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The Gaussian free field and its Liouville measure

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Abstract

This master thesis gives an elementary and detailed construction of the Gaussian free field (GFF) and its corresponding Liouville measure (LM). For simplicity we restrict to bounded domains in $\mathbb{R}^d$, which have a continuous differentiable boundary. We start by building the GFF as the generalisation of the Brownian bridge to higher dimensional index sets. This is not trivial, as we can´t define the GFF pointwise in higher dimensions. We elaborate important properties of the GFF, like its Markov property and the invariance under conformal maps in dimension two. In the second part we show the existence of the LM, which is roughly speaking the exponential of the GFF. We develop the connection to the so called thick points of the GFF, and finish up with its covariance under conformal maps. This thesis should make the GFF and its LM readily accessible to mathematicians with a focus on probability theory.

Kurzfassung

Diese Masterarbeit gibt eine elementare und detaillierte Konstruktion des Gaußschen freien Feldes (GFF) und des zugehörigen Liouville Maßes (LM). Der Einfachheit halber arbeiten wir auf beschränkten Definitionsmengen in $\mathbb{R}^d$ mit stetig differenzierbarem Rand. Wir beginnen damit, das GFF als die Verallgemeinerung einer Brownschen Brücke in höher dimensionale Indexmengen zu sehen. Dies ist nicht trivial, da das GFF in höheren Dimensionen nicht punktweise definiert werden kann. Wir entwickeln anschließend Eigenschaften des GFF, unter anderem seine Markov Eigenschaft, oder die Invarianz unter konformen Abbildungen in Dimension zwei. Im zweiten Teil geben wir einen Beweis für die Existenz des LM, was grob gesagt die Exponentialfunktion des GFF ist. Wir zeigen die Verbindung zu den sogenannten dicken Punkten des GFF, und enden mit der Kovarianz des LM unter konformen Abbildungen. Diese Arbeit soll das GFF und sein LM für Mathematikerinnen mit einem Fokus auf Wahrscheinlichkeitstheorie leicht zugänglich machen.

Contents

1 The Gaussian Free Field

The Gaussian Free Field (GFF) is the most general stochastic process on higher dimensions. It is the scaling limit of many things, like the the domino tiling model (dimer model) as one can see in chapter 4.6 of [Ken03] and of course of the discrete GFF (see [WP20] chapter 1). It is important in many mathematical physics applications, for example in the construction of the Liouville measure. Our aim in the first section, is to construct the GFF as the generalisation of the Brownian Bridge by interpreting it as a Gaussian random vector. See for example [She07] for this approach.

1.1 Preliminaries

In this thesis, we will usually restrict ourselves to the case of open, bounded and simply connected domains with C^1-boundary (one time differential boundary) if not stated differently. For a domain $D \subset \mathbb{R}^d$ we use the notations $C_0(D)$ for continuous functions with compact support, $C^\infty(D)$ for the space of smooth functions, and $L^p(D)$ for the Lebesgue spaces on D. The expression $\langle u, v\rangle$ and $\|u\|$ mean mostly $\int_D u(x)v(x)\,dx$ and $\left(\int_D u(x)^2\,dx\right)^{\frac{1}{2}}$, but will be used for more general settings as well later on.

To get a natural intuition for the upcoming construction of the GFF, we use this section to repeat some basic definitions and properties.

1.1.1 Gaussian random vector

Definition 1 (Standard Gaussian vector on finite dimensional spaces)**.** Let V be a d-dimensional vector space with inner product $\langle , \rangle_V$. Define the d-dimensional standard distribution μ_V with $\mu_V(dw) = \frac{1}{Z}e^{-\frac{\langle w,w\rangle_V}{2}}$. The constant Z is a normalizing constant, such that μ_V is a probability measure. A random vector v having the law μ_V is called a standard Gaussian random vector.

Without proof, we recap some properties of Gaussian vectors.

Proposition 1.1.1. *Let V be a d-dimensional vector space with inner product $\langle , \rangle_V$ and v a random vector on V. The following is equivalent*

(i) v is a standard Gaussian random vector.

(ii) Let $\{e_j\}_{j=1,\dots,d}$ be an orthonormal basis of V and $\{\mathcal{N}_j\}_{j=1,\dots,d}$ independent standard Gaussian random variables. Then v has the same law as $\sum_{j=1}^d \mathcal{N}_j e_j$.

(iii) For $w \in V$ the random variable $\langle v, w\rangle_V$ is $\mathcal{N}(0, \langle w, w\rangle_V)$ distributed.

(iv) The characteristic function of v is

$$\mathbb{E}(e^{i\langle v,w\rangle_V}) = e^{-\frac{\langle w,w\rangle_V}{2}}$$

where $w \in V$.

Remark 1. A natural approach to generalise a Gaussian vector on infinite dimensional spaces, is to construct it like in proposition 1.1.1 *(ii)* by $v = \sum_{j=1}^{\infty} \mathcal{N}_j e_j$ where $\{e_j\}_{j=1}^{\infty}$ is now an infinite orthonormal basis of V. We can see that for v property *(iii)* of proposition 1.1.1 still holds by looking at the martingale $(M_N)_{N=1}^{\infty} := \left(\langle \sum_{j=1}^{\infty} \mathcal{N}_j e_j, w\rangle_V\right)_{N=1}^{\infty}$. The submartingale M_N^2 is bounded by

$$\mathbb{E}(\langle \sum_{j}^{N} \mathcal{N}_j e_j, w\rangle^2) = \sum_{j=1}^{N} \langle e_j, w\rangle_V^2 \leq \|w\|_V^2$$

and therefore the variance of M_N converges by the martingale convergence theorem and hence M_N converges in $L^2(\mathbb{P})$. The limit of M_N is $\mathcal{N}(0, \|w\|_V^2)$ distributed, as $\lim_{N\to\infty} \mathbb{E}(M_N^2) = \|w\|_V^2$. From the fact

$$\mathbb{E}(e^{iX}) = e^{-\frac{\sigma^2}{2}}$$

if and only if $X \sim \mathcal{N}(0, \sigma^2)$ it follows directly, that property *(iv)* is satisfied as well.

This heavily recommends to define infinite standard Gaussian vectors as the random series of a basis. But unluckily

$$\|\sum_{j=1}^{N} \mathcal{N}_j e_j\|_V^2 = \sum_{j=1^N} \mathcal{N}_j^2 \|e_j\|_V^2 = \sum_{j=1}^{N} \mathcal{N}_j^2$$

and therefore the variance of $\|\sum_{j=1}^{N} \mathcal{N}_j e_j\|_V$ tends to infinity for N to infinity. Consequently v is not in V almost surely. Even if v is not in V, Gross showed that one can construct a bigger space around V, such that v is contained in this bigger space. See [Gro67] for the general setting. The following leads to an example of such a construction.

1.1.2 Brownian bridge

Definition 2 (brownian bridge)**.** Let $(W_t)_{t\geq 0}$ be a Brownian motion (Wiener process). For $T > 0$ the stochastic process $(B_t)_{t\in[0,T]}$ defined by

$$B_t^T = W_t - \frac{t}{T} W_T$$

is called the Brownian bridge on the interval $[0, T]$. Further we call $B_t := B_t^1$ the standard Brownian bridge.

The Brownian bridge is a Brownian motion which returns to zero at the end of the interval. We want to study the Brownian motion from the point of view of an infinite Gaussian vector. For that we will make use of the following theorem. A proof can be found for example in [Ale15].

Theorem 1.1.2 (Karhunen-Loeve expansion for Gaussian processes)**.** *Let $(X_t)_{t\in[a,b]}$ be a Gaussian process and $k_X(s,t) = Cov(X_s, X_t)$ the corresponding covariance function. Define the Hilbert Schmidt operator $K_X : L^2([a,b]) \to L^2([a,b])$ as*

$$K_X(u)(t) = \int_a^b k_X(s,t)u(s)ds$$

for $u \in L^2([a,b])$. Then X_t can be presented by the convergent random series

$$X_t = \sum_{j=1}^{\infty} \mathcal{N}_j \sqrt{\lambda_j} e_j$$

where $\{e_j\}_{j=1}^{\infty}$ are the eigenfunctions of K_X, $\{\lambda_j\}_{j=1}^{\infty}$ the (positive) corresponding eigenvalues, and $\{\mathcal{N}_j\}_{j=1}^{\infty}$ are independent standard Gaussian random variables.

With this statement we obtain the the Karhunen-Loeve expansion of the standard Brownian bridge.

Corollary 1.1.3. *The standard Brownian bridge $(B_t)_{t\in[0,1]}$ can be written as*

$$B_t = \sum_{j=1}^{\infty} \mathcal{N}_j \frac{1}{j\pi} \sqrt{2} \sin(jt\pi)$$

Proof. Plugging in the definition of the Brownian bridge, and using $Cov(W_s, W_t) = min(s,t)$ gives us the covariance function $k_B(s,t) = s(1-t)$ for $s \leq t$. An eigenfunction e of K_B have to fulfill

$$\lambda e(t) = K_B(e)(t) = (1-t)\int_0^t se(s)ds + t\int_t^1 (1-s)e(s)ds \tag{1.1.1}$$

for a $\lambda \in \mathbb{R}$. If we differentiate the equation two times, we get

$$\lambda e''(t) = -e(t).$$

The general solution to this differential equation is

$$e(t) = A\sin(\frac{t}{\sqrt{\lambda}}) + B\cos(\frac{t}{\sqrt{\lambda}}).$$

with $A, B \in \mathbb{R}$. If we set $t = 0$ in (1.1.1), we see that $e(0) = 0$ and hence $B = 0$. Doing the same with $t = 1$ we conclude

$$A\sin(\frac{t}{\sqrt{\lambda}}) = 0$$

which leads to $\lambda = (j\pi)^{-2}$, where $j = 1, \ldots$. Hence after normalising, our eigenfunctions are $e_j = \sqrt{2}\sin(tj\pi)$ with the corresponding eigenvalues $\lambda_j = (j\pi)^{-2}$. Using Theorem 1.1.2, we get the desired result. □

Remark 2. Due to remark 1 and corollary 1.1.3, the standard Brownian bridge can be interpreted as the standard Gaussian vector on the vectors

$$\{e_j\}_{j=1}^{\infty} = \{\frac{\sqrt{2}}{j\pi}\sin(tj\pi)\}_{j=1}^{\infty}. \tag{1.1.2}$$

These vectors form an orthonormal basis if we choose the inner product

$$(u, v) \mapsto \langle v', u' \rangle$$

for $v, u \in L^2(D)$ such that v', u' are still in $L^2(D)$.

1.1.3 The Brownian bridge as an infinite Gaussian vector

The right space to construct the Brownian bridge on is the following.

Definition 3 (Dirichlet inner product)**.** Let D be a domain in $\mathbb{R}^d$. For $u, v \in C_0^{\infty}$ we define the Dirichlet inner product by

$$\langle u, v \rangle_{\nabla} := \int_D \nabla u \cdot \nabla v$$

and the corresponding Dirichlet norm as

$$\|u\|_{\nabla} := \left(\int_D |\nabla u|^2 \right)^{\frac{1}{2}}.$$

We denote $H_0^1(D)$ as the completion of $C_0^{\infty}(D)$ with respect to the Dirichlet norm.

Remark 3. Because the boundary values are zero, the Dirichlet inner product and the corresponding norm are positive definite. Dropping this assumption, every constant function would have norm zero.

For now, we define the Laplacian operator as $\Delta = \sum_{j=1}^{d} \frac{\partial}{\partial x_j}^2$ without worrying too much if the derivatives are strong or weak. We will recap the following important theorems.

Theorem 1.1.4. *Let D be an open bounded subset of $\mathbb{R}^d$ with C^1 boundary. Then the following holds:*

(i) (Divergence theorem). Let X be a continuous differentiable vector field on $\overline{D}$. Then

$$\int_D divX dx = \int_{\partial D} X \cdot \nu \; dS$$

where ν is the outwards pointing normal vector.

(ii) (Gauss-Green formula) For $u, v \in C^2(D) \cup C(\bar{D})$ the formula

$$\int_D \nabla u \cdot \nabla v \; dx = -\int_D u \Delta v \; dx + \int_{\partial D} u \nabla v \cdot \nu \; dS$$

holds.

Proof. (i) We refer to classic literature like [R+76] p.273-275.

(ii) Set $X = u\nabla v$. Applying (i) on X we get

$$\int_D \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(u \frac{\partial}{\partial x_j} v) = \int_{\partial D} u \nabla v \cdot \nu \; dS$$

which gives the statement after using the product rule on the left hand side.

□

Remark 4. Let D be an unbounded open set and u or v having compact support. Then we can reduce D to a bounded open set, which contains the compact support, and consequently both formulas in theorem 1.1.4 are still valid under this assumption. Due to the compact support, the function and its gradient is zero on the boundary. Hence the boundary term vanishes, and the Gauss-Green formula reduces in that case to

$$\langle \Delta u, v \rangle = -\langle u, v \rangle_\nabla = \langle u, \Delta v \rangle. \tag{1.1.3}$$

Let $\{e_j\}_{j=1}^\infty$ be defined as in 1.1.2. Using Gauss-Green, we see that

$$\langle e_j, e_k \rangle_\nabla = -\langle e_j, \Delta e_k \rangle = \langle e_j, (k\pi)^2 e_k \rangle = \delta_{jk}$$

as $\{\sqrt{2}\sin(j\pi t)\}_{j=1}^\infty$ is a well known orthonormal basis of $C_0^\infty(D)$ equipped with the $L^2(D)$ norm. Therefore it is easy to believe, that $\{e_j\}_{j=1}^\infty$ forms an orthonormal basis in $H_0^1(D)$. Together with remark 2, we can interpret the Brownian bridge as the standard Gaussian vector on the space $H_0^1(D)$. Note here, that the Brownian bridge is not differentiable, and therefore not an element of $H_0^1((0,1))$ by itself. But it is still an element of $L^2((0,1))$.

The natural idea is, to generalise the Brownian bridge to other (higher dimensional) domains D, by building the standard Gaussian vector on the space $H_0^1(D)$. It turns out that this won't be an element of $L^2(D)$ anymore, because this construction (which will be the Gaussian free field), is too rough to be a function. Therefore we have to work in more general spaces.

Before starting this, we want to lose some vague words on the "density function" of the GFF.

Remark 5. The Brownian motion can be seen (very informally) as a random function f in

$$\{g \in C^1((0,T)) : \lim_{x\to 0} g(x) = 0\}$$

with density function $\frac{1}{Z}e^{-\frac{\|f\|_\nabla^2}{2}}$.This analogy holds for the Brownian bridge, where f has to fulfill additionally $\lim_{x\to T} f(x) = 0$. For the GFF we want the same behaviour, and we can think about it as a random function f in

$$\{g \in C^1(D) : \lim_{x\to y} g(x) = 0 \quad \text{if } y \in \partial D\}$$

with density $\frac{1}{Z}e^{-\frac{\|f\|_\nabla^2}{2}}$. This density distribution states, that the BB and GFF is getting more unlikely if the Dirichlet inner product increases, meaning both try to be harmonic. Of course this is very informal, as neither the BM/BB or the GFF are elements in C^1 and the constant Z is not finite. However, this makes perfect sense, if we restrict to the discrete case. That means we look on just finitely many evaluations of the Brownian bridge or the Gaussian Free Field. And consequently on a finite dimensional Gaussian random vector with some sort of discrete Dirichlet inner product on the underlying vector space. In the higher dimensional case this discretisation could be achieved by taking averages of the GFF over little cubes of D. For readers which are more interested in the discrete GFF can take a look at [WP20].

1.2 Sobolev spaces $H^m(D)$

This section follows [AF03] and [GTGT77]. We will introduce distributions and the Sobolev spaces $W^{m,p}$ for $p = 2$ and $m \in \mathbb{Z}$, which are often denoted by H^m. Readers who are interested in the general case, are referred to [AF03]. The goal of this section is, to construct the inner product for Sobolev spaces on a bounded domain, in terms of the eigenfunctions of the Laplacian.

1.2.1 Weak derivatives and distributions

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $|\alpha| = \sum_{j=1}^d \alpha_j$. For a $|\alpha|$- times differentiable function f we set

$$\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f.$$

Definition 4 (test functions and distributions)**.** Let D be a domain in $\mathbb{R}^d$. We say that a sequence of $(\varphi_j)_{j=1}^\infty$ in $C_0^\infty(D)$ converges to a $\varphi \in C_0^\infty(D)$, if $\bigcup_{j=1}^\infty supp(\varphi_j) \cup supp(\varphi)$ is compact and $\partial^\alpha \varphi_j$ converges uniformly to $\partial^\alpha \varphi$ for all $\alpha \in \mathbb{N}^d$. We call $C_0^\infty(D)$ endowed with this topology as the space of test functions, and denote it by $\mathcal{D}(D)$. The corresponding dual space $\mathcal{D}'(D)$ is called the space of distributions on D.

Remark 6. (i) The space $\mathcal{D}'(D)$ is often called the space of Schwartz distributions. This definition of distribution is not connected with distributions in the sense of probability theory.

(ii) Note that the topology of $\mathcal{D}(D)$ is not related to a metric. Actually $C_0^\infty(D)$ is an example for a space which can have a topology, but no metric on it.

A measurable function u is called locally integrable, if $u \in L^1(K)$ for all K which are relatively compact in D. For that we write $u \in L^1_{\text{loc}}(D)$. By

$$T_u(\varphi) := \int_D u(x)\varphi(x)dx.$$

we can identify every $u \in L^1_{\text{loc}}(D)$ with a distribution T_u. That the inverse does not hold can be seen for example by looking at the Dirac distribution $\varphi \mapsto \varphi(0)$.

Definition 5 (Distributional derivative)**.** Take $T \in \mathcal{D}'(D)$ and $\alpha \in \mathbb{N}^d$. We define the distribution $\partial^\alpha T$ as

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(\partial^\alpha \varphi).$$

Let $u \in L^1_{\text{loc}}(D)$. If there exists a $v \in L^1_{\text{loc}}(D)$ such that $\partial^\alpha T_u = T_v$, we call v the weak or distributional α-partial derivative of u.

One can easily show that $\partial^\alpha T$ is continuous on $\mathcal{D}(D)$, and therefore it is indeed a distribution. Clearly $\partial^\alpha T_u = T_v$ means

$$(-1)^\alpha \int_D u(x) \partial^\alpha \varphi(x) dx = \int_D v(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{D}(D)$. By partial integration we see that the derivative (in the classical sense) is the same as the weak derivative, as soon as u is smooth enough.

1.2.2 Definition and properties of Sobolev spaces

Definition 6 (Sobolev space $H^m(D)$)**.** Let D be a (probably unbounded) domain in $\mathbb{R}^d$ and $m \in \mathbb{N}$. We define the Sobolev space $H^m(D)$ by

$$H^m(D) := \{u \in L^2(D) : \partial^\alpha u \in L^2(D) \text{ for all } |\alpha| \leq m\}.$$

For $u \in H^m(D)$ the corresponding Sobolev norm is defined by

$$\|u\|_{H^m(D)} = (\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2)^{\frac{1}{2}}.$$

Further we denote the dual space of $H^m(D)$ by $H^{-m}(D)$.

Let's recall some easy properties of the Sobolev spaces.

- Sobolev spaces are complete by the Sobolev norm.
- For $u, v \in H^m(D)$ the bilinearform

  $$\langle u, v \rangle_{H^m(D)} = \sum_{|\alpha| \leq m} \int_D u(x) v(x) dx$$

  forms an inner product. Hence $H^m(D)$ is a Hilbert space.
- $H^m(D) \subset H^n(D)$ for $m \geq n$.
- $H^0(D) = L^2(D)$.

We recap a well know statement.

Proposition 1.2.1. *The space $C^\infty(D)$ is dense in $H^m(D)$.*

Proof. We refer to [AF03] p.65-67. □

This motivates the following definition

Definition 7. Let $m \in \mathbb{N}$. We define the Sobolev space with zero boundary $H_0^m(D)$ as the closure of $C_0^\infty(D)$ with respect to the Sobolev norm of $H^m(D)$. On this space we define the norm

$$\|u\|_{H_0^m(D)} = (\sum_{|\alpha|=m} \|\partial^\alpha u\|^2)^{\frac{1}{2}}$$

and set $H_0^{-m}(D)$ to be the dual space of $H_0^m(D)$.

Remark 7. The norm of $H_0^1(D)$ coincides with the Dirichlet norm. Consequently for $m = 1$ this definition is the same as definition 3.

For bounded domains, it is easy to show the equivalence between this norm and the Sobolev norm from definition 6 on $H_0^1(D)$ by the following statement.

Theorem 1.2.2 (Poincare inequality)**.** *Let D be a bounded domain. Then there exists a constant K, such that for all $\varphi \in C_0^\infty(D)$ the inequality*

$$\|\varphi\| \leq K\|\varphi\|_\nabla$$

holds.

Proof. Let $\varphi \in C_0^\infty(D)$. We can assume that $D \subset \mathbb{R}^{d-1} \times [0, c]$ for $c > 0$. The fundamental law of calculus states for $x = (x', x_d)$ where $x' \in \mathbb{R}^{d-1}$ and $x_d \in [0, c]$, that

$$\varphi(x) = \int_0^{x_d} \frac{d}{dt}\varphi(x', t)\ dt$$

as $\varphi(x', 0) = 0$. Applying Cauchy-Schwarz inequality yields

$$\int_0^{x_d} \left|\frac{d}{dt}\varphi(x', t)\right|\ dt \leq \sqrt{x_d}\sqrt{\int_0^{x_d} \left(\frac{d}{dt}\varphi(x', t)\right)^2\ dt}.$$

Putting this together, we see

$$\begin{aligned}
\int_D \varphi(x)^2\ dx &\leq \int_{\mathbb{R}^{d-1}} \int_0^c \left(\int_0^{x_d} \left|\frac{d}{dt}\varphi(x', t)\right|\ dt\right)^2\ dx_d dx' \\
&\leq \int_{\mathbb{R}^{d-1}} \int_0^c x_d \int_0^{x_d} \left(\frac{d}{dt}\varphi(x', t)\right)^2\ dt dx_d dx' \\
&\leq \int_0^c x_d \underbrace{\int_{\mathbb{R}^{d-1}} \int_0^c \left(\frac{d}{dt}\varphi(x', t)\right)^2\ dt dx'}_{\int_D \partial_d\varphi(x)^2 dx}\ dx_d = \frac{c^2}{2}\int_D \partial_d\varphi(x)^2 dx \leq \frac{c^2}{2}\|\varphi\|_\nabla^2
\end{aligned}$$

which gives the statement on $C_0^\infty(D)$, for $K = \frac{c}{\sqrt{2}}$. □

For $\varphi \in C_0^\infty(D)$
$$\|\varphi\|_\nabla \leq \|\varphi\|_{H^1(D)} \leq (K+1)\|\varphi\|_\nabla$$
and by induction, this equivalence extends to the norms $\|.\|_{H_0^m(D)}$ and $\|.\|_{H^m(D)}$ for $m \in \mathbb{N}$. Therefore $C_0^\infty(D)$ is dense in $H_0^m(D)$ with respect to both norms, and so the equivalence of norms extend to the whole of $H_0^m(D)$. From now on, we will just work with the spaces $H_0^m(D)$ equipped with $\|.\|_{H_0^m(D)}$, as soon as D is bounded.

1.2.3 Basis of $L^2(D)$ by eigenbasis of $-\Delta$

This section follows [GTGT77] Chapter 8. The aim of this subsection is to show, that $L^2(D)$ has a basis consisting of the eigenfunctions of the negative Laplace operator. Let D be a (probably unbounded) domain and $g \in L^2(D)$. We say a function u solves the generalised Poisson equation with Dirichlet boundary condition
$$\begin{aligned} -\Delta u &= g \\ u(x) &= 0 \quad \text{for } x \in \partial D \end{aligned}$$
if $u \in H_0^1(D)$ and for all $\varphi \in C_0^\infty(D)$
$$\int_D \nabla u(x).\nabla\varphi(x)dx = \int_D g(x)\varphi(x)dx \tag{1.2.1}$$
holds. Note that by the Gauss-Green formula this is equivalent to the usual Poisson equation, as soon as the function u is smooth enough. For the rest of the section, we will assume D to be bounded.

