

Dissertation

Nonlinear waves in kinetic transport and nonlocal diffusion equations

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Abstract

This thesis consists of two parts. The first one deals with travelling wave solutions of kinetic transport equations. At the beginning a unified framework for studying the existence and stability of kinetic shock profiles for conservation laws is presented. This includes small amplitude waves for the situation when the macroscopic model is a hyperbolic system of conservation laws with genuine nonlinearity. Here we construct kinetic travelling waves close to the profiles for diffusive macroscopic approximations derived by the Chapman-Enskog method. To prove solvability of the differential problem for the correction term we adapt the micro-macro decomposition in the spirit of Caflisch and Nicolaenko for the Boltzmann equation. The stability proof relies on energy estimates, where we again start from the system with diffusive regularisation. The details are carried out for the BGK model for the isentropic gas dynamics. For the case of scalar conservation laws, also large amplitude waves can be understood.

Applications range from BGK-models for general scalar conservation laws and for gas dynamics, to an equation for fermions in a scattering background under the action of an electric field and to the Boltzmann equation of gas dynamics.

We then consider a kinetic transport model for chemical reactions as an approximation for the KPP-Fisher reaction diffusion equation and again construct kinetic profiles close to travelling waves of the KPP-Fisher equation. The major difficulty in this work is caused by the fact that, in contrary to above, the macroscopic problem is not a conservation law. Stability of these travelling waves is shown for perturbations in a weighted L^2 -space, where the weight function is similar to corresponding results on the macroscopic level.

In the second and shorter part of the thesis we analyse scalar conservation laws with a nonlocal diffusion term corresponding to a Riesz-Feller differential operator. Solvability results for the Cauchy problem in H^1 and L^∞ are adapted from the case of the fractional derivative with homogenous symbol. Our main interest is the investigation of travelling waves. In the case of a convex or concave flux function smooth wave profiles exist, which are monotone and satisfy the standard entropy condition. The proof of their asymptotic stability relies on the construction of a Lyapunov functional.

Kurzfassung

Die vorliegende Arbeit besteht aus zwei Teilen. Der erste Teil beschäftigt sich mit wandernden Wellenlösungen von kinetischen Transportgleichungen. Am Beginn führen wir Existenz- und Stabilitätsanalysen von kinetischen Schockprofilen in einem allgemeinen Rahmen durch. Das beinhaltet Wellen mit kleinen Amplituden, wenn das makroskopische Modell ein hyperbolisches System von Erhaltungssätzen mit echter Nichtlinearität ist. In diesem Fall konstruieren wir die kinetischen wandernden Wellen nahe der Profile der diffusiven makroskopischen Näherung resultierend aus der Chapman-Enskog Methode. Um die Lösbarkeit des Differentialproblems für den Korrekturterm zu zeigen, adaptieren wir die mikro-makro Zerlegung im Sinne von Caflisch und Nicolaenko für die Boltzmann Gleichung. Der Stabilitätsbeweis basiert auf Energie-Ungleichungen, wobei wir mit den Abschätzungen wieder beim System mit der diffusiven Regularisierung beginnen. Die Details werden anhand des BGK Modells für das isentrope System der Gasdynamik ausgeführt. Für skalare Erhaltungssätze sind auch Resultate für Wellen mit großen Amplituden gegeben.

Anwendungen reichen von BGK Modellen für skalare Erhaltungssätze und Systeme der Gasdynamik bis zu einer Gleichung für Fermione in einem streuenden Hintergrund unter der Einwirkung eines elektrischen Feldes und hin zur Boltzmann Gleichung der Gasdynamik.

Weiters betrachten wir ein kinetisches Transportmodell für chemische Reaktionen als Annäherung für die KPP-Fisher Reaktions-Diffusions Gleichung und konstruieren wieder kinetische Profile nahe zu wandernden Wellenlösungen der KPP-Fisher Gleichung. Die größere Schwierigkeit verursacht die Tatsache, dass das makroskopische Modell im Gegensatz zu vorhin kein Erhaltungssatz ist. Wir zeigen die Stabilität dieser wandernden Wellenlösungen in einem gewichteten L^2 -Raum, wobei die Gewichtsfunktion dieselbe ist, die für ein entsprechendes Resultat für die makroskopische Gleichung benötigt wird.

Im zweiten und kürzeren Teil der Arbeit analysieren wir skalare Erhaltungssätze mit einem nichtlokalen Diffusionsterm, der einem Riesz-Feller Operator entspricht. Lösbarkeitsresultate für das Cauchy-Problem in H^1 und L^∞ können von der fraktalen Ableitung mit homogenem Symbol übernommen werden. Unser Hauptinteresse gilt den wandernden Wellenlösungen. Im Fall einer konvexen oder konkaven Flussfunktion existieren glatte Wellenlösungen, die monoton sind und die klassische Entropiebedingung erfüllen. Der Beweis der asymptotischen Stabilität basiert auf der Konstruktion eines Lyapunov-Funktional.

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Chapter 1

Introduction

In the first part of the thesis travelling wave solutions of kinetic transport equations are investigated. Chapter 2 provides a review and generalisation of results concerning the existence and stability of travelling wave solutions of kinetic models leading to conservation laws in the macroscopic limit. Shock waves connecting different far-field values by a jump discontinuity are basic weak solutions of nonlinear hyperbolic conservation laws. The main question considered in this work is the existence and dynamic stability of kinetic shock profiles, which are smooth travelling wave solutions of the kinetic equation, sharing the far-field states with the shock wave.

For systems of nonlinear hyperbolic conservation laws only results for small amplitude shock waves are available. In this case the Chapman-Enskog approximation, which amounts to a diffusive regularisation of the conservation laws, can be expected to provide a good approximation for solutions of the kinetic equation. Thus kinetic shock profiles can be constructed close to profiles of the system with diffusive regularisation. The classical result on the existence of small amplitude kinetic shock profiles for the gas dynamics Boltzmann equation is due to Caflisch and Nicolaenko. We present a modified and generalised version of their approach leading to more accurate approximation results. We briefly summarise the main ideas of the constructive existence proof:

The basic idea of the Chapman-Enskog method is to approximate the solution of the kinetic equation in terms of its macroscopic moments. The solution is expanded around the corresponding equilibrium distribution, which solves the system of conservation laws. The leading term of the remainder then provides a diffusive regularisation for the latter. We define an asymptotic solution for kinetic travelling waves as suggested by the formal Chapman-Enskog expansion. In order to get a rigorous result we have to keep error terms and therefore define this refined approximation in terms of the exact profile solutions of the regularised conservation laws. The travelling wave problem for the small error term we treat in a similar fashion to Caflisch and Nicolaenko by decomposing the correction term into a leading term with separated variables, where we prescribe the velocity distribution in an appropriate way, and a small microscopic term. A projection then splits the travelling equation into a system with separated derivatives for the two unknown functions of the decomposition. Proving the solvability of this system by a

fixed point argument requires good estimates for the solution of the leading linear problem, which then allows to include also small nonlinear terms. The leading equation for the microscopic contribution of the decomposition is the inhomogeneous linearised kinetic travelling wave equation. Solvability is made difficult due to the fact that the linearised collision operator is only negative semidefinite. Here one replaces the linearised collision operator by an appropriately chosen negative definite operator and solves the modified equation instead, which can be shown to be equivalent to the original problem. This manipulation again originates from the work of Caflisch and Nicolaenko and provides a good result for the linear problem, such that the contraction argument can be carried out.

The proof for local dynamic stability of the constructed kinetic profiles relies on energy estimates. The argument is again based on a micro-macro decomposition now in the spirit of Liu and Yu for the Boltzmann equation. Proceeding in a similar fashion to the Chapman-Enskog approximation we obtain equations for the moments of the perturbation, which are at leading order the corresponding macroscopic system with diffusive regularisation. We derive energy estimates as one would do for the purely macroscopic case. Combining these estimates with the ones obtained from the kinetic perturbation equation allows to control also the small microscopic part. This way we produce a Lyapunov functional decaying in time.

This above described approach to show the existence and stability of weak kinetic profiles is the core of the first part of this thesis. It follows the work of Cuesta and Schmeiser for the BGK model leading to scalar conservation laws in the macroscopic limit. We first adapted this procedure for the BGK model for the isentropic and isothermal gas dynamics, which in particular required to treat compactly supported equilibrium distributions and is the content of Chapter 3. We then worked out the proofs in a very general setting in Chapter 2. This also contains the stability of travelling waves for the BGK model for the isentropic gas dynamics, which we could simplify later on. This modified version is carried out in Chapter 3. We shall also note here that this thesis is a collection of publications and preprints, hence the notation is only coherent within each Chapter.

For scalar conservation laws stronger results are available in the literature. The general idea for proving existence and stability for big shocks is demonstrated in Chapter 2.

In Chapter 4 we consider chemical reactions of the form $A + B \leftrightarrow 2A$, where the density of species B is assumed to be constant. A reaction-diffusion model for the density of particles A is given by the KPP-Fisher equation, which is a nonlinear prototype model for travelling wave solutions. In a kinetic model for the same situation the diffusion can be replaced by collisions with a nonmoving background medium. Typically this collision operator is linear. We choose BGK-type operators for the linear collisions as well as the reactive term. This kinetic transport equation is in terms of a small parameter scaled such that we recover the KPP-Fisher equation in the macroscopic limit. We show that also the kinetic equation admits travelling wave solutions by constructing them similar to above close to the wave profiles of the KPP-Fisher equation. The main difference to the previous works causes the fact that the macroscopic equation is not a conservation law. We also

point out that here no smallness assumption on the amplitude of the wave is necessary. Applicability of the above proof follows from the scaling of the equation. Deriving the asymptotic stability of wave profiles of the KPP-Fisher equation requires to work with weighted spaces, which force the perturbations to decay faster than the waves. On the kinetic level we have to impose a similar weight function, which enables us to control macroscopic terms in the same fashion as for the KPP-Fisher equation. To bound small microscopic terms we then however have to make the assumption of a compactly supported velocity space.

In the second part of the thesis we consider scalar conservation laws with a nonlocal diffusion term. Fractal conservation laws involving pseudo-differential operators with homogenous symbol already received attention in the literature. Our analysis concerns a different one, which resembles the form of a Weyl derivative and in its Fourier-representation is contained in the class of Riesz-Feller operators. This fractional derivative has been derived as the physically correct viscosity term in two layer shallow water flows by performing formal asymptotic expansions associated to the triple-deck regularization used in fluid mechanics (see e.g. Viertl). Also it appears in the work of Fowler in the context of dune formations.

We first analyse the Cauchy problem. Following the standard approach for parabolic equations we determine the solution of the linear evolution equation to obtain the semigroup generated by the fractional derivative. The nonlinear equation can then be stated in its mild formulation also referred to as the Duhamel formula, which provides a fixed point equation for the solution. Due to the local Lipschitz continuity of the flux function the fixed point operator is shown to be a contraction in a suitable space. Banachs fixed point argument allows to conclude the local existence and uniqueness of a solution. A typical approach for extending this local solvability result for all times is the construction of a Lyapunov functional yielding the decay of the solution. Then the global solvability follows as an immediate consequence by repeating the local existence result.

Our main interest is the investigation of travelling waves connecting different far-field values. The travelling wave version of the evolution equation is a nonlinear integral equation of Volterra type. To show the existence of solutions we proceed in a similar fashion as one would do for the corresponding ordinary differential equation. We first show the existence on an interval from $-\infty$ up to a small number by using a fixed point argument. The local extension then is a generalised version of Picard-Lindelöf's Theorem based on the local Lipschitz continuity of the flux. Due to the assumption of (even a bit less than) convexity of the flux the qualitative properties of the equation imply the monotonicity as well as a maximum principle for the solution. A Gronwall-type inequality can be used to show that also its derivative cannot blow up in finite time. Hence the local existence result can be repeated over the whole real axis.

To show the asymptotic stability of the travelling waves we construct a Lyapunov functional in a similar fashion as one would do in the case of the classical viscous regularisation. Care has to be taken since our diffusion operator provides less regularity. To bound the nonlinearities we use a generalisation of the Gagliardo-Nirenberg estimates for fractional Sobolev spaces. This allows us to deduce the vanishing of initially small perturbations as times goes to ∞ .

Part I

Travelling waves in kinetic transport equations

Chapter 2

Kinetic shock profiles for nonlinear hyperbolic conservation laws^{1 2}

2.1 Introduction

This work contributes to the mathematical theory establishing the connection between kinetic transport equations and hyperbolic systems of conservation laws, occurring as their macroscopic limits. Shock waves are basic weak solutions of nonlinear hyperbolic conservation laws featuring a discontinuity. The main question considered in this work is the existence and dynamic stability of kinetic shock profiles, i.e. smooth travelling wave solutions of the kinetic equation, sharing the far-field states with the shock wave. Several recent results are reviewed and presented in a unified way.

For systems of nonlinear conservation laws only results for small amplitude shock waves are available. In this case, the Chapman-Enskog approximation, i.e. a diffusive regularization of the conservation laws, can be expected to provide a good approximation for solutions of the kinetic equation. Kinetic shock profiles can be constructed close to viscous shock profiles. The classical result on the existence of small amplitude kinetic shock profiles for the gas dynamics Boltzmann equation is due to Caflisch and Nicolaenko [11]. In Section 2.4, a modified and generalized version of their approach is presented, leading to more accurate approximation results. Stability of small amplitude kinetic shock profiles is the issue of Section 2.5. An approach based on energy (actually entropy) estimates in the spirit of the work of Liu and Yu [28] is presented. The main idea is to start from an approach for the system with diffusive regularization. This can actually be extended to proving convergence to rarefaction waves [16].

For scalar conservation laws, stronger results are possible. An approach for the

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construction of large amplitude kinetic shock profiles is presented in Section 2.6. The main ideas originate from the work of Golse [22] on the Perthame-Tadmor model [35]. Finally, dynamical stability is discussed, based on ideas from [5].

We consider a kinetic transport model for plane waves in the form

$$\partial_t f + v \partial_x f = Q(f), \quad (2.1.1)$$

where $f(t, x, \mathbf{v})$ is a particle distribution function at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}$. The components of the 'velocity' vector $\mathbf{v} = (v, w) \in V \subset \mathbb{R} \times [0, \infty)$ can be interpreted as the velocity component v in x -direction, and as an abbreviation w for $(v_2^2 + \dots + v_d^2)^{1/2}$, where $(v, v_2, \dots, v_d) \in \mathbb{R}^d$ is the particle velocity. Thus, we assume x -axisymmetric velocity distributions. The set V of velocities is equipped with a measure $d\mu(\mathbf{v})$. Discrete sets V and, thus, hyperbolic relaxation models are permitted.

The so called *collision operator* Q is assumed to act on the velocity variable $\mathbf{v}$ only. Equations of the form (2.1.1) can be derived from fully d -dimensional kinetic transport equations, if the collision events are invariant under rotations (at least around the x -axis).

The collision operator is assumed to be nonlinear and to have the conservation property

$$\int_V \phi(\mathbf{v}) Q(f)(\mathbf{v}) d\mu(\mathbf{v}) = 0. \quad (2.1.2)$$

The (linearly independent) components of the vector $\phi(\mathbf{v}) \in \mathbb{R}^n$ are called *collision invariants*. As a consequence of (2.1.2), the macroscopic moments of f , collected in the vector

$$U_f(t, x) := \int_V \phi(\mathbf{v}) f(t, x, \mathbf{v}) d\mu(\mathbf{v}),$$

are the macroscopic densities of conserved quantities:

$$\partial_t U_f + \partial_x J_f = 0, \quad \text{with } J_f := \int_V v \phi f d\mu. \quad (2.1.3)$$

As expected, the zero set of Q will be assumed to be n -dimensional and parametrizable by the macroscopic moments:

$$Q(f) = 0 \iff f(\mathbf{v}) = \mathcal{M}(U_f, \mathbf{v}), \quad (\text{implying } U_{\mathcal{M}(U)} = U.)$$

The *generalized Maxwellian* $\mathcal{M}(U)$ is the equilibrium distribution of the collision processes. It is plausible that the dynamics of close-to-equilibrium solutions of (2.1.1) is approximated by the system of conservation laws

$$\partial_t U_f + \partial_x J(U_f) = 0, \quad J(U) := J_{\mathcal{M}(U)}, \quad (2.1.4)$$

obtained by replacing f by $\mathcal{M}(U_f)$ in the second term of (2.1.3). This approximation can only be expected to be physically relevant under a stability condition: We assume the existence of a *kinetic entropy density* $H(f, \mathbf{v})$, satisfying

$$\int_V \partial_f H(f) Q(f) d\mu \leq 0. \quad (2.1.5)$$

where H is continuous in $\mathbf{v}$ and twice differentiable and convex in f . We also assume definiteness in the sense that equality in (2.1.5) only holds if $f = \mathcal{M}(U_f)$. This leads to the kinetic entropy inequality

$$\partial_t \int_V H(f) d\mu + \partial_x \int_V v H(f) d\mu \leq 0. \quad (2.1.6)$$

The macroscopic system (2.1.4) will be assumed to be strictly hyperbolic meaning that for every U , the Jacobian $J'(U)$ of the *macroscopic flux* $J(U)$ has n distinct eigenvalues $\lambda_1(U) < \dots < \lambda_n(U)$, and the corresponding left and right eigenvectors $l_k(U)$ and, respectively, $r_k(U)$, $k = 1, \dots, n$, are assumed to be normalized such that $l_j(U) \cdot r_k(U) = \delta_{jk}$.

Piecewise constant weak solutions of (2.1.4) of the form

$$U(x, t) = \begin{cases} U_- & \text{for } x < st, \\ U_+ & \text{for } x > st, \end{cases}$$

are called *shock waves*. Here s is the shock speed and $U_{\pm}$ are the constant left and right states, which are related by the *Rankine-Hugoniot jump conditions*

$$s(U_+ - U_-) = J(U_+) - J(U_-). \quad (2.1.7)$$

For a fixed left state U_- the *Hugoniot locus* is defined as the set of all U_+ such that (2.1.7) is satisfied for an appropriate s . In a neighbourhood of U_- , the Hugoniot locus consists of n curves intersecting in U_- . At U_- , the k -th curve is tangent to $r_k(U_-)$ and the shock speed s takes the value $\lambda_k(U_-)$ (see, e.g., [26]). If U_+ lies on the k -th curve of the Hugoniot locus, we refer to $\{U_{\pm}, s\}$ as a *k-shock*.

If the k -th field is *genuinely nonlinear*, i.e. $r_k \cdot \nabla \lambda_k \neq 0$, then the *Lax entropy condition*

$$\lambda_k(U_+) < s < \lambda_k(U_-) \quad (2.1.8)$$

is a stability condition for k -shocks (see, e.g., [26]) and we assume a normalization of r_k such that $r_k \cdot \nabla \lambda_k = 1$.

