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Generalized pp-waves: Completeness and Causality

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Abstract

This thesis deals with pp-wave spacetimes, a family of solutions to Eintsein's field equation. We will focus on global results for a very general class of pp-waves, which we call M-PFWs in this work. These spacetimes are defined on a Lorentzian manifold $M \times \mathbb{R}^2$, with M a connected Riemannian manifold. In the first part of our work we study geodesic completeness. As an introduction, we discuss differences between geodesically complete Riemannian manifolds and geodesically complete Lorentzian manifolds, before we start with our research on M-PFWs. We compare our results to those for pp-waves, give examples of geodesic complete M-PFWs and show that all plane waves, which model electromagnetic- and gravitational waves, are complete. Further global properties of spacetimes will be discussed in the second part of this work, namely the causal structure of M-PFWs. We also compare our results on M-PFWs to older results in the literature for pp-waves, e.g. the ones given in the classic paper 'the remarkable property of plane waves' by Roger Penrose.

Vorwort

Diese Arbeit befasst sich mit pp-Wellen, einer Familie von Raumzeiten, welche die Einstein'sche Feldgleichung erfüllen. Wir konzentrieren uns auf globale Eigenschaften einer Verallgemeinerung von pp-Wellen, die wir in dieser Arbeit als M-PFWs bezeichnen. Diese Raumzeiten sind als Lorentz'sche Mannigfaltigkeit $M \times \mathbb{R}^2$ definiert, wobei M eine zusammenhängende Riemann'sche Mannigfaltigkeit ist. Der erste Teil unserer Arbeit befasst sich mit der geodätischen Vollständigkeit. Darum diskutieren wir zu Beginn den Unterschied zwischen gedoätisch vollständigen Riemann'schen Mannigfaltigkeiten und geodätisch vollständigen Lorentz'schen Mannigfaltigkeiten, bevor wir mit der Untersuchung von M-PFW's beginnen. Wir vergleichen unsere Resultate mit denen von pp-Wellen, und geben Beispiele von geodätisch vollständigen M-PFWs. Insbesondere zeigen wir, dass die sogenannten 'plane waves', die als Modelle für elektromagnetische Wellen und Gravitationswellen Anwendung finden, geodätisch vollständig sind. Weitere globale Eigenschaften werden im zweiten Teil der Arbeit untersucht, nämlich die kausale Stuktur von M-PFWs. Wieder vergleichen wir unsere Resultate mit älteren aus der Literatur, wie zum Beispiel jene aus dem klassischen Paper 'the remarkable property of plane waves' von Roger Penrose.

Inhaltsverzeichnis

1 Preface

This master's thesis deals with pp-wave spacetimes, a family of solutions to Einstein's field equation. We will focus on global properties of a very general class of pp-waves, which we call M-PFWs in this work. These spacetimes are defined as $\mathcal{M} = M \times \mathbb{R}^2$, with M a connected Riemannian manifold. The metric on $\mathcal{M}$ is given by

$$\langle ., . \rangle_{\mathcal{M}} = \langle ., . \rangle_M + dudv + H(x, u)du^2,$$

where $\langle ., . \rangle_M$ denotes the Riemannian metric on M, (u, v) are coordinates of $\mathbb{R}^2$ and H is a smooth function $H : M \times \mathbb{R} \to \mathbb{R}$.

In chapter 2 we recall the necessary properties from semi-Riemannian geometry. We focus, in particular, on the differences between geodesically complete Riemannian manifolds and geodesically complete Lorentzian manifolds. We also define the relevant objects to study causality. This section doesn't include proofs (for more details and proofs see for example [ON83]). In chapter 3 we will briefly discuss and define plane-waves, pp-waves and M-PFWs, which are the basic objects we will study in this work.

The main part of this work starts with chapter 4, where we explicitly calculate the curvature quantities and basic properties of pp-waves respectively M-PFWs. Here we mainly follow [CFS03] and [CFS04]. We also discuss the energy conditions for M-PFWs in this section and give necessary conditions of asymptotically vanishing M-PFWs $\mathcal{M}$ with respect to the smooth function H in the metric of $\mathcal{M}$, following [FS03] and [FS06].

The results for geodesics and geodesic completeness in chapter 5, theorem 5.1.1 and theorem 5.1.3, will be the starting point for studying geodesic completeness of M-PFWs in the technical subsection 5.2, following [CRS12] and [CRS13] and later in chapter 7, for the causal structure of M-PFWs as well.

The highlight of this thesis will be an application to pp-waves in chapter 6, where the completeness of any plane wave follows directly from our more general results on M-PFWs discussed in the previous chapter 5. Additionally, we will give examples for geodesically complete M-PFWs in chapter 6 and discuss the difference between the completeness results from chapter 5.

Finally, in chapter 7 we study the causal structure of M-PFWs. Therefore, we discuss the 'remarkable property of plane waves' proven by Roger Penrose in his seminal paper [Pen65b]. Finally, we give necessary conditions of M-PFWs, with respect to the smooth function H, for global hyperbolicity and strong causality, following [FS06].

2 Elements of semi-Riemannian geometry

In this preparatory section, we give an overview on some essential elements of semi-Riemannian geometry thereby fixing our notations and conventions used throughout this thesis. By defining the relevant objects to study causality and geodesic completeness of semi-Riemannian manifolds, we will also discuss basic differences between Riemannian manifolds and Lorentzian manifolds.

Definition 2.0.1 *(Semi-Riemannian manifold)*
A semi-Riemannian manifold (M, g) is a smooth manifold M furnished with a smooth metric tensor g.

A *smooth metric tensor* is a symmetric nondegenerate $(0, 2)-$tensor field on M of constant index. More precisely, g smoothly assigns to each $p \in M$, a symmetric bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$, which is nondegenerate (i.e. $g(v, w) = 0$ for all $v \in T_pM$ if and only if w is the zero-vector), with the same index for all T_pM. Here the index of T_pM is defined as

$$\nu := \max_{W \subseteq T_pM} \{\dim(W) \mid g_{|W} \text{ is negative definite}\}.$$

Definition 2.0.2 *(Lorentzian manifolds and Riemannian manifolds)*
A semi-Riemannian manifold M is called Lorentzian manifold if dim$(M) \geq 2$ with index $\nu = 1$. If $\nu = 0$, then M is called a Riemannian manifold.

Note, that $\nu = 0$ if and only if g is positive definite (i.e. each g_p is a positive definite inner product on T_pM). Instead of $g(v, w)$ we will sometimes write $\langle v, w \rangle$.

Definition 2.0.3 *(Vector field along a curve)*
Let $I \subseteq \mathbb{R}$ be a real interval and $\gamma : I \to TM$ a curve on M. A smooth map $X : I \to M$ is called a vector field along the curve γ, if

$$\big(\pi \circ X\big)(t) = \gamma(t),$$

where $\pi : TM \to M$ the natural projection on M. i.e. $X(t) \in T_{\gamma(t)}M$ for all $t \in I$.

Analogously to the notation for smooth vector fields X on M, where we write $\mathfrak{X}(M)$, we denote the C^∞-module of all vector fields along γ by $\mathfrak{X}(\gamma)$.
Now, we define the 'directional derivative' of a vector field $Y \in \mathfrak{X}(M)$ along another vector field $X \in \mathfrak{X}(M)$ in a semi-Riemannian manifold.

Definition 2.0.4 *(Connection)*
A connection on a C^∞-manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that the following properties hold:

(∇1) $\nabla_X Y$ *is $C^\infty(M)$-linear in X*
i.e. for all $f \in C^\infty(M)$, we have $\nabla_{fX} Y = f\nabla_X Y$,

(∇2) $\nabla_X Y$ *is $\mathbb{R}$-linear in Y*
i.e. for all scalars $c \in \mathbb{R}$, we have $\nabla_X cY = c\nabla_X Y$,

(∇3) $\nabla_X Y$ *satisfies the Leibniz rule (in Y)*
i.e. for all $f \in C^\infty(M)$, we have $\nabla_X fY = X(f)Y + \nabla_X Y$.

We call $\nabla_X Y$ the covariant derivative of Y in direction of X with respect to a connection ∇.

Note, that in 2.0.4, the metric is not involved. The next result shows, that there is exactly one connection, which is compatible with the metric.

Proposition 2.0.5 *(Levi-Civita connection and induced covariant derivative)*
Let ∇ be a connection, where in addition to (∇1)-(∇3), the following two properties hold

(∇4) $[X, Y] = \nabla_X Y - \nabla_Y X$, *(torsion free condition)*

(∇5) $Z\big(\langle X, Y\rangle\big) = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$, *(metric condition)*

for all $X, Y, Z \in \mathfrak{X}(M)$, where $[.,.]$ denotes the Lie bracket on TM.

Then ∇ is unique and we call it the Levi-Civita connection of (M, g).

∇ induces a derivative on $\mathfrak{X}(\gamma)$. More precisely, let $\gamma : I \to M$ be a smooth curve on M. There exists an unique mapping $\mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, with

$$Z \mapsto \nabla_t Z \equiv \frac{\nabla Z}{dt},$$

called the induced covariant derivative, such that the following properties hold

(i) $\nabla_t\big(Z_1 + \lambda Z_2\big) = \nabla_t Z_1 + \lambda \nabla_t Z_2$,

(ii) $\nabla_t\big(fZ\big) = \frac{df}{dt} Z + f\nabla_t Z$,

(iii) $\nabla_t\big(X \circ \gamma\big)(t) = \nabla_{\gamma'(t)} X$,
for all $Z_1, Z_2, Z \in \mathfrak{X}(\gamma)$, $\lambda \in \mathbb{R}$, $f \in C^\infty(I, \mathbb{R})$ and $X \in \mathfrak{X}(M)$.

In addition, we also have the following 'metric condition' for the induced covariant derivative

(iv) $\frac{d}{dt}\langle Z_1, Z_2\rangle = \langle \nabla_t Z_1, Z_2\rangle + \langle Z_1, \nabla_t Z_2\rangle$.

We sometimes write $Z'(t)$ *instead of* $\nabla_t Z$ *and* $Z''(t)$ *for* $\nabla_t Z'$.

Definition and Proposition 2.0.6 *(Christoffel symbols)*
Choosing a local chart $\big((x^1, \cdots, x^n), U\big)$ *in a semi-Riemannian manifold* M *and denote by* $\frac{\partial}{\partial x^i} = \partial_i$ *the base vector fields with respect to the coordinates* $(x^1, \cdots, x^n)$. *We know that every vector field* $X \in \mathfrak{X}(M)$ *can be written as* $X = f^i \partial_i$ *on* U, *for* $f^i \in C^\infty(U)$ *(we use Einstein's summation notation). Since,* $\nabla_{\partial_i}\partial_j$ *is again a vector field, we can write it locally as*

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k,$$

where the C^∞*-functions* Γ^k_{ij} $(1 \leq k \leq n)$ *are the so called Christoffel symbols of* M *with respect to the metric* g.
One can calculate the Christoffel symbols directly from the metric, by the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \Big(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\Big), \tag{2.1}$$

where $(g_{ij})_{1 \leq i,j \leq n}$ *is the matrix with entries* $\langle \partial_i, \partial_j \rangle = g_{ij}$, *while* $(g^{ij})_{1 \leq i,j \leq n}$ *denotes its inverse.*

Definition and Proposition 2.0.7 *(Geodesics)*
Note that for a smooth curve $\gamma(t)$, *the derivative* $\gamma'(t)$ *with respect to* t *is a vector field along* γ $(\gamma'(t) \in \mathfrak{X}(\gamma))$. *We define a geodesic on* M *to be a curve* γ, *such that its tangent vector* γ' *is a parallel vector field along* γ, *i.e.* $\nabla_t \gamma'(t) = 0$, *or for short* $\gamma''(t) = 0$.
Taking a chart $\big((x_1, \cdots, x_n), U\big)$ *in* M *and* $\gamma : I \to U$. *Then* γ *is a geodesic if and only if the local coordinate expressions* $x^k \circ \gamma$ *of* γ *obey the geodesic equation*

$$\frac{d^2(x^k \circ \gamma)}{dt^2} + \Gamma^k_{ij} \circ \gamma \frac{d(x^i \circ \gamma)}{dt} \frac{d(x^j \circ \gamma)}{dt} = 0 \quad (1 \leq k \leq n). \tag{2.2}$$

It is most common to abbreviate the local expressions $x^k \circ \gamma$ of γ by γ^k. Using this notation, the geodesic equation (2.2) takes the form

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0, \quad \text{or even shorter} \quad \ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0 \quad (1 \leq k \leq n). \tag{2.3}$$

Note, that the geodesic equation is a system of nonlinear ODEs of second order and so by basic ODE-theory, we always have an interval I around 0 and an unique geodesic $c : I \to M$ with initial conditions $c(0) = p \in M$ and $c'(0) = v \in T_pM$. We denote by c_v the unique geodesic starting at p with initial velocity v.
For more details and proofs of 2.0.5, 2.0.6 and 2.0.7 see for example [ON83].

Definition 2.0.8 *(Exponential map)*
For $p \in M$, we define the exponential map at p, $\exp_p : \mathcal{D}_p \to M$ by

$$\exp_p(v) \coloneqq c_v(1),$$

where $\mathcal{D}_p \coloneqq \{v \in T_pM : c_v$ is at least defined on $[0,1]\}$.

Definition and Proposition 2.0.9 *(Normal coordinates)*
Let M be a semi-Riemannian manifold and let $V \subseteq T_pM$ be an open neighbourhood of $0 \in T_pM$, such that the exponential map $\exp_p : V \to U \coloneqq \exp_p(V)$ is a diffeomorphism. Let U be a normal neighbourhood of $p \in M$ (i.e. $V \subseteq T_pM$ is star shaped). Taking any orthonormal basis $\{e_1, \dots, e_n\}$ in T_pM, then the Riemannian normal coordinates $\big((x^1, \dots, x^n), U\big)$ are defined, using the exponential map in p, such that for every $q \in U$, we have

$$x^i(q)e_i = \exp_p^{-1}(q).$$

These coordinates have the following properties:

1. $g_{ik}(p) = \delta_{ik}$ *and*
2. $\Gamma^k_{ij}(p) = 0$.

Definition 2.0.10 *(Geodesic completeness)*
Let M be a semi-Riemannian manifold. If every maximal geodesic is defined on all of $\mathbb{R}$, then M is called geodesically complete. Equivalently, M is geodesically complete, if the exponential map is defined on the entire tangent space T_pM for all $p \in M$.

Definition 2.0.11 *(Totally geodesic submanifolds)*
Let $\big(\mathcal{M}, \bar{g}\big)$ be a semi-Riemannian manifold and $\big(M, g\big)$ a semi-Riamannian submanifold (i.e. $M \subseteq \mathcal{M}$ and the metric g induced by $\bar{g}$ is non-degenerate). M is called totally geodesic, if every geodesic $\gamma : I \to M$ with respect to g is also a geodesic in $\mathcal{M}$ with respect to $\bar{g}$.

One can also define a totally geodesic semi-Riemannian submanifold M of $\mathcal{M}$ equivalently by the property that its shape tensor vanishes, i.e. $\mathrm{I\!I} = 0$. This means, that a totally geodesic semi-Riemannian submanifold is extrinsically flat. Note that M itself has not to be intrinsically flat. For example, geodesics on a sphere (i.e. great circles) are totally geodesic submanifolds.

2.1 Geodesic completeness vs. metric completeness

We discuss the differences between geodesically complete Riemannian manifolds and geodesically complete Lorentzian manifolds. We will start with the well-known result for Riemannian manifolds, for which we need the following terminology:

Definition 2.1.1 *(Riemannian distance)*
Let M be a Riemannian manifold and let $p, q \in M$. We define the set of permissible paths connecting p and q by

$$\Omega(p,q) \coloneqq \{\gamma : \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\}.$$

The Riemannian distance $d(p,q)$ between p and q is then defined to be

$$d(p,q) \coloneqq \inf_{\gamma \in \Omega(p,q)} L(\gamma).$$

Furthermore, we define the ε-neighbourhood of a point p in a Riemannian manifold M for all $\varepsilon > 0$ by

$$U_\varepsilon(p) \coloneqq \{q \in M : d(p,q) < \varepsilon\}.$$

Theorem 2.1.2 *(Hopf-Rinow)*
Let M be a connected Riemannian manifold. The following are equivalent:

1. *(M, d) is complete (as a metric space), where d is the Riemannian distance.*
2. *M is geodesically complete.*
3. *There exists a point $p \in M$, such that the exponential map $exp_p : T_pM \to M$ is defined for all $v \in T_pM$.*
4. *Every closed and bounded subset of M is compact.*

Again, a proof of this foundational result can be found in [ON83]. This result shows, that in case of a Riemannian manifold, geodesic completeness is equivalent to completeness as a metric space. The next example shows, that there are topologically compact Lorentzian manifolds, which are not globally geodesic.

Example (The Clifton-Pohl Torus) [ON83]: Let M be the manifold $M := \mathbb{R}^2 \setminus \{0\}$, with the Lorentzian metric

$$ds^2 = \frac{2}{(x^2 + y^2)} dxdy.$$

Define the map $\phi(x, y) = (2x, 2y)$. Clearly, this is an isometry of M. We will show, that the group $\Gamma = \{\phi^n\}$, generated by ϕ, acts properly discontinuous on M (i.e.: every $p \in M$ has a neighbourhood U, such that $\phi(U) \cap U = \emptyset$ for all $\phi \in \Gamma$ with $\phi \neq id$), which implies that $T := M/\Gamma$ is a Lorentz surface (see: [ON83] Corollary 7.12, p191). Topologically, T is isomorphic to a torus (called the Clifton-Pohl torus), since it is a closed annulus of radius $1 \leq r \leq 2$, with boundary points identified by the action of Γ, hence T is compact. We show, that Γ acts properly discontinuous. To see this, fix a point $p = (x_0, y_0) \in M$ and define the neighbourhood of p to be $U = \{(x, y) \in M : e\big((x, y), (x_0, y_0)\big) < 1\}$, where e denotes the Euclidean distance on M. Since $e\big(\phi(x_0, y_0), (x_0, y_0)\big) = 2$, we have $U \cap \phi(U) = \emptyset$.
The matrix representation of the metric on M is given by

$$(g_{ij}) = \begin{pmatrix} 0 & \frac{1}{x^2+y^2} \\ \frac{1}{x^2+y^2} & 0 \end{pmatrix},$$

with its inverse

$$(g^{ij}) = \begin{pmatrix} 0 & (x^2 + y^2) \\ (x^2 + y^2) & 0 \end{pmatrix}.$$

Using formula (2.1), we see, that $\Gamma^1_{11} = -\frac{x}{x^2+y^2}$ and $\Gamma^2_{22} = -\frac{y}{x^2+y^2}$, while all other Christoffel symbols vanish. Thus for a curve $\gamma(t) = (x(t), y(t))$, the geodesic equation (2.2) becomes

$$x''(t) = \ddot{x}(t) + \Gamma^1_{11}\dot{x}^2 = 0,$$
$$y''(t) = \ddot{y}(t) + \Gamma^2_{22}\dot{y}^2 = 0.$$

Consider the curve $\gamma(t) = (\frac{1}{1-t}, 0)$, where we calculate

$$x''(t) = \frac{1}{(1-t)^3} - \frac{\left(\frac{1}{1-t}\right)}{\frac{1}{(1-t)^2} + 0^2}\left(\frac{1}{(1-t)^2}\right)^2 = 0$$

and also $y''(t) = 0$. Therefore, γ is a geodesic in M, hence also in T. But, it is incomplete, since it is not defined for $t = 1$.

Since we don't have an analogously strong result to Hopf-Rinow (theorem 2.1.2) for Lorentzian manifolds, it makes sense, to study geodesic completeness for Lorentzian manifolds separately, which we will do in this work for a special case, called **generalized pp-waves**, see: definition 3.2.1.

Furthermore, the metric on a Riemannian manifold is always positive definite (since the index is 0). Therefore, in a Riemannian manifold, $\langle v, v\rangle$ is always non-negative and its just zero, if and only if $v \equiv 0$. But, on a Lorentzian manifold there exist vectors $v \neq 0$ in TM, with $\langle v, v\rangle = 0$, and also vectors with negative 'inner product'. This motivates the study of the so called *causal character* for Lorentzian manifolds (especially for generalized pp-waves later on).

2.2 Causal character

We define the causal character of a vector (vector field) of Lorentzian manifolds, respectively spacetimes.

