot s_2-s_2\cdot s_1\in\mathfrak{s}$. Since $\lambda\neq-1$, we have $\mathfrak{g}\cdot\mathfrak{g}\subseteq\operatorname{rad}(\mathfrak{g})$ by Lemma 4.37, but this means $s_1\cdot s_2-s_2\cdot s_1=0$, and since $\lambda\neq 1$, we get $[s_1,s_2]=0$. This implies that $\mathfrak{s}$ is both abelian and semisimple, which is a contradiction.

□

We can use these two results now to deduce the following result for λ-SPA-structures on perfect Lie algebras.

Proposition 4.39. *Let $\mathfrak{g}$ be a perfect Lie algebra and assume there exists a λ-SPA-structure $x\cdot y$ on $\mathfrak{g}$. Then $x\cdot y=0$, which corresponds to $\lambda=1$, or $x\cdot y=[x,y]$, corresponding to $\lambda=-1$.*

Proof. We have $\lambda=1$ or $\lambda=-1$, because otherwise $\mathfrak{g}$ is solvable by Corollary 4.38. For $\lambda=1$, we have $x\cdot y=0$ by Theorem 4.18. If $\lambda=-1$, then we can associate a CPA-structure to $\mathfrak{g}$ by $x\circ y=x\cdot y-[x,y]$. However all CPA-structures are trivial, see Theorem 4.18, implying $x\cdot y=[x,y]$. □

4.2.2 Inner λ-SPA-structures

In this section we give an alternative proof to Proposition 4.36 using a decomposition of a Lie algebra admitting a λ-SPA-structure. Let us start with the notion of an inner λ-SPA-structure which goes hand in hand with the notion of an inner post-Lie algebra structure.

Definition 4.40. Let $x\cdot y$ be a λ-SPA-structure on a Lie algebra $\mathfrak{g}$. We call $x\cdot y$ an *inner λ-SPA-structure* if there exists a $\phi\in\operatorname{End}(V)$ such that

$$x\cdot y=\lambda[\phi(x),y]$$

for all $x,y\in V$.

Now let us generalize some results from Section 4.1.

Lemma 4.41. *Assume $(A, \cdot)$ is a λ-SPA-structure on $\mathfrak{g}$. Then the chain $I_0 = 0$ and*

$$I_n = \{x \in A | x \cdot A \subseteq I_{n-1}\}$$

is both a Lie algebra ideal and an algebra ideal for A.

Proof. Note that $I_n \cdot A \subseteq I_{n-1} \subseteq I_n$ by definition. First let us show that I_n defines a Lie algebra ideal. We do this by induction. We have $I_1 = \ker(L)$ which is a Lie algebra ideal. Then by our induction hypothesis, we have

$$\begin{aligned}
[I_n, \mathfrak{g}] \cdot A &\subseteq I_n \cdot (\mathfrak{g} \cdot A) + \mathfrak{g} \cdot (I_n \cdot A) \\
&\subseteq I_n \cdot A + \mathfrak{g} \cdot I_{n-1} \\
&\subseteq I_{n-1} + \mathfrak{g} \cdot I_{n-1} \\
&\subseteq I_{n-1} + I_{n-1} \cdot \mathfrak{g} + (1-\lambda)[\mathfrak{g}, I_{n-1}] \\
&\subseteq I_{n-1} + (1-\lambda)[\mathfrak{g}, I_{n-1}] \\
&\subseteq I_{n-1}
\end{aligned}$$

This implies that $[I_n, \mathfrak{g}] \subseteq I_n$ and is therefore a Lie algebra ideal.

Now we still need to show that I_n is an algebra ideal. We already know that it is a left ideal, i.e. $I_n \cdot A \subseteq I_n$. We still need to show that it is a right ideal. We have

$$A \cdot I_n \subseteq (1-\lambda)[A, I_n] + I_n \cdot A \subseteq I_n.$$

□

For inner λ-SPA structures, we have the following.

Lemma 4.42. *Suppose $(A, \cdot)$ is an inner λ-SPA structure on a Lie algebra $\mathfrak{g}$. Then all Lie algebra ideals of $\mathfrak{g}$ are ideals of A and the ascending chain of ideals I_n are ϕ-invariant. Conversely if the structure is non-degenerate, i.e.* $\ker(L) = 0$*, then all ideals of A are Lie algebra ideals of $\mathfrak{g}$.*

Proof. Let I be an ideal of $\mathfrak{g}$. Let us show first that it is a right ideal, i.e. $A \cdot I \subseteq I$. We have

$$A \cdot I = \mathfrak{g} \cdot I = \lambda[\phi(\mathfrak{g}), I] \subseteq \lambda[\mathfrak{g}, I] \subseteq I.$$

Moreover we have

$$I \cdot A \subseteq (1-\lambda)[A, I] + A \cdot I \subseteq I.$$

Conversely, assume $\ker(L) = 0$, then we also have $\ker(\phi) = 0$. Hence ϕ is invertible and $\phi(\mathfrak{g}) = \mathfrak{g}$. Let I be an algebra ideal, so we have $\mathfrak{g} \cdot I \subseteq I$, then

$$\lambda[\mathfrak{g}, I] = \lambda[\phi(\mathfrak{g}), I] = \mathfrak{g} \cdot I \subseteq I$$