An alternative approach to entropy conditions starts from the kinetic entropy inequality (2.1.6) and uses the close-to-equilibrium approximation

$$\partial_t \eta(U_f) + \partial_x \Psi(U_f) \leq 0, \quad (2.1.9)$$

as a side condition for weak solutions of (2.1.4). Here the macroscopic entropy density and entropy flux (satisfying $\nabla \Psi(U) = \nabla \eta(U) \cdot J'(U)$) are given by

$$\eta(U) = \int_V H(\mathcal{M}(U)) d\mu, \quad \Psi(U) = \int_V v H(\mathcal{M}(U)) d\mu. \quad (2.1.10)$$

It can be shown that for small amplitude shock waves, i.e., small enough values of $|U_+ - U_-|$, the conditions (2.1.8) and (2.1.9) are equivalent.

Remark 2.1.1. Further properties of the kinetic entropy density will be used below. The above implies the minimisation principle

$$\eta(U) = \int_V H(\mathcal{M}(U)) d\mu = \min_{\int_V \phi f d\mu = U} \int_V H(f) d\mu, \quad (2.1.11)$$

which has the further consequence that $\partial_f H(\mathcal{M}(U))$ is linear in the collision invariants, i.e. there exists a vector $b_U \in \mathbb{R}^n$ such that $\partial_f H(\mathcal{M}(U)) = b_U \cdot \phi$. If we take the gradient of the first relation in (2.1.10) it turns out that $b_U = \nabla \eta(U)$, such that $\partial_f H(\mathcal{M}(U)) = \nabla \eta(U) \cdot \phi$.

For later reference, we collect the assumptions on the collision operator made so far.

Assumption 1. The collision operator Q has n linearly independent collision invariants $\phi_1(\mathbf{v}), \dots, \phi_n(\mathbf{v})$, and its zero set is given by $\{\mathcal{M}(U, \mathbf{v}) : U \in \mathbb{R}^n\}$. There exists a strictly convex kinetic entropy density $H(f, \mathbf{v})$ satisfying the inequality (2.1.5) (with equality iff $f = \mathcal{M}(U_f)$). The macroscopic system (2.1.4) is strictly hyperbolic.

2.2 Examples

Not all collision operators can be interpreted as appropriate models for microscopic collision processes. In many cases they are just constructed as relaxation models towards a desired equilibrium. The so called *BGK-models* [7], [9] of the form $Q(f) = \mathcal{M}(U_f) - f$ belong to this class. All the examples presented below satisfy Assumption 1.

2.2.1 BGK-models for scalar conservation laws

A family of generalized Maxwellians for an arbitrary scalar hyperbolic conservation law with flux $J(U)$ is given by

$$\mathcal{M}(U, v) = \int_0^U m(v - J'(r)) dr, \quad v \in V = \mathbb{R},$$

(see [18]), where $m(v) > 0$ can be any even function satisfying $\int_{-\infty}^{\infty} m(v) dv = 1$. The conservation law is conservation of mass with $\phi(v) = 1$ and $U_f = \int_{-\infty}^{\infty} f dv$. Noting that $\mathcal{M}$ is strictly increasing as a function of U , we define kinetic entropy densities by inverting it:

$$\zeta(f, v) = U : \Leftrightarrow \mathcal{M}(U, v) = f, \quad H(f, v) := \int_0^f \eta'(\zeta(g, v)) dg, \quad (2.2.1)$$

where η is an arbitrary convex function. Then the entropy inequality

$$\int_{-\infty}^{\infty} \partial_f H(f) [\mathcal{M}(U_f) - f] dv = \int_{-\infty}^{\infty} [\eta'(\zeta(f)) - \eta'(U_f)] [\mathcal{M}(U_f, v) - f] dv \leq 0$$

holds, since the equality follows from mass conservation and the inequality is a consequence of the monotonicities of $\zeta(f, v)$ with respect to f and of η' . The corresponding macroscopic entropy density is given by $\eta(U)$. So all the macroscopic entropies can be recovered from kinetic entropies.

There are of course also many other choices such as discrete velocity models, the simplest with two velocities: $d\mu(v) = (\delta(v+a) + \delta(v-a))dv$ and

$$\mathcal{M}(U, \pm a) = \frac{1}{2a}(aU \pm J(U)),$$

the corresponding BGK-model being equivalent to the standard relaxation model [24]

$$\partial_t U + \partial_x j = 0, \quad \partial_t j + a^2 \partial_x U = J(U) - j,$$

with $U = f(-a) + f(a)$, $j = a(f(a) - f(-a))$. Kinetic entropy densities can be constructed as above, if the generalized Maxwellian is strictly monotone in U , i.e., if the subcharacteristic condition $|J'(U)| < a$ holds for all relevant values of U .

2.2.2 A BGK-model for isentropic and isothermal gas dynamics

The following class of generalized Maxwellians has been introduced in [27]. Here $n = 2$, $U = (\rho, \rho u)$, $V = \mathbb{R}$, and

$$\mathcal{M}(\rho, u, v) = b \left(\frac{2\gamma}{\gamma-1} \rho^{\gamma-1} - (v-u)^2 \right)_+^\beta,$$

with $1 < \gamma < 3$,

$$\beta = \frac{3-\gamma}{2(\gamma-1)}, \quad b = \frac{1}{J_\beta} \left(\frac{2\gamma}{\gamma-1} \right)^{-1/(\gamma-1)}, \quad J_\beta = \int_{-1}^1 (1-z^2)^\beta dz.$$

The collision invariants $\phi(v) = (1, v)$ correspond to conservation of mass and momentum (in the x -direction), and the macroscopic flux vector is given by

$$J(\rho, m) = \int_{-\infty}^{\infty} v \begin{pmatrix} 1 \\ v \end{pmatrix} \mathcal{M}(\rho, m/\rho, v) dv = \begin{pmatrix} m \\ \frac{m^2}{\rho} + \rho^\gamma \end{pmatrix} = \begin{pmatrix} m \\ j(\rho, m) \end{pmatrix} \quad (2.2.2)$$

Thus, the system (2.1.4) is the p-system of isentropic gas dynamics with adiabatic exponent γ . The eigenvalues of $J'(\rho, m)$ are given by

$$\lambda_{1/2} = u \mp c(\rho), \quad \text{where } c(\rho) = \sqrt{\gamma} \rho^{(\gamma-1)/2}.$$

For $\rho > 0$ (away from vacuum) $\lambda_1 < \lambda_2$ holds everywhere and the system is strictly hyperbolic. With the corresponding right and left eigenvectors

$$r'_1 = (1, \lambda_1), \quad r'_2 = (1, \lambda_2), \quad l'_1 = (\lambda_2, -1), \quad l'_2 = (-\lambda_1, 1),$$

one can see that the system is also genuinely nonlinear. The primes indicate that the eigenvectors are not scaled as assumed in Section 2.1. Moreover one can show that the Lax admissibility condition for a 1-shock reduces to

$$\rho_- < \rho_+ \quad (2.2.3)$$

and for a 2-shock to $\rho_- > \rho_+$.

A kinetic entropy density is given by

$$H(f, v) = \frac{v^2}{2}f + \frac{f^{1+1/\beta}}{2b^{1/\beta}(1+1/\beta)},$$

leading to the macroscopic entropy density

$$\eta(\rho, u) = \frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1},$$

whose physical interpretation is, of course, energy.

From the Maxwellians

$$\mathcal{M}(\rho, u, v) = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{(v-u)^2}{2}}, \quad \text{where } v \in \mathbb{R},$$

we recover the isothermal gas dynamics, where the flux is given by (2.2.2) with $\gamma = 1$. Then the eigenvalues are as above with $c \equiv 1$. Here the kinetic entropy

$$H(f, v) = \frac{v^2}{2}f + f \ln f$$

leads to the macroscopic one

$$\eta(\rho, u) = \frac{\rho u^2}{2} + (\rho \ln \rho - \rho \ln \sqrt{2\pi}),$$

see e.g. also [9].

2.2.3 The gas dynamics BGK-model

In the general d -dimensional Maxwellian

$$\frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{(v-u)^2 + (v_2-u_2)^2 + \dots + (v_d-u_d)^2}{2T}\right)$$

with density ρ , mean velocity $(u, u_2, \dots, u_d)$, and temperature T , the axisymmetry assumption is equivalent to vanishing transversal mean velocities, $u_2 = \dots = u_d = 0$, and leads to

$$\mathcal{M}(\rho, u, T; v, w) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{(v-u)^2 + w^2}{2T}\right).$$

The integration measure is defined as $d\mu(v, w) := w^{d-2} |S^{d-1}| dv dw = d(v, v_2, \dots, v_d)$, where $|S^{d-1}|$ is the surface of the $(d-1)$ -dimensional unit sphere. Conservation of mass, momentum in the x -direction, and energy is required, i.e., $\phi(v, w) = (1, v, (v^2 + w^2)/2)$ and

$$U_f = \begin{pmatrix} \rho_f \\ m_f \\ E_f \end{pmatrix} = \begin{pmatrix} \rho_f \\ \rho_f u_f \\ \rho_f u_f^2/2 + \frac{d}{2} \rho_f T_f \end{pmatrix} = \int_v \begin{pmatrix} 1 \\ v \\ (v^2 + w^2)/2 \end{pmatrix} f d\mu.$$

The macroscopic flux is given by

$$J(\rho, u, T) = \begin{pmatrix} \rho u \\ \rho u^2 + \rho T \\ u(\rho u^2/2 + \frac{d}{2}\rho T + \rho T) \end{pmatrix}.$$

The macroscopic system (2.1.4) are the compressible Euler equations for a d -dimensional ideal gas, reduced to one dimension by assuming plane wave solutions with vanishing transversal velocity components.

The Jacobian

$$J'(U) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma E \frac{u}{\rho} & \gamma \frac{E}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix}, \quad \gamma = \frac{d+2}{d},$$

has the eigenvalues

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c,$$

with the sound speed $c = \sqrt{\gamma T}$. The corresponding right and left eigenvectors are given by

$$\begin{aligned} r_k &= \left(1, \lambda_k, \frac{1}{\gamma-1} \left(\frac{3-\gamma}{2}u^2 - (3-\gamma)u\lambda_k + \lambda_k^2 \right) \right), \\ l_k &= \frac{1}{c^2 d} \left(\frac{1}{\lambda_k} \left(\frac{\gamma-3}{2(\gamma-1)}(\lambda_k - \gamma u)u^2 + (\gamma-1)u^3 - \gamma E \frac{u}{\rho} \right), \frac{1}{\gamma-1}(\lambda_k - \gamma u), 1 \right), \end{aligned}$$

satisfying $(l_k \cdot r_k)_{k=1}^3 = (1, -2, 1)$. In view of further calculations we give r_1 explicitly: $r_1 = (1, u - c, \frac{1}{2}(u - c)^2 + \frac{3-\gamma}{2(\gamma-1)}c^2)$. The first and third field are genuinely nonlinear, whereas the second field is linearly degenerate, i.e, $r_2 \cdot \nabla \lambda_2 \equiv 0$.

The kinetic entropy density is the classical $H(f) = f \ln f$, and

$$\int_V \partial_f H(f) [\mathcal{M}(U_f) - f] d\mu = \int_V [\ln f - \ln \mathcal{M}(U_f)] [\mathcal{M}(U_f) - f] d\mu \leq 0,$$

the equality being a consequence of the fact that the logarithm of the Maxwellian is a linear combination of the collision invariants. The macroscopic entropy density is given by

$$\eta(\rho, T) = \frac{\rho d}{2} \ln \left(\frac{\rho}{(2\pi T)^{d/2}} \right).$$

Subtracting a multiple of the conserved quantity ρ and dividing by a constant factor gives the classical $\hat{\eta} = -\rho \ln(\rho T / \rho^\gamma)$ with $\gamma = (d+2)/d$.

2.2.4 Fermions in a background medium and a constant electric field

Semiclassical modelling of the scattering of fermions with an equilibrium background medium leads to collision operators of the form [30]

$$Q_s(f)(v) = \int_{\mathbb{R}} \sigma(v, v') [f(v')(1 - f(v))M(v) - f(v)(1 - f(v'))M(v')] dv',$$

where the collision cross section $\sigma(v, v') \geq \underline{\sigma} > 0$ is symmetric, $M(v) = (2\pi)^{-1/2} e^{-v^2/2}$ is a normalized Gaussian, and the occurrence of the factors $(1 - f)$ is a consequence of the quantum mechanical Pauli exclusion principle. The zero set of Q_s is one-dimensional (corresponding to the conservation of mass) and consists of the Fermi-Dirac distributions $(1 + c/M)^{-1}$, $c > 0$. The action of a constant electric field with the x -component E is included in the total collision operator

$$Q(f) = Q_s(f) - E \partial_v f.$$

The only conserved quantity is mass ($\phi(v) = 1$), and it has been proven in [3] that the zero set of Q can be parametrized by the density:

$$Q(f) = 0 \iff f(v) = \mathcal{M}(\rho_f, v),$$

where the generalized Maxwellian is a strictly increasing function of $\rho_f = \int_{\mathbb{R}} f dv$. Without the lower bound on the collision cross section the existence of nontrivial equilibrium distributions is not guaranteed (see [37]).

The somewhat surprising result that the definition (2.2.1) yields a kinetic entropy density for the operator Q (including an acceleration term) has been proven in [5].

2.3 Macroscopic and small wave approximations

2.3.1 The hydrodynamic limit

The macroscopic approximation (2.1.4) for the kinetic equation (2.1.1) can be formally derived by rescaling position and time by $x \rightarrow x/\varepsilon$, $t \rightarrow t/\varepsilon$, and passing to the limit $\varepsilon \rightarrow 0$ in

$$\varepsilon \partial_t f^\varepsilon + \varepsilon v \partial_x f^\varepsilon = Q(f^\varepsilon). \quad (2.3.1)$$

Assuming a strong enough convergence $f^\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$, passing to the limit yields $Q(f) = 0$ and, thus, $f(t, x, \mathbf{v}) = \mathcal{M}(U(t, x), \mathbf{v})$ with $U = \lim_{\varepsilon \rightarrow 0} U_{f^\varepsilon}$. Passing to the limit in the conservation laws

$$\partial_t U_{f^\varepsilon} + \partial_x J_{f^\varepsilon} = 0 \quad (2.3.2)$$

leads to the macroscopic system (2.1.4). This limit can be justified for all examples of Section 2.2 in the following sense: If the initial data $f_I(x, \mathbf{v}) = f^\varepsilon(0, x, \mathbf{v})$ are smooth and possess smooth moments $U_{f_I}(x)$, then a unique smooth solution of (2.1.4) taking these initial data exists for a short enough open time interval. The unique solution of the initial value problem for (2.3.1) converges to $\mathcal{M}(U)$, where U is the solution of (2.1.4), on any compact subinterval of the existence interval of the latter (see, e.g., [10] for the case of the Boltzmann equation of gas dynamics).

The harder question of global convergence to weak entropy solutions has also been answered for all the examples except for the gas dynamics BGK-model (Section 2.2.3). Proofs working for all cases in Section 2.2.1 can be found in [33], [34]. For the isentropic gas dynamics model (Section 2.2.2) it has been carried out in [6], and for the fermion model (Section 2.2.4) in [5].

2.3.2 The linearised collision operator

The properties of linearisations of the collision operator at equilibrium distributions will be needed throughout the rest of this work and, in particular, in the following section for the construction of a more accurate macroscopic approximation. For a fixed vector $\hat{U}$, the linearisation of the collision operator around $\hat{\mathcal{M}} := \mathcal{M}(\hat{U})$ is denoted by

$$\mathcal{L}f := Q'(\hat{\mathcal{M}})f.$$

The motivation for introducing a suitable functional analytic framework for the operator $\mathcal{L}$ comes from the entropy inequality

$$\frac{1}{\varepsilon^2} \int_V Q(\hat{\mathcal{M}} + \varepsilon f) \partial_f H(\hat{\mathcal{M}} + \varepsilon f) d\mu \leq 0$$

Since $\partial_f H(\hat{\mathcal{M}})$ is a linear combination of the collision invariants (see Remark 2.1.1), the limit as $\varepsilon \rightarrow 0$ of the left hand side is equal to $\langle \mathcal{L}f, f \rangle_{\mathbf{v}}$ with the weighted scalar product

$$\langle f, g \rangle_{\mathbf{v}} := \int_V f g \partial_f^2 H(\hat{\mathcal{M}}) d\mu. \quad (2.3.3)$$

The induced Hilbert space and its norm are denoted by $(L_{\mathbf{v}}^2, \|\cdot\|_{\mathbf{v}})$. The operator $\mathcal{L}$ is assumed to be bounded and symmetric and, by passing to the limit in the above entropy inequality, it is negative semidefinite in $L_{\mathbf{v}}^2$. By the symmetry assumption, the functions $\phi_j / \partial_f^2 H(\hat{\mathcal{M}})$, $1 \leq j \leq n$, (where ϕ_j is the j -th collision invariant) lie in the null space $\mathcal{N}$ of $\mathcal{L}$. We assume that they span $\mathcal{N}$, but use the alternative basis $\partial_{U_j} \mathcal{M}(\hat{U}, \mathbf{v})$, $j = 1, \dots, n$, having the useful property

$$(U_{\partial_{U_j} \mathcal{M}(\hat{U})})_i = \partial_{U_j} \int_V \phi_i \mathcal{M}(\hat{U}) d\mu = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Since

$$\langle f, \phi / \partial_f^2 H(\hat{\mathcal{M}}) \rangle_{\mathbf{v}} = U_f,$$

$\mathcal{N}^\perp = \{f \in L_{\mathbf{v}}^2 : U_f = 0\}$ holds, where $\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ in $L_{\mathbf{v}}^2$. We assume that $\mathcal{L} : \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp$ is invertible. In other words, $U_g = 0$ is the solvability condition for the equation $\mathcal{L}f = g$, which has a unique solution $f \in \mathcal{N}^\perp$. Finally, the orthogonal projection from $L_{\mathbf{v}}^2$ to $\mathcal{N}$ is given by $f \mapsto U_f \cdot \nabla_U \mathcal{M}(\hat{U})$.

