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Timelike curvature comparison in Lorentzian length spaces

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Abstract

This thesis deals with constructions in the theory of Lorentzian (pre-) length spaces, properties of timelike curvature bounds and rigidity statements in that context. Lorentzian (pre-)length spaces are the Lorentzian analogue of metric and length spaces. First, hyperbolic angles are introduced and basic properties of them are discussed. Second, a globalization result for upper curvature bounds is given, akin to the Alexandrov patchwork, as well as a bound on the diameter for negative lower curvature bounds, mimicking the Bonnet-Myers theorem. Third, two rigidity results are proven for spaces with a lower curvature bound.

Zusammenfassung

In dieser Arbeit werden Konstruktionen in der Theorie der Lorentz-(prä-)Längenräume, Eigenschaften von zeitartigen Krümmungsschranken und Rigiditätsaussagen mit Krümmungssschranken behandelt. Lorentz-(prä-)Längenräume sind das lorentzsche Analogon zu metrischen Räumen bzw. Längenräumen. Zuerst werden hyperbolische Winkel definiert und ihre elementaren Eigenschaften ausgeführt. Als nächstes wird ein Globalisierungsresultat für obere Krümmungsschranken bewiesen, das von Alexandrov's Patchwork angelehnt ist, und eine Schranke an den Durchmesser des Raums gezeigt, falls einen negative untere Krümmungsschranke gegeben ist, das Analogon vom Satz von Bonnet-Myers in diesem Kontext. Als letztes werden zwei Rigiditätsresultate für Lorentz-Längenräume mit unteren Krümmungsschranken besprochen.

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Introduction

The standard mathematical theory of general relativity usually assumes a smooth Lorentzian metric. Most results still hold for a $\mathcal{C}^{1,1}$ metric, see [23], and things start to go horribly wrong for Lorentzian metrics which are only Hölder continuous (causal bubbles form and timelike futures fail to remain open, see [14, 16]). On the other hand, when physicists deal with general relativity, they very often are confronted with Lorentzian metrics where that regularity is not given. Typical examples are matter distributions like stars, shock waves, conical singularities and impulsive gravitational waves, which are described by Lorentzian metrics that are not twice differentiable, not continuous, or even only distributionally defined, see [20, 29, 34, 17, 31, 30]. For the standard local existence result for the Einstein equations, seen as an initial value problem, one usually only assumes very few derivatives are bounded in the sense that the initial condition is in a certain Sobolev space [32]. Even worse, some attempts to form a theory of quantum gravity take the spacetime to be discrete. Therefore, an important part of mathematical general relativity is concerned with extending previous classical results to increasingly low regularity settings to at least get some of these examples into the realm of mathematical relativity, see e.g. [14, 35, 29].

The development of Lorentzian length spaces (LLS) turns this approach around, not assuming a manifold structure but considering which conditions are needed to make some theorems work, making this a "bottom up" approach and incorporating all these physically relevant examples. It works by creating an analogue to metric length spaces in the setting of Lorentzian geometry, making the time separation function the main object, corresponding to the distance in metric spaces (and getting rid of any notion of spacelike distance which was present in the Lorentzian metric). This field is very young: The first work on LLS was published in 2018 [19], and the field has expanded very quickly since then. There are also some slightly modified versions of LLS, see [26, 21, 24], but they all agree on the time separation function as the core notion.

Since it incorporates aspects of both metric length spaces and Lorentzian geometry, there are two main inspirations for results in LLS: carrying over results from metric length spaces (which tend to get more subtle as one has to take into account the genuinely Lorentzian concept of causality), and extending results for Lorentzian manifolds to LLS (where the non-smoothness precludes the use of differential calculus).

In the theory of metric length spaces, a very important notion is that of curvature bounds in the sense of Alexandrov, namely Alexandrov spaces (curvature bounded above) and CAT(K) spaces (curvature bounded below). There are many equivalent formulations of these, see [10], one of them being triangle comparison. In the setting of Riemannian manifolds, they are equivalent to bounds on the sectional curvature. There is also a version of triangle comparison for Lorentzian manifolds: [18] proves that triangle comparison is an equivalent formulation of sectional curvature bounds for Lorentzian manifolds. It even works in the general context of semi-Riemannian manifolds, see [2].

Carrying this notion over to LLS, one arrives at the notion of *timelike triangle comparison*, introduced in [19]. In the setting of Lorentzian manifolds, this is the part of the assumption of triangle comparison that still work after discarding spacelike distances. In Lorentzian manifolds, timelike triangle comparison has recently been proven to be equivalent to sectional curvature bounds on timelike planes, see [7], but the sectional curvature on spacelike planes is now unrestricted. This is still physically relevant: sectional curvature on timelike planes corresponds to the tidal forces experienced by an observer travelling in a direction in the timelike plane in the spacelike direction in the plane.

In the theory of metric length spaces, adding a measure to the space allows one to formulate the so-called CD (curvature dimension) conditions. For Riemannian manifolds with the volume measure, this corresponds to an upper bound on the dimension and a lower bound on the Ricci curvature, see [36, 15]. Recently, there have been developments of a corresponding notion for measured LLS, called TCD (timelike curvature dimension) conditions described by a convexity condition on an entropy functional along Wasserstein geodesics, see [12, 9]. As with the metric measure version, it is using the tools of optimal transport and is equivalent to an upper bound on the dimension and a lower bound on the Ricci curvature for Lorentzian manifolds with the volume measure. Measured LLS satisfying TCD conditions are also physically relevant, as the Ricci tensor appears in the Einstein equations: There is even a description of lower and upper Ricci curvature bounds in [25], bringing a version of the full vacuum Einstein equations into this synthetic framework.

This thesis adapts some approaches from the theory of metric length spaces to the framework of LLS. The most overarching concept across the papers included here is that of hyperbolic angles. They even prove important beyond the scope of this thesis, see e.g. [6]. In the setting of physics on Lorentzian manifolds, hyperbolic angles correspond to speed differences at a point, measured as the so-called rapidity, and they are the analogue of angles in metric length spaces.

The first included paper introduces hyperbolic angles and their properties like semi-continuity in spaces with timelike curvature bounds. Although inspired from the metric side of things, hyperbolic angles behave differently: there is no upper bound for angles (for metric spaces, there is no angle larger than π, here, they extend all the way to infinity). The central statement regarding hyperbolic angles is the triangle inequality of angles. In most cases, it holds under very weak assumptions, but in one case one has to assume a timelike triangle bound from below, otherwise there is a counterexample of a positive angle branching of a geodesic. The triangle inequality of angles allows one to construct, among others, the space of directions at a point, timelike tangent cones (which are a substitute for the tangent space only using timelike related points) and correspondingly the timelike exponential map and its inverse, the logarithmic map. These notions, in turn, allow one to go back and forth between the timelike tangent cone and a neighbourhood in the future of the point. Additionally, the equivalence of the angle monotonicity condition to the timelike triangle condition is established, and work is done toward a third equivalent definition, the angle condition (which is introduced in a different paper, see [4, 8]).

The second paper is concerned with global synthetic geometry and includes two main results: regarding the first one, usually triangle comparison is only done on small enough neighbourhoods. In metric geometry there is the so-called Alexandrov's patchwork, a classical result from Alexandrov spaces that local curvature bounds above in a space with continuously varying geodesics imply a global curvature bound above, i.e. the neighbourhood in which we do triangle comparison can be generalized to be the whole space, see [3]. It uses a very visual approach, repeatedly subdividing a large triangle into two smaller ones until the small ones are each in a local neighbourhood. Transferring this to the setting of LLS is possible, but one has to be very careful to keep the small triangles timelike.

Second, a proof of a bound on the diameter of the space given a negative lower bound on the curvature is given. This corresponds to the well-known Bonnet-Myers theorem in metric geometry, see [11] (there the bound has to be positive, this discrepancy in the sign is due to some technicalities on the sign of the curvature), but the resulting statement is a bit more subtle: in metric geometry, a geodesic is locally minimizing the length, and the Bonnet-Myers theorem shows that it stops minimizing after at most the maximal diameter. In Lorentzian geometry, timelike geodesics are local maximizers of length, and we get that all geodesics stop maximizing after the bound on the diameter, i.e. points are "even further apart" than any geodesic, i.e. have to have infinite distance.

The main result of the next paper is a splitting theorem for LLS. Splitting theorems are rigidity results and typically involve non-positive curvature (like sectional curvature bounds or Ricci bounds) and a line, resulting in the space "splitting", i.e. being isometric to a product space. The history of splitting theorems started long ago with Riemannian manifolds and sectional curvature bounds, and there are versions for any combination of sectional or Ricci bounds and Riemannian or Lorentzian manifolds, see [33, 13, 5, 28], as well as a metric length space version with triangle comparison, see [22, 10]. In this paper, timelike triangle comparison is used for this, and the proof heavily relies on hyperbolic angles, the proof being inspired by both the Lorentzian manifold sectional curvature version and the metric length space version.

The last work unites aspects from its two precursors: it is a rigidity result on the Bonnet-Myers result from the second work and uses among others all the techniques from the splitting paper. It assumes that in the setting of the Bonnet-Myers theorem from the second work, there exists a maximizing geodesic with length as large as it is allowed by the Bonnet-Myers theorem and proves that the space is a warped product with warping function cosine (for metric spaces, one would call this a spherical suspension, here one should think of it as the warped product structure of a part of anti-deSitter space). In the setting of Riemannian manifolds this is already known even for the weaker assumption of Ricci bounds (the Myers theorem, [27]) and also known for metric length spaces, see [11].

In both the last work and the splitting paper, warped products resp. products are dealt with differently than the warped products from [1]: The last work includes an alternate description of warped product LLS which matches the definition of [1] for strictly intrinsic metric spaces, but allows one to get explicit formulae for some relevant warping functions and can return a non-intrinsic Lorentzian pre-length space for a non-intrinsic metric space.

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Hyperbolic angles in Lorentzian length spaces and timelike curvature bounds

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Abstract

Within the synthetic-geometric framework of Lorentzian (pre-)length spaces developed in Kunzinger and Sämann (Ann. Glob. Anal. Geom. 54(3):399–447, 2018) we introduce a notion of a hyperbolic angle, an angle between timelike curves and related concepts like timelike tangent cone and exponential map. This provides valuable technical tools for the further development of the theory and paves the way for the main result of the article, which is the characterization of timelike curvature bounds (defined via triangle comparison) with an angle monotonicity condition. Further, we improve on a geodesic non-branching result for spaces with timelike curvature bounded below.

Keywords: metric geometry, Lorentz geometry, Lorentzian length spaces, hyperbolic angles, synthetic curvature bounds

MSC2020: 28A75, 51K10, 53C23, 53C50, 53B30, 53C80, 83C99

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1 Introduction

In the theory of (metric) length spaces, Alexandrov spaces and CAT(K)-spaces angles play an important role. They are an indispensable tool and in particular allow one to characterize curvature bounds with a monotonicity condition on angles. Thus a fruitful way forward in the theory of *Lorentzian (pre-)length spaces* [KS18] is to introduce the notion of an hyperbolic angle into the Lorentzian synthetic setting. As compared to the metric case, several difficulties have to be overcome, foremost the issue of keeping track of the time orientation and causal relations of the points and curves considered. Thus several proofs become more involved than one would expect from the metric case.

Lorentzian (pre-)length spaces have been developed to allow one to define timelike or causal curvature bounds for spacetimes of low regularity or for situations where one does not even have a Lorentzian metric, like in some approaches to Quantum Gravity, e.g. the causal set approach [BLMS87] or causal Fermion systems [Fin18] (see also [KS18, Subsec. 5.3]). Triangle comparison theorems in Lorentzian geometry were pioneered by Harris in [Har82] and triangle comparison (to model spaces) is the key concept to define timelike and causal curvature bounds in [KS18]. The study of spacetimes (and cone structures) of low regularity is a very active field with connections to General Relativity and in particular to singularity theorems, the cosmic censorship conjecture and (in-)extendibility (cf. [CG12, FS12, Säm16, GL17, Sbi18, GGKS18, GLS18, Min19, GKSS20, Gra20, LLS21, KOSS22]). Questions of (in-)extendibility of spacetimes can be formulated in the setting of Lorentzian length spaces and this approach enabled [GKS19], for the first time, to relate inextendibility to a blow-up of (synthetic) curvature.

Furthermore, as pioneered by Kronheimer and Penrose [KP67] and Busemann [Bus67], causality theory and (parts of) Lorentzian geometry can be studied decoupled from the manifold structure [KS18, Subsec. 3.5], [ACS20, BGH21]. A large class of Lorentzian length spaces have been investigated in [AGKS21], where warped products of a one-dimensional base with metric length space fibers are considered. These so-called *generalized cones* have particularly nice causal properties. Moreover, one can relate Alexandrov curvature bounds of the fibers to timelike curvature bounds of the entire space and vice versa, and a first synthetic singularity theorem was established in this setting. Recently, the gluing of Lorentzian pre-length spaces was developed in [BR22] and applied to gluing of spacetimes, thereby establishing an analogue of a result by Reshetnyak for $\mathrm{CAT}(K)$-spaces, which roughly states that gluing is compatible with upper curvature bounds.

Cavalletti and Mondino [CM20] took the field even further by introducing a synthetic notion of Ricci curvature bounds by using techniques from optimal transport analogous to the Lott-Villani-Sturm theory of metric measure spaces with Ricci curvature bounded below [LV09, Stu06a, Stu06b]. They built on work by McCann [McC20] and Mondino-Suhr [MS21], who characterize timelike Ricci curvature bounds in terms of synthetic notions à la Lott-Villani-Sturm for smooth spacetimes. A related topic was then to find canonical measures in the Lorentzian setting akin to Hausdorff measures in the metric one. This has been achieved recently in [MS22], where also a synthetic dimension for Lorentzian pre-length spaces is given, the compatibility of the Lorentzian measures with the volume measure of continuous spacetimes is shown and its relation to [CM20] is studied.

Finally, there is the related approach of Sormani and Vega [SV16] that introduces a metric on Lorentzian manifolds with a suitable time function, the so-called *null distance*. Its relation to spacetime convergence has been explored in [AB21], while in [KS21] the null distance is introduced on Lorentzian length spaces and a kind of compatibility between the two approaches is established.

In the final stages of preparing the article we have been made aware of the preprint [BMS22] by Barrera, Montes de Oca and Solis, where the authors also introduce angles in Lorentzian length spaces. Compared to our work they are investigating only curvature bounds from below but discuss an angle comparison condition and first variation formula, which we do not. On the other hand in our more comprehensive study we consider bounds from below and above, exponential and logarithmic map, triangle inequality of angles and equivalence of monotonicity comparison and timelike curvature bounds. Thus our two works nicely complement each other and their simultaneous appearance is a clear indication that this direction of research will prove fruitful.

In the following subsection we present the main results of our article and outline the structure of the paper.

1.1 Main results and outline of the article

The plan of the paper is as follows. In Subsection 1.2 we introduce the relevant background on Lorentzian geometry and fix some notations and conventions. Then, to conclude the introduction we give a brief review of the theory of Lorentzian length spaces in Subsection 1.3 and of curvature comparison in Subsection 1.4.

In Section 2 we introduce angles in Lorentzian pre-length spaces in timelike triangles and between timelike curves. A main tool is the Lorentzian version of the law of cosines (Lemma 2.4) and corollaries thereof. As a compatibility check we establish that in a smooth and strongly causal spacetime the angle between continuously differentiable timelike curves agrees with the synthetic angle in the sense of Lorentzian pre-length spaces. We also show that one can alternatively use K-comparison angles for the definition of angles (for any $K \in \mathbb{R}$).

After these technical preparations we study in detail angles between timelike curves that have the same time orientation in Section 3. A central result is the triangle inequality.

Theorem 3.1 (Triangle inequality for (upper) angles)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and locally causally closed Lorentzian pre-length space with τ locally finite-valued and locally continuous. Let $\alpha, \beta, \gamma\colon [0, B) \to X$ be timelike curves with coinciding time orientation starting at $x := \alpha(0) = \beta(0) = \gamma(0)$. Then*

$$\measuredangle_x(\alpha, \gamma) \leq \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma)\,. \tag{1}$$

Two timelike curves of the same time orientation and starting at the same point are defined to be equivalent if they have zero angle. Together with the triangle inequality this makes the space of (future/past) directed timelike directions at a point into a metric space. This allows one to define the *timelike tangent cone* as the Minkowski cone over the metric space of (future/past) directed timelike directions. Also, one can define an exponential and a logarithmic map and we establish foundational properties and relations of these objects.

Moreover, we introduce a monotonicity condition for angles at the past/ future endpoint of a timelike triangle, called *future/past K-monotonicity comparison*. This gives then a bound on the angle between timelike geodesics of the same time orientation (Corollary 3.8), and that angles exist, if finite (Lemma 3.9). Another central result is that timelike curvature bounds imply future/past K-monotonicity comparison as follows.

Theorem 3.10 (Triangle comparison implies future K-monotonicity comparison)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally strictly timelike geodesically connected Lorentzian pre-length space and let $K \in \mathbb{R}$. If X has timelike curvature bounded below (above) by K, it also satisfies future K-monotonicity comparison from below (above).*

In Section 4 we establish the triangle inequality between timelike curves of arbitrary time orientation in Theorem 4.5. Moreover, we improve upon a geodesic non-branching result of [KS18], where we do not need any additional hypotheses, as follows.

Theorem 4.8 (Timelike non-branching)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded below by some $K \in \mathbb{R}$. Then timelike distance realizers cannot branch.*

Finally, in Subsection 4.1, we define a general K-monotonicity comparison condition (Definition 4.9), show that it implies the existence of angles in Lemma 4.10 and, this being the main result of the article, that it characterizes timelike curvature bounds (Definition 1.20) as follows.

Theorem 4.13 (Equivalence of triangle and monotonicity comparison)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally strictly timelike geodesically connected Lorentzian pre-length space and let $K \in \mathbb{R}$. Then X has timelike curvature bounded below (above) by K if and only if it satisfies K-monotonicity comparison from below (above).*

The proof of the Lorentzian law of cosines Lemma 2.4 and of a related lemma is outsourced to the Appendix 4.1.

1.2 Notation and conventions

We fix some notation and conventions as follows. One of our main examples are *spacetimes* (M, g), where the Lorentzian metric g is of different regularity classes. Here M denotes a smooth, connected, second countable Hausdorff manifold and the Lorentzian metric g on M is continuous, $\mathcal{C}^2$ or smooth. Note that our convention is that g is of signature $(- + + + \ldots)$. A vector $v \in TM$ is called

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g(v,v) \quad \begin{cases} < 0\,, \\ = 0 \text{ and } v \neq 0\,, \\ \leq 0 \text{ and } v \neq 0\,, \\ > 0 \text{ or } v = 0\,. \end{cases} \tag{2}$$

Moreover, we assume that (M, g) is time-oriented (i.e., there exists a continuous timelike vector field ξ, that is, $g(\xi, \xi) < 0$ everywhere). We call (M, g) a *continuous/$\mathcal{C}^2$-/smooth spacetime*. Furthermore, a causal vector $v \in TM$ is *future/past directed* if $g(v, \xi) < 0$ (or $g(v, \xi) > 0$, respectively), where ξ is the global timelike vector field giving the time orientation of the spacetime (M, g). Analogously, one defines the causal character and time orientation of sufficiently smooth curves into M. The *(Lorentzian) length* $L^g(\gamma)$ of a causal curve $\gamma\colon [a, b] \to M$ is defined as $L^g(\gamma) := \int_a^b \sqrt{-g(\dot\gamma, \dot\gamma)}$ and the *time separation function* is defined as follows. For $x, y \in M$ set $\tau(x, y) := \sup\{L^g(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\} \cup \{0\}$.

Two events $x, y \in M$ are *timelike* related if there is a future directed timelike curve from x to y, denoted by $x \ll y$. Analogously, $x \leq y$ if there is a future directed causal curve from x to y or $x = y$. The spacetime (M, g) is *strongly causal* if for every point $p \in M$ and for every neighborhood U of p there is a neighborhood V of p such that $V \subseteq U$ and all causal curves with endpoints in V are contained in U, (in which case V is called *causally convex* in U).

Finally, let us recall the notion of sectional curvature on a smooth semi-Riemannian manifold.

Definition 1.1 (Sectional curvature). *Let (M, g) be a semi-Riemannian manifold, $p \in M$ a point. A non-degenerate subspace $P \subseteq T_pM$ is* non-degenerate *if $(g|_p)|_P : P \times P \to \mathbb{R}$ is a non-degenerate symmetric bilinear map. Let $P \subseteq T_pM$ be a non-degenerate 2-dimensional subspace (or plane). Let $v, w \in P$ be linearly independent. Let R be the Riemann curvature tensor of g, then the* sectional curvature *of M at p on the plane P is* $\frac{g(R(v,w)w,v)}{g(v,v)g(w,w)-g(v,w)^2}$.

1.3 A brief introduction to Lorentzian (pre-)length spaces

Kronheimer and Penrose [KP67] studied causality theory from an abstract point-of-view and their basic object is a so-called *causal space*. Here we use a slightly more general version of this.

Definition 1.2 (Causal space). *A set X with two binary relations $\ll, \leq$, where both are transitive, $\leq$ is reflexive, and any timelike related pairs $p \ll q$ are also causally related $p \leq q$ (sometimes denoted as $\ll \subseteq \leq$), is called a* causal space.

We define the chronological and causal futures and pasts and the chronological and causal diamonds as follows.

- $I^+(p) := \{q \in X : p \ll q\}$, $I^-(q) := \{p \in X : p \ll q\}$,
- $J^+(p) := \{q \in X : p \leq q\}$, $J^-(q) := \{p \in X : p \leq q\}$,
- $I(p,q) := I^+(p) \cap I^-(q)$, $J(p,q) := J^+(p) \cap J^-(q)$.

A Lorentzian pre-length space generalizes spacetimes by taking the causal relations and the time separation function as its fundamental objects, thereby foregoing the smooth structure of the manifold completely.

Definition 1.3 (Lorentzian pre-length space). *A* Lorentzian pre-length space *$(X, d, \ll, \leq, \tau)$ is a causal space $(X, \ll, \leq)$ together with a metric d on X and a map $\tau : X \times X \to [0, \infty]$ satisfying*

- *τ is lower semicontinuous (with respect to d),*
- *$\tau(p,r) \geq \tau(p,q) + \tau(q,r)$ for $p \leq q \leq r$ (reverse triangle inequality) and*
- *$\tau(p,q) > 0 \Leftrightarrow p \ll q$.*

To introduce the notion of *intrinsic* time separation functions, we define the length of curves as follows.

Definition 1.4 (Causal, timelike and null curves). *Let $(X, \ll, \leq, d, \tau)$ be a Lorentzian pre-length space. A non-constant locally Lipschitz continuous curve $\gamma\colon I \to X$ is* future directed causal *or* future directed timelike *if $\forall t_1, t_2 \in I, t_1 < t_2$, $\gamma(t_1) \ll \gamma(t_2)$ or $\gamma(t_1) \leq \gamma(t_2)$, respectively. The* causal character *of a future directed causal curve is* timelike *if it is timelike and* null *if $\forall t_1, t_2 \in I, t_1 < t_2$, $\gamma(t_1) \not\ll \gamma(t_2)$.*

For past directed causal and past directed timelike, we reverse these relations. Upon parameter reversal, they are future directed causal / timelike.

Definition 1.5 (Length of curves). *Let $(X, \ll, \leq, d, \tau)$ be a Lorentzian pre-length space. We can define the length of future directed causal curves as in metric spaces, replacing the supremum with an infimum in the definition of the variational length. We define the* length $L_\tau(\gamma) = \inf\{\sum_i \tau(\gamma(t_i), \gamma(t_{i+1})) :$ *(t_i) a partition of $I\}$.*

One of the main examples of an Lorentzian pre-length space is a spacetime (M, g) together with its timelike and causal relations $\ll, \leq$, its time separation function τ and a metric induced by a complete Riemannian background metric, cf. [KS18, Ex. 2.11]. Moreover, spacetimes of low regularity (i.e., with continuous metric and sufficiently well-behaved causality) and Lorentz-Finsler spaces are Lorentzian pre-length spaces as well, cf. [KS18, Prop. 5.8, Prop. 5.14].

Definition 1.6 (Intrinsic space). *A Lorentzian pre-length space is* strictly intrinsic *or* geodesic *if for all $p < q$ there exists a future directed causal curve γ from p to q of length $L_\tau(\gamma) = \tau(p, q)$. Such a curve is called a* distance realizer*.*

A Lorentzian pre-length space is intrinsic *if for all $p < q$ and all $\varepsilon > 0$ there exists a future directed causal curve γ from p to q of length $L_\tau(\gamma) > \tau(p, q) - \varepsilon$. Such a curve is called an* ε-distance realizer*.*

A subset $A \subseteq X$ in a Lorentzian pre-length space is called *(causally) convex* if for any two points $p, q \in A$ their causal diamond is contained in A, i.e., $J(p, q) \subseteq A$.

A key technical tool in smooth semi-Riemannian geometry is the existence of geodesically convex neighborhoods, in which the causality is particularly simple and where one has a complete description of length-maximizing curves. The analogue of this notion in the present context is the following. A Lorentzian pre-length space X is called *localizable* if every $x \in X$ has an open, so-called *localizing* neighborhood Ω_x such that

(i) The d-length of all causal curves contained in Ω_x is uniformly bounded.

(ii) The localizing neighborhood Ω_x can be turned into a Lorentzian pre-length space by using the restrictions $d|_{\Omega_x \times \Omega_x}$, $\ll|_{\Omega_x \times \Omega_x}$ and $\leq|_{\Omega_x \times \Omega_x}$, and define the local time separation function $\omega_x \colon \Omega_x \times \Omega_x \to [0, \infty)$ as follows. For $p, q \in \Omega_x$ we set $\omega_x(p, q) := \sup\{L_\tau(\gamma) : \gamma : [a, b] \to \Omega_x$ is $\leq$-causal from p to $q\} \cup \{0\}$. For $p < q$ we require that there is an ω_x-realizer $\gamma_{p,q}$ in Ω_x from p to q, and that this makes Ω_x into a Lorentzian pre-length space. See e.g. [Ber20, chapter 1.7.2] or [KS18, Def. 3.16].

(iii) For every $y \in \Omega_x$ we have $I^\pm(y) \cap \Omega_x \neq \emptyset$.

If, in addition, the neighborhoods Ω_x can be chosen such that

(iv) Whenever $p, q \in \Omega_x$ satisfy $p \ll q$ then $\gamma_{p,q}$ is timelike and strictly longer than any future-directed causal curve in Ω_x from p to q that contains a null segment,

then $(X, d, \ll, \leq, \tau)$ is called *regularly localizable.*

Locally distance realizing curves can be thought of as an analog of geodesics.

Definition 1.7 (Geodesic)**.** *Let X be a Lorentzian pre-length space. A (say) future directed causal curve $\gamma \colon I \to X$ is a* geodesic *if for each $t \in I$ there is a neighborhood $[a, b]$ of t (i.e., $a < t < b$, but allowing for equality at the endpoints of I) such that $\gamma|_{[a,b]}$ is a* distance realizer*, i.e. $\tau(\gamma(a), \gamma(b)) = L_\tau(\gamma|_{[a,b]})$.*

In localizable spaces, the notion of a geodesic was previously defined using localizable neighborhoods as follows.

Definition 1.8 (Geodesic à la [GKS19, Def. 4.1])**.** *Let X be a localizable Lorentzian pre-length space. A (say) future directed causal curve $\gamma \colon I \to X$ is a* geodesic *if it is* locally maximal*, i.e., for every $t_0 \in I$ there is a localizing neighborhood $\Omega_{\gamma(t_0)}$ and a neighborhood $[a, b]$ of t_0 in I such that $\gamma|_{[a,b]}$ is maximal from $\gamma(a)$ to $\gamma(b)$ in $\Omega_{\gamma(t_0)}$.*

In this article we will use the first definition and the latter notion actually satisfies the first one when everything is considered in the Lorentzian pre-length space Ω_x. Thus, when developing the theory here, it is enough to use the first version, and not employ localizability unless explicitly stated.

Definition 1.9. *We call τ* locally finite-valued *if every point has a neighborhood U such that $\tau < \infty$ on $U \times U$. Similarly, we call τ* locally continuous *if every point has a neighborhood U such that τ is continuous on $U \times U$.*

Moreover, X is locally causally closed *if every point has a neighborhood U such that for all $x_n, y_n \in U$ with $x_n \to x \in \bar{U}$, $y_n \to y \in \bar{U}$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.*

Finally, X is causally path-connected *if for all $x, y \in X$ with $x < y$ (or $x \ll y$) there is a future directed causal (or timelike) curve from x to y.*

Remark 1.10. *Note that by [GKS19, Lem. 4.3] an intrinsic and strongly causal Lorentzian pre-length space is locally finite-valued. On the other hand, if τ is locally finite-valued, then X is chronological, i.e, $\ll$ is irreflexive:*

By [KS18, Prop. 2.14] one has that for all $x \in X$ either $\tau(x,x) = 0$ or $\tau(x,x) = \infty$. The latter is excluded by assumption, so $\tau(x,x) = 0$, which is equivalent to $x \not\ll x$.

The definition of local causal closedness is the original definition given in [KS18, Def. 3.4]. However, in [ACS20] an issue with non-strongly-causal Lorentzian pre-length spaces was pointed out and an alternative definition has been proposed in [ACS20, Def. 2.19]. We opted for the former one as we are only concerned with strongly causal spaces and for compatibility with previous results in [KS18, AGKS21].

Any timelike distance realizer can be parametrized with respect to τ-arclength as long as τ is locally finite-valued and locally continuous, cf. [KS18, Subsec. 3.7].

Definition 1.11 (Lorentzian length space)**.** *A* Lorentzian length space *is a Lorentzian pre-length space which is locally causally closed, causally path connected, intrinsic and localizable.*

Definition 1.12. *An open set $U \subseteq X$ is called* timelike geodesically connected *if whenever $x, y \in U$ with $x \ll y$, there exists a future-directed maximal geodesic in U from x to y. An open set $U \subseteq X$ is called* strictly timelike geodesically connected *if whenever x, $y \in U$ with $x \ll y$, there exists a future-directed maximal geodesic in U from x to y, and that any future-directed maximal geodesic in U from x to y is timelike. X is called* locally strictly timelike geodesically connected *if it is covered by strictly timelike geodesically connected neighborhoods.*

1.4 Curvature comparison for Lorentzian pre-length spaces

Curvature bounds for metric spaces, generalizing sectional curvature bounds of Riemannian manifolds, are defined by comparing distances in triangles to distances in comparison triangles in two-dimensional (Riemannian) manifolds of constant curvature, see e.g. [BH99, BBI01, AKP22]. Timelike and causal curvature bounds were introduced analogously for Lorentzian pre-length spaces in [KS18]. In this setting we measure distances with the time separation, so we restrict to causal triangles.

Definition 1.13 (Geodesic triangles)**.** *A* timelike (geodesic) triangle $\Delta = (p_1, p_2, p_3)$ *in a Lorentzian pre-length space X consists of three points $p_1 \ll p_2 \ll p_3 \in X$ (with $\tau(p_i, p_j) < \infty$ for $i < j$) and three future directed causal distance realizing curves α_{ij} connecting p_i to p_j (for $i < j$). Analogously, we define a* causal triangle.

An admissible causal (geodesic) triangle (p_1, p_2, p_3) *in a Lorentzian pre-length space* X *consists of three points* $p_1 \ll p_2 \leq p_3$ *or* $p_1 \leq p_2 \ll p_3 \in X$ *(with* $\tau(p_i, p_j) < \infty$ *for* $i < j$*) and three possibly constant*[3] *future directed causal distance realizing curves* α_{ij} *connecting* p_i *to* p_j *(for* $i < j$*). We call the sides between two vertices* $p_i \ll p_j$ *a* timelike side *(although it need not be realized via a timelike curve).*

We call p_1 *the* past endpoint *and* p_3 *the* future endpoint *of the triangle. A causal or timelike triangle is called* non-degenerate *if the reverse triangle inequality* $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$ *is strict, and it is called* degenerate in the strict sense *if the sides* α_{12} *and* α_{23} *are (reparametrized) parts of the longest side* α_{13}*.*

1.4.1 Comparison (model) spaces

Here we recall the two-dimensional Lorentzian manifolds of constant curvature $K \in \mathbb{R}$, i.e., (scaled) (anti-)de Sitter spacetime and Minkowski spacetime.

Definition 1.14 (Model spaces)**.** *For integers* $0 \leq m \leq n$ *denote by* $\mathbb{R}^n_m$ *the vector space* $\mathbb{R}^n$ *together with the inner product* $b(v, w) = -\sum_{i=1}^m v_i w_i + \sum_{i=m+1}^n v_i w_i$*, where* $v = (v_1, \ldots, v_n)$*,* $w = (w_1, \ldots, w_n)$*. Let* $K \in \mathbb{R}$*.*

The Lorentzian K-planes *or the* comparison spaces *of constant curvature* K *are the following.*

- *Positive curvature* $K > 0$*: We define* $\mathbb{L}^2(K)$ *to be the universal cover of* $\{v \in \mathbb{R}^3_1 : b(v, v) = \frac{1}{K^2}\}$*.*
- *Negative curvature* $K < 0$*: We define* $\mathbb{L}^2(K)$ *to be the universal cover of* $\{v \in \mathbb{R}^3_2 : b(v, v) = -\frac{1}{K^2}\}$*.*
- *Curvature* $K = 0$ *(flat):* $\mathbb{L}^2(0) := \mathbb{R}^2_1$*, the two-dimensional Minkowski spacetime.*

The (finite timelike) diameter of $\mathbb{L}^2(K)$ *is* $D_K = \frac{\pi}{\sqrt{-K}}$ *for* $K < 0$ *and* $D_K = \infty$ *for* $K \geq 0$[4]*.*

Definition 1.15. *Three numbers* $a_{12}, a_{23}, a_{13} \geq 0$ *satisfying the reverse triangle inequality* $a_{12} + a_{23} \leq a_{13}$ *(making* a_{13} *the largest) are said to satisfy the* timelike size bounds for K *if* $a_{13} < D_K$*.*[5]

[3] Of course, if e.g. $p_1 \leq p_2 \ll p_3$, only $\alpha_{1,2}$ can be constant, and $\alpha_{2,3}$ in the other case.
[4] This is the maximum time separation which is finite.
[5] $a_{13} < D_K$ is trivial if $K \geq 0$.

Andersson-Howard [AH98] introduced semi-Riemannian curvature bounds and Alexander-Bishop developed the theory further in [AB08], by characterizing smooth semi-Riemannian sectional curvature bounds by triangle comparison. In the semi-Riemannian setting it is necessary to distinguish the causal character of the tangent planes considered. Otherwise only spaces of constant curvature would fulfill the sectional curvature bounds, see [O'N83, Prop. 8.28].

Let M be a semi-Riemannian manifold. Let $p \in M$ be a point and $P \subseteq T_pM$ be a plane. Then P is called *spacelike* if $g|_p$ is positive or negative definite on P, *timelike* if $g|_p$ is non-degenerate and indefinite on P.

Definition 1.16 (Sectional curvature comparison). *M satisfies* sectional curvature comparison from below (or above) by $K \in \mathbb{R}$ *if for all points $p \in M$ and all spacelike planes $P \subseteq T_pM$, the sectional curvature is $\geq K$ (or $\leq K$) and for all timelike planes $P \subseteq T_pM$, the sectional curvature is $\leq K$ (or $\geq K$).*

Remark 1.17. *Equivalently, sectional curvature comparison from below can be written in compact form as $g(R(v,w)w,v) \geq K(g(v,v)g(w,w) - g(v,w)^2)$ (and with inequality reversed for sectional curvature comparison from above).*

Remark 1.18. *Sectional curvature comparison is not transitive in dimension larger than two: if M satisfies sectional curvature comparison from below / above by $K \in \mathbb{R}$, it does not automatically satisfy it from below / above for any $\tilde{K} \neq K$. But if M satisfies timelike sectional curvature comparison from below (or above) by K (i.e., the sectional curvature comparison just for timelike planes), it automatically satisfies timelike curvature comparison from below (or above) by any $\tilde{K} \geq K$ (or $\tilde{K} \leq K$) (but note the reversal of the inequality as compared to the Riemannian case). For two dimensional Lorentzian manifolds there are only timelike tangent planes, hence sectional curvature comparison is timelike sectional curvature comparison. We give a detailed account for this in the case of Lorentzian pre-length spaces (and the Lorentzian K-planes) and triangle comparison in Lemma 3.27.*

Note that comparison triangles exist, when the side-lengths satisfy size bounds for $K \in \mathbb{R}$. To be precise [AB08, Lem. 2.1] gives the following.

Proposition 1.19 (Comparison triangles exist). *Given three non-negative reals a_{12}, a_{23}, a_{13} satisfying the reverse triangle inequality $a_{12} + a_{23} \leq a_{13}$ and timelike size bounds for $K \in \mathbb{R}$, there exists a causal triangle $\Delta p_1p_2p_3$ in a normal neighborhood in the K-plane such that $\tau(p_i, p_j) = a_{ij}$ (for $i < j$).*

Any two such triangles $\Delta p_1p_2p_3$, $\Delta q_1q_2q_3$ (for the same side-lengths a_{ij}) in the K-plane are related by an isometry φ mapping one to the other. The *isometry φ is unique unless $a_{13} = 0$ (making all $a_{ij} = 0$) or the reverse triangle inequality is actually an equality.*

1.4.2 Triangle comparison

In this subsection we introduce timelike curvature bounds as in [KS18, Subsec. 4.3].

Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space, $\mathbb{L}^2(K)$ be the K-plane with time separation function $\bar{\tau}$ and $p_1 \ll p_2 \ll p_3$ be three timelike related points in X. A *comparison triangle* of $\Delta p_1 p_2 p_3$ in the Lorentzian K-plane is a timelike triangle with vertices $\overline{p_1} \ll \overline{p_2} \ll \overline{p_3}$ in the K-plane with agreeing side-lengths, i.e., $\tau(p_i, p_j) = \bar{\tau}(\overline{p_i}, \overline{p_j})$ for $i < j$. We say the vertex p_i corresponds to $\overline{p_i}$. Moreover, we say that the side α_{ij} connecting p_i to p_j $(i < j)$ corresponds to the side $\overline{\alpha_{ij}}$ connecting $\overline{p_i}$ to $\overline{p_j}$.

For a point q on some side α_{ij} of the triangle we define the corresponding point $\overline{q}$ on $\overline{\alpha_{ij}}$ by requiring equal distances to the endpoints of the curve it is on, i.e., we require $\tau(p_i, q) = \bar{\tau}(\overline{p_i}, \overline{q})$ (and then automatically, $\tau(q, p_j) = \bar{\tau}(\overline{q}, \overline{p_j})$ as α_{ij} and $\overline{\alpha_{ij}}$ are distance realizing). Note that it might be necessary to specify which side q should be considered to be on (as two sides can partially overlap in X, but not in $\mathbb{L}^2(K)$, unless the triangle is degenerate). If we take two such points q_1, q_2 (usually on different sides), we can compare the time separation of q_1 to q_2 with the time separation of the corresponding points in the comparison situation. This sets the stage for timelike and causal curvature comparison.

Definition 1.20 (Timelike curvature bounds). *Let X be a Lorentzian pre-length space and $K \in \mathbb{R}$. An open subset U is called a* timelike $\geq K$-comparison neighborhood *(or* timelike $\leq K$-comparison neighborhood*) or just* comparison neighborhood *if*

- *τ is finite and continuous on $U \times U$,*
- *U is strict timelike geodesically connected and*
- *for all timelike triangles $\Delta p_1 p_2 p_3$ in U satisfying timelike size bounds for K, q_1, q_2 two points on some sides α and β and all (any) comparison situations $\Delta \overline{p_1 p_2 p_3}, \overline{q_1}, \overline{q_2}$ in the K-plane, the time separation satisfies*

$$\tau(q_1, q_2) \leq \bar{\tau}(\overline{q_1}, \overline{q_2}) \qquad (\text{or } \tau(q_1, q_2) \geq \bar{\tau}(\overline{q_1}, \overline{q_2}))\,.$$

We say X has timelike curvature bounded below by K *if it is covered by timelike $\geq K$-comparison neighborhoods. Likewise, it has* timelike curvature is bounded above by K *if it is covered by timelike $\leq K$-comparison neighborhoods.*

Remark 1.21. *Note that in the case of timelike curvature bounded below, $q_1 \ll q_2$ implies $\overline{q_1} \ll \overline{q_2}$, and in the case of timelike curvature bounded above, $\overline{q_1} \ll \overline{q_2}$ implies $q_1 \ll q_2$. Furthermore, if U is also a locally causally closed neighborhood we also have for curvature bounded below that $q_1 \leq q_2$ implies $\overline{q_1} \leq \overline{q_2}$ and in curvature bounded above the other implication holds.*

In an analogous way we can define causal curvature bounds for admissible causal triangles where two vertices are not necessarily timelike related, cf. [KS18, Subsec. 4.5].

Remark 1.22 (Automatic size bounds)**.** *If X is strongly causal and has timelike curvature bounded above / below, we can restrict ourselves to comparison neighborhoods U where timelike size bounds are automatically satisfied and of the form of timelike diamonds, and there exists a basis of the topology of such neighborhoods, see [Ber20, Rem. 2.1.8].*

2 Angles

In this section we introduce comparison angles, establish foundational properties and define the notion of an angle between timelike curves.

2.1 Definition and basic properties

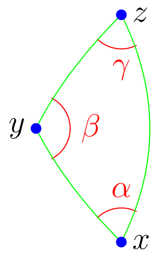
Definition 2.1 (K-comparison angles)**.** *Let $K \in \mathbb{R}$, let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $x, y, z \in X$ be* causally *related, i.e. $x \leq y \leq z$ or a permutation of this, and all time separations between these points finite. If $K < 0$ we demand that the side-lengths of Δxyz satisfy the timelike size bounds for K. Let $x \ll y$ and $x \ll z$ and let $\Delta\bar{x}\bar{y}\bar{z}$ be a comparison triangle of Δxyz in $\mathbb{L}^2(K)$. The K-*comparison (hyperbolic) *angle at x is defined as*

$$\tilde{\measuredangle}_x^K(y, z) := \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}, \bar{z}) \,. \tag{3}$$

Additionally, for notational simplicity we set $\tilde{\measuredangle}_x(y, z) := \tilde{\measuredangle}_x^0(y, z)$.

To make the notation more concise we define the signed angle: If the angle is at a future or past endpoint, we set $\sigma = -1$, if it is not, $\sigma = 1$. We define the signed angle *$\tilde{\measuredangle}_x^{K,\mathrm{S}}(y, z) := \sigma\tilde{\measuredangle}_x^K(y, z)$.*

Remark 2.2. *Note that in case $x \ll y \ll z$ we have $\tilde{\measuredangle}_x^{K,\mathrm{S}}(y, z) = -\tilde{\measuredangle}_x^K(y, z)$, so minus the "usual" angle, see Figure 1. Moreover, we will sometimes refer to the value of σ when we want to distinguish cases depending on the time orientation of the vertices. Note also that $\tilde{\measuredangle}_x^K(y, z) = \tilde{\measuredangle}_x^K(z, y)$.*



$$-\gamma = \tilde{\measuredangle}_r^{K,S}(p,q) \leq 0\,,$$

$$\beta = \tilde{\measuredangle}_q^{K,S}(p,r) \geq 0\,,$$

$$-\alpha = \tilde{\measuredangle}_p^{K,S}(q,r) \leq 0\,.$$

Figure 1: The three signed angles of a triangle.

Remark 2.3 (Angles in comparison spaces)**.** *Let $\bar{x}, \bar{y}, \bar{z} \in \mathbb{L}^2(K)$, such that $\bar{x} \ll \bar{y}$, $\bar{x} \ll \bar{z}$ then $\measuredangle^{\mathbb{L}^2(K)}$ is defined as follows:*

$$\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y},\bar{z}) = \operatorname{arcosh}\Big(\big|\langle \gamma'_{\bar{x},\bar{y}}(0), \gamma'_{\bar{x},\bar{z}}(0)\rangle\big|\Big)\,, \tag{4}$$

where $\gamma_{r,s}$ is the unit speed timelike geodesic from r to s and $\langle\cdot,\cdot\rangle$ is the metric on $\mathbb{L}^2(K)$. We have an analogous equation at the other vertices. Note that $\operatorname{arcosh}\Big(\big|\langle \gamma'_{\bar{x},\bar{y}}(0), \gamma'_{\bar{x},\bar{z}}(0)\rangle\big|\Big)$ is defined as the argument is greater equal than one by the reverse Cauchy Schwarz inequality for timelike vectors: $|\langle v,w\rangle| \geq |v||w|$.

The sign σ of the signed angle $\tilde{\measuredangle}_x^{K,S}(y,z)$ is just the sign of $\langle \gamma'_{\bar{x},\bar{y}}(0), \gamma'_{\bar{x},\bar{z}}(0)\rangle$ (which is the reason to define it this way).

Physically, the angle is a monotonously increasing function in the change of velocity of a particle switching from α to β. Moreover, note that in case $\sigma = 1$, the angles might seem to go against our (spatial) intuition: The angle ω between two parts of a straight line in Minkowski space is 0 here, in contrast to the Euclidean angle taking the maximum value π, as indicated in Figure 2.

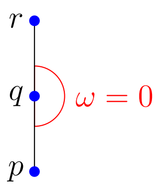


Figure 2: The $\omega = 0$ angle with $\sigma = 1$ appears large to our Euclidean intuition.

A fundamental tool in (semi-)Riemannian and metric geometry is the law of cosines. For the convenience of the reader we include a proof in our setting in the appendix 4.1. Note that it also could be derived from [AB08, Lem. 2.2. and Lem. 2.3]. The definition of the comparison angle leads to the *hyperbolic law of cosines* as follows.

Lemma 2.4 (Law of cosines)**.** *For p, q, r in the Lorentzian K-plane forming a finite causal triangle (not necessarily in this order), let $a = \max(\tau(p,q), \tau(q,p))$, $b = \max(\tau(q,r), \tau(r,q))$, $c = \max(\tau(p,r), \tau(r,p))$ be the side-lengths (note that in each maximum, one of the two arguments is 0) with $a, b > 0$, i.e., q is timelike related to p and r but p and r need only be causally related. Let $\omega = \tilde{\measuredangle}_q^K(p,r)$ be the hyperbolic angle at q, σ be the sign of the signed angle $\tilde{\measuredangle}_q^{K,\mathrm{S}}(p,r)$ and the scaling factor $s = \sqrt{|K|}$. Then we have:*

$$\begin{aligned} a^2 + b^2 &= c^2 - 2ab\sigma\cosh(\omega) && \text{for } K = 0\,, \\ \cos(sc) &= \cos(sa)\cos(sb) - \sigma\cosh(\omega)\sin(sa)\sin(sb) && \text{for } K < 0\,, \\ \cosh(sc) &= \cosh(sa)\cosh(sb) + \sigma\cosh(\omega)\sinh(sa)\sinh(sb) && \text{for } K > 0\,. \end{aligned}$$

Remark 2.5 (Analyticity and monotonicity in the law of cosines)**.** *Note these three formulae join up to an* analytic formula*: For the first equation, use $c^2 = a^2 + b^2 + 2ab\sigma\cosh(\omega)$. For the second equation, use $\frac{1-\cos(sc)}{|K|} = \frac{1-\cos(sa)\cos(sb)}{|K|} + \sigma\cosh(\omega)\frac{\sin(sa)\sin(sb)}{|K|}$. For the third equation, use $\frac{\cosh(sc)-1}{|K|} = \frac{\cosh(sa)\cosh(sb)-1}{|K|} + \sigma\cosh(\omega)\frac{\sinh(sa)\sinh(sb)}{|K|}$. Then the left- resp. right-hand-sides join up to an analytic formula in c and K resp. a, b, ω and K.*

In particular, fixing two sides and varying the third, ω is a strictly increasing function in the longest side and a strictly decreasing function in the other two sides.

In the case $\sigma = -1$, we can even make the side opposite the angle spacelike.

Corollary 2.6 (Extended law of cosines)**.** *Let p, q, r in the Lorentzian K-plane with pairwise finite τ-distances. We assume $p \ll q$ and $p \ll r$, but q and r not causally related and not all three on a single geodesic. Let $a = \tau(p,q)$, $b = \tau(p,r)$ be the side-lengths. Let $\omega = \measuredangle_p^{\mathbb{L}^2(K)}(q,r)$.* [6] *Then we have*

$$\begin{aligned} a^2 + b^2 &< 2ab\cosh(\omega) && \text{for } K = 0\,, \\ 1 &< \cos(sa)\cos(sb) + \cosh(\omega)\sin(sa)\sin(sb) && \text{for } K < 0\,, \\ 1 &> \cosh(sa)\cosh(sb) - \cosh(\omega)\sinh(sa)\sinh(sb) && \text{for } K > 0\,. \end{aligned}$$

Proof. Let q, r be not causally related. Without loss of generality we assume $\tau(p,r) \geq \tau(p,q)$ and consider the timelike geodesic α from p to q. We consider the parameter $t' = \sup\alpha^{-1}(I^-(r))$, then the point $q' := \alpha(t')$ is the point

[6] Note that this angle is still defined as an angle in the model space $\mathbb{L}^2(K)$ even though the side opposite the angle is spacelike.

along α being null related to r, i.e., $q' \leq r$, $q' \not\ll r$. Note $\omega := \measuredangle_p^{\mathbb{L}^2(K)}(q,r) = \measuredangle_p^{\mathbb{L}^2(K)}(q',r)$. We define the side-lengths: $a' = \tau(p,q') < a = \tau(p,q)$, $b = \tau(p,r)$. We apply the law of cosines (Lemma 2.4 with $\sigma = -1$ and a null side) to this situation. We now form a new triangle $\Delta\bar{p}\bar{q}\bar{r}$ with $\tau(\bar{p},\bar{q}) = a$, $\tau(\bar{p},\bar{r}) = b$ and $\bar{q} \leq \bar{r}$ null related. Let $\bar{\omega} := \measuredangle_{\bar{p}}^{\mathbb{L}^2(K)}(\bar{q},\bar{r})$. The triangles $\Delta\bar{p}\bar{q}\bar{r}$ and $\Delta pq'r$ only differ by the value of ω and the side-length $a > a'$. Now we apply the monotonicity statement of the law of cosines to a. Thus ω is a decreasing function of a, thus $\omega < \bar{\omega}$, and the formula in the law of cosines holds with equality for $a, b, c = 0, \bar{\omega}$. Replacing $\bar{\omega}$ by ω, one easily sees whether the right-hand-side increases or decreases. $\square$

Remark 2.7. *The extended law of cosines fits nicely with the monotonicity of the law of cosines, i.e., taking a timelike triangle and making the side opposite the angle null makes the angle larger, and similarly going from a null side to a spacelike side opposite the angle increases the angle.*

The following is helpful for calculations and is an example how one can utilize the law of cosines. The proof is outsourced to the appendix 4.1.

Lemma 2.8 (Calculating one-sided comparison situations)**.** *(1) Let $K \in \mathbb{R}$ and let $a + b + c \leq d$, and d satisfy timelike size bounds for K. Then construct a timelike triangle $\Delta p_1p_2p_3$ in $\mathbb{L}^2(K)$ with sidelengths $a, b+c, d$. We get a point q on the side $[p_2p_3]$ with $\tau(p_2,q) = b$ (see Figure 3). We denote $x = \tau(p_1,q)$ and $s = \sqrt{|K|}$. Then we have*

for $K = 0$

$$x^2 = \frac{bd^2 + a^2c}{b+c} - bc\,,$$

for $K < 0$

$$\cos(sx) = (\cos(sd) - \cos(sa)\cos(s(b+c))) \times \frac{\sin(sb)}{\sin(s(b+c))} + \cos(sa)\cos(sb)\,,$$

for $K > 0$

$$\cosh(sx) = (\cosh(sd) - \cosh(sa)\cosh(s(b+c))) \times \frac{\sinh(sb)}{\sinh(s(b+c))} + \cosh(sa)\cosh(sb)\,, \text{ and}$$

$$x^2 = \frac{bd^2 + a^2c}{b+c} - bc + O(s^2d^4) \text{ for } K \to 0 \text{ or } d \to 0\,.$$

(2) Under the same conditions, we can construct a timelike triangle $\Delta p_1p_2p_3$ in $\mathbb{L}^2(K)$ with sidelengths $a+b, c, d$. We get a point q on the side $[p_1p_2]$ with

$\tau(p_1, q) = a$ *(see Figure 3). We denote* $x = \tau(q, p_3)$ *and* $s = \sqrt{|K|}$. *Then we have*

$$\begin{aligned}
&\textit{for } K = 0 \\
&\qquad x^2 = \frac{bd^2 + c^2 a}{b+a} - ba\,, \\
&\textit{for } K < 0 \\
&\qquad \cos(sx) = (\cos(sd) - \cos(sc)\cos(s(b+a))) \times \\
&\qquad\qquad \frac{\sin(sb)}{\sin(s(b+a))} + \cos(sc)\cos(sb)\,, \\
&\textit{for } K > 0 \\
&\qquad \cosh(sx) = (\cosh(sd) - \cosh(sc)\cosh(s(b+a))) \times \\
&\qquad\qquad \frac{\sinh(sb)}{\sinh(s(b+a))} + \cosh(sc)\cosh(sb)\,, \ \textit{and}
\end{aligned}$$

$$x^2 = \frac{bd^2 + c^2 a}{b+a} - ba + O(s^2 d^4) \textit{ for } K \to 0 \textit{ or } d \to 0\,.$$

(3) Now let $K \in \mathbb{R}$ *and let* $a + b \leq c + d$, *and* $c + d$ *satisfy timelike size bounds for* K. *Then construct a timelike triangle* $\Delta p_1 p_2 p_3$ *in* $\mathbb{L}^2(K)$ *with sidelengths* $a, b, c + d$. *We get a point* q *on the side* $[p_1 p_3]$ *with* $\tau(p_1, q) = c$ *(see Figure 3). We denote* $x = \max(\tau(p_2, q), \tau(q, p_2))$ *and* $s = \sqrt{|K|}$. *Then we have*

$$\begin{aligned}
&\textit{for } K = 0 \\
&\qquad x^2 = \frac{cb^2 + a^2 d}{c+d} - cd\,, \\
&\textit{for } K < 0 \\
&\qquad \cos(sx) = (\cos(sb) - \cos(sa)\cos(s(c+d))) \times \\
&\qquad\qquad \frac{\sin(sc)}{\sin(s(c+d))} + \cos(sa)\cos(sc)\,, \\
&\textit{for } K > 0 \\
&\qquad \cosh(sx) = (\cosh(sb) - \cosh(sa)\cosh(s(c+d))) \times \\
&\qquad\qquad \frac{\sinh(sc)}{\sinh(s(c+d))} + \cosh(sa)\cosh(sc)\,, \ \textit{and}
\end{aligned}$$

$$x^2 = \frac{cb^2 + a^2 d}{c+d} - cd + O(s^2 (c+d)^4) \textit{ for } K \to 0 \textit{ or } c + d \to 0\,.$$

Note that in each case, the equations can be transformed to match the left-hand-sides of the equations in the analyticity statement of the law of cosines 2.4, and then the left- and right-hand-sides join up to an analytic formula.

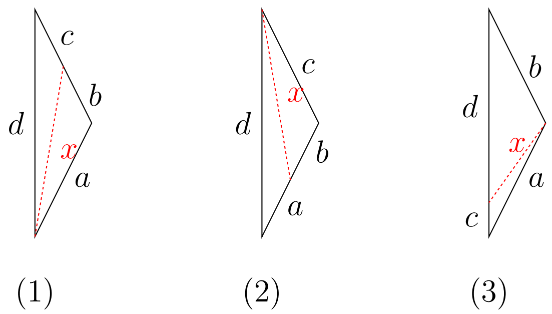


Figure 3: The three one-sided comparison situations (case (3) consists of two cases: one can distinguish whether x points to the future or the past).

At this point we introduce the angle between timelike curves.

Definition 2.9 (Angles between curves)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space, where τ is locally finite-valued. Let $\alpha, \beta\colon [0, \varepsilon) \to X$ be two future or past directed timelike curves with $\alpha(0) = \beta(0) =: x$. We define*

$$\begin{aligned} A_K := \{(s,t) \in (0,\varepsilon)^2 \colon & \qquad (5)\\ \alpha(s) \leq \beta(t) \text{ and } \Delta x\alpha(s)\beta(t) \text{satisfies size bounds for } K \text{ or} & \qquad (6)\\ \beta(t) \leq \alpha(s) \text{ and } \Delta x\beta(t)\alpha(s) \text{ satisfies size bounds for } K\}\,. & \qquad (7) \end{aligned}$$

The upper angle *between α and β at x is*

$$\measuredangle_x(\alpha, \beta) := \limsup_{(s,t)\in A_0;\, s,t\searrow 0} \tilde{\measuredangle}_x(\alpha(s), \beta(t))\,. \qquad (8)$$

Moreover, if $\lim_{(s,t)\in A_0;\, s,t\searrow 0} \tilde{\measuredangle}_x(\alpha(s), \beta(t))$ exists and is finite, then we call $\measuredangle_x(\alpha, \beta)$ the angle *between α and β at x, and say* the angle between α and β at x exists*. Similarly to the signed angle for three points, we define the signed angle between curves: If x is the future or past endpoint, i.e., α and β have the same time orientation, we set $\sigma = -1$ and if it is not, i.e., α and β have different time orientation, we set $\sigma = 1$. We define the signed angle $\measuredangle_x^{\mathrm{S}}(\alpha, \beta) = \sigma \measuredangle_x(\alpha, \beta)$.*

Remark 2.10. *The set A_K is precisely the set of (s,t) for which $\tilde{\measuredangle}_x^K(\alpha(s), \beta(t))$ is defined. Also, note that the signed angle between α and β has the same sign as the signed angle of $\Delta\alpha(s)x\beta(t)$ at x for any $s, t > 0$.*

Note that the definition of the upper angle of two future or past directed timelike curves α, β makes sense, i.e., there are $s, t \to 0$ with $(s,t) \in A_K$: In case α and β have a different time orientation, i.e., $\sigma = 1$, the condition $\alpha(s) \leq \beta(t)$ or conversely is automatically satisfied. In the other case, i.e., $\sigma = -1$, say without loss of generality α, β future directed. Then for fixed $t \in [0, \varepsilon)$, there is a $\delta > 0$ such that $\alpha([0,\delta)) \subseteq I^-(\beta(t))$, i.e., $x \ll \alpha(s) \ll \beta(t)$ for all $s \in [0, \delta)$.

Another useful concept is the one of a hinge:

Definition 2.11 (K-comparison hinges). *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space, where τ is locally finite-valued. We call two future or past directed timelike geodesics $\alpha : [0, a] \to X$, $\beta : [0, b] \to X$ with $\alpha(0) = \beta(0) =: x$ such that the angle between them exists a* hinge*.*

Let (α, β) be a hinge, and let $K \in \mathbb{R}$. Then a comparison hinge *is a hinge $(\bar\alpha, \bar\beta)$ in $\mathbb{L}^2(K)$, where the sides have the same time orientation and with $\measuredangle_x(\alpha, \beta) = \measuredangle_{\bar x}^{\mathbb{L}^2(K)}(\bar\alpha, \bar\beta)$, $L_\tau(\alpha) = L(\bar\alpha)$ and $L_\tau(\beta) = L(\bar\beta)$.*

Remark 2.12. *Comparison hinges always exist and are unique up to (unique) isometry of $\mathbb{L}^2(K)$, if $L_\tau(\alpha)$ and $L_\tau(\beta)$ satisfy timelike size bounds for K, where the isometry is unique if none of $L_\tau(\alpha), L_\tau(\beta)$ and $\measuredangle_x(\alpha, \beta)$ is zero.*

As a first compatibility check we establish below that in a ($\mathcal{C}^2$, strongly causal) spacetime the angle between two timelike curves with respect to the Lorentzian metric and the angle as defined above agree.

Proposition 2.13 (Angles agree). *Let $\alpha, \beta : [0, \varepsilon) \to M$ be future or past directed $\mathcal{C}^1$-regular g-timelike curves in a strongly causal $\mathcal{C}^2$-spacetime (M, g) with $\alpha(0) = \beta(0) = x$. Then the angle $\measuredangle_x(\alpha, \beta)$ in the Lorentzian pre-length space is the same as the Lorentzian angle in (M, g), i.e.,*

$$\measuredangle_x(\alpha, \beta) = \operatorname{arcosh}\Big(\frac{\big|g(\alpha'(0), \beta'(0))\big|}{|\alpha'(0)|\,|\beta'(0)|}\Big)\,. \tag{9}$$

Proof. First, note that by [KS18, Ex. 3.24(i)] (M, g) gives rise to a Lorentzian length space and in particular the time separation function is locally finite. Second, by [KS18, Lem. 2.21(i)] any g-timelike curve is timelike in the relation sense.

Let U be a causally convex normal neighbourhood of x. We now consider $\varphi_\varepsilon : \frac{1}{\varepsilon}\exp_x^{-1}(U) \to U$ given by $\varphi_\varepsilon(v) = \exp_x(\varepsilon v)$. We pullback the curves and the metric via φ_ε, i.e., $\tilde\alpha_\varepsilon(t) := \varphi_\varepsilon^{-1}(\alpha(\varepsilon t))$ and $\tilde\beta_\varepsilon(t) := \varphi_\varepsilon^{-1}(\beta(\varepsilon t))$ for $t \in [0, 1)$ and note that they converge uniformly in $\mathcal{C}^1$ to $\tilde\alpha(t) := t\alpha'(0)$ and $\tilde\beta(t) := t\beta'(0)$ as $\varepsilon \searrow 0$. Similarly, the metric $\tilde g_\varepsilon = \frac{1}{\varepsilon^2}(\varphi_\varepsilon)_* g$ converges locally uniformly in $\mathcal{C}^1$ to g_x, which we can assume to be the Minkowski metric. In particular, we obtain a time separation function $\tilde\tau_\varepsilon$ on $\frac{1}{\varepsilon}\exp_x^{-1}(U)$ from the metric $\tilde g_\varepsilon$. Note that as U is causally convex, we have $\varepsilon\tilde\tau_\varepsilon(p,q) = \tau(\varphi_\varepsilon(p),\varphi_\varepsilon(q))$.

As the (Minkowski-)comparison angle is scale-invariant, we have that $\tilde\measuredangle_x(\alpha(\varepsilon s),\beta(\varepsilon t)) = \tilde\measuredangle_0(\tilde\alpha_\varepsilon(s),\tilde\beta_\varepsilon(t))$, where the latter uses $\tilde\tau_\varepsilon$ to define the comparison angle. Finally, since $\alpha'(0)$ and $\beta'(0)$ are bounded away from the null cone, we have $\frac{\tilde\tau(x,\tilde\alpha(s))}{\tilde\tau_\varepsilon(x,\tilde\alpha_\varepsilon(s))}\to 1$ as well as $\frac{\tilde\tau(x,\tilde\beta(s))}{\tilde\tau_\varepsilon(x,\tilde\beta_\varepsilon(s))}\to 1$, and $\tilde\tau(\tilde\alpha(s),\tilde\beta(t)) - \tilde\tau_\varepsilon(\tilde\alpha_\varepsilon(s),\tilde\beta_\varepsilon(t))\to 0$, where $\tilde\tau$ denotes the Minkowski time separation.

By the law of cosines formula, we see that the comparison angle between $\tilde\alpha_\varepsilon(s)$ and $\tilde\beta_\varepsilon(t)$ converges to the comparison angle between $\tilde\alpha(s)$ and $\tilde\beta(t)$ which is $\operatorname{arcosh}\left(\frac{|g_x(\alpha'(0),\beta'(0))|}{|\alpha'(0)||\beta'(0)|}\right)$.

□

As a consequence we obtain that the angle between timelike curves does not depend on the comparison angle used. This is an analog of [BH99, Prop. II.3.1, p. 184] in the metric case.

Proposition 2.14 (All $\tilde\measuredangle^K$ converge to $\measuredangle$)**.** *Let X be a strongly causal Lorentzian pre-length space with τ locally finite-valued, α,β timelike curves (each can be future or past directed) with $\alpha(0)=\beta(0)=x$. Then for all $K\in\mathbb{R}$, we have that*

$$\limsup_{(s,t)\in A_K;\, s,t\searrow 0} \tilde\measuredangle_x^K(\alpha(s),\beta(t)) = \measuredangle_x(\alpha,\beta)\,, \tag{10}$$

and the limit superior on the left-hand-side is a limit and finite if and only if the angle between α and β exists.

Proof: We restrict to t,s small enough to have the timelike size bounds automatically satisfied (cf. Remark 1.22). Setting $l=\sqrt{|K|}$, $\sigma=\pm1$ the appropriate sign and the side-lengths $a=\max(\tau(x,\alpha(s)),\tau(\alpha(s),x))$, $b=\max(\tau(x,\beta(t)),\tau(\beta(t),x))$, $c=\max(\tau(\alpha(s),\beta(t)),\tau(\beta(t),\alpha(s)))$ we apply the law of cosines (Lemma 2.4): If $K>0$, we obtain

$$\cosh(\tilde\measuredangle_x^K(\alpha(s),\beta(t))) = \sigma\frac{\cosh(la)\cosh(lb)-\cosh(lc)}{\sinh(la)\sinh(lb)} =: (\star)\,.$$

We first look at the denominator of this fraction: Note $\sinh(x)=x+o(x)$ as $x\to0$, so we have

$$\sinh(la)\sinh(lb) = l^2ab + o(la)\cdot lb + la\cdot o(lb) = l^2ab\cdot(1+o(1))\,.$$

Now we have $\frac{1}{\sinh(la)\sinh(lb)} = \frac{1+o(1)}{l^2ab}$ and are done with the denominator.

Now we look at the enumerator of this fraction: We use the Taylor series of cosh, i.e., $\cosh(x) = 1 + \sum_{n\geq 2,\text{ even}} \frac{x^n}{n!}$. For the whole enumerator, we thus have

$$\begin{aligned}
&\cosh(la)\cosh(lb) - \cosh(lc) && (11)\\
&= \sum_{n\geq 2,\text{ even}} \frac{(la)^n}{n!} + \sum_{n\geq 2,\text{ even}} \frac{(lb)^n}{n!} + o(l^2ab) - \sum_{n\geq 2,\text{ even}} \frac{(lc)^n}{n!}\,, && (12)
\end{aligned}$$

where we abbreviated the sum of terms where a^2b^2 appeared by $o(l^2ab)$. We now look at each n separately, together with the denominator: for $n = 2$, we get the desired $\frac{l^2}{l^2} \cdot \frac{a^2+b^2-c^2}{2ab}$. For $n > 2$, we get $\frac{l^n}{n!}$ times

$$\begin{aligned}
&a^n + b^n - c^n\\
&= (a^2+b^2)(a^{n-2} + b^{n-2} - c^{n-2}) + (a^2+b^2-c^2)c^{n-2} - a^2b^{n-2} - a^{n-2}b^2\\
&= o(1)(a^{n-2} + b^{n-2} - c^{n-2}) - (a^2+b^2-c^2)o(1) + o(ab)\,,
\end{aligned}$$

now we use induction on n to get $a^n + b^n - c^n = o(a^2 + b^2 - c^2)$.

In total, we have

$$\begin{aligned}
(\star) &= \sigma \frac{(a^2+b^2-c^2)\cdot\left(1 + \sum_{n\geq 4,\text{ even}} \frac{l^{n-2}}{n!} o(a^2+b^2-c^2)\right) + o(l^2ab)}{2ab} && (13)\\
&\quad\times\ (1+o(1))\,. && (14)
\end{aligned}$$

Note that the sum in the enumerator is

$$(a^2+b^2-c^2)\,o(1) \sum_{n\geq 4,\text{ even}} \frac{l^{n-2}}{n!} \sum_{k=0}^{n-2} c^k\,, \tag{15}$$

which is $o(a^2 + b^2 - c^2)$. So we have

$$(\star) = \sigma \frac{(a^2+b^2-c^2)\cdot(1+o(1)) + o(ab)}{2ab} \cdot (1+o(1))\,.$$

For $a, b, c \to 0$, this converges if and only if

$$\cosh(\tilde{\measuredangle}_x^0(\alpha(s), \beta(t))) = \sigma \frac{a^2+b^2-c^2}{2ab}\,,$$

converges, and they converge to the same value, and similarly for $K < 0$. $\square$

3 Angles between timelike curves of the same time orientation

In this section we study the angle between curves and, in particular, geodesics that have the same time orientation, i.e., are all future or past directed. Some proofs of the results here are given later in Section 4, where these results are proven in full generality, i.e., where the curves can have a different time orientation. However, no logical issues arise from this.

We start by establishing that (upper) angles satisfy the (usual) triangle inequality if all the curves have the same time orientation.

Theorem 3.1 (Triangle inequality for (upper) angles)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and locally causally closed Lorentzian pre-length space with τ locally finite-valued and locally continuous. Let $\alpha, \beta, \gamma \colon [0, B) \to X$ be timelike curves with coinciding time orientation starting at $x := \alpha(0) = \beta(0) = \gamma(0)$. Then*

$$\measuredangle_x(\alpha, \gamma) \leq \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma)\,. \tag{16}$$

Proof: This proof is a direct adaptation of the proof of [BH99, Prop. I.1.14] or of [BBI01, Thm. 3.6.34] in the metric setting, with several adaptions needed to keep track of the causality.

We establish the case where α, β, γ are all future directed. The prospective inequality is only non-trivial if $\measuredangle_x(\alpha, \beta) < \infty$ and $\measuredangle_x(\beta, \gamma) < \infty$ — so let that be the case. Moreover, assume to the contrary that the inequality does not hold, so that there is an $\varepsilon > 0$ such that

$$\measuredangle_x(\alpha, \gamma) > \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma) + 4\varepsilon\,. \tag{17}$$

By definition of the upper angles, there is a $B' \in (0, B)$ small enough such that the initial segments (defined on $[0, B']$) of α, β, γ are contained in a causally closed neighborhood, where τ is finite and continuous, and for all $r, t, s \in (0, B')$ we have the upper bound

$$\tilde{\measuredangle}_x(\alpha(r), \beta(s)) < \measuredangle_x(\alpha, \beta) + \varepsilon\,, \tag{18}$$

$$\tilde{\measuredangle}_x(\beta(s), \gamma(t)) < \measuredangle_x(\beta, \gamma) + \varepsilon\,. \tag{19}$$

whenever the comparison angles exist, i.e., whenever $\alpha(r) \leq \beta(s)$ and $\beta(s) \leq \gamma(t)$ or the other way round.

Furthermore, there are sequences $\tilde{r}_n \searrow 0$ and $t_n \searrow 0$ with

$$\tilde{\measuredangle}_x(\alpha(\tilde{r}_n), \gamma(t_n)) > \measuredangle_x(\alpha, \gamma) - \frac{\varepsilon}{2}\,. \tag{20}$$

if $\measuredangle_x(\alpha, \gamma) < \infty$, otherwise we can have that $\tilde{\measuredangle}_x(\alpha(\tilde{r}_n), \gamma(t_n)) > C$ for any $C > 0$.

In particular $\alpha(\tilde{r}_n), \gamma(t_n)$ are causally related, without loss of generality we assume $\alpha(r_n) \leq \gamma(t_n)$ for all n. For $n \in \mathbb{N}$ let $0 < r_n < \tilde{r}_n$ be so close to $\tilde{r}_n$ that by the (local) continuity of τ and the continuity in the law of cosines we get

$$\tilde{\measuredangle}_x(\alpha(r_n), \gamma(t_n)) > \measuredangle_x(\alpha, \gamma) - \varepsilon\,, \tag{21}$$

and $\alpha(r_n) \ll \gamma(t_n)$ for all $n \in \mathbb{N}$.

We define

$$\begin{aligned} s_-^n &:= \inf\{\beta^{-1}(I^+(\alpha(r_n)))\}\,, \\ s_+^n &:= \sup\{\beta^{-1}(I^-(\gamma(t_n)))\}\,. \end{aligned}$$

For large enough n, we have $s_-^n < B'$ and $s_+^n < B'$: $I^-(\beta(\frac{B'}{2}))$ contains an initial segment of α and γ, thus $\alpha(r_n), \gamma(r_n) \in I^-(\beta(\frac{B'}{2}))$ for large enough n. Then we have $\beta(\frac{B'}{2}) \in I^+(\alpha(r_n)) \cap I^+(\gamma(t_n))$, thus $s_-^n \leq \frac{B'}{2}$. As X is chronological, we have $\beta(s) \not\ll \gamma(t_n)$ for $s \geq \frac{B'}{2}$, thus $s_+^n \leq \frac{B'}{2}$. We pick a particular such n and will drop the subscript n from now on.

We distinguish the cases $s_- < s_+$ and $s_- \geq s_+$. Consider first the case $s_- < s_+$: It is easy to see that $\beta^{-1}(I(\alpha(r), \gamma(t)))$ is the open interval (s_-, s_+). For $s \in (s_-, s_+)$, we get a *timelike tetrad* $x \ll \alpha(r) \ll \beta(s) \ll \gamma(t)$.

Set $a := \alpha(r), b(s) := \beta(s), c := \gamma(t)$ and note that by construction, our indirect assumption reads

$$\tilde{\measuredangle}_x(a, c) > \tilde{\measuredangle}_x(a, b(s)) + \tilde{\measuredangle}_x(b(s), c)\,, \tag{22}$$

for all $s \in (s_-, s_+)$.

We now create a comparison situation where all side lengths and two of the angles are realized: Choose comparison triangles for $\Delta xab(s)$ and $\Delta xb(s)c$ in the Minkowski plane $\mathbb{R}^2_1$, which share the side $[\bar{x}\bar{b}_s]$ and such that the points $\bar{a}_s$ and $\bar{c}_s$ are on different sides of the line $[\bar{x}\bar{b}_s]$, i.e., we pick points $\bar{x}, \bar{a}_s, \bar{b}_s, \bar{c}_s$, such that $\tau(x, a) = \bar{\tau}(\bar{x}, \bar{a}_s)$, $\tau(x, b(s)) = \bar{\tau}(\bar{x}, \bar{b}_s)$, $\tau(x, c) = \bar{\tau}(\bar{x}, \bar{c}_s)$, $\tau(a, b(s)) = \bar{\tau}(\bar{a}_s, \bar{b}_s)$, $\tau(b(s), c) = \bar{\tau}(\bar{b}_s, \bar{c}_s)$, see Figure 4. This realizes the angles $\tilde{\measuredangle}_x(a, b(s))$ and $\tilde{\measuredangle}_x(b(s), c)$. Note that $\bar{a}_s, \bar{b}_s, \bar{c}_s$ may depend on s in general. This choice of points can be made continuously in s, even when one of the sides becomes null (i.e., when extending s to $s = s_-$ or $s = s_+$).