Definition 2.2.1 *(Causal character)*
Let $(\mathcal{M}, \langle ., .\rangle)$ be any Lorentzian manifold and let $0 \neq v \in T_p\mathcal{M}$. We call v

- ***timelike*** *if and only if $\langle v, v\rangle < 0$*
- ***lightlike*** *if and only if $\langle v, v\rangle = 0$*
- ***spacelike*** *if and only if $\langle v, v\rangle > 0$,*

while we define the zero-vector $v = 0$ to be spacelike. We also use the word ***causal*** *to mean both, lightlike or timelike (i.e. $v \neq 0$, with $\langle v, v\rangle \geq 0$). This definition naturally extends to vector fields $X \in \mathfrak{X}(\mathcal{M})$.*
Furthermore, we say that a (piecewise smooth) curve $\alpha : I \to \mathcal{M}$ is causal, if its tangent vector $\alpha'(t)$ is causal, for all $t \in I \subseteq \mathbb{R}$ (analogously for spacelike curves).
We also define the chronological future and causal future of a point (an event) $p \in \mathcal{M}$ to be the sets

$$I^+(p) := \{q \in \mathcal{M} |\ \textit{there exists a future directed timelike curve from } p \textit{ to } q\},$$

$$J^+(p) := \{q \in \mathcal{M} |\ \textit{there exists a causal curve from } p \textit{ to } q\}.$$

The correspondent past sets, are

$$I^-(p) := \{q \in \mathcal{M} |\ \textit{there exists a future directed timelike curve from } q \textit{ to } p\},$$

$$J^-(p) := \{q \in \mathcal{M} |\ \textit{there exists a causal curve from } q \textit{ to } p\}.$$

While by $I(p)$, we denote both, the chronological past and future, hence $I(p) = I^+(p) \cup I^-(p)$ and analogously for the causal past and future, we write $J(p) = J^+(p) \cup J^-(p)$.

Definition 2.2.2 *(Spacetime)*
A semi-Riemannian manifold M is called a spacetime, if M is a Lorentzian manifold equipped with a time orientation, that is a (non-vanishing) timelike vectorfield.

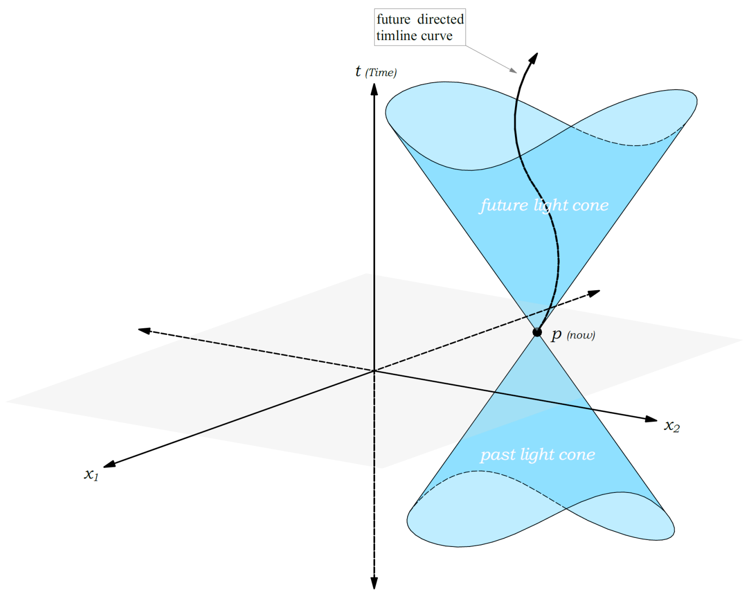

Abbildung 2.1: Subdivision of 3 dimensional Minkowski spacetime with respect to an event p in four disjoint sets. The lightcone, the chronological future $I^{+}(p)$, the chronological past $I^{-}(p)$ and elsewhere.

We now define certain notions that restrict the global properties of a spacetime.

Definition 2.2.3 *(Chronological)*
A spacetime is called chronological, if there exist no closed timelike curves.

Definition 2.2.4 *(Causal)*
A semi-Riemannian manifold M is called causal, if there exist no closed causal curves on M.

Definition 2.2.5 *(Distinguishing)*
A semi-Riemannian manifold M is called distinguishing, if there exists no points $p, q \in M$ with the same chronological past or future. (i.e. If $I(p) = I(q)$, then $p = q$, $\forall p, q \in M$).

Definition and Proposition 2.2.6 *(Strongly causal)*
We use the following two equivalent characterisations for strong causality in this work.

- *A semi-Riemannian manifold M is called strongly causal if for each $p \in M$, given a neighbourhood U of p, there exists a neighbourhood $V \subseteq U$ of p, such that every causal curve segment with endpoints in V lies entirely in U.*
- *A semi-Riemannian manifold M is called strongly causal, if for each $p \in M$, given a neighbourhood U of p, there exists a neighbourhood $p \in V \subseteq U$, such that there exist no timelike curve that passes through V more then once.*

One can think of a strongly causal spacetimes to be a spacetime with no almost closed causal curves. If a causal curve through p leaves V, it can not come back. For more details, and for a proof that the two characterisations of strong causality above are equivalent, see [MS08].

Definition 2.2.7 *(Globally hyperbolic)*
A spacetime M is called globally hyperbolic, if there exists a Cauchy hypersurface S i.e. on $S \subseteq M$, such that every inextensible timelike curve intersect it exactly once.

There is a hierarchy of these conditions, summarized in the last proposition of this section.

Proposition 2.2.8 *(Causal hierarchy of spacetimes)*
The following implications on causal characters of spacetimes hold, while for all converse implications there are counterexamples:

globally hyperbolic $\Rightarrow$ *strongly causal* $\Rightarrow$ *distinguishing* $\Rightarrow$ *causal* $\Rightarrow$ *chronological*

For a proof, see [MS08]. We sometimes call this hierarchy, the causal ladder of spacetimes, visualised in figure 7.2 below on page 60.
Note, that there are several more causality properties in the hierarchy of spacetimes. For example *stable causal*, which lies between *globally hyperbolic* and *strongly causal*. It was first introduced as the causality being a stable property under perturbations, but Hawking showed [HA69], that this is equivalent to the existence of a global time function. However, we don't need this property here.

3 Setup

In this section, we introduce a more general class of pp-waves, which are the basic objects in this work, for which we will study geodesic completeness and causality.
To begin with, we start with the definition of pp-waves and their important subclass, called plane waves.

3.1 The class of pp-waves

We discuss the well-known family of exact solutions of Einstein's field equation, called ***p**lane-fronted waves with **p**arallel propagation* or **pp-waves** for short. Geometrically, they are defined as a 4-dimensional Lorentzian manifold $\big(\mathcal{M}, \langle .,. \rangle\big)$, which admits a covariantly constant null vector field K (i.e. $\langle K, K\rangle = 0$ and $\nabla K = 0$, where ∇ denotes the Levi-Civita connection on $\mathcal{M}$). They may represent gravitational waves, electromagnetic waves, some other forms of matter, or any combination of these.

The family of pp-wave space-times was first discussed by Brinkmann (1925, see [Bri25]), and interpreted physically in terms of gravitational waves by Peres (1959). The geometric definition of a pp-wave is coordinate-free. But it allows to write the metric locally in the so called *Brinkmann coordinates* $(x_1, x_2, v, u) \in \mathbb{R}^4$, in which the metric takes the form

$$ds^2 = dx_1^2 + dx_2^2 + 2dudv + H(x_1, x_2, u)du^2 \ , \tag{3.1}$$

where $H : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function.

To relate the Brinkmann coordinates to the usual Minkowski coordinates $(x, y, z, t) \in \mathbb{R}^4$, where the Minkowski-line element takes the form

$$ds_0^2 = dx^2 + dy^2 + dz^2 - dt^2,$$

let us first set

$$x_1 = x,\ x_2 = y,\ u = \frac{z-t}{\sqrt{2}},\ v = \frac{z+t}{\sqrt{2}}.$$

Then, from this coordinate transformation we get $2dudv = dz^2 - dt^2$, thus

$$ds^2 = ds_0^2 + H(x_1, x_2, u)du^2.$$

This coordinate transformation is visualized in figure 3.1.

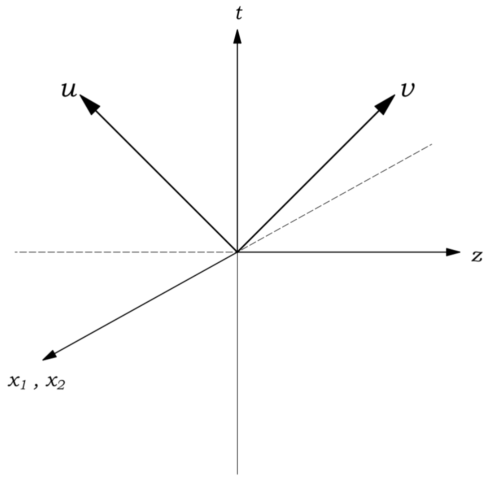

Abbildung 3.1: Brinkmann coordinates

In Minkowski coordinates the metric takes the slightly more complicate form

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{1}{2}H(dz + dt)^2.$$

Note, that the smooth real-valued function H in the pp-wave metric (3.1) doesn't depend on the v-coordinate. Further assumptions on H characterize the following subclasses of pp-waves:

Definition 3.1.1 *(Plane waves)*
*A plane wave is a pp-wave, in which the smooth function $H(x_1, x_2, u)$ in (3.1) is quadratic in x_1 and x_2. A plane wave is called an **exact electromagnetic wave**, if $\Delta_x H(x, u) \equiv 0$, where $\Delta_x H$ denotes the Laplacian of H with respect to $x = (x_1, x_2)$. If in addition, H takes the special form*

$$H(x_1, x_2, u) = f(u)(x_1^2 - x_2^2) + 2g(u)x_1x_2, \tag{3.2}$$

*for some smooth maps f, g and $f^2 + g^2 \not\equiv 0$, we call it an **exact gravitational wave**. In particular, an exact gravitational wave is called **sandwich wave** if both f and g have compact support (see figure 3.2), while we say it is **polarized** if $g \equiv 0$.*

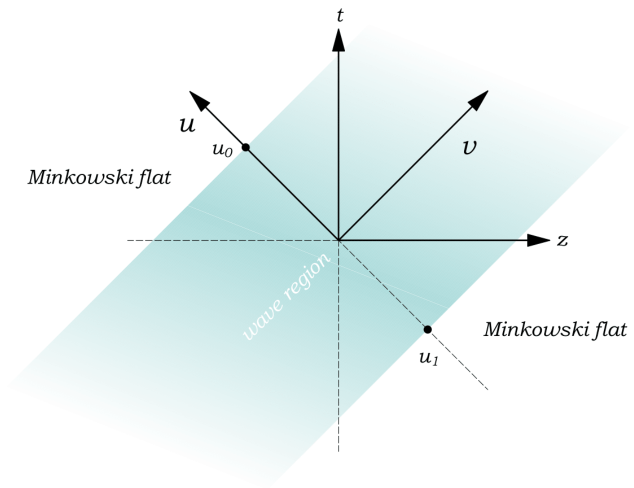

Abbildung 3.2: A sandwich wave, with bounds u_0 and u_1 in u-direction. Wave region simulated by H (respectively f and g).

3.2 More general pp-waves

In this work, we will deal with a slightly more general form of *pp-waves*, where $\mathbb{R}^2$ is replaced by a connected Riemannian manifold of finite dimension $n \geq 1$. More precisely:

Definition 3.2.1 *(M-PFW)*
A general plane fronted wave, M-PFW for short, is a semi-Riemannian manifold $(\mathcal{M}, \langle ., . \rangle_{\mathcal{M}})$, where $\mathcal{M} = M \times \mathbb{R}^2$, with $(M, \langle ., . \rangle_M)$ a connected n-dimensional Riemannian manifold and the metric on $\mathcal{M}$ is defined as

$$\langle ., . \rangle_{\mathcal{M}} = \langle ., . \rangle_M + 2dudv + H(x, u)du^2. \tag{3.3}$$

Here $x = (x^1, \cdots, x^n)$ are coordinates of the Riemannian part M, (v, u) are coordinates of $\mathbb{R}^2$ and $H \not\equiv 0$ a smooth scalar field $H : M \times \mathbb{R} \to \mathbb{R}$ (not depending on v).

Remark 3.2.2 *(M-PFW)*

- *We sometimes just write $\langle ., . \rangle$, if there is no danger of confusion between the metric on the M-PFW $\mathcal{M}$, or on the Riemannian part M.*

- *pp-waves are a spacial case of M-PFWs with* $M = \left(\mathbb{R}^2, ds = dx_1^2 + dx_2^2\right)$, *since then, the metric in* (3.3) *reduces to*

  $$ds^2 = dx_1^2 + dx_2^2 + 2dudv + H(x_1, x_2, u)du^2,$$

  which is (3.1) *of section 3.1.*

- *There is a more general definition (see: [CFS03]), called N-fronted waves, for short Np-waves. In contrast to M-PFWs, Np-waves can also have off-diagonal-terms. The metric of* $\mathcal{M}$ *then takes the form*

  $$\langle .,. \rangle_{\mathcal{M}} = \langle .,. \rangle_M + 2dudv + 2A(x,u)du + H(x,u)du^2.$$

  If we write down this metric in matrix notation, it has the following form:

  $$\begin{pmatrix} & & & 0 & A_1 \\ & \left(g_{ij}^{(R)}\right) & & \vdots & \vdots \\ & & & 0 & A_n \\ 0 & \dots & 0 & 0 & 1 \\ A_1 & \dots & A_n & 1 & H \end{pmatrix}$$ *, where* $g_{ij}^{(R)}$ *denotes the Riemanian metric.*

  It can be shown, that locally, it is always possible to get rid of the off-diagonal-terms $A_1, A_2, \dots, A_n$, *by a coordinate transformation. Here,* $A_i : M \times \mathbb{R} \to \mathbb{R}$ *are smooth functions in coordinates* x *and* u. *So all local results on M-PFWs here, are equivalent to those of Np-waves, while global results can differ.*

4 Curvature and Matter

In this section we calculate all non zero Christoffel symbols for a M-PFW metric, which we summarize in proposition 4.0.1. This we will use to give expressions for the curvature tensors of our M-PFW $\mathcal{M}$ in section 4.1, following [FS03] and we also give conditions of the stress-energy tensor in Einstein's field equation for pp-wave space-times, following [FS06] in section 4.2. Further, we will discuss the role of H in the metric (3.3) on $\mathcal{M}$, for the asymptotic behaviour of the M-PFW.

We start by calculating the Christoffel symbols, which we will use several times in what follows.

First, we write the metric of $\mathcal{M} = M \times \mathbb{R}^2$ in matrix notation, hence

$$(g_{\nu\gamma}) = \begin{pmatrix} & & & 0 & 0 \\ & (g_{ij}^{(R)}) & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & H \end{pmatrix} \quad (g^{\nu\gamma}) = \begin{pmatrix} & & & 0 & 0 \\ & (g^{ij}_{(R)}) & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & -H & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

Here we denote by $\big(g_{ij}^{(R)}\big)$ the matrix representation of the metric on the Riemannian manifold M, while $\big(g^{ij}_{(R)}\big)$ is its inverse.
Note, that the matrix $\big(g_{\nu\gamma}\big)$ representing the metric of $\mathcal{M}$ is of dimension $(n+2)\times(n+2)$.

Conventions:

- For the sake of simplicity we denote the $(n+1)^{th}$-component by v, and the $(n+2)^{th}$-component by u.
- For $\frac{\partial f}{\partial x^{n+1}}$, we also just write $\frac{\partial f}{\partial v}$ and analogously for the partial derivative with respect to u.
- We use Einstein's sum notation, where indices γ, ν, μ and κ run from 1 to $(n+2)$, while we use $1 \leq i, j, k \leq n$ for the 'Riemannian indices'.

To calculate the Christoffel symbols $\Gamma^{\mu}_{\nu\gamma}$, we use the standard formula

$$\Gamma^{\mu}_{\nu\gamma} = \frac{1}{2} g^{\mu\kappa}\Big(\frac{\partial g_{\gamma\kappa}}{\partial x^{\nu}} + \frac{\partial g_{\nu\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial x^{\kappa}}\Big).$$

- We start with the case, where v is one of the lower indices (or both), therefore the formula above reads:

$$\Gamma^{\mu}_{v\gamma} = \frac{1}{2} g^{\mu\kappa} \Big(\frac{\partial g_{\gamma\kappa}}{\partial v} + \frac{\partial g_{v\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{v\gamma}}{\partial x^{\kappa}} \Big).$$

Since no entry in $\big(g_{\gamma\kappa}\big)$ depends on v, we have $\frac{\partial g_{\gamma\kappa}}{\partial v} = 0$. The other two terms in the bracket are 0, since entries $g_{v\mu}$ are either 0 or 1. Thus, by symmetry of the Christoffel symbols, we get $\Gamma^{\gamma}_{v\mu} = \Gamma^{\gamma}_{\nu v} = 0$.

- For $1 \leq k \leq n$, our formula gives

$$\Gamma^{k}_{\nu\gamma} = \frac{1}{2} g^{k\kappa} \Big(\frac{\partial g_{\gamma\kappa}}{\partial x^{\nu}} + \frac{\partial g_{\nu\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial x^{\kappa}} \Big).$$

If $\kappa = v$, then all terms vanish, analogously for $\kappa = u$ and therefore, $\Gamma^{k}_{ij} = \Gamma^{k(R)}_{ij}$ for all $1 \leq i, j, k \leq n$, where $\Gamma^{k(R)}_{ij}$ denotes the Christoffel symbols, with respect to the metric on M.

- Note, that $g^{u\kappa} = 0$ for all κ, except for $g^{uv} = 1$, thus we get

$$\Gamma^{u}_{\nu\gamma} = \frac{1}{2} g^{u\kappa} \Big(\frac{\partial g_{\gamma\kappa}}{\partial x^{\nu}} + \frac{\partial g_{\nu\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial x^{\kappa}} \Big) = \frac{1}{2} \Big(\frac{\partial g_{\mu v}}{\partial v} + \frac{\partial g_{\nu v}}{\partial x^{\gamma}} - \frac{\partial g_{vv}}{\partial v} \Big) = \frac{1}{2} \frac{\partial g_{vv}}{\partial x^{\gamma}} = 0.$$

- By the form of the metric, we have $g^{vv} = -H, \ g^{vu} = 1$, all other entries are 0. So we get

$$\Gamma^{v}_{\nu\gamma} = -\frac{1}{2} H \Big(\frac{\partial g_{\gamma v}}{\partial x^{i}} + \frac{\partial g_{\nu v}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial v} \Big) + \frac{1}{2} \Big(\frac{\partial g_{\gamma u}}{\partial x^{\nu}} + \frac{\partial g_{\nu u}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial u} \Big).$$

The first bracket vanishes, since as in the first point, no entry depends on v and $g_{\gamma v}$ are either 0 or 1, therefore

$$\Gamma^{v}_{\nu\gamma} = \frac{1}{2} \Big(\frac{\partial g_{\gamma u}}{\partial x^{\nu}} + \frac{\partial g_{\nu u}}{\partial x^{\gamma}} - \frac{\partial g_{\nu\gamma}}{\partial u} \Big).$$

- Now, for $1 \leq i, j \leq n$, with analogously argumentation for v-indices terms as before, we have $\Gamma^{v}_{ij} = 0$ and

$$\Gamma^{v}_{uu} = \frac{1}{2} \frac{\partial g_{uu}}{\partial u} = \frac{1}{2} \frac{\partial H}{\partial u}.$$

- To finish calculating the Christoffel symbols, we are left with Γ^{v}_{uj} and Γ^{k}_{uu} for $1 \leq i, j, k \leq n$. For the first one we get

$$\Gamma^{v}_{uj} = \frac{1}{2} \Big(\frac{\partial g_{ju}}{\partial u} + \frac{\partial g_{uu}}{\partial x^{j}} - \frac{\partial g_{uj}}{\partial u} \Big) = \frac{1}{2} \frac{\partial g_{uu}}{\partial x^{j}} = \frac{1}{2} \frac{\partial H}{\partial x^{j}},$$

while for the last one, we have

$$\Gamma^k_{uu} = \frac{1}{2} g^{km}\Big(\frac{\partial g_{um}}{\partial u} + \frac{\partial g_{um}}{\partial u} - \frac{\partial g_{uu}}{\partial x^m}\Big) \underset{1\le k\le n}{=} -\frac{1}{2} g^{km}_{(R)} \frac{\partial g_{uu}}{\partial x^m} = -\frac{1}{2} g^{km}_{(R)} \frac{\partial H}{\partial x^m}.$$

We collect all non zero Christoffel symbols of a M-PFW, in the next proposition.

Proposition 4.0.1 *(Christoffel symbols for M-PFWs) For the M-PFW (3.3) we obtain the following non-vanishing Christoffel symbols:*

1.
$$\Gamma^k_{ij} = \Gamma^{k(R)}_{ij} \quad \big(1 \le i, j, k \le n\big),$$
where again, $\Gamma^{k(R)}_{ij}$ denotes the Christoffel symbols, with respect to the Riemannian part M of the M-PFW $\mathcal{M}$.

