MASTERARBEIT | MASTER'S THESIS

Titel | Title

Local rigidity of an embedding from the real ellipsoid in two-dimensional complex space into the sphere in three-dimensional complex space

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Abstract

Das reelle Ellipsoid im zweidimensionalen komplexen Raum kann holomorph in die Sphäre im dreidimensionalen komplexen Raum eingebettet werden. Die Existenz einer solchen Einbettung wirft die Frage der Eindeutigkeit auf. Nach einer Einführung in die Konzepte der Funktionentheorie mehrerer Veränderlicher und der CR-Geometrie wird in dieser Masterarbeit die lokale Starrheit der Einbettung in einem speziellen Fall durch die Berechnung infinitesimaler Deformationen bewiesen. Dadurch wird gezeigt, dass die Einbettung bis auf Automorphismen des Definitionsbereichs und des Zielbereichs lokal eindeutig ist.

Abstract

The real ellipsoid in two-dimensional complex space can be holomorphically embedded into the sphere in three-dimensional complex space. The existence of this embedding raises the question of its uniqueness. In this master's thesis, after an introduction to the theory of Several Complex Variables and CR Geometry, we will prove the local rigidity of this mapping in a special case by computing its infinitesimal deformations. This means that the embedding is locally unique up to automorphisms of the domain and the target space.

Contents

1 Introduction

The study of embeddings between manifolds is a central topic in differential geometry and complex analysis. Embeddings provide insights into the intrinsic and extrinsic properties of manifolds.

Poincaré [15] set the question of whether, for two given real-analytic hypersurfaces in $\mathbb{C}^2$, one can find holomorphic mappings that send one into the other, giving the intuitive answer that, in the case of biholomorphisms, it is not always possible to find such maps.

In the case of maps $H : \mathbb{C}^2 \to \mathbb{C}^3$, this question regarding hyperquadrics $\mathbb{S}_k^N \subset \mathbb{C}^N$, i.e. hypersurfaces of the form

$$\mathbb{S}_k^N = \left\{(z_1, \cdots, z_N) \in \mathbb{C}^N : |z_1|^2 + \cdots |z_k|^2 - |z_{k+1}|^2 - \cdots |z_N|^2 = 1\right\},$$

has been answered by Faran and Lebl.

Faran [10] classified the non-constant holomorphic mappings $H : \mathbb{S}^2 \to \mathbb{S}^3$ into exactly four equivalence classes of maps, up to automorphisms of the domain and the target space:

(i) $H_1(z, w) = (z, w, 0)$

(ii) $H_2(z, w) = (z, zw, w^2)$

(iii) $H_3(z, w) = (z^2, \sqrt{2}zw, w^2)$

(iv) $H_4(z, w) = (z^3, \sqrt{3}zw, w^3)$.

Lebl [13] classified the non-constant holomorphic mappings $H : \mathbb{S}^2 \to \mathbb{S}_2^3$, showing that they are equivalent to one of the following families of maps:

(i) $H_1(z, w) = (z, w, 0)$

(ii) $H_2(z, w) = (z^2, \sqrt{2}w, w^2)$

(iii) $H_3(z, w) = \left(\frac{1}{z}, \frac{w^2}{z^2}, \frac{w}{z^2}\right)$

(iv) $H_4(z, w) = \frac{1}{w^2+z+\sqrt{3}w-1}\left(z^2 + \sqrt{3}zw + w^2 - z, w^2 + z - \sqrt{3}w - 1, z^2 - \sqrt{3}zw + w^2 - z\right)$

(v) $H_5(z, w) = \frac{1}{w^2+\sqrt{2}iw+1}\left(\sqrt[4]{2}(zw - iz), w^2 - \sqrt{2}iw + 1, \sqrt[4]{2}(zw + iz)\right)$

(vi) $H_6(z, w) = \frac{1}{3z^2+1}\left(2w^3, z(z^2 + 3), \sqrt{3}w(z^2 - 1)\right)$

(vii) $H_7(z, w) = (1, l(z, w), l(z, w))$, for an arbitrary non-constant holomorphic function $l : \mathbb{C}^2 \to \mathbb{C}$.

Reiter [16] gave another proof of the above results using the same techniques for both embeddings. In [17], Reiter and Son determined all CR maps from the unit sphere $\mathbb{S}^2 \in \mathbb{C}^2$ into the tube $\mathcal{T} \times i\mathbb{R}^2 \subset \mathbb{C}^3$, where $\mathcal{C} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^3, x_3 > 0\}$, providing five equivalence classes of maps.

Regarding hyperboloids, Oh [14] showed that all general real ellipsoids in $\mathbb{C}^N$ of the form

$$\rho(z_1, \cdots, z_{N-1}, w) = \sum_{j=1}^{N-1} |z_j|^2 + \sum_{j=1}^{N} a_j\left(z_j^2 + \bar{z}_j^2\right) + |w|^2 + a(w^2 + \bar{w}^2) - 1$$

for $a_j > -\frac{1}{2}$ and $a < \frac{1}{2}$ are locally CR-embeddable into the sphere with CR co-dimension 1.

Furthermore, by the same paper [14], the hyperboloids of the form

$$\rho(z_1, \cdots, z_{N-1}, w) = \sum_{j=1}^{N-1} |z_j|^2 + \sum_{j=1}^{N} a_j\left(z_j^2 + \bar{z}_j^2\right) - |w|^2 - a(w^2 + \bar{w}^2) + 1,$$

are locally, around $z = (z_1, \cdots, z_{N-1}) = 0$, CR-embeddable into spheres with a positive CR-codimension $< N - 2$ if and only if $a_1 = a_2 = \cdots = a_{N-1} = a = 0$.

Oh provided an explicit embedding for the case of the real ellipsoid. For $N = 2$ the embedding can be expressed as a map $F : \mathcal{E} \to \mathbb{H}^3$ from the real ellipsoid

$$\mathcal{E} : |z|^2 + |w|^2 + a(z^2 + \bar{z}^2) + b(w^2 + \bar{w}^2) = 1 \subset \mathbb{C}^2$$

into the Heisenberg hypersurface $\mathbb{H}^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Im} z_3 = |z_1|^2 + |z_2|^2\}$ with

$$F(z,w) = (z, w, i - 2iaz^2 - 2ibw^2).$$

Using the inverse Cayley Transform, it can be expressed as a map

$$H : \mathcal{E} \to \mathbb{S}^3$$

$$H(z,w) = \left(\frac{z}{1 - az^2 - bw^2}, \frac{w}{1 - az^2 - bw^2}, \frac{az^2 + bw^2}{1 - az^2 - bw^2} \right),$$

which is a holomorphic, CR and 2-nondegenerate around every point $(z,w) \in \mathcal{E}$.

The existence of such an embedding raises the question of whether this map is also unique, up to automorphisms of $\mathcal{E}$ and $\mathbb{S}^3$. In this master's thesis, we will show that this embedding is locally rigid in the special case of real ellipsoids with $a = 0$ or $b = 0$. In other words, close enough to the above map H, there is no other holomorphic map.

In order to show the above result, Chapters 2 and 3 provide basic theory about functions of Several Complex Variables and Cauchy-Riemann Geometry, as well as results that will be used afterwards to prove rigidity of the map.

In Chapter 4, after an analysis of the domain, the target space, and the embedding, we will compute the Segre varieties of the real ellipsoid around the point $(0,1)$ for the special case $b = 0$. The local rigidity will be proven by comparing the vector spaces $\mathfrak{aut}(H)$ and $\mathfrak{hol}(H)$.

For the description of the infinitesimal deformations $\mathfrak{hol}(H)$, we will first find their formulas along the first and second Segre varieties. Since the second Segre variety is generically of full rank, this will allow us to generalize them as vector fields with domain the entire ellipsoid. To simplify their formulas, we will use some conditions imposed by the Weierstrass Division Theorem to ensure the holomorphicity of the vector fields' entries. This will result in analytic expressions of the vector fields in $\mathfrak{hol}(H)$.

As a final step, we will use the description of $\mathfrak{hol}(\mathbb{S}^N)$ from [12] in the case $N = 3$ to characterize the vector space $\mathfrak{aut}(H)$. The vector space equality $\mathfrak{aut}(H) = \mathfrak{hol}(H)$ will prove the local rigidity of the map, showing that the map H is locally unique, up to automorphisms of the domain and the target space.

The result of this master's thesis does not prove global uniqueness of the map, even in this special case. However, it indicates that the map may be unique.

2 Functions of Several Complex Variables

This section outlines some fundamental definitions and properties for the functions of several complex variables, as discussed in the book [4] and the lecture notes [12].

The $N-$dimensional complex space $\mathbb{C}^N$ is the vector space

$$\{z = (z_1, \cdots, z_N) : z_j \in \mathbb{C}\},$$

where each z_j is of the form $z_j = x_j + iy_j$ with $x_j := \operatorname{Re} z_j$ and $y_j := \operatorname{Im} z_j$ being real numbers. The complex conjugate of $z \in \mathbb{C}^N$ is defined componentwise as $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_N)$ with $\bar{z}_j = x_j - iy_j$. The space $\mathbb{C}^N$ is a complex vector space with the usual addition and the (complex) scalar multiplication.

We can define a norm in $\mathbb{C}^N$ by

$$\|z\|= \sum_{j=1}^{N} |z_j|^2.$$

The vector space $\mathbb{C}^N$ becomes an inner product space with inner product

$$\langle z, w\rangle = \sum_{j=1}^{N} z_j \bar{w}_j.$$

We can identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ using the map

$$(z_1, \cdots, z_N) \mapsto (x_1, y_1, \cdots, x_N, y_N)$$

where $x_j = \operatorname{Re} z_j = \frac{z_j + \bar{z}_j}{2}$ and $y_j = \operatorname{Im} z_j = \frac{z_j - \bar{z}_j}{2}$ for $j = 1, \cdots, N$.

Hence, we can write a function $f(z)$ defined on a subset of $\mathbb{C}^N$ as $f(x, y)$ with $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$ or, by abuse of notation, as $f(z, \bar{z})$.

2.1 Complexification of vector spaces

Consider a real vector space V equipped with the basis $\{v_1, \ldots, v_n\}$. The complexification of V, denoted as $\mathbb{C}V = \mathbb{C} \otimes V$, consists of all complex linear combinations of the basis vectors

$$\mathbb{C}V = \left\{ \sum_{j=1}^{n} a_j v_j : a_j \in \mathbb{C} \right\}.$$

$\mathbb{C}V$ forms a complex vector space with the standard operations of vector addition and scalar multiplication by complex numbers.

Furthermore, V can be viewed as isomorphic to the real subspace $1 \otimes V$, and $\mathbb{C}V$ can be decomposed as $\mathbb{C}V = V \oplus iV$.

2.2 Implicit function theorem

Definition 1. *A formal power series $A : \mathbb{C}^N \to \mathbb{C}$ centered at $p \in \mathbb{C}^N$ is an expression of the form*

$$A(z) = \sum_{\alpha \in \mathbb{N}^N} A_\alpha (z - p)^\alpha$$

with coefficients $A_\alpha \in \mathbb{C}$, where $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $w^\alpha = (w_1^{\alpha_1}, \cdots, w_N^{\alpha_N})$ for $w \in \mathbb{C}^N$. The set of all formal power series centered at $p \in \mathbb{C}^N$ is denoted by $\mathbb{C}[[z - p]]$.

Definition 2. *A formal power series map $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ is a tuple $H = (H_1, \cdots, H_{N'}) \in \mathbb{C}[[z - p]]^{N'}$.*

Given $p \in \mathbb{C}^N$ and formal maps $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ with $H(p) = p'$ and $G : \mathbb{C}^{N'} \to \mathbb{C}^M$ with $G(p') = q$ their composition $G \circ H : \mathbb{C}$ is well defined as the formal power series map

$$(G \circ H)(z) = \sum_{\beta \in \mathbb{N}^{N'}} G_\beta \left(H(z) - p'\right)^\beta .$$

Differentiation is well-defined in $\mathbb{C}[[z-p]]$. For the partial derivatives of a formal map $A \in \mathbb{C}[[z-p]]$, we have

$$\frac{\partial^{|\beta|}}{\partial z^\beta} A(z) = \sum_{\gamma \in \mathbb{N}^N} \frac{(\gamma+\beta)\,!}{\gamma!} A_{\gamma+\beta}(z-p)^\gamma .$$

For convenience, we will also use the notation $A_{z^\beta}(p) = \frac{\partial^{|\beta|}}{\partial z^\beta} A(p)$.

Theorem 1. *Let $A : \mathbb{C}^N \times \mathbb{C}^M \to \mathbb{C}^M$ be a formal map, $(z_0, w_0) \in \mathbb{C}^N \times \mathbb{C}^M$ such that $A(z_0, w_0) = 0$ and $\det A_w(z_0, w_0) \neq 0$, where*

$$A_w = \begin{bmatrix} A^1_{w_1} & \cdots & A^1_{w_M} \\ \vdots & & \vdots \\ A^M_{w_1} & \cdots & A^M_{w_M} \end{bmatrix} .$$

Then, there exists a unique formal map $\phi : \mathbb{C}^N \to \mathbb{C}^M$, such that $A\left(z, \phi(z)\right) = 0$ and $\phi(z_0) = w_0$.

2.3 Holomorphic and real-analytic functions

Let $\Omega \subset \mathbb{C}^N$ be an open set, $f : \Omega \to \mathbb{C}$ a smooth function and $p \in \Omega$. For $z = (z_1, \cdots, z_N) \in \mathbb{C}^N$, we write $z_j = x_j + iy_j$ and define the differential df of f as

$$df(p) = \sum_{j=1}^N \frac{\partial f}{\partial x_j}(p) dx_j + \frac{\partial f}{\partial y_j}(p) dy_j .$$

We use the complex differentials

$$dz_j = dx_j + idy_j \text{ and } d\bar{z}_j = dx_j - idy_j$$

and the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) .$$

Under these notations, we write

$$df(p) = \sum_{j=1}^N \frac{\partial f}{\partial z_j}(p, \bar{p}) dz_j + \sum_{j=1}^N \frac{\partial f}{\partial \bar{z}_j}(p, \bar{p}) d\bar{z}_j$$

and

$$\partial f(p) = \sum_{j=1}^N \frac{\partial f}{\partial z_j}(p, \bar{p}) dz_j \text{ and } \bar{\partial} f(p) = \sum_{j=1}^N \frac{\partial f}{\partial \bar{z}_j}(p, \bar{p}) d\bar{z}_j .$$

Hence $df(p) = \partial f(p) + \bar{\partial} f(p)$ for all $p \in \mathbb{C}^N$.

Definition 3. *Let $\Omega \subset \mathbb{C}^N$ be an open set. A smooth function $f : \Omega \to \mathbb{C}$ is holomorphic if $\bar{\partial} f = 0$ on Ω. The space of holomorphic functions is denoted by $\mathcal{H}(\Omega)$.*

Definition 4. *Let $U \subset \mathbb{R}^k$ be open. A function $f : U \to \mathbb{C}$ is real-analytic if, for every $p \in U$, there is a neighbourhood V of $U \subset \mathbb{C}^k$ and a holomorphic function $f : V \to \mathbb{C}$, such that $F|_U = f$.*

Any real analytic function $f : U \to \mathbb{C}$ can be written as a power series in $(z, \bar{z})$ as

$$f(z,\bar{z}) = \sum_{\alpha,\beta\in\mathbb{N}^k} f_{\alpha,\beta}(z-p)^{\alpha}(\bar{z}-\bar{p})^{\beta},$$

for any $p \in U$.

Definition 5. *Let $p \in \mathbb{C}^N$ and $f : \mathbb{C}^N \to \mathbb{C}^{N'}$ be a holomorphic map. The $k-$th jet mapping j_p^k is defined as*

$$j_p^k : \mathcal{H}(\mathbb{C}^N, \mathbb{C}^{N'}) \to \mathbb{C}^{N' r(N,k)}$$

$$j_p^k f := \left(\frac{\partial^{|\alpha|} f}{\partial^{\alpha} z}(p) : |\alpha| \leq k \right),$$

where r is a rational function of N and k.

2.4 Holomorphic Implicit Function Theorem

Let $f : U \subset \mathbb{C}^N \to \mathbb{C}^N$ be a holomorphic function. For $z \in U$, consider $w = f(z) \in \mathbb{C}^N$. We can decompose both z and w into real and imaginary parts $z = x + iy$ and $w = u + iv$, respectively, and consider the real map $f_R : U' \subset \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ associated to f such that $f_R(x, y) = (u, v)$.

The holomorphicity of f translates, by Definition 3, to the equations $\frac{\partial f}{\partial \bar{z}_j} = 0$ for $j = 1, \cdots, N$. Writing $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$, f is holomorphic if and only if the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \text{ and } \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

hold for $j = 1, \cdots, N$.

Proposition 1. *If we denote by J_f the (complex) Jacobian of f, and by J_{f_R} the (real) Jacobian of f_R, then*

$$\det(J_{f_R}) = |\det(J_f)|^2$$

Proof. With help of the Cauchy-Riemann equations written in the matrix form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we have

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Hence, using properties of the determinant and the Cauchy-Riemann equations

$$\begin{aligned}
\det(J_{f'}) &= \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} - i\frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} \\
&= \det\begin{bmatrix} \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \\ 0 & \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial f}{\partial z} & \frac{\partial u}{\partial y} \\ 0 & \overline{\frac{\partial f}{\partial z}} \end{bmatrix} = \det\left(\frac{\partial f}{\partial z}\right)\det\left(\overline{\frac{\partial f}{\partial z}}\right) \\
&= \det(J_f)\overline{\det(J_f)} = |\det(J_f)|^2.
\end{aligned}$$

□

Theorem 2. *Let $f : U_1 \times V_1 \to \mathbb{C}^M$ be a holomorphic function, where $U_1 \subset \mathbb{C}^N$ and $V_1 \subset \mathbb{C}^M$ are open sets and $(z_0, w_0) \in U_1 \times V_1$ such that $f(z_0, w_0) = 0$. If*

$$\det \frac{\partial f}{\partial w}(z_0, w_0) \neq 0,$$

then there exist open neighbourhoods $U_2 \subset U_1$ of z and $V_2 \subset V_1$ of w and a holomorphic map $\phi : U_2 \to V_2$ that satisfies $f(z, \phi(z)) = 0$.

Proof. We can consider the holomorphic function $g : \mathbb{C}^M \to \mathbb{C}^M$ with $g_z(w) = f(z, w)$. By the assumption of the theorem, $\det J_{g_z} \neq 0$. Using Proposition 1 and the considerations at the beginning of this subsection for the associated real map g'_z of $g_z, \det(J_{g'_z}) \neq 0$. Hence, we can deduce all the conclusions from the Implicit Function Theorem for real functions, except for the holomorphicity of f.

Thus, there is a function $\phi : U_2 \to V_2$ such that $f(z, \phi(z)) = 0$. Therefore, for $j = 1, \cdots, M$ and $k = 1, \cdots, N$:

$$0 = \frac{\partial f_j}{\partial \bar{z}_k} = \sum_{l=1}^{N} \frac{\partial f_j}{\partial z_l} \frac{\partial z_l}{\partial \bar{z}_k} + \sum_{l=1}^{M} \frac{\partial f_j}{\partial w_l} \frac{\partial \phi_l}{\partial \bar{z}_k} = \sum_{l=1}^{M} \frac{\partial f_j}{\partial w_l} \frac{\partial \phi_l}{\partial \bar{z}_k}$$

Hence, in the form of matrices

$$0 = \left(\frac{\partial g_z}{\partial w}\right)\left(\frac{\partial \phi}{\partial \bar{z}}\right).$$

It holds that $\det\left(\frac{\partial g_z}{\partial w}\right) \neq 0$ in a neighbourhood of (z, w). Without loss of generality, after intersecting with $U_2 \times V_2$, we may assume that this neighbourhood is $U_2 \times V_2$. Multiplying the equality by $\left(\frac{\partial g_z}{\partial w}\right)^{-1}$, we obtain that

$$\frac{\partial \phi}{\partial \bar{z}} = 0.$$

Thus, ϕ is holomorphic on U_2. □

2.5 The Weierstrass Division Theorem

Definition 6. *Let g be a holomorphic function in a neighbourhood of the origin $0 \in \mathbb{C}^N$. We say that g is $k-$regular in the z_N direction if the function $z_N \mapsto g(0, \cdots, 0, z_N)$ has a zero of order k at the origin.*

Theorem 3. *Let f, g be holomorphic functions near 0 with g being $k-$regular in the z_N direction. Then, there exist a neighbourhood of the origin and unique holomorphic functions q, r, where r is a polynomial in z_N of degree at most $k-1$ such that*

$$f(z) = g(z)q(z) + r(z).$$

A proof of the Weierstrass Division Theorem, in a more general version, can be found in [11].

3 CR Geometry

This section is an introduction to complex submanifolds of $\mathbb{C}^N$ and their basic properties, as discussed in the book [4] and the lecture notes [12].

3.1 Real Submanifolds of the complex space and their tangent spaces

Definition 7. *A (smooth) real submanifold of $\mathbb{C}^N$ of codimension d is a subset M of $\mathbb{C}^N$ such that for every point $p \in M$, there is a neighbourhood U of p and a smooth real valued function $\rho = (\rho_1, \cdots, \rho_d)$ defined in U such that*

$$M \cap U = \{Z \in U : \rho(z, \bar{z}) = 0\}$$

with differentials $d\rho_1, \cdots, d\rho_d$ linearly independent in U. Such a function ρ is called local defining function for M near p.

Definition 8. *A real hypersurface of $\mathbb{C}^N$ is a real submanifold of $\mathbb{C}^N$ of codimension 1. The hypersurface is called real-analytic if the defining function ρ can be chosen to be real-analytic.*

Proposition 2. *Let $M \subset \mathbb{C}^N$ be a real hypersurface, $p \in M$, and ρ_1 and ρ_2 two defining functions for M near p. Then, there is a nonvanishing real-valued smooth function a defined in a neighbourhood of p such that $\rho_1 = a\rho_2$.*

Proof. Since $d\rho_2(x) \neq 0$ on M in a neighbourhood of p, after an affine change of coordinates, we may assume $p = 0$ and $\frac{\partial \rho}{\partial x_1}(0) \neq 0$. Then, setting $x_1' = \rho_2, x_j' = x_j, j = 2, \cdots, N$ and $y_j' = y_j, j = 1, \cdots, N$, the defining function ρ_2 takes the form $\rho_2(x) = x_1$ in a neighbourhood of p.

Let $x' = (y_1, x_2, y_2, \cdots, x_N, y_N)$. Then $\rho_1(0, x') = 0$, and using the Fundamental Theorem of Calculus, we obtain

$$\rho_1(x_1, x') = \rho_1(x_1, x') - \rho_1(0, x') = x_1 \int_0^1 \frac{\partial \rho_1}{\partial x_1}(tx_1, x')dt,$$

which implies that $\rho_1 = a\rho_2$ for some smooth function a.

Finally, $d\rho_1(x) = h(x)d\rho_2(x)$ for all $x \in \mathbb{R}^N \cong \mathbb{C}^N$, and since both $d\rho_1(x), d\rho_2(x)$ are nonzero in a neighbourhood of p, we conclude that $h(x)$ does not vanish in this neighbourhood. $\square$

For $p \in \mathbb{C}^N$, we define the real tangent space to $\mathbb{C}^N$ at p as

$$T_p\mathbb{C}^N = T_p\mathbb{R}^{2N} = \left\{ X = \sum_{j=1}^N a_j \frac{\partial}{\partial x_j}\bigg|_p + b_j \frac{\partial}{\partial y_j}\bigg|_p : a_j, b_j \in \mathbb{R} \right\}$$

It is a real vector space of dimension $2N$. If we take as basis of $T_p\mathbb{C}^N$ the vectors

$$\frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_N}\bigg|_p, \frac{\partial}{\partial y_1}\bigg|_p, \cdots, \frac{\partial}{\partial y_N}\bigg|_p$$

and introduce a real linear mapping J from $T_p\mathbb{C}^N$ into $T_p\mathbb{C}^N$ defined by

$$J\left(\frac{\partial}{\partial x_j}\bigg|_p\right) = \frac{\partial}{\partial y_j}\bigg|_p \text{ and } J\left(\frac{\partial}{\partial y_j}\bigg|_p\right) = -\frac{\partial}{\partial x_j}\bigg|_p, \ j = 1, \cdots, N,$$

$T_p\mathbb{C}^N$ becomes a complex vector space of dimension N, denoted by $T_p^c\mathbb{C}^N$. By linearity, J can be extended as a $\mathbb{C}$ linear operator from $\mathbb{C}T_p\mathbb{C}^N$ onto $\mathbb{C}T_p\mathbb{C}^N$, denoted again by J.

Definition 9. *Let M be a smooth real submanifold of $\mathbb{C}^N$. Given $p \in M$, an element*

$$X = \sum_{j=1}^N a_j \frac{\partial}{\partial x_j}\bigg|_p + b_j \frac{\partial}{\partial y_j}\bigg|_p \in T_p\mathbb{C}^N$$

is called tangent to M at p if

$$\sum_{j=1}^N a_j \frac{\partial \rho_k}{\partial x_j}(p, \bar{p}) + b_j \frac{\partial \rho_k}{\partial y_j}(p, \bar{p}) = 0, \ k = 1, \cdots, d$$

for a local defining function $\rho = (\rho_1, \cdots, \rho_d)$ for M near p. By Proposition 2, this definition is independent of the choice of the defining function ρ.

The tangent space of M at p, denoted by T_pM, is the space of all real vectors tangent to M at p.

The complex tangent spaces $\mathbb{C}T_p\mathbb{C}^N$ and $\mathbb{C}T_pM$ are defined similarly by allowing the coefficients a_i and b_i in the above expressions to be complex numbers.

Any $X \in \mathbb{C}T_p\mathbb{C}^N$ can be written uniquely in the form

$$X = \sum_{j=1}^{N} a_j \frac{\partial}{\partial z_j}\bigg|_p + b_j \frac{\partial}{\partial \bar{z}_j}\bigg|_p, a_j, b_j \in \mathbb{C}.$$

Definition 10. *A tangent vector given by the above formula is called $(0,1)$ vector if $b_j = 0$ for $j = 1, \cdots, N$ and $(1,0)$ vector if $a_j = 0$ for $j = 1, \cdots, N$. For $p \in M$ we denote by $T_p^{0,1}\mathbb{C}^N$ the space of $(0,1)$ tangent vectors and by $T_p^{1,0}\mathbb{C}^N$ the space of $(1,0)$ tangent vectors.*

By the chain rule, this definition is independent of holomorphic coordinates. We have

$$\dim_{\mathbb{C}} T_p^{0,1}\mathbb{C}^N = \dim_{\mathbb{C}} T_p^{1,0}\mathbb{C}^N = N.$$

For $p \in M$ we denote by $\mathcal{V}_p$ the space of $(0,1)$ vectors tangent to M at p, i.e.

$$\mathcal{V}_p := T_p^{0,1}M := T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM.$$

If ρ is a local defining function for M, then

$$\dim_{\mathbb{C}} \mathcal{V}_p = N - \operatorname{rank}_{\mathbb{C}} \left(\frac{\partial \rho_k}{\partial \bar{z}_i}(p, \bar{p}) \right)_{1 \le j \le N, 1 \le k \le d}. \tag{3.1}$$

The above is independent of the choice of the holomorphic coordinates in $\mathbb{C}^N$ and the defining function ρ.

3.2 Cauchy Riemann, totally real and generic submanifolds

Definition 11. *A real submanifold $M \subset \mathbb{C}^N$ is called Cauchy Riemann (CR) if $dim\mathcal{V}_p$ is constant for all $p \in M$. This constant dimension $dim\mathcal{V}_p$ is called the CR dimension of M.*

The complex subbundle $\mathcal{V} \subset \mathbb{C}TM$, whose fiber at $p \in M$ is $\mathcal{V}_p$, is called the CR bundle of M.

Definition 12. *A CR submanifold $M \subset \mathbb{C}^N$ with $dim\mathcal{V}_p = 0$ is called totally real.*

Equivalently, M is totally real if the dimension of the complex tangent spaces $dim_{\mathbb{R}}T_p^cM = 0$ or if

$$T_pM \cap J(T_pM) = \{0\}$$

for $p \in M$.

Definition 13. *A real submanifold $M \subset \mathbb{C}^N$ is generic if near every point $p \in M$, there is a local defining function $\rho = (\rho_1, \cdots, \rho_d)$ such that the complex differentials $\partial\rho_1, \cdots, \partial\rho_d$ are $\mathbb{C}-$linearly independent near p.*

By (3.1), it follows that the CR dimension of a generic submanifold of codimension d is $n = N - d$.

Definition 14. *A real submanifold $M \subset \mathbb{C}^N$ is called maximally totally real if it is totally real and generic.*

For generic submanifolds one can find special coordinates near every point, as the following proposition shows:

Proposition 3. *Let $M \subset \mathbb{C}^N$ be a generic submanifold of codimension d through $p \in M$. Then, there exist holomorphic coordinates (z, w) near p, vanishing at p, with $z \in \mathbb{C}^{N-d}$ and $w \in \mathbb{C}^d$, and a real valued smooth function ϕ defined near 0 in $\mathbb{R}^{2N-d}$ with $\phi(0) = 0$ and $d\phi(0) = 0$ such that near p, M is given by*

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w).$$

Proof. Let $\rho(z,\bar{z})$ be a defining function for M near p. Without loss of generality, by applying an affine change of coordinates, we may assume that $p=0$ and that for $k=1,\cdots,d$, $\frac{\partial\rho_k}{\partial y_N}(0)\neq 0$. For $j=1,\cdots,d$, the Taylor expansions of ρ_k around 0 can be written as

$$\rho_k(z',\bar{z'})=\sum_{j=1}^{N}(a_{kj}x_{kj}+b_{kj}y_{kj})+O(2),$$

where $O(2)$ vanishes of order at least 2 at 0. Since every ρ_k is real valued, so are its first and second derivatives. Thus, $a_{kj},b_{kj}\in\mathbb{R}$. We can write

$$\sum_{j=1}^{N}(a_{kj}x_{kj}+b_{kj}y_{kj})=\operatorname{Im}\left(\sum_{j=1}^{N}c_{kj}z'_j\right)$$

with $c_{kj}=b_{kj}+ia_{kj}$.

We take the linear holomorphic change of coordinates $z_j=z'_j$ for $j=1,\cdots,N-d$ and $w_k=\sum_{j=1}^{N}c_{kj}z'_j$ for $k=1,\cdots,d$. A $d-$tuple of defining equations for M near $(0,0)$ is given by solving each of the equations $\rho_k(z,\bar{z})=0$ with respect to $\operatorname{Im}w_k$. This can be done by use of the Impicit Function Theorem, since for $k=1,\cdots,d$,

$$\rho_k=\operatorname{Im}w_k+O(2).$$

The properties $\phi(0)=0$ and $d\phi(0)=0$ are immediate. The equation $\operatorname{Im}w=\phi(z,\bar{z},\operatorname{Re}w)$ has the same solutions as $\rho=(\rho_1,\cdots,\rho_d)=0$ near $(0,0)$ and hence is a defining function for M near p. □

Definition 15. *Let $M\subset\mathbb{C}^N$ be a CR submanifold and $\mathcal{V}$ be its CR bundle. We call a vector field L on M a CR vector field if, for every $p\in M$, L_p is an element of $\mathcal{V}_p$.*

If M is of codimension d and L is a CR vector field on M, we may write

$$L=\sum_{j=1}^{N}a_j(p)\frac{\partial}{\partial\bar{z}_j},$$

where a_j are smooth functions on M and

$$\sum_{j=1}^{N}a_j(p)\frac{\partial\rho_k}{\partial\bar{z}_j}(p)=0,\ 1\leq k\leq d,$$

for all local defining functions ρ on M.

Definition 16. *Let M be a CR submanifold. We say that $f\in C^k(M), k\geq 1$, is a CR function if it is annihilated by any CR vector field L on M.*

Definition 17. *Let $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ be submanifolds. We say that a mapping $H:\mathbb{C}^N\to\mathbb{C}^{N'}$ is a CR map if $H\in C^k(\mathbb{C}^N)$ and for all $p\in M$,*

$$H_*(\mathcal{V}_p)\subset\mathcal{V}'_{H(p)},$$

where H_* denotes the pushforward $H_*:T_pM\to T_{H(p)}M'$ induced by H.

Proposition 4. *Let $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ be CR submanifolds. If $H:M\to M'$ is a C^k mapping, $k\geq 1$, with components $H=(H_1,\cdots H_{N'})$, then H is a CR mapping if and only if each component $H_j, j=1,\cdots N'$ is a CR function.*

A proof can be found in section 2.3 of [4].

Definition 18. *Let $M \subset \mathbb{C}^N$ be a smooth hypersurface, $p \in M$, and $\rho(z, \bar{z})$ a local defining function for M in an open neighbourhood U of p. Then, M is strictly pseudoconvex at p if either*

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p, \bar{p}) a_j \bar{a}_k > 0$$

or

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p, \bar{p}) a_j \bar{a}_k < 0$$

for all $a \in \mathbb{C}^N \setminus \{0\}$ with $\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p, \bar{p}) a_j = 0$. The above form is called Levi form.

