MASTERARBEIT | MASTER'S THESIS

Titel | Title

Solution concepts for the Fornberg-Whitham equation and continuous weak traveling waves

verfasst von | submitted by

Calvin Stanko BSc

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien | Vienna, 2024

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt | Degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von | Supervisor:	ao. Univ.-Prof. Mag. Dr. Günther Hörmann

Abstract

We consider the Fornberg-Whitham equation, a nonlinear dispersive partial differential equation introduced in 1967 by Gerald Whitham as a model to analyze the wave breaking phenomena of water waves. Since traveling waves are a common tool for constructing solutions of partial differential equations, our goal is to answer the question, if wave breaking can be expressed by traveling wave solutions for the Fonberg-Whitham equation. After giving a brief summary on dispersive waves, we proceed defining different notions of solution concepts for the Fornberg-Whitham equation. These include weak solutions, entropy solutions and mild solutions. Along the way, we state some additional conditions on either the solution or the initial data, in order to compare these solution concepts with each other. Next, we introduce two types of solitary traveling waves. First, peaked solitary waves, which solve the equation in a weak sense, but do not describe wave breaking. Secondly, cusped solitary waves, which are determined by the asymptotic behaviour required for wave breaking. However, the cusped solitary waves do not solve the equation described as in the presented notions.

Zusammenfassung

Wir betrachten die Fornberg-Whitham Gleichung, eine nichtlineare dispersive partielle Differentialgleichung, aufgestellt im Jahr 1967 von Gerald Whitham als Modell zur Untersuchung der Wellenbrechung von Wasserwellen. Nachdem wandernde Wellen sich als Werkzeug zur Konstruktion von Lösungen für partielle Differentialgleichungen bewährt haben, ist das Ziel dieser Arbeit die Beantwortung der Frage, ob Wellenbrechung durch Lösungen der Fornberg-Whitham Gleichung, in Form von wandernden Wellen, ausgedrückt werden kann. Nach einer kurzen Einführung zu dispersiven Wellen, werden unterschiedliche Lösungkonzepte der Fornberg-Whitham Gleichung definiert. Diese beinhalten schwache Lösungen, Entropielösungen und milde Lösungen. Parallel dazu stellen wir Bedingungen für die Lösung oder den Anfangswert auf um die Lösungskonzepte in Relation zu bringen. Anschliedßend führen wir zwei Arten von wanderenden Wellen ein. Einerseits die peaked solitary wave, welche eine schwache Lösung der Gleichung ist, aber die Definition der Wellenbrechung erfüllt. Andererseits, die cusped solitary wave, die durch das asymptotische Verhalten, erforderlich für die Wellenbrechung, charakterisiert ist, aber die Fornberg-Whitham Gleichung in keiner der präsentierten Lösungskonzepten erfüllt.

Contents

1 Introduction

The modeling of fluid behavior has fascinated mankind for a lot of centuries. Over time the complexity of these models grew to a large scale. Therefore, these models in form of partial differential equations were not only interesting from a physical point of view, but also from the mathematical side itself. One fundamental aspect of it was the motion of water waves, especially the phenomena of breaking waves, which one can see at beaches. In 1967 Gerald Whitham was one of the first, who derived a comparably simple partial differential equation, that could describe the breaking of (nonlinear dispersive) water waves. The corresponding integro-differential equation for the wave height $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}, (t,x) \mapsto u(t,x)$ is denoted as follows

$$\begin{aligned} \partial_t u + u\partial_x u + K * \partial_x u &= 0 \\ u(0,x) &= u_0(x) \end{aligned} \tag{1.1}$$

and is called Whitham equation. For the specific choice of $K(x) = \frac{1}{2}e^{-|x|}$ it is called Fornberg-Whitham equation.
The aim of this thesis is to develop a weak solution concept and analyze given examples of possible solutions in the realm of this solution theory. However, firstly this PDE has to be put into a specific class of PDEs in order to establish an appropriate theory. A lot of fundamental models, which describe wave motion belong to the class of hyperbolic equations, e.g. wave equation. However, most types of wave motions, especially the ones of water waves, do not belong to this category. These wave motions can be found in the group of dispersive partial differential equations. The wave equation is a rare example of an equation, which falls into both categories. In contrary to most classes of partial differential equations, nonlinear dispersive ones are characterized by the type of solution and not the structure of the equation itself. However, in the linear case it is possible to relate these two.

1.1 Linear dispersive waves

As mentioned in [Whi11] dispersive waves are perceived by the existence of solutions in the form of

$$u(t,x) = Ae^{i\kappa \cdot x - i\omega t}, \tag{1.1}$$

which are also known as sinussoidal wavetrains (or plane waves) with the constants κ for the wave number, ω for the frequency and A for the amplitude. The constants κ and

ω are in a relation also called dispersion relation

$$\omega = W(\kappa).$$

Each solution corresponds to a unique function $W(\kappa)$. These functions are called modes. The phase

$$\theta = \kappa \cdot x - \omega t$$

describes the position on a cycle between a crest. This leads to the corresponding phase velocity

$$c = \frac{\omega}{\kappa}\widehat{\kappa},$$

where $\widehat{\kappa}$ is the unit vector in the direction of κ. Note, that c is not independent of κ and different wave numbers imply different phase speeds. From a physical point of view, this effect is called dispersion. In terms of categorization, c cannot be arbitrary. A first example would be c=const., which does not reveal any kind of dispersion. Secondly, taking the heat equation $\partial_t u = \Delta u$ with $\omega = -i\kappa^2$ as an example it follows, that there exist solutions of the form (1.1). However, these are not wavelike and not dispersive. Therefore, the following restriction on $W(\kappa)$ follows:

$$W(\kappa) \text{ is real and } \det\Big(\Big|\frac{\partial^2 W}{\partial\kappa_i \partial\kappa_j}\Big|\Big) \neq 0.$$

For the one dimensional case the latter condition is equivalent to $W''(\kappa) \neq 0$.

Remark 1.1. In general, PDEs with real coefficients have real dispersion relations, if and only if they possess only even or only odd derivatives. Each derivative yields one factor i. By allowing complex coefficients and complex solutions it is possible to have mixed derivatives and real dispersion relation. An example would be the Schrödinger equation $i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u$ with real dispersion relation $\hbar\omega = \frac{\hbar^2\kappa^2}{2m}$.

Remark 1.2. Identifying κ with ξ_0 and choosing for $W(\kappa)$ a real-valued polynomial $h(\xi_0)$ leads to a different expression for the plane wave:

$$u(t,x) = Ae^{ix\xi_0 + ith(\xi_0)},$$

which will be used later. In the case of the scaled Schrödinger equation $h(\xi_0) = -|\xi_0|^2$.

For dispersion only in the linear case, where $W(\kappa)$ is a real polynomial, it is possible to establish a connection between the solution and the structure of the equation. First of all rewrite the single linear constant coefficient equation to

$$P\Big(\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\Big)u = 0,$$

where P is a polynomial and of order 1 in the first variable. Recall, that upon inserting $Ae^{i\kappa\cdot x-i\omega t}$ into this equation the derivatives $\partial_t, \partial_{x_j}$ produce an additional factor $-i\omega$ and $i\kappa_j$ respectively. The dispersion relation

$$P\Big(-i\omega, i\kappa_1, i\kappa_2, i\kappa_3\Big) = 0$$

follows easily. This leads to the following association

$$\partial_t \leftrightarrow -i\omega, \quad \partial_{x_j} \leftrightarrow i\kappa_j.$$

As a first step linear dispersive partial differential equation, that also include wave behavior seen in the Schrödinger equation, are defined as follows:

Definition 1.3. (Linear dispersive Partial Differential Equation, [Tao06])
The constant coefficient linear partial differential equation

$$\partial_t u - Lu = 0, \quad u : \mathbb{R} \times \mathbb{R}^d \to V, \quad V \in \{\mathbb{R}, \mathbb{C}\} \tag{1.2}$$

is called *dispersive*, if

$$L = ih(D), \quad h(x) = \sum_{|\alpha|\leq k} i^{|\alpha|-1} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{C},$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a real-valued polynomial and $D = \frac{\nabla}{i}$ is the frequency operator. $h(x)$ is also called dispersion relation.

Remark 1.4. If the requirement, that $h(x)$ is a real-valued polynomial, holds, then $L = \sum_{|\alpha|\leq k} c_\alpha \partial_x^\alpha$.

At this point one can see, that the plane wave $e^{ix\xi_0 + ith(\xi_0)}$ solves the linear dispersive PDE (1.2) and can even be used to create more general solutions.

Lemma 1.5. *Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\xi + ith(\xi)} \widehat{u_0}(x)\, d\xi$$

solves the linear dispersive PDE (1.2) with initial condition $u(0,x) = u_0(x)$.

Proof. First of all, take the Fourier transform with respect to the variable x in Equation (1.2):

$$\partial_t \widehat{u}(t,\xi) - ih(\xi)\widehat{u}(t,\xi) = 0.$$

With the initial condition $\widehat{u}(0,\xi) = \widehat{u_0}(\xi)$ this ODE can easily be solved:

$$\widehat{u}(t,\xi) = e^{ith(\xi)} \widehat{u_0}(\xi).$$

Applying the inverse Fourier transform finishes the proof. □

The next step now is to not only take polynomial dispersion into account, but also operators of more general type. Possibilities are for example considering the oscillatory wave motion in one variable or wavelike behavior in all variables represented in the function (1.1). However, one way is the following.
Consider the one-dimensional integro-differential equation

$$\partial_t u(t,x) + \int_{-\infty}^{\infty} K(x-y)\partial_y u(t,y)\, dy = \partial_t u(t,x) + (K * \partial_t u)(t,x) = 0,$$

where $K(x)$ is a convolution kernel, which has to satisfy the following condition.
If $u(t,x) = Ae^{i\kappa \cdot x - i\omega t}$ is a solution, then

$$-i\omega e^{i\kappa x} + \int_{-\infty}^{\infty} K(x-y) i\kappa e^{i\kappa y}\, dy = 0$$

has to hold. Manipulating this equation appropriately one obtains

$$\int_{-\infty}^{\infty} K(y) e^{-i\kappa y}\, dy = \frac{\omega}{\kappa} = c,$$

which is the phase velocity. Since the integral term is the Fourier transform of K, applying the inverse formula gives

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\kappa) e^{i\kappa x}\, d\kappa. \tag{1.3}$$

Note that $c(\kappa) = W(\kappa)$ from before, i.e. the dispersion relation. Therefore, we can extend our notion of dispersive PDEs for every function $c(\kappa)$, that satisfies (1.3). This also does not contradict our definition of linear constant coefficient dispersive PDEs in Definition 1.3. Choosing

$$c(\kappa) = c_0 + c_1\kappa + c_2\kappa^2 + ... + c_n\kappa^n$$

yields

$$K(x) = \sum_{j\le n} i^{j-1} c_j \delta^{(j)}(x)$$

giving

$$\partial_t u - i \sum_{|\alpha|\le n} i^{|\alpha|-1} c_\alpha \Big(\frac{1}{i}\partial_x\Big)^{\alpha} u = 0,$$

which was defined above.

1.2 Derivation of the Fornberg-Whitham Equation

In [FW78] Fornberg and Whitham consider for the study of nonlinear waves the equation

$$\partial_t u + f(u)\partial_x u + K * \partial_x u = 0,$$

where they choose $f(u) = u$ for the nonlinear part additional to the linear dispersive equation. This gives

$$\partial_t u + u\partial_x u + K * \partial_x u = 0,$$

which is called Whitham equation and is the main equation studied in the further analysis of this thesis. The function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}, (t,x) \mapsto u(t,x)$ describes the wave height. Note, that the first two terms of the equation are part of the Burgers' equation. In addition to become the desired Fornberg-Whitham equation the convolution kernel has to be $K(x) = \frac{1}{2}e^{-|x|}$, which corresponds to the dispersion relation $c(\kappa) = \frac{1}{1+\kappa^2}$. This can be seen by

$$K(x) = (\mathcal{F}^{-1}c)(x) = \frac{1}{2}e^{-|x|}.$$

Taking the derivatives leads to

$$\begin{aligned} K(x) &= \frac{1}{2}e^{-|x|}, \\ K'(x) &= -\frac{1}{2}e^{-|x|}\mathrm{sgn}(x), \\ K''(x) &= \frac{1}{2}e^{-|x|} - \delta(x) = K(x) - \delta(x). \end{aligned}$$

It can be seen, that the kernel is also a fundamental solution of the ODE

$$(1 - \partial_x^2)K = K - K'' = \delta.$$

Writing $\partial_t u + u\partial_x u = -K * \partial_x u$ and setting $v := -K * u$, with corresponding derivative $\partial_x v := -K * \partial_x u$, leads to a system of partial differential equations

$$\begin{aligned} \partial_t u + u\partial_x u &= \partial_x v, \\ \partial_x^2 v &= v + u, \end{aligned}$$

that is called Burgers-Poisson (BP)-system ([SF04]) and a different portrayal of the Fornberg-Whitham equation. Another alternative representation can be used by the observation that $(1-\partial_x^2)^{-1}\partial_x u = K * \partial_x u$. As in [Hör18] applying the operator $(1-\partial_x^2)$ on $\partial_t u + u\partial_x u + K * \partial_x u = 0$ gives

$$\partial_t u - \partial_{txx} u - 3\partial_x u \partial_{xx} u - u\partial_{xxx} u + u\partial_x u + \partial_x u = 0.$$

Note that compared to other models for water waves the Fornberg-Whitham equation is not completely integrable. As stated in [CLH12], two examples of completely integrable

water wave models are the following:
Firstly, the Korteweg-de Vries equation, defined by

$$\partial_t u - 6u\partial_x u + u\partial_{xxx} u = 0,$$

is completely integrable with a strong dispersion relation $W(\kappa) = 1 - \kappa^2$. Solutions are given by solitary waves. However, the Korteweg-de Vries equation does not describe wave-breaking.
Secondly, there is the Camassa-Holm equation

$$\partial_t u + 2k\partial_x u - \partial_{xxt} u + 3u\partial_x u = 2\partial_x u\partial_{xx} u + u\partial_{xxx} u,$$

which is again completely integrable with dispersion relation $W(\kappa) = 2k\frac{\kappa}{1+\kappa^2}$. In addition, it also describes wave breaking and possesses solutions that are traveling waves or solitons.