We claim that there is an $s \in (s_-, s_+)$ such that $\bar{a}_s, \bar{b}_s, \bar{c}_s$ lie on a line (which then is timelike). For $s \in (s_-, s_+)$ let L_s be the line connecting $\bar{a}_s$ with $\bar{c}_s$. Setting $s = s_-$, we get that a and $b(s_-)$ are null related, but

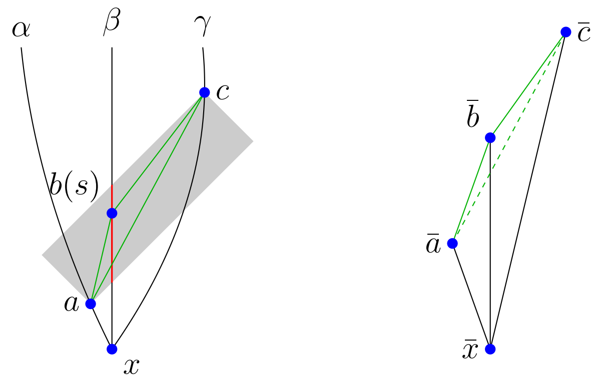


Figure 4: On the left, the configuration in X is shown. The grey area is $I(a, c)$, the red line is $\beta([s_-, s_+])$. In the comparison picture on the right, two points are connected by a line if this line has the same length as the corresponding line in X. The dashed line need not have the same length.

we can still form the comparison situation (denoted as above). In this new comparison configuration, it is obvious that $\bar{b}(s_-) =: \bar{b}_-$ lies below $L_{s_-} =: L_-$ as both $\bar{b}_-$ and $\bar{c}_s$ lie on the same side of $[\bar{x}\bar{a}_s]$, and $\bar{b}_-$ is in the causal future of $\bar{a}_s$ and $\bar{c}_s$ is in the timelike future of $\bar{a}_s$ as the family of lines $(L_s)_s$ are causal and hence cannot tilt too much, see Figure 5.

Similarly, setting $s = s_+$, we get that $b(s_+)$ and c are null related. Here, it is obvious that $\bar{b}_{s_+}$ lies above L_{s_+}.

Now, by continuity of all the points, there is a value $s \in (s_-, s_+)$ such that $\bar{b}_s$ lies on L_s.

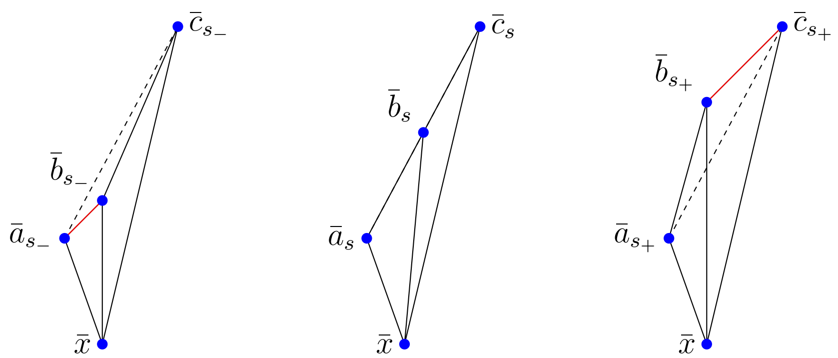


Figure 5: As the red lines are null and the dashed lines are timelike, we deduce that $\bar{b}_{s_-}$ is below L_{s_-} and $\bar{b}_{s_+}$ is above L_{s_+}.

Now we use this s and drop the s-subscript, i.e., $\bar{b} := \bar{b}_s$. Then we have $\bar{\tau}(\bar{a},\bar{c}) = \bar{\tau}(\bar{a},\bar{b}) + \bar{\tau}(\bar{b},\bar{c})$ as they lie on a straight line (a timelike Minkowski-geodesic). We choose $\tilde{c}$ such that $\Delta\bar{x}\bar{a}\tilde{c}$ is a comparison triangle of Δxac, with $\bar{c}$ and $\tilde{c}$ on the same side of $[x\bar{a}]$. In this situation, the angles under consideration are given by

- $\tilde{\measuredangle}_x(a,b) = \measuredangle_{\bar{x}}(\bar{a},\bar{b})$,
- $\tilde{\measuredangle}_x(b,c) = \measuredangle_{\bar{x}}(\bar{b},\bar{c})$,
- $\tilde{\measuredangle}_x(a,c) = \measuredangle_{\bar{x}}(\bar{a},\tilde{c})$.

By the reverse triangle inequality, we know that $\tau(a,c) \geq \tau(a,b) + \tau(b,c)$, so $\bar{\tau}(\bar{a},\tilde{c}) \geq \tau(\bar{a},\bar{b}) + \tau(\bar{b},\bar{c}) = \tau(\bar{a},\bar{c})$. Thus the triangles $\Delta\bar{x}\bar{a}\bar{c}$ and $\Delta\bar{x}\bar{a}\tilde{c}$ have all side lengths equal except possibly $[\bar{a}\bar{c}]$ and $[\bar{a}\tilde{c}]$, where we know $[\bar{a}\tilde{c}]$ is not the shorter one. Consequently, by the monotonicity statement in the law of cosines (Remark 2.5 with $\sigma = -1$, $K = 0$), we conclude that $\measuredangle_{\bar{x}}(\bar{a},\tilde{c}) \leq \measuredangle_{\bar{x}}(\bar{a},\bar{c}) = \measuredangle_{\bar{x}}(\bar{a},\bar{b}) + \measuredangle_{\bar{x}}(\bar{b},\bar{c})$, where the last equality is due to the additivity of hyperbolic angles in the plane. This is a contradiction to the assumption, i.e., Equation (22).

At this point we consider the case $s_- \geq s_+$. Now we may not get a single point on β, but we may get two of them: Set $b_\pm = \beta(s_\pm)$. Then we have $x \ll \alpha(r) \leq b_-$ and $x \ll b_+ \leq \gamma(t)$. We can form the following comparison situation:

For the triangle $\Delta x\alpha(r)\gamma(t)$ we get a comparison triangle $\Delta\bar{x}\bar{a}\bar{c}$. In the same diagram, we add a comparison triangle for $\Delta x\alpha(r)b_-$ and call the additional point $\bar{b}_-$, and add a comparison triangle for $\Delta xb_+\gamma(t)$ and call the additional point $\bar{b}_+$. We set it up so that $\bar{a}$ is to the left of $\bar{c}$, $\bar{b}_-$ is to the right of $\bar{a}$ and $\bar{b}_+$ is to the left of $\bar{c}$, see Figure 6. We again define L as the (future directed timelike) straight line from $\bar{a}$ to $\bar{c}$. For the angles we get that

$$\begin{aligned}
\omega_1 &:= \tilde{\measuredangle}_{\bar{x}}(\bar{a},\bar{b}_-) < \measuredangle_x(\alpha,\beta) + \varepsilon\,, \\
\omega_2 &:= \tilde{\measuredangle}_{\bar{x}}(\bar{b}_+,\bar{c}) < \measuredangle_x(\beta,\gamma) + \varepsilon\,, \\
\omega_3 &:= \tilde{\measuredangle}_{\bar{x}}(\bar{a},\bar{c}) > \measuredangle_x(\alpha,\gamma) - \varepsilon\,.
\end{aligned}$$

By assumption, we obtain $\omega_3 > \omega_1 + \omega_2 + \varepsilon$, thus the line $[\bar{x}\bar{b}_-]$ is to the left of $[\bar{x}\bar{b}_+]$ (this follows from angle additivity in the plane). As $[\bar{a}\bar{b}_-]$ is future directed null and $L = [\bar{a}\bar{c}]$ is future directed timelike and both go to the right, $\bar{b}_-$ lies below L. As $[\bar{b}_+\bar{c}]$ is null and $L = [\bar{a}\bar{c}]$ is timelike and both go to the right, $\bar{b}_+$ lies above L.

We now construct a future directed timelike curve from $\bar{b}_-$ to $\bar{b}_+$ in the comparison space (that means $b_+ \leq b_-$, but $\bar{b}_+ \gg \bar{b}_-$):

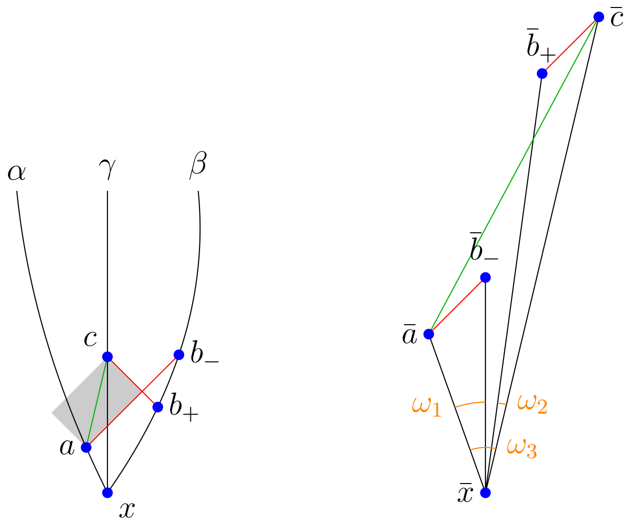


Figure 6: As the red lines are null and the green line is timelike, b_- is below L and b_+ is above L.

We begin at $\bar{b}_-$. As $\bar{b}_-$ lies below L, we use the extension of the line $[\bar{x}\bar{b}_-]$ to reach the line L, then go along a part of L. As $\bar{b}_+$ lies above L, we use the end of the line $[\bar{x}\bar{b}_+]$ to reach $\bar{b}_+$.

In particular, $\bar{\tau}(\bar{b}_-, \bar{b}_+) > 0$, and by the reverse triangle inequality $\tau(x, b_-) = \bar{\tau}(\bar{x}, \bar{b}_-) < \bar{\tau}(\bar{x}, \bar{b}_+) = \tau(x, b_+)$. However, in X we have $\tau(x, b_-) \geq \tau(x, b_+)$, a contradiction.

Finally, note that in case $\measuredangle_x(\alpha, \gamma) = \infty$, one obtains analogous contradictions in the cases $s_- < s_+$ and $s_- \geq s_+$, as the comparison angle $\tilde{\measuredangle}_x(a, c)$ is finite but arbitrarily large. $\square$

Proposition 3.2 (Angle bound implies quantitative timelike relation)**.** *Let X be chronological, τ be locally continuous, and let $\alpha, \beta : [0, \eta) \to X$ be two future directed timelike geodesics parametrized by τ-arclength with $x := \alpha(0) = \beta(0)$ such that $\measuredangle_x(\alpha, \beta) =: \omega < \infty$.*

Then for all $0 < c < e^{-\omega}$ there is a $T > 0$ small enough such that for all $0 < t < T$ we have $\alpha(ct) \ll \beta(t)$.

Proof. By the definition of upper angles, we get that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $s < \delta$, $t < \delta$ such that $\alpha(s) \ll \beta(t)$, $\tilde{\measuredangle}_x(\alpha(s), \beta(t)) < \omega + \varepsilon$.

Let $T_1 = \min(\inf(\beta^{-1}(I^+(\alpha(\delta)))), \delta)$. Then for $t < T_1$ and s such that $\alpha(s) \ll \beta(t)$, we automatically have $s, t < \delta$ (and $T_1 > 0$ as X is chronological and τ continuous).

For each t, we consider the set of parameters s such that $\alpha(s) \ll \beta(t)$, i.e., $\alpha^{-1}(I^-(\beta(t))) = [0, s_t) \subseteq [0, \eta)$. By continuity of τ, we have $\tau(\alpha(s), \beta(t)) \to 0$ as $s \nearrow s_t$.

Now by the definition of comparison angles we obtain

$$\cosh(\omega + \varepsilon) > \cosh(\tilde{\measuredangle}_x(\alpha(s), \beta(t))) \tag{23}$$

$$= \frac{s^2 + t^2 - \tau(\alpha(s), \beta(t))^2}{2\, s\, t}\,. \tag{24}$$

Again by continuity of τ this also holds in the limit $s \nearrow s_t$, i.e.,

$$\cosh(\omega + \varepsilon) \geq \frac{s_t^2 + t^2}{2\, s_t\, t}\,. \tag{25}$$

For simplicity, we set $o := \cosh(\omega + \varepsilon)$ and get the quadratic inequality

$$0 \geq s_t^2 - 2 o s_t t + t^2\,,$$

yielding $t(o - \sqrt{o^2 - 1}) \leq s_t \leq t(o + \sqrt{o^2 - 1})$. Note $o - \sqrt{o^2 - 1} = \cosh(\omega + \varepsilon) - \sqrt{\cosh(\omega + \varepsilon)^2 - 1} = \cosh(\omega + \varepsilon) - \sinh(\omega + \varepsilon) = \exp(-\omega - \varepsilon)$. In particular, we have that

$$s_t \geq t \exp(-\omega - \varepsilon)\,. \tag{26}$$

That is, for $s < t \exp(-\omega - \varepsilon)$ we have $s < s_t$ and thus $\alpha(s) \ll \beta(t)$ by construction.

Finally, for all $c < e^{-\omega}$ we set $\varepsilon := -\log(c e^{\omega}) > 0$ and obtain $\alpha(s) \ll \beta(t)$ for all $0 < t < T_1 = T_1(\varepsilon)$, yielding the claim. $\square$

At this point we introduce the notion of *direction* of a timelike geodesic.

Definition 3.3 (Direction)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space with τ locally finite-valued. For $x \in X$ we define $D_x^+ := \{\alpha\colon [0, \varepsilon) \to X$ future directed timelike geodesic with $\alpha(0) = x$ for some $\varepsilon > 0\}$. We define a relation $\sim$ on D_x^+ by saying $\alpha \sim \beta$ if $\measuredangle_x(\alpha, \beta) = 0$. If $\alpha \sim \beta$, we say that α and β* have the same direction at x. *Similarly, we define $D_x^- := \{\alpha\colon [0, \varepsilon) \to X$ past directed timelike geodesic with $\alpha(0) = x$ for some $\varepsilon > 0\}$, and define a relation on D_x^- by $\alpha \sim \beta$ if $\measuredangle_x(\alpha, \beta) = 0$.*

Lemma 3.4 (Properties of angles and direction)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and locally causally closed Lorentzian pre-length space with τ locally finite-valued.*

1. *Let $\alpha\colon [a, b) \to X$ be a future directed timelike geodesic, let $t_0 \in [a, b)$ and set $x := \alpha(t_0)$, then $\measuredangle_x(\alpha, \alpha) = 0$.*

2. *Let $\alpha\colon (a,b) \to X$ be a future directed timelike geodesic, let $t_0 \in (a,b)$ and set $x := \alpha(t_0)$. Then $\measuredangle_x(\alpha, \alpha^-) = 0$, where $\alpha^-(t) = \alpha(-t)$ ($t \in (-b,-a)$) is the reversed curve, which is a past directed timelike geodesic.*

3. *Let $x \in X$, the relation $\sim$ of having the same direction at x is an equivalence relation on $D_x^\pm$.*

Proof:

(i),(ii) Let $t_0 < s < t < b$ or $a < s < t_0 < t < b$ be close enough to t_0 such that α maximizes on $[\min(t_0,s),t]$. Then the triangle $x \ll \alpha(s) \ll \alpha(t)$ or $\alpha(s) \ll x \ll \alpha(t)$ is degenerate and the corresponding comparison triangle $\Delta\bar{x}\bar{y}\bar{z}$ in $\mathbb{R}^2_1 = \mathbb{L}^2(0)$ is a straight timelike segment, and so $\tilde{\measuredangle}_x(\alpha(s),\alpha(t)) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{y},\bar{z}) = 0$. Thus, the angle between α and itself exists and is zero.

3. Symmetry of $\sim$ is clear from the definition and reflexivity follows from point (i) above. To show transitivity let $\alpha,\beta,\gamma \in D_x^+$ with $\alpha \sim \beta$ and $\beta \sim \gamma$. Then by the triangle inequality for upper angles (Theorem 3.1) we get that

$$0 \leq \measuredangle_x(\alpha,\gamma) \leq \measuredangle_x(\alpha,\beta) + \measuredangle_x(\beta,\gamma) = 0\,, \tag{27}$$

which shows that the angle between α and γ exists and equals zero. So we have $\alpha \sim \gamma$, as required.

□

Let us emphasize point 2 above: Contrary to the metric case the hyperbolic angle between the incoming and outgoing segments of an interior point of a timelike geodesic is zero and not π, cf. Figure 2.

The notion of direction allows us to define the space of (future or past directed) timelike directions at a point.

Lemma 3.5. *Let $(X,d,\ll,\leq,\tau)$ be a strongly causal, locally causally closed Lorentzian pre-length space with τ locally finite-valued and so that angles between (future directed) timelike geodesics always exist, and $x \in X$. Then $\mathcal{D}_x^+ := D_x^+/\sim$ and $\mathcal{D}_x^- := D_x^-/\sim$ are metric spaces with metric $\measuredangle_x$.*

Definition 3.6 (Space of timelike directions)**.** *Let $(X,d,\ll,\leq,\tau)$ be a strongly causal, locally causally closed Lorentzian pre-length space with τ locally finite-valued and so that angles between (future directed) timelike geodesics* *always exist. We call the metric space* $(\mathcal{D}_x^+, \measuredangle_x)$ *the* space of future directed timelike directions *at* x*, and* $(\mathcal{D}_x^-, \measuredangle_x)$ *the* space of past directed timelike directions *at* x*. Furthermore, we denote the metric completion of* $(\mathcal{D}_x^\pm, \measuredangle_x)$ *by* $(\Sigma_x^\pm, \measuredangle_x)$*.*

This definition makes sense as the angle $\measuredangle_x$ is symmetric and satisfies the triangle inequality by Theorem 3.1. Moreover, the angle does not depend on the choice of representative and then by construction $\measuredangle_x$ is positive definite.

For technical reasons it is useful to also introduce the completion $\Sigma_x^\pm$ of the space of timelike directions $\mathcal{D}_x^\pm$. However, we will later show in Proposition 3.15 that for a large class of Lorentzian pre-length spaces with a lower curvature bound the space of timelike directions $\mathcal{D}_x^\pm$ is already complete.

At this point we use angles to introduce another notion of curvature bound by demanding that the comparison angles behave monotonically along the sides of a timelike geodesic triangle. The analogous concept in metric geometry is called *monotonicity condition* in [BBI01, Def. 4.3.1].

Definition 3.7 (Future K-monotonicity comparison)**.** *Let* $(X, d, \ll, \leq, \tau)$ *be a Lorentzian pre-length space and let* $K \in \mathbb{R}$*. We say that* X *satisfies* future timelike K-monotonicity comparison from below (above) *if every point in* X *possesses a neighborhood* U *such that*

1. $\tau|_{U\times U}$ *is finite and continuous.*
2. U *is strictly timelike geodesically connected.*
3. *Let* $x \ll y \ll z$ *be timelike related and forming a timelike geodesic triangle in* U*, whose side lengths satisfy timelike size bounds for* K*. Let* $z' \in [x, z]$ *and* $y' \in [x, y]$ *with* $y' \neq x \neq z'$ *and* z' *causally related to* y'*, then we have*

$$\tilde{\measuredangle}_x^K(y', z') \geq \tilde{\measuredangle}_x^K(y, z) \qquad (\tilde{\measuredangle}_x^K(y', z') \leq \tilde{\measuredangle}_x^K(y, z))\,. \tag{28}$$

We can define *past K-monotonicity comparison* analogously by choosing $x' \in [x, z]$ and $y' \in [y, z]$ using the comparison angles at z instead. Note that y' is on a different side here as compared to the future case. Moreover, let us remark that x', y' or z' are named as such because they *seem to be in the same direction* as x, y or z from the point we are measuring the angle at. For an illustration of K-monotonicity comparison see Figure 7.

A direct consequence of K-monotonicity and Proposition 2.14 is the following corollary.

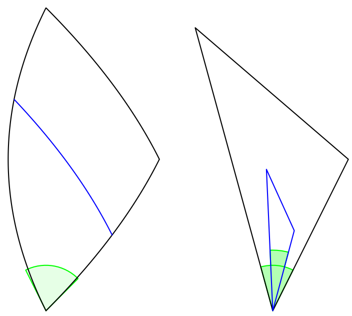


Figure 7: When comparing with $\mathbb{L}^2(K)$ the interior side is shorter than expected — the K-angle has to behave accordingly.

Corollary 3.8 (Monotonicity implies bound on angle of geodesics)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space that satisfies future/past K-monotonicity comparison from below (above) for some $K \in \mathbb{R}$. Then for any $x \in X$ and $\alpha, \beta \colon [0, B] \to X$ future/past directed timelike geodesics with $\alpha(0) = \beta(0) = x$ one has that*

$$\measuredangle_x(\alpha, \beta) \geq \tilde{\measuredangle}_x^K(\alpha(s), \beta(t)) \qquad \left(\measuredangle_x(\alpha, \beta) \leq \tilde{\measuredangle}_x^K(\alpha(s), \beta(t))\right), \tag{29}$$

for all $s, t \in [0, B]$ small enough such that timelike size bounds for K are satisfied and chosen such that $\alpha(s)$ and $\beta(t)$ are causally related.

Moreover, K-monotonicity yields that the limit superior in the definition of the angle between geodesics is actually a (not necessarily finite) limit, see e.g. [BH99, Prop. II.3.1] or [BBI01, Prop. 4.3.2] for an analogous result in the metric case.

Lemma 3.9 (Monotonicity implies existence of angles)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. If it satisfies timelike K-monotonicity comparison from above or below, the angle between any two future directed timelike geodesics starting at the same point x exists if it is finite (the limit superior in the definition of upper angles is a limit, but we do not know if it is finite).*

If it satisfies timelike K-monotonicity comparison from below, we know the limit is finite (in particular, the angle exists).

Proof. This will be proven later in more generality in Lemma 4.10. ☐

The connection of future K-monotonicity comparison to timelike curvature bounds is as follows.

Theorem 3.10 (Triangle comparison implies future K-monotonicity comparison)**.** *Let $(X,d,\ll,\leq,\tau)$ be a locally strictly timelike geodesically connected Lorentzian pre-length space and let $K\in\mathbb{R}$. If X has timelike curvature bounded below (above) by K, it also satisfies future K-monotonicity comparison from below (above).*

Proof. This will be proven later in more generality in Theorem 4.13. □

Analogously to the metric case (cf. [BBI01, Thm. 4.3.11] and [BH99, Prop. II.3.3]), angles are semi-continuous.

Proposition 3.11 (Semi-continuity of angles)**.** *Let $(X,d,\ll,\leq,\tau)$ be a Lorentzian pre-length space with timelike curvature bounded below. Then angles of timelike geodesics of the same time orientation are lower semicontinuous. To be precise, for $x\in X$ and $\alpha,\alpha_n,\beta,\beta_n\colon[0,b]\to X$ all future or all past directed timelike geodesics starting at x with $\alpha_n\to\alpha$, $\beta_n\to\beta$ pointwise, then*

$$\measuredangle_x(\alpha,\beta)\leq\liminf_n\measuredangle_x(\alpha_n,\beta_n)\,. \tag{30}$$

Proof: This will be proven later in more generality in Proposition 4.14. □

In case of timelike curvature bounded above, the angle between geodesics is in fact continuous.

Proposition 3.12 (Angle between geodesics is continuous)**.** *Let $(X,d,\ll,\leq,\tau)$ be a Lorentzian pre-length space with timelike curvature bounded above. Then angles are continuous for geodesics, i.e., for $x\in X$ and $\alpha,\alpha_n,\beta,\beta_n:[0,b]\to X$ all future or all past directed timelike geodesics starting at x with $\alpha_n\to\alpha$, $\beta_n\to\beta$ pointwise, then*

$$\measuredangle_x(\alpha,\beta)=\lim_n\measuredangle_x(\alpha_n,\beta_n)\,. \tag{31}$$

Proof: This will be proven later in more generality in Proposition 4.15. □

If the curves considered are not geodesics, then there is no semi-continuity of angles, as is well-known in the smooth spacetime case. We include an example below for the convenience of the reader.

Example 3.13. *Let $X=\mathbb{R}^2_1$ be two-dimensional Minkowski spacetime and consider the future directed timelike curves $\gamma_n(t)=(t,\frac{1}{2}\frac{t}{1+nt})$ $(t\in[0,\infty))$. They all start at $\gamma_n(0)=(0,0)=:x$ and converge uniformly to the curve $\gamma(t)=(t,0)$. Nevertheless, they do not converge in a $\mathcal{C}^1$-way. We calculate* *the angles in the spacetime sense (note that by Proposition 2.13 these angles agree with the angles in the synthetic sense), so* $\gamma_n'(0) = (1, \frac{1}{2})$*, making* $\measuredangle_x(\gamma_n, \gamma) = \operatorname{arcosh}(\frac{2}{\sqrt{3}}) > 0 = \measuredangle_x(\gamma, \gamma)$*, contradicting upper semicontinuity.*

For $\beta(t) = (t, \frac{2}{3}t)$ *we have* $\measuredangle_x(\gamma_n, \beta) = \operatorname{arcosh}(\frac{4}{\sqrt{15}}) < \operatorname{arcosh}(\frac{3}{\sqrt{5}}) = \measuredangle_x(\gamma, \beta)$*, contradicting lower semicontinuity, but the* γ_n*s are not geodesics (although the limits* β *and* γ *are).*

As there is no automatic local existence of timelike geodesics and, moreover, the time of existence need not be locally uniform, we have to exclude these situations when considering the question of whether the space of timelike directions is complete. For convenience we give a name to the following condition which will be used several times.

Definition 3.14 (Non-lingering)**.** *Let* $(X, d, \ll, \leq, \tau)$ *be a Lorentzian prelength space. A point* $x \in X$ *has the* uniform future/past non-lingering property *if there is an* $s > 0$ *and an* $\varepsilon > 0$ *such that all future/past directed timelike geodesics* γ *starting at* $x = \gamma(0)$*, when parametrized by d-arclength and extended maximally, are defined at least up to parameter* s*, and* $d(x, \gamma(s)) > \varepsilon$*. Moreover, we say that* $(X, d, \ll, \leq, \tau)$ *is* future/past non-lingering *if every point* $x \in X$ *has the uniform future/past non-lingering property.*

By using cylindrical neighborhoods and the fact that the time coordinate is a temporal function on such a neighborhood (cf. [CG12, Prop. 1.10]), it directly follows that any continuous spacetime is future and past non-lingering.

As pointed out before, given a lower curvature bound, we can give sufficient conditions so that the space of timelike directions is complete. We are now ready to provide a proof of this fact.

Proposition 3.15 (Space of timelike directions complete)**.** *Let* $(X, d, \ll, \leq, \tau)$ *be a locally compact, strongly causal, locally causally closed, d-compatible and regularly localizable Lorentzian length space with timelike curvature bounded below. If* $x \in X$ *has the uniform future non-lingering property, then* $\mathcal{D}_x^+$ *is a complete metric space, i.e.,* $\Sigma_x^+ = \mathcal{D}_x^+$*.*

Proof. Note that, by strong causality and local compactness of (X, d), small causal diamonds are compact.

Let $\gamma_n : [0, L^d(\gamma_n)] \to X$ be future directed timelike geodesics, d-unit-speed-parametrized, with $\gamma_n(0) = x$ and $[\gamma_n] \in \mathcal{D}_x^+$ forming a Cauchy sequence. We can assume by the non-lingering property that all γ_n are initially but not completely contained in a compact set $J(x, y)$ (for some $y \gg x$), which is at the same time a d-compatible neighborhood. Denoting the initial segments by $\gamma_n|_{[0,b_n]}$, we can assume that $\gamma_n([0,b_n]) \subseteq J(x,y)$, $\gamma_n(b_n) \in \partial J(x,y) \cap \partial J^-(y)$ for all $n \in \mathbb{N}$. By being in a d-compatible neighborhood, we get that the b_ns stay bounded.

By the limit curve theorem [KS18, Thm. 3.7], we find a subsequence of $(\gamma_n)_n$ which converges uniformly to a future directed causal curve $\gamma\colon [0,b] \to X$ which also has $\gamma(b) \in \partial J(x,y) \cap \partial J^-(y)$. Without loss of generality we denote this subsequence again by $(\gamma_n)_n$ and note that γ is non-constant. By [KS18, Prop. 3.17], the limit curve γ is maximizing. We claim it is timelike and $[\gamma_n] \to [\gamma]$.

Now let $\varepsilon > 0$, and n be large enough such that $\measuredangle_x(\gamma_n, \gamma_m) < \varepsilon$ for all $m \geq n$. Let s,t be such that $\gamma(s) \ll \gamma_n(t)$ (these exist: for each t there is an $s > 0$ such that this holds as γ is future directed causal). As $\gamma_k(s) \to \gamma(s)$, we get that for large enough k, also $\gamma_k(s) \ll \gamma_n(t)$ and we can calculate the comparison angle using the law of cosines (Lemma 2.4 with $\sigma = -1$) as follows (we only do the $K = 0$ case, the others are similar, just a bit more involved).

$$\cosh(\tilde{\measuredangle}_x(\gamma_k(s), \gamma_n(t))) = \tag{32}$$

$$\frac{\tau(x,\gamma_k(s))^2 + \tau(x,\gamma_n(t))^2 - \tau(\gamma_k(s),\gamma_n(t))^2}{2\tau(x,\gamma_k(s))\tau(x,\gamma_n(t))} . \tag{33}$$

Lemma 3.9 implies that X satisfies future K-monotonicity (where K is the timelike curvature bound from below). Moreover, by Corollary 3.8 we conclude that $\varepsilon > \measuredangle_x(\gamma_k, \gamma_n) \geq \tilde{\measuredangle}_x^K(\gamma_k(s), \gamma_n(t))$. When taking the limit as $k \to \infty$, we have to be careful. Denoting the enumerator in the left-hand-side of Equation (32) by x_k and the denominator by y_k, we have $1 \leq \frac{x_k}{y_k} < \cosh(\varepsilon) =: 1 + \tilde{\varepsilon}$. We have that $y_k \to 0$ precisely when $x_k \to 0$. If they do not converge to 0, $\tau(x, \gamma(s)) > 0$ and thus γ is timelike by regular localizability. Thus we have a valid timelike triangle. Taking $k \to \infty$, and then $s,t \to 0$, we get that the upper angle satisfies $\measuredangle_x(\gamma_n, \gamma) \leq \varepsilon$. Letting $\varepsilon \to 0$ yields a $\measuredangle_x$-convergent subsequence of $([\gamma_n])_n$ and hence we are done in this case.

Now we exclude the other cases: If $y_k \to 0$, we have $\tau(x, \gamma_k(s)) \to 0$ and together with $x_k \to 0$ this implies $\tau(\gamma_k(s), \gamma_n(t)) \to \tau(x, \gamma_n(t))$. Note that this also works for $K \neq 0$ (e.g. we have a $\cosh(\tau(x,\gamma_k(s)))$ factor on one of the terms, which converges to 1). Denoting the points $u := \gamma(s)$ and $z := \gamma_n(t)$, we have $x \leq u$ but $x \not\ll u$ and $\tau(u,z) = \tau(x,z) > 0$, as γ_n is timelike. If $x \neq u$, we concatenate the geodesics from x to u and from u to z having length $0 + \tau(u,z) = \tau(x,z)$, i.e., a distance realizer containing a null segment (the segment from x to u), in contradiction to regular localizability.

Thus $x = u$ and hence $s = 0$ — a contradiction to $s > 0$. $\square$

3.1 Timelike tangent cones

In metric geometry the *tangent cone* is a generalization of tangent spaces of smooth manifolds (see e.g. [BBI01, p. 321]) and a valuable tool, especially in Alexandrov spaces with curvature bounded below. Using the space of timelike directions at a point we introduce in an analogous way the *timelike tangent cone* at a point in an Lorentzian pre-length space. To this end we use the concept of the *Minkowski cone* over a metric space, introduced in [AGKS21, Sec. 2].

Definition 3.16 (Minkowski cone). *Let (Y, d) be a metric space. Then the* Minkowski cone $\mathrm{Cone}(Y)$ *of Y is a Lorentzian pre-length space, where the underlying metric space is $([0,\infty) \times Y)/(\{0\} \times Y)$, equipped with the cone metric d_c (cf. [BBI01, Def. 3.6.16])*

$$d_c((s,p),(t,q)) = \begin{cases} \sqrt{s^2 + t^2 - 2st\cos d(p,q)} & \text{if } d(p,q) \leq \pi\,, \\ s + t & \text{if } d(p,q) \geq \pi\,. \end{cases}$$

Moreover, the causal relation $\leq$ and the time separation function τ are defined via

$$(s,p) \leq (t,q) \Leftrightarrow s^2 + t^2 - 2st\cosh d(p,q) \geq 0 \text{ and } s \leq t\,,$$

$$\tau((s,p),(t,q)) = \sqrt{s^2 + t^2 - 2st\cosh d(p,q)} \text{ if } (s,p) \leq (t,q)\,,$$

and $\ll$ induced by τ (i.e., $(s,p) \ll (t,q) \Leftrightarrow \tau((s,p),(t,q)) > 0$). See Section 2 in [AGKS21] for more details, and how Minkowski cones (without the vertex) can be viewed as instances of generalized cones *([AGKS21, Ex. 3.31]).*

Remark 3.17 (Minkowski cone not localizable). *Proposition 2.2 and Corollary 2.4 in [AGKS21] establish that the Minkowski cone over a geodesic length space is a geodesic Lorentzian pre-length space and that the time separation τ is continuous. However, it cannot be localizable as the vertex* 0 *is* isolated *with respect to $\ll$, i.e., there is no point x such that $x \ll 0$.*

However, Minkowski cones have localizable neighborhoods at all points except the vertex 0, which we establish below but first we need the following lemma, whose proof is elementary.

Lemma 3.18 (Useful properties for Minkowski cones). *Let (Y, d) be a metric space and $X := \mathrm{Cone}(Y)$ the Minkowski cone over (Y, d). Let $p = (s, y) \leq q = (t, y')$.*

1. *If* $s > 0$ *one has that* $d(y, y') \leq \log(t) - \log(s)$*.*

2. *If* $d(y, y') \leq \pi$*, then* $d_c(p, q) \leq (t - s) + td(y, y')$*.*

Proposition 3.19 (Minkowski cones nearly localizable)**.** *Let* (Y, d) *be a metric space which is strictly intrinsic. Then any* $0 \neq x \in X := \mathrm{Cone}(Y)$ *has a localizable neighborhood. Moreover,* $0 \in X$ *has a neighborhood* Ω_0 *that satisfies all the conditions of a localizable neighborhood except that* $I^-(0) \cap \Omega_0 = \emptyset$*. We call such a neighborhood* nearly localizable*.*

Proof: As (nearly) localizable neighborhoods, we choose the sets $\{(t, y) : t_- < t < t_+\}$ and $\{(t, y) : 0 \leq t < t_+\}$, respectively, for any $0 < t_- < t_+$. These neighborhoods are causally convex, and any two causally related points can be connected by a τ-distance realizer ([AGKS21, Cor. 2.4]). Furthermore, by [AGKS21, Prop. 2.2] τ is continuous, and except at 0 there are no $\ll$-isolated points. For d-compatibility, the d_c-length of causal curves within such a neighborhood is at most $4t_+$. This can be shown by using Lemma 3.18: Parametrize a (future directed) causal curve γ from p to q as $\gamma(r) = (r, \beta(r))$, where $r \in [s, t]$ and $\beta \colon [s, t] \to Y$. We can without loss of generality assume that $s = 0$, hence $p = 0 \in \mathrm{Cone}(Y)$ (otherwise extend γ to 0 — this is always possible, but not necessarily within the neighborhood). At this point, let $0 = s_0 < s_1 < \ldots < s_M = t$ be a partition of $[0, t]$. We choose a refinement $0 = s_0 = s'_0 < s'_1 < \ldots < s'_{M'} = t$ of $(s_i)_{i=0}^M$ such that $s'_{i+1} \leq 2s'_i$ and $d(\beta(s'_i), \beta(s'_{i+1})) \leq \pi$ for all $i \neq 0$. Thus

$$\sum_{i=0}^{M-1} d_c(\gamma(s_i), \gamma(s_{i+1})) \leq d_c(0, \gamma(s'_1)) + \sum_{i=1}^{M'-1} d_c(\gamma(s'_i), \gamma(s'_{i+1})) \tag{34}$$

$$\leq s'_1 + \sum_{i=1}^{M'-1} (s'_{i+1} - s'_i) + s'_{i+1}\, d(\beta(s'_i), \beta(s'_{i+1})) \tag{35}$$

$$\leq 2t + \sum_{i=1}^{M'-1} s'_{i+1} \left(\log(s'_{i+1}) - \log(s'_i)\right) \tag{36}$$

$$\leq 2t + \sum_{i=1}^{M'-1} \frac{s'_{i+1}}{s'_i} (s'_{i+1} - s'_i) \leq 2t + 2(t - s'_1) \leq 4t\,. \tag{37}$$

where we used the concavity of the logarithm. Taking the supremum over all partitions of $[0, t]$ yields $L^{d_c}(\gamma) \leq 4t$ and $t \leq t_+$ if q lies in such a neighborhood. $\square$

Localizability of the Minkowski cone away from the vertex also works if Y is only a locally strictly intrinsic metric space by using timelike diamonds $I((t_-,y),(t_+,y))$ for $y \in Y$ and t_-, t_+ close enough to each other so that $\mathrm{pr}_Y(I((t_-,y),(t_+,y)))$ is contained in a strictly intrinsic neighborhood.

Remark 3.20 (Minkowski cone geodesic)**.** *The Minkowski cone is geodesic precisely when the base Y is. This can be seen easily from [AGKS21, Lem. 2.3] and adding the vertex. In the metric case an analogous statement holds, see [BBI01, Thm. 3.6.17].*

Remark 3.21 (Cone metric does not induce Euclidean topology)**.** *This is a warning that usually, e.g., for spacetimes, the cone metric d_c induces a coarser topology than one might at first expect. In any case we have $d_c(0,p) = \tau(0,p)$ for all $p \in \mathrm{Cone}(Y)$ and hence*

$$B_1^{d_c}(0) = \{p \in \mathrm{Cone}(Y) : \tau(0,p) < 1\}\,, \tag{38}$$

i.e., the hyperboloid $\{p \in \mathrm{Cone}(Y) : \tau(0,p) = r\}$ of radius $r > 0$ is bounded, closed and not compact for non-compact Y.

For example, if Y is the $n-1$-dimensional hyperbolic space $\mathbb{H}^{n-1}$, then the Minkowski cone $\mathrm{Cone}(Y)$ can be identified with $I^+(0) \cup \{0\}$ in n-dimensional Minkowski spacetime $\mathbb{R}^n_1$, and clearly the topology does not agree with the Euclidean one of $\mathbb{R}^n$. This can be most easily seen (in without loss of generality $n = 1+1$ dimensions) by considering the sequence $p_k := (\frac{1}{k}, y_k)$, where $y_k \in \mathbb{H}^1 \cong \mathbb{R}^1$ is given by $y_k = (k^2, \sqrt{k^4-1})$ in the realization $\mathbb{H}^1 \subseteq \mathbb{R}^2_1$. Then $d_c(0,p_k) = \frac{1}{k}$, hence $p_k \to 0 \in \mathrm{Cone}(Y)$, but $p_k \cong \frac{1}{k} \cdot y_k = (k, \sqrt{k^2 - \frac{1}{k^2}}) \not\to 0$ in $\mathbb{R}^2$ with respect to the Euclidean topology ($\frac{1}{k} \cdot y_k$ even diverges). Here we used the identification of $p = (t,y)$ with $t \cdot \psi(y) \in I^+(0) \cup \{0\} \subseteq \mathbb{R}^n_1$, given in [AGKS21, Rem. 2.1].

Despite Remark 3.21 we will use the cone metric d_c on the Minkowski cone $\mathrm{Cone}(Y)$ over a metric space (Y,d), as it is a *canonical* choice (in contrast to e.g. the usual metric on Minkowski space).

In a Minkowski cone, the space of directions at the vertex is, as expected, essentially the base space.

Proposition 3.22 (Space of timelike directions in Minkowski cone)**.** *Let (Y,d) be any metric space and consider the Minkowski cone $X := \mathrm{Cone}(Y)$ over Y. Then the space of timelike directions at $0 \in X$, $(\mathcal{D}_0, \measuredangle_0)$, is isometric to (Y,d) as metric spaces.*

Proof. We consider the map $\Phi\colon Y \to \mathcal{D}_0$ that maps $y \in Y$ to the (equivalence class of the) curve $\gamma_y(t) := (t, y)$ $(t \in [0,1])$. The curve $\gamma_y\colon [0,1] \to X$ is a maximizer as $\tau((s,y),(t,y)) = \sqrt{s^2 + t^2 - 2st} = |s - t|$. For two such curves $\gamma_{y_1}, \gamma_{y_2}\colon [0,1] \to X$, we have that $\measuredangle_0(\gamma_{y_1}, \gamma_{y_2}) = d(y_1, y_2)$ as the definition of the time separation function τ in the Minkowski cone is made precisely such that $\tilde{\measuredangle}_0(\gamma_{y_1}(s), \gamma_{y_2}(t)) = d(y_1, y_2)$ whenever s, t are such that $\gamma_{y_1}(s), \gamma_{y_2}(t)$ are timelike related (Law of Cosines). Thus, Φ is an isometric embedding.

Conversely, let $\tilde{\gamma}$ be any geodesic in X starting at 0, without loss of generality parametrized by the t-coordinate. Let $[0, \varepsilon]$ be a part of the domain where $\tilde{\gamma}$ maximizes. So let $0 < s < t \leq \varepsilon$. Define $(s, p) := \tilde{\gamma}(s)$ and $(t, q) := \tilde{\gamma}(t)$, we indirectly assume $p \neq q$. Then $\tau(0, (t, q)) = t$, $\tau(0, (s, p)) = s$ and $\tau((s,p),(t,q)) = \sqrt{s^2 + t^2 - 2st\cosh(d(p,q))} < t - s$ as $d(p, q) > 0$, contradicting $\tilde{\gamma}$ being a distance realizer when restricted to $[0, \varepsilon]$. Thus $\tilde{\gamma}$ can only realize τ if $p = q$ for all $0 < s < t \leq \varepsilon$, hence $\tilde{\gamma} \sim \gamma_p$ and so Φ is surjective. $\square$

At this point we are in the position to introduce the notion of a timelike tangent cone.

Definition 3.23 (Timelike tangent cone). *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space with τ locally finite-valued and so that angles between future/past directed timelike geodesics always exist and let $x \in X$. The* future/past timelike tangent cone *$T_x^\pm$ at x is the Minkowski cone* $\mathrm{Cone}(\Sigma_x)$ *over the metric space $\Sigma_x^\pm$ of the completion of future/past directed timelike directions $\mathcal{D}_x^\pm$.*

The following is straightforward:

Lemma 3.24. *Let M be an $n+1$-dimensional strongly causal spacetime with metric of $\mathcal{C}^2$-regularity, and consider it as a Lorentzian length space. Let $p \in M$, then the space of future directed timelike directions at p, $\mathcal{D}_p^+$, is isometric to n-dimensional hyperbolic space $\mathbb{H}^n$ and the tangent cone T_x^+ is isometric to $I^+(0) \subseteq T_xM$.*

Proof. In a $\mathcal{C}^2$-spacetime, geodesics are solutions to the geodesic equation. Thus, future directed timelike geodesics passing through x are in one-to-one correspondence to future directed timelike vectors in $T_xM \cong \mathbb{R}^{n+1}_1$. As angles agree (Lemma 2.13), geodesics have the same angle as the vectors they correspond to, and we can restrict to unit future directed timelike vectors, which form $\mathbb{H}^n$. The usual metric on $\mathbb{H}^n$ is just defined as $d(p,q) = \measuredangle_0(p,q)$, which is the metric on $\mathcal{D}_x^+$. For the tangent cone, we just note that $\mathrm{Cone}(\mathbb{H}^n) \cong I^+(0) \subseteq \mathbb{R}^{n+1}_1$, see [AGKS21, Rem. 2.1]. $\square$

A natural question is that of whether $\Sigma_x^\pm$ is a geodesic length space. Moreover, in the metric case it is not hard to see that the completion of a length space is a length space ([BBI01, Exc. 2.4.18]). We give an example below, where $\mathcal{D}_x^+$ is not intrinsic, and then later in Proposition 3.26 give sufficient conditions for $\mathcal{D}_x^\pm$ to be intrinsic.

Example 3.25. *In essence, this example is the Minkowski cone over a non-intrinsic space. However, it is defined as a subset of Minkowski spacetime and not as a Minkowski cone over a metric space. Let* $X = \{(t,x,y) \in \mathbb{R}^3_1 : x^2 \geq yt\} \subseteq \mathbb{R}^3_1$ *be the closed exterior of a tilted double cone, given by* $yt = \frac{1}{4}((y+t)^2 - (y-t)^2) = x^2$*, see Figure 8. It is a non-convex, conical (with respect to* 0*) subset of Minkowski spacetime, and we consider it as a Lorentzian pre-length space by restricting* τ *from Minkowski space. We only have to consider* $\tilde{\measuredangle}$ *as angles exist and are invariant under scaling. The timelike geodesics through* 0 *are just straight lines. Thus, the space of directions* $\mathcal{D}_0^+$ *at* 0 *is a non-convex subset of* $\mathbb{H}^n$*, and thus not intrinsic. Note* X *was not intrinsic as well.*

If we intrinsify X *(see [Ber20, Thm. 1.7.3]), i.e., considering* X *with the new time separation function* $\hat{\tau}(p,q) := \sup\{L_\tau(\gamma) : \gamma$ *future directed causal from* p *to* $q\} \cup \{0\}$*, the lengths of geodesics stay the same, but angles will change. We need the other geodesics as well: For* p, q *not both on the cone such that the straight line connection is not contained in* X*, the connecting geodesic will be straight outside the cone, will be tangential to the cone where it touches it, and at such points it might switch over to have a part contained in the cone (which is a geodesic in the part of the cone which can be considered as a* $1+1$*-dimensional Lorentzian manifold). Now we have a distance realizer* γ *from* p *to* q*. It is intuitively clear that the straight lines from* 0 *to* $\gamma(t)$ *form a distance realizer in* $\mathcal{D}_0^+$*, making it strictly intrinsic.*

We have the following sufficient criterion for the completion of the space of timelike directions to be intrinsic.

Proposition 3.26 (Sufficient conditions for space of timelike directions to be intrinsic)**.** *Let* $(X, d, \ll, \leq, \tau)$ *be a strongly causal, locally causally closed Lorentzian pre-length space with* τ *locally finite valued and timelike curvature bounded above by* $K_+ \in \mathbb{R}$ *and below by* $K_- \in \mathbb{R}$*. Then* $\Sigma_x^\pm$ *is intrinsic. If additionally* X *is locally compact, d-compatible, regularly localizable and future and past non-lingering, then already* $\mathcal{D}_x^\pm$ *itself is intrinsic.*

Proof. We only establish the future case, the past case is completely analogous.

We want to employ [BBI01, Theorem 2.4.16], which states that a complete metric space which has ε-midpoints is already intrinsic. Note that we only

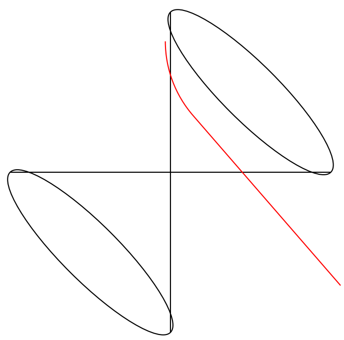


Figure 8: The example where $\mathcal{D}_x^+$ is not intrinsic, but neither is X itself. The red curve maximizes the length of all causal curves connecting the endpoints, while not realizing the time separation.

have to find ε-midpoints for a dense subset, for which we use $\mathcal{D}_x^+ \subseteq \Sigma_x^+$. So let $\varepsilon > 0$, and let $[\alpha], [\beta]$ be two points in $\mathcal{D}_x^+$, with representatives $\alpha \in [\alpha]$, $\beta \in [\beta]$ parametrized by τ-arclength. Note that by Lemma 3.9, the angle between α and β exists and is finite. By strong causality and the curvature bound we find a neighborhood of x such that all τ-distances can be realized locally. Moreover, by the definition of the angle, Proposition 2.14 and Proposition 3.2, for small enough $t > 0$ we have that $s := te^{-\measuredangle_x(\alpha,\beta)-1}$ satisfies $\alpha(s) \ll \beta(t)$ and both $|\tilde{\measuredangle}_x^{K_-}(\alpha(s), \beta(t)) - \measuredangle_x(\alpha, \beta)| < \varepsilon$ and $|\tilde{\measuredangle}_x^{K_+}(\alpha(s), \beta(t)) - \measuredangle_x(\alpha, \beta)| < \varepsilon$. Let $\gamma_t\colon [0, 1] \to X$ be a distance realizer (i.e., a timelike geodesic) from $\alpha(s)$ to $\beta(t)$. For each $r \in [0, 1]$ let $\eta_{r,t}\colon [0, 1] \to X$ be a distance realizer from x to $\gamma_t(r)$. We now claim that, for small enough t and some r, the direction $[\eta_{r,t}]$ is a 4ε-midpoint of $[\alpha]$ and $[\beta]$. We denote the points by $a = \alpha(s)$, $b = \gamma_t(r)$, $c = \beta(t)$. (See Figure 9 for a drawing.) Note that $a \ll b \ll c$.

We realize this in a comparison configuration as follows: Form a comparison triangle $\Delta\bar{x}_-\bar{a}_-\bar{c}_-$ in $\mathbb{L}^2(K_-)$, and analogously form a comparison triangle $\Delta\bar{x}_+\bar{a}_+\bar{c}_+$ in $\mathbb{L}^2(K_+)$. As $b \in [a, c]$ and denoting the corresponding points by $\bar{b}_\pm \in [\bar{a}_\pm, \bar{c}_\pm]$, we conclude by using the curvature bounds, that $\tau(x, b) \leq \underline{\tau}(\bar{x}_-, \bar{b}_-)$ and $\tau(x, b) \geq \bar{\tau}(\bar{x}_+, \bar{b}_+)$, where $\underline{\tau}$ and $\bar{\tau}$ denote the time separation functions on $\mathbb{L}^2(K_-)$ and $\mathbb{L}^2(K_+)$, respectively. By the law of cosines 2.4, we can translate this to statements on the angles and conclude

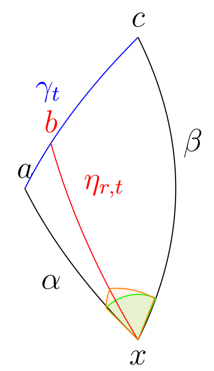


Figure 9: For a suitably chosen point on γ_t, the curve $\eta_{r,t}$ is an 4ε-midpoint.

that

$$\tilde{\measuredangle}_x^{K_-}(a,b) \leq \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-,\bar{b}_-)\,, \tag{39}$$
$$\tilde{\measuredangle}_x^{K_-}(b,c) \geq \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-,\bar{c}_-)\,, \tag{40}$$
$$\tilde{\measuredangle}_x^{K_+}(a,b) \geq \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{a}_+,\bar{b}_+)\,, \tag{41}$$
$$\tilde{\measuredangle}_x^{K_+}(b,c) \leq \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+,\bar{c}_+)\,. \tag{42}$$

Moreover, since the comparison situations are planar we have that

$$\measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-,\bar{b}_-) + \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-,\bar{c}_-) = \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-,\bar{c}_-)\,, \tag{43}$$
$$\measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{a}_+,\bar{b}_+) + \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+,\bar{c}_+) = \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{a}_+,\bar{c}_+)\,. \tag{44}$$

We have by the triangle inequality for angles (Theorem 3.1) that

$$\measuredangle_x(\alpha,\eta_{r,t}) + \measuredangle_x(\eta_{r,t},\beta) \geq \measuredangle_x(\alpha,\beta)\,, \tag{45}$$

and by (39), (42) that

$$\tilde{\measuredangle}_x^{K_-}(a,b) + \tilde{\measuredangle}_x^{K_+}(b,c) \leq \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-,\bar{b}_-) + \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+,\bar{c}_+)\,. \tag{46}$$

At this point we claim that $\measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-,\bar{c}_-) \geq \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+,\bar{c}_+) - \varepsilon$. To this end we compare the following four triangles and their angle at x:

1. the triangle $\Delta\bar{x}_-\bar{b}_-\bar{c}_-$ in $\mathbb{L}^2(K_-)$ with the angle $\omega_1 := \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-,\bar{c}_-)$,

2. a comparison triangle $\Delta\tilde{x}_-\tilde{b}_-\tilde{c}_-$ in Minkowski space $\mathbb{L}^2(0)$ of $\Delta\bar{x}_-\bar{b}_-\bar{c}_-$ and the angle $\measuredangle_{\tilde{x}_-}^{\mathbb{L}^2(0)}(\tilde{b}_-,\tilde{c}_-) = \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(0)}(\bar{b}_-,\bar{c}_-) =: \omega_2$,

3. a comparison triangle $\Delta\tilde{x}_+\tilde{b}_+\tilde{c}_+$ in Minkowski space $\mathbb{L}^2(0)$ of $\Delta\bar{x}_+\bar{b}_+\bar{c}_+$ and the angle $\measuredangle_{\tilde{x}_+}^{\mathbb{L}^2(0)}(\tilde{b}_+,\tilde{c}_+) = \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(0)}(\bar{b}_+,\bar{c}_+) =: \omega_3$, and

4. the triangle $\Delta\bar{x}_+\bar{b}_+\bar{c}_+$ in $\mathbb{L}^2(K_+)$ with the angle $\omega_4 := \measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+,\bar{c}_+)$.

Recall that all the points depend on r, t. Our aim is to have uniform comparison estimates on the differences of the angles $\omega_1, \ldots, \omega_4$, while varying r and t. We claim that for all $\varepsilon > 0$ there is $T > 0$ such that for all $0 < t < T$ we have $|\omega_1 - \omega_2| < \varepsilon$, $|\omega_2 - \omega_3| < \varepsilon$ and $|\omega_3 - \omega_4| < \varepsilon$ hold uniformly in r.

For the first and last inequality, note the triangles are comparison triangles of each other, i.e., they have the same sidelengths but different K, we use the second-to-last equation in the proof of Proposition 2.14 (or the corresponding equation for $K < 0$). We only show $|\omega_1 - \omega_2| < \varepsilon$, the argument for $|\omega_3 - \omega_4| < \varepsilon$ only differs by replacing some $-$ indices by $+$. We temporarily set $u = \tau(x, \bar{b}_-)$, $v = \tau(\bar{b}_-, \bar{c}_-)$. Then the second-to-last equation in the proof of Proposition 2.14 (or the corresponding equation for $K < 0$) (and the law of cosines for $K = 0$) read

$$\begin{aligned}
\cosh(\omega_1) &= -\frac{(u^2+t^2-v^2)(1+o(1)) + o(ut)}{2ut}(1+o(1))\,, \\
\cosh(\omega_2) &= -\frac{u^2+t^2-v^2}{2ut}\,,
\end{aligned}$$

as $u, t, v \to 0$. Here, we have $0 < u, v < t$ and let $t = \tau(x, \beta(t)) = \tau(x, c) \to 0$. As $\frac{u^2+t^2-v^2}{2ut}$ stays bounded (the angle exists), we have $\cosh(\omega_1) - \cosh(\omega_2) = o(1)$ as $t \to 0$, hence $\omega_1 - \omega_2 \to 0$, and similarly $\omega_3 - \omega_4 \to 0$.

For the second inequality, we compare the side-lengths $\lambda_- = \underline{\tau}(\bar{x}_-, \bar{b}_-) = \tilde{\tau}(\tilde{x}_-, \tilde{b}_-)$ and $\lambda_+ = \bar{\tau}(\bar{x}_+, \bar{b}_+) = \tilde{\tau}(\tilde{x}_+, \tilde{b}_+)$, where $\tilde{\tau}$ denotes the time separation function on Minkowski spacetime $\mathbb{L}^2(0)$. We get by the first case of Lemma 2.8 (for K_+ and K_-), that $\frac{|\lambda_+ - \lambda_-|}{t} \to 0$ as $t \to 0$. By the choice of s we obtain

$$0 < te^{-\measuredangle_x(\alpha,\beta)-1} = s < \lambda_- < t\,, \tag{47}$$

and hence we get $\frac{\lambda_+}{\lambda_-} =: q \to 1$. Now plugging this into the law of cosines 2.4 ($K = 0$, $\sigma = -1$) we obtain the following relation between ω_2 and ω_3:

$$\begin{aligned}
\lambda_-^2 + t^2 &= \tau(b,c)^2 + 2\lambda_- t\cosh(\omega_2)\,, \\
\lambda_+^2 + t^2 &= \tau(b,c)^2 + 2\lambda_+ t\cosh(\omega_3)\,.
\end{aligned}$$

This yields

$$\cosh(\omega_3) - \cosh(\omega_2) = \frac{\lambda_-^2(q^2-q) + t^2(1-q) - \tau(b,c)^2(1-q)}{2\lambda_- qt}\,, \tag{48}$$

which goes to 0 by (47), $q \to 1$ and $\tau(b,c) < t$.

In total, we get $|\omega_1 - \omega_4| < 3\varepsilon$ for small t, so we have proven the claim for 4ε and can replace the $\measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+, \bar{c}_+)$ by a $\measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-, \bar{c}_-)$ in equation (46), introducing only an error of 4ε. Then all the points are in the same comparison space and we have $\measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-, \bar{b}_-) + \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-, \bar{c}_-) = \measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{a}_-, \bar{c}_-)$.

To sum up, we have

$$\measuredangle_x(\alpha, \eta_{r,t}) + \measuredangle_x(\eta_{r,t}, \beta) \leq \measuredangle_x(\alpha, \beta) + 5\varepsilon\,,$$

where two ε come from the approximation of angles, and three ε comes from the difference of $\measuredangle_{\bar{x}_+}^{\mathbb{L}^2(K_+)}(\bar{b}_+, \bar{c}_+)$ and $\measuredangle_{\bar{x}_-}^{\mathbb{L}^2(K_-)}(\bar{b}_-, \bar{c}_-)$.

Note that this argument was independent of r. As τ is locally continuous, and we are considering it only in a comparison neighborhood, we get that $\tilde{\measuredangle}_x^{K_+}(a,b) = \tilde{\measuredangle}_x^{K_+}(\alpha(s), \gamma_t(r))$ is a continuous function in r, with values 0 for $r = 0$ and $\tilde{\measuredangle}_x^{K_+}(a,c) > \frac{\measuredangle_x(\alpha,\beta)}{2}$ for $r = 1$ (if ε is small enough). In particular, there is a value of r such that $\tilde{\measuredangle}_x^{K_+}(a,b) = \frac{\measuredangle_x(\alpha,\beta)}{2}$. This then implies that

$$\begin{aligned}
|\measuredangle_x(\alpha, \eta_{r,t}) - \frac{\measuredangle_x(\alpha, \beta)}{2}| &< \varepsilon\,,\\
|\measuredangle_x(\eta_{r,t}, \beta) - \frac{\measuredangle_x(\alpha, \beta)}{2}| &< 6\varepsilon\,.
\end{aligned}$$

As Σ_x^+ is complete, [BBI01, Theorem 2.4.16] shows it is intrinsic.

Finally, if additionally, X is locally compact, d-compatible, regularly localizable and future and past non-lingering, then the spaces of timelike directions $\mathcal{D}_x^{\pm}$ are complete for every $x \in X$ by Proposition 3.15 and hence intrinsic by the above. □

As the previous Proposition 3.26 used two-sided curvature bounds we comment on the relationship of different curvature bounds, which was not spelled out so explicitly before. In particular, a non-trivial space cannot have arbitrary curvature bounds from below and above.

Lemma 3.27 (Relation of different curvature bounds)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space with timelike curvature bounded above by K_+ and below by K_-. If there exists a non-degenerate timelike triangle satisfying timelike size bounds for K_- and K_+ inside a neighborhood, which is simultaneously a comparison neighborhood for K_- and for K_+, then $K_+ \leq K_-$.*

For $(X, d, \ll, \leq, \tau)$ a Lorentzian pre-length space with timelike curvature bounded below by K_-, then it also has timelike curvature bounded below by K for all $K \geq K_-$. And if $(X, d, \ll, \leq, \tau)$ has timelike curvature bounded above by K_+, it also has timelike curvature bounded above by K for all $K \leq K_+$.

In particular, for $K_+ \leq K_-$, $\mathbb{L}^2(K_-)$ has timelike curvature bounded above by K_+, and $\mathbb{L}^2(K_+)$ has timelike curvature bounded below by K_-.

Proof: Let $\Delta = (p_1, p_2, p_3)$ with $p_1 \ll p_2 \ll p_3$ be a non-degenerate timelike triangle inside a curvature comparison neighborhood for both K_- and K_+ satisfying the timelike size bounds for K_- and K_+. Then we find comparison triangles $\bar{\Delta}^{\pm} = (\bar{p}_1^{\pm}, \bar{p}_2^{\pm}, \bar{p}_3^{\pm})$ in $\mathbb{L}^2(K_{\pm})$. For points q_1, q_2 on the sides of Δ, we get comparison points $\bar{q}_1^{\pm}, \bar{q}_2^{\pm}$ on the sides of $\bar{\Delta}^{\pm}$. Note we can also view $\bar{\Delta}^{\pm}, \bar{q}_1^{\pm}, \bar{q}_2^{\pm}$ as a comparison situation for $\bar{\Delta}^{\mp}, \bar{q}_1^{\mp}, \bar{q}_2^{\mp}$, respectively.

By timelike curvature comparison, we get

$$\bar{\tau}_{\mathbb{L}^2(K_+)}(\bar{q}_1^+, \bar{q}_2^+) \leq \tau(q_1, q_2) \leq \bar{\tau}_{\mathbb{L}^2(K_-)}(\bar{q}_1^-, \bar{q}_2^-)\,. \tag{49}$$

We assume indirectly that $K_+ > K_-$ and choose $\bar{q}_1^{\pm} := \bar{p}_1^{\pm}$ and $q_2^{\pm} \in [\bar{p}_2^{\pm}, \bar{p}_3^{\pm}]$. Then [AGKS21, Lem. 6.1] yields that $\bar{\tau}_{\mathbb{L}^2(K_+)}(\bar{p}_1^+, \bar{q}_2^+) > \bar{\tau}_{\mathbb{L}^2(K_-)}(\bar{p}_1^-, \bar{q}_2^-)$ — a contradiction to Equation (49). $\square$

Note this relation between the different curvature bounds is opposite to the direction one would expect, e.g. timelike curvature ≥ 1 implies ≥ 2, although in the metric case it is the other way round, and ≥ 1 and ≤ 2 contradict each other (making all small enough timelike triangles degenerate). However, it was chosen this way to be compatible with [AB08] and hence the smooth spacetime case.

3.2 Exponential and logarithmic map

The notion of a timelike tangent cone allows us to introduce an exponential and logarithmic map into our setting, which are indispensable geometric tools. For exponential and logarithmic maps in the metric case see [BBI01, p. 321ff.]. We first define the logarithmic map, which (locally) assigns points to elements in the timelike tangent cone encoding the time separation and the equivalence class of the connecting geodesic, i.e., the timelike direction.

Definition 3.28 (Logarithmic map). *Let $(X, d, \ll, \leq, \tau)$ be a locally uniquely timelike geodesic Lorentzian pre-length space, i.e., every point has a neighborhood U such that in U two timelike related points can be joined by a unique timelike geodesic contained in U. Let $x \in X$ and let U be such a neighborhood of x. Then, for all $y \in U$ with $x \ll y$ ($y \ll x$) there exists an unique future (past) directed timelike geodesic γ from x to y contained in U. Then set $\log_x^+(y) := (\tau(x, y), [\gamma]) \in T_x^+$ and $\log_x^-(y) := (\tau(y, x), [\gamma]) \in T_x^-$, respectively. So, the logarithmic map $\log_x^{\pm} \colon U \cap I^{\pm}(x) \to T_x^{\pm}$ maps points to elements of the tangent cone.*

Note that the image $\log_x^\pm(U \cap I^\pm(x)) \subseteq T_x^\pm$ is star-shaped with respect to $(0,\star) \in T_x^\pm$ in the sense that for an element $(t,[\gamma]) \in \log_x^\pm(U \cap I^\pm(x))$ and $\lambda \in [0,1]$, also the scaled element $(\lambda t,[\gamma]) \in T_x^\pm$ is in $\log_x^\pm(U \cap I^\pm(x))$.

At this point we introduce the *exponential map*, that projects elements of the timelike tangent cone down to points in the space. Moreover, it is the inverse of the logarithmic map.

Definition 3.29 (Exponential map)**.** *Let $(X,d,\ll,\leq,\tau)$ be a locally uniquely timelike geodesic Lorentzian pre-length space with timelike curvature bounded below, let $x \in X$ and U as in Definition 3.28. Then the* future/past exponential map $\exp_x^\pm \colon T_x^\pm \supseteq \log_x^\pm(U \cap I^\pm(x)) \to U$ *is defined by*

$$(r,[\gamma]) \mapsto \tilde{\gamma}(r)\,, \tag{50}$$

where $\tilde{\gamma} \in [\gamma]$ is parametrized with respect to τ-arclength (cf. [KS18, Cor. 3.35]).

Remark 3.30. *The exponential map is well-defined: Let α,β be two timelike geodesics starting at x, defined on $[0,r]$, parametrized with respect to τ-arclength, and ending at different points $\alpha(r)=p\neq q=\beta(r)$. Then, α,β agree on a closed set and so there is a point $\alpha(s) \in I^-(q)$ which is not on β, and by uniquely timelike geodesicness the concatenation of $\alpha|_{[0,s]}$ and the unique timelike geodesic from $\alpha(s)$ to q is strictly shorter than β. Thus we get a non-degenerate triangle $\Delta x\alpha(s)\beta(r)$. Then we conclude $\measuredangle_x(\alpha,\beta) \geq \tilde{\measuredangle}_x(\alpha(s),\beta(r)) > 0$ by using the timelike curvature bound from below, Corollary 3.8 and Theorem 3.10. Thus $[\alpha]$ and $[\beta]$ are different points in $\mathcal{D}_x^+$. Moreover, as we restrict to the image of U under $\log_x^\pm$ we have $\tilde{\gamma}(r) \in U$.*

As we assume that the space is locally uniquely timelike geodesic and has timelike curvature bounded from below the exponential map is well-defined. In principle, one could define an exponential map also for general Lorentzian pre-length spaces as set-valued maps, cf. the discussion in the metric case [BBI01, Rem. 9.1.43].

Remark 3.31. *Let $X \subseteq Y$ be two Lorentzian pre-length spaces included in each other (i.e. the relations and time separation function on X is just the restriction of those of Y) with logarithm and exponential map defined, and let $x \in X$. We can consider the space of directions with respect to X and with respect to Y. Then $(\mathcal{D}_x^+)^X \subseteq (\mathcal{D}_x^+)^Y$ and in particular, $(T_x^+)^X$ is a subset of $(T_x^+)^Y$. We then have that the logarithmic map and exponential map with respect to Y are extensions of the logarithmic map and exponential map with respect to X.*

Example 3.32. *The image of the logarithm, hence the domain of the exponential map, need not be open, even in well-behaved spaces (as happens in metric geometry). Let* $X = \{(t,x) \in \mathbb{R}^2_1 : t^2 + (x-1)^2 < 1\} \cup \{0\} \subseteq \mathbb{R}^2_1$ *(see Figure 10). As a subset of a spacetime, X is a Lorentzian pre-length space. It is a quite well-behaved space: It is geodesically convex and thus strictly in-*

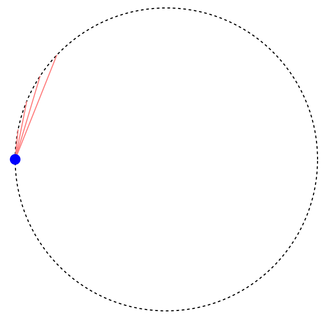


Figure 10: The counterexample where the space of future directed timelike directions $\mathcal{D}^+$ is not complete.

trinsic, τ is continuous, there exist no $\ll$-isolated points, it is a d-compatible set and distance realizing curves do not change their causal character, so the whole space is a regularly localizable neighborhood, and is even strongly regularly localizable. It is strongly causal and even causally simple (i.e., the relation $\leq$ is a closed subset of $X \times X$), but not globally hyperbolic. It has timelike curvature bounded above and below by $K = 0$.

However, the space of directions $\mathcal{D}^+_0$ at $p = 0$ is isometric to $(0, \infty)$ and thus not complete, and the image $\log^+_0(I^+(0))$, which is contained in $T^+_0 = \{0\} \cup I^+(0)$ of Minkowski spacetime, is not open at 0: The sequence $(x_n)_{n \geq 2}$, $x_n \in T^+_0$ corresponding to the points $(\sqrt{1-(1-\frac{1}{n})^2}, \frac{1}{n}) \in I^+(0)$ in $\mathbb{R}^2_1$) (on the considered circle, transferred to T^+_0 by the logarithm on Minkowski space) converges to 0, but is not contained in $\log^+_0(I^+(0))$.

On the one hand, it is necessary that all curves are timelike when considering their angles. On the other hand, the limit curve theorem gives only causal limit curves in general. However, when the sequence of angles is bounded the limit curve has to be timelike, given that one has curvature bounded below.

Lemma 3.33 (Bounded angles imply limit curve timelike)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and regularly localizable Lorentzian pre-length space with timelike curvature bounded from below. Let $\gamma_n \colon [0, \varepsilon] \to X$ be future* *directed timelike geodesics, starting at $\gamma_n(0) = x$ and converging uniformly to a causal curve $\gamma\colon [0,\varepsilon] \to X$. If $(\measuredangle_x(\gamma_n, \gamma_0))_n$ is bounded the limit curve γ is timelike.*

Proof: First, by [KS18, Prop. 3.17] the limit curve γ is maximal and by [KS18, Thm. 3.18] it is either timelike or null. Indirectly assume that γ is null and without loss of generality we can assume that all curve (segments) are in one regularly localizable neighborhood, which is a comparison neighborhood as well. Let $C > 0$ be such that $\measuredangle_x(\gamma_n, \gamma_0) \leq C$ for all $n \in \mathbb{N}$ and let $K \in \mathbb{R}$ be the timelike curvature bound from below. Let $t > 0$ and $s > 0$ such that $\gamma(s) \ll \gamma_0(t)$ and γ_0 is a distance realizer on $[0, t]$. Then for n large enough we have that $\gamma_n(s) \ll \gamma_0(t)$ and hence

$$C \geq \measuredangle_x(\gamma_n, \gamma_0) \geq \tilde{\measuredangle}_x^K(\gamma_n(s), \gamma_0(t)) =: \omega_n \,, \tag{51}$$

by using Corollary 3.8 and Theorem 3.10. At this point we consider the triangle $\Delta_n := \Delta x \gamma_n(s) \gamma_0(t)$, which has side-lengths $a_n := \tau(x, \gamma_n(s))$, $b := \tau(x, \gamma_0(t))$ and $c_n := \tau(\gamma_n(s), \gamma_0(t))$. Note that $a_n \to 0$ and $c_n \to \tau(\gamma(s), \gamma_0(t)) =: c \leq b$, by the reverse triangle inequality. For simplicity we assume that $K = 0$ — the other cases being analogous. By the law of cosines Lemma 2.4 for $K = 0$ and $\sigma = -1$ we conclude that

$$\cosh(\omega_n) = \frac{a_n^2 + b^2 - c_n^2}{2 a_n b} \,, \tag{52}$$

If $c < b$, the enumerator stays positive in the limit, and the denominator converges to 0, yielding $\omega_n \to \infty$, contradicting $\omega_n < C$ for all $n \in \mathbb{N}$.

If $c = b$, this is a contradiction to X being regular (cf. [KS18, Thm. 3.18]): We get that the concatenation of γ from x to $\gamma(s)$ with the null segment $\gamma(s)$ to $\gamma_0(t)$, of overall length $0 + c$ has the same length as the distance realizer $\gamma_0|_{[0,t]}$, which goes from x to $\gamma_0(t)$ and which has the length $b = c$. Note that a maximal causal curve from $\gamma(s)$ to $\gamma_0(t)$ exists by localizability and strong causality. $\square$

With the above result in hand we are able to establish that the exponential map is continuous.

Lemma 3.34 (Exponential map continuous)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally uniquely timelike geodesic, globally hyperbolic, locally causally closed, regularly localizable and future or past non-lingering, respectively, Lorentzian pre-length space with timelike curvature bounded below. Then $\exp_x^\pm$ is continuous away from $0 \in T_x^\pm$ for all $x \in X$.*

Proof: We only establish the future directed case, i.e., for $\exp_x^+$. The past directed case is completely analogous. Let $x \in X$ and U as in definition 3.28. By global hyperbolicity and local timelike geodesic connectedness, we can shrink U to be with compact closure. Let $(t_n, [\gamma_n]) \in \log_x^+(I^+(x) \cap U)$ converge to $(t, [\gamma])$, and let γ_n and γ be defined up to parameter t_n and t. Then by the limit curve theorem [KS18, Thm. 3.7], we can assume (without loss of generality) that $(\gamma_n)_n$ converges to a future directed causal curve $\tilde{\gamma}$. Note that $\tilde{\gamma}$ is non-constant by the non-lingering property (Definition 3.14). We have to show that $[\tilde{\gamma}] = [\gamma]$ and to this end we need to establish that $\tilde{\gamma}$ is timelike. As $[\gamma_n] \to [\gamma]$ the sequence $(\measuredangle_x(\gamma_n, \gamma_0))_n$ is bounded and because $\gamma_n \to \tilde{\gamma}$ uniformly we obtain that $\tilde{\gamma}$ is timelike by Lemma 3.33. Now, by semi-continuity of angles, Proposition 3.11, we get $\measuredangle_x(\gamma, \tilde{\gamma}) \leq \lim_n \measuredangle_x(\gamma, \gamma_n) = 0$, so $[\gamma] = [\tilde{\gamma}]$ and we are done. $\square$

Similarly, we can show that the logarithm is continuous.