2.
$$\Gamma^v_{uj} = \frac{1}{2}\frac{\partial H}{\partial x^j} \quad \big(1 \le j \le n\big),$$

3.
$$\Gamma^v_{uu} = \frac{1}{2}\frac{\partial H}{\partial u},$$

4.
$$\Gamma^k_{uu} = -\frac{1}{2} g^{km}_{(R)} \frac{\partial H}{\partial x^m} \quad \big(1 \le k \le n\big), \tag{4.2}$$
where $g^{km}_{(R)}$ denotes the metric with respect to the Riemannian part M.

As mentioned before, the geometric coordinate-free definition of a pp-wave requires a covariant constant null vector field. We will show that ∂_v, the vector field metric equivalent to the one-form du in our M-PFW, is such a vector field.
Furthermore, we can fix a time orientation, such that ∂_v is past directed. Note, that ∂_{x^i} are spacelike vector fields, while ∂_u can be spacelike, null or timelike (depending on the sign of H). We collect some properties of ∂_v we will need several times below.

Proposition 4.0.2 *(Properties of ∂_v)*
Let $(\mathcal{M}, \langle .,. \rangle)$ be a M-PFW, then we have

1. *∂_v is a parallel null vector field, hence ∂_v is the required covariant constant null vector field of the pp-wave resp. M-PFW.*
2. *$\partial_v = grad(u)$, where $u : \mathcal{M} \to \mathbb{R}$ denotes the smooth natural projection $(x, v, u) \mapsto u$ and grad denotes the gradient with respect to the metric on $\mathcal{M}$.*

3. *The vector field* $X = \partial_u - \frac{1}{2}H(x,u)\partial_v$ *is lightlike and future directed.*

4. *For any future directed causal curve* $z(s) = \big(x(s), v(s), u(s)\big)$*, we have*

$$\dot{u}(s) = \langle \dot{z}(s), \partial_v \rangle \geq 0,$$

with strict inequality, if $z(s)$ *is timelike.*

Proof Again, we choose local coordinates $(x_1, \cdots, x_n)$ for M and (u,v) for $\mathbb{R}^2$.

1. To see, that ∂_v is null, we directly calculate $\langle \partial_v, \partial_v \rangle = \mathbf{0} + 2 \cdot 0 \cdot 1 + 0 = 0$. For ∂_v parallel, we have to show that $\nabla \partial_v = 0$, i.e. $\nabla_Y \partial_v = 0$ for all $Y \in \mathfrak{X}(\mathcal{M})$. Locally, any Y can be written as

$$Y = \sum_{i=1}^{n} Y^i \partial_i + Y^v \partial_v + Y^u \partial_u,$$

where $\partial_i = \partial_{x^i}$ are the basis fields on M. Then, by the properties of the Levi-Civita connection ∇, we obtain

$$\nabla_Y \partial_v = \sum_{i=1}^{n} Y^i \nabla_{\partial_i} \partial_v + Y^v \nabla_{\partial_v} \partial_v + Y^u \nabla_{\partial_u} \partial_v.$$

The first terms $Y^i \nabla_{\partial_i} \partial_v$ on the right hand-side of the equation vanish by the form of the metric (as in proposition 4.0.1) . For the other terms, note that $\nabla_{\partial_v} \partial_v = \Gamma^{\mu}_{vv} \partial_\mu$ and $\nabla_{\partial_u} \partial_v = \Gamma^{\mu}_{uv} \partial_\mu$. As in proposition 4.0.1, it follows, that $\Gamma^{\mu}_{uv} = \Gamma^{\mu}_{uv} = 0$, hence ∂_v is a parallel vector field.

2. $\mathrm{grad}(u)$ is a vector field s.t. $\langle \mathrm{grad}(u), X \rangle = du(X)$ for all $X \in \mathfrak{X}(\mathcal{M})$. Locally, we can write X as

$$X = \sum_{i=1}^{n} X^i \partial_i + X^v \partial_v + X^u \partial_u.$$

On one hand, we can calculate $\langle \mathrm{grad}(u), X \rangle = du(X) = \sum_{i=1}^{n} \frac{\partial u}{\partial x^i} X^i + \frac{\partial u}{\partial v} X^v + \frac{\partial u}{\partial u} X^u = X^u$.

On the other hand, we have $\langle \partial_v, X \rangle = \langle \begin{pmatrix} \mathbf{0} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \boldsymbol{X^i} \\ X^v \\ X^u \end{pmatrix} \rangle = X^u$, as required.

3. By multi-linearity of the metric, we get

$$\langle X, X \rangle = \langle \partial_u, \partial_u \rangle - H \cdot \langle \partial_u, \partial_v \rangle + \frac{1}{4} \cdot \langle \partial_v, \partial_v \rangle = H - H = 0,$$

hence, X is lightlike. To see, that X is future directed, we calculate

$$\langle X, \partial_v \rangle = \langle \partial_u, \partial_v \rangle - \frac{1}{2} H \langle \partial_v, \partial_v \rangle = 1 > 0 \ ,$$

so X and ∂_v are in different equivalence classes of orientation, and since ∂_v is past directed by definition, X is future directed.

4. First observe, that

$$\langle \dot{z}(s), \partial_v \rangle = \langle \dot{x}(s), 0 \rangle + \dot{u}(s) = \dot{u}(s)$$

Now, if z is causal and future directed, we obtain $\dot{u}(s) \leq 0$, since ∂_v is past directed. Finally, in the timelike case, the inequality is strict.

□

4.1 Curvature

Since we already calculated all Christoffel symbols for a M-PFW $\mathcal{M}$, we are now in a position to give coordinate expressions for the Levi-Civita connection ∇_M on M and ∇ on $\mathcal{M}$. This allows us to give a coordinate expression for the Ricci tensor on $\mathcal{M}$ and we relate the Riemann curvature tensor of $\mathcal{M}$ to the Hessian of H, just with respect to the Riemannian part M, as the next proposition shows.

Proposition 4.1.1 *(Curvature properties)*
Let $\big(\mathcal{M} = M \times \mathbb{R}^2, \langle ., . \rangle\big)$ be a M-PFW and $x = (x_1, \cdots, x_n)$ some local coordinates for the Riemannian part M and (v, u) for $\mathbb{R}^2$. Then

1. *M is totally geodesic (see 2.0.11).*

2. *The Riemannian curvature tensor R of $\mathcal{M}$ satisfies:*

$$-2R(., \partial_u, \partial_u, .) = Hess_M H(x, u)(., .) \ ,$$

where $Hess_M$ denotes the Hessian of H with respect to M.

3. *The Ricci tensors Ric of $\mathcal{M}$ and Ric_M of M satisfy*

$$Ric = Ric_M - \frac{1}{2} \Delta_M H du \otimes du. \tag{4.3}$$

Thus, Ric is zero if and only if both, the Riemannian Ricci tensor Ric_M and the spatial Laplacian $\Delta_M H$ vanish.

4. *The scalar curvature at (x, v, u) on $\mathcal{M}$, is equal to the scalar curvature of the Riemannian part M at x.*

Proof

1. This is an easy consequence of the Christoffel symbols, since

$$\Gamma^k_{i,j} = \Gamma^{k(R)}_{ij} \quad \text{for } 1 \leq i, j, k \leq n,$$

Thus, we also get $\nabla_{\partial_i}\partial_j = \nabla^M_{\partial_i}\partial_j$ and the geodesic equations with respect to $\mathcal{M}$ are the same as those with respect to M, hence M is totally geodesic.

2. By use of the Christoffel symbols, we get

$$2\nabla_{\partial_u}\partial_u = \Gamma^\mu_{uu}\partial_\mu = -\mathrm{grad}_M H(x,u) + \partial_u H(x,u)\partial_v, \tag{4.4}$$

while $\nabla_{\partial_v}\partial_u = 0$ and for $1 \leq i \leq n$, we get

$$2\nabla_{\partial_i}\partial_u = \Gamma^\mu_{iu}\partial_\mu = \partial_i H(x,u)\partial_v. \tag{4.5}$$

Furthermore, we have that

$$R(X, \partial_u, \partial_u, Y) = \big(R_{\partial_u, Y}\partial_u\big)X = \big(\nabla_{[\partial_u, Y]}X\big) - \big([\nabla_{\partial_u}, \nabla_Y]\partial_u\big)X.$$

Choosing Riemann normal coordinates $(x^1, \dots, x^n)$ at p, as in proposition 2.0.9, we have $[\partial_u, Y]_p = 0$ for every vector field $Y = f^\mu\partial_\mu \in \mathfrak{X}(\mathcal{M})$, hence $\nabla_{[\partial_u, Y]_p}X$ vanishes, thus the Riemannian curvature tensor reduces at p to

$$R(., \partial_u, \partial_u, Y) = \nabla_Y\nabla_{\partial_u}\partial_u - \nabla_{\partial_u}\nabla_Y\partial_u = -\nabla_{\partial_u}f^\mu\nabla_{\partial_\mu}\partial_u,$$

where the last equality follows by (4.4) and (4.5) and since $\nabla_{\partial_v}\partial_u = 0$, $\nabla_{\partial_v}\partial_v = 0$ and $\nabla_{\partial_v}(\mathrm{grad}_M H) = 0$.

Therefore, the Riemannian curvature tensor R satisfies

$$-2\mathrm{R}\big(., \partial_u, \partial_u, .\big) = \mathrm{Hess}_M H(x,u)\big(., .\big).$$

3. We write the Ricci tensor in local coordinates, thus Ric$= R_{\mu\sigma}dx^\mu \otimes dx^\sigma$, while we can calculate the smooth functions $R_{\mu\sigma}$, by use of the Christoffel symbols and the well-known formula

$$R_{\mu\sigma} = \partial_\kappa\Gamma^\kappa_{\mu\sigma} - \partial_\sigma\Gamma^\kappa_{\mu\kappa} + \Gamma^\kappa_{\kappa\nu}\Gamma^\nu_{\mu\sigma} - \Gamma^\kappa_{\mu\nu}\Gamma^\nu_{\sigma\kappa}.$$

An easy direct computation shows, that $R_{vv} = R_{vu} = 0$, since $\Gamma^\mu_{v\gamma} = 0$ by proposition 4.0.1, while for R_{uu} we have

$$R_{uu} = \partial_\kappa\Gamma^\kappa_{uu} - \partial_u\Gamma^\kappa_{u\kappa} + \Gamma^\kappa_{\kappa\nu}\Gamma^\nu_{uu} - \Gamma^\kappa_{u\nu}\Gamma^\nu_{u\kappa}.$$

The second term vanishes, because $\Gamma^{\kappa}_{u\kappa} = 0$. For the third term $\Gamma^{\kappa}_{\kappa\nu}\Gamma^{\nu}_{uu}$, we have

$$\Gamma^{\kappa}_{\kappa\nu}\Gamma^{\nu}_{uu} = \sum_{m,k=1}^{n} \Gamma^{k}_{km}\Gamma^{m}_{uu} + \Gamma^{v}_{vv}\Gamma^{v}_{uu} + \Gamma^{v}_{vu}\Gamma^{u}_{uu} + \Gamma^{u}_{uv}\Gamma^{v}_{uu} + \Gamma^{u}_{uu}\Gamma^{u}_{uu} =$$

$$= \sum_{m,k=1}^{n} \Gamma^{k}_{km}\Gamma^{m}_{uu} = \sum_{m,k,l=1}^{n} -\frac{1}{2} g^{ml}_{(R)} \frac{\partial H}{\partial x^l}(x,u)\Gamma^{k}_{km},$$

and the last term $\Gamma^{\kappa}_{u\nu}\Gamma^{\nu}_{u\kappa} = 0$. Therefore, we can write R_{uu} as

$$R_{uu} = -\frac{1}{2}\Big(\sum_{k,l=1}^{n} \frac{\partial}{\partial x^k}\big(g^{kl}_{(R)} \frac{\partial H}{\partial x^l}(x,u)\big) + \sum_{m,k,l=1}^{n} g^{ml}_{(R)} \frac{\partial H}{\partial x^l}(x,u)\Gamma^{k}_{km}\Big) = -\frac{1}{2}\Delta_M H(x,u)$$

and the claim follows.

4. The scalar curvature S is defined as the trace (with respect to the metric) of the Ricci tensor (i.e. $S = g^{\nu\kappa}R_{\nu\kappa}$). Thus the claim is direct consequence of (4.3), since the second term just depends on the Laplacian with respect to M, hence just on the x-coordinates.

□

4.2 Matter and the role of H

Now, we will characterise the classical energy conditions for M-PFW to study the interplay between H and reasonable matter sources, following [FS06] and [FS03]. We will assume Einstein's field equation with zero cosmological constant, hence

$$G := \mathrm{Ric} - \frac{1}{2} S g = 8\pi T. \tag{4.6}$$

Note from (4.3) and (4.6), that the strong energy condition, or, equivalently, the timelike convergence condition $\mathrm{Ric}(\xi,\xi) \geq 0$, for all timelike $\xi = (\xi_M, \xi_v, \xi_u)$ holds if and only if

$$\mathrm{Ric}_M(\xi_M,\xi_M) \geq 0, \quad \text{for all } \xi_M \in T_M \qquad \text{and} \qquad \Delta_M H \leq 0,$$

where H is the smooth function from the M-PFW metric in (3.3).

Proposition 4.2.1 *(Space-time energy conditions)*
Let $\mathcal{M} = M \times \mathbb{R}^2$ be a ***four-dimensional*** *M-PFW (i.e. the Riemannian part is of dimension 2) and let $K(x)$ be the Gaussian curvature of the surface M at x. Then, the following are equivalent:*

1. *The strong energy condition ($Ric(\xi,\xi) \geq 0$ for all timelike ξ) is satisfied.*

2. *The weak energy condition ($T(\xi,\xi) \geq 0$ for all timelike ξ) is satisfied.*

3. *The dominant energy condition ($-T^{\alpha}_{\beta}\xi^{\beta}$ is either 0 or causal and future pointing, for all future-pointing timelike $\xi \equiv \xi^{\beta}$) is satisfied.*

4. *The following two inequalities hold:*

$$K(x) \geq 0 \quad \textit{and} \quad \Delta_M H(x,u) \leq 0, \quad \textit{for all } (x,u) \in M \times \mathbb{R} \tag{4.7}$$

Proof Since the dimension of M is 2, we have the following relations between Ric_M and the corresponding scalar curvature S_M:

$$\mathrm{Ric}_M(\xi_M, \xi_M) = K(x)\langle \xi_M, \xi_M \rangle, \quad \forall \xi_M \in T_xM \text{ and } S_M(x) = 2K(x), \ \forall x \in M. \tag{4.8}$$

Now (1.$\Leftrightarrow$ 4.) is clear by this observation (4.8) and the local expression of the Ricci tensor in (4.3).
For (2.$\Leftrightarrow$ 4.) note, that by proposition 4.1.1 (item 4), the scalar curvature of $\mathcal{M}$ and M coincides, thus, for the Einstein tensor G on any tangent vector $\xi = (\xi_M, \xi_v, \xi_u) \in T_{(x,v,u)}\mathcal{M}$, we get by using again (4.8) and (4.3), that

$$G = \mathrm{Ric} - \frac{1}{2}Sg = \mathrm{Ric}_M - \frac{1}{2}\Delta_M H du \otimes du - \frac{1}{2}2K \quad \text{, thus } G \text{ becomes}$$

$$G(\xi.\xi) = K(x)\big(\langle \xi_M, \xi_M \rangle - \langle \xi, \xi \rangle\big) - \frac{1}{2}\Delta_M H(x,u)\xi_u^2. \tag{4.9}$$

Note, that for timelike ξ, we have $\langle \xi, \xi \rangle = \langle \xi_M, \xi_M \rangle + Hu^2 + 2uv < 0$. For the Riemannian part ξ_M we have $\langle \xi_M, \xi_M \rangle \geq 0$, hence $Hu^2 + 2uv < 0$ and therefore, we get $\langle \xi, \xi \rangle + \langle \xi_M, \xi_M \rangle < 0$, thus, the term $\big(\langle \xi_M, \xi_M \rangle - \langle \xi, \xi \rangle\big)$ in equation (4.9) is positive for timelike ξ. This shows (4.$\Rightarrow$ 2.). For the converse, just note, that $\big(\langle \xi_M, \xi_M \rangle - \langle \xi, \xi \rangle\big) > 0$ satisfies

$$\langle \xi_M, \xi_M \rangle - \langle \xi, \xi \rangle = -2\xi_u\xi_v - H\xi_u^2 \ ,$$

thus ξ_v can be chosen in a way, to make the left hand-side, arbitrary large on one hand and arbitrary close to 0 on the other hand.
Finally we show (3.$\Leftrightarrow$ 4.). Recall, that the dominant energy condition (3.) always implies the weak energy condition (2.) (see [MS08]). We have already shown (2.$\Rightarrow$ 4.), therefore, it remains to show the converse (4.$\Rightarrow$ 3.):
Let ξ^{β} be timelike and future pointing. We have to show that

$$-T_{\alpha\beta}\xi^{\beta} = 0 \quad \text{is causal and future pointing.}$$

i.e. we have to show:

(i) $-T_{\alpha\beta}\xi^{\beta}\xi^{\alpha} \leq 0$ and

(ii) $-g^{\alpha\gamma}T_{\alpha\beta}\xi^{\beta}T_{\gamma\delta}\xi^{\delta} \geq 0$.

For (i), we use (4.6) together with (4.8) and the assumption on K and H from item (4.) in the proposition, to get

$$8\pi T_{\alpha\beta}\xi^{\beta}\xi^{\alpha} = 8\pi T(\xi,\xi) = G(\xi,\xi) = K(x)\big(\langle\xi_M,\xi_M\rangle - \langle\xi,\xi\rangle\big) - \frac{1}{2}\Delta_M H\xi_u^2 \geq 0.$$

Now for (ii), a straight forward computation from (4.8), or equivalent from

$$8\pi T = -K(x)\big(du\otimes dv + dv\otimes du\big) - \big(K(x)H(x,u) + \frac{1}{2}\Delta_M H(x,u)\big)du^2, \tag{4.10}$$

shows, that the left hand-side of (ii) becomes

$$2K^2\xi_u\xi_v + \big(K\Delta_M H + K^2 H\big)\xi_u^2 = K^2\big(\langle\xi,\xi\rangle - \langle\xi_M,\xi_M\rangle\big) + K\Delta_M H\xi u^2,$$

which is non-positive by the assumptions in (4.). □

Now, let us discuss the role of the smooth function H in the metric (3.3), for the asymptotic behaviour of M-PFWs. A well-known property of pp-wave spacetimes is, that all scalar curvature invariants are zero. We already saw in proposition 4.1.1, that the most 'characteristic' curvatures of the M-PFW (i.e. this curvatures - taken along a lightlike geodesic), are related to the Hessian of H with respect to the Riemannian part M. So we will focus on that, instead of scalar invariants. TTo move arbitrary farïn a M-PFW, can be thought of, as taking $x \in M$ and $v \in \mathbb{R}$ large for each fixed u. Therefore, any definition of 'finiteness' or 'asymptotic vanishing' of the M-PFW seems to imply that $\mathrm{Hess}_M H(x,u)$ must go (fast) to 0 for large x. More precisely, as in [FS06], we define:

Definition 4.2.2 *(Asymptotically vanishing M-PFW)*
Let $\lambda_i(x,u)$, $i \in \{1,\dots,n\}$ be the eigenvalues of $\mathrm{Hess}_M H(x,u)$ and fix $\tilde{x} \in M$. If the M-PFW $\mathcal{M}$ vanishes asymptotically, then $\lim_{d(x,\tilde{x})\to\infty}\lambda_i(x,u)$ must vanish fast for each u. Again, $d(.,.)$ denotes the Riemannian distance, induced by the metric on M.
Defining $|\lambda(x,u)|$ to be the maximum of all eigenvalues, we can assume the following inequality, as a definition of ***asymptotic vanishing M-PFWs:***

$$|\lambda(x,u)| \leq \frac{A(u)}{d(x,\tilde{x})^{q(u)}}, \tag{4.11}$$

for some continuous functions $A(u)$ and $q(u) > 0$.

We will need the following boundary conditions:

Definition 4.2.3 *(At most quadratic growth)*
Let $\mathcal{M} = M \times \mathbb{R}^2$ be a M-PFW and $V : M \times \mathbb{R} \to \mathbb{R}$ any smooth function. Fixing a point on the Riemannian part $\tilde{x} \in M$, we say that $|V(x,u)|$ behaves ***subquadratically (at most quadratically)*** *at spatial infinity, if there are continuous functions $R_1, R_2 > 0$*

and $q(u) < 2$, ($q(u) \leq 2$), such that

$$|V(x,u)| \leq R_1(u)d^{q(u)}(x,\tilde{x}) + R_2(u) \quad \text{for all } x \in M \text{ and } u \in \mathbb{R}, \tag{4.12}$$

while we say it behaves ***superquadratically*** *if $q(u) > 2$. When we just say* ***quadratic,*** *we mean (4.12) with $q(u) = 2$. Note, that this definition is independent of the chosen $\tilde{x} \in M$.*

Inequality (4.11) implies bounds for the spatial growth of $|H|$, as the next proposition shows.