1.3 Wave breaking

Since Whitham's initial intention was the analysis of the wave breaking phenomena, this has to be formulated mathematically.

Definition 1.6. (Wave breaking,[Con11]) Wave breaking happens for a two dimensional surface wave $u(t, x)$, if $\exists T > 0$ finite, such that

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}} |u(t, x)| < \infty,$$
$$\limsup_{t\to T} (\sup_{x\in\mathbb{R}} |\partial_x u(t, x)|) = \infty.$$

Remark 1.7. The aim of this thesis is not to rigorously analyze the wave breaking of the Fornberg-Whitham equation, which can be seen in [Haz17], but rather develop a weak solution concept and apply it to the traveling wave solutions, derived in [CLH12], in the hope of finding an explicit weak solution, which describes wave breaking.

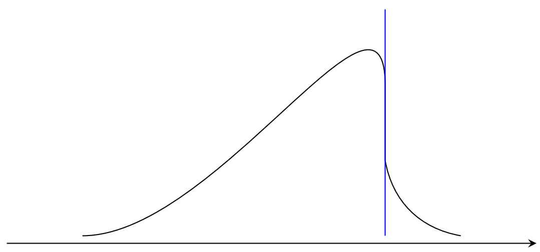

Figure 1.1: Graph of the wave height of a breaking wave.

Remark 1.8. [ZT10], [EK13] and [TWW22] use similar global bifurcation methods to derive several types of solitary wave solutions for the Whitham equation. However, here only the solitary waves, that can be found in [CLH12], are part of the analysis.

2 Solution concepts

In this section three different notions of solution concepts are going to be introduced, including the well known weak solutions, as well as entropy solutions and mild solutions, where the latter two lie between variational and strong solutions. The aim is to not only present these solutions, but also state some conditions in order to relate them with each other. This also leads to a strategy, which can be later seen in Chapter 3, to analyze the already mentioned solitary waves from [CLH12]. First, consider the weak formulation.

2.1 Weak solutions

In order to obtain the weak formulation for the Fornberg-Whitham equation, consider the test space denoted by $\mathcal{D}(\mathbb{R}^2) := C_c^\infty(\mathbb{R}^2)$. First, recall the initial condition $u(0,x) = u_0(x)$. Now suppose u is a classical solution and multiply the equation with an arbitrary test function $\phi \in \mathcal{D}(\mathbb{R}^2)$. After that, integrating with respect to x over $\mathbb{R}$ and t over the interval $[0,\infty)$ leads to

$$\int_0^\infty \int_{-\infty}^\infty \Big(-u(t,x)\partial_t\phi(t,x) - \frac{u^2(t,x)}{2}\partial_x\phi(t,x) + (K' * u(t,\cdot))(x)\phi(t,x)\Big)\, dx\, dt \\ = \int_{-\infty}^\infty u_0(x)\phi(0,x)\, dx. \quad (2.1)$$

Since $K(x) = \frac{1}{2}e^{-|x|}$, one can see, that $K'(x) = -\frac{1}{2}e^{-|x|}\mathrm{sgn}(x)$. This implies, that $K' \in L^1(\mathbb{R})$ and therefore Equation (2.1) is on both sides finite for u, u_0 measurable, where u is bounded on the domain $[0,T] \times \mathbb{R}$, for all $T > 0$, and u_0 is (locally) bounded.

Definition 2.1. (Weak solution, [Hör21]) Let $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be measurable and bounded on $[0,T] \times \mathbb{R}$, for all $T > 0$, and $u_0 \in L^\infty(\mathbb{R})$. If

$$\int_0^\infty \int_{-\infty}^\infty \Big(u(t,x)\partial_t\phi(t,x) + \frac{u^2(t,x)}{2}\partial_x\phi(t,x) - (K' * u(t,\cdot))(x)\phi(t,x)\Big)\, dx\, dt \\ + \int_{-\infty}^\infty u_0(x)\phi(0,x)\, dx = 0 \quad (2.2)$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^2)$, then u is called a weak solution for the Fornberg-Whitham equation.

Note, that u is defined on $[0,\infty) \times \mathbb{R}$. However, it may be in our interest to extend u to all of $\mathbb{R}^2$. This can be done by defining $u(t,x) := 0$, for any $t < 0$. Now consider the two operators $Au := u$ and $Bu := \frac{u^2}{2} + K * u$. Since $K' * u = \partial_x(K * u)$, one can show to obtain the expression of the divergence

$$\nabla_{(t,x)} \cdot (Au, Bu) = u_0 \otimes \delta \quad \text{on } \mathbb{R}^2.$$

Similar to [Daf16], u gets modified on a set of measure zero. For $\mathcal{K} \subset \mathbb{R}$ open and compact we will see, that the map $[0, \infty) \to L^\infty(\mathcal{K})$, $t \mapsto u(t, \cdot)|_{\mathcal{K}}$, with $u(0, \cdot) = u_0$, will be weak*-continuous. Then

$$[0, \infty) \to \mathcal{D}'(\mathbb{R}), \quad t \mapsto u(t)$$

follows to be a continuous map for $u(t)$ defined by its action on a test function $\phi \in \mathcal{D}(\mathbb{R})$

$$\langle u(t), \phi \rangle := \int_{\mathbb{R}} u(t, x)\phi(x)\, dx,$$

for all $t \geq 0$. This expression leads to

$$\lim_{t \to 0} \int_{\mathcal{K}} (u(t, x) - u_0(x))\phi(x)\, dx = 0, \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

In other words,

$$u(t) \to u_0 \text{ in } \mathcal{D}'(\mathbb{R}), \text{ as } t \to 0$$

or

$$u(0) = u_0 \text{ in } \mathcal{D}'(\mathbb{R}),$$

and this clearly shows in which sense the weak solution obtains its initial value $u_0(x)$.

Remark 2.2. As seen in [Daf16] an issue of nonlinear hyperbolic laws for Cauchy problems, especially the Burgers' equation, is non-uniqueness of weak solutions. Recall from Chapter 1, that the Burgers' equation perturbed by the convolution term in a non-local, but linear way, gives the Fornberg-Whitham equation. Therefore, the question of non-uniqueness arises also here. However, this subject will not be discussed in depth here. The focus now is rather on developing methods for conservation laws by using entropy conditions on weak solutions for the Fornberg-Whitham equation.

2.2 Weak entropy solutions

Consider the Fornberg-Whitham equation in the form of a scalar balance law

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = -K' * u$$

and, like in [Daf16], an entropy-entropy flux pair $\eta, Q : \mathbb{R} \to \mathbb{R}$, where η is convex and $Q'(z) = \eta'(z)z$. Suppose now, that u solves the Fornberg-Whitham equation in a classical sense and insert η, Q as follows into the equation

$$\partial_t \eta(u) + \partial_x Q(u) = \eta'(u)\partial_t u + Q'(u)\partial_x u = \eta'(u)(\partial_t u + u\partial_x u) = -\eta'(u)(K' * u).$$

Hence,

$$\partial_t \eta(u) + \partial_x Q(u) + \eta'(u)(K' * u) = 0. \tag{2.1}$$

However, in general this can not hold for any weak solution defined as in Definition 2.1, which can be chosen as a measurable function. The convexity now implies, that η is locally Lipschitz continuous and η' is increasing. Furthermore, η and η' are Borel measurable and locally bounded. Therefore, $Q'(z) = \eta'(z)z$ is measurable and locally bounded, hence Q is also locally Lipschitz continuous. This means for the compositions in the equation, that $\eta \circ u$, $\eta' \circ u$ and $Q \circ u$ are Lebesgue measurable, since the outer functions are all Borel measurable. The above works for strong solutions without any issues. However, one has to perform adaptions in order to apply it for weak solutions. This is generally known as notion of admissibility for weak solution to hyperbolic conservation laws of the form $\partial_t u + \partial_x f(u) + g(u) = 0$ with corresponding entropy-entropy flux pair η and Q with $Q'(z) = \eta'(z)f'(z)$. This can be done by changing the Equality (2.1) to an inequality of the following form

$$\partial_t \eta(u) + \partial_x Q(u) + \eta'(u)g(u) \leq 0,$$

which holds in a weak sense. In other words, $-(\partial_t \eta(u) + \partial_x Q(u) + \eta'(u)g(u))$ is equal to a non-negative measure. Applying this explicitly to the Fornberg-Whitham equation gives the expression

$$\partial_t \eta(u) + \partial_x Q(u) + \eta'(u)(K' * u) \leq 0. \tag{2.2}$$

It is now possible to develop the weak entropy solution concept. First of all, it is sufficient to restrict the entropy-entropy flux pair to Kruzkov entropy conditions

$$\eta(z) = |z - \lambda|, \quad Q(z) = \operatorname{sgn}(z - \lambda)\frac{z^2 - \lambda^2}{2} = \frac{1}{2}|z - \lambda|(z + \lambda),$$

where $\lambda \in \mathbb{R}$, since convex functions can be approximated by linear combinations of linear functions of the form $z \mapsto |z - \lambda|$, as mentioned in [Hör07].

Definition 2.3. (Weak entropy solutions (intermediate version), [Hör21]) Let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be measurable and bounded on $[0, T] \times \mathbb{R}$, for all $T > 0$, and $u_0 \in L^\infty(\mathbb{R})$. If

$$\begin{aligned} 0 \leq \int_0^\infty \int_{-\infty}^\infty \Big(&|u(t,x) - \lambda|\partial_t \phi(t,x) + \operatorname{sgn}(u(t,x) - \lambda)\frac{u^2(t,x) - \lambda^2}{2}\partial_x \phi(t,x) \\ &- \operatorname{sgn}(u(t,x) - \lambda)(K' * u(t,\cdot))(x)\phi(t,x)\Big)\, dx\, dt + \int_{-\infty}^\infty |u_0(x) - \lambda|\phi(0,x)\, dx \end{aligned} \tag{2.3}$$

holds for all $\lambda \in \mathbb{R}$ and for all non-negative $\phi \in \mathcal{D}(\mathbb{R}^2)$, then u is called a weak entropy solution of the Cauchy problem (1.1).

As already mentioned above, weak entropy solutions are in a sense stronger than weak solutions defined as in Definition 2.1. This is stated by the following Lemma.

Lemma 2.4. *Entropy solutions are weak solutions.*

Proof. We need to show, that the Inequality (2.3) implies the Equality (2.2) from Definition 2.1.
First of all, for any given non-negative $\phi \in \mathcal{D}(\mathbb{R}^2)$, the Inequality (2.3) holds for $\lambda = -r$ and $\lambda = r$, where $r > 0$ and $|u| < r$ on $\operatorname{supp}(\phi)$. Consider now these two cases.
First, take $\lambda = r$, then

$$0 \leq \int_0^\infty \int_{-\infty}^\infty \Big(|u(t,x) - r| \partial_t \phi(t,x) + \operatorname{sgn}(u(t,x) - r) \frac{u^2(t,x) - r^2}{2} \partial_x \phi(t,x) \\ - \operatorname{sgn}(u(t,x) - r)(K' * u(t,\cdot))(x) \phi(t,x) \Big) \, dx \, dt + \int_{-\infty}^\infty |u_0(x) - r| \phi(0,x) \, dx.$$

Evaluating the absolute values in the terms of the integrand gives

$$0 \leq \int_0^\infty \int_{-\infty}^\infty \Big((r - u(t,x) \partial_t \phi(t,x) - \frac{u^2(t,x) - r^2}{2} \partial_x \phi(t,x) \\ + (K' * u(t,\cdot))(x) \phi(t,x) \Big) \, dx \, dt + \int_{-\infty}^\infty (r - u_0(x)) \phi(0,x) \, dx.$$

Now one obtains the inequality

$$\int_0^\infty \int_{-\infty}^\infty r \partial_t \phi(t,x) \, dx \, dt + \int_{-\infty}^\infty r \phi(0,x) \, dx + \frac{r^2}{2} \int_0^\infty \int_{-\infty}^\infty \partial_x \phi(t,x) \, dx \, dt \geq \\ \int_0^\infty u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x)}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x) \phi(t,x) \, dx \, dt \\ + \int_{-\infty}^\infty u_0(x) \phi(0,x) \, dx.$$

Having a closer look, the left-hand side of the last inequality reveals for the first two terms

$$\int_0^\infty \int_{-\infty}^\infty r \partial_t \phi(t,x) \, dx \, dt + \int_{-\infty}^\infty r \phi(0,x) \, dx = - \int_{-\infty}^\infty r \phi(0,x) \, dx + \int_{-\infty}^\infty r \phi(0,x) \, dx = 0$$

and the last term

$$\frac{r^2}{2} \int_0^\infty \int_{-\infty}^\infty \partial_x \phi(t,x) \, dx \, dt = \frac{r^2}{2} \int_0^\infty \Big[\phi(t,x) \Big]_{-\infty}^\infty = 0.$$