Definition 19. *Let $M \subset \mathbb{C}^N$ be a submanifold and $p \in M$. M is called Levi-nondegenerate at p if the Levi form is nondegenerate at p.*

Definition 20. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic real hypersurfaces and $U \subset \mathbb{C}^N$ a neighbourhood of $p \in M$. A holomorphic mapping $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ with $H(U \cap M) \subset M'$ is called transversal to M' at $H(p)$ if*

$$T_{H(p)} M' + dH(T_p \mathbb{C}^N) = T_{H(p)} \mathbb{C}^{N'}.$$

Definition 21. *A CR submanifold $M \subset \mathbb{C}^N$ is called minimal at $p \in M$ if there is no germ of a CR submanifold $\tilde{M} \subsetneq M$ in $\mathbb{C}^N$ through p with the same CR dimension as M at p.*

In [1] the following criterion for minimality is proven:

Proposition 5. *Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold of CR dimension n and codimension d, and let $p \in M$. Then M is minimal at p if and only if the generic dimension of the maximal Segre set of M at p is $n + d$. In particular, if M is generic, then M is minimal at p if and only if the above generic dimension is N.*

Another proof of the above result in a more algebraic way can be found in [5].

3.3 Degeneracy of mappings

The following definitions and notations can be found in [16].

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and an $n-$tuple of vector fields $L = (L_1, \cdots, L_n)$, we define

$$L^\alpha := L_1^{\alpha_1} \cdots L_n^{\alpha_n},$$

with $L_j^0 = \mathrm{Id}$. Furthermore, we use the notation $\frac{\partial \rho_j}{\partial z}$ for the vector-valued function

$$\frac{\partial \rho_j}{\partial z}(z, \bar{z}) = \left(\frac{\partial \rho_j}{\partial z_1}(z, \bar{z}), \cdots, \frac{\partial \rho_j}{\partial z_N}(z, \bar{z}) \right).$$

Definition 22. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be generic, real-analytic submanifolds of codimensions d and d' respectively, and denote $n = N - d$ and $n' = N' - d'$. For $p \in M, p' \in M'$ and a neighbourhood $U \subset \mathbb{C}^N$ of p, let $H : U \to \mathbb{C}^{N'}$ be a holomorphic mapping satisfying $H(U \cap M) \subset M'$. Let ρ and ρ' be local defining functions near $p \in M$ and $p' \in M'$, respectively. Write $L_1, \cdots, L_n$ for a basis of CR vector fields of M. Then, after possibly shrinking U, we define for $k \geq 0$ and $q \in U$ the following subspaces of $\mathbb{C}^{N'}$*

$$E'_k(q) = span_{\mathbb{C}} \left\{ L^\alpha \frac{\partial \rho'_j}{\partial z'} \left(H(z), \bar{H}(\bar{z}) \right) \Big|_{z=q} : 0 \leq |\alpha| \leq k, 1 \leq j \leq d' \right\} \subset \mathbb{C}^{N'}.$$

Since for $k \geq 0$ the $E'_k(q)$ form an ascending chain of vector spaces in $\mathbb{C}^{N'}$, there exists a minimal $k_0 \geq 0$ such that $E'_k(p) = E'_{k_0}(p)$ for all $k \geq k_0$ and $E'_{k_0 - 1}(p) \subsetneq E'_{k_0}(p)$ for $p \in U$.

The number

$$s(q) = N' - \dim_{\mathbb{C}} E'_{k_0}(q)$$

is called the degeneracy of H at q, and H is called $(k_0, s(q)) -$ degenerate at $q \in M$.

If $s = s(p)$ is constant for $p \in U$, we say H is constantly $(k_0, s)-$degenerate near $q \in M$ and s is called constant degeneracy of H.

If for some $q \in M$ we have $s(q) = 0$, then $E'_{k_0}(q) = \mathbb{C}^{N'}$ and H is called k_0-nondegenerate.

Definition 23. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be generic, real-analytic submanifolds, and let $U \subset \mathbb{C}^N$ be a neighbourhood of $p \in M$. Let $H : U \to \mathbb{C}^{N'}$ be a holomorphic mapping satisfying $H(U \cap M) \subset M'$. Fix a neighbourhood $V \subset U$ of $p \in M$ such that $\overline{V \cap M} \subset U$. The number*

$$s_H(V) = \min \left\{ s(q) : q \in \overline{V \cap M} \right\}$$

is called generic degeneracy of H in a neighbourhood $V \subset \mathbb{C}^N$ of $p \in M$.

3.4 Segre varieties and Segre maps

Definition 24. *Let $M \subset \mathbb{C}^N$ be a generic, real-analytic submanifold with local defining function $\rho\,(z, \bar{z})$ such that the rank of $\rho_z(z, \bar{z})$ remains maximal (i.e. $= d$) throughout $U \times \bar{U}$ and*

$$M \cap U = \{z \in U : \rho(z, \bar{z}) = 0\}.$$

Then, for $p \in U$, the (first iterated) Segre variety of p is defined as

$$\mathcal{S}_p(U) = \mathcal{S}^1_p(U) = \{z \in U : \rho(z, \bar{p}) = 0\}\,.$$

It is typical to omit the neighbourhood U from the notation and assume that the statements hold in a neighbourhood around the point p.

Given a holomorphic function $h(z)$, we use the notation $\bar{h}(z) := \overline{h(\bar{z})}$.

Lemma 1. *With the notation introduced in the above definition, let $p, q \in U$. Then*

(i) $q \in \mathcal{S}_p \Leftrightarrow p \in \mathcal{S}_q$

(ii) $p \in \mathcal{S}_p \Leftrightarrow p \in M$

Proof. We first observe that since the defining function $\rho\,(z, \bar{z})$ is real-valued, it satisfies

$$\rho\,(z, \bar{z}) = \overline{\rho\,(z, \bar{z})} = \bar{\rho}\,(\,\bar{z}, z)\,.$$

Since the subspace $\{(z, \bar{z}) : z \in \mathbb{C}^N\} \subset \{(z, \zeta) : z, \zeta \in \mathbb{C}^N\}$ is totally real, the holomorphic extension $\rho(z, \zeta)$ of the defining function satisfies

$$\bar{\rho}(z, \zeta) = \rho(\zeta, z).$$

(i) Assume $q \in \mathcal{S}_p$. This means that $\rho(q, \bar{p}) = 0$, or equivalently, $\bar{\rho}(\bar{q}, p) = 0$. Using the considerations at the beginning of the proof, we have $\rho(p, \bar{q}) = 0$, or by definition, $p \in \mathcal{S}_q$.

(ii) Assume $p \in \mathcal{S}_p$. Using the definition of the Segre varieties, this means that $\rho(p, \bar{p}) = 0$, or equivalently, $p \in M$.

□

Lemma 2. *Let M, M' be generic, real-analytic submanifolds with local defining functions ρ and ρ', and open sets $U \subset M$ and $U' \subset M'$. Let $H : U \to U'$ be a holomorphic map from M into M'. Then, for $p \in U$,*

$$H(\mathcal{S}_p) \subset \mathcal{S}'_{H(p)}.$$

Proof. The assumption that H maps M into M' implies the existence of a real-analytic map A such that

$$\rho'\left(H(z), \bar{H}(\zeta)\right) = A\left(z, \zeta\right) \rho\left(z, \zeta\right).$$

If $q \in \mathcal{S}_p$, i.e. $\rho(q, \bar{p}) = 0$, then $\rho\left(H(q), \bar{H}(p)\right) = A\left(q, \bar{p}\right) \rho(q, \bar{p}) = 0$, and hence $H(q) \in \mathcal{S}_{H(p)}$. □

Definition 25. *Let $M \subset \mathbb{C}^N$ be a generic, real analytic submanifold with local defining function $\rho(z, \bar{z})$. We define the k-th iterated Segre variety as*

$$\mathcal{S}_p^k = \bigcup_{q \in \mathcal{S}_p^{k-1}} \mathcal{S}_q.$$

For a set $S \subset \mathbb{C}^N$ we define

$$^*S = \{\zeta \in \mathbb{C}^N : \bar{\zeta} \in S\}.$$

Using the above notation, for $p \in U$ an equivalent definition for the $k-$th Segre variety for $k \geq 2$ is the following

$$\mathcal{S}_p = \left\{ z \in U : \text{ there exists } \zeta \in {}^*\mathcal{S}_p^{k-1}(U) \text{ with } (z, \zeta) \in \mathbb{C}M \right\}.$$

A proof for the equivalence of the above definitions can be found in section 10.2 of [4].

Definition 26. *Let $M \subset \mathbb{C}^N$ be a real-analytic submanifold. We define the complexification $\mathcal{M}$ or $\mathbb{C}M$ of M as the submanifold $\mathcal{M} \subset \mathbb{C}^{2N}$ such that $\mathcal{M} \cap \{(z, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N : \zeta = \bar{z}\} = M$.*

One can show that M is a maximally totally real submanifold of its complexification and that $\mathcal{M}$ is unique near $\{(z, \zeta) \in \mathbb{C}^{2N} : \zeta = \bar{z}\}$.

Definition 27. *Let $M \subset \mathbb{C}^N$ be a generic, real-analytic submanifold, given in the complexified form by*

$$w = Q(z, \chi, \tau),$$

where $Q : \mathbb{C}^{2n+d} \to \mathbb{C}^d$ is a holomorphic function (cf. section 3.5). For $p \in M$ and $x_i \in \mathbb{C}^n$, we define a sequence of formal mappings $S_j : \mathbb{C}^{nj} \to \mathbb{C}^N$ called the iterated Segre mappings of M at p, as follows

$$S_1(x_1; p) = (x_1, Q(x_1, p))$$

$$S_j(x_j, x_{j-1}, \cdots, x_1; p) = (x_j, Q(x_j, \bar{S}_{j-1}(x_{j-1}, \cdots, x_1; p))),$$

for $j \geq 1$, where $\bar{S}_{j-1}$ is the power series with coefficients conjugate to the ones of S_j.

Using the above definition, the Segre varieties can be viewed as images of the Segre maps:

$$\mathcal{S}_p^{2k} = \operatorname{im}\left((z, \chi_1, z_1, \chi_2, \cdots, z_{k-1}, \chi_k) \mapsto S_{2k}(z, \chi_1, z_1, \chi_2, \cdots, z_{k-1}, \chi_k; p)\right)$$

and

$$\mathcal{S}_p^{2k+1} = \operatorname{im}\left((z, \chi_1, z_1, \chi_2, \cdots, \chi_k, z_k) \mapsto S_{2k+1}(z, \chi_1, z_1, \chi_2, \cdots, \chi_k, z_k; p)\right),$$

for $k \in \mathbb{N}$.

Theorem 4. *Let $M \subset \mathbb{C}^N$ be a smooth, generic submanifold of codimension d. Then, the rank of the iterated Segre maps $\operatorname{rank} S_j$ for $j \geq 1$ is an increasing function of j and is independent of the choice of the holomorphic coordinates $z \in \mathbb{C}^N$ and the defining function. In addition, there exists an integer $k_0, 1 \leq k_0 \leq d + 1$, such that $\operatorname{rank} S_j = \operatorname{rank} S_{j+1}$ for $j \geq k_0$. If $k_0 > 1$, then $\operatorname{rank} S_j < \operatorname{rank} S_{j+1}$ for $1 \leq j \leq k_0 - 1$.*

The proof of the above result can be found in [2], as well as in [5], where the Segre varieties and maps together with their properties are introduced in an algebraic way.

3.5 Normal Coordinates

Let $M \subset \mathbb{C}^N$ be a smooth, generic submanifold, and let $p \in M$. We will introduce special coordinates around p in which the Segre varieties are given by equations that are easier to handle.

Let n be the CR dimension of M and d the real codimension of M in $\mathbb{C}^N$. Since M is generic, we have $N = n + d$.

Definition 28. *Under the above notations, a set of holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ near $p \in M$ such that $(z, w)|_p = 0$ are called normal coordinates if there exists a real-analytic function ϕ such that in a neighbourhood of p in $\mathbb{C}^N$, M is given by*

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w)$$

with ϕ satisfying $\phi(z, 0, s) = \phi(0, \bar{z}, s) = 0$.

The equation $\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w)$ can be written in the complexified form as $w - \tau = 2i\phi\left(z, \chi, \frac{w+\tau}{2}\right)$. Applying the Implicit Function Theorem and solving for w, we obtain that $\mathcal{M}$ is given by $w = Q(z, \chi, \tau)$ for a unique holomorphic function Q.

In normal coordinates, we have that, together with $w = Q(z, \chi, \tau)$ and $\tau = \bar{Q}(\chi, z, w)$, we get $w = Q\left(z, \chi, \bar{Q}(\chi, z, w)\right)$ and $\tau = \bar{Q}\left(\chi, z, Q(z, \chi, \tau)\right)$.

Proposition 6. *Let $M \subset \mathbb{C}^{n+d}$ be a generic, real-analytic submanifold with $0 \in M$ and (z, w) any holomorphic coordinates near 0. Then the following are equivalent:*

(i) *(z, w) are normal coordinates for M*

(ii) *There exists a real-analytic defining function ρ valued in $\mathbb{R}^d$ for M near 0 such that $\rho(z, w, 0, w) = 0$.*

(iii) *Normality condition: for the complexification $\mathcal{M}$ of M, there exists a holomorphic function $Q : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}^d$ with $Q(z, 0, \tau) = \tau = Q(0, \chi, \tau)$.*

(iv) *For w_0 near 0, the Segre varieties at $(0, w_0)$ are flat, i.e., $\mathcal{S}_{(0,w_0)} = \{w = \bar{w}_0\}$.*

Furthermore, if the condition in (ii) holds for some real-analytic defining function, then it holds for any such function.

Proof. $(i) \Rightarrow (ii)$: The defining function $\rho(z, \chi, w, \tau) = w - \tau - 2i\phi(z, \chi, \frac{w+\tau}{2})$ satisfies $\rho(z, w, 0, w) = 0$.

$(ii) \Rightarrow (i)$: The condition in (ii) is equivalent to $\rho(z, 0, w, \bar{w}) = K(z, w, \bar{w})(w - \bar{w})$, for a holomorphic function K in a neighbourhood of 0 in $\mathbb{C}^{n+2d}$. Since $\partial\rho_1 \wedge \partial\rho_2 \wedge \cdots \wedge \partial\rho_d(0) \neq 0$, $K(0)$ is invertible. Applying the Implicit Function Theorem, we can solve the equation $\rho = 0$ for $\operatorname{Im} w$ to obtain ϕ such that $\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w)$.

$(i) \Rightarrow (iii)$: Using the considerations preceding this proposition, we can find a holomorphic function Q such that M is described by $w = Q(z, \bar{z}, \bar{w})$. Given that $\phi(z, 0, s) = 0$, and considering the normality of the coordinates, the unique solution for $\bar{z} = 0$ is $w = \bar{w}$, which is equivalent to $Q(z, 0, \tau) = \tau$. Similarly, we obtain $Q(0, \chi, \tau) = 0$.

$(iii) \Rightarrow (ii)$: The conditions in (ii) are fulfilled by the defining function $\rho(z, \chi, w, \tau) = Q(z, \chi, \tau) - w$.

$(iii) \Rightarrow (iv)$: The Segre variety $\mathcal{S}_{(0,w_0)}$ is defined as $\mathcal{S}_{(0,w_0)} = \{(z, w) \in \mathbb{C}^2 : w = Q(z, 0, w_0)\}$. By changing coordinates to $\tilde{w} = w + Q(z, 0, w_0)$, we can express $\mathcal{S}_{(0,w_0)} = \{(z, w) \in \mathbb{C}^2 : w = \bar{w}_0\}$.

$(iv) \Rightarrow (iii)$: Using the form of the Segre varieties in (iv), we have $Q(z, 0, \bar{w}_0) = \bar{w}_0$. $\square$

Theorem 5. *If $M \subset \mathbb{C}^N$ is a real-analytic generic submanifold and $p \in M$, then there exist normal coordinates for M at p.*

The following proof, as detailed in [12], not only establishes the existence of normal coordinates but also provides a way for deriving such coordinates near a given point on a real-analytic generic submanifold.

Proof. Let (z', w') be holomorphic coordinates at p, such that M is given by $Im\, w' = \phi(z', \bar{z}', \mathrm{Re}\, w')$, where ϕ is real-analytic with $\phi(0) = 0$ and $\nabla\phi(0) = 0$. A defining function for $\mathbb{C}M$ is then

$$\rho'(z', w', \chi', \tau') = w' - \tau' - 2i\phi\left(z', \chi', \frac{w + \tau'}{2}\right).$$

Since $\nabla\phi(0) = 0$, there are no linear terms in ϕ, and hence $\rho_{w'}(0) = I_{d\times d} - 2\phi_{w'}(0) = I_{d\times d}$. Using the Implicit Function Theorem, we obtain that $\mathbb{C}M$ is given by $\rho''(z', w', \chi', \tau') = w' - Q(z', \chi', \tau')$. We need to determine a holomorphic change of coordinates $(z', w') = (f(z, w), g(z, w))$ such that in the new coordinates (z, w), we have $Q(z, 0, \tau) = \tau = Q(0, \chi, \tau)$. Equivalently,

$$\rho(z, w, \chi, \tau) := \rho''\left(f(z, w), g(z, w), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau)\right) = g(z, w) - Q\left(f(z, w), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau)\right)$$

needs to satisfy $\rho(z, w, 0, w) = 0$. Since there are no linear terms in ϕ, we write $Q(z, \chi, \tau) = \tau + \tilde{Q}(z, \chi, \tau)$, where in $\tilde{Q}$ only terms of order 2 occur. We require that $\rho(z, w, 0, w) = 0$, which translates to $g(z, w) = Q\left(f(z, w), \bar{f}(0, w), \bar{g}(0, w)\right)$ or

$$g(z, w) - g(0, w) = \tilde{Q}\left(f(z, w), \bar{f}(0, w), \bar{g}(0, w)\right). \tag{3.2}$$

To simplify the above equation, we may assume that $g(z, w) = g(w)$ and $f(z, w) = z$. Using these simplifications in (3.2), we obtain

$$g(w) - \bar{g}(w) = \tilde{Q}(z, 0, \bar{g}(w)). \tag{3.3}$$

Since f and g are transformations, $g_w(0) \neq 0$ and hence we can write $g(w) = w + \tilde{G}(w)$, where $\tilde{G}$ is of order 2. Then, (3.3) is a condition for $\mathrm{Im}\ \tilde{G}$, and hence we can set $\tilde{G} = iG$, where $G(w) = \bar{G}(w)$. To confirm that such a holomorphic G actually exists, we plug our choice for g into (3.2) and set $z = 0$ to get

$$2iG(w) = \tilde{Q}(0, 0, w + iG(w)).$$

Since the left-hand side is of order 2 in $G(w)$, we can solve the above equation using the Implicit Function Theorem to obtain a holomorphic G.

Finally, we need to check whether the condition $G(w) = \bar{G}(w)$ is fulfilled. We plug $g(z, w) = w + iG(w)$ and $f(z, w) = z$ into ρ and set $z = 0$ to get

$$w + iG(w) = Q\left(0, 0, w - iG(w)\right).$$

After conjugating the equation and replacing $\bar{w}$ by w, we obtain

$$w - i\bar{G}(w) = \bar{Q}\left(0, 0, w + i\bar{G}(w)\right).$$

Combining the above two equations and using the reality condition implies $G(w) = \bar{G}(w)$. □

3.6 Infinitesimal deformations

The following section contains definitions and properties of infinitesimal deformations and local rigidity of maps, as detailed in the papers [6], [7] and [9].

Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real submanifolds. Consider the set $\mathcal{H}(M, M')$ of holomorphic mappings defined in a neighbourhood of M in $\mathbb{C}^N$ and valued in $\mathbb{C}^{N'}$, satisfying $H(M) \subset M'$. The group $G = Aut(M) \times Aut(M')$ acts on $\mathcal{H}(M, M')$ as follows: given $(\phi, \psi) \in G$, we define a map

$$\mathcal{H}(M, M') \to \mathcal{H}(M, M')$$

$$H \mapsto (\phi, \psi) \cdot H = \psi \circ H \circ \phi^{-1}.$$

Definition 29. *Two maps $H_1, H_2 \in \mathcal{H}(M, M')$ are called equivalent if there exist $\phi \in Aut(M)$ and $\psi \in Aut(M')$ such that*

$$H_1 = \psi \circ H_2 \circ \phi^{-1}.$$

Definition 30. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be closed, generic, real submanifolds of class C^ω, and let $H \in \mathcal{H}(M, M')$. We say that H is locally rigid if the image of the map H in the quotient space ${}^{\mathcal{H}(M,M')}/_{G}$ is an isolated point.*

The above definition states that all maps close enough to a given map $H \in \mathcal{H}(M, M')$ are equivalent to H under the action of G.

Definition 31. *Given $H \in \mathcal{H}(M, M')$, we say that the restriction X of a holomorphic section of $H^* T^{(0,1)} \mathbb{C}^{N'}$ to M, i.e.*

$$X = \sum_{j=1}^{N'} X_j(z) \frac{\partial}{\partial z'_j}\bigg|_{H(z)}$$

is an infinitesimal deformation of H if $\operatorname{Re} X$ is tangent to M' along $H(M)$.

More precisely, this means that for any point $p \in M$ and any real-valued, real-analytic function $\rho(z', \bar{z}')$ defined near $H(p) \in M'$ and vanishing on M', it holds that

$$(\operatorname{Re} X \rho)(z) := \sum_{j=1}^{N'} \operatorname{Re} X_j(z) \frac{\partial \rho}{\partial z'_j}(z')\bigg|_{H(z)} = 0$$

for $z \in M \cap U$, where U is an open neighbourhood of p. The collection of all infinitesimal deformations of H is denoted by $\mathfrak{hol}(H)$.

In the special cases when H is the identity map, i.e. $H = id_{\mathbb{C}^N}$ or $H = id_{\mathbb{C}^{N'}}$, we use the notations

$$\mathfrak{hol}(M) := \mathfrak{hol}(id_{\mathbb{C}^N}) \text{ and } \mathfrak{hol}(M') := \mathfrak{hol}(id_{\mathbb{C}^{N'}}),$$

respectively, and call these sets infinitesimal automorphisms of M and M', respectively.

We additionally define the set

$$\mathfrak{aut}(H) = H_*(\mathfrak{hol}(M)) + \mathfrak{hol}(M')\Big|_H,$$

which is the space of trivial infinitesimal deformations of H.

In [8] the following is shown as corollary of a more general theorem:

Proposition 7. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be compact, generic, real-analytic submanifolds with M minimal, and let $H : M \to M'$ be a CR map which is finitely nondegenerate at all points of M. If $\mathfrak{hol}(H) = \mathfrak{aut}(H)$, then H is locally rigid.*

In particular, by [7] we have the following result:

Proposition 8. *Let $M \subset \mathbb{C}^2$ be a strictly pseudoconvex real-analytic hypersurface and $M' \subset \mathbb{C}^3$ be a real-analytic hypersurface that is also Levi-nondegenerate. Let $H : \mathbb{C}^2 \to \mathbb{C}^3$ be a 2-nondegenerate, transversal embedding satisfying $\dim_{\mathbb{R}} \mathfrak{hol}(H) = \dim_{\mathbb{R}} \mathfrak{hol}(M')$. Then H is locally rigid.*

We note that, if the conditions of the above propositions on the submanifolds M and M' are satisfied, the group $Aut(M')$ is a finite dimensional Lie group with Lie algebra $\mathfrak{hol}(M')$. The elements of $\mathfrak{hol}(M')$ trivially restrict to elements of $\mathfrak{hol}(H)$ and the restriction map is injective. In general, it holds that $\mathfrak{aut}(H) \subseteq \mathfrak{hol}(H)$ and $\dim_{\mathbb{R}} \mathfrak{hol}(H) \geq \dim_{\mathbb{R}} \mathfrak{hol}(M')$.

4 Local rigidity of an embedding from the real ellipsoid in the 2-dimensional complex space to the sphere in the 3-dimensional complex space

We will now study the holomorphic embeddings from the real ellipsoid in $\mathbb{C}^2$ into the sphere in $\mathbb{C}^3$. In [14] it is shown that the real ellipsoid can be embedded into the Heisenberg hypersurface, which is a biholomorphic image of the sphere and an explicit map is provided.

The central question that arises is whether this embedding is unique, up to automorphisms of the domain and the target space. In the following sections, with help of techniques found in [12] and [16] using the Segre varieties, we will compute the set of infinitesimal deformations of the embedding in order to show that, in one special case, all maps close to the above map are equivalent. Therefore, locally, the given map is the representative of the unique equivalence class of mappings.

4.1 The real ellipsoid in the 2-dimensional complex space

The real ellipsoid $\mathcal{E}$ in $\mathbb{C}^2$ is the hypersurface described in [18] by the equation:

$$\mathcal{E} = \{(z = x + iy, w = u + iv) \in \mathbb{C}^2 : Ax^2 + By^2 + Cu^2 + Dv^2 = 1\},$$

where $A, B, C, D \in \mathbb{R}_{>0}$. Setting $\alpha = \frac{A+B}{2}, \beta = \frac{C+D}{2}, a = \frac{A-B}{2}, b = \frac{C-D}{2}$, the equation takes the form

$$\mathcal{E} : \alpha|z|^2+\beta|w|^2+Re\left(az^2 + bw^2\right) = 1.$$

Applying the biholomorphic transformation $(z, w) \mapsto (\sqrt{\alpha}z, \sqrt{\beta}w)$, we can assume, without loss of generality, that $\alpha = \beta = 1$. This results in the following form for the defining equation:

$$\mathcal{E} : z\bar{z} + w\bar{w} + a(z^2 + \bar{z}^2) + b(w^2 + \bar{w}^2) = 1$$

for $0 \le a, b \le 1$. We may also assume that either $a \neq 0$ or $b \neq 0$, since otherwise we are dealing with a sphere whose embeddings have already been studied. The complexification $\mathbb{C}\mathcal{E}$ of this domain is then

$$\mathbb{C}\mathcal{E} : z\chi + w\tau + a(z^2 + \chi^2) + b(w^2 + \tau^2) = 1.$$

Let

$$\rho(z, \bar{z}, w, \bar{w}) = z\bar{z} + w\bar{w} + a(z^2 + \bar{z}^2) + b(w^2 + \bar{w}^2) - 1$$

be a defining function for $\mathcal{E}$. It is clearly real-analytic and Cauchy-Riemann, and following the notations in section 3.2, the codimension of $\mathcal{E}$ is $d = 1$. Direct computation shows that the matrix

$$\left(\frac{\partial \rho}{\partial z_j}(p, \bar{p})\right)_{j=1,2} = \begin{bmatrix}\frac{\partial \rho}{\partial z}(p, \bar{p}) & \frac{\partial \rho}{\partial w}(p, \bar{p})\end{bmatrix} = \begin{bmatrix}\bar{z} + 2az|_{(p,\bar{p})} & \bar{w} + 2bw|_{(p,\bar{p})}\end{bmatrix}$$

satisfies

$$rank_{\mathbb{C}}\left(\frac{\partial \rho}{\partial z_j}(p, \bar{p})\right)_{j=1,2} = 1$$

for all the points $(z, w) \in \mathcal{E}$. This implies that, for every $p \in \mathbb{C}^2$, $\dim \mathcal{V}_p = N - rank_{\mathbb{C}}\left(\frac{\partial \rho}{\partial z_j}(p, \bar{p})\right)_{j=1,2} = 1.$ Since the complex differential of ρ is nonzero, as shown above, $\mathcal{E}$ is also generic.

Finally, $\mathcal{E}$ is a strictly pseudoconvex hypersurface of $\mathbb{C}^2$. Indeed, $\frac{\partial \rho}{\partial \bar{z}} = z + 2a\bar{z}$, $\frac{\partial \rho}{\partial \bar{w}} = w + 2b\bar{w}$ and

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}} = 1, \frac{\partial^2 \rho}{\partial w \partial \bar{z}} = 0, \frac{\partial^2 \rho}{\partial w \partial \bar{w}} = 1 \text{ and } \frac{\partial \rho}{\partial z \partial \bar{w}} = 0.$$

For all $p \in \mathcal{E}$ and $a = (a_1, a_2) \in \mathbb{C}^2$, the Levi form is positive definite, since

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}} a_1\bar{a}_1 + \frac{\partial^2 \rho}{\partial w \partial \bar{z}} a_2\bar{a}_1 + \frac{\partial \rho}{\partial z \partial \bar{w}} a_1\bar{a}_2 + \frac{\partial^2 \rho}{\partial w \partial \bar{w}} a_2\bar{a}_2 = |a_1|^2+|a_2|^2 \geq 0$$

with equality if and only if $a_1 = a_2 = 0$. Hence, the defining function ρ is strictly plurisubharmonic and this confirms the strict pseudoconvexity of $\mathcal{E}$.

4.2 Coordinate transformations for the ellipsoid

We will now investigate the defining equation of the ellipsoid $\mathcal{E}$, to determine if specific coordinate transformations can result to simpler defining equations that are easier to handle.

4.2.1 Normal Coordinates

The first coordinate transformation that we will apply is normal coordinates. They have the property, as seen in section 3.5, that the Segre Varieties expressed in these coordinates are flat. We will follow the proof of Theorem 5, in order to find a coordinate transformation $(z', w') = (f(z, w), g(z, w))$ for $\mathcal{E}$. Following the notation of this proof, let

$$\rho'(z', w', \bar{z}', \bar{w}') = z'\bar{z}' + w'\bar{w}' + a(z'^2 + \bar{z}'^2) + b(w'^2 + \bar{w}'^2) - 1$$

be a defining equation for $\mathcal{E}$.

We begin by changing coordinates $(z', w') \mapsto \left(z', iw' + \frac{1}{\sqrt{2b+1}}\right)$ in order to achieve $\rho'(0, 0, 0, 0) = 0$. This transformation yields the defining equation:

$$\rho(z', w', \bar{z}', \bar{w}') = |z'|^2 + a(z'^2 + \bar{z}'^2) + w'\bar{w}' + i\sqrt{2b+1}(w' - \bar{w}') - b(w'^2 + \bar{w}'^2).$$

We choose $f(z, w) = z$ and $g(z, w) = g(w) = w + iG(w)$, where G satisfies $G(w) = \bar{G}(w)$. Substituting $z' = z$ and $w' = w + iG(w)$ into the equation and setting $z = 0$ leads to

$$G(w)^2 - \frac{2}{\sqrt{2b+1}}G(w) + \frac{1-2b}{2b+1}w^2 = 0.$$

We choose the solution satisfying $G(0) = 0$:

$$G(w) = \frac{1 - \sqrt{1 - (1-2b)w^2}}{\sqrt{2b+1}}.$$

To derive the full formula for $g(z, w)$, we use the normality condition $\rho(z, w, 0, w) = 0$, or equivalently,

$$\begin{aligned} &f(z, w)\bar{f}(0, w) + a\left(f(z, w)^2 + \bar{f}(0, w)^2\right) + g(z, w)\bar{g}(0, w) \\ &\quad + i\sqrt{2b+1}\,(g(z, w) - \bar{g}(0, w)) - bg(z, w)^2 - b\bar{g}(0, w)^2 = 0. \end{aligned}$$

Substituting $f(z, w) = z$ and $g(0, w) = w + iG(w)$, we can derive the formula for $g(z, w)$ as follows:

$$\begin{aligned} az^2 &+ g(z, w)\left(w - \frac{i}{\sqrt{2b+1}}\sqrt{1 - (1-2b)w^2}\right) \\ &+ i\sqrt{2b+1}\left(g(z, w) - w + \frac{i}{\sqrt{2b+1}}\sqrt{1 - (1-2b)w^2}\right) \\ &- bg(z, w)^2 - b\left(w - \frac{i}{\sqrt{2b+1}}\sqrt{1 - (1-2b)w^2}\right) = 0. \end{aligned}$$

If $b = 0$, the equation simplifies to

$$g(z, w) = i + w - i\sqrt{1 - w^2} - awz^2 + iaz^2\sqrt{1 - w^2}$$

and if $b \neq 0$, with the condition $g(0, 0)$, the solution becomes

$$\begin{aligned} g(z, w) = &\frac{i}{\sqrt{2b+1}} - i\frac{\sqrt{2b + 1 + (4b^2 - 1)w^2}}{2b+1} + \frac{w + i\sqrt{2b + (4b^2 - 1)w^2}}{2b} \\ &+ \frac{\sqrt{2w^2 - 4w^2 b + 2iw(1-2b)\sqrt{1 + 2b + (4b^2-1)w^2} + 4abz^2 - 1 - 2b}}{2b}. \end{aligned}$$

Hence, in the new coordinates, the defining equation for $\mathbb{C}\mathcal{E}$ takes the following form:
If $b = 0$:

$$\begin{aligned}\rho(z,w,\chi,\tau) = &-1 + 2iw + w\tau + a\chi^2 - aw\chi^2\tau + z\chi + az^2 - 2iaz^2w - az^2w\tau \\ &+ a^2z^2w\chi^2\tau + \sqrt{1-w^2}\left(2 - i\tau - ia\chi^2\tau - 2az^2 + iaz^2\tau - ia^2z^2\chi^2\tau\right) \\ &+ \sqrt{1-w^2}\sqrt{1-\tau^2}\left(-1 + az^2 + a\chi^2 - ia^2z^2w\chi^2 - a^2z^2\chi^2\right) \\ &+ w\sqrt{1-\tau^2}\left(-i + iaz^2 + ia\chi^2\right),\end{aligned}$$

If $b \neq 0$:

$$\begin{aligned}\rho(z,w,\chi,\tau) = &\frac{1}{4b^2(1+2b)^2}(-2ib(1+2b)^{3/2}(2ib\bar{k}(\tau) + (2b+1)\tau + i(1+2b)\sqrt{1+2b+(-1+4b^2)\tau^2} \\ &+ (1+2b)\bar{h}(\chi,\tau)) - b(2ib\bar{k}(\tau) + (2b+1)\tau + i(1+2b)\sqrt{1+2b+(-1+4b^2)\tau^2} \\ &+ (1+2b)\bar{h}(\chi,\tau))^2 + 4b^2(1+2b)^2\chi z + 4ab^2(1+2b)^2(\chi^2+z^2) + 2ib(1+2b)^{3/2}(2ibk(w) \\ &+ (2b+1)w + i(1+2b)h(z,w)) + (2ib\bar{k}(\tau) + (2b+1)\tau \\ &+ i(1+2b)\sqrt{1+2b+(-1+4b^2)\tau^2} + (1+2b)\bar{h}(\chi,\tau))(2ibk(w) + (2b+1)w \\ &+ i(1+2b)\sqrt{1+2b+(-1+4b^2)w^2} + (1+2b)h(z,w)) - b(2ibk(w) + (2b+1)w \\ &+ i(1+2b)\sqrt{1+2b+(-1+4b^2)w^2} + (1+2b)h(z,w))^2),\end{aligned}$$

where

$$h(z,w) = \sqrt{-1 - 2b + 4abz^2 + 2w(1-2b)\left(w + i\sqrt{1+2b+(-1+4b^2)\,w^2}\right)}$$

and

$$k(w) = \sqrt{1+2b} - \sqrt{1+2b+(-1+4b^2)\,w^2}.$$

4.2.2 Projective Space

The second coordinate transfomation employs the projective space for the coordinates $(z,w) \in \mathcal{E}$. We set $z = \frac{z_1}{z_3}$ and $w = \frac{z_2}{z_3}$ in the defining equation

$$\mathcal{E} : |z|^2 + |w|^2 + a(z^2 + \bar{z}^2) + b(w^2 + \bar{w})^2 = 1. \tag{4.1}$$

After multiplying both sides of (4.1) by $|z_3|^4 = z_3^2\bar{z}_3^2$, we obtain

$$|z_1|^2|z_3|^2 + |z_2|^2|z_3|^2 + a(z_1^2\bar{z}_3^2 + \bar{z}_1^2z_3^2) + b(z_2^2\bar{z}_3^2 + \bar{z}_2^2z_3^2) = z_3^2\bar{z}_3^2.$$

We now set:

$$\begin{pmatrix} -\frac{1}{\sqrt{2a+1}} \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} \frac{1}{\sqrt{2a+1}} \\ 0 \\ -1 \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Thus, the defining equation takes the form:

$$\begin{aligned}\rho(z,w,\bar{z},\bar{w}) = &-\frac{2a}{a+1}(z^2+\bar{z}^2) - 4\frac{2a+1}{a+1}z\bar{z} - 2\bar{z} - 2z - 2\bar{z}^2z - 2z^2\bar{z} \\ &+ |w|^2(z+1)(\bar{z}+1) + \frac{b}{2}(\bar{z}+1)^2w^2 + \frac{b}{2}(z+1)^2\bar{w}^2\end{aligned}$$

We also use a second transformation:

$$\begin{pmatrix} \frac{1}{\sqrt{2a+1}} \\ 0 \\ -1 \end{pmatrix} + z\begin{pmatrix} \frac{i}{\sqrt{2a+1}} \\ 0 \\ i \end{pmatrix} + w\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 . \\ z_3 \end{pmatrix}$$

Substituting these formulas for $z_i, i = 1,2,3$, the defining equation takes the form:

$$\begin{aligned}\rho(z,w,\bar{z},\bar{w}) = \frac{2}{a+1}(\bar{z}^2+z^2) - 4\frac{2a+1}{a+1}z\bar{z} \\ - |w|^2(iz-1)(i\bar{z}+1) + \frac{b}{2}(i\bar{z}+1)^2w^2 + \frac{b}{2}(iz-1)^2\bar{w}^2 - 2i\bar{z} + 2iz - 2iz\bar{z}^2 + 2i\bar{z}z^2.\end{aligned}$$

The above coordinate transformations will not be used in the next sections of this thesis, since the resulting defining equations are not sufficiently simplified for our purposes. However, they may be useful for other applications.