Theorem 1.2.3. *The generalised Poisson equation with Dirichlet boundary condition has an unique solution in $H_0^1(D)$.*

Proof. Let $g \in L^2(D)$. For $v \in H_0^1(D)$, we set $L_g(v) := \int_D g(x)v(x)dx$. By Cauchy-Schwarz and Poincare inequality we have
$$|L_g(v)| \leq \|g\|\|v\| \leq K\|g\|\|v\|_\nabla \tag{1.2.2}$$
and hence L_g is a linear functional on $H_0^1(D)$. By the Riez representation theorem there exists a unique $u \in H_0^1(D)$ which fulfills
$$\langle u, v\rangle_\nabla = L_g(v)$$
for all $v \in H_0^1(D)$. Clearly this holds uniquely as well for all $v \in C_0^\infty(D)$. This is equivalent to 1.2.1, and further u fulfills the boundary condition by being an element of $H_0^1(D)$. □

The element $u \in H_0^1(D)$ which represents L_g fulfills $\|u\|_{H_0^1(D)} = \|L_g\|_{(H_0^1(D))'}$. Consequently by 1.2.2 the map which assigns every $g \in L^2(D)$ to an element of $H_0^1(D)$ (denoted by $-\Delta^{-1}$) is a bounded linear operator. From the Rellich-Kondrachov theorem (look for example at [AF03] theorem 6.3) we know that the imbedding $\iota : H_0^1(D) \to L^2(D)$

is compact, and therefore $\iota \circ -\Delta^{-1} : L^2(D) \to L^2(D)$ is a compact linear operator. Further let $f, g \in C_0^\infty$ and take $\tilde{f} := \iota \circ -\Delta^{-1}(f)$ and $\tilde{g} := \iota \circ -\Delta^{-1}(g)$. We see by the Gauss-Green formular, that

$$\langle \iota \circ -\Delta^{-1}(f), g\rangle = \langle \tilde{f}, \Delta \tilde{g}\rangle = \langle \Delta \tilde{f}, \tilde{g}\rangle = \langle f, \iota \circ -\Delta^{-1}(g)\rangle$$

and therefore the operator $\iota \circ -\Delta^{-1}$ is self adjoint on $C_0^\infty(D)$. As $C_0^\infty(D)$ is dense in $L^2(D)$, $\iota \circ -\Delta^{-1}$ is self adjoint on $L^2(D)$. We can apply the following theorem.

Theorem 1.2.4 (Hilbert-Schmidt Theorem)**.** *Let H be a Hilbert space, and T be a linear, compact and self adjoint operator on H. Then there exist a countable set of eigenfunctions of T which form an orthonormal basis of $ran(T)$. The corresponding eigenvalues are real valued, and if the dimension of $ran(T)$ is infinite, they tend to zero. Further $H = \overline{ran(T)} \oplus ker(T)$.*

Proof. Look at [RR06] Theorem 7.94, p.267-268. □

Since by injectivity the kernel of $\iota \circ -\Delta^{-1}$ is zero, there is an orthonormal basis on $L^2(D)$ consisting of eigenfunctions of $\iota \circ -\Delta^{-1}$. This eigenfunctions are eigenfunctions of $-\Delta$ as well, and so we get our desired result.

Corollary 1.2.5. *The eigenfunctions of the negative Laplace operator form an orthonormal basis of $L^2(D)$.*

For the rest of the chapter, we denote $\{e_j\}_{j=1}^\infty$ as the eigenfunctions of $-\Delta$, and $\{\lambda_j\}_{j=1}^\infty$ as the corresponding eigenvalues. Note that the basis $\{e_j\}$ is the basis we use, when writing a Fourier series of a function. By the following theorem this basis is even smooth.

Theorem 1.2.6. *Let D be a (eventually unbounded) domain in $\mathbb{R}^d$. Let L be a linear differential operator of the form $L = \sum_{j,k=1}^d a_{ij}\partial_j\partial_k + b$ where a_{jk} and b are smooth functions. Further let L be a strictly elliptic operator, meaning there exists some $c > 0$, such that for all $\xi_1, \dots, \xi_d \in \mathbb{R}$*

$$\sum_{j,k=1}^d \xi_j a_{jk} \xi_k \geq c \sum_{j=1}^d \xi_j^2.$$

Then for every solution $u \in H^1(D)$ of the generalised equation $Lu = 0$, meaning u solves

$$\sum_{k,j=1}^d \langle a_{jk}\partial_j u, \partial_k \varphi\rangle = \langle bu, \varphi\rangle$$

for every $\varphi \in C_0^\infty(D)$, we get $u \in C^\infty(D)$.

Proof. For a proof we refer to [GTGT77] Corollary 8.11. □

Note that this statement is not just for $u \in H_0^1(D)$, as we did not specify the boundary conditions. We will need this more general condition later on. Looking at $L = \Delta + \lambda_j$ for an eigenvalue λ_j, we can apply theorem 1.2.6 to show that e_j is smooth.

1.2.4 Norm and inner product of $H_0^m(D)$

Definition 8. Let D be a bounded domain and $m \in \mathbb{N}$. For $u, v \in C_0^\infty(D)$ we define the bilinearform

$$\langle u, v\rangle_{\mathbf{H}_0^m(D)} = \sum_{j=1}^{\infty} \lambda_j^m \langle u, e_j\rangle\langle v, e_j\rangle$$

with the corresponding norm

$$\|u\|_{\mathbf{H}_0^m(D)} = (\sum_{j=1}^{\infty} \lambda_j^m \langle u, e_j\rangle^2)^{\frac{1}{2}}.$$

We define the space $\mathbf{H}_0^m(D)$ as the closure of $C_0^\infty(D)$ with respect to the norm $\|.\|_{\mathbf{H}_0^m(D)}$.

Remark 8. (i) It is well defined on $C_0^\infty(D)$, as

$$\|u\|_{\mathbf{H}_0^m(D)} = \sum_{j=1}^{\infty} \langle \lambda_j^m u, e_j\rangle\langle u, e_j\rangle = \sum_{j=1}^{\infty} \langle (-\Delta)^m u, e_j\rangle\langle u, e_j\rangle = \langle (-\Delta)^m u, u\rangle$$

is finite for $u \in C_0^\infty(D)$.

(ii) The space $\mathbf{H}_0^m(D)$ forms a Hilbert space.

Let $m \in \mathbb{N}$ and $u, v \in C_0^\infty$. We can write

$$\begin{aligned}
\langle u, v\rangle_{H_0^m(D)} &= \sum_{|\alpha|=m-1} \sum_{j=1}^{d} \langle \partial_j \partial^\alpha u, \partial_j \partial^\alpha v\rangle \\
&= \sum_{|\alpha|=m-1} \int_D \nabla\partial^\alpha u . \nabla \partial^\alpha v = \sum_{|\alpha|=m-1} \int_D \partial^\alpha (-\Delta) u \partial^\alpha v \\
&= \langle -\Delta u, v\rangle_{H_0^{m-1}(D)} = \dots \\
&= \langle (-\Delta)^m u, v\rangle
\end{aligned}$$

by applying Gauss-Green. As e_j is an eigenbasis of $L^2(D)$, we get again by Gauss-Green

$$\langle -\Delta^m u, v\rangle = \sum_{j=1}^{\infty} \langle (-\Delta)^m u, e_j\rangle\langle v, e_j\rangle = \sum_{j=1}^{\infty} \langle u, (-\Delta)^m e_j\rangle\langle v, e_j\rangle = \sum_{j=1}^{\infty} \lambda_j^m \langle u, e_j\rangle\langle v, e_j\rangle$$

which is the same as the inner product of $\mathbf{H}_0^m(D)$. Therefore $\langle , \rangle_{H_0^m(D)}$ and $\langle , \rangle_{\mathbf{H}_0^m(D)}$ coincide on $C_0^\infty(D)$. But $C_0^\infty(D)$ is dense in $H_0^m(D)$ and $\mathbf{H}_0^m(D)$, so the spaces coincide and the inner products are the same on the entirety of both spaces. Using $\lambda_j^m \langle u, e_j\rangle = \langle u, e_j\rangle_{H_0^m(D)}$, we have

$$\|u\|^2_{H_0^m(D)} = \sum_{j=1}^{\infty} \lambda_j^m \langle u, e_j\rangle^2 = \sum_{j=1}^{\infty} \langle u, \lambda_j^{-\frac{m}{2}} e_j\rangle^2_{H_0^m(D)}.$$

This means that $\left\{\lambda_j^{-\frac{m}{2}} e_j\right\}_{j=1}^{\infty}$ fulfills the Parseval identity for all $u \in H_0^m(D)$, and it is clearly an orthonormal system of $H_0^m(D)$. Hence it forms an orthonormal basis on $H_0^m(D)$.

Let L be in the dual space of $H_0^m(D)$, which we denoted as $H_0^{-m}(D)$. Then by the Riesz representation theorem there exists a unique $v \in H_0^m(D)$ such that

$$L(u) = \langle v, u \rangle_{H_0^m(D)} = \sum_{j=1}^{\infty} \lambda_j^m \langle v, e_j \rangle \langle u, e_j \rangle$$

for all $u \in H_0^m(D)$. We define the element $\tilde{v} = \sum_{j=1}^{\infty} \lambda_j^m \langle v, e_j \rangle e_j$ which is in $\mathbf{H}_0^{-m}(D)$. The map $L \mapsto \tilde{v}$ leads to an isomorphism between $H_0^{-m}(D)$ and $\mathbf{H}_0^{-m}(D)$ as

$$L(u) = \langle v, u \rangle_{H_0^m(D)} = \langle \tilde{v}, u \rangle$$

holds for all $u \in H_0^m(D)$. Note that the expression $\langle \tilde{v}, u \rangle$ still makes sense for $\tilde{v} \in \mathbf{H}_0^{-m}(D)$ and $u \in H_0^m(D)$. Consequently $H_0^{-m}(D)$ can be identified with $\mathbf{H}_0^{-m}(D)$. As $C_0^{\infty}(D)$ is a dense subspace of $H_0^m(D)$, every $\tilde{v} \in \mathbf{H}_0^{-m}(D) \cong H_0^{-m}(D)$ defines uniquely a distribution by $\langle T_{\tilde{v}}, \varphi \rangle = \langle \tilde{v}, \varphi \rangle$ for $\varphi \in \mathcal{D}(D)$. Meaning that even if not every element in $H_0^{-m}(D)$ can be represented by a $L^2(D)$ function, it can be represented by a distribution. For $n, m \in \mathbb{Z}$ with $n \leq m$ get

$$\mathcal{D}(D) \subset H_0^m(D) \subset H_0^n(D) \subset \mathcal{D}'(D).$$

1.3 Construction of the Gaussian Free Field

Having covered background on Sobolev spaces, we can continue with generalising the Brownian bridge for domains in higher dimensions. Like the standard Brownian bridge is a standard Gaussian vector on $H_0^1((0,1))$, we consider a standard Gaussian vector on $H_0^1(D)$ by

$$\mathbf{h} = \sum_{j=1}^{\infty} \mathcal{N}_j u_j \tag{1.3.1}$$

where $\{\mathcal{N}_j\}_{j=1}^{\infty}$ are iid standard Gaussian variables and $\{u_j\}_{j=1}^{\infty}$ is an arbitrary orthonormal basis on $H_0^1(D)$. As previously mentioned, the variance of the norm of $\mathbf{h}$ is infinite in $H_0^1(D)$. But we can show, that for D bounded, $\mathbf{h}$ is an element of the space $H_0^m(D)$ with $m \in \mathbb{Z}$ small enough. To show this, we need the following lemma.

Lemma 1.3.1 (Weyl´s law). *Let $D \subset \mathbb{R}^d$ be bounded. Then the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the negative Laplacian satisfy*

$$\frac{\lambda_j}{j^{\frac{2}{d}}} \to \frac{(2\pi)^2}{(\omega_d Vol(D))^{\frac{2}{d}}} \qquad (j \to \infty)$$

where ω_d is the volume of the unit ball in dimension d, and $Vol(D)$ is the volume of D with respect to do the Lebesgue measure.

Proof. For a proof in 2 and 3 dimensions look at [Str07] chapter 11.6. . In the general case we refer to [Cha84] page 155. □

Theorem 1.3.2 (**h** is a distribution)**.** *Let D be a bounded domain in $\mathbb{R}^d$ and $\boldsymbol{h}$ be defined as in 1.3.1. For $m < 1 - \frac{d}{2}$, the random series $\boldsymbol{h}$ is almost surely an element in $H_0^m(D)$.*

Proof. Take $m < 1 - \frac{d}{2}$. We want to show that $\|\mathbf{h}\|_{H_0^m(D)} < \infty$ almost surely. As this is an implication of $\mathbb{E}(\|\mathbf{h}\|^2_{H_0^m(D)})$ being finite, we look at

$$\mathbb{E}(\|\sum_{j=1}^{\infty} \mathcal{N}_j u_j\|^2_{H_0^m(D)}) = \sum_{j,k=1}^{\infty} \mathbb{E}(\mathcal{N}_j \mathcal{N}_k)\langle u_j, u_k\rangle_{H_0^m(D)} = \sum_{j=1}^{\infty} \|u_j\|^2_{H_0^m(D)}.$$

The summands can be written as

$$\|u_j\|^2_{H_0^m(D)} = \sum_{k=1}^{\infty} \lambda^m \langle u_j, e_k\rangle^2 = \sum_{k=1}^{\infty} \lambda_k^m (\lambda_k^{-1}\langle u_j, e_k\rangle_\nabla)^2 = \sum_{k=1}^{\infty} \lambda^{m-1}\langle u_j, \lambda^{-\frac{1}{2}} e_k\rangle^2_\nabla$$

where we used $\langle u_j, e_k\rangle_\nabla = \lambda_k \langle u_j, e_k\rangle$. Plugging this in, we can exchange the sums as all terms are positive. We get

$$\mathbb{E}(\|\mathbf{h}\|^2_{H_0^m(D)}) = \sum_{k=1}^{\infty} \lambda_k^{m-1} \sum_{j=1}^{\infty} \langle u_j, \lambda_k^{-\frac{1}{2}} e_k\rangle^2_\nabla = \sum_{j=1}^{\infty} \lambda_j^{m-1}$$

which is finite for $m < 1 - \frac{d}{2}$ by Weyl´s law. □

Remark 9. If we allow m to be an element in $\mathbb{R}$, one can extend the definition 8 of $\mathbf{H}_0^m(D)$ straight away. After extending the spaces $H_0^m(D)$ to $m \in \mathbb{R}$ as well (which is not straightforward), both spaces are equivalent again. Doing so, theorem 1.3.2 generalises to $m < 1 - \frac{d}{2}$ for $m \in \mathbb{R}$.

It is important to recognise, that the GFF is a L^2-function in one dimension (i.e. the Brownian bridge), but gets too rough in higher dimensions to be a function. But even if it can´t be evaluated in a single point, it can be evaluated against test functions, as it is almost surely in a distribution space. By looking at remark 1, we already know that $\langle h, v\rangle_\nabla$ is $\mathcal{N}(0, \|v\|_\nabla)$ distributed for $v \in H_0^1(D)$. With the same arguments we can say something similar about the law of $\langle h, v\rangle$.

Proposition 1.3.3. *Let $v \in H_0^1(D)$. Then $\langle h, v\rangle$ is almost surely finite and $\mathcal{N}(0, \|v\|^2_{H_0^{-1}(D)})$ distributed.*

Proof. Set $M_N := \langle \sum_j^N \mathcal{N}_j u_j, v\rangle$. The process $(M_N)_{N=1}^\infty$ is clearly a martingale, and by Jensen's inequality $(M_N^2)_{N=1}^\infty$ is a submartingale. Using Gauss-Green formula states $\langle u_j, v\rangle = \langle \nabla u_j, \nabla(-\Delta^{-1} v)\rangle = \langle u_j, -\Delta^{-1} v\rangle_\nabla$ which shows

$$\mathbb{E}(M_N^2) = \sum_{j=1}^{N} \langle u_j, v\rangle^2 = \sum_{j=1}^{N} \langle u_j, -\Delta^{-1} v\rangle^2_\nabla \leq \| - \Delta^{-1} v\|^2_\nabla.$$

The right hand side is again by Gauss-Green equal to

$$\sum_{j=1}^{\infty}\lambda_j\langle e_j,-\Delta^{-1}v\rangle^2=\sum_{j=1}^{\infty}\lambda_j\langle -\Delta^{-1}e_j,v\rangle^2=\sum_{j=1}^{\infty}\lambda_j^{-1}\langle e_j,v\rangle^2=\|v\|^2_{H_0^{-1}(D)}.$$

Because $v\in H_0^1(D)\subset H_0^{-1}(D)$, this is a finite bound simultaneously. Therefore the expectation of the submartingale M_N^2 is bounded for all $N\in\mathbb{N}$. Hence by the martingale convergence theorem, M_N converges in $L^2(\mathbb{P})$. The limit of the centred and normal distributed martingale M_N is $\mathcal{N}(0,\|v\|^2_{H_0^{-1}(D)})$ distributed, as $\lim_{N\to\infty}\mathbb{E}((\sum_j^N\mathcal{N}_ju_j)^2)=\|v\|^2_{H_0^{-1}(D)}$. □

Even if the norm with respect to $H_0^1(D)$ of $\mathbf{h}$ does not exist, the evaluation $\langle\mathbf{h},v\rangle$ against a function v of $H_0^1(D)$ exists almost surely. The same naturally holds for evaluating against a test function.

Remark 10. Choose $-m<1-\frac{d}{2}$. As $\mathbf{h}$ is an element in $H_0^{-m}(D)$ and $\lambda_k^{\frac{m}{2}}e_k$ is an eigenbasis for this space, we can write

$$\begin{aligned}\mathbf{h}&=\sum_{k=1}^{\infty}\langle\mathbf{h},\lambda_k^{\frac{m}{2}}e_k\rangle_{H_0^{-m}}\lambda_k^{\frac{m}{2}}e_k\\&=\sum_{k=1}^{\infty}\lambda_k^{-m}\langle\mathbf{h},\lambda_k^{\frac{m}{2}}e_k\rangle\lambda_k^{\frac{m}{2}}e_k\\&=\sum_{k=1}^{\infty}\langle\mathbf{h},\lambda_k^{\frac{1}{2}}e_k\rangle\lambda_k^{-\frac{1}{2}}e_k.\end{aligned}$$

From 1.3.3 we know that $\langle\mathbf{h},\lambda_k^{\frac{1}{2}}e_k\rangle$ is standard Gaussian distributed. Further

$$\mathbb{E}\left(\langle\sum_{j=1}^{N}\mathcal{N}_ju_j,e_m\rangle\langle\sum_{j=1}^{N}\mathcal{N}_ju_j,e_n\rangle\right)=\sum_{j=1}^{N}\langle u_j,e_m\rangle\langle u_j,e_n\rangle=0$$

if $m\neq n$. Letting $j\to\infty$ yields that $\left\{\langle\mathbf{h},\lambda_k^{\frac{1}{2}}e_k\rangle\right\}_{k=1}^{\infty}$ is a collection of independent random variables. As a result, different $\mathbf{h}$ (meaning that they are built on different bases of $H_0^1(D)$) can be written as a random series of the same basis.

This states that the law of $\mathbf{h}$ does not depend on the specific basis we choose. Therefore the following definition of the GFF on bounded domains is well defined.

Definition 9 (GFF on bounded domains)**.** Let D be a bounded domain in $\mathbb{R}^d$ and $m<1-\frac{d}{2}$. Let $\{\mathcal{N}_j\}_{j=1}^{\infty}$ be iid standard Gaussian variables and $\{u_j\}_{j=1}^{\infty}$ an orthonormal basis on $H_0^1(D)$. A random distribution $\mathbf{h}$ mapping from a probability space to $H_0^m(D)$ is called a Gaussian free field, if $\mathbb{P}(\mathbf{h}\in A)=\mathbb{P}(\sum_j^{\infty}\mathcal{N}_ju_j\in A)$ for all A in the Borel algebra of $H_0^m(D)$ (meaning that $\mathbf{h}$ has the same law as $\sum_j^{\infty}\mathcal{N}_ju_j$).

Note that by the same argument as in remark 10, every GFF can be written as a random series of $\lambda_j^{-\frac{1}{2}} e_j$ or any other orthonormal basis of $H_0^1(D)$.

Remark 11. Even if $\mathbf{h}$ is not a function, we can plot $\sum_{j=1}^N \mathcal{N}_j u_j$ for some $N \in \mathbb{N}$. In the case of $D = (0, 2\pi) \times (0, 2\pi)$ the eigenbasis of $H_0^1(D)$ produced by the eigenfunctions of the negative Laplacian are the functions

$$u_{k,l}(x, y) = \frac{1}{\pi\sqrt{k^2 + l^2}} \sin(kx) \sin(ly)$$

for $k, l \in \mathbb{N}$. The following plot should give an intuition, of how to imagine the GFF (or at least its approximation).

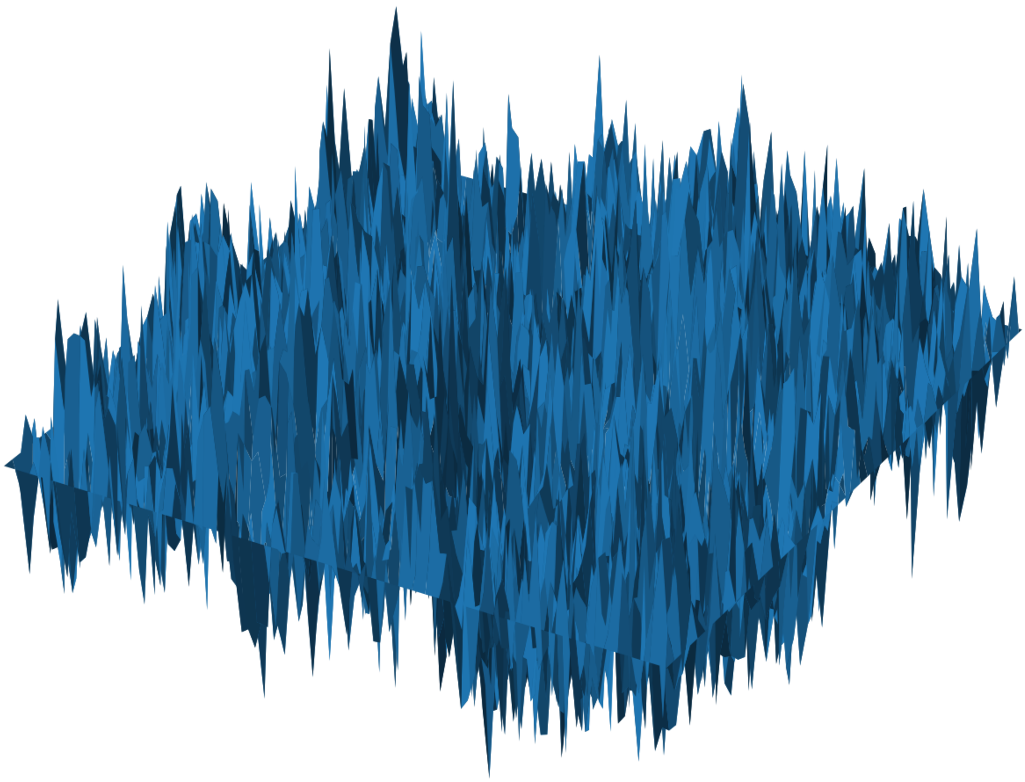

Figure 1.1: Approximation of the GFF on $D = (0, 2\pi) \times (0, 2\pi)$ by $\sum_{j,k=1}^N \mathcal{N}_{i,j} u_{i,j}$ with $N = 100$.

1.4 Markov property of the GFF

For a process with one time dimension $(X_t)_{t\geq 0}$ the Markov property states, that for all $s \geq 0$ the process $(X_t - X_s)_{t\geq s}$ is independent of $(X_t)_{t\in[0,s]}$. Meaning roughly speaking, that the future behaviour of the process depends only on the current state of the process, and does not care for what happened before. Transferring this to the higher dimensional case, we speak of the Markov property if the behaviour inside an area of a random field does not depend on what is happening outside of the area. In the case of the GFF, this property is fulfilled.