This implies the following inequality

$$0 \geq \int_0^\infty u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x)}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x) \phi(t,x) \, dx \, dt \\ + \int_{-\infty}^\infty u_0(x) \phi(0,x) \, dx \quad (2.4)$$

for any $\phi \geq 0$ and ϕ Lipschitz.
Now in the second case consider $\lambda = -r$, then

$$0 \leq \int_0^\infty \int_{-\infty}^\infty \Big(|u(t,x) + r| \partial_t \phi(t,x) + \operatorname{sgn}(u(t,x) + r) \frac{u^2(t,x) - r^2}{2} \partial_x \phi(t,x) - \operatorname{sgn}(u(t,x) + r)(K' * u(t,\cdot))(x)\phi(t,x) \Big) \, dx \, dt + \int_{-\infty}^\infty |u_0(x) + r| \phi(0,x) \, dx.$$

Similar as before, we get

$$0 \leq \int_0^\infty \int_{-\infty}^\infty \Big((r + u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x) - r^2}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x)\phi(t,x) \Big) \, dx \, dt + \int_{-\infty}^\infty (r + u_0(x)) \phi(0,x) \, dx.$$

Bringing every term not including r to the other side results in

$$\int_0^\infty \int_{-\infty}^\infty r \partial_t \phi(t,x) \, dx \, dt + \int_{-\infty}^\infty r \phi(0,x) \, dx - \frac{r^2}{2} \int_0^\infty \int_{-\infty}^\infty \partial_x \phi(t,x) \, dx \, dt \geq - \Big(\int_0^\infty u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x)}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x)\phi(t,x) \, dx \, dt + \int_{-\infty}^\infty u_0(x) \phi(0,x) \, dx \Big).$$

With exactly the same arguments as above for the left-hand side, one can conclude, that

$$0 \leq \int_0^\infty u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x)}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x)\phi(t,x) \, dx \, dt + \int_{-\infty}^\infty u_0(x) \phi(0,x) \, dx, \quad (2.5)$$

for any $\phi \geq 0$ and ϕ Lipschitz.
Now combining the results (2.4) and (2.5) gives

$$0 = \int_0^\infty u(t,x) \partial_t \phi(t,x) + \frac{u^2(t,x)}{2} \partial_x \phi(t,x) - (K' * u(t,\cdot))(x)\phi(t,x) \, dx \, dt + \int_{-\infty}^\infty u_0(x) \phi(0,x) \, dx, \quad (2.6)$$

for any $\phi \geq 0$. For more general $\phi \in \mathcal{D}(\mathbb{R}^2)$ consider the decomposition $\phi = \phi_1 - \phi_2$, where $\phi_1 = \max(\phi, 0)$, $\phi_2 = \max(-\phi, 0)$ and clearly $\phi_1, \phi_2 \geq 0$ and Lipschitz.
This finishes the proof. □

Remark 2.5. Since $K' * \lambda = K * \lambda' = 0$ for λ being a constant, this convolution term can be added to the equation. To be precise, this means, that

$$\operatorname{sgn}(u(t,x) - \lambda)(K' * u(t,\cdot))(x)\phi(t,x) = \operatorname{sgn}(u(t,x) - \lambda)(K' * (u(t,\cdot) - \lambda))(x)\phi(t,x)$$

and the calculations above can be carried out in a similar way.

Recall the left-hand side from (2.2). The negative sum of the three terms is equal to a non-negative measure. Given a weak entropy solution, the term $\eta'(u)(K' * u)$ is bounded and measurable, such that the other terms $\partial_t \eta(u) + \partial_x Q(u)$ are equal to a signed measure. In addition, the map $t \mapsto u(t)$ is weak*-continuous. This will be a consequence of one of the following results, which can be found in [Daf16], where the flux gets identified through surfaces of co-dimension one. This allows us to retrieve the balance law in its original form.
Before, we need to develop a weak version of Helly's selection theorem, to be precise Helly's second theorem, since we will need it in a proof later. We are going to start with the selection principle.

Theorem 2.6. *(Helly's selection principle, [Car00]) Let $\{f_n\}$ be a uniformly bounded sequence of real-valued functions defined on X and let $\mathcal{K} \subset X$ be countable. Then there exists a subsequence $\{f_{n_k}\}$, which converges pointwise on $\mathcal{K}$.*

Proof. Assume $|f_n(x)| \leq C$, for all $n \in \mathbb{N}$ and $x \in X$. Define $\mathcal{K} := \{x_k : k \geq 1\}$. Since the sequence $\{f_n(x_1)\}$ is bounded, there exists a subsequence $\{f_{n_1}\} \subset \{f_n\}$, such that $\{f_{n_1}(x_1)\}$ converges. Analogously, the sequence $\{f_n(x_2)\}$ is also bounded. hence there exists a subsequence $\{f_{n_2}\} \subset \{f_{n_1}\}$, such that $\{f_{n_2}(x_2)\}$ converges, as well as $\{f_{n_2}(x_1)\}$ converges. By induction, one can always extract a subsequence $\{f_{n_{m+1}}\} \subset \{f_{n_m}\}$, such that $\{f_{n_{m+1}}(x_k)\}_{n\in\mathbb{N}}$ converges, for all $1 \leq k \leq m+1$. Since, for all k, the tail sequence $\{f_{n_n}(x_k)\}_{n\geq k}$ is a subsequence of $\{f_{n_k}(x_k)\}_{n\in\mathbb{N}}$, it follows, that the diagonal sequence $\{f_{n_n}(x_k)\}_{n\in\mathbb{N}}$ converges for all $k \in \mathbb{N}$. $\square$

Instead of using Helly's selection principle, we use a generalization for BV functions. Now recall the total variation

$$Var(f, [a, b]) := \sup_{P\in\mathcal{P}} \sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)|,$$

where $\mathcal{P} = \{P = \{x_0, ..., x_N\} \mid P \text{ is a partition of [a,b]}\}$, and, that the norm in the space of functions of bounded variations is defined as

$$\|f\|_{BV([a,b])} := \|f\|_{L^1([a,b])} + Var(f, [a, b]),$$

which is needed in Helly's first theorem.

Theorem 2.7. *(Helly's first theorem, [Car00]) Let $\{f_n\}$ be a bounded sequence in $BV([a, b])$ such that $\|f_n\|_{BV} \leq C$, for all $n \in \mathbb{N}$. Then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$, that converges pointwise to a function $f \in BV([a, b])$, which satisfies $\|f\|_{BV} \leq C$.*

Proof. Since $\|f_n\| \leq \|f_n\|_{BV} \leq C$, for all $n \in \mathbb{N}$, the sequence $\{f_n\}$ is uniformly bounded. Now define $v_n(x) := Var(f_n, [a, x])$, for $x \in [a, b]$, then $|v_n(x)| \leq Var(f_n, [a, b]) \leq C$ and $|v_n(x) - f_n(x)| \leq 2C$, for all n. Hence, $\{f_n\}$ is the difference of the two uniformly bounded sequences of increasing functions $\{v_n\}$ and $\{v_n - f_n\}$. By applying Helly's selection principle on countable dense subsets over again, there exists a subsequence n_k such that the limits

$$g(x) := \lim_{k\to\infty} v_{n_k}(x), \quad h(x) := \lim_{k\to\infty} (v_{n_k}(x) - f_{n_k}(x))$$

exist for all $x \in [a, b]$, since g and h being increasing functions, implies, that $f = g - h$ is of bounded variation. Therefore, $f(x) = \lim_{k\to\infty} f_{n_k}(x)$, for all $x \in [a, b]$. Finally, conclude, that $\|f\|_{BV} \leq C$. □

Furthermore, Barbu and Precupanu developed a generalization for functions of bounded variations in Banach spaces. In some literature this is also known as a weak form of Helly's first theorem.

Theorem 2.8. *(Helly's second theorem, [BP12]) Let X be a reflexive separable Banach space with separable dual space X^*. Consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset BV([a, b], X)$ such that $\|f_n(t)\| \leq C$ for $t \in [a, b]$ and $Var(f_n, [a, b]) \leq C$ for all $n \in \mathbb{N}$.*
Then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ and a limit function $f \in BV([a, b], X)$ such that, for $k \to \infty$,

$$f_{n_k}(t) \to f(t), \quad \textit{weakly in } X, \forall t \in [a, b],$$

$$\int_a^b \phi \, df_{n_k}(t) \to \int_a^b \phi \, df(t), \quad \textit{weakly in } X, \forall \phi \in C([a, b]), \forall t \in [a, b].$$

Proof. Let $x^* \in X^*$ be given. By Helly's first theorem extract a subsequence also denoted by $\{f_n\}$ and $f_{x^*} \in BV([a, b])$ such that

$$\langle f_n(t), x^*\rangle \to f_{x^*}(t), \quad t \in [a, b].$$

Let $\mathcal{K}$ be a countable and dense subset of X^*. By applying a diagonal process, one obtains a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that, for $n_k \to \infty$,

$$\langle f_{n_k}(t), x^*\rangle \to f_{x^*}(t), \quad \forall x^* \in \mathcal{K}, t \in [a, b].$$

Therefore, by the density of $\mathcal{K}$ it follows, that, for $n_k \to \infty$,

$$\langle f_{n_k}(t), x^*\rangle \to f_{x^*}(t), \quad \forall x^* \in X^*, t \in [a, b]$$

and

$$\int_a^b \phi \, d\langle f_{n_k}(t), x^*\rangle \to \int_a^b \phi \, df_{x^*}(t), \quad \text{weakly in X}, \forall \phi \in C([a, b]), t \in [a, b].$$

As a consequence by the Banach-Steinhaus theorem there exists $f \in BV([a, b], X)$, such that

$$\langle f(t), x^*\rangle = f_{x^*}(t), \quad \forall t \in [a, b], x^* \in X^*.$$

□

Now Helly's second theorem will be used in the proof of the announced result from [Daf16].

Lemma 2.9. *([Daf16]) Let $A \in L^\infty(\mathcal{K};\mathbb{M}^{1\times k})$ and $P \in \mathcal{M}(\mathcal{K})$ satisfy*

$$\nabla \cdot A = P \tag{2.7}$$

in a distributional sense, where $\mathcal{K} = (\alpha,\beta)\times\mathcal{B}$ is a cylindrical domain with $\mathcal{B}$ as a ball in $\mathbb{R}^{k-1}$. Let E_k be the k-base vector in $\mathbb{R}^k$ and $X = (t,x)$ with $t \in (\alpha,\beta)$ and $x \in \mathcal{B}$. Then, after a possible necessary modification of A on a set of measure zero, the function $a(t,x) = A(t,x)E_k$ acquires the following properties:
The one-sided limits $a(\tau\pm,\cdot) \in L^\infty(\mathcal{B})$, with respect to the weak-topology, exist, for all $\tau \in (\alpha,\beta)$, and can be determined by*

$$\begin{cases} a(\tau-,x) = \operatorname*{ess\,lim}_{t\uparrow\tau} A(t,x)E_k = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{\tau-\epsilon}^{\tau} A(t,x)E_k\,dt, \\ a(\tau+,x) = \operatorname*{ess\,lim}_{t\downarrow\tau} A(t,x)E_k = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{\tau}^{\tau+\epsilon} A(t,x)E_k\,dt, \end{cases} \tag{2.8}$$

where the limits are taken in $L^\infty(\mathcal{B})$, with respect to the weak-topology.*
Furthermore, for all $\tau \in (\alpha,\beta)$ and any Lipschitz continuous function ϕ with compact support in $\mathcal{K}$,

$$\begin{cases} \int_{\mathcal{B}} a(\tau-,x)\phi(\tau,x)\,dx = \int_{(\alpha,\tau)\times\mathcal{B}} A(X)\nabla\phi(X)\,dX + \langle P,\phi\rangle_{(\alpha,\tau)\times\mathcal{B}}, \\ -\int_{\mathcal{B}} a(\tau+,x)\phi(\tau,x)\,dx = \int_{(\tau,\beta)\times\mathcal{B}} A(X)\nabla\phi(X)\,dX + \langle P,\phi\rangle_{(\tau,\beta)\times\mathcal{B}} \end{cases} \tag{2.9}$$

holds. Therefore, $a(\tau-,\cdot) = a(\tau+,\cdot) = a(\tau,\cdot)$, unless τ is an element of an (at most) countable set of points with $|P|(\{\tau\}\times\mathcal{B}) > 0$ Especially, if P is absolutely continuous with respect to the Lebesgue measure, then the map $\tau \mapsto a(\tau,\cdot)$ is continuous on (α,β) in the weak-topology of $L^\infty(\mathcal{B})$.*