Proposition 4.2.4 *(Asymptotically vanishing of M-PFWs)*
If the M-PFW $\mathcal{M}$ vanishes asymptotically as in (4.11), then $|H(x,u)|$ behaves subquadratically at spatial infinity (i.e. $q(u) < 2$).

Proof If M is a complete Riemannian manifold. Fix a $\tilde{x} \in M$ and define

$$H_0(u) \coloneqq \max\{|H(x,u)| : d(x,\tilde{x}) \leq 1\}.$$

Now for $x \in M$, let γ_x be a minimizing geodesic from x to $\tilde{x}$, parametrized with unit speed. Also assume, without loss of generality, that $0 < \delta < 1$ and the distance from x to $\tilde{x}$ to be bigger than 1, hence $d(x,\tilde{x}) > 1$. By the fundamental theorem of calculus, we have

$$H(x,u) = H(\tilde{x},u) + \int_0^{d(x,\tilde{x})} \int_0^{\sigma} \frac{d^2}{ds^2} H(\gamma_x(s),u)ds\; d\sigma.$$

Thus, by the definition of H_0 and the assumption $d(x,\tilde{x}) > 1$, we get

$$|H(x,u)| \leq H_0(u) + \int_1^{d(x,\tilde{x})} \int_1^{\sigma} |\mathrm{Hess}_M H(\gamma_x(s),u)| ds\; d\sigma$$

$$\leq H_0(u) + \int_1^{d(x,\tilde{x})} \int_1^{\sigma} |\lambda(\gamma_x(s),u)| ds\; d\sigma.$$

Now, we use the definition of asymptotic vanishing of a M-PFW (4.11), to get

$$
\begin{aligned}
|H(x,u)| &\leq H_0(u) + A(u) \int_1^{d(x,\tilde{x})} \int_1^{\sigma} \frac{1}{d(\gamma_x(s),\tilde{x})^{\delta}} ds\ d\sigma \\
&\leq H_0(u) + A(u) \int_1^{d(x,\tilde{x})} \int_1^{\sigma} \frac{1}{s^{\delta}} ds\ d\sigma = H_0(u) + A(u) \int_1^{d(x,\tilde{x})} \frac{1}{1-\delta}\big(\sigma^{1-\delta}-1\big) d\sigma \\
&= H_0(u) + A(u) \frac{1}{1-\delta}\frac{1}{2-\delta}[\sigma^{2-\delta}]_1^{d(x,\tilde{x})} + \frac{1}{1-\delta}[-s]_1^{d(x,\tilde{x})} \\
&= H_0(u) + \frac{A(u)}{(1-\delta)(2-\delta)} d(x,\tilde{x})^{2-\delta} + \frac{A(u)}{(1-\delta)(2-\delta)} \\
&+ \frac{A(u)}{\delta-1} d(x,\tilde{x}) - \frac{A(u)}{1-\delta} \leq H_0(u) + C(u) + C(u) d(x,\tilde{x})^{2-\delta} \\
&\leq R_1(u) d(x,\tilde{x})^{2-\delta} + R_2(u).
\end{aligned}
$$

And for $q(u) < 1-\delta$, the claim follows for complete Riemannian manifolds M. In the case of an incomplete manifold M, we can approximate any curve from x to $\tilde{x}$ by broken geodesics and reproduce the above prove (taken from [EE93], Prop. 5.3). □

Remark 4.2.5 *(Asymptotic behaviour of M-PFWs)*

(i) *This proposition shows, that the condition of asymptotic vanishing for a M-PFW* (4.11) *implies subquadratic growth for $|H(x,u)|$, but the converse is not true, as we will see in chapter 7. Also, in chapter 7 and chapter 5, we will use only this more general subquadratic condition on $|H|$, or even only the subquadraticity of just H or $-H$, such that the range of applications will be wider.*

(ii) *Inequality* (4.11) *is compatible with the energy conditions in 4.2.1. There are complete Riemannian surfaces with $K(x) > 0$, for example the sphere or the paraboloid. For our examples, we will assume $M = \mathbb{R}^2$ with the flat metric.*

(1) Take $H(x_1, x_2, u) = H_u(x_1)$ (independent of x_2) and put $x \equiv x_1$, then the energy conditions 4.2.1 will hold if and only if

$$\frac{d^2 H_u}{dx^2} \leq 0.$$

Second, if the more restrictive condition

$$\frac{d^2 H_u}{dx^2} \leq -\varepsilon < 0, \text{ for an } \varepsilon > 0,$$

also holds, then $-H_u(x)$ behaves either quadratic or superquadratic at spatial infinity.

If

$$-\frac{A(u)}{|x|^{q(u)}} \leq \frac{d^2 H_u}{dx^2}(x) \leq 0,$$

for some smooth functions $A(u), q(u) \leq 0$, *then it will be subquadratic.*

(2) Recall, that a classical gravitational wave is a pp-wave, where H *is given by*

$H(x_1, x_2, u) = f(u)(x_1^2 - x_2^2) + 2g(u)x_1, x_2$, *for some real-valued smooth functions* f, g.

Now, a simple computation shows, that for $f, g > 0$, *we get* $\Delta_M H = 0$, *thus the energy conditions hold.*

(iii) *For plane waves (see definition 3.1.1), neither* H *nor* $-H$ *have subquadratic behaviour, since* $|H(x,u)|$ *behaves (at most) quadratically at spatial infinity. So, they lie in the critical quadratic situation, which we will focus on, in chapter 5 and chapter 7.*

5 Geodesic completeness

In this section, we focus on geodesic completeness. First, we will describe geodesics on $\mathcal{M} = M \times \mathbb{R}^2$ in theorem 5.1.1, before we give criteria for geodesic completeness on a M-PFW $\mathcal{M}$ in theorem 5.1.3, following [CFS03] (also see [CFS04]). This ODE turns out, to be a purely Riemannian problem just on M, depending on the smooth function H (according to definition 3.2.1) and has been studied systematically in the early 70's by several authors, for example [AM78] or [Gor70]. Therefore, in the technical section 5.2, we start to study the completeness of a more general differential equation, just on the Riemannian part M of the M-PFW $\mathcal{M}$. This we will use later, for completeness results on our Lorentzian Manifold $\mathcal{M}$ in chapter 6, following [CRS12] and [CRS13].

5.1 Geodesics in a M-PFW

We start with a result, which shows how geodesics in a M-PFW $\mathcal{M}$ look like. We will need this result in both, chapter 5 for study of geodesic completeness and in chapter 7 for causality of $\mathcal{M}$.

Theorem 5.1.1 *(Geodesic in a M-PFW)*
Let $z(s) = (x(s), v(s), u(s))$ be a curve in $\mathcal{M} = M \times \mathbb{R}^2$ of constant energy

$$E_z = \langle z'(s), z'(s) \rangle_{\mathcal{M}}, \quad \textit{where } z : I \to \mathcal{M} \textit{ and } 0 \in I.$$

Then z is a geodesic on $\mathcal{M}$ if and only if the following three conditions on x, u and v hold:

(i) u is affine, i.e. $u(s) = u_0 + su_1$ for all $s \in I$, where $u(0) = u_0$ and $\dot{u}(0) = u_1$.

(ii) x solves the following 'equation of motion':

$$\nabla_s x'(s) = \frac{u_1^2}{2} \cdot grad_M H(x(s), u_o + su_1). \tag{5.1}$$

(iii) If $\dot{u}(0) = 0$, then v is affine (i.e. $v(s) = v_0 + sv_1$ for all $s \in I$, where $v(0) = v_0$ and $\dot{v}(0) = v_1$). Otherwise, if $\dot{u}(0) \neq 0$ for all $s \in I$, we have

$$v(s) = v_0 + \frac{1}{2u_1} \int_0^s \Big(E_z - \langle \dot{x}(t), \dot{x}(t) \rangle_M - u_1^2 \cdot H\big(x(t), u_0 + tu_1\big)\Big) dt. \tag{5.2}$$

Proof Fix some local coordinates $(x^1, \cdots, x^n)$ in the Riemannian part M, and coordinates (v, u) of $\mathbb{R}^2$. By the Christoffel symbols (4.2) on $\mathcal{M}$, the geodesic equations $\nabla_{z'(s)} z'(s) = z''(s) = 0$, where ∇ denotes the Levi-Civita conection on $\mathcal{M}$ and $z(s) = (x(s), v(s), u(s))$, become:

$$(x'')^i = \ddot{x}^i + \sum_{j,k=1}^{n} \Gamma_{jk}^{i(R)} \dot{x}^j \dot{x}^i + \Gamma_{uu}^i \dot{u}^2 = 0 \quad \text{for all } i \in \{1, \cdots, n\}, \tag{5.3}$$

$$v'' = \ddot{v} + 2 \sum_{j=1}^{n} \Gamma_{ju}^v \dot{x}^j \dot{u} + \Gamma_{uu}^v \dot{u}^2 = 0, \tag{5.4}$$

$$u'' = \ddot{u} = 0. \tag{5.5}$$

Now *(i)* easily follows from (5.5).
(ii) follows by (5.3) , since

$$\Gamma_{uu}^i = -\frac{1}{2} \sum_{m=1}^{n} g_{(R)}^{km} \frac{\partial H}{\partial x^m}(x, u) \quad \text{for all } i \in \{1, \cdots, n\},$$

thus, by the definition of the gradient with respect to the metric on $\mathcal{M}$, we get

$$\Gamma_{uu}^i = -\frac{1}{2} \mathrm{grad}_M H(x, u).$$

(iii) It's easy, if $u_1 = 0$, since then by *(i)* $\dot{u} \equiv 0$, thus by (5.4), we get $\ddot{v} = 0$. Otherwise, if $u_1 \neq 0$, the expression of the energy E_z yields

$$E_z = \langle z', z' \rangle_{\mathcal{M}} = \langle \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}, \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix} \rangle_{\mathcal{M}} = \langle \dot{x}, \dot{x} \rangle_M + 2\dot{u}\dot{v} + H\dot{u}^2.$$

So we get

$$2\dot{u}\dot{v} = E_z - \langle \dot{x}, \dot{x} \rangle_M - H(x, u)\dot{u}^2.$$

Since $\dot{u}$ is constant by *(i)*, integrating $\dot{v}$ from 0 to s gives (5.2).

$\square$

Remark 5.1.2 *Note, that in (ii), the gradient is taken with respect to the Riemannian part M.*

The next result shows, that for completeness of a M-PFW $\mathcal{M}$, one has just to study the gradient (with respect to the complete Riemannian part M) of the smooth function H from the metric on $\mathcal{M}$.

Theorem 5.1.3 *(Completeness of M-PFW)*
A M-PFW $\mathcal{M} = M \times \mathbb{R}^2$ is geodesically complete if and only if M is a complete Riemannian manifold and any solution of

$$\nabla_s x'(s) = \frac{1}{2} \cdot grad_M H(x(s), s), \tag{5.6}$$

can be extended so, as to be defined on all of $\mathbb{R}$ (i.e. the solution is complete).

Here, ∇_s denotes the induced covariant derivative by ∇ on $\mathcal{M}$.

Proof ($\Rightarrow$) If $z = (x, 0, 0)$ is a complete geodesic in $\mathcal{M}$, then $x(s)$ is a geodesic in M, so M has to be complete.
If $z = (x, v, u)$ is a complete geodesic in $\mathcal{M}$, then by theorem 5.1.1 part *(ii)*, it is a solution of (5.1), which is equivalent to (5.6), with $u_0 = 0$ and $u_1 = 1$.
($\Leftarrow$) Let $x(s)$ be a solution of (5.1), then either $u_1 = 0$ and x is geodesic in the complete manifold M, or $u_1 \neq 0$. For the second case take

$$y(t) = x\Big(\frac{t - u_0}{u_1}\Big),$$

which is a geodesic, since x is, and geodesics are invariant under affine reparametrization. Then by (5.1), we see that

$$\nabla_t y'(t) = \frac{1}{u_1^2} \nabla_t x'(t) = \frac{1}{2} \mathrm{grad}_M H(y, t)$$

is a solution of (5.6). By assumption it is complete, hence the M-PFW $\mathcal{M}$ is geodesically complete. $\square$

We see that completeness of $\mathcal{M} = M \times \mathbb{R}^2$ for a complete Riemannian manifold M, just depends on $H(x, u)$ and its M-gradient.
Before we use a more general approach, to find criteria on H for geodesic completeness, we will give a natural condition for completeness of solutions for (5.6) in the special case when H is autonomous (i.e. independent of u, $H(x, u) \equiv H(x)$). We assume, that H is controlled at infinity (in a suitable way) by a so called *positively complete* function U_0. More precisely:

Definition 5.1.4 *(Positively complete function)*
A non-increasing C^2-function $U_0 : [0, \infty) \to \mathbb{R}$, such that

$$\int_0^s \frac{dt}{\sqrt{\alpha - U_0(t)}} = +\infty \ , \tag{5.7}$$

*for a constant $\alpha > U_0(0)$, for all $s \in [0, +\infty)$, is called **positively complete**.*

Proposition 5.1.5 *(Natural completeness condition for H autonomous)*
Let M be a complete Riemannian manifold and H a smooth map on M.
If there exist $r > 0$ and $\overline{x} \in M$ and a positively complete function U_0, such that

$$H(x) \leq -U_0\big(d(x, \overline{x})\big) \quad \textit{for all } x \in M \textit{ with } r \leq d(x, \overline{x}).$$

Then, all solutions of (5.6) are complete. Here, $d(x, \overline{x})$ denotes the Riemannian distance from x to $\overline{x}$ (in M), induced by the Riemannian metric on M.

For a proof, see [WM70], Theorem 3.7.15. We skip it here, because this result will be a direct consequence of our later one, where H can also be non-autonomous (time dependent). Moreover, the proof techniques are similar to those we will introduce below.

Example (Positively complete function)
Note, that if U_0 is a positively complete function and V_0 is a non-increasing C^1-function, such that $V_0(0) = U_0(0)$ and $V_0 \geq U_0$, then V_0 is also a positively complete function. So, we are interested in the case, where the limit $\lim_{s \to +\infty} U_0(s)$ goes to $-\infty$, as fast as possible. Therefore, the relevant limit equivalent to (5.7) for any non-increasing function on $[0, +\infty)$ is

$$\int_{s_0}^{+\infty} \frac{ds}{\sqrt{-U_0(0)}} = +\infty \ ,$$

where s_0 is any point with $U_0(s_0) < 0$.
In particular, $U_0(t) = -R_0 t^p$ with $R_0 > 0$ and $0 \leq p \leq 2$ is such a positively complete function, while it is not positively complete if $p > 2$. Hence, by Theorem 5.1.3 together with Proposition 5.1.5, we get the following.

Theorem 5.1.6 *(Completeness in case of H autonomous)*
Let $\mathcal{M} = M \times \mathbb{R}^2$ be a M-PFW, such that M is complete and $H(x, u) \equiv H(x)$. If there exist $r > 0$, $\overline{x} \in M$, $R_0 > 0$, such that

$$H(x) \leq R_0 d^2(x, \overline{x}) \quad \textit{for all } x \in M \textit{ s.t. } r \leq d(x, \overline{x})$$

Then, $\mathcal{M}$ is complete.

5.2 Extended Lagrangian and Hamiltonian systems

The problem in theorem 5.1.3, has been studied by several authors in the 70's [Gor70], [WM70] and they obtained accurate results, when the potential H is autonomous, for example theorem 5.1.6. More precisely, in classical mechanics, one of the most interesting equations on a (connected) Riemannian manifold $(M, \langle .,. \rangle)$ is

$$\nabla_t \gamma'(t) = -grad_M V(\gamma(t), t),$$

where again, ∇_t denotes the induced covariant derivative and γ' is the velocity vector field along γ, while $V : M \times \mathbb{R} \to \mathbb{R}$ is a smooth time-dependent potential. In fact, when $M = \mathbb{R}^3$, this ODE is just Newton's second law for forces, for an external time-dependent potential (in our case $V = -\frac{1}{2}H$, where H is the smooth map coming from the M-PFW metric in (3.3)). As mentioned before, this problem is classical but extensions to the non-autonomous and velocity depending forces are quiet recent, see: [CRS12], [CRS13]) from Spanish mathematicians M. Sanchez, A. Romero and Italian mathematician A.M. Candela. We will collect their completeness results and work out the proofs in detail. Therefore, this section is mainly based on [CFS03], [CFS04] and [CRS12], [CRS13].
First, we will need some terminology to formulate the main results in this section. Then, in section 5.2.1, we will give results for the case if H is autonomous. Then, in section 5.2.2, we will focus on the non-autonomous problem and in section 5.2.3 for the case of a potential $V = -\frac{1}{2}H$. At the end, we will compare the different results and give examples, demonstrating the sharpness of the bounds in our results. Note, that throughout this section, $(M, \langle .,. \rangle)$ is always assumed to be a connected Riemannian manifold and $\langle .,. \rangle$ denotes the (positive definite) metric tensor on M.

Definition 5.2.1 *(Non-autonomous ODE)*
Let $(M, \langle .,. \rangle)$ be a connected n-dimensional Riemannian manifold and let $\pi : M \times \mathbb{R} \to M$ be the natural projection. Given a smooth $(1,1)$-tensor field F along π, a smooth vector field X along π and let γ be a curve in M. We consider the second order differential equation

$$\nabla_t \gamma'(t) = F_{(\gamma(t),t)} \gamma'(t) + X_{(\gamma(t),t)}, \tag{5.8}$$

where again, ∇_t denotes the induced covariant derivative on M.

Note, that ODE (5.8) describes the dynamics of a classical particle under the action of a force field. This force field linearly depends on its velocity, and an external force field X. For purpose of comparison with previous results in the literature, recall that in equation (5.8),

(i) the ODE also depends on t.

(ii) X is not necessarily the gradient of a potential.

(iii) forces, which depend linearly on the velocities, are allowed.

To study extensibility, respectively completeness of geodesics in a M-PFW, we see, that equation (5.6) in Theorem 5.1.3, is a spacial case of (5.8), where $F \equiv 0$ and X is the M-gradient of a potential, depending on the scalar field H in the metric. γ' is a vector field (velocity field) along γ. If both, F and X are time independent, then (5.8) simplifies to

$$\nabla_t \gamma'(t) = F_{\gamma(t)} \gamma'(t) + X_{\gamma(t)}. \tag{5.9}$$

This is called the **autonomous** equation, using the term **non-autonomous** for equation (5.8), when one or both F or X, are time-dependent.

Remark 5.2.2 *(ODE)*

1. *By ODE theory we know that for (5.8) with initial conditions $\gamma(0) = p$, $\dot{\gamma}(0) = v$, there exists an unique solution γ with an inextensible maximal open interval $I = (a, b)$.*
   If $I = (a, \infty)$ (resp. $I = (-\infty, b)$), we say that the solution is forward (resp. backward) complete. If $I = \mathbb{R}$ we call the curve γ complete.

2. *Pointwise, a $(1,1)$-tensor F can be described as a matrix. We know from linear algebra, that we can decompose every matrix F into a self-adjoint part S and a skew-adjoint part A, with respect to the metric $\langle ., . \rangle$, hence*

$$F = S + A.$$

   Since, $\langle v, Fv \rangle = \langle v, Sv + Av \rangle = \langle v, Sv \rangle + \langle v, Av \rangle = \langle v, Sv \rangle - \langle Av, v \rangle$ and $\langle ., . \rangle$ is symmetric we have $\langle v, Av \rangle = 0$. So $\langle v, Fv \rangle = \langle v, Sv \rangle$.

First we will prove a result for the autonomous case (5.9) and then we show, how to adapt this, for the non-autonomous case (5.8).
Before we do so, we need some terminology to formulate the main results about extensibility of solutions of (5.9), respectively (5.8).

Definition 5.2.3 *(Bounds on F, respectively S and X)*
We define bounds for $F \in \mathcal{T}^1_1(M)$, or equivalent, a bound to its self-adjoint part S and a bound on $X \in \mathcal{X}(M)$.

$$S_{\sup} := \sup_{\substack{v \in TM \\ \|v\|=1}} \langle v, Sv \rangle, \quad S_{\inf} := \inf_{\substack{v \in TM \\ \|v\|=1}} \langle v, Sv \rangle \quad \|S\| := \max\{|S_{\sup}|, |S_{\inf}|\}$$

For the non-autonomous equation, consider the analogous notations

$$S_{\sup}(t), S_{\inf}(t), \|S(t)\|,$$

computed for each slice $M \times \{t\}$.