4.3 The sphere in the 3-dimensional complex space

The sphere $\mathbb{S}^3 \subset \mathbb{C}^3$ is the hypersurface defined as

$$\mathbb{S}^3 = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^2+|z_2|^2+|z_3|^2 = 1\}.$$

The defining equation for $\mathbb{S}^3$

$$\rho'(z_1,z_2,z_3) = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - 1$$

is clearly real-analytic and hence $\mathbb{S}^3$ is a real-analytic hypersurface. Since $(0,0,0) \notin \mathbb{S}^3$, the rank of the matrix

$$\left(\frac{\partial\rho'}{\partial z_j}(p,\bar{p})\right)_{j=1,2,3} = \begin{bmatrix}\bar{z}_1|_{(p,\bar{p})} & \bar{z}_2|_{(p,\bar{p})} & \bar{z}_3|_{(p,\bar{p})}\end{bmatrix}$$

is 1 everywhere. Hence, for every $p \in \mathbb{C}^3$

$$\dim \mathcal{V}'_p = N' - \operatorname{rank}_{\mathbb{C}}\left(\frac{\partial\rho'}{\partial z_j}(p,\bar{p})\right)_{j=1,2,3} = 2$$

and $\mathbb{S}^3$ is a Cauchy Riemann submanifold of $\mathbb{C}^3$ and generic, since $\left(\frac{\partial\rho'}{\partial z_j}\right)_{j=1,2,3}$ is nonzero everywhere.

Finally, $\mathbb{S}^3$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^3$, since $\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k} = \begin{cases}1, & j=k \\ 0, & j\neq k\end{cases}$. Thus, for the Levi form, we have

$$\sum_{j,k=1}^{3}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(p,\bar{p})a_j\bar{a}_k = |a_1|^2+|a_2|^2+|a_3|^2 \geq 0$$

for all $a = (a_1,a_2,a_3) \in \mathbb{C}^3$ with equality if and only if $a_1 = a_2 = a_3 = 0$.

We will show that there is a biholomorphic map between the sphere $\mathbb{S}^3$ and the Heisenberg hypersurface $\mathbb{H}^3 = \{(z_1,z_2,w) \in \mathbb{C}^3 : \operatorname{Im} w = |z_1|^2+|z_2|^2\}$.

For the one direction will use the Cayley Transform

$$F : \mathbb{S}^3 \to \mathbb{H}^3, (z_1,z_2,w) \mapsto \left(\frac{z_1}{1+w}, \frac{z_2}{1+w}, i\frac{1-w}{1+w}\right).$$

Indeed, this holomorphic map satisfies

$$i\frac{1-w}{1+w} = i\frac{(1-w)(1+\bar{w})}{(1+w)(1+\bar{w})} = i\frac{1-w-\bar{w}-w\bar{w}}{(1+w)(1+\bar{w})} = i\frac{1-2i\operatorname{Im}\, w+|w|^2}{(1+w)(1+\bar{w})}$$

and hence

$$\operatorname{Im}\left(i\frac{1-w}{1+w}\right) = \frac{1-|w|^2}{(1+w)(1+\bar{w})} = \frac{|z_1|^2+|z_2|^2}{(1+w)(1+\bar{w})} = \frac{z_1}{1+w}\frac{\bar{z}_1}{1+\bar{w}} + \frac{z_2}{1+w}\frac{\bar{z}_2}{1+\bar{w}}.$$

Finally, the inverse of the Cayley Transform,

$$F^{-1} : \mathbb{H}^3 \to \mathbb{S}^3, (z_1,z_2,w) \mapsto \left(\frac{2iz_1}{w+i}, \frac{2iz_2}{w+i}, \frac{i-w}{w+i}\right)$$

is again a holomorphic map.

4.4 Definition of the holomorphic embedding

An embedding $H : \mathcal{E} \to \mathbb{S}^3$ is a map $H = (f_1, f_2, g) : \mathcal{E} \to \mathbb{S}^3$ such that

$$|f_1(z,w)|^2 + |f_2(z,w)|^2 + |g(z,w)|^2 = 1$$

for $|z|^2 + |w|^2 + a(z^2 + \bar{z}^2) + b(w^2 + \bar{w}^2) = 1$. The complexified version of the above equation is

$$f_1(z,w)\bar{f}_1(\chi,\tau) + f_2(z,w)\bar{f}_2(\chi,\tau) + g(z,w)\bar{g}(\chi,\tau) = 1$$

for $z\chi + w\tau + a(z^2 + \chi^2) + b(w^2 + \tau^2) = 1$.

One checks easily that the map

$$(z,w) \mapsto (z, w, i - 2iaz^2 - 2ibw^2)$$

by [14] holomorphically embedds the ellipsoid $\mathcal{E} \subset \mathbb{C}^2$ into the Heisenberg hypersurface $\mathbb{H}^3$. This can be translated to a map $\mathcal{E} \to \mathbb{S}^3$ defined as

$$(z,w) \mapsto \left(\frac{z}{1 - az^2 - bw^2}, \frac{w}{1 - az^2 - bw^2}, \frac{az^2 + bw^2}{1 - az^2 - bw^2} \right)$$

by use of the inverse Cayley Transform defined in section 4.3. Indeed, using the defining equation of the ellipsoid, we obtain

$$\begin{aligned} |z|^2 + |w|^2 + \left|az^2 + bw^2\right|^2 &= z\bar{z} + w\bar{w} + (az^2 + bw^2)(a\bar{z}^2 + b\bar{w}^2) \\ &= z\bar{z} + w\bar{w} + a^2 z^2 \bar{z}^2 + abz^2\bar{w}^2 + ab\bar{z}^2 w^2 + b^2 w^2 \bar{w}^2 \\ &= 1 - az^2 - a\bar{z}^2 - bw^2 - b\bar{w}^2 + az^2\bar{z}^2 + abz^2\bar{w}^2 + ab\bar{z}^2w^2 + b^2w^2\bar{w}^2 \\ &= \left(1 - az^2 - bw^2\right)\left(1 - a\bar{z}^2 - b\bar{w}^2\right) \\ &= \left|1 - az^2 - bw^2\right|^2 . \end{aligned}$$

As shown in the sections 4.1 and 4.3, both $\mathcal{E}$ and $\mathbb{S}^3$ are real-analytic, strictly pseudoconvex hypersurfaces. A basis of CR vector fields for the ellipsoid $\mathcal{E}$ is given by

$$\bar{L} = \rho_{\bar{z}} \frac{\partial}{\partial \bar{w}} - \rho_{\bar{w}} \frac{\partial}{\partial \bar{z}} = (z + 2a\bar{z}) \frac{\partial}{\partial \bar{w}} - (w + 2b\bar{w}) \frac{\partial}{\partial \bar{z}},$$

whereas

$$\left\{ \bar{L}'_1 = \rho'_{\bar{z}_1} \frac{\partial}{\partial \bar{z}_1} - \rho'_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_2} = z_1 \frac{\partial}{\partial \bar{z}_1} - z_2 \frac{\partial}{\partial \bar{z}_2}, \bar{L}'_2 = \rho'_{\bar{z}_2} \frac{\partial}{\partial \bar{z}_2} - \rho'_{\bar{z}_3} \frac{\partial}{\partial \bar{z}_3} = z_2 \frac{\partial}{\partial \bar{z}_2} - z_3 \frac{\partial}{\partial \bar{z}_3} \right\}$$

forms a basis for the CR vector fields of $\mathbb{S}^3$. Using Proposition 4, $H : \mathcal{E} \to \mathbb{S}^3$ is clearly a CR mapping.

Furthermore, around every point $(z,w) \in \mathcal{E}$ the map H is 2-nondegenerate. Indeed, we have

$$\left(\frac{\partial \rho'}{\partial z'} \right) = \left(\frac{\partial \rho'}{\partial z'_1}, \frac{\partial \rho'}{\partial z'_2}, \frac{\partial \rho'}{\partial z'_3} \right) = (\bar{z}_1, \bar{z}_2, \bar{z}_3),$$

and hence

$$\left(\frac{\partial \rho'}{\partial z'} \left(H(z,w), \bar{H}(z,w)\right) \right) = \left(\frac{\bar{z}}{1 - a\bar{z}^2 - b\bar{w}^2}, \frac{\bar{w}}{1 - a\bar{z}^2 - b\bar{w}^2}, \frac{a\bar{z}^2 + b\bar{w}^2}{1 - a\bar{z}^2 - b\bar{w}^2} \right).$$

Around every point $(z,w) \in \mathcal{E}$, the set

$$\left\{ \frac{\partial \rho'}{\partial z'} \left(H(z,w), \bar{H}(z,w)\right), \bar{L} \frac{\partial \rho'}{\partial z'} \left(H(z,w), \bar{H}(z,w)\right), \bar{L}^2 \frac{\partial \rho'}{\partial z'} \left(H(z,w), \bar{H}(z,w)\right) \right\}$$

clearly spans $\mathbb{C}^3$. Using the notation of section 3.3, $E'_1(z,w) \subsetneq \mathbb{C}^3$, $E'_2(z,w) = \mathbb{C}^3$ and $s(z,w) = 0$ for all $(z,w) \in \mathcal{E}$.

We will prove that $\mathfrak{hol}(H) = \mathfrak{aut}(H)$ in the special case $b = 0$. In other words, the map H is locally rigid, i.e. locally unique up to automorphisms of $\mathcal{E}$ and $\mathbb{S}^3$. We will thus consider ellipsoids with defining equation

$$\rho(z,w,\bar{z},\bar{w}) = |z|^2 + |w|^2 + a\left(z^2 + \bar{z}^2\right) - 1.$$

4.5 Segre Varieties of the real ellipsoid

The Segre varieties are a useful tool for finding the formulas of mappings involved in a complexified equation. To this end, will calculate the first and second Segre variety around the point $(0,1) \in \mathcal{E}$. Let U be a neighbourhood of $(0,1)$. Then

$$\begin{aligned} S^1_{(0,1)} &= \{(z,w) \in U : \rho(z,w,0,1) = 0\} \\ &= \left\{(z,w) \in U : w = 1 - az^2\right\} \\ &= \left\{(z, 1-az^2) \in U : z \in \mathbb{C}\right\} \end{aligned} \tag{4.2}$$

$$\begin{aligned} S^2_{(0,1)} &= \left\{(z,w) \in U : \rho(z,w,\chi,1-a\chi^2) = 0, (\chi, 1-a\chi^2) \in {}^*U\right\} \\ &= \left\{(z,w) \in U : z\chi + w - aw\chi^2 + az^2 + a\chi^2 = 1, (\chi, 1-a\chi^2) \in {}^*U\right\} \\ &= \left\{(z,w) \in U : w(1-a\chi^2) = 1 - az^2 - a\chi^2 - z\chi, (\chi, 1-a\chi^2) \in {}^*U\right\} \\ &= \left\{\left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right) \in U : z, \chi \in \mathbb{C}, (\chi, 1-a\chi^2) \in {}^*U\right\} \end{aligned} \tag{4.3}$$

The rank of the first Segre map $S_1 : z \mapsto (z, 1-az^2)$ for $z \in U$ is 1 everywhere.

For the rank of the second Segre map $S_2 : (z,\chi) \mapsto \left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right)$, we consider the matrix $\left(\frac{\partial^2 S_2}{\partial(z,\chi)}\right)$ with determinant

$$\left|\frac{\partial^2 S_2}{\partial(z,\chi)}\right| = \det\left(\begin{bmatrix} 1 & 0 \\ * & -\frac{z(1+a\chi^2+2a^2z\chi)}{(1-a\chi^2)^2}\end{bmatrix}\right) = -\frac{z(1+a\chi^2+2a^2z\chi)}{(1-a\chi^2)^2}.$$

Since the above determinant is not zero everywhere, the generic rank of S_2 is 2. This is the maximum rank that such a map can achieve and, by view of Theorem 4, the generic rank of S_k is 2 for every $k \geq 2$.

By view of Proposition 5, $\mathcal{E}$ is minimal, since the generic dimension of the maximal Segre set is 2.

4.6 Infinitesimal deformations of the map

In this subsection, as a first step, we will compute a subclass of the infinitesimal deformations of the map, showing that it contains only the zero vector fields.

As a second step, we will compute the infinitesimal deformations of the map. We will begin by writing an equation that such vector fields have to fulfill in the complexified form. We will then derive formulas for the vector fields along the first Segre variety. Using the reflection principle, we will extend these formulas to the second Segre variety.

Since the second Segre variety is generically of full rank, this will allow us to generalize the maps to the entire ellipsoid. The holomorphicity assumptions will impose conditions on the vector fields, simplifying their formulas. Finally, using a non-standard argument involving the Weierstrass Division Theorem, we will further simplify the formulas and provide an explicit description of $\mathfrak{hol}(H)$.

As already mentioned, we will consider a subclass of the infinitesimal deformations $\mathfrak{hol}(H)$ of H, namely the vector fields that are tangent to $\mathbb{S}^3$ along $H(\mathcal{E})$. The set of such vector fields is denoted by:

$$\mathfrak{ht}(H) = \left\{ X = \sum_{j=1}^{3} X_j(z,w)\frac{\partial}{\partial z_j} : X\left(\|z\|^2 - 1\right)\Big|_{H(\mathcal{E})} = 0 \text{ for } z = (z_1,z_2,z_3) \in \mathbb{S}^3 \right\}.$$

We have $\mathfrak{ht}(H) \subset \mathfrak{hol}(H)$, and furthermore, we observe

$$\begin{aligned} \mathfrak{hol}(H) \cap i\mathfrak{hol}(H) &= \{X \in \mathfrak{hol}(H)\} \cap \{iX \in \mathfrak{hol}(H)\} \\ &= \left\{Re\, X\left(\|z\|^2-1\right)|_{H(\mathcal{E})} = 0\right\} \cap \left\{Re(iX)\left(\|z\|^2-1\right)|_{H(\mathcal{E})} = 0\right\} \\ &= \left\{X \in \mathfrak{hol}(H) : X\left(\|z\|^2-1\right)|_{H(\mathcal{E})} = 0\right\} \\ &= \mathfrak{ht}(H). \end{aligned}$$

Thus, if $\mathfrak{h}\mathfrak{t}(H) \neq \{0\}$, it implies that $\mathfrak{hol}(H)$ contains also other maps besides the zero map.

Let $X \in \mathfrak{h}\mathfrak{t}(H)$. The condition $X\left(\|z\|^2-1\right)\Big|_{H(\mathcal{E})} = 0$ can be expressed as

$$\sum_{j=1}^{3} X_j(z,w)\frac{\partial}{\partial z_j}\left(\sum_{j=1}^{3} z_j\bar{z}_j - 1\right)\Bigg|_{H(\mathcal{E})} = 0,$$

which simplifies to

$$X_1(z,w)\bar{z}_1 + X_2(z,w)\bar{z}_2 + X_3(z,w)\bar{z}_3|_{H(\mathcal{E})} = 0.$$

Using the definition of the map H, we obtain

$$X_1(z,w)\bar{z} + X_2(z,w)\bar{w} + X_3(z,w)\left(a\bar{z}^2 + b\bar{w}^2\right) = 0$$

for $|z|^2+|w|^2+a\left(z^2+\bar{z}^2\right)+b\left(w^2+\bar{w}^2\right)=1$, or equivalently, in the complexified version

$$X_1(z,w)\chi + X_2(z,w)\tau + X_3(z,w)\left(a\chi^2 + b\tau^2\right) = 0 \tag{4.4}$$

for $z\chi + w\tau + a\left(z^2+\chi^2\right) + b\left(w^2+\tau^2\right) = 1$.

In order to compute all three vector fields X_1, X_2 and X_3, we need two more equations. Therefore, we apply the CR vector field

$$\bar{L} = \rho_\tau\frac{\partial}{\partial\chi} - \rho_\chi\frac{\partial}{\partial\tau} = (\tau+2bw)\frac{\partial}{\partial\chi} - (z+2a\chi)\frac{\partial}{\partial\tau}.$$

to the equation (4.4). This yields

$$(w+2b\tau)X_1(z,w) - (z+2a\chi)X_2(z,w) + 2(w+2b\tau)a\chi X_3(z,w) - 2(z+2a\chi)b\tau X_3(z,w) = 0.$$

Furthermore, applying

$$\bar{L}^2 = (\tau+2bw)^2\frac{\partial^2}{\partial\chi^2} - 2(z+2a\chi)(\tau+2bw)\frac{\partial^2}{\partial\chi\partial\tau} + (z+2a\chi)^2\frac{\partial^2}{\partial\tau^2},$$

we obtain

$$\left[2a(\tau+2bw)^2 + 2b(z+2a\chi)^2\right]X_3(z,w) = 0.$$

Hence, we get the following system of equations

$$\begin{cases} \chi X_1(z,w) + \tau X_2(z,w) + \left(a\chi^2+b\tau^2\right)X_3(z,w) = 0 \\ (w+2b\tau)X_1(z,w) - (z+2a\chi)X_2(z,w) + \left[2a\chi(w+2b\tau) - 2b\tau(z+2a\chi)\right]X_3(z,w) = 0 \\ \left[2a(w+2b\tau)^2 + 2b(z+2a\chi)^2\right]X_3(z,w) = 0 \end{cases}$$

or equivalently,

$$\begin{bmatrix} \chi & \tau & a\chi^2+b\tau^2 \\ w+2b\tau & -(z+2a\chi) & 2a\chi(w+2b\tau)-2b\tau(z+2a\chi) \\ 0 & 0 & 2a(w+2b\tau)^2+2b(z+2a\chi)^2 \end{bmatrix} \cdot \begin{bmatrix} X_1(z,w) \\ X_2(z,w) \\ X_3(z,w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the above system

$$-\left[2a(\tau+2bw)^2 + 2b(z+2a\chi)^2\right]\left(\chi z + 2a\chi^2 + w\tau + 2b\tau^2\right)$$

does not vanish identically on $\mathbb{C}\mathcal{E}$. Hence, the generic rank of the above matrix is 3, and the solution for X_1, X_2, and X_3 is

$$X_1(z,w) = X_2(z,w) = X_3(z,w) = 0.$$

As a second step, we compute the set of infinitesimal deformations of H defined in section 3.6 as

$$\mathfrak{hol}(H) = \left\{ X = \sum_{j=1}^{3} X_j(z,w)\frac{\partial}{\partial z_j} : \operatorname{Re} X \left(\|z\|^2 - 1\right)\Big|_{H(M)} = 0 \text{ for } z = (z_1, z_2, z_3) \in \mathbb{S}^3 \right\},$$

in the special case $b = 0$. Let $X \in \mathfrak{hol}(H)$. The condition $\operatorname{Re} X \left(\|z\|^2 - 1\right)\Big|_{H(\mathcal{E})} = 0$ translates to

$$\sum_{j=1}^{3} \operatorname{Re}\left(X_j(z,w)\frac{\partial}{\partial z_j}\right)\left(\sum_{j=1}^{3} z_j\bar{z}_j - 1\right)\Bigg|_{H(\mathcal{E})} = 0,$$

or, equivalently,

$$\begin{aligned} &\Bigg(X_1(z,w)\frac{\partial}{\partial z_1} + \bar{X}_1(z,w)\frac{\partial}{\partial \bar{z}_1} + X_2(z,w)\frac{\partial}{\partial z_2} + \bar{X}_2(z,w)\frac{\partial}{\partial \bar{z}_2} \\ &+ X_3(z,w)\frac{\partial}{\partial z_3} + \bar{X}_3(z,w)\frac{\partial}{\partial \bar{z}_3}\Bigg)\left(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - 1\right)|_{H(\mathcal{E})} = 0 \end{aligned}$$

which yields

$$X_1(z,w)\bar{z}_1 + \bar{X}_1(\bar{z},\bar{w})z_1 + X_2(z,w)\bar{z}_2 + \bar{X}_2(\bar{z},\bar{w})z_2 + X_3(z,w)\bar{z}_3 + \bar{X}_3(\bar{z},\bar{w})\Big|_{H(\mathcal{E})} = 0.$$

Using the definition of the map H, in the case $b = 0$, we have

$$\begin{aligned} &X_1(z,w)\frac{\bar{z}}{1 - a\bar{z}^2} + X_2(z,w)\frac{\bar{w}}{1 - a\bar{z}^2} + X_3(z,w)\frac{a\bar{z}^2}{1 - a\bar{z}^2} \\ &+ \bar{X}_1(\bar{z},\bar{w})\frac{z}{1 - az^2} + \bar{X}_2(\bar{z},\bar{w})\frac{w}{1 - az^2} + \bar{X}_3(\bar{z},\bar{w})\frac{az^2}{1 - az^2} = 0, \end{aligned} \tag{4.5}$$

for $|z|^2 + |w|^2 + a\left(z^2 + \bar{z}^2\right) = 1$.

In order to simplify matters, we multiply both sides of (4.5) by $(1 - az^2)(1 - a\bar{z}^2)$ to get

$$\begin{aligned} &(1 - az^2)X_1(z,w)\bar{z} + (1 - az^2)X_2(z,w)\bar{w} + (1 - az^2)X_3(z,w)a\bar{z}^2 \\ &+ (1 - a\bar{z}^2)\bar{X}_1(\bar{z},\bar{w})z + (1 - a\bar{z}^2)\bar{X}_2(\bar{z},\bar{w})w + (1 - a\bar{z}^2)\bar{X}_3(\bar{z},\bar{w})az^2 = 0. \end{aligned} \tag{4.6}$$

Setting $Y_j(z,w) = (1 - az^2)X_j(z,w)$ for $j = 1,2,3$ in (4.6) and complexifying yields

$$Y_1(z,w)\chi + Y_2(z,w)\tau + Y_3(z,w)a\chi^2 + \bar{Y}_1(\chi,\tau)z + \bar{Y}_2(\chi,\tau)w + \bar{Y}_3(\chi,\tau)az^2 = 0, \tag{4.7}$$

for $z\chi + w\tau + a\left(z^2 + \chi^2\right) = 1$.

We will first compute the maps along the first Segre Variety

$$S^1_{(0,1)} = \left\{(z, 1 - az^2) \in U : z \in \mathbb{C}\right\},$$

where U is a neighbourhood of $(0,1) \in \mathcal{E}$. Applying

$$L = \tau\frac{\partial}{\partial z} - (\chi + 2az)\frac{\partial}{\partial w}$$

and

$$L^2 = \tau^2\frac{\partial^2}{\partial z^2} - 2(\chi + 2az)\tau\frac{\partial^2}{\partial z\partial w} + (\chi + 2az)^2\frac{\partial^2}{\partial w^2}$$

to (4.7) yield a system of three equations for Y_1, Y_2 and Y_3

$$\begin{aligned}Y_1(z,w)\chi + Y_2(z,w)\tau + Y_3(z,w)a\chi^2 + \bar{Y}_1(\chi,\tau)z + \bar{Y}_2(\chi,\tau)w + \bar{Y}_3(\chi,\tau)az^2 = 0\\ LY_1(z,w)\chi + LY_2(z,w)\tau + LY_3(z,w)a\chi^2 + \bar{Y}_1(\chi,\tau)L(z) + \bar{Y}_2(\chi,\tau)L(w) + \bar{Y}_3(\chi,\tau)L(az^2) = 0\\ L^2Y_1(z,w)\chi + L^2Y_2(z,w)\tau + L^2Y_3(z,w)a\chi^2 + \bar{Y}_1(\chi,\tau)L^2(z) + \bar{Y}_2(\chi,\tau)L^2(w) + \bar{Y}_3(\chi,\tau)L^2(az^2) = 0.\end{aligned} \tag{4.8}$$

Setting $(z,w) = (0,1)$ and $(\chi,\tau) = (\chi, 1-a\chi^2) \in S^1_{(0,1)}$ in the above equations, we can solve to deduce formulas for Y_1, Y_2, and Y_3 along $S^1_{(0,1)}$:

$$\begin{aligned}\bar{Y}_1(\chi,1-a\chi^2) = \frac{1}{a\chi^2-1}\Big(&Y_2^{(1,0)}(0,1) + \chi Y_2(0,1) - \chi Y_2^{(0,1)}(0,1) + \chi Y_1^{(1,0)}(0,1) + \chi^2 Y_1(0,1)\\ &- \chi^2 Y_1^{(0,1)}(0,1) - 2a\chi^2 Y_2^{(1,0)}(0,1) + a\chi^2 Y_3^{(1,0)}(0,1) - a\chi^3 Y_2(0,1) + a\chi^3 Y_3(0,1)\\ &+ a\chi^3 Y_2^{(0,1)}(0,1) - a\chi^3 Y_3^{(0,1)}(0,1) - a\chi^3 Y_1^{(1,0)}(0,1) + a^2\chi^4 Y_2^{(1,0)}(0,1)\\ &- a^2\chi^4 Y_3^{(1,0)}(0,1)\Big)\end{aligned} \tag{4.9}$$

$$\bar{Y}_2(\chi,1-a\chi^2) = -Y_2(0,1) - \chi Y_1(0,1) + a\chi^2 Y_2(0,1) - a\chi^2 Y_3(0,1) \tag{4.10}$$

$$\begin{aligned}\bar{Y}_3(\chi,1-a\chi^2) = &-Y_2(0,1) + Y_2^{(0,1)}(0,1) - \frac{1}{2a}Y_2^{(2,0)}(0,1) + \frac{\chi}{a}Y_2^{(1,1)}(0,1) - \frac{\chi}{2a}Y_1^{(2,0)}(0,1)\\ &+ \frac{1}{a\chi^2-1}\Big(\chi Y_1(0,1) - \chi Y_1^{(0,1)}(0,1) + a\chi^2 Y_3(0,1) - a\chi^2 Y_3^{(0,1)}(0,1) + \frac{\chi^2}{2a}Y_2^{(0,2)}(0,1)\\ &- \frac{\chi^2}{a}Y_1^{(1,1)}(0,1) - \chi^2 Y_2^{(2,0)}(0,1) + \frac{\chi^2}{2}Y_3^{(2,0)}(0,1) + \frac{a\chi^4}{2}Y_2^{(2,0)}(0,1) - \frac{a\chi^4}{2}Y_3^{(2,0)}(0,1)\Big)\\ &+ \frac{1}{(a\chi^2-1)^2}\left(\frac{\chi^3}{2a}Y_1^{(0,2)}(0,1) + \chi^3 Y_3^{(1,1)}(0,1) - \frac{\chi^4}{a}Y_3^{(0,2)}(0,1) - a\chi^5 Y_3^{(1,1)}(0,1)\right).\end{aligned} \tag{4.11}$$