Therefore I is also a Lie algebra ideal. We show now that I_n is ϕ-invariant, i.e. $\phi(I_n) \subseteq I_n$. We have

$$\begin{aligned}\phi(I_n) \cdot \mathfrak{g} &\subseteq \mathfrak{g} \cdot \phi(I_n) + (1-\lambda)[\phi(I_n), \mathfrak{g}] \\ &\subseteq \mathfrak{g} \cdot \phi(I_n) + (1-\lambda) I_n \cdot \mathfrak{g} \\ &\subseteq \mathfrak{g} \cdot \phi(I_n) + I_{n-1} \\ &\subseteq \lambda[\phi(\mathfrak{g}), \phi(I_n)] + I_{n-1} \\ &\subseteq -I_n \cdot \phi(\mathfrak{g}) + I_{n-1} \\ &\subseteq I_n \cdot \mathfrak{g} + I_{n-1} \\ &\subseteq I_{n-1}.\end{aligned}$$

Hence, we obtain $\phi(I_n) \subseteq I_n$. □

Lemma 4.43. *Suppose $x \cdot y = \lambda[\phi(x), y]$ is an inner λ-SPA-structure on a Lie algebra $\mathfrak{g}$ with $\phi \in \operatorname{End}(V)$ and let $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ϕ. Then we have*

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha\beta\lambda}$

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ *implies*

$$\alpha + \beta = \frac{1}{\lambda} - 1.$$

Proof. (1) To prove this, we first show that

$$(\phi - \alpha\beta\lambda \operatorname{id})^n([x, y]) = \lambda^n \sum_{i=0}^{n} f(n, i)\alpha^i[(\phi - \alpha \operatorname{id})^{n-i}(x), \phi^{n-i}((\phi - \beta \operatorname{id})^i(y))],$$

where $f(n, i)$ denotes the coefficient at $\alpha^i[(\phi - \alpha \operatorname{id})^{n-i}(x), \phi^{n-i}((\phi - \beta \operatorname{id})^i(y))]$ which is defined recursively via $f(n, i) = f(n-1, i-1) + f(n-1, i)$ for $i \geq 1$ with $f(0, 0) = f(n, n) = 1 = f(n, 0)$ and $f(n, k) = 0$ for all $k \geq n$. We do this by induction. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. For $n = 1$, we have since $\phi([x, y]) = \lambda[\phi(x), \phi(y)]$

$$(\phi - \alpha\beta\lambda \operatorname{id})([x, y]) = \lambda[(\phi - \alpha \operatorname{id})(x), \phi(y)] + \lambda[\alpha x, (\phi - \beta \operatorname{id})(y)].$$

Hence $(\phi - \alpha\beta\lambda \operatorname{id})([x, y]) = \lambda^1 \sum_{i=0}^{1} f(1, i)\alpha^i[(\phi - \alpha \operatorname{id})^{1-i}(x), \phi^{1-i}(\phi - \beta \operatorname{id})^i(y)]$ with $f(1, 0) = 1, f(1, 1) = 1$.

Our induction hypothesis is that

$$(\phi - \alpha\beta\lambda \operatorname{id})^n([x, y]) = \lambda^n \sum_{i=0}^{n} f(n, i)\alpha^i[(\phi - \alpha \operatorname{id})^{n-i}(x), \phi^{n-i}((\phi - \beta \operatorname{id})^i(y))]$$

with $f(n, i) = f(n-1, i-1) + f(n-1, i)$ with $f(0, 0) = f(n, n) = 1 = f(n, 0)$ and $f(n, k) = 0$ for all $k \geq n$.

Now let us show that this holds for $(\phi - \alpha\beta\lambda\,\mathrm{id})^{n+1}$. Indeed, we have

$$
\begin{aligned}
&(\phi - \alpha\beta\lambda\,\mathrm{id})^{n+1}([x,y]) \\
&= (\phi - \alpha\beta\lambda\,\mathrm{id})^{n}(\lambda[(\phi - \alpha\,\mathrm{id})(x), \phi(y)] + \lambda[\alpha x, (\phi - \beta\,\mathrm{id})(y)]) = \\
&+ \lambda^{n+1}\sum_{i=0}^{n} f(n,i)\alpha^i[(\phi - \alpha\,\mathrm{id})^{n-i}((\phi - \alpha\,\mathrm{id})(x)), \phi^{n-i}((\phi - \beta\,\mathrm{id})^i(\phi(y)))] \\
&+ \lambda^{n+1}\sum_{i=0}^{n} f(n,i)\alpha^i[(\phi - \alpha\,\mathrm{id})^{n-i}(\alpha x), \phi^{n-i}((\phi - \beta\,\mathrm{id})^i((\phi - \beta\,\mathrm{id})(y)))] \\
&= \lambda^{n+1}\sum_{i=0}^{n+1} f(n+1,i)\alpha^i[(\phi - \alpha\,\mathrm{id})^{n+1-i}(x), \phi^{n+1-i}((\phi - \beta\,\mathrm{id})^i(y))]
\end{aligned}
$$

where $f(n+1,i) = f(n,i-1) + f(n,i)$ for $i \geq 1$, $f(n+1,n+1) = f(n,n) = 1 = f(n+1,0)$ and $f(n+1,k) = 0$ for all $k \geq n+1$.