*We say that S is **bounded along finite times**, if for each $T > 0$, there exists a constant N_T, such that for all $t \in [-T, T]$, we have*

$$\|S(t)\| < N_T. \tag{5.10}$$

*For the autonomous case, we say that the norm of a vector field X on M **grows at most linearly in** M, if there exist some constants $A, C > 0$, such that*

$$\|X\|_p := \sqrt{g(X_p, X_p)} \leq A \cdot d(p, p_0) + C \quad \textit{for all } p \in M \textit{ and a fixed } p_0 \in M, \tag{5.11}$$

where d denotes the distance induced by the Riemannian metric $\langle ., . \rangle$ on M.

*Analogously, in the non-autonomous case, we say that X **grows at most linearly in** M **along finite times**, if for each $T > 0$, there exists $p_0 \in M$ and constants $A_T, C_T > 0$, such that*

$$\|X\|_{(p,t)} \leq A_T \cdot d(p, p_0) + C_T, \quad \textit{for all } (p, t) \in M \times [-T, T]. \tag{5.12}$$

We are now in a position to formulate the main theorems in this section:

Theorem 5.2.4 *(Main result for the non-autonomous case)*
Let $(M, \langle ., . \rangle)$ be a complete Riemannian manifold and consider (5.8). If the following two properties hold:

(i) X grows at most linearly in M along finite times and

(ii) the self-adjoint part S of F is bounded along finite times.

Then, each inextensible solution of (5.8) must be complete.
In particular, if M is compact, then any inextensible solution of (5.8) is complete for any F and X.

Note, that the relevant case for us is (according to equation (5.6) in theorem 5.1.3), when X comes from the gradient of a potential. In this case, we need the following definition about growth of a scalar field (potential). We already defined in chapter 4 (definition 4.2.3) what it means for a smooth real-valued function, to behaves subquadratically or quadratically, but we will do it here again, with the notations we need in our proof(s) and a little variation, because we are working on $M \times \mathbb{R}$, not just on M.

Definition 5.2.5 *(At most quadratic growth along finite times)*

*A function (potential) $H : M \times \mathbb{R} \to \mathbb{R}$ **grows at most quadratically along finite times**, if for each $T > 0$, there exist $p_0 \in M$ and some constants $A_T, C_T > 0$ such that*

$$H(p, t) \leq A_T \cdot d(p, p_0)^2 + C_T \quad \textit{for all } (p, t) \in M \times [-T, T]. \tag{5.13}$$

Theorem 5.2.6 *(Main result for gradients of a potential)*
Let $(M, \langle ., . \rangle)$ be a complete Riemannian manifold. Let S be the self-adjoint part of F and $H : M \times \mathbb{R} \to \mathbb{R}$ a smooth map. If the following three hold:

(i) H grows at most quadratically along finite times,

(ii) $|\frac{\partial H}{\partial t}|$ grows at most quadratically along finite times,

(iii) S is bounded along finite times.

Then, every inextensible solution of

$$\nabla_t \gamma'(t) = F_{(\gamma(t),t)}\gamma'(t) + \frac{1}{2} gradH(\gamma(t), t) \tag{5.14}$$

is complete.

Here, again grad denotes the gradient with respect to the metric on M.

We will first prove a similar statement to theorem 5.2.4 for the autonomous case (5.9), while for the non-autonomous case (5.8) we give an argument, how to adapt the autonomous case to the product manifold $M \times \mathbb{R}$, to prove the non-autonomous case. Finally, we come back to our problem (theorem 5.1.3 which is a special case of theorem 5.2.6), where $F \equiv 0$ and X is the gradient of some potential. Further, we also discuss the optimality of the bounds on X (respectively H) and F (respectively S) of the main theorems in the examples at the end of section 5.2

5.2.1 The autonomous case

In this technical section, we focus on equation (5.9), that is

$$\nabla_t \gamma'(t) = F_{\gamma(t)}\gamma'(t) + X_{\gamma(t)}, \tag{5.15}$$

where tensor fields X and F are defined on the tangent bundle TM (independent of t) and M is a connected Riemannian manifold.
Similar to the geodesic flow in Riemannian geometry (see: [ON83], S70 Prop. 28), there exists a vector field G^0 on TM, such that its integral curves are precisely the velocities $s \mapsto \dot{\gamma}$ of curves γ, which solves (5.9). Recall from ODE theory, that an integral curve c of a vector field defined on some bounded interval $[a, b)$ with $b < \infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, with $\lim_{n\to\infty} t_n = b$, such that $\{c(t_n)\}_n$ converges. Thus, to show completeness, one has just to ensure, that the integral curves of G^0, restricted to a bounded interval, lie in a compact subset of TM.
The following lemma is a direct consequence of this fact.

Lemma 5.2.7 *(Extensibility of curves on M)*
Let $\gamma : [0, b) \to M$ be a solution of (5.15) with $b < \infty$. Then γ can be extended to b as

a solution of (5.15) if and only if there exists a sequence $\{t_n\}_n$ such that $\lim_{n\to\infty} t_n = b$ and the sequence of velocities $\gamma'(t_n)\}_n$ converges in TM.

We also recall an ODE result about subsolutions, we will use more then once in our proofs.

Lemma 5.2.8 *(Subsolution argument)*
Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$and let $w(t)$ be a subsolution of the differential equation $u'(t) = f(t,u)$ on $[t_0, T)$ (i.e. $w'(t) < f(t,w)$). Then, for every solution u of the differential equation

$$u'(t) = f(t,u),$$

with $w(t_0) \leq u(t_0)$, we have

$$w(t) < u(t) \quad \textit{for all } t \in (t_0, T).$$

Theorem 5.2.9 *(Extensibility for the autonomous case)*
Let $(M, \langle .,. \rangle)$ be a complete Riemannian manifold, $X \in \mathfrak{X}(M)$ and S the self-adjoint part of $F \in \mathfrak{T}^1_1(M)$. If $\gamma : I \to M$ is an inextensible solution of (5.15), satisfying the following two assumptions:

(a1) There exists a constant $c_\gamma > 0$, such that

$$\langle \gamma'(t), S\gamma'(t)) \rangle \leq c_\gamma \cdot \langle \gamma'(t), \gamma'(t)) \rangle, \quad \textit{for all } t \in I.$$

(a2) $\|X\|_{\gamma(t)}$ grows at most linear in $|\gamma| := d(\gamma(t), p_0)$. i.e. there exists a scalar $r_\gamma > 0$, such that

$$\|X\|_{\gamma(t)} \leq r_\gamma \cdot (1 + |\gamma|) \quad \textit{for all } t \in I,$$

where $d(\gamma(t), p_0)$ is the Riemannian distance induced by $\langle .,. \rangle$, with respect to some fixed $p_o \in M$.

Then, γ must be forward complete.

Proof Without loss of generality, let $I = [0, b)$, $0 < b < \infty$, be an inextensible (forward) solution γ of (5.15), and let $p_0 = \gamma(0)$, such that $|\gamma(t)| = d(\gamma(t), \gamma(0))$.
By assumption, $(M, \langle .,. \rangle)$ is complete. Therefore, every bounded metric ball is compact in M by 2.1.2. We will establish a bound on $\gamma'(I)$ and together with lemma 5.2.7 we show, that γ can be extended to b in contraction with its maximality assumption, and we are done.
We start, by defining

$$u(t) := \langle \gamma'(t), \gamma'(t) \rangle,$$

thus, it is enough to show, that there is a constant $k > 0$, such that

$$u(t) \leq k \quad \text{for all } t \in I = [0, b). \tag{5.16}$$

For any $t \in [0, b)$, equation (5.15) implies that

$$\begin{aligned} u'(t) &= \nabla_t\big(\langle \gamma'(t), \gamma'(t)\rangle\big) \\ &= 2\langle \gamma'(t), \nabla_t \gamma'(t)\rangle = 2\langle \gamma'(t), F\gamma'(t)\rangle + 2\langle \gamma'(t), X_{\gamma(t)}\rangle \\ &= 2\langle \gamma'(t), S\gamma'(t)\rangle + 2\langle \gamma'(t), X_{\gamma(t)}\rangle. \end{aligned}$$

Now we use assumption $(a1)$, $(a2)$ and the Cauchy-Schwarz inequality to get

$$\begin{aligned} u'(t) &\leq 2c_\gamma \cdot \langle \gamma'(t), \gamma'(t))\rangle + 2r_\gamma \cdot (1 + |\gamma|)\sqrt{\langle \gamma'(t), \gamma'(t)\rangle} \\ &\leq 2c_\gamma \cdot \langle \gamma'(t), \gamma'(t))\rangle + r_\gamma^2 \cdot (1 + |\gamma|^2) + \langle \gamma'(t), \gamma'(t))\rangle \\ &= 2c_\gamma \cdot u(t) + r_\gamma^2 \cdot (1 + |\gamma|^2) + u(t). \end{aligned}$$

Hence, we get the following inequality

$$u'(t) \leq (2c_\gamma + 1)u(t) + r_\gamma^2 + r_\gamma^2 |\gamma(t)|^2. \tag{5.17}$$

Again, by Cauchy-Schwarz, we get

$$|\gamma(t)|^2 \leq \Big(\int_0^t \sqrt{u(s)}\, ds\Big)^2 \leq b\int_0^t u(s)\, ds$$

and putting $v(t) = \int_0^t u(s)\, ds$, we see that inequality (5.17) yields

$$v'' = k_1 v' + k_2 v + k_3. \tag{5.18}$$

Now, we just need to apply the subsolution argument from lemma 5.2.8. To this end note, that any solution w of (5.18) with equality (instead of inequality) is a complete C^2-function. Choosing such a w with $w(0) = v(0) = 0$, $w'(0) = u(0)$ and by applying the subsolution lemma 5.2.8 twice, we have that

$$v(t) < w(t), \quad v'(t) < w'(t), \quad \text{for all } t \in (0, b).$$

Thus (5.16) holds with $k = \max\{w'(t) : t \in [0, b)\}$. $\square$

The case of autonomous backward completeness is know an easy consequence of the previous result.

Corollary 5.2.10 *(Autonomous backward completeness)*
If we use $(a1')$ *instead of* $(a1)$*, in Theorem 5.2.9, where*

(a1') There exist $c_\gamma > 0$*, such that*

$$-\langle \gamma'(t), S\gamma'(t)\rangle \leq c_\gamma \langle \gamma'(t), \gamma'(t)\rangle, \quad \textit{for all } t \in I.$$

Then, γ *is backward complete.*

Proof Let $\gamma : (-b, 0] \to M$ be an inextensible (backward) solution of (5.15). The following reparametrization $\gamma^*(t) = \gamma(-t)$ on $[0, b)$ is an inextensible (forward) solution of

$$\nabla_t \gamma^*(t)' = -F_{\gamma^*(t)}\gamma^*(t)' + X_{\gamma^*(t)}.$$

With assumption $(a1')$, we see that $-F$ satisfies $(a1)$. Then by theorem 5.2.9 γ^* is forward-complete, hence γ is backward-complete. $\square$

Finally, theorem 5.2.9 and corollary 5.2.10 give the following statement for completeness of equation (5.15):

Theorem 5.2.11 *(Autonomous completeness)*
Let $(M, \langle ., . \rangle)$ be a complete Riemannian manifold, $X \in \mathfrak{X}(M)$ a smooth vector field on M and S the self-adjoint part of the $(1,1)$-tensor field F on M. If

1. *X grows at most linearly in M and*

2. *$\|S\|$ is bounded,*

then all inextensible solutions of (5.15) are complete. In particular, this holds if M is compact.

For the next results, we need the following lemma:

Lemma 5.2.12 *(Comparison lemma)*
Let $\varphi : [a, \infty) \to \mathbb{R}$ be a locally Lipschitz monotone increasing function, such that

$$\varphi(s) > 0, \quad \text{for all } s \geq a \quad \text{and} \quad \int_a^{+\infty} \frac{ds}{\varphi(s)} = +\infty.$$

If a C^1-function f is an inextensible solution of

$$f'(t) = \varphi\big(f(t)\big), \quad \text{with } f(0) \geq a, \tag{5.19}$$

then f is defined for all $t \geq 0$.

Furthermore, if $h : [0, b) \to \mathbb{R}$ is a continuous function, such that $h(t) \geq 0$ for all $t \in [0, b)$ and the following two properties hold

(i) $a \leq h(t) \leq h(0) + \int_0^t \varphi\big(h(s)\big)\, ds$ *for all* $t \in [0, b)$,

(ii) $h(0) \leq f(0)$.

Then, we have $h(t) \leq f(t)$ for all $t \in [0, b)$.

Proof Let f be an inextensible solution of (5.19) in an interval $[0, \tilde{b})$. Since $\varphi > 0$ per definition, we have

$$f(t) \geq f(0) \geq a, \quad \text{for all } t \in [0, \tilde{b}). \tag{5.20}$$

Therefore, $\varphi(f(t))(> 0)$ becomes strictly monotone increasing and is well defined for all $t \in [0, \tilde{b})$. Thus, dividing equation (5.19) by $\varphi(f(t))$ and integrating from 0 to t, where $t \in [0, \tilde{b})$, we get

$$\int_0^t \frac{f'(\tau)}{\varphi(f)(\tau)} d\tau = t,$$

hence, $f = f(t)$ is the inverse map of

$$t(f) = \int_{f(0)}^{f} \frac{ds}{\varphi(s)}. \tag{5.21}$$

Now, $\lim_{f \to +\infty} t(f)$ in (5.21) is equal to $\tilde{b}$, hence by the assumption on the term on the right-hand side of (5.21), we get $b = +\infty$, as required.

Now, let h be a function as in the statement and define another function v by

$$v(t) := h(0) + \int_0^t \varphi\big(h(s)\big)\, ds.$$

Note, that $v(0) = f(0)$ and $v'(t) = \varphi(f(t))$ for all $t \in [0, b)$ and v is continuous. Moreover, from the assumption on h it follows that

$$a \leq h(t) \leq v(t) \quad \text{for all } t \in [0, b), \tag{5.22}$$

while the monotonicity of φ implies $v'(t) \leq \varphi(f(t))$ for all $t \in [0, b)$. This, together with (5.19) and the assumptions φ gives

$$\frac{v'(t)}{\varphi(v(t))} \leq 1 = \frac{f'(t)}{\varphi(f(t))} \quad \text{for all } t \in [0, b).$$

Integrating on both sides gives

$$\int_{f(0)=v(0)}^{v(t)} \frac{ds}{\varphi(s)} \leq \int_{f(0)}^{f(t)} \frac{ds}{\varphi(s)} \quad \text{for all } t \in [0, b). \tag{5.23}$$

Now, assume that there exists a $t_0 \in [0, b)$ such that $v(t_0) > f(t_0)$. By the assumption on φ and inequality (5.20), we would have

$$\int_{f(0)}^{v(t_0)} \frac{ds}{\varphi(s)} > \int_{f(0)}^{f(t_0)} \frac{ds}{\varphi(s)},$$

in contradiction to (5.23). Thus, we have $v(t) \leq f(t)$ for all $t \in [0, b)$ and the claim follow by inequality (5.22). $\square$

Now, we can prove the autonomous version of theorem 5.2.6, more precisely:

Theorem 5.2.13 *(Completeness for autonomous potential)*
Let $(M, \langle ., . \rangle)$ be a complete Riemannian manifold, $H : M \to \mathbb{R}$ a smooth map and S the self-adjoint part of F, satisfying
$(a1)$ There exists a constant $c_\gamma > 0$, such that

$$\langle \gamma'(t), S\gamma'(t)) \rangle \leq c_\gamma \cdot \langle \gamma'(t), \gamma'(t)) \rangle, \quad \text{for all } t \in I.$$

$(a1')$ There exist $c_\gamma > 0$, such that

$$-\langle \gamma'(t), S\gamma'(t) \rangle \leq c_\gamma \langle \gamma'(t), \gamma'(t) \rangle, \quad \text{for all } t \in I.$$

Let $\gamma : I \to M$ be an inextensible solution of

$$\nabla_t \gamma'(t) = F_{\gamma(t)} \gamma'(t) + \frac{1}{2} gradH_{\gamma(t)},$$

and U_0 a positively complete function, such that

$$H\big(\gamma(t)\big) \leq U_0\Big(d\big(\gamma(t), p_0\big)\Big) \quad \text{for all } t \in I \text{ and a fixed } p_0 \in M.$$

Then γ must be complete.

Proof Analogously to the proof of theorem 5.2.9, we construct a contraction. So let $I = [0, b)$ with $b < \infty$ be the maximal interval of the inextensible forward solution. Again, we will show that $\|\gamma'\|$ is bounded (i.e. there exists $k > 0$, such that $u(t) \leq k$, with u defined as in the proof of theorem 5.2.9, hence $u(t) := \langle \gamma'(t), \gamma'(t) \rangle = \|\gamma'\|^2$).
For all $s \in I$, we have

$$\begin{aligned}
\frac{1}{2} \nabla_s \langle \gamma'(s), \gamma'(s) \rangle &= \langle \gamma'(s), \nabla_s \gamma'(s) \rangle \\
&= \langle \gamma'(s), F_{\gamma(s)} \gamma'(s) + \frac{1}{2} \mathrm{grad} H_{\gamma(s)} \rangle \\
&= \langle \gamma'(s), F_{\gamma(s)} \gamma'(s) \rangle + \langle \gamma'(s), \frac{1}{2} \mathrm{grad} H_{\gamma(s)} \rangle \\
&= \langle \gamma'(s), S_{\gamma(s)} \gamma'(s) \rangle + \langle \gamma'(s), \frac{1}{2} \mathrm{grad} H_{\gamma(s)} \rangle
\end{aligned}$$

On one hand, by assumption (a1), for all $t \in I$ we get

$$\frac{1}{2} \frac{d}{ds} \Big(u(s) - H\big(\gamma(s)\big) \Big) = \langle \gamma'(s), S_{\gamma(s)} \gamma'(s) \rangle \leq c_\gamma \cdot u(s). \tag{5.24}$$

Define v as the integral of u, that is

$$v(t) := \int_0^t u(s)ds, \quad \textit{thus } v'(t) = u(t)\ , v(0) = 0.$$

Then, for all $t \in I$, we get

$$v'(t) - 2c_\gamma v(t) \leq \alpha_\gamma + H\big(\gamma(t)\big), \tag{5.25}$$

where $\alpha_\gamma = u(0) - \frac{1}{2}H\big(\gamma(0)\big)$. On the other hand, define

$$l_\gamma(t) := d\big(p_0, \gamma(0)\big) + \int_0^t \sqrt{\langle \gamma'(s), \gamma'(s)\rangle} ds. \tag{5.26}$$

By the triangle inequality, we get (for all $t \in I$)

$$d\big(p_0, \gamma(t)\big) \leq d\big(p_0, \gamma(0)\big) + d\big(\gamma(0), \gamma(t)\big) \leq l_\gamma(t).$$

Together with (5.25) and the assumption on the positively complete function U_0, we obtain

$$v'(t) - 2c_\gamma v(t) \leq \alpha_\gamma - U_0\big(\gamma(t)\big) \leq \alpha_\gamma - U_0\big(l_\gamma(t)\big) \quad \textit{for all } t \in I,$$

where the last inequality holds, because U_0 is non-increasing.
Taking $\alpha > \max\{\alpha_\gamma, U_0\big(l_\gamma(t)\big)\}$, we even get

$$v'(t) - 2c_\gamma v(t) < \alpha - U_0\big(l_\gamma(t)\big) \quad \textit{for all } t \in [0, b) = I. \tag{5.27}$$

Note, that the right-hand side of this inequality is positive. Taking v_0 as the solution of the ODE (5.27) with equality ('=' instead of '<'), i.e. a solution of $v'(t) - 2c_\gamma v(t) = \alpha - U_0\big(l_\gamma(t)\big)$, we see, that v is a subsolution of v_0. Thus, applying the subsolution argument (lemma 5.2.8), we get

$$v(t) < v_0(t) \quad \textit{and} \quad u(t) = v'(t) < v_0'(t) \quad \textit{for all} \quad t \in (0, b).$$

Explicitly, v_0 can be written as

$$v_0(t) = e^{2c_\gamma t} \int_0^t e^{-2c_\gamma s}\Big(\alpha - U_0\big(l_\gamma(t)\big)\Big) ds\ , \quad \textit{with } v_0(0) = 0.$$

Thus, for its derivative, we get

$$v_0'(t) = 2c_\gamma e^{2c_\gamma t} \int_0^t e^{-2c_\gamma s}\Big(\alpha - U_0\big(l_\gamma(s)\big)\Big) ds + e^{2c_\gamma t}\Big(\alpha - U_0\big(l_\gamma(t)\big)\Big).$$

Choosing $t \in [0, b)$, then the properties of the positively complete function U_0 imply