Proof. Let $\epsilon > 0$ be fixed and r the radius of $\mathcal{B}$. Define $\mathcal{B}_\epsilon$ as the ball in $\mathbb{R}^{k-1}$ with radius $r-\epsilon$ and the same center as $\mathcal{B}$. Now the functions A and P are getting mollified on $(\alpha+\epsilon,\beta-\epsilon)\times\mathcal{B}$ by $\psi_\epsilon(X) = \epsilon^{-k}\psi(\epsilon^{-1}X)$, where $\psi \in C_0^\infty(\mathbb{R}^k)$, resulting in

$$A_\epsilon = \psi_\epsilon * A, \quad p_\epsilon = \psi_\epsilon * P,$$

which satisfy

$$\nabla \cdot A_\epsilon(x) = p_\epsilon(X). \tag{2.10}$$

Then set $a_\epsilon(t,x) = A_\epsilon(t,x)E_k$. Now multiply (2.10) with a Lipschitz function ϕ on $\mathbb{R}^{k-1}$ with compact support on $\mathcal{B}_\epsilon$ and integrate the equation over $(r,s)\times\mathcal{B}_\epsilon$, $\alpha+\epsilon < r < s < \beta-\epsilon$. Integration by parts yields the following expression

$$\int_{\mathcal{B}_\epsilon} a_\epsilon(s,x)\phi(x)\,dx - \int_{\mathcal{B}_\epsilon} a_\epsilon(r,x)\phi(x)\,dx = \int_r^s\int_{\mathcal{B}_\epsilon} (A_\epsilon(t,x)\pi_k\nabla\phi(x) + p_\epsilon(t,x)\phi(x))\,dx\,dt, \tag{2.11}$$

where π_k is the projection of $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$. Now the total variation of the map $t \mapsto \int_{\mathcal{B}_\epsilon} a_\epsilon(t,x)\phi(x)\,dx$ is bounded over the interval $(\alpha+\epsilon, \beta-\epsilon)$. Take a countable family $\{\phi_l\}$ of test functions with compact support in $\mathcal{B}$, which is dense in $L^1(\mathcal{B})$. By Helly's second theorem extract a subsequence $\{a_{\epsilon_m}\}$, where also $\epsilon_m \to 0$, as $m \to \infty$, resulting in the same limit function as for a_ϵ defined above. Hence, for any $l \in \mathbb{N}$, the sequence $\{\int_{\mathcal{B}} a_{\epsilon_m}(t,x)\phi_l(x)\,dx\}$ converges weakly, as $m \to \infty$, for $t \in (a,b) \setminus G$, where $G \subset (\alpha,\beta)$ is countable and the limit has bounded variation over (α,β). For all $l \in \mathbb{N}$ the limit function is defined by $\int_{\mathcal{B}} a(t,x)\phi_l(x)\,dx$, where the map $t \mapsto a(t,\cdot)$ is in $L^\infty(\mathcal{B})$. Since $a(t,x) = A(t,x)E_k$ a.e. in $\mathcal{K}$, the function a does not depend on the sequence $\{\epsilon_m\}$, which is needed for its construction, and (2.8) holds $\forall \tau \in (\alpha,\beta)$.
Take $\tau \in (\alpha,\beta)$ and an arbitrary Lipschitz function ϕ with compact support in $\mathcal{K}$, then multiply (2.10) with ϕ and integrate the equation over $(\alpha+\epsilon, s) \times \mathcal{B}_\epsilon$, where $s \in (\alpha+\epsilon, \tau) \setminus G$. Again after integration by parts one obtains

$$\int_{\mathcal{B}_\epsilon} a_\epsilon(s,x)\phi(s,x)\,dx = \int_{(a+\epsilon,s)\times\mathcal{B}_\epsilon} (A_\epsilon(X)\nabla\phi(X) + p_\epsilon\phi(X))\,dX. \tag{2.12}$$

Taking the limits $\epsilon \downarrow 0$ and $s \uparrow \tau$ give the first equation in (2.9). By similar procedure the second equation in (2.9) follows as well.
If P is absolutely continuous with respect to the Lebesgue measure, Equation (2.11) implies that $\{\int_{\mathcal{B}_\epsilon} a_\epsilon(t,x)\phi_l(x)\,dx\}$ is equicontinuous, where ϵ serves as a parameter. Therefore, $\int_{\mathcal{B}} a(t,x)\phi_l(x)\,dx$ is continuous on (α,β) for $l \in \mathbb{N}$.
Finally, the map $t \mapsto a(t,\cdot)$ is continuous on (α,β) in $L^\infty(\mathcal{B})$, with respect to the weak*-topology, which finishes the proof. □

Now replace a with $\eta(u)$, being convex, in Lemma 2.9. As a result, since Q is also a quadratic function, the two maps $[0,\infty) \to L^\infty(\mathcal{B}), t \mapsto u(t)$ and $[0,\infty) \to L^\infty(\mathcal{B}), t \mapsto u^2(t)$ are now both weak*-continuous. Additionally, this can even be extended to norm continuity.

Corollary 2.10. *([Hör21]) The map $[0,\infty) \to L^1(\mathcal{B}), t \mapsto u(t)$ is norm continuous.*

Proof. Take $t_n, t_0 \geq 0$, where $t_n \to t_0$ as $n \to \infty$. It is sufficient to show, that $v_n := u(t_n)|_{\mathcal{B}}$ converges to $v_0 := u(t_0)|_{\mathcal{B}}$ in $L^1(\mathcal{B})$. By the weak*-convergence of $v_n \to v_0$ and $v_n^2 \to v_0^2$, as well as the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}\Big(\int_{\mathcal{B}} |v_n - v_0|\,dx\Big)^2 &\leq \int_{\mathcal{B}} 1^2\,dx \cdot \int_{\mathcal{B}} |v_n - v_0|^2\,dx \\ &= |\mathcal{B}|\Big(\int_{\mathcal{B}} v_n^2\,dx - 2\int_{\mathcal{B}} v_n v_0\,dx + \int_{\mathcal{B}} v_0^2\,dx\Big) \to 0\end{aligned}$$

□

Let the initial value u_0 be in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the continuity in $L^1(\mathbb{R})$ developed above might lead for the weak entropy solution u to be in $C([0,\infty), L^1(\mathbb{R}))$. [Daf16] shows, that this is possible for scalar conservation laws, in particular [GN16] has shown this for the Fornberg-Whitham equation. Taking this adaptions into account and removing the term including the initial condition in the Inequality (2.3) give a new definition for the weak entropy solution.

Definition 2.11. (Weak entropy solutions, [Hör21]) Let $u \in C([0,\infty), L^1(\mathbb{R}))$ be bounded on $[0,T] \times \mathbb{R}$, for all $T > 0$, and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If $u(0) = u_0$ and

$$0 \leq \int_0^\infty \int_{-\infty}^\infty \Big(|u(t,x) - \lambda| \partial_t \phi(t,x) + \operatorname{sgn}(u(t,x) - \lambda) \frac{u^2(t,x) - \lambda^2}{2} \partial_x \phi(t,x) \\ - \operatorname{sgn}(u(t,x) - \lambda)(K' * u(t,\cdot))(x) \phi(t,x) \Big) \, dx \, dt, \quad (2.13)$$

holds for all $\lambda \in \mathbb{R}$ and any non-negative $\phi \in \mathcal{D}((0,\infty) \times \mathbb{R})$, then u is called a weak entropy solution of the Fornberg-Whitham equation.

One attempt to address the issue of non-uniqueness is by using the method of vanishing viscosity. We start with a result, which proves uniqueness for the entropy inequality (2.2) with initial condition $u_0 \in BV(\mathbb{R})$.

Theorem 2.12. *(Global weak solution, [SF04]) Let $u_0 \in BV(\mathbb{R})$. Then there exists a unique global solution*

$$u \in L^\infty_{loc}([0,\infty), BV(\mathbb{R})) \quad (2.14)$$

of (1.1), such that the entropy inequality

$$\partial_t \eta(u) + \partial_x Q(u) + \eta'(u)(K' * u) \leq 0 \quad (2.15)$$

is fulfilled in a distributional sense for $\eta \in C^1(\mathbb{R})$ convex and Q such that $Q'(u) = \eta'(u)u$. The function u obtains its initial value in the sense of

$$\int_{\mathbb{R}} |u(t,x)| \, dx \to \int_{\mathbb{R}} |u_0(x)| \, dx, \quad t \to 0.$$

Proof. Similar as in ([Tay13], Chapter 16.6) we use the method of vanishing viscosity. Therefore, instead of (1.1) consider the regularized equation

$$\partial_t u + u \partial_x u = -K * \partial_x u + \nu \partial^2_{xx} u, \quad (2.16)$$

where $\nu > 0$. By standard arguments it can be shown, that there exist local solutions of the initial value problem (2.16) with $u(t=0) = u_0 \in BV(\mathbb{R})$. Hence, we continue with the development of the L^1-stability for (2.16). Let u_1, u_2 be solutions of (2.16) with corresponding initial data $f_1, f_2 \in L^1(\mathbb{R})$. Therefore, the difference $v = u_1 - u_2$ satisfies the equation

$$\partial_t v + \partial_x (wv) = -K * \partial_x v + \nu \partial^2_{xx} v, \quad v(t=0) = f_1 - f_2, \quad (2.17)$$

where $w = \frac{u_1 + u_2}{2}$. Now introduce $abs_\delta(\cdot)$ as a convex regularization of the modulus, which converges uniformly to $|\cdot|$, for $\delta \to 0$, and satisfies $|abs'_\delta(v)| \leq 1$. Now after multiplying (2.17) with $abs'_\delta(v)$ and integrating with respect to the variable x one obtains the expression

$$\frac{d}{dt}\int_{\mathbb{R}} abs_\delta(v)\,dx = \int_{\mathbb{R}} w\,v\,abs''_\delta(v)\,\partial_x v\,dx + \int_{\mathbb{R}} abs'_\delta(v)(K * \partial_x v)\,dx \\ - \nu\int_{\mathbb{R}} abs''_\delta(v)(\partial_x v)^2)\,dx \quad (2.18)$$

Note that $\int_0^v s, abs''_\delta(s)\,ds \to 0$ uniformly, for $\delta \to 0$. Therefore, we get for the first term of the right-hand side of (2.18)

$$\int_{\mathbb{R}} w\,v\,abs''_\delta(v)\,\partial_x v\,dx = -\int_{\mathbb{R}} \partial_x w \int_0^v s abs''_\delta(s)\,ds\,dx \to 0, \quad \delta \to 0.$$

The operator $K' * u$ is bounded (This is contained in the proof of Lemma 2.35). Considering the properties of abs_δ, it follows, that after taking the limit, as $\delta \to 0$, in (2.18),

$$\frac{d}{dt}\|v\|_{L^1(\mathbb{R})} \leq C\|v\|_{L^1(\mathbb{R})}. \quad (2.19)$$

Similar as in ([Tay13], Lemma 6.1), we obtain

$$\frac{d}{dt}\|v\|_{BV(\mathbb{R})} \leq C\|v\|_{BV(\mathbb{R})}$$

for the solution of (2.16) following by the Inequality (2.19). Using this property is enough to show global existence and uniformly boundedness in $L^\infty_{loc}([0,\infty, BV(\mathbb{R}))$ in ν for the solution of (2.16). This is sufficient to prove the a.e. pointwise convergence on $(0,\infty)\times\mathbb{R}$, as $\nu \to 0$. The details can be found in [Tay13].
As a next step we show that the solution derived as above fulfills the entropy inequality (2.15). For that consider a smooth convex entropy density η, as well as a smooth Q, satisfying $Q' = u\eta'$, and multiply Equation (2.16) with $\eta'(u(t,x))$. Let $\phi \in C_0^\infty((0,\infty)\times\mathbb{R})$, then after integration by parts we get for the weak formulation

$$-\int_{\mathbb{R}}\int_{(0,\infty)} (\eta\partial_t\phi + Q\partial_x)\,dt\,dx = \int_{\mathbb{R}}\int_{(0,\infty)} ((-K * \partial_x u)\eta'(u) - \nu\eta''(\partial_x u)^2 + \nu\partial^2_{xx}\eta)\phi\,dt\,dx.$$

Similar as above in the vanishing viscosity limit, the terms $\eta(u), Q(u)$ and $(-K * \partial_x u)\eta'(u)$ converge pointwise a.e. on $(0,\infty)\times\mathbb{R}$, as $\nu \to 0$, since η and Q are both smooth. Additionally, in [Tay13] it was shown that $\nu\partial^2_{xx}\eta \to 0$ weakly, while $-\nu\eta''(\partial_x u)^2 \leq 0$ by the convexity of η. The latter implies the sign in the Inequality (2.16). Using an approximation argument it follows, that (2.16) is satisfied for $\eta \in C^1(\mathbb{R})$.
If $\eta \in W^{1,\infty}(\mathbb{R})$ is convex, then η' is defined a.e. and is monotonically increasing. Therefore, the number of jumps is at most countable. After defining η' pointwise everywhere in a way such that $|\eta'|(x) \leq \|\phi'\|_{L^\infty}$, there are smooth approximations $\eta_\delta \to \eta$ in $W^{1,\infty}(\mathbb{R})$ such that $|\eta'_\delta|(x) \leq |\eta'|(x)$ and $\eta'_\delta \to \eta'$ pointwise everywhere. As a consequence, using this smoothing η_δ yields, that the limit in the right-hand side of (2.16)

is well-defined by the theorem of dominated convergence, where the upper bound is $(-K * \partial_x u)\|\eta'(u)\|_{L^\infty(\mathbb{R})}$.
For showing uniqueness in the case of the Kruzkov entropy density $\eta(u) = |u - \lambda|$, for $\lambda \in \mathbb{R}$, choose as smoothing η_δ such that $\lim_{\delta\to 0} \eta'_\delta(0) = \operatorname{sgn}(0) = 0$. In addition consider the two solutions $u(t,x)$ and $\overline{u}(s,y)$ and take the two sums of (2.15), where $\lambda = \overline{u}(s,y)$ in the first sum and $\lambda = u(t,x)$ in the second sum. Since $|\operatorname{sgn}(u-\lambda)| \leq 1$, the right-hand side can be estimated such that we obtain

$$-\int\int\int\int |u-\overline{u}|(\partial_t\phi + \partial_s\phi)\, dt\, dx\, ds\, dy - \int\int\int\int Q(u,\overline{u})(\partial_x\phi + \partial_y\phi)\, dt\, dx\, ds\, dy$$
$$\leq \int\int\int\int |K * \partial_x u - K * \partial_y \overline{u}|\phi\, dt\, dx\, ds\, dy,$$

where $Q(u,\overline{u}) = \frac{1}{2}(u+\overline{u})|u-\overline{u}|$ and $\phi = \phi(t,x,s,y)$ is a non-negative test function. We choose ϕ such as

$$\phi(t,x,s,y) = f(t)\delta_\epsilon\Big(\frac{t-s}{2}\Big)g(x)\delta_\epsilon\Big(\frac{x-y}{2}\Big),$$

where $f, \delta_\epsilon, g_\epsilon \in C_0^\infty(\mathbb{R})$ are non-negative. Furthermore, $\delta_\epsilon(\cdot)$ converges to the delta distribution, as $\to 0$, and g_ϵ is a so-called plateau function, which satisfies the properties, that $0 \leq g_\epsilon \leq 1$ and $g_e \equiv 1$ on the interval $(-\frac{1}{\epsilon}, \frac{1}{\epsilon})$. Since

$$\partial_t\phi + \partial_s\phi = f'\delta_\epsilon\Big(\frac{t-s}{2}\Big)g\delta_\epsilon\Big(\frac{x-y}{2}\Big), \quad \partial_x\phi + \partial_y\phi = f\delta_\epsilon\Big(\frac{t-s}{2}\Big)g'\delta_\epsilon\Big(\frac{x-y}{2}\Big),$$

taking the limit, as $\epsilon \to 0$, yields

$$-\int_{(0,\infty)}\int_{\mathbb{R}} |u(t,x) - \overline{u}(t,x)|f'\, dx\, dt \leq \int_{(0,\infty)}\int_{\mathbb{R}} |K * \partial_x u - K * \partial_y \overline{u}|f\, dx\, dt.$$