Lemma 3.35 (Logarithmic map continuous)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally uniquely timelike geodesic, globally hyperbolic, locally causally closed, regularly localizable Lorentzian pre-length space with timelike curvature bounded above. Then $\log_x^\pm$ is continuous for all $x \in X$.*

Proof: We only establish the future directed case, i.e., for $\log_x^+$. The past directed case is completely analogous. Let $x \in X$ and U as in Definition 3.28, and by global hyperbolicity and local timelike geodesic connectedness we can assume without loss of generality that U is relatively compact and by shrinking U further that U is a regularly localizing neighborhood. Indirectly, assume that $\log_x^+$ is not continuous, so there is a sequence $(y_n)_n$ in $U \cap I^+(x)$ with $y_n \to y \in U \cap I^+(x)$ and $\log_x^+(y_n) \not\to \log_x^+(y)$. Thus, by definition, we have $\log(y_n) = (t_n, [\gamma_n])$ and $\log_x^+(y) = (t, [\gamma])$, where $t_n = \tau(x, y_n)$, $t = \tau(x, y)$, γ_n are future directed timelike geodesics from x to y_n and γ is a future directed timelike geodesic from x to y. As $t_n = \tau(x, y_n)$ and τ is continuous (on U), $t_n \to t = \tau(x, y)$. By assumption we know that $\gamma_n(t_n) = y_n \to y$. By the limit curve theorem, we get a limit curve $\tilde{\gamma}$ of a subsequence of $(\gamma_n)_n$ from x to y and we indirectly assume $[\gamma] \neq [\tilde{\gamma}]$, hence $[\gamma_n] \not\to [\gamma]$. As $x \ll y$ and X is chronological, $\tilde{\gamma}$ is non-constant. By [KS18, Prop. 3.17] $\tilde{\gamma}$ is also maximizing and still contained in $\bar{U}$. From regular localizability and [KS18, Thm. 3.18] we conclude that $\tilde{\gamma}$ is timelike, hence by uniqueness of geodesics in $\bar{U}$, γ and $\tilde{\gamma}$ are reparametrizations of each other. Finally, by Proposition 3.11 we estimate

$$0 = \measuredangle_x(\tilde{\gamma}, \gamma) \geq \limsup_n \measuredangle_x(\gamma_n, \gamma)\,, \tag{53}$$

a contradiction to $[\gamma_n] \not\to [\gamma]$. $\square$

Combining the last two results, we obtain

Corollary 3.36. *The exponential map* $\exp_x^\pm$ *is a homeomorphism away from* 0 *for locally uniquely timelike geodesic, globally hyperbolic, locally causally closed, regularly localizable, future or past non-lingering Lorentzian pre-length spaces with timelike curvature bounded above and below.*

4 Angles between timelike curves of arbitrary time orientation

In this final section we consider angles between curves of different time orientation. This is needed to complete the characterization of (timelike) curvature bounds via K-angle monotonicity and for the triangle inequality of angles.

In fact, for a special case of the triangle inequality of angles, we need the following corollary of the straightening lemma for shoulder angles [AB08, Lemma 2.4] (see also [Kir18, Lemma 3.3.2]).

Corollary 4.1 (Straightening lemma)**.** *Let* $K \in \mathbb{R}$*. Let* $p \ll m \ll q \leq r$ *be points in* $\mathbb{L}^2(K)$ *such that* $\tau(p,q)+\tau(q,r) \leq \tau(p,m)+\tau(m,r)$[7] *and such that* p *and* r *lie on opposite sides of the geodesic segment* $[mq]$*. Also assume that* $\tau(p,q)+\tau(q,r) < D_K$ *to ensure that all relevant triangles satisfy timelike size bounds for* K*. We consider the causal triangles* $\Delta_1 = \Delta pmq$*,* $\Delta_2 = \Delta mqr$ *and an additional causal triangle* $\Delta' = \Delta p'q'r'$ *in* $\mathbb{L}^2(K)$ *with side-lengths*

$$\begin{aligned}\tau(p',q') &= \tau(p,q)\,,\\ \tau(q',r') &= \tau(q,r)\,,\\ \tau(p',r') &= \tau(p,m)+\tau(m,r)\,,\end{aligned}$$

and a point m' *on the side* $[p'r']$ *with* $\tau(p',m') = \tau(p,m)$*. For an illustration of the set-up see Figure 11.*

Then we have that $\measuredangle_m^{\mathbb{L}^2(K)}(p,q) > \measuredangle_m^{\mathbb{L}^2(K)}(q,r)$ *holds if and only if* $\tau(m,q) < \tau(m',q')$ *holds. Moreover, the same equivalence holds for the reversed inequalities and equality on one side holds if and only if equality holds on the other.*

Proof. First we assume that $q \ll r$. We want to apply the straightening Lemma for shoulder angles (or Alexandrov's Lemma), cf. [AB08, Lemma

[7] Note this is needed for the existence of $\tilde{\Delta}$.

2.4]. For this, we have to make sense of the required inequality in [AB08, Lemma 2.4], i.e.,

$$(1-\lambda)\angle pmq + \lambda\angle rmq \geq 0\,. \tag{54}$$

To this end let $\gamma_{p',r'} : [0,1] \to \mathbb{L}^2(K)$ be the timelike geodesic from p' to r', then there is a parameter λ such that $m' = \gamma_{p',r'}(\lambda)$, which we calculate as $\lambda = \frac{\tau(p,m)}{\tau(p,m)+\tau(m,r)}$. Then the required inequality (54) translates to

$$\tau(m,r)\tau(p,m)\tau(m,q)\cosh(\measuredangle_m^{\mathbb{L}^2(K)}(p,q)) \tag{55}$$

$$-\,\tau(p,m)\tau(m,r)\tau(m,q)\cosh(\measuredangle_m^{\mathbb{L}^2(K)}(q,r)) \geq 0\,, \tag{56}$$

where we used the formula of the non-normalized angle including the two side-lengths, the cosh of the angle and the sign of the angle. This reduces to

$$\measuredangle_m^{\mathbb{L}^2(K)}(p,q) \geq \measuredangle_m^{\mathbb{L}^2(K)}(q,r)\,,$$

and similarly for the inequality reversed. Under this condition, the straightening Lemma yields that

$$\langle \dot\gamma_{p',q'}(0), \dot\gamma_{p',m'}(0)\rangle =: \angle q'p'm' \geq \angle qpm = \langle \dot\gamma_{p,q}(0), \dot\gamma_{p,m}(0)\rangle\,,$$

which in our terminology means that

$$\measuredangle_{p'}^{\mathbb{L}^2(K)}(m',q') \leq \measuredangle_p^{\mathbb{L}^2(K)}(m,q)\,.$$

Now we note that $\Delta p'm'q'$ and Δpmq have two equal sides (all but the $[mq]$ side) and use the monotonicity statement in the law of cosines 2.5 ($\sigma = -1$, varying a short side) to get $\tau(m',q') \geq \tau(m,q)$.

Similarly, we get $\tau(m',q') \leq \tau(m,q)$ when $\measuredangle_m^{\mathbb{L}^2(K)}(p,q) \leq \measuredangle_m^{\mathbb{L}^2(K)}(q,r)$. Finally, observe that the equality $\measuredangle_m^{\mathbb{L}^2(K)}(p,q) = \measuredangle_m^{\mathbb{L}^2(K)}(q,r)$ implies that p,m,r lie on a straight line, making $p'=p$, $m'=m$, $q'=q$ and $r'=r$ satisfy all the requirements on Δ'.

Finally, we establish the general case where $q \leq r$. Let $\varepsilon > 0$ be small enough. We consider $\tilde q \in [mq]$ with $\tau(\tilde q, q) = \varepsilon$. Then $\tilde q \ll r$, so we can apply the above: Let the causal triangle $\tilde\Delta' = \Delta\tilde p'\tilde q'\tilde r'$ in $\mathbb{L}^2(K)$ with side-lengths $\tau(\tilde p',\tilde q') = \tau(p,\tilde q)$, $\tau(\tilde q',\tilde r') = \tau(\tilde q, r)$, $\tau(\tilde p',\tilde r') = \tau(p,m)+\tau(m,r)$ and a point $\tilde m'$ on the side $[\tilde p'\tilde r']$ with $\tau(\tilde p',\tilde m') = \tau(p,m)$. Thus we get that

$$\measuredangle_m^{\mathbb{L}^2(K)}(p,q) = \measuredangle_m^{\mathbb{L}^2(K)}(p,\tilde q) > \measuredangle_m^{\mathbb{L}^2(K)}(\tilde q,r) = \measuredangle_m^{\mathbb{L}^2(K)}(q,r) \tag{57}$$

$$\Leftrightarrow \tau(m,\tilde q) < \tau(\tilde m',\tilde q')\,, \tag{58}$$

the same for flipped inequalities and equality on one side holds if and only if equality holds on the other. By continuity of τ and $\measuredangle^{\mathbb{L}^2(K)}$ we get as $\varepsilon \to 0$ that

$$\measuredangle_m^{\mathbb{L}^2(K)}(p,q) > \measuredangle_m^{\mathbb{L}^2(K)}(q,r) \Rightarrow \tau(m,q) \leq \tau(m',q')\,, \tag{59}$$

$$\tau(m,q) < \tau(m',q') \Rightarrow \measuredangle_m^{\mathbb{L}^2(K)}(p,q) \geq \measuredangle_m^{\mathbb{L}^2(K)}(q,r)\,, \tag{60}$$

or with reversed inequalities, so we still need to show strictness in either of these. For this, we consider what happens in case of equality.

If $\tau(m,q) = \tau(m',q')$, we get that Δ_1 and $\Delta p'm'q'$ have the same side-lengths and Δ_2 and $\Delta m'q'r'$ have the same side-lengths. Thus, angle additivity in $\mathbb{L}^2(K)$ gives us that $\measuredangle_m^{\mathbb{L}^2(K)}(p,q) = \measuredangle_m^{\mathbb{L}^2(K)}(p',q') = \measuredangle_m^{\mathbb{L}^2(K)}(q',r') = \measuredangle_m^{\mathbb{L}^2(K)}(q,r)$, so equality in the other inequality.

If $\measuredangle_m^{\mathbb{L}^2(K)}(p,q) = \measuredangle_m^{\mathbb{L}^2(K)}(q,r)$ we use angle additivity in $\mathbb{L}^2(K)$ to get $\measuredangle_m^{\mathbb{L}^2(K)}(p,r) = 0$ and thus p,m,r lie on a geodesic. Consequently, $p' = p$, $q' = q$, $r' = r$, $m' = m$ is actually a valid choice for Δ' and m' and thus $\tau(m,q) = \tau(m',q')$. $\square$

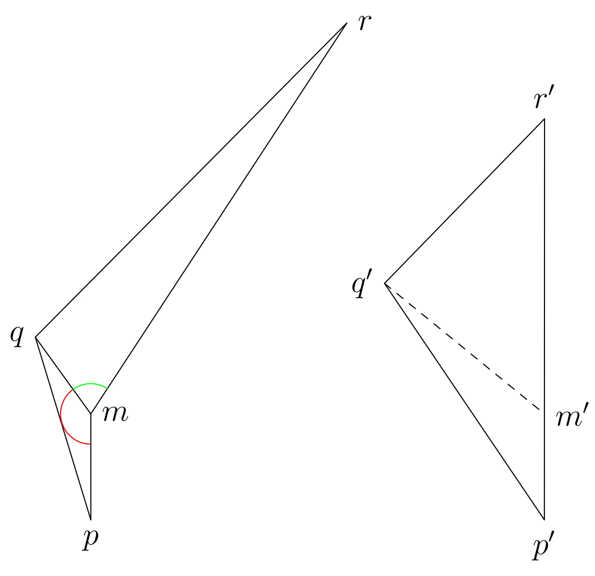


Figure 11: The set-up of the straightening lemma.

Definition 4.2 (Geodesic prolongation)**.** *A Lorentzian pre-length space* $(X,d,\ll,\leq,\tau)$ *is said to satisfy* geodesic prolongation *if all causal geodesics can be prolonged (extended) to a causal geodesic which has an open domain,* *i.e., a geodesic defined on a closed interval* $[b, c]$ *can be prolonged (as a geodesic) to* (a, e)*, where* $a < b < c < e$*.*

Remark 4.3. *Note we do not require this extension of a geodesic to be unique. Although under some assumptions, it will be, see e.g. [KS18, Theorem 4.12].*

This definition is complementary to [GKS19, Def. 4.5], where extendibility of geodesics was introduced. Thus we opted for calling it *prolongation* instead of *extendibility* to easily distinguish them.

Smooth spacetimes satisfy geodesic prolongation. However, smooth spacetimes with boundary do not satisfy geodesic prolongation: There, timelike geodesics which pass non-tangentially into the boundary have (half-)closed domains and cannot be extended. For details see the following example.

Example 4.4. *Closed half-Minkowski spacetime* $H = \{(t, x) : x \geq 0\} \subseteq \mathbb{R}^2_1$ *has very nice properties. In fact, it has timelike curvature bounded below and above by* 0*, it is globally hyperbolic, causally path connected, strongly regularly localizable and it is non-lingering at each point. However, the distance realizer* $\alpha(t) = (t, \frac{t_0 - t}{2})$ *is only defined on* $(-\infty, t_0]$*, has* $\alpha(t_0) = (t_0, 0)$ *and cannot be extended as a geodesic. Thus* H *fails to satisfy geodesic prolongation at each point on the boundary, see Figure 12.*

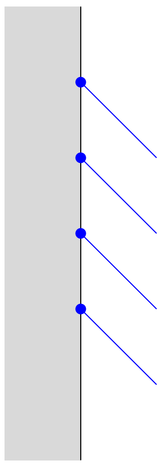


Figure 12: Four timelike geodesics in the closed half-Minkowski spacetime that cannot be extended.

Finally, we are in a position to establish the triangle inequality for angles for arbitrarily time oriented timelike curves. This complements Theorem 3.1.

Theorem 4.5 (Triangle inequality for (upper) angles — the case of mixed time orientation)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal Lorentzian pre-length space with τ locally finite-valued and locally continuous and let $\alpha, \beta, \gamma\colon [0, B) \to X$ be timelike curves starting at $x := \alpha(0) = \beta(0) = \gamma(0)$.*

1. *If α, β are future directed, γ is past directed and the angle $\measuredangle_x(\alpha, \gamma)$ exists, we get the* triangle inequality of angles
$$\measuredangle_x(\alpha, \gamma) \leq \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma)\,.$$

2. *If we assume that X is locally causally closed, has timelike curvature bounded below and satisfies geodesic prolongation, the triangle inequality of angles also holds for α, γ being future directed curves and β being a past directed geodesic.*

See Figure 13 for an illustration of the two cases.

By time reversal and switching the roles of α and γ, this statement and that of Theorem 3.1 cover all possible combinations of time orientation.

Proof: (i): We restrict to a neighborhood of x such that τ is finite and continuous and all curve segments are contained in it. Let α, β be future directed, γ past directed and such that the angle $\measuredangle_x(\alpha, \gamma)$ exists. By definition (and existence) of the angle $\measuredangle_x(\alpha, \gamma)$ we have that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\forall r, s, t \in (0, \delta)$ we have the upper bound

$$\begin{aligned}
\tilde{\measuredangle}_x(\alpha(r), \beta(s)) &< \measuredangle_x(\alpha, \beta) + \varepsilon\,,\\
\tilde{\measuredangle}_x(\beta(s), \gamma(t)) &< \measuredangle_x(\beta, \gamma) + \varepsilon\,,\\
\tilde{\measuredangle}_x(\alpha(r), \gamma(t)) &> \measuredangle_x(\alpha, \gamma) - \varepsilon\,,
\end{aligned}$$

whenever $\alpha(r) \leq \beta(s)$ or the other way around ($\gamma(t) \ll \beta(s)$ and $\gamma(t) \ll \alpha(r)$ are always satisfied, as γ is past directed).

We denote $a = \alpha(r)$, $b = \beta(s)$ and $c = \gamma(t)$. We let $a \leq b$ (fixing $s > 0$, there is an r such that this is satisfied).

We form a comparison situation in $\mathbb{L}^2(0)$ as follows: Let $\bar{x}, \bar{a}, \bar{b}, \bar{c}$ containing comparison triangles for Δcxb and Δxab, where we let $\bar{a}$ and $\bar{c}$ on the same side of (the line given by the segment) $[\bar{x}\bar{b}]$. We consider the segment $L = [\bar{c}\bar{b}]$ and its relation to $\bar{a}$. We vary r and will get two cases either $\bar{a}$ lying on L is possible or if not, we approximate appropriately. For this, we use continuity of $\bar{a}$ when varying r. For small r, it is clear $\bar{a}$ will lie below L (as $\bar{x}$ is below L). As $r \to \sup \alpha^{-1}(J^-(b)) =: r^+$ we have $\tau(a, b) \to 0$ and thus $\bar{a}$ has to approach $\partial I^-(\bar{b})$. As the part of $\partial I^-(\bar{b})$ lying on the same side of $[\bar{x}\bar{b}]$

as $\bar{c}$ lies entirely above or on L except for $\bar{b}$ itself. Thus either $\bar{a}$ is eventually above L or $\bar{a} \to \bar{b}$ (which is on L).

First, we discuss the case where $\bar{a}$ is eventually on or above L, so we have a value for r where $\bar{a}$ is on L. Now we have the situation where $\bar{c} \ll \bar{a} \leq \bar{b}$ all lie on a straight line, i.e., $\bar{\tau}(\bar{c}, \bar{b}) = \bar{\tau}(\bar{c}, \bar{a}) + \bar{\tau}(\bar{a}, \bar{b})$. As we have $\tau(c, b) \geq \tau(c, a) + \tau(a, b)$ and the side-lengths agree with the ones in the comparison triangles $\Delta\bar{c}\bar{x}\bar{b}$, $\Delta\bar{x}\bar{a}\bar{b}$ except $\tau(c, a)$, we get that $\bar{\tau}(\bar{c}, \bar{a}) \geq \tau(c, a)$. This yields for the angles that

$$\begin{aligned}
\tilde{\measuredangle}_x(a, b) &= \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a}, \bar{b}) \,, \\
\tilde{\measuredangle}_x(c, b) &= \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{c}, \bar{b}) \,, \\
\tilde{\measuredangle}_x(c, a) &\leq \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{c}, \bar{a}) \,,
\end{aligned}$$

where the last inequality holds by the monotonicity of the law of cosines (Remark 2.5, varying the longest side, $\bar{\tau}(\bar{c}, \bar{a}) \geq \tau(c, a)$). In the comparison configuration, i.e., in Minkowski spacetime we can use additivity of angles, and thus we get that

$$\tilde{\measuredangle}_x(c, a) \leq \measuredangle_{\bar{x}}(\bar{c}, \bar{a}) = \measuredangle_{\bar{x}}(\bar{a}, \bar{b}) + \measuredangle_{\bar{x}}(\bar{c}, \bar{b}) = \tilde{\measuredangle}_x(a, b) + \tilde{\measuredangle}_x(c, b) \,,$$

which is what we need. We have proven that the triangle inequality of angles is not violated by more than 3ε.

Now let $\bar{a}$ be under L for all r[8], then $\bar{a} \to \bar{b}$ as $r \to r^+$. We adapt the above proof to this case, and can assume that we have the above construction except for the location of $\bar{a}$. Now, we can take r close enough to r^+ such that $\bar{\tau}(\bar{c}, \bar{a}) \geq \tau(c, a) - \tilde{\varepsilon}$ for some constant $\tilde{\varepsilon} > 0$ to be specified later. This is possible as $\bar{a} \to \bar{b}$, $\tau(c, b) \geq \tau(c, a)$ and τ is continuous. Using the law of cosines (2.4), we get

$$\begin{aligned}
\cosh(\tilde{\measuredangle}_x(a, c)) &= \frac{\tau(c, a)^2 - \tau(x, a)^2 - \tau(c, x)^2}{2\tau(x, a)\tau(c, x)} \,, \\
\cosh(\measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a}, \bar{c})) &= \frac{\bar{\tau}(\bar{c}, \bar{a})^2 - \tau(x, a)^2 - \tau(c, x)^2}{2\tau(x, a)\tau(c, x)} \,,
\end{aligned}$$

where we used the equality of all distances except $\tau(c, a)$ and $\bar{\tau}(\bar{c}, \bar{a})$ in the comparison situation. The difference is

$$\begin{aligned}
\cosh(\measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a}, \bar{c})) - \cosh(\tilde{\measuredangle}_x(a, c)) &= \frac{\bar{\tau}(\bar{c}, \bar{a})^2 - \tau(c, a)^2}{2\tau(x, a)\tau(c, x)} && (61) \\
&\geq \frac{-(\bar{\tau}(\bar{c}, \bar{a}) + \tau(c, a))\tilde{\varepsilon}}{2\tau(x, a)\tau(c, x)} \,. && (62)
\end{aligned}$$

[8]This case will not occur if the neighborhood is causally closed, since then the limit value for $r = r^+$ is allowed, for which $\tau(a, b) = 0$, so $\bar{a} = \bar{b}$ or $\bar{a}$ is above L.

We have that $\tau(c,a), \bar{\tau}(\bar{c},\bar{a}) \leq \tau(c,b)$ stay bounded as $r \to r^+$ and $\tau(x,a)$ is increasing and thus bounded away from 0. Thus the right-hand-side goes to 0 as $r \to r^+$ and thus $\tilde{\varepsilon} \to 0$. As the inverse function of cosh is uniformly continuous, we find a small enough $\tilde{\varepsilon} > 0$ for each $\varepsilon > 0$ such that

$$\measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a},\bar{c}) - \tilde{\measuredangle}_x(a,c) \geq -\varepsilon\,.$$

Now we have a chain of inequalities as above

$$\tilde{\measuredangle}_x(a,c) - \varepsilon \leq \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a},\bar{c}) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{a},\bar{b}) + \measuredangle_{\bar{x}}^{\mathbb{L}^2(0)}(\bar{b},\bar{c}) = \tilde{\measuredangle}_x(a,b) + \tilde{\measuredangle}_x(b,c)\,,$$

which gives analogous to the above that the triangle inequality of angles is not violated by more than 4ε.

(ii): At this point we assume that X is locally causally closed, has timelike curvature bounded below by $K \in \mathbb{R}$ and satisfies geodesic prolongation. Let α, γ be future directed timelike curves and β a past directed geodesic.

First, we reduce this to the case where β and γ join up to a geodesic. By geodesic prolongation, we can extend β to the future of x by a future directed geodesic $\tilde{\beta}$. By Lemma 3.4.2, we have that $\measuredangle_x(\beta,\tilde{\beta}) = 0$. We claim that the following triangle inequalities of angles hold

$$\measuredangle_x(\alpha,\tilde{\beta}) \leq \measuredangle_x(\alpha,\beta) + \underbrace{\measuredangle_x(\beta,\tilde{\beta})}_{=0}\,,$$
$$\measuredangle_x(\tilde{\beta},\gamma) \leq \underbrace{\measuredangle_x(\tilde{\beta},\beta)}_{=0} + \measuredangle_x(\beta,\gamma)\,.$$

Note these claimed triangle inequalities are just like the desired triangle inequality, i.e., a future-past-future case. Assuming this, we get the desired triangle inequality

$$\measuredangle_x(\alpha,\gamma) \leq \measuredangle_x(\alpha,\tilde{\beta}) + \measuredangle_x(\tilde{\beta},\gamma) \leq \measuredangle_x(\alpha,\beta) + \measuredangle_x(\beta,\gamma)\,,$$

where the first inequality is by the future-only triangle inequality of angles, i.e., Theorem 3.1.

So we are left with (without loss of generality) establishing $\measuredangle_x(\alpha,\beta) \geq \measuredangle_x(\alpha,\tilde{\beta})$. Let U be a comparison neighborhood of x, and by strong causality we can assume that all the points and geodesic segments are contained in U. Let $\varepsilon > 0$ and choose some parameters $r, s, t > 0$ small enough such that

$$\tilde{\measuredangle}_x^K(\alpha(r),\beta(s)) < \measuredangle_x(\alpha,\beta) + \varepsilon\,,$$
$$\tilde{\measuredangle}_x^K(\alpha(r),\tilde{\beta}(t)) > \measuredangle_x(\alpha,\tilde{\beta}) - \varepsilon\,,$$
$$\alpha(r) \leq \tilde{\beta}(t)\,,$$

where we used Proposition 2.14 to replace the Minkowski comparison angle with the K-comparison angle and without loss of generality choose the time orientation in the third line (the other case being analogous). We denote $a := \alpha(r)$, $b := \beta(s)$ and $c := \tilde{\beta}(t)$.

We now construct the situation of the straightening lemma 4.1: We form comparison triangles $\Delta\bar{x}\bar{a}\bar{c}$ for Δxac and $\Delta\bar{b}\bar{x}\bar{a}$ for Δbxa in $\mathbb{L}^2(K)$ such that they share the side $[\bar{x}\bar{a}]$ and $\bar{b}$, $\bar{c}$ lie on different sides of (the line extending) $[\bar{x}\bar{a}]$. Note that $\bar{\tau}(\bar{b},\bar{a}) + \bar{\tau}(\bar{a},\bar{c}) = \tau(b,a) + \tau(a,c) \leq \tau(b,c) = \tau(b,x) + \tau(x,c) = \bar{\tau}(\bar{b},\bar{x}) + \bar{\tau}(\bar{x},\bar{c})$, as required by the straightening lemma. These triangles realize the K-comparison angles $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a},\bar{c}) = \tilde{\measuredangle}_x^K(a,c)$ and $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{b},\bar{a}) = \tilde{\measuredangle}_x^K(b,a)$. Let $\Delta\bar{a}'\bar{b}'\bar{c}'$ be a comparison triangle for Δabc and let $\bar{x}' \in [\bar{b}',\bar{c}']$ with $\bar{\tau}(\bar{b}',\bar{x}') = \bar{\tau}(\bar{b},\bar{x}) = \tau(b,x)$. Also note that $\bar{\tau}(\bar{b}',\bar{a}') = \bar{\tau}(\bar{b},\bar{a})$, $\bar{\tau}(\bar{a}',\bar{c}') = \bar{\tau}(\bar{a},\bar{c})$ and $\bar{\tau}(\bar{b}',\bar{c}') = \tau(b,c) = \tau(b,x) + \tau(x,c) = \bar{\tau}(\bar{b},\bar{x}) + \bar{\tau}(\bar{x},\bar{c})$.

At this point we use the lower curvature bound to obtain $\tau(x,a) = \bar{\tau}(\bar{x},\bar{a}) \leq \bar{\tau}(\bar{x}',\bar{a}')$. Thus by the straightening lemma 4.1, we get that $\tilde{\measuredangle}_x^K(b,a) \geq \tilde{\measuredangle}_x^K(a,c)$ which implies

$$\measuredangle_x(\alpha,\beta) + \varepsilon > \tilde{\measuredangle}_x^K(a,b) \geq \tilde{\measuredangle}_x^K(a,c) > \measuredangle_x(\alpha,\tilde{\beta}) - \varepsilon\,.$$

As this holds for all $\varepsilon > 0$ we conclude that $\measuredangle_x(\alpha,\tilde{\beta}) \leq \measuredangle_x(\alpha,\beta)$ and we are done. □

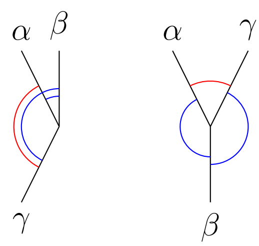


Figure 13: The configuration of the three geodesics in the triangle inequality for angles.

Example 4.6 (Necessity of the assumptions in the triangle inequality with mixed time orientations). *If α, γ are future directed and β is past directed, the triangle inequality may fail as the following example shows. We consider a causal funnel (cf. [KS18, Ex. 3.19]) with a (Minkowski-)timelike curve β, i.e., $X = \beta((-\infty,0]) \cup J^+(\beta(0))$ in n-dimensional Minkowski spacetime*

$\mathbb{R}^n_1$, and set $x := \beta(0)$, see Figure 14. It exhibits timelike branching, say β branches into future directed timelike geodesics α and γ at x. Then α and γ have $\measuredangle_x(\alpha, \gamma) > 0$. However, as both the concatenation of β and α and the concatenation of β and γ is a geodesic, we have that $\measuredangle_x(\alpha, \beta) = \measuredangle_x(\beta, \gamma) = 0$ by Lemma 3.4,2. Thus the triangle inequality of angles is not satisfied: $0 = \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \gamma) \ngeq \measuredangle_x(\alpha, \gamma)$. As a causal funnel has timelike curvature unbounded below, this shows the necessity of the timelike curvature bound below in Theorem 4.5.

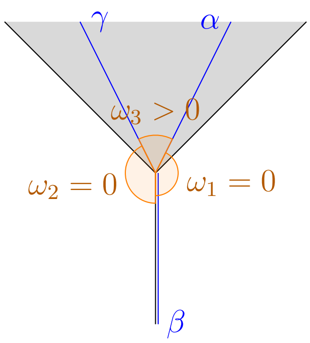


Figure 14: When timelike geodesics branch with a positive angle (e.g. the causal funnel here), one of the triangle inequalities for angles does not hold.

At this point, we use the triangle inequality of angles to get an analogue of the metric fact that the angles along a geodesic add to π.

Corollary 4.7 (Triangle equality along geodesics)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and locally causally closed Lorentzian pre-length space with timelike curvature bounded below. Let $\alpha_+ \colon [0, \varepsilon) \to X$ and $\alpha_- \colon [0, \varepsilon) \to X$ be a future and a past directed geodesic with $x := \alpha_+(0) = \alpha_-(0)$ which join up to a geodesic. Let $\beta \colon [0, \varepsilon) \to X$ be a future directed timelike curve. If the angle $\measuredangle_x(\beta, \alpha_-)$ exists, then*

$$\measuredangle_x(\beta, \alpha_+) = \measuredangle_x(\beta, \alpha_-)\,.$$

Proof. As the angle $\measuredangle_x(\beta, \alpha_-)$ exists, we use the triangle inequality of angles (Theorem 4.5,1) to get that

$$\measuredangle_x(\beta, \alpha_-) \leq \measuredangle_x(\beta, \alpha_+) + \underbrace{\measuredangle_x(\alpha_+, \alpha_-)}_{=0}\,.$$

Here we used that $\measuredangle_x(\alpha_+, \alpha_-) = 0$ by Lemma 3.4,2. We also apply the triangle inequality of angles to $\alpha_+, \alpha_-, \beta$: We assume a timelike curvature bound below, but technically, we did not assume geodesic prolongation, however for the proof of 4.5,2 we only assumed α_- to be extendible as a geodesic, which it is (with extension α_+). Thus we conclude that

$$\measuredangle_x(\beta, \alpha_+) \leq \measuredangle_x(\beta, \alpha_-) + \underbrace{\measuredangle_x(\alpha_-, \alpha_+)}_{=0} .$$

In total, we have $\measuredangle_x(\beta, \alpha_+) = \measuredangle_x(\beta, \alpha_-)$. $\square$

In [KS18, Thm. 4.12] we established that given a timelike lower curvature bound timelike distance realizers do not branch using either one of two additional hypotheses. The first being that the space is timelike locally uniquely geodesic and the second being that locally in X maximal future directed timelike curves are strictly longer than any future directed causal curve with the same endpoints that contains a null segment. These two hypotheses are only used to establish the existence of a non-degenerate timelike triangle. Using angles we were able to remove these two hypotheses and therefore establish that K-monotonicity comparison from below implies timelike non-branching. However, after the fact we were able to avoid the use of angles completely, and therefore make the result even stronger because we do not need local timelike geodesicness for applying Theorem 3.10, i.e., that a lower timelike curvature bound implies future K-monotonicity comparison from below. Thus the result obtained is now completely analogous to the metric case, see e.g. [Shi93, Lem. 2.4] where a lower bound on curvature (in the Alexandrov sense) implies non-branching of distance realizers.

Theorem 4.8 (Timelike non-branching)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded below by some $K \in \mathbb{R}$. Then timelike distance realizers cannot branch.*

Proof. Let indirectly $\alpha, \beta : [-B, B] \to X$ be timelike distance realizers parametrized by τ-arclength such that $\alpha|_{[-B,0]} = \beta|_{[-B,0]}$ and $\alpha((0, B)) \neq \beta((0, B))$. Define $x = \alpha(0) = \beta(0)$ the last known common point.

We distinguish two cases. First, we assume that for all $s, t > 0$, such that $\alpha(s) \ll \beta(t)$, we have $\tau(\alpha(s), \beta(t)) = t - s$ and for $\beta(t) \ll \alpha(s)$, we have $\tau(\beta(t), \alpha(s)) = s - t$. Thus, e.g. in the former case we can form the timelike triangle $x \ll \alpha(s) \ll \beta(t)$ and for any $r > 0$ the degenerate timelike triangle $\alpha(-r) = \beta(-r) \ll \alpha(s) \ll \beta(t)$. Similarly, in the latter case. Now for $t > 0$ consider $s_t := \sup\{s : \alpha(s) \ll \beta(t)\}$. Then $\lim_{s \nearrow s_t} \tau(\alpha(s), \beta(t)) = t - s_t$ and $\tau(\alpha(s), \beta(t)) = 0$ for $s > s_t$, so $s \mapsto \tau(\alpha(s), \beta(t))$ has a jump of size $= t - s_t$ at $s = s_t$. As τ is continuous, the jump-size has to be $t - s_t = 0$. Consequently, for all $s < t$ we have $\alpha(s) \ll \beta(t)$ and hence $\tau(\alpha(s), \beta(t)) = t - s$. Analogously we have for all $t < s$ that $\beta(t) \ll \alpha(s)$ and $\tau(\beta(t), \alpha(s)) = s - t$. Now, we get that $\alpha(s-\varepsilon) \ll \beta(s) \ll \alpha(s+\varepsilon))$, i.e., $\beta(s) \in I(\alpha(s-\varepsilon), \alpha(s+\varepsilon))$ for all $\varepsilon > 0$. By strong causality these timelike diamonds form a neighborhood base of $\alpha(s)$, and as this being along a timelike curve, we conclude that $\alpha(s) = \beta(s)$. As this holds for all s we obtain $\alpha = \beta$ — a contradiction.

In the second case, there are $s, t \in (0, B)$ such that, without loss of generality, $x \ll \alpha(s) \ll \beta(t)$ is a non-degenerate timelike triangle. Thus also for every $r > 0$ the timelike triangle $p := \alpha(-r) = \beta(-r) \ll \alpha(s) =: q \ll \beta(t) =: z$ is non-degenerate. At this point we can argue as in the proof of [KS18, Thm. 4.12]. For the sake of completeness we include the argument here. Let $\bar{\Delta} = \Delta\bar{p}\bar{q}\bar{z}$ be a comparison triangle of Δpqz in $\mathbb{L}^2(K)$. Let $\bar{x}_1 \in [\bar{p}, \bar{q}]$ correspond to x and $\bar{x}_2 \in [\bar{p}, \bar{z}]$ correspond to x as well, i.e., $\bar{\tau}(\bar{p}, \bar{x}_i) = \tau(p, x) = r$ for $i = 1, 2$. As $\bar{\Delta}$ is non-degenerate we have $\bar{x}_1 \notin [\bar{p}, \bar{z}]$. Consequently, $\bar{\tau}(\bar{x}_1, \bar{z}) < \bar{\tau}(\bar{x}_2, \bar{z})$ as otherwise the broken geodesic going from $\bar{p}$ to $\bar{x}_1$ to $\bar{z}$ is as long as the unbroken geodesic $[\bar{p}, \bar{z}]$. Finally, we arrive at

$$\tau(x, z) = \bar{\tau}(\bar{x}_2, \bar{z}) > \bar{\tau}(\bar{x}_1, \bar{z}) \geq \tau(x, z)\,, \tag{63}$$

where we used the timelike curvature bound from below in the last inequality — a contradiction. □

4.1 Monotonicity comparison

In this final subsection we introduce a version of K-monotonicity which holds for an arbitrary angle in a timelike geodesic triangle by using the signed angle $\measuredangle^{K,\mathrm{S}}$. The main result is that K-monotonicity is equivalent to curvature bounded by K.

Definition 4.9 (K-monotonicity)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $K \in \mathbb{R}$. We say that X satisfies* timelike K-monotonicity comparison from below (above) *if every point in X possesses a neighborhood U such that*

1. *$\tau|_{U\times U}$ is finite and continuous.*
2. *Whenever x, $y \in U$ with $x \ll y$, there exists a future-directed maximal timelike geodesic α in U from x to y.*
3. *Whenever $\alpha : [0, a] \to U, \beta : [0, b] \to U$ are timelike distance realizers in U with $x := \alpha(0) = \beta(0)$, we define the function $\theta\colon (0, a] \times (0, b] \supseteq D \to \mathbb{R}$ by*

$$\theta(s, t) := \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t)), \tag{64}$$

where $(s,t) \in D$ precisely when $\alpha(s), \beta(t)$ are causally related. We require this to be monotonically increasing (decreasing) in s and t for timelike K-monotonicity comparison from below (above).

A direct consequence of K-monotonicity comparison is that it implies the existence of angles between timelike geodesics depending on monotonicity comparison from below or above and the time orientation of the geodesics. The analogous result in metric geometry (see e.g. [BBI01, Prop. 4.3.2] or [BH99, Prop. II.3.1]) does not have this conditional dependence, of course.

Lemma 4.10 (Monotonicity implies existence of angles between geodesics of arbitrary time orientation)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. If X satisfies timelike K-monotonicity comparison from above or below, the angle between any two timelike geodesics starting at the same point x (both future or past directed or one future, one past directed) exists if it is finite[9].*

Moreover, in case X satisfies timelike K-monotonicity comparison from below and one of the two timelike geodesics is future directed and one is past directed (i.e., the $\sigma = 1$ case), the angle between them exists.

Similarly, in case X satisfies timelike K-monotonicity comparison from above and the two timelike geodesics are both future directed or both past directed (i.e., the $\sigma = -1$ case), the angle between them exists.

Proof: Let U be a comparison neighborhood given by Definition 4.9 above and let $\alpha, \beta : [0, \varepsilon) \to U$ be two future or past directed timelike geodesics with $x := \alpha(0) = \beta(0)$. Without loss of generality we can assume that either both are future directed or α is past directed and β is future directed. Let $s > 0$ and $t > 0$ such that $\alpha(s)$ and $\beta(t)$ are causally related. Then by K-monotonicity comparison from below (or above) we get that $\theta(s,t) = \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t))$ is monotone (decreasing if we have K-monotonicity comparison from below, increasing if we have it from above), and so this sequence has a limit. By Corollary 2.14 this limit is $\measuredangle_x^{\mathrm{S}}(\alpha, \beta)$, but we do not yet know it is finite. Now if $\sigma = 1$, we have 0 as a lower bound for all signed comparison angles, and if $\sigma = -1$, we have 0 as an upper bound. So the angles are finite (and thus exist) in two cases: In case of K-monotonicity comparison from below and $\sigma = 1$, and in case of K-monotonicity comparison from above and $\sigma = -1$. $\square$

Analogous to Corollary 3.8 a direct consequence of K-monotonicity comparison and Proposition 2.14 is that, given a curvature bound, K-comparison angles bound the angle between geodesics.

[9] The limit superior in the definition of upper angles is a limit, but we do not know if it is finite

Corollary 4.11 (Monotonicity implies bound on signed angle of geodesics)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space that satisfies K-monotonicity comparison from below (above) for some $K \in \mathbb{R}$. Then for any $x \in X$ and $\alpha, \beta \colon [0, B] \to X$ timelike geodesics with $\alpha(0) = \beta(0) = x$ one has that*

$$\measuredangle_x^{\mathrm{S}}(\alpha, \beta) \leq \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t)) \qquad \left(\measuredangle_x^{\mathrm{S}}(\alpha, \beta) \geq \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t))\right), \tag{65}$$

for all $s, t \in [0, B]$ small enough (with respect to timelike size bounds for K) such that $\alpha(s)$ and $\beta(t)$ are causally related.

Monotonicity comparison also directly influences the length of the third side in a comparison hinge.

Corollary 4.12 (Monotonicity comparison implies hinge comparison)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space that satisfies K-monotonicity comparison from below (above) for some $K \in \mathbb{R}$. Let (α, β) be a hinge, i.e., $x \in X$ and $\alpha \colon [0, A] \to X$, $\beta \colon [0, B] \to X$ timelike geodesics of arbitrary time orientation with $\alpha(0) = \beta(0) = x$, which satisfies the timelike size bounds for K. Then one can form a comparison hinge $(\bar{\alpha}, \bar{\beta})$ in $\mathbb{L}^2(K)$ and we have*

$$\tau(\alpha(A), \beta(B)) \geq \bar{\tau}(\bar{\alpha}(A), \bar{\beta}(B)) \qquad \left(\tau(\alpha(A), \beta(B)) \leq \bar{\tau}(\bar{\alpha}(A), \bar{\beta}(B))\right). \tag{66}$$

Proof. We only establish the case of K-monotonicity comparison from below, in the other case the previous Corollary 4.11 yields a reversed inequality and so gives an analogous proof.

We denote the points by x, $a = \alpha(A)$ and $b = \beta(B)$. We first assume that they are causally related, say $a \leq b$. We consider the comparison hinge $(\bar{\alpha}, \bar{\beta})$ in $\mathbb{L}^2(K)$, and consider it as a triangle $\bar{\Delta} = \Delta\bar{x}\bar{a}\bar{b}$, where $\bar{x} := \bar{\alpha}(0) = \bar{\beta}(0)$, $\bar{a} := \bar{\alpha}(A)$ and $\bar{b} := \bar{\beta}(B)$. We assume $\bar{a} \leq \bar{b}$. Furthermore, we consider a comparison triangle $\bar{\Delta}' = \Delta\bar{x}'\bar{a}'\bar{b}'$ for Δxab. These two triangles have two agreeing side-lengths, i.e.,

$$\bar{\tau}(\bar{x}', \bar{a}') = \tau(x, a) = \bar{\tau}(\bar{x}, \bar{a}) \text{ (or with arguments flipped if } \alpha \text{ is past directed)},$$
$$\bar{\tau}(\bar{x}', \bar{b}') = \tau(x, b) = \bar{\tau}(\bar{x}, \bar{b}) \text{ (or with arguments flipped if } \beta \text{ is past directed)}.$$

For the angles we have

$$\measuredangle_{\bar{x}'}^{\mathbb{L}^2(K)}(\bar{a}', \bar{b}') = \tilde{\measuredangle}_x^K(a, b),$$
$$\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a}, \bar{b}) = \measuredangle_x(\alpha, \beta).$$

By the previous Corollary 4.11, we know that $\measuredangle_x^{\mathrm{S}}(\alpha,\beta) \leq \tilde{\measuredangle}_x^{K,\mathrm{S}}(a,b)$.

Now we distinguish the cases where the angle is at. If $\sigma = 1$, we have that $\measuredangle_{\bar{x}'}^{\mathbb{L}^2(K)}(\bar{a}',\bar{b}')$ is at least as big as $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a},\bar{b})$, and the monotonicity statement of the law of cosines (Remark 2.5, varying the longest side) gives that

$$\bar{\tau}(\bar{a},\bar{b}) \leq \bar{\tau}(\bar{a}',\bar{b}') = \tau(a,b)\,. \tag{67}$$

If $\sigma = -1$, we have that $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a},\bar{b})$ is at least as big as $\measuredangle_{\bar{x}'}^{\mathbb{L}^2(K)}(\bar{a}',\bar{b}')$, and the monotonicity statement of the law of cosines (Remark 2.5, varying a non-longest side) gives that

$$\bar{\tau}(\bar{a},\bar{b}) \leq \bar{\tau}(\bar{a}',\bar{b}') = \tau(a,b)\,. \tag{68}$$

Now for the case that a and b are not causally related (thus $\sigma = -1$). In case of monotonicity comparison from above there is nothing to do, so let X satisfy monotonicity comparison from below. We need to establish that $\bar{\tau}(\bar{a},\bar{b}) = 0$. Let $\varepsilon > 0$ small enough, then $s_\varepsilon = \sup\{s \in [0,A] : \tau(\alpha(s),b) > \varepsilon\} < A$, so $a_\varepsilon := \alpha(s_\varepsilon)$ is the last point on α with $\tau(a_\varepsilon,b) = \varepsilon$. Then as above we get a point $\bar{a}_\varepsilon$ in the comparison hinge, and the above argument yields that $\varepsilon = \tau(a_\varepsilon,b) \geq \bar{\tau}(\bar{a}_\varepsilon,\bar{b}) \geq \bar{\tau}(\bar{a},\bar{b})$. Now let $\varepsilon \to 0$ to get $\bar{\tau}(\bar{a},\bar{b}) = 0$.

Now for the case that $\bar{a}$ and $\bar{b}$ are not causally related (thus $\sigma = -1$). In case of monotonicity comparison from below there is nothing to do, so let X satisfy monotonicity comparison from above. We need to establish that a,b are not timelike related, so indirectly assume $a \ll b$. We have two triangles with two equal side-lengths: $\bar{\Delta}$ with a non-timelike side and $\bar{\Delta}'$ which is timelike. Now the monotonicity statement of the extended law of cosines (Corollary 2.6) yields that the angle in the non-causal triangle is bigger, i.e., $\measuredangle_{\bar{x}'}^{\mathbb{L}^2(K)}(\bar{a}',\bar{b}') < \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a},\bar{b})$. But by monotonicity comparison from above we get that $\measuredangle_x(\alpha,\beta) \leq \tilde{\measuredangle}_x^{K}(a,b)$, i.e., $\measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{a},\bar{b}) \leq \measuredangle_{\bar{x}'}^{\mathbb{L}^2(K)}(\bar{a}',\bar{b}')$, a contradiction. $\square$

We are now in a position to prove the main result of this final section. We establish that timelike curvature bounds via triangle comparison (Definition 1.20) are equivalent to monotonicity comparison (Definition 4.9).

Theorem 4.13 (Equivalence of triangle and monotonicity comparison)**.** *Let $(X,d,\ll,\leq,\tau)$ be a locally strictly timelike geodesically connected Lorentzian pre-length space and let $K \in \mathbb{R}$. Then X has timelike curvature bounded below (above) by K if and only if it satisfies K-monotonicity comparison from below (above).*

Proof: The first two conditions to be satisfied in timelike triangle comparison and K-monotonicity comparison are the same under the condition of local timelike geodesic connectedness when shrinking U to be a strictly timelike geodesically connected neighborhood. Moreover, let U be a comparison neighborhood given either by the timelike triangle comparison or by K-monotonicity comparison.

First, let X have timelike curvature bounded from below (above) by K and let $\Delta = \Delta xyz$ be a timelike geodesic triangle in U that satisfies timelike size bounds for K. We only prove the case where the vertex at which we consider the angle at is x, and we will point out where the other cases are not analogous. Let $z' \in [x,z]$, $y' \in [x,y]$ with $y' \neq x \neq z'$. We assume $y' < z'$, the case $z' < y'$ is analogous. Moreover, let $\bar{\Delta} = \Delta\bar{x}\bar{y}\bar{z}$ be a comparison triangle of Δ in $\mathbb{L}^2(K)$ and let $\bar{y}', \bar{z}'$ be the corresponding points of y', z' assuming first that $\bar{y}' \leq \bar{z}'$. Then by the timelike curvature bound we get that $\tau(y',z') \leq \bar{\tau}(\bar{y}',\bar{z}')$ (or $\tau(y',z') \geq \bar{\tau}(\bar{y}',\bar{z}')$). Set $\alpha := \tilde{\measuredangle}_x^K(y,z) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y},\bar{z}) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}',\bar{z}')$. Let $(\bar{x},\bar{y}'',\bar{z}'')$ be a comparison triangle of $\Delta xy'z'$, see Figure 15. If indirectly $\tilde{\measuredangle}_x^K(y',z') < \alpha$ for $\sigma = -1$ or $\tilde{\measuredangle}_x^K(y',z') > \alpha$ for $\sigma = 1$, the monotonicity statement of the law of cosines (Remark 2.5) implies that $\tau(y',z') = \bar{\tau}(\bar{y}'',\bar{z}'') > \tau(\bar{y}',\bar{z}')$ — a contradiction to $\tau(y',z') \leq \bar{\tau}(\bar{y}',\bar{z}')$. Analogously for the case of timelike curvature bounded above.

However, in case of timelike curvature bounded above there is an issue if $\bar{y}'$ is not causally related to $\bar{z}'$, so $\sigma = -1$. Then the monotonicity statement of the extended law of cosines (Corollary 2.6) gives $\alpha = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y},\bar{z}) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}',\bar{z}') > \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}'',\bar{z}'')$, which is what we wanted to prove.

In total, X satisfies K-monotonicity comparison from below (or above, respectively).

Second, let X satisfy K-monotonicity comparison from below (above), let $\Delta = \Delta xyz$ be a timelike geodesic triangle in U satisfying timelike size bounds for K and let $\bar{\Delta} = \Delta\bar{x}\bar{y}\bar{z}$ be a comparison triangle of Δ in $\mathbb{L}^2(K)$. Let $p \in [x,y]$, $q \in [x,z]$, the other cases work nearly analogously by considering the angles at the other vertices. Moreover, if $p = x$ or $q = x$ there is nothing to do so let $p \neq x \neq q$. First we assume that p and q are causally related, say $p < q$. Let $\bar{p}, \bar{q}$ be corresponding points of p, q on $\bar{\Delta}$, and we assume $\bar{p} \leq \bar{q}$. By assumption we have $\tilde{\measuredangle}_x^{K,\mathrm{S}}(p,q) \leq \tilde{\measuredangle}_x^{K,\mathrm{S}}(y,z)$ (or $\tilde{\measuredangle}_x^{K,\mathrm{S}}(p,q) \geq \tilde{\measuredangle}_x^{K,\mathrm{S}}(y,z)$). Let $(\bar{x},\bar{p}',\bar{q}')$ be a comparison triangle of Δxpq. Note that $\tilde{\measuredangle}_x^K(p,q) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p}',\bar{q}')$ and $\tilde{\measuredangle}_x^K(y,z) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p},\bar{q})$. Then the law of cosines (Remark 2.5) implies that $\tau(p,q) = \bar{\tau}(\bar{p}',\bar{q}') \leq \bar{\tau}(\bar{p},\bar{q})$ (or $\tau(p,q) \geq \bar{\tau}(\bar{p},\bar{q})$, respectively).

Now we consider the case where $\bar{p}$ and $\bar{q}$ not causally related, which can only happen if $\sigma = -1$. We need to exclude $p \ll q$, so let indirectly $p \ll q$. For the timelike curvature bounded above case, we need $\tau(p,q) \geq \bar{\tau}(\bar{p},\bar{q}) = 0$ which is automatically satisfied. For the timelike curvature bounded below case, we need $\tau(p,q) \leq \bar{\tau}(\bar{p},\bar{q}) = 0$, i.e. we need to exclude this case. We compare the triangles $\Delta\bar{x}\bar{p}\bar{q}$ and $\Delta\bar{x}\bar{p}'\bar{q}'$. Two of the sides are equal, and in the first triangle the other side is spacelike. Thus the monotonicity statement of the extended law of cosines (Corollary 2.6) gives us that $\tilde{\measuredangle}_x^K(y,z) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p},\bar{q}) > \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p}',\bar{q}') = \tilde{\measuredangle}_x^K(p,q)$, in contradiction to K-monotonicity comparison from below (note $\sigma = -1$).

In case p and q are not causally related ($\sigma = -1$) we cannot form timelike triangles, so the main argument does not work. For the timelike curvature bounded below case, we need $0 = \tau(p,q) \leq \bar{\tau}(\bar{p},\bar{q})$ which is automatically satisfied. For the timelike curvature bounded above case, we need $0 = \tau(p,q) \geq \bar{\tau}(\bar{p},\bar{q})$, i.e. $\bar{p} \not\ll \bar{q}$. First assume U is a causally closed neighbourhood and we indirectly assume the weaker $\bar{p} \leq \bar{q}$. We get that there exists a point $\hat{p} \in [x,y]$ with $\hat{p} \leq q$ null related. We compare the triangles $\Delta(x,\hat{p},q)$ and $\Delta(\bar{x},\bar{p},\bar{q})$: we have $\tau(x,\hat{p}) < \tau(x,p)$, $\tau(\hat{p},q) = 0 \leq \bar{\tau}(\bar{p},\bar{q})$ and the longest side-length is equal. Thus the strict law of cosines monotonicity (Remark 2.5) gives that $\tilde{\measuredangle}_x^K(\hat{p},q) > \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p},\bar{q})$, which contradicts the inequality given by K-monotonicity comparison from above, i.e., $\tilde{\measuredangle}_x^K(\hat{p},q) \leq \tilde{\measuredangle}_x^K(p,q) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{p},\bar{q})$. Without U being a causally closed neighbourhood, one does not necessarily get a null related point $\hat{p}$ but a sequence $\hat{p}_n$ with $\tau(\hat{p}_n,q) \searrow 0$. Indirectly assuming $\bar{p} \ll \bar{q}$, a limiting argument gives the claim.

In total, X has timelike curvature bounded below (above) by K. $\square$

We now give a proof of the fact that signed angles are semi-continuous, given a curvature bound by using the equivalence of timelike curvature bounds and K-monotonicity comparison established in Theorem 4.13. In particular, this implies the case of geodesics with the same time orientation, i.e., Proposition 3.11. In the metric setting the situation is of course easier as one does not have different time orientations and thus there is no need for signed angles, cf. [BBI01, Thm. 4.3.11] and [BH99, Prop. II.3.3].

Proposition 4.14 (Semi-continuity of angles)**.** *Let $(X,d,\ll,\leq,\tau)$ be a locally strictly timelike geodesically connected Lorentzian pre-length space with timelike curvature bounded below (above). Then signed angles of timelike geodesics are upper (lower) semicontinuous, i.e., $\alpha,\alpha_n,\beta,\beta_n\colon [0,B] \to X$ future*

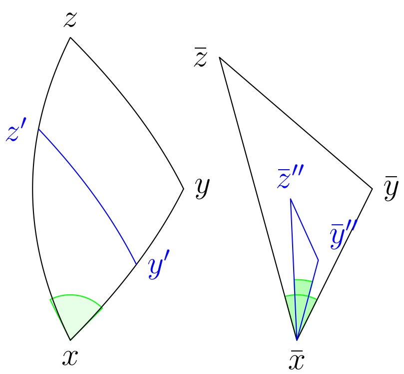


Figure 15: When, comparing with $\mathbb{L}^2(K)$, the interior side is shorter than expected, the K-angle has to behave accordingly.

or past directed timelike geodesics at x with $\alpha_n \to \alpha$, $\beta_n \to \beta$ pointwise, then

$$\measuredangle_x^{\mathrm{S}}(\alpha,\beta) \geq \limsup_n \measuredangle_x^{\mathrm{S}}(\alpha_n,\beta_n) \qquad (\measuredangle_x^{\mathrm{S}}(\alpha,\beta) \leq \liminf_n \measuredangle_x^{\mathrm{S}}(\alpha_n,\beta_n))\,. \tag{69}$$

Proof: First, we use the fact that the timelike curvature bound implies the corresponding K-monotonicity comparison by Theorem 4.13. We will prove the case of timelike curvature bounded below by $K \in \mathbb{R}$ — the other case is completely analogous.

For $s,t > 0$ such that $\alpha_n(s)$ and $\beta_n(t)$ are causally related set $\theta_n(s,t) := \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha_n(s),\beta_n(t))$ and analogously for $s,t > 0$ such that $\alpha(s)$ and $\beta(t)$ are causally related we define $\theta(s,t) := \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s),\beta(t))$. As the side lengths of the (comparison) triangles converge we get $\theta_n \to \theta$ pointwise if both are defined.

Without loss of generality we assume that β is future directed. To see that the comparison angles of the approximating sequence exist pick sequences $s_k, t_k \searrow 0$ with $\alpha(s_k) \ll \beta(t_k)$. By the openness of the timelike relation $\ll$ (cf. [KS18, Prop. 2.13]) we get that $\alpha_n(s_k) \ll \beta_n(t_k)$ for large enough $n \in \mathbb{N}$ (depending on k), so both θ and θ_n are defined at (s_k,t_k).

Thus Corollary 4.11 gives for n,k large enough that

$$\measuredangle_x^{\mathrm{S}}(\alpha_n,\beta_n) \leq \theta_n(s_k,t_k)\,. \tag{70}$$

Letting $n \to \infty$ gives

$$\limsup_{n\to\infty} \measuredangle_x^{\mathrm{S}}(\alpha_n,\beta_n) \leq \theta(s_k,t_k)\,, \tag{71}$$

and taking $k \to \infty$ then yields $\limsup_{n\to\infty} \measuredangle_x^{\mathrm{S}}(\alpha_n, \beta_n) \leq \measuredangle_x^{\mathrm{S}}(\alpha, \beta)$, as required.

$\square$

Under weak additional assumptions we even get continuity of angles.

Proposition 4.15 (Continuity of angles)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally strictly timelike geodesically connected and locally causally closed Lorentzian pre-length space. Then angles are continuous between geodesics in the following cases. Let $x \in X$ and $\alpha, \alpha_n, \beta, \beta_n \colon [0, B] \to X$ future or past directed timelike geodesics starting at x with $\alpha_n \to \alpha$, $\beta_n \to \beta$ pointwise (in particular, α_n is future directed if and only if α is and similarly for β).*

1. *The space X has timelike curvature bounded above.*
   - *(a) The geodesics α, β have the same time orientation or*
   - *(b) if α, β have different time orientations we assume that X is strongly causal and the angles $\measuredangle_x(\alpha, \beta)$, $\measuredangle_x(\alpha, \beta_n)$ and $\measuredangle_x(\alpha_n, \beta_n)$ are finite for all $n \in \mathbb{N}$.*
2. *The space X has timelike curvature bounded below, is strongly causal and satisfies geodesic prolongation.*

Then

$$\measuredangle_x(\alpha, \beta) = \lim_n \measuredangle_x(\alpha_n, \beta_n) \,. \tag{72}$$

Proof: In both cases the timelike curvature bound implies K-monotonicity comparison by Theorem 4.13 and all geodesic segments are assumed to lie in a K-monotonicity comparison neighborhood.

1. Let X have timelike curvature bounded above. By Proposition 4.14 we have that $\measuredangle_x(\alpha_n, \alpha) \to 0$ and $\measuredangle_x(\beta_n, \beta) \to 0$ as $\measuredangle_x(\alpha, \alpha) = \measuredangle_x(\beta, \beta) = 0$ by Lemma 3.4,1 and the fact that α_n and α have the same time orientation, as well as β_n and β.

   At this point we use the triangle inequality of angles. In case (a) we use Theorem 3.1 and in case (b) we use Theorem 4.5.2. In case (b) we have by assumption that the angles $\measuredangle_x(\alpha, \beta)$, $\measuredangle_x(\alpha, \beta_n)$ and $\measuredangle_x(\alpha_n, \beta_n)$ are finite and by Lemma 4.10 these angles exist for all $n \in \mathbb{N}$. Thus in both cases we estimate

$$\measuredangle_x(\alpha_n, \beta_n) \leq \measuredangle_x(\alpha_n, \alpha) + \measuredangle_x(\alpha, \beta) + \measuredangle_x(\beta, \beta_n) \,, \tag{73}$$
$$\measuredangle_x(\alpha, \beta) \leq \measuredangle_x(\alpha, \alpha_n) + \measuredangle_x(\alpha_n, \beta_n) + \measuredangle_x(\beta_n, \beta) \,. \tag{74}$$

Consequently, in both cases we obtain

$$|\measuredangle_x(\alpha_n, \beta_n) - \measuredangle_x(\alpha, \beta)| \leq \measuredangle_x(\alpha, \alpha_n) + \measuredangle_x(\beta, \beta_n) \to 0\,, \tag{75}$$

as claimed.

2. Let X have timelike curvature bounded below. By geodesic prolongation, there is a timelike geodesic $\check{\alpha}\colon [0, \varepsilon) \to X$ having the opposite time orientation as α and such that the concatenation of the time reversed $\check{\alpha}$ and α is a timelike geodesic. In particular, $\measuredangle_x(\check{\alpha}, \alpha) = 0$. Similarly, we find $\check{\beta}\colon (-\varepsilon, 0] \to X$ with $\measuredangle_x(\check{\beta}, \beta) = 0$. By Proposition 4.14 we have that $\measuredangle_x(\alpha_n, \check{\alpha}) \to 0$ and $\measuredangle_x(\beta_n, \check{\beta}) \to 0$.

   At this point we use the triangle inequality of angles (Theorem 4.5). Note that angles between geodesics of different time orientation always exist by Lemma 4.10. Thus we estimate

$$\measuredangle_x(\alpha_n, \beta_n) \leq \measuredangle_x(\alpha_n, \check{\alpha}) + \measuredangle_x(\check{\alpha}, \check{\beta}) + \measuredangle_x(\check{\beta}, \beta_n)\,, \tag{76}$$

$$\measuredangle_x(\check{\alpha}, \check{\beta}) \leq \measuredangle_x(\check{\alpha}, \alpha_n) + \measuredangle_x(\alpha_n, \beta_n) + \measuredangle_x(\beta_n, \check{\beta})\,. \tag{77}$$

   Consequently, in both cases we obtain

$$|\measuredangle_x(\alpha_n, \beta_n) - \measuredangle_x(\check{\alpha}, \check{\beta})| \leq \measuredangle_x(\check{\alpha}, \alpha_n) + \measuredangle_x(\check{\beta}, \beta_n) \to 0\,. \tag{78}$$

   Finally, we apply Corollary 4.7 twice to obtain $\measuredangle_x(\check{\alpha}, \check{\beta}) = \measuredangle_x(\alpha, \beta)$, which gives the claim.

□

Conclusion

In this article we introduced hyperbolic angles into the setting of Lorentzian length spaces and characterized timelike curvature bounds by an angle monotonicity property. We also introduced a diversity of metric tools, and thereby furthering the development of the theory considerably. Moreover, these tools and methods have further applications in Lorentzian geometry and General Relativity: The most striking example is the recently established *splitting theorem for Lorentzian length spaces with non-negative curvature* in [BORS23]. The splitting theorem can be understood as a rigidity statement of the classical Hawking-Penrose singularity theorem [HP70] and as such is of fundamental importance in Einstein's theory of gravity.

Finally, let us point out several directions of future research. First, hyperbolic angles could be used to define coordinates in Lorentzian (pre-) length spaces with timelike curvature bounded below or above. Second, a natural continuation is to establish the variation formula and an angle comparison condition as in [BMS22] also for curvature bounded from above (and for curvature bounded from below by a non-zero $K \in \mathbb{R}$ for the former). Further, we expect angles to play a significant role in globalization theorems for curvature bounds and in a Lorentzian version of billiards (cf. [BFK98]) with possible applications to particle collisions.

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A.1 Law of Cosines

Proof. (of Lemma 2.4 and Remark 2.5) We first establish the case where the triangle is timelike, i.e., $a, b, c > 0$. For $K = 0$, we are in Minkowski space and the sides of this triangle are just straight lines. Equation (2.1) of [AB08], gives us $-c^2 = -a^2 - b^2 - 2ab\sigma \cosh(\omega)$ by noting that their c is $-c^2$ here etc., which is the desired formula.

If $K \neq 0$, the sides of the triangle are future or past directed timelike geodesics $\gamma_{pq}, \gamma_{qr}, \gamma_{pr}$, which we assume to be defined on the domain $[0, 1]$. The energy $E(\gamma_{p_i p_j})$ is given by $-\tau(p_i, p_j)^2$ if $p_i \ll p_j$. Now we apply [Kir18, Thm. 3.1.3] to get

$$
\begin{aligned}
\cos(\sqrt{KE(\gamma_{pr})}) = \cos(\sqrt{KE(\gamma_{pq})}) \cos(\sqrt{KE(\gamma_{qr})}) \\
+ \langle \gamma_{qp}'(0), \gamma_{qr}'(0) \rangle \frac{\sin(\sqrt{KE(\gamma_{pq})})}{\sqrt{E(\gamma_{pq})}} \frac{\sin(\sqrt{KE(\gamma_{qr})})}{\sqrt{E(\gamma_{qr})}}.
\end{aligned}
$$

Note that this uses imaginary numbers, and $\cos(ix) = \cosh(x)$, $\sin(ix) = i\sinh(x)$.
For $K < 0$, this equation reads

$$\cos(sc) = \cos(sa)\cos(sc) + \sigma ab\cosh(\omega)\frac{\sin(sa)}{ia}\frac{\sin(sb)}{ib}\,, \tag{79}$$

$$\cos(sc) = \cos(sa)\cos(sb) - \sigma\cosh(\omega)\sin(sa)\sin(sb)\,, \tag{80}$$

where σ is the sign of the inner product (which is as described in the statement) and the minus comes from an i^2 in the denominator.
For $K > 0$, this equation reads

$$\cos(isc) = \cos(isa)\cos(isb) + \sigma ab\cosh(\omega)\frac{\sin(isa)}{ia}\frac{\sin(isb)}{ib}\,, \tag{81}$$

$$\cosh(sc) = \cosh(sa)\cosh(sb) + \sigma\cosh(\omega)\sinh(sa)\sinh(sb)\,, \tag{82}$$

where the is in the denominator get canceled by the is appearing due to $\sin(ix) = i\sinh(x)$.

At this point, we establish the case where $c = 0$, i.e., $\sigma = -1$ and without loss of generality $q \ll p \leq r$ with p, r null related. We consider the timelike geodesic α from q to p. For each $\varepsilon > 0$ we consider the point $p_\varepsilon := \alpha(1-\varepsilon)$, then $\tau(p_\varepsilon, r) > 0$ and $\tau(p_\varepsilon, r) \to 0$ as $\varepsilon \to 0$. We also have $\measuredangle_q^{\mathbb{L}^2(K)}(p_\varepsilon, r) = \measuredangle_q^{\mathbb{L}^2(K)}(p, r)$. We now apply the timelike case above to the timelike triangle $q \ll p_\varepsilon \ll r$. We note that the side-lengths converge as follows: $a_\varepsilon = \tau(q, p_\varepsilon) \to a = \tau(q, p)$, $b = \tau(q, r)$ is constant and $c_\varepsilon = \tau(p_\varepsilon, r) \to 0$. As the term containing ω stays finite, the law of cosines continues to hold in the limit $\varepsilon \to 0$, giving the claimed equality.

For the monotonicities, we without loss of generality assume that $K \in \{-1, 0, 1\}$. We look at each side varying separately. In each case, we say that ω and one of the side-lengths depend on some additional variable t (i.e., we insert $\omega(t)$ and $a(t)$ or $b(t)$ or $c(t)$, respectively, into the law of cosines equations) and take the derivative of the law of cosines equations. Then $\operatorname{sgn}(a'(t)) = \operatorname{sgn}(\omega'(t))$ (or $\operatorname{sgn}(c'(t))$) makes ω monotonically increasing in a (or c) and $\operatorname{sgn}(a'(t)) = -\operatorname{sgn}(\omega'(t))$ monotonically decreasing.
For $K = 0$ and varying a, we get the equation

$$a'(2a + 2b\sigma\cosh(\omega)) = \omega'(-2ab\sigma\sinh(\omega))\,.$$

If c is the longest side, $\sigma = 1$ and the signs of the factors are clear, we get $\operatorname{sgn}(a') = -\operatorname{sgn}(\omega')$. Otherwise $\sigma = -1$. If b is the longest side, it is clear that $2b\cosh(\omega) \geq 2a$ and so $\operatorname{sgn}(a') = \operatorname{sgn}(\omega')$. If a is the longest side, we have to be more careful. We use the law of cosines formula and transform it to

$$\begin{aligned} a^2+b^2 &= c^2+2ab\cosh(\omega)\,, \\ 2a^2-2ab\cosh(\omega) = 2a(a-b\cosh(\omega)) &= c^2-b^2+a^2 \geq 0\,, \end{aligned}$$

thus the positive $2a$ term dominates over the negative $2b\sigma\cosh(\omega)$ and so $\operatorname{sgn}(a') = \operatorname{sgn}(\omega')$.

For $K=-1$, we get the equation

$$\begin{aligned} & a'(-\sinh(a)\cosh(b) - \sigma\cosh(\omega)\cosh(a)\sinh(b)) && (83) \\ &\quad = \omega'(\sigma\sinh(\omega)\sinh(a)\sinh(b))\,. && (84) \end{aligned}$$

If c is the longest side, $\sigma = 1$ and the signs of the factors are clear, giving $\operatorname{sgn}(a') = -\operatorname{sgn}(\omega')$. Otherwise $\sigma = -1$. If b is the longest side, $\cosh(a)\sinh(b) - \sinh(a)\cosh(b) = \sinh(b-a) \geq 0$, thus $\operatorname{sgn}(a') = \operatorname{sgn}(\omega')$. If a is the longest side, we have to be more careful. We multiply the law of cosines formula by $\cosh(a)$ and transform it into

$$\begin{aligned} \cosh(c) &= \cosh(a)\cosh(b) + \cosh(\omega)\sinh(a)\sinh(b)\,, \\ \cosh(a)\cosh(c) &= \cosh(a)^2\cosh(b) \\ &\quad + \cosh(\omega)\sinh(a)\cosh(a)\sinh(b)\,, \\ \cosh(a)\cosh(c) &= (1+\sinh(a)^2)\cosh(b) \\ &\quad + \cosh(\omega)\sinh(a)\cosh(a)\sinh(b)\,, \\ \cosh(a)\cosh(c) - \cosh(b) &= \sinh(a)(\sinh(a)\cosh(b) \\ &\quad + \cosh(\omega)\cosh(a)\sinh(b))\,. \end{aligned}$$

So the sign of the factor of a' is minus the sign of the left-hand-side here (as $\sinh(a) > 0$). Now we use the reverse triangle inequality $b \leq a - c$ and monotonicity of cosh to get $\cosh(a)\cosh(c) - \cosh(b) \geq \cosh(a)\cosh(c) - \cosh(a-c) = \sinh(a)\sinh(c) \geq 0$, thus the factor of a' is positive and $\operatorname{sgn}(a') = \operatorname{sgn}(\omega')$.

For $K=1$, we get the equation

$$a'(\sin(a)\cos(b) + \sigma\cosh(\omega)\cos(a)\sin(b)) = \omega'(-\sigma\sinh(\omega)\sin(a)\sin(b))\,.$$

If c is the longest side, $\sigma = 1$ and the signs of the factors are clear, $\operatorname{sgn}(a') = -\operatorname{sgn}(\omega')$. Otherwise $\sigma = -1$. If b is the longest side, $\cos(a)\sin(b) - \sin(a)\cos(b) = \sin(b-a) \geq 0$, thus $\operatorname{sgn}(a') = \operatorname{sgn}(\omega')$. If a is the longest side, we have to be more careful. We multiply the law of cosines formula by

$\cos(a)$ and transform it into

$$\begin{aligned}
\cos(c) &= \cos(a)\cos(b) - \cosh(\omega)\sin(a)\sin(b)\,,\\
\cos(a)\cos(c) &= \cos(a)^2\cos(b) - \cosh(\omega)\sin(a)\cos(a)\sin(b)\,,\\
\cos(a)\cos(c) &= (1-\sin(a)^2)\cos(b) - \cosh(\omega)\sin(a)\cos(a)\sin(b)\,,\\
\cos(a)\cos(c) - \cos(b) &= -\sin(a)(\sin(a)\cos(b) + \cosh(\omega)\cos(a)\sin(b))\,.
\end{aligned}$$

So the sign of the factor of a' is minus the sign of the left-hand-side here ($0 < a < \pi$, so $-\sin(a) < 0$). Now we use the reverse triangle inequality $b \leq a-c$ and monotonicity of cos to get $\cos(a)\cos(c)-\cos(b) \leq \cos(a)\cos(c)-\cos(a-c) = -\sin(a)\sin(c) \leq 0$, thus the factor of a' is positive and $\mathrm{sgn}(a') = \mathrm{sgn}(\omega')$. □

Proof. (of Lemma 2.8) The analyticity and order of convergence statements directly follow from inserting the power series of cos, sin, cosh and sinh.

For (1), we consider the triangles $\Delta p_1p_2p_3$ and Δp_1p_2q: They share one side-length and the angle $\omega = \measuredangle_{p_2}^{\mathbb{L}^2(K)}(p_1,p_3) = \measuredangle_{p_2}^{\mathbb{L}^2(K)}(p_1,q)$. We only do the $K=0$ case since the other cases work analogously. By the law of cosines 2.4, we get the two equations

$$\begin{aligned}
a^2 + (b+c)^2 &= d^2 - 2a(b+c)\cosh(\omega)\,,\\
a^2 + b^2 &= x^2 - 2ab\cosh(\omega)\,.
\end{aligned}$$

Canceling ω and solving for x, we get

$$x^2 = \frac{bd^2 + a^2c}{b+c} - bc = \frac{a^2+d^2}{2} + \frac{(a^2-d^2)(c-b)}{2(b+c)} - bc\,.$$

Note (2) is just the time-reversed statement of (1).

For (3), we have two cases: $q \ll p_2$ and $p_2 \ll q$. We consider the triangles $\Delta p_1p_2p_3$ and Δp_1qp_2: They share one side-length and the angle $\omega = \measuredangle_{p_1}^{\mathbb{L}^2(K)}(p_2,p_3) = \measuredangle_{p_1}^{\mathbb{L}^2(K)}(q,p_2)$ agrees by the law of cosines, which does not distinguish the two cases. First, we establish the $K=0$ case. By the law of cosines 2.4, we get two equations

$$\begin{aligned}
a^2 + (c+d)^2 &= b^2 + 2a(c+d)\cosh(\omega)\,,\\
a^2 + c^2 &= x^2 + 2ac\cosh(\omega)\,.
\end{aligned}$$

Canceling ω and solving for x, we get

$$x^2 = \frac{b^2c + a^2d}{c+d} - cd\,.$$

Now for the case $K > 0$. By the law of cosines 2.4, we get the two equations (where $s = \sqrt{|K|}$)

$$\begin{aligned}
\cosh(sb) &= \cosh(sa)\cosh(s(c+d)) + \sinh(sa)\sinh(s(c+d))\cosh(\omega)\,,\\
\cosh(sx) &= \cosh(sa)\cosh(sc) + \sinh(sa)\sinh(sc)\cosh(\omega)\,.
\end{aligned}$$

Canceling ω and solving for x, we get

$$\cosh(sx) = \frac{\cosh(sb)\sinh(sc) - \cosh(sa)\cosh(s(c+d))\sinh(sc)}{\sinh(s(c+d))} \tag{85}$$

$$+ \cosh(sa)\cosh(sc)\,. \tag{86}$$

Now for the case $K < 0$. By the law of cosines 2.4, we get the two equations (where $s = \sqrt{|K|}$)

$$\begin{aligned}
\cos(sb) &= \cos(sa)\cos(s(c+d)) + \sin(sa)\sin(s(c+d))\cosh(\omega)\,,\\
\cos(sx) &= \cos(sa)\cos(sc) + \sin(sa)\sin(sc)\cosh(\omega)\,.
\end{aligned}$$

Canceling ω and solving for x, we get

$$\cos(sx) = \frac{\cos(sb)\sin(sc) - \cos(sa)\cos(s(c+d))\sin(sc)}{\sin(s(c+d))} + \cos(sa)\cos(sc)\,.$$

$\square$

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Alexandrov's Patchwork and the Bonnet–Myers Theorem for Lorentzian length spaces

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Abstract

We present key initial results in the study of global timelike curvature bounds within the Lorentzian pre-length space framework. Most notably, we construct a Lorentzian analogue to Alexandrov's Patchwork, thus proving that suitably nice Lorentzian pre-length spaces with local upper timelike curvature bound also satisfy a corresponding global upper bound. Additionally, for spaces with global lower bound on their timelike curvature, we provide a Bonnet–Myers style result, constraining their finite diameter. Throughout, we make the natural comparisons to the metric case, concluding with a discussion of potential applications and ongoing work.