In this section, we have posed further assumptions on the collision operator:

Assumption 2. *The linearised collision operator $Q'(\hat{\mathcal{M}})$ is symmetric with respect to the scalar product (2.3.3). Its kernel $\mathcal{N}$ is n -dimensional, and its restriction to $\mathcal{N}^\perp$ is invertible.*

For BGK-models the linearised collision operator is given by $\mathcal{L}f = U_f \cdot \nabla_U \hat{\mathcal{M}} - f$, immediately showing $\dim(\mathcal{N}) = n$, $\mathcal{L}|_{\mathcal{N}^\perp} = -\text{id}$, and implying

$$\langle \mathcal{L}f, g \rangle_{\mathbf{v}} - \langle \mathcal{L}g, f \rangle_{\mathbf{v}} = \int_V (g U_f - f U_g) \cdot \nabla_U \hat{\mathcal{M}} \partial_f^2 H(\hat{\mathcal{M}}) d\mu.$$

Symmetry of $\mathcal{L}$ is now a consequence of the identity

$$\nabla_U \mathcal{M}(U) \partial_f^2 H(\mathcal{M}(U)) = \nabla^2 \eta(U) \phi, \quad (2.3.4)$$

derived by computing the gradient of $\partial_f H(\mathcal{M}(U)) = \nabla \eta(U) \cdot \phi$ (see Remark 2.1.1). The identity (2.3.4) has other useful consequences. Taking its tensor product with $v \nabla_U \mathcal{M}(U)$ and integrating it with respect to $\mathbf{v}$ shows that the matrix $\nabla^2 \eta(U) J'(U)$ is symmetric. This in turn implies that $\nabla^2 \eta(U) r_k(U)$ is a left eigenvector of $J'(U)$ corresponding to the eigenvalue λ_k . Thus,

$$\nabla^2 \eta(U) r_k(U) = \kappa_k(U) l_k(U), \quad \text{with } \kappa_k(U) = \nabla^2 \eta(U)(r_k(U), r_k(U)) > 0,$$

implying the following relation between elements of $\mathcal{N}$:

$$r_k(\hat{U}) \cdot \nabla_U \hat{\mathcal{M}} = \kappa_k(\hat{U}) l_k(\hat{U}) \cdot \frac{\phi}{\partial_f^2 H(\hat{\mathcal{M}})}. \quad (2.3.5)$$

2.3.3 The Chapman-Enskog approximation

There are two basic strategies for improving the approximation quality of the macroscopic limit $\mathcal{M}(U)$ as an approximation for f^ε . The idea of the *Hilbert expansion* [23] is rather straightforward and amounts to constructing an asymptotic expansion for f^ε in terms of powers of ε :

$$f^\varepsilon(t, x, \mathbf{v}) = \mathcal{M}(U(t, x), \mathbf{v}) + \sum_{j=1}^n \varepsilon^j f_j(t, x, \mathbf{v}) + O(\varepsilon^{n+1}).$$

Substitution of this ansatz in (2.3.1), (2.3.2), and in the initial conditions, and comparing coefficients of ε leads to equations determining the sequence $\{f_n\}$ recursively.

The second approach does not concentrate on solving arbitrary initial value problems for (2.3.1), but to approximate a solution manifold parametrized by the macroscopic moments $U^\varepsilon = U_{f^\varepsilon}$. It starts with the *micro-macro decomposition*

$$f^\varepsilon = \mathcal{M}(U^\varepsilon) + \varepsilon f^\perp,$$

and tries to compute $f^\perp$ in terms of U^ε and the dynamics of U^ε , such that f^ε solves (2.3.1). When, in this program, $O(\varepsilon^2)$ -errors are accepted in the equation for U_ε , an approximation up to $O(\varepsilon)$ -errors is needed for $f^\perp$. From (2.3.1) we obtain

$$Q'(\mathcal{M}(U^\varepsilon)) f^\perp = \nabla_U \mathcal{M}(U^\varepsilon) \cdot (\partial_t U^\varepsilon + v \partial_x U^\varepsilon) = \nabla_U \mathcal{M}(U^\varepsilon) \cdot (v - J'(U^\varepsilon)) \partial_x U^\varepsilon,$$

where $O(\varepsilon)$ -terms have been neglected. Computing the gradient with respect to U of the relation $J(U) = \int_V v \phi \mathcal{M}(U) d\mu$ shows that the right hand side satisfies the solvability condition mentioned at the end of the previous section such that $f^\perp$ can be computed uniquely in terms of the approximation U of U^ε :

$$f^\perp[U] = Q'(\mathcal{M}(U))^{-1} [\nabla_U \mathcal{M}(U) \cdot (v - J'(U))] \partial_x U. \quad (2.3.6)$$

Using this in (2.3.2) gives the *Chapman-Enskog approximation* [12]

$$\partial_t U + \partial_x J(U) = \varepsilon \partial_x (D(U) \partial_x U), \quad (2.3.7)$$

with the diffusivity matrix

$$D(U) = - \int_V v \phi \otimes Q'(\mathcal{M}(U))^{-1} [\nabla_U \mathcal{M}(U) \cdot (v - J'(U))] d\mu,$$

where the symbol $\otimes$ denotes the tensor product. For BGK-models, $Q'|_{\mathcal{N}^\perp} = -\text{id}$ and a more explicit representation can be found:

$$D(U) = \int_V v^2 \phi \otimes \nabla_U \mathcal{M}(U) d\mu - J'(U)^2.$$

The diffusion dissipates the macroscopic entropy, which is reflected by the fact that

$$\nabla^2 \eta(U) D(U) = -\langle Q'(\mathcal{M}(U)) \chi \otimes \chi \rangle_v \geq 0$$

holds, where the identity (2.3.4) and the notation $\chi = Q'(\mathcal{M}(U))^{-1} \nabla_U \mathcal{M}(U) \cdot (v - J'(U))$ have been used.

Remark 2.3.1. For the BGK-models for scalar macroscopic equations ($\phi = 1$) in Section 2.2.1, $\partial_U \mathcal{M} > 0$ holds, implying

$$J'(U)^2 = \left(\int_V v \partial_U \mathcal{M}(U) d\mu \right)^2 < \int_V v^2 \partial_U \mathcal{M}(U) d\mu,$$

since $\int_V \partial_U \mathcal{M}(U) d\mu = 1$. Thus, the diffusivity $D(U)$ is strictly positive.

For the BGK-model for isentropic gas dynamics in Section 2.2.2,

$$D(\rho, u) = (3 - \gamma) \rho^{\gamma-1} \begin{pmatrix} 0 & 0 \\ -u & 1 \end{pmatrix}$$

holds, leading to the Navier-Stokes model

$$D(\rho, u) \partial_x \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(\rho) \partial_x u \end{pmatrix},$$

with the viscosity $\mu(\rho) = (3 - \gamma) \rho^\gamma$. This example shows that the diffusivity is in general not regular, such that diffusion does not act on all components of U .

The gas dynamics BGK-model in Section 2.2.3 gives

$$D(\rho, u, T) = T \begin{pmatrix} 0 & 0 & 0 \\ -\frac{2(d-1)}{d} u & \frac{2(d-1)}{d} u & 0 \\ -\frac{(d+2)}{2} T - \frac{3(d-2)}{2d} u^2 & \frac{(d-4)}{d} u & \frac{d+2}{d} \end{pmatrix},$$

leading to

$$D(\rho, u, T) \partial_x \begin{pmatrix} \rho \\ \rho u \\ \rho(u^2 + dT)/2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(\rho, T) \partial_x u \\ u\mu(\rho, T) \partial_x u + \kappa(\rho, T) \partial_x T \end{pmatrix},$$

with viscosity $\mu(\rho, T) = \frac{2(d-1)}{d}\rho T$ and heat conductivity $\kappa(\rho, T) = \frac{d+2}{2}\rho T$. Note that with $\gamma = \frac{d+2}{d}$ and with the isentropic relation $\rho T = \rho^\gamma$, the viscosity of the previous example is recovered.

Remark 2.3.2. Similar results as those described in Section 2.3.1 are also available for the vanishing diffusion limit $\varepsilon \rightarrow 0$ in (2.3.7) (see, e.g., [20], or [8] for a recent result).

2.3.4 Weakly nonlinear approximation for small waves

In this section we consider the slow modulation of travelling wave solutions of the linearization of the hydrodynamic system (2.1.4) at a constant state U_- . Modulations are caused by nonlinearity and by the dissipative terms in the Chapman-Enskog system (2.3.7).

We choose k such that the k -th field is strictly nonlinear ($r_k \cdot \nabla \lambda_k = 1$) and introduce a moving reference frame and a long time scale by $x = \eta + \lambda_k(U_-)t$ and $t = \tau/\varepsilon$ in (2.3.7):

$$\varepsilon \partial_\tau U + (J'(U) - \lambda_k(U_-))\partial_\eta U = \varepsilon \partial_\eta (D(U)\partial_\eta U). \quad (2.3.8)$$

This motivates the ansatz

$$U(\tau, \eta) = U_- + \varepsilon y(\tau, \eta)r_k(U_-) + \varepsilon^2 U_2(\tau, \eta) + O(\varepsilon^3),$$

which annihilates the $O(1)$ - and $O(\varepsilon)$ -terms in (2.3.8). At $O(\varepsilon^2)$, we obtain

$$\partial_\tau y r_k + y \partial_\eta y J''(r_k, r_k) + (J' - \lambda_k)\partial_\eta U_2 = D r_k \partial_\eta^2 y,$$

where all functions of U are evaluated at U_- . A solvability condition for this equation for U_2 is obtained by taking the scalar product with the left eigenvector l_k of J' (satisfying $l_k \cdot r_k = 1$):

$$\partial_\tau y + y \partial_\eta y = D_k(U_-)\partial_\eta^2 y, \quad (2.3.9)$$

with $D_k(U) = l_k(U) \cdot D(U)r_k(U)$, where the relation

$$\begin{aligned} 1 &= r_k \cdot \nabla \lambda_k = r_k \cdot \nabla (l_k \cdot J' r_k) = l_k \cdot J''(r_k, r_k) + r_k \cdot (\lambda_k r_k \cdot \nabla l_k + \lambda_k l_k \cdot \nabla r_k) \\ &= l_k \cdot J''(r_k, r_k) + \lambda_k r_k \cdot \nabla (l_k \cdot r_k) = l_k \cdot J''(r_k, r_k) \end{aligned} \quad (2.3.10)$$

has been used.

Approximately, the modulation of travelling wave solutions is described by the viscous Burgers equation (2.3.9). Positivity of the scalar diffusivity D_k will be assumed. It has to be checked example by example.

Remark 2.3.3. For scalar conservation laws we obviously have $D_1(U) = D(U)$. For the isentropic gas dynamics model we obtain $D_k = (3 - \gamma)\rho^{\gamma-1}/2$ for both $k = 1$ and $k = 2$, and for the full gas dynamics BGK-model $D_k = T$ for the genuinely nonlinear fields $k = 1$ and $k = 3$.

2.3.5 Viscous shock profiles for weak shocks

Let the k -th field of (2.1.4) be genuinely nonlinear and let $\{U_{\pm}, s\}$ denote a k -shock, where the shock speed can be written as

$$s = \lambda_- + \varepsilon \sigma,$$

with $\sigma < 0$ and a small perturbation parameter $0 < \varepsilon \ll 1$. The sign conditions are due to the Lax entropy condition (2.1.8). Then the difference of the far field states has an asymptotic expansion

$$U_+ - U_- = 2\varepsilon \sigma r_- + O(\varepsilon^2),$$

where $r_- := r_k(U_-)$ and $\lambda_- := \lambda_k(U_-)$. A travelling wave solution $U = U_{vsp}(\xi)$, $\xi = x - st$ of the Chapman-Enskog equations (2.3.7), satisfying the far-field conditions $U_{vsp} \rightarrow U_{\pm}$ for $\xi \rightarrow \pm\infty$, will be called a *viscous shock profile*. It can be seen as an heteroclinic orbit of the ODE system

$$\varepsilon D(U) \partial_{\xi} U = J(U) - J(U_-) - s(U - U_-). \quad (2.3.11)$$

General results on the existence of viscous shock profiles are not available (even for artificial viscosity of the form $D(U) = I$). For small shocks, i.e. ε small enough, U_{vsp} can be expected to stay close to the constant state U_- , and therefore the asymptotics of the previous section can be used for an approximation. This leads to a travelling wave problem for the viscous Burgers equation (2.3.9) with wave speed σ and with the far-field values $y_- = 0$ and $y_+ = 2\sigma$. A travelling wave solution $y_{vsp}(\eta - \sigma\tau) = y_{vsp}(\xi)$ can be computed explicitly.

To make this approximation rigorous is a nontrivial problem of the theory of singularly perturbed ODEs. The details of the justification depend on the properties of the diffusivity matrix $D(U)$. A general rigorous treatment is, thus, impossible and we state the result as an assumption.

Assumption 3. *Let the k -th field of the macroscopic flux $J(U)$ be genuinely nonlinear, and let $D_k(U_-) = l_k(U_-) \cdot D(U_-) r_k(U_-) > 0$. Let, for ε small enough, (2.3.11) have a solution $U_{vsp}(\xi) = U_- + \varepsilon y_{vsp}(\xi) r_k(U_-) + \varepsilon^2 U_2^{\varepsilon}(\xi)$, satisfying $\lim_{\xi \rightarrow \pm\infty} U_{vsp}(\xi) = U_{\pm}$, such that U_2^{ε} and all its derivatives are uniformly bounded with respect to ε .*

We verify the assumption for the BGK-models of Section 2.2. For the case of a scalar conservation law (Section 2.2.1), genuine nonlinearity, w.l.o.g. $J'' > 0$, has to be assumed. In this case, by $D(U) > 0$, viscous shock profiles obviously exist, iff the entropy condition $U_+ < U_-$ is satisfied. Smallness of the shock is not needed.

For the case of isentropic gas dynamics (Section 2.2.2), both fields are genuinely nonlinear. By the form of $D(\rho, m)$ the first equation in the system (2.3.11) is algebraic, and the system can be reduced to the scalar ODE

$$\frac{\varepsilon \mu(\rho)}{\rho} \partial_{\xi} \rho = (\rho - \rho_-)(u_- - s) + \frac{\rho^{\gamma+1} - \rho \rho_-^{\gamma}}{\rho_-(s - u_-)}.$$

It is easily seen that $s - u_-$ is negative for a 1-shock and positive for a 2-shock. Obviously, this sign determines the convexity of the right hand side, and a viscous

shock profile exists, whenever the entropy condition ($\rho_+ > \rho_-$ for a 1-shock, $\rho_+ < \rho_-$ for a 2-shock) is satisfied. Again smallness of the shock is not needed.

For the full gas dynamics BGK-model (Section 2.2.3) again one equation in (2.3.11) is algebraic. However, after elimination of one unknown, a singularly perturbed second order system remains. Existence of viscous profiles for small shocks (in the genuinely nonlinear fields $k = 1, 3$) has been shown for various applications.

2.4 Existence of kinetic profiles for weak shocks

In this section we shall present an approach for the construction of small amplitude travelling wave solutions of the kinetic equation (2.1.1). The macroscopic moments of their far field limits are connected by genuinely nonlinear entropic shock waves of the hyperbolic system (2.1.4). The main ideas are generalizations of the work of Caflisch and Nicolaenko [11] on the gas dynamics Boltzmann equation. Our approach is slightly different in several details. In particular, starting from a formal asymptotic approximation, a perturbation equation for the correction term is considered. This leads to a sharper error bound in the final result. Also the problem is in general not linearized around the far-field state. This is necessary for treating problems with equilibrium velocity distributions with compact support (like the BGK-model for isentropic gas dynamics in Section 2.2.2), in order to guarantee that the support of the state we linearize around contains the support of the travelling wave.

In the following two subsections, the general procedure is outlined. Applications to several examples are contained in the last subsection.

A kinetic shock profile is a solution $f = f(\xi, \mathbf{v})$ of

$$\varepsilon(v-s)\partial_\xi f = Q(f), \quad \lim_{\xi \rightarrow \pm\infty} f(\xi, \mathbf{v}) = \mathcal{M}_\pm(\mathbf{v}) := \mathcal{M}(U_\pm, \mathbf{v}). \quad (2.4.1)$$

Considering (2.3.11) as an approximation for (2.4.1), an approximative kinetic profile for a small k -shock is given by

$$f_{as} := \mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}], \quad (2.4.2)$$

where the microscopic correction term is defined in (2.3.6).

2.4.1 The micro-macro decomposition of the correction term

We start by analyzing the formal approximation properties of (2.4.2). The residual is given by

$$\begin{aligned} \varepsilon^3 h &:= \varepsilon(v-s)\partial_\xi f_{as} - Q(f_{as}) \\ &= \varepsilon(v-s)\partial_\xi \mathcal{M}(U_{vsp}) + \varepsilon^2(v-s)\partial_\xi f^\perp[U_{vsp}] \\ &\quad - Q(\mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}]). \end{aligned} \quad (2.4.3)$$

Using the asymptotic expansion of U_{vsp} given in Assumption 3, it is straightforward to show that the scaling of the residual is justified in the sense that, as a function of ξ , h and its derivatives are bounded uniformly with respect to ε .

Also the system (2.3.11) implies that the macroscopic moments of the residual vanish: $U_h = 0$. Finally, f_{as} satisfies the far-field conditions in (2.4.1) exactly.