Since $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, there exists some $k > 0$ such that $(\phi - \alpha\,\mathrm{id})^k(x) = 0$ and there exists some $l > 0$ such that $(\phi - \beta\,\mathrm{id})^l(y) = 0$. Then we have

$$
(\phi - \alpha\beta\lambda\,\mathrm{id})^{l+k}([x,y]) = \lambda^{l+k}\sum_{i=0}^{l+k} f(l+k,i)\alpha^i[(\phi - \alpha\,\mathrm{id})^{l+k-i}(x), \phi^{l+k-i}(\phi - \beta\,\mathrm{id})^i(y)]
$$

We have that the terms from $i = 0, ..., l$ vanish since $(\phi - \alpha\,\mathrm{id})^{l+k-i}(x) = 0$. The terms from $i = l+1, ..., l+k$ also vanish since $(\phi - \beta\,\mathrm{id})^i(y) = 0$. Hence

$$
(\phi - \alpha\beta\lambda\,\mathrm{id})^{l+k}([x,y]) = 0,
$$

which implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha\beta\lambda}$.

(2) For the second we have for $x, y \in \mathfrak{g}$

$$
\begin{aligned}
\lambda[\phi(x), \phi(y)] &= -\lambda[\phi(y), \phi(x)] \\
&= -y \cdot \phi(x) = (1-\lambda)[\phi(x), y] - \phi(x) \cdot y \\
&= (1-\lambda)[\phi(x), y] - \lambda[\phi^2(x), y] \\
&= -\lambda[\phi^2(x), y] + (1-\lambda)[\phi(x), y].
\end{aligned}
$$

Hence we get

$$
\phi([x,y]) = (-1)[(\lambda\phi^2 - (1-\lambda)\phi)(x), y].
$$

Using induction, we obtain

$$
(\phi + \gamma\,\mathrm{id})^k([x,y]) = (-1)^k[(\lambda\phi^2 - (1-\lambda)\phi - \gamma\,\mathrm{id})^k(x), y]. \tag{5}
$$

Indeed, for $k = 1$ we have

$$
\begin{aligned}
(\phi + \gamma\,\mathrm{id})([x,y]) &= (-1)[(\lambda\phi^2 - (1-\lambda)\phi)(x), y] + \gamma\,\mathrm{id}[x,y] \\
&= (-1)[(\lambda\phi^2 - (1-\lambda)\phi - \gamma\,\mathrm{id})(x), y]
\end{aligned}
$$

Using the induction hypothesis, we get

$$\begin{aligned}(\phi+\gamma\,\mathrm{id})^{k+1}([x,y]) &= (\phi+\gamma\,\mathrm{id})(\phi+\gamma\,\mathrm{id})^{k}([x,y])\\ &= (\phi+\gamma\,\mathrm{id})((-1)^k[(\lambda\phi^2-(1-\lambda)\phi-\gamma\,\mathrm{id})^k(x),y])\\ &= (-1)^{k+1}[(\lambda\phi^2-(1-\lambda)\phi-\gamma\,\mathrm{id})((\lambda\phi^2-(1-\lambda)\phi-\gamma\,\mathrm{id})^k(x),y]\\ &= (-1)^{k+1}[(\lambda\phi^2-(1-\lambda)\phi-\gamma\,\mathrm{id})^{k+1}(x),y]\end{aligned}$$

Let x be a generalized eigenvector of ϕ and α the corresponding generalized eigenvalue, so there exists some $n>0$, such that $(\phi-\alpha\,\mathrm{id})^n(x)=0$. We have

$$(\phi^2-\alpha^2\,\mathrm{id})^n(x)=((\phi+\alpha\,\mathrm{id})(\phi-\alpha\,\mathrm{id}))^n(x)=0.$$

Hence α^2 is a generalized eigenvalue for ϕ^2.

Therefore the right-hand side of (5) vanishes for $\gamma:=\lambda\alpha^2-(1-\lambda)\alpha$ and k large enough. So we get $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{-(\lambda\alpha^2-(1-\lambda)\alpha)}$ and similiarly we get $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{-(\lambda\beta^2-(1-\lambda)\beta)}$ and by (1) we get

$$[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{-(\lambda\alpha^2-(1-\lambda)\alpha)}\cap\mathfrak{g}_{-(\lambda\beta^2-(1-\lambda)\beta)}\cap\mathfrak{g}_{\alpha\beta\lambda}.$$

Since $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\neq 0$, the spaces on the right-hand side coincide, and we get

$$-(\lambda\alpha^2-(1-\lambda)\alpha)=-(\lambda\beta^2-(1-\lambda)\beta)=\alpha\beta\lambda,$$

which implies:

$$\alpha+\beta=\frac{1}{\lambda}-1$$

□

The case where $\lambda=-1$, seems to be a special case as the following result shows.