Keywords: Lorentzian length spaces, synthetic curvature bounds, globalization, triangle comparison, metric geometry, Lorentzian geometry, hyperbolic angles

MSC2020: 53C50, 53C23, 53B30, 51K10

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1 Introduction

By utilising the theory of metric length spaces, the scope of many results in differential geometry can be extended beyond the setting of smooth manifolds. In particular, metric length spaces are a key tool in the abstraction of the fundamental properties of Riemannian manifolds, to structures of lower regularity [BBI01, BH99]. In this so-called synthetic approach, curvature bounds (locally/in the small) are constructed via the comparison properties of geodesic triangles and are used to tame some of the more pathological behaviour of such length spaces. A semi-Riemannian extension of these comparison methods has also been developed in [AB08, Har82].

The properties that arise from supplementing length spaces with global curvature bounds (producing spaces of Alexandrov or CAT(k) type) have also been significant for the application of metric length spaces to problems in a variety of fields, from dynamical systems to group theory. An illustrative result in the case of curvature bounded below is given by the stability, under Gromov–Hausdorff limits, of global lower curvature bounds on metric spaces [BGP92, Kap02]. We can see the impact of spaces with global upper curvature bounds by looking to algebraic topology: the fundamental group of any complete metric space with curvature bounded above by zero has no non-trivial finite subgroups [BBI01, Corollary 9.3.2].

As such, a salient question asks under which conditions can curvature bounds, which are imposed locally, be extended to hold in the large? For spaces with curvature bounded above, Alexandrov [Ale57] demonstrated that, provided we also have unique geodesics which vary continuously with their endpoints, local CAT(k) spaces are CAT(k). Extending the work of Toponogov [Top59] on Riemannian manifolds, it was shown by Perelman [BGP92] that for curvature bounded below, complete length spaces are sufficient. In fact, a generalization of the Bonnet–Myers Theorem [BBI01] follows as a natural corollary of the aforementioned result, bounding the diameter of complete length spaces with local curvature bounded below. A further globalization result appears in [Pet16], where the author treats completions of geodesic spaces with curvature bounded below.

Analysis of low regularity Lorentzian metrics has become increasingly pertinent in the study of general relativity and physically relevant space-times, which may feature cosmic strings and gravitational waves, see for example [Ren05, Vic90, PSSu16]. Hence, the introduction of the Lorentzian pre-length space [KS18] as an extension of the basic objects described by causal spaces [KP67], has led to the rapid development of a synthetic Lorentzian framework, mirroring the growth of the theory of metric length spaces several decades ago [AGKS21, GKS19, KS22]. In particular, [KS18] develop comparison methods for synthetic Lorentzian geometry, in order to impose curvature bounds on Lorentzian pre-length spaces. Equivalent approaches have also been proposed by [BS22, BMS22], however none of these works relate curvature bounds enforced globally to those imposed on neighbourhoods. Furthermore, while the development of metric length spaces was guided by disciplines such as group theory and the study of partial differential equations, alongside its purely geometric origin, research into Lorentzian pre-length spaces has, for the most part, focused on their apparent necessity in general relativity.

This paper, continuing from the work of [BR22, Rot22, BS22, 1], aims to develop suitable Lorentzian analogues to some of the fundamental results of metric length spaces, in order to facilitate the wider application of the Lorentzian length space framework. In particular, given their myriad of applications in the metric setting, we focus on the notion of spaces with global timelike curvature bounds.

An outline of the paper is as follows. We begin in Section 2 by re-iterating some basic definitions regarding Lorentzian pre-length spaces, τ-length, and the causal ladder. We also introduce the notion of a regular Lorentzian pre-length space, the technique of triangle comparison, and both local and global timelike curvature bounds. Similar definitions in the context of metric spaces are also provided for convenience and comparison. In Section 3, we provide a summary of some globalization results from the metric setting that we wish to mirror with our 'Lorentzified' constructions. In particular, we give explicit statements of the so-called Alexandrov's Patchwork, Toponogov's Theorem, and the Bonnet–Myers Theorem. The main results of this paper are proven within Section 4, which is split into two parts. The first part concerns globalization of upper timelike curvature bounds via gluing of timelike triangles and culminates in the following theorem:

Theorem 4.6 (Alexandrov's Patchwork Globalization, Lorentzian version)**.** *Let X be a strongly causal, non-timelike locally isolating, and regular Lorentzian pre-length space which has (local) timelike curvature bounded above by $K \in \mathbb{R}$. Suppose that X satisfies* (i) *and* (ii) *in Definition 2.7, i.e., $\tau^{-1}([0, D_K))$ is open and τ is continuous on that set, and for all $x \ll y$ in X with $\tau(x, y) < D_K$, there exists a geodesic joining them. Additionally assume that the geodesics between timelike related points with τ-distance less than D_K are unique. Let G be the geodesic map of X restricted to the set $\{(x, y, t) \in \ll \times [0, 1] \mid \tau(x, y) < D_K\} = \tau^{-1}((0, D_K)) \times [0, 1]$ and assume that G is continuous. Then X also satisfies Definition 2.7.*(iii)*, in particular it is a $(\leq K)$-comparison neighbourhood and X has global curvature bounded above by K.*

The second part concerns a result akin to the Bonnet–Myers Theorem, bounding the finite timelike diameter of Lorentzian pre-length spaces, using global lower timelike curvature bounds. More precisely:

Theorem 4.11 (Bound on the finite diameter)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic Lorentzian pre-length space which has global curvature bounded below by K. Assume $K < 0$. Assume that X possesses the following non-degeneracy condition: for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle. Then* $\mathrm{diam}_{\mathrm{fin}}(X) \leq D_K$.

We conclude the paper with a discussion of ongoing research into the globalization of timelike curvature bounded below, as well as highlighting a potential application of Lorentzian pre-length spaces and global curvature bounds to the theory of causal sets.

2 Preliminaries

In this section we collect basic results from the theory of Lorentzian length spaces that will be of use in this article. For more details, we refer the interested reader to [KS18]. We also recall the corresponding elementary concepts from metric geometry, as a showcase of the tools used in the globalization of metric curvature bounds. For details regarding their precise application, see [BH99, BBI01].

2.1 Introduction to Lorentzian pre-length spaces

Let us begin by summarising the fundamentals of the Lorentzian length space framework, pioneered by Kunzinger and Sämann in [KS18]. In particular, we present rungs from the causal ladder [ACS20, Rot22] which will be necessary in later proofs, in addition to describing the use of triangle comparison to test for curvature bounds. First, let us define a Lorentzian pre-length space:

Definition 2.1 (Lorentzian pre-length space)**.** *Let (X, d) be a metric space, $\ll, \leq$ two relations on X, and $\tau : X \times X \to [0, \infty]$ a function. The quintuple $(X, d, \ll, \leq, \tau)$ is then called a* Lorentzian pre-length space *if it satisfies the following:*

(i) $(X, \ll, \leq)$ *is a* causal space*, i.e., $\leq$ is a reflexive and transitive relation and $\ll$ is a transitive relation contained in $\leq$.*

(ii) τ *is lower semi-continuous with respect to d.*

(iii) $\tau(x, z) \geq \tau(x, y) + \tau(y, z)$ *for $x \leq y \leq z$ and $\tau(x, y) > 0 \iff x \ll y$.*

In this case, τ is called the time separation function*, with $\ll$ and $\leq$ referred to as the* timelike *and* causal relations*, respectively. All of these concepts are motivated by the corresponding notions in spacetimes.*

A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ will usually be denoted simply by X, where the latter is clear. When referring to causal (or timelike) pasts and futures, we shall use the standard notation, e.g., $I^+(x) := \{y \in X \mid x \ll y\}$ and $J^-(x) := \{y \in X \mid y \leq x\}$. In particular, for diamonds, we shall use the notation $J(x, z) := J^+(x) \cap J^-(z) = \{y \in X \mid x \leq y \leq z\}$.

Definition 2.2 (Causal and timelike curves)**.** *Let X be a Lorentzian pre-length space. A locally Lipschitz curve $\gamma : [a, b] \to X$ is called* future-directed causal *(respectively* timelike*), if $\gamma(s) \leq \gamma(t)$ (respectively $\gamma(s) \ll \gamma(t)$) for all $s < t$ in $[a, b]$. A* past-directed *curve is defined analogously with the relations in X reversed. To make our terminology less cumbersome and avoid repeated reference to time orientation, we assume causal curves are future-directed, unless it is explicitly stated otherwise.*

Definition 2.3 (τ-length and geodesics)**.** *Let $\gamma : [a, b] \to X$ be a causal curve from x to y in a Lorentzian pre-length space X.*

(i) We define its τ-length *as*

$$L_\tau(\gamma) := \inf\left\{ \sum_{i=0}^{n-1} \tau(\gamma(t_i), \gamma(t_{i+1})) \,\middle|\, a = t_0 < t_1 < \ldots < t_n = b, n \in \mathbb{N} \right\}. \tag{87}$$

(ii) *By definition we always have $L_\tau(\gamma) \leq \tau(x,y)$. In the case of equality, we call γ a* distance-realizer *or* geodesic.

(iii) *X is called* geodesic *if, for each pair of causally related points, there exists a geodesic connecting them. X is called* uniquely geodesic *if it is geodesic and the geodesic between each pair of points is unique. If a geodesic γ between $x \ll y$ is unique, or if it is not unique and a choice of geodesic has either been made or is inconsequential, then we denote the geodesic by γ_{xy}. Unless otherwise mentioned, we shall assume γ_{xy} is parameterized by $[0,1]$ and has constant speed, i.e., $\tau(\gamma(s),\gamma(t)) = \tau(x,y)|t-s|$ for all $s < t \in [0,1]$.*

The additional assumptions given above, that geodesics are parameterized by $[0,1]$ and have constant speed, pose no technical issues when dealing with geodesics (between timelike related points) which are timelike. In formulating the main results of this paper, we consider Lorentzian pre-length spaces which are well-behaved, in the sense that geodesics between timelike related points are always timelike. In order to encode this as a property of the space, we now introduce the notion of regularity.

Technically, this is very similar to the definition of a regularly localizable Lorentzian length space, cf. [KS18, Definition 3.16]. As we do not require much of the additional structure provided by Lorentzian length spaces, we prefer to formulate everything in terms of Lorentzian pre-length spaces.

Definition 2.4 (Regular Lorentzian pre-length space)**.** *A Lorentzian pre-length space X is called* regular *if for all $x, y \in X$ such that $x \ll y$ all geodesics connecting x and y are timelike.*

In general, a geodesic between timelike related points may contain a null piece, see for example the causal funnel [KS18, Example 3.19].

Before providing a definition of timelike curvature bounds, it is now necessary to introduce the notion of triangle comparison:

Definition 2.5 (Model spaces and triangle comparison)**.** *Let X be a Lorentzian pre-length space. We define the following:*

(i) *A* timelike triangle *$\Delta(x,y,z)$ in X is a collection of three timelike related points $x \ll y \ll z$ and three pairwise connecting geodesics γ_{xy}, γ_{yz} and γ_{xz} between them. To indicate a point p lies on the triangle, we write $p \in \Delta(x,y,z)$, with $p \in \gamma_{xy}$ used to specify which side.*

(ii) *By $\mathbb{L}^2(K)$ we denote the Lorentzian model space of constant sectional curvature K. That is, $\mathbb{L}^2(0)$ is the Minkowski plane and $\mathbb{L}^2(K)$, for*

$K > 0$ and $K < 0$, is an appropriately scaled version of 2-dimensional deSitter or anti-deSitter spacetime, respectively, see [KS18, Definition 4.5].

(iii) *Given a timelike triangle $\Delta(x, y, z)$ in X, we call a timelike triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{L}^2(K)$ whose sides have the same sidelengths a* comparison triangle *for $\Delta(x, y, z)$. We assume that all timelike triangles in the remainder of the paper satisfy appropriate size bounds,*[13] *unless it is explicitly stated otherwise.*

(iv) *Let $p \in \gamma_{xy}$ (analogously for γ_{xz} and γ_{yz}) be a point on some side of the triangle $\Delta(x, y, z)$ in X and let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle for $\Delta(x, y, z)$. The* comparison point *for p, in $\Delta(\bar{x}, \bar{y}, \bar{z})$, is the (unique) point $\bar{p} \in \gamma_{\bar{x}\bar{y}}$, whose τ-distance to the endpoints of $\gamma_{\bar{x}\bar{y}}$ is the same as the τ-distance from p to the respective endpoints of γ_{xy}.*

(v) *We call a timelike triangle $\Delta(x, y, z)$* non-degenerate *if the inequality $\tau(x, z) > \tau(x, y) + \tau(y, z)$ holds. In this case, any associated comparison triangle is non-degenerate in the visual sense, i.e. not just a geodesic segment.*

Before finally providing the definition of timelike curvature bounds, we raise the following technical detail. We feel that in general, but especially in the context of this paper, the original definition of timelike curvature bounds given in [KS18] should be modified slightly. More precisely, the defining properties of a comparison neighbourhood (see [KS18, Definition 4.7]) should only hold wherever τ is "not too large" for the comparison space. This is mainly as a result of exotic behaviour exhibited by anti de-Sitter space (AdS), the model space for constant negative timelike curvature. In AdS, geodesics (in the smooth sense) stop being maximizing when they exceed length π (when $K = -1$, otherwise this bound is appropriately scaled).

Visually, this can be explained as follows: place two points x and y on the "equator" of AdS such that they are not antipodal and connect them via the longer geodesic (recall that geodesics in AdS arise by intersecting the space with a plane through 0 and the two endpoints). Then we can create curves of increasing lengths by going out to infinity, say to the right of the equator. In particular, there is no longest curve joining x and y and $\tau(x, y) = \infty$. This is related to AdS not being globally hyperbolic; indeed, the maximal globally hyperbolic subset of AdS has (ordinary) diameter π. This is clearly

[13] This is an equivalent condition for the existence of a comparison triangle in a chosen model space, see Definition 2.6.

pathological behaviour exclusive to the Lorentzian case, however there is some similarity with metric model spaces of positive curvature, namely that between points which are exactly a distance π apart there exist infinitely many geodesics (compare the antipodal points on AdS with those on the sphere).

In order to provide our updated definition, we have to introduce the so-called finite diameter of a space. This is essentially the diameter, i.e., the supremum of all values of τ, but we explicitly exclude ∞ as a value because of the nature of AdS. Note that, despite the nomenclature, the finite diameter of a Lorentzian pre-length space need not be finite.

Definition 2.6. *Let X be a Lorentzian pre-length space.*

(i) The finite diameter *of X is*

$$\mathrm{diam}_{\mathrm{fin}}(X) = \sup(\{\tau(x,y) \mid x \ll y\} \setminus \{\infty\}), \tag{88}$$

i.e., the supremum of all values τ takes except ∞.

(ii) By D_K we denote the finite diameter of $\mathbb{L}^2(K)$. In particular,

$$D_K = \mathrm{diam}_{\mathrm{fin}}(\mathbb{L}^2(K)) = \begin{cases} \infty, & \text{if } K \geq 0, \\ \frac{\pi}{\sqrt{-K}}, & \text{if } K < 0. \end{cases} \tag{89}$$

Note that a triangle $\Delta(x,y,z)$ satisfies size-bounds for $\mathbb{L}^2(K)$ precisely if $\tau(x,z) < D_K$, cf. [KS18, Lemma 4.6].

Let us now make our point concrete: *all* properties of a comparison neighbourhood should respect the appropriate range of values of τ. In particular, we do not care whether τ is continuous near points separated by a distance which cannot be realized in the model space, and we should also not require such points to possess a joining geodesic. On the one hand, this refined definition is somehow more in alignment with its metric counterpart. For example, in the definition of CAT(k) spaces, cf. [BH99, Definition II.1.1], the authors explicitly only require that there exists a geodesic between points which are less than the diameter of the corresponding model space apart. On the other hand, curvature bounds should morally not be concerned with behaviour which cannot be realized in the model space.

Definition 2.7 (Timelike curvature bounds)**.** *Let X be a Lorentzian pre-length space. An open subset U is called a* timelike ($\geq K$)-comparison neighbourhood *(or* timelike ($\leq K$)-comparison neighbourhood*) if:*

1. *τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$ and $(U \times U) \cap \tau^{-1}([0, D_K))$ is open.*

2. *For all $x, y \in U$ with $x \ll y$ and $\tau(x, y) < D_K$ there exists a geodesic connecting them which is contained entirely in U.*

3. *Let $\Delta(x, y, z)$ be a timelike triangle in U, with p, q two points on the sides of $\Delta(x, y, z)$. Let $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and $\bar{p}, \bar{q}$ comparison points for p and q, respectively. Then*

$$\tau(p, q) \leq \tau(\bar{p}, \bar{q}) \quad (\text{or } \tau(p, q) \geq \tau(\bar{p}, \bar{q})). \tag{90}$$

We say X has timelike curvature bounded below by K *if it is covered by timelike $(\geq K)$-comparison neighbourhoods. Likewise, X has* timelike curvature bounded above by K *if it is covered by timelike $(\leq K)$-comparison neighbourhoods.*

We say X has global timelike curvature bounded below by K *if X itself is a $(\geq K)$-comparison neighbourhood. Similarly, X has* global timelike curvature bounded above by K *if X is a $(\leq K)$-comparison neighbourhood.*

Note that within a $(\geq K)$ comparison neighbourhood, $p \ll q$ implies $\bar{p} \ll \bar{q}$, and within a $(\leq K)$ comparison neighbourhood, $\bar{p} \ll \bar{q}$ implies $p \ll q$.

Remark 2.8 (Global curvature bound of AdS)**.** *Note that with the above definition, $\mathbb{L}^2(-1)$ satisfies global curvature bounds both above and below. In contrast, $\mathbb{L}^2(-1)$ does not satisfy a global curvature bound with respect to the original definition of [KS18], since it does not satisfy their conditions for a comparison neighbourhood: τ is neither finite nor continuous, and for x, y with $\tau(x, y) > \pi$, there is no geodesic joining them.*

When treating local curvature bounds, we consider spaces covered by comparison neighbourhoods. In establishing a globalization theorem, a precise description of the aforementioned covering will be useful. To this end, one step of the causal ladder, namely strong causality, is crucial:

Definition 2.9 (Strong causality)**.** *A Lorentzian pre-length space X is called* strongly causal *if $\mathcal{I} := \{I(x, y) \,|\, x, y \in X\}$ is a subbasis for the topology induced by d.*

It turns out, however, that a finite intersection of diamonds inside an arbitrary neighbourhood is not sufficient; we will actually require the existence of a timelike diamond inside any neighbourhood, i.e., $\mathcal{I}$ must be a basis for the topology. This is possible under the additional assumption of non-timelike local isolation:

Definition 2.10 (Non-timelike local isolation)**.** *A Lorentzian pre-length space X is said to be* non-timelike locally isolating *if for all $x \in X$ and all neighbourhoods U of x in X there exist $x_-, x_+ \in U$ such that $x_- \ll x \ll x_+$.*

Proposition 2.11 (Diamonds form basis)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space. Then $\mathcal{I}$ forms a basis for the topology. In particular, given any neighbourhood of any point, we can construct a timelike diamond containing the point, such that the diamond and its governing points are also contained in the neighbourhood.*

Proof. See [Rot22, Lemma 3.5]. □

Conveniently, the previous proposition also proves to be the perfect tool for highlighting the relationship between comparison neighbourhoods in the sense of [KS18, Definition 4.7] and the modification we propose in Definition 2.7. Indeed, under assumptions of strong causality and non-timelike local isolation (as in AdS, for example), the two definitions may be used interchangeably and one may construct comparison neighbourhoods in the sense of [KS18] via the following lemma:

Lemma 2.12 (Automatic size bounds)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space which has timelike curvature bounded below (above) by some $K \in \mathbb{R}$. Then X is covered by timelike $(\geq K)$-comparison (resp. $(\leq K)$-comparison) neighbourhoods U where $\tau|_{U \times U} < D_K$. In particular, these U are curvature comparison neighbourhoods in the sense of the old definition [KS18, Definition 4.7] and all timelike triangles are realizable.*

Moreover, all curvature comparison neighbourhoods in the sense of [KS18, Definition 4.7] are $(\leq K)$-comparison neighbourhoods in the sense of Definition 2.7. In particular, a Lorentzian pre-length space has (local) curvature bounds in the sense of 2.7 if and only if it has curvature bounds in the sense of [KS18, Definition 4.7].

Proof. Let $x \in X$ and $\tilde{U}$ be a $(\geq K)$-comparison (resp. $(\leq K)$-comparison) neighbourhood of x. We have that $\tau(x, x) = 0$. $(\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$ is open and contains (x, x), thus we find a small neighbourhood V of x such that $V \times V \subseteq (\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$. By Proposition 2.11, we find $x_-, x_+ \in V$ such that $x_- \ll x \ll x_+$ and $x \in I(x_-, x_+) \subset V$, and set $U = I(x_-, x_+)$. It follows that $\tau(x_-, x_+) < D_k$, and hence $\tau|_{U \times U} < D_k$.

We now verify that U is a $(\geq K)$-comparison (resp. $(\leq K)$-comparison) neighbourhood in the sense of [KS18, Definition 4.7]: Clearly τ is finite and continuous on $U \times U$. Furthermore, by causal convexity[14] of timelike diamonds, geodesics between points in U remain in U. Finally, U inherits property *(iii)* of Definition 2.7 from the comparison neighbourhood $\tilde{U}$.

If τ is continuous on $U \times U$ it is also continuous on $(\tilde{U} \times \tilde{U}) \cap \tau^{-1}([0, D_K))$, and this set is open by the (local) continuity of τ. □

In metric geometry, there are several reformulations of curvature bounds expressed via classical triangle comparison, using angles, for example. Alternative versions also exist for timelike curvature bounds (see [BS22, BMS22]) and several of these characterizations will prove useful in our context. Before we state these explicitly, let us introduce some more terminology:

Definition 2.13 (K-comparison angles and sign)**.** *Let X be a Lorentzian pre-length space, $K \in \mathbb{R}$, $\Delta(x, y, z)$ a timelike triangle in X, and $\Delta(\bar{x}, \bar{y}, \bar{z})$ a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$.*

(i) *The* K-comparison angle *at x is defined as the ordinary hyperbolic angle at $\bar{x}$ between $\bar{y}$ and $\bar{z}$:*

$$\tilde{\measuredangle}_x^K(y, z) := \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{y}, \bar{z}) = \operatorname{arcosh}(|\langle \gamma'_{\bar{x}\bar{y}}(0), \gamma'_{\bar{x}\bar{z}}(0) \rangle|), \tag{91}$$

where we assume the mentioned geodesics to be unit speed parameterized.

(ii) *The* sign σ *of a K-comparison angle is the sign of the corresponding inner product (in the $-, +, \cdots, +$ convention). That is, in this notation, the sign is -1 if the angle is measured at x or z and 1 if the angle is measured at y.*

(iii) *The* signed K-comparison angle *is defined as $\tilde{\measuredangle}_x^{K,S}(y, z) := \sigma \tilde{\measuredangle}_x^K(y, z)$.*

Definition 2.14 (Angles and hinges)**.** *Let X be a Lorentzian pre-length space and let α and β be two timelike curves of arbitrary time orientation emanating at $\alpha(0) = \beta(0) =: x$.*

(i) *The* angle *between α and β, if it exists, is defined as*

$$\measuredangle_x(\alpha, \beta) := \lim_{s,t \to 0} \tilde{\measuredangle}_x^0(\alpha(s), \beta(t)), \tag{92}$$

where the limit only considers values of s and t for which the triple $(x, \alpha(s), \beta(t))$ (or some permutation thereof) forms a timelike triangle.

[14] Recall that a subset A of a Lorentzian pre-length space X is called *causally convex* if for all $p, q \in A$ it holds that $J(p, q) \subseteq A$. Causal and timelike diamonds are among the most prominent examples of causally convex sets.

(ii) The sign σ *of an angle is* -1 *if* α *and* β *have the same time orientation and* 1 *otherwise. The* signed angle *is defined as* $\measuredangle_x^S(\alpha,\beta) \coloneqq \sigma\measuredangle_x(\alpha,\beta)$.

(iii) An angle at a point $x \in X$ *and the associated geodesics are called a* hinge*, which we will denote by* (α,β).

(iv) Given $K \in \mathbb{R}$ *and a hinge* (α,β) *at* x *in* X*, we call a hinge* $(\bar{\alpha},\bar{\beta})$ *at* $\bar{x}$ *in* $\mathbb{L}^2(K)$ *whose corresponding sides have the same lengths and time orientations and satisfy* $\measuredangle_x(\alpha,\beta) = \measuredangle_{\bar{x}}^{\mathbb{L}^2(K)}(\bar{\alpha},\bar{\beta})$ *a* K*-comparison hinge for* (α,β).

As we will never work with different model spaces simultaneously and as the limit in (92), cf. [BS22, Proposition 2.14], is the same regardless of the model space in which it is considered, we will usually drop the superscript in the comparison angle and just write $\tilde{\measuredangle}_x(y,z)$.

Now we highlight some alternative formulations of curvature bounds, beginning with monotonicity comparison. The remaining results in this subsection are also valid when considering upper curvature bounds, where inequalities are switched in the obvious way. However, as we will only need our reformulations in the setting of lower curvature bounds, we shall not state the former case herein.

As provided in [BS22, Definition 4.8], monotonicity comparison (specifically point *(ii)*) requires the additional technical assumption that the neighbourhoods considered are *strictly timelike geodesic*, meaning that for any two close enough timelike related points there is a *timelike* geodesic joining them. This can be omitted by instead assuming that our Lorentzian pre-length space is regular as in Definition 2.4; monotonicity comparison then takes the following form:

Definition 2.15 (K-Monotonicity comparison)**.** *Let* $K \in \mathbb{R}$ *and let* X *be a regular Lorentzian pre-length space.* X *is said to satisfy* K-monotonicity comparison from below *if every point in* X *possesses an open neighbourhood* U *such that:*

(i) τ *is continuous on* $(U \times U) \cap \tau^{-1}([0, D_K))$ *and* $(U \times U) \cap \tau^{-1}([0, D_K))$ *is open.*

(ii) For all $x, y \in U$ *with* $x \ll y$ *and* $\tau(x,y) < D_K$ *there is a (timelike) geodesic joining them which is contained entirely in* U.

(iii) *Given two timelike geodesics $\alpha, \beta : [0,1] \to X$ of arbitrary time orientation emanating at $\alpha(0) = \beta(0) = x$, we have that $\tilde{\measuredangle}_x^{K,S}(\alpha(s), \alpha(t))$ is a monotonically increasing function in s and t (where it is defined).*

Note that by assuming that our space is regular, points *(i)* and *(ii)* above are precisely those given in Definition 2.7 of timelike curvature bounds and only point *(iii)* differs.

Currently, monotonicity comparison is the only formulation which is equivalent to triangle comparison in reasonable generality, with other formulations being implied by, but not implying, the monotonicity condition.[15]

Theorem 2.16 (Triangle and monotonicity comparison are equivalent)**.** *Let $K \in \mathbb{R}$ and let X be a regular Lorentzian pre-length space. Then X has timelike curvature bounded below by K in the sense of Definition 2.7 if and only if it satisfies K-monotonicity comparison from below.*

Proof. See [BS22, Theorem 4.13]. □

Theorem 2.17 (Curvature bounds imply angle and hinge comparison)**.** *Let X be a regular Lorentzian pre-length space with timelike curvature bounded below by $K \in \mathbb{R}$. Let $x \in X$ and let $\alpha, \beta : [0,1] \to X$ be any two timelike geodesics emanating from x.*

(i) *It holds that*

$$\measuredangle_x^S(\alpha, \beta) \leq \tilde{\measuredangle}_x^S(\alpha(s), \beta(t)) \tag{93}$$

for all s, t which form a timelike triangle with x.

(ii) *Let $(\bar{\alpha}, \bar{\beta})$ be a comparison hinge in $\mathbb{L}^2(K)$. Then*

$$\tau(\alpha(1), \beta(1)) \geq \bar{\tau}(\bar{\alpha}(1), \bar{\beta}(1)). \tag{94}$$

Proof. See [BS22, Corollaries 4.11 and 4.12]. □

We conclude this chapter with the following useful fact about angles.

Proposition 2.18. *Let X be a strongly causal and locally causally closed Lorentzian pre-length space with timelike curvature bounded below by $K \in \mathbb{R}$, and let $\alpha : [0,1] \to X$ be a timelike geodesic. Let $x = \alpha(t)$ for $t \in (0,1)$ and consider the restrictions $\alpha_- = \alpha|_{[0,t]}$ and $\alpha_+ = \alpha|_{[t,1]}$ as past-directed and future-directed geodesics emanating from x, respectively. Let β be a timelike geodesic emanating from x. Then $\measuredangle_x(\alpha_-, \beta) = \measuredangle_x(\alpha_+, \beta)$.*

[15] [BMS22, Theorem 14] shows that classical triangle comparison can be deduced from angle comparison in the case of lower timelike curvature bounds using additional assumptions on the behaviour of angles.

Proof. See [BS22, Corollary 4.7] (and [BS22, Lemma 4.10] for the existence of the angle). $\square$

2.2 Elementary concepts from metric geometry

We now turn to presenting some basic definitions from the realm of metric geometry. Most importantly, we would like to highlight the fundamental differences in the definitions of lengths and curvature bounds in the metric setting when compared to those Lorentzian pre-length spaces.

Definition 2.19 (Length of a curve and geodesics)**.** *Let (X, d) be a metric space. The* length *of a curve $\gamma : [a, b] \to X$ from x to y is defined as*

$$L_d(\gamma) := \sup\left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \,\middle|\, a = t_0 < t_1 < \ldots < t_n = b, n \in \mathbb{N} \right\}. \tag{95}$$

If $L_d(\gamma) = d(x, y)$, then γ is called a distance-realizer *or* geodesic*. As we will generally not consider metric and Lorentzian geodesics simultaneously, context should be sufficient to deduce which of the two concepts is being applied and there should be no confusion between them.*

We define triangles, comparison triangles, and comparison points in complete analogy to Definition 2.5. Again we assume that all triangles satisfy size bounds. As discussed for geodesics, we will generally not use Lorentzian and Riemannian model spaces simultaneously, however, we shall denote the Riemannian model spaces by M_k, cf. [CE75].

Definition 2.20 (Metric triangle comparison)**.** *Let X be a metric space. An open subset U is called a* $(\geq K)$-comparison neighbourhood *(or* $(\leq k)$-comparison neighbourhood*) if $(U, d|_{U\times U})$ is geodesic for pairs of points with distance less than* $\mathrm{diam}(M_k)$*, and for all triangles $\Delta(x, y, z)$ in U, and all $p, q \in \Delta(x, y, z)$, the following is satisfied: let $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in M_k for $\Delta(x, y, z)$ (satisfying size bounds) and let $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ be comparison points for p and q respectively. Then*

$$d(p, q) \geq d(\bar{p}, \bar{q}) \quad (\textit{or } d(p, q) \leq d(\bar{p}, \bar{q}))\,. \tag{96}$$

We say X has curvature bounded below by *k if it is covered by $(\geq k)$-comparison neighbourhoods. Likewise, its* curvature is bounded above by *k if it is covered by $(\leq k)$-comparison neighbourhoods.*

We say X has global curvature bounded below (or above) by k if X is a $(\geq k)$ (or $(\leq k)$) comparison neighbourhood. Spaces with global curvature bounded above by k are called CAT(k) spaces.

3 Metric spaces with global curvature bounds

The globalization of curvature bounds in metric spaces serves as a natural motivation for investigating analogous results in the Lorentzian case. Hence, we include here a brief discussion of the metric picture in order to familiarize ourselves and the reader with the techniques and constraints we wish to transfer. For details of the wider metric setting, we refer the reader to [BBI01, BH99, AKP19].

3.1 Curvature bounded above

Let us first consider the case of curvature bounded above. The following example demonstrates that globalization is not automatic in this case, and that some additional assumptions are required.

Example 3.1 (The circle)**.** *Consider the unit circle $X := \mathbb{S}^1$ with its intrinsic metric. Then locally, X is isometric to a line segment, which clearly has curvature bounded above by $k = 0$. However, X itself is not a ≤ 0-comparison neighbourhood: consider a large triangle defined by three equidistant points in X, such that it covers the whole circle. The corresponding comparison triangle in the Euclidean plane is also equilateral. Given any two points p, q on different sides of the triangle in X, the triangle inequality yields equality when going along the shorter of the two arcs between each pair of points, see Figure 3.1. However, in the plane, triangle equality is only obtained when the two points lie on the same side of the triangle. It follows that* (96) *is not satisfied.*

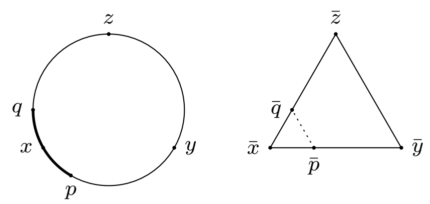

Figure 16: Triangle comparison fails for too large triangles in the circle.

The following theorem specifies sufficient additional assumptions under which upper curvature bounds may be globalized:

Theorem 3.2 (Alexandrov's Patchwork)**.** *Let X be a metric space with (local) curvature bounded above by k and suppose that there is a unique geodesic* *joining each pair of points that are a distance less than* $\operatorname{diam}(M_k)$ *apart. If these geodesics vary continuously with their endpoints,*[16] *then X is a CAT(k) space.*

Proof. See [BH99, Proposition II.4.9]. □

On a related note, it turns out that these additional assumptions under which the globalization of curvature bounds does work are rather mild, in the sense that they always hold in a CAT(k) space. In other words, a metric space with curvature bounded above by k is CAT(k) if and only if it is uniquely geodesic (for points with distance less than $\operatorname{diam}(M_k)$) and these geodesics vary continuously with their endpoints.

Proposition 3.3 (Elementary properties of CAT(k) spaces)**.** *Let X be a CAT(k) space. Then geodesics in X are unique (for points with distance less than* $\operatorname{diam}(M_k)$*) and these vary continuously with their endpoints.*

Proof. See [BH99, Proposition II.1.4]. □

It should be immediately apparent that the circle with its intrinsic metric does not have continuously varying geodesics, let alone unique geodesics. Therefore, by the above proposition, it cannot be CAT(0).

3.2 Curvature bounded below

Concerning curvature bounded below, there is also a globalization result, which is perhaps even more iconic than the Alexandrov's Patchwork approach for curvature bounded above. It is known as the theorem of Toponogov and was first proven for general complete length spaces by Perelman in [BGP92].

Note that for Theorem 3.4 and Theorem 3.5 in the form stated, we follow the authors of [BBI01] and explicitly exclude 1-dimensional spaces. More precisely, if $k > 0$, then X must not be isometric to $\mathbb{R}$, $(0, \infty)$, $[0, B]$ for any $B > \frac{\pi}{\sqrt{k}}$, or any circle with radius greater than $\frac{2\pi}{\sqrt{k}}$.

Theorem 3.4 (Toponogov's Globalization Theorem)**.** *Let X be a complete length space with curvature bounded below by k, which is not one of the aforementioned* 1*-dimensional spaces. Then X has global curvature bounded below by k.*

[16] This means that if $x_n \to x, y_n \to y$, then $\gamma_{x_n y_n} \to \gamma_{xy}$ uniformly.

Proof. See [BBI01, Theorem 10.3.1] for a proof under a local compactness assumption. See also [LS13, AKP19] for more general proofs, as well as a timeline of refinements. □

So far, we have been unable to transport this result to the synthetic Lorentzian setting, but, as we shall discuss a little further in the outlook, we are actively working on formulating a result.

In the metric case, there is an addendum to Toponogov's Theorem, generalizing the Bonnet–Myers Theorem from Riemannian geometry to the setting of Alexandrov geometry. In essence, the theorem bounds the diameter of a complete length space with local, positive, lower curvature bound in such a way that comparison triangles exist, thus eliminating concern about the existence of triangles in X which are too large.

Theorem 3.5 (Lower curvature bounds imply finite diameter)**.** *Let X be a complete length space with (local) curvature bounded below by some $k > 0$, which is not one of the aforementioned 1-dimensional spaces. Then* $\mathrm{diam}(X) \leq \frac{\pi}{\sqrt{k}}$.

Proof. See [BBI01, Theorem 10.4.1]. □

Since the metric Bonnet–Myers Theorem is a direct result of Toponogov's Globalization Theorem 3.4, we will not be able to precisely follow the derivation given in [BBI01, Therem 10.3.1] when producing a Lorentzian analogue. However, we will derive a diameter bound on Lorentzian pre-length spaces, under the slightly stricter assumption of global curvature being bounded below by some $K < 0$, cf. Theorem 4.11. The corresponding result assuming only local curvature bounds is then a natural candidate for future research (see Section 5 for more detail).

4 Lorentzian pre-length spaces with global curvature bounds

We now return fully to the setting of synthetic Lorentzian geometry. As previously mentioned, the main task of this work is to provide Lorentzian versions of the Alexandrov's Patchwork approach to globalizing upper curvature bounds and the Bonnet–Myers Theorem constraining the diameter of spaces with positive lower curvature bounds on sectional curvature, cf. Theorem 3.2 and Theorem 3.5, respectively. Let us first discuss Alexandrov's Patchwork.

4.1 Timelike curvature bounded above

As it turns out, the proof of the metric globalization result, Theorem 3.2, may be very straightforwardly adapted to the Lorentzian setting if we respect some minor technicalities. We first provide a few preparatory results, before diving into the proof proper. In particular, we will need the Gluing Lemma for timelike triangles:

Lemma 4.1 (Gluing Lemma for timelike triangles, Case I)**.** *Let X be a Lorentzian pre-length space and let $U \subseteq X$ be an open subset satisfying (i) and (ii) in the definition for a comparison neighbourhood in X, cf. Definition 2.7. Let $T_3 := \Delta(x, y, z)$ be a timelike triangle in U satisfying size bounds for $\mathbb{L}^2(K)$, with $K \in \mathbb{R}$ fixed but arbitrary. Let $p \in [x, z]$ be such that $p \ll y$ (or $y \ll p$). In other words, $T_1 := \Delta(x, p, y)$ and $T_2 := \Delta(p, y, z)$ are again timelike triangles (if $y \ll p$, it is necessary to interchange the order of y and p in the triangles), see Figure 17. Let $\overline{T}_1 := \Delta(\bar{x}, \bar{p}, \bar{y})$ and $\overline{T}_2 := \Delta(\bar{p}, \bar{y}, \bar{z})$ be comparison triangles for T_1 and T_2 in $\mathbb{L}^2(K)$, respectively. Suppose T_1 and T_2 satisfy timelike curvature bounds from above for K, i.e., for all $a, b \in T_i$ and corresponding comparison points $\bar{a}, \bar{b} \in \overline{T}_i, i = 1, 2$, we have $\tau(a, b) \geq \bar{\tau}(\bar{a}, \bar{b})$ (cf. Definition 2.7.3). Then T_3 satisfies the same timelike curvature bound from above.*

Proof. See [BR22, Lemma 4.3.1]. Note that this still works with the new definition of timelike curvature bounds. □

Lemma 4.2 (Gluing Lemma for timelike triangles, Case 2)**.** *Let X and U be as in Lemma 4.1 and let $\Delta(x, y, z)$ be a timelike triangle in U. Let p be a point on γ_{xy} (or γ_{yz}) and consider the two resulting sub-triangles that share the (timelike) segment γ_{pz} (or γ_{xp}). If the sub-triangles satisfy a timelike curvature bound from above, then so does the original triangle.*

Proof. See [BR22, Corollary 4.3.2]. □

So now we know that, if we can triangulate each big timelike triangle into smaller ones, with each of the small sub-triangles contained in a $(\leq K)$-comparison neighbourhood, then we can reconstruct the big timelike triangle step-by-step, using the Gluing Lemma to get that Definition 2.7.3 is satisfied by the big triangle.

In order to globalize curvature bounds, we now require conditions which guarantee such a triangulation of arbitrary big triangles, with sub-triangles contained in comparison neighbourhoods. Using Proposition 2.11, we have the following elegant description of comparison neighbourhoods in strongly

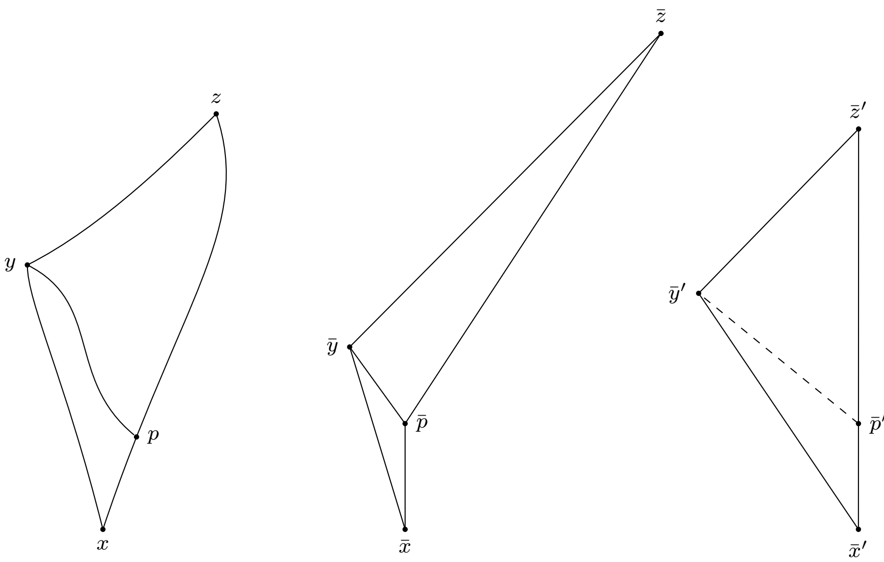

Figure 17: A timelike triangle in X subdivided into two timelike triangles, the comparison triangles for the smaller triangles and the comparison triangle for the big triangle.

causal Lorentzian pre-length spaces with curvature bound, cf. [Ber20, Remark 2.2.12], which shall turn out to be sufficient:

Proposition 4.3 (Timelike diamonds form neighbourhood basis of comparison neighbourhoods)**.** *Let X be a strongly causal and non-timelike locally isolating Lorentzian pre-length space with curvature bounded above (or below) by $K \in \mathbb{R}$. Then each point has a neighbourhood basis of timelike diamonds which are also comparison neighbourhoods.*

Proof. Let $x \in X$ and let U be a comparison neighbourhood of x. Any timelike diamond $D := I(p, q)$ containing x and contained in U (we can even assume $p, q \in U$) is a comparison neighbourhood: Indeed, points *(i)* and *(iii)* of Definition 2.7 are directly inherited when passing to open subsets, and *(ii)* follows by causal convexity of D, as in Lemma 2.12.

As X is strongly causal and non-timelike local isolating, Proposition 2.11 yields that timelike diamonds form a neighbourhood basis. Hence, for each neighbourhood V of x, there exists a timelike diamond containing x, which is contained in $U \cap V$. By the above, such a timelike diamond is also a comparison neighbourhood. $\square$

Definition 4.4 (Geodesic map)**.** *Let X be a uniquely geodesic and regular Lorentzian pre-length space. Viewing the timelike relation $\ll$ as a subset of*

$X \times X$, *the* geodesic map[17] *of* X *is formally defined as*

$$G : \ll \times [0,1] \to X\,, \quad G(x,y,t) := \gamma_{xy}(t)\,. \tag{97}$$

We say that geodesics vary continuously *if* G *is continuous.*

The definition above appears to be a different notion of continuous variation of geodesics to that required in Theorem 3.2. As it turns out, however, they are equivalent (note that of course in the Lorentzian formulation of the version used in Theorem 3.2 we must restrict to sequences and limits of timelike related points). The following proposition does not make use of τ at all, but we still formulate it in the Lorentzian context.

Proposition 4.5 (Equivalent notions of continuously varying geodesics)**.** *Let X be a uniquely geodesic and regular Lorentzian pre-length space and assume $\tau^{-1}([0,D_K))$ is open. Then $G|_{\tau^{-1}([0,D_K))\times[0,1]}$ is continuous if and only if (timelike) geodesics with length less than D_K vary continuously in the sense of Theorem 3.2.*

Proof. First assume we have continuously varying geodesics in the sense of Theorem 3.2. Note that since geodesics are continuous by definition, we know that G is continuous in t. We have to show $G(x_n,y_n,t_n) \to G(x,y,t)$ for sequences $x_n \to x, y_n \to y, t_n \to t, x_n \ll y_n, x \ll y, \tau(x,y) < D_K, \tau(x_n,y_n) < D_K$, i.e., $\gamma_{x_n y_n}(t_n) \to \gamma_{xy}(t)$. We have $d(\gamma_{x_n y_n}(t_n), \gamma_{xy}(t)) \leq d(\gamma_{x_n y_n}(t_n), \gamma_{xy}(t_n)) + d(\gamma_{xy}(t_n), \gamma_{xy}(t))$ and conclude that the right hand goes to 0 using the uniform convergence $\gamma_{x_n y_n} \to \gamma_{xy}$ and the fact that geodesics are continuous.

Conversely, suppose (the suitable restriction of) G is continuous. Given sequences $x_n \to x, y_n \to y$ with $\tau(x,y) < D_K$, note that by the openness of $\tau^{-1}([0,D_K))$, we also have $\tau(x_n,y_n) < D_K$ for large enough n. The functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(s) := d(\gamma_{x_n y_n}(s), \gamma_{xy}(s))$ are continuous on a compact domain, hence f_n attains its maximum at least once on $[0,1]$. Fix $s_n := \operatorname{argmax}(f_n(s))$ to be one such maximizing parameter (the precise choice is not important). We know $f_n(s) \to 0$ pointwise in s as $n \to \infty$ and we want to show that this convergence is uniform. As $\{s_n\}_n$ is contained in the compact set $[0,1]$, we may assume without loss of generality that $s_n \to s'$ for some $s' \in [0,1]$. Then

$$\begin{aligned} d(\gamma_{x_n y_n}(s), \gamma_{xy}(s)) = f_n(s) \leq f_n(s_n) &= d(\gamma_{x_n y_n}(s_n), \gamma_{xy}(s_n)) \\ &\leq d(\gamma_{x_n y_n}(s_n), \gamma_{xy}(s')) + d(\gamma_{xy}(s'), \gamma_{xy}(s_n)), \end{aligned}$$

which goes to 0 as $n \to \infty$ uniformly in s, using that both the geodesic map G and geodesics γ_{xy} are continuous in turn. □

[17] This is closely related to the *line-of-sight map*, cf. [AKP19, Definition 9.32].

Theorem 4.6 (Alexandrov's Patchwork Globalization, Lorentzian version)**.** *Let X be a strongly causal, non-timelike locally isolating, and regular Lorentzian pre-length space which has (local) timelike curvature bounded above by $K \in \mathbb{R}$. Suppose that X satisfies* (i) *and* (ii) *in Definition 2.7, i.e., $\tau^{-1}([0, D_K))$ is open and τ is continuous on that set, and for all $x \ll y$ in X with $\tau(x, y) < D_K$, there exists a geodesic joining them. Additionally assume that the geodesics between timelike related points with τ-distance less than D_K are unique. Let G be the geodesic map of X restricted to the set $\{(x, y, t) \in \ll \times [0, 1] \mid \tau(x, y) < D_K\} = \tau^{-1}((0, D_K)) \times [0, 1]$ and assume that G is continuous. Then X also satisfies Definition 2.7.*(iii)*, in particular it is a $(\leq K)$-comparison neighbourhood and X has global curvature bounded above by K.*

Proof. By our assumptions, it is only left to show triangle comparison. Let $\Delta(x, y, z)$ be a timelike triangle in X. Given $t \in [0, 1]$, let $\beta_t := \gamma_{x\gamma_{yz}(t)} = G(x, \gamma_{yz}(t), \cdot) : [0, 1] \to X$ be the geodesic from x to $\gamma_{yz}(t)$. By the continuity of G, we can regard the map $F(s, t) := \beta_t(s)$ as a geodesic variation with starting point x that "spans" the timelike triangle $\Delta(x, y, z)$. In particular, this "filled in" triangle is compact as the continuous image under F of the compact set $[0, 1] \times [0, 1]$. Fix $t \in [0, 1]$. For each $s \in [0, 1]$, we find a timelike diamond $I(x_s, y_s)$ that is a comparison neighbourhood of $\beta_t(s)$, cf. Proposition 2.11 and Proposition 4.3. Since β_t is continuous, there is a neighbourhood N_s of s in $[0, 1]$ such that $\beta_t(N_s) \subseteq I(x_s, y_s)$. In particular, for $s \in (0, 1)$, we find $s^- < s < s^+$ in N_s. By the causal convexity of diamonds, we then obtain that $I_s := I(\beta_t(s^-), \beta_t(s^+)) \subseteq I(x_s, y_s)$ is also a comparison neighbourhood of $\beta_t(s)$. The point is that we can choose the comparison neighbourhood diamonds in such a way that the governing points are situated on the geodesic. For the parameters 0 and 1 this will not be possible as these are the endpoints of the geodesic, however, we may still force one of the governing points to be on β_t, i.e., we set $I_0 := I(x_0, \beta_0(0^+))$ and $I_1 := I(\beta_1(1^-), y_1)$. Clearly, $\bigcup_s I_s$ is an open cover of $\beta_t([0, 1])$. By compactness, we can extract a finite subcover[18] say $\bigcup_{k=0}^n I_{s_k} \supseteq \beta_t([0, 1])$. Now order these diamonds with respect to, say, (the parameters of) their future governing points, i.e., $s_k^+ < s_{k+1}^+$ for all k. Further assume that the cover is minimal in the sense that no diamond can be removed from the cover; in particular, no diamond is entirely contained inside another one. This then immediately implies that the bottom governing points are ordered similarly and that subsequence diamonds overlap and only subsequent ones do so, i.e.,

$$I_i \cap I_j \neq \emptyset \iff |i - j| \leq 1. \tag{98}$$

[18]Note that because of the way we chose the governing points of each I_s, the first and the last diamonds I_0 and I_1 are always included (because the endpoints are not inside any other I_s).

Clearly, $F(\cdot, t) = \beta_t$. Since F is continuous and $\bigcup_{k=0}^n I_{s_k}$ is a neighbourhood of $\beta_t([0,1])$, it follows that there exists an open neighbourhood of t, denote it by J_t, such that $F([0,1], J_t) \subseteq \bigcup_{k=0}^n I_{s_k}$. By shrinking J_t if necessary, we can assume that $\gamma_{yz}(J_t) \subseteq I_1$. Visually, $\bigcup_{k=0}^n I_{s_k}$ covers $\beta_{t'}$ for all t' in a neighbourhood of t and each $\beta_{t'}$ ends in the top diamond I_1.

Now we let t vary: doing the above described procedure for each $t \in [0,1]$, we end up with an open cover $\bigcup_t J_t$[19] of $[0,1]$. Again by a compactness argument, we can extract a finite subcover from $\bigcup_t J_t$, say $\bigcup_{l=0}^m J_{t_l} \supseteq [0,1]$. Order these set in the same way as the diamonds, i.e., increasing with respect to, say, the right endpoint, and remove unnecessary ones. Then also the left endpoints are ordered in an increasing fashion and subsequent sets do overlap and these are the only ones which overlap. In total, we obtain, modifying the above notation slightly, that

$$\bigcup_{l=0}^{m} \bigcup_{k=0}^{n_l} I_{s_k}^{t_l} \supseteq F([0,1],[0,1]), \tag{99}$$

where I_s^t is the diamond "around" $\beta_t(s)$, i.e., the t emphasizes that this diamond belongs to the cover of β_t. The reward for this tedious construction is now the following: The triangle may be viewed as a fan consisting of m pieces, and the covers of subsequent "fan-geodesics" share some geodesics between them. More precisely, as $J_{t_l} \cap J_{t_{l+1}} \neq \emptyset, l = 0, \ldots, m-1$, and all geodesics ending in $\gamma_{yz}(J_{t_l})$ are contained in $\bigcup_{k=0}^{n_l} I_{s_k}^{t_l}$ (and the same for $l+1$), we know there exists some $\tilde{t}_l$ such that

$$\beta_{\tilde{t}_l}([0,1]) \subseteq (\bigcup_{k=0}^{n_l} I_{s_k}^{t_l}) \cap (\bigcup_{k=0}^{n_{l+1}} I_{s_k}^{t_{l+1}}). \tag{100}$$

A sketch of this process is depicted in Figure 18. We continue with the process of triangulation as follows: given l, consider the timelike triangle $\Delta(x, \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$. By construction, both $\beta_{\tilde{t}_l}$ and $\beta_{t_{l+1}}$ end in $\gamma_{yz}(J_{t_{l+1}}) \subseteq I_{s_{n_{l+1}}}^{t_{l+1}}$ and enter that set via $I_{s_{n_{l+1}}}^{t_{l+1}} \cap I_{s_{n_{l+1}-1}}^{t_{l+1}}$, i.e., they pass through the intersection of the ultimate and penultimate diamonds covering $\beta_{t_{l+1}}$. In particular, we can choose $\tilde{r}_1$ such that $\beta_{\tilde{t}_l}(\tilde{r}_1)$ is in said intersection. Note that the top governing point of the second to last diamond is timelike after the chosen point on $\beta_{\tilde{t}_l}$, i.e.,

$$\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(s_{n_{l+1}-1}^+). \tag{101}$$

[19] Note that similar to I_0 and I_1 from above, the sets J_0 and J_1 contain 0 and 1 at the boundary, respectively. Visually, this means that at the edges of the original triangle, nearby geodesics can only be "on one side" of these edges.

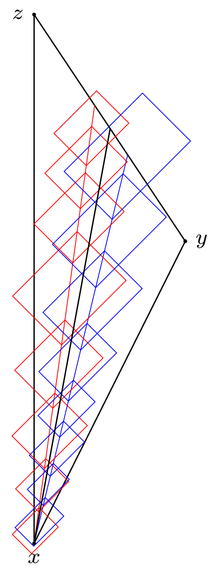

Figure 18: J_{t_l} (t_l represented by the blue geodesic β_{t_l}) and $J_{t_{l+1}}$ (t_{l+1} represented by the red geodesic $\beta_{t_{l+1}}$) overlap: $\tilde{t}_l \in J_{t_l} \cap J_{t_{l+1}}$ ($\tilde{t}_l$ represented by the black geodesic $\beta_{\tilde{t}_l}$).

By the openness of $\ll$, we can move a bit below the top governing point and still retain a timelike relation to the chosen point on $\beta_{\tilde{t}_l}$. In particular, both of these points are then contained in the intersection of the last two diamonds, and by the causal convexity also their connecting geodesic is entirely contained therein. More precisely, we find r_1 such that $\beta_{t_{l+1}}(r_1) \in I^{t_{l+1}}_{s_{n_{l+1}}} \cap I^{t_{l+1}}_{s_{n_{l+1}-1}}$ and $\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(r_1)$. Essentially, we constructed a quadrilateral consisting of $\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{t_{l+1}}(r_1), \beta_{\tilde{t}_l}(1)$ and $\beta_{t_{l+1}}(1)$, which is completely contained in $I^{t_{l+1}}_{s_{n_{l+1}}}$. By transitivity of $\ll$, also the "past most" and "future most" points of this quadrilateral are timelike related, so we can split this into two timelike triangles $\Delta(\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{t_{l+1}}(r_1), \beta_{t_{l+1}}(1))$ and $\Delta(\beta_{\tilde{t}_l}(\tilde{r}_1), \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$, see Figure 19. Both of these triangles are entirely contained in the last timelike diamond $I^{t_{l+1}}_{s_{n_{l+1}}}$, so they satisfy the curvature bound by assumption. Iteratively continuing this procedure, we end up at x after a finite amount of steps. As $\beta_{\tilde{t}_l}(\tilde{r}_1) \ll \beta_{t_{l+1}}(r_1)$ are also contained in $I^{t_{l+1}}_{s_{n_{l+1}-1}}$, we can continue this procedure iteratively. After $n_{l+1} - 1$ steps, we end up with $\beta_{\tilde{t}_l}(\tilde{r}_{n_{l+1}-1}) \ll \beta_{t_{l+1}}(r_{n_{l+1}-1})$ lying in $I^{t_{l+1}}_{s_1}$, where also x lies.

That is, in the end we have $n_{l+1} - 1$ quadrilaterals, each of which we can split into two timelike triangles, and one additional timelike triangle at the bottom ending in x. Each of these timelike triangles is contained in one

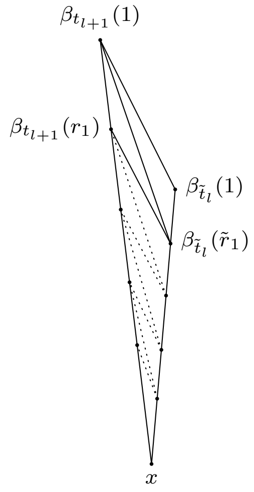

Figure 19: The process of subdividing a slim triangle.

of the comparison neighbourhoods $\{I_{s_i}^{t_{l+1}} : i = 1, \cdots, n_{l+1}\}$ (i.e. the chain of comparison diamonds covering the geodesic $\beta_{t_{l+1}}$ and $\beta_{\tilde{t}_l}$), so satisfies the curvature bound, hence several applications of the Gluing Lemma 4.1 yield that the "long and slim" triangle $\Delta(x, \beta_{\tilde{t}_l}(1), \beta_{t_{l+1}}(1))$ satisfies the curvature bound. Note that in a triangle of the form $\Delta(x, \beta_{t_l}(1), \beta_{\tilde{t}_{l+1}}(1))$ the top side has a different time orientation from the point of view of the geodesic β_{t_l} around which the covering is centered, but this changes nothing for the above described process. So we can do this for all of the $2m-1$ long and slim triangles and apply the Gluing Lemma $2m-2$ times to obtain that the original triangle $\Delta(x, y, z)$ obeys the desired curvature bound. $\square$

At first glance, our proof appears to be quite similar to the metric version in [BH99, Proposition II.4.9]. There are however, some technical details we have to be wary of. In particular, in an arbitrary covering of the triangle, even if we use timelike diamonds, it is generally not true that we can achieve a subdivision consisting of timelike triangles such that each sub-triangle is contained within a comparison neighbourhood. We have to carefully construct the covering and then construct the sub-triangles in a seemingly complicated way as well.

As with the circle in the metric case (see Example 3.1), it is possible to construct counterexamples to the automatic globalization of upper curvature bounds on Lorentzian pre-length spaces. The following is one such example, where the space has (local) curvature bounded above, but has neither unique nor continuously varying geodesics:

Example 4.7 (The Lorentzian cylinder)**.** *We set the Lorentzian cylinder to be the spacetime $X = \mathbb{R} \times \mathbb{S}^1$, i.e., take a strip $\mathbb{R} \times [0, 2\pi]$ in Minkowski space and glue the boundary as depicted by the arrows in Figure 20(which is not to be confused with the totally vicious cylinder $\mathbb{S}^1 \times \mathbb{R}$!).*

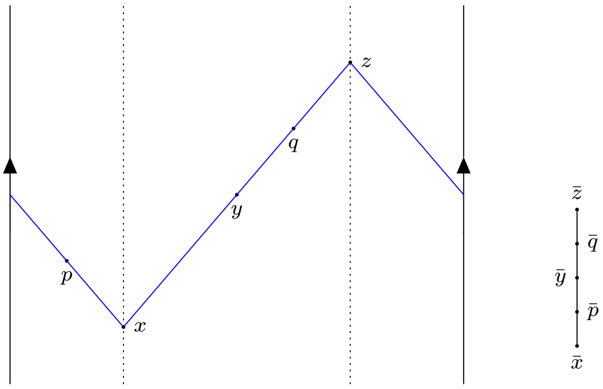

Figure 20: The Lorentzian cylinder. The depicted triangle fails to satisfy an upper curvature bound since its comparison triangle is degenerate.

This space is locally isometric to Minkowski space, hence clearly has (local) timelike curvature bounded above by 0.

Take two (dotted) vertical lines on the cylinder, which are directly opposite each other (as in Figure 20). Given two timelike related points, with one point on each of the two vertical lines, there exist precisely two geodesics between these points (wrapping around to the right and to the left on the cylinder, respectively). Consider any such pair of points, denoted x and z, and the corresponding pair of geodesics, and choose as a third vertex some point y on one of the two geodesics. Then the comparison triangle for $\Delta(x, y, z)$ is clearly degenerate. Choosing two points p and q on the two different geodesics (and at different parameters, say p occurs at an earlier parameter than q) sufficiently far away from the endpoints of the two geodesics, p and q will not be timelike related, i.e., $\tau(p, q) = 0$. However, as the comparison triangle is essentially a line segment and the two points are at different parameters, we clearly have $\tau(\bar{p}, \bar{q}) > 0$, violating global upper curvature bounds.

However, as in the metric case, some of our additional assumptions are relatively mild and can be recovered from the curvature bounds. More specifically, while we have the following result concerning the uniqueness of geodesics, we do not yet know under which conditions the geodesic map G is continuous.

Proposition 4.8 (Unique geodesics in upper curvature bounds)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded above by $K \in \mathbb{R}$. Let $x \ll y$ be in a comparison neighbourhood $U \subseteq X$ and suppose $\tau(x, y) < D_K$. Then there exists a unique geodesic from x to y contained in U. In particular, if X satisfies a global upper curvature bound, geodesics between timelike related points in X with τ-distance less than D_K are unique.*

Proof. Suppose towards a contradiction that there is another geodesic from x to y. Denote the curves corresponding to two of these geodesic segments by α_1 and α_2, respectively, and parameterize them with constant speed on $[0, 1]$. Let $p \in \alpha_1((0, 1))$ and consider the timelike triangle $\Delta(x, p, y)$, which satisfies size-bounds for $\mathbb{L}^2(K)$. Clearly the comparison triangle in $\mathbb{L}^2(K)$ is degenerate. As $\alpha_1 \neq \alpha_2$, there exists $t \in (0, 1)$ such that $\alpha_1(t) \neq \alpha_2(t)$. Then there exist neighbourhoods V_1 and V_2 of $\alpha_1(t)$ and $\alpha_2(t)$, respectively, such that $V_1 \cap V_2 = \emptyset$. As X is strongly causal, we find points $x_1, y_1, \ldots, x_n, y_n$ and $p_1, q_1, \ldots, p_m, q_m$ such that $\alpha_1(t) \in U_1 := \bigcap_{i=1}^n I_1(x_i^1, y_i^1) \subseteq V_1$ and $\alpha_2(t) \in U_2 := \bigcap_{j=1}^m I_1(p_j^2, q_j^2) \subseteq V_2$. By the continuity of α_1, there is some neighbourhood I_1 of t such that $\alpha_1(I_1) \subseteq U_1$. As U_1 is causally convex by definition, all diamonds with endpoints inside U_1 are contained in U_1. Similarly for U_2. In particular, there exists $\varepsilon > 0$ such that $D_1 := I(\alpha_1(t-\varepsilon), \alpha_1(t+\varepsilon)) \subseteq U_1$ and $D_2 := I(\alpha_2(t-\varepsilon), \alpha_2(t+\varepsilon)) \subseteq U_2$. Then $\alpha_2(t) \notin D_1$ (and vice versa), so either $\alpha_1(t-\varepsilon) \not\ll \alpha_2(t)$ or $\alpha_2(t) \not\ll \alpha_1(t+\varepsilon)$. However, in $\mathbb{L}^2(K)$ we have $\overline{\alpha}_1(s) = \overline{\alpha}_2(s)$ for all $s \in [0, 1]$. In particular, $\overline{\alpha}_1(t-\varepsilon) \ll \overline{\alpha}_2(t) \ll \overline{\alpha}_1(t+\varepsilon)$, a contradiction to upper timelike curvature bounds. $\square$

Remark 4.9 (Globalization of continuity)**.** *If we strengthen the requirements on X and the geodesic map, instead imposing that geodesics between any $x \ll y$ exist and are unique, and that the geodesic map G on its full domain $\ll \times [0, 1]$ is continuous, it follows that τ is globally continuous (and indeed finite by [KS18, Lemma 2.25]). In particular, if τ is globally continuous, then X automatically satisfies condition* (i) *of Definition 2.7 and we need not assume so in Theorem 4.6.*

As X has timelike curvature bounded above in the sense of Definition 2.7, it follows that τ is locally continuous[20] *(with respect to the covering of comparison neighbourhoods constructed in Lemma 2.12). By using that X is geodesic and slightly adapting the proof of [KS18, Proposition 3.17], it can be shown that L_τ is upper semi-continuous. Combining this with the continuity*

[20] In essence, we pass from local curvature bounds in the sense of Definition 2.7 to those in the sense of [KS18, Definition 4.7], where globalization of continuity is straightforward under our new assumptions.

of G, we find $L_\tau(G(x,y,\cdot)) = L_\tau(\gamma_{xy}) = \tau(x,y)$, hence τ is both upper and lower semi-continuous on X and is therefore globally continuous.

4.2 Timelike curvature bounded below

Finally, we show that a bound may be placed on the (finite) diameter (see Definition 2.6) of a Lorentzian pre-length space with negative lower timelike curvature bound. This result is in the spirit of the Bonnet–Myers Theorem (Theorem 3.5) from Riemannian geometry, however, as for previous results, we need to be careful about the Lorentzian subtleties.

First, note that due to the way timelike curvature bounds were introduced in [KS18], the hierarchy of curvature bound implications is reversed. More precisely, recall that if a metric space has curvature bounded above by some k, then it also has curvature bounded above by all $k' \geq k$. Similarly, if it has k as a lower curvature bound, it also has any $k' \leq k$ as a lower curvature bound. In the Lorentzian case, it turns out that any Lorentzian pre-length space satisfying timelike curvature bounded below by K, does so for all $K' \geq K$. Hence, we shall be required to assume a negative *lower* bound. In addition, due to the behaviour of Anti-deSitter discussed before Definition 2.6, we consider the finite diameter of Lorentzian pre-length spaces, rather than the ordinary diameter as in the metric case. Before diving into the theorem, we mention a lemma, giving a non-degeneracy condition for sub-triangles.

Lemma 4.10 (Non-degeneracy condition)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic Lorentzian pre-length space X and let U be a comparison neighbourhood in X. Let $a \ll b$ in U and let α be a geodesic in U starting at a and ending at b. Let $x = \alpha(t)$ and let $y \in I(x,b)$. Assume that both $\Delta(a,x,y)$ and $\Delta(x,y,b)$ satisfy size bounds. Let β be a timelike geodesic starting in x and ending in y, and denote by $\alpha_- = \alpha|_{[0,t]}$ and $\alpha_+ = \alpha|_{[t,1]}$ the parts of α in the past and future of y, respectively.*

1. *If X has timelike curvature bounded below by K and $\Delta(x,y,b)$ is non-degenerate, then $\Delta(a,x,y)$ is also non-degenerate, and $\measuredangle_x(\alpha_+,\beta)$ and $\measuredangle_x(\alpha_-,\beta)$ are equal and positive.*

2. *If X has timelike curvature bounded above by K and the angle $\measuredangle_x(\alpha_-,\beta)$ exists and $\Delta(a,x,y)$ is non-degenerate, then $\Delta(x,y,b)$ and (if it satisfies size bounds) $\Delta(a,y,b)$ are also non-degenerate, and both $\measuredangle_x(\alpha_\pm,\beta) > 0$ though they need not be equal.*

Proof. (i) As $\Delta(x,y,b)$ is non-degenerate, also the corresponding comparison triangle is non-degenerate and hence $\tilde{\measuredangle}_x(y,b) > 0$. By angle comparison, cf.

Theorem 2.17.*(i)*, we get $0 < \tilde{\measuredangle}_x(y,b) \leq \measuredangle_x(\alpha_+,\beta)$. As X is locally causally closed, strongly causal and has timelike curvature bounded below, we can apply Proposition 2.18, from which it follows that $\measuredangle_x(\alpha_+,\beta) = \measuredangle_x(\alpha_-,\beta) > 0$, and again by angle comparison, we have $\tilde{\measuredangle}_x(a,y) \geq \measuredangle_x(\alpha_-,\beta) > 0$. In particular, also $\Delta(a,x,y)$ is non-degenerate.

(ii) For the curvature bounded above case, the arguments of the curvature bounded below case reverse (we only get $\tilde{\measuredangle}_x(a,y) \leq \measuredangle_x(\alpha_-,\beta)$ from the triangle inequality of angles, see [BS22, Theorem 4.5.(i)]), and monotonicity comparison at the angle at a yields the statement on the big triangle.

□

The result of Theorem 4.11 should be closely compared to [CM20, Proposition 5.10]. In this pioneering work, the authors introduce synthetic Ricci curvature bounds using optimal transport methods, hence their result might be even closer in spirit to the original Bonnet–Myers Theorem than the one shown below. Moreover, should it prove true that Ricci curvature bounds (using optimal transport) are weaker than sectional curvature bounds (using triangle comparison) in the Lorentzian picture, as is the case for metric curvature comparison, cf. [Pet19], then our result has narrower scope. However, as the hierarchy of curvature bounds is not yet known and our method is distinct, the proof of the following theorem is valuable in its own right.

Theorem 4.11 (Bound on the finite diameter)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic*[21] *Lorentzian pre-length space which has global curvature bounded below by K. Assume $K < 0$. Assume that X possesses the following non-degeneracy condition: for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x,y,z)$ is a non-degenerate timelike triangle.*[22] *Then* $\mathrm{diam}_{\mathrm{fin}}(X) \leq D_K$.

Proof. Without loss of generality, we only consider $K = -1$. Let indirectly $a, b \in X$ with $\tau(a,b) = \pi + \varepsilon$ for some small enough $\varepsilon > 0$, and let $\alpha : [0, \pi + \varepsilon] \to X$ be a timelike distance realizer from a to b parameterized by τ-arclength. Let $x = \alpha(t_-)$ and $z = \alpha(t_+)$ for $t_- = \frac{\pi}{2} + \frac{\varepsilon}{2}$ (i.e., the midpoint), $t_+ = \frac{\pi}{2} + \frac{\pi}{8}$. Note that the specific value of t_+ is not important, any $t_+ \in (t_-, \pi)$ suffices. The corresponding point z is mainly used further on in the proof to ensure that a triangle with longest side shorter than $\tau(a,z) < \pi$,

[21] Global curvature bounds guarantee the existence of a geodesic for all $a \ll b$ with $\tau(a,b) < D_K$. In this context, however, we will need the existence of geodesics for all timelike related pairs of points slightly further apart. X being geodesic is a sufficient condition for this.

[22] This condition ensures the space is not locally 1-dimensional; Compare this to the discussion before Theorem 3.4 in the metric setting.

is realizable in $\mathbb{L}^2(-1)$. Denote by $\alpha_- = \alpha|_{[0,t_-]}$ and $\alpha_+ = \alpha|_{[t_-,\pi+\varepsilon]}$ the parts of α in the past and future of ~~y~~ x, respectively.

By the non-degeneracy assumption on X, we find a point $y \in I(x, z)$ such that $\tau(x, z) > \tau(x, y) + \tau(y, z)$, i.e., $\Delta(x, y, z)$ is non-degenerate. Let β be a distance realizer from x to y. By Lemma 4.10.*(i)*, we get that both $\Delta(a, x, y)$ and $\Delta(x, y, z)$ are non-degenerate, and $\omega \coloneqq \measuredangle_x(\alpha_-, \beta) = \measuredangle_x(\alpha_+, \beta) > 0$. We now claim that $\tau(a, b) < \tau(a, y) + \tau(y, b)$, contradicting the reverse triangle inequality.

We name the lengths: $t_- = \tau(a, x) = \tau(x, b) =: t$, $\tau(a, y) =: p$, $\tau(y, b) =: q$ and $\tau(x, y) =: m$, so the claim reads

$$2t < p + q. \tag{102}$$

We create a situation in $\mathbb{L}^2(K)$ consisting of comparison hinges for (α_-, β) and (β, α_+), giving the triangles $\Delta(\tilde{a}, \tilde{x}, \tilde{y})$ and $\Delta(\tilde{x}, \tilde{y}, \tilde{b})$, see Figure 21. Note that these triangles are non-degenerate since $\omega > 0$. We name the side-lengths $\tau(\tilde{a}, \tilde{y}) = \tilde{p}$, $\tau(\tilde{y}, \tilde{b}) = \tilde{q}$.

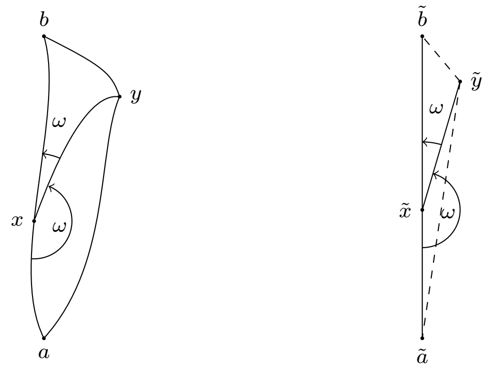

Figure 21: The construction in X and comparison hinges.

By hinge comparison, cf. Theorem 2.17.(ii), we get $p = \tau(a, x) \geq \tau(\tilde{a}, \tilde{x}) = \tilde{p}$ and $q = \tau(x, b) \geq \tau(\tilde{x}, \tilde{b}) = \tilde{q}$. We claim that $2t < \tilde{p} + \tilde{q}$. As $p \geq \tilde{p}$ and $q \geq \tilde{q}$, this implies the above claim.