The problem (2.4.1) rewritten in terms of the correction term $\varepsilon^2 g = f - f_{as}$ reads

$$\varepsilon(v - s)\partial_\xi g - \mathcal{L}_{as}g = \varepsilon^2 R(g) - \varepsilon h, \quad (2.4.4)$$

with $\mathcal{L}_{as} = Q'(f_{as})$ and $R(g) = \varepsilon^{-4}(Q(f_{as} + \varepsilon^2 g) - Q(f_{as}) - \varepsilon^2 \mathcal{L}_{as}g)$, subject to

$$g(\pm\infty, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in V. \quad (2.4.5)$$

By computing the moments and integration with respect to ξ , we derive the property

$$\int_V (v - s)\phi g d\mu = 0. \quad (2.4.6)$$

The collision operator has been linearized around the approximation f_{as} . This has the inconvenience to depend on the spatial variable ξ . Therefore we shall also use the linearization $\mathcal{L} := Q'(\mathcal{M}(\hat{U}))$ around the constant-in- ξ state $\mathcal{M}$, chosen such that $\hat{U} = U_- + \varepsilon \tilde{U}$ (with $\tilde{U}$ bounded uniformly in ε) and, consequently, $f_{as} = \mathcal{M} + O(\varepsilon)$ and $\mathcal{L}_{as} = \mathcal{L} + O(\varepsilon)$. Here and in the following, we use the abbreviations

$$\hat{\mathcal{M}} = \mathcal{M}(\hat{U}), \quad \hat{\lambda} = \lambda_k(\hat{U}), \quad \hat{r} = r_k(\hat{U}), \quad \hat{l} = l_k(\hat{U}).$$

The correction term is split into a macroscopic and a microscopic part:

$$g(\xi, \mathbf{v}) = z(\xi)\Phi(\mathbf{v}) + \varepsilon w(\xi, \mathbf{v}), \quad (2.4.7)$$

where the macroscopic variable is scalar and corresponds only to contributions from the k -th field. The choice of the profile function Φ is motivated by the work of Caflisch and Nicolaenko for the Boltzmann equation [11]. It is chosen such that it approximately solves a generalized eigenvalue problem:

$$\mathcal{L}\Phi = \varepsilon\tau(v - s)\Phi + O(\varepsilon^2),$$

for a constant τ and, additionally has the moment property of g :

$$\int_V (v - s)\phi \Phi d\mu = 0 \quad \implies \quad \int_V (v - s)\phi w d\mu = 0. \quad (2.4.8)$$

Hence expanding $\Phi = \Phi_0 + \varepsilon\Phi_1$ and decomposing the wave speed as $s = \hat{\lambda} + \varepsilon\hat{\sigma}$ with $\hat{\sigma} = \sigma - \tilde{U} \cdot \nabla \hat{\lambda} + O(\varepsilon)$, we determine the components Φ_0 and Φ_1 and the eigenvalue τ such that

$$\mathcal{L}\Phi_0 = 0 \quad \text{and} \quad \int_V (v - \hat{\lambda})\phi \Phi_0 d\mu = 0, \quad (2.4.9)$$

$$\mathcal{L}\Phi_1 = \tau(v - \hat{\lambda})\Phi_0 \quad \text{and} \quad \int_V (v - s)\phi \Phi_1 d\mu = \hat{\sigma} \int_V \phi \Phi_0 d\mu. \quad (2.4.10)$$

Considering that the null space of $\mathcal{L}$ is spanned by the components of $\nabla_U \mathcal{M}$, the problem (2.4.9) is solved by $\Phi_0 = \hat{r} \cdot \nabla_U \mathcal{M}$. Note that the second equation in

(2.4.9) is the solvability condition for the first equation in (2.4.10). We choose a solution of the form

$$\Phi_1 = \tau \mathcal{L}^{-1}[(v - \hat{\lambda})\Phi_0] + \sum_{j \neq k} \beta_j r_j(\hat{U}) \cdot \nabla_U \mathcal{M}.$$

The second equation in (2.4.10) then becomes

$$-\tau D(\hat{U})\hat{r} + \sum_{j \neq k} \beta_j (\lambda_j(\hat{U}) - s) r_j(\hat{U}) = \hat{\sigma} \hat{r},$$

which can be solved for τ and the β_j by

$$\tau = -\frac{\hat{\sigma}}{D_k(\hat{U})}, \quad \beta_j = \frac{\tau}{\lambda_j(\hat{U}) - s} l_j(\hat{U}) \cdot D(\hat{U})\hat{r}, \quad j \neq k.$$

As a consequence of (2.3.5) and of (2.4.9)

$$\psi = \mathcal{L}^{-1} \left((v - \hat{\lambda}) \frac{\hat{l} \cdot \phi}{\partial_f^2 H(\mathcal{M})} \right) \in \mathcal{N}^\perp$$

is well defined. In order to make the decomposition (2.4.7) unique, we pose the orthogonality condition

$$\langle (v - s)\psi, w \rangle_v = 0. \quad (2.4.11)$$

The computation

$$\begin{aligned} -\tilde{D} &= \langle (v - s)\psi, \Phi \rangle_v = \langle v\psi, \Phi_0 \rangle_v + O(\varepsilon) \\ &= \hat{l} \cdot \int_V v \phi \mathcal{L}^{-1}[(v - \hat{\lambda})\hat{r} \cdot \nabla_U \mathcal{M}] d\mu + O(\varepsilon) = -D_k(\hat{U}) + O(\varepsilon) \end{aligned}$$

shows that, for ε small enough, $\tilde{D} > 0$ and the decomposition (2.4.7) is well defined.

We now write the perturbation equation (2.4.4) in terms of the decomposition (2.4.7) and divide by ε :

$$(v - s)\Phi \partial_\xi z - z \frac{1}{\varepsilon} \mathcal{L}_{as} \Phi + \varepsilon(v - s)\partial_\xi w - \mathcal{L}_{as} w = \varepsilon R(z\Phi + \varepsilon w) - h, \quad (2.4.12)$$

It is part of the method of Caflisch and Nicolaenko that for projecting the equation to its macroscopic and microscopic parts, the alternative decomposition

$$g = P g - \frac{(v - s)\Phi}{\tilde{D}} \Pi g, \quad \text{with } \Pi g = \langle \psi, g \rangle_v,$$

is used. This definition and the property (2.4.8) of Φ imply $U_{Pg} = U_g$. Application of Π to (2.4.12) gives the macroscopic equation

$$-\tilde{D} \partial_\xi z + \Psi(\xi)z = \varepsilon \Gamma w + \varepsilon \Pi R - \Pi h, \quad (2.4.13)$$

where

$$\Psi = -\frac{1}{\varepsilon} \Pi \mathcal{L}_{as} \Phi \quad \text{and} \quad \Gamma w = \frac{1}{\varepsilon} \Pi \mathcal{L}_{as} w.$$

These terms are (formally) $O(1)$. This might not be obvious for the last one: The construction of ψ , the symmetry of $\mathcal{L}$ and (2.4.8) imply

$$\Pi \mathcal{L}_{as} w = \langle \mathcal{L} \psi, w \rangle_v + O(\varepsilon) = \hat{l} \cdot \int_V (v - s) \phi w d\mu + O(\varepsilon) = O(\varepsilon).$$

In the limit $\varepsilon \rightarrow 0$, z satisfies a linear equation independent from the microscopic solution component w . The following result shows that it is a small perturbation of the linearization of the travelling wave version of the viscous Burgers equation (2.3.9).

Lemma 2.4.1. *Let Assumption 3 hold. Then, formally, $\Psi(\xi) = y_{vsp}(\xi) - \sigma + O(\varepsilon)$.*

Proof. We shall use the formula

$$Q''(\mathcal{M}(U))(\nabla_U \mathcal{M}(U), \nabla_U \mathcal{M}(U)) = -Q'(\mathcal{M}(U)) \nabla_U^2 \mathcal{M}(U), \quad (2.4.14)$$

which can be derived by computing the Hessian with respect to U of $Q(\mathcal{M}(U)) = 0$. Since Φ_0 is in the null space of $\mathcal{L}$,

$$\frac{1}{\varepsilon} \mathcal{L}_{as} \Phi = \frac{\mathcal{L}_{as} - \mathcal{L}}{\varepsilon} \Phi_0 + \mathcal{L}_{as} \Phi_1 \quad (2.4.15)$$

holds. By the definition of Φ_1 ,

$$\Pi \mathcal{L}_{as} \Phi_1 = \langle \psi, \mathcal{L} \Phi_1 \rangle_v + O(\varepsilon) = \tau \langle \psi, (v - s) \Phi \rangle_v + O(\varepsilon) = \hat{\sigma} + O(\varepsilon). \quad (2.4.16)$$

The definition of f_{as} and the expansion of the viscous shock profile U_{vsp} imply $f_{as} = U_- + \varepsilon y_{vsp} r_k(U_-) + O(\varepsilon^2)$ and, thus,

$$\begin{aligned} \frac{\mathcal{L}_{as} - \mathcal{L}}{\varepsilon} &= \frac{Q'(\mathcal{M}(U_- + \varepsilon y_{vsp} r_k(U_-))) - Q'(\hat{\mathcal{M}})}{\varepsilon} + O(\varepsilon) \\ &= Q''(\hat{\mathcal{M}})(y_{vsp} \hat{r} - \tilde{U}) \cdot \nabla_U \hat{\mathcal{M}} + O(\varepsilon). \end{aligned}$$

With (2.4.14) we therefore obtain

$$\frac{\mathcal{L}_{as} - \mathcal{L}}{\varepsilon} \Phi_0 = -\mathcal{L} \nabla_U^2 \hat{\mathcal{M}}(y_{vsp} \hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon),$$

implying, with the symmetry of $\mathcal{L}$,

$$\begin{aligned} \Pi \frac{\mathcal{L}_{as} - \mathcal{L}}{\varepsilon} \Phi_0 &= -\hat{l} \cdot \int_V (v - \hat{\lambda}) \phi \nabla_U^2 \hat{\mathcal{M}} d\mu(y_{vsp} \hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon) \\ &= -\hat{l} \cdot J''(\hat{U})(y_{vsp} \hat{r} - \tilde{U}, \hat{r}) + O(\varepsilon) = -y_{vsp} + \tilde{U} \cdot \nabla \hat{\lambda} + O(\varepsilon), \end{aligned}$$

where the last equality requires a computation analogous to (2.3.10). Combining this with (2.4.15) and (2.4.16) and recalling $\hat{\sigma} = \sigma - \tilde{U} \cdot \nabla \hat{\lambda} + O(\varepsilon)$ completes the proof. $\square$

The microscopic projection P is used to derive from (2.4.12) an equation for w . The linearized collision operator is now approximated by $\mathcal{L}$:

$$\varepsilon(v - s) \partial_\xi w - \mathcal{L} w = z \frac{1}{\varepsilon} P \mathcal{L}_{as} \Phi + \varepsilon \tilde{\Gamma} w + \varepsilon P R - P h, \quad (2.4.17)$$

where

$$\tilde{\Gamma}w = -\frac{1}{\varepsilon} \frac{(v-s)\Phi}{\tilde{D}} \Pi \mathcal{L}w + \frac{1}{\varepsilon} P(\mathcal{L}_{as} - \mathcal{L})w.$$

The operator Γ is formally $O(1)$. Like g , its micro- and macro-components z and, respectively, w have to satisfy homogeneous far-field conditions

$$z(\pm\infty) = w(\pm\infty, \mathbf{v}) = 0. \quad (2.4.18)$$

The problem (2.4.4), (2.4.5) for g is equivalent to the (z, w) -problem (2.4.13), (2.4.17), (2.4.18).

The basic idea for the solution is to produce a fixed point problem by considering the right hand sides of (2.4.13) and of (2.4.17) as given. This is made difficult by the nondefiniteness of $\mathcal{L}$. The following procedure of *removal of the null space* is again based on ideas from Caflisch and Nicolaenko [11].

We introduce a negative definite perturbation of $\mathcal{L}$, which coincides with $\mathcal{L}$ on the set of functions w satisfying the moment conditions (2.4.8) and the orthogonality condition (2.4.11):

$$\mathcal{K}w = \mathcal{L}w - (v-s)\psi \langle (v-s)\psi, w \rangle_{\mathbf{v}} - (v-s) \frac{\phi}{\partial_f^2 H(\hat{\mathcal{M}})} \cdot \int_V (v-s)\phi w d\mu. \quad (2.4.19)$$

We already know that $\mathcal{L}$ is negative definite on $\mathcal{N}^\perp$. Writing the general element of the null space of $\mathcal{L}$ as $w = \sum_{j=1}^n \alpha_j r_j(\hat{U}) \cdot \nabla_U \hat{\mathcal{M}} \in \mathcal{N}$, we compute

$$-\langle \mathcal{K}w, w \rangle_{\mathbf{v}} = \left(\sum_{j=1}^n \alpha_j \hat{l} \cdot D(\hat{U}) r_j(\hat{U}) \right)^2 + \left| \sum_{j=1}^n \alpha_j (\lambda_j(\hat{U}) - s) r_j(\hat{U}) \right|^2.$$

The limits $(\lambda_j(U_-) - \lambda_k(U_-))r_j(U_-)$, $j \neq k$, as $\varepsilon \rightarrow 0$ of the vectors $(\lambda_j(\hat{U}) - s)r_j(\hat{U})$, $j \neq k$, in the last term are linearly independent. Therefore this term controls the coefficients α_j , $j \neq k$. The coefficient of α_k in the first term on the right hand side is equal to $D_k(\hat{U})$, whose limit $D_k(U_-)$ as $\varepsilon \rightarrow 0$ is positive. So this term controls α_k , showing that $\mathcal{K}$ is negative definite.

We now replace the operator $\mathcal{L}$ in (2.4.17) by $\mathcal{K}$:

$$\varepsilon(v-s)\partial_\xi w - \mathcal{K}w = z \frac{1}{\varepsilon} P \mathcal{L}_{as} \Phi + \varepsilon \tilde{\Gamma}w + \varepsilon PR - Ph, \quad (2.4.20)$$

and look for a solution of (2.4.13), (2.4.18), (2.4.20) in the following. The equivalence of the problems is not obvious:

Lemma 2.4.2. *For given $z(\xi)$, the problems (2.4.17), (2.4.18) and (2.4.20), (2.4.18) for w are equivalent.*

Proof. Since any solution of (2.4.17), (2.4.18) satisfies (2.4.8) and (2.4.11), it also solves (2.4.20), (2.4.18).

Let on the other hand w be a solution of (2.4.20), (2.4.18). Multiplication of (2.4.20) by the components of ϕ and integration with respect to velocity as well as taking the scalar product of (2.4.20) with ψ results in a system of $n + 1$ linear homogeneous first order ODEs for the quantities

$$\int_V (v - s) \phi w d\mu \quad \text{and} \quad \langle (v - s) \psi, w \rangle_v.$$

Due to the homogeneous far-field conditions, these quantities vanish for all ξ , implying (2.4.8) and (2.4.11) and, thus, (2.4.17). $\square$

2.4.2 The existence result

The solvability of the nonlinear problem (2.4.13), (2.4.18), (2.4.20) is deduced by using a fix-point argument. Hence we first consider the leading linear system, where we regard the right hand sides of (2.4.13) and (2.4.20) as given inhomogeneities:

$$\varepsilon(v - s) \partial_\xi w - \mathcal{K} w = h_w, \quad \text{subject to } w(\pm\infty, \mathbf{v}) = 0, \quad (2.4.21)$$

$$\tilde{D} \partial_\xi z - \Psi(\xi) z = h_z, \quad \text{subject to } z(\pm\infty) = 0. \quad (2.4.22)$$

Taking the scalar product of (2.4.21) with w and integrating with respect to ξ gives

$$- \int_{-\infty}^{\infty} \langle \mathcal{K} w, w \rangle_v d\xi = \int_{-\infty}^{\infty} \langle h_w, w \rangle_v d\xi. \quad (2.4.23)$$

This shows that the definiteness of $\mathcal{K}$ implies uniqueness of the solution of (2.4.21), whereas equation (2.4.22) has a one parameter set of solutions, which reflects the translational invariance of the travelling wave problem. Therefore we pose the initial condition

$$z(0) = z_0, \quad (2.4.24)$$

with an arbitrary $z_0 \in \mathbb{R}$. Lemma 2.4.1 and the far-field behaviour of y_{vsp} imply $\Psi(\infty) < 0$ and $\Psi(-\infty) > 0$. Therefore the fundamental solution Z satisfying

$$\tilde{D} \partial_\xi Z - \Psi(\xi) Z = 0, \quad Z(0) = 1,$$

decays exponentially for $\xi \rightarrow \pm\infty$, and the solution

$$z(\xi) = Z(\xi) z_0 + \frac{1}{\tilde{D}} \int_0^\xi \frac{Z(\xi)}{Z(\eta)} h_z(\eta) d\eta$$

of (2.4.22), (2.4.24) is bounded for bounded h_z .

At this point it is necessary to choose a functional analytic framework for the further development. Different choices are possible and have been made in different situations in the past. In this general treatment, we stay abstract and assume that two norms $\|\cdot\|_\xi^*$ and $\|\cdot\|_\xi^{**}$ for functions of the spatial variable ξ have been

chosen, where the first one is used for solutions of (2.4.22) and the second one for the right hand sides. Similarly, the norms $\|\cdot\|_{\xi, \mathbf{v}}^*$ and $\|\cdot\|_{\xi, \mathbf{v}}^{**}$ for functions of $(\xi, \mathbf{v})$ are used for solutions of (2.4.21) and, respectively, right hand sides. In the following, C denotes (possibly different) ε -independent constants.

Assumption 4. *A solution of (2.4.21) exists and the solutions of (2.4.22), (2.4.24) and of (2.4.21) satisfy estimates of the form*

$$\|z\|_{\xi}^* \leq C(|z_0| + \|h_z\|_{\xi}^{**}), \quad \|w\|_{\xi, \mathbf{v}}^* \leq C\|h_w\|_{\xi, \mathbf{v}}^{**}.$$

Caflisch and Nicolaenko use weighted L^∞ -norms for the Boltzmann equation [11], whereas for BGK-models, as we will see later, L^2 -based norms turn out to be convenient. In view of (2.4.23), an L^2 -approach seems natural. However, for the control of the nonlinearities regularity with respect to ξ is needed. Control of nonlinearities is straightforward in a L^∞ -approach. Estimating the solution of (2.4.21) in terms of L^∞ -norms on the other hand, requires much more sophistication than the derivation of L^2 -estimates.

The approach for the existence proof of a solution of (2.4.21) is based on spectral theory in [11]. In [17] the proof relies on a discretisation of the velocity component.

The existence and uniqueness proof of solutions of the nonlinear problem (2.4.13), (2.4.20) and (2.4.24) is now a contraction argument. Therefore we need estimates for the right-hand sides of (2.4.13) and (2.4.20). Corresponding to the spaces of the solutions and inhomogenities of the linear problem we define the norms

$$\|(z, w)\|^* := \|z\|_{\xi}^* + \varepsilon\|w\|_{\xi, \mathbf{v}}^*, \quad \|(h_z, h_w)\|^{**} := \|h_z\|_{\xi}^{**} + \varepsilon\|h_w\|_{\xi, \mathbf{v}}^{**}, \quad (2.4.25)$$

weighted according to the decomposition $g = \Phi z + \varepsilon w$. In the following we identify g with the pair (z, w) , i.e., $\|g\|^* = \|(z, w)\|^*$.