Because

$$\int_{\mathbb{R}} |K * \partial_x(u - \overline{u})|\, dx \leq \|K'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u - \overline{u}|\, dx$$

with $\|K'\|_{L^\infty(\mathbb{R})} = \frac{1}{2}$ and taking f arbitrary, it can be concluded after integration by parts, that

$$\frac{d}{dt}\int_{\mathbb{R}} |u-\overline{u}|\, dx \leq \frac{1}{2}\int_{\mathbb{R}} |u-\overline{u}|\, dx.$$

Uniqueness now is implied by the Gronwall Lemma and the proof is finished. $\square$

Remark 2.13. Note, that the original statement of this theorem in [SF04] does not give a relation between the solution u and the initial value u_0, since there the initial condition only appears implicitly in the last step of the proof applying the Gronwall Lemma. Let

$\phi(t) := \int_{\mathbb{R}} |u-\overline{u}|\,dx$, then the last inequality from the proof can be expressed as $\frac{d}{dt}\phi \le \frac{1}{2}\phi$ and the Gronwall Lemma yields $\phi(t) \le \phi(0)e^{\frac{1}{2}t}$. Taking the limit as $t \to 0$ gives, that $\phi(t) \to 0$ or in other words $\phi(0) = 0$. Since $\phi(t) \le \phi(0)e^{\frac{1}{2}t} = 0$, it follows, that $\phi(t) = 0$ for all t and uniqueness is the consequence.

Additionally, we can state another version of this theorem, which can also be found in [Hör21], in the sense of weak entropy solutions using the Kruzkov entropy condition, seen in Definition 2.11, but still with initial condition $u_0 \in BV(\mathbb{R})$.

Theorem 2.14. *([Hör21]) Let $u_0 \in BV(\mathbb{R})$. Then there exists a unique weak entropy solution u in the sense of Definition 2.11 for the Fornberg-Whitham equation. Additionally, $u \in L^\infty_{loc}([0,\infty), BV(\mathbb{R}))$.*

The uniqueness in Theorem 2.14 follows by the proof of Theorem 2.12. The next theorem, mentioned in [Hör21], extends the previous result to initial values $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as in Definition 2.11, giving also another important tool, which will be especially needed for our analysis in Chapter 3, the Oleinik-type inequality, which is satisfied by weak entropy solutions.

Theorem 2.15. *([Hör21]) Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a unique weak entropy solution u for the Fornberg-Whitham equation described as in Definition 2.11, satisfying the* ***Oleinik type inequality:*** *$\forall t > 0, \forall x, y \in \mathbb{R}, x < y$:*

$$u(t,y) - u(t,x) \le \Big(\frac{1}{t} + 2 + 2t(1 + 2e^t \|u_0\|_{L^1(\mathbb{R})})\Big)(y - x). \tag{2.20}$$

Furthermore, if v is the weak entropy solution for the initial value $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then

$$\|u(t) - v(t)\|_{L^1(\mathbb{R})} \le e^t \|u_0 - v_0\|_{L^1(\mathbb{R})}, \quad \forall t > 0. \tag{2.21}$$

In particular, setting $v_0 = 0$ leads to the following estimate

$$\|u(t)\|_{L^1(\mathbb{R})} \le e^t \|u_0\|_{L^1(\mathbb{R})}, \quad \forall t \ge 0. \tag{2.22}$$

Remark 2.16. [GN16] presents the same result as Theorem 2.15, with the slight change of $u_0 \in L^1(\mathbb{R})$. However, $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is what will be assumed, similar to [Hör21].

Remark 2.17. We will not prove Theorem 2.15 here, whose proof is developed in [GN16] using the following strategy. Relying on the flux-splitting method, approximate solutions are used to prove the Oleinik-type inequality. On that account the decay estimates of Burgers' semi-groups and Lipschitz continuity of solutions for the Poisson equation are used. Convergence, as well as pre-compactness in $L^1_{loc}(\mathbb{R})$ are the consequence. By energy estimates one obtains Hölder continuity, that implies a tightness property for the sequence of approximate solutions. The continuity of the solution u follows. The Inequality (2.21) is also known as L^1-stability, which can be derived from the boundedness of weak entropy solutions and the L^1-continuity of u. This L^1-continuity with respect to the variable t gives the integral inequality. In addition to the L^1-stability, via the same tools, uniqueness follows as well.

2.3 Mild solutions

The aim of this section is to develop the concept of mild solutions. The idea is to generate a solution semi-group of nonlinear contractions in the space $L^1(\mathbb{R})$, since, without the convolution term, Equation (1.1) is the Burgers' equation. The main approach is to extend the map $C_c^1(\mathbb{R}) \to L^1(\mathbb{R}), v \mapsto \partial_x\left(\frac{v^2}{2}\right)$ to an accretive operator in $L^1(\mathbb{R})$. In addition, requiring the initial condition u_0 to be in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies the mild solution to be also a weak entropy solution, which is one of the consequences. First of all, we introduce some fundamental notions in order to establish this solution concept. Recall the linear case.

Definition 2.18. (C_0-semi-groups, [Nee22]) Let X be a Banach space and $S = \{S(t)\}_{t\geq 0}$ a family of bounded linear operators acting on X. S is called a C_0-semi-group, if

(i) $S(0) = I$,

(ii) (semi-group property) $S(t)S(s) = S(t+s), \quad \forall t, s \geq 0$,

(iii) (strong continuity) $\lim_{t\downarrow 0} \|S(t)x - x\| = 0, \quad \forall x \in X$,

hold. The generator is the linear operator defined by

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x), \quad x \in D(A),$$

where $D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X\}$ is the domain of A.

Then

$$u(t) := S(t)u_0$$

is interpreted as the solution of the linear problem

$$\begin{cases} u'(t) = Au(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

for $u_0 \in D(A)$. However, the Fornberg-Whitham equation is not a linear problem. Therefore, because of the nonlinearity the appropriate semi-group needs to be generated by a nonlinear operator. This is done in the special setting, where the space is $X := L^1(\mathbb{R})$ and the considered nonlinear operator is denoted by G on X. Also define the map $J : X \to 2^{X^*}$ such that

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

In our case, G needs to be an accretive operator. We recall some basic notions first.

Definition 2.19. A general (multi-valued) nonlinear operator $G : X \to X$ is defined by a relation $G \subseteq X \times X$. The value of G at an element $x \in X$ is defined by $G(x) := \{y \in X | (x, y) \in G\}$. The domain of G is defined by $D(G) := \{x \in X : G(x) \neq \varnothing\}$ and the range of G gets defined by $R(G) := \bigcup_{x \in D(G)} G(x)$.

Remark 2.20. Since G is defined by a relation $G \subseteq X \times X$, we can equivalently view G as a subset $G = \{(x, y) | x \in D(G), y \in G(x)\} \subseteq X \times X$.

Remark 2.21. If the sets $G(x)$ are single-valued, for all $x \in D(G)$, then $G = \{(u, v) | u \in D(G), v \in G(u)\}$ is the identification of the map with its graph $\mathrm{gr}(G) = \{(x, y) | y \in G(x)\} \subseteq X \times X$.

In addition, some basic operations can be formulated for this type of nonlinear operator.

Definition 2.22. If $G, F \subseteq X \times X$ and $\lambda \in \mathbb{R}$, then define the operations

(i) (scalar multiplication) $\lambda G := \{(x, \lambda y) | (x, y) \in G\}$,

(ii) (sum) $G + F := \{(x, y + z) | (x, y) \in G, (x, z) \in F\}$,

(iii) (composition) $G \circ F := \{(x, z) | \exists y \in X : (x, y) \in F, (y, z) \in G\}$,

(iv) (inverse) $G^{-1} := \{(y, x) | (x, y) \in G\}$.

Now, it is possible to define an accretive operator.

Definition 2.23. (Accretive operator, [Bar10]) A subset $G \subseteq X \times X$ (equivalently, a multi-valued operator $G : X \to X$) is called accretive, if for all $(x_1, y_1), (x_2, y_2) \in G$, there exists a $w \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, w \rangle \geq 0.$$

Remark 2.24. If $-G$ is accretive, then G is called dissipative.

We introduce some notions for accretive operators.

Definition 2.25. An accretive set is maximal accretive, if it is not completely contained in any accretive subset of $X \times X$.

Definition 2.26. An accretive operator G is m-accretive, if

$$R(I + G) = X,$$

where I is the identity on X.

Definition 2.27. Let $\omega \in \mathbb{R}$. G is *ω-accretive*, if $G + \omega I$ is *accretive*. In the case, where $G + \omega I$ is *m-accretive*, G is called *ω-m-accretive*.

Remark 2.28. Operators, that are *ω-accretive* (respectively, *ω-m-accretive*), are also called *quasi-accretive* (respectively, *quasi-m-accretive*).

The next step is to develop a geometric property for accretive operators. This can be done using the following Lemma.

Lemma 2.29. *(Kato's Lemma, [Bar10]) Let $x, y \in X$ be given. Then there exists $w \in J(x)$ such that $\langle y, w\rangle \geq 0$ if and only if*

$$\|x\| \leq \|x + \lambda y\|, \quad \forall \lambda > 0. \tag{2.1}$$

Proof. For the first implication, let $x, y \in X$ be given, such that $\langle y, w\rangle \geq 0$ for some $w \in J(x)$. By the definition of J it follows, that

$$\|x\|^2 = \langle x, w\rangle \leq \langle x + \lambda y, w\rangle \leq \|x + \lambda y\|\|w\| = \|x + \lambda y\|\|x\|, \quad \forall \lambda > 0.$$

Dividing both sides by $\|x\|$ gives (2.1).
Now suppose (2.1) holds. For $\lambda > 0$, let w_λ be an arbitrary element of $J(x+\lambda y)$. Without loss of generality: $x \neq 0$. Therefore, $w_\lambda \neq 0$ for small λ. Now define $f_\lambda := w_\lambda \|w_\lambda\|^{-1}$. Note, that the weak* compactness of $\{f_\lambda\}_\lambda$ in X^* gives a net $f_\lambda \rightharpoonup f$.
Now for the second implication, consider the inequality

$$\|x\| \leq \|x + \lambda y\| = \langle x + \lambda y, f_\lambda\rangle \leq \|x\| + \lambda\langle y, f_\lambda\rangle,$$

then subtract $\|x\|$ and divide by λ in order to obtain

$$\langle y, f_\lambda\rangle \geq 0, \quad \forall \lambda > 0.$$

Therefore, $\langle y, f\rangle \geq 0$ and $\|x\| \leq \langle x, f\rangle$. Since $\|f\| \leq 1$, it follows that $\|x\| = \langle x, f\rangle$, $\|f\| = 1$. Finally one gets $w = f\|x\| \in J(x)$, and $\langle y, w\rangle \geq 0$. □

Corollary 2.30. *([Bar10]) A subset G of $X \times X$ is accretive if and only if*

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| \tag{2.2}$$

holds for all $\lambda > 0$ and $(x_i, y_i) \in G, i = 1, 2$.

Remark 2.31. (i) Corollary 2.30 follows directly from Kato's Lemma and the definition of an accretive operator.

(ii) An analogous result can be carried out for ω-accretive operators, i.e. ω-accretiveness for G is equivalent to

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq (1 - \lambda\omega)\|x_1 - x_2\|,$$

for $0 < \lambda < \frac{1}{\omega}$ and $(x_i, y_i) \in G, i = 1, 2$.

In order to develop the mild solution concept, we need to define a continuous semi-group of nonlinear operators (respectively, contractions) first.

Definition 2.32. (Continuous semi-group of nonlinear operators (respectively, contractions), [Bar10]) A continuous semi-group of nonlinear operators (respectively, contractions) on X is a family $S = \{S(t)\}_{t\geq 0}$ of maps $S(t) : X \to X$ such that:

(i) $S(0) = I$.

(ii) $S(t+s) = S(t) \circ S(s), \quad \forall t, s \geq 0$.

(iii) The map $t \mapsto S(t)x$ is continuous on $[0, \infty)$, $\forall x \in X$.

(iv) In case of contractions,

$$\|S(t)x - S(t)y\| \leq \|x - y\|, \quad \forall t \geq 0, \forall x, y \in X.$$

In this case $S(t)$ is called a continuous ω-quasi-contractive semi-group. The operator $A : D(A) \to X$, defined by,

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \quad x \in D(A), \tag{2.3}$$

with domain $D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X\}$, is called the infinitesimal generator of the semi-group $S(t)$.

Remark 2.33. (i) As mentioned in [Bar10], there is a relation between the continuous semi-group of contractions and accretive operators, since it can be seen that $-A$ from Definition 2.32 is accretive in $X \times X$. Moreover, if $S(t)$ is quasi-contractive, then $-A$ is ω-accretive.