In order to determine the maps Y_j along the second Segre variety, we need to compute the $2-$jet $j^2_{(0,1)}$ of the Y_j along $S^1_{(0,1)}$. For this purpose, we apply

$$N_1 = \frac{\partial}{\partial\chi} - \frac{\partial w}{\partial\chi}\frac{\partial}{\partial w} = \frac{\partial}{\partial\chi} - \frac{z+2a\chi}{\tau}\frac{\partial}{\partial w},$$

and

$$N_2 = \frac{\partial}{\partial\tau} - \frac{\partial w}{\partial\tau}\frac{\partial}{\partial w} = \frac{\partial}{\partial\tau} + \left(\frac{a\chi^2 + az^2 - z\chi - 1}{\tau^2}\right)\frac{\partial}{\partial w}$$

to (4.7) to obtain equations for $\bar{Y}_j^{(1,0)}(\chi,1-a\chi^2)$ and $\bar{Y}_j^{(0,1)}(\chi,1-a\chi^2)$, $j=1,2,3$, respectively. In both cases, we use the vector fields L and L^2, as before, to obtain two systems of three equations for $\bar{Y}_j^{(1,0)}(\chi,1-a\chi^2)$ and $\bar{Y}_j^{(0,1)}(\chi,1-a\chi^2)$, and then substitute $(z,w)=(0,1)$ and $(\chi,\tau)=(\chi,1-a\chi^2)\in\mathcal{S}^1_{(0,1)}$. Solving the above systems and using the formulas (4.9), (4.10) and (4.11), we can deduce the following expressions:

$$\begin{aligned}\bar{Y}_1^{(1,0)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^2}\Big(&2a^3\chi^5 Y_2^{(1,1)}(0,1) - 2a^3\chi^5 Y_3^{(1,0)}(0,1) - 2a^3\chi^5 Y_3^{(1,1)}(0,1)\\ &- a^2\chi^4 Y_1^{(1,0)}(0,1) - 2a^2\chi^4 Y_1^{(1,1)}(0,1) - a^2\chi^4 Y_2^{(0,1)}(0,1) + 2a^2\chi^4 Y_2^{(0,2)}(0,1)\\ &+ a^2\chi^4 Y_2(0,1) - a^2\chi^4 Y_3^{(0,1)}(0,1) - 2a^2\chi^4 Y_3^{(0,2)}(0,1) + a^2\chi^4 Y_3(0,1)\\ &- 4a^2\chi^3 Y_2^{(1,1)}(0,1) + 4a^2\chi^3 Y_3^{(1,0)}(0,1) + 2a^2\chi^3 Y_3^{(1,1)}(0,1) - 2a\chi^3 Y_1^{(0,2)}(0,1)\\ &+ 2a\chi^2 Y_1^{(1,0)}(0,1) + 2a\chi^2 Y_1^{(1,1)}(0,1) - 2a\chi^2 Y_2^{(0,2)}(0,1) + 3a\chi^2 Y_3^{(0,1)}(0,1)\\ &- 3a\chi^2 Y_3(0,1) + 2a\chi Y_2^{(1,1)}(0,1) - 2a\chi Y_3^{(1,0)}(0,1) + 2\chi Y_1^{(0,1)}(0,1) - 2\chi Y_1(0,1)\\ &- Y_1^{(1,0)}(0,1) + Y_2^{(0,1)}(0,1) - Y_2(0,1)\Big)\end{aligned}$$

$$\bar{Y}_2^{(1,0)}(\chi, 1-a\chi^2) = \frac{1}{a\chi^2-1}\Big(2a^2\chi^3 Y_2^{(0,1)}(0,1) - 2a^2\chi^3 Y_2(0,1) - 2a^2\chi^3 Y_3^{(0,1)}(0,1) - 2a\chi^2 Y_1^{(0,1)}(0,1)$$
$$+ a\chi^2 Y_1(0,1) - 2a\chi Y_2^{(0,1)}(0,1) + 2a\chi Y_2(0,1) + 2a\chi Y_3(0,1) + Y_1(0,1)\Big)$$

$$\bar{Y}_3^{(1,0)}(\chi, 1-a\chi^2) = \frac{1}{2a(a\chi^2-1)^3}\Big(-2a^4 Y_3^{(2,0)}(0,1)\chi^7 + 2a^4 Y_2^{(2,1)}(0,1)\chi^7 - 2a^4 Y_3^{(2,1)}(0,1)\chi^7$$
$$+ 2a^3 Y_2^{(1,1)}(0,1)\chi^6 - 6a^3 Y_3^{(1,1)}(0,1)\chi^6 + 4a^3 Y_2^{(1,2)}(0,1)\chi^6 - 4a^3 Y_3^{(1,2)}(0,1)\chi^6$$
$$- a^3 Y_1^{(2,0)}(0,1)\chi^6 - 2a^3 Y_1^{(2,1)}(0,1)\chi^6 + 4a^4 Y_2(0,1)\chi^5 - 4a^4 Y_2^{(0,1)}(0,1)\chi^5$$
$$+ 4a^4 Y_2^{(0,2)}(0,1)\chi^5 + 2a^2 Y_2^{(0,2)}(0,1)\chi^5 - 4a^4 Y_3^{(0,2)}(0,1)\chi^5 - 4a^2 Y_3^{(0,2)}(0,1)\chi^5$$
$$+ 2a^2 Y_2^{(0,3)}(0,1)\chi^5 - 2a^2 Y_3^{(0,3)}(0,1)\chi^5 - 4a^2 Y_1^{(1,1)}(0,1)\chi^5 - 4a^2 Y_1^{(1,2)}(0,1)\chi^5$$
$$+ 6a^3 Y_3^{(2,0)}(0,1)\chi^5 - 6a^3 Y_2^{(2,1)}(0,1)\chi^5 + 4a^3 Y_3^{(2,1)}(0,1)\chi^5 - 2a^3 Y_1(0,1)\chi^4$$
$$+ 2a^3 Y_1^{(0,1)}(0,1)\chi^4 - 4a^3 Y_1^{(0,2)}(0,1)\chi^4 - 3a Y_1^{(0,2)}(0,1)\chi^4 - 2a Y_1^{(0,3)}(0,1)\chi^4$$
$$- 6a^2 Y_2^{(1,1)}(0,1)\chi^4 + 12a^2 Y_3^{(1,1)}(0,1)\chi^4 - 8a^2 Y_2^{(1,2)}(0,1)\chi^4 + 4a^2 Y_3^{(1,2)}(0,1)\chi^4$$
$$+ 3a^2 Y_1^{(2,0)}(0,1)\chi^4 + 4a^2 Y_1^{(2,1)}(0,1)\chi^4 - 8a^3 Y_2(0,1)\chi^3 - 4a^3 Y_3(0,1)\chi^3$$
$$+ 8a^3 Y_2^{(0,1)}(0,1)\chi^3 + 4a^3 Y_3^{(0,1)}(0,1)\chi^3 - 8a^3 Y_2^{(0,2)}(0,1)\chi^3 - 4a Y_2^{(0,2)}(0,1)\chi^3$$
$$+ 4a^3 Y_3^{(0,2)}(0,1)\chi^3 + 4a Y_3^{(0,2)}(0,1)\chi^3 - 2a Y_2^{(0,3)}(0,1)\chi^3 + 8a Y_1^{(1,1)}(0,1)\chi^3$$
$$+ 4a Y_1^{(1,2)}(0,1)\chi^3 - 6a^2 Y_3^{(2,0)}(0,1)\chi^3 + 6a^2 Y_2^{(2,1)}(0,1)\chi^3 - 2a^2 Y_3^{(2,1)}(0,1)\chi^3$$
$$+ 4a^2 Y_1^{(0,2)}(0,1)\chi^2 + 3Y_1^{(0,2)}(0,1)\chi^2 + 6a Y_2^{(1,1)}(0,1)\chi^2 - 6a Y_3^{(1,1)}(0,1)\chi^2$$
$$+ 4a Y_2^{(1,2)}(0,1)\chi^2 - 3a Y_1^{(2,0)}(0,1)\chi^2 - 2a Y_1^{(2,1)}(0,1)\chi^2 + 4a^2 Y_2(0,1)\chi$$
$$+ 4a^2 Y_3(0,1)\chi - 4a^2 Y_2^{(0,1)}(0,1)\chi - 4a^2 Y_3^{(0,1)}(0,1)\chi + 4a^2 Y_2^{(0,2)}(0,1)\chi$$
$$+ 2Y_2^{(0,2)}(0,1)\chi - 4Y_1^{(1,1)}(0,1)\chi + 2a Y_3^{(2,0)}(0,1)\chi - 2a Y_2^{(2,1)}(0,1)\chi + 2a Y_1(0,1)$$
$$- 2a Y_1^{(0,1)}(0,1) - 2Y_2^{(1,1)}(0,1) + Y_1^{(2,0)}(0,1)\Big)$$

and

$$\bar{Y}_1^{(0,1)}(\chi, 1-a\chi^2) = \frac{1}{(a\chi^2-1)^2}\Big(-a^2\chi^4 Y_2^{(1,0)}(0,1) + a^2\chi^4 Y_2^{(1,1)}(0,1) - a^2\chi^4 Y_3^{(1,1)}(0,1) - a\chi^3 Y_1^{(1,1)}(0,1)$$
$$- a\chi^3 Y_2^{(0,1)}(0,1) + a\chi^3 Y_2^{(0,2)}(0,1) + 2a\chi^2 Y_2^{(1,0)}(0,1) - 2a\chi^2 Y_2^{(1,1)}(0,1)$$
$$+ a\chi^3 Y_2(0,1) - a\chi^3 Y_3^{(0,2)}(0,1) + a\chi^2 Y_3^{(1,1)}(0,1) - \chi^2 Y_1^{(0,2)}(0,1) + \chi Y_1^{(1,1)}(0,1)$$
$$+ \chi Y_2^{(0,1)}(0,1) - \chi Y_2^{(0,2)}(0,1) - Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1) - \chi Y_2(0,1)\Big)$$

$$\bar{Y}_2^{(0,1)}(\chi, 1-a\chi^2) = \frac{1}{a\chi^2-1}\Big(a\chi^2 Y_2^{(0,1)}(0,1) - 2a\chi^2 Y_2(0,1) - a\chi^2 Y_3^{(0,1)}(0,1) + a\chi^2 Y_3(0,1)$$
$$- \chi Y_1^{(0,1)}(0,1) + \chi Y_1(0,1) - Y_2^{(0,1)}(0,1) + 2Y_2(0,1)\Big)$$

$$\bar{Y}_3^{(0,1)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^3}\Big(a^3\chi^6Y_2^{(2,0)}(0,1) - a^3\chi^6Y_2^{(2,1)}(0,1) + 2a^3\chi^4Y_2^{(0,1)}(0,1)$$
$$- 2a^3\chi^4Y_2^{(0,2)}(0,1) - 2a^3\chi^4Y_2(0,1) + a^3\chi^6Y_3^{(2,1)}(0,1) + 2a^3\chi^4Y_3^{(0,2)}(0,1)$$
$$+ a^2\chi^5Y_1^{(2,1)}(0,1) + 2a^2\chi^3Y_1^{(0,2)}(0,1) - 2a^2\chi^5Y_2^{(1,2)}(0,1) - 3a^2\chi^4Y_2^{(2,0)}(0,1)$$
$$+ 3a^2\chi^4Y_2^{(2,1)}(0,1) - 4a^2\chi^2Y_2^{(0,1)}(0,1) + 4a^2\chi^2Y_2^{(0,2)}(0,1) + 4a^2\chi^2Y_2(0,1)$$
$$+ 2a^2\chi^5Y_3^{(1,1)}(0,1) + 2a^2\chi^5Y_3^{(1,2)}(0,1) - 2a^2\chi^4Y_3^{(2,1)}(0,1) - 2a^2\chi^2Y_3^{(0,2)}(0,1)$$
$$+ 2a\chi^4Y_1^{(1,1)}(0,1) + 2a\chi^4Y_1^{(1,2)}(0,1) - 2a\chi^3Y_1^{(2,1)}(0,1) - 2a\chi Y_1^{(0,2)}(0,1)$$
$$- a\chi^4Y_2^{(0,2)}(0,1) - a\chi^4Y_2^{(0,3)}(0,1) + 4a\chi^3Y_2^{(1,2)}(0,1) + 3a\chi^2Y_2^{(2,0)}(0,1)$$
$$- 3a\chi^2Y_2^{(2,1)}(0,1) + 2aY_2^{(0,1)}(0,1) - 2aY_2^{(0,2)}(0,1) - 2aY_2(0,1) + 2a\chi^4Y_3^{(0,2)}(0,1)$$
$$+ a\chi^4Y_3^{(0,3)}(0,1) - 2a\chi^3Y_3^{(1,1)}(0,1) - 2a\chi^3Y_3^{(1,2)}(0,1) + a\chi^2Y_3^{(2,1)}(0,1)$$
$$+ 2\chi^3Y_1^{(0,2)}(0,1) + \chi^3Y_1^{(0,3)}(0,1) - 2\chi^2Y_1^{(1,1)}(0,1) - 2\chi^2Y_1^{(1,2)}(0,1) + \chi Y_1^{(2,1)}(0,1)$$
$$+ \chi^2Y_2^{(0,2)}(0,1) + \chi^2Y_2^{(0,3)}(0,1) - 2\chi Y_2^{(1,2)}(0,1) - Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)\Big).$$

In order to compute the second order derivatives of the maps, we apply the vector fields

$$N_1^2 = \frac{\partial^2}{\partial\chi^2} - 2\frac{z+2a\chi}{\tau}\frac{\partial^2}{\partial\chi\partial w} + \left(\frac{z+2a\chi}{\tau}\right)^2,$$

$$N_1N_2 = \frac{\partial^2}{\partial\chi\partial\tau} - \frac{z+2a\chi}{\tau}\frac{\partial^2}{\partial w\partial\tau} + \frac{az^2+a\chi^2-z\chi-1}{\tau^2}\frac{\partial^2}{\partial\chi\partial w} - (z+2a\chi)\frac{az^2+a\chi^2-z\chi-1}{\tau^2}\frac{\partial^2}{\partial\chi\partial\tau},$$

and

$$N_2^2 = \frac{\partial^2}{\partial\tau^2} + 2\frac{az^2+a\chi^2+z\chi-1}{\tau^2}\frac{\partial^2}{\partial\chi\partial w} + \left(\frac{az^2+a\chi^2-z\chi-1}{\tau^2}\right)^2\frac{\partial^2}{\partial w^2}$$

to (4.7). To each of these three equations, we further apply L and L^2, in order to obtain formulas for $\bar{Y}_j^{(2,0)}(\chi,1-a\chi^2)$, $\bar{Y}_j^{(1,1)}(\chi,1-a\chi^2)$, and $\bar{Y}_j^{(0,2)}(\chi,1-a\chi^2)$, respectively, along $\mathcal{S}^1_{(0,1)}$:

$$\bar{Y}_1^{(2,0)}(\chi,1-a\chi^2) = -\frac{1}{(a\chi^2-1)^3}\Big(-2a^4\chi^6Y_2^{(1,1)}(0,1) - 4a^4\chi^6Y_2^{(1,2)}(0,1) + 2a^4\chi^6Y_3^{(1,0)}(0,1)$$
$$+ 10a^4\chi^6Y_3^{(1,1)}(0,1) + 4a^4\chi^6Y_3^{(1,2)}(0,1) + 6a^3\chi^5Y_1^{(1,1)}(0,1) + 4a^3\chi^5Y_1^{(1,2)}(0,1)$$
$$- 2a^3\chi^5Y_2^{(0,1)}(0,1) - 2a^3\chi^5Y_2^{(0,2)}(0,1) - 4a^3\chi^5Y_2^{(0,3)}(0,1) + 6a^3\chi^4Y_2^{(1,1)}(0,1)$$
$$+ 8a^3\chi^4Y_2^{(1,2)}(0,1) + 2a^3\chi^5Y_2(0,1) + 10a^3\chi^5Y_3^{(0,2)}(0,1) + 4a^3\chi^5Y_3^{(0,3)}(0,1)$$
$$- 6a^3\chi^4Y_3^{(1,0)}(0,1) - 20a^3\chi^4Y_3^{(1,1)}(0,1) - 4a^3\chi^4Y_3^{(1,2)}(0,1) + 6a^2\chi^4Y_1^{(0,2)}(0,1)$$
$$+ 4a^2\chi^4Y_1^{(0,3)}(0,1) - 12a^2\chi^3Y_1^{(1,1)}(0,1) - 4a^2\chi^3Y_1^{(1,2)}(0,1) - 4a^2\chi^3Y_2^{(0,1)}(0,1)$$
$$+ 8a^2\chi^3Y_2^{(0,2)}(0,1) + 4a^2\chi^3Y_2^{(0,3)}(0,1) - 6a^2\chi^2Y_2^{(1,1)}(0,1) - 4a^2\chi^2Y_2^{(1,2)}(0,1)$$
$$+ 4a^2\chi^3Y_2(0,1) + 2a^2\chi^3Y_3^{(0,1)}(0,1) - 14a^2\chi^3Y_3^{(0,2)}(0,1) + 6a^2\chi^2Y_3^{(1,0)}(0,1)$$
$$+ 10a^2\chi^2Y_3^{(1,1)}(0,1) - 2a^2\chi^3Y_3(0,1) + 6a\chi^2Y_1^{(0,1)}(0,1) - 10a\chi^2Y_1^{(0,2)}(0,1)$$
$$+ 6a\chi Y_1^{(1,1)}(0,1) - 6a\chi^2Y_1(0,1) + 6a\chi Y_2^{(0,1)}(0,1) - 6a\chi Y_2^{(0,2)}(0,1)$$
$$+ 2aY_2^{(1,1)}(0,1) - 6a\chi Y_2(0,1) + 6a\chi Y_3^{(0,1)}(0,1) - 2aY_3^{(1,0)}(0,1) - 6a\chi Y_3(0,1)$$
$$+ 2Y_1^{(0,1)}(0,1) - 2Y_1(0,1)\Big)$$

$$\bar{Y}_2^{(2,0)}(\chi,1-a\chi^2)=\frac{1}{(a\chi^2-1)^2}\Big(-6a^3\chi^4Y_2^{(0,1)}(0,1)+4a^3\chi^4Y_2^{(0,2)}(0,1)+6a^3\chi^4Y_2(0,1)
-2a^3\chi^4Y_3^{(0,1)}(0,1)-4a^3\chi^4Y_3^{(0,2)}(0,1)+2a^2\chi^3Y_1^{(0,1)}(0,1)-4a^2\chi^3Y_1^{(0,2)}(0,1)
-2a^2\chi^3Y_1(0,1)+4a^2\chi^2Y_2^{(0,1)}(0,1)-4a^2\chi^2Y_2^{(0,2)}(0,1)-4a^2\chi^2Y_2(0,1)
+10a^2\chi^2Y_3^{(0,1)}(0,1)-6a^2\chi^2Y_3(0,1)+6a\chi Y_1^{(0,1)}(0,1)-6a\chi Y_1(0,1)
+2aY_2^{(0,1)}(0,1)-2aY_2(0,1)-2aY_3(0,1)\Big)$$

$$\bar{Y}_3^{(2,0)}(\chi,1-a\chi^2)=-\frac{1}{a(a\chi^2-1)^4}\Big(a^5Y_3^{(2,0)}(0,1)\chi^8-a^5Y_2^{(2,1)}(0,1)\chi^8+5a^5Y_3^{(2,1)}(0,1)\chi^8
-2a^5Y_2^{(2,2)}(0,1)\chi^8+2a^5Y_3^{(2,2)}(0,1)\chi^8+6a^4Y_3^{(1,1)}(0,1)\chi^7-6a^4Y_2^{(1,2)}(0,1)\chi^7
+14a^4Y_3^{(1,2)}(0,1)\chi^7-4a^4Y_2^{(1,3)}(0,1)\chi^7+4a^4Y_3^{(1,3)}(0,1)\chi^7+3a^4Y_1^{(2,1)}(0,1)\chi^7
+2a^4Y_1^{(2,2)}(0,1)\chi^7+6a^5Y_2(0,1)\chi^6-6a^5Y_2^{(0,1)}(0,1)\chi^6+2a^5Y_2^{(0,2)}(0,1)\chi^6
-a^3Y_2^{(0,2)}(0,1)\chi^6+6a^5Y_3^{(0,2)}(0,1)\chi^6+6a^3Y_3^{(0,2)}(0,1)\chi^6-4a^5Y_2^{(0,3)}(0,1)\chi^6
-5a^3Y_2^{(0,3)}(0,1)\chi^6+4a^5Y_3^{(0,3)}(0,1)\chi^6+9a^3Y_3^{(0,3)}(0,1)\chi^6-2a^3Y_2^{(0,4)}(0,1)\chi^6
+2a^3Y_3^{(0,4)}(0,1)\chi^6+2a^3Y_1^{(1,1)}(0,1)\chi^6+10a^3Y_1^{(1,2)}(0,1)\chi^6+4a^3Y_1^{(1,3)}(0,1)\chi^6
-4a^4Y_3^{(2,0)}(0,1)\chi^6+4a^4Y_2^{(2,1)}(0,1)\chi^6-15a^4Y_3^{(2,1)}(0,1)\chi^6+6a^4Y_2^{(2,2)}(0,1)\chi^6
-4a^4Y_3^{(2,2)}(0,1)\chi^6-2a^4Y_1(0,1)\chi^5+2a^4Y_1^{(0,1)}(0,1)\chi^5+2a^4Y_1^{(0,2)}(0,1)\chi^5
+3a^2Y_1^{(0,2)}(0,1)\chi^5+4a^4Y_1^{(0,3)}(0,1)\chi^5+7a^2Y_1^{(0,3)}(0,1)\chi^5+2a^2Y_1^{(0,4)}(0,1)\chi^5
-18a^3Y_3^{(1,1)}(0,1)\chi^5+18a^3Y_2^{(1,2)}(0,1)\chi^5-28a^3Y_3^{(1,2)}(0,1)\chi^5+8a^3Y_2^{(1,3)}(0,1)\chi^5
-4a^3Y_3^{(1,3)}(0,1)\chi^5-9a^3Y_1^{(2,1)}(0,1)\chi^5-4a^3Y_1^{(2,2)}(0,1)\chi^5-10a^4Y_2(0,1)\chi^4
-6a^4Y_3(0,1)\chi^4+10a^4Y_2^{(0,1)}(0,1)\chi^4+6a^4Y_3^{(0,1)}(0,1)\chi^4-2a^4Y_2^{(0,2)}(0,1)\chi^4
+3a^2Y_2^{(0,2)}(0,1)\chi^4-16a^4Y_3^{(0,2)}(0,1)\chi^4-12a^2Y_3^{(0,2)}(0,1)\chi^4+8a^4Y_2^{(0,3)}(0,1)\chi^4
+10a^2Y_2^{(0,3)}(0,1)\chi^4-4a^4Y_3^{(0,3)}(0,1)\chi^4-9a^2Y_3^{(0,3)}(0,1)\chi^4+2a^2Y_2^{(0,4)}(0,1)\chi^4
-6a^2Y_1^{(1,1)}(0,1)\chi^4-20a^2Y_1^{(1,2)}(0,1)\chi^4-4a^2Y_1^{(1,3)}(0,1)\chi^4+6a^3Y_3^{(2,0)}(0,1)\chi^4
-6a^3Y_2^{(2,1)}(0,1)\chi^4+15a^3Y_3^{(2,1)}(0,1)\chi^4-6a^3Y_2^{(2,2)}(0,1)\chi^4+2a^3Y_3^{(2,2)}(0,1)\chi^4
-4a^3Y_1(0,1)\chi^3+4a^3Y_1^{(0,1)}(0,1)\chi^3-8a^3Y_1^{(0,2)}(0,1)\chi^3-6aY_1^{(0,2)}(0,1)\chi^3
-4a^3Y_1^{(0,3)}(0,1)\chi^3-7aY_1^{(0,3)}(0,1)\chi^3+18a^2Y_3^{(1,1)}(0,1)\chi^3-18a^2Y_2^{(1,2)}(0,1)\chi^3
+14a^2Y_3^{(1,2)}(0,1)\chi^3-4a^2Y_2^{(1,3)}(0,1)\chi^3+9a^2Y_1^{(2,1)}(0,1)\chi^3+2a^2Y_1^{(2,2)}(0,1)\chi^3
+2a^3Y_2(0,1)\chi^2+4a^3Y_3(0,1)\chi^2-2a^3Y_2^{(0,1)}(0,1)\chi^2-4a^3Y_3^{(0,1)}(0,1)\chi^2
-2a^3Y_2^{(0,2)}(0,1)\chi^2-3aY_2^{(0,2)}(0,1)\chi^2+10a^3Y_3^{(0,2)}(0,1)\chi^2+6aY_3^{(0,2)}(0,1)\chi^2
-4a^3Y_2^{(0,3)}(0,1)\chi^2-5aY_2^{(0,3)}(0,1)\chi^2+6aY_1^{(1,1)}(0,1)\chi^2+10aY_1^{(1,2)}(0,1)\chi^2
-4a^2Y_3^{(2,0)}(0,1)\chi^2+4a^2Y_2^{(2,1)}(0,1)\chi^2-5a^2Y_3^{(2,1)}(0,1)\chi^2+2a^2Y_2^{(2,2)}(0,1)\chi^2
+6a^2Y_1(0,1)\chi-6a^2Y_1^{(0,1)}(0,1)\chi+6a^2Y_1^{(0,2)}(0,1)\chi+3Y_1^{(0,2)}(0,1)\chi
-6aY_3^{(1,1)}(0,1)\chi+6aY_2^{(1,2)}(0,1)\chi-3aY_1^{(2,1)}(0,1)\chi+2a^2Y_2(0,1)+2a^2Y_3(0,1)
-2a^2Y_2^{(0,1)}(0,1)-2a^2Y_3^{(0,1)}(0,1)+2a^2Y_2^{(0,2)}(0,1)+Y_2^{(0,2)}(0,1)-2Y_1^{(1,1)}(0,1)
+aY_3^{(2,0)}(0,1)-aY_2^{(2,1)}(0,1)\Big)$$

$$
\begin{aligned}
\bar{Y}_1^{(0,2)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^3}\Big(& a^2\chi^4 Y_2^{(1,2)}(0,1) - 2a^2\chi^4 Y_3^{(1,1)}(0,1) - a^2\chi^4 Y_3^{(1,2)}(0,1) \\
& - 2a\chi^3 Y_1^{(1,1)}(0,1) - a\chi^3 Y_1^{(1,2)}(0,1) + a\chi^3 Y_2^{(0,2)}(0,1) + a\chi^3 Y_2^{(0,3)}(0,1) \\
& - 2a\chi^2 Y_2^{(1,2)}(0,1) - 3a\chi^3 Y_3^{(0,2)}(0,1) - a\chi^3 Y_3^{(0,3)}(0,1) + 2a\chi^2 Y_3^{(1,1)}(0,1) \\
& + a\chi^2 Y_3^{(1,2)}(0,1) - 3\chi^2 Y_1^{(0,2)}(0,1) - \chi^2 Y_1^{(0,3)}(0,1) + 2\chi Y_1^{(1,1)}(0,1) + \chi Y_1^{(1,2)}(0,1) \\
& - \chi Y_2^{(0,2)}(0,1) - \chi Y_2^{(0,3)}(0,1) + Y_2^{(1,2)}(0,1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Y}_2^{(0,2)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^2}\Big(& -2a\chi^2 Y_2^{(0,1)}(0,1) + a\chi^2 Y_2^{(0,2)}(0,1) + 2a\chi^2 Y_2(0,1) - a\chi^2 Y_3^{(0,2)}(0,1) \\
& - \chi Y_1^{(0,2)}(0,1) + 2Y_2^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) - 2Y_2(0,1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Y}_3^{(0,2)}(\chi,1-a\chi^2) = -\frac{1}{2a(a\chi^2-1)^4}\Big(& -a^3\chi^6 Y_2^{(2,2)}(0,1) - 2a^3\chi^4 Y_2^{(0,2)}(0,1) - 2a^3\chi^4 Y_2^{(0,3)}(0,1) \\
& + 2a^3\chi^6 Y_3^{(2,1)}(0,1) + a^3\chi^6 Y_3^{(2,2)}(0,1) + 6a^3\chi^4 Y_3^{(0,2)}(0,1) + 2a^3\chi^4 Y_3^{(0,3)}(0,1) \\
& + 2a^2\chi^5 Y_1^{(2,1)}(0,1) + a^2\chi^5 Y_1^{(2,2)}(0,1) + 6a^2\chi^3 Y_1^{(0,2)}(0,1) + 2a^2\chi^3 Y_1^{(0,3)}(0,1) \\
& - 4a^2\chi^5 Y_2^{(1,2)}(0,1) - 2a^2\chi^5 Y_2^{(1,3)}(0,1) + 3a^2\chi^4 Y_2^{(2,2)}(0,1) + 4a^2\chi^2 Y_2^{(0,2)}(0,1) \\
& + 4a^2\chi^2 Y_2^{(0,3)}(0,1) + 4a^2\chi^5 Y_3^{(1,1)}(0,1) + 8a^2\chi^5 Y_3^{(1,2)}(0,1) + 2a^2\chi^5 Y_3^{(1,3)}(0,1) \\
& - 4a^2\chi^4 Y_3^{(2,1)}(0,1) - 2a^2\chi^4 Y_3^{(2,2)}(0,1) - 6a^2\chi^2 Y_3^{(0,2)}(0,1) - 2a^2\chi^2 Y_3^{(0,3)}(0,1) \\
& + 4a\chi^4 Y_1^{(1,1)}(0,1) + 8a\chi^4 Y_1^{(1,2)}(0,1) + 2a\chi^4 Y_1^{(1,3)}(0,1) - 4a\chi^3 Y_1^{(2,1)}(0,1) \\
& - 2a\chi^3 Y_1^{(2,2)}(0,1) - 6a\chi Y_1^{(0,2)}(0,1) - 2a\chi Y_1^{(0,3)}(0,1) - 2a\chi^4 Y_2^{(0,2)}(0,1) \\
& - 4a\chi^4 Y_2^{(0,3)}(0,1) - a\chi^4 Y_2^{(0,4)}(0,1) + 8a\chi^3 Y_2^{(1,2)}(0,1) + 4a\chi^3 Y_2^{(1,3)}(0,1) \\
& - 3a\chi^2 Y_2^{(2,2)}(0,1) - 2aY_2^{(0,2)}(0,1) - 2aY_2^{(0,3)}(0,1) + 6a\chi^4 Y_3^{(0,2)}(0,1) \\
& + 6a\chi^4 Y_3^{(0,3)}(0,1) + a\chi^4 Y_3^{(0,4)}(0,1) - 4a\chi^3 Y_3^{(1,1)}(0,1) - 8a\chi^3 Y_3^{(1,2)}(0,1) \\
& - 2a\chi^3 Y_3^{(1,3)}(0,1) + 2a\chi^2 Y_3^{(2,1)}(0,1) + a\chi^2 Y_3^{(2,2)}(0,1) + 6\chi^3 Y_1^{(0,2)}(0,1) \\
& + 6\chi^3 Y_1^{(0,3)}(0,1) + \chi^3 Y_1^{(0,4)}(0,1) - 4\chi^2 Y_1^{(1,1)}(0,1) - 8\chi^2 Y_1^{(1,2)}(0,1) \\
& - 2\chi^2 Y_1^{(1,3)}(0,1) + 2\chi Y_1^{(2,1)}(0,1) + \chi Y_1^{(2,2)}(0,1) + 2\chi^2 Y_2^{(0,2)}(0,1) \\
& + 4\chi^2 Y_2^{(0,3)}(0,1) + \chi^2 Y_2^{(0,4)}(0,1) - 4\chi Y_2^{(1,2)}(0,1) - 2\chi Y_2^{(1,3)}(0,1) + Y_2^{(2,2)}(0,1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Y}_1^{(1,1)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^3}\Big(& 2a^3\chi^5 Y_2^{(1,2)}(0,1) - 4a^3\chi^5 Y_3^{(1,1)}(0,1) - 2a^3\chi^5 Y_3^{(1,2)}(0,1) \\
& - 3a^2\chi^4 Y_1^{(1,1)}(0,1) - 2a^2\chi^4 Y_1^{(1,2)}(0,1) + a^2\chi^4 Y_2^{(0,1)}(0,1) + a^2\chi^4 Y_2^{(0,2)}(0,1) \\
& + 2a^2\chi^4 Y_2^{(0,3)}(0,1) - 4a^2\chi^3 Y_2^{(1,2)}(0,1) - a^2\chi^4 Y_2(0,1) - 5a^2\chi^4 Y_3^{(0,2)}(0,1) \\
& - 2a^2\chi^4 Y_3^{(0,3)}(0,1) + 6a^2\chi^3 Y_3^{(1,1)}(0,1) + 2a^2\chi^3 Y_3^{(1,2)}(0,1) - 4a\chi^3 Y_1^{(0,2)}(0,1) \\
& - 2a\chi^3 Y_1^{(0,3)}(0,1) + 4a\chi^2 Y_1^{(1,1)}(0,1) + 2a\chi^2 Y_1^{(1,2)}(0,1) - 2a\chi^2 Y_2^{(0,2)}(0,1) \\
& - 2a\chi^2 Y_2^{(0,3)}(0,1) + 2a\chi Y_2^{(1,2)}(0,1) + 3a\chi^2 Y_3^{(0,2)}(0,1) - 2a\chi Y_3^{(1,1)}(0,1) \\
& + 2\chi Y_1^{(0,2)}(0,1) - Y_1^{(1,1)}(0,1) - Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1) + Y_2(0,1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Y}_2^{(1,1)}(\chi,1-a\chi^2) = \frac{1}{(a\chi^2-1)^2}\Big(& -4a^2\chi^3 Y_2^{(0,1)}(0,1) + 2a^2\chi^3 Y_2^{(0,2)}(0,1) + 4a^2\chi^3 Y_2(0,1) \\
& - 2a^2\chi^3 Y_3^{(0,2)}(0,1) + a\chi^2 Y_1^{(0,1)}(0,1) - 2a\chi^2 Y_1^{(0,2)}(0,1) - a\chi^2 Y_1(0,1) \\
& + 4a\chi Y_2^{(0,1)}(0,1) - 2a\chi Y_2^{(0,2)}(0,1) - 4a\chi Y_2(0,1) + 2a\chi Y_3^{(0,1)}(0,1) - 2a\chi Y_3(0,1) \\
& + Y_1^{(0,1)}(0,1) - Y_1(0,1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Y}_3^{(1,1)}(\chi, 1-a\chi^2) = \frac{1}{2a(a\chi^2-1)^4}\Big(&-4a^4Y_3^{(2,1)}(0,1)\chi^7 + 2a^4Y_2^{(2,2)}(0,1)\chi^7 - 2a^4Y_3^{(2,2)}(0,1)\chi^7 \\
&- 6a^3Y_3^{(1,1)}(0,1)\chi^6 + 6a^3Y_2^{(1,2)}(0,1)\chi^6 - 14a^3Y_3^{(1,2)}(0,1)\chi^6 + 4a^3Y_2^{(1,3)}(0,1)\chi^6 \\
&- 4a^3Y_3^{(1,3)}(0,1)\chi^6 - 3a^3Y_1^{(2,1)}(0,1)\chi^6 - 2a^3Y_1^{(2,2)}(0,1)\chi^6 - 4a^4Y_2(0,1)\chi^5 \\
&+ 4a^4Y_2^{(0,1)}(0,1)\chi^5 + 2a^2Y_2^{(0,2)}(0,1)\chi^5 - 8a^4Y_3^{(0,2)}(0,1)\chi^5 - 8a^2Y_3^{(0,2)}(0,1)\chi^5 \\
&+ 4a^4Y_2^{(0,3)}(0,1)\chi^5 + 6a^2Y_2^{(0,3)}(0,1)\chi^5 - 4a^4Y_3^{(0,3)}(0,1)\chi^5 - 10a^2Y_3^{(0,3)}(0,1)\chi^5 \\
&+ 2a^2Y_2^{(0,4)}(0,1)\chi^5 - 2a^2Y_3^{(0,4)}(0,1)\chi^5 - 4a^2Y_1^{(1,1)}(0,1)\chi^5 - 12a^2Y_1^{(1,2)}(0,1)\chi^5 \\
&- 4a^2Y_1^{(1,3)}(0,1)\chi^5 + 10a^3Y_3^{(2,1)}(0,1)\chi^5 - 6a^3Y_2^{(2,2)}(0,1)\chi^5 + 4a^3Y_3^{(2,2)}(0,1)\chi^5 \\
&- 6a^3Y_1^{(0,2)}(0,1)\chi^4 - 6aY_1^{(0,2)}(0,1)\chi^4 - 4a^3Y_1^{(0,3)}(0,1)\chi^4 - 9aY_1^{(0,3)}(0,1)\chi^4 \\
&- 2aY_1^{(0,4)}(0,1)\chi^4 + 12a^2Y_3^{(1,1)}(0,1)\chi^4 - 14a^2Y_2^{(1,2)}(0,1)\chi^4 + 20a^2Y_3^{(1,2)}(0,1)\chi^4 \\
&- 8a^2Y_2^{(1,3)}(0,1)\chi^4 + 4a^2Y_3^{(1,3)}(0,1)\chi^4 + 7a^2Y_1^{(2,1)}(0,1)\chi^4 + 4a^2Y_1^{(2,2)}(0,1)\chi^4 \\
&+ 8a^3Y_2(0,1)\chi^3 - 8a^3Y_2^{(0,1)}(0,1)\chi^3 - 4aY_2^{(0,2)}(0,1)\chi^3 + 12a^3Y_3^{(0,2)}(0,1)\chi^3 \\
&+ 8aY_3^{(0,2)}(0,1)\chi^3 - 8a^3Y_2^{(0,3)}(0,1)\chi^3 - 8aY_2^{(0,3)}(0,1)\chi^3 + 4a^3Y_3^{(0,3)}(0,1)\chi^3 \\
&+ 4aY_3^{(0,3)}(0,1)\chi^3 - 2aY_2^{(0,4)}(0,1)\chi^3 + 8aY_1^{(1,1)}(0,1)\chi^3 + 16aY_1^{(1,2)}(0,1)\chi^3 \\
&+ 4aY_1^{(1,3)}(0,1)\chi^3 - 8a^2Y_3^{(2,1)}(0,1)\chi^3 + 6a^2Y_2^{(2,2)}(0,1)\chi^3 - 2a^2Y_3^{(2,2)}(0,1)\chi^3 \\
&+ 8a^2Y_1^{(0,2)}(0,1)\chi^2 + 6Y_1^{(0,2)}(0,1)\chi^2 + 4a^2Y_1^{(0,3)}(0,1)\chi^2 + 3Y_1^{(0,3)}(0,1)\chi^2 \\
&- 6aY_3^{(1,1)}(0,1)\chi^2 + 10aY_2^{(1,2)}(0,1)\chi^2 - 6aY_3^{(1,2)}(0,1)\chi^2 + 4aY_2^{(1,3)}(0,1)\chi^2 \\
&- 5aY_1^{(2,1)}(0,1)\chi^2 - 2aY_1^{(2,2)}(0,1)\chi^2 - 4a^2Y_2(0,1)\chi + 4a^2Y_2^{(0,1)}(0,1)\chi \\
&+ 2Y_2^{(0,2)}(0,1)\chi - 4a^2Y_3^{(0,2)}(0,1)\chi + 4a^2Y_2^{(0,3)}(0,1)\chi + 2Y_2^{(0,3)}(0,1)\chi \\
&- 4Y_1^{(1,1)}(0,1)\chi - 4Y_1^{(1,2)}(0,1)\chi + 2aY_3^{(2,1)}(0,1)\chi - 2aY_2^{(2,2)}(0,1)\chi \\
&- 2aY_1^{(0,2)}(0,1) - 2Y_2^{(1,2)}(0,1) + Y_1^{(2,1)}(0,1)\Big).
\end{aligned}
$$