Theorem 1.4.1 (Markov property of the GFF)**.** *Let D be a bounded domain in $\mathbb{R}^d$, and $\boldsymbol{h}$ a GFF on D. For every subdomain $U \subset D$ we can split $\boldsymbol{h}$ into*

$$\boldsymbol{h} = \boldsymbol{h}_0 + \psi$$

where

(i) $\boldsymbol{h}_0$ *is a GFF on* U

(ii) ψ *is harmonic on* U

(iii) $\boldsymbol{h}_0$ *and* φ *are independent.*

Proof. We define the subspaces of $H_0^1(D)$

$$\begin{aligned} Supp(U) &:= \{u \in H_0^1(D) : supp(u) \subset U\} \equiv H_0^1(U) \\ Harm(U) &:= \{u \in H_0^1(D) : u|_U \text{ is harmonic in } U\}. \end{aligned}$$

The first step is to show, that

$$H_0^1(D) = Supp(U) \oplus Harm(U). \tag{1.4.1}$$

Choose $u \in Supp(U)$, $v \in Harm(U)$, and a sequence $(u_n)_{n=1}^{\infty}$ in $C_0^{\infty}(U)$ which tends to u by the Dirichlet norm. Because $v|_U$ is harmonic as well in the weak sense on U, we have $\langle u_n, v\rangle_{\nabla} = 0$. Letting n tend to infinity, yields $\langle u, v\rangle_{\nabla} = 0$. Now take $u \in H_0^1(D)$ and u_0 its orthogornal projection on $Supp(U)$. The function $\psi = u - u_0$ is per definition orthogonal to $C_0^{\infty}(U) \subset Supp(U)$, and therefore $\langle \psi, \varphi\rangle_{\nabla} = 0$ for all $\varphi \in C_0^{\infty}(D)$. Meaning $\psi|_U$ solves the Laplace equation in the weak sense on U. Using theorem 1.2.6, the function $\psi|_U$ is smooth in U, and consequently $\psi|_U$ is harmonic in the usual sense as well. Therefore 1.4.1 is shown.

Now consider an orthonormal basis $\{u_j\}_{j=1}^{\infty}$ of $Supp(U)$ and an orthonormal basis $\{v_j\}_{j=1}^{\infty}$ of $Harm(U)$. Further let $\{\mathcal{X}_j\}_{j=1}^{\infty}$ and $\{\mathcal{Y}_j\}_{j=1}^{\infty}$ be independent standard Gaussian random variables. We set

$$\begin{aligned} h_0 &= \sum_{j=1}^{\infty} \mathcal{X}_j u_j \\ \psi &= \sum_{j=1}^{\infty} \mathcal{Y}_j v_j. \end{aligned}$$

By 1.4.1, the union of $\{u_j\}$ and $\{v_j\}$ forms an orthonormal basis of $H_0^1(D)$. Consequently $h_0 + \psi$ is the GFF on D. Clearly h_0 is the GFF on U, which fulfills (i). For every $\varphi \in C_0^\infty(U)$ the sum $\langle \sum_{j=1}^N \mathcal{Y}_j v_j, \varphi \rangle_\nabla$ is equal zero for $N \in \mathbb{N}$. Therefore $\langle \psi, \varphi \rangle_\nabla = 0$, meaning $\psi|_U$ is harmonic in the weak sense on U. As above using theorem 1.2.6 yields that $\psi|_U$ is harmonic in the usual sense. This shows (ii), and (iii) is obvious by the construction of h_0 and ψ. $\square$

1.5 GFF in dimension 2

The GFF has a special property in the second dimension, which we will elaborate in this section.

1.5.1 Conformal maps

Definition 10 (conformal maps)**.** Let D_1 and D_2 be domains in $\mathbb{C} \cong \mathbb{R}^2$. Let $\phi : D_1 \to D_2$ be a bijective function. If for every $z_0 \in D_1 \subset \mathbb{C}$ the limit

$$\phi'(z_0) := \lim_{z \to z_0} \frac{\phi(z) - \phi(z_0)}{z - z_0}$$

exists and does not depend on the choice of z (i.e. ϕ is holomorphic), we call ϕ a conformal map from D_1 to D_2.

Remark 12. We recall some basic properties of conformal maps. Let ϕ be a holomorphic function, mapping D_1 to D_2. Further choose u, v real valued functions, such that $\phi(z) = u(z) + v(z)i$ for all $z \in D_1$. The following holds.

(i) The inverse of ϕ is a conformal map.

(ii) The map ϕ fulfills the Chauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This directly implies, that u and v are harmonic.

(iii) A conformal map is angle preserving under the Dirichlet inner product. This means that for $f, g \in H_0^1(D_2)$

$$\langle f \circ \phi, g \circ \phi \rangle_\nabla = \langle f, g \rangle_\nabla$$

holds. To obtain this, look at

$$\langle f \circ \phi, g \circ \phi \rangle_\nabla = \int_{D_1} \nabla f \circ \phi(z) \cdot \nabla g \circ \phi(z) dz = \int_{D_1} \nabla f(\phi(z)) D\phi(z) D\phi(z)^T \nabla g(\phi(z))^T dz. \tag{1.5.1}$$

By the Riemann-Cauchy equation

$$D\phi(z) = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}$$

which turns 1.5.1 into

$$\int_{D_1} \nabla f(\phi(z)) \cdot \nabla g(\phi(z)) |\det(D\phi(z))| dz = \int_{D_2} \nabla f(\tilde{z}) \cdot \nabla g(\tilde{z}) d\tilde{z} = \langle f, g \rangle_\nabla .$$

Hence we see that this definition of conformal maps coincides with the definition of conformal maps in Riemann geometry.

(iv) Let $F \in C^2(D_2)$. Then

$$\Delta(F \circ \phi) = |\phi'|^2 \Delta F(\phi).$$

This can be seen by a straight forward calculation. Additionally we can show the statement in the week sense, which implies the usual sense as well. For $\varphi \in \mathcal{D}(D_1)$ we have by transformation formula

$$\langle \Delta F \circ \phi, \varphi \rangle = \langle \Delta F \circ \phi(\phi^{-1}) |\det D\phi^{-1}|, \varphi \circ \phi^{-1} \rangle.$$

On the other hand we know by Gauss-Green and invariance under Dirichlet inner product, that

$$\langle \Delta F \circ \phi, \varphi \rangle = -\langle F \circ \phi, \varphi \rangle_\nabla = -\langle F, \varphi \circ \phi^{-1} \rangle_\nabla = \langle \Delta F, \varphi \circ \phi^{-1} \rangle.$$

Assuming without loss of generality that D_1 is bounded, we see that every test function on D_2 can be written as $\varphi \circ \phi^{-1}$ for a $\varphi \in \mathcal{D}(D_1)$. Therefore

$$\Delta F \circ \phi(\phi^{-1}) = \Delta F |\det D\phi(\phi^{-1})|$$

in the sense of $\mathcal{D}'(D_2)$, which gives (after considering $|\det D\phi| = |\phi'|^2$ and plugging in ϕ) the statement.

Theorem 1.5.1 (Riemann mapping theorem)**.** *Let D be a simply connected domain in $\mathbb{C}$ with $D \neq \mathbb{C}$. For every $z \in D$ there exists, up to rotation, a unique conformal map $\phi : \mathbb{D} := \{z \in : |z| < 1\} \to D$ such that $\phi(0) = z$.*

Proof. Look for example at [Kod07] chapter 5. □

We will use the following quite a lot later on.

Definition 11 (conformal radius)**.** Let D be a simply connected domain in $\mathbb{C}$ with $D \neq \mathbb{C}$. The function $R(.;D) : D \to \mathbb{R}_+$ with $R(z;D) = |\phi'(0)|$ is called the conformal radius of D, where ϕ is the conformal map taking $\mathbb{D}$ to D and 0 to z.

By the Riemann mapping theorem this function exists, and is well defined as the conformal map taking $\mathbb{D}$ to D and 0 to a fixed $z \in D$ is unique up to rotation.

Remark 13. (i) The Koebe quarter theorem states, that for every conformal map $\phi : \mathbb{D} \to D$ the ball $B_{\frac{1}{4}|\phi'(0)|}(f(0))$ is contained in D. Therefore

$$\frac{1}{4}R(z;D) \leq dist(z,\partial D)$$

and hence the conformal radius is bounded, as soon as the domain D is bounded.

(ii) Let $\phi : D_1 \to D_2$ be a conformal map, and fix $z \in D_2$. Set $\tilde{z} = \phi^{-1}(z)$ and take $\tilde{\phi}$ as the conformal map taking $\mathbb{D}$ to D_1, such that $\tilde{\phi}(0) = \tilde{z}$. As the map $\phi \circ \tilde{\phi}$ is a conformal map taking 0 to z, we can write

$$R(z, D_2) = |(\phi \circ \tilde{\phi})'(0)| = |\phi'(\tilde{z})\tilde{\phi}'(0)| = |\phi'(\tilde{z})|R(\tilde{z}, D_1).$$

1.5.2 Conformal invariance of the GFF

Theorem 1.5.2 (Conformal invariance of the GFF)**.** *Let D_1 and D_2 be bounded domains in $\mathbb{R}^2$ and $\phi : D_1 \to D_2$ a conformal map. Further let $\boldsymbol{h}^{D_1}$ be a GFF on D_1. Then the distribution $\boldsymbol{h}^{D_1} \circ \phi^{-1}$ which acts as*

$$\langle \boldsymbol{h}^{D_1} \circ \phi^{-1}, \varphi\rangle = \langle \boldsymbol{h}^{D_1}, |\phi'|^2 \varphi \circ \phi\rangle$$

on $\varphi \in \mathcal{D}(D_2)$, is a GFF on D_2.

Proof. We know that we can write $\mathbf{h}^{D_1} = \sum_{j=1}^{\infty} \mathcal{N}_j u_j$ where $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis of $H_0^1(D_1)$. By the invariance of conformal maps under the Dirichlet inner product, the functions $\{u_j \circ \phi^{-1}\}_{j=1}^{\infty}$ form an orthonormal basis for $H_0^1(D_2)$ and hence $\mathbf{h}^D \circ \phi^{-1} = \sum_{j=1}^{\infty} \mathcal{N}_j u_j \circ \phi^{-1}$ is a GFF on D_2. For a $\varphi \in \mathcal{D}(D_2)$ we have

$$\langle \sum_{j=1}^{\infty} \mathcal{N}_j u_j \circ \phi^{-1}, \varphi\rangle = \sum_{j=1}^{\infty} \mathcal{N}_j \langle u_j \circ \phi^{-1}, \varphi\rangle = \sum_{j=1}^{\infty} \mathcal{N}_j \langle u_j, |\phi'|^2 \varphi \circ \phi\rangle = \langle \sum_{j=1}^{\infty} \mathcal{N}_j u_j, |\phi'|^2 \varphi \circ \phi\rangle$$

by the transformation formula, which shows that $\mathbf{h}^{D_1} \circ \phi^{-1}$ acts in the stated way. □

1.6 The GFF as stochastic process

Often it is beneficial to think of the GFF as a stochastic process on $C_0^\infty(D)$ instead of a random distribution. For this, we look at the covariance of the evaluation of $\mathbf{h}$ on two different functions $\varphi_1, \varphi_2 \in C_0^\infty(D)$. Without loss of generality, assume $\mathbf{h} = \sum_{j=1}^{\infty} \mathcal{N}_j \lambda_j^{-\frac{1}{2}} e_j$. As the expectation of $\langle h, \varphi_1\rangle$ and $\langle h, \varphi_2\rangle$ is zero by proposition 1.3.3, we have

$$\begin{aligned}
Cov(\langle h, \varphi_1\rangle, \langle h, \varphi_2\rangle) = \mathbb{E}(\langle h, \varphi_1\rangle\langle h, \varphi_2\rangle) &= \mathbb{E}(\sum_{j,k=1}^{\infty} \mathcal{N}_j \mathcal{N}_k \langle \lambda_j^{-\frac{1}{2}} e_j, \varphi_1\rangle\langle \lambda_k^{-\frac{1}{2}} e_k, \varphi_2\rangle) \\
&= \sum_{j=1}^{\infty} \langle \lambda_j^{\frac{-1}{2}} e_j, \varphi_1\rangle\langle \lambda_j^{\frac{-1}{2}} e_j, \varphi_2\rangle = \sum_{j=1}^{\infty} \lambda_j^{-1}\langle e_j, \varphi_1\rangle\langle e_j, \varphi_2\rangle \\
&= \langle \varphi_1, \varphi_2\rangle_{H_0^{-1}(D)}.
\end{aligned}$$

It's easily seen that $\langle \varphi_1, \varphi_2 \rangle_{H_0^{-1}(D)} = \langle -\Delta^{-1}\varphi_1, \varphi_2 \rangle$ and therefore

$$Cov(\langle h, \varphi_1 \rangle, \langle h, \varphi_2 \rangle) = \langle -\Delta^{-1}\varphi_1, \varphi_2 \rangle. \tag{1.6.1}$$

Fo find an explicit expression for the negative of the inverse Laplacian, we have to recap the following classical construction for solving PDE's.

1.6.1 Green´s function for the Poisson equation

Definition 12. Let $D \subset \mathbb{R}^d$ be an arbitrary domain, and $L : \mathcal{D}(D) \to \mathcal{D}(D)$ be a linear differential operator on the test functions of D. Then the expression $G^D(x, y)$ satisfying

$$LG^D(x, y) = \delta_x(y)$$

in the distributional sense is called the Green´s function of the operator L.

The Green's function is used to find solutions to differential equations of the form $Lu = f$ by $u(x) := \langle G(x, .), f \rangle$ (at least this gives a solution if the boundary conditions are zero). Note that in general $G(x, .)$ is a distribution and not necessarily a function. If the kernel of L is trivial, then the Green´s function is unique.

In this subsection, we will develop the Green´s function for L being the negative Laplacian by solving the Poisson equation

$$-\Delta u(x) = f(x) \qquad \text{for } x \in D \tag{1.6.2}$$
$$u(x) = g(x) \qquad \text{for } x \in \partial D \tag{1.6.3}$$

on a bounded domain $D \subset \mathbb{R}^d$ for $d \geq 2$, where ∂D is a C^1 boundary. It will be enough for our purpose, to select $f, g \in \mathbb{C}^\infty(D)$. For simplicity, we will write sometimes u_j instead of $\frac{\partial}{\partial x_j} u$. We will mainly follow the classical approach used in [Eva22] chapter 2.2.

I.) Fundamental solution

The first step is to find a general solution for the Poisson equation

$$\Delta u = 0$$

on $\mathbb{R}^d$. One can easily check, that $\Delta u \circ O = \Delta u$ for any orthonormal matrix O. Hence it is natural, to search for a radially symmetric solution u. After this assumption we can define $v : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ which fulfills $v(r) := u(|x|)$ for $r = |x|$. Excluding $x = 0$ reduces the Poisson equation to

$$\Delta v(|x|) = \sum_{j=1}^{d} \partial_j (v'(|x|)\frac{x_j}{|x|}) = \sum_{j=1}^{d} v''(|x|)\frac{x_j^2}{|x|^2} + v'(|x|)(\frac{1}{|x|} - \frac{x_j^2}{|x|^3}) = v''(|x|) + \frac{d-1}{|x|}v'(|x|).$$

Hence solving $\Delta u(x) = 0$ is equivalent to solve

$$v''(r) + \frac{d-1}{r}v'(r) = 0 \tag{1.6.4}$$

for $r = |x| \neq 0$. Assuming $v' \neq 0$, this leads to solving

$$\frac{d}{dr}\log(v'(r)) = \frac{v''(r)}{v'(r)} = \frac{1-d}{r}$$

which gives after integrating and applying the exponential function

$$v(r)' = a\frac{1}{r^{d-1}}$$

for a constant $a \in \mathbb{R}$. Hence for 1.6.4 we get the general solution

$$v(r) = \begin{cases} a\log(r) + b & \text{for } d = 2 \\ a(2-d)\frac{1}{r^{d-2}} + b & \text{for } d \geq 3 \end{cases}$$

for $a, b \in \mathbb{R}$. This motivates the following definition.

Definition 13 (Fundamental solution of the Laplace equation)**.** Let ω_d be the volume of the unit ball in dimension d. The function $\Phi : \mathbb{R}^d \setminus 0 \to \mathbb{R}$, defined by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & \text{for } d = 2 \\ \frac{1}{\omega_d d(d-2)}\frac{1}{|x|^{d-2}} & \text{for } d \geq 3 \end{cases}$$

is called the fundamental solution of the Laplace equation.

The specific choice of the coefficients are necessary for the following theorem.

Theorem 1.6.1. *Let $f \in C_0^\infty(\mathbb{R}^d)$. Then*

$$u(x) := \int_{\mathbb{R}^d} \Phi(x-y)f(y)\,dy$$

solves $-\Delta u = f$.

Proof. As usual for convolutions over the whole space, we can write

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy. \tag{1.6.5}$$

We have

$$\partial_j u(x) = \lim_{h\to 0}\int_{\mathbb{R}^d} \Phi(y)\frac{f(x+he_j-y)-f(x-y)}{h}\ dy = \int_{\mathbb{R}^d} \Phi(y)\partial_j f(x-y)\ dy$$

where we could put the limit inside as $\frac{f(x+he_j-y)-f(x-y)}{h} \to \partial_j f(x)$ uniformly by compactness of $supp(f)$. Therefore

$$\Delta u(x) = \int_{\mathbb{R}^d} \Phi(y)\Delta_x f(x-y)\ dy = \int_{\mathbb{R}^d} \Phi(y)\Delta_y f(x-y)\ dy.$$

We separate $\Delta u(x)$ into

$$\int_{B_\varepsilon(0)} \Phi(y)\Delta_y f(x-y)\; dy + \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \Phi(y)\Delta_y f(x-y)\; dy =: I_\varepsilon + J_\varepsilon.$$

The first term can be bounded by

$$|I_\varepsilon| \le \max_{j=1,\dots,d} \|\partial_j f(x)\|_\infty \int_{B_\varepsilon(0)} |\Phi(y)|\; dy$$

where $\|f\|_\infty = \sup_{x\in supp(f)} |f|$ as usual. Taking $\tilde{\Phi}(r) = \Phi(x)$ for $r = |x|$ and applying polar coordinates yields

$$\int_{B_\varepsilon(0)} |\Phi(y)|\; dy = \int_0^\varepsilon |\tilde{\Phi}(r)| d\omega_d r^{d-1}\; dr = \begin{cases} -\frac{1}{2}\varepsilon^2\log(\varepsilon) + \frac{\varepsilon^2}{4} & \text{for } d=2 \text{ and } \varepsilon < 1 \\ \frac{\varepsilon^2}{2(d-2)} & \text{for } d \ge 3 \end{cases}$$

and hence I_ε tends to zero as ε tends to zero.

By the compactness of *supp*(f) we can apply the Gauss-Green formula for J_ε. Doing this twice gives

$$\begin{aligned} J_\varepsilon = &\int_{\partial(\mathbb{R}^d\setminus B_\varepsilon(0))} \Phi(y)\frac{\partial_y f(x-y)}{\partial\nu(y)}\; dS(y) - \int_{\partial(\mathbb{R}^d\setminus B_\varepsilon(0))} f(x-y)\frac{\partial\Phi(y)}{\partial\nu(y)}\; dS(y) \\ &+ \int_{\mathbb{R}^d\setminus B_\varepsilon(0)} \Delta\Phi(y) f(x-y)\; dy \end{aligned}$$

where the last term cancels because Φ is harmonic outside of the origin. By compactness of *supp*(f) it further reduces to

$$J_\varepsilon = \int_{\partial B_\varepsilon(0)} \Phi(y)\frac{\partial_y f(x-y)}{\partial\nu(y)}\; dS(y) - \int_{\partial B_\varepsilon(0)} f(x-y)\frac{\partial\Phi(y)}{\partial\nu(y)}\; dS(y) =: K_\varepsilon - L_\varepsilon$$

where ν is the inwards oriented normal vector. The first term can be bounded by

$$|K_\varepsilon| \le \max_{j=1,\dots,d} \|\partial_j f(x)\|_\infty \tilde{\Phi}(\varepsilon) \int_{\partial B_\varepsilon(0)} 1\; dS(y)$$

and since

$$\tilde{\Phi}(\varepsilon)\int_{\partial B_\varepsilon(0)} 1 dS(y) = \tilde{\Phi}(\varepsilon)\omega_d d\varepsilon^{d-1} \begin{cases} -\log(\varepsilon)\varepsilon & d=2, \varepsilon<1 \\ (d-2)\varepsilon & d\ge 3 \end{cases}$$

the expression K_ε vanishes when ε goes to zero. As $\nabla\Phi(y) = -\frac{1}{d\omega_d}\frac{y}{|y|^d}$ and $\nu(y) = -\frac{y}{|y|}$ we end up with

$$L_\varepsilon = -\frac{1}{d\omega_d}\int_{\partial B_\varepsilon(0)} f(x-y)\frac{y}{|y|^d}\cdot\frac{y}{|y|}\; dS(y) = \frac{1}{\varepsilon^{d-1} d\omega_d}\int_{\partial B_\varepsilon(0)} f(x-y)\; dS(y) \to f(x)$$

for $\varepsilon \to 0$ by continuity of f. Putting all these estimates together gives

$$\Delta u(x) = I_\varepsilon + K_\varepsilon - L_\varepsilon \to -f(x) \qquad \varepsilon \to 0$$

returning the statement. □

We see from the proof that the value of $\nabla\Phi$ explains the specific choice of the scalars of Φ. Note further, that by the representation 1.6.5 it is easy to see that the solution u is infinitely differentiable.

II.) Green´s function and representation formula

For D bounded and $u \in \mathbb{C}^\infty(D)$, we rewrite again by applying Gauss Green formula twice and the harmonicity of Φ outside of zero

$$-\int_{D\setminus B_\varepsilon(x)} \Phi(x-y)\Delta u(y)\, dy = \int_{\partial D\setminus B_\varepsilon(x)} \Phi(x-y)\frac{\partial u(y)}{\partial\nu(y)} - u(y)\frac{\partial_y\Phi(x-y)}{\partial\nu(y)}\, dS(y) \quad (1.6.6)$$

where the right hand side is equal to

$$\int_{\partial D} \Phi(x-y)\frac{\partial u(y)}{\partial\nu(y)} - u(y)\frac{\partial_y\Phi(x-y)}{\partial\nu(y)}\, dS(y) + \int_{\partial B_\varepsilon(x)} \Phi(x-y)\frac{\partial u(y)}{\partial\nu(y)} - u(y)\frac{\partial_y\Phi(x-y)}{\partial\nu(y)}\, dS(y).$$

From the proof of Theorem 1.6.1, we know that equation 1.6.6 tends to

$$-\int_{D} \Phi(x-y)\Delta u(y)\, dy = \int_{\partial D} \Phi(x-y)\frac{\partial u(y)}{\partial\nu(y)} - u(y)\frac{\partial_y\Phi(x-y)}{\partial\nu(y)}\, dS(y) + u(x). \quad (1.6.7)$$

We want to get rid of the integral on the boundary of D. For this, we define for a fixed $x \in D$ the function $\phi_x(y)$ as the unique solution to the Laplace equation

$$\begin{aligned} \Delta\phi_x(y) &= 0 && y \in D \\ \phi_x(y) &= \Phi(x-y) && y \in \partial D. \end{aligned}$$

We see that

$$-\int_{D} \phi_x(y)\Delta u(y)\, dy = \int_{\partial D} \phi_x(y)\frac{\partial u(y)}{\partial\nu(y)} - u(y)\frac{\partial \phi_x(y)}{\partial\nu(y)}\, dS(y) \quad (1.6.8)$$

after two times Gauss-Green formula. Using equation 1.6.7 and 1.6.8 yields

$$u(x) = -\int_{D} (\Phi(x-y) - \phi_x(y))\Delta u(y)\, dy + \int_{\partial D} u(y)\frac{\partial_y(\Phi(x-y) - \phi_x(y))}{\partial\nu(y)}\, dS(y)$$

because $\phi_x(y)$ is chosen to coincide with $\Phi(x-y)$ on the boundary. We define the function $G : D^2 \setminus \{(x,x) : x \in D\} \to \mathbb{R}$ by

$$G(x,y) := \Phi(x-y) - \phi_x(y).$$

The statement directly follows.