(ii) Additionally, since X is a Banach space, it is mentioned in [Bar10], that in the case, where G is an ω-m-accretive operator, the range condition

$$R(I + \lambda G) \supset \overline{D(G)}$$

is satisfied by G. Therefore, the semi-group $S(t)$ from Definition 2.32 is said to be generated by the ω-m-accretive nonlinear operator G, if

$$S(t)(u_0) = \lim_{n\to\infty} \left(I + \frac{t}{n} G\right)^{-n} (u_0) \tag{2.4}$$

holds for all $u_0 \in \overline{D(G)}$, the closure of the domain of G. It can be also shown explicitly, that Formula (2.4) satisfies the properties (i)-(iv) in Definition 2.32.

Finally the anticipated mild solution concept for the Fornberg-Whitham equation can be formulated in connection with semi-groups. Consider the two operators A, the generator of the solution semi-group of contractions for the Burgers' equation, and B, the continuous linear convolution operator $L^1 \to L^1, u \mapsto Bu := K' * u$.

Definition 2.34. (Mild solution, [Hör21]) Let $A + B$ be an ω-m-accretive operator and generator of the continuous semi-group $\{S(t)\}_{t\geq 0}$ on $L^1(\mathbb{R})$. If $u_0 \in L^1(\mathbb{R})$, then $u(t) := S(t)(u_0)$ is in $C([0,\infty), L^1(\mathbb{R}))$, for $t \geq 0$, and is called the mild solution for the Fornberg-Whitham equation.

However, the well-posedness of a mild solution still needs to be shown. But before, some properties for the convolution operator and a result for m-accretive operators will be formulated.

Lemma 2.35. *([Hör21]) Let $Bu := K' * u$. Then the following properties hold:*

(i) For all $p, q \in \mathbb{R}$ with $1 \leq q \leq p \leq \infty$ holds, that

$$B \text{ is bounded from } L^q(\mathbb{R}) \text{ to } L^p(\mathbb{R}).$$

(ii) The space $BV(\mathbb{R})$ gets mapped into $W^{1,\infty}(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$.

(iii) For all $u \in L^\infty(\mathbb{R})$ it holds, that

$$\sup |\partial_x (K' * u)| \leq 2\|u\|_\infty.$$

Proof. (i) Since $K' \in L^r(\mathbb{R})$, for all $r \in [1, \infty]$, it follows consequently by Young's convolution inequality, that

$$\|K' * u\|_p \leq \|K'\|_r \|u\|_q,$$

if $1 \leq q := \frac{rp}{r+rp-p} \leq p$.

(ii) Let $u \in BV(\mathbb{R})$. Then Du, the BV derivative of u, is a finite measure. Consider $\partial_x (K' * u) = K' * Du$ and again apply Young's convolution inequality, but now for measures. Therefore, one obtains the inequality

$$\|K' * Du\|_p \leq |Du|(\mathbb{R}) \cdot \|K'\|_p.$$

Here $|Du|$ is the total variation measure related to Du.

(iii) As already seen in Chapter 1.2,

$$K'' = K - \delta$$

with δ being the delta-distribution. Hence,

$$\begin{aligned} \sup |\partial_x (K' * u)| &= \|K'' * u\|_\infty \leq \|K * u\|_\infty + \|u\|_\infty \\ &\leq (\|K\|_1 + 1)\|u\|_\infty = 2\|u\|_\infty. \end{aligned}$$

□

Now define the operators J_μ and G_μ as follows:

$$\begin{aligned} J_\mu x &:= (I + \mu G)^{-1} x, \quad x \in R(I + \mu G), \\ G_\mu x &:= \mu^{-1}(x - J_\mu x), \quad x \in R(I + \mu G). \end{aligned}$$

Proposition 2.36. *([Bar10]) G is m-accretive, if and only if*

$$R(I+\lambda G)=X, \quad \forall \lambda>0,$$

and equivalently, there exist some $\lambda>0$, such that $R(I+\lambda G)=X$.

Proof. For the first implication let G be m-accretive, $y \in X$ and $\lambda>0$ fixed. The inclusion equation

$$y \in x+\lambda G x \tag{2.5}$$

can be rewritten to

$$x=J_1\Big(\frac{y}{\lambda}+\Big(1-\frac{1}{\lambda}\Big)x\Big). \tag{2.6}$$

By Banach's fixed-point theorem, Equation (2.6) is solvable for $\frac{1}{2}<\lambda<\infty$.
Fix $\lambda_0>\frac{1}{2}$ and rewrite again the equation to

$$x=(I+\lambda_0 G)^{-1}\Big(\Big(1-\frac{\lambda_0}{\lambda}\Big)x+\frac{\lambda_0}{\lambda}y\Big) \tag{2.7}$$

Since $J_{\lambda_0}=(I+\lambda_0 G)^{-1}$ is non-expansive, (2.7) is solvable for $\frac{\lambda_0}{2}<\lambda<\infty$. This procedure can be repeated to gain $R(I+\lambda G)=X$.
For the other implication, assume now there exists $\lambda_0>0$, such that $R(I+\lambda_0 G)=X$. After transforming (2.5) again to (2.7), it can be concluded as before, that $R(I+\lambda G)=X$, for $\frac{\lambda_0}{2}<\lambda<\infty$, hence for all $\lambda>0$. □

At this point, we need to define, what it means for a subset, equivalently viewed as an operator in the sense of Definition 2.19, to be closed.

Definition 2.37. A subset G is said to be closed, if $x_n \to x$, $y_n \to y$ and $(x_n, y_n) \in G$, for all $n \in \mathbb{N}$, imply that $(x, y) \in G$.

Additionally, it can be shown that m-accretive sets are closed.

Proposition 2.38. *([Bar10]) Let G be an m-accretive subset of $X \times X$. Then G is closed and, if $\lambda_n \in \mathbb{R}, x_n \in X$ are such that $\lambda_n \to 0$ and*

$$x_n \to x, G_{\lambda_n} x_n \to y, \quad \textit{for } n \to \infty, \tag{2.8}$$

then $(x, y) \in G$.

Proof. Let $x_n \to x$, $y_n \to y$, $(x_n, y_n) \in G$. Since G is accretive, it follows, that

$$\|x_n-u\| \leq \|x_n+\lambda y_n-(u+\lambda v)\|, \quad \forall (u, v) \in G, \lambda>0.$$

Therefore,

$$\|x-u\| \leq \|x+\lambda y-(u+\lambda v)\|, \quad \forall (u, v) \in G, \lambda>0.$$

The m-accretiveness of G implies, that there exist $(u, v) \in G$ such that $u+\lambda v=x+\lambda y$. Inserting this into the last inequality gives that $x=u$ and $y=v \in G x$, as claimed.
If (2.8) is satisfied by λ_n and x_n, then $\{G_{\lambda_n} x_n\}$ is bounded and $J_{\lambda_n} x_n-x_n \to 0$. Since $G_{\lambda_n} x_n \in G J_{\lambda_n} x_n$ with $J_{\lambda_n} x_n \to x_n$ and G is closed, it follows, that $(x, y) \in G$. □

Proposition 2.39. *([Bar10]) If $G : X \to X$ is an m-accretive operator, then G is densely defined, i.e. $\overline{D(G)} = X$.*

Proof. Let $y \in X$ be given. For all $\lambda > 0$, the equation $x_\lambda + \lambda G(x_\lambda) = y$ has the unique solution $x_\lambda \in D(G)$. Since $\|x_\lambda\| \leq \|y\|$, for all $\lambda > 0$, there exists a subsequence $\lambda_n \to 0$, such that

$$x_{\lambda_n} \rightharpoonup x, \lambda_n G(x_{\lambda_n}) \rightharpoonup y - x \text{ in } X.$$

By definition, G is closed, hence its graph is weakly closed in $X \times X$. Therefore, $\lambda_n x_{\lambda_n} \to 0$ and $\lambda_n G(\lambda_n x_{\lambda_n}) \rightharpoonup y - x$ imply $y - x = 0$. As a consequence,

$$(1 + \lambda_n G)^{-1}(y) \rightharpoonup y.$$

This proves, that $y \in \overline{D(G)}$, because the weak closure of $D(G)$ equals the strong closure, since $D(G)$ is a convex subset. □

Corollary 2.40. *([Bar10]) If G is ω-m-accretive, then for all $u_0 \in \overline{D(G)}$ and $f \in L^1(0, T; X)$ there exists a unique mild solution u to*

$$\begin{cases} f(t) \in \dfrac{du}{dt}(t) + G(u(t)), & t \in [0, T], \\ u(0) = u_0, \end{cases}$$

Corollary 2.40 is a consequence of ([Bar10], Theorem 4.2). Finally, we can prove well-posedness for a mild solution of the Fornberg-Whitham equation.

Theorem 2.41. *([Hör21]) Let $u_0 \in L^1(\mathbb{R})$. Then there exists a unique mild solution $u \in C([0, \infty), L^1(\mathbb{R}))$ of the Cauchy problem (1.1).*

Proof. Recall, that the left-hand side of the Fornberg-Whitham equation can be split into a non-local term $K' * u$, which is bounded in L^1, and the Burgers' term $\partial_t u + u \partial_x u$, which possesses a generator of a nonlinear semi-group of contractions in L^1. Now consider the first term as a perturbation of the second term in order to use the established theory of semi-groups with a nonlinear generator. In other words, define A as the generator of the solution semi-group for the Burgers' equation and $Bu := K' * u$ on $L^1(\mathbb{R})$. By Lemma 2.35 the operator B is bounded. In order to apply Corollary 2.40, the condition, that $A + B$ is ω-m-accretive on $L^1(\mathbb{R})$, needs to be checked.
First, let $b := \|B\| < \infty$. Since A is an accretive operator and B is bounded it holds, that for all $v_1, v_2 \in L^1(\mathbb{R})$ and $0 < \lambda < \frac{1}{b}$:

$$\begin{aligned} \|v_1 - v_2 + &\lambda(A(v_1) + Bv_1 - A(v_2) - Bv_2)\|_1 \\ &\geq \|v_1 - v_2 + \lambda(A(v_1) - A(v_2))\|_1 - \lambda \|B(v_1 - v_2)\|_1 \\ &\geq (1 - \lambda b)\|v_1 - v_2\|_1 \end{aligned}$$

Therefore, $A + B$ is ω-accretive.
By Proposition 2.36 it suffices to prove, that $I + \lambda(A + B)$ is surjective for small $\lambda > 0$,

to get ω-m-accretiveness for $A + B$. In other words, it needs to be shown, that for a given v the equation

$$(I + \lambda G)^{-1}(v) - (I + \lambda G)^{-1}(\lambda B u) = u \tag{2.9}$$

has a solution $u \in L^1(\mathbb{R})$.
If $\lambda < \frac{1}{b}$, the left part of (2.9) is a contraction. Using the accretiveness of A, one obtains

$$\|(I + \lambda A)^{-1}(\lambda B u_1) - (I + \lambda A)^{-1}(\lambda B u_2)\|_1 \leq \|\lambda B u_1 - \lambda B u_2\|_1 \leq \lambda b \|u_1 - u_2\|_1$$

and the Equation (2.9) has a solution. Now for $A + B$ being ω-m-accretive it follows, that its domain $D(A+B)$ is dense in $L^1(\mathbb{R})$ by Proposition 2.39. By Corollary 2.40 there exists a unique mild solution for initial value $u_0 \in \overline{D(A + B)} = L^1(\mathbb{R})$. □

Additionally, the well-posedness for weak entropy solutions can as well be established by a generator of a nonlinear semi-group. In order to reach that goal one has to make a slight change to the initial condition, i.e. $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Theorem 2.42. *([Hör21]) Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The global mild solution $u \in C([0, \infty), L^1(\mathbb{R}))$ of the Fornberg-Whitham equation is also a weak entropy solution.*

Proof. In order to obtain the nonlinear semi-group, as well as the entropy solution, one follows the same procedure as in the proof of ([Bar10], Theorem 5.6). However, some adaptions have to be made:
By Proposition 2.36 it holds, that $R(I + \lambda(A + B)) = L^1(\mathbb{R})$, for $\lambda > 0$ sufficiently small. Next use (2.4) to get a formula for the mild solution, which converges uniformly on compact intervals,

$$u(t) = \lim_{n\to\infty} \Big(I + \frac{t}{n}(A + B)\Big)^{-n}(u_0).$$

Because the convolution operator and the unperturbed operator A are bounded in an arbitrary L^p-space and the estimates for $A + B$ do not use properties from L^1, which are not valid in any other L^p-space, the estimates from the above proof can also be developed in any L^p-space, where $1 \leq p \leq \infty$ holds. In our case, similar as for ([Bar10], Eq. (5.125)), one obtains an approximated solution u_ϵ for the ϵ- regularized equation

$$\begin{cases} \dfrac{(u_\epsilon(t) - u_\epsilon(t - \epsilon))}{\epsilon} + A(u_\epsilon(t)) + Bu_\epsilon(t) = 0, & t > \epsilon, \\ u_\epsilon(t) = u_0, & t < 0, \end{cases}$$

that satisfies the estimate

$$\|u_\epsilon(t)\|_{L^p(\mathbb{R})} \leq e^{bt}\|u_0\|_{L^p(\mathbb{R})},$$

uniformly in $\epsilon > 0$ and

$$u_\epsilon(t) \to u(t), \quad \text{as } \epsilon \to 0,$$

uniformly for t on a compact interval. From here on the proof continues as in ([Bar10], Theorem 5.6) from the Inequality (5.128) onwards. One also has to use the operator $A + B$ instead of A. □

3 Continuous weak traveling waves

After developing different concepts of solving the Cauchy problem (1.1), we now want to find either explicit solutions, which satisfy the Fornberg-Whitham equation in one of these senses, or a description of their asymptotic behaviour. One way of doing that, is by using the idea of traveling waves, defined as in [Eva22], which are solutions u of the form

$$u(t,x) = v(x-ct), \quad t \in \mathbb{R}, x \in \mathbb{R},$$

where c is the velocity and v the wave profile. First, this helps us to deepen the connection between the weak solution concept and the entropy solution concept.