Now, we have all the necessary formulas to determine the functions $Y_j, j = 1,2,3$, along the second Segre variety

$$
\mathcal{S}^2_{(0,1)} = \left\{ \left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2} \right) \in U : z, \chi \in \mathbb{C}, (\chi, 1-a\chi^2) \in {}^*U \right\}.
$$

We apply

$$
\bar{L} = w\frac{\partial}{\partial \chi} - (z+2a\chi)\frac{\partial}{\partial \tau}
$$

and

$$
\bar{L}^2 = w^2\frac{\partial^2}{\partial \chi^2} - 2(z+2a\chi)w\frac{\partial^2}{\partial\chi\partial\tau} + (z+2a\chi)^2\frac{\partial^2}{\partial\tau^2}
$$

to (4.7) and set $(z,w) = \left(z, \frac{1-a\chi^2-az^2-z\chi}{1-a\chi^2}\right) \in \mathcal{S}^2_{(0,1)}$ and $(\chi,\tau) = (\chi, 1-a\chi^2) \in \mathcal{S}^1_{(0,1)}$. Using the formulas for the $2-$jet of $(\bar{Y}_1, \bar{Y}_2, \bar{Y}_3)$ along $\mathcal{S}^1_{(0,1)}$, we can find formulas for Y_1, Y_2 and Y_3 along $\mathcal{S}^2_{(0,1)}$ of the form

$$
Y_j\left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right) = \frac{1}{1-az^2}\sum_{(k,l)\in I_k} z^k \left(\frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right)^l f^j_{kl}(\chi) := \frac{1}{1-az^2}B_j(z,\chi)
\tag{4.12}
$$

where $I_j \subset \mathbb{N}\times\mathbb{Z}, j = 1,2,3$ are finite sets and the f^j_{kl} are rational functions of χ. The full formulas can be found in the Appendix.

Since the second Segre map is generically of full rank, we can generalize the maps

$$Y_j\left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right)$$

to maps $Y_j(z,w)$ for $j = 1,2,3$. For this purpose, we need to write χ as a holomorphic function $\phi(z,w)$ for (z,w) in a neighbourhood U of $(0,1)$. The uniqueness of the function ϕ in U is justified by the Implicit Function Theorem.

Indeed, using the proof of Proposition 2.11 in [3], we express $w-1$ as a power series in χ

$$w-1 = -\frac{az^2+z\chi}{1-a\chi^2} =: Q(z,\chi,1-a\chi^2) = \sum_{j=1}^{\infty} a_j(z)\chi^j,$$

with $a_j(0) = 0$ for all $j \in \mathbb{N}$. Since $Q_\chi(z,0,1) = -z$, we have $a_1(z) = -z$. Therefore, dividing by z^2, we obtain

$$\frac{w-1}{z^2} = -\frac{\chi}{z} + \sum_{j=2}^{\infty} a_j(z)z^{j-2}\left(\frac{\chi}{z}\right)^j.$$

Setting $b_j(z) = a_j(z)z^{j-2}$, for $j \geq 2$, $t = \frac{w-1}{z^2}$, and $u = \frac{\chi}{z}$, yields

$$t = -u + \sum_{j=2}^{\infty} b_j(z)u^j.$$

The Implicit Function Theorem implies then that there is a unique holomorphic function ϕ such that $u = \phi(t,z)$ with $\phi(0,0) = 0$, and hence $\chi = z\phi\left(z, \frac{w-1}{z^2}\right)$. The function ϕ can be found explicitly by solving the equation

$$t = -\frac{a+u}{1-az^2u^2}$$

for u to obtain

$$u = \frac{1 \pm \sqrt{1+4a^2tz^2+4at^2z^2}}{2atz^2}.$$

We express both solution as power series in t :

$$\frac{1-\sqrt{1+4a^2tz^2+4at^2z^2}}{2atz^2} = -a + (a^3z^2-1)t - 2(a^5z^4-a^2z^2)t^2 + O(t^3)$$

and

$$\frac{1+\sqrt{1+4a^2tz^2+4at^2z^2}}{2atz^2} = \frac{1}{az^2t} + a - (a^3z^2-1)t + (2a^5z^4-2a^2z^2)t^2 + O(t^3),$$

and choose the solution with no singularities at $t = 0$.

Hence, $\phi(t,z) = \frac{1-\sqrt{1+4a^2tz^2+4at^2z^2}}{2atz^2}$ Substituting $\chi = z\phi(t,z)$ in the formulas (4.12) and expanding as power series in t, we have

$$\begin{aligned} Y_j(z,w) := Y_j\left(z, \frac{1-az^2-a\chi^2-z\chi}{1-a\chi^2}\right) &= \frac{1}{1-az^2}B_j(z,\chi) \\ = \frac{1}{1-az^2}B_j(z,\phi(t,z)z) &= \frac{1}{1-az^2}\sum_{k=0}^{\infty} b_{j,k}(z)t^k, \end{aligned}$$

for $j = 1,2,3$. Using the Weierstrass Division Theorem, we express $b_{j,k}(z) = z^{2k}q_{j,k}(z) + r_{j,k}(z)$ for all $k \in \mathbb{N}$ and $j = 1,2,3$. Since $t = \frac{w-1}{z^2}$, in order to ensure the holomorphicity of the functions $Y_j(z,w)$ in a neighbourhood of $(0,1)$, $r_{j,k}(z)$ must vanish. This implies $b_{j,k}(z) = z^{2k}q_{j,k}(z)$.

This condition can simplify the lengthy formulas for $Y_j(z,w)$ from the Appendix. For computational simplicity, we introduce the functions $Z_j(z,w)=(1-az^2)Y_j(z,w)$ for $j=1,2,3$, and focus on the conditions imposed by the first ten powers of t. We express

$$u(t,z)=\sum_{l=0}^{10}u_k(z)t^k+O(t^{11}).$$

We substitute the above formula for $u(t,z)$ into $B_j(z,u(t,z)z)$, and then expand as a power series

$$B_j(z,u(t,z)z)=\sum_{k=0}^{10}b_{j,k}(z)t^k+O(t^{11}).$$

The functions $b_{j,1}(z)t, b_{j,2}(z)t^2$ and $b_{j,3}(z)t^3$ have no singularities near $z=0$. For the power series expansion of the coefficient $b_{j,4}(z)$ of t^4 :

$$b_{j,4}(z)=\sum_{l=0}^{7}b_{j,4,l}z^l+O(z^8),$$

we use the Weierstrass Division Theorem for $k=4$ to obtain $\sum_{l=0}^{7}b_{j,4,l}z^l=0$ for all z in a neighbourhood of 0. Hence $b_{j,4,l}=0$ for $j=1,2,3$ and $l=0,\cdots,7$. In particular, we have the following conditions

$$\begin{aligned}&3a^3\left(-3Y_1^{(1,1)}(0,1)+3Y_2^{(0,2)}(0,1)+8Y_3^{(0,2)}(0,1)\right)-\frac{9}{2}a^2Y_3^{(2,1)}(0,1)\\&+\frac{1}{2}a\left(24Y_1^{(1,2)}(0,1)-8Y_2^{(0,3)}(0,1)-5Y_3^{(0,3)}(0,1)\right)=0\end{aligned}$$

$$3a^3Y_3^{(1,1)}(0,1)-6a^2Y_1^{(0,2)}(0,1)-\frac{3}{2}a\left(-Y_1^{(2,1)}(0,1)+Y_2^{(1,2)}(0,1)+Y_3^{(1,2)}(0,1)\right)+\frac{1}{2}Y_1^{(0,3)}(0,1)=0$$

$$6a^3Y_3^{(1,1)}(0,1)-27a^2Y_1^{(0,2)}(0,1)+a\left(3Y_1^{(2,1)}(0,1)-3Y_2^{(1,2)}(0,1)-\frac{21}{2}Y_3^{(1,2)}(0,1)\right)+6Y_1^{(0,3)}(0,1)=0.$$

Similarly, for $k=5$, we express

$$b_{j,5}(z)=\sum_{l=0}^{9}b_{j,5,l}z^l+O(z^{10}).$$

By the Weierstrass Division Theorem, $\sum_{l=0}^{9}b_{j,5,l}z^l=0$ for all z in a neighbourhood of 0, and hence, $b_{j,5,l}=0$ for $j=1,2,3$ and $l=0,\cdots,9$. In particular, we obtain the conditions:

$$-24a^3Y_1^{(0,2)}(0,1)-12a^2Y_3^{(1,2)}(0,1)+8aY_1^{(0,3)}(0,1)=0$$

$$\begin{aligned}&\frac{1}{2}a^5\left(60Y_1^{(1,1)}(0,1)-60Y_2^{(0,2)}(0,1)+156Y_3^{(0,2)}(0,1)\right)+15a^4Y_3^{(2,1)}(0,1)\\&-a^3\left(-12Y_1^{(1,1)}(0,1)-36Y_1^{(1,2)}(0,1)+12Y_2^{(0,2)}(0,1)+18Y_2^{(0,3)}(0,1)+39Y_3^{(0,2)}(0,1)\right.\\&\left.+55Y_3^{(0,3)}(0,1)\right)+\frac{3}{2}a^2\left(4Y_3^{(2,1)}(0,1)+Y_3^{(2,2)}(0,1)\right)+\frac{1}{2}a\left(-39Y_1^{(1,2)}(0,1)\right.\\&\left.-8Y_1^{(1,3)}(0,1)+13Y_2^{(0,3)}(0,1)+4Y_2^{(0,4)}(0,1)+7Y_3^{(0,3)}(0,1)+Y_3^{(0,4)}(0,1)\right)=0\end{aligned}$$

$$\begin{aligned}&3a^3\left(-Y_1^{(1,1)}(0,1)+Y_2^{(0,2)}(0,1)+3Y_3^{(0,2)}(0,1)\right)-\frac{3}{2}a^2Y_3^{(2,1)}(0,1)\\&+\frac{1}{2}a\left(9Y_1^{(1,2)}(0,1)-3Y_2^{(0,3)}(0,1)-2Y_3^{(0,3)}(0,1)\right)=0\end{aligned}$$

$$-54a^4Y_1^{(0,2)}(0,1)-\frac{3}{2}a^3\left(Y_1^{(2,1)}(0,1)+2Y_3^{(1,1)}(0,1)+18Y_3^{(1,2)}(0,1)\right)$$
$$+\frac{1}{4}a^2\left(48Y_1^{(0,2)}(0,1)+86Y_1^{(0,3)}(0,1)\right)$$
$$+\frac{1}{4}a\left(-6Y_1^{(2,1)}(0,1)-3Y_1^{(2,2)}(0,1)+6Y_2^{(1,2)}(0,1)+6Y_2^{(1,3)}(0,1)+6Y_3^{(1,2)}(0,1)+2Y_3^{(1,3)}(0,1)\right)$$
$$+\frac{1}{4}\left(-4Y_1^{(0,3)}(0,1)-Y_1^{(0,4)}(0,1)\right)=0$$

and

$$-12a^5Y_3^{(1,1)}(0,1)-63a^4Y_1^{(0,2)}(0,1)-6a^3\left(Y_1^{(2,1)}(0,1)-Y_2^{(1,2)}(0,1)+2Y_3^{(1,1)}(0,1)+6Y_3^{(1,2)}(0,1)\right)$$
$$+a^2\left(60Y_1^{(0,2)}(0,1)+\frac{119}{2}Y_1^{(0,3)}(0,1)\right)$$
$$+a\left(-6Y_1^{(2,1)}(0,1)-\frac{3}{2}Y_1^{(2,2)}(0,1)+6Y_2^{(1,2)}(0,1)+3Y_2^{(1,3)}(0,1)+18Y_3^{(1,2)}(0,1)+\frac{5}{2}Y_3^{(1,3)}(0,1)\right)$$
$$-\frac{3}{2}\left(8Y_1^{(0,3)}(0,1)+Y_1^{(0,4)}(0,1)\right)=0.$$

Analogously, for $k=6,\cdots,10$, using the Weierstrass Division Theorem, $b_{j,k,l}=0$ for $j=1,2,3$ and $l=0,\cdots,2k-1$, we obtain a list of conditions for the holomorphicity of the functions $Y_j(z,w)$. Together with the above conditions, they yield a system of equations for

$$\{Y_1(0,1),Y_2(0,1),Y_3(0,1),Y_1^{(0,1)}(0,1),Y_2^{(0,1)}(0,1),Y_3^{(0,1)}(0,1),Y_1^{(0,2)}(0,1),Y_2^{(0,2)}(0,1),$$
$$Y_3^{(0,2)}(0,1),Y_1^{(0,3)}(0,1),Y_2^{(0,3)}(0,1),Y_3^{(0,3)}(0,1),Y_1^{(0,4)}(0,1),Y_2^{(0,4)}(0,1),Y_3^{(0,4)}(0,1),$$
$$Y_1^{(1,0)}(0,1),Y_2^{(1,0)}(0,1),Y_3^{(1,0)}(0,1),Y_1^{(1,1)}(0,1),Y_2^{(1,1)}(0,1),Y_3^{(1,1)}(0,1),Y_1^{(1,2)}(0,1),$$
$$Y_2^{(1,2)}(0,1),Y_3^{(1,2)}(0,1),Y_1^{(1,3)}(0,1),Y_2^{(1,3)}(0,1),Y_3^{(1,3)}(0,1),Y_1^{(2,0)}(0,1),Y_2^{(2,0)}(0,1),$$
$$Y_3^{(2,0)}(0,1),Y_1^{(2,1)}(0,1),Y_2^{(2,1)}(0,1),Y_3^{(2,1)}(0,1),Y_1^{(2,2)}(0,1),Y_2^{(2,2)}(0,1),Y_3^{(2,2)}(0,1)\}.$$

The solution to the above system, with the additional condition $a>0$, is:

- $Y_1^{(0,2)}(0,1)=0$
- $Y_3^{(0,2)}(0,1)=0$
- $Y_1^{(0,3)}(0,1)=0$
- $Y_2^{(0,3)}(0,1)=0$
- $Y_3^{(0,3)}(0,1)=0$
- $Y_1^{(0,4)}(0,1)=0$
- $Y_3^{(0,4)}(0,1)=0$
- $Y_1^{(1,1)}(0,1)=\frac{1}{2}Y_2^{(0,2)}(0,1)$
- $Y_3^{(1,1)}(0,1)=0$
- $Y_1^{(1,2)}(0,1)=\frac{1}{3}Y_2^{(0,3)}(0,1)$
- $Y_2^{(1,2)}(0,1)=0$
- $Y_3^{(1,2)}(0,1)=0$
- $Y_1^{(1,3)}(0,1)=\frac{1}{2}Y_2^{(0,4)}(0,1)$
- $Y_3^{(1,3)}(0,1)=0$
- $Y_1^{(2,1)}(0,1)=Y_2^{(1,2)}(0,1)$
- $Y_3^{(2,1)}(0,1)=2\left(aY_2^{(0,2)}(0,1)-aY_1^{(1,1)}(0,1)\right)$
- $Y_1^{(2,2)}(0,1)=2Y_2^{(1,3)}(0,1)$
- $Y_2^{(2,2)}(0,1)=4\left(aY_2^{(0,2)}(0,1)-aY_1^{(1,1)}(0,1)\right)$
- $Y_3^{(2,2)}(0,1)=0.$

We substitute the above conditions into the formulas for $B_j(z,\chi)$ and set

$$\chi=z\phi(t,z)=z\phi\left(\frac{w-1}{z^2}\right).$$

The resulting candidates for the functions $Y_j(z,w)$ are the following rational functions in (z,w) :

$$Y_1(z,w) = \frac{1}{1-az^2}\Big(-awz^2Y_1^{(0,1)}(0,1) - \frac{1}{2}az^4Y_1^{(2,0)}(0,1) - az^3Y_1^{(1,0)}(0,1) + az^2Y_1^{(0,1)}(0,1) - az^2Y_1(0,1) + az^4Y_2^{(1,1)}(0,1) - \frac{1}{2}az^3Y_2^{(0,2)}(0,1) + wY_1^{(0,1)}(0,1) + \frac{1}{2}wzY_2^{(0,2)}(0,1) + \frac{1}{2}z^2Y_1^{(2,0)}(0,1) + zY_1^{(1,0)}(0,1) - Y_1^{(0,1)}(0,1) + Y_1(0,1) + \frac{1}{2}z^3Y_2^{(2,1)}(0,1) - \frac{1}{2}zY_2^{(0,2)}(0,1)\Big), \tag{4.13}$$

$$Y_2(z,w) = \frac{1}{1-az^2}\Big(-\frac{1}{2}a^2z^4Y_2^{(0,2)}(0,1) - awz^2Y_2^{(0,1)}(0,1) - \frac{1}{2}az^4Y_2^{(2,0)}(0,1) + \frac{1}{2}az^4Y_2^{(2,1)}(0,1) - az^3Y_2^{(1,0)}(0,1) + az^3Y_2^{(1,1)}(0,1) + az^2Y_2^{(0,1)}(0,1) - az^2Y_2(0,1) + \frac{1}{2}w^2Y_2^{(0,2)}(0,1) + \frac{1}{2}wz^2Y_2^{(2,1)}(0,1) + wzY_2^{(1,1)}(0,1) + wY_2^{(0,1)}(0,1) - wY_2^{(0,2)}(0,1) + \frac{1}{2}z^2Y_2^{(2,0)}(0,1) - \frac{1}{2}z^2Y_2^{(2,1)}(0,1) + zY_2^{(1,0)}(0,1) - zY_2^{(1,1)}(0,1) - Y_2^{(0,1)}(0,1) + \frac{1}{2}Y_2^{(0,2)}(0,1) + Y_2(0,1)\Big), \tag{4.14}$$

$$Y_3(z,w) = \frac{1}{1-az^2}\Big(-\frac{1}{2}a^2z^4Y_2^{(0,2)}(0,1) + \frac{1}{2}awz^2Y_2^{(0,2)}(0,1) - awz^2Y_3^{(0,1)}(0,1) + \frac{1}{2}az^4Y_2^{(2,1)}(0,1) + az^3Y_2^{(1,1)}(0,1) - \frac{1}{2}az^2Y_2^{(0,2)}(0,1) - \frac{1}{2}az^4Y_3^{(2,0)}(0,1) - az^3Y_3^{(1,0)}(0,1) + az^2Y_3^{(0,1)}(0,1) - az^2Y_3(0,1) + wY_3^{(0,1)}(0,1) + \frac{1}{2}z^2Y_3^{(2,0)}(0,1) + zY_3^{(1,0)}(0,1) - Y_3^{(0,1)}(0,1) + Y_3(0,1)\Big). \tag{4.15}$$

We need to verify if the candidate formulas (4.13), (4.14), and (4.15) satisfy the equation we started with, i.e., using Proposition 2, if there exists some holomorphic function $A(z,w,\chi,\tau)$ such that

$$\begin{aligned} &Y_1(z,w)\chi + Y_2(z,w)\tau + Y_3(z,w)a\chi^2 + \bar{Y}_1(\chi,\tau)z + \bar{Y}_2(\chi,\tau)w + \bar{Y}_3(\chi,\tau)az^2 \\ &= A(z,w,\chi,\tau)\left(1 - z\chi - w\tau - az^2 - a\chi^2\right). \end{aligned} \tag{4.16}$$

Equivalently, multiplying both sides of (4.16) by $(1-az^2)(1-a\chi^2)$ and using $Z_j(z,w) = (1-az^2)Y_j(z,w)$, the above equation can be expressed as

$$\begin{aligned} &Z_1(z,w)\chi(1-a\chi^2) + Z_2(z,w)\tau(1-a\chi^2) + Z_3(z,w)a\chi^2(1-a\chi^2) + \bar{Z}_1(\chi,\tau)z(1-az^2) \\ &+ \bar{Z}_2(\chi,\tau)w(1-az^2) + \bar{Z}_3(\chi,\tau)az^2(1-az^2) = A(z,w,\chi,\tau)\left(1 - z\chi - w\tau - az^2 - a\chi^2\right). \end{aligned} \tag{4.17}$$

We keep the same notation $A(z,w,\chi,\tau)$ for the corresponding holomorphic function on the right-hand side. The left-hand side of (4.17) is a polynomial in (z,w,χ,τ). We may also write

$$A(z,w,\chi,\tau) = \sum_{k=0}^{\infty} A_k(z,w,\chi,\tau)$$

with A_k of weight k, which means that $A_k(\lambda z, \lambda w, \lambda\chi, \lambda\tau) = \lambda^k A_k(z,w,\chi,\tau)$, for all $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$.

Terms of the same weight on both sides must be equal. This implies

$$A_0(z,w,\chi,\tau) = 0 \tag{4.18}$$

$$A_1(z,w,\chi,\tau)=\tau\left(-Y_2^{(0,1)}(0,1)+\frac{1}{2}Y_2^{(0,2)}(0,1)+Y_2(0,1)\right)+w\left(-\bar{Y}_2^{(0,1)}(0,1)+\frac{1}{2}\bar{Y}_2^{(0,2)}(0,1)+\bar{Y}_2(0,1)\right)$$
$$+\chi\left(Y_1(0,1)-Y_1^{(0,1)}(0,1)\right)+z\left(\bar{Y}_1(0,1)-\bar{Y}_1^{(0,1)}(0,1)\right) \tag{4.19}$$

$$\begin{aligned}A_2(z,w,\chi,\tau) &= \chi^2\left(aY_3(0,1)-aY_3^{(0,1)}(0,1)\right)+z^2\left(a\bar{Y}_3(0,1)-a\bar{Y}_3^{(0,1)}(0,1)\right)\\ &+w\chi\left(Y_1^{(0,1)}(0,1)+\bar{Y}_2^{(1,0)}(0,1)-\bar{Y}_2^{(1,1)}(0,1)\right)\\ &+\tau w\left(Y_2^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+\bar{Y}_2^{(0,1)}(0,1)-\bar{Y}_2^{(0,2)}(0,1)\right)\\ &+\chi z\left(Y_1^{(1,0)}(0,1)+\bar{Y}_1^{(1,0)}(0,1)-\frac{1}{2}Y_2^{(0,2)}(0,1)-\frac{1}{2}\bar{Y}_2^{(0,2)}(0,1)\right)\\ &+\tau z\left(\bar{Y}_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)-Y_2^{(1,1)}(0,1)\right)\end{aligned} \tag{4.20}$$

$$\begin{aligned}&A_3(z,w,\chi,\tau)-A_1(z,w,\chi,\tau)\left(z\chi+w\tau+az^2+a\chi^2\right)\\ &=w\chi^2\left(a\bar{Y}_2^{(0,1)}(0,1)-a\bar{Y}_2(0,1)+aY_3^{(0,1)}(0,1)+\frac{1}{2}\bar{Y}_2^{(2,0)}(0,1)-\frac{1}{2}\bar{Y}_2^{(2,1)}(0,1)\right)\\ &+wz^2\left(a\bar{Y}_2^{(0,1)}(0,1)-\frac{1}{2}a\bar{Y}_2^{(0,2)}(0,1)-a\bar{Y}_2(0,1)\right)+\chi^3\left(aY_1^{(0,1)}(0,1)-aY_1(0,1)\right)\\ &+\chi z^2\left(aY_1^{(0,1)}(0,1)-aY_1(0,1)+a\bar{Y}_3^{(1,0)}(0,1)+\frac{1}{2}Y_1^{(2,0)}(0,1)\right)\\ &+\chi^2 z\left(a\bar{Y}_1^{(0,1)}(0,1)-a\bar{Y}_1(0,1)+aY_3^{(1,0)}(0,1)+\frac{1}{2}\bar{Y}_1^{(2,0)}(0,1)\right)\\ &+z^3\left(a\bar{Y}_1^{(0,1)}(0,1)-a\bar{Y}_1(0,1)\right)+\tau\chi^2\left(aY_2^{(0,1)}(0,1)-\frac{1}{2}aY_2^{(0,2)}(0,1)-aY_2(0,1)\right)\\ &+\tau z^2\left(aY_2^{(0,1)}(0,1)-aY_2(0,1)+a\bar{Y}_3^{(0,1)}(0,1)+\frac{1}{2}Y_2^{(2,0)}(0,1)-\frac{1}{2}Y_2^{(2,1)}(0,1)\right)+\frac{1}{2}\tau w^2Y_2^{(0,2)}(0,1)\\ &+\tau wzY_2^{(1,1)}(0,1)+\frac{1}{2}w\chi zY_2^{(0,2)}(0,1)+\frac{1}{2}\tau^2w\bar{Y}_2^{(0,2)}(0,1)+\tau w\chi\bar{Y}_2^{(1,1)}(0,1)+\frac{1}{2}\tau\chi z\bar{Y}_2^{(0,2)}(0,1)\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&A_4(z,w,\chi,\tau) - A_2(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right)\\
&= \chi^2 z^2 \left(-\frac{1}{2}a^2 Y_2^{(0,2)}(0,1) - \frac{1}{2}a^2\bar{Y}_2^{(0,2)}(0,1) + a^2 Y_3^{(0,1)}(0,1) - a^2 Y_3(0,1) + a^2\bar{Y}_3^{(0,1)}(0,1)\right.\\
&\qquad \left. + a^2(-\bar{Y}_3(0,1)) + \frac{1}{2}aY_3^{(2,0)}(0,1) + \frac{1}{2}a\bar{Y}_3^{(2,0)}(0,1)\right) + \chi^4\left(a^2 Y_3^{(0,1)}(0,1) - a^2 Y_3(0,1)\right)\\
&\quad + z^4\left(a^2\bar{Y}_3^{(0,1)}(0,1) - a^2\bar{Y}_3(0,1)\right) + w\chi^3\left(-aY_1^{(0,1)}(0,1) - a\bar{Y}_2^{(1,0)}(0,1) + a\bar{Y}_2^{(1,1)}(0,1)\right)\\
&\quad + w\chi z^2\left(-aY_1^{(0,1)}(0,1) - a\bar{Y}_2^{(1,0)}(0,1) + a\bar{Y}_2^{(1,1)}(0,1)\right)\\
&\quad + \tau w\chi^2\left(-aY_2^{(0,1)}(0,1) + aY_2^{(0,2)}(0,1) - a\bar{Y}_2^{(0,1)}(0,1) + \frac{1}{2}\bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + \tau w z^2\left(-aY_2^{(0,1)}(0,1) - a\bar{Y}_2^{(0,1)}(0,1) + a\bar{Y}_2^{(0,2)}(0,1) + \frac{1}{2}Y_2^{(2,1)}(0,1)\right)\\
&\quad + \chi z^3\left(-aY_1^{(1,0)}(0,1) - a\bar{Y}_1^{(1,0)}(0,1) - \frac{1}{2}aY_2^{(0,2)}(0,1) + \frac{1}{2}a\bar{Y}_2^{(0,2)}(0,1) + \frac{1}{2}Y_2^{(2,1)}(0,1)\right)\\
&\quad + \chi^3 z\left(-aY_1^{(1,0)}(0,1) - a\bar{Y}_1^{(1,0)}(0,1) + \frac{1}{2}aY_2^{(0,2)}(0,1) - \frac{1}{2}a\bar{Y}_2^{(0,2)}(0,1) + \frac{1}{2}\bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + \tau z^3\left(-a\bar{Y}_1^{(0,1)}(0,1) - aY_2^{(1,0)}(0,1) + aY_2^{(1,1)}(0,1)\right)\\
&\quad + \tau\chi^2 z\left(-a\bar{Y}_1^{(0,1)}(0,1) - aY_2^{(1,0)}(0,1) + aY_2^{(1,1)}(0,1)\right)
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&A_5(z,w,\chi,\tau) - A_3(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right)\\
&= w\chi^2 z^2\left(\frac{1}{2}a^2Y_2^{(0,2)}(0,1) - a^2\bar{Y}_2^{(0,1)}(0,1) + a^2\bar{Y}_2(0,1) - a^2Y_3^{(0,1)}(0,1) - \frac{1}{2}a\bar{Y}_2^{(2,0)}(0,1) + \frac{1}{2}a\bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + w\chi^4\left(-\frac{1}{2}a^2\bar{Y}_2^{(0,2)}(0,1) - a^2Y_3^{(0,1)}(0,1) - \frac{1}{2}a\bar{Y}_2^{(2,0)}(0,1) + \frac{1}{2}a\bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + \chi z^4\left(-a^2\bar{Y}_3^{(1,0)}(0,1) - \frac{1}{2}aY_1^{(2,0)}(0,1) + aY_2^{(1,1)}(0,1)\right)\\
&\quad + \chi^3 z^2\left(-a^2Y_1^{(0,1)}(0,1) + a^2Y_1(0,1) + a^2\bar{Y}_2^{(1,1)}(0,1) - a^2\bar{Y}_3^{(1,0)}(0,1) - \frac{1}{2}aY_1^{(2,0)}(0,1)\right)\\
&\quad + \chi^2 z^3\left(-a^2\bar{Y}_1^{(0,1)}(0,1) + a^2\bar{Y}_1(0,1) + a^2Y_2^{(1,1)}(0,1) - a^2Y_3^{(1,0)}(0,1) - \frac{1}{2}a\bar{Y}_1^{(2,0)}(0,1)\right)\\
&\quad + \chi^4 z\left(-a^2Y_3^{(1,0)}(0,1) - \frac{1}{2}a\bar{Y}_1^{(2,0)}(0,1) + a\bar{Y}_2^{(1,1)}(0,1)\right) + \tau\chi^2 z^2\left(-a^2Y_2^{(0,1)}(0,1) + a^2Y_2(0,1)\right.\\
&\qquad\qquad \left. + \frac{1}{2}a^2\bar{Y}_2^{(0,2)}(0,1) - a^2\bar{Y}_3^{(0,1)}(0,1) - \frac{1}{2}aY_2^{(2,0)}(0,1) + \frac{1}{2}aY_2^{(2,1)}(0,1)\right)\\
&\quad + \tau z^4\left(-\frac{1}{2}a^2Y_2^{(0,2)}(0,1) - a^2\bar{Y}_3^{(0,1)}(0,1) - \frac{1}{2}aY_2^{(2,0)}(0,1) + \frac{1}{2}aY_2^{(2,1)}(0,1)\right)\\
&\quad - \frac{1}{2}a\tau w^2\chi^2Y_2^{(0,2)}(0,1) - a\tau w\chi^2 zY_2^{(1,1)}(0,1) - \frac{1}{2}aw\chi^3 zY_2^{(0,2)}(0,1)\\
&\quad - \frac{1}{2}a\tau^2 wz^2\bar{Y}_2^{(0,2)}(0,1) - a\tau w\chi z^2\bar{Y}_2^{(1,1)}(0,1) - \frac{1}{2}a\tau\chi z^3\bar{Y}_2^{(0,2)}(0,1)
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&A_6(z,w,\chi,\tau) - A_4(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right)\\
&= w\chi^3 z^2 \left(a^2 Y_1^{(0,1)}(0,1) + a^2 \bar{Y}_2^{(1,0)}(0,1) - a^2 \bar{Y}_2^{(1,1)}(0,1)\right)\\
&\quad + \tau w \chi^2 z^2 \left(a^2 Y_2^{(0,1)}(0,1) + a^2 \bar{Y}_2^{(0,1)}(0,1) - \frac{1}{2} a Y_2^{(2,1)}(0,1) - \frac{1}{2} a \bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + \chi^3 z^3 \left(a^2 Y_1^{(1,0)}(0,1) + a^2 \bar{Y}_1^{(1,0)}(0,1) + \frac{1}{2} a^2 Y_2^{(0,2)}(0,1) + \frac{1}{2} a^2 \bar{Y}_2^{(0,2)}(0,1) - \frac{1}{2} a Y_2^{(2,1)}(0,1)\right.\\
&\qquad\qquad \left. - \frac{1}{2} a \bar{Y}_2^{(2,1)}(0,1)\right) + \tau \chi^2 z^3 \left(a^2 \bar{Y}_1^{(0,1)}(0,1) + a^2 Y_2^{(1,0)}(0,1) - a^2 Y_2^{(1,1)}(0,1)\right)\\
&\quad + \chi^2 z^4 \left(-\frac{1}{2} a^3 Y_2^{(0,2)}(0,1) + \frac{1}{2} a^3 \bar{Y}_2^{(0,2)}(0,1) - a^3 \bar{Y}_3^{(0,1)}(0,1) + a^3 \bar{Y}_3(0,1) + \frac{1}{2} a^2 Y_2^{(2,1)}(0,1)\right.\\
&\qquad \left. - \frac{1}{2} a^2 Y_3^{(2,0)}(0,1) - \frac{1}{2} a^2 \bar{Y}_3^{(2,0)}(0,1)\right) + \chi^4 z^2 \left(\frac{1}{2} a^3 Y_2^{(0,2)}(0,1) - \frac{1}{2} a^3 \bar{Y}_2^{(0,2)}(0,1) - a^3 Y_3^{(0,1)}(0,1)\right.\\
&\qquad\qquad \left. + a^3 Y_3(0,1) + \frac{1}{2} a^2 \bar{Y}_2^{(2,1)}(0,1) - \frac{1}{2} a^2 Y_3^{(2,0)}(0,1) - \frac{1}{2} a^2 \bar{Y}_3^{(2,0)}(0,1)\right)
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&A_7(z,w,\chi,\tau) - A_5(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right)\\
&= \tau \chi^2 z^4 \left(\frac{1}{2} a^3 Y_2^{(0,2)}(0,1) - \frac{1}{2} a^3 \bar{Y}_2^{(0,2)}(0,1) + a^3 \bar{Y}_3^{(0,1)}(0,1) + \frac{1}{2} a^2 Y_2^{(2,0)}(0,1) - \frac{1}{2} a^2 Y_2^{(2,1)}(0,1)\right)\\
&\quad + w \chi^4 z^2 \left(-\frac{1}{2} a^3 Y_2^{(0,2)}(0,1) + \frac{1}{2} a^3 \bar{Y}_2^{(0,2)}(0,1) + a^3 Y_3^{(0,1)}(0,1) + \frac{1}{2} a^2 \bar{Y}_2^{(2,0)}(0,1) - \frac{1}{2} a^2 \bar{Y}_2^{(2,1)}(0,1)\right)\\
&\quad + \chi^4 z^3 \left(-a^3 Y_2^{(1,1)}(0,1) + a^3 Y_3^{(1,0)}(0,1) + \frac{1}{2} a^2 \bar{Y}_1^{(2,0)}(0,1) - a^2 \bar{Y}_2^{(1,1)}(0,1)\right)\\
&\quad + \chi^3 z^4 \left(-a^3 \bar{Y}_2^{(1,1)}(0,1) + a^3 \bar{Y}_3^{(1,0)}(0,1) + \frac{1}{2} a^2 Y_1^{(2,0)}(0,1) - a^2 Y_2^{(1,1)}(0,1)\right)
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
&A_8(z,w,\chi,\tau) - A_6(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right)\\
&= \chi^4 z^4 \left(\frac{1}{2} a^4 Y_2^{(0,2)}(0,1) + \frac{1}{2} a^4 \bar{Y}_2^{(0,2)}(0,1) - \frac{1}{2} a^3 Y_2^{(2,1)}(0,1) - \frac{1}{2} a^3 \bar{Y}_2^{(2,1)}(0,1) + \frac{1}{2} a^3 Y_3^{(2,0)}(0,1)\right.\\
&\qquad\qquad \left. + \frac{1}{2} a^3 \bar{Y}_3^{(2,0)}(0,1)\right),
\end{aligned} \tag{4.26}
$$