By the reverse triangle inequality, we have $t > m + \tilde{q}$ and $\tilde{p} > t + m > 2m + \tilde{q}$, thus $\tilde{p} - \tilde{q} > 2m$. Note that the reverse triangle inequality yields strict inequalities since the triangles are non-degenerate. Recall that $\tilde{p} \leq p = \tau(a, y) \leq \tau(a, z) = \frac{\pi}{2} + \frac{\pi}{8}$ and $\tilde{q} \geq q = \tau(y, b) \geq \tau(z, b) = \frac{\pi}{2} + \varepsilon - \frac{\pi}{8} > \frac{\pi}{2} - \frac{\pi}{8}$. Thus, we get $0 < 2m < \tilde{p} - \tilde{q} < \frac{\pi}{4}$. In particular, as cosine is decreasing, we have

$$0 < \cos(\frac{\tilde{p} - \tilde{q}}{2}) < \cos(m). \tag{103}$$

We now write down the equations for $\omega = \measuredangle_{\tilde{x}}(\tilde{a}, \tilde{y}) = \measuredangle_{\tilde{x}}(\tilde{y}, \tilde{b})$ in the law of cosines (cf. [BS22, Lemma 2.4] and remember $K = -1$):

$$\cos(m)\cos(t) - \sin(m)\sin(t)\cosh(\omega) = \cos(\tilde{p}),$$
$$\cos(m)\cos(t) + \sin(m)\sin(t)\cosh(\omega) = \cos(\tilde{q}).$$

We add these two equations to eliminate ω and then use the cosine addition formula:

$$\cos(m)\cos(t) = \frac{1}{2}(\cos(\tilde{p}) + \cos(\tilde{q})) = \cos(\frac{\tilde{p}+\tilde{q}}{2})\cos(\frac{\tilde{p}-\tilde{q}}{2}). \tag{104}$$

We know further reformulate and end up with

$$\cos(t)\frac{\cos(m)}{\cos(\frac{\tilde{p}-\tilde{q}}{2})} = \cos(\frac{\tilde{p}+\tilde{q}}{2}). \tag{105}$$

As $0 < \cos(\frac{\tilde{p}-\tilde{q}}{2}) < \cos(m)$, we know that the fraction on the left hand side is bigger than one, and since $\frac{\pi}{2} < t = \tau(a,x) = \tau(x,b) < \pi$, we have $\cos(t) < 0$. In total, we get

$$\cos(\frac{\tilde{p}+\tilde{q}}{2}) < \cos(t), \tag{106}$$

and as cos is monotonically decreasing, we obtain

$$\tilde{p} + \tilde{q} > 2t, \tag{107}$$

which by above arguments implies the original claim (102). □

Remark 4.12 (Global hyperbolicity and spacetimes)**.** *If we additionally assume that X is a globally hyperbolic Lorentzian length space, then the result can be extended to apply to the diameter, in addition to the finite diameter. Indeed, τ is then finite, cf. [KS18, Theorem 3.28], so the diameter and the finite diameter agree. Our result may then be viewed as an extension of [BE79, Theorem 9.5], in which a bound is derived for the diameter of a globally hyperbolic spacetime with timelike sectional curvature bounded below by $K < 0$.*

Remark 4.13 (An implication of Theorem 4.11)**.** *There is an immediate corollary to the metric version of Theorem 3.5, which states that the perimeter of any triangle in a space with curvature bounded below by k cannot be greater than $\frac{2\pi}{\sqrt{k}}$, see [BBI01, Corollary 10.4.2]. This can be argued using hinge comparison. Typically, deriving Lorentzian analogues of metric results is at least as difficult as the original metric derivation, so it is noteworthy that this corollary is easier to show in the Lorentzian world. In fact, the result immediately follows from the reverse triangle inequality: let $\Delta(x,y,z)$ be a timelike triangle, then by Theorem 4.11, we know $D_K > \tau(x,z) \geq \tau(x,y)+\tau(y,z)$, hence $\tau(x,y) + \tau(y,z) + \tau(x,z) < 2D_K$ as required.*

5 Outlook

Finally, let us discuss potential future research stemming from this paper. In particular, we wish to highlight the possibility of a Lorentzian version of the famous Toponogov Globalization Theorem (Theorem 3.4) for lower curvature bounds. In the smooth Lorentzian case, this was achieved in [Har82], however, despite attempts by the authors and several other researchers, a synthetic Lorentzian equivalent has not yet been obtained. Given the recent work of [BMS22, BS22] formulating curvature bounds on Lorentzian pre-length spaces in terms of monotonicity and angles, it is worth re-examining how well the techniques utilized in the proof of [Har82] adapt to our setting. As [BS22] also introduces curvature comparison of Lorentzian pre-length spaces via hinges, the proof of the metric statement provided in [AKP19, Theorem 8.31] may also be a good point of ingress. Most promisingly, the first and third author of this work are currently part of a collaboration developing a Lorentzian description of (lower) curvature bounds using the so-called four point condition, see [BBI01, Proposition 10.1.1]. This condition is used in Perelman's proof of the Toponogov Theorem for complete length spaces, cf. [BGP92, Section 3.4] (see also [BBI01, Theorem 10.3.1] for more clarification) so is likely to be a strong tool in the arsenal of synthetic Lorentzian geometry.

The authors of the current work have also been investigating generalizations of Theorem 4.11 which require only local curvature bounds, in the spirit of the metric Bonnet–Myers Theorem 3.5. Direct approaches have not yet proven fruitful. However, in the metric case, such a bound on the diameter of a (complete) length space is a direct consequence of Toponogov's Theorem 3.4, hence the space having local curvature bounded below is sufficient [BBI01, Theorem 10.4.1]. It is therefore plausible that the the statement of Theorem 4.11 could be strengthened in a similar manner, once a Toponogov style result is obtained in the Lorentzian pre-length space setting.

Another area of interest is the study of Lorentzian polyhedral spaces. In metric geometry, it is known that one-dimensional polyhedral spaces, or (metric) graphs, which are additionally locally finite and connected, satisfy a local non-positive upper curvature bound. Furthermore, compact length spaces can be described by the Gromov–Hausdorff limit of finite graphs [BBI01, Proposition 7.5.5], performing a discretization of a continuous space. In the theory of quantum gravity, locally finite sets, now equipped with a partial order, are considered similarly when modelling discrete spacetimes [BLMS87, Sur19, DHS04]. Such 'causal sets' are represented by locally finite, transitively reduced, directed, acyclic graphs called Hasse diagrams, where the additional qualifiers reflect the properties of the inferred Lorentzian geometry. It is known that not all causal sets have continuum approximation which is given by a spacetime (see the excellent review paper [Sur19] for more details), however, it is plausible that a wider range of causal sets are approximated by a Lorentzian pre-length space. Conversely, under some additional constraints, it is expected that Lorentzian pre-length spaces are given by the Gromov–Hausdorff limit of causal sets, interpreted as polyhedral spaces, and that causal sets satisfy a timelike curvature bound. This would drive the development of the Lorentzian pre-length space and causal set frameworks forward in parallel, enabling consolidation and verification of derived results.

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The splitting theorem for globally hyperbolic Lorentzian length spaces with non-negative timelike curvature

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Abstract

In this work, we prove a synthetic splitting theorem for globally hyperbolic Lorentzian length spaces with global non-negative timelike curvature containing a complete timelike line. Inspired by the proof for smooth spacetimes [11], we construct complete, timelike asymptotes which, via triangle comparison, can be shown to fit together to give timelike lines. To get a control on their behaviour, we introduce the notion of parallelity of timelike lines in the spirit of the splitting theorem for Alexandrov spaces as proven in [17] and show that asymptotic lines are all parallel. This helps to establish a splitting of a neighbourhood of the given line. We then show that this neighbourhood has the *timelike completeness* property and is hence inextendible by a result in [40], which globalises the local result.

Keywords: Lorentzian length spaces, synthetic curvature bounds, splitting theorems

MSC2020: 53C50, 53C23, 53B30

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1 Introduction

As originally formulated in the context of Riemannian geometry, splitting theorems are a class of results which establish that a Riemannian manifold (M, g) subject to certain hypotheses on its curvature and its geodesic structure actually *splits*, i.e. it is isometric to a product. In most cases, the curvature assumption involves a weak inequality (usually on sectional or Ricci curvature) while the geodesic assumption is related to the existence of a *line*, that is, a globally distance realising complete geodesic. Since lines are conjugate point free and positive curvature promotes the appearance of conjugate points, both features are expected to hold only under very special circumstances.

The very first attempt to establish the metric product structure of a Riemannian manifold with non-negative curvature and having a line is attributed to Cohn-Vossen [27], who proved that a non-compact complete manifold of dimension 2 with sectional curvature $K \geq 0$ is either diffeomorphic to $\mathbb{R}^2$ or flat. However, the proof of this result relied on the Gauss-Bonnet Theorem and therefore was not suitable for generalisation to arbitrary dimensions. The breakthrough that allowed such a generalisation was due to Toponogov, who in the mid 1960s proved his celebrated Triangle Comparison Theorem, which asserts that "the angles of an arbitrary triangle in a Riemannian space made up of minimising arcs are no greater than the corresponding angles of the plane Euclidean triangle with sides of the same length" [62]. Armed with this tool, Toponogov established the archetype of a splitting theorem [61, 62]:

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold with $K \geq 0$ containing a line, then (M, g) is isometric to $(\mathbb{R}^k \times N, g_0 \oplus h)$ where $(\mathbb{R}^k, g_0)$ is the standard Euclidean space and (N, h) is complete and does not have any lines.*

Shortly after Toponogov's splitting theorem was published, Cheeger and Gromoll made substantial progress by generalising it under the less stringent condition $\mathrm{Ric} \geq 0$ [24]. Although their original motivation was rooted in the investigation of topological obstructions on manifolds of non-negative curvature [25], their result sparked the interest in the study of splitting theorems in many different contexts, like in the theory of orbifolds [14] or in the study of curvature inequalities relating other tensors, (like the curvature operator [56] or Bakry-Emery tensor [29, 64]). Furthermore, some more general versions of the Cheeger-Gromoll splitting theorem have been proven. For example, the curvature condition can be replaced by averaged Ricci inequalities [31], or by almost positivity of the Ricci tensor [21, 57]. There are also some local versions in which the splitting occurs either on tubular neighbourhoods or the complement of a compact set disjoint from a line [20, 22].

Remarkably, Yau posed in 1982 the problem of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem [65], thus inaugurating a very active field in which different versions of the Lorentzian splitting theorem were established [11, 10, 28, 30, 32, 55][25], and some important problems have remained unsolved even to this day, like the famous Bartnik Conjecture [8, 33]. In analogy to its Riemannian counterparts, the first Lorentzian splitting theorem was proven for globally hyperbolic spacetimes with non-positive sectional curvature on timelike planes. In [11, 10] Beem, Ehrlich, Markvorsen and Galloway proved what can be thought as the Lorentzian analogue of Toponogov's splitting theorem:

Theorem 1.2. *Let (M, g) be a spacetime of dimension $n \geq 2$ that satisfies the following conditions:*

1. *(M, g) is globally hyperbolic.*
2. *$K(\Pi) \leq 0$ for all timelike planes Π.*
3. *M has a timelike line.*

Then (M, g) splits isometrically as $(M, g) \simeq (\mathbb{R} \times N, -dt^2 \oplus h)$, where (N, h) is a complete Riemannian manifold.

Notice that global hyperbolicity can be seen as a more suitable analogue of Riemannian completeness than timelike geodesic completeness, since the latter fails to guarantee connectability by distance realising geodesic segments. The original proof of this result not only uses techniques related to Toponogov's theorem —a Lorentzian triangle comparison due to Harris [43]—, but also tools used in Cheeger-Gromoll's theorem, like Busemann functions and Hessian estimates.

Some of the key techniques used in Toponogov's original proof can be abstracted into an axiomatic setting rather than derived, thus expanding its range of applications to non-smooth contexts, particularly, to the realm of metric length spaces. Recall that a length space (X, d) is a metric space whose distance is intrinsic, in other words, the distance $d(p, q)$ can be recovered as the infimum of the lengths of curves joining p to q. This is the setting for the so-called synthetic methods in geometry, that have proven instrumental in the development of recent mathematical landmarks such as

[25] Refer to chapter 14 in [9] for a detailed account.

the study of geometry in the large [41, 42], precompactness theorems [17, 18], geometric flows [39, 6] and optimal transport [63, 60]. An *Alexandrov space with curvature bounded from below by* k —or Alex(k) for short— is a locally compact, complete and path connected (metric) length space (X, d) on which the triangle comparison theorem holds [17]. For Alex(0) spaces an analogue of Theorem 1.1 was first proved by Milka [53]. In this result, instead of the existence of a line, the existence of an m-affine function, (i.e. a function $g\colon M \to \mathbb{R}$ that when restricted to any unit speed geodesics satisfies the differential equation $g'' + mg = 0$) is assumed. A warped product $I \times_g F$ with g an m-affine function is called a *cone* [18]. Cones naturally have m-affine functions, and conversely, an Alex(k) space with a non-constant m-affine function splits as a cone, provided that a boundary condition is met [3, 51]. In the particular case of a complete Riemannian manifold, Innami [46] showed that a 0-affine function exists if and only if M is isometric to a product $\mathbb{R} \times N$. Moreover, under the hypothesis of Theorem 1.1 the Busemann functions associated to a line are affine and thus Innami's theorem implies Toponogov's splitting theorem. Finally, let us note that in [17] there is a proof of Milka's splitting theorem for Alexandrov spaces that resembles more closely the original works of Toponogov. The precise statement is as follows:

Theorem 1.3. *Let* (X, d) *be an* Alex(0) *space containing a line. Then* X *splits as a metric product* $\mathbb{R} \times Y$*, where* Y *is an* Alex(0) *space.*

In their seminal work [48] Kunzinger and Sämann set the foundations for a synthetic approach to Lorentzian geometry. Their novel notion of Lorentzian (pre-)length space is suited to accommodate several different non-smooth scenarios such as cones [5, 54], spacetimes with C^0 metrics [26, 50], contact structures [45] or causal boundaries [1]. Moreover, in [48] the authors also introduced a notion of timelike curvature bounds in the same spirit of Alex(k) spaces. Among the recent developments in this fast growing field we have detailed analyses of the causal structure [2, 36], extendibility [40, 35], convergence [49], gluing techniques [12, 58] and the basis for a comparison theory [7, 13].

In this work, we aim at proving a splitting theorem for Lorentzian length spaces in the spirit of Theorems 1.1, 1.2 and 1.3. In precise terms, we establish the following splitting result for Lorentzian length spaces:

Theorem 1.4 (Splitting). *Let* $(X, d, \ll, \leq, \tau)$ *be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric* d *and global non-negative timelike curvature satisfying timelike geodesic prolongation and containing a complete timelike line* $\gamma : \mathbb{R} \to X$*. Then there is a*

τ and causality preserving homeomorphism $f : \mathbb{R} \times S \to X$, where S is a proper, strictly intrinsic metric space of Alexandrov curvature ≥ 0.

The paper is organised as follows: In Section 2, we collect some well-known facts about Lorentzian (pre-)length spaces while focusing on (versions of) results best suited for our needs. We review the basic theory of Lorentzian (pre-)length spaces and discuss concepts like timelike curvature bounds, angles and extensions. Section 3 is dedicated to the study of product Lorentzian pre-length spaces of the form $\mathbb{R} \times X$, where (X, d) is a metric space. Section 4 is the heart of the proof and deals with the construction of co-rays and asymptotes, where we follow [11]. In order to have a control on the behaviour of lines formed as asymptotes to a given line, we introduce the notion of parallelity of timelike lines which is motivated by the splitting theorem for Alexandrov spaces as treated in [17] and show that asymptotic lines are always timelike, of infinite τ-length, and parallel to each other. In Section 5 we first establish a local splitting result by endowing a cross section S of the parallel asymptotes spanning $I(\gamma)$ with a natural metric and then showing that $I(\gamma)$ splits as $\mathbb{R} \times S$. Then we show that $I(\gamma)$ has the timelike completeness (TC) property, from which, via an inextendibility argument, $I(\gamma) = X$ follows. Finally, in Section 6 we note some classes of spacetimes which naturally satisfy the technical assumptions in our results, and hence split synthetically. We then give an outlook on open problems in the context of synthetic splitting theorems that may be addressed in future projects.

Notation and conventions

Let us collect some notation and conventions that will be used throughout the paper.

$A \subset B$ means A is a subset of B (not necessarily a proper one). $\mathbb{R}^{1,1}$ is two dimensional Minkowski space with the Lorentzian metric $-dt^2 + dx^2$ and coordinates (t, x). $\overline{\tau}$ denotes the time separation on $\mathbb{R}^{1,1}$. A *proper* metric space (X, d) is a metric space such that all closed balls are compact. A metric space is called *length space* or *intrinsic* if the distance between two points is the infimum of lengths of curves running between them, and *strictly intrinsic* if that infimum is a minimum, i.e. between any two points there exists a distance-realising curve.

We denote an open ball with radius R around a point x in a metric space by $B_R(x)$, and the corresponding closed ball by $\bar{B}_R(x)$. Although in general this ball is not the closure $\overline{B_R(x)}$ of the open ball, notice that for length spaces and proper metric spaces we do have the equality $\bar{B}_R(x) = \overline{B_R(x)}$.

2 Basic theory of Lorentzian (pre-)length spaces

2.1 Lorentzian (pre-)length spaces

We give a brief review of the theory of Lorentzian (pre-)length spaces. We focus on the specific results that we need and give proofs if they cannot be found in the literature, otherwise we give precise references. In particular, we refer to [48] for a detailed treatment. We start with a few basic definitions.

Definition 2.1 (Lorentzian pre-length spaces)**.** *A* Lorentzian pre-length space *$(X, d, \ll, \leq, \tau)$ consists of a metric space (X, d), a reflexive and transitive relation $\leq$ (the causal relation), a transitive relation $\ll$ (the timelike relation) contained in $\leq$, and a lower semi-continuous map (time separation) $\tau : X \times X \to [0, \infty]$ with the following properties: $\tau(x, y) = 0$ if $x \not\leq y$, and $\tau(x, y) > 0$ if and only if $x \ll y$. Moreover, if $x \leq y \leq z$, then the following* reverse triangle inequality *holds:*

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z).$$

In the synthetic theory, we employ the usual nomenclature and well-known notation from Lorentzian geometry, such as $I^{\pm}(x)$, $J^{\pm}(x)$ for timelike and causal pasts and futures, as well as $I(x, y) := I^+(x) \cap I^-(y)$ and $J(x, y) := J^+(x) \cap J^-(y)$ for timelike and causal diamonds, respectively.

Lemma 2.2 (Openness of timelike futures and push-up)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space and $x \ll y \leq z$ or $x \leq y \ll z$. Then $x \ll z$. In particular, for any $x \in X$, the sets $I^{\pm}(x)$ are open. Moreover, the relation $\ll$ is open in $X \times X$.*

Proof. See [48, Lem. 2.10, Lem. 2.12, Prop. 2.13]. □

Definition 2.3 (Causal curves)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A non-constant curve $\gamma : I \to X$, I an interval, is called* future directed causal *(resp.* future directed timelike*) if it is locally Lipschitz and for all $t_1, t_2 \in I$ with $t_1 < t_2$ we have that $\gamma(t_1) \leq \gamma(t_2)$ (resp. $\gamma(t_1) \ll \gamma(t_2)$). It is called* future directed null *if it is future directed causal and no two points on the curve are $\ll$-related.* Past directed causal/timelike/null *curves are defined dually. From now on, unless explicitly stated otherwise, we assume all causal curves to be future directed.*

We will sometimes deal with causal curves which are not locally Lipschitz. For these curves, some important results (e.g. the limit curve theorem) will not hold in general, however, many important properties (especially those that are purely causal-theoretic in nature) will continue to be true.

Definition 2.4 (Continuous causal curves)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A non-constant curve $\gamma : I \to X$, I an interval, is called a* future directed continuous causal curve *(resp.* future directed continuous timelike curve*) if it is continuous and for all $t_1, t_2 \in I$ with $t_1 < t_2$ we have that $\gamma(t_1) \leq \gamma(t_2)$ (resp. $\gamma(t_1) \ll \gamma(t_2)$). It is called a* future directed continuous null curve *if it is a future directed continuous causal curve and no two points on the curve are $\ll$-related. Past versions of these notions are defined dually. From now on, unless explicitly stated otherwise, we assume all continuous causal curves to be future directed.*

We emphasise that a *causal/timelike/null curve* is always understood to be locally Lipschitz. If *continuous* causal/timelike/null curves are meant, we will state that explicitly. Wherever possible, we will state definitions and results for continuous causal curves instead of (locally Lipschitz) causal curves for greater generality. In most cases, the proofs given in the literature for these results apply word for word to the continuous case.

Definition 2.5 (Extensions of causal curves)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $\gamma : I \to X$ be a continuous causal curve, where I is any interval. We say that γ is* (continuously) extendible *if there exists a (continuous) causal curve $\tilde{\gamma} : J \to X$, J an interval with $J \supsetneq I$, such that $\tilde{\gamma}|_I = \gamma$. Otherwise, we say γ is* (continuously) inextendible*. We refer to (in)extendibility to points in the future resp. past directions of γ as future/past (in)extendibility.*[26]

Definition 2.6 (τ-length)**.** *Let $\gamma : [a, b] \to X$ be a future directed continuous causal curve. Then its* τ-length *is defined by*

$$L_\tau(\gamma) := \inf \left\{ \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) \mid a = t_0 < t_1 < \cdots < t_N = b \right\}.$$

If γ is past directed causal, then $L_\tau(\gamma)$ is defined analogously. For continuous causal curves defined on half-open intervals, e.g. $[a, b)$, one takes the limit of $L_\tau(\gamma|_{[a,c]})$ as $c \to b$, and similarly in the other cases.

The τ-length of a continuous causal curve is invariant under reparametrisations. We will have more to say on these topics later.

Definition 2.7 (Maximising causal curves)**.** *A future directed (continuous) causal curve $\gamma : [a, b] \to X$ is called* maximising *(or τ-*maximising*) if $\tau(\gamma(a), \gamma(b)) = L_\tau(\gamma)$, and analogously for past directed (continuous) causal curves. We also refer to such curves as* (continuous) distance realisers.

[26] Geodesics defined on intervals $[a, b]$ with $a = -\infty$ or $b = \infty$ should be thought of as properly reparametrised.

Lorentzian pre-length spaces where between each pair of causally related points there is a maximising (locally Lipschitz) causal curve are referred to as *strictly intrinsic* or *geodesic*.

Next, we define some of the steps on the causal ladder for Lorentzian pre-length spaces.

Definition 2.8 (Causality conditions)**.** *A Lorentzian pre-length space* $(X, d, \ll, \leq, \tau)$ *is called*

1. non-totally imprisoning *if for every compact set* $K \subset X$ *there is* $C > 0$ *such that the d-lengths of all causal curves in* K *is bounded by* C*,*
2. strongly causal *if the Alexandrov topology generated by the subbasis of timelike diamonds* $\{I(x, y) \mid x, y \in X\}$ *coincides with the metric topology,*
3. globally hyperbolic *if it is non-totally imprisoning and all* causal diamonds $J(x, y)$ *are compact.*

Lemma 2.9. *Let* $(X, d, \ll, \leq, \tau)$ *be a strongly causal Lorentzian pre-length space. Then each* $x \in X$ *has a neighbourhood basis consisting of causally convex*[27] *neighbourhoods. Even more is true:* X *has a topological basis of causally convex neighbourhoods.*

Proof. This is trivial since by definition finite intersections of timelike diamonds are a basis for the topology, but these sets are causally convex. □

Definition 2.10 (Causal path-connectedness)**.** *A Lorentzian pre-length space* X *is called* causally path-connected *if for all* $x < y$ *there is a causal curve from* x *to* y *and for all* $x \ll y$ *there is a timelike curve from* x *to* y*.*

For the next definition, we use the version given in [2, Def. 2.16] rather than [48, Def. 3.4] as it more closely resembles the case of smooth spacetimes. Before doing so, note that on causally path-connected Lorentzian pre-length spaces X, we can introduce a *local causal relation*: Let $U \subset X$ be open and $x, y \in U$, then we say $x \leq_U y$ if there exists a future causal curve from x to y entirely contained in U.

Definition 2.11 (Local causal closedness)**.** *Let* X *be causally path-connected and* $U \subset X$ *be open. We say that* U *is* causally closed *if for all sequences* p_n, q_n *in* U *converging to* $p, q \in U$ *and* $p_n \leq_U q_n$*, we have that* $p \leq_U q$*. We say that* X *is* locally causally closed *if every point in* X *has a causally closed neighbourhood.*

[27] A set $V \subset X$ is called *causally convex* if for all $p, q \in V$ we have $J(p, q) \subset V$.

Definition 2.12 (Global causal closedness)**.** *A Lorentzian pre-length space X is called globally causally closed if $\leq$ is a closed relation.*

Recall that in the theory of smooth spacetimes, a globally causally closed and causal spacetime is *causally simple.* Such spacetimes are one step below global hyperbolicity in the classical causal ladder.

Definition 2.13 (d-compatibility)**.** *A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is called* d-compatible *if for every $x \in X$ there is a neighbourhood U of x and a constant $C > 0$ such that the d-lengths of all causal curves contained in U are bounded above by C.*

Lemma 2.14. *Let $(X, d, \ll, \leq, \tau)$ be a locally causally closed, d-compatible, globally hyperbolic Lorentzian pre-length space. Then no compact set in X contains an inextendible causal curve.*

Proof. See [48, Cor. 3.15]. □

Definition 2.15 (Localisability)**.** *A Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is called* localisable *if for each $x \in X$ there exists a neighbourhood (called* localising neighbourhood*) Ω_x containing x with the following properties:*

1. *Causal curves in Ω_x have uniformly bounded d-length, i.e., Ω_x is d-compatible.*

2. *For each $y \in \Omega_x$, $I^{\pm}(y) \cap \Omega_x \neq \emptyset$.*

3. *There is a continuous map (*local time separation*) $\omega_x : \Omega_x \times \Omega_x \to [0, \infty)$ such that Ω_x is a Lorentzian pre-length space upon restricting $d, \ll, \leq$.*

4. *For all $p, q \in \Omega_x$ with $p < q$ there exists a causal curve γ_{pq} from p to q entirely in Ω_x with maximal τ-length among all causal curves from p to q contained in Ω_x, as well as $L_\tau(\gamma_{pq}) = \omega_x(p, q)$.*

Note that this necessarily means that ω_x is of the form

$$\omega_x(p, q) := \max\{L_\tau(\gamma) \mid \gamma \text{ is a causal curve from } p \text{ to } q \text{ in } \Omega_x.\}. \tag{108}$$

If in addition, for $p, q \in \Omega_x$ with $p \ll q$ the curve γ_{pq} is timelike and strictly longer than any causal curve from p to q in Ω_x containing a null segment, then Ω_x is called a regular localising neighbourhood, *and X* regularly localisable *if any point has a regular localising neighbourhood. Moreover, X is called* strongly localisable *if each point has a neighbourhood base of localising neighbourhoods and* SR-localisable *if each point has a neighbourhood base of regular localising neighbourhoods.*

Lemma 2.16. *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal Lorentzian pre-length space.*

1. *If X is localisable, then it is strongly localisable.*

2. *If X is regularly localisable, then it is SR-localisable.*

Proof. By Lemma 2.9, each point in X has a neighbourhood basis of causally convex neighbourhoods. Now suppose X is localisable, take $x \in X$ and let Ω_x be a localising neighbourhood of x and let $U \subset \Omega_x$ be any neighbourhood of x. Then there is a causally convex neighbourhood V of x such that $V \subset U$. We are done if we show that V is a localising neighbourhood. But this is easy to see since any causal curve starting and ending in V is entirely contained in there. If X is regularly localisable, then V is even a regular localising neighbourhood. $\square$

Regularly localisable spaces have the following important property:

Theorem 2.17 (Causal character of maximising causal curves)**.** *In a regularly localisable Lorentzian pre-length space, maximising causal curves are either timelike or null.*

Proof. See [48, Thm. 3.18]. $\square$

Note that in localisable Lorentzian pre-length spaces, the sets $I^{\pm}(y)$ are never empty for any $y \in X$.

Definition 2.18 (Geodesic)**.** *Let X be a localisable Lorentzian pre-length space and $\gamma : I \to X$ a (continuous) causal curve. Then γ is a (future-directed causal)* (continuous) geodesic *if for each $t_0 \in I$ there exists a localising neighbourhood Ω of $\gamma(t_0)$ and a neighbourhood $[c, d] \subset I$ of t_0 such that $\gamma([c, d]) \subset \Omega$ and $\gamma|_{[c,d]}$ is maximising in Ω with respect to the local time separation.*

We will sometimes need the notion of *geodesic (in)extendibility* which is defined in terms of the existence of a geodesic extending a given geodesic. Note that inextendibility implies geodesic inextendibility but the converse need not hold in general.

For many arguments in the proof of the splitting theorem, the property that geodesics can always be extended to open domains will be essential. While this is a consequence of the ODE theory of the geodesic equation in the case of smooth spacetimes, we will need to assume it here:

Definition 2.19 (Timelike geodesic prolongation)**.** *A localisable Lorentzian pre-length space X is said to have the* timelike geodesic prolongation *property if any maximising timelike segment $\gamma : [a, b] \to X$ can be extended to a timelike geodesic with an open domain, i.e. there is an $\varepsilon > 0$ and a timelike geodesic $\bar{\gamma} : (a - \varepsilon, b + \varepsilon) \to X$ with $\bar{\gamma}|_{[a,b]} = \gamma$.*

It is clear that this is satisfied in smooth strongly causal spacetimes considered as a Lorentzian pre-length space. One easily sees that spacetimes with timelike boundary do not satisfy this.

Definition 2.20 (Locally quasiuniformly maximising)**.** *A localisable Lorentzian pre-length space X is called* locally quasiuniformly maximising *if for each $x \in X$ there exists a localisable neighbourhood U containing x such that each causal geodesic entirely contained in U is maximising in U with respect to the local time separation.*

It is unclear to us how the property of being locally quasiuniformly maximizing is related to other notions for Lorentzian pre-length spaces, such as being intrinsic, strongly causal or (locally) causally closed.

Proposition 2.21 (Upper semicontinuity of Lorentzian arclength)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal and localisable Lorentzian pre-length space. Then L_τ is upper semi-continuous, i.e. if $\gamma_n : [a, b] \to X$ are causal curves converging d-uniformly to a causal curve $\gamma : [a, b] \to X$, then*

$$L_\tau(\gamma) \geq \limsup_{n\to\infty} L_\tau(\gamma_n).$$

Proof. See [48, Prop. 3.17]. □

Proposition 2.22. *Let X be a strongly causal, localisable, locally quasiuniformly maximising Lorentzian pre-length space and let $\gamma : [a, b) \to X$ be a causal geodesic. Then γ is geodesically extendible to $[a, b]$ if and only if it is extendible as a continuous curve to $[a, b]$.*

Proof. See [40, Prop. 4.6] and note that the proof really requires the space to be locally quasiuniformly maximising. □

We now state the limit curve theorem in the case of strongly causal Lorentzian pre-length spaces, which will be sufficient for our needs. Note that in this case, by [2, Prop. 2.21], Definition 2.11 is equivalent to [48, Def. 3.4].

Theorem 2.23 (Limit curve theorem)**.** *Let $(X, d, \ll, \leq, \tau)$ be a locally causally closed, localisable, strongly causal, d-compatible Lorentzian pre-length space with proper metric d. Let $\gamma_n : [0, L_n] \to X$ be a sequence of causal curves parametrised by d-arclength, and suppose that $L_n = L^d(\gamma_n) \to \infty$. If the sequence $\gamma_n(0)$ has an accumulation point $x \in X$, then some subsequence of the γ_n converges locally uniformly to a future inextendible causal curve $\gamma : [0, \infty) \to X$. Moreover, if the γ_n are maximising, then so is γ. In this case, for any $T > 0$, $L_\tau(\gamma|_{[0,T]}) = \lim_n L_\tau(\gamma_n|_{[0,T]})$.*

Proof. The statement about the existence of the limit curve is shown in [48, Thm. 3.14]. Now suppose that γ_n are maximising and w.l.o.g. let γ_n itself converge locally uniformly to γ. Consider any $T > 0$. Then by upper semicontinuity of L_τ and lower semicontinuity of τ,

$$\limsup_n L_\tau(\gamma_n|_{[0,T]}) \leq L_\tau(\gamma|_{[0,T]}) \leq \tau(\gamma(0), \gamma(T)) \leq \liminf_n \tau(\gamma_n(0), \gamma_n(T)) = \liminf_n L_\tau(\gamma_n|_{[0,T]}).$$

Thus, equality holds everywhere. In particular, γ is maximising and $\lim_n L_\tau(\gamma_n|_{[0,T]}) = L_\tau(\gamma|_{[0,T]})$. □

In the context of the splitting theorem, we are interested in Lorentzian pre-length spaces with significantly more structure, which we will discuss now.

Definition 2.24 (Lorentzian length spaces)**.** *A locally causally closed, causally path-connected and localisable Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is called* Lorentzian length space *if for all $x \leq y$, $x \neq y$,*

$$\tau(x, y) = \sup\{L_\tau(\gamma) \mid \gamma \textit{ future directed causal from } x \textit{ to } y\}.$$

Note that since by definition $L_\tau(\gamma) \leq \tau(p, q)$ for any (even continuous) causal curve γ from p to q, one can equivalently take the supremum over continuous causal curves to obtain the time separation in the case of Lorentzian length spaces.

Proposition 2.25 (Causality of Lorentzian length spaces)**.**

1. *A Lorentzian length space is strongly causal if and only if each point has a neighbourhood base of causally convex neighbourhoods.*

2. *Strongly causal Lorentzian length spaces are non-totally imprisoning.*

3. *Globally hyperbolic Lorentzian length spaces are strongly causal. Moreover, τ is continuous and finite and for any two causally related points there exists a maximising causal curve connecting them.*

Proof. See [48, Thm. 3.26, Thm. 3.28, Thm. 3.30]. □

Lemma 2.26. *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal Lorentzian pre-length space such that for each $x \in X$ there exists a timelike curve $\gamma : [-\varepsilon, \varepsilon] \to X$ with $\gamma(0) = x$. Then $\{I(x, y) \mid x, y \in X\}$ is a base for the topology on X. Moreover, for each $x \in X$, each timelike curve $\gamma : [-\varepsilon, \varepsilon] \to X$ with $\gamma(0) = x$ and parameters $0 < s_n, t_n \to 0$, $\{I(\gamma(-s_n), \gamma(t_n)) \mid n \in \mathbb{N}\}$ is a neighbourhood base at x.*

Proof. Let $x \in X$ with $x \in U$, $U \subset X$ open. By the definition of the Alexandrov topology, finite intersections of the sets $I(y, z)$ with $y, z \in X$ are a base. Thus, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ such that

$$x \in V := I(x_1, y_1) \cap \cdots \cap I(x_n, y_n) \subset U.$$

Now let $\gamma : [-\varepsilon, \varepsilon] \to X$ be a timelike curve with $\gamma(0) = x$. By continuity, maybe after making ε smaller, we may assume that $\gamma([-\varepsilon, \varepsilon]) \subset V$. Then $W := I(\gamma(-\varepsilon), \gamma(\varepsilon))$ is an open set containing x and we claim that $W \subset V$. To see this, let $z \in W$, then for any $i = 1, \ldots, n$ we have $x_i \ll \gamma(-\varepsilon) \ll z$, hence $z \in I^+(x_i)$. Similarly, $z \ll \gamma(\varepsilon) \ll y_i$, hence $z \in I^-(y_i)$. Thus $z \in V$. This proves that $\{I(x, y) \mid x, y \in X\}$ is a base for the topology of X, and the second claim then easily follows from the arguments above. □

Lorentzian pre-length spaces where each point lies in the interior of a timelike curve are future and past approximating in the sense of [19, Def. 2.17] (see also [19, Lem. 2.18])

Lemma 2.27 (Geodesics in strongly causal Lorentzian length spaces)**.** *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal Lorentzian length space. A (continuous) causal curve $\gamma : I \to X$ is a (continuous) geodesic if and only if for each $t_0 \in I$ there exists a subinterval $[c, d] \subset I$ with $t_0 \in [c, d]$ such that $\gamma|_{[c,d]}$ is τ-maximising.*

Proof. This follows from [40, Lem. 4.3]. □

To conclude this subsection, we discuss d-arclength and τ-arclength reparametrisations of causal curves.

Remark 2.28 (On d-arclength parametrisations). Let $(X, d, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian length space with proper metric d and $\gamma : [a, b) \to X$ a causal curve that is inextendible at b. Then γ has infinite d-length: By Lemma 2.14, γ has to leave every compact set, in particular it leaves each compact ball $\overline{B}_n(\gamma(a))$, which shows that $L^d(\gamma) = \infty$. If we reparametrise γ by d-arclength (note that this is in general not possible for continuous causal curves), it is thus defined on $[0, \infty)$. Similarly, if we have a doubly inextendible causal curve $\tilde{\gamma} : (a, b) \to X$, then upon reparametrising it by d-arclength, it is defined on $\mathbb{R}$, so that $L^d(\tilde{\gamma}|_{[c,d]}) = |d - c|$ (see also [19, Rem. 2.5]). Note that d-arclength parametrisations are globally 1-Lipschitz.

Lemma 2.29 (τ-arclength parametrisations for inextendible curves)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space with continuous time separation τ which satisfies $\tau(x, x) = 0$ for all $x \in X$. Let $\gamma : [a, b) \to X$ be a future inextendible continuous timelike curve defined on a half-open interval with $L_\tau(\gamma|_{[a,c]}) < \infty$ for each $c \in [a, b)$ (notice we might have $L_\tau(\gamma) = \infty$). Then the map $\phi : [a, b) \to [0, L_\tau(\gamma))$, $t \mapsto L_\tau(\gamma|_{[a,t]})$ is continuous and strictly increasing. Moreover, $\tilde{\gamma} := \gamma \circ \phi^{-1} : [0, \infty) \to X$ is a weak reparametrisation of γ with respect to τ-arclength. Similarly, a doubly inextendible continuous timelike curve $\gamma : (a, b) \to X$ can be weakly parametrised by τ-arclength and is then defined on $\big(-L_\tau(\gamma|_{(a,t_0]}), L_\tau(\gamma|_{[t_0,b)})\big)$, where $t_0 \in (a, b)$ is arbitrary.*

Proof. This can be seen by applying [48, Lem. 3.33, Lem. 3.34] to each $\gamma|_{[a,\tilde{b}]}$ with $\tilde{b} < \infty$. $\square$

Note that, in general, the τ-arclength parametrisation results in a continuous timelike curve, even if the original curve is locally Lipschitz.

In the following, whenever τ-arclength parametrised curves are mentioned, it is always implicitly understood that those curves have finite τ-length on compact subintervals. In localisable spaces this is anyway the case.

2.2 Timelike curvature bounds

Establishing an analogue to sectional curvature bounds in semi-Riemannian manifolds à la [4] was one of the main goals in the first work on Lorentzian length spaces [48]. Historically, splitting theorems always used some type of curvature condition, either Ricci or sectional curvature. By M_k we denote the two-dimensional Lorentzian model space of constant sectional curvature k and, in this subsection only, $\bar{\tau}$ denotes the time separation in M_k (in general $\bar{\tau}$ will only be used as the time separation in $M_0 = \mathbb{R}^{1,1}$). For more details on this, see [48, Definition 4.5]. We will always assume that all mentioned triangles satisfy size bounds for M_k, cf. [48, Lemma 4.6]. However, since we are mainly working with $k = 0$, we do not have to worry about this.

We call three points $x_1, x_2, x_3 \in X$ that are timelike related together with maximisers between them a *timelike triangle*. We denote such triangles by $\Delta(x_1, x_2, x_3)$.

Definition 2.30. *Let X be a Lorentzian pre-length space and $\Delta(x_1, x_2, x_3)$ be a timelike triangle. Then a timelike triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_k such that $\tau(x_i, x_j) = \bar{\tau}(\bar{x}_i, \bar{x}_j)$ is called a* comparison triangle *for $\Delta(x_1, x_2, x_3)$. It always exists (if the size bounds are satisfied) and is unique up to isometries of M_k.*

Definition 2.31 (Timelike curvature bounds)**.** *A Lorentzian pre-length space X is said to satisfy a* timelike curvature bound from below by k*, if each point in X has a so-called comparison neighbourhood U, satisfying the following:*

1. *$\tau|_{U \times U}$ is finite and continuous.*
2. *For all $x, y \in U$ with $x \ll y$ there exists a causal maximiser γ from x to y which is entirely contained in U.*
3. *Let $\Delta(x, y, z)$ be a triangle in U and let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be its comparison triangle in M_k. Let $p, q \in \Delta(x, y, z)$ and let $\bar{p}, \bar{q}$ be the corresponding comparison points in $\Delta(\bar{x}, \bar{y}, \bar{z})$. Then $\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q})$.*

For curvature bounds from above, the inequality in (iii) is reversed, but we will not need these types of curvature bounds. If $k = 0$, we also say that X has non-negative timelike curvature *resp.* non-positive timelike curvature*.*

Evidently, curvature bounds are only formulated locally as of yet. There is, however, a very natural way of globalising this concept.

Definition 2.32 (Global curvature bound)**.** *A Lorentzian pre-length space X is said to have* global curvature bounded below by *k if X itself is a comparison neighbourhood in the sense of Definition 2.31.*

Remark 2.33 (Continuous triangles vs. Lipschitz triangles). If X is a globally hyperbolic Lorentzian length space with global nonnegative timelike curvature, then in fact curvature comparison even holds for timelike triangles where the maximisers are continuous: Indeed, suppose $\Delta := \Delta_{C^0}(x, y, z)$ is a continuous timelike triangle, and let $p, q \in \Delta$. Due to the properties of X, we find Lipschitz maximisers from the endpoints of Δ to p, q, respectively. The concatenations at p and q respectively of those maximisers are again maximisers because the sides on Δ are (continuous) maximisers. Hence we have realised p, q on a Lipschitz triangle. If X is in addition regularly localisable, then in fact every continuous timelike maximiser is Lipschitz reparametrisable [52], so the above discussion is superfluous in this case.

One of the most commonly used implications of lower (timelike) curvature bounds is the prohibition of branching of distance-realisers. A formulation of this result for Lorentzian pre-length spaces was first given in [48, Theorem 4.12]. However, with the introduction of hyperbolic angles in [13] it was possible to generalise this result by omitting some of the additional assumptions:

Theorem 2.34 (Timelike non-branching)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded below by $k \in \mathbb{R}$. Then timelike distance realisers cannot branch, i.e., if $\alpha, \beta : [-\varepsilon, \varepsilon] \to X$ are timelike distance realisers such that there exists $t_0 \in (-\varepsilon, \varepsilon)$ with $\alpha|_{[-\varepsilon,t_0]} = \beta|_{[-\varepsilon,t_0]}$, then $\alpha = \beta$.*

Proof. See [13, Theorem 4.7]. □

The non-branching of timelike distance realisers is a key property of spaces with lower curvature bounds and will appear in various forms in our proof of the splitting theorem.

2.3 Angles and comparison angles

Hyperbolic angles in Lorentzian pre-length spaces were introduced in [13] and [7], where the latter puts a bigger focus on comparison results. We will follow the conventions of the former reference.

Lemma 2.35 (The law of cosines ($k = 0$))**.** *Let $X = \mathbb{R}^{1,1}$ be two-dimensional Minkowski space and x_1, x_2, x_3 be three points which are timelike related (an (unordered) timelike triangle). Let $a_{ij} = \max(\tau(x_i, x_j), \tau(x_j, x_i))$ (note one of these is zero anyway). Let ω be the hyperbolic angle between the straight lines x_1x_2 and x_2x_3 at x_2. Set $\sigma = 1$ if x_2 is not a time endpoint of the triangle (i.e., $x_1 \ll x_2 \ll x_3$ or $x_3 \ll x_2 \ll x_1$) and $\sigma = -1$ if x_2 is a time endpoint of the triangle (i.e., $x_2 \ll x_1, x_3$ or $x_1, x_3 \ll x_2$). Then we have:*

$$a_{13}^2 = a_{12}^2 + a_{23}^2 + \sigma a_{12} a_{23} \cosh(\omega).$$

In particular, when only changing one side-length, the angle ω is a monotonously increasing function of the longest side-length and monotonously decreasing in the other side-lengths.

Proof. See [13, Appendix A]. □

Note that for $a_{ij} > 0$ satisfying a reverse triangle inequality and choosing an appropriate $\sigma = \pm 1$, we can always solve this equation for ω.

Definition 2.36 (Comparison angles)**.** *Let X be a Lorentzian pre-length space and x_1, x_2, x_3 three timelike related points. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{R}^{1,1}$ be a comparison triangle for x_1, x_2, x_3. We define the* comparison angle $\tilde{\measuredangle}_{x_2}(x_1, x_3)$

as the hyperbolic angle between the straight lines $\bar{x}_1\bar{x}_2$ and $\bar{x}_2\bar{x}_3$ at $\bar{x}_2$. It can be calculated with the law of cosines using $a_{ij} = \max(\tau(x_i, x_j), \tau(x_j, x_i))$ and σ, where we set $\sigma = 1$ if x_2 is not a time endpoint of the triangle and $\sigma = -1$ if x_2 is a time endpoint of the triangle. σ is called the sign *of the comparison angle (even though we always have $\tilde{\measuredangle}_{x_2}(x_1, x_3) > 0$). For a reduction of the number of case distinctions we also define the* signed comparison angle $\tilde{\measuredangle}^{\mathrm{S}}_{x_2}(x_1, x_3) = \sigma\tilde{\measuredangle}_{x_2}(x_1, x_3)$.

Definition 2.37 (Angles)**.** *Let X be a Lorentzian pre-length space and $\alpha, \beta : [0, \varepsilon) \to X$ be two timelike curves (future or past directed or one of each) with $x := \alpha(0) = \beta(0)$. Then we define the* upper angle

$$\measuredangle_x(\alpha, \beta) = \limsup_{(s,t)\in D, s,t\to 0} \tilde{\measuredangle}_x(\alpha(s), \beta(t)),$$

where $D = \{(s, t) : s, t > 0,\ \alpha(s), \beta(t)$ timelike related$\}$. If the limes superior is a limit and finite, we say the angle exists *and call it an* angle.

Note that the sign of the comparison angle is independent of $(s, t) \in D$. We define the sign *of the (upper) angle σ to be that sign, and define the* signed (upper) angle *to be $\measuredangle^{\mathrm{S}}_x(\alpha, \beta) = \sigma\measuredangle_x(\alpha, \beta)$.*

If maximisers between any two timelike related points are unique (as e.g. in $\mathbb{R}^{1,1}$), then we simply write $\measuredangle_p(x, y)$ for the angle at p between the maximisers from p to x and p to y.

We give an alternative definition of timelike curvature bounds in the case $k = 0$ following Definition 4.9 in [13]:

Definition 2.38. *Let X be a Lorentzian pre-length space. We say that X has* timelike curvature bounded below by $k = 0$ (bounded above by $k = 0$) in the sense of monotonicity comparison *if every point in X has a neighbourhood U such that*

(i) *$\tau|_{U\times U}$ is finite and continuous.*

(ii) *U is* timelike geodesically connected*, i.e. whenever $x, y \in U$ with $x \ll y$, there exists a future-directed timelike distance realiser α in U from x to y, and any distance realiser from x to y contained in U is timelike.*

(iii) *Whenever $\alpha : [0, a] \to U$, $\beta : [0, b] \to U$ are timelike distance realisers in U with $x := \alpha(0) = \beta(0)$, we define the function $\theta : (0, a] \times (0, b] \supset D \to \mathbb{R}$ by $\theta(s, t) := \tilde{\measuredangle}^{\mathrm{S}}_x(\alpha(s), \beta(t))$, where $(s, t) \in D$ precisely when $\alpha(s), \beta(t)$ are timelike related. We require this to be monotonically increasing (decreasing) in s and t.*

Additionally, in the case of non-positive timelike curvature, we consider $0 < s' \leq s$ and $0 < t' \leq t$ and require that if $\theta(s', t')$ is not defined (i.e. $\alpha(s'), \beta(t')$ are not timelike related) but $\theta(s, t)$ is, then also the comparison points for $\alpha(s'), \beta(t')$ in a comparison triangle for $\Delta(x, \alpha(s), \beta(t))$ are not timelike related.

We call the curvature bound global if X is a comparison neighbourhood in the above sense.

Theorem 2.39 (Timelike curvature: Equivalence of definitions)**.** *Let X be a locally timelike geodesically connected Lorentzian pre-length space. Then X has non-negative (non-positive) timelike curvature in the sense of Definition 2.31 if and only if it has non-negative (non-positive) timelike curvature in the sense of monotonicity comparison (see Definition 2.38).*

Proof. See [13, Theorem 4.12]. □

Remark 2.40. Note that if X is *globally timelike geodesically connected*, i.e. between any two timelike related points there is a timelike distance realiser connecting them, and any distance realiser connecting them is timelike (which is certainly true if X is a regularly localisable and globally hyperbolic Lorentzian length space), then X has global non-negative (non-positive) timelike curvature in the sense of Definition 2.32 if and only if it has global non-negative (non-positive) timelike curvature in the sense of monotonicity comparison.

The timelike geodesic prolongation property plays a technical role in our proof of the splitting theorems mainly because of the following result.

Proposition 2.41 (Continuity of angles in spaces with timelike curvature bounded below)**.** *Let X be a strongly causal, localisable, timelike geodesically connected and locally causally closed Lorentzian pre-length space with global non-negative timelike curvature and which satisfies timelike geodesic prolongation. Let $\alpha_n, \alpha, \beta_n, \beta$ be future or past directed timelike geodesics all starting at $\alpha_n(0) = \alpha(0) = \beta_n(0) = \beta(0) =: x$ and with $\alpha_n \to \alpha$ and $\beta_n \to \beta$ pointwise (in particular, α_n is future directed if and only if α is, and similarly for β_n and β). Then*

$$\measuredangle_x(\alpha, \beta) = \lim_n \measuredangle_x(\alpha_n, \beta_n)$$

Proof. See [13, Proposition 4.14]. □

The following two propositions will be proven together.

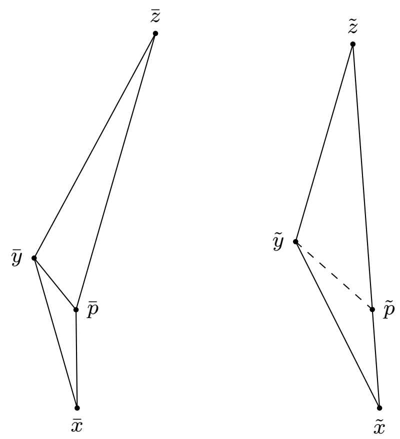

Figure 22: A concave situation in the across version.

Proposition 2.42 (Alexandrov lemma: across version)**.** *Let X be a Lorentzian pre-length space. Let $\Delta := \Delta(x, y, z)$ be a timelike triangle (in particular the distance realisers between the endpoints exist). Let p be a point on the side xz with $p \ll y$, such that the distance realiser between p and y exists. Then we can consider the smaller triangles $\Delta_1 := \Delta(x, p, y)$ and $\Delta_2 := \Delta(p, y, z)$. We construct a comparison situation consisting of a comparison triangle $\bar{\Delta}_1$ for Δ_1 and $\bar{\Delta}_2$ for Δ_2, with $\bar{x}$ and $\bar{z}$ on different sides of the line through $\bar{p}\bar{y}$ and a comparison triangle $\tilde{\Delta}$ for Δ with a comparison point $\tilde{p}$ for p on the side xz. This contains the subtriangles $\tilde{\Delta}_1 := \Delta(\tilde{x}, \tilde{y}, \tilde{p})$ and $\tilde{\Delta}_2 := \Delta(\tilde{p}, \tilde{y}, \tilde{z})$ which are timelike if $\tilde{p} \ll \tilde{y}$, see Figure 22.*

Then the situation $\bar{\Delta}_1$, $\bar{\Delta}_2$ is convex (concave) at p (i.e. $\tilde{\measuredangle}_p(x, y) = \measuredangle_{\bar{p}}(\bar{x}, \bar{y}) \geq \measuredangle_{\bar{p}}(\bar{y}, \bar{z}) = \tilde{\measuredangle}_p(y, z)$ (or $\leq$)) if and only if $\tau(p, y) \leq \bar{\tau}(\tilde{p}, \tilde{y})$ (or $\geq$). If this is the case and $\tilde{p} \ll \tilde{y}$, we have that

- *each angle in the triangle $\bar{\Delta}_1$ is $\geq$ (or $\leq$) than the corresponding angle in the triangle $\tilde{\Delta}_1$,*
- *each angle in the triangle $\bar{\Delta}_2$ is $\geq$ (or $\leq$) than the corresponding angle in the triangle $\tilde{\Delta}_2$.*

In any case, we have that

- $\measuredangle_{\bar{y}}(\bar{x}, \bar{z}) \geq \measuredangle_{\tilde{x}}(\tilde{x}, \tilde{z}) = \tilde{\measuredangle}_y(x, z)$.

The same is true if p is a point on the side xz such that $y \ll p$. Note that if X has non-negative (non-positive) timelike curvature and Δ is within a comparison neighbourhood, the condition is satisfied, i.e. $\tau(p, y) \leq \bar{\tau}(\tilde{p}, \tilde{y})$ (or $\geq$).

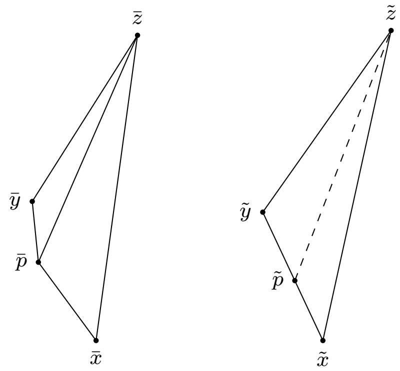

Figure 23: A convex situation in the future version.

Proposition 2.43 (Alexandrov lemma: future version)**.** *Let X be a Lorentzian pre-length space. Let $\Delta := \Delta(x,y,z)$ be a timelike triangle (in particular the distance realisers between the endpoints exist). Let p be a point on the side xy, such that the distance realiser between p and z exists. Then we can consider the smaller triangles $\Delta_1 := \Delta(x,p,z)$ and $\Delta_2 := \Delta(p,y,z)$. We construct a comparison situation consisting of a comparison triangle $\bar{\Delta}_1$ for Δ_1 and $\bar{\Delta}_2$ for Δ_2, with $\bar{x}$ and $\bar{y}$ on different sides of the line through $\bar{p}\bar{z}$ and a comparison triangle $\tilde{\Delta}$ for Δ with a comparison point $\tilde{p}$ for p on the side $\tilde{x}\tilde{y}$. This contains the subtriangles $\tilde{\Delta}_1 := \Delta(\tilde{x},\tilde{p},\tilde{z})$ and $\tilde{\Delta}_2 := \Delta(\tilde{p},\tilde{y},\tilde{z})$, see Figure 23.*

Then the situation $\bar{\Delta}_1$,$\bar{\Delta}_2$ is convex (concave) at p (i.e. $\measuredangle_{\bar{p}}(\bar{y},\bar{z}) \leq \measuredangle_{\bar{p}}(\bar{x},\bar{z})$ (or $\geq$)) if and only if $\tau(p,z) \leq \bar{\tau}(\tilde{p},\tilde{z})$ (or $\geq$). If this is the case, we have that

- *each angle in the triangle $\bar{\Delta}_1$ is $\geq$ (or $\leq$) than the corresponding angle in the triangle $\tilde{\Delta}_1$,*
- *each angle in the triangle $\bar{\Delta}_2$ is $\leq$ (or $\geq$) than the corresponding angle in the triangle, $\tilde{\Delta}_2$.*

In any case, we have that

- $\measuredangle_{\bar{z}}(\bar{x},\bar{y}) \leq \measuredangle_{\tilde{z}}(\tilde{x},\tilde{y}) = \tilde{\measuredangle}_z(x,y)$.

Note that if X has non-negative (non-positive) timelike curvature and Δ is within a comparison neighbourhood, the condition is satisfied, i.e. $\tau(p,z) \leq \bar{\tau}(\tilde{p},\tilde{z})$ (or $\geq$).

Proof for both Alexandrov lemmas. It is sufficient to show the "if" part of the statement. Indeed, under the assumption that the triangles Δ_1 and Δ_2

are non-degenerate, the "if" part holds with strict inequalities, which then also implies the "only if" part of the statement. The degenerate cases are trivial.

For the triangles $\bar{\Delta}_1$ vs. $\tilde{\Delta}_1$, only one side-length changes, and it is not the longest side of the triangle in both versions. Thus, the monotonicity statement in the law of cosines, cf. Lemma 2.35, implies that in the across version, the relation between the desired angle $\measuredangle_{\bar{p}}(\bar{x}, \bar{y})$ in $\bar{\Delta}_1$ to the corresponding angle $\measuredangle_{\tilde{p}}(\tilde{x}, \tilde{y})$ in $\tilde{\Delta}_1$ is pointing in the opposite direction than the relation of the changing side-length $\tau(p, y) = \bar{\tau}(\bar{p}, \bar{y})$ to the side-length $\bar{\tau}(\tilde{p}, \tilde{y})$. In the future version, we obtain that the relation between the desired angle $\measuredangle_{\bar{p}}(\bar{x}, \bar{z})$ in $\bar{\Delta}_1$ to the corresponding angle $\measuredangle_{\tilde{p}}(\tilde{x}, \tilde{z})$ in $\tilde{\Delta}_1$ is pointing in the other direction than the relation of the changing side-length $\tau(p, z) = \bar{\tau}(\bar{p}, \bar{z})$ to the side-length $\bar{\tau}(\tilde{p}, \tilde{z})$.

Similarly, for the triangles $\bar{\Delta}_2$ vs. $\tilde{\Delta}_2$, only one side-length changes. In the across version, it is not the longest side, and for the future version, it is. Thus, in the future version, the monotonicity statement in the law of cosines yields that the relation between the angle $\measuredangle_{\bar{p}}(\bar{y}, \bar{z})$ in $\bar{\Delta}_1$ to the corresponding angle $\measuredangle_{\tilde{p}}(\tilde{y}, \tilde{z})$ in $\tilde{\Delta}_1$ is pointing in the same direction as the relation of the changing side-length $\bar{\tau}(\bar{p}, \bar{y})$ to the side-length $\bar{\tau}(\tilde{p}, \tilde{y})$. Similarly, in the across version, we get that the relations between the angles $\measuredangle_{\bar{p}}(\bar{y}, \bar{z})$ in $\bar{\Delta}_1$ and $\measuredangle_{\tilde{p}}(\tilde{y}, \tilde{z})$ in $\tilde{\Delta}_1$ points in the opposite direction as the relation of the changing side-length $\bar{\tau}(\bar{p}, \bar{y})$ to the side-length $\bar{\tau}(\tilde{p}, \tilde{y})$.

For the "split" angle, we use the triangle equality along lines and the reverse triangle inequality on the broken side, giving $\bar{\tau}(\bar{x}, \bar{z}) \geq \bar{\tau}(\bar{x}, \bar{p}) + \bar{\tau}(\bar{p}, \bar{z}) = \tau(x, p) + \tau(p, z) = \tau(x, z) = \bar{\tau}(\tilde{x}, \tilde{z})$ in the across statement and $\bar{\tau}(\bar{x}, \bar{y}) \geq \bar{\tau}(\bar{x}, \bar{p}) + \bar{\tau}(\bar{p}, \bar{y}) = \tau(x, p) + \tau(p, y) = \tau(x, y) = \bar{\tau}(\tilde{x}, \tilde{y})$ in the future statement. For the triangles $\bar{\Delta} = \Delta(\bar{x}, \bar{y}, \bar{z})$ and $\tilde{\Delta} = \Delta(\tilde{x}, \tilde{y}, \tilde{z})$, only one side-length changes. In the across version it is the longest side of the triangle, and in the future version, it is not. Thus, in the future version, the monotonicity statement in the law of cosines implies that the relation between the angle $\measuredangle_{\bar{z}}(\bar{x}, \bar{y})$ in $\bar{\Delta}$ to the corresponding angle $\measuredangle_{\tilde{z}}(\tilde{x}, \tilde{y})$ in $\tilde{\Delta}$ is pointing in the opposite direction than the relation of the changing side-lengths $\tau(x, y) = \bar{\tau}(\bar{x}, \bar{y})$ and $\bar{\tau}(\tilde{x}, \tilde{y})$. Similarly, in the across version, we obtain that the relation of the angle $\measuredangle_{\bar{y}}(\bar{x}, \bar{z})$ in $\bar{\Delta}$ and $\measuredangle_{\tilde{y}}(\tilde{x}, \tilde{z})$ in $\tilde{\Delta}$ points in the same direction as the inequality between $\tau(x, z) = \bar{\tau}(\bar{x}, \bar{z})$ and $\bar{\tau}(\tilde{x}, \tilde{z})$.

For the future version, note that $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ together form the triangle $\tilde{\Delta}$, so we have $\measuredangle_{\tilde{p}}(\tilde{x}, \tilde{z}) = \measuredangle_{\tilde{p}}(\tilde{y}, \tilde{z})$. Together with the angle inequalities in Δ_1 and Δ_2, the convexity/concavity follows.

For the across version, we go the other direction: assume that $\measuredangle_{\bar{p}}(\bar{x}, \bar{y}) \geq \measuredangle_{\bar{p}}(\bar{y}, \bar{z})$ (or $\leq$). Consider a point $\bar{x}'$ such that $\bar{x}'\bar{p}\bar{z}$ lie on a distance realizer, with $\bar{\tau}(\bar{x}', \bar{p}) = \bar{\tau}(\bar{x}, \bar{p}) = \tau(x, p)$ (note that then $\bar{\tau}(\bar{x}', \bar{z}) = \bar{\tau}(\tilde{x}, \tilde{z})$). Then we have that $\measuredangle_{\bar{p}}(\bar{x}', \bar{y}) = \measuredangle_{\bar{p}}(\bar{y}, \bar{z})$. Then by law of cosines monotonicity, we get that $\bar{\tau}(\tilde{x}, \tilde{y}) = \bar{\tau}(\bar{x}, \bar{y}) \leq \bar{\tau}(\bar{x}', \bar{y})$ (or $\geq$). By law of cosines monotonicity, we have that $\measuredangle_{\bar{z}}(\bar{p}, \bar{y}) = \measuredangle_{\bar{z}}(\bar{x}', \bar{y}) \leq \measuredangle_{\tilde{z}}(\tilde{x}, \tilde{y}) = \measuredangle_{\tilde{z}}(\tilde{p}, \tilde{y})$ (or $\geq$). We again use law of cosines monotonicity to get $\tau(p, y) = \bar{\tau}(\bar{p}, \bar{y}) \geq \bar{\tau}(\tilde{p}, \tilde{y})$ (or $\leq$). For the across version, we modify the barred configuration: Consider a point $\bar{x}'$ such that $\bar{x}'\bar{p}\bar{z}$ lie on a distance realizer, with $\bar{\tau}(\bar{x}', \bar{p}) = \bar{\tau}(\bar{x}, \bar{p}) = \tau(x, p)$ (note that then $\bar{\tau}(\bar{x}', \bar{z}) = \bar{\tau}(\tilde{x}, \tilde{z})$). Now assume that the first line of the following is true:

$$\begin{aligned}
\tau(p, y) = \bar{\tau}(\bar{p}, \bar{y}) &\geq \bar{\tau}(\tilde{p}, \tilde{y}) && \text{(or } \leq) \\
\measuredangle_{\bar{z}}(\bar{p}, \bar{y}) &\leq \measuredangle_{\tilde{z}}(\tilde{p}, \tilde{y}) && \text{(or } \geq) \\
\measuredangle_{\bar{z}}(\bar{x}', \bar{y}) &\leq \measuredangle_{\tilde{z}}(\tilde{x}, \tilde{y}) && \text{(or } \geq) \\
\bar{\tau}(\bar{x}', \bar{y}) &\geq \bar{\tau}(\tilde{x}, \tilde{y}) && \text{(or } \leq) \\
\bar{\tau}(\bar{x}', \bar{y}) &\geq \bar{\tau}(\bar{x}, \bar{y}) && \text{(or } \leq) \\
\measuredangle_{\bar{p}}(\bar{x}', \bar{y}) &\leq \measuredangle_{\bar{p}}(\bar{x}, \bar{y}) && \text{(or } \geq) \\
\measuredangle_{\bar{p}}(\bar{z}, \bar{y}) &\leq \measuredangle_{\bar{p}}(\bar{x}, \bar{y}) && \text{(or } \geq)
\end{aligned}$$

The second line follows from law of cosines monotonicity, the third as the left and right sides are equal to those of the previous lines, the fourth follows from law of cosines monotonicity again. The right hand side of the fifth is equal to the previous line as they are both comparison sides, the sixth follows by law of cosines monotonicity, for the left hand side of the last note that $\bar{x}', \bar{y}, \bar{z}$ is straight, so the angles agree.

Finally, note that all monotonicity arguments work the same if the time-like relation between y and p is reversed in the across version. $\square$

2.4 Extensions of Lorentzian length spaces

As in [11], in the proof of the splitting theorem we will first show that $I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$ splits, and then we will use an inextendibility argument to show that $I(\gamma) = X$. To this end, let us recall the notion of extensions of Lorentzian (pre-)length spaces as introduced in [40, Def. 3.1].

Definition 2.44 (Extensions)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A Lorentzian pre-length space $(\tilde{X}, \tilde{d}, \tilde{\ll}, \tilde{\leq}, \tilde{\tau})$ is called an* extension *of $(X, d, \ll, \leq, \tau)$ if*

1. *$(\tilde{X}, \tilde{d})$ is connected,*
2. *there exists an isometric embedding $\iota : (X, d) \to (\tilde{X}, \tilde{d})$,*

3. *$\iota(X) \subsetneq \tilde{X}$ is open,*

4. *ι preserves $\leq$ and $\ll$,*

5. *ι preserves τ-lengths of causal curves.*

Remark 2.45 (Convex extensions). An important class of examples (and the only one we will need) is the case of open, causally convex subsets of a suitable Lorentzian length space. In this case, ι even preserves τ.

Next, we recall the definition of *timelike completeness* for (localisable) Lorentzian pre-length spaces, cf. [40, Def. 5.1]. However, since continuous extendibility, extendibility as a causal curve and geodesic extendibility are in general inequivalent concepts, we will define the timelike completeness property precisely in such a way that it is sufficient for the subsequent inextendibility result.

Definition 2.46 (Timelike completeness (TC)). *Let $(X, d, \ll, \leq, \tau)$ be a localisable Lorentzian pre-length space. We say X satisfies the* timelike completeness (TC) property *if each (future or past directed) timelike geodesic $\alpha : [a, b) \to X$ that is inextendible to $[a, b]$ as a continuous curve has infinite τ-length.*

Theorem 2.47 (Inextendibility of (TC) spaces). *Let $(X, d, \ll, \leq, \tau)$ be a strongly causal Lorentzian length space satisfying the (TC) property. Then X is inextendible as a regularly localisable Lorentzian length space.*

Proof. See [40, Thm. 5.3]. □

3 Lorentzian products

This section is dedicated to the treatment of Lorentzian product spaces. In the context of Lorentzian pre-length spaces, (warped) products were first introduced in [5]. On the one hand, this investigation is too general for our purposes, we only need products instead of warped products. On the other hand, the formulation we use also works for non-intrinsic spaces.

Definition 3.1 (Lorentzian product). *Let (X, d) be a metric space. Define the space $Y := \mathbb{R} \times X$. Equip it with the product metric*

$$D : Y \times Y \to \mathbb{R},\ D((s, x), (t, y)) := \sqrt{|t - s|^2 + d(x, y)^2}.$$

Define the timelike relation as $(s,x) \ll (t,y) :\iff t-s > d(x,y)$ and the causal relation as $(s,x) \leq (t,y) :\iff t-s \geq d(x,y)$. Define the product time separation $\tau : Y \times Y \to \mathbb{R}$ via

$$\tau((s,x),(t,y)) := \sqrt{(t-s)^2 - d(x,y)^2}$$

if $(s,x) \leq (t,y)$, and 0 otherwise.

Then $(Y, D, \ll, \leq, \tau)$ is called the Lorentzian product of X with $\mathbb{R}$. If it is clear that the Lorentzian product is meant, we simply denote it by $\mathbb{R} \times X$.

The following is more or less immediate from the definition:

Proposition 3.2 (Properties of Lorentzian products). *Let (X,d) be a metric space. Then the Lorentzian product $(Y, D, \ll, \leq, \tau)$ is a Lorentzian pre-length space. Moreover, τ is continuous and $\leq$ is closed on $Y \times Y$.*

Proof. τ is continuous by definition, so in particular, lower semi-continuous. The inclusion of $\ll$ in $\leq$ is clear from the definition as well. The transitivity of both relations as well as the reverse triangle inequality of τ for $\leq$-related triples follow (via an elementary calculation) from the triangle inequality for d. The reflexivity of $\leq$ is clear from the definition. Finally, the definition of $\ll$ is a reformulation of $\tau > 0$.

Note that for converging real sequences $n_i \to n, m_i \to m$ such that $n_i \leq m_i$ for all i, we have $n \leq m$. The closedness of $\leq$ on $Y \times Y$ follows immediately from this fact. □

It is easily seen that generalised cones in the sense of [5] with warping function $f \equiv 1$ are Lorentzian products in the sense of Definition 3.1. Indeed, in this case we have, in the notation of [5, Def. 3.9], that $m_{s,t} := \min_{r \in [s,t]} f(r) = 1$ for all $s, t \in \mathbb{R}$. Then [5, Lem. 3.10] shows that a causal curve in the sense of [5, Def. 3.2] is a causal curve in the usual sense in our setting, with timelikeness being inherited as well. Moreover, the variational length of [5, Def. 3.9] is precisely the τ-length of a causal curve in the sense of Definition 2.6. It is easily seen that if X is strictly intrinsic, then the product $\mathbb{R} \times X$ in our sense can be canonically identified with the warped product $\mathbb{R} \times_f X$ with $f \equiv 1$.

We now show that Lorentzian products are always strongly causal and non-totally imprisoning.

Proposition 3.3 (Diamonds form basis). *Let $Y := \mathbb{R} \times X$ be a Lorentzian product. Then $\{I(p,q) \mid p, q \in Y\}$ forms a basis for the topology. In particular, Y is strongly causal.*

Proof. Consider the open set $O := (a,c) \times B_R(x)$ for $a, c \in \mathbb{R}, R > 0, x \in X$ and let $(b, y) \in O$. Choose $\varepsilon = \min(b-a, c-b, R-d(x,y)) > 0$, then $p := (b-\varepsilon, y), q := (b+\varepsilon, y) \in \bar{O}$. Then we have that $I(p,q) \subseteq O$: If $(s,z) \in I(p,q)$, then $s \in (b-\varepsilon, b+\varepsilon) \subseteq (a,c)$. In particular, at least one of $|s-(b-\varepsilon)| \leq \varepsilon$ and $|s-(b+\varepsilon)| \leq \varepsilon$ holds. By definition of $\ll$, we then also have $d(x,z) \leq d(x,y) + d(y,z) < R$, so $y \in B_R(x)$. Thus, $(s,z) \in O$. $\square$

Proposition 3.4 (Non-total imprisonment of Lorentzian products)**.** *Any Lorentzian product $Y = \mathbb{R} \times X$, where (X,d) is a metric space, is non-totally imprisoning.*

Proof. For any two points $(s,x) \leq (t,y) \in Y$ we have $d(x,y) \leq t-s$, so $D((s,x),(t,y)) = \sqrt{(t-s)^2 + d(x,y)^2} \leq \sqrt{2}(t-s)$. In particular, for any future directed causal curve $\gamma : [a,b] \to Y$ starting at $\gamma(a) = (s,x)$ and ending at $\gamma(b) = (t,y)$ we have $L_D(\gamma) \leq \sqrt{2}(t-s)$: In any partition of $[a,b]$, the above bound in any of the subintervals forms a telescopic sum, making $\sqrt{2}(t-s)$ an upper bound on the length of the polygon approximation of γ corresponding to the partition. Let now K be compact, then we can enclose it in a bounded set: $K \subseteq [s,t] \times \bar{B}_R(x)$ for some $s < t$, $R > 0$ and $x \in X$. We know that the D-length of future directed causal curves contained in $[s,t] \times \bar{B}_R(x)$ is bounded by $\sqrt{2}(t-s)$, so we have that Y is non-totally imprisoning. $\square$

The next result characterises distance realisers in $Y = \mathbb{R} \times X$. Note that we make use of the following: A distance minimiser in a metric space can always be parametrised by unit speed, which is evidently a Lipschitz parametrisation, cf. [17, Proposition 2.5.9].