Theorem 1.6.2 (Green´s representation formula). *Let $D \subset \mathbb{R}^d$ bounded and $f, g \in C^\infty(D)$. Then the Poisson equation 1.6.2 has the unique solution*

$$u(x) = \int_{D} G(x,y)f(y)\, dy + \int_{\partial D} g(y)\frac{\partial_y G(x,y)}{\partial\nu(y)}\, dS(y).$$

Taking $f \in \mathcal{D}(D)$ and $g \equiv 0$, we see that G is the unique Green´s function for the negative Laplacian, and we write $G_D(x,y)$ if we want to specify that it is the Green´s function on the domain D. Further, G as a function is unique when we assume continuity, as then uniqueness in the distributional sense implies uniqueness in the general sense. From here on, we mean the Green´s function for the negative Laplacian when we talk about the Green´s function.

III.) Properties of the Green´s function and extension

Proposition 1.6.3. *Let $D \subset \mathbb{R}^d$ be bounded. Then the Green´s function fulfills*

(i) $\Delta_y G(x,y) = 0$ *for* $x \neq y$.

(ii) $G(x,y) = 0$ *for* $y \in \partial D$.

(iii) $G(x,y) = G(y,x)$.

Proof. (i) and (ii) are clear from the construction. For (iii) set $V_\varepsilon := D \setminus \{B_\varepsilon(x) \cup B_\varepsilon(y)\}$ and $v(z) := G(x,z)$ and $w(z) := G(y,z)$. Applying Gauss Green on V_ε gives

$$\int_{\partial V_\varepsilon} v(z) \frac{\partial w(z)}{\partial \nu(z)} \, dS(z) = \int_{\partial V_\varepsilon} w(z) \frac{\partial v(z)}{\partial \nu(z)} \, dS(z).$$

As both functions are zero on the boundary of D, this is equivalent to

$$\begin{aligned} &\int_{\partial B_\varepsilon(x)} v(z) \frac{\partial w(z)}{\partial \nu(z)} dS(z) + \int_{\partial B_\varepsilon(y)} v(z) \frac{\partial w(z)}{\partial \nu(z)} \, dS(z) \\ = &\int_{\partial B_\varepsilon(x)} w(z) \frac{\partial v(z)}{\partial \nu(z)} dS(z) + \int_{\partial B_\varepsilon(y)} w(z) \frac{\partial v(z)}{\partial \nu(z)} \, dS(z) \end{aligned} \tag{1.6.9}$$

where ν is the inward pointing normal vector. Like in the proof of theorem 1.6.1, the first term on the left hand side vanishes for $\varepsilon \to 0$. The second term on the left hand side can be split into

$$\int_{\partial B_\varepsilon(y)} v(z) \frac{\partial \Phi(y-z)}{\partial \nu(z)} \, dS(z) + \int_{\partial B_\varepsilon(y)} v(z) \frac{\partial \phi_x(z)}{\partial \nu(z)} \, dS(z)$$

which from the proof of theorem 1.6.1 we know tends to $v(y)$, as $\phi_x(z)$ is a smooth function. Applying the same arguments on the right hand side of 1.6.9 and $\varepsilon \to 0$, we end up with

$$v(y) = w(x)$$

which gives the statement. $\square$

Proposition 1.6.4 (conformal invariance of Green´s function)**.** *Let D and $\tilde{D}$ be subset domains of $\mathbb{R}^2$, where $\tilde{D}$ is bounded. Let $\phi : \tilde{D} \to D$ be a conformal map (which exists by theorem 1.5.1). Then*

$$G^D(x,y) = G^{\tilde{D}}(\phi^{-1}(x), \phi^{-1}(y)).$$

Proof. Let $u \in \mathcal{D}(D)$. Clearly the map $u \circ \phi$ is in $\mathcal{D}(\tilde{D})$ and hence can be written as

$$u \circ \phi(\tilde{x}) = \int_{\tilde{D}} G^{\tilde{D}}(\tilde{x}, \tilde{y}) \Delta(u \circ \phi)(\tilde{y}) \, d\tilde{y}. \tag{1.6.10}$$

Using remark 12 (iv) and substituting $\tilde{y} = \phi^{-1}(y)$ gives

$$\begin{aligned} u(x) = u \circ \phi(\phi^{-1}(x)) &= - \int_{\tilde{D}} G^{\tilde{D}}(\phi^{-1}(x), \tilde{y}) \Delta u(\phi(\tilde{y})) |\phi'(\tilde{y})|^2 \, d\tilde{y} \\ &= - \int_D G^{\tilde{D}}(\phi^{-1}(x), \phi^{-1}(y)) \Delta u(y) \, dy \end{aligned}$$

which shows the statement by the definition and the uniqueness of the Green´s function. □

Proposition 1.6.5 (growth conditions on the diagonal).

(i) For $d \geq 2$ the Greens function fulfills

$$G(x, y) = \Phi(x - y) + \mathcal{O}(1)$$

as $y \to x$.

(ii) For $d = 2$ the Greens function satisfies

$$G(x, y) = -\frac{1}{2\pi} \log(|x - y| + \frac{1}{2\pi} \log(R(x; D)) + o(1)$$

as $y \to x$, where $R(x; D)$ is the conformal radius defined in 11.

Proof. (i) is clear as $\phi_x(y) = \mathcal{O}(1)$. (ii) The Green´s function on $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ satisfies

$$G^{\mathbb{D}}(0, y) = -\frac{1}{2\pi} \log(|y|)$$

as $\phi_0(y) \equiv 0$. Fix some $x, y \in D$. Choosing a conformal map $\phi : \mathcal{D} \to D$ which maps 0 to x. Denoting $\tilde{y} = \phi^{-1}(y)$ and using proposition 1.6.4, we get

$$G^D(x, y) = G^{\mathbb{D}}(0, \tilde{y}) = -\frac{1}{2\pi} \log(|\tilde{y}|).$$

We can split

$$-\log(|\tilde{y}|) = -\log(|y - x|) + \log(|\phi'(0)|) + \log(\frac{1}{|\phi'(0)|} \frac{|\phi(\tilde{y}) - \phi(0)|}{|\tilde{y}|})$$

and as $\frac{\phi(\tilde{y}) - \phi(0)}{\tilde{y}}$ tends to $\phi'(0)$ as $y \to x$, the statement is fulfilled. □

1.6.2 Construction of the GFF as a stochastic process

Knowing what we will need about the Green´s function, we continue with the observation 1.6.1, and with the intention to see the GFF as a Gaussian stochastic process on $C_0^\infty(D)$. This leads us to define the GFF as the centred Gaussian stochastic process with covariance $Cov(\langle \mathbf{h}, \varphi_1\rangle, \langle \mathbf{h}, \varphi_2\rangle) = \int_{D^2} G(x,y)\varphi_1(x)\varphi_2(y)dxdy$. Before doing so, we extend $\mathbf{h}$ to a bigger indexset.

Definition 14. Let $\mathfrak{M}_D^+$ be the set of all finite measures ρ, which are acting on D and satisfy

$$\int_{D^2} G(x,y)\ \rho(dx)\rho(dy) < \infty.$$

Denote the space of signed measures as $\mathfrak{M}_D = \{\rho_+ - \rho_- : \rho_+, \rho_- \in \mathfrak{M}_D^+\}$. We define the bilinear form $\Gamma : \mathfrak{M}_D^2 \to \mathbb{R}$ by

$$\Gamma(\rho_1, \rho_2) = \int_{D^2} G(x,y)\ \rho_1(dx)\rho_2(dy)$$

and write $\Gamma(\rho) = \Gamma(\rho, \rho)$.

Note that we do not know so far, if $\mathfrak{M}_D$ is a vector space, and if $\Gamma(\rho_1, \rho_2)$ is finite for $\rho_1, \rho_2 \in \mathfrak{M}_D$. To show that this is actually the case, we start with some observations.

Remark 14. (i) If $x \in D$ is an atom of a measure ρ on D, we get

$$\int_{D^2} G^D(x,y)\ \rho(dx)\rho(dy) \geq G^D(x,x)\underbrace{\rho(\{x\})^2}_{\neq 0} = \infty.$$

Therefore measures in $\mathfrak{M}_D^+$ and $\mathfrak{M}_D$ do not have any atoms.

(ii) Let $K \subset D$ be a compact set. We can find a lower bound B_K for G on $K \times K$, which is strictly positive. Hence for each $\rho \in \mathfrak{M}_D^+$

$$\Gamma(\rho) \geq \int_{K\times K} B_K\, \rho(dx)\rho(dy) = B_K \rho(K)^2.$$

and consequently ρ is finite on all compact sets in D (same holds for $\rho \in \mathfrak{M}_D$). This states that $\mathfrak{M}_D^+$ is a set of Radon measures.

(iii) The point above implies that $\rho \in \mathfrak{M}_D$ is finite on the open set $D_\varepsilon := \{z \in D : diam(z, \partial D) > \varepsilon\}$, because the closure of it is compact and contained in D.

For a positive function $\varphi \in C_0^\infty(D)$ define the measure ρ_φ such that $\rho_\varphi(dx) = \varphi(x)dx$. It is an easy application of the Gauss-Green formula, that

$$\Gamma(\rho_\varphi) = \langle \varphi, \Delta^{-1}\varphi\rangle = \sum_{j=1}^{\infty} \langle \varphi, e_j\rangle\langle \varphi, \Delta^{-1} e_j\rangle = \|\varphi\|^2_{H_0^{-1}(D)} < \infty \tag{1.6.11}$$

and hence $\rho_\varphi \in \mathfrak{M}_D^+$. The main message of the following statement will be that these measures, produced by test functions, are dense in $\mathfrak{M}_D$ with respect to Γ.

Proposition 1.6.6. *For all $\rho_1, \rho_2 \in \mathfrak{M}_D$ there exist sequences $(\varphi_n^1)_{n=1}^\infty$ and $(\varphi_n^2)_{n=1}^\infty$ in $C_0^\infty(D)$, such that $\Gamma(\rho_{\varphi_n^1}, \rho_{\varphi_n^2})$ tends to $\Gamma(\rho_1, \rho_2)$ for $n \to \infty$.*

Proof. Take $\rho \in \mathfrak{M}_D$ and $\rho_+, \rho_- \in \mathfrak{M}_D^+$ such that $\rho = \rho_+ - \rho_-$. Define for $n \in \mathbb{N}$ the functions

$$\begin{aligned} \varphi_n^+(x) &:= \int_D n^d \varphi(n(x-z)) \mathbb{1}_{\{d(z,\partial D) > \frac{2}{n}\}} \, \rho_+(dz) \\ \varphi_n^-(x) &:= \int_D n^d \varphi(n(x-z)) \mathbb{1}_{\{d(z,\partial D) > \frac{2}{n}\}} \, \rho_-(dz) \end{aligned} \tag{1.6.12}$$

where φ is a positive test function supported on the unit ball, with $\int \varphi = 1$. Clearly φ_n^+ and φ_n^- are positive functions. We see that $\varphi_n^+(x) = \varphi_n^-(x) = 0$ if $d(x, \partial D) \leq \frac{1}{n}$, meaning they have compact support. Further φ_n^+ and φ_n^- are smooth as φ is smooth with compact support (which implies that we can put derivatives inside of the integral of 1.6.12), and consequently φ_n^+ and φ_n^- are in $\mathcal{D}(D)$. Write

$$\begin{aligned} &\Gamma(\rho_{\varphi_n^+}, \rho_{\varphi_n^-}) = \int_{D^2} G(x,y) \varphi_n^+(x) \, dx \, \varphi_n^-(y) \, dy \\ &= \int_{D^2} \int_{D^2} G(x,y) n\varphi(n(x-z_1)) \mathbb{1}_{\{d(z_1,\partial D) > \frac{2}{n}\}} n\varphi(n(y-z_2)) \mathbb{1}_{\{d(z_2,\partial D) > \frac{2}{n}\}} \, dxdy \, \rho_+(dz_1) \rho_-(dz_2) \\ &= \lim_{\varepsilon \to 0} \int_{D_\varepsilon^2} g_n^\varepsilon(z_1, z_2) \, \rho_+(dz_1) \rho_-(dz_2) \end{aligned} \tag{1.6.13}$$

where

$$g_n^\varepsilon(z_1, z_2) := \\ \mathbb{1}_{\{d(z_1,\partial D) > \frac{2}{n}\}} \mathbb{1}_{\{d(z_2,\partial D) > \frac{2}{n}\}} \int_{D^2} \mathbb{1}_{\{|x-y| > \varepsilon\}}(x,y) G(x,y) n\varphi(n(x-z_1)) n\varphi(n(y-z_2)) \, dxdy.$$

Note that we used that ρ has no atoms (hence there is no mass on the diagonal) and monotone convergence for the last equation of 1.6.13. Clearly

$$g_n^\varepsilon(z_1, z_2) \to \mathbb{1}_{\{|z_1 - z_2| > \varepsilon\}} G(z_1, z_2)$$

for $n \to \infty$ and

$$g_n^\varepsilon(z_1, z_2) \leq \sup_{\{|x-y| > \varepsilon\}} \{G(x,y)\} \int_{D^2} n\varphi(n(x-z_1)) n\varphi(n(y-z_2)) \, dxdy = \sup_{\{|x-y| > \varepsilon\}} \{G(x,y)\}$$

which is finite, because G is a continuous function on the compact set $\overline{D^2 \setminus \{(x,y) : |x-y| \leq \varepsilon\}}$. From remark 14 (iii) we see that $\int_{D_\varepsilon^2} \rho_+(dx) \rho_-(dy)$ is finite. Therefore we can use dominated convergence to obtain

$$\lim_{n \to \infty} \int_{D_\varepsilon^2} g_n(z_1, z_2) \, \rho_+(dz_1) \rho_-(dz_2) = \int_{D_\varepsilon^2} \mathbb{1}_{\{|x-y| > \varepsilon\}}(z_1, z_2) G(z_1, z_2) \, \rho_+(dz_1) \rho_-(dz_2). \tag{1.6.14}$$

Combining 1.6.13 and 1.6.14 and exchanging limits yields

$$\begin{aligned}\lim_{n\to\infty} \Gamma(\rho_{\varphi_n^+}, \rho_{\varphi_n^-}) &= \lim_{\varepsilon\to 0} \int_{D_\varepsilon^2} \mathbb{1}_{\{|x-y|>\varepsilon\}}(z_1,z_2) G(z_1,z_2)\, \rho_+(dz_1)\rho_-(dz_2) \\ &= \int_{D^2} G(z_1,z_2)\, \rho_+(dz_1)\rho_-(dz_2) = \Gamma(\rho_+,\rho_-)\end{aligned}$$

by monotone convergence. In the same way it follows that $\lim_{n\to\infty} \Gamma(\rho_{\varphi_n^+}) = \Gamma(\rho_+)$ and $\lim_{n\to\infty} \Gamma(\rho_{\varphi_n^-}) = \Gamma(\rho_-)$. Hence for $\varphi_n = \varphi_n^+ - \varphi_n^- \in C_0^\infty(D)$ we get

$$\begin{aligned}\lim_{n\to\infty} \Gamma(\rho_{\varphi_n}) &= \lim_{n\to\infty} \Gamma(\rho_{\varphi_n^+}) - 2\Gamma(\rho_{\varphi_n^+}, \rho_{\varphi_n^-}) + \Gamma(\rho_{\varphi_n^-}) \\ &= \Gamma(\rho_+) - 2\Gamma(\rho_+,\rho_-) + \Gamma(\rho_-) = \Gamma(\rho).\end{aligned}$$

Following the same procedure to obtain sequences $(\varphi_n^1)_{n=1}^\infty$ for ρ_1 and $(\varphi_n^2)_{n=1}^\infty$ for ρ_2, yields analogously to the last equation

$$\lim_{n\to\infty} \Gamma(\rho_{\varphi_n^1}, \rho_{\varphi_n^2}) = \Gamma(\rho_1,\rho_2).$$

□

We introduce the equivalence relation $\approx$ on $\mathfrak{M}_D$ by $\rho_1 \approx \rho_2$ if $\Gamma(\rho_1 - \rho_2) = 0$. Write $\mathfrak{M}_D/\approx$ for the quotient space of $\mathfrak{M}_D$ with respect to $\approx$.

Proposition 1.6.7. *The space $\mathfrak{M}_D/\approx$ is a Hilbert space with inner product Γ.*

Proof. Clearly $r.\rho$ is an element of $\mathfrak{M}_D$ for $r \in \mathbb{R}$ and $\rho \in \mathfrak{M}_D$. Assume $\rho_1, \rho_2 \in \mathfrak{M}_D$ and by splitting

$$\Gamma(\rho_1+\rho_2) = \Gamma(\rho_1) + 2\Gamma(\rho_1,\rho_2) + \Gamma(\rho_2)$$

we see that we just have to show that $\Gamma(\rho_1,\rho_2)$ is finite. By proposition 1.6.6 we can choose sequences $(\varphi_1^n)_{n=1}^\infty$ and $(\varphi_2^n)_{n=1}^\infty$ in $C_0^\infty(D)$ such that

$$\Gamma(\rho_{\varphi_1^n}) \to \Gamma(\rho_1) \qquad \Gamma(\rho_{\varphi_2^n}) \to \Gamma(\rho_2) \qquad \Gamma(\rho_{\varphi_1^n}, \rho_{\varphi_2^n}) \to \Gamma(\rho_1,\rho_2).$$

By Cauchy-Schwarz and a variation of 1.6.11

$$\begin{aligned}\Gamma(\rho_1,\rho_2) &= \lim_{n\to\infty} \Gamma(\rho_{\varphi_1^n}, \rho_{\varphi_2^n}) = \lim_{n\to\infty} \langle \varphi_1^n, \varphi_2^n \rangle_{H_0^{-1}(D)} \\ &\le \lim_{n\to\infty} \|\varphi_1^n\|_{H_0^{-1}(D)} \|\varphi_2^n\|_{H_0^{-1}(D)} = \lim_{n\to\infty} \sqrt{\Gamma(\rho_{\varphi_1^n})\Gamma(\rho_{\varphi_2^n})} = \sqrt{\Gamma(\rho_1)\Gamma(\rho_2)} < \infty\end{aligned}$$

and hence $\rho_1 + \rho_2 \in \mathfrak{M}_D$. Consequently $\mathfrak{M}_D$ is a (complete) vector space and clearly Γ is an inner product on $\mathfrak{M}_D/\approx$. □

This result shows that definition 14 is indeed well-defined. We can subsequently make the following observation.

Corollary 1.6.8. *The space $\mathfrak{M}_D/\approx$ can be identified with $H_0^{-1}(D)$.*

Proof. Define the map $I : C_0^\infty(D) \to \mathfrak{M}_D/\approx$, $\varphi \mapsto \rho_\varphi$. This map is clearly linear and injective, and by 1.6.11 norm preserving. Due to proposition 1.2.1, the test functions are dense in $H_0^{-1}(D)$, and so is the image of the test functions under I by proposition 1.6.6. Therefore we can extend I to an isomorphism taking $H_0^{-1}(D)$ to $\mathfrak{M}_D$. □

We can now state the main theorem of the section.

Theorem 1.6.9. *The Gaussian free field can be identified with the centred Gaussian process* $\boldsymbol{h}$ *on the index set* $\mathfrak{M}_D$*, which has covariance function* $Cov(\boldsymbol{h}_{\rho_1}, \boldsymbol{h}_{\rho_2}) = \Gamma(\rho_1, \rho_2)$.

Proof. The inner product Γ is a symmetric and positive bilinear form on a finite collection of measures in $\mathfrak{M}_D$. Clearly it is consistent on different choices of any finite collections, and by Kolmogorov's extension theorem, there exists a stochastic process $(\mathbf{h}_\rho)_{\rho \in \mathfrak{M}_D}$ with the desired property. □

Extension of the GFF to unbounded domains

Although if we are not interested in the unbounded case in this thesis (as it is not necessary for the construction of the Liouville measure), we include a brief discussion for the sake of completeness.

(i) Seeing the GFF as a Gaussian process on $\mathfrak{M}_D$ with covariance function Γ, we can extend the definition to all domains, where the Green´s function is finite outside the diagonal. This includes most unbounded domains as well. However, it will no longer be an element in $H_0^1(D)$.

(ii) In the second dimension, the extension of the GFF to unbounded simply connected domains D is straightforward by using the conformal map $\phi : \mathbb{D} \to D$ to construct the Green´s function on D, as in proposition 1.6.4. Further, the conformal invariance of the GFF (theorem 1.5.2) still holds, and therefore $\mathbf{h}^D = \mathbf{h}^{\mathbb{D}} \circ \phi^{-1}$ gives the corresponding random distribution on D.

1.7 Circle averages of the GFF

This section follows [DS11], as we consider the average on spheres of the GFF. The aim is to show that decreasing the radius of a sphere gives a Brownian motion after the right scaling of the radius. For simplicity (and because sufficient for our purpose later on), we develop the statement only for bounded domains D. For $\varepsilon > 0$ we use again $D_\varepsilon := \{z \in D : diam(z, \partial D > \varepsilon\}$.

Definition 15 (Circle averages)**.** Let $\varepsilon > 0$ and $z \in D_\varepsilon$. We write ρ_ε^z as the uniform measure on the sphere $\partial B_\varepsilon(z)$. We then call

$$\mathbf{h}_\varepsilon^z := \langle \mathbf{h}, \rho_\varepsilon^z \rangle$$

the sphere average (or circle average if $d = 2$) around z of the GFF, with respect to the radius ε.

We start with the following observation.

Lemma 1.7.1 (sphere average of the Green´s function). *For $\varepsilon > 0$ and $z \in \overline{D}_\varepsilon$ we set*

$$\xi_\varepsilon^z(x) := \int_D G(x,y)\,\rho_z^\varepsilon(dy).$$

Then the function $\xi_\varepsilon^z : D \to \mathbb{R}_+$ has values

$$\xi_\varepsilon^z(x) = \begin{cases} G(x,z) & x \notin B_\varepsilon(z) \\ \Phi(\varepsilon) - \phi_z(x) & x \in B_\varepsilon(z) \end{cases}$$

where Φ and ϕ_z are defined like in subsection 1.6.1.

Proof. The Green´s function $G(x,.)$ is harmonic in $B_\varepsilon(z)$ if $\|x-z\| > \varepsilon$. Hence we deduce

$$\xi_\varepsilon^z(x) = \fint_{\partial B_\varepsilon(z)} G(x,y)\,dS(y) = G(x,z)$$

and get the statement for $x \notin \overline{B_\varepsilon(z)}$. For $x \in B_\varepsilon(z)$ split ξ_ε^z into

$$\xi_\varepsilon^z(x) = \fint_{\partial B_\varepsilon(z)} \Phi(x-y)\,dS(y) - \fint_{\partial B_\varepsilon(z)} \phi_x(y)\,dS(y) =: I_\varepsilon^z(x) - J_\varepsilon^z(x).$$

That $J_\varepsilon^z(x) = \phi_x(z)$ is straightforward by the harmonicity of ϕ_x. Observe that for $x \in B_\varepsilon(z)$

$$\frac{\Phi(x+he_j-y) - \Phi(x-y)}{h}$$

converges uniformly on the compact set $\partial B_\varepsilon(z)$ as $h \to 0$. Therefore, we can include the Laplace operator inside the integral in

$$\Delta I_\varepsilon^z(x) = \fint_{\partial B_\varepsilon(z)} \Delta_x \Phi(x-y)\,dS(y)$$

and as $\Phi(x-y)$ is harmonic outside of y, it follows that I_ε^z is harmonic in $B_\varepsilon(z)$. Note that the rotational invariance of Φ implies rotational invariance of I_ε^z around z, and consequently I_ε^z is constant on $B_\varepsilon(z)$. By recognising

$$I_\varepsilon^z(z) = \fint_{\partial B_\varepsilon(z)} \Phi(\varepsilon)\,dS(y) = \Phi(\varepsilon)$$

we get the desired result on $B_\varepsilon(z)$. The value for $x \in \partial B_\varepsilon(z)$ follows by continuity. $\square$

Remark 15. (i) From theorem 1.6.9 we know, that $\mathbf{h}_\varepsilon^z$ is a Gaussian centred variable. Lemma 1.7.1 states

$$Cov(\mathbf{h}_{\varepsilon_1}^{z_1}, \mathbf{h}_{\varepsilon_2}^{z_2}) = \fint_{\partial B_{\varepsilon_2}(z_2)} \xi_{\varepsilon_1}^{z_1}(x)\,dS(x) = \fint_{\partial B_{\varepsilon_1}(z_1)} \xi_{\varepsilon_2}^{z_2}(x)\,dS(x)$$

which yields
$$Var(\mathbf{h}_\varepsilon^z) = \Phi(\varepsilon) - \phi_z(z)$$
and
$$Var(\mathbf{h}_\varepsilon^z) = -\frac{1}{2\pi}\log(\varepsilon) + \frac{1}{2\pi}\log(R(z;D))$$
in the second dimension.