$U_0\big(l_\gamma(s)\big) \geq U_0\big(l_\gamma(t)\big)$ for all $s \in [0,t)$ and hence

$$\int_0^t e^{-2c_\gamma s}\Big(\alpha - U_0\big(l_\gamma(s)\big)\Big)ds \leq \frac{1}{c_\gamma}\big(1-e^{-c_\gamma t}\big)\big(\alpha - U_0(\gamma(t))\big).$$

Using this and (5.27), we get

$$\begin{aligned} v_0'(t) &\leq e^{2c_\gamma t}\big(1-e^{-c_\gamma t}\big)\big(\alpha - U_0(\gamma(t))\big) + e^{2c_\gamma t}\Big(\alpha - U_0\big(l_\gamma(t)\big)\Big) \\ &\leq e^{2c_\gamma b}\big(\alpha - U_0(l_\gamma(t))\big). \end{aligned}$$

Note, that $\alpha - U_0\big(l_\gamma(t)\big)$ is positive, whence

$$\sqrt{u(t)} \leq \sqrt{e^{2c_\gamma b}\big(\alpha - U_0(l_\gamma(t))\big)} \quad \textit{for all } t \in [0,b). \tag{5.28}$$

On the other hand, we have by definition of l_γ in (5.26) that

$$\frac{dl_\gamma}{dt}(t) = \sqrt{u(t)} \quad \textit{and} \;\; l_\gamma(0) = d\big(\gamma(0), p_0\big).$$

Thus by (5.28), we get

$$\frac{dl_\gamma}{dt}(t) \leq \sqrt{e^{2c_\gamma b}\big(\alpha - U_0(l_\gamma(t))\big)} =: \varphi\big(l_\gamma(t)\big) \quad \textit{for all } t \in [0,b),$$

which implies

$$l_\gamma(t) \leq l_\gamma(0) + \int_0^t \varphi\big(l_\gamma(s)\big)ds \quad \textit{for all } t \in [0,b). \tag{5.29}$$

To end the prove, let $f(t)$ be the unique inextensible solution of the Cauchy problem:

$$\frac{df}{dt} = \varphi\big(f(t)\big), \quad \textit{with initial condition } f(0) = d\big(\gamma(0), p_0\big).$$

As U_0 is positively complete, a direct check of the properties of φ implies that the first part of lemma 5.2.12 applies, so f is defined for all $t \geq 0$. Moreover, from (5.29) and by the second part of lemma 5.2.12 it follows that $l_\gamma(t) \leq f(t)$, for all $t \in [0,b)$. Thus, $l_\gamma(t)$ is bounded in $[0,b)$ and so is $u(t)$ by inequality (5.28). Now, as required, we have shown a contradiction (to the assumption that γ is inextensible beyond b).
The case when $\gamma : (-b, 0] \to M$ is backward incomplete, follows analogously as in the proof of Corollary 5.2.10 (with property $(a1')$ instead of $(a1)$). □

The autonomous version of theorem 5.2.6 is now a straight forward consequence of the previous theorem, by use of the positive complete (quadratic) function $U_0 = -R_0 s^2$. More precisely:

Theorem 5.2.14 *(Autonomous version of theorem 5.2.6)*

Let $(M, \langle ., . \rangle)$ be a complete Riemannian manifold, S the self-adjoint component of a smooth time-independent $(1,1)$-tensor field F on M and $H : M \to \mathbb{R}$ a smooth time-independent scalar function.

1. *If S is bounded, i.e. $\|S\| < \infty$ and*
2. *H grows at most quadratically,*

then each inextensible solution of

$$\nabla_t \gamma'(t) = F_{\gamma(t)} \gamma'(t) + \frac{1}{2} gradH_{\gamma(t)}$$

must be complete. In particular, this is true if M is compact.

Now, we are in the position to prove the main theorems (i.e. the non-autonomous equations (5.8) and (5.14)), which we will do in the next subsection.

5.2.2 The non-autonomous case

The main theorems in section 5.2, theorem 5.2.4 and theorem 5.2.6, will be proven here in this subsection. To do so, we will reduce the non-autonomous case, given by:

$$\nabla_t \gamma'(t) = F_{(\gamma(t),t)} \gamma'(t) + X_{(\gamma(t),t)} \tag{5.30}$$

to the previous autonomous one (5.15), by working on the product manifold $M \times \mathbb{R}$, instead of M.
By ODE theory, we have the standard existence and uniqueness result of solutions to second order differential equations with initial conditions. In our case, for each $(v_p, t_0) \in TM \times \mathbb{R}$ ($p \in M$, $v_p \in T_pM$), we consider the unique inextensible solution $\gamma_{(v_p,t_0)}$ of (5.30), which satisfies $\gamma_{(v_p,t_0)}(t_0) = p$ and $\gamma'_{(v_p,t_0)}(t_0) = v_p$. As mentioned at the beginning of this section, we will now give a similar argument to the geodesic flow in Riemannian geometry, before we prove the main theorem 5.2.4.

Lemma 5.2.15 *(Unique solution flow)*
There exists a unique vector field G on $TM \times \mathbb{R}$ such that the curves $t \mapsto \left(\gamma'_{(v_p,t_0)}(t), t\right)$ are the integral curves of G.

Proof If such a G exists, it must be defined as

$$G_{(v_p,t_0)} = \frac{d^2}{dt^2}|_{t=t_0} \gamma_{(v_p,t_0)}(t) + \frac{\partial}{\partial t}|_{(v_p,t_0)}.$$

To check that its integral curves satisfy the required property, let us consider $\mathcal{D} \subseteq \mathbb{R} \times (TM \times \mathbb{R})$ to be the maximal domain of definition of the map:

$$\psi : \mathcal{D} \to TM \times \mathbb{R}$$

$$\big(s, (v_p, t_0)\big) \mapsto \big(\gamma'_{(v_p,t_0)}(t_0 + s), t_0 + s\big).$$

Recall, that $\mathcal{D} \cap (\mathbb{R} \times \{(v_p, t_0)\})$ is always open and contains $0 \times \{(v_p, t_0)\}$. Clearly, ψ defines a local action on $TM \times \mathbb{R}$ (namely, $\psi_{s+t} = \psi_s \circ \psi_t$ whenever the right- and left hand-side is defined), thus the result follows by ODE theory.

□

Proof of the main theorem 5.2.4:
Now, in order to be able to use the previous result (theorem 5.2.9) of the autonomous case (5.15), consider the trajectories of Equation (5.30) as some of the integral curves of a vector field $\tilde{G}$ on the tangent bundle $T(M \times \mathbb{R})$. First, note that the time-dependent tensor fields X and F on M naturally induces tensor fields $\tilde{X}$ and $\tilde{F}$ on the product Riemannian manifold $\tilde{M} := M \times \mathbb{R}$, namely:

$$\tilde{X}_{(p,t_0)} = \big(X_{(v_p,t_0)}, 0\big) \equiv X_{p,t_0}$$

$$\tilde{F}_{(p,t_0)}\Big(v_p, s\frac{d}{dt}|_{t=t_0}\Big) = \big(F_{(v_p,t_0)}(v_p), 0\big) \equiv F_{p,t_0}(v_p). \tag{5.31}$$

Second, consider the natural product Riemannian metric $\tilde{g} = g \oplus dt^2$ on $M \times \mathbb{R}$, and denote by $\tilde{\nabla}$ the corresponding covariant derivative, while $\tilde{\nabla}_t$ denotes the induced covariant derivative for vector fields along a curve on $\mathcal{M}$. To finally proof the main theorem 5.2.4, we need the following technical result, on the product manifold $M \times \mathbb{R} = \mathcal{M}$.

Proposition 5.2.16 *(Extended solutions on $M \times \mathbb{R}$)*
A curve $\tilde{\gamma}(t) = \big(\gamma(t), \tau(t)\big)$ on $M \times \mathbb{R}$ solves

$$\tilde{\nabla}_t \tilde{\gamma}'(t) = \tilde{F}_{\tilde{\gamma}}(t) \cdot \tilde{\gamma}'(t) + \tilde{X}_{\tilde{\gamma}(t)} \tag{5.32}$$

if and only if γ solves (5.30) and $\tau(t)$ is affine (i.e. $\tau(t) = at + b$ for some $a, b \in \mathbb{R}$). Therefore, if the inextensible solutions $\tilde{\gamma}$ of (5.32) are complete, then are the trajectories for the time-dependent equation (5.30).

Proof The first assertion follows from $\tilde{\nabla}_t \tilde{\gamma}' = (\nabla_t \gamma', \tau'')$ and equation (5.31).
For the second one, if γ is any inextensible solution of equation (5.30), the corresponding curve $\tilde{\gamma}(t) = (\gamma(t), t)$ is an inextensible solution of (5.32) and by assumption, $\tilde{\gamma}$ is complete and so is γ. □
Now we are in the position to prove theorem 5.2.4:

Proof of theorem 5.2.4
If a trajectory $\gamma : [0, b) \to M$ would be forward inextensible with $0 < b < +\infty$, then it should lie in a region such as $M \times [-T, T]$ with $b < T$. By proposition 5.2.16, a contradiction will follow by proving the extensibility to b of the trajectory $\tilde{\gamma}(t) = (\gamma(t), t)$ for $\tilde{M}, \tilde{g}, \tilde{X}$ and $\tilde{F}$. Recall, that in the region $M \times [-T, T]$, the g-bound assumptions

(5.10) on F (resp. for the self-adjoint part S) and (5.12) on X are equivalent to the corresponding $\tilde{g}$-bounds on $\tilde{F}$ and $\tilde{X}$ (note, that the $\tilde{g}$-distance can be easily bounded in terms of g and dt^2, and the distance on the dt^2 part is bounded by $2T$). Thus, forward extensibility of $\tilde{\gamma}$ follows from theorem 5.2.9, as required. Analogous arguments apply for proving backward completeness. □

5.2.3 Trajectories under the gradient of a non-autonomous potential

Throughout this subsection, the non-autonomous problem, where X is the gradient of some smooth function H, is studied. Again it is

$$\nabla_t \gamma'(t) = F_{\gamma(t),t}\gamma'(t) + \frac{1}{2}\mathrm{grad}_M H(\gamma(t), t). \tag{5.33}$$

In contrast to the previous theorem, the main result in this case (theorem 5.2.6) will not be reduced to the autonomous case. The reason is, that if we consider $\tilde{M}$, $\tilde{g}$ and $\tilde{F}$ as in the previous subsection and just put $\tilde{H} : M \times \mathbb{R}$ simply equal to H, then we have

$$\mathrm{grad}_{M\times\mathbb{R}}\tilde{H} = \big(\mathrm{grad}_M H, \frac{\partial H}{\partial t}\big).$$

But, the component $\frac{\partial H}{\partial t}$ makes the trajectories for $\mathrm{grad}_{M\times\mathbb{R}}\tilde{H}$ intrinsically different from those of $\mathrm{grad}_M H$, hence proposition 5.2.16 doesn't hold. Therefore, we will directly modify the proof of theorem 5.2.13.

Recall, that S is the self-adjoint part of F, while $H : M \times \mathbb{R} \to \mathbb{R}$ is a smooth map and we have the following assumptions of theorem 5.2.6:

(i) $|\frac{\partial H}{\partial t}|$ grows at most quadratically along finite times,

(ii) H grows at most quadratically along finite times,

(iii) S is bounded along finite times.

Proof of Theorem 5.2.6
We will show, that if (i), (ii) and (iii) hold, every inextensible solution of

$$\nabla_t \gamma'(t) = F_{\gamma(t),t}\gamma'(t) + \frac{1}{2}\mathrm{grad}_M H(\gamma(t), t) \quad \text{is complete.} \tag{5.34}$$

Let γ be an inextensible solution of the ODE (5.34), with $0 < b < +\infty$ included in some region $M \times [-T, T]$, where $b < T$. By (i) we have $N_T \in \mathbb{R}$, such that $\|S(t)\| < N_T$ for all $t \in [-T, T]$. By (iii) we have constants A_T and C_T determining the allowed quadratic growth of $\frac{\partial H}{\partial t}$. By this assumptions, together with the definition of l_γ in (5.26) (in the proof of theorem 5.2.13) and using the triangle inequality, we get

$$\frac{1}{2}\frac{d}{ds}\Big(u(s) - H\big(\gamma(s), s\big)\Big) \leq N_T u(s) + \frac{\partial H}{\partial s}\big(\gamma(s), s\big) \tag{5.35}$$
$$\leq N_T u(s) + A_T d^2\big(\gamma(s), p_0\big) + C_T \tag{5.36}$$
$$\leq N_T u(s) + A_T l_\gamma^2(s) + C_T. \tag{5.37}$$

As
$$\int_0^t l_\gamma^2(s)ds \leq T l_\gamma^2(t) \quad \text{for all } t \in [0, b),$$
integrating (5.37) changes Equation (5.25) from the proof of theorem 5.2.13 into

$$v'(t) - 2N_T v(t) \leq \alpha_\gamma + H\big(\gamma(t), t\big) + 2TA_T l_\gamma^2(t), \tag{5.38}$$

for all $t \in [0, b)$, with $\alpha_\gamma = u(0) - H\big(\gamma(0), 0\big) + TC_T$. Taking into account, also, the at-most-quadratic bound for H, we can choose constants $A, C > 0$ and construct the function $U_0(s) = -As^2 - C$ so that (5.38) yields:

$$v'(t) - 2N_T v(t) < \alpha - U_0(l_\gamma(t)) \quad \text{for all } t \in [0, b), \tag{5.39}$$

where α =max$\{\alpha_\gamma, U_0(0)\}$. Note that the inequality (5.39) is formally equal to (5.27) in the proof of theorem 5.2.13. So, arguing as in proof of theorem 5.2.13, we show the extensibility of γ through b, a contradiction.

On the other hand, if $\gamma : (-b, 0] \to M$ is a backward inextensible solution with $0 < b < +\infty$, we can consider $T > b$ and $\tilde{\gamma}(t) := \gamma(-t)$ in $[0, b)$ and, as from the lower boundedness of S and the quadratic growth of $\frac{\partial H}{\partial t}$ along finite times, it follows

$$\frac{1}{2}\frac{d}{ds}\Big(u(-s) - H\big(\gamma(-s), -s\big)\Big) = -\langle S_{(\gamma(-s),-s)}\gamma'(-s), \gamma'(-s)\rangle + \frac{\partial H}{\partial s}\big(\gamma(-s), -s\big)$$
$$\leq N_T u(-s) + A_T d^2\big(\gamma(-s), p_0\big) + C_T,$$

where from now on, we just repeat the previous proof for $\tilde{\gamma}(t)$.

□

5.2.4 Examples

To end this section, we discuss the optimality of the bounds for X or F in the main theorems in the next two examples (as in [CRS12]):

Example 1 (Optimality of bound for $\|X\|$)
We will discuss the assumption on X in theorem 5.2.9, where X grows at most linearly on $|\gamma|$. Thus, we have a scalar $r_\gamma > 0$, such that

$$\|X\|_{\gamma(t)} \leq r_\gamma(1 + |\gamma(t)|) \quad \text{for all } t \in I,$$

where $|\gamma(t)| := d\big(\gamma(t), p_0\big)$ is the Riemannian distance to some fixed point $p_0 \in M$.
We establish the optimality of the bound on $\|X\|$ by taking $M = \mathbb{R}$, $\langle .,. \rangle = dx^2$.
Put $F \equiv 0$ and the vector field $X(x) = \mu_\epsilon(x)\frac{d}{dx}$, where $\mu_\epsilon(x) = (1+\epsilon)x^{1+2\epsilon}$ for all $x \geq 1$ and some $\epsilon > 0$. Thus, in a region where $x(t) \geq 1$, equation (5.9) reads

$$x''(t) = (1+\epsilon)x^{1+2\epsilon}(t). \tag{5.40}$$

Multiplying by x', integrating with respect to t and considering the initial conditions $x(0) = 1,\ x'(0) = 0$, the solution of (5.40) satisfies

$$\big(x'(t)\big)^2 = x^{2(1+\epsilon)}(t) - 1.$$

The solution of this Cauchy problem is the inverse map of

$$t(x) = \int_1^x \frac{d\sigma}{\sqrt{\sigma^{2(1+\epsilon)} - 1}},$$

which is defined for $t \in [0, b)$, and the maximal b is

$$b = \lim_{x \to +\infty} t(x).$$

Thus, (5.40) is incomplete, whenever the power of the growth of $\|X\|$ becomes greater than the permitted linear one.

Example 2 (Optimality of the bound for $\|F\|$)
Again, take $M = \mathbb{R}$ and $\langle .,. \rangle = dx^2$. For some $\epsilon > 0$ put $F\frac{d}{dx} = \nu_\epsilon(x)\frac{d}{dx}$, with

$$\nu_\epsilon(x) = (1+\epsilon)x^\epsilon,$$

for $x \geq 1$ and $X \equiv 0$. Then equation (5.9) reduces to

$$x''(t) = (1+\epsilon)x^\epsilon(t)x'(t), \tag{5.41}$$

whenever $x(t) \geq 1$. Thus, integrating with respect to t and considering the initial conditions: $x(0) = x'(0) = 0$. Then, the solution of equation (5.41) satisfies

$$x'(t) = x^{1+\epsilon}(t).$$

Hence the solution of this Cauchy problem in this region is the inverse of

$$t(x) = \int_1^x \frac{d\sigma}{\sigma^{1+\epsilon}} \ ,$$

which shows the incompleteness of $x(t)$, as $\lim_{x\to+\infty} t(x) < +\infty$. That is, incompleteness appears, when $\|F\|$ is not bounded (even under slow power growth).

We also discuss the role of the bound on $\|F\|$ for forward/backward completeness here. For this, take $X \equiv 0$ and $F\frac{d}{dx} = -\mu(x)\frac{d}{dx}$, where μ is defined on all of $\mathbb{R}$ with $\mu(x) = |x|$ for $|x| > 1$. Equation (5.9) reduces to

$$x''(t) = -|x(t)|x'(t) \ , \text{ whenever } |x| > 1. \tag{5.42}$$

Since F is self-adjoint and satisfies assumption (a1) of theorem 5.2.9, it follows, that the solutions of (5.42) are forward complete. However, $x(t) = \frac{2}{t+1}$, with $t \in (-1, 0]$, yields a backward inextensible and incomplete solution of (5.42).

6 Examples of geodesically complete M-PFWs

In this section, we discuss examples of pp-waves, respectively M-PFWs, which are geodesically complete, using our completeness results from chapter 5. We also discuss the relation between the different results from section 5.2, again following [CRS13] and [CRS12].

6.1 Geodesic completeness of pp-waves

Recall, that pp-waves are the relativistic space-times on $\mathbb{R}^4$, endowed with the Lorentzian metric

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x, y, u)du^2,$$

where $(x, y, v, u) \in \mathbb{R}^4$. Now, theorem 5.1.3 shows, that geodesic completeness of that space-times is equivalent to the completeness of solutions $\gamma(u) = (x(u), y(u))$ for the non-autonomous problem on $\mathbb{R}^2$:

$$\gamma''(u) = \frac{1}{2}\mathrm{grad}_{\mathbb{R}^2} H\big(\gamma(u), u\big).$$

Therefore, we can use our results of section 5.2.
In the particular case of plane waves (see definition 3.1.1), the expression of H is quadratic in x and y, that is

$$H(x, y, u) = f_{11}(u)x^2 - f_{22}y^2 + 2f_{12}xy, \tag{6.1}$$

for some smooth functions f_{11}, f_{12} and f_{22}. Although, completeness of geodesic in this case was already known (since direct integration is possible) or the result in theorem 5.1.6 from the 70's, we can now also easily see this by our main results in section 5.2. Since $\frac{1}{2}H$ grows at most quadratically along finite times, we can use theorem 5.2.6 and on the other side, we can use theorem 5.2.4, since the gradient $\frac{1}{2}\mathrm{grad}_{\mathbb{R}^2} H\big(\gamma(u), u\big)$ grows at most linearly. In both, cases we again see, that plane waves are geodesically complete. Moreover

Theorem 6.1.1 *(Geodesic completeness of pp-waves)*
Any pp-wave, such that the metric behaves qualitatively as in (6.1) is geodesically complete.

Now, we will study geodesic completeness of M-PFWs, where we will see familiar results to those of plane waves, respectively pp-waves, deduced from our results in section 5.2. But, we will also discuss some differences that appear, by using our results for the more general M-PFWs.