Proposition 3.5 (Characterisation of distance realisers)**.** *Let (X,d) be a metric space and let $Y := \mathbb{R} \times X$ the Lorentzian product. Then a continuous causal curve $\gamma = (\alpha, \beta) : [a,b] \to Y$ is a distance realiser if and only if either $\beta = const$ or $\beta : [a,b] \to X$ is a (metric) distance realiser and when reparametrising γ such that β is unit speed parametrised, $\alpha : [a,b] \to \mathbb{R}$ is affine, i.e. of the form $\alpha(t) = ct + d$ with $c = 1$ (which implies that γ future directed null) or $c > 1$ (which implies that γ is future directed timelike). In particular, γ has causal character and a reparametrisation as a (Lipschitz) causal curve. If in addition Y is localisable, then the same is true for continuous geodesics.*

Proof. γ is a distance realiser if and only if for all $r < s < t$ we have

$$\tau(\gamma(r), \gamma(t)) = \tau(\gamma(r), \gamma(s)) + \tau(\gamma(s), \gamma(t)). \quad (\star)$$

We denote $\gamma(r) = (t_1, x_1)$, $\gamma(s) = (t_2, x_2)$ and $\gamma(t) = (t_3, x_3)$, then $(\star)$ reads:

$$\sqrt{(t_3 - t_1)^2 - d(x_1, x_3)^2} = \sqrt{(t_2 - t_1)^2 - d(x_1, x_2)^2} + \sqrt{(t_3 - t_2)^2 - d(x_2, x_3)^2}$$

We now define $\lambda = \frac{t_2 - t_1}{t_3 - t_1}$ and the function $f(x) = \sqrt{1 - x^2}$. Then $(\star)$ simplifies to

$$f\left(\frac{d(x_1, x_3)}{t_3 - t_1}\right) = \lambda f\left(\frac{d(x_1, x_2)}{\lambda(t_3 - t_1)}\right) + (1 - \lambda) f\left(\frac{d(x_2, x_3)}{(1 - \lambda)(t_3 - t_1)}\right). \quad (\star\star)$$

We use that f is concave and monotonously decreasing on $[0, 1]$ and $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$ on the right hand side of $(\star\star)$:

$$\begin{aligned} &\lambda f\left(\frac{d(x_1, x_2)}{\lambda(t_3 - t_1)}\right) + (1 - \lambda) f\left(\frac{d(x_2, x_3)}{(1 - \lambda)(t_3 - t_1)}\right) \\ &\leq f\left(\frac{d(x_1, x_2) + d(x_2, x_3)}{(t_3 - t_1)}\right) \leq f\left(\frac{d(x_1, x_3)}{(t_3 - t_1)}\right). \end{aligned}$$

So we get that $(\star\star)$ is equivalent to having equality here. As f is strictly concave, the first inequality has equality if and only if

$$\frac{d(x_1, x_2)}{t_2 - t_1} = \frac{d(x_2, x_3)}{t_3 - t_2}, \tag{109}$$

where we got rid of λ again for generality. As f is strictly monotonously decreasing, the second inequality has equality if and only if

$$d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3). \tag{110}$$

We translate this back to what happens for general $r < s < t$: Equation 110 reads $d(\beta(r), \beta(s)) + d(\beta(s), \beta(t)) = d(\beta(r), \beta(t))$, i.e., it holds if and only if β is a distance realiser. Equation 109 reads $\frac{d(\beta(r),\beta(s))}{\alpha(s)-\alpha(r)} = \frac{d(\beta(s),\beta(t))}{\alpha(t)-\alpha(s)}$, i.e. that quantity is a constant $\frac{1}{c}$ (if $\frac{1}{c}$ is 0 we get that β is constant, in this case the result is trivial). In particular, if we reparametrise γ such that β is unit speed (i.e. $d(\beta(s), \beta(t)) = t - s$), $\alpha(t) - \alpha(s) = c(t - s)$, i.e. α is affine.

Now we look at what happens for different parameters c: If $c > 1$, we easily see that $f(c) > 0$, so $\tau(\gamma(r), \gamma(t)) = (\alpha(t) - \alpha(r)) f(c) > 0$ and the curve is timelike. For $c = 1$, $f(c) = f(1) = 0$, so $\alpha(t) - \alpha(r) = d(\beta(r), \beta(t))$ and the curve is null.

Now if Y is localisable and γ is only a geodesic, we can cover the domain with intervals J_i that are open in $[a, b]$ where it is distance realising. Then on each of these J_i we apply the above. As the J_i overlap, the constant c agrees, and we get that $\beta|_{J_i}$ is distance realising, thus β is a geodesic. The converse follows equivalently. We even get that the subintervals of the domain where the curves are distance realising agree. $\square$

Corollary 3.6 (τ-arclength parametrisations of distance realisers)**.** *Any continuous timelike maximiser $\gamma : [a, b] \to \mathbb{R} \times X$ has a Lipschitz τ-arclength parametrisation, which is of the form (α, β), with $\alpha : [0, L_\tau(\gamma)] \to \mathbb{R}$, $\alpha(t) = ct + d$, and $\beta : [0, L_\tau(\gamma)] \to X$ a constant speed minimiser of speed $\sqrt{c^2 - 1}$. If $\mathbb{R} \times X$ is localisable, then the same is true for timelike geodesics (with β constant speed geodesic).*

Note that the results above continue to hold for maximisers resp. geodesics defined on any interval I, not just a closed one.

To finish this section, we show that global hyperbolicity of the Lorentzian product $Y = \mathbb{R} \times X$ is equivalent to the metric properness of X.

Proposition 3.7 (Global hyperbolicity and properness)**.** *A Lorentzian product $Y := \mathbb{R} \times X$ is globally hyperbolic if and only if X is a proper metric space.*

Proof. Since products are always non-totally imprisoning by Proposition 3.4, we only need to check that causal diamonds in Y are compact if and only if (X, d) is a proper metric space. First, suppose that Y is globally hyperbolic. Let $(t, x) \in Y$ and fix $R > 0$. Consider the points $p := (t - R, x), q := (t + R, x)$. Then $(t, x) \in J(p, q)$. By definition of $\leq$, any point (t, y) with the same $\mathbb{R}$-coordinate as (t, x) is in $J(p, q)$ if and only if $d(x, y) \leq R$. Thus, the set $J(p, q) \cap \{(t, x) \mid x \in X\}$, which is compact by assumption, can be identified with the closed ball $\bar{B}_R(x)$ in X of radius R around x, establishing the fact that closed balls in X are compact.

Conversely, suppose that X is proper and let $p = (r, x), q = (t, z) \in Y$ with $p \leq q$. By definition of $\leq$, if $(s, y) \in J(p, q)$, then $r \leq s \leq t$. Moreover, we have $|s| \leq |r| + |t|$. Set $R := 2|r| + 2|t|$. Then for $(s, y) \in J(p, q)$, we also have $d(x, y) \leq |s - r| \leq |s| + |r| \leq R$. Thus, $y \in \bar{B}_R(x)$, which is compact by assumption. In total, we conclude that $J(p, q) \subseteq [r, t] \times \bar{B}_R(x)$, which is a compact set as the Cartesian product of two compact sets. The fact that $J(p, q)$ is closed follows immediately from the closedness of $\leq$, so $J(p, q)$ is compact as well. $\square$

4 Rays, lines, co-rays and asymptotes

4.1 Rays, lines, co-rays, asymptotes, timelike co-ray condition

In this subsection, we study causal rays and lines and show how to obtain them as limits of causal maximisers. Moreover, we analyse triangles where one side is a segment on a timelike line and show that the angles adjacent to the line are equal to their comparison angles. This in fact follows from the more general principle that one can stack triangle comparisons of nested triangles with two endpoints on a timelike line. The latter situation arises in the construction of asymptotes, and via the stacking principle, one can show any future directed and any past directed asymptote from a common point to a given timelike line fit together to give a (timelike) asymptotic line. In constructing co-rays and asymptotes we follow [11], whereas the stacking principle and equality of angles can be viewed as Lorentzian analogues of [17, Lem. 10.5.4].

Definition 4.1 (Rays, Lines)**.** *Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A* future directed causal ray *is a future inextendible, future directed causal curve $c : I \to X$ that maximises the time separation between any of its points, where I is either a closed interval $[a, b]$ or a half-open interval $[a, b)$. A* future directed causal line *is a (doubly) inextendible, future directed causal curve $\gamma : I \to X$ that maximises the time separation between any of its points. Here I can in general be open, closed, or half-open. More generally, for $S \subset X$, a* future directed causal S-ray *is a future directed, future inextendible causal curve starting in S and satisfying $\tau(S, c(t)) := \sup_{p \in S} \tau(p, c(t)) = L_\tau(c|_{[0,t]})$. Past directed rays and lines are defined analogously. A future directed causal ray $c : I \to X$ is called* complete *if $L_\tau(c) = \infty$. Similarly, a future directed causal line γ is called* complete *if there is $t_0 \in I$ such that the past and future directed rays obtained from dividing γ at $\gamma(t_0)$ are complete. Unless explicitly stated otherwise, all causal rays and lines are understood to be future directed.*

Remark 4.2.

1. Clearly, any ray c is a $\{c(0)\}$-ray. Conversely, if $c : [a, b) \to X$ is a future directed S-ray for some $S \subset X$, then $\tau(c(a), c(t)) \leq \tau(S, c(t)) = L_\tau(c|_{[a,t]}) \leq \tau(c(a), c(t))$, so c is a ray. A similar statement is true for past directed rays.

2. If X is regularly localisable, then any causal ray/line c is either timelike or null by Theorem 2.17.

3. If X is localisable and causally path-connected, then any ray c is defined on a half-open interval and any line γ is defined on an open interval.

In the following, unless we specify the assumptions on X, we always take X to be as in the splitting theorem.

Proposition 4.3. *Let $z_n \to z$ in X. Let $p_n \in I^+(z_n)$ and let $c_n : [0, a_n] \to X$ be a sequence of future directed, maximising causal curves from z_n to p_n in*

d-arclength parametrisation. If $\tau(z_n, p_n) \to \infty$, then there is a limit causal ray $c : [0, \infty) \to X$ from z.

Proof. We only have to show $a_n \to \infty$, the rest follows from the limit curve theorem. Suppose $a_n \to a < \infty$. This implies that all p_n are contained in a common compact set K: Indeed, if we assume this to be false, then for each $n > 0$ there would be a p_k, $k = k(n)$, such that $p_k \notin \overline{B}_n(z) = \{x \in X : d(x, z) \leq n\}$ (recall that (X, d) is proper), which in particular means that

$$a_{k(n)} = L^d(c_{k(n)}) \geq d(z_{k(n)}, p_{k(n)}) \geq d(p_{k(n)}, z) - d(z_{k(n)}, z) \geq n - d(z_{k(n)}, z),$$

which is a contradiction since the right hand side tends to ∞. So, up to a choice of subsequence, we may assume that $\gamma_n(a_n) = p_n \to p$. But then, by continuity of τ,

$$\tau(z, p) = \lim_n \tau(z_n, p_n) = \infty,$$

a contradiction to the finiteness of τ on all of $X \times X$. $\square$

Let now $c : [0, \infty) \to X$ be a complete timelike S-ray, and fix $z \in I^-(c) \cap I^+(S)$. Let $z_n \to z$ and $p_n := c(r_n)$ for some sequence of parameters $r_n \to \infty$. Then it is easily seen that $\tau(z_n, p_n) \to \infty$: Indeed, let $r > 0$ be large and assume w.l.o.g. that all z_n are in $I^-(c(r))$, then

$$\tau(z_n, p_n) \geq \tau(z_n, c(r)) + \tau(c(r), c(r_n)) \to \infty.$$

Hence, by the above result, we may construct causal rays as follows: Let μ_n be maximising timelike curves from z_n to p_n and let μ be a limit causal ray from z whose existence we just proved.

Definition 4.4 (Co-rays, asymptotes)**.** *Any causal ray μ constructed by the above method is called a* co-ray *to the ray c at z. If $z_n = z$ for all n, then μ is called an* asymptote *to c at z.*

Since maximising causal curves have causal character, a co-ray is either timelike or null.

Definition 4.5 (Timelike co-ray condition)**.** *Let $c : [0, \infty) \to X$ be a complete timelike S-ray and let $p \in I^-(c) \cap I^+(S)$. We say the* timelike co-ray condition (TCRC) *holds at p if every co-ray to c at p is timelike.*

We turn to the treatment of triangles adjacent to lines and begin with the aforementioned stacking principle. It will be an essential technical tool for controlling the behaviour of asymptotes to a line. The following two results are true for rather general Lorentzian pre-length spaces, and the exact conditions are specified.

Proposition 4.6 (Comparison situations stack along a geodesic)**.** *Let X be a timelike geodesically connected Lorentzian pre-length space with global timelike curvature bounded below by* 0 *and $\gamma : \mathbb{R} \to X$ be a complete timelike line. Let $p \in X$ be a point not on γ. Let $t_1 < t_2 < t_3$ such that all $y_i := \gamma(t_i)$ are timelike related to p, see Figure 24. Let $\bar{\Delta}_{12} := \Delta(\bar{p}, \bar{y}_1, \bar{y}_2)$ be a comparison triangle for $\Delta_{12} := \Delta(p, y_1, y_2)$ and extend the side $\bar{p}, \bar{y}_2$ to a comparison triangle $\bar{\Delta}_{23} := \Delta(\bar{p}, \bar{y}_2, \bar{y}_3)$ for $\Delta_{23} := \Delta(p, y_2, y_3)$. We choose it in such a way that $\bar{y}_1$ and $\bar{y}_3$ lie on opposite sides of the line through $\bar{y}_1, \bar{y}_2$. Then $\bar{y}_1, \bar{y}_2, \bar{y}_3$ are collinear. That makes $\bar{\Delta}_{13} := \Delta(\bar{p}, \bar{y}_1, \bar{y}_3)$ a comparison triangle for $\Delta_{13} := \Delta(p, y_1, y_3)$.*

Proof. We set $s_- = \sup(\gamma^{-1}(I^-(p)))$ and $s_+ = \inf(\gamma^{-1}(I^+(p)))$. Then the set of s where $\gamma(s)$ is timelike related to p is $(-\infty, s_-) \cup (s_+, +\infty)$. We assume $p \ll y_2$, the other case $y_2 \ll p$ can be reduced to this one by flipping the time orientation of the space.

We create comparison situations for an Alexandrov situation: We take the triangles $\bar{\Delta}_{12} = \Delta(\bar{p}, \bar{y}_1, \bar{y}_2)$ and $\bar{\Delta}_{23} = \Delta(\bar{p}, \bar{y}_2, \bar{y}_3)$ as in the statement. We also create a comparison triangle $\tilde{\Delta}_{13} = \Delta(\tilde{p}, \tilde{y}_1, \tilde{y}_3)$ for (p, y_1, y_3) and get a comparison point $\tilde{y}_2$ for y_2 on the side $y_1 y_3$. Then we can apply curvature comparison to get $\bar{\tau}(\tilde{p}, \tilde{y}_2) \geq \tau(p, y_2) = \bar{\tau}(\bar{p}, \bar{y}_2)$. By Lemmas 2.42 and 2.43[28], this means the situation is convex, i.e.

$$\tilde{\measuredangle}_{y_2}(p, y_1) \leq \tilde{\measuredangle}_{y_2}(p, y_3)\,. \tag{111}$$

If $p \ll y_1$ we also get

$$\tilde{\measuredangle}_{y_1}(p, y_2) \leq \tilde{\measuredangle}_{y_1}(p, y_3) \tag{112}$$

and if $y_1 \ll p$, this inequality flips.

Now we apply equation (112) to the situation where we fix y_2 and move y_3 and y_1: We get $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$ is monotonously increasing in t_3. Similarly, the time-reversed situation gives that if $p \ll \gamma(t_1)$, $\tilde{\measuredangle}_{y_2}(p, \gamma(t_1))$ is monotonously increasing in t_1 and if $\gamma(t_1) \ll p$ it is also monotonously increasing in t_1 (note it is also increasing when switching from one case to the other). Then remember $\tilde{\measuredangle}_{y_2}(p, \gamma(t_1)) \leq \tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$ by the convexity of the Alexandrov situation (equation (111)). Note that the left hand side is decreasing with decreasing t_1 and the right hand side is increasing with increasing t_3.

<u>Claim</u>: For each t_1 and t_3, equation (111) has equality.

<u>Then</u>: The comparison situation is straight, i.e. $\bar{y}_1, \bar{y}_2, \bar{y}_3$ are collinear, and by triangle equality along straight lines we have $\tau(y_1, y_3) = \tau(y_1, y_2) +$

[28] If $p \ll y_1 \ll y_2 \ll y_3$, we use Lemma 2.43 and if $y_1 \ll p \ll y_2 \ll y_3$ we use Lemma 2.42.

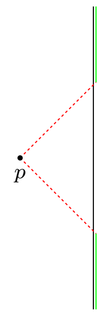

Figure 24: The domain where the y_i can lie in is in green.

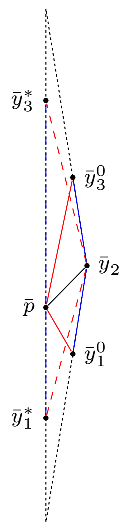

Figure 25: We assume that the path $\bar{y}_1^0\bar{y}_2\bar{y}_3^0$ is longer than the path $\bar{y}_1^0\bar{p}\bar{y}_3^0$. If the angle at $\bar{y}_2$ is not straight, we extend some lines and as points further from $\bar{p}$ are more on the left, we find some t_1^* and t_3^* such that $\bar{y}_1^*\bar{p}\bar{y}_3^*$ are collinear. But then the path $\bar{y}_1^0\bar{y}_2\bar{y}_3^0$ is shorter than $\bar{y}_1^0\bar{p}\bar{y}_3^0$, which then yields a contradiction to γ being maximising.

$\tau(y_2, y_3) = \bar{\tau}(\bar{y}_1, \bar{y}_2) + \bar{\tau}(\bar{y}_2, \bar{y}_3) = \bar{\tau}(\bar{y}_1, \bar{y}_3)$, i.e., $\bar{p}, \bar{y}_1, \bar{y}_3$ is a comparison triangle for p, y_1, y_3.

Proof: We indirectly assume there is a t_1^0, t_3^0 such that the comparison situation $\bar{\Delta}_{12}, \bar{\Delta}_{23}$ is strictly convex. We realise this situation such that $\bar{p} = 0$, $\bar{y}_2$ is to the right of the t-axis, such that the side $\bar{y}_2\bar{y}_3^0$ slopes to the left and $\bar{y}_1^0\bar{y}_2$ slopes to the right (i.e., $\frac{x(\bar{y}_3^0 - \bar{y}_2)}{t(\bar{y}_3^0 - \bar{y}_2)} < 0$, $\frac{x(\bar{y}_2 - \bar{y}_1^0)}{t(\bar{y}_2 - \bar{y}_1^0)} > 0$). When we vary t_1 and t_3, the comparison situation is chosen such that $\bar{p}$ and $\bar{y}_2$ stay fixed. As the comparison angle $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$ is monotonously increasing in t_3, we get that for $t_3 \geq t_3^0$ the slope $\frac{x(\bar{y}_3 - \bar{y}_2)}{t(\bar{y}_3 - \bar{y}_2)}$ is smaller than the slope of $\bar{y}_2\bar{y}_3$. In particular, $\bar{y}_3$ lies to the left of the extension of the side $\bar{y}_2\bar{y}_3^0$. Thus, for large enough t_3, $\bar{y}_3$ lies to the left of the t-axis, and for a certain t_3^* $\bar{y}_3^*$ lies on it. Similarly, we find a t_1^* such that $\bar{y}_1^*$ lies on the t-axis. See Figure 25 for a visualisation of the construction. But then

$$\tau(y_1^*, p)+\tau(p, y_3^*) = \bar{\tau}(\bar{y}_1^*, \bar{p})+\bar{\tau}(\bar{p}, \bar{y}_3^*) > \bar{\tau}(\bar{y}_1^*, \bar{y}_2)+\bar{\tau}(\bar{y}_2, \bar{y}_3^*) = \tau(y_1^*, y_2)+\tau(y_2, y_3^*) = \tau(y_1^*, y_3^*)$$

in contradiction to the reverse triangle inequality. Thus we get the claim. □

Remark 4.7. The proof of the statement can also be used in more general situations: Let X be a locally timelike geodesically connected Lorentzian pre-length space with local timelike curvature bounded below by 0, and let γ be a (possibly extendible) timelike maximiser. Let $p \ll x = \gamma(t_2)$ be points in a comparison neighbourhood. Assume the statement is not true (with certain $t_1 < t_2 < t_3$). Then we get $\omega_1 = \tilde{\measuredangle}_{y_2}(p, y_1)$, $\omega_3 = \tilde{\measuredangle}_{y_2}(p, y_3)$. Let $d_1 + d_2 = \omega_3 - \omega_1$ and $e = \tau(p, x)\sinh(d_1 + \omega_1)$. Then for the parameters $t_1 = -\frac{e}{\sinh(d_1)}$ and $t_3 = \frac{e}{\sinh(d_3)}$ we have one of the following:

- γ is not defined at/inextendible to one of them,
- γ stops being distance realising between t_1^* and t_3^* (and we have found a longer curve),
- there is no curvature comparison neighbourhood containing both $\gamma(t_i^*)$ $(i = 1, 3)$.

For this, realise the above with $\measuredangle_{\bar{y}_2}(\partial_t, \bar{y}_3) = d_3$ and $\measuredangle_{\bar{y}_2}(\partial_t, \bar{y}_1) = d_1$. Then e is the difference of the x-coordinates of p and x.

Note the similarity of this result and classical arguments for conjugate points along geodesics in Riemannian/Lorentzian geometry.

Proposition 4.8 (Angle = comparison angle). *Let X be a timelike geodesically connected Lorentzian pre-length space with global timelike curvature bounded below by* 0 *and $\gamma : \mathbb{R} \to X$ be a complete timelike line and $x := \gamma(t_0)$*

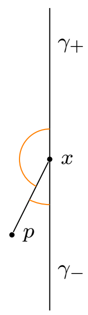

Figure 26: Illustration: These angles are the same, and have the same value as if they are considered as comparison angles.

a point on it. We split γ into the future part $\gamma_+ = \gamma|_{[t_0,+\infty)}$ and the past part $\gamma_- = \gamma|_{(-\infty,t_0]}$. Let $p \in X$ be a point not on γ with x and p timelike related and $\alpha : x \rightsquigarrow p$ a connecting distance realiser. Then for all $s \neq t$ such that p and $\gamma(s)$ are timelike related, we have:

$$\tilde{\measuredangle}_x(p, \gamma(s)) = \measuredangle_x(\alpha, \gamma_+) = \measuredangle_x(\alpha, \gamma_-)\,,$$

i.e. the comparison angle is equal to the angle, and the same in both directions.

Proof. First, we check that the comparison angle $\tilde{\measuredangle}_x(p, \gamma(s))$ is constant in s: For s_1 and s_2 for which this is defined, we have three parameters on γ involved: s_1, s_2, t_0. The previous result (Proposition 4.6) tells us we can construct a comparison situation for all three triangles at once. As the comparison situations have the comparison angle $\tilde{\measuredangle}_x(p, \gamma(s_1))$ resp. $\tilde{\measuredangle}_x(p, \gamma(s_2))$ as the angle in $\bar{x}$, they are equal.

Now we look at the angle $\measuredangle_x(\alpha, \gamma_\pm)$: We assume $p \ll x$. We already know $\tilde{\measuredangle}_x(\alpha(s), \gamma(t))$ is constant in t. We now have to look at its dependence on s: By Theorem 2.39, $\tilde{\measuredangle}^{\mathrm{S}}_x(\alpha(s), \gamma(t))$ is monotonously increasing in s. Note now that for $t < t_0$, the sign of this angle is $\sigma = -1$, and for $t > t_0$, the sign of this angle is $\sigma = +1$. So choose some $t_- < t_0$ and $t_+ > t_0$ for which all the necessary angles exist (i.e. $\gamma(t_-) \ll p$ and $t_+ > t_0$) we have that $\tilde{\measuredangle}_x(\alpha(s), \gamma(t_-)) = \tilde{\measuredangle}_x(\alpha(s), \gamma(t_+))$ is both a monotonously decreasing and increasing function in s. Thus it is constant, and $\tilde{\measuredangle}_x(\alpha(s), \gamma(t)) = \measuredangle_x(\alpha, \gamma_-) = \measuredangle_x(\alpha, \gamma_+)$, which includes the desired equalities. $\square$

Corollary 4.9. *Let X be a timelike geodesically connected, globally causally closed Lorentzian pre-length space satisfying $J^\pm(p) = \overline{I^\pm(p)}$ for all $p \in X$*[29]

[29] Approximating and globally causally closed Lorentzian pre-length spaces satisfy this, cf. [19, Def. 2.17].

and having global timelike curvature bounded below by 0 *and let* $\gamma : \mathbb{R} \to X$ *be a complete timelike line. Then for any point* $p \in X$ *and two points* $x_1 = \gamma(t_1)$ *and* $x_2 = \gamma(t_2)$ *which are timelike related to* p*, we get a timelike triangle* $\Delta = \Delta(x_1, x_2, p)$*. Let* q_1, q_2 *be any points on* Δ*, one of them lying on the side* $x_1 x_2$*. We form a comparison triangle* $\bar{\Delta}$ *for* Δ *and find comparison points* $\bar{q}_1, \bar{q}_2$ *for* q_1, q_2*. Then* $q_1 \leq q_2$ *if and only if* $\bar{q}_1 \leq \bar{q}_2$ *and* $\tau(q_1, q_2) = \bar{\tau}(\bar{q}_1, \bar{q}_2)$*, where* $\bar{\tau}$ *denotes the time separation of two-dimensional Minkowski space.*

Proof. First assume $q_1 \leq q_2$. As global curvature is bounded below by 0, we get that $\tau(q_1, q_2) \leq \bar{\tau}(\bar{q}_1, \bar{q}_2)$, and a continuity argument then shows that $\bar{q}_1 \leq \bar{q}_2$.

Conversely, let $\bar{q}_1 \leq \bar{q}_2$. We distinguish which sides the q_i lie on: Note we assumed one of them is on the side $x_1 x_2$. We only prove the case where q_1 is on the side α connecting $x_1 x_2$ (say $q_1 = \alpha(s)$) and q_2 is on the side β connecting x_1, p (say $q_2 = \beta(t)$), the proof of the other cases is easily adapted.

Consider now the hinge $(\alpha|_{[0,s]}, \beta|_{[0,t]})$, which has base-point x_1 and two tips at q_1, q_2. By the previous Proposition 4.8, we get that $\omega := \measuredangle_{x_1}(\alpha, \beta) = \tilde{\omega} := \tilde{\measuredangle}_{x_1}(p, x_2)$. We now have two comparison situations: $\bar{\Delta}$ has angle $\tilde{\measuredangle}_{x_1}(p, x_2)$ at x_1, and a comparison hinge $(\tilde{\alpha}, \tilde{\beta})$ with tips $\tilde{q}_1, \tilde{q}_2$. But note that the angles and distances at $\bar{x}_1$ resp. $\tilde{x}_1$ are the same, so we have that $\bar{\tau}(\bar{q}_1, \bar{q}_2) = \bar{\tau}(\tilde{q}_1, \tilde{q}_2)$. But for the comparison hinge, [13, Cor. 4.11] gives that $\tau(q_1, q_2) \geq \bar{\tau}(\tilde{q}_1, \tilde{q}_2) = \bar{\tau}(\bar{q}_1, \bar{q}_2)$. In total, this gives that $\tau(q_1, q_2) = \bar{\tau}(\bar{q}_1, \bar{q}_2)$, as desired. A similar continuity argument then gives that $q_1 \leq q_2$, thus completing the proof. □

We return to the situation of the splitting theorem. Let $\gamma : \mathbb{R} \to X$ be the complete timelike line whose existence we assume, and let it be in any locally Lipschitz parametrisation (e.g. parametrisation by d-arclength) defined on $\mathbb{R}$. We call any co-ray to any of its past or future directed subrays a co-ray to the line γ, so in particular, we can construct past and future directed co-rays from all points on $I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$. The next result shows that the timelike co-ray condition holds on $I(\gamma)$, i.e. all co-rays from all points in $I(\gamma)$ are timelike. For our purposes it would be sufficient to show this for asymptotes, since we only work with those in the proof.

Proposition 4.10. *The timelike co-ray condition holds on* $I(\gamma)$*.*

Proof. Suppose there is a point $x \in I(\gamma)$ such that the TCRC does not hold at x, so w.l.o.g. there is a sequence $x_n \to x$, $r_n \to \infty$ and maximal timelike curves σ_n from x_n to $\gamma(r_n)$ such that σ_n converge to a future directed null ray σ. Choose some $q \in \gamma \cap I^-(x)$ and let μ_n be maximal timelike curves from q to x_n (assuming n to be large enough, $q \ll x_n$). Suitably pre- and post-composing μ_n and then applying the limit curve theorem, it is easily seen

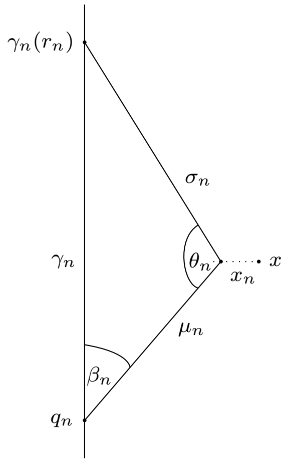

Figure 27: The construction of maximal curves along $I(\gamma)$.

that (up to a choice of subsequence) the μ_n converge locally uniformly to a maximising limit causal curve μ from q to x, which is timelike since $q \ll x$. Denote by γ_n the piece of γ that runs between q and $\gamma(r_n)$. Set $a_n := L_\tau(\mu_n)$, $b_n := L_\tau(\sigma_n)$ and $c_n := L_\tau(\gamma_n)$. Let $\beta_n := \measuredangle_q(\mu_n, \gamma_n)$ and $\theta_n := \measuredangle_{x_n}(\mu_n, \sigma_n)$ (see Figure 27). Then $a_n \to a := \tau(q, x)$ and by the continuity of angles (cf. Proposition 2.41), $\beta_n \to \beta$, where β is the angle between μ and γ. Consider the comparison triangle $(\overline{\mu}_n, \overline{\sigma}_n, \overline{\gamma}_n)$ in $\mathbb{R}^{1,1}$, then the angle $\overline{\beta}_n$ between $\overline{\mu}_n$ and $\overline{\gamma}_n$ satisfies $\overline{\beta}_n \leq \beta_n$ and similarly $\overline{\theta}_n \geq \theta_n$. Since $\beta_n \to \beta$, there is some $C > 0$ such that $\overline{\beta}_n \leq C$ for all n. The hyperbolic law of cosines in $\mathbb{R}^{1,1}$ (see Lemma 2.35) gives

$$b_n^2 = a_n^2 + c_n^2 - 2a_n c_n \cosh(\overline{\beta}_n),$$
$$c_n^2 = a_n^2 + b_n^2 + 2a_n b_n \cosh(\overline{\theta}_n).$$

Using these two equations and solving for $\cosh(\overline{\theta}_n)$, we get

$$\cosh(\overline{\theta}_n) = -\frac{a_n}{b_n} + \frac{c_n}{b_n}\cosh(\overline{\beta}_n).$$

By the initial equation for b_n, it is easy to see that $b_n/c_n \to 1$ and $b_n \to \infty$ (using that $a_n \to a$ and $\overline{\beta}_n \leq C$), hence there is some constant $\tilde{C} > 0$ such that $\cosh(\overline{\theta}_n) \leq \tilde{C}$ and thus also $\theta_n \leq C'$ for some constant C'. Using the monotonicity condition, we get that $\tilde{\measuredangle}_{x_n}(\sigma_n(s), \mu_n(t))) \leq \tilde{\measuredangle}_{x_n}(q, \gamma(r_n)) = \bar{\theta}_n$ for each s and t, so this is bounded too. We will see that this is incompatible with σ_n "getting more and more null": Note that we get the following estimate for some constant C'':

$$C'' \geq \cosh(\tilde{\measuredangle}_{x_n}(\sigma_n(s), \mu_n(t))) = \frac{\tau(\mu_n(t), \sigma_n(s))^2 - \tau(\mu_n(t), x_n)^2 - \tau(x_n, \sigma_n(s))^2}{2\tau(x_n, \sigma_n(s))\tau(\mu_n(t), x_n)}.$$

Since the denominator goes to 0 for $n \to \infty$ (as $\tau(x_n, \sigma_n(s)) \to \tau(x, \sigma(s)) = 0$ and $\tau(\mu_n(t), x_n) \to \tau(\mu(t), x) > 0$), the numerator has to go to 0 as well, in particular this means

$$\tau(\mu(t), \sigma(s)) = \tau(\mu(t), x)$$

for all s, t. This implies that running along μ from $\mu(t)$ to x and then along σ to $\sigma(s)$ gives a maximiser, but this curve has a timelike and a null piece, a contradiction to Theorem 2.17. $\square$

Next, we show that any asymptote to γ (which we now know to be timelike) is complete.

Proposition 4.11. *Let $x \in I(\gamma)$ and let η be a timelike asymptote to γ at x. Then $L_\tau(\eta) = \infty$.*

Proof. W.l.o.g. let η be future-directed. Suppose $L := L_\tau(\eta) < \infty$. By construction, $\eta : [0, \infty) \to X$ arises as a locally uniform limit of timelike maximisers $\eta_n : [0, a_n] \to X$ from x to $\gamma(t_n)$, where $t_n \to \infty$. By continuity of angles (see Proposition 2.41), $\measuredangle_x(\eta, \eta_n) \to 0$ for $n \to \infty$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for $n \geq N$, $\measuredangle_x(\eta, \eta_n) < \varepsilon$ (in fact, any finite bound would suffice here, we do not necessarily need an arbitrarily small one). Since $\tau(x, \gamma(t_n)) \to \infty$, we may assume that $\tau(x, \gamma(t_n)) \geq 3L\cosh(\varepsilon)$ (upon possibly choosing a larger N). Now note that for any t_n with $n \geq N$, $\partial J^-(\gamma(t_n)) \cap \eta = (J^-(\gamma(t_n)) \setminus I^-(\gamma(t_n))) \cap \eta$ is non-empty: Certainly, $x \in I^-(\gamma(t_n))$, so if this intersection were empty, then η would be imprisoned in the compact set $J^+(x) \cap J^-(\gamma(t_n))$, which cannot happen. So we find a point $y_0 \in \eta$ that is null-related to $\gamma(t_n)$, i.e. $y_0 < \gamma(t_n)$ and $\tau(y_0, \gamma(t_n)) = 0$. By continuity, there is y on η near y_0 such that $0 < \tau(y, \gamma(t_n)) < 3L\cosh(\varepsilon)/2$. Let now ν be the part of η from x to y with length $L_\tau(\nu) =: a$, σ a timelike maximiser from y to $\gamma(t_n)$ with length $b := L_\tau(\sigma) = \tau(y, \gamma(t_n))$. Moreover, we write $c := L_\tau(\eta_n)$. Then (ν, σ, η_n) forms a timelike triangle with angle $\beta := \measuredangle_x(\eta, \eta_n) < \varepsilon$ at x, consider a corresponding comparison triangle $(\overline{\nu}, \overline{\sigma}, \overline{\eta}_n)$ with angle $\overline{\beta}$ at $\overline{x}$. By the law of cosines, we get

$$b^2 = a^2 + c^2 - 2ac\cosh(\overline{\beta}). \tag{113}$$

Our curvature assumption gives that $\overline{\beta} \leq \beta < \varepsilon$. Moreover, as we have argued, $c \geq 3L\cosh(\varepsilon)$ and $b < c/2$ and $a < L$. Inserting all of this, we get

$$\begin{aligned} b^2 &\geq a^2 + c^2 - 2ac\cosh(\varepsilon) \geq a^2 + c^2 - 2Lc\cosh(\varepsilon) \geq c^2 - 2Lc\cosh(\varepsilon) \\ &= c^2\left(1 - \frac{2L\cosh(\varepsilon)}{c}\right) \geq c^2/3, \end{aligned}$$

contradicting $b < c/2$. □

To conclude this subsection, we show that any future directed and any past directed (timelike) asymptote to γ from a common point fit together to give a timelike line.

Proposition 4.12 (Asymptotic lines)**.** *Let $p \in I(\gamma)$ and consider any future and past rays $\sigma^+ : [0,\infty) \to X$ and $\sigma^- : (-\infty, 0] \to X$ from p to γ. Then $\sigma := \sigma^-\sigma^+ : \mathbb{R} \to X$ is a complete timelike line.*

Proof. Let σ_n^+ and σ_n^- be two sequences of timelike maximisers from p to $\gamma(r_n)$ and $\gamma(-r_n)$, respectively, such that σ^+ and σ^- arise as limits of these sequences as $r_n \to \infty$. To show that σ is a line, it is sufficient to show that for any $t > 0$, $\tau(\sigma(-t), \sigma(t)) = L_\tau(\sigma|_{[-t,t]})$. To see this, let $q_+ := \sigma(t)$ and $q_- := \sigma(-t)$, and $q_+^n := \sigma_n^+(t)$, $q_-^n := \sigma_n^-(-t)$. Then $q_\pm = \lim_n q_\pm^n$. Consider the triangle going from x via σ_n^+ to the endpoint of σ_n^+ , then following γ down to the endpoint of σ_n^-, and finally running the latter up to x again. Consider a comparison triangle with points $\overline{q_\pm^n}$ corresponding to $q_\pm^n$. Sending $n \to \infty$, we see that the stacked comparison triangles in $\mathbb{R}^{1,1}$ converge to a vertical line (here we use Proposition 4.6 and the completeness of γ), hence our curvature bound gives

$$\tau(q_-, q_+) = \lim_{n\to\infty} \tau(q_-^n, q_+^n) \leq \lim_{n\to\infty} \overline{\tau}(\overline{q_-^n}, \overline{q_+^n}) = \lim_n L_\tau(\sigma_n^-\sigma_n^+|_{[-t,t]}) = L_\tau(\sigma|_{[-t,t]}),$$

which is what we wanted to show, as the other inequality is trivial. □

4.2 Parallel lines

This subsection introduces the notion of parallelity for complete timelike lines. This approach is better suited for the synthetic case as it circumvents the analysis of Busemann functions. In the following, we call a map $f : Y_1 \to Y_2$ between Lorentzian pre-length spaces $(Y_1, d_1, \ll_1, \leq_1, \tau_1)$ and $(Y_2, d_2, \ll_2, \leq_2, \tau_2)$ *τ-preserving* if for all $p, q \in Y_1$, $\tau_1(p,q) = \tau_2(f(p), f(q))$, and we call f *causality preserving* if $p \leq_1 q$ if and only if $f(p) \leq_2 f(q)$.

Definition 4.13 (Parallel lines)**.** *Let α, β be two complete timelike lines in a Lorentzian pre-length space X defined on open intervals, w.l.o.g. on $\mathbb{R}$. They are called* parallel *if there exists a τ- and causality preserving map $f : (\alpha(\mathbb{R}) \cup \beta(\mathbb{R})) \to \mathbb{R}^{1,1}$ such that $f(\alpha(\mathbb{R}))$ and $f(\beta(\mathbb{R}))$ are parallel timelike lines in Minkowski (in the sense of parallel lines in affine spaces). We call such a map f a* parallel realisation *of α and β.*

Remark 4.14. Note that by post-composing this by an isometry of Minkowski space, we can always achieve that $f(\alpha(\mathbb{R})) = \{(t, 0) : t \in \mathbb{R}\} \subseteq \mathbb{R}^{1,1}$ and $f(\beta(\mathbb{R})) = \{(t, c) : t \in \mathbb{R}\}$ for some $c \geq 0$. In this form, if α and β are parallel and both parametrized by τ-arclength (this is possible if τ is continuous and $\tau(x, x) = 0$ for all $x \in X$, cf. Lemma 2.29), we get that $f(\alpha(t)) = (t + a, 0)$ and $f(\beta(t)) = (t + b, c)$. By doing a shift in Minkowski, we can make a to be 0 (changing b to $b - a$).

Lemma 4.15 (Properties of parallel realisations)**.** *Let X be a strongly causal Lorentzian pre-length space with continuous time separation, let $\alpha, \beta : \mathbb{R} \to X$ be two complete timelike lines and let $f : \alpha(\mathbb{R}) \cup \beta(\mathbb{R}) \to \mathbb{R}^{1,1}$ be a parallel realisation. Then f is a homeomorphism onto its image.*

Proof. We first show that f is injective. Certainly, due to the fact that f preserves time separations, $f|_{\alpha(\mathbb{R})}$ and $f|_{\beta(\mathbb{R})}$ are injective: Indeed suppose that e.g. $f(\alpha(t)) = f(\alpha(s)) = (r, x)$ for some $s < t$, then $0 < \tau(\alpha(s), \alpha(t)) = \overline{\tau}((r, x), (r, x)) = 0$, a contradiction. Now suppose that the lines $f(\alpha(\mathbb{R}))$ and $f(\beta(\mathbb{R}))$ in $\mathbb{R}^{1,1}$ are different, then we are done. Otherwise, by nature of lines in Minkowski space, they have to be equal if they intersect. In that case, each point on that line is reached by one point on α and one point on β via f. So suppose that $(r, x) = f(\alpha(t)) = f(\beta(s))$. Since f is τ-preserving and τ is continuous, for each $\varepsilon > 0$ there are $\delta, \tilde{\delta} > 0$ such that $f(\beta(s + \delta)) = (r + \varepsilon, x)$ and $f(\beta(s - \tilde{\delta})) = (r - \varepsilon, x)$. But this means that $\alpha(t) \in \bigcap_{\delta,\tilde{\delta}\downarrow 0} I(\beta(s - \tilde{\delta}), \beta(s + \delta))$. By strong causality, these sets are a neighbourhood basis at $\beta(s)$, hence $\alpha(t) = \beta(s)$. This concludes the proof that f is injective.

Clearly, $\alpha(\mathbb{R}) \cup \beta(\mathbb{R})$ is a strongly causal Lorentzian pre-length space with the restriction of the structure of X, and similarly for their images in Minkowski space. Moreover, f maps the timelike diamonds in $\alpha(\mathbb{R}) \cup \beta(\mathbb{R})$ to the timelike diamonds in $f(\alpha(\mathbb{R})) \cup f(\beta(\mathbb{R}))$. As these form topological bases due to strong causality, we conclude that f is a homeomorphism onto its image. $\square$

Note that in the above setting, any parallel realisation f gives us a τ-arclength parametrisation of the parallel lines α, β: Let $\tilde{\alpha}$ be a τ-arclength parametrisation of the line $f(\alpha(\mathbb{R}))$ in Minkowski, then $f^{-1} \circ \tilde{\alpha}$ is τ-arclength parametrised, similarly for β.

Definition 4.16 (Synchronised parallel lines)**.** *Let X be a Lorentzian pre-length space with continuous time separation τ satisfying $\tau(x,x) = 0$ for all $x \in X$. Let $\alpha : \mathbb{R} \to X$, $\beta : \mathbb{R} \to X$ be two τ-arclength parametrised, parallel and complete timelike lines. They are called* synchronised parallel[30] *if the parallel realisation $f : (\alpha(\mathbb{R}) \cup \beta(\mathbb{R})) \to \mathbb{R}^{1,1}$ can be chosen to be of the form $f(\alpha(t)) = (t, 0)$ and $f(\beta(t)) = (t, c)$ for some $c \geq 0$. The constant c is a well-defined property of (α, β) and is called the* distance *of the parallel lines. For any two parallel lines α and β parametrised by τ-arclength, one can shift β such that $(\alpha, \beta \circ (t \mapsto t - a))$ is synchronised parallel.*

Lemma 4.17 (The c-criterion for parallel lines)**.** *Let X be a Lorentzian pre-length space with continuous time separation τ satisfying $\tau(x,x) = 0$ for all $x \in X$. Let $\alpha : \mathbb{R} \to X$ and $\beta : \mathbb{R} \to X$ be two τ-arclength parametrised complete timelike lines. We define the following partial functions:*

- $c_{\alpha\beta}(s,t) = \sqrt{(t-s)^2 - \tau(\alpha(s), \beta(t))^2} (\in \mathbb{C})$ *if $\alpha(s) \leq \beta(t)$ (otherwise undefined),*
- $c_{\beta\alpha}(s,t) = \sqrt{(s-t)^2 - \tau(\beta(t), \alpha(s))^2} (\in \mathbb{C})$ *if $\beta(t) \leq \alpha(s)$ (otherwise undefined),*
- $c^N_{\alpha+}(s) = \inf\{t - s : \alpha(s) \leq \beta(t)\}$,
- $c^N_{\beta+}(s) = \inf\{t - s : \beta(s) \leq \alpha(t)\}$.

These are all constant where defined, have the same value c and the infima are minima if and only if α and β are synchronised parallel. This constant c is the distance between α and β.

If X is additionally globally causally closed and $\alpha \cap I^+(\beta) \neq \emptyset$ and $\beta \cap I^+(\alpha) \neq \emptyset$, then the infima in $c^N_{\alpha+}$ and $c^N_{\beta+}$ are automatically minima and the conditions that $c^N_{\alpha+} = c^N_{\beta+} = c$ are automatically satisfied if $c_{\alpha\beta} = c_{\beta\alpha} = c$. In that case, all of the constants agree.

Proof. We define f as in the definition of synchronised parallel with distance c: $f(\alpha(s)) := (s, 0)$, $f(\beta(s)) := (s, c)$. We will prove that f is causality preserving if and only if $c^N_{\alpha+}$ and $c^N_{\beta+}$ are constantly c, and under the assumption that this is the case f is τ-preserving if and only if $c_{\alpha\beta}$ and $c_{\beta\alpha}$ are constantly c wherever defined.

[30] This agrees with the usual notion of synchronising clocks in the sense that this is the parametrisation coming from synchronising the clocks on the particles α and β and parametrising α and β by their clock values.

As α and β are future directed τ-arclength parametrised timelike lines, it is clear that f is causality and τ-preserving along α and along β, i.e. we only have to check the conditions on $f(\alpha(s)) \leq f(\beta(t))$, their τ-distance and the same for α, β reversed.

So first, for causality preserving: We know that $f(\alpha(s)) \leq f(\beta(t)) \Leftrightarrow (s, 0) \leq (t, c) \Leftrightarrow (t-s) \geq c$. On the other hand, $\alpha(s) \leq \beta(t) \Leftrightarrow t-s \geq c^N_{\alpha+}(s)$ if this is a minimum and $\alpha(s) \leq \beta(t) \Leftrightarrow t-s > c^N_{\alpha+}(s)$ if it is only an infimum. In particular, these conditions are the same precisely when $c^N_{\alpha+}(s) = c$ and it is a minimum. Thus, $\alpha(s) \leq \beta(t) \Leftrightarrow f(\alpha(s)) \leq f(\beta(t))$ if and only if $c^N_{\alpha+}$ is constantly c and is always a minimum. Analogously, we get that $\beta(s) \leq \alpha(t) \Leftrightarrow f(\beta(s)) \leq f(\alpha(t))$ if and only if $c^N_{\beta+}$ is constantly c and is always a minimum.

We already proved that f is causality preserving. Next, we show that it preserves τ. On the Minkowski side, we have: $\bar{\tau}(f(\alpha(s)), f(\beta(t))) = \sqrt{(t-s)^2 - c^2}$ (if $f(\alpha(s)) \leq f(\beta(t))$). On the X side, we solve the defining equation for $c_{\alpha\beta}$ for $\tau(\alpha(s), \beta(t))$ to get: $\tau(\alpha(s), \beta(t)) = \sqrt{(t-s)^2 - c_{\alpha\beta}(s,t)^2}$ (if $\alpha(s) \leq \beta(t)$). Now note that as f is causality preserving, the side conditions when these equations can be applied match up (if they are not applied, $\tau(\alpha(s), \beta(t)) = 0 = \bar{\tau}(f(\alpha(s)), f(\beta(t)))$ anyway). If the side conditions are satisfied, we note that the equation differ only in one place: $\tau(\alpha(s), \beta(t))$ uses $c_{\alpha\beta}(s,t)$ and $\bar{\tau}(f(\alpha(s)), f(\beta(t)))$ uses c. Thus, we easily see that $\tau(\alpha(s), \beta(t)) = \bar{\tau}(f(\alpha(s)), f(\beta(t)))$ if and only if $c_{\alpha\beta}(s,t) = c$.

Analogously, we get that $\tau(\beta(t), \alpha(s)) = \bar{\tau}(f(\beta(t)), f(\alpha(s)))$ if and only if $c_{\beta\alpha}(s,t) = c$.

For the additional claim, fix s. If X satisfies the additional assumptions, $\{t : \alpha(s) \leq \beta(t)\}$ is closed and bounded (as is easily seen from the fact that $c_{\alpha\beta} \in \mathbb{R}$ when defined due to the reverse triangle inequality), thus it has a minimum automatically. We consider t such that $\alpha(s) \ll \beta(t)$, that is $t \in (t_s, +\infty) = \beta^{-1}(I^+(\alpha(s)))$. We get that $\tau(\alpha(s), \beta(t)) \to 0$ as $t \searrow t_s$. We transform the equation $c = \sqrt{(t-s)^2 - \tau(\alpha(s), \beta(t))^2}$ to $\tau(\alpha(s), \beta(t)) = \sqrt{(t-s)^2 - c^2}$. By continuity of τ, this still holds in the limit $t \searrow t_s$, i.e. $\sqrt{(t_s - s)^2 - c^2} = \tau(\alpha(s), \beta(t_s)) = 0$, so $t_s - s = c$ and $\{t : \alpha(s) \leq \beta(t)\} = [t_s, +\infty)$ as claimed. $\square$

Lemma 4.18 (The strong causality trick)**.** *Let X be a strongly causal Lorentzian pre-length space with continuous τ and $\tau(x,x) = 0$ for all $x \in X$. Let $\alpha, \beta : [0, b) \to \mathbb{R}$ be two τ-arclength parametrised timelike distance realisers with $x := \alpha(0) = \beta(0)$. Assume that for all s, t such that $\alpha(s)$ and $\beta(t)$ are timelike related, the comparison angle is $\tilde{\measuredangle}_x(\alpha(s), \beta(t)) = 0$. Then $\alpha = \beta$.*

Proof. We set $f_+(s,t) = \tau(\alpha(s), \beta(t))$ and $f_-(s,t) = \tau(\beta(s), \alpha(t))$. They are both monotonically increasing in t and continuous. We want to describe the set where $f_+ > 0$ resp. $f_- > 0$. We set $t_s^+ = \inf\{t : f_+(s,t) > 0\}$, then $\lim_{t\searrow t_s^+} f_+(s,t) = 0$. Thus in the law of cosines (Lemma 2.35), we get $\lim_{t\searrow t_s^+} \cosh(\tilde{\measuredangle}_x(\alpha(s),\beta(t))) = 1 = \lim_{t\searrow t_s^+} \frac{s^2+t^2-f_+(s,t)}{2st} = \frac{s^2+(t_s^+)^2}{2st_s^+}$, so $s = t_s^+$. Analogously, we get $s = \inf\{t : f_-(s,t) > 0\}$, in total $f_\pm(s,t) > 0$ for $s < t$.

That means that whenever $s_- < t < s_+$ we have that $\alpha(t) \in I(\beta(s_-),\beta(s_+))$. By Lemma 2.26, the $I(\beta(s_-),\beta(s_+))$ form a neighbourhood basis of the point $\beta(t)$, and $\alpha(t)$ is inside of all of these neighbourhoods. As X is Hausdorff, we get that $\alpha(t) = \beta(t)$ for all $t \in (0,b)$, so $\alpha = \beta$. □

Lemma 4.19 (Parallel lines are unique)**.** *Let X be a strongly causal, timelike geodesically connected Lorentzian pre-length space with continuous τ, $\tau(x,x) = 0$ for all $x \in X$. Suppose that X has global timelike curvature bounded below by 0, let $\alpha : \mathbb{R} \to X$ be a complete timelike line and $p \in X$ a point. Then there is (up to reparametrisation) at most one parallel line to α through p.*

Proof. We indirectly assume there are two parallel lines to α through p, namely $\beta : \mathbb{R} \to X$ and $\tilde{\beta} : \mathbb{R} \to X$. We assume all three curves are parametrised by τ-arclength, such that $\beta(0) = \tilde{\beta}(0) = p$, then α, β are synchronised parallel. Then also $\alpha, \tilde{\beta}$ are parametrised in a way that they are synchronised parallel, and both α, β and $\alpha, \tilde{\beta}$ have the same distance: set $s_+ = \inf \alpha^{-1}(I^+(p))$ and $s_- = \sup \alpha^{-1}(I^-(p))$ the boundary of the points on α timelike related to p. Then the distance of α, β and that of $\alpha, \tilde{\beta}$ can be calculated as $\frac{s_+-s_-}{2}$, so they are the same, and the necessary shift to make β and $\tilde{\beta}$ synchronised parallel to α can be calculated by $\frac{s_++s_-}{2}$, so they are the same too. As α, β are assumed to be synchronised already, this shift is 0.

We construct a comparison situation for all three lines at once: We choose the parallel realisation f for α and β given by $f(\alpha(s)) = (s,0)$ and $f(\beta(s)) = (s,c)$, and similarly we choose $\tilde{f}$ for α and $\tilde{\beta}$ given by $\tilde{f}(\tilde{\beta}(s)) = (s,-c)$ and $\tilde{f}(\alpha(s)) = (s,0)$, i.e. we realise β and $\tilde{\beta}$ on opposite sides of α.

We calculate the angle $\omega := \measuredangle_p(\beta|_{[0,+\infty)}, \tilde{\beta}|_{[0,+\infty)})$: By Proposition 4.8, this is equal to any comparison angle $\tilde{\measuredangle}_p(\beta(s),\tilde{\beta}(t))$ as long as $\beta(s) \ll \tilde{\beta}(t)$ or conversely. We know that $\beta(s) \ll \alpha(s+c+\varepsilon) \ll \tilde{\beta}(s+2c+2\varepsilon)$ and set $t = s+2c+2\varepsilon$. We now look at the law of cosines to estimate the comparison angle $\omega = \tilde{\measuredangle}_p(\beta(s),\tilde{\beta}(t))$: $\cosh(\omega) = \frac{s^2+t^2-\tau(\beta(s),\tilde{\beta}(t))^2}{2st} \leq \frac{2t^2}{2s^2}$ which converges to 1 as $s \to +\infty$. In particular, $\omega = 0$. We can now apply the strong causality trick to get $\beta = \tilde{\beta}$ (so they are even equal in this parametrisation), so there is only one line parallel to α through p. □

Let us now show for X as in the splitting theorem that asymptotic lines to γ constructed in Proposition 4.12 are parallel to γ.

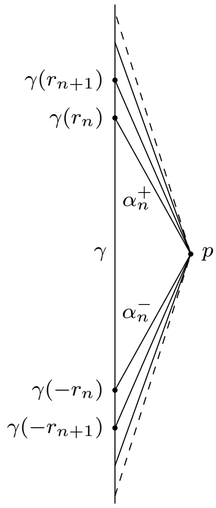

Figure 28: Stacking comparison triangles.

Lemma 4.20. *Let $\alpha : \mathbb{R} \to X$ be a complete timelike asymptotic line to γ. Then α, γ are parallel.*

Proof. By construction, α arises as the limit of timelike maximising segments α_n^+ from $p := \alpha(0)$ to $\gamma(r_n)$ and α_n^- from p to $\gamma(-r_n)$, where $r_n \to \infty$. We will use the stacking principle (cf. Proposition 4.6) to show that α and γ are parallel. Indeed, the triangles $\Delta(\gamma(-r_n), p, \gamma(r_n))$ stack in $\mathbb{R}^{1,1}$, hence by orienting the side corresponding to the segment on γ vertically (see Figure 28), we may assume that for $n \to \infty$ the comparison triangles converge to two vertical lines in $\mathbb{R}^{1,1}$.

We now argue that the map sending $\alpha(\mathbb{R})$ and $\gamma(\mathbb{R})$ to the corresponding limit lines $\bar{\alpha}$ and $\bar{\gamma}$ is a parallel realisation. For any t, s, consider $\tau(\alpha(t), \gamma(s)) = \lim_n \tau(\alpha_n(t), \gamma(s))$. For n so large that $-r_n < s < r_n$, $\gamma(s)$ is part of the triangle $\Delta(\gamma(-r_n), p, \gamma(r_n))$. From Corollary 4.9 we conclude that $\alpha_n(t) \ll \gamma(s)$ if and only if $\overline{\alpha}_n(t) \ll \overline{\gamma}(s)$, and $\tau(\alpha_n(t), \gamma(s)) = \overline{\tau}(\overline{\alpha}_n(t), \overline{\gamma}(s))$, where the bars denote the corresponding points on the comparison side. This shows that the realisation of α as $\bar{\alpha}$ and γ as $\bar{\gamma}$ is a parallel realisation. $\square$

Lemma 4.21 (Shifting asymptotes)**.** *Let $\alpha : \mathbb{R} \to X$ be a complete timelike asymptote to γ through $\alpha(0)$. Then the asymptote to γ through $\alpha(s)$ is $\alpha(\cdot - s)$ after parametrising both by τ-arclength.*

Proof. Let $\tilde{\alpha}$ be the asymptotic line through $\alpha(s)$. Then the previous result shows that both γ and α as well as γ and $\tilde{\alpha}$ are parallel and they meet at $\alpha(s) = \tilde{\alpha}(0)$. Thus, as parallel lines are unique (see Lemma 4.19), we have that (after synchronising) $\alpha = \tilde{\alpha}$. To get the correct parameters before shifting, we know $\alpha(s) = \tilde{\alpha}(0)$, which fixes the shift parameter. $\square$

Corollary 4.22 (Asymptotes stay in $I(\gamma)$)**.** *Asymptotic lines to γ from points in $I(\gamma)$ stay in $I(\gamma)$.*

Proof. Since asymptotic lines to γ are parallel to γ by Lemma 4.20, this readily follows. $\square$

Now that we have established the parallelity of γ and its asymptotes, we may reparametrise all of them by τ-arclength and fix a shift on each asymptote so that they are synchronised to γ. Note that we fix a τ-arclength parametrisation of γ, namely the one keeping $\gamma(0)$ the same.

Definition 4.23 (Busemann parametrisation)**.** *Let $\alpha : \mathbb{R} \to X$ be a complete timelike asymptote to γ. Then we call the (unique) τ-arclength parametrisation of α that synchronises it to γ its* Busemann parametrisation*.*

Remark 4.24 (Busemann functions). Up to this point, we have avoided the use of Busemann functions in our treatment. Though it will not really be necessary in what follows, let us discuss them briefly here. Let X be as in the splitting theorem and γ the given timelike line in τ-arclength parametrisation. Then we define the (*future*) *Busemann function* of γ as $b^+ : I(\gamma) \to \mathbb{R} \cup \{\pm\infty\}$ via $b^+(x) := \lim_{t\to\infty}(t - \tau(x, \gamma(t)))$. If $\alpha : \mathbb{R} \to X$ is any asymptotic line to γ in τ-arclength parametrisation and we choose the parallel realisation between α and γ so that $\gamma(t)$ is mapped to $(t, 0) \in \mathbb{R}^{1,1}$ (which we always do) and $\alpha(s)$ is mapped to $(s + S, c) \in \mathbb{R}^{1,1}$, then

$$b^+(\alpha(s)) = \lim_{t\to\infty}(t - \tau(\alpha(s), \gamma(t))) = \lim_{t\to\infty}(t - \sqrt{(t - (s + S))^2 - c^2}) = s + S.$$

Hence, b^+ indicates the shift in the time parameter along any τ-arclength parametrised asymptote. In particular, b^+ is finite-valued on all of $I(\gamma)$ since any point lies on an asymptote, and in fact it is not hard to show that b^+ is continuous on $I(\gamma)$ in a similar fashion as [11, Lem. 3.3], but we will not need this. Note that if α is in Busemann parametrisation, then $b^+(\alpha(s)) = s$ for all $s \in \mathbb{R}$, hence the name.

Lemma 4.25 (Parallelity is weakly transitive)**.** *Let X be a strongly causal, timelike geodesically connected Lorentzian pre-length space with continuous τ, $\tau(x, x) = 0$ for all x, and global non-negative timelike curvature. Let $\alpha, \beta, \gamma : \mathbb{R} \to X$ be timelike lines such that α, β and β, γ are parallel. Assume that there is a point p on γ such that there is a parallel line to α through p. Then this parallel line is already γ.*

Proof. We assume the parallel lines are even synchronised parallel in a suitable τ-arclength parametrisation. Let $\tilde{\gamma}$ be the synchronised parallel line to α through p. Let a be the distance between α, β, b be the distance between β, γ and c be the distance between $\alpha, \tilde{\gamma}$. We have two lines through p: γ and $\tilde{\gamma}$. We assume the parameters for p are $p = \gamma(t_0) = \tilde{\gamma}(\tilde{t}_0)$. By Proposition 4.8, we know that $\tilde{\measuredangle}_p(\gamma(s), \tilde{\gamma}(t))$ is constant in s and t as long as $\gamma(s) \ll \tilde{\gamma}(t)$ (or conversely). We know that $\gamma(s) \ll \beta(s+b+\varepsilon) \ll \alpha(s+a+b+2\varepsilon) \ll \tilde{\gamma}(s+a+b+c+3\varepsilon)$ and set $t = s+a+b+c+3\varepsilon$. We now look at the law of cosines to estimate the comparison angle $\omega = \tilde{\measuredangle}_p(\gamma(s), \tilde{\gamma}(t))$: $\cosh(\omega) = \frac{(s-t_0)^2+(t-\tilde{t}_0)^2-\tau(\gamma(s),\tilde{\gamma}(t))}{2(s-t_0)(t-\tilde{t}_0)} \leq \frac{2(t-\tilde{t}_0)^2}{2(s-t_0)^2}$ which converges to 0 as $s \to +\infty$. In particular, $\omega = 0$. We can now apply the strong causality trick to get that γ and $\tilde{\gamma}$ are just shifts of each other, so α and γ are parallel.

By doing the same argument backwards as well, i.e. $\alpha(s-a-b-2\varepsilon) \ll \gamma(s) \ll \alpha(s+a+b+2\varepsilon)$, we get that the synchronised version only differs by a shift of at most $a+b$ (upon taking $\varepsilon \to 0$). □

Lemma 4.26 (Verticality in Minkowski space)**.** *Let $a = (0,0) \ll b$ be points in Minkowski space $\mathbb{R}^{1,1}$. Let the t-coordinate difference be $\Delta t = t(b) - t(a)$. Let c_n be points with $b \ll c_n$, the t-coordinate $t(c_n) \to +\infty$, c_n above the straight line through a, b and $\tau(a, c_n) - \tau(b, c_n) \to \Delta t$ as $n \to +\infty$. Then $\frac{x(c_n)}{t(c_n)} \to 0$, or equivalently, $\frac{c_n}{\|c_n\|} \to \partial_t$.*[31]

Proof. We show that $\forall \varepsilon > 0$ there are constants $T > 0$ and $\delta > 0$ such that for all p with $\tau(a,p) \geq T$ and $|\tau(a,p) - \tau(b,p) - (t_b - t_a)| < \delta$, we have that the angle of ap with the t-axis satisfies $\measuredangle_a(\partial_t, p) < \varepsilon$.

We set $\omega_1 = \measuredangle_a(\partial_t, b)$ the angle between the vertical and ab, and $\omega_2 = \measuredangle_a(p, b)$ the angle between the vertical and ap. We will prove $|\omega_1 - \omega_2| < \varepsilon$. By assumption, p and ∂_t lie on the same side of the straight line through ab, so by angle additivity in the plane we also get that $\measuredangle_a(\partial_t, p) < \varepsilon$.

We apply the law of cosines:

$$\tau(b,p)^2 - \tau(a,p)^2 = \tau(a,b)^2 - 2\tau(a,b)\tau(a,p)\cosh(\omega_2)$$

and use that $\cosh(\omega_1)\tau(a,b) = t_b - t_a$ to get:

$$\tau(a,p) - \tau(b,p) = (t_b - t_a) \underbrace{\frac{(2\tau(a,p)\cosh(\omega_2) - 1)}{\cosh(\omega_1)(\tau(b,p) + \tau(a,p))}}_{=:F} .$$

We now have to check that whenever $|\omega_1 - \omega_2| \geq \varepsilon$, the factor F is bounded away from 1.

[31]So c_n converges to the point $[\partial_t]$ on the limit sphere.

We have two cases to cover: First, we indirectly assume $\omega_1 > \omega_2 + \varepsilon$ (then $1 - \frac{\cosh(\omega_2)}{\cosh(\omega_1)} > 1 - \cosh(\varepsilon)$). (Assuming b lies to the right of the t-axis, this is the case where p also lies on the right.) We will show $F < 1 - \delta$ (if $\tau(a,p)$ is large enough and $\tau(a,p), \tau(b,p)$ are close enough). This is equivalent to:

$$(2\tau(a,p)\cosh(\omega_2) - 1) < (1-\delta)\cosh(\omega_1)(\tau(b,p) + \tau(a,p))$$

Dividing by $\tau(a,p)$, this is obvious: $\frac{\tau(b,p)}{\tau(a,p)} \to 1$ as $\tau(a,p) \to +\infty$ and the difference is bounded. We can e.g. choose

$$\delta < \cosh(\varepsilon) - 1$$

and T large enough.

Now we indirectly assume $\omega_1 < \omega_2 - \varepsilon$ (then $\frac{\cosh(\omega_2)}{\cosh(\omega_1)} - 1 > \cosh(\varepsilon) - 1$) (Assuming b lies to the right of the t-axis, this is the case where p lies on the left.) We will show $F > 1 + \delta$ (if $\tau(a,p)$ is large enough and $\tau(a,p), \tau(b,p)$ are close enough). We need to check:

$$(2\tau(a,p)\cosh(\omega_2) - 1) > (1+\delta)\cosh(\omega_1)(\tau(b,p) + \tau(a,p))$$

Dividing by $\tau(a,p)$, this is obvious: $\frac{\tau(b,p)}{\tau(a,p)} \to 1$ as $\tau(a,p) \to +\infty$ and the difference is bounded. We can e.g. choose

$$\delta < \cosh(\varepsilon) - 1$$

and T large enough. □

We now run into a subtlety: being parallel is in general only weakly transitive, but being synchronised parallel is probably not. Luckily, for X as in the splitting theorem, the fact that we consider asymptotic lines to γ simplifies the situation:

Lemma 4.27 (Two asymptotes are parallel)**.** *Let $x, y \in I(\gamma)$ and α, β the asymptotes to γ through x resp. y in Busemann parametrisation. Then α, β are synchronised parallel.*

Proof. We dive into the construction of asymptotes: We first assume $x \ll y$. We set $s_0 = b^+(x)$ and $t_0 = b^+(y)$, then there are maximisers α_n from x to $\gamma(T_n)$ parametrised in τ-arclength such that $\alpha_n(s_0) = x$ and β_n from y to $\gamma(T_n)$ parametrised in τ-arclength such that $\beta_n(t_0) = y$, which converge (pointwise) to the upper parts of α and β in Busemann parametrisation for $T_n \to \infty$, respectively. Note that $\alpha_n(s_0 + \tau(x, \gamma(T_n))) = \beta_n(t_0 + \tau(y, \gamma(T_n))) = \gamma(T_n)$.

We try to prove the c-criterion (Lemma 4.17): Clearly, due to our assumptions on X, we do not need to consider the null functions. We look at $c_{\alpha\beta}(s_0, t_0)$ and $c_{\alpha\beta}(s', t')$ or $c_{\beta\alpha}(s', t')$ for $s_0 \leq s'$, $t_0 \leq t'$: We require $a := x = \alpha(s_0) \ll b := y = \beta(t_0)$. We get points converging to these parameters: $a'_n := \alpha_n(s') \to a' := \alpha(s')$ and $b'_n := \beta_n(t') \to b' := \beta(t')$ (note that $\alpha_n(s_0) = a$ and $\beta_n(t_0) = b$ anyway). We get a timelike triangle $\Delta_n = \Delta(a, b, c_n := \gamma(T_n))$ containing the points (a'_n, b'_n). We form a comparison situation for this in Minkowski space $\mathbb{R}^{1,1}$: $\bar{\Delta}_n = \Delta(\bar{a}, \bar{b}, \bar{c}_n)$ with comparison points $\bar{a}'_n, \bar{b}'_n$. As X has global curvature bounded below by 0, we get that $\tau(a'_n, b'_n) \leq \bar{\tau}(\bar{a}'_n, \bar{b}'_n)$ and $\tau(b'_n, a'_n) \leq \bar{\tau}(\bar{b}'_n, \bar{a}'_n)$.

We would now like to let $T_n \to +\infty$, so we need control over what the comparison triangle $\bar{\Delta}_n$ converges to. For this, we select which way to realise $\bar{\Delta}_n$ in $\mathbb{R}^{1,1}$. We choose:

- $\bar{a} = (s_0, 0)$ constant in n, i.e. at the right t-coordinate and with x-coordinate 0,
- $\bar{b} = (t_0, c_{\alpha\beta}(s_0, t_0))$ constant in n. Note this has the right τ-distance to $\bar{a}$ by definition of $c_{\alpha\beta}$.
- $\bar{c}_n$ above the straight line $\bar{a}$ and $\bar{b}$ lie on.

We claim that in the limit $T_n \to +\infty$, the line $\bar{a}\bar{c}_n$ becomes vertical (the line $\bar{b}\bar{c}_n$ also becomes vertical). For this, we look at what $\bar{\tau}(\bar{a}, \bar{c}_n) - \bar{\tau}(\bar{b}, \bar{c}_n)$ does as $T_n \to +\infty$: Using Landau small-oh notation, we get that $\bar{\tau}(\bar{a}, \bar{c}_n) = T_n - b^+(a) + o(1)$ and $\bar{\tau}(\bar{b}, \bar{c}_n) = T_n - b^+(b) + o(1)$ as $T_n \to +\infty$, so the difference is $b^+(b) - b^+(a) + o(1)$, which approaches the t-coordinate-difference $t_0 - s_0$ of $\bar{a}$ and $\bar{b}$. We can now apply Lemma 4.26 to get that $\frac{x(c_n)}{t(c_n)} \to 0$ and $t(c_n) \to +\infty$, thus these lines become vertical.

We have proven that the comparison points converge, so we have limit comparison points: $\bar{a} = (s_0, 0)$ and $\bar{b} = (t_0, c_{\alpha\beta}(s_0, t_0))$ stay fixed anyway, $\bar{a}'_n \to \bar{a}' = (s', 0)$, $\bar{b}'_n \to \bar{b}' = (t', c_{\alpha\beta}(s_0, t_0))$. We also set the comparison sides $\bar{\alpha}(s) = (s, 0)$ and $\bar{\beta}(t) = (t, c_{\alpha\beta}(s_0, t_0))$. By continuity of τ and $\bar{\tau}$, our curvature assumption gives $\tau(a', b') \leq \bar{\tau}(\bar{a}', \bar{b}')$ in the limit (similarly with arguments flipped). Now we compare the definition of $c_{\alpha\beta}(s', t')$ resp. $c_{\beta\alpha}(s', t')$ with the c-functions for $\bar{\alpha}$ and $\bar{\beta}$ at the same parameters:

$$
\begin{aligned}
c_{\alpha\beta}(s', t') &= \sqrt{(t' - s')^2 - \tau(a', b')^2}, \\
c_{\bar{\alpha}\bar{\beta}}(s', t') &= \sqrt{(t' - s')^2 - \bar{\tau}(\bar{a}', \bar{b}')^2} = x(\bar{b}') - x(\bar{a}') = c_{\alpha\beta}(s_0, t_0),
\end{aligned}
$$

whenever defined. The last equality holds because we know $\bar{\alpha}$ and $\bar{\beta}$ are synchronised parallel with distance $c_{\alpha\beta}(s_0, t_0)$. Note that $a' \leq b'$ implies $\bar{a}' \leq$

$\bar{b}'$ by the curvature bound and a simple continuity argument, so whenever the first line is defined so is the second. Notice these equations only differ in the τ term, and we know $\tau(a', b') \leq \bar{\tau}(\bar{a}', \bar{b}')$, so we get $c_{\alpha\beta}(s', t') \geq c_{\alpha\beta}(s_0, t_0)$ whenever the former is defined. Similarly, if $b' \leq a'$ we get $c_{\beta\alpha}(s', t') \geq c_{\alpha\beta}(s_0, t_0)$ for all $s' \geq s_0$ and $t' \geq t_0$. We can also do this if $a \gg b$, giving $c_{\alpha\beta}(s', t') \geq c_{\beta\alpha}(s_0, t_0)$ and $c_{\beta\alpha}(s', t') \geq c_{\beta\alpha}(s_0, t_0)$.

Doing the same construction towards the past, we similarly get $c_{\alpha\beta}(s', t') \geq c_{\alpha\beta}(s_0, t_0)$ and $c_{\beta\alpha}(s', t') \geq c_{\alpha\beta}(s_0, t_0)$ resp. $c_{\alpha\beta}(s', t') \geq c_{\beta\alpha}(s_0, t_0)$ and $c_{\beta\alpha}(s', t') \geq c_{\beta\alpha}(s_0, t_0)$ (depending on whether $a \ll b$ or $b \ll a$) for all $s' \leq s_0$ and $t' \leq t_0$.

Now we use Lemma 4.21 to get that the (Busemann parametrised) asymptote to γ through $\alpha(s)$ is α, and similarly the asymptote to γ through $\beta(t)$ is β. In particular, we can use the above argument again for $\alpha(s)$ instead of x and $\beta(t)$ instead of y. We know that $b^+(\alpha(s)) = s$ and $b^+(\beta(t)) = t$. The above (to the future, $a \ll b$) then gives: $c_{\alpha\beta}(s', t') \geq c_{\alpha\beta}(s, t)$ for $s \leq s'$, $t \leq t'$ such that $a \ll b$ and $a' \ll b'$, extending continuously, $c_{\alpha\beta}$ is monotonously increasing where defined. On the other hand, the above (to the past, $a \ll b$) gives: $c_{\alpha\beta}(s', t') \geq c_{\alpha\beta}(s, t)$ for $s \geq s'$, $t \geq t'$ such that $a \ll b$ and $a' \ll b'$, extending continuously, $c_{\alpha\beta}$ is monotonously decreasing where defined. Via a two-step process, we see that $c_{\alpha\beta}$ is constant where defined. Similarly, we get that also $c_{\beta\alpha}$ is constant where defined, and that they have the same value.

Thus, the c-criterion (Lemma 4.17) yields that α and β are synchronised parallel. $\square$

This result establishes the transitivity of synchronised parallel lines: Any two asymptotes to γ in Busemann parametrisation are synchronised parallel.

5 Proof of the main result

Let us summarise what we have shown so far: Let $(X, d, \ll, \leq, \tau)$ be a connected, regularly localisable, globally hyperbolic Lorentzian length space satisfying timelike geodesic prolongation, with proper metric d and global non-negative timelike curvature containing a complete timelike line $\gamma : \mathbb{R} \to X$. Then from each point in $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$ we can construct (unique) asymptotic rays to γ, all of which are timelike with infinite τ-length. Future directed and past directed rays from a common point fit together to give a timelike line which is parallel to γ. We fix a τ-arclength parametrisation of γ fixing $\gamma(0)$, and always consider it to be mapped to the t-axis in $\mathbb{R}^{1,1}$ in any parallel realisation. Then, by way of the Busemann parametrisation, all asymptotic lines become synchronised parallel to γ and to each other. This synchronisation selects a "spacelike" slice (namely $\{b^+ = 0\}$) in X which will provide the metric part of the splitting.

In this section, we complete the proof of the splitting theorem in two steps: First, we show that $I(\gamma)$ splits, then we show that $I(\gamma)$ has the (TC) property, which implies $I(\gamma) = X$ due to Theorem 2.47.

Definition 5.1 (Spacelike slice). *We call the set of all $\alpha(0)$, where α is a Busemann parametrised timelike asymptotic line to γ, the* spacelike slice *and denote it by S. For $\alpha(0), \beta(0) \in S$, we define $d_S(\alpha(0), \beta(0))$ to be the distance between α and β in the sense of parallel lines.*

Lemma 5.2. *(S, d_S) is a metric space.*

Proof. As asymptotes to γ are parallel to γ and parallel lines are unique, d_S is well-defined.