(ii) The increments of the process $(\mathbf{h}_\varepsilon^z)_{\varepsilon\in(0,diam(z,\partial D)]}$ are normally distributed as for $0 < t \le s \le diam(z,\partial D)$
$$\mathbf{h}_t^z - \mathbf{h}_s^z = \langle \mathbf{h}, \rho_t^z - \rho_s^z\rangle$$
which is $\mathcal{N}(0,\Gamma(\rho_t^z - \rho_s^z))$ distributed. The variance is
$$Var(\mathbf{h}_t^z - \mathbf{h}_s^z) = Var(\mathbf{h}_t^z) + Var(\mathbf{h}_s^z) - 2Cov(\mathbf{h}_t^z, \mathbf{h}_s^z) = \Phi(t) - \Phi(s)$$
as
$$Cov(\mathbf{h}_t^z, \mathbf{h}_s^z) = \fint_{B_t(z)} \xi_s^z(x)\, dS(x) = \xi_s^z(z) = \Phi(s) - \phi_z(z).$$
Further the increments are independent, as it is easy to see, that the covariance of two different increments is zero.

This remark suggests that we can create a Brownian motion out of the sphere averages. However, the continuity of this process is still missing. To show this we will need the following statement.

Theorem 1.7.2 (Kolmogorov's continuity theorem for random fields)**.** *Let $(X_t)_{t\in A}$ be a random process/ random field on a bounded set $A \subset \mathbb{R}^d$. If there are positive constants α, β, C such that*
$$\mathbb{E}(|X_s - X_t|^\alpha) \le C\|s-t\|^{d+\beta}$$
for all $s,t \in A$, then there exists a version of X which is almost surely η-Hölder continuous for $\eta \in (0, \frac{\beta}{\alpha})$.

Proof. Without loss of generality, assume $A \subset \mathbb{R}_+^d$. We fix T large enough such that $A \subset [0,T]^d$ and extend X to be a process on $[0,T]^d$, where $X_t = 0$ for $t \notin A$. Take
$$P_n := \{(k_1 2^{-n}, \dots, k_d 2^{-n}) \in \mathbb{R}^d : k_j = 0, \dots, 2^n T \text{ for } j = 1,\dots,d\} \subset [0,T]^d$$
as the dyadic points in $[0,T]^d$ and write $x \sim y$ for points $x, y \in P_n$ if they are direct neighbours of each other. By Kolmogorov's inequality
$$\begin{aligned}\mathbb{P}(\sup_{\{x,y\in P_n: x\sim y\}} |X_x - X_y| > \underbrace{2^{-n\eta}}_{=\|x-y\|^\eta}) &\le \sum_{\{x,y\in P_n:x\sim y\}} \mathbb{P}(|X_x - X_y|^\alpha > 2^{-n\eta\alpha}) \\ &\le \sum_{\{x,y\in P_n:x\sim y\}} 2^{n\eta\alpha}\mathbb{E}(|X_x - X_y|^\alpha) \\ &\le 2^{nd}2^{n\eta\alpha}C2^{-n(d+\beta)} = 2^{n(\eta\alpha-\beta)}C\end{aligned}$$

which tells that

$$\sum_{n=1}^{\infty} \mathbb{P}(\sup_{\{x,y \in P_n : x \sim y\}} |X_x - X_y| > 2^{-n\eta}) < \infty$$

because $\eta < \frac{\beta}{\alpha}$. Therefore by Borel-Cantelli lemma, the events

$$\left\{ \sup_{\{x,y \in P_n : x \sim y\}} |X_x(\omega) - X_y(\omega)| > 2^{-n\eta} \right\}_{n=1}^{\infty}$$

occur finitely often almost surely. This means that there exists an almost surely finite random variable M, such that

$$\sup_{n \in \mathbb{N}} \sup_{\{x,y \in P_n : x \sim y\}} |X_x(\omega) - X_y(\omega)| 2^{n\eta} \leq M(\omega). \tag{1.7.1}$$

Set $P := \bigcup_{n=1}^{\infty} P_n$. We want to show that there is a constant $\tilde{M}(\omega)$ such that

$$|X_s(\omega) - X_t(\omega)| \leq \tilde{M}(\omega) \|t - s\|^{\eta} \tag{1.7.2}$$

for all $s, t \in P$. We start with the case $s, t \in P$ such that $s_1 < t_1$ and $s_k = t_k$ for $k = 2, \dots, d$. There exists $r, k \in \mathbb{N}$ such that $2^{-r} \leq t_1 - s_1 < 2^{-(r-1)}$ and $k2^{-r} \in [s_1, t_1]$. Further there are $l, m \in \mathbb{N}$ and $\{\varepsilon_j\}_{j=0}^{l}$, $\{\varepsilon'_j\}_{j=0}^{,}$ which are equal zero or one, such that

$$s_1 = k2^{-r} - \sum_{j=0}^{l} \varepsilon_j 2^{-r+j}$$
$$t_1 = k2^{-r} + \sum_{j=0}^{m} \varepsilon'_j 2^{-r+j}$$

and we define

$$s_1^i = k2^{-r} - \sum_{j=0}^{i} \varepsilon_j 2^{-r+j} \qquad \text{for } i = 0, \dots, l$$
$$t_1^i = k2^{-r} - \sum_{j=0}^{i} \varepsilon'_j 2^{-r+j} \qquad \text{for } i = 0, \dots, m.$$

Then we can split

$$\begin{aligned} &|X_s - X_t| \\ \leq &|X_{t_1^1, \dots, t_d} - X_{s_1^1, \dots, t_d}| + \sum_{i=1}^{m-1} \varepsilon_i |X_{s_1^{i+1}, \dots, s_d} - X_{s_1^i, \dots, s_d}| + \sum_{i=1}^{l-1} \varepsilon'_i |X_{t_1^{i+1}, \dots, t_d} - X_{t_1^i, \dots, t_d}| \end{aligned}$$

and using equation 1.7.1 yields

$$
\begin{aligned}
|X_s(\omega) - X_t(\omega)| &\leq M(\omega)(2^{-(r+1)\eta} + \sum_{i=1}^{m-1} \varepsilon_i 2^{-(r+i+1)\eta} + \sum_{i=1}^{l-1} \varepsilon_i' 2^{-(r+i+1)\eta}) \\
&\leq M(\omega)\left(2^{-(r+1)\eta} + 2^{-(r+2)\eta}\frac{1}{1-2^{-\eta}} + 2^{-(r+2)\eta}\frac{1}{1-2^{-\eta}}\right) \\
&= \bar{M}(\omega)2^{-(r-1)\eta} \leq \bar{M}(\omega)|t_1 - s_1|^\eta
\end{aligned}
$$

where $\bar{M}(\omega) = M(\omega)\left(2^{-2\eta} + 2.2^{-(r+3)\eta}\frac{1}{1-2^{-\eta}}\right)$, which is an independent bound of the choice of s_1, t_1. We can do the same procedure and obtain the same bound for $t, s \in P$ when they differ in another coordinate. Hence we can write for arbitrary $s, t \in P$

$$
\begin{aligned}
&|X_t - X_s| \\
\leq &|X_{t_1,t_2,\ldots,t_d} - X_{s_1,t_2,\ldots,t_d}| + |X_{s_1,t_2,\ldots,t_d} - X_{s_1,s_2,\ldots,t_d}| + \cdots + |X_{t_1,s_2,\ldots,t_d} - X_{s_1,s_2,\ldots,s_d}| \\
\leq &\bar{M}d \max_{j=1,\ldots,d}\{|t_j - s_j|^\eta\} \leq \bar{M}d\|t-s\|^\eta
\end{aligned}
$$

and taking $\tilde{M} = \bar{M}d$ gives 1.7.2 and so $(X_t)_{t\in P}$ is η-Hölder continuous almost surely. Let the random field $(\tilde{X}_t)_{t\in A}$ be the continuous extension of $(X_t)_{t\in P\cap A}$. In the zero probability case when $(X_t)_{t\in P\cap A}$ is not η-Hölder continuous, set $\tilde{X}_t \equiv 0$, such that $\tilde{X}$ is always η-Hölder continuous. By Fatou's lemma

$$
\begin{aligned}
\mathbb{E}(|X_t - \tilde{X}_t|^\alpha) =&\mathbb{E}(\liminf_{t_n\to t}|X_t - \tilde{X}_{t_n}|^\alpha) \\
&\leq \liminf_{t_n\to t}\mathbb{E}(|X_t - \underbrace{\tilde{X}_{t_n}}_{=X_{t_n}}|^\alpha) \leq \liminf_{t_n\to t} C\|t_n - t\|^{d+\beta} = 0
\end{aligned}
$$

where the sequence t_n is chosen to be in P. Therefore $\tilde{X}$ is indeed a version of X. □

Proposition 1.7.3. *There exists a version of the GFF, such that the random field $\boldsymbol{h}_\varepsilon^z$ acting on $A := \{(z,\varepsilon) : z \in D, \varepsilon \in (0, diam(z,D)]\}$ is almost surely locally η-Hölder continuous for $\eta \in (0, \frac{1}{2})$.*

Proof. Take $\varepsilon_0 > 0$ and set $A_{\varepsilon_0} := \{(z,\varepsilon) : z \in D_{2\varepsilon_0}, \varepsilon \in (\varepsilon_0, diam(z,D)]\}$. Our first aim is to show, that there is a constant K_{ε_0} such that

$$
Var(\mathbf{h}_{\varepsilon_1}^{z_1} - \mathbf{h}_{\varepsilon_2}^{z_2}) \leq K_{\varepsilon_0}(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|) \tag{1.7.3}
$$

for all $(z_1,\varepsilon_1), (z_2,\varepsilon_2) \in A_{\varepsilon_0}$. For fixed $x \in D$ the function $(z,\varepsilon) \mapsto \xi_\varepsilon^z(x)$ is continuous on the compact set $\overline{A_{\varepsilon_0}}$ and hence Lipschitz continuous. Consequently there exists a constant L_x, such that

$$
|\xi_{\varepsilon_1}^{z_1}(x) - \xi_{\varepsilon_2}^{z_2}(x)| \leq L_x(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|).
$$

Let $L_{\varepsilon_0} := \max\{L_x : x \in \overline{D_{\varepsilon_0}}\}$. For $z_1, z_2 \in D_{2\varepsilon_0}$ we estimate by remark 15 (i)

$$\begin{aligned} Var(\mathbf{h}^{z_1}_{\varepsilon_1} - \mathbf{h}^{z_2}_{\varepsilon_2}) =& Cov(\mathbf{h}^{z_1}_{\varepsilon_1} - \mathbf{h}^{z_2}_{\varepsilon_2}, \mathbf{h}^{z_1}_{\varepsilon_1}) - Cov(\mathbf{h}^{z_1}_{\varepsilon_1} - \mathbf{h}^{z_2}_{\varepsilon_2}, \mathbf{h}^{z_2}_{\varepsilon_2}) \\ \leq& \fint_{\partial B_{\varepsilon_1}(z_1)} |\xi^{z_1}_{\varepsilon_1}(x) - \xi^{z_2}_{\varepsilon_2}(x)| \, dS(x) + \fint_{\partial B_{\varepsilon_2}(z_2)} |\xi^{z_1}_{\varepsilon_1}(x) - \xi^{z_2}_{\varepsilon_2}(x)| \, dS(x) \\ \leq& \fint_{\partial B_{\varepsilon_1}(z_1)} L_{\varepsilon_0}(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|) \, dS(x) \\ &+ \fint_{\partial B_{\varepsilon_2}(z_2)} L_{\varepsilon_0}(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|) \, dS(x) \\ =& 2L_{\varepsilon_0}(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|) \end{aligned}$$

and for taking $K_{\varepsilon_0} = 2L_{\varepsilon_0}$ the inequality 1.7.3 is fulfilled. For n being an even number

$$\mathbb{E}(|\mathcal{N}(0, \sigma^2)|^n) = n!!\sigma^n$$

holds, which is easy to check by partial integration. Applying this, we conclude

$$\mathbb{E}((\mathbf{h}^{z_1}_{\varepsilon_1} - \mathbf{h}^{z_2}_{\varepsilon_2})^{2n}) = n!!Var(\mathbf{h}^{z_1}_{\varepsilon_1} - \mathbf{h}^{z_2}_{\varepsilon_2})^n \leq n!!K^n_{\varepsilon_0}(\|z_1 - z_2\| + |\varepsilon_1 - \varepsilon_2|)^n. \tag{1.7.4}$$

We can write this equation in light of theorem 1.7.2 with $C = n!!(K_{\varepsilon_0}\sqrt{2})^n$, $\alpha = 2n$ and $\beta = n - d$. Letting n tend to infinity, shows that $\frac{\beta}{\alpha}$ tends to $\frac{1}{2}$ and hence there exists a version $\mathbf{h}$ such that $\mathbf{h}^z_\varepsilon$ is η-Hölder continuous on A_{ε_0} for $\eta \in (0, \frac{1}{2})$. We can assume that the versions of the GFF coincides with respect to different ε_0. Therefore there exists a version of the GFF, which is η-Hölder continuous on A_{ε_0} for all $\varepsilon_0 > 0$. Since A_{ε_0} goes to A, we obtain the statement. □

Remark 16. Note that we have just shown locally Hölder continuity, as the Hölder constant depends on A_{ε_0}.

Proposition 1.7.3 implies that there exists a version of the GFF, such that for all $z \in D$ the process $(\mathbf{h}^z_\varepsilon)_{\varepsilon \in (0, diam(z, \partial D)]}$ is continuous almost surely. (It can be shown that this is the same version which is almost surely an element in $H^m_0(D)$). Having this in hand together with remark 15, gives the main statement of the section.

Theorem 1.7.4 (circle averages form a Brownian motion)**.** *Let $\varepsilon_0 > 0$, $z \in D_{\varepsilon_0}$ and $t_0 = \Phi(\varepsilon_0)$. Then the process $(X_t)_{t \in [0,\infty)}$ defined by*

$$X_t = \boldsymbol{h}^z_{\Phi^{-1}(t+t_0)} - \boldsymbol{h}^z_{\varepsilon_0}$$

is a Brownian motion. In the two dimensional case this is

$$X_t = \boldsymbol{h}^z_{e^{-2\pi t}\varepsilon_0} - \boldsymbol{h}^z_{\varepsilon_0}.$$

For the rest of the thesis, we are just interested in bounded domains in $\mathbb{R}^2$. To make things easier, we switch to the standard scale of the GFF used in Liouville quantum gravity. Therefore from now on, if we talk about the GFF, we will think of

$$h = \sqrt{2\pi}\mathbf{h}$$

where $\mathbf{h}$ is the GFF on a bounded domain D of $\mathbb{R}^2$ from definition 9. We summarize the following properties which we will use quite frequently.

Corollary 1.7.5. *(i) For $\varepsilon \leq dist(z, \partial D)$ the variance of the circle average is given by*

$$Var(h_\varepsilon(z)) = -\log(\varepsilon) + \log(R(z; D)).$$

(ii) Take $z \in D$ and $\varepsilon_0 \leq dist(z, \partial D)$. Then the process $(X_t)_{t \geq 0}$ with

$$X_t = h^z_{e^{-t}\varepsilon_0} - h^z_{\varepsilon_0}$$

is a Brownian motion.

(iii) The process $(X_t)_{t \geq 0}$ is independent of what happens outside of $B_{\varepsilon_0}(z)$.

Proof. (i) follows directly from 15 (ii). For (ii) consider theorem 1.7.4. The property (iii) can be seen by writing h as in theorem 1.4.1 by $h = h_0 + \varphi$ where h_0 is the GFF on $B_{\varepsilon_0}(z)$. Because φ is harmonic in $B_{\varepsilon_0}(z)$, we know that $\langle \varphi, \rho^z_{e^{-(t+t_0)}} \rangle = \langle \varphi, \rho^z_{e^{-t_0}} \rangle$ and hence

$$X_t = \langle h_0, \rho^z_{e^{-(t+t_0)}} \rangle - \langle h_0, \rho^z_{e^{-t_0}} \rangle$$

which is independent of φ. □

2 The Liouville measure

This chapter follows in many aspects the structure of [BP21] Chapter 2. The Liouville measure (we will use the shortcut LM) is roughly speaking the measure which gives a meaning to the expression

$$e^{\gamma h(x)}$$

where $\gamma \in [0, 2\pi]$ and h is the GFF. The LM is used for quantum surfaces (see [DMS14]) and appears when looking at the Liouville field (see [HRV18]). In general it seems to be important in theoretical physics, e.g. in string theory, or in two dimensional quantum field theory (for example in Liouville quantum gravity).

2.1 Liouville measure and Liouville quantum gravity

This section is ment to raise some interest in the Liouville measure, by taking an example of its appearance in Liouville quantum gravity. This section is informal and is not needed for the rest of the thesis. This example comes from [HRV18], [DKRV16] and [R$^+$16].

In Liouville quantum gravity, one is interested in random fields on a domain D with "density" function characterised by the Liouville action. The Liouville action S is defined for a field X and metric g on D by

$$S(X,g) := \frac{1}{4\pi}\int_D (|\nabla X|^2 + QR_g X + 4\pi\mu e^{\gamma X})\, d\lambda_g + \frac{1}{2\pi}\int_{\partial D}(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X})\, d\lambda_{\partial g}$$

where R_g and K_g are the Ricci Scalar curvature and Geodesic curvature, λ_g and $\lambda_{\partial g}$ the surface and line measures on D with respect to the metric g, and Q, μ, μ_∂ are positive constants. For a functional F we write informally

$$\mathbb{E}(F((X)_{t\in D})) = \frac{1}{Z}\int F(X) e^{-S(X,g)}\, dX$$

where Z is a normalisation constant, and the integral encloses all possible shapes which X can take. Note that this expression indeed makes sense, when we look at a discrete version of the field X. (Compare this with remark 5, and consider that in the case of a flat manifold and $\mu = \mu_\partial = 0$ this gives the GFF, as in this case the Liouville action coincides with the Dirichlet energy.) Roughly speaking, the field wants to be a field which has minimal Liouville action. Plugging in the definition of the Liouville action, we obtain

$$\mathbb{E}(F((X_t)_{t\in D})) =$$
$$\frac{1}{Z}\int F(X) e^{-\frac{1}{4\pi}\int_D (QR_g X + 4\pi\mu e^{\gamma X})\, d\lambda_g - \frac{1}{2\pi}\int_{\partial D}(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X})\, d\lambda_{\partial g}} e^{-\frac{1}{2}\frac{1}{2\pi}\int_D |\nabla X|\, d\lambda_g}\, dX.$$

We have seen in remark 5 that the law of the $\sqrt{2\pi}$ scaled GFF is intuitively given by

$$\mu_{GFF}(dX) = e^{-\frac{1}{2}\frac{1}{2\pi}\int_D |\nabla X|^2 \, d\lambda_g} \, dX$$

and therefore by denoting h as the GFF yields

$$\begin{aligned}&\mathbb{E}(F((X_t)_{t\in D}))\\ =&\frac{1}{Z}\int F(h)e^{-\frac{1}{4\pi}Q\langle h,R_g\rangle_g}e^{-\frac{1}{2\pi}Q\langle h,K_g\rho_{\partial D}\rangle_g}e^{-\mu\int_D e^{\gamma h}\, d\lambda_g}e^{-\mu_\partial \int_{\partial D} e^{\frac{\gamma}{2}h}\, d\lambda_{\partial g}}\, dh\end{aligned}$$

where all integration is with respect to the metric g. One can deal with $\langle h, R_g\rangle_g$ and $\langle h, K_g\rho_{\partial D}\rangle_g$ as they are Gaussian variables as in theorem 1.6.9. What we have to make sense of is

$$e^{\gamma h(z)} \, d\lambda_g(z)$$

which is not straightforward, as the GFF is not defined pointwise. Still for $\gamma \in [0, 2]$ it is possible to construct a measure μ_γ, which satisfies heuristically

$$\mu_\gamma(A) = \int_A e^{\gamma h(z)} dz.$$

Proving this for $\gamma \in [0, 2)$ will be the scope of this chapter.

2.2 Preparation

Before we start with the content of this chapter, we recap some statements from probability theory, which we will shortly make frequent use of.

Lemma 2.2.1 (Standard bound for Gaussian distribution)**.** *Let X be $\mathcal{N}(0, 1)$ distributed. Then*

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-\frac{x^2}{2}} \leq \mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}$$

for $x > 0$.

Proof. The statement follows by using $e^{-\frac{t^2}{2}} = -\frac{1}{t}\frac{d}{dt}e^{-\frac{t^2}{2}}$ and two times partial integration for

$$\begin{aligned}\sqrt{2\pi}\mathbb{P}(X \geq x) &= \int_x^\infty e^{-\frac{t^2}{2}}\, dt = \frac{1}{x}e^{-\frac{x^2}{2}} - \int_x^\infty \frac{1}{t^2}e^{-\frac{t^2}{2}}\\ &= \frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} + \int_x^\infty \frac{3}{t^4}e^{-\frac{t^2}{2}}.\end{aligned}$$

□

Lemma 2.2.2. *Let X be a d-dimensional random vector on some probability space. Then the following is equivalent:*

(i) X is $\mathcal{N}(\mu,\Sigma)$ distributed.

(ii) $\mathbb{E}(e^{\langle\alpha,X\rangle}) = e^{\langle\alpha,\mu\rangle+\frac{1}{2}\alpha^T\Sigma\alpha}$ for all $\alpha\in\mathbb{R}^d$.

Proof. We start by showing the result for a normal distributed random variable X with mean $\mu\in\mathbb{R}$ and variance $\sigma^2\in\mathbb{R}$.
$(i)\Rightarrow(ii)$. First we assume $X\sim\mathcal{N}(0,1)$. Write

$$\mathbb{E}(e^{\alpha X}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\alpha x}e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{\alpha^2}{2}}e^{-\frac{(\alpha-x)^2}{2}}\,dx = e^{\frac{\alpha^2}{2}}$$

where we used the binomial formula. In the general case $X\sim\mathcal{N}(\mu,\sigma^2)$

$$\mathbb{E}(e^{\alpha X}) = \mathbb{E}(e^{\alpha\mu+\alpha\sigma\mathcal{N}(0,1)})$$

yields the statement.
$(ii)\Rightarrow(i)$. For $z\in\mathbb{C}$ the complex functions $z\mapsto\mathbb{E}(e^{zX})$ and $z\mapsto e^{z\mu+\frac{1}{2}z^2\sigma^2}$ are both analytic. As they coincide for $z\in\mathbb{R}$, they coincide for $z\in\mathbb{C}$ as well. Therefore

$$\mathbb{E}(e^{i\alpha X}) = e^{i\alpha\mu-\frac{\alpha^2\sigma^2}{2}}$$

and hence the characteristic function of X is the same as the characteristic function of a $\mathcal{N}(\mu,\sigma^2)$ distributed random variable.

It is well known, that a Gaussian vector X with mean $\mu\in\mathbb{R}^d$ and covariance matrix $\Sigma\in\mathbb{R}^{d\times d}$ can be uniquely characterised by being the random vector with

$$\langle\alpha,X\rangle\sim\mathcal{N}(\langle\alpha,\mu\rangle,\alpha^T\Sigma\alpha)$$

for all $\alpha\in\mathbb{R}^d$. Applying the result in the one dimensional case for $\langle\alpha,X\rangle$ gives the statement. $\square$

The next result is a simplifies version of Girsanow's theorem.

Lemma 2.2.3. *Let $\mathbb{P}$ be the law of a Gaussian random vector X with mean $\mu\in\mathbb{R}^d$ and covariance matrix Σ. For $\alpha\in\mathbb{R}^d$ define another law $\mathbb{Q}$ for X by*

$$\mathbb{Q}(dX) = \frac{e^{\langle\alpha,X\rangle}}{\mathbb{E}(e^{\langle\alpha,X\rangle})}\,\mathbb{P}(dX).$$

Then X is $\mathcal{N}(\mu+\Sigma\alpha,\Sigma)$ distributed with respect to $\mathbb{Q}$.