The following assumption contains rigorous statements concerning the formal properties of the terms on the right hand sides of (2.4.13), (2.4.20):

Assumption 5. (i) *The linear terms appearing in the right hand sides of (2.4.13) and (2.4.20) can be bounded as follows:*

$$\frac{1}{\varepsilon}\|P\mathcal{L}_{as}\Phi z\|_{\xi, \mathbf{v}}^{**} \leq C\|z\|_{\xi}^*, \quad \|\Gamma w\|_{\xi}^{**} + \|\tilde{\Gamma}w\|_{\xi, \mathbf{v}}^{**} \leq C\|w\|_{\xi, \mathbf{v}}^*. \quad (2.4.26)$$

(ii) *The residual terms are uniformly bounded:*

$$\|\Pi h\|_{\xi}^{**} + \|Ph\|_{\xi, \mathbf{v}}^{**} \leq C. \quad (2.4.27)$$

(iii) *The nonlinear term $R(g)$ is quadratic:*

$$\begin{aligned} & \|\Pi R(g_1) - \Pi R(g_2)\|_{\xi}^{**} + \|PR(g_1) - PR(g_2)\|_{\xi, \mathbf{v}}^{**} \\ & \leq C(\|g_1\|^* + \|g_2\|^*)\|g_1 - g_2\|^*, \quad \text{for } \|g_1\|^*, \|g_2\|^* \leq \frac{C_0}{\varepsilon}. \end{aligned} \quad (2.4.28)$$

Before stating the existence and uniqueness result we note that in terms of the original unknown $f = f_{as} + \varepsilon^2 g$, the condition $z(0) = z_0$ reads

$$\langle (v-s)\psi, f - f_{as} \rangle_v(\xi=0) = -\varepsilon^2 z_0 \tilde{D}. \quad (2.4.29)$$

Theorem 2.4.1. *Let the Assumptions 1 – 5 hold. Then for every $z_0 \in \mathbb{R}$ and every small enough $\varepsilon > 0$ there exists a solution of (2.4.1), (2.4.29), which is unique in a ball $\{f : \|f - f_{as}\|^* \leq \varepsilon \delta\}$ with δ independent of ε . It satisfies*

$$\|f - \mathcal{M}(U_{vsp})\|^* \leq C\varepsilon^2,$$

or, more precisely,

$$f = \mathcal{M}(U_{vsp}) + \varepsilon f^\perp[U_{vsp}] + \varepsilon^2 \Phi z + \varepsilon^3 w,$$

where U_{vsp} is the solution of (2.3.11) and $\|z\|_\xi^*$ and $\|w\|_{\xi,v}^*$ are uniformly bounded as $\varepsilon \rightarrow 0$.

Proof. It remains to prove the existence and uniqueness of the full nonlinear problem (2.4.13), (2.4.20), (2.4.24). As a consequence of assumption (2.4.26), the estimates from assumption 4 can be extended to the full linear problem

$$\begin{aligned} \tilde{D}\partial_\xi z - \Psi(\xi)z &= \varepsilon \Gamma w + h_z, \\ \varepsilon(v-s)\partial_\xi w - \mathcal{K}w - z\frac{1}{\varepsilon}P\mathcal{L}_{as}\Phi &= \varepsilon \tilde{\Gamma}w + h_w, \end{aligned}$$

with given inhomogenities h_z, h_w and $z(0) = z_0$. In terms of the norms defined in (2.4.25) the estimate on the solution of the linear problem can be written as

$$\|(z, w)\|^* \leq C(|z_0| + \|(h_z, h_w)\|^{**}).$$

Applying the solution operator for this system to (2.4.13), (2.4.20) implies a fixed point problem of the form

$$z = \varepsilon R_z(z, w) + \tilde{h}_z, \quad (2.4.30)$$

$$w = \varepsilon R_w(z, w) + \tilde{h}_w, \quad (2.4.31)$$

where R_z and R_w share the property in (2.4.28), and $\tilde{h}_z, \tilde{h}_w$ are the terms containing the residual, hence given and bounded due to (2.4.27). Using (2.4.28), the fix-point operator can be estimated by

$$\|(\varepsilon R_z(z, w) + \tilde{h}_z, \varepsilon R_w(z, w) + \tilde{h}_w)\|^* \leq c(1 + \varepsilon(\|(z, w)\|^*)^2),$$

for a constant $c > 0$.

This implies that for ε small enough both the ball with radius $2c$ and the ball with radius $\varepsilon^{-1} \min\{1/(2c), C_0\}$ are mapped into themselves by the right hand side of (2.4.30), (2.4.31). Due to the properties of the nonlinearity, the fix-point operator is a contraction on a ball with an $O(\varepsilon^{-1})$ radius. And we conclude that for ε small, (2.4.30), (2.4.31) has a solution with $\|(z, w)\|^* \leq 2c$, which is unique in a ball with an $O(\varepsilon^{-1})$ radius. Knowing this and returning to (2.4.31), also the boundedness of $\|w\|_{\xi,v}^*$ follows. $\square$

Lemma 2.4.3. *Let the assumptions of Theorem 2.4.1 hold and let the norm $\|\cdot\|_{\xi, \mathbf{v}}^*$ be such that $\|U_w\|_\infty \leq C\|w\|_{\xi, \mathbf{v}}^*$. Then the macroscopic moments $U_{f,j}(\xi)$, $j = 1, \dots, n$, of the solution f of (2.4.1), (2.4.29) are strictly monotone. Due to the asymptotic expansion of the travelling wave solution f , $\text{sgn}(\partial_\xi U_{f,j}) = \text{sgn}(r_k(U_-)_j \partial_\xi y_{vsp})$ follows.*

Proof. We proceed as in [17]. One can easily extend the proof of Theorem 2.4.1 to show that the difference of two solutions (z, w) and $(\hat{z}, \hat{w})$ is depending Lipschitz continuously on the initial data

$$\|z - \hat{z}\|_\xi^* \leq C|z_0 - \hat{z}_0|, \quad \|w - \hat{w}\|_{\xi, \mathbf{v}}^* \leq C|z_0 - \hat{z}_0|.$$

For the corresponding solutions f and $\hat{f}$ of (2.4.1), (2.4.29) the relation

$$U_{f,k}(0) - U_{\hat{f},k}(0) = \varepsilon^2 U_{\Phi,k}(z_0 - \hat{z}_0) + \varepsilon^3 (U_{w,k}(0) - U_{\hat{w},k}(0))$$

holds. The assumption $\|U_{w,k} - U_{\hat{w},k}\|_\infty \leq C\|w - \hat{w}\|_{\xi, \mathbf{v}}^*$ now implies

$$|U_{w,k}(0) - U_{\hat{w},k}(0)| \leq C|z_0 - \hat{z}_0|.$$

Since $U_{\Phi,k} \neq 0$, the map $z_0 \mapsto U_{f,k}(0)$ is invertible for ε small, meaning that the travelling wave can also be made locally unique by prescribing the value of $U_{f,k}(0)$ instead of z_0 . This argument can of course be repeated with $U_{f,k}(\xi_0)$ for every $\xi_0 \in \mathbb{R}$ instead of the origin.

Now assume $U_{f,k}(\xi)$ is not strictly monotone. Then there exist two ξ -values ξ_0 and $\xi_0 + \delta$ with an arbitrarily small positive δ , such that $U_{f,k}(\xi_0) = U_{f,k}(\xi_0 + \delta)$. Now also $\tilde{f}(\xi, \nu) = f(\xi + \delta, \nu)$ is a travelling wave with $U_{\tilde{f},k}(\xi_0) = U_{f,k}(\xi_0)$. By the uniqueness result we obtain $f \equiv \tilde{f}$. Consequently f must be periodic, which is a contradiction to the far-field conditions. $\square$

2.4.3 Examples

For the BGK-models introduced in Section 2, it only remains to check Assumptions 4 and 5. The standard norms and spaces of functions of ξ we denote by $(L_\xi^2, \|\cdot\|_\xi)$, $(H_\xi^m, \|\cdot\|_{H_\xi^m})$, $(L_\xi^\infty, \|\cdot\|_\infty)$ and recall the definition of the inner product in $\mathbf{v}$ in (2.3.3). Then the Hilbert space $L_{\xi, \mathbf{v}}^2$ is naturally defined by the scalar product

$$\langle f, g \rangle_{\xi, \mathbf{v}} = \int_{\mathbb{R}} \langle f, g \rangle_{\mathbf{v}} d\xi, \quad \text{where } \text{supp } f, \text{supp } g \subset V,$$

with the induced norm $\|\cdot\|_{\xi, \mathbf{v}}$. Similarly the spaces $H_\xi^k(L_{\mathbf{v}}^2)$ of functions, whose derivatives in ξ up to order k are in $L_{\mathbf{v}}^2$, are defined by

$$\|f\|_{H_\xi^k(L_{\mathbf{v}}^2)} = \left(\|f\|_{\xi, \mathbf{v}}^2 + \dots + \|\partial_\xi^k f\|_{\xi, \mathbf{v}}^2 \right)^{\frac{1}{2}}.$$

In terms of these norms we have to make some assumptions on the Maxwellians and kinetic entropies. We require that for a fixed $\mathbf{v} \in V$ the equilibrium distribution

$\mathcal{M}(U, \mathbf{v})$ with support in V is five times continuously differentiable in U and

$$\int |\phi(\mathbf{v})^\alpha \nabla_U^\beta \mathcal{M}(U, \mathbf{v})| d\mu < C, \quad \int |\phi(\mathbf{v})^\alpha| \left(\nabla_U^\beta \mathcal{M}(U, \mathbf{v}) \right)^2 \partial_f^2 H(\mathcal{M}) d\mu < C, \quad (2.4.32)$$

where α and β are multiindices with $|\alpha| \leq 4$ and $|\beta| \leq 5$ and U is $O(\varepsilon)$ -close to $\mathcal{M}$. Moreover we assume

$$\left| \int \phi(\mathbf{v})^\alpha f d\mu \right| \leq C \|f\|_{\mathbf{v}}, \quad \text{for } |\alpha| \leq 3, \quad (2.4.33)$$

implying

$$\|U_{f,k}\|_{H_\xi^m} \leq C \|f\|_{H_\xi^m(L_V^2)}, \quad k = 1, \dots, n. \quad (2.4.34)$$

We are now prepared to investigate the existence of a solution to the linear problem corresponding to (2.4.21), (2.4.22). We shall mention that for the existence proof H_ξ^1 -based norms are sufficient. In this case we would only need the moment conditions in (2.4.32) up to $|\beta| = 3$. But since showing the asymptotic stability requires L^∞ -bounds on the macroscopic profiles of the travelling wave and also on their derivatives up to second order, we shall rather look for solutions in the spaces H_ξ^3 , respectively $H_\xi^3(L_V^2)$:

$$\varepsilon(v-s)\partial_\xi w - \mathcal{K}w = h_w, \quad \text{with } h_w \in H_\xi^3(L_V^2), \quad (2.4.35)$$

$$\tilde{D}\partial_\xi z - \Psi(\xi)z = h_z, \quad \text{with } h_z \in H_\xi^2. \quad (2.4.36)$$

As we have already indicated before, there exist constants $\gamma, \bar{\xi} > 0$ such that

$$\Psi(\xi) \leq -\gamma \quad \text{for } \xi \geq \bar{\xi}, \quad \Psi(\xi) \geq \gamma \quad \text{for } \xi \leq \bar{\xi}. \quad (2.4.37)$$

Using this property of Ψ it was shown in [17] that the solution z of (2.4.36) with $z(0) = z_0$ satisfies the estimate

$$\|z\|_{H_\xi^3} \leq C(|z_0| + \|h_z\|_{H_\xi^2}).$$

Additionally, based on a discretisation of the velocity component v , it was proven in [17] that there exists a unique solution $w \in H_\xi^3(L_V^2)$ of (2.4.21) satisfying

$$\|\partial_\xi^k w\|_{\xi, \mathbf{v}} \leq \frac{1}{\kappa} \|\partial_\xi^k h_w\|_{\xi, \mathbf{v}}, \quad \text{for } k = 0, \dots, 3.$$

Here the positive constant κ is the one from the coercivity estimate $-\langle \mathcal{K}w, w \rangle_{\mathbf{v}} \geq \kappa \|w\|_{\mathbf{v}}^2$. This coercivity estimate in particular holds for the negative definite operators $\mathcal{K}$ appearing in the examples under consideration.

If we can also verify the bounds on the linear and nonlinear terms in assumption 5 the contraction argument can be carried out. Using the boundedness of $\Pi : L_V^2 \rightarrow \mathbb{R}$ and $P : L_V^2 \rightarrow L_V^2$, the moment conditions (2.4.32), (2.4.33) and the smoothness of U_{vsp} , we obtain the desired bounds on the linear terms

$$\frac{1}{\varepsilon} \|P \mathcal{L}_{as} \Phi z\|_{H_\xi^3(L_V^2)} \leq C \|z\|_{H_\xi^3}, \quad \|\Gamma w\|_{H_\xi^3} + \|\tilde{\Gamma} w\|_{H_\xi^3(L_V^2)} \leq C \|w\|_{H_\xi^3(L_V^2)}. \quad (2.4.38)$$

We observe that for the particular examples under consideration the behaviour of U_{vsp} is exponential as $\xi \rightarrow \pm\infty$. This allows us to integrate the derivatives in ξ , and enables us to deduce the boundedness of the residual-terms

$$\|Ph\|_{H_\xi^3(L_v^2)} + \|\Pi h\|_{H_\xi^3} \leq C. \quad (2.4.39)$$

Here we have again additionally used the smoothness of U_{vsp} , the boundedness of P and Π and the moment conditions (2.4.32), (2.4.33).

Now it only remains to control the nonlinear term

$$\begin{aligned} R(g) &= \frac{1}{\varepsilon^4} [\mathcal{M}(U_{vsp} + \varepsilon^2 U_g) - \mathcal{M}(U_{vsp}) - \varepsilon^2 \nabla \mathcal{M}(U_{vsp}) \cdot U_g] \\ &= U_g \mathcal{M}''(U_{vsp} + \varepsilon^2 \vartheta U_g) U_g, \end{aligned}$$

for a $\vartheta \in (0, 1)$. By differentiation, the moment conditions in (2.4.32) and the one-dimensional Sobolev imbedding, the estimate

$$\|R(g_1) - R(g_2)\|_{H_\xi^k(L_v^2)} \leq C(\|g_1\|_{H_\xi^k(L_v^2)} + \|g_2\|_{H_\xi^k(L_v^2)})\|g_1 - g_2\|_{H_\xi^k(L_v^2)} \quad (2.4.40)$$

can be deduced to hold for all g_1, g_2 with $\|g_1\|_{H_\xi^k(L_v^2)}, \|g_2\|_{H_\xi^k(L_v^2)} \leq C\varepsilon^{-2}$ in general. If V is compact, the ball of admissible functions g_1, g_2 has to be reduced to a ball with a radius $C_0\varepsilon^{-1}$. Due to the construction of V this guarantees that the supports of the Maxwellians resulting from Taylor expansions stay in V .

We shall give the norm of g according to (2.4.25) explicitly:

$$\|g\|^* = \|(z, w)\|^* = \|z\|_{H_\xi^3} + \varepsilon\|w\|_{H_\xi^3(L_v^2)}. \quad (2.4.41)$$

Hence obviously $\|g\|_{H_\xi^3(L_v^2)} \leq C\|g\|^*$, and the existence and uniqueness result is an immediate consequence from Theorem 2.4.1.

For the oncoming examples it now only remains to give a concrete setting for the Maxwellians and the kinetic entropies, such that (2.4.32)-(2.4.33) hold.

BGK-models for scalar conservation laws

We have already mentioned that the monotonicity condition on the Maxwellian, $\partial_U \mathcal{M} > 0$, provides a kinetic entropy. Considering $V = \mathbb{R}$, we linearize around the left states and hence the inner product in v can be written as

$$\langle f, g \rangle_v = \int \frac{f g}{\partial_U \mathcal{M}_-} d\mu.$$

As long as the Maxwellians satisfy the conditions corresponding to (2.4.32) and (2.4.33), the existence result is an immediate consequence.

The BGK-model for the isothermal system and the gas dynamics

In both cases we have smooth Maxwellians with $V = \mathbb{R}$. The conditions (2.4.32) and (2.4.33) can be checked by direct calculations.

The BGK-model for the isentropic system

In this example the Maxwellians under consideration have a compact support. Hence we have to construct a Maxwellian $\mathcal{M} := \mathcal{M}(\hat{\rho}, \hat{u})$ with a bigger support than all other functions appearing in our calculations. For simplicity we denote in the following $\gamma = 1 + 2\alpha$. Then the support of $\mathcal{M}(\rho_f, u_f)$ is bounded by

$$u_f - \frac{c_f}{\sqrt{\alpha}} \leq v \leq u_f + \frac{c_f}{\sqrt{\alpha}}. \quad (2.4.42)$$

As we have already seen, the macroscopic profiles of the travelling wave will be monotone, i.e. $\partial_\xi \rho_f > 0$ and

$$\partial_\xi u_f = \frac{s\rho_f - \rho_f u_f}{\rho_f^2} \partial_\xi \rho_f = \frac{s\rho_- - \rho_- u_-}{\rho_f^2} \partial_\xi \rho_f = -\frac{(c_- - \varepsilon\sigma)\rho_-}{\rho_f^2} \partial_\xi \rho_f < 0.$$

Now one can see that the left hand side of (2.4.42) is strictly decreasing. An expansion shows that also the right hand side of (2.4.42) is decreasing, and hence neither $\mathcal{M}_-$ nor $\mathcal{M}_+$ provide a large enough support. We choose

$$\hat{u} = u_-, \quad \hat{c} = c_+(1 + \varepsilon/\rho_+), \quad (2.4.43)$$

defining $\hat{\rho}$ and $\hat{u}$ uniquely. Then for ε small $\mathcal{M}$ has the desired properties, i.e. the support of $\mathcal{M}$ includes the supports of all $\mathcal{M}(\rho_f, u_f)$ plus an additional range of order ε . And thus we linearize from now on around the Maxwellian $\mathcal{M}$ with the support $V := \left[\hat{u} - \frac{\hat{c}}{\sqrt{\alpha}}, \hat{u} + \frac{\hat{c}}{\sqrt{\alpha}} \right]$. The inner product (2.3.3) reads

$$\langle f, g \rangle_v := \frac{1}{2\beta b^{\frac{1}{\beta}}} \int f g \mathcal{M}^{\frac{1}{\beta}-1} dv, \quad \text{for } \text{supp } f, \text{supp } g \subset V.$$

Now it only remains to check (2.4.32), i.e. for $\mathcal{M}(\rho, u)$ with $\text{supp } \mathcal{M}(\rho, u) \subset V$:

$$\sup_\xi \left| \int \left(\partial_\rho^j \partial_{\rho u}^k \mathcal{M}(\rho, u) \right)^2 \mathcal{M}^{\frac{1}{\beta}-1} dv \right| < C, \quad (2.4.44)$$

for $j + k = 0, \dots, 5$. In order to guarantee that this holds, we have to make a technical assumption and restrict in the following α to the values

$$0 < \alpha < \frac{1}{17}, \quad \text{or equivalently} \quad 1 < \gamma < 1 + \frac{2}{17}.$$

It is sufficient to show the uniform boundedness of

$$\int_{\text{supp } \mathcal{M}(\rho, u)} \left(\frac{c^2}{\alpha} - (v - u)^2 \right)^{2(\beta-n)} \left(\frac{\hat{c}^2}{\alpha} - (v - \hat{u})^2 \right)^{1-\beta} dv, \quad \text{for } n = 0, \dots, 5. \quad (2.4.45)$$

The assumption $\text{supp } \mathcal{M}(\rho, u) \subset V$ implies

$$\left(\frac{c}{\sqrt{\alpha}} + u - v \right) \left(\frac{c}{\sqrt{\alpha}} - u + v \right) \leq \left(\frac{\hat{c}}{\sqrt{\alpha}} + \hat{u} - v \right) \left(\frac{\hat{c}}{\sqrt{\alpha}} - \hat{u} + v \right),$$

for all $v \in \text{supp } \mathcal{M}(\rho, u)$ and $\xi \in \mathbb{R}$, and hence, assuming for the moment $\beta > 1$, the integral in (2.4.45) is bounded by

$$\int \left(\frac{c^2}{\alpha} - (v - u)^2 \right)_+^{\beta+1-2n} dv.$$

A transformation of variable leads to the Beta-function and hence (2.4.32) is valid only if $\beta + 1 - 2n > -1$, i.e. $\beta > 8$ or equivalently $0 < \alpha < 1/17$.