Theorem 4.44. *Let $\mathfrak{g}$ be a Lie algebra and $x\cdot y=\lambda[\phi(x),y]$ an inner λ-SPA-structure with $\phi\in\mathrm{End}(V)$ and $\lambda\neq-1$. Then $\mathfrak{g}$ decomposes into the sum of ϕ-invariant ideals*

$$\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}$$

with the following properties

(1) $\phi_{|\mathfrak{n}}$ is a nilpotent endomorphism of $\mathfrak{n}$.

(2) $\phi_{|\mathfrak{h}}$ is an automorphism of $\mathfrak{h}$ and $\mathfrak{h}$ is metabelian.

Proof. For $\lambda=1$, this has already been done, see Theorem 4.16. So let $\lambda\neq1$. Consider $\mathfrak{g}=\oplus_\alpha\mathfrak{g}_\alpha=\mathfrak{g}_0\oplus(\oplus_{\alpha\neq0}\mathfrak{g}_\alpha)$ of $\mathfrak{g}$ with respect to ϕ with $\mathfrak{n}=\mathfrak{g}_0$ and $\mathfrak{h}=\oplus_{\alpha\neq0}\mathfrak{g}_\alpha$. Both $\mathfrak{h}$ and $\mathfrak{n}$ are Lie algebra ideals. Indeed, let $\alpha\neq0$, and suppose $[\mathfrak{g}_\alpha,\mathfrak{g}_0]\neq0$, then we have $\alpha=\frac{1}{\lambda}-1$ and $[\mathfrak{g}_\alpha,\mathfrak{g}_0]\subseteq\mathfrak{g}_0$ by Lemma 4.43. Moreover we also have

$[\mathfrak{g}_\alpha, \mathfrak{g}_0] \subseteq \mathfrak{g}_{-\lambda\alpha^2-(1-\lambda)\alpha}$ see the proof of Lemma 4.43 and since we assumed $[\mathfrak{g}_\alpha, \mathfrak{g}_0] \neq 0$, we get $[\mathfrak{g}_\alpha, \mathfrak{g}_0] \subseteq \mathfrak{g}_0 \cap \mathfrak{g}_{-\lambda(\frac{1}{\lambda}-1)^2-(1-\lambda)(\frac{1}{\lambda}-1)}$, so that $-\lambda(\frac{1}{\lambda}-1)^2-(1-\lambda)(\frac{1}{\lambda}-1) = 0$, hence $\lambda = 1$, which we have excluded. By Lemma 4.42 they are also algebra ideals. Furthermore, we have for $x \in \mathfrak{g}_\alpha$, that also $\phi(x) \in \mathfrak{g}_\alpha$. Hence both $\mathfrak{n}$ and $\mathfrak{h}$ are ϕ-invariant. By definition we have that $\phi|_\mathfrak{n}$ is a nilpotent endomorphism and since all generalized eigenvalues of $\mathfrak{h}$ are nonzero, the restriction of ϕ to $\mathfrak{h}$ is an automorphism.

Now we need to show that $\mathfrak{h}$ is metabelian, i.e.

$$[[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]] = 0$$

for all $\alpha, \beta, \gamma, \delta \neq 0$. Suppose

$$[[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]] \neq 0.$$

By Lemma 4.43, we have

$$\begin{aligned} \alpha + \beta &= \frac{1}{\lambda} - 1 \\ \gamma + \delta &= \frac{1}{\lambda} - 1 \\ \alpha\beta + \gamma\delta &= \frac{1}{\lambda^2} - \frac{1}{\lambda}. \end{aligned}$$

The Jacobi identity gives us:

$$[[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]] \subseteq [\mathfrak{g}_\gamma, [\mathfrak{g}_\delta, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]]] + [\mathfrak{g}_\delta, [[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], \mathfrak{g}_\gamma]]$$

and

$$[[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]] \subseteq [[\mathfrak{g}_\beta, [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]], \mathfrak{g}_\alpha] + [[[\mathfrak{g}_\gamma, \mathfrak{g}_\delta], \mathfrak{g}_\alpha], \mathfrak{g}_\beta].$$

This yields in the first case $[\mathfrak{g}_\delta, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]] \subseteq [\mathfrak{g}_\delta, \mathfrak{g}_{\alpha\beta\lambda}] \neq 0$, so $\delta + \alpha\beta\lambda = \frac{1}{\lambda} - 1$ or $[[\mathfrak{g}_\alpha, \mathfrak{g}_\beta], \mathfrak{g}_\gamma] \subseteq [\mathfrak{g}_{\alpha\beta\lambda}, \mathfrak{g}_\gamma] \neq 0$, so $\alpha\beta\lambda + \gamma = \frac{1}{\lambda} - 1$.
The second case yields $[[\mathfrak{g}_\gamma, \mathfrak{g}_\delta], \mathfrak{g}_\alpha] \subseteq [\mathfrak{g}_{\gamma\delta\lambda}, \mathfrak{g}_\alpha] \neq 0$, so $\gamma\delta\lambda + \alpha = \frac{1}{\lambda} - 1$ or $[\mathfrak{g}_\beta, [\mathfrak{g}_\gamma, \mathfrak{g}_\delta]] \subseteq [\mathfrak{g}_\beta, \mathfrak{g}_{\gamma\delta\lambda}] \neq 0$, so $\beta + \gamma\delta\lambda = \frac{1}{\lambda} - 1$.