Remark 17. Note that $\mathbb{Q}$ is indeed a probability measure.

Proof. For $\xi\in\mathbb{R}^d$ we use lemma 2.2.2 to calculate

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(e^{\langle\xi,X\rangle}\right) &= \frac{1}{\mathbb{E}(e^{\langle\alpha,X\rangle})}\int_{\mathbb{R}^d} e^{\langle\xi,X\rangle}e^{\langle\alpha,X\rangle}\,\mathbb{P}(dX) = \frac{\mathbb{E}(e^{\langle(\xi+\alpha),X\rangle})}{\mathbb{E}(e^{\langle\alpha,X\rangle})}\\ &= \frac{e^{\langle(\xi+\alpha),\mu\rangle+\frac{1}{2}(\xi+\alpha)^T\Sigma(\xi+\alpha)}}{e^{\langle\alpha,\mu\rangle+\frac{1}{2}\alpha^T\Sigma\alpha}} = e^{\langle\xi,\mu+\Sigma\alpha\rangle}e^{\frac{1}{2}\xi^T\Sigma\xi}\end{aligned}$$

and the result follows again by lemma 2.2.2. $\square$

2.3 Approximation measure and the main statement

Having this in hand, we start thinking about how to approximate $e^{\gamma h(z)}$. A natural approach is to see it as the limit of the exponential of the circle averages around z which is well defined. But from lemma 1.7.1 and lemma 2.2.2 we know that

$$\mathbb{E}(e^{\gamma h_\varepsilon(z)}) = \varepsilon^{-\frac{\gamma^2}{2}} R(z;D)^{\frac{\gamma^2}{2}}$$

and so we expect $e^{\gamma h_\varepsilon(z)}$ to tend to infinity, which coincides with what we would expect by an approximation of $e^{\gamma h(z)}$. As we don´t want our target measure blowing up in each point to infinity, we have to scale our approximation by a factor $\varepsilon^{\frac{\gamma^2}{2}}$. Consequently we define the approximation measure $\mathcal{M}_\varepsilon$ for a $\gamma \in [0,2)$ and $\varepsilon > 0$ as

$$\mathcal{M}_\varepsilon(dz) = \tilde{\varepsilon}^{\frac{\gamma^2}{2}} e^{\gamma h_{\tilde{\varepsilon}}(z)} \, dz$$

where $\tilde{\varepsilon} = \min\{\varepsilon, dist(z, \partial D)\}$.

The main statement of this chapter, is to show the weak convergence of these measures.

Theorem 2.3.1. *Let D be a bounded domain in $\mathbb{R}^2$ and $\gamma \in [0,2)$. Then the sequence of measures $(\mathcal{M}_{2^{-k}})_{k=1}^\infty$ converges weakly almost surely against a measure $\mathcal{M}$ (which we will call the Liouville measure).*

Before starting the proof, we want to take a further look at the density function of $\mathcal{M}_\varepsilon$ which we will denote by m_ε.

2.3.1 Density of $\mathcal{M}_\varepsilon$ is a martingale at every point

From stochastic calculus we recap

Lemma 2.3.2. *Let $(W_t)_{t\geq 0}$ be a Brownian motion. Then*

$$M_t = e^{W_t - \frac{t}{2}}$$

is a martingale.

Proof. We use lemma 2.2.2 to get

$$\mathbb{E}(e^{W_t - \frac{t}{2}}) = 1$$

and therefore M_t is an integrable process. Applying Ito´s formula and cancelling all terms which equal zero yields

$$M_t = M_0 + \int_0^t e^{W_s - \frac{s}{2}} \, dW_s + \int_0^t -\frac{1}{2} e^{W_s - \frac{s}{2}} \, ds + \frac{1}{2}\int_0^t e^{W_s - \frac{s}{2}} \, d[W]_s = 1 + \int_0^t e^{W_s - \frac{s}{2}} \, dW_s$$

and it is a classic result from the theory of Ito´s integration that $\int_0^t e^{W_s - \frac{s}{2}} \, dW_s$ is a local martingale. The quadratic variation of M_t is

$$[M]_t = \int_0^t (e^{W_s - \frac{s}{2}})^2 \, d[W]_s = \int_0^t e^{2W_s - s} \, ds$$

and by Fubini and again lemma 2.2.2

$$\mathbb{E}([M]_t) = \int_0^t \mathbb{E}(e^{2W_s})e^{-s}\,ds = e^t$$

which states that for all $t \geq 0$, the process $(M_{\min\{s,t\}})_{s\geq 0}$ has bounded quadratic variation and therefore is bounded in $L^2(\mathbb{P})$. Consequently $(M_{\min\{s,t\}})_{s\geq 0}$ is a real martingale for all $t \geq 0$, and hence $(M_t)_{t\geq 0}$ is a real martingale as well. $\square$

Remark 18. Note that clearly lemma 2.3.2 stays valid for time scaled Brownian motions as well.

Proposition 2.3.3. *For $z \in D$ let $\varepsilon_0 \leq dist(z, \partial D)$. Then the process $(m_\varepsilon(z))_{\varepsilon\leq\varepsilon_0}$ is a martingale with $\mathbb{E}(m_\varepsilon(z)) = R(z; D)^{\frac{\gamma^2}{2}}$.*

Proof. Assume $\varepsilon \leq \varepsilon_0 \leq dist(z, \partial D)$. We can write

$$m_\varepsilon(z) = e^{\gamma(h_\varepsilon(z)-h_{\varepsilon_0}(z))-\frac{\gamma^2}{2}\log(\frac{\varepsilon_0}{\varepsilon})}e^{\gamma h_{\varepsilon_0}(z)+\frac{\gamma^2}{2}\log(\varepsilon_0)}.$$

From corollary 1.7.5 (i) we know that $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is a time scaled Brownian motion with variance $\log(\frac{\varepsilon_0}{\varepsilon})$. Therefore by Lemma 2.3.2

$$e^{\gamma(h_\varepsilon(z)-h_{\varepsilon_0}(z))-\frac{\gamma^2}{2}\log(\frac{\varepsilon_0}{\varepsilon})}$$

is a martingale. Further this term is independent of $e^{\gamma h_{\varepsilon_0}(z)+\frac{\gamma^2}{2}\log(\varepsilon_0)}$, and consequently $m_\varepsilon(z)$ fulfills the martingale property. Additionally

$$\mathbb{E}(e^{\gamma h_{\varepsilon_0}(z)+\frac{\gamma^2}{2}\log(\varepsilon_0)}) = \varepsilon_0^{\frac{\gamma^2}{2}}\,\mathbb{E}(e^{\gamma h_{\varepsilon_0}(z)}) = R(z; D)^{\frac{\gamma^2}{2}}$$

is finite, meaning $m_\varepsilon(z)$ is integrable for all $\varepsilon \leq \varepsilon_0$ which gives the result. $\square$

Remark 19. Note that by Fubini we obtain

$$\mathbb{E}(\mathcal{M}_\varepsilon(S)) = \int_S \mathbb{E}(m_\varepsilon(z))\,dz = \int_S R(z; D)^{\frac{\gamma^2}{2}}\,dz$$

meaning that the expectation of the approximation measure is constant and does not depend on ε.

Doob´s martingale convergence theorem says, that the positive martingale $m_\varepsilon(z)$ converges almost surely for all $z \in D$. We could hope that for all $S \subset D$ the process $\mathcal{M}_\varepsilon(S)$ is a martingale as well, which would be positive and therefore convergent for $\varepsilon \to 0$. But we won´t be able to find a filtration, such that $m_\varepsilon(z)$ is a martingale with respect to this filtration for all $z \in S$ at the same time, because this filtration would be too fine, and destroys the martingale property of m_ε in each point. Therefore we should not hope that $\mathcal{M}_\varepsilon(S)$ is indeed a martingale, and we need some more investigations to proof theorem 2.3.1.

2.4 Proof of the main statement

The important ingredient to prove theorem 2.3.1, is that for all $S \subset D$ which have its closure contained in D, the sequence $\mathcal{M}_{2^{-k}}(S)$ converges in $L^1(\mathbb{P})$. For easier notation we will use

$$\begin{aligned}\overline{h}_\varepsilon(z) &:= \gamma h_\varepsilon(z) - \frac{1}{2} Var(\gamma h_\varepsilon(z)) \\ &= \gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\left(\log\left(\frac{1}{\varepsilon}\right) + \log(R(z;D))\right) \\ \sigma(dz) &:= R(z;D)^{\frac{\gamma^2}{2}} dz \\ R_D &:= \max_{z\in D}\{R(z;D)\}\end{aligned}$$

where we recall that R_D exists by remark 13 (i). If $S \in D_\varepsilon$ for $\varepsilon > 0$, we can write

$$\mathcal{M}_\varepsilon(S) = \int_S \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)}\, dz = \int_S e^{\overline{h}_\varepsilon(z)} \sigma(dz)$$

such that we don´t have to deal with points having a distance smaller than ε to the boundary. Note that for $z \in D_\varepsilon$ the process $(e^{\overline{h}_\delta(z)})_{\delta\le\varepsilon}$ is by proposition 2.3.3 a martingale with expectation 1 and satisfies

$$\begin{aligned}\mathbb{E}\left(e^{2\overline{h}_\delta(z)}\right) &= \mathbb{E}\left(e^{2\gamma h_\delta(z)}\right)\delta^{\gamma^2} R(z;D)^{-\gamma^2} = \delta^{-\gamma^2} R(z;D)^{\gamma^2} \\ \mathbb{E}\left(e^{2\overline{h}_{\delta_2}(z) - 2\overline{h}_{\delta_1}(z)}\right) &= \mathbb{E}\left(e^{2\gamma\left(h_{\delta_2}(z) - h_{\delta_1}(z)\right)}\right)\frac{\delta_2}{\delta_1}^{\gamma^2} = \frac{\delta_1}{\delta_2}^{\gamma^2}\end{aligned} \tag{2.4.1}$$

by lemma 2.2.2. We start with our first step.

2.4.1 $L^2(\mathbb{P})$ convergence of $\mathcal{M}_{2^{-k}}(S)$ for $\gamma \in [0, \sqrt{2})$

We recap a frequently used lemma from stochastic calculus.

Lemma 2.4.1. *Let $(M_t)_{t\ge 0}$ be a martingale adapted by the filtration $(\mathcal{F}_t)_{t\ge 0}$. Then for $t \ge s$*

$$\mathbb{E}((M_t - M_s)^2 | \mathcal{F}_s) = \mathbb{E}(M_t^2 - M_s^2 | \mathcal{F}_s).$$

Proof. This is straight forward by

$$\mathbb{E}(M_t^2 - 2M_tM_s - M_s^2|\mathcal{F}_s) = \mathbb{E}(M_t^2|\mathcal{F}_s) - 2M_s \underbrace{\mathbb{E}(M_t|\mathcal{F}_s)}_{M_s} + M_s^2 = \mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s).$$

□

Proposition 2.4.2. *There exists a constant C depending on γ and $Vol(D)$, such that*

$$\mathbb{E}((\mathcal{M}_\varepsilon(S) - \mathcal{M}_{\frac{\varepsilon}{2}}(S))^2) \le C\varepsilon^{2-\gamma^2}$$

for all $S \in \overline{D}_\varepsilon$.

Proof. For simplifying notation, set $\delta = \frac{\varepsilon}{2}$. Write

$$\begin{aligned}\mathbb{E}\left((\mathcal{M}_\varepsilon(S)-\mathcal{M}_\delta(S))^2\right) =&\mathbb{E}\left(\int_{S^2}(e^{\overline{h}_\varepsilon(x)}-e^{\overline{h}_\delta(x)})(e^{\overline{h}_\varepsilon(y)}-e^{\overline{h}_\delta(y)})\,\sigma(dx)\sigma(dy)\right)\\ =&\int_{S^2}\mathbb{E}\left(e^{\overline{h}_\varepsilon(x)+\overline{h}_\varepsilon(y)}(1-e^{\overline{h}_\delta(x)-\overline{h}_\varepsilon(x)})(1-e^{\overline{h}_\delta(y)-\overline{h}_\varepsilon(y)})\right)\,\sigma(dx)\sigma(dy).\end{aligned}$$

The random variable $\overline{h}_\delta(x)-\overline{h}_\varepsilon(x)$ is independent from $D\setminus B_\varepsilon(x)$ by corollary 1.7.5 (ii), and therefore $(1-e^{\overline{h}_\delta(x)-\overline{h}_\varepsilon(x)})$ is independent from $e^{\overline{h}_\varepsilon(x)+\overline{h}_\varepsilon(y)}$ and $(1-e^{\overline{h}_\delta(y)-\overline{h}_\varepsilon(y)})$ if $B_\varepsilon(x)$ and $B_\varepsilon(y)$ don´t intersect. The same argument holds for $\overline{h}_\delta(y)-\overline{h}_\varepsilon(y)$. Meaning we can split up the expectation in the integral if $\|x-y\|\geq 2\varepsilon$, and applying the martingale property yields

$$\begin{aligned}\mathbb{E}\left(1-e^{\overline{h}_\delta(x)-\overline{h}_\varepsilon(x)}\right)&=1-\mathbb{E}\left(\mathbb{E}\left(e^{\overline{h}_\delta(x)-\overline{h}_\varepsilon(x)}|\overline{h}_\varepsilon(x)\right)\right)=1-\mathbb{E}\left(e^{-\overline{h}_\varepsilon(x)}(\mathbb{E}\left(e^{\overline{h}_\delta(x)}|\overline{h}_\varepsilon(x)\right)\right)\\ &=1-\mathbb{E}\left(e^{-\overline{h}_\varepsilon(x)}e^{\overline{h}_\varepsilon(x)}\right)=0\end{aligned}$$

and therefore the expectation in the integral is zero if $\|x-y\|\geq 2\varepsilon$. Consequently we can reduce to

$$\begin{aligned}&\mathbb{E}\left((\mathcal{M}_\varepsilon(S)-\mathcal{M}_\delta(S))^2\right)\\ &=\int_{\{x,y\in S:\|x-y\|<2\varepsilon\}}\mathbb{E}\left((e^{\overline{h}_\varepsilon(x)}-e^{\overline{h}_\delta(x)})(e^{\overline{h}_\varepsilon(y)}-e^{\overline{h}_\delta(y)})\right)\,\sigma(dx)\sigma(dy)\\ &\leq\int_{\{x,y\in S:\|x-y\|<2\varepsilon\}}\sqrt{\mathbb{E}\left((e^{\overline{h}_\varepsilon(x)}-e^{\overline{h}_\delta(x)})^2\right)}\sqrt{\mathbb{E}\left((e^{\overline{h}_\varepsilon(y)}-e^{\overline{h}_\delta(y)})^2\right)}\,\sigma(dx)\sigma(dy)\end{aligned}\tag{2.4.2}$$

where we obtain the inequality by applying Cauchy-Schwarz on the expectation. By lemma 2.4.1 and using independence again, we can write

$$\begin{aligned}\mathbb{E}\left(\left(e^{\overline{h}_\varepsilon(x)}-e^{\overline{h}_\delta(x)}\right)^2\right)&=\mathbb{E}\left(e^{2\overline{h}_\varepsilon(x)}-e^{2\overline{h}_\delta(x)}\right)=\mathbb{E}\left(e^{2\overline{h}_\varepsilon(x)}\right)\mathbb{E}\left(e^{2(\overline{h}_\delta(x)-\overline{h}_\varepsilon(x))}-1\right)\\ &=\varepsilon^{-\gamma^2}R(x;D)^{\gamma^2}(2^{\gamma^2}-1)\end{aligned}\tag{2.4.3}$$

where the last equation comes after considering (2.4.1). Combining (2.4.2) and (2.4.3) states

$$\begin{aligned}\mathbb{E}\left((\mathcal{M}_\varepsilon(S)-\mathcal{M}_\delta(S))^2\right)\leq&\varepsilon^{-\gamma^2}R_D^{\gamma^2}(2^{\gamma^2}-1)\int_{\{x,y\in S:\|x-y\|<2\varepsilon\}}1\,\sigma(dx)\sigma(dy)\\ \leq&\varepsilon^{-\gamma^2}R_D^{\gamma^2}(2^{\gamma^2}-1)\int_S(2\varepsilon)^2\pi R_D^{\gamma^2}\,dx\\ \leq&C\varepsilon^{2-\gamma^2}\end{aligned}$$

for $C=R_D^{2\gamma^2}(2^{\gamma^2}-1)4\pi Vol(D)$. $\square$

For each $S \subset D_\varepsilon$ and $k \leq l$ such that $2^{-k} \leq \varepsilon$, the estimate

$$\mathbb{E}\left((\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-l}}(S))^2\right) \leq \sum_{m=k}^{l-1} \mathbb{E}\left((\mathcal{M}_{2^{-m}}(S) - \mathcal{M}_{2^{-(m-1)}}(S))^2\right) \leq C \sum_{m=k}^{l-1} 2^{-k(2-\gamma^2)}$$

holds. The expression on the right is bounded for all $l \geq k$, as long as $k(2-\gamma^2)$ is positive. In other words, $\mathcal{M}_{2^{-k}}(S)$ is a Cauchy sequence in $L^2(\mathbb{P})$ if $\gamma \in [0, \sqrt{2})$.

2.4.2 Cutting out thick points

We would like to strengthen the bound in proposition 2.4.2, such that $\mathcal{M}_{2^{-k}}(S)$ becomes a Cauchy sequence in $L^2(\mathbb{P})$ even for $\gamma \in [\sqrt{2}, 2)$. This won´t be possible for $\mathcal{M}_{2^{-k}}(S)$, but it is doable if we cut out very rare and "heavy" points.

Definition 16 (thick points)**.** Let $\alpha > 0$. A point $z \in D$ is called α-thick, if

$$\lim_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log(\frac{1}{\varepsilon})} = \alpha.$$

Remark 20. For a $z \in D$ and $\varepsilon_0 \leq dist(z, \partial D)$ we can write

$$\frac{h_\varepsilon(z)}{\log(\frac{1}{\varepsilon})} = \frac{h_\varepsilon(z) - h_{\varepsilon_0}(z)}{\log(\frac{\varepsilon_0}{\varepsilon})} \frac{\log(\frac{\varepsilon_0}{\varepsilon})}{\log(\frac{1}{\varepsilon})} + \frac{h_{\varepsilon_0}(z)}{\log(\frac{1}{\varepsilon})}$$

and we know from corollary 1.7.5 (ii) that $h_\varepsilon(z) - h_{\varepsilon_0}(z)$ is a logarithmic time scaled Brownian motion for $\varepsilon \leq \varepsilon_0$. Consequently

$$\frac{h_\varepsilon(z) - h_{\varepsilon_0}(z)}{\log(\frac{\varepsilon_0}{\varepsilon})} \to 0$$

almost surely (by the law of large numbers) and clearly

$$\frac{\log(\frac{\varepsilon_0}{\varepsilon})}{\log(\frac{1}{\varepsilon})} \to 1 \qquad \text{and} \qquad \frac{h_{\varepsilon_0}(z)}{\log(\frac{1}{\varepsilon})} \to 0$$

for $\varepsilon \to 0$. This yields that a point $z \in D$ is not thick almost surely, and hence the Lebesgue measure on the set

$$\tau_\alpha := \{z \in D : z \text{ is } \alpha\text{- thick}\}$$

is zero. Still, thick points are existing for $\alpha \in (0, 2]$, and in [HMP10] it is proven, that the Hausdorff-dimension of τ_α is given by $(2 - \frac{\alpha^2}{2})$.

The idea is to cut out some thick points, and hope that they don´t contribute significantly to $\mathcal{M}_\varepsilon$ because of their rare dimension, even if they are very "heavy". For this, we define for $\alpha > 0$ the event

$$G_\varepsilon^\alpha(z) := \left\{ \frac{h_\varepsilon(z)}{\log(\frac{1}{\varepsilon})} \leq \alpha \right\}$$

and for $S \subset D$ the random set

$$S_\varepsilon^\alpha = S \setminus \left\{ z : \frac{h_\varepsilon(z)}{\log(\frac{1}{\varepsilon})} \leq \alpha \right\}.$$

Proposition 2.4.3. *Let $\varepsilon > 0$ and $\alpha > \gamma$. Then for all $z \in D_\varepsilon$ and $\delta \in (0, \varepsilon]$*

$$\mathbb{E}\left(e^{\overline{h}_\delta(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} \right) \geq 1 - p(\varepsilon)$$

where there exists a constant $K > 0$, such that for all $\varepsilon \leq \min\{1, dist(z, \partial D)\}$ the function p is a polynomial function of the form

$$p(\varepsilon) = K \varepsilon^{\frac{(\alpha-\gamma)^2}{2}}.$$

Proof. We can restrict to the case $\delta = \varepsilon$, as by proposition 2.3.3

$$\begin{aligned}
\mathbb{E}\left(e^{\overline{h}_\delta(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} \right) &= \mathbb{E}\left(\mathbb{E}\left(e^{\overline{h}_\delta(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} | h_\varepsilon(z) \right) \right) \\
&= \mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(z)} \mathbb{E}\left(e^{\overline{h}_\delta(z)} | h_\varepsilon(z) \right) \right) = \mathbb{E}\left(e^{\overline{h}_\varepsilon(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} \right).
\end{aligned}$$

By writing the probability measure

$$d\mathbb{Q} := e^{\overline{h}_\varepsilon(z)} d\mathbb{P} = \frac{e^{\gamma h_\varepsilon(z)}}{\mathbb{E}\left(e^{\gamma h_\varepsilon(z)}\right)} d\mathbb{P}$$

we can apply lemma 2.2.3 to write

$$\mathbb{E}\left(e^{\overline{h}_\varepsilon(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} \right) = \mathbb{Q}\left(G_\varepsilon^\alpha(z) \right) = \mathbb{Q}\left(h_\varepsilon(z) \leq \alpha \log\left(\frac{1}{\varepsilon}\right) \right)$$

where $h_\varepsilon(z)$ is $\mathcal{N}\left(\gamma Var(h_\varepsilon(z)), Var(h_\varepsilon(z))\right)$ distributed. For ε small enough such that $(\alpha - \gamma)\log\left(\frac{1}{\varepsilon}\right) - \gamma \log(R(z; D)) \geq 0$ holds, we can estimate

$$\begin{aligned}
\mathbb{Q}\left(h_\varepsilon(z) \leq \alpha \log\left(\frac{1}{\varepsilon}\right) \right) &= \mathbb{P}\left(\mathcal{N}\left(0, \log\left(\frac{1}{\varepsilon}\right) + \log(R(z; D)) \right) \leq (\alpha - \gamma)\log\left(\frac{1}{\varepsilon}\right) - \gamma \log(R(z; D)) \right) \\
&\geq \mathbb{P}\left(\mathcal{N}\left(0, \log\left(\frac{1}{\varepsilon}\right) \right) \leq (\alpha - \gamma)\log\left(\frac{1}{\varepsilon}\right) - \gamma \log(R_D) \right) \\
&= \mathbb{P}\left(\mathcal{N}(0,1) \leq (\alpha - \gamma)\sqrt{\log\left(\frac{1}{\varepsilon}\right)} - \gamma \log(R_D) \log\left(\frac{1}{\varepsilon}\right)^{-\frac{1}{2}} \right).
\end{aligned}$$

The standard bound from lemma 2.2.1 yields

$$\mathbb{E}\left(e^{\overline{h}_\varepsilon(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)} \right) \geq 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\left((\alpha-\gamma)^2 \log\left(\frac{1}{\varepsilon}\right) + 2(\alpha-\gamma)\gamma \log(R_D) + (\gamma \log(R_D))^2 \log\left(\frac{1}{\varepsilon}\right)^{-1} \right)}}{(\alpha - \gamma)\sqrt{\log\left(\frac{1}{\varepsilon}\right)} - \gamma \log(R_D) \log\left(\frac{1}{\varepsilon}\right)^{-\frac{1}{2}}}$$

and choosing $K > 0$ such that for all $\varepsilon \leq 1$

$$K \geq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\left(2(\alpha-\gamma)\gamma \log(R(z;D))+\gamma \log(R_D)^2 \log\left(\frac{1}{\varepsilon}\right)^{-1}\right)}}{(\alpha-\gamma)\sqrt{\log\left(\frac{1}{\varepsilon}\right)} - \gamma \log(R_D) \log\left(\frac{1}{\varepsilon}\right)^{-\frac{1}{2}}}$$

holds (which we can as the term on the right hand side is decreasing for $\varepsilon \to 0$) gives the statement. □

Remark 21. The proposition states that if $\varepsilon \leq 1$ and $S \in D_\varepsilon$, then for $\delta \leq \varepsilon$

$$\mathbb{E}\left(|\mathcal{M}_\delta(S) - \mathcal{M}_\delta(S_\varepsilon^\alpha)|\right) = \int_S \mathbb{E}\left(e^{\overline{h}_\delta(z)} - e^{\overline{h}_\delta(z)} \mathbb{1}_{G_\varepsilon^\alpha(z)}\right) \sigma(dx) \leq \int_S K\varepsilon^{\frac{(\alpha-\gamma)^2}{2}} \sigma(dx) \leq C\varepsilon^{\frac{(\alpha-\gamma)^2}{2}}$$

with $C = Vol(D) R_D K$. Meaning that for $\alpha > \gamma$ the two terms tend against each other in $L^1(\mathbb{P})$ polynomially fast. Therefore the contribution of points tending to be bigger than γ-thick, goes to zero.