and for $k \geq 7$,

$$
A_{k+2}(z,w,\chi,\tau) - A_k(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right) = 0. \tag{4.27}
$$

The power series $\sum_{k=0}^{\infty} A_k(z,w,\chi,\tau)$ needs to converge in a neighbourhood of $(0,1,0,1)$. We have

$$
\begin{aligned}
\sum_{k=7}^{\infty} A_k(z,w,\chi,\tau) &= \sum_{k=3}^{\infty} A_{2k+1}(z,w,\chi,\tau) + \sum_{k=4}^{\infty} A_{2k}(z,w,\chi,\tau)\\
&= A_7(z,w,\chi,\tau) \sum_{k=0}^{\infty} \left(z\chi + w\tau + az^2 + a\chi^2\right)^k + A_8(z,w,\chi,\tau) \sum_{k=0}^{\infty} \left(z\chi + w\tau + az^2 + a\chi^2\right)^k\\
&= \left(A_7(z,w,\chi,\tau) + A_8(z,w,\chi,\tau)\right) \sum_{k=0}^{\infty} \left(z\chi + w\tau + az^2 + a\chi^2\right)^k.
\end{aligned}
$$

Thus, $A_8(z,w,\chi,\tau) = A_7(z,w,\chi,\tau) = 0$ for (z,w) in a neighbourhood of $(0,1,0,1)$. For $k \in \{3,\cdots,8\}$ the equations (4.21),(4.22),(4.23),(4.24),(4.25) and (4.26) can be written as

$$A_k(z,w,\chi,\tau) - A_{k-2}(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right) = p_k(z,w,\chi,\tau),$$

where p_k is a polynomial of degree k in (z,w,χ,τ). Thus,

$$A_{k-2}(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right) = A_k(z,w,\chi,\tau) - p_k(z,w,\chi,\tau). \tag{4.28}$$

Since all functions in (4.28) are polynomials, the right-hand side is divisible by $z\chi + w\tau + az^2 + a\chi^2$. This means that if

$$A_k(z,w,\chi,\tau) - p_k(z,w,\chi,\tau) = q_k(z,w,\chi,\tau)\left(z\chi + w\tau + az^2 + a\chi^2\right) + r_k(z,w,\chi,\tau), \tag{4.29}$$

then

$$r_k(z,w,\chi,\tau) = 0 \text{ and } A_{k-2}(z,w,\chi,\tau) = q_k(z,w,\chi,\tau),$$

for all (z,w,χ,τ) in a neighbourhood of $(0,1,0,1)$ and $k \in \{3,\cdots,8\}$. Hence, starting backwards from $k=8$ until $k=3$ and equating $q_1(z,w,\chi,\tau)$ and $q_2(z,w,\chi,\tau)$ with the right-hand sides in (4.19) and (4.20), respectively, we get further conditions for the $Y_i^{(j,k)}(0,1)$, $i = 1,2,3$, $j,k = 1,2$ that appear in the above equations:

$$\begin{cases}
\bar{Y}_2(0,1) = -Y_2(0,1) \\
\bar{Y}_2^{(0,1)}(0,1) = Y_2^{(0,1)}(0,1) - 2Y_2(0,1) \\
\bar{Y}_2^{(0,2)}(0,1) = 2Y_2^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) - 2Y_2(0,1) \\
\bar{Y}_1^{(1,0)}(0,1) = -Y_1^{(1,0)}(0,1) + Y_2^{(0,1)}(0,1) - Y_2(0,1) \\
Y_2^{(1,0)}(0,1) = -\bar{Y}_1(0,1) \\
Y_2^{(1,1)}(0,1) = \bar{Y}_1^{(0,1)}(0,1) - \bar{Y}_1(0,1) \\
Y_1^{(2,0)}(0,1) = -2\left(-aY_1^{(0,1)}(0,1) + aY_1(0,1) + a\bar{Y}_3^{(1,0)}(0,1) - \bar{Y}_1^{(0,1)}(0,1) + \bar{Y}_1(0,1)\right) \\
Y_2^{(2,0)}(0,1) = -2\left(-aY_2^{(0,1)}(0,1) + aY_2(0,1) + a\bar{Y}_3(0,1)\right) \\
\bar{Y}_3^{(2,0)}(0,1) = 2aY_2^{(0,1)}(0,1) - 2aY_2(0,1) + 2aY_3^{(0,1)}(0,1) - 2aY_3(0,1) \\
\qquad\qquad + 2a\bar{Y}_3^{(0,1)}(0,1) - 2a\bar{Y}_3(0,1) - Y_3^{(2,0)}(0,1) \\
Y_2^{(2,1)}(0,1) = -2\left(-aY_2^{(0,2)}(0,1) - a\bar{Y}_3^{(0,1)}(0,1) + a\bar{Y}_3(0,1)\right) \\
\bar{Y}_2^{(2,1)}(0,1) = -2a\left(-2Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1) + 2Y_2(0,1) - Y_3^{(0,1)}(0,1) + Y_3(0,1)\right)
\end{cases}$$

Hence, we obtain the parametrization

$$\begin{cases} Y_1(0,1) = \varepsilon + i\zeta \\ Y_2(0,1) = i\alpha \\ Y_3(0,1) = \mu + i\nu \\ Y_1^{(0,1)}(0,1) = \eta + i\theta \\ Y_2^{(0,1)}(0,1) = \beta + i\alpha \\ Y_3^{(0,1)}(0,1) = \sigma + i\psi \\ Y_2^{(0,2)}(0,1) = \beta + i\gamma \\ Y_1^{(1,0)}(0,1) = \frac{\beta}{2} + i\delta \\ Y_2^{(1,0)}(0,1) = -\varepsilon + i\zeta \\ Y_3^{(1,0)}(0,1) = \kappa + i\lambda \\ Y_2^{(1,1)}(0,1) = (\eta - \varepsilon) + i(\zeta - \theta) \\ Y_1^{(2,0)}(0,1) = -2a\varepsilon + i(-2a\zeta + 2a\theta + 2a\lambda + 2\zeta - 2\theta) + 2a\eta - 2a\kappa - 2\varepsilon + 2\eta \\ Y_2^{(2,0)}(0,1) = 2a\beta - 2a\mu + 2ia\nu \\ Y_3^{(2,0)}(0,1) = a\beta - 2a\mu + 2a\sigma + i\varphi \\ Y_2^{(2,1)}(0,1) = 2a\beta + i(2a\gamma + 2a\nu - 2a\psi) - 2a\mu + 2a\sigma \end{cases} \tag{4.30}$$

for $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \sigma, \varphi, \psi \in \mathbb{R}$.

Substituting into (4.13), (4.14), and (4.15), the functions $Z_j(z,w)$ take the form:

$$\begin{aligned} Z_1(z,w) = {} & z^4\left(a^2\varepsilon + ia^2\zeta - a^2\eta - ia^2\theta + a^2\kappa - ia^2\lambda\right) + z^3\left(\frac{ia\gamma}{2} - ia\delta - a\mu + ia\nu + a\sigma - ia\psi\right) \\ & + wz^2(-a\eta - ia\theta) + z^2(-2a\varepsilon - 2ia\zeta + 2a\eta + 2ia\theta - a\kappa + ia\lambda - \varepsilon + i\zeta + \eta - i\theta) \\ & + wz\left(\frac{\beta}{2} + \frac{i\gamma}{2}\right) + z\left(i\delta - \frac{i\gamma}{2}\right) + w(\eta + i\theta) + \varepsilon + i\zeta - \eta - i\theta, \end{aligned}$$

$$\begin{aligned} Z_2(z,w) = {} & z^4\left(-\frac{a^2\beta}{2} + \frac{1}{2}ia^2\gamma + a^2\sigma - ia^2\psi\right) + z^3(a\eta - ia\theta) \\ & + wz^2(-ia\alpha + ia\gamma - a\mu + ia\nu + a\sigma - ia\psi) + z^2(a\beta - ia\gamma - a\sigma + ia\psi) \\ & + wz(-\varepsilon + i\zeta + \eta - i\theta) + w^2\left(\frac{\beta}{2} + \frac{i\gamma}{2}\right) + z(-\eta + i\theta) + w(i\alpha - i\gamma) - \frac{\beta}{2} + \frac{i\gamma}{2}, \end{aligned}$$

and

$$\begin{aligned} Z_3(z,w) = {} & z^4\left(\frac{1}{2}ia^2\gamma + ia^2\nu - ia^2\psi - \frac{ia\varphi}{2}\right) + z^3(-a\varepsilon + ia\zeta + a\eta - ia\theta - a\kappa - ia\lambda) \\ & + wz^2\left(\frac{a\beta}{2} + \frac{ia\gamma}{2} - a\sigma - ia\psi\right) + z^2\left(-\frac{1}{2}ia\gamma - 2a\mu - ia\nu + 2a\sigma + ia\psi + \frac{i\varphi}{2}\right) \\ & + z(\kappa + i\lambda) + w(\sigma + i\psi) + \mu + i\nu - \sigma - i\psi. \end{aligned}$$

Indeed, the above formulas satisfy the equation (4.17) with

$$\begin{aligned} A(z,w,\chi,\tau) = \frac{1}{2}\big(& \chi^2 z^2\left(4a^2\mu - 4a^2\sigma\right) + \chi^2(-2a\mu - 2ia\nu + 2a\sigma + 2ia\psi) + w\chi^2(-a\beta - ia\gamma) \\ & + \tau z^2(-a\beta + ia\gamma) + \chi z^2(2a\varepsilon + 2ia\zeta - 2a\eta - 2ia\theta) + z^2(-2a\mu + 2ia\nu + 2a\sigma - 2ia\psi) \\ & + \chi^2 z(2a\varepsilon - 2ia\zeta - 2a\eta + 2ia\theta) + \tau(\beta - i\gamma) + \chi(-2\varepsilon - 2i\zeta + 2\eta + 2i\theta) + w(\beta + i\gamma) \\ & + z(-2\varepsilon + 2i\zeta + 2\eta - 2i\theta)\big). \end{aligned}$$

Hence, $\mathfrak{hol}(H)$ can be described as follows

$$\mathfrak{hol}(H) = \left\{ X = X_1 \frac{\partial}{\partial z_1} + X_2 \frac{\partial}{\partial z_2} + X_3 \frac{\partial}{\partial z_3} \right\},$$

where

$$X_j(z, w) = \frac{1}{(1 - az^2)^2} Z_j(z, w),$$

for $j = 1, 2, 3$.

4.7 Trivial infinitesimal deformations of the map

Our next target is to determine the space of trivial infinitesimal deformations of H :

$$\mathfrak{aut}(H) = H_* \left(\mathfrak{hol}(\mathcal{E}) \right) + \mathfrak{hol}(\mathbb{S}^3) \Big|_{H(\mathcal{E})}.$$

Formulas for the infinitesimal automorphisms of quadrics, i.e. hypersurfaces with defining equation

$$\operatorname{Im} w = zAz^*,$$

where A is a hermitian $n \times n$ matrix or spheres with defining equations

$$zz^* = 1$$

can be found in [12].

We will include in this chapter part of the computations from [12] in the special case of the sphere $\mathbb{S}^3$. Therefore, we will compute $\mathfrak{hol}(\mathbb{S}^3)$ by using the projective space of the defining equation

$$\mathbb{S}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1.$$

For this purpose, we set $z_1 = \xi_1 \nu$, $z_2 = \xi_2 \nu$, and $z_3 = \xi_3 \nu$. With these notations, the defining equation can be written in the form of matrix multiplication:

$$\Lambda A \Lambda^* := \begin{bmatrix} \nu & \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\nu} \\ \bar{\xi}_1 \\ \bar{\xi}_2 \\ \bar{\xi}_3 \end{bmatrix} = 0. \tag{4.31}$$

Let

$$\frac{\partial}{\partial \Lambda} = \begin{bmatrix} \frac{\partial}{\partial \nu} \\ \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \xi_3} \end{bmatrix}$$

and consider a linear vector field

$$X_P = \Lambda P \frac{\partial}{\partial \Lambda},$$

where P is a 4×4 matrix. Then,

$$\begin{aligned} 2ReX_P \Lambda^* A \Lambda &= \left(\Lambda P \frac{\partial}{\partial \Lambda} + \frac{\partial}{\partial \Lambda}^* P^* \lambda^* \right) \Lambda A \Lambda^* \\ &= \Lambda \left(PA + AP^* \right) \Lambda^*, \end{aligned}$$

and hence ReX_P is tangent to the sphere if and only if

$$\Lambda \left(PA + AP^* \right) \Lambda^* = c\left(\Lambda, \Lambda^* \right) \Lambda A \Lambda^*.$$

By homogeneity, c has to be a real constant, which implies the equation $PA + AP^* = cA$. Since the action of matrices cI is trivial in the projective space, we derive the following equation:

$$PA + AP^* = 0.$$

Equivalently, when expressed in the form of block matrices, this becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} = 0.$$

Hence

- $A + \bar{A} = 0$
- $C = B^*$
- $B = C^*$
- $D = -D^*$.

Thus, the matrix P takes the block form

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$$

with A purely imaginary and $D = -D^*$ a skew-hermitian 3×3 matrix. Using the notation

$$\frac{\partial}{\partial \xi} = \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \xi_3} \end{bmatrix}, \quad \frac{\partial}{\partial Z} = \begin{bmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial z_3} \end{bmatrix}, \quad \Xi = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}, \tag{4.32}$$

and the rules

$$\frac{\partial}{\partial z_k} = \nu \frac{\partial}{\partial \xi_k} \quad \nu \frac{\partial}{\partial \nu} = -\sum_{j=1}^{3} z_j \frac{\partial}{\partial z_j} \tag{4.33}$$

for $k = 1, 2, 3$, we obtain the following three families of vector fields

$$\begin{array}{ccc} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} & A\nu \frac{\partial}{\partial \nu} & -A \sum_{j=1}^{3} z_j \frac{\partial}{\partial z_j} \\ \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} & \nu B \frac{\partial}{\partial \Xi} + \Xi B^* \frac{\partial}{\partial \nu} & B \frac{\partial}{\partial Z} - Z B^* Z \frac{\partial}{\partial Z} \\ \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} & \Xi D \frac{\partial}{\partial \Xi} & Z D \frac{\partial}{\partial Z}. \end{array}$$

Hence, writing

$$A = is, \quad B = \begin{bmatrix} c_1 + id_1 & c_2 + id_2 & c_3 + id_3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} is_2 & -a_1 + ib_1 & -a_2 + ib_2 \\ a_1 + ib_1 & is_3 & -a_3 + ib_3 \\ a_2 + ib_2 & a_3 + ib_3 & is_3 \end{bmatrix}$$

with $s, s_i, a_i, b_i, c_i, d_i \in \mathbb{R}$ for $i = 1, 2, 3$, we obtain a list of the vector fields that together with their linear combinations form $\mathfrak{hol}(\mathbb{S}^3)$:

- $K_1(z, w) = (is_1 z_1, is_2 z_2, is_3 z_3)$
- $K_2(z, w) = ((a_1 + ib_1)z_2, (-a_1 + ib_1)z_1, 0)$
- $K_3(z, w) = ((a_2 + ib_2)z_3, 0, (-a_2 + ib_2)z_1)$
- $K_4(z, w) = (0, (a_3 + ib_3)z_3, (-a_3 + ib_3)z_2)$
- $K_5(z, w) = (c_1 + id_1 - z_1^2\bar{c}_1 - z_1 z_2 \bar{c}_2 - z_1 z_3 \bar{c}_3 + iz_1^2 \bar{d}_1 + iz_1 z_2 \bar{d}_2 + iz_1 z_3 \bar{d}_3, c_2 + id_2 - z_1 z_2 \bar{c}_1 - z_2^2 \bar{c}_2 - z_2 z_3 \bar{c}_3 + iz_1 z_2 \bar{d}_1 + iz_2^2 \bar{d}_2 + iz_2 z_2 \bar{d}_3, c_3 + id_3 - z_1 z_3 \bar{c}_1 - z_2 z_3 \bar{c}_2 - z_3^3 \bar{c}_3 + iz_1 z_3 \bar{d}_1 + iz_2 z_3 \bar{d}_2 + iz_3^2 \bar{d}_3)$,

where we use the notation $F = (F_1, F_2, F_3)$ for a vector field of the form

$$F = F_1 \frac{\partial}{\partial z_1} + F_2 \frac{\partial}{\partial z_2} + F_3 \frac{\partial}{\partial z_3}.$$

Setting

$$(z_1, z_2, z_3) = \left(\frac{z}{1 - az^2}, \frac{w}{1 - az^2}, \frac{az^2}{1 - az^2} \right) = H(z, w),$$

we can describe $\mathfrak{hol}(\mathbb{S}^3)|_{H(\mathcal{E})}$ as

$$\mathfrak{hol}(\mathbb{S}^3)\Big|_{H(\mathcal{E})} = \{F(z, w) = (F_1(z, w), F_2(z, w), F_3(z, w))\} \tag{4.34}$$

with

$$\begin{aligned} F_1(z, w) = &\frac{w(a_1 + ib_1)}{1 - az^2} + \frac{az^2(a_2 + ib_2)}{1 - az^2} \\ &- \frac{z^2 (c_1 - id_1) + wz (c_2 - id_2) + az^3 (c_3 - id_3)}{(1 - az^2)^2} + \frac{is_1 z}{1 - az^2} + c_1 + id_1 \end{aligned}$$

$$\begin{aligned} F_2(z, w) = &\frac{z(-a_1 + ib_1)}{1 - az^2} + \frac{az^2(a_3 + ib_3)}{1 - az^2} \\ &- \frac{zw (c_1 - id_1) + w^2 (c_2 - id_2) + awz^2 (c_3 - id_3)}{(1 - az^2)^2} + \frac{is_2 w}{1 - az^2} + c_2 + id_2 \end{aligned}$$

$$\begin{aligned} F_3(z, w) = &\frac{z(-a_2 + ib_2)}{1 - az^2} + \frac{w(-a_3 + ib_3)}{1 - az^2} \\ &- \frac{az^3 (c_1 - id_1) + awz^2 (c_2 - id_2) + a^2z^4 (c_3 - id_3)}{(1 - az^2)^2} + \frac{ias_3 z^2}{1 - az^2} + c_3 + id_3 \end{aligned}$$

for $s, s_i, a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2, 3$.

Regarding the automorphisms of the domain $\mathcal{E}$, they are rotations of the form $\phi(z, w) = (\pm z, e^{i\theta} w)$, where $\theta \in \mathbb{R}$. Then,

$$(H \circ \phi)(z, w) = \left(\frac{\pm z}{1 - az^2}, \frac{e^{i\theta} w}{1 - az^2}, \frac{az^2}{1 - az^2} \right) = (\psi \circ H),$$

where $\psi(z_1, z_2, z_3) = (\pm z_1, e^{i\theta} z_2, z_3)$ is an automorphism of the sphere $\mathbb{S}^3$. Therefore, these rotations do not produce any new vector fields that we have not already considered.

4.8 Local rigidity of the map

By section 3.3, for the vector spaces $\mathfrak{aut}(H)$ and $\mathfrak{hol}(H)$, we have $\mathfrak{aut}(H) \subseteq \mathfrak{hol}(H)$. In the previous section, we described all the vector fields in $\mathfrak{aut}(H)$. We will now verify that any vector field in $\mathfrak{hol}(H)$ can be written in the form described in (4.34). Indeed, solving the equations

$$X_j(z, w) = F_j(z, w) \tag{4.35}$$

for $j = 1, 2, 3$ and all (z, w) in a neighbourhood of $(0, 1)$ we obtain a unique solution for the parameters $s, s_j, a_j, b_j, c_j, d_j, j = 1, 2, 3$ with respect to the parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \sigma, \varphi, \psi$ used in the description of the vector fields in $\mathfrak{hol}(H)$:

- $s_1 = -\frac{\gamma}{2} + \delta$
- $s_2 = \alpha - \gamma$
- $s_3 = -\frac{\gamma}{2} + \nu + \frac{\phi}{2a} - \psi$
- $a_1 = \eta$
- $a_2 = -\kappa$
- $a_3 = -\sigma$
- $b_1 = \theta$
- $b_2 = \lambda$
- $b_3 = \psi$
- $c_1 = \epsilon - \eta$
- $c_2 = \frac{\beta}{2}$
- $c_3 = \mu - \sigma$
- $d_1 = \zeta - \theta$
- $d_2 = \frac{\gamma}{2}$
- $d_3 = \nu - \psi$.

Thus, $\mathfrak{hol}(H) = \mathfrak{aut}(H)$, and in light of Propositions 7 and 8, this proves the local rigidity of the map H. This means that close to the mapping

$$H : \mathcal{E} \to \mathbb{S}^3$$

$$H(z, w) = \left(\frac{z}{1 - az^2}, \frac{w}{1 - az^2}, \frac{az^2}{1 - az^2} \right)$$

there is no other holomorphic map, up to automorphisms of the ellipsoid $\mathcal{E}$ and the sphere $\mathbb{S}^3$.

The same procedure done so far can be applied similarly for real ellipsoids of the form

$$\mathcal{E} : |z|^2 + |w|^2 + b\left(w^2 + \bar{w}^2\right) = 1$$

and map

$$H(z, w) = \left(\frac{z}{1 - bw^2}, \frac{w}{1 - bw^2}, \frac{bw^2}{1 - bw^2} \right),$$

by symmetry.

Hence, in the cases $a = 0$ or $b = 0$, we proved local uniqueness of the map H, up to automorphisms of the real ellipsoid $\mathcal{E} \subset \mathbb{C}^2$ and the sphere $\mathbb{S}^3 \subset \mathbb{C}^3$.

5 Appendix

The full formulas for the functions $Y_j, j = 1, 2, 3$ described by (4.12) are the following:

$$
\begin{aligned}
&Y_1\left(z,\frac{z\chi+a\left(z^2+\chi^2\right)-1}{a\chi^2-1}\right)=\frac{1}{4\left(az^2-1\right)\left(a\chi^2-1\right)^4\left(z\chi+a(z^2+\chi^2)-1\right)^2}(a^4(4z^2Y_2^{(1,0)}(0,1)a^3\\
&+4(z(-2Y_2^{(0,1)}(0,1)+Y_3^{(0,1)}(0,1)+z(Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))-Y_2^{(1,0)}(0,1))a^2\\
&+2(2(z^2-1)Y_1^{(0,1)}(0,1)-2(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))+z(-2Y_2^{(0,1)}(0,1)+2Y_1^{(1,1)}(0,1)\\
&+z(z(-2Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)-Y_3^{(0,2)}(0,1))+2(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1)))a+z(-((2(2Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)+Y_2^{(2,0)}(0,1)+2Y_2^{(2,1)}(0,1))+Y_2^{(2,2)}(0,1))z^2)\\
&+4(Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))z-2(2Y_2^{(0,1)}(0,1)\\
&+Y_2^{(0,2)}(0,1))))\chi^{12}-2a^3z(2z^2(-4Y_1^{(0,1)}(0,1)+3z(2Y_3^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1))\\
&-6Y_2^{(1,0)}(0,1)-4Y_2^{(1,1)}(0,1))a^3+(3Y_2^{(2,2)}(0,1)z^3-2(6Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1)\\
&+z(4Y_1^{(0,2)}(0,1)+6Y_2^{(1,0)}(0,1)+14Y_2^{(1,1)}(0,1)+4Y_2^{(1,2)}(0,1)-3z(2Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+2Y_2^{(2,1)}(0,1))))z-2(6z^2+5)Y_1^{(0,1)}(0,1)+6Y_2^{(1,0)}(0,1)+2Y_2^{(1,1)}(0,1))a^2\\
&+((Y_1^{(0,2)}(0,1)+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))z^2-(4Y_2^{(0,1)}(0,1)+4Y_2^{(0,2)}(0,1)+12Y_1^{(1,1)}(0,1)\\
&+2Y_1^{(1,2)}(0,1)+6Y_2^{(2,0)}(0,1)+8Y_2^{(2,1)}(0,1)+Y_2^{(2,2)}(0,1))z+12Y_1^{(0,1)}(0,1)+2(Y_1^{(0,2)}(0,1)\\
&+6Y_2^{(1,0)}(0,1)+8Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))a+z(-4Y_2^{(0,1)}(0,1)-2(4Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1))\\
&+z(2Y_1^{(0,2)}(0,1)+Y_1^{(0,3)}(0,1)+4Y_2^{(1,1)}(0,1)+5Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1))))\chi^{11}\\
&+a^2(-8z^4(-2Y_1^{(0,1)}(0,1)+3z(2Y_3^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1))-3Y_2^{(1,0)}(0,1)\\
&-2Y_2^{(1,1)}(0,1))a^5-4z^2(3Y_2^{(2,2)}(0,1)z^3-2(6Y_2^{(0,1)}(0,1)+9Y_3^{(0,1)}(0,1)+2(Y_2^{(0,2)}(0,1)\\
&+Y_3^{(0,2)}(0,1))+z(2Y_1^{(0,2)}(0,1)+3Y_2^{(1,0)}(0,1)+7Y_2^{(1,1)}(0,1)+2Y_2^{(1,2)}(0,1)-3z(2Y_1^{(1,1)}(0,1)\\
&+Y_1^{(1,2)}(0,1)+Y_2^{(2,0)}(0,1)+2Y_2^{(2,1)}(0,1))))z-2(3z^2+4)Y_1^{(0,1)}(0,1)+4(3Y_2^{(1,0)}(0,1)\\
&+Y_2^{(1,1)}(0,1)))a^4-4((z^4+15z^2+12)Y_1^{(0,1)}(0,1)-6Y_2^{(1,0)}(0,1)+z(3(3Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))\\
&+z((3Y_1^{(0,2)}(0,1)+7Y_2^{(1,1)}(0,1)+3Y_2^{(1,2)}(0,1))z^2-(14Y_2^{(0,1)}(0,1)+8Y_2^{(0,2)}(0,1)+18Y_1^{(1,1)}(0,1)\\
&+4Y_1^{(1,2)}(0,1)+9Y_2^{(2,0)}(0,1)+13Y_2^{(2,1)}(0,1)+2Y_2^{(2,2)}(0,1))z+4Y_1^{(0,2)}(0,1)+15Y_2^{(1,0)}(0,1)\\
&+23Y_2^{(1,1)}(0,1)+4Y_2^{(1,2)}(0,1))))a^3+2(2(10z^2+9)Y_1^{(0,1)}(0,1)+18(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\\
&-z((12-26z^2)Y_2^{(0,1)}(0,1)+6Y_2^{(0,2)}(0,1)+z(z(-6Y_3^{(0,1)}(0,1)-7(5Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1))\\
&+2z(7Y_1^{(0,2)}(0,1)+3Y_1^{(0,3)}(0,1)+14Y_2^{(1,1)}(0,1)+16Y_2^{(1,2)}(0,1)+3Y_2^{(1,3)}(0,1)))-2(Y_1^{(0,2)}(0,1)\\
&+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))+9(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))))a^2+2z((-2Y_2^{(0,2)}(0,1)\\
&-Y_2^{(0,3)}(0,1)+3(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))z^2+2(-3Y_1^{(0,1)}(0,1)+6Y_1^{(0,2)}(0,1)\\
&+Y_1^{(0,3)}(0,1)-3Y_2^{(1,0)}(0,1)+9(Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))+Y_2^{(1,3)}(0,1))z-3(8Y_2^{(0,1)}(0,1)\\
&+7Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1)))a-z^3(6(Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1))+Y_2^{(0,4)}(0,1)))\chi^{10}\\
&-2a^2(8a^6(2Y_3^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1))z^6+4a^5((2(2Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+2Y_2^{(2,1)}(0,1))+Y_2^{(2,2)}(0,1))z^2-8Y_2^{(0,1)}(0,1)-8Y_3^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)\\
&-2Y_3^{(0,2)}(0,1))z^4-(-8Y_2^{(0,1)}(0,1)+20Y_2^{(0,2)}(0,1)+12Y_2^{(0,3)}(0,1)+Y_2^{(0,4)}(0,1)+z(4Y_1^{(0,2)}(0,1)\\
&-Y_1^{(0,3)}(0,1)+8Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)-Y_2^{(1,3)}(0,1)))z^2+4a^4((2Y_1^{(0,1)}(0,1)+3Y_1^{(0,2)}(0,1)
\end{aligned}
$$