2.5 Stability of kinetic shock profiles for weak shocks

2.5.1 Stability of viscous shock profiles

Goodman [21] shows the asymptotic stability of viscous shock profiles for hyperbolic conservation laws with a positive definite viscosity. Kawashima and Matsumura investigated the asymptotic stability of travelling wave solutions of some systems for one-dimensional gas motion [25]. In particular a decay rate for the scalar conservation law and a stability proof for the Navier-Stokes equations in Lagrangian coordinates are given. The stability for the isentropic gas dynamics in Lagrangian coordinates was derived by Matsumura and Nishihara in [31].

We consider a viscous regularization of the conservation law in terms of travelling wave coordinates and of a parabolic time scale:

$$\varepsilon \partial_t U + (J'(U) - s) \partial_\xi U = \varepsilon \hat{D} \partial_\xi^2 U,$$

where, for simplicity, the diffusivity matrix is considered constant. A viscous profile U_{vsp} satisfies the stationary version

$$(J'(U_{vsp}) - s) \partial_\xi U_{vsp} = \varepsilon \hat{D} \partial_\xi^2 U_{vsp}.$$

We introduce the perturbation by $\varepsilon U_G(t, \xi) := U(t, \xi) - U_{vsp}(\xi)$ and assume the 'well-preparedness' condition

$$\int_{-\infty}^{\infty} U_G(0, \xi) d\xi = 0, \quad (2.5.1)$$

for the initial data. This should fix the shift of the asymptotic travelling wave such that we expect convergence of U_G to zero. The equation for U_G can be written as

$$\partial_t U_G + \frac{1}{\varepsilon} \partial_\xi [(J'(U_{vsp}) - s) U_G] + \partial_\xi r(U_G) = \hat{D} \partial_\xi^2 U_G, \quad (2.5.2)$$

with the nonlinearity $r(U) = [J(U_{vsp} + \varepsilon U) - J(U_{vsp}) - \varepsilon J'(U_{vsp})U]/\varepsilon^2$. One of the basic assumptions of the analysis we present here, will be the existence of a symmetric, positive definite, U -dependent matrix $\Lambda(U)$, such that $\Lambda(U)J'(U)$ is symmetric and such that $\Lambda(U_{vsp})\hat{D} \geq \kappa > 0$ is positive definite. A possible candidate is the Hessian $\nabla^2 \eta(U)$ of the entropy density, which satisfies the symmetrization property, and the matrix $\nabla^2 \eta(U)D(U)$ with the Chapman-Enskog diffusivity is always symmetric and positive semidefinite (compare to [29]). Positive definiteness

cannot be expected in general, as the examples in Section 2.3.3 show. For the case of non-definiteness, the details of the stability estimates will depend on the structure of $D(U)$. An example is carried out below.

Positive definiteness of $\nabla^2 \eta(U_{vsp})D(U_{vsp})$ is of course preserved, when $D(U_{vsp})$ is replaced by a constant approximation, say $\hat{D} = D(\hat{U})$.

Assumption 6. For every $U \in \mathbb{R}^n$ there exists a symmetric positive definite matrix $\Lambda(U)$, smoothly depending on U , such that $\Lambda(U)J'(U)$ is symmetric and $\Lambda(U_{vsp})\hat{D} \geq \kappa > 0$.

Taking the scalar product in L_ξ^2 of (2.5.2) with $\Lambda(U_{vsp})U_G$ gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|U_G\|_\Lambda^2 - \frac{1}{2\varepsilon} \langle U_G, [\partial_\xi \Lambda(J' - s) - \Lambda \partial_\xi J'] U_G \rangle_\xi - \langle \partial_\xi (\Lambda U_G), r(U_G) \rangle_\xi \\ &= -\langle \partial_\xi (\Lambda U_G), \hat{D} \partial_\xi U_G \rangle_\xi, \end{aligned} \quad (2.5.3)$$

where we used the weighted L^2 -norm $\|U\|_\Lambda^2 := \langle \Lambda U, U \rangle_\xi$. It is well known that stability cannot be proven based only on this equation. The main reason is that the bracket in the second term has the unfavourable definiteness in general. An example is the scalar case, where $\Lambda = 1$ and J' is a decreasing function of ξ along a shock profile.

We shall still extract some information from an estimate based on (2.5.3). Using the fact that $\partial_\xi U_{vsp} = O(\varepsilon)$, and that r is quadratic in the sense that $|r(U)| \leq C(|U|)U^2$ (with an increasing function C), standard estimation leads to

$$\frac{d}{dt} \|U_G\|_\Lambda^2 + \kappa \|\partial_\xi U_G\|_\xi^2 \leq C(\|U_G\|_\infty) \|U_G\|_\xi^2. \quad (2.5.4)$$

It is by now a standard method to introduce the primitive $W(t, \xi) = \int_{-\infty}^\xi U_G(t, \xi') d\xi'$. The assumption (2.5.1) on the initial data and the conservation property imply the far field conditions

$$W(t, \pm\infty) = 0. \quad (2.5.5)$$

Integration of (2.5.2) gives

$$\partial_t W + \frac{1}{\varepsilon} (J'(U_{vsp}) - s) \partial_\xi W + r(U_G) = \hat{D} \partial_\xi^2 W. \quad (2.5.6)$$

As above, we test with $\Lambda(U_{vsp})W$:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|W\|_\Lambda^2 - \frac{1}{2\varepsilon} \langle W, \partial_\xi [\Lambda(J' - s)] W \rangle_\xi + \langle \Lambda W, r(U_G) \rangle_\xi \\ &= -\langle \partial_\xi W, \Lambda \hat{D} \partial_\xi W \rangle_\xi - \langle W, \partial_\xi \Lambda \hat{D} \partial_\xi W \rangle_\xi. \end{aligned} \quad (2.5.7)$$

Now it is reasonable to assume that the second term has the favourable sign. The last term we estimate as

$$\left| \langle W, \partial_\xi \Lambda \hat{D} \partial_\xi W \rangle_\xi \right| \leq \frac{\kappa}{2} \|\partial_\xi W\|_\xi^2 + c \|\partial_\xi \Lambda W\|_\xi^2.$$

A somewhat stronger version of the above assumption is that

$$-\frac{1}{2\varepsilon}\langle W, \partial_\xi[\Lambda(J' - s)]W \rangle_\xi - c\|\partial_\xi \Lambda W\|_\xi^2 \geq 0.$$

With the properties of the nonlinearity we obtain

$$\frac{d}{dt}\|W\|_\Lambda^2 + \left(\kappa - C(\|U_G\|_\infty)\|W\|_\infty\right)\|U_G\|_\xi^2 \leq 0. \quad (2.5.8)$$

For our estimates (2.5.4) and (2.5.8) to be useful we need pointwise-in-time control of $\|U_G\|_\infty$. This will be provided by an L^2 -estimate on $V := \partial_\xi U_G$ and Sobolev imbedding. The derivative of (2.5.2) with respect to ξ can be written as

$$\partial_t V + \frac{1}{\varepsilon}\partial_\xi \left((J'(U_{vsp}) - s)V + \partial_\xi J' U_G \right) + \partial_\xi^2 r(U_G) = \hat{D}\partial_\xi^2 V, \quad (2.5.9)$$

We treat this equation similarly to (2.5.2) and (2.5.6), but omit the details. The result is the estimate

$$\frac{d}{dt}\|\partial_\xi U_G\|_\Lambda^2 + \kappa\|\partial_\xi^2 U_G\|_\xi^2 \leq C(\|U_G\|_\infty)(\|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2). \quad (2.5.10)$$

The stability proof is completed by a combination of (2.5.4), (2.5.8), and (2.5.10). For positive constants γ_1, γ_2 , we define

$$I(t) := \|W\|_\Lambda^2 + \gamma_1\|U_G\|_\Lambda^2 + \gamma_2\|\partial_\xi U_G\|_\Lambda^2.$$

Then, by Sobolev imbedding,

$$\|W\|_\infty^2 + \|U_G\|_\infty^2 \leq cI.$$

With $M := \sqrt{cI(0)}$, we assume that M is small enough, so γ_1 and γ_2 can be chosen such that

$$\kappa > C(M)(M + \gamma_1 + \gamma_2), \quad \kappa\gamma_1 > C(M)\gamma_2.$$

Then there is a positive constant λ such that

$$\frac{dI}{dt} \leq -\lambda\|U_G\|_{H_\xi^2}^2.$$

Thus, I is a Lyapunov functional. By integration with respect to time, U_G converges to zero as $t \rightarrow \infty$ in the sense that

$$\int_0^\infty \|U_G\|_{H_\xi^2}^2 dt < \infty.$$

2.5.2 A Lyapunov functional for BGK-models

Now the ideas of the preceding section will be carried over to kinetic shock profiles. Here the L^2 -energy methods for the macroscopic system will be extended to also control the microscopic part. Similar techniques have been used by Liu and Yu for the Boltzmann equation [28].

We start with the kinetic equation, written in travelling wave variables and a macroscopic diffusion scaling:

$$\varepsilon^2 \partial_t f + \varepsilon(v-s) \partial_\xi f = \mathcal{M}(U_f) - f.$$

Let φ denote a kinetic shock profile:

$$\varepsilon(v-s) \partial_\xi \varphi = \mathcal{M}(U_\varphi) - \varphi.$$

The perturbation $\varepsilon G = f - \varphi$ satisfies

$$\varepsilon^2 \partial_t G + \varepsilon(v-s) \partial_\xi G = \frac{1}{\varepsilon} [\mathcal{M}(U_\varphi + \varepsilon U_G) - \mathcal{M}(U_\varphi)] - G. \quad (2.5.11)$$

The micro-macro decomposition of the perturbation is defined by $G = U_G \cdot \nabla_U \hat{\mathcal{M}} + \varepsilon g$, where $\hat{U}$ in $\hat{\mathcal{M}} = \mathcal{M}(\hat{U})$ is a constant approximation of U_φ , and $\varepsilon g = -\mathcal{L}G$ is the microscopic projection with the linearization $\mathcal{L}$ of the collision operator around $\hat{\mathcal{M}}$. Computing the macroscopic moments of (2.5.11) gives

$$\partial_t U_G + \frac{1}{\varepsilon} (J'(\hat{U}) - s) \partial_\xi U_G + \partial_\xi J_g = 0. \quad (2.5.12)$$

Like in the previous section, we obtain an equation for $W(t, \xi) = \int_{-\infty}^\xi U_G(t, \xi') d\xi'$ by integration:

$$\partial_t W + \frac{1}{\varepsilon} (J'(\hat{U}) - s) \partial_\xi W + J_g = 0. \quad (2.5.13)$$

An equation for the microscopic part is derived by applying the microscopic projection to (2.5.11):

$$\begin{aligned} & \varepsilon^2 \partial_t g - \varepsilon \mathcal{L}((v-s) \partial_\xi g) + \nabla_U \hat{\mathcal{M}} \cdot (v - J'(\hat{U})) \partial_\xi U_G \\ &= -g + \frac{1}{\varepsilon} U_G \cdot [\nabla_U \mathcal{M}(U_\varphi) - \nabla_U \hat{\mathcal{M}}] + R(U_G), \end{aligned} \quad (2.5.14)$$

with

$$R(U) = \frac{1}{\varepsilon^2} [\mathcal{M}(U_\varphi + \varepsilon U) - \mathcal{M}(U_\varphi) - \varepsilon U \cdot \nabla_U \mathcal{M}(U_\varphi)].$$

The next step is to compute the last term in (2.5.13) in the spirit of the Chapman-Enskog approximation by computing g from (2.5.14):

$$\partial_t W + \frac{1}{\varepsilon} (J'(U_\varphi) - s) \partial_\xi W - \hat{D} \partial_\xi^2 W = \varepsilon^2 \partial_t J_g - \varepsilon \partial_\xi J_{\mathcal{L}((v-s)g)} - r(U_G), \quad (2.5.15)$$

with $\hat{D} = D(\hat{U})$ and

$$r(U) = J_{R(U)} = \frac{1}{\varepsilon^2} [J(U_\varphi + \varepsilon U) - J(U_\varphi) - \varepsilon J'(U_\varphi)U].$$

In the same way we derived (2.5.8) in the previous section, we obtain

$$\frac{d}{dt} \|W\|_\Lambda^2 + [\kappa - C(\|U_G\|_\infty) \|W\|_\infty] \|U_G\|_\xi^2 \leq \varepsilon^2 \langle \Lambda W, \partial_t J_g \rangle_\xi - \varepsilon \langle \Lambda W, \partial_\xi J_{\mathcal{L}((v-s)g)} \rangle_\xi.$$

The first term on the right hand side we rewrite using (2.5.13):

$$\langle \Lambda W, \partial_t J_g \rangle_\xi = \frac{d}{dt} \langle \Lambda W, J_g \rangle_\xi + \frac{1}{\varepsilon} \langle \Lambda(J'(\hat{U}) - s)U_G, J_g \rangle_\xi + \|J_g\|_\Lambda^2,$$

leading to

$$\begin{aligned} & \frac{d}{dt} \left(\|W\|_\Lambda^2 - \varepsilon^2 \langle \Lambda W, J_g \rangle_\xi \right) + [\kappa - C(\|U_G\|_\infty) \|W\|_\infty] \|U_G\|_\xi^2 \\ & \leq \varepsilon \langle \Lambda(J'(\hat{U}) - s)U_G, J_g \rangle_\xi + \varepsilon^2 \|J_g\|_\Lambda^2 + \varepsilon \langle \Lambda U_G, J_{\mathcal{L}((v-s)g)} \rangle_\xi, \end{aligned} \quad (2.5.16)$$

where an integration by parts has been carried out in the last term. An estimate for the microscopic part of the perturbation is derived by taking the $L_{\xi,v}^2$ -scalar product of the full perturbation equation (2.5.11) with G :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|U_G\|_\Lambda^2 + \varepsilon^2 \|g\|_{\xi,v}^2 \right] + \|g\|_{\xi,v}^2 \\ & = \left\langle U_G \cdot \frac{\nabla_U \mathcal{M}(U_\varphi) - \nabla_U \hat{\mathcal{M}}}{\varepsilon}, g \right\rangle_{\xi,v} + \langle R(U_G), g \rangle_{\xi,v}, \end{aligned} \quad (2.5.17)$$

with $\hat{\Lambda} = \nabla^2 \eta(\hat{U})$. Now we assume that the factors in the scalar products on the right hand sides of (2.5.16) and (2.5.17) are bounded linear maps of U_G and of g with the exception of the quadratic term $R(U_G)$:

$$\begin{aligned} & \frac{d}{dt} \left[\|W\|_\Lambda^2 - \varepsilon^2 \langle \Lambda W, J_g \rangle_\xi \right] + \left(\frac{\kappa}{2} - C(\|U_G\|_\infty) \|W\|_\infty \right) \|U_G\|_\xi^2 \leq \varepsilon^2 c \|g\|_{\xi,v}^2, \\ & \frac{d}{dt} \left[\|U_G\|_\Lambda^2 + \varepsilon^2 \|g\|_{\xi,v}^2 \right] + \|g\|_{\xi,v}^2 \leq C(\|U_G\|_\infty) \|U_G\|_\xi^2. \end{aligned}$$

Adding these inequalities after multiplying the second by a positive constant δ gives

$$\begin{aligned} & \frac{d}{dt} \left[\|W\|_\Lambda^2 - \varepsilon^2 \langle \Lambda W, J_g \rangle_\xi + \varepsilon^2 \delta \|g\|_{\xi,v}^2 + \delta \|U_G\|_\Lambda^2 \right] \\ & + \left(\frac{\kappa}{2} - (\delta + \|W\|_\infty) C(\|U_G\|_\infty) \right) \|U_G\|_\xi^2 + (\delta - \varepsilon^2 c) \|g\|_{\xi,v}^2 \leq 0. \end{aligned}$$

For fixed δ and ε small enough, the term under the time derivative can be bounded from below by

$$c \left[\|W\|_\xi^2 + \|U_G\|_\xi^2 + \varepsilon^2 \|g\|_{\xi,v}^2 \right],$$

with a positive constant c . So it controls $\|W\|_\infty$, but not $\|U_G\|_\infty$.