6.2 Geodesic completeness of M-PFWs

Again, in order to study geodesic completeness of a Lorentzian manifold $\mathcal{M} = M \times \mathbb{R}^2$, where M is a complete Riemannian manifold, we know by theorem 5.1.3, it is sufficient to study the completeness of the inextensible solutions of:

$$\nabla_t \gamma'(t) = \frac{1}{2} \mathrm{grad}_M H\big(\gamma(t), t\big), \tag{6.2}$$

where H is the smooth potential (according to definition 3.2.1) and γ a smooth curve on $\mathcal{M}$. Therefore, the main theorem in the general *non-autonomous case* form section 5.2, which is theorem 5.2.4, gives the following result for geodesic completeness of M-PFWs:

Corollary 6.2.1 *(Geodesic completeness of M-PFWs, part 1)*
A M-PFW $\mathcal{M} = M \times \mathbb{R}^2$ is geodesically complete if M is a complete Riemannian manifold and the gradient of H, with respect to the Riemannian part M $grad_M H(x,t)$, grows ***at most linearly*** *along finite times.*

Note, that there are no assumptions on the smooth function H itself, but just on its M-gradient. Let us take a look of how theorem 5.2.6, for the *non-autonomous potential*, can help us. We first discuss the relation between corollary 6.2.1 and theorem 5.2.6, which in our case gives the following result for ODE (6.2) above, and hence for M-PFWs:

Corollary 6.2.2 *(Geodesic completeness of M-PFWs, part 2)*
A M-PFW $\mathcal{M} = M \times \mathbb{R}^2$ is geodesically complete if M is a complete Riemannian manifold and H grows ***at most quadratically*** *along finite times, as well as both, $\frac{\partial H}{\partial t}$ and $-\frac{\partial H}{\partial t}$.*

To see the relation between corollary 6.2.1 and corollary 6.2.2 (resp. theorem 5.2.6), consider the following Example:

In corollary 6.2.1, we have the assumption, that $\mathrm{grad}_M H$ grows at most linearly along finite times (i.e. $\exists\ \alpha, \beta : [0, \infty) \to [0, \infty]$ (at least C^1), such that

$$\mathrm{grad}_M H\big(x(t), t\big) \le \alpha(t) \cdot d\big(x(t), t\big) + \beta(t), \quad \text{for all } (x(t), t) \in M \times \mathbb{R}.$$

Recall from Riemannian geometry, that by Hopf-Rinow (theorem 2.1.2), the exponential map $\exp_{p_0}$ is defined for all $v \in T_{p_0}M$, for a p_0 in a complete Riemannian manifold M. For each $p \in M$ let $\gamma_p : [0, l_p] \to M$ be the unique geodesic, parametrised by arclength, from p_0 to p (i.e. $l_p = d(p, p_0)$). Therefore,

$$\begin{aligned} H(p,l_p) - H(p_0,0) &= \int_0^{l_p} \frac{dH}{ds}(\gamma_p(s),s)ds \\ &= \int_0^{l_p} \langle \operatorname{grad}_M H(\gamma_p(s),s), \gamma_p'(s)\rangle_M + \frac{\partial H}{\partial s}(\gamma_p(s),s)ds \\ &\leq \int_0^{l_p} \|\operatorname{grad}_M H(\gamma(s),s)\| ds + \int_0^{l_p} \frac{\partial H}{\partial s}(\gamma_p(s),s)ds \\ &\leq \alpha(l_p)l_p^2 + C(l_p)l_p + \int_0^{l_p} \frac{\partial H}{\partial s}(\gamma_p(s),s)ds. \end{aligned}$$

We see, that in the case of H autonomous (i.e. $\frac{\partial H}{\partial s} = 0$), that corollary 6.2.1 is a special case of corollary 6.2.2 (resp. theorem 5.2.6), while in the non-autonomous case, both results are independent, since corollary 6.2.1 doesn't have a bound on $\frac{\partial H}{\partial s}$ as needed in corollary 6.2.2. Moreover, a bound on $\frac{\partial H}{\partial s}$ is independent of the relation between the bound of H or between the bound on the gradient $\operatorname{grad}_M H$.

We give a further result, similar to corollary 6.2.2, taken from [CRS13] about completeness of curtain ODEs, with a stronger restriction on H and an alternative bound for $-\frac{\partial H}{\partial s}$ resp. $\frac{\partial H}{\partial s}$. A proof can be reconstructed by an analogous argument as in the proof of theorem 5.2.6 (or see: [CRS12], Theorem 2.3. for a proof in detail).

Corollary 6.2.3 *(Geodesic completeness of M-PFWs, part 3)*
A M-PFW $\mathcal{M} = M\times\mathbb{R}^2$ with M a complete Riemannian manifold is geodesically complete if there exists two C^1-functions $\alpha,\beta : \mathbb{R}\to\mathbb{R}$, such that

1. $H(x,u) \leq \beta_(u)$ *and*
2. $|\frac{\partial H}{\partial u}| \leq \alpha(u)\cdot\big(\beta(u) - H(x,u)\big)$,

for all $(x,u)\in M\times\mathbb{R}$.

Note that the conditions of this corollary also allows the following property of H: *If H grows fast (for example in a superquadratic way), then it also allows such a rapid growth for $\frac{\partial H}{\partial u}$, respectively $-\frac{\partial H}{\partial u}$.*

6.3 Ehlers-Kundt conjecture

We will mention a conjecture which fits in this context, namely the Ehlers-Kundt conjecture. It is a physical assertion about the fundamental role of plane waves for the description of gravitational waves. Ehlers and Kundt proved in [EK62], that pp-waves are geodesically complete (see: theorem 6.1.1). Also in this seminal paper, they posed the following question:

Are plane waves the only complete gravitational pp-waves (i.e. Ricci flat), no matter which topology one choose?

Mathematically, it becomes equivalent to a NNewtonian problemön the Euclidean plane $\mathbb{R}^2$ with a simple formulation:

Given a non-necessarily autonomous potential $H(x,u)$, $(x,u) \in \mathbb{R}^2 \times \mathbb{R}$, harmonic and a polynomial in $x = (x_1, x_2)$ (i.e. source-free), then the trajectories of its associated dynamical system

$$x''(s) = -\mathrm{grad}_x H(x(s), s)$$

are complete if and only if $H(x_1, x_2, u)$ is of degree at most 2.

This conjecture is open to this day. A partial positive resolution is given in [FS18], where $H(x,u)$ is assumed to be a polynomial bounded for finite values of u.

7 Causality

In this section, we discuss the exact position of M-PFWs on their causal ladder. We already know from proposition 4.2.4, that the smooth function $H(x,u)$ from the metric on the M-PFW $\mathcal{M}$ has subquadratic growth, if $\mathcal{M}$ is asymptotically vanishing (according to (4.11)). Here, in this section, we focus on the 'opposite direction', where we study, how different assumptions on H determines the causal character of $\mathcal{M}$.
First, we will look at plane waves (see definition 3.1.1), to model gravitational waves or electromagnetic waves, where $M = \mathbb{R}^2$ and $H(x,u) = \sum_{i,i} h_{ij}(u)x_i x_j \big(\neq 0\big)$, for functions $h_{ij}(u)$ (at least C^2). This spacetimes are known to be strongly causal, but not globally hyperbolic, if H has quadratic behaviour, see [EK62]. We will discuss the *remarkable Property of Plane Waves in General Relativity*, proven by Roger Penrose in his seminal paper [Pen65b], which shows the non-existence of a Cauchy hypersurface for (sandwich) plane waves, hence, they are not globally hyperbolic.
Second, we will see, how stronger (or less stronger) assumptions on H, determine the causal character of M-PFWs, following [FS03] and [FS06]. We will see, that a quadratically behaviour of H, will still guarantee strong causality and a subquadratic growth will ensure, that $\mathcal{M}$ is even globally hyperbolic, while in the other direction (superquadratic growth of H), causality gets totally destroyed, such that we can't even guarantee the distinguishing property for a M-PFW. At the end of this final section, we will discuss and summarize our results.

7.1 A remarkable property of plane waves

Roughly, Penrose showed in [Pen65b], that plane waves (especially he showed it for sandwich waves see 3.1.1) have a certain focusing property on null geodesics, which as a consequence leads to the non-existence of a Cauchy hyperserface. Therefore, they are not globally hyperbolic. Let us define the relevant objects, to formulate the focusing property:

Again, as in definition 3.1.1, a plane wave is a 4-dimensional pp-wave in adapted coordinates $(x_1, x_2, v, u) \in \mathcal{M} = \mathbb{R}^4$, where H is quadratic in x_1 and x_2. That is, the space-time $\big(\mathcal{M} = \mathbb{R}^4, \langle .,. \rangle\big)$, where the metric takes the form

$$ds^2 = dx_1^2 + dx_2^2 + 2dudv + H(x_1, x_2, u)du^2,$$

while we can write H (since it is quadratic in x_1 and x_2) as

$$H(x_1, x_2, u) = h_{11}(u)x_1^2 + h_{12}(u)x_1 x_2 + h_{22}(u)x_2^2 \ ,$$

where $h_{ij}(u)$ is a symmetric matrix of smooth functions. We call it a *sandwich wave*, if in addition, the gravitational or electromagnetic amplitudes $h_{ij}(u)$ vanish outside an interval (a, b), hence the space-time is flat for $u < a$ and $u > b$ (see Figure 7.1 on the left). We also define the null cone of a point:

Definition 7.1.1 *(Null cone)*
The null cone $k_3 \subset \mathcal{M}$ at a point $Q \in \mathcal{M}$ is defined to be the set of points lying on all null geodesics through Q.

Now, we have all terminology to formulate the lemma, which as a consequence leads to the non-existence of Cauchy hypersurfaces for plane waves.

Lemma 7.1.2 *(Focusing null cones of sandwich waves)*
The past (or future) null cone of any point R in a plane wave $\mathcal{M}$ is forced to a single point except for one line $\mathcal{R}_1$ (see figure 7.1 on the right).

Proof A proof can be found in [Pen65b]. □

To see, why this result shows the non-existence of a Cauchy hypersurface, consider a candidate for a Cauchy hypersurface C (i.e. a connected spacelike surface without boundary, where every null-geodesic of $\mathcal{M}$ intersect it exactly once) threw a point Q in the plane wave $\mathcal{M}$. Such a hypersurface has to intersect also the null line $\mathcal{R}_1$ throw R. To see, that this is not possible, we visualized the situation in figure 7.1 on the right. The surface C threw Q must lie 'below' the null cone of Q (i.e. in the past of Q). By the above lemma 7.1.2, this cone focus again at a point R. Now, if the hypersurface C would intersect the line $\mathcal{R}_1$, or any of the propagation lines beyond $\mathcal{R}_1$. But C it cannot cross the null cone and remain spacelike everywhere. Thus, we arrive at a contradiction and therefore, no Cauchy hypersurface exists, hence the wave is not globally hyperbolic.

7.2 The causal ladder for M-PFWs

We discuss the position on the causal ladder for M-PFWs in this subsection. The following figure visualizes the causal hierarchy of spacetime from proposition 2.2.8.
Since Penrose showed, that plane waves are not globally hyperbolic, it became interesting to study the exact position of pp-waves and also the more general M-PFWs, on the causal ladder. This was done in great detail by P. E. Ehrlich and G. G. Emch in [EE92] and [EE93] in the case of pp-waves and for M-PFWs by J.L. Flores and M. Sanchez in [FS03] and in [FS06]. We collect their results in this section, but before we do so. We begin with the causal classification of M-PFWs (with respect to the causal ladder, see proposition 2.2.8 respectively figure 7.2), starting with some simple results.

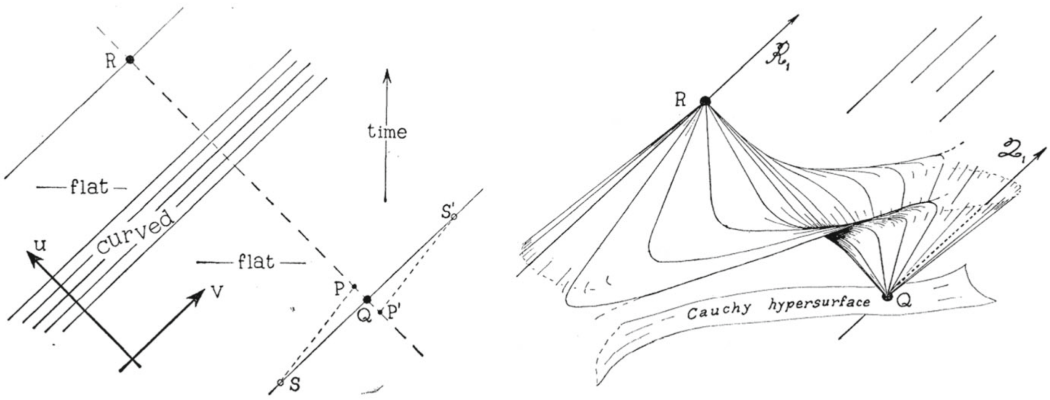

Abbildung 7.1: The picture on the left visualises the profile of a sandwich wave in which the u-coordinate range of the 'curved' region is an bounded interval. The graphic on the right visualises the focusing property of the future null cone. Both figures taken from [Pen65b].

Proposition 7.2.1 *(Chronological)*
All M-PFWs are chronological.

Proof By Proposition 4.0.2, any future directed causal curve $z(s) = \big(x(s), v(s), u(s)\big)$ satisfies the following inequality:

$$\langle \dot{z}(s), \partial_v \rangle = \dot{u}(s) \geq 0,$$

where $\partial_v = \text{grad}(u)$ is the covariantly constant, null vector field, metric equivalent to the one-form du. Such an inequality prevents the existence of closed timelike curves, hence the spacetime is chronological. $\square$

The next result shows, that a M-PFW is also causal. This will be done (proof taken from [EE92]), by showing, that any M-PFW admits a so called *quasi-time function*, which implies that it is causal. More precisely:

Definition 7.2.2 *(Quasi-time function)*
Let $\big(\mathcal{M}\,\langle .,. \rangle\big)$ be a Lorentzian Manifold. A smooth function $f : \mathcal{M} \to \mathbb{R}$ is called a quasi-time function on $\mathcal{M}$ if the following two item hold

(i) ∇f is everywhere causal and past-directed,

(ii) Every null geodesic segment γ such that $f \circ \gamma$ is constant, is injective.

Proposition 7.2.3 *(Quasi-time function vs. causal)*
Any spacetime admitting a quasi-time function is causal.

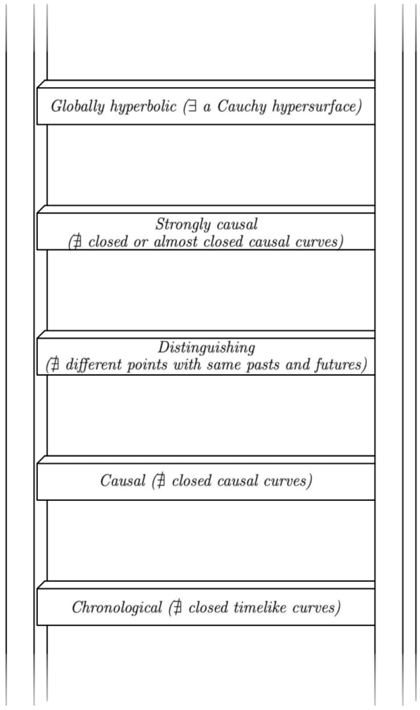

Abbildung 7.2: Causal ladder. Every step up in the ladder is a special case of the rung before.

Proof Assume that f is a quasi-time function. Then by *(i)* we have that f is strictly increasing along all future-directed timelike curves in $\mathcal{M}$, hence $\big(\mathcal{M}\,\langle .,.\rangle\big)$ is chronological. Now we will prove causality by contradiction. For this, assume that $\big(\mathcal{M}\,\langle .,.\rangle\big)$ is not causal. Then $\mathcal{M}$ would contain ([EE92],4.10) a non-trivial, smooth, future-directed null geodesic segment $\tilde{\gamma} : [0,1] \to \mathcal{M}$ with $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ and $\dot{\tilde{\gamma}}(0) = \dot{\tilde{\gamma}}(1)$. Furthermore $\tilde{\gamma}$ may be extended to an inextensible geodesic $\gamma : \mathbb{R} \to \mathcal{M}$ by $\gamma(s) = \tilde{\gamma}(s \bmod 1)$. Again because of *(i)* and by continuity of all the relevant properties, f is non-decreasing along γ, hence $f \circ \gamma(s) = \gamma_0$ for all $s \in \mathbb{R}$, for a constant $\gamma_0 \in \mathbb{R}$, which would contradict *(ii)*, since $\gamma(0) = \gamma(1)$. Thus, $\big(\mathcal{M}\,\langle .,.\rangle\big)$ has to be causal. $\square$

Theorem 7.2.4 *(Causal)*
All M-PFWs are causal.

Proof By the last proposition, it is enough to show that there exists a quasi-time function on any M-PFW. We will, show that the coordinate function u is a such a function.
To see *(i)* from the definition of a quasi-time function, recall, that by proposition 4.0.2, we have that $\operatorname{grad}(u) = \partial_v$, which is the covariantly constant null vector field and by definition it is past directed. Next, note that the restriction of the metric to the null hypersurface

$$\Pi_{u_0} := u^{-1}(u_0), \quad \text{for some } u_0 \in \mathbb{R},$$

is independent of the smooth function H and M is spacelike. Thus, the null geodesic generator of Π_{u_0} are of the form

$$\gamma : v \in \mathbb{R} \mapsto (x_0, v, u_0) \in \Pi_{u_0},$$

which is clearly injective. Since all other geodesics in Π_{u_0} are spacelike, *(ii)* in the definition of a quasi-time function holds too. □

7.2.1 M-PFWs are not necessarily distinguishing

We already know, that M-PFWs cannot contain closed timelike curves, since by proposition 7.2.1 they are chronological and by theorem 7.2.4 they cannot contain closed causal curves. Nevertheless, M-PFWs are not necessarily strongly causal, as the next proposition shows. This result also provides examples of M-PFWs, which are not even distinguishing, following [FS03].

Proposition 7.2.5 *(M-PFWs are not necessarily distinguishing)*
A M-PFW $\mathcal{M} = M \times \mathbb{R}^2$ is non-distinguishing if M is complete, H is non-positive (i.e. $H \leq 0$) and $-H$ has ***superquadratic growth*** *in the following sense: There exists a sequence $\{y_n\}_n \subseteq M$, with $d(y_n, \tilde{x}) \to \infty$ for a fixed $\tilde{x} \in M$, such that*

$$-H(y_n, u) \geq R_1 d^{2+\varepsilon}(y_n, \tilde{x}) + R_2,$$

for all $u \in \mathbb{R}$, with scalars $R_1, \varepsilon > 0$, $R_2 \in \mathbb{R}$.

Proof We will show, that we can find distinct points $z_0 \neq z_1 \in \mathcal{M}$, possessing the same future, i.e. $I^+(z_0) = I^+(z_1)$. Thus, fixing $z_0 = (x_0, v_0, u_0) \in \mathcal{M}$, it is enough to show that

$$I^+(z_0) = \{(x, v, u) : x \in M, v \in \mathbb{R}, u_0 < u\} = M \times \mathbb{R} \times (u_0, \infty), \quad \text{for all } x_0 \in M, v_0 \in \mathbb{R}.$$

Recall, as in proposition 4.1.1, that for any future-directed timelike curve $z(s) = \big(x(s), v(s), u(s)\big)$, we always have $\dot{u}(s) > 0$, and thus,

$$I^+(z_0) \subseteq M \times \mathbb{R} \times (u_0, \infty).$$

In the reminder of this proof, we will show the converse, that is, for fixed $z_1 = (x_1, v_1, u_1)$ with $u_0 < u_1$, we will construct a piecewise smooth, future directed and timelike curve

$$z(s) = \big(x(s), v(s), u_0 + s\Delta u\big),$$

(where $\Delta u \coloneqq u_1 - u_0$ and $s \in [0,1]$), from z_0 to z_1. Therefore, for $0 < \delta < 1$ and any natural number n, we define the piecewise smooth curve

$$x_n(s) \coloneqq \begin{cases} \alpha_n(s) & \text{if} \quad s \in [0, \frac{1}{2} - \frac{\delta}{2}] \\ y_n & \text{if} \quad s \in [\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} + \frac{\delta}{2}] \\ \beta_n(s) & \text{if} \quad s \in [\frac{1}{2} + \frac{\delta}{2}, 1], \end{cases} \tag{7.1}$$

where $\alpha_n : [0, \frac{1}{2} - \frac{\delta}{2}] \to M$ and $\beta_n : [\frac{1}{2} + \frac{\delta}{2}, 1] \to M$ are minimizing geodesics for the Riemannian distance on M joining x_0 with y_n and y_n with x_1 respectively.