2.4.3 $L^1(\mathbb{P})$ convergence of $\mathcal{M}_{2^{-k}}(S)$ for $\gamma \in [0, 2)$

After the consideration in remark 21 we see that

$$\begin{aligned}\mathbb{E}\left(|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)|\right) \leq\ & \mathbb{E}\left(|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-k}}(S_{2^{-k}}^\alpha)|\right) \\ & + \mathbb{E}\left(|\mathcal{M}_{2^{-k}}(S_{2^{-k}}^\alpha) - \mathcal{M}_{2^{-(k+1)}}(S_{2^{-k}}^\alpha)|\right) \\ & + \mathbb{E}\left(|\mathcal{M}_{2^{-(k+1)}}(S_{2^{-k}}^\alpha) - \mathcal{M}_{2^{-(k+1)}}(S)|\right)\end{aligned}$$

can be bounded by

$$\leq 2C(2^{-k})^{\frac{(\alpha-\gamma)^2}{2}} + \mathbb{E}\left(|\mathcal{M}_{2^{-k}}(S_{2^{-k}}^\alpha) - \mathcal{M}_{2^{-(k+1)}}(S_{2^{-k}}^\alpha)|\right). \tag{2.4.4}$$

Hence if there is a good bound for $\mathbb{E}\left(|\mathcal{M}_{2^{-k}}(S_{2^{-k}}^\alpha) - \mathcal{M}_{2^{-(k+1)}}(S_{2^{-k}}^\alpha)|\right)$, the sequence $(\mathcal{M}_{2^{-k}})_{k\geq 0}$ will be Cauchy in $L^1(\mathbb{P})$ for $\alpha > \gamma$.

Proposition 2.4.4. *Let $\alpha \in (\gamma, 2\gamma]$. Then there is a constant C depending on γ and D, such that for all $S \in \overline{D}_\varepsilon$*

$$\mathbb{E}\left((\mathcal{M}_\varepsilon(S_\varepsilon^\alpha) - \mathcal{M}_{\frac{\varepsilon}{2}}(S_\varepsilon^\alpha))^2\right) \leq C\varepsilon^{2-\gamma^2+\frac{1}{2}(2\gamma-\alpha)^2}.$$

Proof. Again for simplicity, set $\delta = \frac{\varepsilon}{2}$. Because $\mathbb{1}_{G_\varepsilon^\alpha(z)}$ is $h_\varepsilon(z)$-measurable for all $z \in D_\varepsilon$, one can proceed with the same arguments as in the proof of proposition 2.4.2 until equation 2.4.2, which looks then like

$$\begin{aligned}&\mathbb{E}\left((\mathcal{M}_\varepsilon(S_\varepsilon^\alpha) - \mathcal{M}_\delta(S_\varepsilon^\alpha))^2\right) \\ &\leq \int_{\{x,y\in S:\|x-y\|<2\varepsilon\}} \sqrt{\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(x)}(e^{\overline{h}_\varepsilon(x)} - e^{\overline{h}_\delta(x)})^2\right)}\sqrt{\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(y)}(e^{\overline{h}_\varepsilon(y)} - e^{\overline{h}_\delta(y)})^2\right)}\, \sigma(dx)\sigma(dy).\end{aligned}$$

Again we can use lemma 2.4.1 to get

$$\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(x)}(e^{\overline{h}_\varepsilon(x)}-e^{\overline{h}_\delta(x)})^2\right)=\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(x)}(e^{2\overline{h}_\varepsilon(x)}-e^{2\overline{h}_\delta(x)})\right)=(2^{\gamma^2}-1)\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(x)}e^{2\overline{h}_\varepsilon(x)}\right)$$

where for the second equality the same arguments as in 2.4.3 are made. One should grasp, that the only thing differing from the proof of proposition 2.4.2 until equation 2.4.3, by cutting out the points tending to be more than α-thick, is that the term $\mathbb{E}\left(e^{2\overline{h}_\varepsilon(x)}\right)$ is replaced by $\mathbb{E}\left(\mathbb{1}_{G_\varepsilon^\alpha(x)}e^{2\overline{h}_\varepsilon(x)}\right)$. Set

$$d\mathbb{Q}:=\frac{e^{2\gamma h_\varepsilon(x)}}{\mathbb{E}\left(e^{2\gamma h_\varepsilon(x)}\right)}d\mathbb{P}.$$

and write

$$\mathbb{E}\left(e^{2\overline{h}_\varepsilon(x)}\mathbb{1}_{G_\varepsilon^\alpha(x)}\right)=\frac{\mathbb{E}\left(e^{2\gamma h_\varepsilon(x)}\mathbb{1}_{G_\varepsilon^\alpha(x)}\right)}{\mathbb{E}\left(e^{2\gamma h_\varepsilon(x)}\right)}\varepsilon^{-\gamma^2}R(x;D)^{\gamma^2}=\mathbb{Q}(G_\varepsilon^\alpha(z))\varepsilon^{-\gamma^2}R(z;D)^{\gamma^2}.$$

According to lemma 2.2.3 the random variable $h_\varepsilon(x)$ is normally distributed with mean $2\gamma(\log\left(\frac{1}{\varepsilon}\right)+\log(R(z;D)))$ and variance $\log\left(\frac{1}{\varepsilon}\right)+\log(R(z;D))$ with respect to the law $\mathbb{Q}$. Consequently

$$\begin{aligned}\mathbb{Q}(G_\varepsilon^\alpha(z))=&\mathbb{P}\left(\mathcal{N}\left(2\gamma(\log\left(\frac{1}{\varepsilon}\right)+\log(R(z;D)),\log\left(\frac{1}{\varepsilon}\right)+\log(R(z;D))\right)\le\alpha\log\left(\frac{1}{\varepsilon}\right)\right)\\=&\mathbb{P}\left(\mathcal{N}\left(0,\log\left(\frac{1}{\varepsilon}\right)+\log(R(z;D))\right)\ge(2\gamma-\alpha)\log\left(\frac{1}{\varepsilon}\right)+2\gamma\log(R(z;D))\right)\end{aligned}$$

which can be bound from above uniformly for all $z\in D$ (as $2\gamma-\alpha\ge 0$) by

$$\begin{aligned}&\mathbb{P}\left(\mathcal{N}\left(0,\log\left(\frac{1}{\varepsilon}\right)+\log(R_D)\right)\ge(2\gamma-\alpha)\log\left(\frac{1}{\varepsilon}\right)\right)\\=&\mathbb{P}\left(\mathcal{N}(0,1)\ge\frac{(2\gamma-\alpha)\log\left(\frac{1}{\varepsilon}\right)}{\sqrt{\log\left(\frac{1}{\varepsilon}\right)+\log(R_D)}}\right)\\\le&\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\log\left(\frac{1}{\varepsilon}\right)+\log(R_D)}}{(2\gamma-\alpha)\log\left(\frac{1}{\varepsilon}\right)}e^{-\frac{1}{2}\frac{(2\gamma-\alpha)^2\log\left(\frac{1}{\varepsilon}\right)^2}{\log\left(\frac{1}{\varepsilon}\right)+\log(R_D)}}=:\frac{1}{\sqrt{2\pi}}P_\varepsilon e^{Q_\varepsilon}\end{aligned}$$

where the bound is obtained from lemma 2.2.1. We see that $P_\varepsilon\to 0$ and $Q_\varepsilon+\frac{1}{2}(2\gamma-\alpha)^2\log(\frac{1}{\varepsilon})\to 1$ for $\varepsilon\to 0$. Hence we can deduce that

$$\mathbb{Q}(G_\varepsilon^\alpha(z))e^{\frac{1}{2}(2\gamma-\alpha)^2\log(\frac{1}{\varepsilon})}=\mathbb{Q}(G_\varepsilon^\alpha(z))\varepsilon^{-\frac{1}{2}(2\gamma-\alpha)^2}\to 0.$$

This means, we can choose $\tilde{C}>0$ such that

$$\mathbb{Q}(G_\varepsilon^\alpha(z))\le\tilde{C}\varepsilon^{\frac{1}{2}(2\gamma-\alpha)^2}$$

for all $\varepsilon \leq 1$. Putting everything together, we get

$$\begin{aligned}\mathbb{E}\left((\mathcal{M}_\varepsilon(S_\varepsilon^\alpha) - \mathcal{M}_{\frac{\varepsilon}{2}}(S_\varepsilon^\alpha))^2\right) &\leq (2^{\gamma^2}-1)\int_{\{x,y\in S:\|x-y\|\leq 2\varepsilon\}} \tilde{C}\varepsilon^{\frac{1}{2}(2\gamma-\alpha)^2}\varepsilon^{-\gamma^2}R_D^{\gamma^2}\,\sigma(dx)\sigma(dy)\\ &\leq (2^{\gamma^2}-1)\tilde{C}\varepsilon^{\frac{1}{2}(2\gamma-\alpha)^2-\gamma^2}R_D^{\gamma^2}\int_{\{x,y\in S:\|x-y\|\leq 2\varepsilon\}} R_D^{\gamma^2}\,dxdy\\ &= C\varepsilon^{\frac{1}{2}(2\gamma-\alpha)^2+2-\gamma^2}\end{aligned}$$

for $C = (2^{\gamma^2}-1)\tilde{C}R_D^{2\gamma^2}4\pi Vol(S)$ and $\varepsilon \leq 1$. □

As for all bounded measure spaces, a bound in L^2 implies a bound in L^1 by using Cauchy-Schwarz. In our case this gives

$$\mathbb{E}\left(|\mathcal{M}_\varepsilon(S_\varepsilon^\alpha) - \mathcal{M}_{\frac{\varepsilon}{2}}(S_\varepsilon^\alpha)|\right) \leq \sqrt{\mathbb{E}(1)\mathbb{E}\left((\mathcal{M}_\varepsilon(S_\varepsilon^\alpha) - \mathcal{M}_{\frac{\varepsilon}{2}}(S_\varepsilon^\alpha))^2\right)} \leq C\varepsilon^{\frac{1}{2}(2-\gamma^2+\frac{1}{2}(2\gamma-\alpha)^2)}.$$

Considering this and remark 21, means that for fixed D and γ we can choose C big enough, such that (2.4.4) can be bounded from above by

$$2C(2^{-k})^{\frac{(\alpha-\gamma)^2}{2}} + C(2^{-k})^{\frac{1}{2}(2-\gamma^2+\frac{1}{2}(2\gamma-\alpha)^2)}.$$

Further we can choose $\alpha > \gamma$ small enough, such that $\frac{1}{2}(2-\gamma^2+\frac{1}{2}(2\gamma-\alpha)^2) > 0$, and therefore we can set $r := \min\left\{\frac{(\alpha-\gamma)^2}{2}, \frac{1}{2}(2-\gamma^2+\frac{1}{2}(2\gamma-\alpha)^2)\right\}$ greater than zero, and clearly

$$\mathbb{E}(|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)|) \leq 3C(2^{-k})^r. \tag{2.4.5}$$

Since $\sum_{k=0}^\infty (2^{-r})^k < \infty$, we can follow that the sequence $\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)$ is Cauchy in $L^1(\mathbb{P})$ and hence converges in $L^1(\mathbb{P})$.

2.4.4 Weak convergence of $\mathcal{M}_{2^{-k}}$

This subsection follows [Ber17] chapter 6. To make the proof of the almost sure weak convergence of the LM more convenient, we subdivide this subsection.

I.) Almost sure convergence on $S \subset D$ with $\overline{S} \subset D$

For a set $S \subset D$ which has the closure contained in D, we can apply equation 2.4.5 which implies

$$\mathbb{E}\left(\sum_{k=0}^\infty |\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)|\right) \leq \sum_{k=0}^\infty \mathbb{E}(|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)|) < \infty. \tag{2.4.6}$$

This states that $\sum_{k=0}^\infty |\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)|$ is finite almost surely, and $\mathcal{M}_{2^{-k}}(S)$ converges almost surely for all $\gamma \in [0,2)$. In this case we denote the almost sure limit of $\mathcal{M}_{2^{-k}}(S)$ as $\mathcal{M}(S)$.

II.) Convergence in probability on general $S \subset D$

Take $S \subset D$ and define $l = \lim_{n\to\infty} \mathcal{M}(S \cap D_{\frac{1}{n}})$. We see that by monotone convergence

$$\mathbb{E}(l) = \lim_{n\to\infty} \mathbb{E}\left(\mathcal{M}(S \cap D_{\frac{1}{n}})\right) = \lim_{n\to\infty} \int_{D_{\frac{1}{n}}} R(x;D)^{\frac{\gamma^2}{2}}\, dx = \int_D R(x;D)^{\frac{\gamma^2}{2}}\, dx < \infty$$

and therefore l indeed exists, and is almost surely finite. Fix $\delta > 0$ and split up

$$\begin{aligned}
&|\mathcal{M}_{2^{-k}}(S) - l| \\
&\leq \left|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-k}}(S \cap D_{\frac{1}{n}})\right| + \left|\mathcal{M}_{2^{-k}}(S \cap D_{\frac{1}{n}}) - \mathcal{M}(S \cap D_{\frac{1}{n}})\right| + \left|\mathcal{M}(S \cap D_{\frac{1}{n}}) - l\right| \\
&=: I_{n,k} + J_{n,k} + K_n.
\end{aligned}$$

Choose $N \in \mathbb{N}$ small enough such that

$$\mathbb{E}(I_{N,k}) = \int_{S \setminus D_N} R(z;D)^{\frac{\gamma^2}{2}} < \delta^2 \tag{2.4.7}$$

and

$$\mathbb{P}(K_N > \delta) < \delta.$$

Note that 2.4.7 is independent of k, and we see by the Markov inequality

$$\mathbb{P}(I_{N,k} > \delta) \leq \frac{\mathbb{E}(I_{N,k})}{\delta} < \delta.$$

Further choose $K \in \mathbb{N}$ such that for all $k \geq K$

$$\mathbb{P}(J_{N,k} > \delta) < \delta$$

which again we can do, since $\mathcal{M}_{2^{-k}}(S \cap D_{\frac{1}{N}})$ converges almost surely to $\mathcal{M}(S \cap D_{\frac{1}{N}})$ by Step I.). Putting all of this together gives for all $k \geq K$

$$\mathbb{P}(|\mathcal{M}_{2^{-k}}(S) - l| > 3\delta) \leq \mathbb{P}(I_{N,k} > \delta) + \mathbb{P}(J_{N,k} > \delta) + \mathbb{P}(K_N > \delta) \leq 3\delta$$

and because δ was arbitrarily chosen, $\mathcal{M}_{2^{-k}}(S)$ converges to l in probability. From now on we denote this limit as $\mathcal{M}(S)$ instead of l (this clearly coincides for S with closure contained in D with $\mathcal{M}(S)$ from the step before).

III.) Last step

In the following, we will make use of Prokorov´s theorem and the Portmanteau theorem. They are usually stated for probability measures, but by scaling they hold in our case as well.

By the convergence of the sequence $\mathcal{M}_{2^{-k}}(D)$ to $\mathcal{M}(D)$ in probability, there exists (by Borell Cantelli lemma) a subsequence $(\mathcal{M}_{2^{-k_l}})_{l=0}^{\infty}(D)$ converging almost surely to $\mathcal{M}(D)$. As D is bounded and $\mathcal{M}(D)$ is finite almost surely, $\{\mathcal{M}_{2^{-k_l}}\}_{l=0}^{\infty}$ is tight almost surely in the space of finite measures on $\overline{D}$. By Prokorov´s theorem there exists a subsequence $(\mathcal{M}_{2^{-k_{l_m}}})_{m=0}^{\infty}$ which is weakly convergent, and we denote this weak limit as $\overline{\mathcal{M}}$. If we show, that $\overline{\mathcal{M}} = \mathcal{M}$ almost surely, we are done, there cannot exist a subsequence of $(\mathcal{M}_{2^{-k}})_{k=0}^{\infty}$ which is not converging to $\overline{\mathcal{M}}$. From Dunkin´s lemma the equivalence of the measures follows from the equivalence of the measures on the π-set

$$\mathcal{A}_{[,)} := \{[x_1, y_1) \times [x_2, y_2) \subset D : \; x_1, y_1, x_2, y_2 \in \mathbb{Q} \quad \text{and} \quad [x_1, y_1] \times [x_2, y_2] \subset D\}$$

which is a generator for the Borel algebra on D. Further define

$$\begin{aligned}\mathcal{A}_{[,]} &:= \{[x_1, y_1] \times [x_2, y_2] \subset D : x_1, y_1, x_2, y_2 \in \mathbb{Q}\}\\ \mathcal{A}_{(,)} &:= \{(x_1, y_1) \times (x_2, y_2) \subset D : \; x_1, y_1, x_2, y_2 \in \mathbb{Q} \quad \text{and} \quad [x_1, y_1] \times [x_2, y_2] \subset D\}.\end{aligned}$$

From step I.) we know that

$$\mathbb{P}(\{\forall A \in \mathcal{A}_{[,)} \cup \mathcal{A}_{[,]} \cup \mathcal{A}_{(,)} : \mathcal{M}_{2^{-k}}(A) \to \mathcal{M}(A)\}) = 1$$

as the set $\mathcal{A}_{[,)} \cup \mathcal{A}_{[,]} \cup \mathcal{A}_{(,)}$ is countable. We claim that for all $A \in \mathcal{A}_{[,)}$

$$\begin{aligned}\mathcal{M}(A) &= \sup\{\mathcal{M}(C) : C \in \mathcal{A}_{[,]} \text{ and } C \subset A\}\\ &= \inf\{\mathcal{M}(O) : O \in \mathcal{A}_{(,)} \text{ and } A \subset O\}.\end{aligned} \tag{2.4.8}$$

Clearly

$$\mathcal{M}(A) \geq \sup\{\mathcal{M}(C) : C \in \mathcal{A}_{[,]} \text{ and } C \subset A\}$$

and

$$\begin{aligned}\mathbb{E}\left(|\mathcal{M}(A) - \sup_{\{C \in \mathcal{A}_{[,]} : C \subset A\}} \{\mathcal{M}(C)\}|\right) &= \inf_{\{C \in \mathcal{A}_{[,]} : C \subset A\}} \{\mathbb{E}\left(\mathcal{M}(A) - \mathcal{M}(C)\right)\}\\ &= \inf_{\{C \in \mathcal{A}_{[,]} : C \subset A\}} \left\{\int_{A \setminus C} R(x; D)^{\frac{\gamma^2}{2}} dx\right\}\end{aligned}$$

where the first equality holds, as $\mathcal{M}(A) - \sup_{\{C \in \mathcal{A}_{[,]} : C \subset A\}} \{\mathcal{M}(C)\} \geq 0$ is dominated by $\mathcal{M}(A)$. The second equation is obtained by remark 19 and the $L^1(\mathbb{P})$ convergence of $\mathcal{M}_{2^{-k}}(A \setminus C)$ to $\mathcal{M}(A \setminus C)$. The right hand side is zero as $R(x; D)$ is bounded, and therefore the first equality in 2.4.8 holds almost surely. The second equality follows in the same way, except that we have to use $\mathcal{M}(D) - \mathcal{M}(A)$ to dominate $\sup_{\{O \in \mathcal{A}_{[,]} : O \subset A\}} \{\mathcal{M}(O)\} - \mathcal{M}(A)$. By Portmanteau´s theorem for every set $C \in \mathcal{A}_{[,]}$ with $C \subset A$

$$\overline{\mathcal{M}}(A) \geq \overline{\mathcal{M}}(C) \geq \lim_{m \to \infty} \mathcal{M}_{2^{-k_{l_m}}}(C) = \mathcal{M}(C)$$

and consequently by 2.4.8

$$\overline{\mathcal{M}}(A) \geq \mathcal{M}(A).$$

By analogy, one obtains by using open sets to approximate A from above

$$\overline{\mathcal{M}}(A) \leq \mathcal{M}(A)$$

and therefore $\overline{\mathcal{M}}(A) = \mathcal{M}(A)$ for all $A \in \mathcal{A}_{[,)}$ which finishes the proof of theorem 2.3.1.

2.5 Properties of the LM

We use the last section, to finish up with some important properties of the Liouville measure.

2.5.1 First consequences

Proposition 2.5.1. *The LM fulfills the following conditions:*

(i) The expectation of the LM is given by

$$\mathbb{E}(\mathcal{M}(S)) = \int_S R(x;D)^{\frac{\gamma^2}{2}}\, dx.$$

(ii) On an open $S \subset D$ the LM is positive almost surely.

Proof. (i) By monotone convergence one can write

$$\mathbb{E}(\mathcal{M}(S)) = \lim_{n\to\infty} \mathbb{E}(\mathcal{M}(S \cap D_{\frac{1}{n}}).$$

The set $S \cap D_{\frac{1}{n}}$ has its closure contained in D, and therefore $\mathcal{M}_{2^{-k}}(S \cap D_{\frac{1}{n}})$ converges to $\mathcal{M}(S \cap D_{\frac{1}{n}})$ in $L^1(\mathbb{P})$. By remark 19

$$\mathbb{E}(\mathcal{M}_{2^{-k}}(S \cap D_{\frac{1}{n}}) = \int_{S\cap D_{\frac{1}{n}}} R(x;D)^{\frac{\gamma^2}{2}}\, dx$$

and hence

$$\mathbb{E}(\mathcal{M}(S)) = \lim_{n\to\infty} \int_{S\cap D_{\frac{1}{n}}} R(x;D)^{\frac{\gamma^2}{2}}\, dx = \int_S R(x;D)^{\frac{\gamma^2}{2}}\, dx.$$

(ii) For each open $S \subset D$ we can find an open subset in S which has its closure contained in D. Therefore it is enough to proof the case when $\overline{S} \subset D$. Write the GFF as $h = 2\pi \sum_{j=1}^{\infty} \mathcal{N}_j u_j$ where $\{u_j\}_{j=1}^{\infty}$ is a smooth basis of $H^1(D)$, and $\{\mathcal{N}_j\}_{j=1}^{\infty}$ are independent centred Gaussian random variables with variance 2π (note that we need this variance because $h = \sqrt{2\pi}\mathbf{h}$). Let $\mathcal{F}_n := \sigma(\mathcal{N}_n, \mathcal{N}_{n+1}, \dots)$ be the σ-algebra generated by $\mathcal{N}_n, \mathcal{N}_{n+1}, \dots$ and $\mathcal{F}_\infty := \bigcap_{n=1}^{\infty} \mathcal{F}_n$ the corresponding tail algebra. Fix $N \in \mathbb{N}$ and write

$$\mathcal{M}(S) = \lim_{k\to\infty} \int_S 2^{-k\frac{\gamma^2}{2}} e^{\gamma \sum_{j=1}^{N-1} \mathcal{N}_j \langle u_j, \rho_{2^{-k}}(x)\rangle} e^{\gamma \sum_{j=N}^{\infty} \mathcal{N}_j \langle u_j, \rho_{2^{-k}}(x)\rangle}\, dx. \tag{2.5.1}$$

As u_j is continuous and zero on the boundary for all $j \in \mathbb{N}$, we find bounds u_u and u_o such that

$$u_u \le u_j(x) \le u_o$$

for all $j = 1, \dots, N-1$ and $x \in D$. By setting

$$J := \lim_{k\to\infty} \int_S 2^{-k\frac{\gamma^2}{2}} e^{\gamma \sum_{j=N}^{\infty} X_j \langle u_j, \rho_{2^{-k}}(x)\rangle} \, dx$$

the equation 2.5.1 can be bounded with

$$J e^{\gamma u_u \sum_{j=1}^{N-1} \mathcal{N}_j} \le \mathcal{M}(S) \le J e^{\gamma u_o \sum_{j=1}^{N-1} \mathcal{N}_j}$$

which implies

$$\{J e^{\gamma u_o \sum_{j=1}^{N-1} \mathcal{N}_j} > 0\} \subset \{\mathcal{M}(S) > 0\} \subset \{J e^{\gamma u_u \sum_{j=1}^{N-1} \mathcal{N}_j} > 0\}.$$

Further $e^{\gamma u_u \sum_{j=1}^{N-1} \mathcal{N}_j}$ and $e^{\gamma u_o \sum_{j=1}^{N-1} \mathcal{N}_j}$ are greater than zero almost surely, and consequently

$$\{J > 0\} = \{J e^{\gamma u_u \sum_{j=1}^{N-1} \mathcal{N}_j} > 0\} = \{J e^{\gamma u_o \sum_{j=1}^{N-1} \mathcal{N}_j} > 0\}$$

almost surely. The event $\{J > 0\}$ is clearly in $\mathcal{F}_N$, therefore $\{\mathcal{M}(S) > 0\} \in \mathcal{F}_N$, and consequently $\{\mathcal{M}(S) > 0\} \in \mathcal{F}_\infty$ almost surely. Applying Komlogorov´s zero-one law yields that $\mathbb{P}(\mathcal{M}(S) > 0) \in \{0, 1\}$. But as $\mathbb{E}(\mathcal{M}(S)) > 0$ for $S \subset D$ open, we know that $\mathbb{P}(\mathcal{M}(S) > 0) > 0$, which shows that $\mathcal{M}(S)$ is positive almost surely.