By taking the derivatives of (2.5.11) and (2.5.15) with respect to ξ , we obtain equations for G and for $H := \partial_\xi G = \partial_\xi U_G \cdot \nabla_U \hat{\mathcal{M}} + \varepsilon h$:

$$\begin{aligned} & \partial_t U_G + \frac{1}{\varepsilon} \partial_\xi \left((J'(U_\varphi) - s) U_G \right) - \hat{D} \partial_\xi^2 U_G = \varepsilon^2 \partial_t J_h - \varepsilon \partial_\xi J_{\mathcal{L}((v-s)h)} - \partial_\xi r(U_G), \\ & \varepsilon^2 \partial_t H + \varepsilon(v-s) \partial_\xi H = \frac{1}{\varepsilon} \partial_\xi \left(\mathcal{M}(U_\varphi + \varepsilon U_G) - \mathcal{M}(U_\varphi) \right) - H. \end{aligned}$$

Treating the first equation like in the previous section and the second like (2.5.11), we obtain

$$\begin{aligned} \frac{d}{dt} \left[\|U_G\|_\Lambda^2 - \varepsilon^2 \langle \Lambda U_G, J_h \rangle_\xi \right] + \kappa \|\partial_\xi U_G\|_\xi^2 &\leq C(\|U_G\|_\infty) \|U_G\|_\xi^2 + \varepsilon^2 c \|h\|_{\xi, \mathbf{v}}^2, \\ \frac{d}{dt} \left[\|\partial_\xi U_G\|_\Lambda^2 + \varepsilon^2 \|h\|_{\xi, \mathbf{v}}^2 \right] + \|h\|_{\xi, \mathbf{v}}^2 &\leq C(\|U_G\|_\infty) (\|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2). \end{aligned}$$

Now we take a linear combination of these inequalities like above:

$$\begin{aligned} \frac{d}{dt} \left[\|U_G\|_\Lambda^2 - \varepsilon^2 \langle \Lambda U_G, J_{\partial_\xi g} \rangle_\xi + \varepsilon^2 \delta \|\partial_\xi g\|_{\xi, \mathbf{v}}^2 + \delta \|\partial_\xi U_G\|_\Lambda^2 \right] \\ + (\kappa - \delta C(\|U_G\|_\infty)) \|\partial_\xi U_G\|_\xi^2 + (\delta - \varepsilon^2 c) \|\partial_\xi g\|_{\xi, \mathbf{v}}^2 \leq C(\|U_G\|_\infty) \|U_G\|_\xi^2. \end{aligned}$$

Again, the term under the time derivative is positive definite. Finally, with $\gamma > 0$ we define the Lyapunov functional by

$$\begin{aligned} I(t) := & \|W\|_\Lambda^2 - \varepsilon^2 \langle \Lambda W, J_g \rangle_\xi + \varepsilon^2 \delta \|g\|_{\xi, \mathbf{v}}^2 + \delta \|U_G\|_\Lambda^2 \\ & + \gamma \left[\|U_G\|_\Lambda^2 - \varepsilon^2 \langle \Lambda U_G, J_{\partial_\xi g} \rangle_\xi + \varepsilon^2 \delta \|\partial_\xi g\|_{\xi, \mathbf{v}}^2 + \delta \|\partial_\xi U_G\|_\Lambda^2 \right], \end{aligned}$$

and obtain

$$\begin{aligned} \frac{dI}{dt} + \left(\frac{\kappa}{2} - (\delta + \gamma + \|W\|_\infty) C(\|U_G\|_\infty) \right) \|U_G\|_\xi^2 + (\delta - \varepsilon^2 c) \|g\|_{\xi, \mathbf{v}}^2 \\ + \gamma \left(\kappa - \delta C(\|U_G\|_\infty) \right) \|\partial_\xi U_G\|_\xi^2 + \gamma (\delta - \varepsilon^2 c) \|\partial_\xi g\|_{\xi, \mathbf{v}}^2 \leq 0. \end{aligned}$$

The functional I controls $\|W\|_\xi^2 + \|U_G\|_\xi^2 + \|\partial_\xi U_G\|_\xi^2 + \varepsilon^2 \|g\|_{\xi, \mathbf{v}}^2 + \varepsilon^2 \|\partial_\xi g\|_{\xi, \mathbf{v}}^2$. So, by Sobolev imbedding,

$$\|W\|_\infty^2 + \|U_G\|_\infty^2 \leq cI$$

holds. With $M := \sqrt{cI(0)}$, I is indeed a Lyapunov functional, if

$$\frac{\kappa}{2} > (\delta + \gamma + M) C(M) \quad \text{and} \quad \delta > \varepsilon^2 c.$$

This can of course be achieved by choosing δ , γ , M , and ε small enough.

2.5.3 Stability of weak kinetic profiles for the isentropic gas dynamics BGK-model

The model of Section 2.2.2 satisfies the assumptions used in the previous section except the regularity of the Chapman-Enskog diffusivity. Therefore the main steps of the analysis will be recalled from [15].

To derive estimates for the macroscopic part we adapt ideas from [31], where the stability of travelling waves for the isentropic system for a compressible viscous gas in Lagrangian coordinates is proven by L^2 -energy estimates. Control of the microscopic terms will be obtained like in the previous section.

As in the previous section, we start with the kinetic equation in diffusion scaling:

$$\varepsilon^2 \partial_t f + \varepsilon(v-s) \partial_\xi f = \mathcal{M}(\rho_f, m_f) - f, \quad (2.5.18)$$

with the far-field conditions $f(t, \xi = \pm\infty, v) = \mathcal{M}(\rho_\pm, m_\pm, v)$. As in Section 2.2.2 we shall switch between the momentum density and the mean velocity, connected by $m_f = \rho_f u_f$, as second macroscopic variable. Let φ be the travelling wave solution. The well-preparedness condition for the initial data now reads

$$\int_{\mathbb{R}} (\rho_{f_0} - \rho_\varphi) d\xi = 0, \quad \int_{\mathbb{R}} (m_{f_0} - m_\varphi) d\xi = 0. \quad (2.5.19)$$

Introducing the perturbation G

$$\varepsilon G = f - \varphi, \quad \rho := \rho_G, \quad m := m_G,$$

we obtain

$$\varepsilon^2 \partial_t G + \varepsilon(v-s) \partial_\xi G = \frac{1}{\varepsilon} [\mathcal{M}(\rho_\varphi + \varepsilon \rho, m_\varphi + \varepsilon m) - \mathcal{M}(\rho_\varphi, m_\varphi)] - G. \quad (2.5.20)$$

As in [15] we apply a micro-macro decomposition to the deviation G

$$G = \nabla_U \mathcal{M}(\hat{\rho}, \hat{m}) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \varepsilon g, \quad (2.5.21)$$

where as before $U = (\rho, m)$. Observe that $-\mathcal{L}G = \varepsilon g$. Then the norm of G satisfies

$$\|G\|_{\xi, v}^2 = \frac{1}{\hat{\rho}} [\hat{c}^2 \|\rho\|_\xi^2 + \|m - \rho \hat{u}\|_\xi^2] + \varepsilon^2 \|g\|_{\xi, v}^2. \quad (2.5.22)$$

Macroscopic equations for ρ and m are obtained by computing the zeroth and first order moments of equation (2.5.20)

$$\varepsilon \partial_t \rho + \partial_\xi (m - \rho s) = 0, \quad (2.5.23)$$

$$\varepsilon \partial_t m + \partial_\xi \left(\nabla_U j(\hat{\rho}, \hat{m}) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} - sm \right) + \varepsilon \partial_\xi \int v^2 g dv = 0. \quad (2.5.24)$$

Next we apply $-\mathcal{L}$ to (2.5.20) to get an equation for g

$$\varepsilon^2 \partial_t g - \partial_\xi \left(\nabla_U \mathcal{M} \cdot \left[(J'(\hat{\rho}, \hat{m}) - v) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} \right] \right) - \varepsilon \partial_\xi \mathcal{L}((v-s)g) = R(\rho, m) - g, \quad (2.5.25)$$

with the nonlinearity

$$R(\rho, m) = \frac{1}{\varepsilon^2} \left[\mathcal{M}(\rho_\varphi + \varepsilon \rho, m_\varphi + \varepsilon m) - \mathcal{M}(\rho_\varphi, m_\varphi) - \varepsilon \nabla_U \mathcal{M} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} \right].$$

Using equation (2.5.25) we calculate

$$\int_{\mathbb{R}} v^2 g dv = q(\rho, m) - \varepsilon S(g) - \hat{D} \partial_\xi (m - \rho \hat{u}), \quad (2.5.26)$$

with the constant $\hat{D} := (3 - \gamma)\hat{\rho}^{\gamma-1} > 0$, the nonlinearity $q(\rho, m) := \int_{\mathbb{R}} v^2 R dv$ and

$$S(g) = \int_{\mathbb{R}} v^2 (\varepsilon \partial_t g - \mathcal{L}((v - s)\partial_\xi g)) dv. \quad (2.5.27)$$

The stability of the shock profiles will be investigated by introducing primitives of the macroscopic variables. According to (2.5.22) and the diffusion term in (2.5.26), it is convenient to use

$$W_\rho(t, \xi) = \int_{-\infty}^{\xi} \rho(t, \xi') d\xi', \quad W_u(t, \xi) = \int_{-\infty}^{\xi} (m(t, \xi') - \rho(t, \xi')\hat{u}) d\xi'.$$

Integrating (2.5.23), (2.5.24) with respect to ξ gives the macroscopic equations

$$\partial_t W_\rho + \frac{1}{\varepsilon} [\partial_\xi W_u + (\hat{c} - \varepsilon \hat{\sigma}) \partial_\xi W_\rho] = 0, \quad (2.5.28)$$

$$\partial_t W_u + \frac{1}{\varepsilon} [(\hat{c} - \varepsilon \hat{\sigma}) \partial_\xi W_u + \hat{c}^2 \partial_\xi W_\rho] + q - \hat{D} \partial_\xi^2 W_u = \varepsilon S(g). \quad (2.5.29)$$

Observe that the second equation is obtained by a linear combination of (2.5.23), (2.5.24). We expand q as follows

$$\begin{aligned} q(\rho, m) &= \frac{1}{\varepsilon} (\nabla_U j(\rho_\varphi, m_\varphi) - \nabla_U j(\hat{\rho}, \hat{m})) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \tilde{q}(\rho, m), \\ \tilde{q}(\rho, m) &= \frac{1}{\varepsilon^2} \left(j(\rho_\varphi + \varepsilon \rho, m_\varphi + \varepsilon m) - j(\rho_\varphi, m_\varphi) - \varepsilon \nabla_U j(\rho_\varphi, m_\varphi) \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} \right). \end{aligned}$$

and note that $\tilde{q}$ is purely quadratic in (ρ, m) .

Now the system (2.5.28), (2.5.29) can equivalently be stated as

$$\partial_t W_\rho + \frac{1}{\varepsilon} [\partial_\xi W_u + (\hat{c} - \varepsilon \hat{\sigma}) \partial_\xi W_\rho] = 0, \quad (2.5.30)$$

$$\partial_t W_u + \frac{1}{\varepsilon} [K_2(\varphi) \partial_\xi W_u + K_1(\varphi) \partial_\xi W_\rho] + \tilde{q} - \hat{D} \partial_\xi^2 W_u = \varepsilon S(g), \quad (2.5.31)$$

where

$$K_1(\varphi) := c_\varphi^2 - (u_\varphi - \hat{u})^2, \quad K_2(\varphi) := \hat{c} - \varepsilon \hat{\sigma} + 2(u_\varphi - \hat{u}). \quad (2.5.32)$$

We will need the signs of K_1, K_2 and of their derivatives. From Lemma 2.4.3 we know that the travelling wave is strictly increasing, which also implies $\partial_\xi u_\varphi < 0$. Then for ε small we get

$$\frac{\hat{c}^2}{2} < K_1(\varphi) < 2\hat{c}^2, \quad \partial_\xi K_1(\varphi) > 0, \quad \partial_\xi (K_1(\varphi)^{-1}) < 0, \quad (2.5.33)$$

$$\frac{\hat{c}}{2} < K_2(\varphi) < 2\hat{c}, \quad \partial_\xi K_2(\varphi) < 0. \quad (2.5.34)$$

Recall from Theorem 2.4.1 that $\partial_\xi K_1(\varphi), \partial_\xi K_2(\varphi)$ are $O(\varepsilon)$ uniformly in ξ .

We start with the derivation of estimates for the macroscopic parts. For controlling the nonlinear terms, L_ξ^∞ -bounds of ρ, m are needed, which we shall control in

H_ξ^1 . This means we need to control the H_ξ^2 -norm of W_ρ, W_u and therefore we give integral estimates for their derivatives up to second order in the following.

Expanding $(\rho_\varphi, m_\varphi)$ around $(\hat{\rho}, \hat{m})$ gives $\rho_\varphi = \rho_- + \varepsilon y = \hat{\rho} + \varepsilon \hat{y}_1$ and $m_\varphi = m_- + \varepsilon y = \hat{m} + \varepsilon \hat{y}_2$ and we can write the nonlinearity as

$$R(\rho, m) = (\hat{y}_1, \hat{y}_2) \cdot \nabla^2 \mathcal{M}_1 \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + (\rho, m) \cdot \nabla^2 \mathcal{M}_2 \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad (2.5.35)$$

where $\mathcal{M}_1 = \mathcal{M}(\hat{\rho} + \varepsilon \vartheta_1 \hat{y}_1, \hat{m} + \varepsilon \vartheta_1 \hat{y}_2)$, $\mathcal{M}_2 = \mathcal{M}(\rho_\varphi + \varepsilon \vartheta_2 \rho, m_\varphi + \varepsilon \vartheta_2 m)$ and $0 \leq \vartheta_1, \vartheta_2 \leq 1$. For $\|R\|_\nu$ to be well defined we have to guarantee that $\text{supp } \mathcal{M}_1, \text{supp } \mathcal{M}_2 \subset \text{supp } \hat{\mathcal{M}}$. Due to the construction of $\hat{\mathcal{M}}$ this holds for $\mathcal{M}_1$. For $\mathcal{M}_2$ this is only true for sufficiently small $\|\rho\|_\infty, \|m\|_\infty$. We make this smallness assumption for the moment and prove it in the stability result at the end of this section. By differentiating (2.5.35) and using (2.4.32), we obtain

$$\|R\|_{H_\xi^k(L_\nu^2)}^2 \leq \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right] \quad \text{for } k = 0, 1, 2, \quad (2.5.36)$$

implying together with (2.4.33) the same bound for q

$$\|q\|_{H_\xi^k}^2 \leq \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right] \quad \text{for } k = 0, 1, 2. \quad (2.5.37)$$

Here and in the following $\tilde{C}$ depends on $\|\rho\|_\infty, \|m\|_\infty$.

Lemma 2.5.1. *Let W_ρ, W_u be the solution of the system (2.5.28), (2.5.29). Then there exists a constant C and $\tilde{C}(\|\rho\|_\infty, \|m\|_\infty)$ such that the following two estimates hold for any $\alpha_k > 0, k = 0, 1, 2$:*

$$\begin{aligned} & \frac{d}{dt} J_0 + \int_{\mathbb{R}} \kappa(\varphi) W_u^2 d\xi + \left(\alpha_0 \left(\frac{\hat{c}^2}{2} - \varepsilon \tilde{C} \right) - \tilde{C} \|W_u\|_\infty \right) \|\partial_\xi W_\rho\|_\xi^2 \\ & + \left(\frac{\hat{D}}{2\hat{c}^2} - \varepsilon C - \tilde{C} \|W_u\|_\infty - \alpha_0 (1 + \varepsilon \tilde{C}) \right) \|\partial_\xi W_u\|_\xi^2 \\ & \leq \varepsilon \int_{\mathbb{R}} \left(K_1(\varphi)^{-1} W_u + \varepsilon \alpha_0 \partial_\xi W_\rho \right) S(g) d\xi, \end{aligned} \quad (2.5.38)$$

where

$$J_0 = \frac{1}{2} \int_{\mathbb{R}} \left[W_\rho^2 + K_1(\varphi)^{-1} W_u^2 + \varepsilon \alpha_0 (\varepsilon \hat{D} (\partial_\xi W_\rho)^2 + 2(\partial_\xi W_\rho) W_u) \right] d\xi \quad (2.5.39)$$

and

$$\kappa(\phi) = \frac{1}{2\varepsilon} \left[-\partial_\xi (K_2(\varphi) K_1(\varphi)^{-1}) - 2\varepsilon \hat{D} |\partial_\xi (K_1(\varphi)^{-1})| \right] > 0,$$

and accordingly for the higher order derivatives $k = 1, 2$:

$$\begin{aligned} & \frac{d}{dt} J_k + \alpha_k \left(\frac{\hat{c}^2}{2} - \varepsilon^2 \tilde{C} \right) \|\partial_\xi^{k+1} W_\rho\|_\xi^2 + \left(\frac{\hat{D}}{2} - \alpha_k (1 + \varepsilon^2 \tilde{C}) \right) \|\partial_\xi^{k+1} W_u\|_\xi^2 \\ & - \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^{k-1}}^2 + \|\partial_\xi W_u\|_{H_\xi^{k-1}}^2 \right] \leq \varepsilon \int_{\mathbb{R}} \left(\partial_\xi^k W_u + \varepsilon \alpha_k \partial_\xi^{k+1} W_\rho \right) \partial_\xi^k S(g) d\xi, \end{aligned} \quad (2.5.40)$$

where

$$J_k = \frac{1}{2} \int_{\mathbb{R}} \left[\hat{c}^2 (\partial_{\xi}^k W_{\rho})^2 + (\partial_{\xi}^k W_u)^2 + \varepsilon \alpha_k (\varepsilon \hat{D} (\partial_{\xi}^{k+1} W_{\rho})^2 + 2 \partial_{\xi}^{k+1} W_{\rho} \partial_{\xi}^k W_u) \right] d\xi. \quad (2.5.41)$$

Proof. We start with the proof of (2.5.38) and split it into two steps. First we derive estimate (2.5.38) with $\alpha_0 = 0$ and in the second step we prove the inequality for the remaining terms containing α_0 .

Step 1: We test (2.5.30) with W_{ρ} and (2.5.31) with $K_1(\varphi)^{-1} W_u$ such that the integrals containing $W_{\rho} \partial_{\xi} W_u$ and $W_u \partial_{\xi} W_{\rho}$ cancel out. Here we also take advantage of the properties of K_1 and K_2 , see (2.5.33), (2.5.34):

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left[W_{\rho}^2 + K_1(\varphi)^{-1} W_u^2 \right] d\xi + \frac{1}{\varepsilon} \int \partial_{\xi} \left(-K_2(\varphi) K_1(\varphi)^{-1} \right) \frac{W_u^2}{2} d\xi \\ & + \int K_1(\varphi)^{-1} W_u \tilde{q} d\xi + \hat{D} \int \left[K_1(\varphi)^{-1} (\partial_{\xi} W_u)^2 + \partial_{\xi} \left(K_1(\varphi)^{-1} \right) \partial_{\xi} \frac{W_u^2}{2} \right] d\xi \\ & = \varepsilon \int K_1(\varphi)^{-1} W_u S(g) d\xi. \end{aligned}$$

In the third term we estimate the quadratic term by

$$\int |\tilde{q}| d\xi \leq \tilde{C} \left[\|\partial_{\xi} W_{\rho}\|_{\xi}^2 + \|\partial_{\xi} W_u\|_{\xi}^2 \right]. \quad (2.5.42)$$

The triangle inequality is used for the fourth term

$$\left| \int \partial_{\xi} \left(K_1(\varphi)^{-1} \right) (\partial_{\xi} W_u) W_u d\xi \right| \geq - \int \left[|\partial_{\xi} \left(K_1(\varphi)^{-1} \right)| W_u^2 + \varepsilon C (\partial_{\xi} W_u)^2 \right] d\xi.$$

From this estimate we cannot control $\|\partial_{\xi} W_{\rho}\|_{\xi}^2$. Therefore we will combine it with the next one.