Theorem 3.1. *([Hör21]) Let $u(t,x) = v(x-ct)$ be a weak traveling wave solution with a bounded and absolutely continuous wave profile v, which is almost everywhere differentiable and has a locally integrable derivative. Then u is an entropy solution defined as in Definition 2.3.*

Proof. Since $u(t,x) = v(x-ct)$ is a weak solution by assumption, the following equation holds in the weak sense and also pointwise a.e. on $\mathbb{R}$:

$$\Big(\frac{(v-c)^2}{2} + K * v\Big)' = 0.$$

Now we want to show, that the equation

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty |v(x-ct) - \lambda| \partial_t \phi(t,x)\, dx\, dt \\
&\quad + \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct) - \lambda) \frac{v^2(x-ct) - \lambda^2}{2} \partial_x \phi(t,x)\, dx\, dt \\
&\quad + \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct) - \lambda)(K' * v(\cdot - ct))(x) \phi(t,x)\, dx\, dt \\
&\quad + \int_{-\infty}^\infty |v(x) - \lambda| \phi(0,x)\, dx = 0
\end{aligned} \tag{3.1}$$

holds for all $\lambda \in \mathbb{R}$ and non-negative $\phi \in C_c^\infty(\mathbb{R}^2)$. The four integral terms will be denoted in the following way:

$$\begin{aligned}
I_1 &:= \int_0^\infty \int_{-\infty}^\infty |v(x-ct)-\lambda| \partial_t \phi(t,x)\, dx\, dt, \\
I_2 &:= \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda) \frac{v^2(x-ct)-\lambda^2}{2} \partial_x \phi(t,x)\, dx\, dt, \\
I_3 &:= \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda)(K' * v(\cdot - ct))(x)\phi(t,x)\, dx\, dt, \\
I_4 &:= \int_{-\infty}^\infty |v(x)-\lambda|\phi(0,x)\, dx.
\end{aligned}$$

One obtains an alternative expression for Equation (3.1):

$$I_1 + I_2 - I_3 + I_4 = 0.$$

Start with I_1 and apply Fubini's theorem with respect to the variable t:

$$\begin{aligned}
I_1 &= \int_0^\infty \int_{-\infty}^\infty |v(x-ct)-\lambda| \partial_t \phi(t,x)\, dx\, dt \\
&= \int_{-\infty}^\infty \Big(\int_0^\infty \Big(\operatorname{sgn}(v(x-ct)-\lambda) c v'(x-ct)\phi(t,x) \Big)\, dt + \Big[|v(x-ct)-\lambda|\phi(t,x) \Big]_{t=0}^\infty \Big)\, dx \\
&= \int_{-\infty}^\infty \int_0^\infty \Big(\operatorname{sgn}(v(x-ct)-\lambda) c v'(x-ct)\phi(t,x) \Big)\, dt\, dx - \int_{-\infty}^\infty |v(x)-\lambda|\phi(0,x)\, dx.
\end{aligned}$$

Note, that the last term cancels with I_4. Using integration by parts, I_2 can be transformed to

$$\begin{aligned}
I_2 &= \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda) \frac{v^2(x-ct)-\lambda^2}{2} \partial_x \phi(t,x)\, dx\, dt \\
&= -\int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda) v(x-ct) v'(x-ct)\phi(t,x)\, dx\, dt.
\end{aligned}$$

Combining all these terms again gives

$$\begin{aligned}
&I_1 + I_2 - I_3 + I_4 \\
&= \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda) \Big((c - v(x-ct)) v'(x-ct) - (K' * v)(x-ct) \Big) \phi(t,x)\, dx\, dt \\
&= \int_0^\infty \int_{-\infty}^\infty \operatorname{sgn}(v(x-ct)-\lambda) \Big(-\Big(\frac{(v(x-ct)-c)^2}{2} \Big)' - (K' * v)(x-ct) \Big) \phi(t,x)\, dx\, dt \\
&= 0.
\end{aligned}$$

□

Before we start the analysis of the solutions from [CLH12], the different types of traveling waves, as well as their classification and asymptotic behaviour, will be presented. But, first an important remark has to be made.

Remark 3.2. Since we analyze solutions of the form $u(t,x) = v(\xi)$ with $\xi = x - ct$, we want to express the Fornberg-Whitham equation only with respect to the variable ξ. First, recall form Chapter 1, that $(1-\partial_x^2)^{-1}\partial_x u = K * \partial_x u$. Applying the inverse operator $(1-\partial_x^2)$ to (1.1) yields

$$\begin{aligned} 0 &= (1-\partial_x^2)(\partial_t u) + (1-\partial_x^2)(u\partial_x u) + \partial_x u \\ &= \partial_t u - \partial_t\partial_x^2 u + u\partial_x - \partial_x^2(u\partial_x u) + \partial_x u \\ &= \partial_t\partial_x^2 u + u\partial_x - 3\partial_x u\partial_x^2 u - u\partial_x^3 u + \partial_x u. \end{aligned}$$

Finally, by the chain rule the equation for $u(t,x) = u(\xi)$ is formulated as

$$-c\partial_\xi u + c\partial_\xi^3 u + \partial_\xi u - u\partial_\xi^3 u + u\partial_\xi u - 3\partial_\xi u\partial_\xi^2 u = 0$$

with respect to the variable ξ. In other words,

$$-c\partial_\xi u + c\partial_\xi^3 u + \partial_\xi u = u\partial_\xi^3 u - u\partial_\xi u + 3\partial_\xi u\partial_\xi^2 u, \tag{3.2}$$

which can be found in the literature also as the Fornberg-Whitham equation.

Definition 3.3. (Single peak solitary wave, [CLH12]) A function $u(\xi)$ is called single peak solitary wave solution for the Fornberg-Whitham equation, if the following properties hold:

(i) $u(\xi)$ is continuous on $\mathbb{R}$ with a unique peak point ξ_0, where either the maximum or the minimum of $u(\xi)$ is attained.

(ii) $u(\xi) \in C^3(\mathbb{R}\setminus\{\xi_0\})$ satisfies (3.2) on $\mathbb{R}\setminus\{\xi_0\}$.

(iii) $\lim_{\xi\to\pm\infty} u(\xi) = A$, with A being the constant $A = c - 1 \pm \sqrt{(c-1)^2 - 2c_1}$, where the parameter c_1 is an integration constant.

Definition 3.4. (Peaked solitary wave, [CLH12]) A wave function $u(\xi)$ is called a peaked solitary wave or peakon, if u is locally smooth on $\mathbb{R}\setminus\{\xi_0\}$ and

$$\lim_{\xi\uparrow\xi_0} \partial_\xi u(\xi) = -\lim_{\xi\downarrow\xi_0} \partial_\xi u(\xi) = a,$$

where $a \neq 0$ and $a \neq \pm\infty$.

Definition 3.5. (Cusped solitary wave, [CLH12]) A wave function $u(\xi)$ is called a cusped solitary wave or cuspon, if u is locally smooth on $\mathbb{R}\setminus\{\xi_0\}$ and

$$\lim_{\xi\uparrow\xi_0} \partial_\xi u(\xi) = -\lim_{\xi\downarrow\xi_0} \partial_\xi u(\xi) = \pm\infty.$$

Without loss of generality, assume for simplicity that the peak point is $\xi_0 = 0$. Before classifying the solutions by their asymptotic behaviour, some properties for single peak solitary wave solutions will be introduced.

Theorem 3.6. *([CLH12]) If $u(\xi)$ is a single peak solitary wave solution with peak point $\xi_0 = 0$ for the Fornberg-Whitham equation, then the following properties hold:*

(i) If $c > A + \frac{4}{3}$, then $u(0) = c$.

(ii) If $c \leq A + \frac{4}{3}$, then $u(0) = c$ or $u(0) = B_1$ or $u(0) = B_2$, where $B_1 = 2c - A - \frac{4}{3} + \frac{2\sqrt{4-3c+3A}}{3}$ and $B_2 = 2c - A - \frac{4}{3} - \frac{2\sqrt{4-3c+3A}}{3}$.

Proof. Consider (3.2) and integrate the equation, such that

$$(u(\xi) - c)\partial_\xi^2 u(\xi) = c_1 - (c-1)u(\xi) + \frac{1}{2}u^2(\xi) - (\partial_\xi u(\xi))^2 \tag{3.3}$$

follows with c_1 being an integration constant. Moreover,

$$(\partial_\xi u(\xi))^2 = \frac{c_2}{(u(\xi)-c)^2} + \frac{(u(\xi)-c)^2}{4} + \frac{2(u(\xi)-c)}{3} + \frac{2c-c^2}{2} + c_1, \tag{3.4}$$

where c_2 is another integration constant. Now introduce the following boundary condition for (3.4):

$$\lim_{\xi \to \pm\infty} u(\xi) = A, \tag{3.5}$$

where $A = c - 1 \pm \sqrt{(c-1)^2 - 2c_1}$. Equation (3.4) can be rewritten to the following ordinary differential equation

$$(\partial_\xi u(\xi))^2 = \frac{(u(\xi)-A)^2 N(u(\xi))}{12(u(\xi)-c)^2} \tag{3.6}$$

with

$$N(u(\xi)) = 3u(\xi)^2 + 2(4 - 6c + 3A)u(\xi) - 12Ac + 12c^2 - 12c + 4A + 3A^2. \tag{3.7}$$

Since both sides of (3.6) are non-negative, as a consequence $N(u(\xi)) \geq 0$. Additionally, if $4 - 3c + 3A \geq 0$, then (3.6) is equivalent to

$$(\partial_\xi u(\xi))^2 = \frac{(u(\xi)-A)^2(u(\xi)-B_1)(u(\xi)-B_2)}{4(u(\xi)-c)^2}, \tag{3.8}$$

where $B_1 = 2c - A - \frac{4}{3} + \frac{2\sqrt{4-3c+3A}}{3}$ and $B_2 = 2c - A - \frac{4}{3} - \frac{2\sqrt{4-3c+3A}}{3}$. Note, that $B_1 \geq B_2$.
Suppose now, that $u(0) \neq 0$. Because $u(\xi) \in C^3(\mathbb{R} \setminus \{0\})$, it follows that $u(\xi) \neq c$, for all $\xi \in \mathbb{R}$. Taking the derivative on both sides of (3.4) gives $u(\xi) \in C^\infty(\mathbb{R})$, leading now to the two properties:

(i) Consider $c > A + \frac{4}{3}$. If $u(\xi) \neq c$, then $u(\xi) \in C^\infty(\mathbb{R})$. On one side, since u is a single peak solitary wave, $u'(0) = 0$. One the other side, $u(0) = A$ must hold by Equation (3.6). This is a contradiction to 0 being the unique peak point.

(ii) Now consider $c \leq A + \frac{4}{3}$. If $u(0) \neq c$, similar as above $u'(0) = 0$ by (3.6). It follows from Equation (3.8), that either $u(0) = B_1$ or $u(0) = B_2$. Finally, $u(0) = A$ would contradict again, that 0 is the unique peak point.

□

After that it is now possible to classify the traveling wave solutions for the Fornberg-Whitham equation.

Theorem 3.7. *([CLH12]) Let $u(\xi)$ be a single peak solitary wave solution for the Fornberg-Whitham equation (3.2) with peak point $\xi_0 = 0$. Then the traveling wave solutions can be classified in the following way:*

(i) $u(\xi)$ is a solitary wave solution, if $u(0) \neq 0$.

(ii) $u(\xi)$ is the peaked solitary wave solution $\frac{4}{3}e^{-\frac{1}{2}|x-ct|} + \frac{3c-4}{3}$, if $u(0) = c$ and $A = c - \frac{4}{3}$.

(iii) $u(\xi)$ is a cusped solitary wave solution, if $u(0) = c$ and $A \neq c - \frac{4}{3}$. Its asymptotic behaviour is given by

$$u(\xi) = c + \mu|\xi|^{\frac{1}{2}} + \mathcal{O}(|\xi|), \quad \xi \to 0, \tag{3.9}$$

$$u'(\xi) = \frac{\mu}{2}|\xi|^{-\frac{1}{2}} sgn(\xi) + \mathcal{O}(1), \quad \xi \to 0, \tag{3.10}$$

where $\mu = \pm\frac{1}{3}\sqrt{3|A-c|\sqrt{9c^2 - 12c - 12Ac + 12A + 9A^2}}$. Therefore, $u(\xi) \notin H^1_{loc}(\mathbb{R})$.

Proof. (i) According to the proof of Theorem 3.6, $u(\xi)$ is a smooth solitary wave solution, if $u(0) \neq 0$.