Now, for any arbitrary function $v_n(s)$, the piecewise smooth curve

$$z_n(s) = \big(x_n(s), v_n(s), u_0 + s\Delta u\big)$$

joins z_0 with (x_1, v, u_1) for some (uncontrolled) v.
To obtain a constant speed timelike curve $v_n(s)$, we use the expression of v for affine u in theorem 5.1.1, by taking the constant energy $E < 0$, to get

$$v_n(s) - v_0 = \frac{1}{2\Delta u} \int_0^s \big(E - \langle \dot{x}_n(\sigma), \dot{x}_n(\sigma)\rangle_M - (\Delta u)^2 \cdot H(x_n(\sigma), \sigma)\big) d\sigma. \tag{7.2}$$

Note, that even though $z_n(s)$ is only piecewise smooth, it satisfies $\langle \partial_v, \dot{z}_n(s)\rangle = \dot{u}(s) = \Delta u > 0$ by proposition 4.0.2, hence it is future directed. Therefore, using (7.1), the assumption on the growth on $-H$ and the fact that $H(x,u) \leq 0$, implies

$$\begin{aligned} v_n(1) - v_0 \geq{}& \frac{E}{2\Delta u} - \frac{1}{2\Delta u}\Big(\int_0^{\frac{1}{2}-\frac{\delta}{2}} \langle \dot{x}_n(\sigma), \dot{x}_n(\sigma)\rangle_M d\sigma + \int_{\frac{1}{2}+\frac{\delta}{2}}^1 \langle \dot{x}_n(\sigma), \dot{x}_n(\sigma)\rangle_M d\sigma\Big) \\ &- \frac{\Delta u}{2} \int_{\frac{1}{2}-\frac{\delta}{2}}^{\frac{1}{2}+\frac{\delta}{2}} H\big(y_n, \sigma\big) d\sigma \\ \geq{}& \frac{E}{2\Delta u} - \frac{d^2(y_n, x_0) + d^2(y_n, x_1)}{2\Delta u} + \frac{\Delta u}{2}\delta(R_1 d^{2+\varepsilon}(y_n, \tilde{x}) + R_2). \end{aligned} \tag{7.3}$$

Note, that for fixed $n \in \mathbb{N}$, v_n goes to $-\infty$, for $E \to -\infty$. On the other hand, we see by (7.3), that for fixed $E \in (-\infty, 0)$, $v_n(1)$ goes to $+\infty$ for $n \to +\infty$. Therefore, for large enough n and varying E, the value of $v_n(1)$ can reach v_1, as required.

$\square$

This proposition includes the following **Example:**
Take $M = \mathbb{R}^2$ and $H(x,u) = -\|x\|^{2+\varepsilon}$, $\varepsilon > 0$ (if required, H can be modified around $x = 0$ in order to obtain a smooth function).

Now, since we have seen, how bad causality can get, we will give criteria for strong causality or even global hyperbolicity for M-PFWs in the next two subsections. Again, we will see, that only assumptions on H, will effect the causal behaviour of M-PFWs.

7.2.2 A sufficient hypothesis for strong causality

In this section, we will show, how the quadratic behaviour of $-H$ at spatial infinity is enough, to ensure strong causality. Before we do so, we need two lemmas, which we will also use for the subquadratic case in section 7.2.3 below.

Lemma 7.2.6 *(Energy correspondence)*
Let M be a connected Riemannian manifold and fix an $\varepsilon > 0$. Then, for any piecewise smooth curve $x : [u_0, u_1] \to M$ with $0 < u_1 - u_0 < \varepsilon$, the following inequality holds:

$$\int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s) \rangle ds \geq \frac{1}{\varepsilon} \int_{u_0}^{u} d^2\big(x(s), x(u_0)\big) ds, \quad \textit{for all } u \in [u_0, u_1]. \tag{7.4}$$

Proof First note, that for all $u \in [u_0, u_1]$, by Cauchy-Schwarz inequality, we get

$$d\big(x(u), x(u_0)\big) \leq \int_{u_0}^{u} \sqrt{\langle \dot{x}(s), \dot{x}(s) \rangle} ds \leq \sqrt{u - u_0} \cdot \sqrt{\int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s) \rangle ds}$$

and therefore,

$$d^2\big(x(u), x(u_0)\big) \leq (u - u_0) \int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s) \rangle ds$$

Note, that this is also true for $s \in [u_0, u]$, hence

$$d^2\big(x(s), x(u_0)\big) \leq (s - u_0) \int_{u_0}^{s} \langle \dot{x}(\bar{s}), \dot{x}(\bar{s}) \rangle d\bar{s} \leq (u - u_0) \int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s) \rangle ds$$

Thus, integrating (in s) the left and right terms of the above inequality yields

$$\int_{u_0}^{u} d^2\big(x(s), x(u_0)\big) ds \leq (u - u_0)^2 \int_{u_0}^{s} \langle \dot{x}(\bar{s}), \dot{x}(\bar{s}) \rangle d\bar{s} \leq \varepsilon^2 \int_{u_0}^{u} \langle \dot{x}(s), \dot{x}(s) \rangle ds$$

and we arrive at (7.4). $\square$

This lemma basically shows, the smaller we choose $\varepsilon > 0$, the greater the the integral of the energy of a u-parameter curve x on a Riemannian manifold M gets (with respect to the integral of the square distance of $x(s)$). The next crucial lemma, which will be the key point for the following two main results on causality, basically says, that for causal curves $\alpha(u) = \big(x(u), v(u), u\big)$ on a M-PFW $\mathcal{M} = M \times \mathbb{R}^2$, with domain $[u_0, u_0 + \varepsilon]$ and fixed $x(u_0) = x_0$, the integral of the energy of x-component is bounded above by the extremes of the other components of α. Moreover, the bound is independent of the chosen curve, if $-H$ behaves quadratically, resp. subquadratically. More precise

Lemma 7.2.7 *(Control between endpoints)*
Let M be a connected Riemannian manifold and let $H : M \times \mathbb{R} \to \mathbb{R}$ be a smooth function. Assume, that $-H$ behaves at most quadratically and fix $u_0 \in \mathbb{R}$ and a bounded subset $B \subset M$.
Then there exist $\varepsilon > 0$ and $R_2' > 0$, such that, for any u-parametrized causal curve $\alpha(u) = \big(x(u), v(u), u\big)$, $u \in [u_0, u_1]$ with $0 < u_1 - u_0 \leq \varepsilon$ and $x(u_0) = x_0 \in B$, we have

$$v(u) - v(u_0) - R_2'(u - u_0) < -\frac{1}{4}\int_{u_0}^{u} \langle \dot{x}(\tilde{u}), \dot{x}(\tilde{u})\rangle d\tilde{u} \leq 0. \tag{7.5}$$

Moreover, if $-H$ behaves subquadratically, then the same conclusion holds for any $\varepsilon > 0$ (i.e. For all $\epsilon > 0$ there is a $R_2'(\varepsilon) > 0$, such that (7.5) holds for any such curve α).

Remark 7.2.8 *(The endpoint case)*
The constants ε, $R_2' > 0$ in the above lemma can be chosen such that, for fixed $u_0 \in \mathbb{R}$ and for any causal curve $\alpha(u) = \big(x(u), v(u), u\big)$, $u \in [u_0, u_1]$ in $\mathcal{M}$, with $u_1 - u_0 \leq \varepsilon$ and $x(u_1) = x_1$ in a bounded subset $B_1 \subseteq M$ (instead of $x(u_0) = x_0 \in B$), one also has

$$v(u_1) - v(u) - R_2'(u_1 - u) < -\frac{1}{4}\int_{u}^{u_1} \langle \dot{x}(\tilde{u}), \dot{x}(\tilde{u})\rangle d\tilde{u} \leq 0. \tag{7.6}$$

Proof For the energy of the u-parametrized curve α define

$$E_\alpha(u) := \langle \dot{\alpha}(u), \dot{\alpha}(u)\rangle = \langle \dot{x}(u), \dot{x}(u)\rangle + 2\dot{v}(u) + H(x, u).$$

As in the proof of theorem 5.1.1 we get

$$v(u) - v(u_0) = \frac{1}{2}\int_{u_0}^{u} \big(E_\alpha(\tilde{u}) - \langle \dot{x}(\tilde{u}), \dot{x}(\tilde{u})\rangle - H(x(\tilde{u}), \tilde{u})\big) d\tilde{u} \quad \text{for all } u \in [u_0, u_1]. \tag{7.7}$$

On the other hand, the at most quadratic growth of $-H$, which is

$$-H(x, u) \leq R_1(u) d^2(x, x_0) + R_2(u), \tag{7.8}$$

gives

$$-\int_{u_0}^{u} H(x(\tilde{u}), \tilde{u}) d\tilde{u} \leq R_1^{\max} \int_{u_0}^{u} d^2(x(\tilde{u}), x_0) d\tilde{u} + R_2^{\max}(u - u_0), \tag{7.9}$$

where $R_i^{\max}$ are the maximum of the $R_i(u)$'s in $[u_0, u_0 + \varepsilon]$. Thus, choosing $\varepsilon > 0$ small enough and taking into account, that x_0 belongs to the bounded set $B \subseteq M$, we get (for some $R_2^{'} > 0$ and all $u \in [u_0, u_1]$)

$$-\int_{u_0}^{u} \left(H(x(\tilde{u}), \tilde{u}) + R_2^{'}\right) d\tilde{u} < \frac{1}{2\varepsilon^2} \int_{u_0}^{u} d^2(x(\tilde{u}), x_0) d\tilde{u} \le \frac{1}{2} \int_{u_0}^{u} \langle \dot{x}(\tilde{u}), \dot{x}(\tilde{u}) \rangle d\tilde{u}, \quad (7.10)$$

where the second inequality follows by lemma 7.2.6. Therefore, we obtain

$$-\frac{1}{2} \int_{u_0}^{u} \Big(\langle \dot{x}(\tilde{u}), \dot{x}(\tilde{u}) \rangle + H\big(x(\tilde{u}), \tilde{u}\big) \Big) d\tilde{u} < R_2^{'}(u - u_0), \quad \text{for all } u \in [u_0, u_1]. \quad (7.11)$$

Now, the result for the quadratic case follows by using (7.11) and (7.7) and taking into account that $E_\alpha \le 0$.

For the subquadratic case, (where we have to replace 2 by $p(u) < 2$ in (7.8)), note, that we can't directly get inequality (7.11) with $p < 2$ instead of 2, because $d(x(u), x_0)^2 < d(x(u), x_0)^p$ for $d(x(u), x_0) \le 1$. Therefore, observe that for $u \in [u_0, u_0 + \varepsilon]$ we have

$$d^{p(u)}\big(x(u), x_0\big) \le \begin{cases} 1 & \text{if } d(x(u), x_0) \le 1 \\ d^p(x(u), x_0) & \text{else} \end{cases},$$

where $p = \max_{u_0 \le u \le u_0 + \varepsilon} p(u)$ and thus, instead of (7.8), we get

$$-H(x, u) \le R_1(u) d^p(x(u), x_0) + R_1(u) + R_2(u),$$

and so we obtain again (7.9), with $R_1^{\max}$ as above and $R_2^{\max} = \max_{[u_0, u_0+\varepsilon]} \{R_1(u) + R_2(u)\}$.
Now, assume that the desired estimate is wrong, i.e.

$$\frac{1}{4\varepsilon^2} d^2(x(u), x_0) < R_1^{\max} d^p(x(u), x_0). \quad (7.12)$$

Note, that for fixed $\varepsilon > 0$, also $R_1^{\max}$ is fixed. This gives

$$d^{2-p}(x, x_0) < 4\varepsilon^2 R_1^{\max},$$

which implies $d(x, x_0) < (4\varepsilon^2 R_1^{\max})^{\frac{1}{2-p}}$. So we have

$$R_1^{\max} d^p\big(x(u), x_0\big) \le \begin{cases} (2\varepsilon R_1^{\max})^{\frac{2p}{2-p}} & \text{in case of (7.12)}, \\ \frac{1}{4\varepsilon^2} d\big(x(u), x_0\big) & \text{else} \end{cases}$$

From this, we get

$$\begin{aligned} R_1^{\max} \int_{u_0}^{u} d^p\big(x(\tilde{u}), x_0\big) d\tilde{u} &\le \frac{1}{4\varepsilon^2} \int_{u_0}^{u} d^2\big(x(\tilde{u}), x_0\big) d\tilde{u} + \big(2\varepsilon R_1^{\max}\big)^{\frac{2p}{2-p}} \\ &\le \frac{1}{4\varepsilon^2} \int_{u_0}^{u} 2\Big(d^2\big(x(\tilde{u}), x_0\big) + d^2\big(x_0, x(\tilde{u})\big)\Big) d\tilde{u} + \big(2\varepsilon R_1^{\max}\big)^{\frac{2p}{2-p}} \end{aligned}$$

Define $A := \big(2\varepsilon R_1^{\max}\big)^{\frac{2p}{2-p}}$ and $B := d^2\big(x_0, x(\tilde{u})\big)$, to get

$$R_1^{\max}\int_{u_0}^{u} d^p\big(x(\tilde{u}), x_0\big)d\tilde{u} \leq \frac{1}{2\varepsilon^2}\int_{u_0}^{u} d^2\big(x(\tilde{u}, x_0\big)d\tilde{u} + \frac{1}{2\varepsilon^2}B(u-u_0) + A$$

Finally, we arrive at

$$R_1^{\max}\int_{u_0}^{u} d^p\big(x(\tilde{u}), x_0\big)d\tilde{u} + R_2^{\max}(u-u_0) \leq \frac{1}{2\varepsilon^2}\int_{u_0}^{u} d^2\big(x(\tilde{u}), x_0\big)d\tilde{u} + \tilde{R}_2^{\max(u-u_0)}, \quad (7.13)$$

where $\tilde{R}_2^{\max}(u-u_0) = \big(\frac{1}{2\varepsilon^2}B + R_2^{\max}\big)(u-u_0) + A$. Therefore, we can replace the right-hand side of (7.9) by the left-hand side of (7.13). From here on, the proof for the subquadratic case follows by the same argumentation as in the quadratic case before. □

Theorem 7.2.9 *(Strongly causal M-PFWs)*
Any M-PFW $\big(\mathcal{M}, \langle .,. \rangle\big)$ such that $-H(x,u)$ grows at most quadratically is strongly causal.

Proof We have to proof the non-existence almost closed causal curves. We will do this by showing, that for a fixed $\varepsilon' > 0$, there always exists an $\varepsilon \in (0, \varepsilon']$ such that any causal curve with start- and endpoint in an neighbourhood $U_\varepsilon \subseteq U_{\epsilon'}$, lies entirely in $U_{\epsilon'}$. More precisely, fixing $z_0 = (x_0, v_0, u_0) \in \mathcal{M}$ we can consider the basis of neighbourhoods $\{U_{\varepsilon'} : \varepsilon' > 0\}$ of z_0, where

$$U_{\varepsilon'} = B_{\varepsilon'}(x_0) \times (v_0 - \varepsilon', v_0 + \varepsilon') \times (u_0 - \varepsilon', u_0 + \varepsilon'),$$

where $B_{\varepsilon'}(x_0)$ denotes the Riemannian metric ball in M centered at x_0, with radius ε'. Now, take any causal curve $\alpha : [0,1] \to U_\varepsilon$ from z_0 to z_1 , with the endpoint $\alpha(1) = z_1 = (x_1, v_1, u_1) \in U_\varepsilon$. If $\alpha(s)$ is *null*, then we have, that the Riemannian part $\dot{x}(s) = 0$, as well as $\dot{u}(s) = 0$ (for all $s \in [0,1]$), therefore $\dot{v}(s) \neq 0$. But then, $v(s)$ has to be either increasing or decreasing. In both cases, if v leaves $(v_0 - \varepsilon, v_0 + \varepsilon)$, it cannot come back. So without loss of generality, let us assume that $\alpha(s)$ is *timelike* with $\dot{u}(s) > 0$ (the case $\dot{u}(s) < 0$ is analogous) and we use the u-coordinate to parametrize α, such that $\alpha(u) = \big(x(u), v(u), u\big)$ (as in lemma 7.2.7), with $0 < u_1 - u_0 < \varepsilon \leq \varepsilon'$.
Now, for fixed $\varepsilon' > 0$, we will see that some smaller $\varepsilon > 0$ can be chosen, such that $x(u)$ lies entirely in $B_{\varepsilon'}(x_0)$. First, when $u = u_1$, the left-hand side of (7.5) is equal to $(v_1 - v_0) - R_2'(u_1 - u_0)$, and since both, $(v_1 - v_0) < \varepsilon$ and $(u_1 - u_0) < \varepsilon$ by assumption, by using (7.5) , we get

$$\int_{u_0}^{u_1} \langle \dot{x}(s), \dot{x}(s) \rangle ds < 4(1 + R_2')\varepsilon,$$

for small enough ε. This inequality is a bound for the energy and thus, for the length of $x(u)$, as required.
Finally, let us show that $v(u)$ remains in $(v_0 - \varepsilon', v_0 + \varepsilon')$ for small enough ε. Note that, from (7.5), we can get

$$v(u) - v_0 < R_2'(u_1 - u_0) < R'2\varepsilon.$$

on the other hand, from (7.6) we get

$$v_1 - v(u) < R_2'\varepsilon,$$

thus the bound for $v(u)$ follows from the last two inequalities. As a consequence, the causal curve $\alpha(s)$ lies entirely in $U_{\varepsilon'}$, as required. Thus, strong causality for M-PFWs follows in this case. $\square$

7.2.3 A sufficient hypothesis for global hyperbolicity

To complete our study of causality for M-PFWs, we proof the following statement about global hyperbolicity in this section.

Theorem 7.2.10 *(Global hyperbolic M-PFWs)*
Any M-PFW $\mathcal{M}$ with a complete Riemannian part M and such that $-H(x,u)$ has subquadratic growth, is globally hyperbolic.

We need the following lemma:

Lemma 7.2.11 *(Strongly causal closure)*
Let $\mathcal{M}$ be a strongly causal spacetime. If $J(p,q) := J^+(p) \cap J^-(q)$ is included in some compact subset $K \subset M$ then $J(p,q)$ is closed (and thus, compact).

Proof This is a direct consequence of [ON83], page 104, lemma 14. $\square$

Lemma 7.2.12 *(Bound for $J(z_0, z_1)$)*
If $-H(x,u)$ behaves subquadratically, then the natural projections of $J(z_0, z_1) \subset M \times \mathbb{R}^2$ ($z_0 < z_2$), on M and $\mathbb{R}^2$ are bounded.

Proof Recall, that we always have, that $J(z_0, z_1) \subseteq \overline{I(z_0, z_1)}$, for the distance d induced by the Riemannian metric on M and the usual distance $du^2 + dv^2$ on $\mathbb{R}^2$, respectively. Therefore, it suffices to prove the result for $I(z_0, z_1) =: I^+(z_0) \cap I^-(z_1)$. Consider a point $r \in I(z_0, z_1)$. We have $\dot{u}(s) >$ for any future directed causal curve by proposition 4.0.2, hence $u(z_0) \leq u(r) \leq u(z_1)$ and, therefore, the points in $I(z_0, z_1)$ have a bound in the u-component.
In order to bound the v-component, consider any future directed timelike curve α joining z_0 and r (respectively, r and z_1), and use the second assertion of lemma 7.2.7 plus remark 7.2.8, to obtain:

$$v(r) - v(z_0) < R_2'(u(r) - u(z_0)), \quad v(z_1) - v(r) < R_2'(u(z_1) - u(r)),$$

and $v(r)$ is also bounded. Finally, the bound of the projection of r on M follows, by using again 7.2.7 and 7.2.8 plus the boundness of $v(u)$ and u. $\square$

Proof of theorem 7.2.10
Because we already know by theorem 7.2.9, that any M-PFW $\mathcal{M}$ with a subquadratic behaviour of $-H$ is already strongly causal, it suffices to prove that $J(z_0, z_1)$ is compact for any $z_1 > z_0$. Now, the Riemannian metric $g_R = \langle ., . \rangle_M + dv^2 + du^2$ on $M \times \mathbb{R}^2$ is complete, because of the completeness of the Riemannian distance d induced by $\langle ., . \rangle_M$ on M and the usual distance on $\mathbb{R}^2$. Thus, lemma 7.2.12 implies, that each $J(z_0, z_1)$ is included in a compact subset $K \subset \mathcal{M}$ and, by lemma 7.2.11, we see, that $J(z_0, z_1)$ is compact in $\mathcal{M}$, as required. □

7.2.4 Causality results/summary

We collect/summarize the causality results for M-PFWs from this section.

Theorem 7.2.13 *(Causal ladder of M-PFWs)*
Any M-PFW $\mathcal{M} = M \times \mathbb{R}^2$ with the metric given by

$$\langle ., . \rangle_{\mathcal{M}} = \langle ., . \rangle_M + 2dudv + H(x, u)du^2,$$

is causal.
The behaviour of the smooth function $H : M \times \mathbb{R} \to \mathbb{R}$ determines the position in the causal ladder.

1. *If $-H$ grows subquadratically and M is a complete Riemannian manifold, then $\mathcal{M}$ is globally hyperbolic.*

2. *If $-H$ grows at most quadratically, then $\mathcal{M}$ is strongly causal.*

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