Let $p, q \in S$. It is obvious from the definition of the distance of two parallel lines that $d_S(p, q) \geq 0$. If $d_S(p, q) = 0$, we get that the asymptotes through p and q are the same curve α. As the Busemann parametrisation is unique, we get that $p = q$.

For the triangle inequality, let $p, q, r \in S$. We get asymptotes α through p, β through q and γ through r, and by Lemma 4.27, they are pairwise synchronised parallel. Let $d_1 = d_S(p, q)$ and $d_2 = d_S(q, r)$, then $\alpha(0) \leq \beta(d_1)$ by the $c^N_{\alpha+}(0)$ criterion (see Lemma 4.17) for α, β and $\beta(d_1) \leq \gamma(d_1 + d_2)$ by the $c^N_{\beta+}(d_1)$ criterion for β, γ. Thus, we get that $\alpha(0) \leq \gamma(d_1 + d_2)$, giving $d_S(p, r) \leq d_1 + d_2$ by the $c^N_{\alpha+}(0)$ criterion for α, γ. □

Definition 5.3 (The splitting map). *We define $f : \mathbb{R} \times S \to I(\gamma) \subset X$ by $f(s, p) = \alpha_p(s)$, where α_p is the asymptote to γ through p in Busemann parametrisation, that is $\alpha_p(0) = p$.*

Proposition 5.4 (Local splitting). *Let X be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric d and global non-negative timelike curvature satisfying timelike geodesic prolongation and containing a complete timelike line $\gamma : \mathbb{R} \to X$. Then $I(\gamma) \subset X$ is a causally convex open set that is itself a path-connected, regularly localisable, globally hyperbolic Lorentzian length space of global non-negative timelike curvature with the metric, relations and time separation induced from X. Moreover, the spacelike slice S is a proper (hence complete), strictly intrinsic metric space, the Lorentzian product $\mathbb{R} \times S$ is a path-connected, regularly localisable, globally hyperbolic Lorentzian length space and the splitting map $f : \mathbb{R} \times S \to I(\gamma)$ is a τ- and causality preserving homeomorphism.*

Proof. First, it is clear that $I(\gamma)$ is path-connected, causally convex in X and has global non-negative timelike curvature. It is hence causally path-connected since X is and it is trivially locally causally closed. Moreover, if

$x \in I(\gamma)$ and U is a regular localising neighbourhood of x in X, then $U \cap I(\gamma)$ is a regular localising neighbourhood of x in $I(\gamma)$, hence $I(\gamma)$ is regularly localisable. By causal convexity, the time separation between causally related points in $I(\gamma)$ is achieved as the supremum of lengths of causal curves running between them which have to stay inside $I(\gamma)$. The causal diamonds in $I(\gamma)$ are precisely those in X, since they must be contained in $I(\gamma)$. Finally, $I(\gamma)$ is non-totally imprisoning (it inherits this from X), thus we have shown all the claims on $I(\gamma)$.

Next, we argue that f is τ- and causality preserving: Let $p, q \in S$. Let α be the asymptote to γ through p and let β be the asymptote to γ through q. Then α and β are synchronised parallel with distance $d_S(p, q)$, so there is a parallel realisation $\tilde{f} : \alpha(\mathbb{R}) \cup \beta(\mathbb{R}) \to \mathbb{R}^{1,1}$ defined by $\tilde{f}(\alpha(s)) = (s, 0)$ and $\tilde{f}(\beta(t)) = (t, d_S(p, q))$. In particular, $\alpha(s) \leq_X \beta(t) \Leftrightarrow t - s \geq d_S(p, q)$ and if this is true, $\tau(\alpha(s), \beta(t)) = \sqrt{(t-s)^2 - d_S(p, q)^2}$. But that is just the definition of $(s, p) \leq_{\mathbb{R}\times S} (t, q)$ resp. $\tau_{\mathbb{R}\times S}((s, p), (t, q))$ in $\mathbb{R} \times S$. This is true for all $p, q \in S$ and $s, t \in \mathbb{R}$, so f is τ- and causality preserving. f is injective as asymptotes to γ are parallel to γ and parallel lines are unique, and surjective since any point in $I(\gamma)$ lies on an asymptote.

From the discussion above, it is easy to see that f maps timelike (and causal) diamonds in $I(\gamma)$ to timelike (and causal) diamonds in $\mathbb{R}\times S$. As both sides are strongly causal, this implies that f is a continuous open bijection, hence a homeomorphism. Since $\mathbb{R} \times S$ is always non-totally imprisoning (cf. Proposition 3.4) and its causal diamonds are compact (as continuous images of compact causal diamonds in $I(\gamma)$), we conclude that $\mathbb{R} \times S$ is globally hyperbolic, hence S is proper by Proposition 3.7. To see that S is a strictly intrinsic space, fix $p, q \in S$ and connect any two timelike related points on the corresponding asymptotes by a distance realiser in $I(\gamma)$. The image of that distance realiser under f is a continuous distance realiser in $\mathbb{R} \times S$, hence by Proposition 3.5 the projection onto S gives a distance minimiser in S between p and q.

Finally, we need to prove the remaining claimed properties of $\mathbb{R}\times S$. Path-connectedness is inherited from $I(\gamma)$ via f, and products are always globally causally closed (cf. Proposition 3.2). Note that $I(x, y)$ for $x, y \in I(\gamma)$ are regular localising neighbourhoods in $I(\gamma)$. Since $f(I(x, y)) = I(f(x), f(y))$ and the d-lengths of causal curves in timelike diamonds can always be uniformly bounded in products (cf. the proof of Proposition 3.4), timelike diamonds in $\mathbb{R} \times S$ are in fact (regular) localising neighbourhoods: For the local time separation, take the restriction of $\tau_{\mathbb{R}\times S}$, and note that maximisers in $I(\gamma)$ (or in any $I(x, y) \subset I(\gamma)$) map to continuous maximisers in $\mathbb{R} \times S$, which are always Lipschitz reparametrisable and are hence causal curves (cf. Proposition 3.5). $\square$

Proposition 5.5 (Global splitting)**.** *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions in Proposition 5.4. Then $I(\gamma) = X$, i.e. the splitting in Proposition 5.4 is global.*

Proof. Suppose $\alpha : [a, b) \to I(\gamma)$ is a future directed timelike geodesic with finite τ-length. We reparametrise α by τ-arclength and denote the reparametrised curve again by α, it is then defined on $[0, L)$, where $L := L_\tau(\alpha) < \infty$. Then $f^{-1} \circ \alpha$ is a continuous timelike geodesic in $\mathbb{R} \times S$, where f is the splitting map. We decompose it into time- and space-component: $f^{-1} \circ \alpha = (\alpha^t, \alpha^x)$. If α^x is constantly p, we see that $\alpha = \alpha_p$, a contradiction since asymptotes have infinite τ-length. Note that since the τ-lengths of $f^{-1} \circ \alpha$ and α agree, $f^{-1} \circ \alpha$ also has finite τ-length in $\mathbb{R} \times S$. By Corollary 3.6, α^x has constant speed and domain $[0, L)$, thus finite d_S-length.

α^t can certainly be extended to L. As S is a complete metric space, we can extend α^x continuously to L. Mapping this extension back to $I(\gamma)$ via f, we get a continuous extension in $I(\gamma)$ of α to b (in its original parametrisation). This shows that $I(\gamma)$ has the (TC) property, thus $I(\gamma) = X$ by Theorem 2.47. This concludes the proof of the splitting in Theorem 1.4. $\square$

Recall the notion of Cauchy sets and (Cauchy) time functions on Lorentzian pre-length spaces from [19, Sec. 5.1]: A Cauchy set is any subset that is met exactly once by doubly inextendible causal curves, and a Cauchy time function is a continuous function $t : X \to \mathbb{R}$ such that $x < y$ implies $t(x) < t(y)$ and the image under t of any doubly inextendible causal curve is all of $\mathbb{R}$.

Corollary 5.6. *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions of Theorem 1.4 and let $f : \mathbb{R} \times S \to X$ be the splitting. Then the sets $S_t := f(\{t\} \times S)$ are Cauchy sets in X that are all homeomorphic to S. Moreover, the map $pr_1 \circ f^{-1}$ is a Cauchy time function. Moreover, all Cauchy sets in X are homeomorphic to S.*

Proof. Let $\alpha : (a, b) \to X$ be any doubly inextendible causal curve in X meeting any S_t twice, then its image under f^{-1} meets $\{t\} \times S$ twice, which cannot happen since $\{t\} \times S$ is acausal in $\mathbb{R} \times S$. Next we argue that any such α meets S_t: For simplicity let $t = 0$, and suppose that α does not meet S, so w.l.o.g. we may assume that $\alpha \subset I^+(S)$ (this is because $X = I^-(S) \cup S \cup I^+(S)$ due to the splitting, and this union is disjoint). Let $t_0 \in (a, b)$, then $\alpha((a, t_0]) \subset J^-(\alpha(t_0)) \cap J^+(S)$ which is easily seen to be a compact set by considering the corresponding situation in $\mathbb{R} \times S$. But this is a contradiction, since X is non-totally imprisoning. Hence S (and any S_t) is a Cauchy set. They are homeomorphic to S since X has the topology of $\mathbb{R} \times S$.

We now argue that $pr_1 \circ f^{-1}$ is a Cauchy time function for X. It is clearly a time function, as it is continuous and the time separation on $\mathbb{R} \times S$ is more or less a reformulation of that. Now let $\alpha : (a, b) \to X$ be a doubly inextendible causal curve, we need to show that $pr_1 \circ f^{-1} \circ \alpha((a, b)) = \mathbb{R}$. Suppose not, so there is some time value t_0 that is not attained. W.l.o.g. suppose $t_0 \geq 0$, so the image is contained in $(-\infty, t_0]$. Similarly to before, this would imply that $\alpha|_{[t_0,b)}$ is contained in the compact set $J^+(\alpha(t_0)) \cap J^-(S_{t_0})$, a contradiction.

Finally, let C be any Cauchy set in X. Then the projection $C \to S$ (via the splitting) is continuous. Its inverse is given by sending each $p \in S$ to the unique point on α_p meeting C. This is a continuous map since C is achronal. This shows that C is homeomorphic to S. □

Note that in general, Cauchy sets in globally hyperbolic Lorentzian pre-length spaces need not be homeomorphic, as [19, Ex. 5.7, Ex. 5.8] show.

Corollary 5.7. *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions of Theorem 1.4. Then (S, d_S) has Alexandrov curvature ≥ 0.*

Proof. Due to the splitting, the space $\mathbb{R} \times S$ has global nonnegative timelike curvature (cf. Remark 2.33; the splitting maps Lipschitz triangles in $\mathbb{R} \times S$ to continuous triangles in X, but this does not lead to any issues), and then it follows from [5, Thm. 5.7] that S has Alexandrov curvature ≥ 0. □

6 Applications and Outlook

Let us now give several reformulations of our splitting result for spacetimes. We start with smooth spacetimes, where we get a weaker version of the smooth splitting result proven in [11].

Corollary 6.1 (Smooth spacetimes)**.** *Let (M, g) be a smooth, globally hyperbolic spacetime with non-positive timelike sectional curvature containing a complete timelike line $\gamma : \mathbb{R} \to M$. Then the Lorentzian length space induced by (M, g) splits in the sense of Theorem 1.4.*

Proof. Fix any complete Riemannian metric h on M and let d_h be the induced distance, then $(M, d_h, \ll, \leq, \tau)$ is a connected, regularly localisable (via convex neighbourhoods) Lorentzian length space and d_h is a proper metric (note that the τ-length of causal curves agrees with the usual notion of length in (M, g) due to [48, Prop. 2.32]). The fact that (M, g) has global non-negative timelike curvature in the sense of Definition 2.32 is a consequence of the Lorentzian Toponogov theorem (see [43, p. 303]). The remaining technical assumptions are easily seen to hold. Hence the Lorentzian length space induced by (M, g) can be split according to Theorem 1.4. □

Remark 6.2. In the setting of Corollary 6.1, once one has shown that the spacelike slice S is a spacelike hypersurface in M and the normal asymptotic distance d_S is the Riemannian distance on S (e.g. via an analysis of the gradient of the Busemann function as in [11]), the Hawking–King–McCarthy theorem [44], which states that any τ-preserving homeomorphism between strongly causal spacetimes is a smooth isometry, provides a somewhat alternative proof of the smooth splitting result proven in [11].

Next, we turn to $C^{1,1}$-spacetimes. First note that the Lorentzian prelength space induced by any strongly causal Lipschitz spacetime is in fact a Lorentzian length space by [48, Thm. 5.12]. For $C^{1,1}$-metrics, the notion of sectional curvature is not available any more, so we have to assume synthetic curvature bounds. However, most of the other technical assumptions are easily seen to be satisfied due to the existence of convex neighbourhoods. For details on low regularity spacetimes, see e.g. the recent review article [59] and the references therein.

Corollary 6.3 ($C^{1,1}$-spacetimes)**.** *Let (M, g) be a globally hyperbolic $C^{1,1}$-spacetime with global non-negative timelike curvature containing a complete timelike line $\gamma : \mathbb{R} \to M$. Then the Lorentzian length space induced by (M, g) splits in the sense of Theorem 1.4.*

Finally let us mention the case of C^1-spacetimes, which are substantially different. Notably, the Lorentzian length space induced by a globally hyperbolic C^1-spacetime need not satisfy the timelike geodesic prolongation property. This is because solutions of the g-geodesic equation need not maximise locally, nor does a maximising segment necessarily extend as a local maximiser beyond its endpoints. However, points in C^1-spacetimes still have neighbourhood bases consisting of globally hyperbolic, causally convex neighbourhoods, so regular localisability is satisfied. Assuming the necessary synthetic properties as well, we have the following (note that geodesics in the synthetic sense are locally maximising curves, which are necessarily C^2-solutions of the geodesic equation even if the metric is just C^1):

Corollary 6.4 (C^1-spacetimes)**.** *Let (M, g) be a globally hyperbolic C^1-spacetime with global non-negative timelike curvature containing a complete timelike line $\gamma : \mathbb{R} \to M$. Suppose that each maximising timelike geodesic segment extends to a locally maximising timelike geodesic on an open domain. Then the Lorentzian length space induced by (M, g) splits in the sense of Theorem 1.4.*

In this work, we have established a synthetic splitting result for Lorentzian length spaces using triangle comparison. Ideas for future projects include the generalisation to Lorentzian length spaces lower on the causal ladder (e.g. (TC) spaces instead of globally hyperbolic ones, an approach to this that seems to be adaptable to the synthetic situation is [34]), an analysis of the aforementioned splittings of low regularity spacetimes (regarding regularity of the splitting, level set, etc.), as well as an investigation into synthetic timelike Ricci curvature bounds introduced in [23] and expanded in [15]. In the case of low regularity spacetimes, it is not yet clear how these different notions of curvature bounds relate to each other (some recent progress in this direction includes [47, 16]). For metric measure spaces, it is known that the CD-condition is not sufficient and the full RCD-condition is needed for a splitting result (see [37, 38]). An analogue of the RCD-condition is currently unavailable for Lorentzian length spaces and will probably have to be developed before a splitting theorem with synthetic timelike Ricci curvature bounds can be considered.

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Declarations

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- **Availability of data and materials** No data was collected nor produced in this work.

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A Bonnet-Myers rigidity theorem for globally hyperbolic Lorentzian length spaces

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Abstract

We prove a synthetic Bonnet-Myers rigidity theorem for globally hyperbolic Lorentzian length spaces with global curvature bounded below by $K < 0$ and an open distance realizer of length $L = \frac{\pi}{\sqrt{|K|}}$. In the course of the proof, we show that the space necessarily is a warped product with warping function $\cos : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}_+$.

Keywords: Lorentzian length spaces, synthetic curvature bounds, warped product spaces

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1 Introduction

Since our approach is closely related to the one in [9], we begin by recalling and extending some notions from the introduction to that paper.

In the setting of Riemannian manifolds, the Bonnet-Myers theorem states that a complete Riemannian manifold M with sectional curvature bounded below by some $K > 0$ has diameter less than $\frac{\pi}{\sqrt{K}}$. Similarly, Myers theorem states that a complete n-dimensional Riemannian manifold M with Ricci curvature bounded below by some $(n-1)K > 0$ has diameter less than $\frac{\pi}{\sqrt{K}}$ (see [25, Thm. I]). As a Riemannian manifold with sectional curvature bounded below by K has Ricci curvature bounded below by $(n-1)K$, the Myers theorem implies the Bonnet-Myers theorem.

For geodesic metric spaces, there is a Bonnet-Myers-style theorem, giving that if Alexandrov curvature bounded below by $K > 0$, its diameter is bounded above by $\frac{\pi}{\sqrt{K}}$, see [15, Thm. 3.6].

The corresponding rigidity theorems talk about the case where the bound on the diameter is achieved by a minimizing geodesic. A Myers rigidity result was proven in [18, Thm. 3.1] : a complete Riemannian manifold with Ricci curvature bounded below by some $(n-1)K > 0$ and diameter equal to $\frac{\pi}{\sqrt{K}}$ is isometric to a sphere, i.e. $M \cong \frac{1}{\sqrt{K}} S^n$. In particular, this can be seen as a (weaker) Bonnet-Myers rigidity theorem.

But when removing the completeness or replacing complete Riemannian manifolds by metric length spaces, one immediately faces a counterexample:

Example 1.1. *Let $r > 0$ and set X to be the warped product $X = [-\frac{\pi}{2}, \frac{\pi}{2}] \times_{\cos} rS^1$. Note at $t = \pm\frac{\pi}{2}$, $\cos(t)$ is 0, so we need to take the quotient identifying $\{-\frac{\pi}{2}\} \times rS^1$ and $\{+\frac{\pi}{2}\} \times rS^1$ to a point, respectively. These points are called the* poles *and will be denoted by $\pm\frac{\pi}{2}$.*

Only for $r = 1$ this is a Riemannian manifold ($X \cong S^2$), otherwise only $X \setminus \{\pm\frac{\pi}{2}\}$ is a Riemannian manifold, and the neighbourhood of $\pm\frac{\pi}{2}$ infinitesimally looks like a cone (to be precise $Cone(rS^1)$, a "conical singularity"). It has Ricci curvature bounded below by 1, and synthetically, only for $r \leq 1$ the Ricci curvature is bounded below by 1 at $\pm\frac{\pi}{2}$.

It has diameter π, thus X is a counterexample for the length space case (for $r < 1$ and using synthetic Ricci curvature bounds, see [3, Thm. 1.4]) and $X \setminus \{\pm\frac{\pi}{2}\}$ is a counterexample for the incomplete Riemannian manifold case (for any $r \neq 1$).

With additional assumptions, one can recover the rigidity result, see [22, Thm. 1.4 and Cor. 1.6].

Without additional assumptions and assuming Alexandrov curvature bounds, it is well-known (but we were unable to find a reference for this) that the following holds:

Folklore 1.1. Let X be a complete geodesic metric space with global Alexandrov curvature bounded below by $K = 1$ and assume there exist $p, q \in X$ with $d(p, q) = \pi$. Then X is a spherical suspension of a complete geodesic metric space Y with global Alexandrov curvature bounded below by $K = 1$, i.e. X is the warped product $[-\frac{\pi}{2}, \frac{\pi}{2}] \times_{\cos} Y$.

A proof of this can be derived from the proof in this paper.

Similarly, in the setting of Lorentzian manifolds, there is a Myers style theorem which can be found in [28, Thm. 3.4.2]. It states that a globally hyperbolic n-dimensional Lorentzian manifold M with Ricci curvature bounded below by some $(n-1)K > 0$ automatically has a diameter less than $\frac{\pi}{\sqrt{K}}$. In the case of synthetic Ricci curvature bounds, there is a synthetic Myers theorem (see [17, Prop. 5.10], [13, Thm. 5.5] [11, Cor. 3.14]), and this has recently been used to solve the low regularity manifold Myers theorem (see [12, Cor. 3.10]). To the best of our knowledge, there is no known result giving strict Myers rigidity in both the smooth and synthetic Lorentzian Ricci comparison (i.e. a result implying that the space is higher dimensional anti-deSitter space). As timelike sectional curvature bounds imply timelike Ricci curvature bounds in the smooth case, the spacetime Myers style theorem [28, Thm. 3.4.2] implies a sectional curvature version. In the setting of synthetic sectional curvature bounds (see [23]), there also is a Bonnet-Myers theorem [8, Thm. 4.11, Rem. 4.12]:

Theorem 1.2 (Synthetic Lorentzian Bonnet-Myers)**.** *Let X be a strongly causal, locally causally closed, regular, and geodesic Lorentzian pre-length space which has global curvature bounded below by K. Assume $K < 0$. Assume that X possesses the following non-degeneracy condition: for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle.*

Then $\tau(p, q) > D_K$ can only hold if $\tau(p, q) = +\infty$.

We will prove a rigidity result for the synthetic Lorentzian Bonnet-Myers theorem, stating the following:

Theorem 5.4. *Let $(X, d, \ll, \leq, \tau)$ be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric d and global timelike curvature bounded below by $K = -1$ satisfying timelike geodesic prolongation and containing a τ-arclength parametrized distance realizer γ :*

$(-\frac{\pi}{2}, \frac{\pi}{2}) \to X$. *Assume that for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle.*

Then there is a proper (hence complete), strictly intrinsic metric space S such that the Lorentzian warped product $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ is a path-connected, regularly localisable, globally hyperbolic Lorentzian length space and there is a map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S \to I(\gamma)$ which is a τ- and $\leq$-preserving homeomorphism.

Both the metric and Lorentzian Bonnet-Myers rigidity theorem will follow the rough guide given by the proof of the splitting theorem for Lorentzian length spaces with non-negative timelike curvature, see [9].

The paper is organised as follows: In section 2.1, we review the concepts of timelike curvature bounds, angles and comparison angles, with their basic implications. In section 3, we give an alternate description of warped product spaces which were first introduced in [2] which is better suited for our needs. In subsection 3.1, we give a description of a part of Anti-deSitter spacetime as such a warped product, and define AdS-suspensions. In subsection 4.1, we discuss AdS-lines and asymptotes (including the heart of the proof, the stacking principle), and in subsection 4.2, we introduce the concept of AdS-parallelity and relate asymptotes with parallelity. Finally, in section 5 we piece the parts together to a proof and further implications.

Notation and conventions

Let us collect some notation and conventions that will be used throughout the paper.

A *proper* metric space (X, d) is a metric space such that all closed balls are compact.

For Lorentzian manifolds, we choose the signature $(-, +, \cdots, +)$ (but this only has side effects and is not directly used in this paper). For Lorentzian pre-length space, as we work with strongly causal spaces, geodesics can be defined either by the definition for localizable spaces or by requiring them to be covered by domains which are globally distance realizing. Mostly, we will only use distance realizers anyway. AdS is two-dimensional anti-deSitter spacetime, which will be introduced in Def. 2.1, and AdS$'$ is the warped product inside it, see Lem. 3.5. $\overline{\tau}$ denotes the time separation on AdS$'$ (and AdS). $\tilde{\measuredangle}_x(y, z) = \tilde{\measuredangle}_x^{-1}(y, z)$ is the comparison angle calculated in AdS.

2 Preliminaries

2.1 Basic theory of Lorentzian (pre-)length spaces

We follow the definitions of Lorentzian (pre-)length spaces in [9], deviating only in curvature bounds and not using timelike geodesic completeness and extensibility. One can also find these definitions in [23].

2.2 Timelike curvature bounds

The timelike curvature bounds were first introduced in [23], and slightly modified in [8]. To describe timelike curvature bounds, we will compare certain distances to distances in comparison spaces: the *Lorentzian model spaces* $\mathbb{L}^2(K)$ of constant sectional curvature K.

We will work with timelike curvature bounds below by $K < 0$ exclusively. As other negative curvature bounds follow easily by scaling the space, we only need $K = -1$. For self-containedness, we include the case $K < 0$ here. For more details on the other cases, see [23, Def. 4.5].

A word of warning though: In the Lorentzian case (in our signature) having sectional curvature bounded below by K as in [1, Chp. 1] actually requires an *upper* bound on the sectional curvature of timelike planes. Our timelike curvature bounds stem from bounds of sectional curvature for timelike planes and thus *the inequalities intuitively point in the wrong direction*, e.g. flat Minkowski space does not have timelike curvature bounded below by $K = -1$, but above. Keeping the inequalities in this direction makes the behaviour mostly match the metric case (e.g. lower curvature bounds prohibit branching, some lower curvature bounds make the diameter finite in some way). One could of course then change the sign of K, but following [23] we will not do that.

Definition 2.1. Let $(\mathbb{R}^{1,2}, b)$ be the 3-dimensional semi-Riemannian space of signature $-, -, +$. We define *anti-deSitter space* $\mathbb{L}^2(-1) = \mathrm{AdS}$ as the universal cover of the set $\{p \in \mathbb{R}^{1,2} : b(p,p) = -1\}$. Equipping the tangent space with the restriction of b makes this a 2-dimensional Lorentzian manifold, with appropriately chosen time orientation. Making this space into a Lorentzian pre-length space, we get the anti-deSitter time separation function $\bar{\tau}$. Scaling this space, we get the Lorentzian model space of constant sectional curvature $K < 0$: $\mathbb{L}^2(K) = (\mathrm{AdS}, \frac{1}{\sqrt{-K}}\bar{\tau})$.

In any of the $\mathbb{L}^2(K)$ $(K < 0)$ there are points of infinite τ-distance. We define $D_K = \frac{\pi}{\sqrt{-K}}$, this is the maximal value τ will take before it becomes infinite.

We call three points $x_1, x_2, x_3 \in X$ that are timelike related together with maximisers between them a *timelike triangle*. We denote such triangles by $\Delta(x_1, x_2, x_3)$.

Definition 2.2. Let X be a Lorentzian pre-length space and $\Delta(x_1, x_2, x_3)$ be a timelike triangle. We say it satisfies the *size bounds for* K if $\tau(x_i, x_j) < D_K$ (note by the timelike relations, this only needs to be checked for one pair). If they do, a timelike triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{L}^2(K)$ such that $\tau(x_i, x_j) = \bar{\tau}(\bar{x}_i, \bar{x}_j)$ is called a *comparison triangle* for $\Delta(x_1, x_2, x_3)$. It always exists if the size bounds are satisfied and is unique up to isometries of $\mathbb{L}^2(K)$.

Definition 2.3 (Timelike curvature bounds by triangle comparison). Let X be a Lorentzian pre-length space. An open subset U is called a timelike ($\geq K$)-comparison neighbourhood in the sense of *triangle comparison* if:

1. τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.

2. For all $x, y \in U$ with $x \ll y$ and $\tau(x, y) < D_K$ there exists a geodesic connecting them which is contained entirely in U.

3. Let $\Delta(x, y, z)$ be a timelike triangle in U satisfying size bounds for K, with p, q two points on the sides of $\Delta(x, y, z)$. Let $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle in $\mathbb{L}^2(K)$ for $\Delta(x, y, z)$ and $\bar{p}, \bar{q}$ comparison points for p and q, respectively. Then
$$\tau(p, q) \leq \tau(\bar{p}, \bar{q}). \tag{114}$$

We say X has timelike curvature bounded below by K (in the sense of *triangle comparison*) if it is covered by timelike ($\geq K$)-comparison neighbourhoods (in the sense of *triangle comparison*).

We say X has global timelike curvature bounded below by K (in the sense of *triangle comparison*) if X itself is a ($\geq K$)-comparison neighbourhood (in the sense of *triangle comparison*).

Remark 2.1 (Continuous triangles vs. Lipschitz triangles)**.** *If X is a globally hyperbolic Lorentzian length space with global timelike curvature bounded below / above by K, then in fact curvature comparison even holds for timelike triangles where the maximisers are continuous: Indeed, suppose $\Delta := \Delta_{C^0}(x, y, z)$ is a continuous timelike triangle, and let $p, q \in \Delta$. Due to the second condition of timelike curvature bounds, we find (Lipschitz) maximisers from the endpoints of Δ to p, q, respectively. The concatenations at p resp. q of those maximisers are again maximisers because the sides on Δ are (continuous) maximisers. Hence we have realised a Lipschitz triangle $\Delta(x, y, z)$ with p, q on its sides.*

One of the most commonly used implications of lower (timelike) curvature bounds is the prohibition of branching of distance-realisers. A formulation of this result for Lorentzian pre-length spaces was first given in [23, Thm. 4.12]. However, with the introduction of hyperbolic angles in [10] it was possible to generalise this result by omitting some of the additional assumptions:

Theorem 2.2 (Timelike non-branching)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded below. Then timelike distance realisers cannot branch, i.e., if $\alpha, \beta : [-\varepsilon, \varepsilon] \to X$ are timelike distance realisers such that there exists $t_0 \in \mathbb{R}$ with $\alpha|_{[-\varepsilon,t_0]} = \beta|_{[-\varepsilon,t_0]}$, then α is a reparametrization of a part of β or conversely.*

Proof. See [10, Thm. 4.7]. □

The non-branching of timelike distance realisers is a key property of spaces with lower curvature bounds and will appear in various forms in this proof of Bonnet-Myers rigidity.

2.3 Angles and comparison angles

Hyperbolic angles in Lorentzian pre-length spaces were introduced in [10] and [4], where the latter puts a bigger focus on comparison results. We will follow the conventions of the former reference.

Lemma 2.3 (The law of cosines ($K = -1$))**.** *Let $X = \mathrm{AdS}$ be anti-deSitter space and x_1, x_2, x_3 be three points which are timelike related (an (unordered) timelike triangle). Let $a_{ij} = \max(\bar{\tau}(x_i, x_j), \bar{\tau}(x_j, x_i))$ (note one of these is zero anyway). Let ω be the hyperbolic angle between the straight lines x_1x_2 and x_2x_3 at x_2. Set $\sigma = 1$ if x_2 is not a time endpoint of the triangle (i.e., $x_1 \ll x_2 \ll x_3$ or $x_3 \ll x_2 \ll x_1$) and $\sigma = -1$ if x_2 is a time endpoint of the triangle (i.e., $x_2 \ll x_1, x_3$ or $x_1, x_3 \ll x_2$). Then we have:*

$$\cos(a_{13}) = \cos(a_{12})\cos(a_{23}) - \sigma \sin(a_{12})\sin(a_{23})\cosh(\omega).$$

In particular, when only changing one side-length, the angle ω is a monotonously increasing function of the longest side-length and monotonously decreasing in the other side-lengths.

Proof. See [10, Appendix A]. □

For $a_{ij} > 0$ satisfying a reverse triangle inequality and choosing an appropriate $\sigma = \pm 1$, we can always solve this equation for ω.

Definition 2.4 (Comparison angles). Let X be a Lorentzian pre-length space and x_1, x_2, x_3 three timelike related points. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathrm{AdS}$ be a comparison triangle for x_1, x_2, x_3. We define the *comparison angle* $\tilde{\measuredangle}_{x_2}^{-1}(x_1, x_3)$ as the hyperbolic angle between the straight lines $\bar{x}_1\bar{x}_2$ and $\bar{x}_2\bar{x}_3$ at $\bar{x}_2$. It can be calculated with the law of cosines using $a_{ij} = \max(\tau(x_i, x_j), \tau(x_j, x_i))$ and σ, where we set $\sigma = 1$ if x_2 is not a time endpoint of the triangle and $\sigma = -1$ if x_2 is a time endpoint of the triangle. σ is called the *sign* of the comparison angle (even though we always have $\tilde{\measuredangle}_{x_2}^{-1}(x_1, x_3) > 0$). For a reduction of the number of case distinctions we also define the *signed comparison angle* $\tilde{\measuredangle}_{x_2}^{\mathrm{S},-1}(x_1, x_3) = \sigma\tilde{\measuredangle}_{x_2}^{-1}(x_1, x_3)$.

The -1 in the exponent stands for the $K = -1$ in $M_{-1} = \mathrm{AdS}$. We will drop it throughout this document (note that [10] drops it if $K = 0$ instead).

Definition 2.5 (Angles). Let X be a Lorentzian pre-length space and $\alpha, \beta : [0, \varepsilon) \to X$ be two timelike curves (future or past directed or one of each) with $x := \alpha(0) = \beta(0)$. Then we define the *upper angle*

$$\measuredangle_x(\alpha, \beta) = \limsup_{(s,t)\in D, s,t\to 0} \tilde{\measuredangle}_x(\alpha(s), \beta(t)) ,$$

where $D = \{(s, t) : s, t > 0,\ \alpha(s), \beta(t)$ timelike related$\} \cap \{(s, t) : \alpha(s), \beta(t), x$ satisfies size bounds for $K = -1\}$. If the limes superior is a limit and finite, we say *the angle exists* and call it an *angle*.

Note that the sign of the comparison angle is independent of $(s, t) \in D$. We define the *sign* of the (upper) angle σ to be that sign, and define the *signed (upper) angle* to be $\measuredangle_x^{\mathrm{S}}(\alpha, \beta) = \sigma\measuredangle_x(\alpha, \beta)$.

If maximisers between any two timelike related points are unique (as e.g. in AdS' (defined in Lem. 3.5)), then we simply write $\measuredangle_p(x, y)$ for the angle at p between the maximisers from p to x and p to y.

For giving an alternative definition of timelike curvature bounds in the case $K = -1$ we need:

Definition 2.6 (Regularity). A Lorentzian pre-length space X is called *regular* if every distance realizer between timelike related points is timelike, i.e., does not contain a null segment.

Definition 2.7 (Timelike curvature bounds by monotonicity comparison). Let X be a regular Lorentzian pre-length space. An open subset U is called a timelike $(\geq K)$-comparison neighbourhood in the sense of *monotonicity comparison* if:

1. τ is continuous on $(U \times U) \cap \tau^{-1}([0, D_K))$, and this set is open.

2. For all $x, y \in U$ with $x \ll y$ and $\tau(x, y) < D_K$ there exists a geodesic connecting them which is contained entirely in U.

3. Let $\alpha : [0,a] \to U, \beta : [0,b] \to U$ be distance realizers such that $x := \alpha(0) = \beta(0)$ and such that $L(\alpha), L(\beta) < D_K$. Define the function $\theta : D \to [0,+\infty)$ by $\theta(s,t) := \tilde{\measuredangle}_x^{K,\mathrm{S}}(\alpha(s), \beta(t))$ ($D \subseteq (0,a] \times (0,b]$ is the set where this is defined). Then θ is monotonously increasing.

We say X has timelike curvature bounded below by K (in the sense of *monotonicity comparison*) if it is covered by timelike ($\geq K$)-comparison neighbourhoods (in the sense of *monotonicity comparison*).

We say X has global timelike curvature bounded below by K (in the sense of *monotonicity comparison*) if X itself is a ($\geq K$)-comparison neighbourhood (in the sense of *monotonicity comparison*).

Theorem 2.4 (Timelike curvature: Equivalence of definitions)**.** *Let X be a regular Lorentzian pre-length space. Then X has timelike curvature bounded below (above) by $K = -1$ in the sense of Def. 2.3 if and only if it has timelike curvature bounded below (above) by $K = -1$ in the sense of monotonicity comparison (see Def. 2.7).*

Proof. See [10, Thm. 4.12] or a more complete picture in [7, Thm. 5.1]. □

For the relation of local and global curvature bounds, note we only need weak additional assumptions to get they are equivalent:

Lemma 2.5 (Equivalence of local and global lower curvature bounds)**.** *Let X be a connected, globally hyperbolic, regular Lorentzian length space with a time function T and curvature bounded below by $K \in \mathbb{R}$ in the sense of triangle comparison such that any timelike related points are connected by a distance realizer. Then it has global curvature bounded below by K in the sense of triangle comparison.*

Proof. This follows from [6, Thm. 3.6] and [7, Thm. 5.1 and Prop. 4.18]. □

The timelike geodesic prolongation property plays a technical role in our proof of the Bonnet-Myers rigidity theorem mainly because of the following result.

Proposition 2.6 (Continuity of angles in spaces with timelike curvature bounded below)**.** *Let X be a strongly causal, localisable, timelike geodesically connected and locally causally closed Lorentzian pre-length space with timelike curvature bounded below and which satisfies timelike geodesic prolongation. Let $\alpha_n, \alpha, \beta_n, \beta$ be future or past directed timelike geodesics all starting at $\alpha_n(0) = \alpha(0) = \beta_n(0) = \beta(0) =: x$ and with $\alpha_n \to \alpha$ and $\beta_n \to \beta$ pointwise (in particular, α_n is future directed if and only if α is, and similarly for β_n and β). Then*

$$\measuredangle_x(\alpha, \beta) = \lim_n \measuredangle_x(\alpha_n, \beta_n)$$

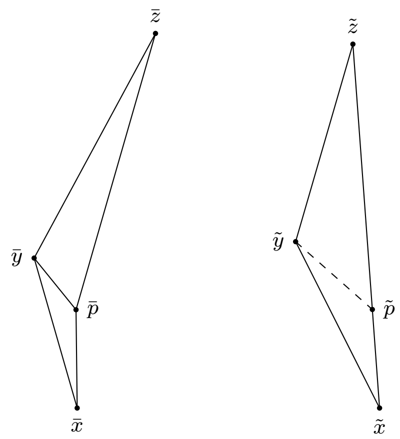

Figure 29: A concave situation in the across version.

Proof. See [10, Prop. 4.14]. $\square$

The following two results seem very similar to [9, Prop. 2.42, Prop. 2.43], except that here the comparison triangles are created in AdS instead of Minkowski space. For the proof, we refer the reader to the proof of [9, Prop. 2.42, Prop. 2.43] which is for $K = 0$, but easily generalizes to general K when additionally assuming size bounds for K.

Proposition 2.7 (Alexandrov lemma: across version)**.** *Let X be a Lorentzian pre-length space. Let $\Delta := \Delta(x, y, z)$ be a timelike triangle satisfying size bounds for $K = -1$ (in particular the distance realisers between the endpoints exist). Let p be a point on the side xz with $p \ll y$, such that the distance realiser between p and y exists. Then we can consider the smaller triangles $\Delta_1 := \Delta(x, p, y)$ and $\Delta_2 := \Delta(p, y, z)$. We construct a comparison situation consisting of a comparison triangle $\bar{\Delta}_1$ for Δ_1 and $\bar{\Delta}_2$ for Δ_2, with $\bar{x}$ and $\bar{z}$ on different sides of the line through $\bar{p}\bar{y}$ and a comparison triangle $\tilde{\Delta}$ for Δ with a comparison point $\tilde{p}$ for p on the side xz. This contains the subtriangles $\tilde{\Delta}_1 := \Delta(\tilde{x}, \tilde{y}, \tilde{p})$ and $\tilde{\Delta}_2 := \Delta(\tilde{p}, \tilde{y}, \tilde{z})$, see Figure 29.*

Then the situation $\bar{\Delta}_1$, $\bar{\Delta}_2$ is convex (concave) at p (i.e. $\tilde{\measuredangle}_p(x, y) = \measuredangle_{\bar{p}}(\bar{x}, \bar{y}) \geq \measuredangle_{\bar{p}}(\bar{y}, \bar{z}) = \tilde{\measuredangle}_p(y, z)$ (or $\leq$)) if and only if $\tau(p, y) \leq \bar{\tau}(\bar{p}, \bar{y})$ (or $\geq$). If this is the case, we have that

- *each angle in the triangle $\bar{\Delta}_1$ is $\geq$ (or $\leq$) than the corresponding angle in the triangle $\tilde{\Delta}_1$,*
- *each angle in the triangle $\bar{\Delta}_2$ is $\geq$ (or $\leq$) than the corresponding angle in the triangle $\tilde{\Delta}_2$.*

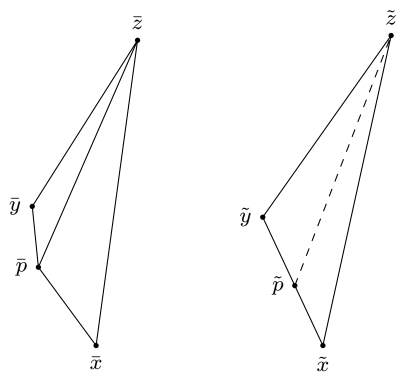

Figure 30: A convex situation in the future version.

In any case, we have that

- $\measuredangle_{\bar y}(\bar x, \bar z) \geq \measuredangle_{\tilde x}(\tilde x, \tilde z) = \tilde\measuredangle_y(x, z)$.

The same is true if p *is a point on the side* xz *such that* $y \ll p$*. Note that if* X *has timelike curvature bounded below (above) by* $K = -1$ *and* Δ *is within a comparison neighbourhood, the condition is satisfied, i.e.* $\tau(p, y) \leq \bar\tau(\tilde p, \tilde y)$ *(or* $\geq$*).*

Proposition 2.8 (Alexandrov lemma: future version)**.** *Let* X *be a Lorentzian pre-length space. Let* $\Delta := \Delta(x, y, z)$ *be a timelike triangle satisfying size bounds for* $K = -1$ *(in particular the distance realisers between the endpoints exist). Let* p *be a point on the side* xy*, such that the distance realiser between* p *and* z *exists. Then we can consider the smaller triangles* $\Delta_1 := \Delta(x, p, z)$ *and* $\Delta_2 := \Delta(p, y, z)$*. We construct a comparison situation consisting of a comparison triangle* $\bar\Delta_1$ *for* Δ_1 *and* $\bar\Delta_2$ *for* Δ_2*, with* $\bar x$ *and* $\bar y$ *on different sides of the line through* $\bar p\bar z$ *and a comparison triangle* $\tilde\Delta$ *for* Δ *with a comparison point* $\tilde p$ *for* p *on the side* xy*. This contains the subtriangles* $\tilde\Delta_1 := \Delta(\tilde x, \tilde p, \tilde z)$ *and* $\tilde\Delta_2 := \Delta(\tilde p, \tilde y, \tilde z)$*, see Figure 30.*

Then the situation $\bar\Delta_1$*,*$\bar\Delta_2$ *is convex (concave) at* p *(i.e.* $\measuredangle_{\bar p}(\bar y, \bar z) \leq \measuredangle_{\bar p}(\bar x, \bar z)$ *(or* $\geq$*)) if and only if* $\tau(p, z) \leq \bar\tau(\bar p, \bar z)$ *(or* $\geq$*). If this is the case, we have that*

- *each angle in the triangle* $\bar\Delta_1$ *is* $\geq$ *(or* $\leq$*) than the corresponding angle in the triangle* $\tilde\Delta_1$*,*

- *each angle in the triangle* $\bar\Delta_2$ *is* $\leq$ *(or* $\geq$*) than the corresponding angle in the triangle,* $\tilde\Delta_2$*.*

In any case, we have that

- $\measuredangle_{\bar{z}}(\bar{x},\bar{y}) \leq \measuredangle_{\tilde{z}}(\tilde{x},\tilde{y}) = \tilde{\measuredangle}_z(x,y)$.

Note that if X has timelike curvature bounded below (above) by $K = -1$ and Δ is within a comparison neighbourhood, the condition is satisfied, i.e. $\tau(p,z) \leq \bar{\tau}(\tilde{p},\tilde{z})$ (or $\geq$).

3 Lorentzian warped products

This section is dedicated to the treatment of Lorentzian warped product spaces. In the context of Lorentzian pre-length spaces, warped products were first introduced in [2]. Here, we give a different treatment which also works for non-intrinsic spaces, and is in addition more suited for our needs. First, for technical reasons, we define the following:

Definition 3.1 (Lorentzian warped product comparison space). Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to (0,+\infty)$ be continuous. The *Lorentzian warped product comparison space for f* or *with warping function f* is the warped product $X = I \times_f \mathbb{R}$ as a spacetime, i.e. equipping the manifold $I \times \mathbb{R}$ with the continuous metric $g = -\mathrm{d}t^2 + (f \circ t)^2 \mathrm{d}x^2$ and the time orientation given by ∂_t. We will use this as a Lorentzian pre-length space, with the product metric D as metric. We set $t : X \to I$ the first and $x : X \to \mathbb{R}$ the second projection.

Proposition 3.1 (Properties of Lorentzian warped product comparison spaces)**.** *Let $f : I \to (0,+\infty)$ be continuous. Then the Lorentzian warped product comparison space $X = I \times_f \mathbb{R}$ has the following properties:*

1. *The reparametrized null distance realizers are given as:*
$$\gamma(t) = \left(t, \pm \int_{t_0}^{t} \frac{1}{f(\lambda)} \mathrm{d}\lambda\right)$$

2. *In particular, $(s,x) \leq (t,y) \iff |x-y| \leq \int_s^t \frac{1}{f(\lambda)} \mathrm{d}\lambda$ and $s \leq t$, and $(s,x) \ll (t,y)$ iff these inequalities are strict.*

3. *It is causally plain (see [19, Def. 1.16]). In particular it forms a Lorentzian length space.*

4. *It is globally hyperbolic, globally causally closed, strongly localizable and strictly intrinsic. $\tau_{I \times_f \mathbb{R}}$ is continuous.*

5. *If f is Lipschitz, it is even a regular Lorentzian length space.*
6. *If f is C^1, we can describe timelike geodesics with the ODE $\gamma = (\alpha, \beta)$, $\alpha'' = g(\beta', \beta')(ff') \circ \alpha$, $\beta'' = \frac{-2(f\circ\alpha)'}{f\circ\alpha}$.*

Proof. (i): Note that these are null curves, and as the space is 2-dimensional, they are reparametrized null geodesics. As any other causal curve parametrized w.r.t. t-coordinate has less speed in the x-coordinate, it is a distance realizer.

(ii): The description of the causal relation follows immediately (there is only this one causal curve between null related points), and one can easily modify γ to be timelike for the strict inequality, see part (iii) for more details.

(iii): We now show that X is causally plain: Let $p_0 = (t_0, x_0) \in X$. As f is continuous, there is a neighbourhood $(t_-, t_+) \ni t_0$ such that $f(s) \in (\frac{f(t_0)}{2}, 2f(t_0))$ for all $s \in (t_-, t_+)$. Using the coordinates t and $\frac{x}{f(t_0)}$, we see that $U := (t_-, t_+) \times \mathbb{R}$ is a cylindrical neighbourhood (as defined in [19, Def. 1.8]). One checks that $p_0 \ll q_1 = (t_1, x_1)$ if and only if $|x_1 - x_0| < \int_{t_0}^{t_1} \frac{1}{f(\lambda)} \mathrm{d}\lambda =: x(t_0, t_1)$. There now is a smooth function $\tilde{f}$ such that $f < \tilde{f} < \frac{x(t_0,t_1)}{|x_1 - x_0|} f$. Then $\tilde{\gamma}(t) = (t, x_0 \pm \int_{t_0}^{t} \frac{|x_1-x_0|}{x(t_0,t_1)} \frac{1}{f(\lambda)} \mathrm{d}\lambda)$ connects p_0 with p_1 and is timelike with respect to the smooth metric $-\mathrm{d}t^2 + \tilde{f}(t)^2 \mathrm{d}x^2$ which has narrower lightcones than g. In particular, one sees that $\check{I}^+(p_0, U) = I(p_0) \cap U$. Now $\partial \check{I}(p, U) = \partial J(p, U)$ follows from $J(p, U) = \overline{I(p, U)} \cap U$.

(iv): Global causal closedness follows from the description of the causal relation (note that f is locally bounded away from 0, so the integral stays finite).

It is non-totally imprisoning: the d-length of a causal curve $\gamma = (\mathrm{id}, \beta) : [a, b] \to X$ is bounded above by $(b - a)\sqrt{1 + \frac{1}{\min\{f(t): t \in [a,b]\}^2}}$, in particular we have a uniform bound on sets of the form $[a, b] \times \mathbb{R}$.

Causal diamonds are compact: A causal diamond $J(p_0, p_1)$ is closed by global causal closedness, and it is compact as $\min\{f(t) : t \in [t_0, t_1]\} > 0$, so it is contained in the compact $[t_0, t_1] \times [x_0 - \frac{1}{\min\{f(t):t\in[t_0,t_1]\}}, x_0 + \frac{1}{\min\{f(t):t\in[t_0,t_1]\}}]$ (as $\min\{f(t) : t \in [t_0, t_1]\} > 0$). In particular, we have global hyperbolicity, continuity of τ and strict intrinsicness.

It is strongly localizable: every timelike diamond U is strictly intrinsic, has continuous τ, there are no $\ll$-isolated points in U and it is a d-compatible neighbourhood.

(v): We have a globally hyperbolic Lorentzian manifold with Lipschitz metric, so can apply [24, Prop. 1.2].

(vi): If f is C^1, the geodesic equation follows from [26, Prop. 7.38]. $\square$

Definition 3.2 (Lorentzian warped product). Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to (0, +\infty)$ be continuous. Let (Y, d) be a metric space. Define the space $X := I \times Y$. Equip it with the product metric

$$D : X \times X \to \mathbb{R},\ D((s,x),(t,y)) := \sqrt{|t-s|^2 + d(x,y)^2}.$$

Define the timelike relation as $(s,x) \ll (t,y)$ if and only if in the Lorentzian warped product comparison space $I \times_f \mathbb{R}$, we have that $(s,0) \ll_{I\times_f\mathbb{R}} (t, d(x,y))$. Similarly, they are causally related if these points in $I \times_f \mathbb{R}$ are causally related, and set

$$\tau((s,x),(t,y)) = \tau_{I\times_f\mathbb{R}}((s,0),(t,d(x,y)))$$

Then $(X, D, \ll, \leq, \tau)$ is called the Lorentzian warped product of Y with $\mathbb{R}$ with warping function f. It will be denoted by $\mathbb{R} \times_f Y$. We set $t : X \to I$ the first and $x : X \to Y$ the second projection.

Proposition 3.2 (Properties of Lorentzian warped products)**.** *Let (Y, d) be a metric space and $f : I \to (0, +\infty)$ be continuous. Set $X := I \times_f Y$ to be the Lorentzian warped product. Then:*

1. *X is a globally causally closed and non-totally imprisoning Lorentzian pre-length space with continuous τ.*
2. *X is strongly causal (the timelike diamonds even form a basis for the topology).*
3. *All causal curves in t-parametrization on some $[s,t]$ are uniformly Lipschitz, in particular X is d-compatible and any causal curve is locally Lipschitz.*
4. *A causal curve $\gamma = (\alpha, \beta) : [0, L] \to X$ is a distance realizer if and only if β is a distance realizer and $\tilde{\gamma}(\lambda) = (\alpha(\lambda), d(\beta(0), \beta(\lambda)))$, $\tilde{\gamma} : [0, L] \to I \times_f \mathbb{R}$ is a distance realizer into the warped product comparison space.*
5. *In particular, if Y is strictly intrinsic, X is strictly intrinsic.*
6. *If Y is strictly intrinsic, X is strongly localizable. If this is the case and f is Lipschitz, X is regularly localizable.*
7. *If Y is proper X is globally hyperbolic, and the converse holds if $\int_I \frac{1}{f(\lambda)} \mathrm{d}\lambda = +\infty$.*

Proof. (i) τ is continuous since $\tau_{I\times_f\mathbb{R}}$ is. Similarly, $\leq$ is closed on $X\times X$. The other properties of Lorentzian pre-length space follow analogously, except for the transitivity of the relations and the reverse triangle inequality. The transitivity follows from the reverse triangle inequality. The reverse triangle inequality can either be seen directly from a $2+1$-dimensional Lorentzian warped product comparison space, or as follows:

For the reverse triangle inequality, let $p=(r,x)\leq q=(s,y)\leq r=(t,z)$. Then we have the triangle inequality of Y: $d(x,z)\leq d(x,y)+d(y,z)$. We find the time separations between the points using points in the Lorentzian warped product comparison space: $\tilde{p}=(r,0)$, $\tilde{q}=(s,d(x,y))$, $\tilde{r}=(t,d(x,y)+d(y,z))$ and $\hat{r}=(t,d(x,z))$. Then we have

$$\begin{aligned}\tau(p,q)&=\tau_{I\times_f\mathbb{R}}(\tilde{p},\tilde{q})\\ \tau(q,r)&=\tau_{I\times_f\mathbb{R}}(\tilde{q},\tilde{r})\\ \tau(p,r)&=\tau_{I\times_f\mathbb{R}}(\tilde{p},\hat{r})\end{aligned}$$

By the reverse triangle inequality we have $\tau_{I\times_f\mathbb{R}}(\tilde{p},\tilde{r})\geq\tau_{I\times_f\mathbb{R}}(\tilde{p},\tilde{q})+\tau_{I\times_f\mathbb{R}}(\tilde{q},\tilde{r})$, and as the x-coordinate difference is smaller between $\tilde{p}\hat{r}$ than between $\tilde{p}\tilde{r}$ (with equal t-coordinates), we can take a distance realizer $\gamma=(\alpha,\beta)$ from $\tilde{p}$ to $\tilde{r}$ and convert it to a longer causal curve $\gamma_2=(\alpha,\frac{d(x,z)}{d(x,y)+d(y,z)}\beta)$ from $\tilde{p}$ to $\hat{r}$.

For non-total imprisonment, let $\tilde{K}$ be compact, so it is contained in some $K=[a,c]\times Y$. Set $\inf f|_{[a,c]}=:f_->0$, then for any two points $(s,y)\leq(t,z)$ we have $d(y,z)\leq\frac{t-s}{f_-}$, so $D((s,y),(t,z))=\sqrt{(t-s)^2+d(y,z)^2}\leq\sqrt{1+\frac{1}{f_-}}(t-s)$. In particular, for any causal curve γ starting at (s,y) and ending at (t,z) we have $L_D(\gamma)\leq\sqrt{1+\frac{1}{f_-}}(t-s)$, and if γ stays in K this is less than $\sqrt{1+\frac{1}{f_-}}(c-a)$ and X is non-totally imprisoning.

(ii) To show that the diamonds form a basis, let $(b,y)\in O=(a,c)\times B_R(x)$ (for some $a<c\in I$, $R>0$, $x,y\in Y$) with $[a,c]\subseteq I$. Then we have that $\inf f|_{[a,c]}=:f_->0$. Take $0<\varepsilon=\min(b-a,c-b,(R-d(x,y))f_-/2)$, set $p:=(b-\varepsilon,y),q:=(b+\varepsilon,y)\in\bar{O}$. We claim $(b,y)\in I(p,q)\subseteq O$: As $\int_a^c\frac{1}{f(\lambda)}\mathrm{d}\lambda\leq\frac{(c-a)}{f_-}=\frac{2\varepsilon}{f_-}\leq R-d(x,y)$ and this is the maximum distance a null related point can be away in the Y coordinate, we get that $I(p,q)\subseteq O$.

(iii) Any causal curve in t-parametrization $\gamma(t)=(t,\beta(t))$ is locally Lipschitz: Note that by 3.1.2, we get that $d(\beta(s),\beta(t))\leq\int_s^t\frac{1}{f(\lambda)}\mathrm{d}\lambda$, thus we get that $D(\gamma(s),\gamma(t))\leq\sqrt{(t-s)^2+d(\beta(s),\beta(t))^2}\leq(t-s)\sqrt{1+\frac{1}{\min_{[s,t]}f}}$, and the Lipschitz constant only depends on s,t.

(iv) Now for the equivalent condition for distance realizers and their existence. First, let $\gamma = (\alpha, \beta) : [0, L] \to X$ be a distance realizer. Set $p = (r, \beta(r))$, $q = (s, \beta(s))$, $r = (t, \beta(t))$. We are now in the same situation as in the proof for the reverse triangle inequality, and note that in he proof of the reverse triangle inequality, equality can only be achieved if $d(\beta(r), \beta(t)) = d(\beta(r), \beta(s)) + d(\beta(s), \beta(t))$, so β has to be a distance realizer. Now note that by definition of τ, the homeomorphism from the two-dimensional subset $\{(t, \beta(\lambda)) : t, \lambda\} \subseteq X$ to $I \times [0, L(\beta)] \subseteq I \times_f \mathbb{R}$ given by $(t, x) \mapsto (t, d(x, \beta(0)))$ is τ-preserving, so they are in bijection as required.

(v) As $I \times_f \mathbb{R}$ is globally hyperbolic, there is a distance realizer $\tilde{\gamma}$ between any causally related points ([27, Prop. 6.4]), so the existence is established if β exists.

(vi) For the strong localizability, we claim that every timelike diamond is a localizable neighbourhood, and the proof is as for the Lorentzian warped product comparison space in Prop. 3.1.3. X being regular is also done in Prop. 3.1.5.

(vii) First let Y be proper, for global hyperbolicity we still need to check the compactness of causal diamonds. This works as for the Lorentzian warped product comparison space, but instead of the closed interval in space, we now need a closed ball (which is compact by properness of Y).

Now let X be globally hyperbolic and $\int_I \frac{1}{f(\lambda)} \mathrm{d}\lambda = +\infty$. We need that Y is proper. So let $x \in Y$ and $R > 0$. Take $t_0 < t_1 < t_2 \in I$ be such that $\int_{t_0}^{t_1} \frac{1}{f(\lambda)} \mathrm{d}\lambda = \int_{t_1}^{t_2} \frac{1}{f(\lambda)} \mathrm{d}\lambda = R$, then $x(J((t_0, x), (t_2, x))) = \bar{B}_R(x) \subseteq Y$: First note that $(t_0, x) \leq (t, y)$ if and only if $d(x, y) \leq \int_{t_0}^{t} \frac{1}{f(\lambda)} \mathrm{d}\lambda$ by Prop. 3.1.2, similarly for $(t, y) \leq (t_2, x)$. Thus one gets that $x(J((t_0, x), (t_2, x))) \subseteq \bar{B}_R(x)$, and for $t = t_1$ we have that $x(J((t_0, x), (t_2, x)) \cap \{(t_1, y) : y \in Y\}) = \bar{B}_R(x)$. □

Proposition 3.3 (The warped product definition agrees with the known notion)**.** *Let (Y, d) be an intrinsic metric space and $f : I \to (0, +\infty)$ be continuous. Then we can view $I \times_f Y$ as a warped product in two ways: As in Def. 3.2, or as in [2, Def. 3.17,3.18].*

The time separation τ and the timelike relation $\ll$ agree, and the causal relation as in [2, Def. 3.18] is contained in the causal relation defined in Def. 3.2.

Moreover, the causal relation agrees if and only if Y is strictly intrinsic.

Proof. We work in the setup of [2], and let $\tilde{\tau}$ and $\tilde{\leq}$ be the notions from Def. 3.2 for now. Let Y be strictly intrinsic and $\gamma : [0, b] \to X$, $\gamma(t) = (\alpha(t), \beta(t))$ be a causal curve from (s, x) to (t, y). If $\beta : [0, b] \to Y$ is not the shortest curve between x and y, we find a timelike curve from (s, x) to (t, y) strictly longer than γ: Let $L(t) = L(\beta|_{[0,t]})$. Let $\tilde{\beta}$ be a shorter curve from x to y with parametrization such that $L(\tilde{\beta}|_{[0,t]}) = \frac{d(x,y)}{L(\beta)} L(t)$. As $\frac{d(x,y)}{L(\beta)} < 1$, we have that also $\tilde{\gamma} = (\alpha, \tilde{\beta})$ is timelike (as of [2, Def. 3.4]) from (s,x) to (t,y), and that $\tilde{\gamma}$ is strictly longer. In particular, the relevant case is to take a minimizer in Y as β (or, if that doesn't exist, a minimizing sequence).

First, for simplicity assume that Y is strictly intrinsic, and take β from x to y to be a τ-distance realizer: by the above, we can restrict ourselves to causal curves of the form (α, β) for some α. We define a curve in the Lorentzian warped product comparison space for f: $\bar{\gamma}(t) = (\alpha(t), d(x, \beta(t)))$, and it has the same length as γ. Now we have $\tau = \tilde{\tau}$: both are the supremum of such $\bar{\gamma}$ in the Lorentzian warped product comparison space (and this supremum is achieved). The same argument also proves $\leq \Leftrightarrow \tilde{\leq}$.

In any case, for any $\leq$-causal curve $\gamma_0 = (\alpha_0, \beta_0)$ from (s,x) to (t,y), we know $d(x,y) \leq L(\beta_0)$. Now γ has the same length as $\tilde{\gamma}_0 = (\alpha_0, t \mapsto L(\beta_0, 0, t))$ in the Lorentziann warped product comparison space for f, and $\tilde{\gamma}_1 = (\alpha_0, t \mapsto \frac{d(x,y)}{L(\beta_0)} L(\beta_0, 0, t))$ is causal and longer (strictly if $d(x,y) \leq L(\beta_0)$ was strict). In particular, $\tau \leq \tilde{\tau}$ and $\leq \Rightarrow \tilde{\leq}$.

If we have that Y is only intrinsic and we have $L = \tilde{\tau}((s,x),(t,y)) > 0$, we get $\tilde{\gamma} = (\tilde{\alpha}, \tilde{\beta})$ in the Lorentzian warped product comparison space with $L_{\tilde{\tau}}(\tilde{\gamma}) = L$, and we can assume $\tilde{\alpha}(t) = t$ (so $\tilde{\gamma}$ is parametrized by t-coordinate and $\tilde{\gamma}$ is Lipschitz). We have that

$$0 < L(\tilde{\gamma}) = \int \sqrt{(\tilde{\alpha}')^2 - (f \circ \tilde{\alpha})^2 |\tilde{\beta}'|^2}$$

In particular, for all $\varepsilon > 0$ there is a small enough $\delta > 0$ and an almost distance realizer $\beta_0 : x \rightsquigarrow y$ in Y with $L(\beta_0) = (1+\delta) d(x,y)$ and a parametrization of β_0 satisfying

$$\frac{\varepsilon}{(1-\varepsilon)(f \circ \tilde{\alpha})^2} (\tilde{\alpha}')^2 \geq (|\beta_0'|)^2 - (|\tilde{\beta}'|)^2$$

so

$$(f \circ \tilde{\alpha})^2 (1-\varepsilon)(|\tilde{\beta}'|)^2) - (f \circ \tilde{\alpha})^2 (1-\varepsilon)(|\beta_0'|)^2 + \varepsilon (\tilde{\alpha}')^2 \geq 0$$

We add this inside the square root of the following:

$$\begin{aligned}
\sqrt{(1-\varepsilon)} L(\tilde{\gamma}) &= \int \sqrt{(1-\varepsilon)((\tilde{\alpha}')^2 - (f \circ \tilde{\alpha})^2 |\tilde{\beta}'|^2} \\
&= \int \sqrt{(1-\varepsilon)(\tilde{\alpha}')^2 - (f \circ \tilde{\alpha})^2 (1-\varepsilon)(|\tilde{\beta}'|)^2)} \\
&\leq \int \sqrt{(\tilde{\alpha}')^2 - (f \circ \tilde{\alpha})^2 (1-\varepsilon)(|\beta_0'|)^2} \leq L_\tau(\gamma_0)
\end{aligned}$$

for the curve $\gamma_0 = (\tilde{\alpha}, \beta_0)$, i.e. we have $L_\tau(\gamma_0) \geq \sqrt{1-\varepsilon} L_{\tilde{\tau}}((\tilde{\alpha}, \tilde{\beta}))$. In particular, the definitions of $\ll$ and τ agree.

If $x, y \in Y$ are not connectible by a distance realizer, $(s,x) \tilde{\leq} (t,y)$ null related, we have that $(s,x) \not\leq (t,y)$: Any $\leq$-causal curve would be improvable, so we would have $\tilde{\tau}((s,x),(t,y)) > 0$, but they were null related. $\square$

Definition 3.3. Let $I \times_f Y$ be a warped product with $0 \in I$. *Conformal time* is given by $\eta(t) = \int_0^t \frac{1}{f(s)} \mathrm{d}s$.

Lemma 3.4 (Conformal time). *We have that for $p, q \in Y$ and $s \in \eta^{-1}(I)$, $(\eta^{-1}(s), p) \leq (\eta^{-1}(s + d(p,q)), q)$ are null related.*

Proof. We have to verify that the points $\tilde{p} = (\eta^{-1}(s), 0)$ and $\tilde{q} = (\eta^{-1}(s + d(p,q)), d(p,q))$ are null related. For this, note that the curve $t \mapsto (\eta^{-1}(s + t), t)$ is a reparametrization of $\gamma(t) = (s + t, \int_s^{s+t} \frac{1}{f(\lambda)} \mathrm{d}\lambda)$ which is a null distance realizer by 3.1.1, thus all points along it are null related. $\square$

3.1 A warped product in Anti-deSitter

Lemma 3.5 (A warped product inside Anti-deSitter). *Take the points $p_- = (-1,0,0)$ and $p_+ = (1,0,0)$ in* AdS *(in the universal cover such that they are timelike related, but as close as possible), and set $X = I(p_-, p_+)$. Then X is globally hyperbolic and maximal as such (i.e. there is no $X \subsetneq \tilde{X} \subseteq \mathrm{AdS}$ which is connected and globally hyperbolic). Furthermore, X is isomorphic to the warped product (even warped product comparison space) $\mathrm{AdS}' := (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{R}$. Timelike triangles (in any space) satisfying the size bounds for $K = -1$ are realizable in AdS', with vertical longest side.*

Proof. In AdS, the $c + \frac{\pi}{2}$-level set of $\tau(p_-, \cdot)$ is given by $\mathrm{AdS} \cap \{(s_1, s_2, z) : s_1 = \sin(c)\}$. All timelike geodesics through p_-, p_+ are given by $t \mapsto (\sin(t), \cos(t)\cosh(x), \cos(t)\sinh(x))$ (the x parametrizes the angles at p), and they are distance realizing, thus pass at right angles through the level sets.

So we define $f : \mathrm{AdS}' \to X$ by setting

$$f(t,x) = (\sin(t), \cos(t)\cosh(x), \cos(t)\sinh(x)) .$$

This is an isometry: The vertical unit speed distance realizers $t \mapsto (t, x_0)$ map to the unit speed distance realizers $t \mapsto (\sin(t), \cos(t)\cosh(x_0), \cos(t)\sinh(x_0))$ (which cover all AdS' resp. X), and the horizontal spacelike geodesic $x \mapsto (t_0, x)$ has speed $\cos(t_0)$ and maps to $x \mapsto (\sin(t_0), \cos(t_0)\cosh(x), \cos(t_0)\sinh(x))$ which is also a spacelike geodesic of speed $\cos(t_0)$ (these are parametrisations of the level sets of $\tau(p_-, \cdot)$, thus cover all AdS' resp. X), and the geodesics of the first family intersect the ones in the second family at right angles as described before.

That X is globally hyperbolic can be seen in the warped product picture.

For realizing timelike triangles satisfying the size bonds for $K = -1$, let $a + b \leq c < \pi$. Choose a realization with a timelike triangle $\Delta(p, q, r)$ in AdS. By applying a suitable isometry, we can w.l.o.g. assume that $p = (\cos(-c/2), \sin(-c/2), 0)$, $r = (\cos(c/2), \sin(c/2), 0) \in X$ (these have the right τ-distance). Then $q \in X$ by causal convexity of X, so $\Delta(p, q, r)$ is realized in X, or equivalently in AdS' where p, r have the same x-coordinate ("vertical"). □

Remark 3.6. *Note that this subset is invariant under some, but not all isometries of* AdS*: all isometries of* AdS *are described as Lorentz transformations of the ambient* $\mathbb{R}^{1,2}$*. Isometries fixing* X *are those fixing* p_- *(and thus automatically* p_+*). Those have dimension* 1 *and codimension* 2*, and are Lorentz boosts in the* s_2, z *directions. In the warped product setting, they are described as translations in the* x*-direction.*

For AdS , we have an explicit solution of the geodesic equation 3.1.6:

Lemma 3.7. *The timelike geodesics in* AdS'*, parametrized by* τ*-arclength and such that at parameter* $\lambda = 0$ *they cross* $t = 0$*, are given by:*

$$\alpha(\lambda) = \left(\arcsin(\sin(\lambda)\cosh(\omega)), \sinh^{-1}\left(\frac{\sin(\lambda)\sinh(\omega)}{\sqrt{1 - \sin(\lambda)^2 \cosh(\omega)^2}} \right) + c \right)$$

for $\omega, c \in \mathbb{R}$*, and these are distance realizers on their domain.* c *is the* x*-coordinate at* $t = \lambda = 0$*,* ω *that of the angle to the vertical at* $t = \lambda = 0$*.*

For $\omega = 0$*, this is a vertical line* $\lambda \mapsto (\lambda, c)$*. For* $\omega > 0$*, set* $\lambda_\pm = \pm\arcsin(\frac{1}{\cosh(\omega)})$*, then* $\lim_{\lambda \to \lambda_-} \alpha(\lambda) = (-\frac{\pi}{2}, -\infty)$ *and* $\lim_{\lambda \to \lambda_+} \alpha(\lambda) = (+\frac{\pi}{2}, +\infty)$*. For* $\omega < 0$*, the signs of the* x*-coordinates of the limits flip. In particular, the* x*-coordinate only stays bounded for* $\omega = 0$*.*

Proof. This follows from transforming the geodesics in AdS to warped product coordinates. □

We state the following lemma only for referencing and self-containedness of this document:

Lemma 3.8 (τ in AdS')**.** *We have that*

$$\begin{aligned}&(t_1, x_1) \leq_{\mathrm{AdS}'} (t_2, x_2) \Leftrightarrow \\ &\qquad \sin(t_1)\sin(t_2) + \cos(t_1)\cos(t_2)\cosh(x_2 - x_1) \leq 1 \text{ and } t_1 \leq t_2\end{aligned}$$

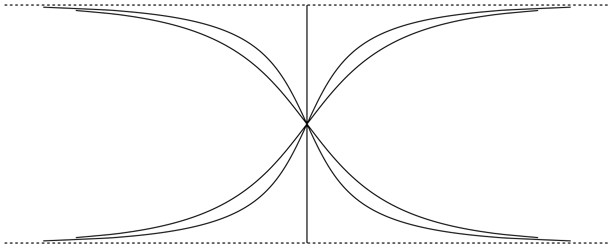

Figure 31: Some geodesics in AdS$'$.

and

$$\tau_{\mathrm{AdS}'}((t_1,x_1),(t_2,x_2)) = \arccos(\sin(t_1)\sin(t_2) + \cos(t_1)\cos(t_2)\cosh(x_2 - x_1)) \qquad \text{if } t_1 \le t_2$$

We expect spaces satisfying the assumptions of the 5.4 to be very similar to $\mathrm{AdS}' = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{R}$, namely of the form $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} Y$. By Lemma 3.7 and by Proposition 3.2.4, we have an explicit description of timelike distance realizers in such spaces.

Lemma 3.9 (Conformal time in AdS$'$)**.** *Conformal time for* $\mathrm{AdS}' = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{R}$ *is given by*

$$\eta(t) = \log\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)$$
$$\eta^{-1}(s) = 2\arctan(e^s) - \frac{\pi}{2}$$

Definition 3.4. A Lorentzian warped product $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} Y$ is called the *anti-deSitter suspension* of Y and is the Lorentzian analogue of spherical suspensions in metric spaces. The corresponding Lorentzian warped product comparison space is AdS$'$.

Lemma 3.10 (The future causal boundary of AdS$'$ is always at distance π from the bottom point)**.** *Let* $p = (t_1, 0) \in \mathrm{AdS}'$ *and let* $\alpha : [0,a) \to \mathrm{AdS}'$ *be a future directed causal curve starting at* p *with* $\lim_{\lambda\to a} t(\alpha(\lambda)) = \frac{\pi}{2}$. *Then for all* $\varepsilon > 0$ *there is a* t_0 *such that* $\lim_{\lambda\to a} \tau((t_0,0),\alpha(\lambda)) > \pi - \varepsilon$.

Proof. W.l.o.g. assume $x(\alpha(\lambda)) \ge 0$ for all λ. Note that $\tau((t_0,0),(t,x))$ is monotonously decreasing in $|x|$, thus the worst case is attained when α is null, i.e. we can assume $\alpha(t) = \left(t, \int_{t_1}^{t} \frac{1}{\cos(t)} \mathrm{d}t\right)$.

Now we calculate $\lim_{t\to a} \tau((t_0,0),\alpha(t))$: For the calculation, we switch to the 3-dimensional picture in Def. 2.1, there we can use the formula

$$\tau(p,q) = \arccos(-\langle p,q\rangle)$$

where the inner product is in the ambient semi-Riemannian vector space, see [20, (2.7)]. The point $\lim_{t\to a}\alpha(t)$ exists and is given by the intersection of the two null geodesics:

$$\{(0,1,0)+s(1,0,1) : s \in \mathbb{R}\} \cap \{(\cos(t_1),\sin(t_1),0)+s(-\sin(t_1),\cos(t_1),1) : s \in \mathbb{R}\}$$

yielding the point

$$\lim \alpha(t) = \left(\frac{1-\sin(t_1)}{\cos(t_1)}, 1, \frac{1-\sin(t_1)}{\cos(t_1)}\right)$$

so

$$\tau((\cos(t_0),\sin(t_0),0), \lim \alpha(t)) = \arccos\left(\frac{\cos(t_0)(1-\sin(t_1))}{\cos(t_1)} + \sin(t_0)\right)$$

where we used the inner product formula for τ.

as $t_0 \to -\frac{\pi}{2}$, this goes to π:

$$\arccos(-1) = \pi$$

□

4 Rays, lines, co-rays and asymptotes

4.1 Timelike co-ray condition

In this subsection, we study causal rays and lines and show how to obtain them as limits of causal maximisers. An AdS-line is then a line of length π, parametrized by τ-unit speed on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then, we analyse triangles where one side is a segment on a timelike AdS-line and show that the angles adjacent to the AdS-line are equal to their comparison angles. This in fact follows from the more general principle that one can stack triangle comparisons of nested triangles with two endpoints on a timelike AdS-line. The latter situation arises in the construction of asymptotes, and via the stacking principle, one can show that any future directed and any past directed asymptote from a common point to a given timelike AdS-line fit together to give a (timelike) asymptotic AdS-line. This approach is nearly equal to the approach in [9], where the construction of co-rays and asymptotes is based on [5] and the stacking principle and equality of angles are based on [14, Lem. 10.5.4].