(ii) Let $u(0) = c$ and $A = c - \frac{4}{3}$, then Equation (3.6) is equivalent to

$$(\partial_\xi u(\xi))^2 = \frac{(3u(\xi) - 3c + 4)^2}{36}. \tag{3.11}$$

Equation (3.11) can be solved and the solution is the peaked solitary wave

$$u(\xi) = u(t, x) = \frac{4}{3}e^{-\frac{1}{2}|x-ct|} + \frac{3c-4}{3}. \tag{3.12}$$

(iii) Let $u(0) = c$ and $A \neq c - \frac{4}{3}$, then $A \neq c$, since $u(\xi)$ is a single peak solitary wave. As a result $N(u(\xi))$ does not include the factor $u - c$. Similar as above (3.6) becomes

$$\partial_\xi u(\xi) = \text{sgn}(c - A)\frac{u(\xi) - A}{u(\xi) - c}\sqrt{\frac{N(u(\xi))}{12}}\text{sgn}(\xi).$$

Define $h(u) := \frac{\sqrt{12}}{(A-u)\sqrt{N(u)}}$, then $h(c) := \frac{\sqrt{12}}{(A-c)\sqrt{N(c)}}$ and

$$\int \operatorname{sgn}(c-A)(u-c)h(u)\,du = \int \operatorname{sgn}(\xi)\,d\xi.$$

Since $h(u) = h(c) + \mathcal{O}(u-c)$, take the initial condition $u(0) = c$ into account in order to obtain

$$\frac{1}{2}(u-c)^2|h(c)|(1+\mathcal{O}(u-c)) = |\xi|.$$

As a consequence,

$$u - c = \pm\sqrt{\frac{2}{|h(c)|}}|\xi|^{\frac{1}{2}}(1+\mathcal{O}(u-c))^{-\frac{1}{2}}$$

implies $u - c = \mathcal{O}(|\xi|^2)$. Finally, the asymptotic behaviour is expressed by

$$u(\xi) = c + \mu|\xi|^{\frac{1}{2}} + \mathcal{O}(|\xi|), \quad \xi \to 0$$

and

$$u'(\xi) = \frac{\mu}{2}|\xi|^{-\frac{1}{2}}\operatorname{sgn}(\xi) + \mathcal{O}(1), \quad \xi \to 0$$

with $\mu = \pm\frac{1}{3}\sqrt{3|A-c|\sqrt{9c^2-12c-12Ac+12A+9A^2}}$, leading to $u(\xi) \notin H^1_{\mathrm{loc}}(\mathbb{R})$. $\square$

Remark 3.8. Theorem 3.7 only gives a classification of weak traveling waves in terms of asymptotic behaviour. ([CLH12], Chapter 3) presents all possible peak solitary wave solutions with corresponding implicit solutions, derived via the standard phase portrait analytical technique.

However, for our further analysis we are using the solutions (3.12) and (3.9). Hence, the following two sections are devoted to them. The main question, which arises, is, if and in which sense (3.12) and (3.9) fulfill the Fornberg-Whitham equation.

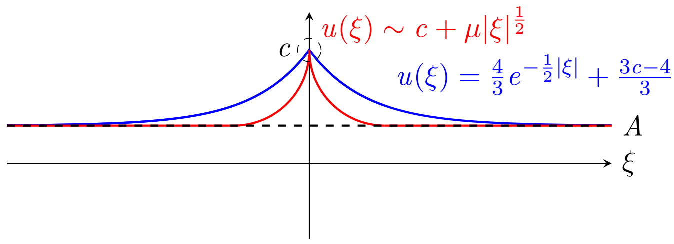

Figure 3.1: Solitary wave solutions.

3.1 Peaked Solitary Wave

For this section, consider the peaked solitary wave (3.12) with the special choice of the constant $c = \frac{4}{3}$.

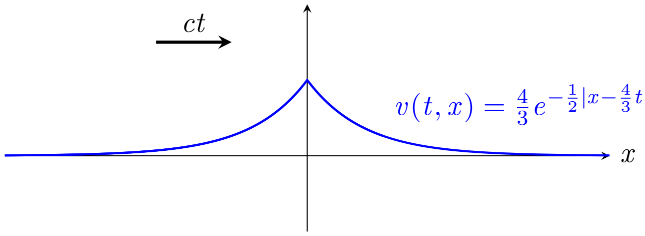

Figure 3.2: Peaked solitary wave solution.

Lemma 3.9. *The peaked solitary wave, defined by*

$$v(t,x) := \frac{4}{3}e^{-\frac{1}{2}|x-\frac{4}{3}t|},$$

gives a weak solution $u(t,x) := v(t,x)H(t)$ with initial value $u_0(x) = \frac{4}{3}e^{-\frac{1}{2}|x|}$ for the Fornberg-Whitham equation in the sense of Definition 2.1, where $H(t)$ is the Heaviside function.

Proof. Let $v(t,x) = \frac{4}{3}e^{-\frac{1}{2}|x-\frac{4}{3}t|}$. By direct calculation it can be seen, that

$$(1-\partial_x^2)v = \frac{3}{4}v + \frac{4}{3}\delta(x - \frac{4}{3}t)$$

and

$$(1-\partial_x^2)(v^2) = \frac{32}{9}\delta(x - \frac{4}{3}t)$$

hold in the distributional sense. Both imply the equation

$$(1-\partial_x^2)\Big(\partial_t v + \partial_x\Big(\frac{v^2}{2}\Big)\Big) = \partial_t((1-\partial_x^2)v) + \frac{1}{2}\partial_x(1-\partial_x^2)(v^2) = -\partial_x v,$$

which holds in the weak sense on $\mathbb{R}^2$. However, it is important to insist, that v^2 is carried out as a pointwise product of functions prior to differentiation. Note, that after $(1-\partial_x^2)^{-1}$ gets convoluted with K, the Equation (1.1) holds for v.
Define $u(t,x) := v(t,x)H(t)$, where $H(t)$ is the Heaviside function. This implies, that

$\operatorname{supp}(u) \subseteq \{t \geq 0\}$. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$, then observe

$$
\begin{aligned}
\langle \partial_t u, \phi \rangle &= -\langle u, \partial_t \phi \rangle = -\langle v \cdot H(t), \partial_t \phi \rangle \\
&= -\int_{\mathbb{R}^2} v(t,x) H(t) \partial_t \phi(t,x)\, d(t,x) = \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} (-v(t,x)\partial_t \phi(t,x))\, dx\, dt \\
&= \int_{-\infty}^{\infty} \Big(\int_0^{\infty} \partial_t v(t,x)\phi(t,x)\, dt - \Big[v(t,x)\phi(t,x) \Big]_0^{\infty} \Big)\, dx \\
&= \int_{-\infty}^{\infty} \Big(\int_0^{\infty} \partial_t v(t,x)\phi(t,x)\, dt - v(0,x)\phi(0,x) \Big)\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_t v(t,x) H(t) \phi(t,x)\, dt\, dx + \int_{-\infty}^{\infty} u_0(x)\phi(0,x)\, dx \\
&= \langle \partial_t v H(t), \phi \rangle + \langle u_0 \otimes \delta, \phi \rangle
\end{aligned}
$$

and therefore,

$$
\partial_t (v(t,x)H(t)) = v(0,x)\delta(t) + \partial_t v(t,x) H(t)
$$

in the distributional sense. Note, that $\partial_t v + \partial_x \Big(\frac{v^2}{2} \Big) + K' * v = 0$. As a consequence,

$$
\begin{aligned}
\partial_t u + \partial_x \Big(\frac{u^2}{2} \Big) + K' * u &= v(0,x)\delta(t) + \Big(\partial_t v + \partial_x \Big(\frac{v^2}{2} \Big) + K' * v \Big) H(t) \\
&= u_0 \otimes \delta + 0 = u_0 \otimes \delta
\end{aligned}
$$

holds in $\mathcal{D}'(\mathbb{R}^2)$. After writing everything out explicitly one obtains, for all $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$
\begin{aligned}
&\int_0^{\infty} \int_{-\infty}^{\infty} \Big(- u(t,x)\partial_t \phi(t,x) - \frac{u^2(t,x)}{2} \partial_x \phi(t,x) + (K' * u(\cdot,t))(x)\phi(t,x) \Big)\, dx\, dt \\
&= \int_{\mathbb{R}^2} \Big(- u(t,x)\partial_t \phi(t,x) - \frac{u^2(t,x)}{2} \partial_x \phi(t,x) + (K' * u(\cdot,t))(x)\phi(t,x) \Big)\, d(t,x) \\
&= \int_{\mathbb{R}^2} \Big(\partial_t u(t,x)\phi(t,x) + \partial_x \Big(\frac{u^2(t,x)}{2} \Big) \phi(t,x) + (K' * u(\cdot,t))(x)\phi(t,x) \Big)\, d(t,x) \\
&= \langle \partial_t u + \partial_x \Big(\frac{u^2}{2} \Big) + K' * u, \phi \rangle = \langle u_0 \otimes \delta, \phi \rangle = \langle u_0, \phi(0,\cdot) \rangle \\
&= \int_{-\infty}^{\infty} u_0(x)\phi(0,x)\, dx
\end{aligned}
$$

the weak formulation of the problem by comparing the first and last line. Therefore, the peaked solitary wave is a weak solution for the Fornberg-Whitham equation in the sense of Definition 2.1. □

Remark 3.10. Note, that $v(t,x) \in H^s(\mathbb{R})$, for all $s < \frac{3}{2}$. Additionally, the $L^\infty(\mathbb{R})$-norm of $\partial_x v(t,x)$ does not blow up. This can easily be seen, since the distributional derivative of v is

$$\partial_x v(t,x) = -\frac{4}{9} e^{-\frac{1}{2}|x-\frac{4}{3}t|} \mathrm{sgn}(x - \frac{4}{3}t), \quad t \in \mathbb{R}, x \in \mathbb{R}$$

and therefore, wave breaking does not occur.

3.2 Cusped Solitary Wave

Now recall the cusped solitary wave (3.9) with derivative (3.10) and consider $\xi = x - ct$. By Definition 3.5 and Theorem 3.7 cusped solitary waves are good candidates in solving the Fornberg-Whitham equation in one of the presented solution concepts, while preserving the wave breaking property. However, due to their asymptotic behaviour traveling waves with a cusp do not fulfill the Fornberg-Whitham equation in a weak sense, hence they are also not a weak entropy solution. This is formulated by the next Lemma. Before, without loss of generality, replace μ by the constant $-2b$, where $b := 4|c|^{\frac{3}{2}}\sqrt{c - \frac{4}{3}} > 0$, similar as in [Hör21].

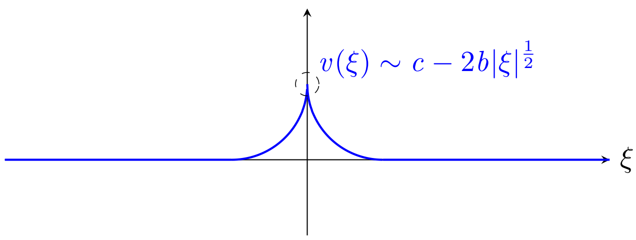

Figure 3.3: Cusped solitary wave solution.

Lemma 3.11. *The cusped solitary wave with asymptotic behaviour*

$$v(\xi) = c - 2b|\xi|^{\frac{1}{2}} + \mathcal{O}(|\xi|), \quad \xi \to 0,$$
$$v'(\xi) = -b|\xi|^{-\frac{1}{2}} sgn(\xi) + \mathcal{O}(1), \quad \xi \to 0,$$

where $b := 4|c|^{\frac{3}{2}}\sqrt{c - \frac{4}{3}} > 0$, *is neither a weak solution, nor a weak entropy solution for the Fornberg-Whitham equation.*

Proof. Assume $u(t,x) := v(x - ct)$ is a weak solution. The traveling wave $u(t,x)$, which is bounded and absolutely continuous, because v' is locally integrable, is also a weak entropy solution by Theorem 3.1. Therefore, by Theorem 2.13 the function $u(t,x)$ has to satisfy the Oleinik type inequality (2.20). Let $\xi = x - ct$ and without loss of generality consider $\xi < 0$. By (2.20) one obtains

$$v(0) - v(\xi) \le \Big(\frac{1}{t} + 2 + 2t\Big(1 + 2e^t \|u_0\|_{L^1(\mathbb{R})}\Big)\Big)(0 - \xi)$$

Using the definition of v and noting, that $v(0) = c$, gives for the left-hand side of the inequality: $v(0) - v(\xi) = 2b|\xi|^{\frac{1}{2}} \pm \mathcal{O}(|\xi|)$. Therefore,

$$2b|\xi|^{\frac{1}{2}} \pm \mathcal{O}(|\xi|) \leq \Big(\frac{1}{t} + 2 + 2t\Big(1 + 2e^{t}\|u_0\|_{L^1(\mathbb{R})}\Big)\Big)|\xi|$$

Note, that $\mathcal{O}(|\xi|) \to 0$ and $|\xi|^{\frac{1}{2}} \to 0$ for $\xi \to 0$, as well as the decay of the left-hand side is controlled by a constant multiplied by $|\xi|$. Taking the limit $\xi \to 0$, the left-hand side must fulfill the expression

$$2b|\xi|^{\frac{1}{2}} \pm \mathcal{O}(|\xi|) = 0, \quad (\xi \to 0).$$

In other words,

$$2b|\xi|^{\frac{1}{2}} = \pm\mathcal{O}(|\xi|), \quad (\xi \to 0).$$

This can not hold and therefore contradicts the Oleinik type inequality. Hence, $u(t, x)$ is neither a weak nor an entropy solution. $\square$

4 Conclusion

Motivated by the results in [CLH12], we wanted to answer the question, if there exists a traveling wave, which includes the phenomena of wave breaking and fulfills the Fornberg-Whitham equation in one of the notions seen in [Hör21]. These include weak solutions, weak entropy solutions and mild solutions. The next step was to develop conditions in order to relate them to each other. The results were the following. If a continuous bounded traveling wave is a weak solution, then we are getting a weak entropy solution. Speaking of them, weak entropy solutions are always weak solutions and finally under the condition, that the initial condition $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, mild solutions are weak entropy solutions.
Afterwards, we analyzed two different forms of single peak solitary wave, derived in [CLH12], regarding the solution concepts. We concluded, that the peaked solitary wave, which does not describe the wave breaking phenomena, is a weak solution and the cusped solitary wave, which posses the required asymptotic behaviour of wave breaking, does not fit into our developed solution concepts.
However, as an outlook, even though traveling waves neither fulfill one of the presented solution concepts and at the same time possess a cusp, this does not exclude the possibility to have breaking wave solutions in a weaker sense. One of the results in [GN16] is a condition on wave breaking for weak entropy solutions. However, the question, if these weak entropy solutions and the corresponding condition end up being strong solutions with wave breaking, seen in most of the literature, is not answered in that paper.

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