$$
\begin{aligned}
&+ Y_2^{(1,0)}(0,1) + 8Y_2^{(1,1)}(0,1) + 3Y_2^{(1,2)}(0,1))z^3 - (8Y_2^{(0,1)}(0,1) + 4Y_2^{(0,2)}(0,1) + 8Y_1^{(1,1)}(0,1) \\
&+ 2Y_1^{(1,2)}(0,1) + 4Y_2^{(2,0)}(0,1) + 6Y_2^{(2,1)}(0,1) + Y_2^{(2,2)}(0,1))z^2 + 4(3Y_2^{(0,1)}(0,1) + Y_3^{(0,1)}(0,1) \\
&+ Y_2^{(0,2)}(0,1)))z^2 + 2a^3(2(Y_1^{(0,1)}(0,1) + 8Y_1^{(0,2)}(0,1) + 3Y_1^{(0,3)}(0,1) + Y_2^{(1,0)}(0,1) + 17(Y_2^{(1,1)}(0,1) \\
&+ Y_2^{(1,2)}(0,1)) + 3Y_2^{(1,3)}(0,1))z^5 - (14Y_2^{(0,1)}(0,1) + 16Y_3^{(0,1)}(0,1) + 7Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) \\
&+ Y_2^{(0,3)}(0,1))z^4 - 2(11Y_1^{(0,1)}(0,1) + 2Y_1^{(0,2)}(0,1) - 7Y_2^{(1,0)}(0,1) + 3Y_2^{(1,1)}(0,1) + 2Y_2^{(1,2)}(0,1))z^3 \\
&+ 4(6Y_2^{(0,1)}(0,1) + 2Y_2^{(0,2)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))z^2 - 8Y_2^{(0,1)}(0,1)) \\
&+ a(z(-8(z^2+4)Y_1^{(0,1)}(0,1) + 16(Y_1^{(0,2)}(0,1) - 2Y_2^{(1,0)}(0,1) + Y_2^{(1,2)}(0,1)) + z(8(3z^2-1)Y_2^{(0,2)}(0,1) \\
&+ (20z^2-2)Y_2^{(0,3)}(0,1) + z(3zY_2^{(0,4)}(0,1) - 2(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1))) + 16(2Y_1^{(1,1)}(0,1) \\
&+ Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))) - 16(2Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1))) + 2a^2(z((14z^2+39)Y_1^{(0,1)}(0,1) \\
&- 7Y_2^{(1,0)}(0,1) + 3Y_2^{(1,1)}(0,1) + z((8Y_2^{(0,2)}(0,1) + 3Y_2^{(0,3)}(0,1) - 2(8Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) \\
&+ 4Y_2^{(2,0)}(0,1) + 5Y_2^{(2,1)}(0,1)) - Y_2^{(2,2)}(0,1))z^2 - 2(9Y_1^{(0,2)}(0,1) + 2Y_1^{(0,3)}(0,1) - 7Y_2^{(1,0)}(0,1) \\
&+ 11Y_2^{(1,1)}(0,1) + 15Y_2^{(1,2)}(0,1) + 2Y_2^{(1,3)}(0,1))z + 22Y_2^{(0,1)}(0,1) + 16Y_3^{(0,1)}(0,1) + 23Y_2^{(0,2)}(0,1) \\
&+ 3Y_2^{(0,3)}(0,1))) - 8Y_2^{(0,1)}(0,1)))\chi^9 - 2a(2a^6(4Y_1^{(0,1)}(0,1) + 4Y_1^{(0,2)}(0,1) + 3Y_2^{(1,0)}(0,1) \\
&+ 12Y_2^{(1,1)}(0,1) + 4Y_2^{(1,2)}(0,1))z^6 + 2a^5((3z^2-8)Y_1^{(0,1)}(0,1) - 4Y_1^{(0,2)}(0,1) + 15Y_2^{(1,0)}(0,1) \\
&- 4(Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) + z(6Y_2^{(0,1)}(0,1) - 33Y_3^{(0,1)}(0,1) + 34Y_2^{(0,2)}(0,1) - 8Y_3^{(0,2)}(0,1) \\
&+ 6Y_2^{(0,3)}(0,1) + z(12Y_1^{(0,2)}(0,1) + 4Y_1^{(0,3)}(0,1) + 3(Y_2^{(1,0)}(0,1) + 9Y_2^{(1,1)}(0,1) + 8Y_2^{(1,2)}(0,1)) \\
&+ 4Y_2^{(1,3)}(0,1))))z^4 + (-3Y_2^{(0,2)}(0,1) + 3Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1))z^3 + a^4(2(9z^2+40)Y_1^{(0,1)}(0,1) \\
&- 2(33Y_2^{(1,0)}(0,1) + 8Y_2^{(1,1)}(0,1)) + z((40Y_2^{(0,2)}(0,1) + 12Y_2^{(0,3)}(0,1) - 66Y_1^{(1,1)}(0,1) - 16Y_1^{(1,2)}(0,1) \\
&- 33Y_2^{(2,0)}(0,1) - 49Y_2^{(2,1)}(0,1) - 8Y_2^{(2,2)}(0,1))z^2 - 2(16Y_1^{(0,2)}(0,1) + 4Y_1^{(0,3)}(0,1) - 9Y_2^{(1,0)}(0,1) \\
&+ 23Y_2^{(1,1)}(0,1) + 28Y_2^{(1,2)}(0,1) + 4Y_2^{(1,3)}(0,1))z + 2(5z^2+12)Y_2^{(0,1)}(0,1) + 96Y_3^{(0,1)}(0,1) \\
&- 36Y_2^{(0,2)}(0,1) + 8Y_3^{(0,2)}(0,1) - 8Y_2^{(0,3)}(0,1)))z^2 + a((-2Y_2^{(0,1)}(0,1) + Y_2^{(0,3)}(0,1) + 6Y_1^{(1,1)}(0,1) \\
&- 2Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1) - Y_2^{(2,2)}(0,1))z^2 - 2(3Y_1^{(0,1)}(0,1) - 12Y_1^{(0,2)}(0,1) \\
&+ Y_1^{(0,3)}(0,1) + 3Y_2^{(1,0)}(0,1) - 21Y_2^{(1,1)}(0,1) - 9Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1))z + 11(-4Y_2^{(0,1)}(0,1) \\
&+ Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1)))z + a^2((74z^2+36)Y_1^{(0,1)}(0,1) + 36(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)) \\
&- z((28-26z^2)Y_2^{(0,1)}(0,1) - 8Y_2^{(0,2)}(0,1) + z(z(-6Y_3^{(0,1)}(0,1) + 58Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) \\
&+ 41Y_2^{(0,3)}(0,1) + 4Y_2^{(0,4)}(0,1) + 2z(11Y_1^{(0,2)}(0,1) - Y_1^{(0,3)}(0,1) + 22Y_2^{(1,1)}(0,1) + 8Y_2^{(1,2)}(0,1) \\
&- Y_2^{(1,3)}(0,1))) - 4(3Y_2^{(1,0)}(0,1) + 5Y_2^{(1,1)}(0,1))) + 18(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))) \\
&+ a^3(-2(13z^4+30z^2+36)Y_1^{(0,1)}(0,1) + 30Y_2^{(1,0)}(0,1) + z(2(5z^4+22z^2+1)Y_2^{(0,1)}(0,1) \\
&- 36Y_3^{(0,1)}(0,1) + 8Y_2^{(0,2)}(0,1) + z(-4(z^2-2)Y_1^{(0,2)}(0,1) - 60Y_2^{(1,0)}(0,1) - 44Y_2^{(1,1)}(0,1) \\
&+ 8Y_2^{(1,2)}(0,1) + z(6Y_2^{(0,4)}(0,1)z^2 - 4(4Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))z + 5(13z^2-4)Y_2^{(0,2)}(0,1) \\
&+ (44z^2-8)Y_2^{(0,3)}(0,1) + 8(12Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + 6Y_2^{(2,0)}(0,1) + 7Y_2^{(2,1)}(0,1)) \\
&+ 4Y_2^{(2,2)}(0,1))))))\chi^8 - 2a(8a^6(2Y_2^{(0,1)}(0,1) - 6Y_3^{(0,1)}(0,1) + 7Y_2^{(0,2)}(0,1) - 2Y_3^{(0,2)}(0,1) \\
&+ Y_2^{(0,3)}(0,1))z^6 + 8a^5((Y_2^{(0,3)}(0,1) - 6Y_1^{(1,1)}(0,1) - 2Y_1^{(1,2)}(0,1) - 3Y_2^{(2,0)}(0,1) - 5Y_2^{(2,1)}(0,1) \\
&- Y_2^{(2,2)}(0,1))z^2 + 2(z^2+3)Y_2^{(0,1)}(0,1) + 12Y_3^{(0,1)}(0,1) + (4z^2-5)Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) \\
&- Y_2^{(0,3)}(0,1))z^4 + (-4Y_2^{(0,1)}(0,1) + 16Y_2^{(0,2)}(0,1) - 2(Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1)) + z(2Y_1^{(0,2)}(0,1)
\end{aligned}
$$

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\begin{aligned}
&- Y_1^{(0,3)}(0,1) + 4Y_2^{(1,1)}(0,1) - Y_2^{(1,2)}(0,1) - Y_2^{(1,3)}(0,1)))z^2 + 2a^4(2(4Y_2^{(0,1)}(0,1) + 14Y_2^{(0,2)}(0,1) \\
&+ 8Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1))z^4 - (17Y_1^{(0,1)}(0,1) + 8Y_1^{(0,2)}(0,1) + 3Y_2^{(1,0)}(0,1) + 27Y_2^{(1,1)}(0,1) \\
&+ 8Y_2^{(1,2)}(0,1))z^3 + 4(6Y_2^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) - Y_2^{(0,3)}(0,1) + 12Y_1^{(1,1)}(0,1) + 2Y_1^{(1,2)}(0,1) \\
&+ 6Y_2^{(2,0)}(0,1) + 8Y_2^{(2,1)}(0,1) + Y_2^{(2,2)}(0,1))z^2 - 8(7Y_2^{(0,1)}(0,1) + 3Y_3^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1)))z^2 \\
&- 2a(-3(3z^2+4)Y_1^{(0,1)}(0,1) + 2(7Y_1^{(0,2)}(0,1) - 6Y_2^{(1,0)}(0,1) + 8Y_2^{(1,1)}(0,1) + 7Y_2^{(1,2)}(0,1)) \\
&+ z(-4Y_2^{(0,1)}(0,1) - 8Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) + 12Y_1^{(1,1)}(0,1) - 2Y_1^{(1,2)}(0,1) + z(Y_1^{(0,2)}(0,1) \\
&+ 6zY_2^{(0,2)}(0,1) - 2z(3Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1)) - 3Y_2^{(1,0)}(0,1) - Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) \\
&+ 6Y_2^{(2,0)}(0,1) + 4Y_2^{(2,1)}(0,1) - Y_2^{(2,2)}(0,1)))z + 24a(2Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1)) \\
&- 2a^3(z^2(56Y_2^{(0,1)}(0,1) + 8Y_2^{(0,2)}(0,1) + z((6z^2-76)Y_1^{(0,1)}(0,1) - 4(Y_1^{(0,2)}(0,1) - 3Y_2^{(1,0)}(0,1) \\
&+ 6Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) + z(-2Y_2^{(0,1)}(0,1) - 12Y_3^{(0,1)}(0,1) + 32Y_2^{(0,2)}(0,1) + 4Y_3^{(0,2)}(0,1) \\
&+ 19Y_2^{(0,3)}(0,1) + 2Y_2^{(0,4)}(0,1) + z(27Y_1^{(0,2)}(0,1) + 2Y_1^{(0,3)}(0,1) + 6Y_2^{(1,0)}(0,1) + 60Y_2^{(1,1)}(0,1) \\
&+ 33Y_2^{(1,2)}(0,1) + 2Y_2^{(1,3)}(0,1)))) + 12(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))) - 24Y_2^{(0,1)}(0,1)) \\
&+ 2a^2(24Y_2^{(0,1)}(0,1) + z(-2(3z^2+37)Y_1^{(0,1)}(0,1) + 6Y_2^{(1,0)}(0,1) - 14Y_2^{(1,1)}(0,1) \\
&+ z(2(z^2-8)Y_2^{(0,1)}(0,1) - 12Y_3^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) + 3Y_2^{(0,3)}(0,1) \\
&+ z(38Y_1^{(0,2)}(0,1) - 6Y_2^{(1,0)}(0,1) + 70Y_2^{(1,1)}(0,1) + 38Y_2^{(1,2)}(0,1) + z(-2Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) \\
&+ 12Y_1^{(1,1)}(0,1) - 4Y_1^{(1,2)}(0,1) + 6Y_2^{(2,0)}(0,1) + 2Y_2^{(2,1)}(0,1) - 2Y_2^{(2,2)}(0,1)))))))\chi^7 + \\
&(16a^6(3Y_1^{(0,1)}(0,1) + 2Y_1^{(0,2)}(0,1) + 2Y_2^{(1,0)}(0,1) + 7Y_2^{(1,1)}(0,1) + 2Y_2^{(1,2)}(0,1))z^6 \\
&+ 4a^5(9(z^2-4)Y_1^{(0,1)}(0,1) - 4(2Y_1^{(0,2)}(0,1) - 3Y_2^{(1,0)}(0,1) + 7Y_2^{(1,1)}(0,1) + 2Y_2^{(1,2)}(0,1)) \\
&+ z(8Y_2^{(0,1)}(0,1) - 27Y_3^{(0,1)}(0,1) + 23Y_2^{(0,2)}(0,1) + 2(Y_3^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1)) + z(24Y_1^{(0,2)}(0,1) \\
&+ 4Y_1^{(0,3)}(0,1) + 9Y_2^{(1,0)}(0,1) + 57Y_2^{(1,1)}(0,1) + 36Y_2^{(1,2)}(0,1) + 4Y_2^{(1,3)}(0,1))))z^4 \\
&+ 2a^4(6(20-3z^2)Y_1^{(0,1)}(0,1) - 80Y_2^{(1,0)}(0,1) + z((34Y_2^{(0,2)}(0,1) + 4Y_2^{(0,3)}(0,1) - 54Y_1^{(1,1)}(0,1) \\
&+ 4Y_1^{(1,2)}(0,1) - 27Y_2^{(2,0)}(0,1) - 23Y_2^{(2,1)}(0,1) + 2Y_2^{(2,2)}(0,1))z^2 - 2(44Y_1^{(0,2)}(0,1) + 4Y_1^{(0,3)}(0,1) \\
&+ 9Y_2^{(1,0)}(0,1) + 97Y_2^{(1,1)}(0,1) + 56Y_2^{(1,2)}(0,1) + 4Y_2^{(1,3)}(0,1))z + 4(2z^2-3)Y_2^{(0,1)}(0,1) \\
&+ 72Y_3^{(0,1)}(0,1) - 44Y_2^{(0,2)}(0,1) - 8Y_3^{(0,2)}(0,1)))z^2 + 2a((-4Y_2^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) \\
&+ Y_2^{(0,3)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))z^2 - 2(Y_1^{(0,1)}(0,1) - 2Y_1^{(0,2)}(0,1) \\
&+ Y_1^{(0,3)}(0,1) + Y_2^{(1,0)}(0,1) - 3Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1))z - 8Y_2^{(0,1)}(0,1) \\
&+ 17Y_2^{(0,2)}(0,1) + 7Y_2^{(0,3)}(0,1))z - z^3(4Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1)) + 2a^2((82z^2+20)Y_1^{(0,1)}(0,1) \\
&+ 20(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)) - z(4(7z^2+6)Y_2^{(0,1)}(0,1) - 28Y_2^{(0,2)}(0,1) + z(4(Y_1^{(0,2)}(0,1) \\
&- 4Y_2^{(1,0)}(0,1) - 5Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) + z(-2Y_3^{(0,1)}(0,1) + 66Y_2^{(0,2)}(0,1) + 3Y_2^{(0,3)}(0,1) \\
&- 4Y_2^{(0,4)}(0,1) + 2z(Y_1^{(0,2)}(0,1) - 3Y_1^{(0,3)}(0,1) + 2Y_2^{(1,1)}(0,1) - 8Y_2^{(1,2)}(0,1) - 3Y_2^{(1,3)}(0,1)))) \\
&+ 10(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))) + 2a^3(-4(12z^4+5z^2+18)Y_1^{(0,1)}(0,1) \\
&+ 40Y_2^{(1,0)}(0,1) + z(4(5z^4+7z^2+4)Y_2^{(0,1)}(0,1) - 20Y_3^{(0,1)}(0,1) + 28Y_2^{(0,2)}(0,1) \\
&+ z(8(z^2+5)Y_1^{(0,2)}(0,1) - 20(Y_2^{(1,0)}(0,1) - 3Y_2^{(1,1)}(0,1) - 2Y_2^{(1,2)}(0,1)) + z((49z^2-44)Y_2^{(0,2)}(0,1) \\
&+ z(-3zY_2^{(0,3)}(0,1) - 4zY_2^{(0,4)}(0,1) + 4Y_2^{(1,1)}(0,1) + 8Y_2^{(1,2)}(0,1)) + 4(18Y_1^{(1,1)}(0,1) - 2Y_1^{(1,2)}(0,1) \\
&+ 9Y_2^{(2,0)}(0,1) + 7Y_2^{(2,1)}(0,1) - Y_2^{(2,2)}(0,1)))))))\chi^6 + 2(8a^6(4Y_2^{(0,1)}(0,1) - 6Y_3^{(0,1)}(0,1)
\end{aligned}
$$

$$+ 9Y_2^{(0,2)}(0,1) - Y_3^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1))z^6 + 4a^5((8Y_2^{(0,1)}(0,1) + 12Y_2^{(0,2)}(0,1) + 2Y_2^{(0,3)}(0,1)$$
$$- 2(6Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) + 4Y_2^{(2,1)}(0,1)) - Y_2^{(2,2)}(0,1))z^2 + 2(12Y_3^{(0,1)}(0,1)$$
$$- 11Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1) - Y_2^{(0,3)}(0,1)))z^4 + a((12Y_2^{(0,2)}(0,1) - Y_2^{(0,4)}(0,1))z^2 + 6(2Y_1^{(0,1)}(0,1)$$
$$+ Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1))z + 8Y_2^{(0,2)}(0,1) - 2Y_2^{(0,3)}(0,1))z^2 + (-12Y_2^{(0,2)}(0,1) + Y_2^{(0,4)}(0,1)$$
$$+ z(Y_1^{(0,3)}(0,1) + 3Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1)))z^2 + 2a^4((-32Y_2^{(0,2)}(0,1) - 4Y_2^{(0,3)}(0,1)$$
$$+ z(-21Y_1^{(0,1)}(0,1) + 2Y_1^{(0,2)}(0,1) + 8z(2Y_2^{(0,1)}(0,1) + 4Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1)) + 3Y_2^{(1,0)}(0,1)$$
$$- 5Y_2^{(1,1)}(0,1) + 2Y_2^{(1,2)}(0,1)) + 4(12Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + 6Y_2^{(2,0)}(0,1) + 7Y_2^{(2,1)}(0,1))$$
$$+ 2Y_2^{(2,2)}(0,1))z^2 + 8(-5Y_2^{(0,1)}(0,1) - 3Y_3^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1)))z^2 - 2a^3(z^2(40Y_2^{(0,1)}(0,1)$$
$$- 8Y_2^{(0,2)}(0,1) + z((6z^2 - 68)Y_1^{(0,1)}(0,1) + 4(Y_1^{(0,2)}(0,1) + 3Y_2^{(1,0)}(0,1) - 3Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))$$
$$+ z(40Y_2^{(0,1)}(0,1) + 54Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) + 13Y_2^{(0,3)}(0,1) + 2z(3Y_1^{(0,2)}(0,1) - 2Y_1^{(0,3)}(0,1)$$
$$+ 3Y_2^{(1,0)}(0,1) + 9Y_2^{(1,1)}(0,1) - 3Y_2^{(1,2)}(0,1) - 2Y_2^{(1,3)}(0,1)))) + 12(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1)$$
$$+ Y_2^{(2,1)}(0,1))) - 24Y_2^{(0,1)}(0,1)) + 2a^2(24Y_2^{(0,1)}(0,1) + z((6z^2 - 50)Y_1^{(0,1)}(0,1) + 6Y_2^{(1,0)}(0,1)$$
$$- 14Y_2^{(1,1)}(0,1) + z(4(z^2 + 9)Y_2^{(0,1)}(0,1) + 34Y_2^{(0,2)}(0,1) + 7Y_2^{(0,3)}(0,1) + z(6Y_1^{(0,2)}(0,1)$$
$$- 4Y_1^{(0,3)}(0,1) + 6Y_2^{(1,0)}(0,1) + 18Y_2^{(1,1)}(0,1) - 6Y_2^{(1,2)}(0,1) - 4Y_2^{(1,3)}(0,1) + z(4Y_2^{(0,2)}(0,1)$$
$$+ 3Y_2^{(0,3)}(0,1) - 2(Y_1^{(1,2)}(0,1) + Y_2^{(2,1)}(0,1)) - Y_2^{(2,2)}(0,1)))))))\chi^5 - (8a^5(6Y_1^{(0,1)}(0,1)$$
$$+ 2Y_1^{(0,2)}(0,1) + 3Y_2^{(1,0)}(0,1) + 10Y_2^{(1,1)}(0,1) + 2Y_2^{(1,2)}(0,1))z^6 + 4a^4(3(4Y_1^{(0,2)}(0,1) + 3Y_2^{(1,0)}(0,1)$$
$$+ 11Y_2^{(1,1)}(0,1) + 4Y_2^{(1,2)}(0,1))z^2 - (Y_2^{(0,1)}(0,1) + 3Y_3^{(0,1)}(0,1) + 12Y_2^{(0,2)}(0,1) - 4Y_3^{(0,2)}(0,1)$$
$$+ 4Y_2^{(0,3)}(0,1))z + (9z^2 - 32)Y_1^{(0,1)}(0,1) - 4Y_1^{(0,2)}(0,1) + 6Y_2^{(1,0)}(0,1) - 4(7Y_2^{(1,1)}(0,1)$$
$$+ Y_2^{(1,2)}(0,1)))z^4 + Y_2^{(2,2)}(0,1)z^3 - 2a^3((42z^2 - 64)Y_1^{(0,1)}(0,1) + 54Y_2^{(1,0)}(0,1) - 16Y_2^{(1,1)}(0,1)$$
$$+ z((12Y_2^{(0,2)}(0,1) + 8Y_2^{(0,3)}(0,1) + 6Y_1^{(1,1)}(0,1) - 8Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) - 5Y_2^{(2,1)}(0,1)$$
$$- 4Y_2^{(2,2)}(0,1))z^2 + 6(8Y_1^{(0,2)}(0,1) + 7Y_2^{(1,0)}(0,1) + 23Y_2^{(1,1)}(0,1) + 8Y_2^{(1,2)}(0,1))z$$
$$+ 2(7z^2 + 2)Y_2^{(0,1)}(0,1) - 20Y_2^{(0,2)}(0,1) + 8Y_3^{(0,2)}(0,1) - 8Y_2^{(0,3)}(0,1)))z^2 - 2(2(z^2 - 5)Y_2^{(0,1)}(0,1)$$
$$+ 16Y_2^{(0,2)}(0,1) + 7Y_2^{(0,3)}(0,1) + z(6Y_1^{(0,2)}(0,1) - 2(Y_1^{(0,3)}(0,1) - 6Y_2^{(1,1)}(0,1) + Y_2^{(1,3)}(0,1))$$
$$+ z(4Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) - Y_1^{(1,2)}(0,1) - Y_2^{(2,1)}(0,1))))z + 2a((32z^2 - 6)Y_1^{(0,1)}(0,1)$$
$$- 6(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)) + z(-2(15z^2 + 11)Y_2^{(0,1)}(0,1) + 8Y_2^{(0,2)}(0,1) + z(6Y_2^{(1,0)}(0,1)$$
$$+ 14Y_2^{(1,1)}(0,1) + z(27Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1) + 13Y_2^{(0,3)}(0,1) + 2z(3Y_1^{(0,2)}(0,1) - Y_1^{(0,3)}(0,1)$$
$$+ 6Y_2^{(1,1)}(0,1) - Y_2^{(1,3)}(0,1)))) + 3(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))))$$
$$+ 2a^2((-26z^4 + 30z^2 - 24)Y_1^{(0,1)}(0,1) + 30Y_2^{(1,0)}(0,1) + z(2(5z^4 + 18z^2 + 4)Y_2^{(0,1)}(0,1)$$
$$+ 6Y_3^{(0,1)}(0,1) + 8Y_2^{(0,2)}(0,1) + z(4(z^2 + 6)Y_1^{(0,2)}(0,1) + 6(5Y_2^{(1,0)}(0,1) + 13Y_2^{(1,1)}(0,1)$$
$$+ 4Y_2^{(1,2)}(0,1)) - z(4(4z^2 - 1)Y_2^{(0,2)}(0,1) + (7z^2 - 8)Y_2^{(0,3)}(0,1) - 4zY_2^{(1,2)}(0,1) + 8(Y_1^{(1,2)}(0,1)$$
$$+ Y_2^{(2,1)}(0,1)) + 4Y_2^{(2,2)}(0,1))))))\chi^4 - 2(8a^3(2(a^2 + a + 1)Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1)$$
$$- a(2aY_3^{(0,1)}(0,1) - 3aY_2^{(0,2)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))z^6$$
$$+ 2a^2(-((7a + 2)Y_1^{(0,1)}(0,1)) + (4a + 5)Y_1^{(0,2)}(0,1) + 7aY_2^{(1,0)}(0,1) - 2Y_2^{(1,0)}(0,1) + 7aY_2^{(1,1)}(0,1)$$
$$+ 8Y_2^{(1,1)}(0,1) + (4a + 5)Y_2^{(1,2)}(0,1))z^5 + a(Y_2^{(2,2)}(0,1) - 2((2a(4a(a + 1) + 15) - 2)Y_2^{(0,1)}(0,1)$$

$$+ 2Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) + 2Y_1^{(1,1)}(0,1) - Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) + a((2-16a^2)Y_3^{(0,1)}(0,1)$$
$$+ (4a(5a+4)+13)Y_2^{(0,2)}(0,1) - Y_3^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) - 8a(2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1)$$
$$+ Y_2^{(2,1)}(0,1)))))z^4 + ((8a(4a+1)+2)Y_1^{(0,1)}(0,1) + (1-4a(2a+5))Y_1^{(0,2)}(0,1) + (4Y_2^{(0,2)}(0,1)$$
$$+ 2(Y_2^{(0,3)}(0,1) + 2Y_1^{(1,1)}(0,1) - Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1)) - 2((8a(a^2+a-4)+2)Y_2^{(0,1)}(0,1)$$
$$+ a((8a^2-2)Y_3^{(0,1)}(0,1) - (8a(a+1)+15)Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1) - Y_2^{(0,3)}(0,1) + 4a(2Y_1^{(1,1)}(0,1)$$
$$+ Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))) - Y_2^{(2,2)}(0,1))z^2 - 2((9a+2)Y_1^{(0,1)}(0,1) - 5Y_1^{(0,2)}(0,1)$$
$$+ (2-7a)Y_2^{(1,0)}(0,1) + (3a-8)Y_2^{(1,1)}(0,1) - 5Y_2^{(1,2)}(0,1))z + 16(a^2+a-1)Y_2^{(0,1)}(0,1)$$
$$- 8Y_2^{(0,2)}(0,1))\chi^3 + 2(2a^3(3Y_2^{(1,0)}(0,1) + (4a+3)Y_2^{(1,1)}(0,1))z^6 + a^2(-4(a+3)Y_2^{(0,1)}(0,1)$$
$$+ 6aY_3^{(0,1)}(0,1) - 14aY_2^{(0,2)}(0,1) + 3(-2Y_2^{(0,2)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)))z^5$$
$$+ 2a(-Y_1^{(0,2)}(0,1) + 3a(2a-3)Y_2^{(1,0)}(0,1) - a(8a+9)Y_2^{(1,1)}(0,1) - 3Y_2^{(1,1)}(0,1) - Y_2^{(1,2)}(0,1))z^4$$
$$+ ((4a(a+6)-2)Y_2^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) - 2a(6aY_3^{(0,1)}(0,1) - 10aY_2^{(0,2)}(0,1) - 6Y_2^{(0,2)}(0,1)$$
$$+ 6Y_1^{(1,1)}(0,1) + 3Y_2^{(2,0)}(0,1) + 3Y_2^{(2,1)}(0,1)))z^3 + 2(Y_1^{(0,2)}(0,1) + 3a(3-4a)Y_2^{(1,0)}(0,1)$$
$$+ a(4a+9)Y_2^{(1,1)}(0,1) + 3Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))z^2 + 3(-4Y_2^{(0,1)}(0,1) + 2aY_3^{(0,1)}(0,1)$$
$$- 2aY_2^{(0,2)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))z$$
$$+ 2(az^2-1)(a^2(4a+3)z^4 - (4a^2+6a+1)z^2 + 3)Y_1^{(0,1)}(0,1) + 6(2a-1)Y_2^{(1,0)}(0,1)$$
$$- 6Y_2^{(1,1)}(0,1))\chi^2 + 4z(az^2-1)(3aY_2^{(1,0)}(0,1)z^2 - 2Y_2^{(0,1)}(0,1)z - Y_2^{(0,2)}(0,1)z$$
$$+ (az^2-1)Y_1^{(0,1)}(0,1) - 3Y_2^{(1,0)}(0,1) + (az^2-1)Y_2^{(1,1)}(0,1))\chi$$
$$+ 4(az^2-1)(a\chi^2-1)^4(z\chi + a(z^2+\chi^2)-1)^2 Y_1(0,1)$$
$$+ 4(az^2-1)^2((az^2-1)Y_2^{(1,0)}(0,1) - zY_2^{(0,1)}(0,1)))$$

$$Y_2\left(z, \frac{z\chi + a\left(z^2+\chi^2\right) - 1}{a\chi^2 - 1}\right) = Y_2(0,1)$$
$$+ \frac{1}{4\left(az^2-1\right)\left(a\chi^2-1\right)^5(z\chi + a(z^2+\chi^2)-1)}(\chi(a^4(-((2(2Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1)$$
$$+ Y_2^{(2,0)}(0,1) + 2Y_2^{(2,1)}(0,1) + a(2(Y_2^{(0,1)}(0,1) - Y_3^{(0,1)}(0,1))a^2 - (2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1)$$
$$+ Y_2^{(2,1)}(0,1))a + 2Y_3^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1))) + Y_2^{(2,2)}(0,1))z^2)$$
$$+ 4((a+1)Y_1^{(0,1)}(0,1) + Y_1^{(0,2)}(0,1) + (a+1)Y_2^{(1,0)}(0,1) + (a+3)Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))z$$
$$- 4(a^2+a+1)Y_2^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1))\chi^{11} - 2a^3(2z^2(-3Y_1^{(0,1)}(0,1) + z(Y_2^{(0,1)}(0,1)$$
$$+ 3Y_3^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1)) - 3(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)))a^3$$
$$- (6(z^2+2)Y_1^{(0,1)}(0,1) + z(-((6Y_1^{(1,1)}(0,1) + 4Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) + 7Y_2^{(2,1)}(0,1)$$
$$+ 2Y_2^{(2,2)}(0,1))z^2) + 6(Y_1^{(0,2)}(0,1) + Y_2^{(1,0)}(0,1) + 3Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))z + 2Y_2^{(0,1)}(0,1)$$
$$+ 6Y_3^{(0,1)}(0,1) + 4Y_2^{(0,2)}(0,1)))a^2 + (6Y_1^{(0,1)}(0,1) + 6(Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1))$$
$$+ z(-4Y_2^{(0,1)}(0,1) - 4Y_2^{(0,2)}(0,1) + z(Y_1^{(0,2)}(0,1) + 2Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) - 3(2Y_1^{(1,1)}(0,1)$$
$$+ Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))))a + z(-4Y_2^{(0,1)}(0,1) - 2(4Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1))$$
$$+ z(2Y_1^{(0,2)}(0,1) + Y_1^{(0,3)}(0,1) + 4Y_2^{(1,1)}(0,1) + 5Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1))))\chi^{10}$$
$$- a^2(4z^4(Y_2^{(0,1)}(0,1) + 3Y_3^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1))a^5 + 2z^2((6Y_1^{(1,1)}(0,1)$$