In each cases this is only possible for $\lambda = -1$, which is a contradiction to our assumption. □

Now we can give an alternative proof to Proposition 4.36.

Lemma 4.45. *Suppose $\mathfrak{g}$ is semisimple. Then $\mathfrak{g}$ only admits the following λ-SPA-structures, $x \cdot y = 0$ where $\lambda = 1$ or $x \cdot y = [x, y]$ where $\lambda = -1$.*

Proof. Suppose $\mathfrak{g}$ admits a λ-SPA-structure. We know that the λ-SPA-structure is inner, since $\mathfrak{g}$ is complete. For $\lambda = 1$, we already know that $x \cdot y = 0$, see Proposition 4.2. Now let $\lambda \neq 1$ and suppose in addition that $\lambda \neq -1$. Then by Theorem 4.44, we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} = \mathfrak{n}$, since $\mathfrak{g}$ is semisimple. Without loss of generality we can assume $\ker(\phi) \neq 0$. We have $\ker(\phi)$ is an ideal of $\mathfrak{g}$. Let $x, y \in \ker(\phi)$, then we have by the first axiom

$$(1 - \lambda)[x, y] = \lambda[\phi(x), y] + \lambda[x, \phi(y)] = 0,$$

and since $\lambda \neq 1$, we have $[x, y] = 0$. This means that $\ker(\phi)$ is an abelian ideal of $\mathfrak{g}$, which is a contradiction to $\mathfrak{g}$ semisimple. Hence there exists no $\lambda \neq -1, 1$ such that $\mathfrak{g}$ admits a λ-SPA-structure. Now if $\lambda = -1$, we can associate a CPA-structure to $\mathfrak{g}$ via $x \circ y = x \cdot y - [x, y]$ and since every CPA-structure on a semisimple Lie algebra is trivial by Proposition 4.2, we get $x \circ y = 0$ and therefore $x \cdot y = [x, y]$. □

4.2.3 λ-SPA-structures for complete Lie algebras

As in the case of commutative post-Lie algebra structures on complete Lie algebras, it suffices to reduce it to simply-complete Lie algebras. We show analogous results for λ-SPA-structures. This is also studied in Chapter 4.3 in [49]. However, it does not use decomposition statements for inner PA-structures. We give a different approach using Section 4.2.2.

Lemma 4.46. *Let $\mathfrak{g}_1, ...\mathfrak{g}_n$ be simply-complete Lie algebras, and suppose a λ-SPA-structure exists on all $\mathfrak{g}_i$ with the same λ. Then the direct Lie algebra sum admits a λ-SPA-structure which is given componentwise*

$$(x_1, ..., x_n) \cdot (y_1, ..., y_n) = (x_1 \cdot y_1, ..., x_n \cdot y_n).$$

Furthermore, for any complete Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ that decomposes into simply complete ideals, any λ-SPA-structure is given as above.

Proof. For the first part, we show that the above defined product is indeed a λ-SPA-structure. We have

$$\begin{aligned}
&(1)\ \ (x_1, ..., x_n) \cdot (y_1, ..., y_n) - (y_1, ..., y_n) \cdot (x_1, ..., x_n)\\
&= (x_1 \cdot y_1, ..., x_n \cdot y_n) - (y_1 \cdot x_1, ..., y_n \cdot x_n)\\
&= (x_1 \cdot y_1 - y_1 \cdot x_1, ..., x_n \cdot y_n - y_n \cdot x_n)\\
&= ((1 - \lambda)[x_1, y_1], ..., (1 - \lambda)[x_n, y_n])\\
&= (1 - \lambda)[(x_1, ..., x_n), (y_1, ..., y_n)]\\
&(2)\ \ [(x_1, ...x_n), (y_1, ..., y_n)] \cdot (z_1, ..., z_n) = ([x_1, y_1], ..., [x_n, y_n]) \cdot (z_1, ..., z_n)\\
&= ([x_1, y_1] \cdot z_1, ..., [x_n, y_n] \cdot z_n)\\
&= (x_1, ..., x_n) \cdot ((y_1, ..., y_n) \cdot (z_1, ..., z_n)) - (y_1, ..., y_n) \cdot ((x_1, ..., x_n) \cdot (z_1, ..., z_n))\\
&(3)\ \ (x_1, ..., x_n) \cdot [(y_1, ..., y_n), (z_1, ..., z_n)]\\
&= (x_1, ..., x_n) \cdot ([y_1, z_1], ..., [y_n, z_n])\\
&= (x_1 \cdot [y_1, z_1], ..., x_n \cdot [y_n, z_n])\\
&= ([x_1 \cdot y_1, z_1], ..., [x_n \cdot y_n, z_n]) + ([y_1, x_1 \cdot z_1], ..., [y_n, x_n \cdot z_n])\\
&= [(x_1, ..., x_n) \cdot (y_1, ..., y_n), (z_1, ..., z_n)] + [(y_1, ..., y_n), (x_1, ..., x_n) \cdot (z_1, ..., z_n)]
\end{aligned}$$