Definition 4.1 (Rays, Lines, AdS-lines). Let $(X, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A *(future directed) causal ray* is a future inextendible, future directed causal curve $c : I \to X$ that maximises the time separation between any of its points, where I is either a closed interval $[a, b]$ or a half-open interval $[a, b)$. A *(future directed) causal line* is a (doubly) inextendible, future directed causal curve $\gamma : I \to X$ that maximises the time separation between any of its points. Here I can in general be open, closed, or half-open. It is a AdS-*line* if γ is timelike and $L(\gamma) = \pi$, parametrized in τ-unit speed parametrization on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark 4.1. *If X is regularly localisable, then any causal ray/line c is either timelike or null by Thm. [23, Thm. 3.18].*

Lemma 4.2 (Limit of distance realizers is distance realizer)**.** *Let X be a localizable Lorentzian pre-length space and let $\gamma_n : [a_n, b_n] \to X$ be causal distance realizers converging pointwise to a causal curve $\gamma : [a, b] \to X$ (and $a_n \to a$, $b_n \to b$). Then γ is a distance realizer, with $L_\tau(\gamma) = \lim_n L_\tau(\gamma_n)$. In particular, if both γ_n and γ are timelike and the convergence is uniform, a choice of τ-arclength parametrizations converge to γ too.*

Proof. This is a part of [9, Thm. 2.23]. □

Note that if the limit curve is only defined on an open domain, "we can apply this to each compact subinterval, but the limit curve can lose length". We will address this in Prop. 4.3 and Prop. 4.10.

Definition 4.2 (Corays, Asymptotes). Let $c : [0, b) \to X$ be future causal and future inextensible. Let $x_n \to x$ and $t_n \nearrow b$ such that $x_n \in I^-(c(t_n))$. Let $\alpha_n : [0, a_n] \to X$ be a sequence of future directed, maximising causal curves from x_n to $c(t_n)$ in d-arclength parametrisation. Any limit curve α of α_n (see [23, Thm. 3.7]) is called a *(future) coray* to c at x. By Lem. 4.2, α is automatically maximizing, although it might be null.

If we choose $x_n = x$ constant, α is called a *(future) asymptote.*

A lot of the time, we will be working in the *setup of the local Bonnet-Myers rigidity theorem*, assuming X to be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric d and global timelike curvature bounded below by $K = -1$ satisfying timelike geodesic prolongation and containing an AdS-line $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ in τ-arclength parametrization.

We can construct past and future directed co-rays from all points on $I(\gamma) := I^+(\gamma) \cap I^-(\gamma)$.

Since maximising causal curves have causal character, a future coray is either timelike or null, but in our case they will always be timelike:

Proposition 4.3. *In the setting of the local Bonnet-Myers rigidity, all co-rays from points in $I(\gamma)$ are timelike.*

Proof. Suppose there is a point $p \in I(\gamma)$ such that the TCRC does not hold at p, so w.l.o.g. there is a sequence $p_n \to p$, $t_n \to +\frac{\pi}{2}$ and maximal timelike curves σ_n from p_n to $y_n := \gamma(t_n)$ such that σ_n converge to a future directed null ray σ. Choose some $x \in \gamma \cap I^-(p)$ and let μ_n be maximal timelike curves from x to p_n (assuming n to be large enough, $x \ll p_n$). Suitably pre- and post-concatenating μ_n and then applying the limit curve theorem, it is easily seen that (up to a choice of subsequence) the μ_n converge locally uniformly to a maximising limit causal curve μ from x to p. μ is timelike since $x \ll p$. Denote by γ_n the piece of γ that runs between x and y_n. Set $a_n := L_\tau(\mu_n)$, $b_n := L_\tau(\sigma_n)$ and $c_n := L_\tau(\gamma_n)$. Let $\beta_n := \measuredangle_x(\mu_n, \gamma_n)$ and $\theta_n := \measuredangle_{p_n}(\mu_n, \sigma_n)$. Then $a_n \to a := \tau(x,p)$ and by the continuity of angles (cf. Prop. 2.6), $\beta_n \to \beta$, where β is the angle between μ and γ. Consider the comparison triangles for $\Delta(x, p_n, y_n)$ in AdS and call its sides $(\overline{\mu}_n, \overline{\sigma}_n, \overline{\gamma}_n)$, then the angle $\overline{\beta}_n$ between $\overline{\mu}_n$ and $\overline{\gamma}_n$ satisfies $\overline{\beta}_n \leq \beta_n$ and similarly $\overline{\theta}_n \geq \theta_n$. Since $\beta_n \to \beta$, there is some $C > 0$ such that $\overline{\beta}_n \leq C$ for all n. The hyperbolic law of cosines in AdS (see Lem. 2.3) gives

$$\begin{aligned}
\cos(b_n) &= \cos(a_n)\cos(c_n) + \sin(a_n)\sin(c_n)\cosh(\overline{\beta}_n),\\
\cos(c_n) &= \cos(a_n)\cos(b_n) - \sin(a_n)\sin(b_n)\cosh(\overline{\theta}_n).
\end{aligned}$$

Using the first equation, noting that $\overline{\beta}_n$ stays bounded and that a_n and c_n stay bounded away from zero and π, we get that also b_n stays bounded away from zero and π. This means that also $\overline{\theta}_n$ stays bounded.

From the monotonicity condition we get that

$$\tilde{\measuredangle}_{p_n}(\sigma_n(s), \mu_n(t))) \leq \tilde{\measuredangle}_{p_n}(x, y_n) = \bar{\theta}_n$$

for each s and t, so this is bounded too. We will see that this is incompatible with σ_n "getting more and more null": Note that we get the following estimate for some constant C'':

$$\begin{aligned}
C'' &\geq \cosh(\tilde{\measuredangle}_{p_n}(\sigma_n(s), \mu(t)))\\
&= \frac{\cos(\tau(\mu_n(t), p_n))\cos(\tau(p_n, \sigma_n(s))) - \cos(\tau(\mu_n(t), \sigma_n(s)))}{\sin(\tau(p_n, \sigma_n(s))\sin(\tau(\mu_n(t), p_n))}.
\end{aligned}$$

Since the denominator goes to 0 for $n \to \infty$ (as $\tau(p_n, \sigma_n(s)) \to \tau(p, \sigma(s)) = 0$ and $\tau(\mu_n(t), p_n) \to \tau(\mu(t), p) > 0$), the enumerator has to go to 0 as well, in particular this means

$$\tau(\mu(t), \sigma(s)) = \tau(\mu(t), p)$$

for all s, t. This implies that running along μ from $\mu(t)$ to p and then along σ to $\sigma(s)$ gives a maximiser, but this curve has a timelike and a null piece, a contradiction to [23, Thm. 3.18]. $\square$

Proposition 4.4. *Let X be a d-compatible, locally causally closed and causal Lorentzian pre-length space with proper d and $c : [0, b) \to X$ future inextensible, and let $\alpha : [0, a) \to X$ be a timelike asymptote to c. Then α is future inextensible.*

Proof. Let $\alpha_n : [0, a_n] \to X;\ p \rightsquigarrow c(t_n)$ be the sequence converging to α, it is parametrized in d-arclength parametrization. As d is proper, we have that $a_n \to +\infty$[33]. Now we can apply the Limit curve theorem for inextensible curves ([23, Thm. 3.14]) to get that α is inextensible. □

We will be in particular interested in future and past asymptotes of an AdS-line $c : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$.

We turn to the treatment of triangles adjacent to AdS-lines and begin with the aforementioned stacking principle. It will be an essential technical tool for controlling the behaviour of asymptotes to an AdS-line. The following two results are true for rather general Lorentzian pre-length spaces, the exact conditions are specified.

Proposition 4.5 (Comparison situations stack along an AdS-line)**.** *Let X be a timelike geodesically connected Lorentzian pre-length space with global timelike curvature bounded below by -1 and $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ be a timelike* AdS*-line. Let $p \in X$ be a point not on γ. Let $t_1 < t_2 < t_3$ such that all $y_i := \gamma(t_i)$ are timelike related to p, see Figure 33. Let $\bar{\Delta}_{12} := \Delta(\bar{p}, \bar{y}_1, \bar{y}_2)$ be a comparison triangle for $\Delta_{12} := \Delta(p, y_1, y_2)$ and extend the side $\bar{y}_1, \bar{y}_2$ to a comparison triangle $\bar{\Delta}_{23} := \Delta(\bar{p}, \bar{y}_2, \bar{y}_3)$ for $\Delta_{23} := \Delta(p, y_2, y_3)$, all in* AdS*. We choose it in such a way that $\bar{y}_1$ and $\bar{y}_3$ lie on opposite sides of the line through $\bar{y}_1, \bar{y}_2$. Then $\bar{y}_1, \bar{y}_2, \bar{y}_3$ are collinear. That makes $\bar{\Delta}_{13} := \Delta(\bar{p}, \bar{y}_1, \bar{y}_3)$ a comparison triangle for $\Delta_{13} := \Delta(p, y_1, y_3)$. In particular, all this can be realized in* AdS$'$ *such that $\bar{y}_i = (t_i, 0)$.*

Proof. We set $s_- = \sup(\gamma^{-1}(I^-(p)))$ and $s_+ = \inf(\gamma^{-1}(I^+(p)))$. Then the set of s where $\gamma(s)$ is timelike related to p is $(-\infty, s_-) \cup (s_+, +\infty)$. We assume $p \ll y_2$, the other case $y_2 \ll p$ can be reduced to this one by flipping the time orientation of the space.

We create comparison situations for an Alexandrov situation: We take the triangles $\bar{\Delta}_{12} = \Delta(\bar{p}, \bar{y}_1, \bar{y}_2)$ and $\bar{\Delta}_{23} = \Delta(\bar{p}, \bar{y}_2, \bar{y}_3)$ as in the statement. We also create a comparison triangle $\tilde{\Delta}_{13} = \Delta(\tilde{p}, \tilde{y}_1, \tilde{y}_3)$ for (p, y_1, y_3) and get a comparison point $\tilde{y}_2$ for y_2 on the side $y_1 y_3$. Then we can apply curvature

[33]Otherwise by properness, $c(t_n)$ would converge to some P_1, and if the whole curve didn't converge to P_1 other subsequence would converge to some other point P_2 within a local causal closed neighbourhood of P_1, and we get $P_1 \leq P_2 \leq P_1$, contradicting causality of X.

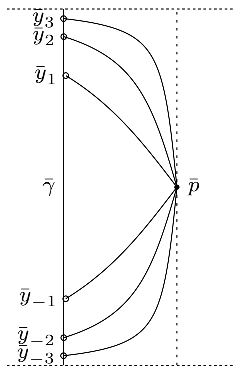

Figure 32: Stacking comparison triangles. Every three subsequent $\bar{y}_k$ together with $\bar{p}$ form two triangles as in Prop. 4.5.

comparison to get $\bar{\tau}(\tilde{p}, \tilde{y}_2) \geq \tau(p, y_2) = \bar{\tau}(\bar{p}, \bar{y}_2)$. By Lemmas 2.7 and 2.8[34], this means the situation is convex, i.e.

$$\tilde{\measuredangle}_{y_2}(p, y_1) \leq \tilde{\measuredangle}_{y_2}(p, y_3)\,. \tag{115}$$

If $p \ll y_1$ we also get

$$\tilde{\measuredangle}_{y_1}(p, y_2) \leq \tilde{\measuredangle}_{y_1}(p, y_3) \tag{116}$$

and if $y_1 \ll p$, inequality (116) flips.

Note that inserting back $y_2 = \gamma(t_2)$ and $y_3 = \gamma(y_3)$ in (116), varying t_2, t_3 and replacing y_1 by y_2 gives that $\tilde{\measuredangle}_{y_2}(p, \gamma(t))$ is monotonously increasing in t for $t > t_2$. Similarly, the time-reversed situation gives that for all t with $p \ll \gamma(t)$, $\tilde{\measuredangle}_{y_2}(p, \gamma(t))$ is monotonously increasing in t and similarly for t with $\gamma(t) \ll p$. Note that inserting the definitions of y_1 and y_3 in (115) and varying t_1, t_3 gives us the last piece to say $\tilde{\measuredangle}_{y_2}(p, \gamma(t))$ is monotonically increasing on its whole domain. We revisit (115): $\tilde{\measuredangle}_{y_2}(p, \gamma(t_1)) \leq \tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$. Note that the left hand side is decreasing with decreasing t_1 and the right hand side is increasing with increasing t_3.

Claim: For each t_1 and t_3, equality holds in (115).

Then: The comparison situation is straight, i.e. $\bar{y}_1, \bar{y}_2, \bar{y}_3$ lie on a distance realizer, and by triangle equality along distance realizers we have $\tau(y_1, y_3) = \tau(y_1, y_2) + \tau(y_2, y_3) = \bar{\tau}(\bar{y}_1, \bar{y}_2) + \bar{\tau}(\bar{y}_2, \bar{y}_3) = \bar{\tau}(\bar{y}_1, \bar{y}_3)$, i.e., $\bar{p}, \bar{y}_1, \bar{y}_3$ is a comparison triangle for p, y_1, y_3.

Proof: For any t_1, t_3 we can draw $\bar{\Delta}_{12}$, $\bar{\Delta}_{23}$ simultaneously in AdS. We indirectly assume there are t_1, t_3 such that the inequality (115) is strict. We first want to show that there are such t_1, t_3 where $\bar{\Delta}_{12}$, $\bar{\Delta}_{23}$ can be drawn

[34] If $p \ll y_1 \ll y_2 \ll y_3$, we use Lem. 2.8 and if $y_1 \ll p \ll y_2 \ll y_3$ we use Lem. 2.7.

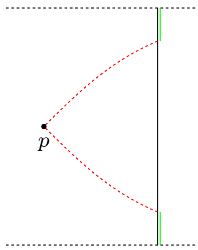

Figure 33: The domain where the y_i can lie in is in green.

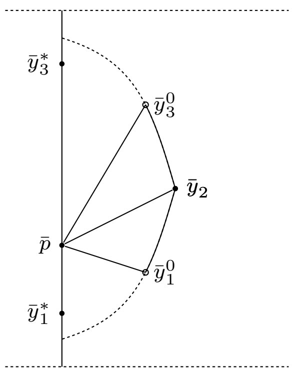

Figure 34: We assume that the path $\bar{y}_1^0\bar{y}_2\bar{y}_3^0$ is longer than the path $\bar{y}_1^0\bar{p}\bar{y}_3^0$. If the angle at $\bar{y}_2$ is not straight, we extend some lines and as points further from $\bar{p}$ are more on the left, we find some t_1^* and t_3^* such that $\bar{y}_1^*\bar{p}\bar{y}_3^*$ are collinear. But then the path $\bar{y}_1^0\bar{y}_2\bar{y}_3^0$ is shorter than $\bar{y}_1^0\bar{p}\bar{y}_3^0$, which then yields a contradiction to γ being maximising.

simultaneously in the warped product picture AdS$'$. As a first case, let inequality (115) always be strict. Then we take the limit as $t_1 \nearrow t_2$ and $t_3 \searrow t_2$, as the sides are monotonous the limits of the sides are both finite, thus for t_1, t_3 close enough to t_2, we have that $\bar{y}_1, \bar{y}_3$ are realizable in AdS$'$ (by triangle equality of angles in AdS, we take an angle a little bit bigger than the limits, then small distances can be realized at that angle or lower from a prescribed side $\bar{p}\bar{y}_2$). The other case is where inequality (115) has equality for t_1, t_3 close to t_2, but one of the sides changes when taking t_1 resp. t_3 further away from t_2. W.l.o.g., say the right side changes, so there is a $t_3^0 > t_3$ such that $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3^0)) > \tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$. Note that as long as (115) has equality, we have that $\tau(\bar{y}_1, \bar{y}_3) = \tau(\bar{y}_1, \bar{y}_2) + \tau(\bar{y}_2, \bar{y}_3) < L(\gamma) = \pi$. As $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$ is continuous in t_3, there also exists a t_3^0 such that $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3^0)) > \tilde{\measuredangle}_{y_2}(p, \gamma(t_3))$ and $\tau(\bar{y}_1, \bar{y}_3) < \pi$. In particular, it is realizable, with $\bar{y}_1^0, \bar{y}_3^0$ on a vertical line, and all of $\bar{y}_1^0, \bar{y}_3^0, \bar{p}$ to the left of the vertical line with x-coordinate $x(\bar{y}_2)$. In any case, we now fix this realization, i.e. fix the points $\bar{y}_1^0, \bar{y}_2, \bar{y}_3^0$ and $\bar{p}$.

Now consider $t_1^* < t_1^0$ and $t_3^* > t_3^0$. On the side $\bar{y}_2\bar{p}$, we can construct comparison points for $\gamma(t_1^*)$ and $\gamma(t_3^*)$, call them $\bar{y}_1^*, \bar{y}_3^*$, if they are still contained in AdS$'$. Note that $\bar{y}_1^*, \bar{y}_3^*$ vary continuously with t_1^*, t_3^*, so if $\bar{y}_1^*, \bar{y}_3^*$ are not defined for some t_i^* we get a $t_1^{\min}, t_3^{\max}$ such that it is defined before that value, and $\bar{y}_1^*$ or $\bar{y}_3^*$ converge to infinity in the limit to $t_1^{\min}$ resp. $t_3^{\max}$, otherwise set $t_1^{\min} = -\frac{\pi}{2}$ and / or $t_3^{\max} = \frac{\pi}{2}$, then similar limits hold. As $\tilde{\measuredangle}_{y_2}(p, \gamma(t_3^*)) > \tilde{\measuredangle}_{y_2}(p, \gamma(t_3^0))$, we get that $\bar{y}_3^*$ is to the left of the (extended) geodesic connecting $\bar{y}_2 t_3^0$, and as $\tau(\bar{y}_2, \bar{y}_3^*) > \tau(\bar{y}_2, \bar{y}_3^0)$, it has much more negative x-coordinate. As the (extended) geodesic connecting $\bar{y}_2 t_3^0$ is going to the left, it has x-coordinate converging to $-\infty$ towards the boundary, and thus so has $\bar{y}_3^*$, similarly $\bar{y}_1^*$. In particular, as $\bar{y}_1^*, \bar{y}_3^*$ depend continuously on t_1^*, t_3^*, there are parameters such that $\bar{y}_1^*, \bar{y}_3^*$ have the same x-coordinate as p, thus by triangle equality along distance realizers and strict triangle inequality we have that

$$\tau(\bar{y}_1^*, \bar{p}) + \tau(\bar{p}, \bar{y}_3^*) = \tau(\bar{y}_1^*, \bar{y}_3^*) > \tau(\bar{y}_1^*, \bar{y}_2) + \tau(\bar{y}_2, \bar{y}_3^*) \,,$$

in contradiction to the fact that $\gamma(t_1^*), \gamma(t_2), \gamma(t_3^*)$ lie on a distance realizer. Thus we get the claim. See Figure 34 for a visualisation of the construction. $\square$

Proposition 4.6 (Angle = comparison angle)**.** *Let X be a timelike geodesically connected Lorentzian pre-length space with global timelike curvature bounded below by $K = -1$ and $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ be a complete timelike* AdS*-line and $x := \gamma(t_0)$ a point on it. We split γ into the future part $\gamma_+ = \gamma|_{[t_0, \frac{\pi}{2})}$ and the past part $\gamma_- = \gamma|_{(-\frac{\pi}{2}, t_0]}$. Let $p \in X$ be a point not on γ with x and*

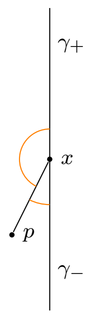

Figure 35: Illustration: These angles are the same, and have the same value as if they are considered as comparison angles.

p timelike related and $\alpha : x \rightsquigarrow p$ a connecting distance realiser. Then for all $s \neq t$ such that p and $\gamma(s)$ are timelike related, we have:

$$\tilde{\measuredangle}_x(p, \gamma(s)) = \measuredangle_x(\alpha, \gamma_+) = \measuredangle_x(\alpha, \gamma_-)\,,$$

i.e. the comparison angle is equal to the angle, and the same in both directions.

Proof. First, we check that the comparison angle $\tilde{\measuredangle}_x(p, \gamma(s))$ is constant in s: For s_1 and s_2 for which this is defined, we have three parameters on γ involved: s_1, s_2, t_0. The previous result (Prop. 4.5) tells us we can construct a comparison situation for all three triangles at once. As the comparison situations have the comparison angle $\tilde{\measuredangle}_x(p, \gamma(s_1))$ resp. $\tilde{\measuredangle}_x(p, \gamma(s_2))$ as the angle in $\bar{x}$, they are equal.

Now we look at the angle $\measuredangle_x(\alpha, \gamma_\pm)$: We assume $p \ll x$. We already know $\tilde{\measuredangle}_x(\alpha(s), \gamma(t))$ is constant in t. We now have to look at its dependence on s: By Thm. 2.4, $\tilde{\measuredangle}^{\mathrm{S}}_x(\alpha(s), \gamma(t))$ is monotonously increasing in s. Note now that for $t < t_0$, the sign of this angle is $\sigma = -1$, and for $t > t_0$, the sign of this angle is $\sigma = +1$. So choose some $t_- < t_0$ and $t_+ > t_0$ for which all the necessary angles exist (i.e. $\gamma(t_-) \ll p$ and $t_+ > t_0$). Then the above applied to $\alpha(s)$ instead of p gives that $\tilde{\measuredangle}_x(\alpha(s), \gamma(t_-)) = \tilde{\measuredangle}_x(\alpha(s), \gamma(t_+))$ is both a monotonously decreasing and increasing function in s. Thus it is constant, and $\tilde{\measuredangle}_x(\alpha(s), \gamma(t)) = \measuredangle_x(\alpha, \gamma_-) = \measuredangle_x(\alpha, \gamma_+)$, which includes the desired equalities. $\square$

Note the situation we are in is very rigid: as soon as a side of a triangle is part of an AdS-line, one-sided configurations where the point is on the line satisfy both the inequality for curvature bounds above and below:

Corollary 4.7. *Let X be a timelike geodesically connected, globally causally closed Lorentzian pre-length space with global timelike curvature bounded below by $K = -1$ and let $\gamma : \mathbb{R} \to X$ be a timelike* AdS*-line. Then for any point $p \in X$ and two points $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ which are timelike related to p, we get an unordered timelike triangle $\Delta = \Delta(x_1, x_2, p)$. Let q_1, q_2 be any points on Δ, one of them lying on the side x_1x_2. We form a comparison triangle $\bar{\Delta}$ for Δ and find comparison points $\bar{q}_1, \bar{q}_2$ for q_1, q_2. Then $q_1 \leq q_2$ if and only if $\bar{q}_1 \leq \bar{q}_2$; and $\tau(q_1, q_2) = \bar{\tau}(\bar{q}_1, \bar{q}_2)$.*

Proof. We distinguish which sides the q_i lie on: Note we assumed one of them is on the side x_1x_2. We only prove the case where q_1 is on the side γ_+ connecting x_1x_2 (say $q_1 = \gamma_+(s)$) and q_2 is on the side β connecting x_1, p (say $q_2 = \beta(t)$), the proof of the other cases is easily adapted.

Now the comparison triangle $\tilde{\Delta}$ for $\Delta(x_1, q_1, q_2)$ has angle $\tilde{\measuredangle}_{x_1}(q_1, q_2) = \measuredangle_{x_1}(\gamma_+, \beta)$ at the point corresponding to x_1, and the comparison triangle for $\Delta = \Delta(x_1, p, x_2)$ has the same angle $\tilde{\measuredangle}_{x_1}(p, x_2) = \measuredangle_{x_1}(\gamma_+, \beta)$ at the point corresponding to x_1. As the comparison points $\bar{q}_1$ and $\bar{q}_2$ lie on the sides enclosing the angle, also $\Delta(\bar{x}_1, \bar{q}_1, \bar{q_2})$ has the angle $\measuredangle_{x_1}(\gamma_+, \beta)$ at $\bar{x}_1$. Now the triangles $\tilde{\Delta}$ and $\Delta(\bar{x}_1, \bar{q}_1, \bar{q_2})$ have two sides and the angle between them equal, thus also the opposite side is equal, establishing $\tau(q_1, q_2) = \bar{\tau}(\bar{q}_1, \bar{q}_2)$. A continuity argument (moving q_1 a bit into the past) then gives that $q_1 \leq q_2$ if and only if $\bar{q}_1 \leq \bar{q}_2$. □

In the following, unless we specify the assumptions on X, we always take X to be as in the local Bonnet-Myers rigidity theorem.

Although in the model for this paper, the splitting theorem, it was possible to circumvent the Busemann functions, this is not possible here: In the warped product the Busemann function becomes the t-coordinate and as the radius function varies, this is important when making comparison situations.

Definition 4.3. We define the *past Busemann function* w.r.t. γ, $b_- : I(\gamma) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ as $b_-(x) = \frac{\pi}{2} - \lim_{t \to -\frac{\pi}{2}} \tau(\gamma(t), x) = \lim_{t \to \frac{\pi}{2}} (t - \tau(\gamma(-t), x))$. Similarly, the *future Busemann function* w.r.t. γ is defined by setting $b_+(x) = +\frac{\pi}{2} - \lim_{t \to +\frac{\pi}{2}} \tau(x, \gamma(t)) = \lim_{t \to +\frac{\pi}{2}} (t - \tau(x, \gamma(t)))$. Note that the arguments in both limits are monotonically decreasing.

Lemma 4.8. *We have that $b_+ + b_- \geq 0$.*

Proof. Let $x \in I(\gamma)$. We dive into the limits: We have that for t_1 small enough and t_2 large enough that

$$\tau(\gamma(t_1), \gamma(t_2)) - \pi \geq \underbrace{\tau(\gamma(t_1), x) - \frac{\pi}{2}}_{\to -b_-(x)} + \underbrace{\tau(x, \gamma(t_2)) - \frac{\pi}{2}}_{\to -b_+(x)},$$

and that the left hand side converges to 0 as $t_1 \to -\frac{\pi}{2}$ and $t_2 \to \frac{\pi}{2}$. □

Next, we show that any future directed asymptote to γ (which we now know to be timelike) has the right length.

Proposition 4.9. *Let $x \in I(\gamma)$ and let α be a future directed timelike asymptotic ray to γ at x, and let $t = b_+(x)$. Then $L_\tau(\alpha) = \frac{\pi}{2} - t$.*

Proof. By construction, $\alpha : [0, b) \to X$ arises as a locally uniform limit of timelike maximisers $\alpha_n : [0, a_n] \to X$ from x to $z_n := \gamma(t_n)$, where $t_n \to +\frac{\pi}{2}$. For each compact subinterval of the domain of α we can apply Lem. 4.2 to get $L(\alpha|_{[0,c]}) = \lim_n L(\alpha_n|_{[0,c]}) \leq \lim_n L(\alpha_n)$, thus $L := L_\tau(\alpha) \leq \frac{\pi}{2} - t$. Now indirectly suppose that $L := L_\tau(\alpha) < \frac{\pi}{2} - t$. By continuity of angles (see Prop. 2.6), $\omega_n := \measuredangle_x(\alpha, \alpha_n) \to 0$ for $n \to \infty$.

Now note that for any t_n with $n \geq N$, $\partial J^-(z_n) \cap \alpha = (J^-(z_n) \setminus I^-(z_n)) \cap \alpha$ is non-empty: Certainly, $x \in I^-(z_n)$, so if this intersection were empty, then α would be imprisoned in the compact set $J^+(x) \cap J^-(z_n)$, which cannot happen by Prop. 4.4. So we find a point $y_n \in \alpha$ that is null-related to z_n, i.e. $y_n < z_n$ and $\tau(y_n, z_n) = 0$.

We now consider the triangle given by the vertices x, y_n, z_n. Two of the sides have natural curves: From x to y set ν_n to be the part of α from x to y, and from x to z_n take α_n. We name the side-lengths: $c_n = \tau(x, z_n) = L(\alpha_n)$, $a_n = \tau(x, y_n) = L(\nu_n)$ and $b_n = \tau(y_n, z_n) = 0$. We know a lot about this triangle as $n \to +\infty$:

- By the definition of $t = b_+(x)$, $c_n \to \frac{\pi}{2} - t$. Note this is the longest side-length and c_n is bounded away from π as $b_+(x) \geq -b_-(x) > -\frac{pi}{2}$.
- $b_n = 0$ by the choice of y_n.
- $a_n = \tau(x, y_n) < L(\alpha)$, so $\limsup_n a_n \leq L < \frac{\pi}{2} - t$.
- Our curvature assumption gives that $\bar{\omega}_n := \tilde{\measuredangle}_x(y_n, z_n) \leq \omega_n \to 0$.

But this cannot be: we claim we can take a comparison situation $\Delta(\bar{x}, \bar{y}_n, \bar{z}_n)$ which converges as $n \to +\infty$: Fix $\bar{x}$, choose $\bar{z}_n$ along a fixed distance realizer $\bar{\alpha}_n$ (independent of n!) through $\bar{x}$, take $\bar{y}_n$ on a distance realizer $\bar{\alpha}$ (dependent on n!) through $\bar{x}$ at the correct angle $\measuredangle_{\bar{x}}(\bar{\alpha}_n, \bar{\alpha}) = \bar{\omega}_n$. Then $\bar{x}$ and $\bar{z}_n$ automatically converge, and $\bar{y}_n$ does so too as $\bar{\omega}_n$ converges.

Note the limiting situation is degenerate, as $\bar{\omega}_n \to 0$, c_n is bounded away from 0 and $a_n > 0$ is increasing, so bounded away from 0 as well. Thus, the triangle inequality of angles has equality in the limit, so

$$0 = \lim_n (c_n - a_n - b_n) = \left(\frac{\pi}{2} - t\right) - L - 0$$

which contradicts $L < \frac{\pi}{2} - t$. $\square$

To conclude this subsection, we show that any future directed and any past directed (timelike) asymptote to γ from a common point fit together to give an AdS-line.

Proposition 4.10 (Asymptotic AdS-lines)**.** *Let $p \in I(\gamma)$ and consider any future and past asymptotes $\sigma^+ : [0, a_+) \to X$ and $\sigma^- : (a_-, 0] \to X$ from p to γ. Then $\sigma := \sigma^-\sigma^+ : (a_-, a_+) \to X$ is a complete, timelike distance realizer of length π. In particular $b_+ + b_- = 0$ and it can be reparametrized in τ-arclength to be an* AdS*-line. The reparametrization is called a (full)* asymptote *σ to γ through p. In particular, for an asymptote σ we have $b_+(\sigma(t)) = t$ ("the Busemann parametrization is a τ-arclength parametrization").*

Proof. Let σ_n^+ and σ_n^- be two sequences of timelike maximisers from p to $\gamma(r_n)$ and $\gamma(-r_n)$, respectively, such that σ^+ and σ^- arise as limits of these sequences as $r_n \to \infty$. To show that σ is a line, it is sufficient to show that for any $s < 0 < t$, $\tau(\sigma(s), \sigma(t)) = L_\tau(\sigma|_{[s,t]})$. To see this, let $q_+ := \sigma(t)$ and $q_- := \sigma(s)$, and $q_+^n := \sigma_n^+(t)$, $q_-^n := \sigma_n^-(s)$. Then $q_\pm = \lim_n q_\pm^n$. Consider the triangle $\Delta(q_-^n, p, q_+^n)$ with sides σ_n^+, σ_n^+ and a part of γ. Consider a comparison triangle with points $\overline{q_\pm^n}$ corresponding to $q_\pm^n$. Sending $n \to \infty$, we see that the stacked comparison triangles in AdS$'$ converge to a vertical line (here we use Prop. 4.5 and that γ has length π), hence our curvature bound gives

$$\tau(q_-, q_+) = \lim_{n\to\infty} \tau(q_-^n, q_+^n) \le \lim_{n\to\infty} \bar\tau(\overline{q_-^n}, \overline{q_+^n}) = \lim_n L_\tau(\sigma_n^-\sigma_n^+|_{[s,t]}) = L_\tau(\sigma|_{[s,t]}),$$

which is what we wanted to show, as the other inequality is trivial.

$a_+ - a_- = \pi$ also follows as the stacked comparison triangles in AdS$'$ converge to a vertical line. For the future and past Busemann functions, we also look at this picture: we get a vertical line $\bar\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathrm{AdS}'$ and $\bar p \in \mathrm{AdS}'$ such that any $\Delta(\bar\gamma(t_1), \bar p, \bar\gamma(t_2))$ is a comparison triangle. In particular, $b_+(p) = \frac{\pi}{2} - \lim_{t_2 \to \frac{\pi}{2}} \bar\tau(\bar p, \bar\gamma(t_2)) = t(p) = \cdots = -b_-(p)$ is the t-coordinate. $\square$

4.2 AdS-Parallel lines

This subsection introduces the notion of parallelity for complete timelike AdS-lines. In the following, we call a map $f : Y_1 \to Y_2$ between Lorentzian pre-length spaces $(Y_1, d_1, \ll_1, \le_1, \tau_1)$ and $(Y_2, d_2, \ll_2, \le_2, \tau_2)$ *τ-preserving* if for all $p, q \in Y_1$, $\tau_1(p, q) = \tau_2(f(p), f(q))$, and we call f *$\le$-preserving* if $p \le_1 q$ if and only if $f(p) \le_2 f(q)$.

Definition 4.4 (Realizing w.r.t. Busemann). Let γ be a AdS-line and $x \in I(\gamma)$. Then a point $\bar{x} \in \mathrm{AdS}'$ is *future-Busemann-realized* if $t(\bar{x}) = b_+(x)$. Similarly, for an asymptote $\alpha : I \to X$ to γ parametrized in τ-arclength parametrization, $\bar{\alpha} : I \to \mathrm{AdS}'$ is *Busemann-realized* if it is vertical (i.e. $(\bar{\alpha}(t))_x$ is constant) and $t(\bar{\alpha}(t)) = b_+(\alpha(t))$. Note that by Prop. 4.10, this is always possible, and α is in τ-arclength parametrization.

Definition 4.5 (AdS-Parallel lines). Let α, β be two AdS-lines in a Lorentzian pre-length space X. They are called *(*AdS-*)parallel* if there exists a τ- and $\leq$-preserving map $f : (\alpha((-\frac{\pi}{2}, \frac{\pi}{2})) \cup \beta((-\frac{\pi}{2}, \frac{\pi}{2}))) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{R}$ such that $f \circ \alpha$ and $f \circ \beta$ are Busemann-realizations of α and β. We call such a map f a *(*AdS-*)parallel realisation* of α and β. The *spacelike distance* between α and β is the x-coordinate difference in the realization. Note that one can similarly define the spacelike distance between an AdS-line and a point (with the stacking principle 4.5 ensuring that it always works out).

Remark 4.11. *Note that by post-composing this by a translation in the x-direction, we can always achieve that $f(\alpha(\mathbb{R})) = \{(t, 0) : t \in \mathbb{R}\} \subseteq \mathrm{AdS}'$ and $f(\beta(\mathbb{R})) = \{(t, c) : t \in \mathbb{R}\}$ for some $c \geq 0$.*

Definition 4.6. Let X be a Lorentzian pre-length space with continuous time separation τ satisfying $\tau(x, x) = 0$ for all $x \in X$. Let $\alpha, \beta : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ be two τ-arclength parametrised AdS-lines. For each $s < t$ with $\alpha(s) \leq \beta(t)$ there is a $c \geq 0$ such that the AdS-parallel lines $\bar{\alpha}, \bar{\beta} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathrm{AdS}'$ with distance c so that the partial map $\alpha(s) \mapsto \bar{\alpha}(s)$ and $\beta(t) \mapsto \bar{\beta}(t)$ is (Busemann- and) τ-preserving. Set:

- $c_{\alpha\beta}(s, t)$ to be that c,
- $c_{\beta\alpha}(s, t)$ to be the analogous thing with $\alpha(s)$ and $\beta(t)$ reversed.

Similarly, for each s, we find a $c \geq 0$ such that the closure of the set $\{t : \alpha(s) \leq \beta(t)\}$ is the same as $\{t : \bar{\alpha}(s) \leq \bar{\beta}(t)\}$. Set:

- $c^N_{\alpha+}(s)$ to be that c,
- $c^N_{\beta+}(s)$ to be the analogous thing with $\alpha(s)$ and $\beta(t)$ reversed.

By Lem. 3.8, we get explicit formulae:

- $c_{\alpha\beta}(s, t) = \operatorname{arcosh}\left(\frac{\cos(\tau(\alpha(s),\beta(t))) - \sin(s)\sin(t)}{\cos(s)\cos(t)}\right)$ if $\alpha(s) \leq \beta(t)$ (otherwise undefined, note the formula is strictly decreasing in $\tau(\alpha(s), \beta(t))$)),
- $c_{\beta\alpha}(s, t) = \operatorname{arcosh}\left(\frac{\cos(\tau(\beta(t),\alpha(s))) - \sin(s)\sin(t)}{\cos(s)\cos(t)}\right)$ if $\beta(t) \leq \alpha(s)$ (otherwise undefined),

- $c^N_{\alpha+}(s) = \operatorname{arcosh}\left(\frac{1-\sin(s)\sin(t)}{\cos(s)\cos(t)}\right)$ for $t = \inf\{t : \alpha(s) \leq \beta(t)\}$ (note the formula is strictly increasing in t),
- $c^N_{\beta+}(s) = \operatorname{arcosh}\left(\frac{1-\sin(s)\sin(t)}{\cos(s)\cos(t)}\right)$ for $t = \inf\{t : \beta(s) \leq \alpha(t)\}$.

We say the *c-criterion for parallel* AdS*-lines* is satisfied if all these are constant where defined, have the same value, and the infima are minima.

The following is easy to see from the fact that most timelike geodesics have the x-coordinates go to infinity before they leave AdS$'$ (see Lem. 3.7):

Lemma 4.12 (Communicability)**.** *Let X be globally causally closed and α, β be* AdS*-parallel. Then for each $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there is a $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\alpha(s) \ll \beta(t)$ (and similarly a t with $\alpha(s) \gg \beta(t)$).*

Proof. We consider the sets $I = \{(s,t) : \alpha(s) \ll \beta(t)\}$ (open) and $J = \{(s,t) : \alpha(s) \leq \beta(t)\} = \bar{I}$ (as X is globally causally closed). They are non-empty: as $\beta \cap I^+(\alpha) \neq \emptyset$, we find $\alpha(s_0) \ll \beta(t_0)$, so $(s_0, t_0) \in I$. We claim that the first projection of I is full, i.e. $pr_1(I) = \{s : \exists t : \alpha(s) \ll \beta(t)\} = (-\frac{\pi}{2}, \frac{\pi}{2})$. We know s_0 is in the left hand side, and with it all smaller values $s < s_0$. So let indirectly $s_n \nearrow s_+ < \frac{\pi}{2}$ and $t_n \nearrow \frac{\pi}{2}$ such that $(s_n, t_n) \in I$, but $s_+ \notin pr_1(I)$. Then by continuity of τ, $\tau(\alpha(s_n), \beta(t_n)) \to 0$. Looking at the formula for $c_{\alpha\beta}(s_n, t_n)$, one sees that the enumerator stays away from 0, whereas the denominator approaches 0. Thus $c_{\alpha\beta}(s_n, t_n) \nearrow +\infty$ which cannot be as it is constant, so $pr_1(I) = (-\frac{\pi}{2}, \frac{\pi}{2})$. □

Lemma 4.13 (The c-criterion for AdS-parallel lines)**.** *Let X be a Lorentzian pre-length space with continuous time separation τ satisfying $\tau(x,x) = 0$ for all $x \in X$. Let $\alpha, \beta : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ be two τ-arclength parametrised* AdS*-lines. Then they satisfy the c-criterion for parallel* AdS*-lines if and only if they are parallel* AdS*-lines. If this is the case, the constant c is the distance between α and β.*

If X is additionally globally causally closed and $\alpha \cap I^+(\beta) \neq \emptyset$ and $\beta \cap I^+(\alpha) \neq \emptyset$, the c-criterion simplifies to only checking $c_{\alpha\beta}$ and $c_{\beta\alpha}$ for being constant and equal.

Proof. We define f as in the definition of parallel AdS-lines with distance c: $f(\alpha(s)) := (s, 0)$, $f(\beta(s)) := (s, c)$ in AdS$'$. We will prove that f is $\leq$-preserving if and only if $c^N_{\alpha+}$ and $c^N_{\beta+}$ are constantly c, and under the assumption that this is the case f is τ-preserving if and only if $c_{\alpha\beta}$ and $c_{\beta\alpha}$ are constantly c wherever defined.

As α and β are future directed τ-arclength parametrised AdS-lines, it is clear that f is $\leq$- and τ-preserving along α and along β, i.e. we only have to check the conditions on $f(\alpha(s)) \bar{\leq} f(\beta(t))$ and their τ-distance, and both also for α, β reversed.

So first, for $\leq$-preserving: For a fixed s, we describe the set of t such that $\alpha(s) \leq \beta(t)$ resp. $f(\alpha(s)) \bar{\leq} f(\beta(t))$, the situation switching α and β is analogous (using $c^N_{\beta+}$ instead of $c^N_{\alpha+}$). By transitivity of $\leq$ resp. $\bar{\leq}$, these sets will both be an interval of the form $[t_{min}, +\frac{\pi}{2})$ or $(t_{min}, +\frac{\pi}{2})$, resp. $[\tilde{t}_{min}, +\frac{\pi}{2})$ (always closed). Note that inserting $\tilde{t}_{min}$ instead of t into the definition of $c^N_{\alpha+}(s)$ yields c by the geometric definition. As the formula of $c^N_{\alpha+}(s)$ is strictly increasing in t, we have that $t_{min} = \tilde{t}_{min}$ if and only if $c = c^N_{\alpha+}(s)$. Thus $\{t : \alpha(s) \leq \beta(t)\} = [\tilde{t}_{min}, +\frac{\pi}{2})$ if and only if $c^N_{\alpha+}(s) = c$ and the infimum in the definition of $c^N_{\alpha+}(s)$ is achieved. The same argument works out for α and β exchanged, thus proving that f is $\leq$-preserving if and only if $c^N_{\alpha+}$ and $c^N_{\beta+}$ are constant and equal to c, and the infima appearing are minima.

For τ-preserving, we assume that f is already $\leq$-preserving. We will only prove $\tau(\alpha(s), \beta(t)) = \bar{\tau}(f(\alpha(s)), f(\beta(t)))$ for $\alpha(s) \leq \beta(t)$, the case where they are causally unrelated is covered by $\leq$-preserving, and the case where α and β are interchanged follows analogously. Note that inserting $\bar{\tau}(f(\alpha(s)), f(\beta(t)))$ for $\tau(\alpha(s), \beta(t))$ into the definition of $c_{\alpha\beta}(s,t)$ yields c by the geometric definition. As the formula of $c_{\alpha\beta}(s,t)$ is strictly decreasing in $\tau(\alpha(s), \beta(t))$, we have that $\tau(\alpha(s), \beta(t)) = \bar{\tau}(f(\alpha(s)), f(\beta(t)))$ if and only if $c_{\alpha\beta}(s,t) = c$. The same argument works out for α and β exchanged, thus proving that if f is $\leq$-preserving, f is τ-preserving if and only if $c_{\alpha\beta}$ and $c_{\beta\alpha}$ are constant and equal to c.

For the additional statement, fix s. By global causal closedness, the infimum in the formula of $c^N_{\alpha+}$ is automatically a minimum, and it is non-void by Lem. 4.12. For the value of $c^N_{\alpha+}(s)$, let t_0 be the minimum. Then for any $t > t_0$ we have $\alpha(s) \ll \beta(t)$, and by continuity of τ we have that $\tau(\alpha(s), \beta(t_0)) = 0$. Now note that plugging this into $c_{\alpha\beta}(s, t_0)$ gives the formula for $c^N_{\alpha+}$. $\square$

Lemma 4.14 (The strong causality trick)**.** *Let X be a strongly causal Lorentzian pre-length space with τ continuous on U and $\tau(x,x) = 0$ for all $x \in X$. Let $\alpha, \beta : [0, b) \to U$ be two τ-arclength parametrised timelike distance realisers with $x := \alpha(0) = \beta(0)$. Assume that for all s, t such that $\alpha(s)$ and $\beta(t)$ are timelike related, the comparison angle $\tilde{\measuredangle}_x(\alpha(s), \beta(t)) = 0$. Then $\alpha = \beta$.*

Proof. We set $f_+(s,t) = \tau(\alpha(s), \beta(t))$ and $f_-(s,t) = \tau(\beta(s), \alpha(t))$. They are both monotonically increasing in t and continuous. We want to describe the set where $f_+ > 0$ resp. $f_- > 0$. We set $t_s^+ = \inf\{t : f_+(s,t) > 0\}$, then $\lim_{t \searrow t_s^+} f_+(s,t) = 0$. Thus in the law of cosines (Lem. 2.3), we get $\lim_{t \searrow t_s^+} \cosh(\tilde{\measuredangle}_x(\alpha(s), \beta(t))) = 1 = \lim_{t \searrow t_s^+} \frac{\cos(f_+(s,t)) - \cos(s)\cos(t)}{\sin(s)\sin(t)}$, so $s = t_s^+$. Analogously, we get $s = \inf\{t : f_-(s,t) > 0\}$, in total $f_\pm(s,t) > 0$ for $s < t$.

That means that whenever $s_- < t < s_+$ we have that $\alpha(t) \in I(\beta(s_-), \beta(s_+))$. By strong causality the $I(\beta(s_-), \beta(s_+))$ form a neighbourhood basis of the point $\beta(t)$, and $\alpha(t)$ is inside of all of these neighbourhoods. As X is Hausdorff, we get that $\alpha(t) = \beta(t)$ for all $t \in (0, b)$, so $\alpha = \beta$. $\square$

Lemma 4.15 (AdS-Parallel lines are unique)**.** *Let X be a strongly causal, timelike geodesically connected Lorentzian pre-length space with τ continuous on U, $\tau(x, x) = 0$ for all $x \in X$. Suppose that X has global timelike curvature bounded below by -1, let $\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \to U$ be an* AdS*-line and $p \in U$ a point. Then there is at most one* AdS*-parallel line to α through p.*

Proof. We indirectly assume there are two AdS-parallel lines to α through p, namely $\beta : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ and $\tilde{\beta} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$, with $p = \beta(t_0) = \tilde{\beta}(\tilde{t}_0)$. We know that $t_0 = \tilde{t}_0$ and that both $\alpha, \tilde{\beta}$ have the same distance, as there is only one way (up to isometry) of realizing α vertically in AdS' and putting p at the right distance.

We construct a comparison situation for all three lines at once: We choose the parallel realisation f for α and β given by $f(\alpha(s)) = (s, 0)$ and $f(\beta(s)) = (s, c)$, and similarly we choose $\tilde{f}$ for α and $\tilde{\beta}$ given by $\tilde{f}(\tilde{\beta}(s)) = (s, -c)$ and $\tilde{f}(\alpha(s)) = (s, 0)$, i.e. we realise β and $\tilde{\beta}$ on opposite sides of α.

We calculate the angle $\omega := \measuredangle_p(\beta|_{[t_0, \frac{\pi}{2})}, \tilde{\beta}|_{[t_0, \frac{\pi}{2})})$: By Prop. 4.6, this is equal to any comparison angle $\tilde{\measuredangle}_p(\beta(s), \tilde{\beta}(t))$ as long as $\beta(s) \ll \tilde{\beta}(t)$ or conversely.

We know by Lem. 4.12 that given $r \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we find an $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\beta(r) \ll \alpha(s)$ and a $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\alpha(s) \ll \tilde{\beta}(t)$. In particular, for all $r \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there is a $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\beta(r) \ll \tilde{\beta}(t)$. We now look at the law of cosines to estimate the comparison angle $\omega = \tilde{\measuredangle}_p(\beta(r), \tilde{\beta}(t))$: $\cosh(\omega) = \frac{\cos(\tau(\beta(r), \tilde{\beta}(t))) - \cos(r - t_0)\cos(t - t_0)}{\sin(r - t_0)\sin(t - t_0)} \leq \frac{1 - \cos(r - t_0)\cos(t - t_0)}{\sin(r - t_0)\sin(t - t_0)}$ which converges to 1 as s and t both approach the maximal value $\frac{\pi}{2}$. In particular, $\omega = 0$. We can now apply the strong causality trick (Lem. 4.14) to get $\beta = \tilde{\beta}$, so there is only one line parallel to α through p. $\square$

Let us now show for X as in the main theorem that asymptotic AdS-lines to γ constructed in Prop. 4.10 are parallel to γ.

Lemma 4.16. *Let $\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$ be a complete timelike asymptotic* AdS*-line to γ. Then α, γ are* AdS*-parallel.*

Proof. By construction, α arises as the limit of timelike maximising segments α_n^+ from $p := \alpha(t_0)$ to $\gamma(r_n)$ and α_n^- from p to $\gamma(t_n)$, where $r_n \to -\frac{\pi}{2}$ and $t_n \to \frac{\pi}{2}$. We will use the stacking principle (cf. Prop. 4.5) to show that α and γ are parallel. Indeed, the triangles corresponding to $\Delta(\gamma(r_n), p, \gamma(t_n))$

stack in AdS$'$ (see Figure 32), and we have that the sides corresponding to α_n^+ and α_n^- in AdS$'$ converge to the vertical line through $\bar{p}$.

We now argue that the map sending $\alpha((-\frac{\pi}{2},\frac{\pi}{2}))$ and $\gamma((-\frac{\pi}{2},\frac{\pi}{2}))$ to the corresponding limit lines $\bar{\alpha}$ and $\bar{\gamma}$ is a parallel realisation.

For any t, s, consider $\tau(\alpha(t),\gamma(s)) = \lim_n \tau(\alpha_n(t),\gamma(s))$. For n so large that $-r_n < s < r_n$, $\gamma(s)$ is part of the triangle $\Delta(\gamma(-r_n), p, \gamma(r_n))$. From Cor. 4.7 we conclude that $\alpha_n(t) \ll \gamma(s)$ if and only if $\overline{\alpha}_n(t) \ll \overline{\gamma}(s)$, and $\tau(\alpha_n(t),\gamma(s)) = \overline{\tau}(\overline{\alpha}_n(t),\overline{\gamma}(s))$, where the bars denote the corresponding points on the comparison side. This shows that the realization of α as $\bar{\alpha}$ and γ as $\bar{\gamma}$ is a parallel realisation. $\square$

Lemma 4.17 (Asymptotes are asymptotes through any point on them)**.** *Let $\alpha : (-\frac{\pi}{2},\frac{\pi}{2}) \to X$ be an asymptote to γ through $\alpha(t_0)$. Then the asymptote to γ through $\alpha(s)$ is also given by α.*

Proof. Let $\tilde{\alpha}$ be the asymptotic line through $\alpha(s)$. Then the previous result shows that both γ and α as well as γ and $\tilde{\alpha}$ are parallel and they meet at $\alpha(t_0) = \tilde{\alpha}(t_0)$. Thus, as parallel lines are unique (see Lem. 4.15), we have that $\alpha = \tilde{\alpha}$. $\square$

Corollary 4.18 (Asymptotes stay in $I(\gamma)$)**.** *Asymptotic lines to γ from points in $I(\gamma)$ stay in $I(\gamma)$.*

Proof. Since asymptotic lines to γ are parallel to γ by Lem. 4.16, this readily follows. $\square$

We can piece parallelity and asymptotes together using the following strengthening of Lem. 4.16.

Lemma 4.19 (Two asymptotes are parallel)**.** *Let $x, y \in I(\gamma)$ and α, β the asymptotes to γ through x resp. y. Then α, β are* AdS*-parallel.*

Proof. We dive into the construction of asymptotes: We first assume $x \ll y$. We set $s_0 = b^+(x)$ and $t_0 = b^+(y)$, then there are maximisers α_n from x to $\gamma(T_n)$ parametrised in τ-arclength such that $\alpha_n(s_0) = x$ and β_n from y to $\gamma(T_n)$ parametrised in τ-arclength such that $\beta_n(t_0) = y$, which converge (pointwise) to the upper parts of α and β in τ-arclength parametrization for $T_n \to \infty$, respectively. Note that $\alpha_n(s_0+\tau(x,\gamma(T_n))) = \beta_n(t_0+\tau(y,\gamma(T_n))) = \gamma(T_n)$.

We try to prove the c-criterion (Lem. 4.13): Clearly, due to our assumptions on X, we do not need to consider the null functions. First, we look at $c_{\alpha\beta}(s_0,t_0)$ and $c_{\alpha\beta}(s',t')$ or $c_{\beta\alpha}(s',t')$ for $s_0 \leq s'$, $t_0 \leq t'$: We require $a := x = \alpha(s_0) \ll b := y = \beta(t_0)$. We get points converging to the primed parameters: $a'_n := \alpha_n(s') \to a' := \alpha(s')$ and $b'_n := \beta_n(t') \to b' := \beta(t')$ (note that $\alpha_n(s_0) = a$ and $\beta_n(t_0) = b$ anyway). We get a timelike triangle $\Delta_n = \Delta(a, b, c_n := \gamma(T_n))$ containing the points (a'_n, b'_n). We form a comparison situation for this in AdS$'$: $\bar{\Delta}_n = \Delta(\bar{a}, \bar{b}, \bar{c}_n)$ with comparison points $\bar{a}'_n, \bar{b}'_n$. As X has global curvature bounded below by $K = -1$, we get that $\tau(a'_n, b'_n) \leq \bar{\tau}(\bar{a}'_n, \bar{b}'_n)$ and $\tau(b'_n, a'_n) \leq \bar{\tau}(\bar{b}'_n, \bar{a}'_n)$.

We would now like to let $T_n \to +\frac{\pi}{2}$, so we need control over what the comparison triangle $\bar{\Delta}_n$ converges to. For this, we select which way to realise $\bar{\Delta}_n$ in AdS$'$. We choose:

- $\bar{a} = (s_0, 0)$ constant in n, i.e. at the right t-coordinate and with x-coordinate 0,
- $\bar{b} = (t_0, c_{\alpha\beta}(s_0, t_0))$ constant in n. Note this has the right τ-distance to $\bar{a}$ by definition of $c_{\alpha\beta}$, and it has the right t-coordinate.
- $\bar{c}_n$ with $\bar{\tau}(\bar{a}, \bar{c}_n) = \tau(a, c_n)$ and $\bar{\tau}(\bar{b}, \bar{c}_n) = \tau(b, c_n)$.
  - Note that $\bar{c}_n$ need not be realizable in AdS$' \subseteq$ AdS, but in AdS it is.
  - But still, the initial segments of the sides $\bar{\alpha}_n$ connecting $\bar{a}\bar{c}_n$ and $\bar{\beta}_n$ connecting $\bar{b}\bar{c}_n$ are contained in AdS$'$.
  - Note that the future tip $\bar{c} = (1, 0, 0) \in$ AdS of AdS$'$ satisfies $\bar{\tau}(\bar{a}, \bar{c}) = \lim_n \tau(a, c_n)$ and $\bar{\tau}(\bar{b}, \bar{c}) = \lim_n \tau(b, c_n)$.
  - Assuming $c_{\alpha\beta}(s_0, t_0) \neq 0$, the function $g : \mathrm{AdS} \to \mathbb{R}^2$, $g(\bar{p}) = (\bar{\tau}(\bar{a}, \bar{p}), \bar{\tau}(\bar{b}, \bar{p}))$ is continuous near $\bar{c}$ and maps a neighbourhood of $\bar{c}$ to a neighbourhood of $(\frac{\pi}{2} - s_0 = \lim_n \tau(a, c_n), \frac{\pi}{2} - t_0 = \lim_n \tau(b, c_n))$. In particular, for n large we can assume $\bar{c}_n$ lie in this neighbourhood and $\bar{c}_n \to \bar{c}$.
  - Assuming $c_{\alpha\beta}(s_0, t_0) = 0$, we have that $\bar{b} = (t_0, 0)$, and as $\tau(a, c_n)$ stays bounded away from π, the triangle $\Delta(\bar{a}, \bar{b}, \bar{c}_n)$ converges (in AdS) to a triangle which satisfies the size bounds (thus realizable uniquely up to isometries in AdS, giving one or two realizations given $\bar{a}$ and $\bar{b}$), and $\bar{\tau}(\bar{a}, \bar{c}_n) \to \frac{\pi}{2} - s_0$ and $\bar{\tau}(\bar{b}, \bar{c}_n) \to \frac{\pi}{2} - t_0$ makes the limit triangle degenerate, thus $\bar{c}_n$ converge to $\bar{c}$ (as this point makes $\bar{a}, \bar{b}, \bar{c}$ degenerate and is at the right distance).
  - In particular, $\bar{\alpha}_n \to \bar{\alpha}$ and $\bar{\beta}_n \to \bar{\beta}$ in τ-arclength parametrization. Thus the initial segments of $\bar{\alpha}_n$ and $\bar{\beta}_n$ become vertical in the limit.

In the AdS$'$ picture, the limit comparison sides have the form $\bar{\alpha}(s) = (s, 0)$ and $\bar{\beta}(t) = (t, c_{\alpha\beta}(s_0, t_0))$. By continuity of τ and $\bar{\tau}$, our curvature assumption gives $\tau(a',b') \leq \bar{\tau}(\bar{a}',\bar{b}')$ in the limit (similarly with arguments flipped). Now we compare the definition of $c_{\alpha\beta}(s',t')$ resp. $c_{\beta\alpha}(s',t')$ with the c-functions for $\bar{\alpha}$ and $\bar{\beta}$ at the same parameters:

$$\begin{aligned}
c_{\alpha\beta}(s',t') &= \operatorname{arcosh}\left(\frac{\cos(\tau(a',b')) - \sin(s')\sin(t')}{\cos(s')\cos(t')}\right),\\
c_{\bar{\alpha}\bar{\beta}}(s',t') &= \operatorname{arcosh}\left(\frac{\cos(\bar{\tau}(\bar{a}',\bar{b}') - \sin(s')\sin(t')}{\cos(s')\cos(t')}\right) = c_{\alpha\beta}(s_0,t_0),
\end{aligned}$$

whenever defined. The last equality holds because we know $\bar{\alpha}$ and $\bar{\beta}$ are AdS-parallel with distance $c_{\alpha\beta}(s_0,t_0)$. Note that $a' \leq b'$ implies $\bar{a}' \leq \bar{b}'$ by the curvature bound and a simple continuity argument, so whenever the first line is defined so is the second. Notice these equations only differ in the τ term, and we know $\tau(a',b') \leq \bar{\tau}(\bar{a}',\bar{b}')$, so we get $c_{\alpha\beta}(s',t') \geq c_{\alpha\beta}(s_0,t_0)$ whenever the former is defined.

Similarly, if $b' \leq a'$ we get $c_{\beta\alpha}(s',t') \geq c_{\alpha\beta}(s_0,t_0)$ for all $s' \geq s_0$ and $t' \geq t_0$. We can also do this if $a \gg b$, giving $c_{\alpha\beta}(s',t') \geq c_{\beta\alpha}(s_0,t_0)$ and $c_{\beta\alpha}(s',t') \geq c_{\beta\alpha}(s_0,t_0)$.

Doing the same construction towards the past, we similarly get $c_{\alpha\beta}(s',t') \geq c_{\alpha\beta}(s_0,t_0)$ and $c_{\beta\alpha}(s',t') \geq c_{\alpha\beta}(s_0,t_0)$ resp. $c_{\alpha\beta}(s',t') \geq c_{\beta\alpha}(s_0,t_0)$ and $c_{\beta\alpha}(s',t') \geq c_{\beta\alpha}(s_0,t_0)$ (depending on whether $a \ll b$ or $b \ll a$) for all $s' \leq s_0$ and $t' \leq t_0$.

Now we use Lem. 4.17 to get that the AdS-asymptote to γ through $\alpha(s)$ is α, and similarly the asymptote to γ through $\beta(t)$ is β. In particular, we can use the above argument again for $\alpha(s)$ instead of $x = \alpha(s_0)$ and $\beta(t)$ instead of $y = \beta(t_0)$. The above (to the future, $a \ll b$) then gives: $c_{\alpha\beta}(s',t') \geq c_{\alpha\beta}(s,t)$ for $s \leq s'$, $t \leq t'$ such that $a \ll b$ and $a' \ll b'$, extending continuously, $c_{\alpha\beta}$ is monotonously increasing where defined. On the other hand, the above (to the past, $a \ll b$) gives: $c_{\alpha\beta}(s',t') \geq c_{\alpha\beta}(s,t)$ for $s \geq s'$, $t \geq t'$ such that $a \ll b$ and $a' \ll b'$, extending continuously, $c_{\alpha\beta}$ is monotonously decreasing where defined. Via a two-step process, we see that $c_{\alpha\beta}$ is constant where defined. Similarly, we get that also $c_{\beta\alpha}$ is constant where defined, and that they have the same value.

Thus, the c-criterion (Lem. 4.13) yields that α and β are synchronised parallel. □

Thus any two asymptotes to γ are parallel.

5 Proof of the main result

Let us summarise what we have shown so far: Let $(X, d, \ll, \leq, \tau)$ be a connected, regularly localisable, globally hyperbolic Lorentzian length space satisfying timelike geodesic prolongation, with proper metric d and global timelike curvature bounded below by $K = -1$ containing an AdS-line $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$. Then from each point in $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$ we can construct a (unique) future asymptotic ray to γ, being timelike and having b_+ on them go to $\frac{\pi}{2}$, and a (unique) past asymptotic ray to γ, being timelike and having b_- on them go to $\frac{\pi}{2}$. $b_+ = -b_-$ and future directed and past directed rays from a common point fit together to give a timelike line which is parallel to γ and can be parametrized in τ-arclength on $(-\frac{\pi}{2}, \frac{\pi}{2})$. The Busemann function selects a "spacelike" slice $\{b^+ = 0\}$ in X containing precisely one point on each of the asymptotes which will provide the metric part of the warped product.

In this section, we first prove that $I(\gamma)$ is a warped product and then that $X = I(\gamma)$, establishing that X is a warped product.

Definition 5.1 (Spacelike slice). We call the set $S = (b_+)^{-1}(0)$ the *spacelike slice*. For $p, q \in S$ we find the asymptotes α, β to γ through p resp. q, so $\alpha(0) = p$ and $\beta(0) = q$, then they are AdS-parallel. We define $d_S(p, q)$ to be the spacelike distance between α and β in the sense of AdS-parallel lines, i.e. the constant c from the c-criterion.

Lemma 5.1. *(S, d_S) is a metric space.*

Proof. As asymptotes to γ are parallel to γ and parallel lines are unique, d_S is well-defined.

Let $p, q \in S$. It is obvious from the definition of the distance of two parallel lines that $d_S(p, q) \geq 0$. If $d_S(p, q) = 0$, the last step in Lem. 4.14 we get that the asymptotes through p and q are the same curve α, so we get that $p = \alpha(0) = q$.

For the triangle inequality, let $p, q, r \in S$. We get asymptotes α through p, β through q and γ through r, and by Lem. 4.19, they are pairwise AdSn-parallel. Let $d_1 = d_S(p, q)$ and $d_2 = d_S(q, r)$ and $d_3 = d_S(p, r)$. Then we can make a situation in AdS$'$ which at the same time is a parallel realization for α, β and for β, γ: $\bar\alpha(t) = (t, 0)$, $\bar\beta(t) = (t, d_1)$, $\bar\gamma(t) = (t, d_1 + d_2)$.

Now for any r, t such that $\bar\alpha(r) \leq \bar\gamma(t)$, the distance realizer connecting them crosses $\bar\beta$, so we find an s such that $\bar\alpha(r) \leq \bar\beta(s) \leq \bar\gamma(t)$. But this means $\alpha(r) \leq \beta(s) \leq \gamma(t)$, so they are also causally related in X. Let $t' = \inf\{t' : \alpha(s) \leq \gamma(t')\}$, then $t' \leq t$. Now note that the equation $c^N_{\alpha+}(s) = \operatorname{arcosh}\left(\frac{1-\sin(s)\sin(t)}{\cos(s)\cos(t)}\right)$ is increasing in t if $t > s$, so we know

$$d_3 = \operatorname{arcosh}\left(\frac{1 - \sin(s)\sin(t')}{\cos(s)\cos(t')}\right) \leq \operatorname{arcosh}\left(\frac{1 - \sin(s)\sin(t)}{\cos(s)\cos(t)}\right) = d_1 + d_2$$

as desired, where the last follows as $\bar\alpha$ and $\bar\gamma$ have spacelike distance $d_1 + d_2$. □

Definition 5.2 (The warped product map). We define $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S \to I(\gamma) \subset X$ by $f(s,p) = \alpha_p(s)$, where α_p is the AdS-asymptote to γ through p.

Proposition 5.2 (Local splitting). *Let X be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric d and global timelike curvature bounded below by $K = -1$ satisfying timelike geodesic prolongation and containing an* AdS*-line $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$. Then $I(\gamma) \subset X$ is a causally convex open set that is itself a path-connected, regularly localisable, globally hyperbolic Lorentzian length space of global timelike curvature bounded below by $K = -1$ with the metric, relations and time separation induced from X. Moreover, the spacelike slice S is a proper (hence complete), strictly intrinsic metric space, the Lorentzian warped product $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ (a anti-deSitter suspension) is a path-connected, regularly localisable, globally hyperbolic Lorentzian length space and the splitting map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S \to I(\gamma)$ is a τ- and $\leq$-preserving homeomorphism.*

Proof. First, it is clear that $I(\gamma)$ is path-connected, causally convex in X and has global timelike curvature bounded below by $K = -1$. It is hence causally path-connected since X is and it is trivially locally causally closed. Moreover, if $x \in I(\gamma)$ and U is a regular localising neighbourhood of x in X, then $U \cap I(\gamma)$ is a regular localising neighbourhood of x in $I(\gamma)$, hence $I(\gamma)$ is regularly localisable. By causal convexity, the time separation between causally related points in $I(\gamma)$ is achieved as the supremum of lengths of causal curves running between them which have to stay inside $I(\gamma)$. The causal diamonds in $I(\gamma)$ are precisely those in X, since they must be contained in $I(\gamma)$. Finally, $I(\gamma)$ is non-totally imprisoning (it inherits this from X), thus we have shown all the claims on $I(\gamma)$.

Next, we argue that f is τ- and $\leq$-preserving. For now, denote the causal relation and time separation function in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ by $\bar{\leq}$ and $\bar{\tau}$. Let $(s_0, p), (t_0, q) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$, we need to check $(s_0, p) \bar{\leq} (t_0, q) \Leftrightarrow f(s_0, p) \leq f(t_0, q)$ and $\bar{\tau}((s_0, p), (t_0, q)) = \tau(f(s_0, p), f(t_0, q))$. Let α be the AdS-asymptote to γ through p and let β be the AdS-asymptote to γ through q. Then α and β are AdS-parallel with distance $d_S(p, q)$, so there is a AdS-parallel realisation $\tilde{f} : \alpha \cup \beta \to \mathrm{AdS}'$ defined by $\tilde{f}(\alpha(s)) = (s, 0)$ and $\tilde{f}(\beta(t)) = (t, d_S(p, q))$. In particular, $\alpha(s) \leq_X \beta(t) \Leftrightarrow \tilde{f}(\alpha(s)) \leq \tilde{f}(\beta(t))$ and if this is true, $\tau(\alpha(s), \beta(t)) = \bar{\tau}(\tilde{f}(\alpha(s)), \tilde{f}(\beta(t)))$.

But the definition of $\leq$ and τ in the Lorentzian warped product just asks us to realize the points by $(s, 0)$ and $(t, d_S(p, q))$, so the τ-distance and causal relation are the same as in the parallel realization, and thus the same as in X, so f is τ- and $\leq$-preserving. f is injective as AdS-asymptotes to γ

are AdS-parallel to γ and AdS-parallel lines don't meet (Lemma 4.15), and surjective since any point in $I(\gamma)$ lies on an AdS-asymptote.

From the discussion above, it is easy to see that f maps timelike (and causal) diamonds in $I(\gamma)$ to timelike (and causal) diamonds in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$. As both sides are strongly causal, this implies that f is a continuous open bijection, hence a homeomorphism. Since $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ is always non-totally imprisoning (cf. Prop. 3.2.1) and its causal diamonds are compact (as continuous images of compact causal diamonds in $I(\gamma)$), we conclude that $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ is globally hyperbolic, hence S is proper by Prop. 3.2.7. To see that S is a strictly intrinsic space, fix $p, q \in S$ and connect any two timelike related points on the corresponding asymptotes by a distance realiser in $I(\gamma)$. The image of that distance realiser under f is a continuous distance realiser in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$, hence by Prop. 3.2.4 the projection onto S gives a distance minimiser in S between p and q.

Finally, we need to prove the remaining claimed properties of $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$. Path-connectedness is inherited from $I(\gamma)$ via f, and warped products are always globally causally closed (cf. Prop. 3.2.1). Note that $I(x, y)$ for $x, y \in I(\gamma)$ are regular localising neighbourhoods in $I(\gamma)$. Since $f(I(x, y)) = I(f(x), f(y))$ and the d-lengths of causal curves in timelike diamonds can always be uniformly bounded in warped products (cf. 3.2.3), timelike diamonds in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ are in fact (regular) localising neighbourhoods: For the local time separation, take the restriction of $\bar{\tau}$, and note that maximisers in $I(\gamma)$ (or in any $I(x, y) \subset I(\gamma)$) map to continuous maximisers in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$, which are always Lipschitz reparametrisable and are hence causal curves (cf. Prop. 3.2.3). $\square$

Proposition 5.3 (Globalize the splitting)**.** *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions in Prop. 5.2 and such that for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle. Then $X = I(\gamma)$.*

Proof. Using Thm. 5.2, we only need to prove $I(\gamma) = X$. We indirectly assume a point in $X \setminus I(\gamma)$: First, if there is a point $p \in I^+(\gamma) \setminus I^-(\gamma)$, we find a t such that $\gamma(t) \ll p$, and a timelike distance realizer α_t connecting them. By geodesic prolongation, we have a point $p \ll p_+$ and set $\varepsilon = 2\tau(p, p_+) > 0$. α_t is initially contained in $I(\gamma)$, so we find a point $\alpha_t(s_t) \in \partial I(\gamma)$, i.e. in $\partial I^-(\gamma)$.

By Lem. 3.10 we get that for all $\varepsilon > 0$ there is $\delta > 0$ small enough such that $\tau(\gamma(r), \alpha_t(s_t - \delta)) \to \pi - \varepsilon$ as $r \to -\frac{\pi}{2}$. In particular, $\tau(\gamma(r), p_+) > \pi$. By the synthetic Bonnet-Myers theorem 1.2, we know that this forces $\tau(\gamma(r), p_+) = +\infty$, contradicting finiteness of τ (see [23, Thm. 3.28]).

The situation where there is a point $p \in I^-(\gamma) \setminus I^+(\gamma)$ is analogous.

Finally, assume that all points in $X \setminus I(\gamma)$ are in neither $I^+(\gamma)$ nor in $I^-(\gamma)$. Take any such point p, then by strong causality, we have $p_- \ll p \ll p_+$. Then also p_-, p_+ are neither in $I^+(\gamma)$ nor in $I^-(\gamma)$. Thus, $I(p_-, p_+) \subseteq X \setminus I(\gamma)$ is an open neighbourhood of p. This works for any p, so both $I(\gamma)$ and $X \setminus I(\gamma)$ are open, contradicting connectedness of X. $\square$

Theorem 5.4. *Let $(X, d, \ll, \leq, \tau)$ be a connected, regularly localisable, globally hyperbolic Lorentzian length space with proper metric d and global timelike curvature bounded below by $K = -1$ satisfying timelike geodesic prolongation and containing a τ-arclength parametrized distance realizer $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to X$. Assume that for each pair of points $x \ll z$ in X we find $y \in X$ such that $\Delta(x, y, z)$ is a non-degenerate timelike triangle.*

Then there is a proper (hence complete), strictly intrinsic metric space S such that the Lorentzian warped product $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$ is a path-connected, regularly localisable, globally hyperbolic Lorentzian length space and there is a map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S \to I(\gamma)$ which is a τ- and $\leq$-preserving homeomorphism.

Proof. With Prop. 5.3, this follows from Prop. 5.2 immediately. $\square$

Recall the notion of Cauchy sets and (Cauchy) time functions on Lorentzian pre-length spaces from [16, Sec. 5.1]: A Cauchy set is any subset that is met exactly once by doubly inextendible causal curves, and a Cauchy time function is a continuous function $t : X \to \mathbb{R}$ such that $x < y$ implies $t(x) < t(y)$ and the image under t of any doubly inextendible causal curve is all of $\mathbb{R}$.

Corollary 5.5. *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions of Theorem 5.4 and let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S \to X$ be the splitting. Then the sets $S_t := f(\{t\} \times S)$ are Cauchy sets in X that are all homeomorphic to S. Moreover, let $\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be a monotonously increasing bijection, then the map $\varphi \circ pr_1 \circ f^{-1}$ is a Cauchy time function. Moreover, all Cauchy sets in X are homeomorphic to S.*

Proof. As $\{t\} \times S$ is acausal in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$, no causal curve can meet it twice. Next we argue that any doubly inextendible causal curve α meets S_t: Suppose that α does not meet $\{t\} \times S$, so w.l.o.g. we may assume that $\alpha \subset I^+(\{t\} \times S)$ (as $X = I^-(\{t\} \times S) \cup \{t\} \times S \cup I^+(\{t\} \times S)$ due to the splitting, and this union is disjoint). Let $t_0 \in (a, b)$, then $\alpha((a, t_0]) \subset J^-(\alpha(t_0)) \cap J^+(\{t\} \times S)$ which is easily seen to be a compact set by considering the corresponding situation in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} S$. But this is a contradiction, since X is non-totally imprisoning. Hence $\{t\} \times S$ is a Cauchy set. They are obviously homeomorphic to S.

Now for $pr_1 \circ f^{-1}$: It is clearly a time function. Now let $\alpha : (a, b) \to X$ be a doubly inextendible causal curve, we need to show that $pr_1 \circ f^{-1} \circ \alpha((a, b)) = (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose not, so there is some time value t_0 that is not attained. W.l.o.g. suppose $t_0 \geq 0$, so the image is contained in $(-\frac{\pi}{2}, t_0]$. Similarly to before, this would imply that $\alpha|_{[t_0,b)}$ is contained in the compact set $J^+(\alpha(t_0)) \cap J^-(S_{t_0})$, a contradiction.

Finally, let C be any Cauchy set in X. Then the projection $C \to S$ (via the splitting) is continuous. Its inverse is given by sending each $p \in S$ to the unique point on α_p meeting C. This is a continuous map since C is achronal. This shows that C is homeomorphic to S. □

Note that in general, Cauchy sets in globally hyperbolic Lorentzian pre-length spaces need not be homeomorphic, as [16, Ex. 5.7, Ex. 5.8] show.

Corollary 5.6. *Let $(X, d, \ll, \leq, \tau)$ satisfy the assumptions of Thm. 5.4. Then (S, d_S) has Alexandrov curvature bounded below by -1.*

Proof. Note that $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos} \mathbb{M}^2(-1)$ is a part of $2 + 1$-dimensional anti-deSitter space (compare to Lem. 3.5) and thus has timelike curvature bounded above and below by $K = -1$, so we can use [2, Thm. 5.7] to get the desired curvature bound on S. □

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