□

2.5.2 LM is supported on γ-thick points

We have already seen in proposition 2.4.3 that the points contributing to the Liouville measure are not more than γ-thick, because they get too rare. On the other hand intuitively speaking, the density of the Liouville measure can be approximated by

$$\mathcal{M}_\varepsilon(dz) = m_\varepsilon(z) = \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)}$$

which tends to zero, if $h_\varepsilon(z)$ grows slower than $\gamma \log(\frac{1}{\varepsilon})$. Therefore we have reason to believe that the points which are sampled by the (normalised) Liouville measure, are γ-thick. We will show this rigorously in this subsection.

Define the probability measures $\mathbb{Q}$ and $\mathbb{Q}_\varepsilon$ on $D \times H_0^1(D)$ by

$$\mathbb{Q}(dz, dh) = \frac{1}{Z}\mathcal{M}(dz)\mathbb{P}(dh)$$
$$\mathbb{Q}_\varepsilon(dz, dh) = \frac{1}{Z}\mathcal{M}_\varepsilon(dz)\mathbb{P}(dh)$$

where $Z = \mathbb{E}(\mathcal{M}(D))$ and $\mathbb{P}$ is the law of the GFF on $H_0^1(D)$. Further make the convention, that for every probability measure μ and $X \in L^1(\mu)$ we write $\mathbb{E}_\mu(X)$ and $\mathbb{E}_\mu(X|\mathcal{A})$ for the expectation and the conditioned expectation on a σ-Algebra $\mathcal{A}$, with respect to μ. We start with a first observation.

Lemma 2.5.2. *The probability measure $\mathbb{Q}_{2^{-k}}$ tends weakly to $\mathbb{Q}$.*

Proof. Fix a bounded continuous function $F : D \times H_0^1(D) \to \mathbb{R}$. Write

$$\mathbb{E}_{\mathbb{Q}_{2^{-k}}}(F) = \frac{1}{Z}\mathbb{E}_{\mathbb{P}}\left(\int_D F(z,h)\mathcal{M}_{2^{-k}}(dz)\right)$$

and as almost surely $\mathcal{M}_{2-k}$ tends to $\mathcal{M}$ weakly, we know

$$\int_D F(z,h)\mathcal{M}_{2^{-k}}(dz) \to \int_D F(z,h)\mathcal{M}(dz)$$

almost surely for $k \to \infty$. Bound $|\int_D F(z,h)\mathcal{M}_{2^{-k}}(dz)|$ by the in $L^1(\mathbb{P})$ converging sequence $\|F\|_\infty \mathcal{M}_{2^{-k}}(D)$, and hence we can apply dominated convergence to obtain

$$\lim_{k\to\infty} \frac{1}{Z}\mathbb{E}_{\mathbb{P}}\left(\int_D F(z,h)\mathcal{M}_{2^{-k}}(dz)\right) = \frac{1}{Z}\mathbb{E}_{\mathbb{P}}\left(\int_D F(z,h)\mathcal{M}(dz)\right) = \mathbb{E}_{\mathbb{Q}}(F)$$

which gives the statement. □

Proposition 2.5.3. *Let $\rho_1, \dots, \rho_n \in \mathfrak{M}_D$ be an arbitrary finite collection of measures and $(h_{\rho_k})_{k=1}^n$ the restriction of the GFF (from the point of view of a stochastic process) to a finite dimensional Gaussian vector on the measures $\rho_1, \dots, \rho_n$. Then for all $\xi \in \mathbb{R}^n$*

$$\mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n \xi_k\langle h,\rho_k\rangle}|\mathcal{B}(D)\right) = \mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \xi_k(\langle h,\rho_k\rangle + \gamma\Gamma(\rho_k,\delta_z))}\right) \tag{2.5.2}$$

holds, where $\mathcal{B}(D)$ is the Borel algebra on D.

This statement says, that the restriction of the GFF to a finite number of measures in $\mathfrak{M}_D$ conditioned to sample a point $z \in D$, is $\mathcal{N}(\gamma\Gamma(.,z),\Gamma)$ distributed. By Kolmogorov´s extension theorem, we can conclude that the whole GFF conditioned on sampling a point $z \in D$, is $\mathcal{N}(\gamma\Gamma(.,z),\Gamma)$ distributed.

Proof. Since $\xi_k\langle h,\rho_k\rangle = \langle h, \xi_k\rho_k\rangle$ and $\xi_k\rho_k$ is an element in $\mathfrak{M}_D$, we can restrict to the case that $\xi_k = 1$ for $k = 1,\dots,n$. Fix a continuous bounded function $\psi : D \to \mathbb{R}$. By lemma 2.5.2, we can write

$$\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}|\mathcal{B}(D)\right)\psi\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\psi\right) = \lim_{k\to\infty}\mathbb{E}_{\mathbb{Q}_{2^{-k}}}\left(e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\psi\right)$$

as $e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\psi$ is bounded by $\|\psi\|_\infty$. Use Fubini to write

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\varepsilon}(e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\psi) &= \frac{1}{Z}\int_{H_0^1(D)}\int_D e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\psi(z)\varepsilon^{\frac{\gamma^2}{2}}e^{\gamma h_\varepsilon(z)}\,dz\,\mathbb{P}(dh)\\
&= \frac{1}{Z}\int_D\int_{H_0^1(D)} e^{i\sum_{k=1}^n\langle h,\rho_k\rangle}\varepsilon^{\frac{\gamma^2}{2}}e^{\gamma h_\varepsilon(z)}\,\mathbb{P}(dh)\,\psi(z)\,dz\\
&= \int_D \mathbb{E}_{\mathbb{Q}_{\varepsilon,z}}(e^{i\sum_{k=1}^n\langle h,\rho_k\rangle})\psi(z)\,dz
\end{aligned}$$

where we denoted the probability measure $\mathbb{Q}_{\varepsilon,z}(dh) = \frac{1}{Z}\varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)}\, \mathbb{P}(dh)$ (this is actually the conditional law $\mathbb{Q}_\varepsilon(dh|z)$). Using lemma 2.2.3 yields that the Gaussian vector

$$(\langle h, \rho_\varepsilon^z\rangle, \langle h, \rho_1\rangle, \dots, \langle h, \rho_n\rangle)$$

has mean

$$(\gamma\Gamma(\rho_\varepsilon^z, \rho_\varepsilon^z), \gamma\Gamma(\rho_\varepsilon^z, \rho_1), \dots, \gamma\Gamma(\rho_\varepsilon^z, \rho_n))$$

and unchanged covariance with respect to the law $\mathbb{Q}_{\varepsilon,z}$ (in terms of lemma 2.2.3 the vector α is equal to $(\gamma, 0, \dots, 0)$). Meaning that

$$\mathbb{E}_{\mathbb{Q}_{\varepsilon,z}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle}\right) = \mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \gamma\Gamma(\rho_\varepsilon^z,\rho_k)}\right).$$

Note that by lemma 1.7.1

$$\Gamma(\rho_\varepsilon^z, \rho_k) = \int_D \xi_\varepsilon^z(x)\,\rho_k(dx)$$

and clearly $\xi_\varepsilon^z(x)$ tends to $G(x,z)$ from below for $\varepsilon \to 0$. Hence by monotone convergence $\Gamma(\rho_\varepsilon^z, \rho_k) \to \Gamma(\delta_z, \rho_k)$, and consequently by dominated convergence

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \gamma\Gamma(\rho_\varepsilon^z,\rho_k)}\right) = \mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \gamma\Gamma(\delta_z,\rho_k)}\right).$$

Putting all together yields

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle}|\mathcal{B}(D)\right)\psi\right) &= \int_D \mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \Gamma(\rho_k,\delta_z)}\right)\psi(z)\,dz \\ &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \Gamma(\rho_k,\delta_.)}\right)\psi\right)\end{aligned}$$

after using dominated convergence. Now fix an open rectangle $A \subset D$ and let $(\psi_n)_{n=1}^\infty$ be a sequence of bounded continuous functions, which tend pointwise to $\mathbb{1}_A$. By dominated convergence we obtain

$$\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle}|\mathcal{B}(D)\right)\psi_n\right) \to \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle}|\mathcal{B}(D)\right)\mathbb{1}_A\right)$$

and

$$\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \Gamma(\rho_k,\delta_.)}\right)\psi_n\right) \to \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{P}}\left(e^{i\sum_{k=1}^n \langle h,\rho_k\rangle + \Gamma(\rho_k,\delta_.)}\right)\mathbb{1}_A\right)$$

which concludes the proof. □

We see, that if a point $z \in D$ gets sampled by the normalised LM, the law of the GFF which generates LM gets shifted by the mean $\gamma\Gamma(., z)$. This seems reasonable, because we expect a GFF which is very high close to z, more likely to produce a LM which samples z. The random variable

$$\frac{h_\varepsilon(z)}{\log(\frac{1}{\varepsilon})}$$

with the condition that h produced a LM measure which sampled z, is then

$$\mathcal{N}\left(\gamma\frac{\log(\frac{1}{\varepsilon}) + \log(R(z;D))}{\log(\frac{1}{\varepsilon})}, \frac{\log(\frac{1}{\varepsilon}) + \log(R(z;D))}{\log(\frac{1}{\varepsilon})^2}\right)$$

distributed. This tends to $\mathcal{N}(\gamma, 0)$ for $\varepsilon \to 0$, which means that z is almost surely γ-thick.

2.5.3 Conformal covariance

In this subsection we ask ourselves the question, how the LM changes under conformal mappings. For this subsection, take two domains $D, \tilde{D} \subset \mathbb{R}^2$ and the function ϕ as a conformal map taking D to $\tilde{D}$. Denote further h and $\mathcal{M}$ as the GFF and LM on D, and $\tilde{h} = h \circ \phi^{-1}$ and $\widetilde{\mathcal{M}}$ as the GFF and LM on $\tilde{D}$. A first hope could be, that the LM is conformal invariant (like the GFF), meaning that $\widetilde{\mathcal{M}} = \mathcal{M} \circ \phi^{-1}$. This is clearly wrong since by proposition 2.5.1 (ii) for $S \subset D$ and $\tilde{S} := \phi(S) \subset \tilde{D}$

$$\mathbb{E}(\widetilde{\mathcal{M}}(\tilde{S})) = \int_{\tilde{S}} R(\tilde{z}; \tilde{D})^{\frac{\gamma^2}{2}} \, d\tilde{z} = \int_S R(z; D)^{\frac{\gamma^2}{2}} |\phi'(z)|^{2+\frac{\gamma}{2}} \, dz$$

and

$$\mathbb{E}(\mathcal{M} \circ \phi^{-1}(\tilde{S})) = \int_S R(z; D)^{\frac{\gamma^2}{2}} dz$$

do not coincide in general. Even if this is not that simple, we can relate the two measures by adding a scaling function.

Theorem 2.5.4 (LM is conformally covariant)**.** *Let $\phi : D \to \tilde{D}$ be a conformal bounded map. Then the LM on $\tilde{D}$ is given by*

$$\widetilde{\mathcal{M}}(d\tilde{z}) = (|\phi'|^{\gamma Q} \cdot \mathcal{M}) \circ \phi^{-1}(d\tilde{z})$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Remark 22. (i) We have to make the assumption that ϕ is bounded, because we need $\tilde{D}$ to be bounded to make sense out of the LM on $\tilde{D}$.

(ii) The constant Q is the central charge of Liouville quantum gravity. The central charge characterises different conformal field theories, and describes how the field theory changes under conformal maps. (See for example [R+16] chapter 3.2. for a very brief introduction)

(iii) The statement yields, that the LM on $\tilde{D}$ can be written as

$$\widetilde{\mathcal{M}}(d\tilde{z}) = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma(Q \log|\phi'| + h_\varepsilon(z))} dz$$

where $z = \phi^{-1}(\tilde{z})$. Meaning that $\widetilde{\mathcal{M}}(\tilde{S})$ is obtained by looking at the mass of $\phi^{-1}(\tilde{S})$ with respect to the measure, which is constructed by the GFF on D with the mean shifted by the deterministic field $Q \log|\phi'(z)|$.

(iv) A first idea for the proof could be to write

$$\begin{aligned}\widetilde{\mathcal{M}}(\phi(S)) &= \lim_{\varepsilon \to 0} \widetilde{\mathcal{M}}_\varepsilon(\phi(S)) = \lim_{\varepsilon \to 0} \int_{\phi(S)} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma \langle \tilde{h}, \rho_\varepsilon^{\tilde{z}} \rangle} \, d\tilde{z} \\ &= \lim_{\varepsilon \to 0} \int_S \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma \langle h, |\phi'| \rho_\varepsilon^{\phi(z)} \circ \phi \rangle} |\phi'(z)|^2 \, dz.\end{aligned}$$

In the case $\phi(z) = re^{i\theta}z + c$ where $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi)$ and $c \in \mathbb{C}$, this actually gives the statement, because then

$$\langle h, |\phi'|\rho_\varepsilon^{\phi(z)} \circ \phi\rangle = \langle h, \rho^z_{\frac{\varepsilon}{r}}\rangle$$

which yields

$$\widetilde{\mathcal{M}}(\phi(S)) = \lim_{\varepsilon\to 0}\int_S r^{\frac{\gamma^2}{2}}\left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2}} e^{\gamma\langle h, \rho^z_{\frac{\varepsilon}{r}}\rangle} r^2\, dz = (r^{\gamma Q}\cdot\mathcal{M})(S).$$

In the general case we run into trouble, because $\langle h, |\phi'|\rho_\varepsilon^{\phi(z)} \circ \phi\rangle$ is not necessarily a circle average.

The last remark suggests, that we need another approximation than by the circle average to proof theorem 2.5.4. For this, we take a smooth basis $\{u_j\}_{j=1}^\infty$ of $H_0^1(D)$ (which exists by subsection 1.2.3) and independent centred Gaussian variables $\{\mathcal{N}_j\}_{j=1}^\infty$ with variance 2π. We define

$$h^N := \sum_{j=1}^N \mathcal{N}_j u_j$$

$$\mathcal{M}^N(dz) := e^{\gamma h^N(z) - \frac{\gamma^2}{2}Var(h^N(z))}\sigma(dz)$$

where we remind, that $\sigma(dz) = R(z; D)^{\frac{\gamma^2}{2}}$. For the filtration $(\mathcal{F}_N := \sigma(\mathcal{N}_1, \dots, \mathcal{N}_N))_{N=1}^\infty$ and $M \le N$

$$\begin{aligned}
&\mathbb{E}\left(\mathcal{M}^N(S)|\mathcal{F}_M\right)\\
=&\int_S e^{\gamma h^M(z) - \frac{\gamma^2}{2}Var(h^M(z))}\underbrace{\mathbb{E}\left(e^{\gamma\sum_{j=M+1}^N \mathcal{N}_j u_j(z)}|\mathcal{F}_M\right)}_{=\mathbb{E}\left(e^{\gamma\sum_{j=M+1}^N \mathcal{N}_j u_j(z)}\right)} e^{\frac{\gamma^2}{2}Var\left(\sum_{j=M+1}^N \mathcal{N}_j u_j(z)\right)}\sigma(dz)\\
=&\mathcal{M}^M(S)
\end{aligned}$$

for $S \subset D$, and therefore $\mathcal{M}^N(S)$ forms a martingale. By the martingale convergence theorem, the positive martingale $\mathcal{M}^N(S)$ converges almost surely to a limit, which we denote by $\mathcal{M}^*(S)$. We show now, that this coincides almost surely with the LM.

Lemma 2.5.5. *The sequence $(\mathcal{M}^N(S))_{N=1}^\infty$ converges to $\mathcal{M}(S)$ almost surely, for all $S \in \mathcal{B}(D)$ having its closure contained in D. Further $\mathcal{M}^N$ converges weakly to $\mathcal{M}$ for $N \to \infty$ almost surely.*

Proof. Let us fix a $S \in \mathcal{B}(D)$, such that $dist(S, \partial D) > 0$. For $\varepsilon \le dist(S, \partial D)$ we can write

$$\begin{aligned}
\mathbb{E}\left(\mathcal{M}_\varepsilon(S)|\mathcal{F}_N\right) &= \int_S e^{\gamma h_\varepsilon^N(z) - \frac{\gamma^2}{2}Var(h_\varepsilon^N(z))}\mathbb{E}\left(e^{\gamma\sum_{j=N+1}^\infty \mathcal{N}_j\langle u_j, \rho_\varepsilon^z\rangle}\right) e^{-\frac{\gamma^2}{2}Var(\sum_{j=N+1}^\infty \mathcal{N}_j\langle u_j,\rho_\varepsilon^z\rangle)}dz\\
&= \int_S e^{\gamma h_\varepsilon^N(z) - \frac{\gamma^2}{2}Var(h_\varepsilon^N(z))}dz =: \mathcal{M}_\varepsilon^N(S)
\end{aligned}$$

which tends to $\mathcal{M}_N(S)$ for $\varepsilon \to 0$, because the functions $\{u_j\}_{j=1}^\infty$ are smooth. Observe that by Jensen´s inequality

$$\begin{aligned}\mathbb{E}\left(\left|\mathcal{M}_\varepsilon^N(S) - \mathcal{M}_{\frac{\varepsilon}{2}}^N(S)\right|\right) \leq & \mathbb{E}\left(\mathbb{E}\left(\left|\mathcal{M}_\varepsilon(S) - \mathcal{M}_{\frac{\varepsilon}{2}}(S)\right| \middle| \mathcal{F}_N\right)\right) \\ = & \mathbb{E}\left(\left|\mathcal{M}_\varepsilon(S) - \mathcal{M}_{\frac{\varepsilon}{2}}(S)\right|\right).\end{aligned}$$

Hence

$$\mathbb{E}\left(\sum_{k=0}^\infty \left|\mathcal{M}_{2^{-k}}^N(S) - \mathcal{M}_{2^{-(k+1)}}^N(S)\right|\right) \leq \sum_{k=0}^\infty \mathbb{E}\left(\left|\mathcal{M}_{2^{-k}}(S) - \mathcal{M}_{2^{-(k+1)}}(S)\right|\right)$$

and by equation 2.4.5 this is finite. Meaning $\sum_{k=0}^\infty \left|\mathcal{M}_{2^{-k}}^N(S) - \mathcal{M}_{2^{-(k+1)}}^N(S)\right|$ is finite almost surely and consequently $\mathcal{M}_{2^{-k}}^N(S)$ converges almost surely. Clearly this limit is $\mathbb{E}(\mathcal{M}(S)|\mathcal{F}_N)$. Further $\mathcal{M}(S)$ is $\bigcup_{N=1}^\infty \mathcal{F}_N$ measurable, and consequently $\mathbb{E}\left(\mathcal{M}(S)|\mathcal{F}_N\right) \to \mathcal{M}(S)$ almost surely. Putting everything together, we obtain almost surely

$$\mathcal{M}^N(S) = \lim_{k\to\infty} \mathcal{M}_{2^{-k}}^N(S) = \lim_{k\to\infty} \mathbb{E}(\mathcal{M}_{2^{-k}}(S)|\mathcal{F}_N) = \mathbb{E}(\mathcal{M}(S)|\mathcal{F}_N) \to \mathcal{M}(S)$$

for $N \to \infty$ and therefore $\mathcal{M}^*(S) = \mathcal{M}(S)$. Choose a generating system A of the Borel algebra on D, which is countable and fulfills $dist(S, \partial D) > 0$ for all $S \in A$ (for example the discrete open Balls in $\mathbb{R}^2$ intersected with $D_{\frac{1}{n}}$ for $n \in \mathbb{N}$). Then $\mathcal{M}^*(S) = \mathcal{M}(S)$ almost surely for all $S \in A$. By Dynkin´s lemma, we extend this property to $\mathcal{B}(D)$, which yields the first part of the statement. For the weak convergence note that $\mathcal{M}^*(D) = \mathcal{M}(D)$ almost surely, and therefore $\mathcal{M}^*(D)$ is finite. Hence the measures $\{\mathcal{M}^N\}_{N=1}^\infty$ are tight almost surely, and we can proceed like in subsection 2.4.4 to obtain the weak convergence of $\mathcal{M}^N \to \mathcal{M}$. □

Remark 23. Clearly the lemma does not depend on the specific choice of $\{u_j\}_{j=1}^\infty$.

Having all we need in hand, we start the proof of theorem 2.5.4.

Proof. Assume that $\{u_j\}_{j=1}^\infty$ is the smooth basis on $H_0^1(D)$, which constructs h^N. We already observed in the proof of theorem 1.5.2 that $\{u_j \circ \phi^{-1}\}_{j=1}^\infty$ is a (smooth) conformal basis on $H_0^1(\tilde{D})$. Hence we can write $\tilde{h}^N = h^N \circ \phi^{-1} = \sum_{j=1}^\infty \mathcal{N}_j u_j \circ \phi^{-1}$. Consider fir a bounded continuous function F on $\tilde{D}$

$$\begin{aligned}\int_{\tilde{D}} F(\tilde{z})\, \widetilde{M}^N(d\tilde{z}) &= \int_{\tilde{D}} F(\tilde{z}) e^{\gamma h^N \circ \phi^{-1}(\tilde{z}) - \frac{\gamma^2}{2} Var(h^N \circ \phi^{-1}(\tilde{z})} R(\tilde{z}, \tilde{D})^{\frac{\gamma^2}{2}}\, d\tilde{z} \\ &= \int_D F(\phi(z)) e^{\gamma h^N(z) - \frac{\gamma^2}{2} Var(h^N(z))} (|\phi'(z)| R(z; D))^{\frac{\gamma^2}{2}} |\phi'(z)|^2\, dz \\ &= \int_D F(\phi(z)) |\phi'(z)|^{\gamma Q}\, \mathcal{M}^N(dz)\end{aligned}$$

and by weak convergence the left hand side tends to $\int_{\tilde{D}} F(\tilde{z})\, \widetilde{M}(d\tilde{z})$ and the right hand side (where we use that $|\phi'(z)|^{\gamma Q}$ is bounded and continuous) to $\int_D F(\phi(z))|\phi'(z)|^{\gamma Q}\, \mathcal{M}(dz)$ for $N \to \infty$. As F was chosen arbitrarily the statement follows. □

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