$$
\begin{aligned}
&+4Y_1^{(1,2)}(0,1)+3Y_2^{(2,0)}(0,1)+7Y_2^{(2,1)}(0,1)+2Y_2^{(2,2)}(0,1))z^2-26Y_2^{(0,1)}(0,1)-10Y_2^{(0,2)}(0,1))a^4\\
&+4z^2(-8Y_2^{(0,1)}(0,1)-5Y_2^{(0,2)}(0,1)+2z(Y_1^{(0,2)}(0,1)+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))a^3\\
&+4z(z(-7Y_2^{(0,1)}(0,1)-3Y_3^{(0,1)}(0,1)-10Y_2^{(0,2)}(0,1)-2Y_2^{(0,3)}(0,1)+2z(2Y_1^{(0,2)}(0,1)\\
&+Y_1^{(0,3)}(0,1)+4Y_2^{(1,1)}(0,1)+5Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1)))-9Y_1^{(0,1)}(0,1))a^2+2((2Y_2^{(0,2)}(0,1)\\
&+Y_2^{(0,3)}(0,1)-3(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))z^2+6(Y_1^{(0,1)}(0,1)-Y_1^{(0,2)}(0,1)\\
&+Y_2^{(1,0)}(0,1)-Y_2^{(1,1)}(0,1)-Y_2^{(1,2)}(0,1))z+6(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)))a+z^2(6(Y_2^{(0,2)}(0,1)\\
&+Y_2^{(0,3)}(0,1))+Y_2^{(0,4)}(0,1)))\chi^9-2a^2(2a^4(Y_1^{(0,1)}(0,1)+2Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+5Y_2^{(1,1)}(0,1)\\
&+2Y_2^{(1,2)}(0,1))z^4+2a^3((z^2+2)Y_1^{(0,1)}(0,1)+8(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))+z(-3Y_2^{(0,1)}(0,1)\\
&-8Y_3^{(0,1)}(0,1)+9Y_2^{(0,2)}(0,1)-2Y_3^{(0,2)}(0,1)+2Y_2^{(0,3)}(0,1)+z(5Y_1^{(0,2)}(0,1)+2Y_1^{(0,3)}(0,1)\\
&+Y_2^{(1,0)}(0,1)+11(Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))+2Y_2^{(1,3)}(0,1))))z^2+(8Y_2^{(0,1)}(0,1)-11Y_2^{(0,2)}(0,1)\\
&-5Y_2^{(0,3)}(0,1)+z(-4Y_1^{(0,2)}(0,1)+Y_1^{(0,3)}(0,1)-8Y_2^{(1,1)}(0,1)-Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1)))z\\
&+a(-8(z^2+2)Y_1^{(0,1)}(0,1)-16(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))+z((15z^2-2)Y_2^{(0,2)}(0,1)\\
&+z(z(13Y_2^{(0,3)}(0,1)+2Y_2^{(0,4)}(0,1))-2(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))+8(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))))+2a^2((7z^2+18)Y_1^{(0,1)}(0,1)+z(-5Y_2^{(0,1)}(0,1)+8Y_3^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)\\
&+z(Y_1^{(0,2)}(0,1)+7Y_2^{(1,0)}(0,1)+9Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)+z(5Y_2^{(0,2)}(0,1)+2Y_2^{(0,3)}(0,1)\\
&-2(4Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)+2Y_2^{(2,0)}(0,1)+3Y_2^{(2,1)}(0,1))-Y_2^{(2,2)}(0,1))))))\chi^8\\
&+2a(2z^4(Y_2^{(0,1)}(0,1)+9Y_3^{(0,1)}(0,1)-13Y_2^{(0,2)}(0,1)+4Y_3^{(0,2)}(0,1)-2Y_2^{(0,3)}(0,1))a^5\\
&+z^2((-4Y_2^{(0,3)}(0,1)+18Y_1^{(1,1)}(0,1)+8Y_1^{(1,2)}(0,1)+9Y_2^{(2,0)}(0,1)+17Y_2^{(2,1)}(0,1)+4Y_2^{(2,2)}(0,1))z^2\\
&-2(3z^2+26)Y_2^{(0,1)}(0,1)-12Y_3^{(0,1)}(0,1)-14(z^2+1)Y_2^{(0,2)}(0,1))a^4-(z^2(32Y_2^{(0,1)}(0,1)\\
&+14Y_2^{(0,2)}(0,1)+z(-6Y_1^{(0,1)}(0,1)+z(6Y_2^{(0,1)}(0,1)+24Y_2^{(0,2)}(0,1)+15Y_2^{(0,3)}(0,1)+2Y_2^{(0,4)}(0,1))\\
&-4(Y_1^{(0,2)}(0,1)+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))+6(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))\\
&-12Y_2^{(0,1)}(0,1))a^3+(12Y_2^{(0,1)}(0,1)+z(-54Y_1^{(0,1)}(0,1)-12(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\\
&+z(-10Y_2^{(0,1)}(0,1)-6Y_3^{(0,1)}(0,1)-6Y_2^{(0,2)}(0,1)+2Y_3^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1)+12z(Y_1^{(0,2)}(0,1)\\
&+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))))a^2+(12(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))+z(6Y_1^{(0,1)}(0,1)\\
&-2(7Y_1^{(0,2)}(0,1)-3Y_2^{(1,0)}(0,1)+11Y_2^{(1,1)}(0,1)+7Y_2^{(1,2)}(0,1))+z(2Y_2^{(0,1)}(0,1)-Y_2^{(0,3)}(0,1)\\
&-6Y_1^{(1,1)}(0,1)+2Y_1^{(1,2)}(0,1)-3Y_2^{(2,0)}(0,1)-Y_2^{(2,1)}(0,1)+Y_2^{(2,2)}(0,1))))a+z^2(3Y_2^{(0,2)}(0,1)\\
&-3Y_2^{(0,3)}(0,1)-Y_2^{(0,4)}(0,1)))\chi^7+2a(2a^4(3Y_1^{(0,1)}(0,1)+4Y_1^{(0,2)}(0,1)+3Y_2^{(1,0)}(0,1)\\
&+11Y_2^{(1,1)}(0,1)+4Y_2^{(1,2)}(0,1))z^4+2a^3(3(z^2-2)Y_1^{(0,1)}(0,1)+6(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\\
&+z(-4Y_2^{(0,1)}(0,1)-6Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)+2Y_3^{(0,2)}(0,1)+z(10Y_1^{(0,2)}(0,1)+2Y_1^{(0,3)}(0,1)\\
&+3Y_2^{(1,0)}(0,1)+23Y_2^{(1,1)}(0,1)+16Y_2^{(1,2)}(0,1)+2Y_2^{(1,3)}(0,1))))z^2+(4Y_2^{(0,1)}(0,1)-2(5Y_2^{(0,2)}(0,1)\\
&+2Y_2^{(0,3)}(0,1))+z(-2Y_1^{(0,2)}(0,1)+Y_1^{(0,3)}(0,1)-4Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1)))z\\
&-2a((9z^2+6)Y_1^{(0,1)}(0,1)+6(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))+z(4Y_2^{(0,1)}(0,1)+7Y_2^{(0,2)}(0,1)\\
&+z(-Y_1^{(0,2)}(0,1)+z(-3Y_2^{(0,2)}(0,1)+3Y_2^{(0,3)}(0,1)+Y_2^{(0,4)}(0,1))+3Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)\\
&-Y_2^{(1,2)}(0,1))-3(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))))+2a^2(Y_2^{(2,2)}(0,1)z^3\\
&-(2(z^2+7)Y_2^{(0,1)}(0,1)-6Y_3^{(0,1)}(0,1)+7Y_2^{(0,2)}(0,1)+z(7Y_1^{(0,2)}(0,1)-3Y_2^{(1,0)}(0,1)
\end{aligned}
$$

$$
\begin{aligned}
&+11Y_2^{(1,1)}(0,1)+7Y_2^{(1,2)}(0,1)+z(-Y_2^{(0,2)}(0,1)+6Y_1^{(1,1)}(0,1)-2Y_1^{(1,2)}(0,1)+3Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))))z+3(z^2+6)Y_1^{(0,1)}(0,1)))\chi^6+(4z^4(2(8Y_2^{(0,2)}(0,1)-Y_3^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1))\\
&-9Y_3^{(0,1)}(0,1))a^5+2z^2((4Y_2^{(0,3)}(0,1)-18Y_1^{(1,1)}(0,1)-4Y_1^{(1,2)}(0,1)-9Y_2^{(2,0)}(0,1)-13Y_2^{(2,1)}(0,1)\\
&-2Y_2^{(2,2)}(0,1))z^2+12(z^2+3)Y_2^{(0,1)}(0,1)+16Y_3^{(0,1)}(0,1)+(20z^2-2)Y_2^{(0,2)}(0,1))a^4\\
&+2(z^2((12Y_2^{(0,1)}(0,1)+27Y_2^{(0,2)}(0,1)+7Y_2^{(0,3)}(0,1))z^2+4(-3Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)\\
&+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))z+16Y_2^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)+8(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1)))-16Y_2^{(0,1)}(0,1))a^3+2(z(46Y_1^{(0,1)}(0,1)+16(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\\
&+z(-24Y_2^{(0,1)}(0,1)+2Y_3^{(0,1)}(0,1)-25Y_2^{(0,2)}(0,1)-6Y_2^{(0,3)}(0,1)+4z(Y_1^{(0,3)}(0,1)+3Y_2^{(1,2)}(0,1)\\
&+Y_2^{(1,3)}(0,1))))-16Y_2^{(0,1)}(0,1))a^2+2((-4Y_2^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1)+2Y_1^{(1,1)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))z^2+2(-Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)-Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)\\
&+Y_2^{(1,2)}(0,1))z-2(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)))a-z^2(4Y_2^{(0,3)}(0,1)+Y_2^{(0,4)}(0,1)))\chi^5\\
&-2(2a^4(3Y_1^{(0,1)}(0,1)+2Y_1^{(0,2)}(0,1)+3Y_2^{(1,0)}(0,1)+7Y_2^{(1,1)}(0,1)+2Y_2^{(1,2)}(0,1))z^4\\
&+2a^3(3(z^2-2)Y_1^{(0,1)}(0,1)+z(-2Y_2^{(0,1)}(0,1)-9Y_2^{(0,2)}(0,1)+2Y_3^{(0,2)}(0,1)-2Y_2^{(0,3)}(0,1)\\
&+z(5Y_1^{(0,2)}(0,1)+3Y_2^{(1,0)}(0,1)+13Y_2^{(1,1)}(0,1)+5Y_2^{(1,2)}(0,1))))z^2-3a((3Y_2^{(0,2)}(0,1)\\
&+Y_2^{(0,3)}(0,1))z^2+2(2Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))z+2Y_2^{(0,2)}(0,1))z-(-9Y_2^{(0,2)}(0,1)\\
&-3Y_2^{(0,3)}(0,1)+z(Y_1^{(0,3)}(0,1)+3Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1)))z-2a^2(3(z^2-2)Y_1^{(0,1)}(0,1)\\
&+z(2(2z^2+5)Y_2^{(0,1)}(0,1)+3Y_2^{(0,2)}(0,1)+z(5Y_1^{(0,2)}(0,1)+3Y_2^{(1,0)}(0,1)+13Y_2^{(1,1)}(0,1)\\
&+5Y_2^{(1,2)}(0,1)+z(5Y_2^{(0,2)}(0,1)+2Y_2^{(0,3)}(0,1)-2(Y_1^{(1,2)}(0,1)+Y_2^{(2,1)}(0,1))-Y_2^{(2,2)}(0,1))))))\chi^4\\
&+(4a^4(Y_2^{(0,1)}(0,1)+3Y_3^{(0,1)}(0,1)-5Y_2^{(0,2)}(0,1))z^4-2a^3(-3(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))z^2+2(3z^2+5)Y_2^{(0,1)}(0,1)+6Y_3^{(0,1)}(0,1)+6(z^2-1)Y_2^{(0,2)}(0,1))z^2+(8(Y_1^{(0,2)}(0,1)\\
&+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))+z(4Y_2^{(0,1)}(0,1)+8Y_2^{(0,2)}(0,1)+2Y_2^{(0,3)}(0,1)-2(Y_1^{(1,2)}(0,1)\\
&+Y_2^{(2,1)}(0,1))-Y_2^{(2,2)}(0,1)))z-6(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))-2a(z(12Y_1^{(0,1)}(0,1)+6(Y_2^{(1,0)}(0,1)\\
&+Y_2^{(1,1)}(0,1))+z(-22Y_2^{(0,1)}(0,1)-11Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1)-Y_2^{(0,3)}(0,1)+4z(Y_1^{(0,2)}(0,1)\\
&+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))))-6Y_2^{(0,1)}(0,1))-2a^2(z^2(z(-6Y_1^{(0,1)}(0,1)+3z(2Y_2^{(0,1)}(0,1)\\
&+Y_2^{(0,2)}(0,1))+4(Y_1^{(0,2)}(0,1)+2Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1)))+3(-2Y_2^{(0,2)}(0,1)+2Y_1^{(1,1)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))-6Y_2^{(0,1)}(0,1)))\chi^3+2(2a^2(a+1)(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))z^4\\
&+a(2(a-2)Y_2^{(0,1)}(0,1)+2aY_3^{(0,1)}(0,1)-6aY_2^{(0,2)}(0,1)-2Y_2^{(0,2)}(0,1)+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))z^3-(2Y_2^{(1,0)}(0,1)(a+1)^2+Y_1^{(0,2)}(0,1)+2(a(a+2)+2)Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))z^2\\
&-((6a-4)Y_2^{(0,1)}(0,1)+2aY_3^{(0,1)}(0,1)-2aY_2^{(0,2)}(0,1)-2Y_2^{(0,2)}(0,1)+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))z+2((a+1)(a(az^2-1)-1)z^2+1)Y_1^{(0,1)}(0,1)+2(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))\chi^2\\
&+2z^2(-2((az^2-1)a^2+1)Y_2^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1))\chi-4z(az^2-1)Y_2^{(0,1)}(0,1)))
\end{aligned}
$$

$$
\begin{aligned}
&Y_3\left(z,\frac{z\chi+a\left(z^2+\chi^2\right)-1}{a\chi^2-1}\right)=\frac{1}{4a\left(az^2-1\right)\left(a\chi^2-1\right)^4\left(z\chi+a(z^2+\chi^2)-1\right)^2}((a(\chi-z)(z+\chi)\\
&+1)(4a(z\chi+a(z^2+\chi^2)-1)Y_2(0,1)(a\chi^2-1)^5-4a(z\chi+a(z^2+\chi^2)-1)Y_2^{(0,1)}(0,1)(a\chi^2-1)^5\\
&+4a(z\chi+a(z^2+\chi^2)-1)^2(2aY_2^{(0,2)}(0,1)\chi^2+(a\chi^2-1)Y_2^{(0,1)}(0,1))(a\chi^2-1)^3
\end{aligned}
$$

$$
\begin{aligned}
&+4(z\chi+a(z^2+\chi^2)-1)^2(Y_2^{(0,1)}(0,1)+a\chi((2-2a\chi^2)Y_1^{(0,1)}(0,1)-3\chi Y_2^{(0,1)}(0,1)+2(Y_2^{(1,0)}(0,1)\\
&+Y_2^{(1,1)}(0,1))-2\chi(Y_2^{(0,2)}(0,1)+a\chi(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))))(a\chi^2-1)^2\\
&-2(z\chi+a(z^2+\chi^2)-1)^2(2a^3Y_3(0,1)\chi^4+(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))\chi^2+a^2((2Y_1^{(1,0)}(0,1)\\
&+Y_2^{(2,0)}(0,1))\chi^2-4Y_3(0,1)+2Y_2^{(0,1)}(0,1))\chi^2-2(Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\chi\\
&+2Y_1^{(1,0)}(0,1)+2a((-2Y_1^{(1,0)}(0,1)+\chi(Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))-Y_2^{(2,0)}(0,1))\chi^2\\
&+Y_3(0,1)-Y_2^{(0,1)}(0,1))+Y_2^{(2,0)}(0,1))(a\chi^2-1)^2\\
&-4a\chi(1-a\chi^2)^3(-z\chi-a(z^2+\chi^2)+1)((1-a\chi^2)Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)\\
&-\chi(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)+a\chi(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))))\\
&+4a\chi(1-a\chi^2)(z\chi+a(z^2+\chi^2)-1)^2(Y_1^{(0,1)}(0,1)(a\chi^2-1)^2+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)\\
&+\chi(3(a\chi^2-1)Y_2^{(0,1)}(0,1)-3Y_2^{(0,2)}(0,1)+a\chi(2(a\chi^2-1)Y_1^{(0,2)}(0,1)-2(Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)\\
&+Y_2^{(1,2)}(0,1))+\chi(7Y_2^{(0,2)}(0,1)+2Y_2^{(0,3)}(0,1)+a\chi(Y_2^{(1,0)}(0,1)+5Y_2^{(1,1)}(0,1)+2Y_2^{(1,2)}(0,1))))))\\
&+2a(\chi-a\chi^3)^2(-z\chi-a(z^2+\chi^2)+1)(-a^2(2aY_3^{(0,1)}(0,1)+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1))\chi^4-2a(Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi^3\\
&-(4(a^2+1)Y_2^{(0,1)}(0,1)+5Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1)-2a(2aY_3^{(0,1)}(0,1)-aY_2^{(0,2)}(0,1)\\
&+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))\chi^2+2(Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)\\
&+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi+2a(2Y_2^{(0,1)}(0,1)-Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))-2Y_1^{(1,1)}(0,1)\\
&-Y_2^{(2,0)}(0,1)-Y_2^{(2,1)}(0,1))+8\chi(1-a\chi^2)(z\chi+a(z^2+\chi^2)-1)^2(Y_1^{(0,1)}(0,1)(a\chi^2-1)^2\\
&+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)+\chi(2(a^2+1)(a\chi^2-1)Y_2^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+a(2a^3Y_3^{(0,1)}(0,1)\chi^4\\
&+(4Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1))\chi^2+a^2((2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))\chi^2-4Y_3^{(0,1)}(0,1)\\
&+2Y_2^{(0,2)}(0,1))\chi^2-2(Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi+2Y_1^{(1,1)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)+a((\chi(2Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+5Y_2^{(1,1)}(0,1)+2Y_2^{(1,2)}(0,1))\\
&-2(2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))\chi^2+2Y_3^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)))))\\
&-2(z+2a\chi)(1-a\chi^2)(-z\chi-a(z^2+\chi^2)+1)(-a^3(2(a(2Y_3^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1))\\
&+2Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)+Y_2^{(2,0)}(0,1)+2Y_2^{(2,1)}(0,1))+Y_2^{(2,2)}(0,1))\chi^7-a^2((2a+7)Y_1^{(0,2)}(0,1)\\
&+2Y_1^{(0,3)}(0,1)+3(a+1)Y_2^{(1,0)}(0,1)+(7a+17)Y_2^{(1,1)}(0,1)+2aY_2^{(1,2)}(0,1)+13Y_2^{(1,2)}(0,1)\\
&+2Y_2^{(1,3)}(0,1))\chi^6-a(8(a^2+a+1)Y_2^{(0,1)}(0,1)+19Y_2^{(0,2)}(0,1)+9Y_2^{(0,3)}(0,1)+Y_2^{(0,4)}(0,1)\\
&-a(-10Y_2^{(0,2)}(0,1)+2a(5Y_3^{(0,1)}(0,1)-8Y_2^{(0,2)}(0,1)+2Y_3^{(0,2)}(0,1)-Y_2^{(0,3)}(0,1))-2Y_2^{(0,3)}(0,1)\\
&+10Y_1^{(1,1)}(0,1)+4Y_1^{(1,2)}(0,1)+5Y_2^{(2,0)}(0,1)+9Y_2^{(2,1)}(0,1)+2Y_2^{(2,2)}(0,1)))\chi^5\\
&+a(2(2a+5)Y_1^{(0,2)}(0,1)+2Y_1^{(0,3)}(0,1)+(7a+6)Y_2^{(1,0)}(0,1)+(15a+26)Y_2^{(1,1)}(0,1)\\
&+4aY_2^{(1,2)}(0,1)+2(8Y_2^{(1,2)}(0,1)+Y_2^{(1,3)}(0,1)))\chi^4+(4(3a(a+1)+2)Y_2^{(0,1)}(0,1)+2(5Y_2^{(0,2)}(0,1)\\
&+Y_2^{(0,3)}(0,1))-a(2(-6Y_2^{(0,2)}(0,1)+a(4Y_3^{(0,1)}(0,1)-9Y_2^{(0,2)}(0,1)+Y_3^{(0,2)}(0,1)-Y_2^{(0,3)}(0,1))\\
&-Y_2^{(0,3)}(0,1)+4Y_1^{(1,1)}(0,1)+Y_1^{(1,2)}(0,1)+2Y_2^{(2,0)}(0,1)+3Y_2^{(2,1)}(0,1))+Y_2^{(2,2)}(0,1)))\chi^3\\
&-((2a+3)Y_1^{(0,2)}(0,1)+(5a+3)Y_2^{(1,0)}(0,1)+9(a+1)Y_2^{(1,1)}(0,1)+(2a+3)Y_2^{(1,2)}(0,1))\chi^2\\
&+(-4(a+1)Y_2^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)+2a(Y_3^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1))+2Y_1^{(1,1)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))\chi+((a(-3a(a+1)\chi^4+(4a+6)\chi^2-2)-3)\chi^2+1)Y_1^{(0,1)}(0,1)
\end{aligned}
$$

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\begin{aligned}
&+ Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)) - 2\chi^2(z\chi + a(z^2+\chi^2) - 1)^2(2a^5(Y_3^{(0,1)}(0,1) + 2Y_3^{(0,2)}(0,1))\chi^6 \\
&+ a^4((2Y_1^{(1,1)}(0,1) + 4Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) + 5Y_2^{(2,1)}(0,1) + 2Y_2^{(2,2)}(0,1))\chi^2 - 6Y_3^{(0,1)}(0,1) \\
&+ 10Y_2^{(0,2)}(0,1) - 8Y_3^{(0,2)}(0,1) + 4Y_2^{(0,3)}(0,1))\chi^4 + a^3(2(3Y_1^{(0,2)}(0,1) + 2Y_1^{(0,3)}(0,1) + 6Y_2^{(1,1)}(0,1) \\
&+ 9Y_2^{(1,2)}(0,1) + 2Y_2^{(1,3)}(0,1))\chi^3 - (6Y_1^{(1,1)}(0,1) + 8Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) + 11Y_2^{(2,1)}(0,1) \\
&+ 4Y_2^{(2,2)}(0,1))\chi^2 + 6Y_3^{(0,1)}(0,1) - 4(3Y_2^{(0,2)}(0,1) - Y_3^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1)))\chi^2 \\
&+ 2(a^2+1)(a\chi^2-1)^2Y_2^{(0,1)}(0,1) + Y_2^{(0,2)}(0,1) - a((17Y_2^{(0,2)}(0,1) + 5Y_2^{(0,3)}(0,1))\chi^2 - 6(Y_1^{(0,2)}(0,1) \\
&+ 2Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))\chi + 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1)) + a^2(((16Y_2^{(0,2)}(0,1) \\
&+ 13Y_2^{(0,3)}(0,1) + 2Y_2^{(0,4)}(0,1))\chi^2 - 4(3Y_1^{(0,2)}(0,1) + Y_1^{(0,3)}(0,1) + 6(Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1)) \\
&+ Y_2^{(1,3)}(0,1))\chi + 6Y_1^{(1,1)}(0,1) + 4Y_1^{(1,2)}(0,1) + 3Y_2^{(2,0)}(0,1) + 7Y_2^{(2,1)}(0,1) + 2Y_2^{(2,2)}(0,1))\chi^2 \\
&- 2Y_3^{(0,1)}(0,1) + 2Y_2^{(0,2)}(0,1))) \\
&- (z+2a\chi)(1-a\chi^2)(2(-z\chi - a(z^2+\chi^2)+1)(-a^3(2(a(2Y_3^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1)) \\
&+ 2Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) + 2Y_2^{(2,1)}(0,1)) + Y_2^{(2,2)}(0,1))\chi^7 - a^2((2a+7)Y_1^{(0,2)}(0,1) \\
&+ 2Y_1^{(0,3)}(0,1) + 3(a+1)Y_2^{(1,0)}(0,1) + (7a+17)Y_2^{(1,1)}(0,1) + 2aY_2^{(1,2)}(0,1) + 13Y_2^{(1,2)}(0,1) \\
&+ 2Y_2^{(1,3)}(0,1))\chi^6 - a(8(a^2+a+1)Y_2^{(0,1)}(0,1) + 19Y_2^{(0,2)}(0,1) + 9Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1) \\
&- a(-10Y_2^{(0,2)}(0,1) + 2a(5Y_3^{(0,1)}(0,1) - 8Y_2^{(0,2)}(0,1) + 2Y_3^{(0,2)}(0,1) - Y_2^{(0,3)}(0,1)) - 2Y_2^{(0,3)}(0,1) \\
&+ 10Y_1^{(1,1)}(0,1) + 4Y_1^{(1,2)}(0,1) + 5Y_2^{(2,0)}(0,1) + 9Y_2^{(2,1)}(0,1) + 2Y_2^{(2,2)}(0,1)))\chi^5 \\
&+ a(2(2a+5)Y_1^{(0,2)}(0,1) + 2Y_1^{(0,3)}(0,1) + (7a+6)Y_2^{(1,0)}(0,1) + (15a+26)Y_2^{(1,1)}(0,1) \\
&+ 4aY_2^{(1,2)}(0,1) + 2(8Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1)))\chi^4 + (4(3a(a+1)+2)Y_2^{(0,1)}(0,1) \\
&+ 2(5Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1)) - a(2(-6Y_2^{(0,2)}(0,1) + a(4Y_3^{(0,1)}(0,1) - 9Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1) \\
&- Y_2^{(0,3)}(0,1)) - Y_2^{(0,3)}(0,1) + 4Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + 2Y_2^{(2,0)}(0,1) + 3Y_2^{(2,1)}(0,1)) \\
&+ Y_2^{(2,2)}(0,1)))\chi^3 - ((2a+3)Y_1^{(0,2)}(0,1) + (5a+3)Y_2^{(1,0)}(0,1) + 9(a+1)Y_2^{(1,1)}(0,1) \\
&+ (2a+3)Y_2^{(1,2)}(0,1))\chi^2 + (-4(a+1)Y_2^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1) + 2a(Y_3^{(0,1)}(0,1) - 2Y_2^{(0,2)}(0,1)) \\
&+ 2Y_1^{(1,1)}(0,1) + Y_2^{(2,0)}(0,1) + Y_2^{(2,1)}(0,1))\chi \\
&+ ((a(-3a(a+1)\chi^4 + (4a+6)\chi^2 - 2) - 3)\chi^2 + 1)Y_1^{(0,1)}(0,1) + Y_2^{(1,0)}(0,1) + Y_2^{(1,1)}(0,1)) \\
&- (z+2a\chi)(1-a\chi^2)(-2Y_2^{(0,2)}(0,1) - \chi(a^2(2(a(2Y_3^{(0,1)}(0,1) - Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1)) \\
&+ 2Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) + 2Y_2^{(2,1)}(0,1)) + Y_2^{(2,2)}(0,1))\chi^5 + 2a((a+4)Y_1^{(0,2)}(0,1) \\
&+ Y_1^{(0,3)}(0,1) + 2(a+1)Y_2^{(1,0)}(0,1) + 2(2a+5)Y_2^{(1,1)}(0,1) + aY_2^{(1,2)}(0,1) + 7Y_2^{(1,2)}(0,1) \\
&+ Y_2^{(1,3)}(0,1))\chi^4 + (2(-4Y_3^{(0,1)}(0,1) + 9Y_2^{(0,2)}(0,1) - 2Y_3^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1))a^2 \\
&+ 2(6Y_2^{(0,2)}(0,1) + Y_2^{(0,3)}(0,1) - 2(2Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) + 2Y_2^{(2,1)}(0,1)) \\
&- Y_2^{(2,2)}(0,1))a + 12(a^2+a+1)Y_2^{(0,1)}(0,1) + 24Y_2^{(0,2)}(0,1) + 10Y_2^{(0,3)}(0,1) + Y_2^{(0,4)}(0,1))\chi^3 \\
&- 2(2(a+2)Y_1^{(0,2)}(0,1) + Y_1^{(0,3)}(0,1) + (4a+2)Y_2^{(1,0)}(0,1) + 2(4a+5)Y_2^{(1,1)}(0,1) + 2aY_2^{(1,2)}(0,1) \\
&+ 7Y_2^{(1,2)}(0,1) + Y_2^{(1,3)}(0,1))\chi^2 + (-12(a+1)Y_2^{(0,1)}(0,1) + 2(-6Y_2^{(0,2)}(0,1) + a(2Y_3^{(0,1)}(0,1) \\
&- 9Y_2^{(0,2)}(0,1) + Y_3^{(0,2)}(0,1) - Y_2^{(0,3)}(0,1)) - Y_2^{(0,3)}(0,1) + 2Y_1^{(1,1)}(0,1) + Y_1^{(1,2)}(0,1) + Y_2^{(2,0)}(0,1) \\
&+ 2Y_2^{(2,1)}(0,1)) + Y_2^{(2,2)}(0,1))\chi + 4(a\chi^2-1)((a+1)\chi^2-1)Y_1^{(0,1)}(0,1) + 2(Y_1^{(0,2)}(0,1) \\
&+ 2Y_2^{(1,0)}(0,1) + 4Y_2^{(1,1)}(0,1) + Y_2^{(1,2)}(0,1))))))
\end{aligned}
$$

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\begin{aligned}
&-2a(1-a\chi^2)(-z\chi-a(z^2+\chi^2)+1)((1-a\chi^2)(-z\chi-a(z^2+\chi^2)+1)(-2Y_2(0,1)(a\chi^2-1)^3\\
&+2\chi((a\chi^2-1)Y_1(0,1)+\chi Y_2^{(0,1)}(0,1)+(a\chi^2-1)Y_2^{(1,0)}(0,1))(a\chi^2-1)+\chi^2(2a^3Y_3(0,1)\chi^4\\
&+(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))\chi^2+a^2((2Y_1^{(1,0)}(0,1)+Y_2^{(2,0)}(0,1))\chi^2-4Y_3(0,1)+2Y_2^{(0,1)}(0,1))\chi^2\\
&-2(Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\chi+2Y_1^{(1,0)}(0,1)+2a((-2Y_1^{(1,0)}(0,1)+\chi(Y_1^{(0,1)}(0,1)\\
&+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))-Y_2^{(2,0)}(0,1))\chi^2+Y_3(0,1)-Y_2^{(0,1)}(0,1))+Y_2^{(2,0)}(0,1)))\\
&+\chi(2(z+2a\chi)Y_2(0,1)(a\chi^2-1)^4-2(z+2a\chi)Y_2^{(0,1)}(0,1)(a\chi^2-1)^4\\
&+4a\chi(z\chi+a(z^2+\chi^2)-1)Y_2^{(0,1)}(0,1)(a\chi^2-1)^3-2(z\chi+a(z^2+\chi^2)-1)((a\chi^2-1)Y_1(0,1)\\
&+\chi Y_2^{(0,1)}(0,1)+(a\chi^2-1)Y_2^{(1,0)}(0,1))(a\chi^2-1)^2-2\chi(z+2a\chi)((1-a\chi^2)Y_1^{(0,1)}(0,1)\\
&+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)-\chi(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1)+a\chi(Y_2^{(1,0)}(0,1)\\
&+Y_2^{(1,1)}(0,1))))(a\chi^2-1)^2+2\chi(1-a\chi^2)(-z\chi-a(z^2+\chi^2)+1)(Y_2^{(0,1)}(0,1)\\
&+a\chi((2-2a\chi^2)Y_1^{(0,1)}(0,1)-3\chi Y_2^{(0,1)}(0,1)+2(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))-2\chi(Y_2^{(0,2)}(0,1)\\
&+a\chi(Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)))))-2\chi(1-a\chi^2)(-z\chi-a(z^2+\chi^2)+1)(2a^3Y_3(0,1)\chi^4\\
&+(2Y_2^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))\chi^2+a^2((2Y_1^{(1,0)}(0,1)+Y_2^{(2,0)}(0,1))\chi^2-4Y_3(0,1)+2Y_2^{(0,1)}(0,1))\chi^2\\
&-2(Y_1^{(0,1)}(0,1)+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))\chi+2Y_1^{(1,0)}(0,1)+2a((-2Y_1^{(1,0)}(0,1)+\chi(Y_1^{(0,1)}(0,1)\\
&+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1))-Y_2^{(2,0)}(0,1))\chi^2+Y_3(0,1)-Y_2^{(0,1)}(0,1))+Y_2^{(2,0)}(0,1))\\
&+\chi^2(z+2a\chi)(1-a\chi^2)(-a^2(2aY_3^{(0,1)}(0,1)+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))\chi^4\\
&-2a(Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi^3-(4(a^2+1)Y_2^{(0,1)}(0,1)\\
&+5Y_2^{(0,2)}(0,1)+Y_2^{(0,3)}(0,1)-2a(2aY_3^{(0,1)}(0,1)-aY_2^{(0,2)}(0,1)+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1)))\chi^2+2(Y_1^{(0,1)}(0,1)+Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi\\
&+2a(2Y_2^{(0,1)}(0,1)-Y_3^{(0,1)}(0,1)+Y_2^{(0,2)}(0,1))-2Y_1^{(1,1)}(0,1)-Y_2^{(2,0)}(0,1)-Y_2^{(2,1)}(0,1))\\
&+2\chi^2(-z\chi-a(z^2+\chi^2)+1)(Y_1^{(0,1)}(0,1)(a\chi^2-1)^2+Y_2^{(1,0)}(0,1)+Y_2^{(1,1)}(0,1)\\
&+\chi(2(a^2+1)(a\chi^2-1)Y_2^{(0,1)}(0,1)-Y_2^{(0,2)}(0,1)+a(2a^3Y_3^{(0,1)}(0,1)\chi^4+(4Y_2^{(0,2)}(0,1)\\
&+Y_2^{(0,3)}(0,1))\chi^2+a^2((2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1))\chi^2-4Y_3^{(0,1)}(0,1)+2Y_2^{(0,2)}(0,1))\chi^2\\
&-2(Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+3Y_2^{(1,1)}(0,1)+Y_2^{(1,2)}(0,1))\chi+2Y_1^{(1,1)}(0,1)+Y_2^{(2,0)}(0,1)\\
&+Y_2^{(2,1)}(0,1)+a((\chi(2Y_1^{(0,2)}(0,1)+Y_2^{(1,0)}(0,1)+5Y_2^{(1,1)}(0,1)+2Y_2^{(1,2)}(0,1))-2(2Y_1^{(1,1)}(0,1)\\
&+Y_2^{(2,0)}(0,1)+Y_2^{(2,1)}(0,1)))\chi^2+2Y_3^{(0,1)}(0,1)-2Y_2^{(0,2)}(0,1)))))))).
\end{aligned}
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