For the second one, we claim that $\mathfrak{g}_i \cdot \mathfrak{g}_j \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j$. Since $\mathfrak{g}$ is complete, we have $x \cdot y = \lambda[\phi(x), y]$ for some $\phi \in \operatorname{End}(V)$ by Proposition 1.91. We get

$$\mathfrak{g}_i \cdot \mathfrak{g}_j \subseteq \lambda[\phi(\mathfrak{g}_i), \mathfrak{g}_j] \subseteq \mathfrak{g}_j.$$

On the other hand we have

$$\mathfrak{g}_i \cdot \mathfrak{g}_j \subseteq \mathfrak{g}_j \cdot \mathfrak{g}_i + (1-\lambda)[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \lambda[\phi(\mathfrak{g}_j), \mathfrak{g}_i] + (1-\lambda)[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i.$$

Therefore we get $\mathfrak{g}_i \cdot \mathfrak{g}_j \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = 0$ for $i \neq j$. $\square$

Remark 14. This means that if a λ-SPA-structure exists on a complete Lie algebra, then also a λ-SPA-structure exists on its simply complete ideals.

Remark 15. If $\mathfrak{g}_i$ are simply-complete Lie algebras each equipped with a λ-SPA-structure, say λ_i is the corresponding one for $\mathfrak{g}_i$. Then we can define a PA-structure on the direct sum of the $\mathfrak{g}_i$'s by

$$x \cdot y = \begin{cases} x \cdot y & \text{if } x, y \in \mathfrak{g}_i \\ 0 & \text{else} \end{cases}$$

and the restriction of the PA-structure on the $\mathfrak{g}_i$'s is a λ_i-SPA-structure.

Hence we can focus only on simply-complete Lie algebras. Here as in the case of CPA-structures, we make a case distinction between metabelian and non-metabelian Lie algebras.

Lemma 4.47. *Suppose $\mathfrak{g}$ is a simply-complete non-metabelian Lie algebra and suppose $\mathfrak{g}$ admits a λ-SPA-structure. Then $\lambda \in \{-1, 1\}$.*

Proof. Suppose there exists a $\lambda \neq -1, 1$ such that $\mathfrak{g}$ admits a λ-SPA-structure. We have, since $\mathfrak{g}$ is complete, that every λ-SPA-structure is inner by Proposition 1.91 and by Theorem 4.44 we have that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. By Lemma 3.1 in [72] we have that both $\mathfrak{n}$ and $\mathfrak{h}$ are complete and since $\mathfrak{g}$ is simply-complete, we have that it is indecomposable. Hence either $\mathfrak{n} = 0$ or $\mathfrak{h} = 0$. Since we assume $\mathfrak{g}$ to be non-metabelian we have $\mathfrak{h} = 0$, and therefore $\mathfrak{g} = \mathfrak{n}$. We can assume $\ker(\phi) \neq 0$, since ϕ is a nilpotent endomorphism on $\mathfrak{g}$. Let $y \in \ker(\phi)$, $x \in \mathfrak{g}$ and suppose ϕ is n-step nilpotent.

Then we have by the first axiom of a λ-SPA-structure and since $\lambda \neq 1$

$$\begin{aligned}
[\phi^{n-1}(x), y] &= \frac{\lambda}{1-\lambda}[\phi^n(x), y] + \frac{\lambda}{1-\lambda}[\phi^{n-1}(x), \phi(y)] = 0, \\
[\phi^{n-2}(x), y] &= \frac{\lambda}{1-\lambda}[\phi^{n-1}(x), y] + \frac{\lambda}{1-\lambda}[\phi^{n-2}(x), \phi(y)] = 0 \\
&\dots \\
[\phi(x), y] &= 0, \\
[x, y] &= 0.
\end{aligned}$$

But this means that $y \in Z(\mathfrak{g})$ and therefore $y = 0$, which is a contradiction to $\ker(\phi) \neq 0$. Hence $\lambda \in \{1, -1\}$. $\square$

For the metabelian case, we only have to look at $\mathfrak{r}_2(\mathbb{C})$, see Proposition 4.24. We classify the λ-SPA-structures on $\mathfrak{r}_2(\mathbb{C})$, which we have included in the next chapter.

4.3 Classification of λ-SPA-structures in lower dimension

This section is dedicated to the classification of λ-SPA-structures for lower-dimensional solvable Lie algebras. We first look at the Borel subalgebra of $\mathfrak{sl}_2(\mathbb{C})$, which is isomorphic to $\mathfrak{r}_2(\mathbb{C})$.

Example 21. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_2(\mathbb{C})$, with Lie bracket $[e_1, e_2] = -2e_1$. Let us classify the λ-SPA-structures:

We have that the post-Lie algebra structure is inner by Corollary 1.92. In particular, the λ-SPA-structure is inner and we therefore have

$$\begin{aligned} \lambda[\phi(x), y] + \lambda[x, \phi(y)] &= (1-\lambda)[x, y] \\ \phi([x, y]) &= \lambda[\phi(x), \phi(y)]. \end{aligned}$$

Let $\phi(e_1) = \alpha_1 e_1 + \alpha_2 e_2$, $\phi(e_2) = \beta_1 e_1 + \beta_2 e_2$. Then we get

$$\begin{aligned} \lambda[\phi(e_1), e_2] + \lambda[e_1, \phi(e_2)] &= -2(1-\lambda)e_1, \\ \lambda[\alpha_1 e_1 + \alpha_2 e_2, e_2] + \lambda[e_1, \beta_1 e_1 + \beta_2 e_2] &= -2(1-\lambda)e_1, \\ -2\lambda\alpha_1 e_1 - 2\lambda\beta_2 e_1 &= -2(1-\lambda)e_1 \end{aligned}$$

which implies $\lambda\alpha_1 + \lambda\beta_2 = 1 - \lambda$. The second condition gives us

$$\begin{aligned} \phi([e_1, e_2]) &= \lambda[\phi(e_1, \phi(e_2)], \\ -2\phi(e_1) &= \lambda[\alpha_1 e_1 + \alpha_2 e_2, \beta_1 e_1 + \beta_2 e_2], \\ -2\alpha_1 e_1 - 2\alpha_2 e_2 &= -2\lambda\alpha_1\beta_2 e_1 + 2\lambda\alpha_2\beta_1 e_1. \end{aligned}$$

This implies $\alpha_2 = 0$ and therefore $-2\alpha_1 = -2\lambda\alpha_1\beta_2$. Now we get the following two cases depending on α_1.

If $\alpha_1 = 0$, then

$$\phi = \begin{pmatrix} 0 & \beta_1 \\ 0 & \frac{1-\lambda}{\lambda} \end{pmatrix}.$$

If $\alpha_1 \neq 0$, then we get $\beta_2\lambda = 1$ and by the first condition we get $\alpha_1 = -1$. Hence

$$\phi = \begin{pmatrix} -1 & \beta_1 \\ 0 & \frac{1}{\lambda} \end{pmatrix}.$$

In dimension 3, we have the following results for λ-SPA-structures on solvable Lie algebras.

Example 22. Let $\mathfrak{g} = \mathfrak{n}_3(\mathbb{C})$ be the 3-dimensional Heisenberg Lie algebra with Lie bracket $[e_1, e_2] = e_3$ and $\mathfrak{n} \cong \mathfrak{g}$ with $\{x, y\} = \lambda[x, y]$, so $\{e_1, e_2\} = \lambda e_3$.

Then there exist λ-SPA-structures given by

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ \lambda - 1 + \beta & \delta & 0 \end{pmatrix}$$

with $\alpha\gamma = 0$ or

$$L(e_1) = \begin{pmatrix} \alpha & \frac{-\alpha^2}{\beta} & 0 \\ \beta & -\alpha & 0 \\ \gamma & \delta & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} \frac{-\alpha^2}{\beta} & \frac{\alpha^3}{\beta^2} & 0 \\ -\alpha & \frac{\alpha^2}{\beta} & 0 \\ \lambda - 1 + \delta & \frac{\alpha(\beta(1-\lambda) - \alpha\gamma - 2\beta\delta}{\beta^2} & 0 \end{pmatrix}$$

with $\beta \neq 0$. This goes hand in hand with Proposition 5.2 in [34].

We end this chapter with the Lie algebra $\mathfrak{r}_3(\mathbb{C})$. Of course one can continue with say $\mathfrak{r}_{3,\mu}(\mathbb{C})$ with Lie bracket $[e_1, e_2] = e_2$ and $[e_1, e_3] = \mu e_3$ with $\mu \neq 0$ and other solvable Lie algebras in lower dimension.

Example 23. Consider the Lie algebra $\mathfrak{r}_3(\mathbb{C})$ with Lie bracket $[e_1, e_2] = e_2$, $[e_1, e_3] = e_2 + e_3$. Then we have the following λ-SPA-structures for $\alpha, \beta \in \mathbb{C}$

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & 0 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}$$

or

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 1-\lambda & 1-\lambda \\ \beta & 0 & 1-\lambda \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

5 Open questions

Throughout the chapters we mentioned some open questions and conjectures. We want to collect them here and also give possible ideas to pursue when tackling these questions.

5.1 Questions on decompositions of Lie algebras

In Chapter 2, we gave a survey on decomposition questions of Lie algebras. We mentioned some open questions regarding the derived length of a dinilpotent Lie algebra. For disemisimple Lie algebras, which are the Lie algebras that we have given more attention to, we have observed that the radical is abelian in most cases.

Question 5. Let $\mathfrak{g}$ be a disemisimple Lie algebra. Is it true that $\mathrm{rad}(\mathfrak{g})$ is abelian?

We have seen how this is true for Lie algebras where the Levi subalgebra has no simple quotient isomorphic to $\mathfrak{sl}_n(\mathbb{C})$, see [42]. It is not clear how one can generalize this result. It suffices to show that there exists no disemisimple Lie algebra where the radical is 2-step nilpotent, then one can already give a positive answer to the question.

Since for the existence of post-Lie algebra structures on pairs of Lie algebras one always gets a vector space sum decomposition, see Lemma 3.8, it is natural to study these questions.

Question 6. Let $\mathfrak{g} = \mathfrak{g}_1 \dot{+} \mathfrak{g}_2$ be a Lie algebra where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are reductive in $\mathfrak{g}$. Is it true that $\mathfrak{g}$ is reductive?

We have not looked at this question more closely, since we only studied those decomposition questions that were relevant for the existence of post-Lie algebra structures. However this question, naturally arose, since this would mean a generalization of the semisimple case, see Section 3.1. For post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is reductive, this might be useful. However, the problem here is that the subalgebra becomes just a subalgebra that is reductive as a Lie algebra. We can illustrate this a bit more precisely by looking at Example 14. We have a reductive Lie algebra $\mathfrak{g}$ and a perfect Lie algebra $\mathfrak{n}$. In addition there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Therefore there also exists a direct vector space decomposition by Lemma 3.8

$$\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \dot{+} \mathfrak{h}.$$

The subalgebras $\phi(\mathfrak{g})$ and $\mathfrak{h}$ are reductive as Lie algebras, but not reductive in $\mathfrak{n} \rtimes \mathfrak{h}$. This is because $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$ contains an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module.
If we assume that $\mathrm{rad}(\mathfrak{n}) = 0$, so $\mathfrak{n}$ is semisimple, we do not believe that such a construction works.

5.2 Existence questions

For semisimple Lie algebras $\mathfrak{n}$, as mentioned above, we do not believe that a construction of a post-Lie algebra structure as in the case where $\mathfrak{g}$ is reductive non-semisimple and $\mathfrak{n}$ is perfect non-semisimple works.

Question 7. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{n}$ be a semisimple Lie algebra. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Is it true that $\mathfrak{g} \cong \mathfrak{n}$?

We believe that the answer to this question is positive and that we can reduce this question to decomposition questions of semisimple Lie algebras. For simple Lie algebras this was done by using Onishchik's work on decompositions of simple Lie algebras, see [38]. We believe that this can be generalized to semisimple Lie algebras and that we can reduce it in the case where $\mathfrak{g}$ is reductive and $\mathfrak{n}$ semisimple to decompositions of compact Lie algebras.

We have also studied the case of $\mathfrak{g}$ reductive with 1-dimensional center and $\mathfrak{n}$ nilpotent non-abelian in Section 3.1. We have seen for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{n}$ nilpotent non-abelian no post-Lie algebra structure exists. This can be generalized to $\mathfrak{g}$ reductive with 1-dimensional center and simple Levi subalgebra. If the Levi subalgebra is semisimple non-simple, this cannot be done so easily, because there is no classification of pre-Lie algebra structures, as in the case of $\mathfrak{gl}_n(\mathbb{C})$.

In Chapter 3, we have also studied post-Lie algebra structures on pairs of Lie algebras where one of them is perfect. We have observed, that in a lot of cases no post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is nilpotent exists.

Question 8. Let $\mathfrak{g}$ be a perfect Lie algebra and $\mathfrak{n}$ a nilpotent Lie algebra. Is it true that there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$?

We have started a study on the most common example of a perfect Lie algebra, namely $\mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$ where $V(n)$ stands for the standard irreducible representation of $\mathfrak{sl}_n(\mathbb{C})$ considered as an abelian Lie algebra, which we have partly mentioned in Chapter 3. Here we believe that no post-Lie algebra structure exists. The problem however for perfect Lie algebras is that the cohomology of them is quite hard. However, we believe that one ingredient can be that there exists no pre-Lie algebra structure on perfect Lie algebras. The proof that there exists no post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is solvable, see Theorem 1.81, uses Lemma 1.80 where the main argument of the proof of this lemma is that there exists no pre-Lie algebra structure on a semisimple Lie algebra, see [23]. Lemma 1.80 can thus be generalized to perfect Lie algebras. However, the proof of Theorem 1.81, uses Levi-Malcev, which we cannot use for perfect Lie algebras.

For pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is semisimple, we pose the following question.

Question 9. Let $\mathfrak{g}$ be a perfect Lie algebra and $\mathfrak{n}$ be a semisimple Lie algebra. Suppose $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Is it true that $\mathfrak{g} \cong \mathfrak{n}$?

We have already seen in lower dimensions that no post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is perfect non-semisimple and $\mathfrak{n}$ is semisimple exists. We believe that one can use the theory of Rota-Baxter operators to study this pair of Lie algebras by studying the ideal $I = \ker(R) \oplus \ker(R + \mathrm{id})$ and the possible subalgebras of $\mathfrak{g}/I$.

In Chapter 4, we looked at commutative post-Lie algebra structures and λ-SPA-structures. We have partially generalized results for CPA-structures to λ-SPA-structures. However, we have not studied λ-SPA-structures on nilpotent Lie algebras. We have also not looked at Lie algebras over fields of characteristic $p > 0$. It is also interesting to look at this more closely and generalize here the results from [41].

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