Step 2: Testing the first derivative of (2.5.30) with $\partial_{\xi} W_{\rho}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int (\partial_{\xi} W_{\rho})^2 d\xi + \frac{1}{\varepsilon} \int \partial_{\xi}^2 W_u \partial_{\xi} W_{\rho} d\xi = 0,$$

and we observe that

$$\frac{d}{dt} \int (\partial_{\xi} W_{\rho}) W_u d\xi = \int (\partial_t W_u \partial_{\xi} W_{\rho} - \partial_t W_{\rho} \partial_{\xi} W_u) d\xi.$$

By combining the equations in the corresponding way we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left[\varepsilon^2 \hat{D} (\partial_{\xi} W_{\rho})^2 + 2 \varepsilon (\partial_{\xi} W_{\rho}) W_u \right] d\xi + \int K_1(\varphi) (\partial_{\xi} W_{\rho})^2 d\xi - \int (\partial_{\xi} W_u)^2 d\xi \\ & + \int (K_2(\varphi) - (\hat{c} - \varepsilon \hat{\sigma})) \partial_{\xi} W_u \partial_{\xi} W_{\rho} d\xi + \varepsilon \int \partial_{\xi} W_{\rho} \tilde{q} d\xi = \varepsilon^2 \int \partial_{\xi} W_{\rho} S(g) d\xi. \end{aligned}$$

For the fourth term on the left hand side we use the triangle inequality together with $K_2(\varphi) - (\hat{c} - \varepsilon \hat{\sigma}) = 2(u_\varphi - \hat{u}) = O(\varepsilon)$. Finally applying (2.5.42) gives the estimate.

For the bounds on the higher order derivatives we proceed as above. First we differentiate the system (2.5.28), (2.5.29) k times and test it with $\hat{c}^2 \partial_\xi^k W_\rho$, respectively $\partial_\xi^k W_u$. Applying

$$\int \partial_\xi^k q \partial_\xi^k W_u d\xi = - \int \partial_\xi^{k-1} q \partial_\xi^{k+1} W_u d\xi \geq -\frac{\hat{D}}{2} \left[\|\partial_\xi^{k+1} W_u\|_\xi^2 + \frac{1}{\hat{D}^2} \|q\|_{H_\xi^{k-1}}^2 \right]$$

the inequality for $\alpha_k = 0$ is straightforward. The remaining part is analogous as Step 2 above. $\square$.

Now we concentrate on bounds for the small terms on the right hand sides in (2.5.38) and (2.5.40). Here the estimates from [17] are extended.

Lemma 2.5.2. *Let W_ρ, W_u, g and $S(g)$ satisfy (2.5.26)-(2.5.29). Then there exists a constant C such that*

$$\begin{aligned} & \varepsilon \int K_1(\varphi)^{-1} W_u S(g) d\xi - \int \kappa(\varphi) W_u^2 d\xi \\ & \leq \varepsilon^2 \frac{d}{dt} \int K_1(\varphi)^{-1} W_u \int v^2 g dv d\xi + \varepsilon C \left[\|g\|_{\xi, v}^2 + \|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2 \right], \end{aligned} \quad (2.5.43)$$

and additionally for $k = 0, 1, 2$

$$\begin{aligned} & \varepsilon \int \partial_\xi^k W_u \partial_\xi^k S(g) d\xi \\ & \leq \varepsilon^2 \frac{d}{dt} \int \partial_\xi^k W_u \int v^2 \partial_\xi^k g dv d\xi + \varepsilon C \left[\|\partial_\xi^k g\|_{\xi, v}^2 + \|\partial_\xi^{k+1} W_\rho\|_\xi^2 + \|\partial_\xi^{k+1} W_u\|_\xi^2 \right]. \end{aligned} \quad (2.5.44)$$

Proof.

$$\begin{aligned} & \int K_1(\varphi)^{-1} W_u S(g) d\xi \\ & = \varepsilon \frac{d}{dt} \int K_1(\varphi)^{-1} W_u \int v^2 g dv d\xi + \varepsilon \int K_1(\varphi)^{-1} \left(\int v^2 g dv \right)^2 d\xi \\ & \quad + \int K_1(\varphi)^{-1} (\hat{c}^2 \partial_\xi W_\rho + (\hat{c} - \varepsilon \hat{\sigma}) \partial_\xi W_u) \int v^2 g dv d\xi \\ & \quad + \int \left[K_1(\varphi)^{-1} \partial_\xi W_u + \partial_\xi (K_1(\varphi)^{-1}) W_u \right] \int v^2 \mathcal{L}((v-s)g) dv d\xi \\ & \leq \varepsilon \frac{d}{dt} \int K_1(\varphi)^{-1} W_u \int v^2 g dv d\xi + C \left(\|g\|_{\xi, v}^2 + \|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2 \right) \\ & \quad + \int \partial_\xi \left(-\frac{1}{2K_1(\varphi)} \right) \left[\left(\int v^2 \mathcal{L}((v-s)g) dv \right)^2 + W_u^2 \right] d\xi. \end{aligned}$$

For the first equality we used (2.5.26) and (2.5.29), moreover (2.4.32) and (2.4.33), which also implies $\int v^2 \mathcal{L}((v-s)\partial_\xi^k g) dv \leq C \|\partial_\xi^k g\|_v$. Finally to control the last term the function $\kappa(\varphi)$ is needed. We multiply the above inequality with ε and use $\kappa(\varphi) + \varepsilon \partial_\xi (K_1(\varphi)^{-1})/2 \geq 0$.

The proof for the second estimate is similar. $\square$

Lemma 2.5.3. *Let W_ρ, W_u, g and $S(g)$ satisfy (2.5.26)-(2.5.29). Then there exists a constant C and a $\tilde{C}$ such that for $k = 0, 1, 2$*

$$\begin{aligned} & \varepsilon^2 \int \partial_\xi^{k+1} W_\rho \partial_\xi^k S(g) d\xi \\ & \leq \varepsilon^2 \frac{\hat{D}}{2} \frac{d}{dt} \|\partial_\xi^{k+1} W_\rho\|_\xi^2 + \varepsilon \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right] + \varepsilon C \|\partial_\xi^k g\|_{\xi,v}^2. \end{aligned}$$

Proof. These estimates cannot be derived in the same way as before, since the derivatives in the bounds would get too high and could not be controlled by the macroscopic estimates anymore. Here we take advantage of ε^2 . We use equation (2.5.26) for $S(g)$

$$\varepsilon^2 \partial_\xi^k S(g) = \varepsilon \left(\partial_\xi^k q - \hat{D} \partial_\xi^{k+2} W_u - \int v^2 \partial_\xi^k g dv \right).$$

Now we substitute $\partial_\xi^{k+2} W_u$ according to (2.5.28) implying

$$\begin{aligned} \varepsilon^2 \int \partial_\xi^{k+1} W_\rho \partial_\xi^k S(g) d\xi &= \varepsilon \int \partial_\xi^k q \partial_\xi^{k+1} W_\rho d\xi + \varepsilon^2 \int \hat{D} \partial_t \frac{(\partial_\xi^{k+1} W_\rho)^2}{2} d\xi \\ &\quad - \varepsilon \int \int v^2 \partial_\xi^k g \partial_\xi^{k+1} W_\rho dv d\xi. \end{aligned}$$

$\square$

For getting control of the microscopic terms we derive estimates from the full kinetic perturbation equation.

Lemma 2.5.4. *Let G , decomposed as in (2.5.21), be the solution of (2.5.20). Then there exists a $\tilde{C}$ such that for $k = 0, 1, 2$*

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{\hat{\rho}} \left(\hat{c}^2 \|\partial_\xi^{k+1} W_\rho\|_\xi^2 + \|\partial_\xi^{k+1} W_u\|_\xi^2 \right) + \varepsilon^2 \|\partial_\xi^k g\|_{\xi,v}^2 \right] + \|\partial_\xi^k g\|_{\xi,v}^2 \\ & \leq \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right]. \end{aligned}$$

Proof. The k th derivative of (2.5.20) is tested with $\partial_\xi^k G$. For more details see [17]. $\square$

Now we are able to prove the main result of this section.

Theorem 2.5.1. *Let the assumptions of Theorem 2.4.1 hold and let φ be the travelling wave solution. Let $f_0(\xi, v)$ be the initial datum for (2.5.18) and let*

$$\begin{aligned} W_{\rho,0}(\xi) &= \frac{1}{\varepsilon} \int_{-\infty}^{\xi} [\rho_{f_0}(\xi') - \rho_{\varphi}(\xi')] d\xi', \\ W_{u,0}(\xi) &= \frac{1}{\varepsilon} \int_{-\infty}^{\xi} [(m_{f_0}(\xi') - m_{\varphi}(\xi')) - \hat{u}(\rho_{f_0}(\xi') - \rho_{\varphi}(\xi'))] d\xi'. \end{aligned}$$

Moreover we assume $f_0 - \varphi \in H_{\xi}^2(L_v^2)$ (implying $f_0(\pm\infty, v) = \varphi(\pm\infty, v)$) and $W_{\rho,0}, W_{u,0} \in L_{\xi}^2$, which ensures assumption (2.5.19). Let

$$\|W_{\rho,0}\|_{L_{\xi}^2} + \|W_{u,0}\|_{L_{\xi}^2} + \frac{1}{\varepsilon} \|f_0 - \varphi\|_{H_{\xi}^2(L_v^2)} \leq \delta \quad (2.5.45)$$

for a δ small enough, which is independent from ε . Then for ε small enough equation (2.5.18) with initial data f_0 has a unique global solution. In particular, small amplitude travelling waves are locally stable in the sense that

$$\lim_{t \rightarrow \infty} \int_t^{\infty} \|f(s, \cdot) - \varphi(\cdot)\|_{H_{\xi}^2(L_v^2)}^2 ds = 0.$$

Proof. The main idea is to construct a Lyapunow functional, which is decaying in time. Recall (2.5.39), (2.5.41) and define

$$I := I_0 + \gamma_1 I_1 + \gamma_2 I_2,$$

where

$$\begin{aligned} I_0 &:= J_0 + \varepsilon C_0 \left[\frac{1}{\hat{\rho}} \left(\hat{c}^2 \|\partial_{\xi} W_{\rho}\|_{\xi}^2 + \|\partial_{\xi} W_u\|_{\xi}^2 \right) + \varepsilon^2 \|g\|_{\xi,v}^2 \right] \\ &\quad - \varepsilon^2 \left[\int \int K_1(\varphi)^{-1} v^2 g W_u dv d\xi + \alpha_0 \frac{\hat{D}}{2} \|\partial_{\xi} W_{\rho}\|_{\xi}^2 \right], \end{aligned}$$

and for $k = 1, 2$

$$\begin{aligned} I_k &:= J_k + \varepsilon C_k \left[\frac{1}{\hat{\rho}} \left(\hat{c}^2 \|\partial_{\xi}^{k+1} W_{\rho}\|_{\xi}^2 + \|\partial_{\xi}^{k+1} W_u\|_{\xi}^2 \right) + \varepsilon^2 \|\partial_{\xi}^k g\|_{\xi,v}^2 \right] \\ &\quad - \varepsilon^2 \left[\int \int v^2 \partial_{\xi}^k g \partial_{\xi}^k W_u dv d\xi + \alpha_k \frac{\hat{D}}{2} \|\partial_{\xi}^{k+1} W_{\rho}\|_{\xi}^2 \right]. \end{aligned}$$

Here the constants $C_j, j = 0, 1, 2$, are positive and independent from ε . Then for any $\gamma_1, \gamma_2 > 0$ the functional $I(t)$ is bounded from above and below by

$$\|W_{\rho}\|_{H_{\xi}^2}^2 + \|W_u\|_{H_{\xi}^2}^2 + \varepsilon \left[\|\partial_{\xi}^3 W_{\rho}\|_{\xi}^2 + \|\partial_{\xi}^3 W_u\|_{\xi}^2 \right] + \varepsilon^3 \|g\|_{H_{\xi}^2(L_v^2)}^2,$$

respectively by

$$\|W_{\rho}\|_{H_{\xi}^2}^2 + \|W_u\|_{H_{\xi}^2}^2 + \varepsilon \|G\|_{H_{\xi}^2(L_v^2)}^2.$$

We combine the estimates from Lemmata 2.5.1-2.5.4 to get a final one for the time derivative of I and write all terms on the left hand side. For an initial data and ε small enough, one can show that there exist constants $\gamma_1, \gamma_2 > 0$ and $\alpha_j > 0$, $j = 0, 1, 2$, such that the coefficients of $\|\partial_\xi^{k+1} W_\rho\|_\xi^2$, $\|\partial_\xi^{k+1} W_u\|_\xi^2$, $k = 0, 1, 2$, are positive initially. Since by the Sobolev-Imbedding I controls the L_ξ^∞ -norms of W_ρ, W_u and their derivatives, these coefficients stay positive, if $I(0)$ is small enough, which is guaranteed by assumption (2.5.45). Hence

$$\frac{d}{dt} I(t) \leq -\lambda \|G\|_{H_\xi^2(L_v^2)}^2, \quad \text{for all } t \geq 0, \text{ where } \lambda > 0,$$

and the proof is completed by integrating with respect to t . $\square$

Remark 2.5.1 (Isothermal Case). *There is one difference in the isothermal case important to be mentioned. Since the sound speed $c = 1$ is constant, the derivative of $K_1(\varphi)$ corresponding to (2.5.32) is now of $O(\varepsilon^2)$ and has a different sign*

$$\partial_\xi K_1(\varphi) = -2(u_\varphi - u_-)\partial_\xi u_\varphi < 0.$$

Therefore the macroscopic estimate corresponding to (2.5.38) with $\alpha_0 = 0$ has to be derived differently. We test equation (2.5.30) with $K_1(\varphi)W_\rho$ and (2.5.31) with W_u to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (K_1(\varphi)W_\rho^2 + W_u^2) d\xi - \tilde{C} \|W_u\|_\infty (\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2) + D \|\partial_\xi W_u\|_\xi^2 \\ & \leq \varepsilon \int S(g) W_u d\xi + \frac{1}{2\varepsilon} \int \left((1 - \varepsilon\sigma) \partial_\xi K_1 W_\rho^2 + 2\partial_\xi K_1 W_\rho W_u + \partial_\xi K_2 W_u^2 \right) d\xi \\ & \leq \varepsilon \int S(g) W_u d\xi - \int \kappa(\varphi) W_u^2 d\xi \end{aligned}$$

for a $\kappa(\varphi) \geq 0$, where we have used Young's inequality. Observe that here we do not need estimate (2.5.43).

2.6 Shock profiles for strong shocks of scalar conservation laws

The first existence proof of large kinetic shock profiles is due to Golse [22] for the Perthame-Tadmor model. The proof for other kinetic models for scalar conservation laws follows similar steps. It consists of obtaining the shock profile as the limit of profiles for ξ on a finite interval $[-L, L]$ as $L \rightarrow \infty$. Since the shock profile problem is translation invariant in the ξ direction, care has to be taken with fixing the profiles before taking this limit.

In this section we consider (2.1.1) and assume that its macroscopic limit is a scalar conservation law. Furthermore we assume throughout this section that $V = \mathbb{R}$ and $d\mu = dv$.

We start by giving some ingredients that are common to the examples that follow and that allow a way of proving existence and stability of strong shock

profiles. A description of the proofs is then done in Section 2.6.1 and 2.6.2. This section is completed in Section 2.6.3 by presenting the program for the examples in sections 2.2.1 and 2.2.4.

We assume that the equilibrium distribution $\mathcal{M}(U, v)$ is continuous in v and has continuous derivatives with respect to U . The macroscopic flux is assumed to be genuinely nonlinear and, without loss of generality, strictly concave:

$$J''(U) < 0. \quad (2.6.1)$$

A key property of $\mathcal{M}$ is its invertibility with respect to U : we assume that there exists a $\zeta(f, v)$ such that

$$U = \zeta(f, v) \iff f = \mathcal{M}(U, v). \quad (2.6.2)$$

To be more precise, we shall assume that the function $\mathcal{M}(v) : U \rightarrow \mathcal{M}(U, v)$ is $C^2(\mathbb{R})$ and that there exist $U_+, U_- \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\partial_U \mathcal{M}(U, v) > 0 \quad \text{for all } U \in [U_-, U_+]. \quad (2.6.3)$$

We continue by briefly describing some of the additional features of the equations.

Existence and uniqueness and the maximum principle:

In general, local (in time) existence and uniqueness of the initial value problem

$$\partial_t f + v \partial_x f = Q(f), \quad \text{on } \mathbb{R}^+ \times \mathbb{R} \times V \quad (2.6.4)$$

$$f(0, x, v) = f_{init}(x, v) \quad \text{for } (x, v) \in \mathbb{R} \times V \quad (2.6.5)$$

follows by considering the mild formulation of (2.1.1)

$$f(t, \cdot, \cdot) = T(t)f_{init}(\cdot, \cdot) + \int_0^t T(t-s)Q(f(s, \cdot, \cdot))ds, \quad (2.6.6)$$

where $T(t)$ denotes the continuous group generated by the linear transport operator $v \partial_x$. Thus, well-posedness follows if $Q(f)$ is Lipschitz continuous in the domain of $T(t)$ by a fixed point argument.

We shall assume that $Q(f)$ allows a form of comparison principle, which relates the solution to the distribution $\mathcal{M}$ at constant values of U : Let $U_-, U_+ \in \mathbb{R}$ be given, then if the initial condition satisfies

$$\mathcal{M}(U_-, v) \leq f_{init}(x, v) \leq \mathcal{M}(U_+, v) \quad (2.6.7)$$

then the solution of (2.6.4)-(2.6.5) satisfies

$$\mathcal{M}(U_-, v) \leq f(t, x, v) \leq \mathcal{M}(U_+, v) \quad \text{for all } t > 0 \quad (2.6.8)$$

and thus also

$$U_- \leq U_f(t, x) \leq U_+ \quad \text{for all } t > 0. \quad (2.6.9)$$

The general result for smooth initial data is the following:

Proposition 2.6.1. *Let $f_{init} \in C_0^1(\mathbb{R} \times V)$, such that there exists $U_-, U_+ \in \mathbb{R}$ with (2.6.7), then there exists a unique solution $f \in C^1(\mathbb{R}^+ \times \mathbb{R} \times V)$ satisfying (2.6.8) and (2.6.9).*

Kinetic entropy inequality For any increasing function χ the following holds

$$\int_V \chi(\zeta(f, v)) Q(f) dv \leq 0. \quad (2.6.10)$$

I.e. (2.1.5) holds with $H(f, v) = \int_{\bar{f}}^f \chi(\zeta(g, v)) dg$. All convex entropies η are recovered from a kinetic entropy density H by taking $\chi = \eta'$, and so (2.1.10) holds.

Additionally, we assume that for $\eta(U) = U^2/2$ the entropy dissipation can be quantified. Namely, that there exists a constant $C > 0$ independent of ε such that

$$\int_V \zeta(f, v) Q(f) dv \leq -C \int_V (f - \mathcal{M}(U_f))^2 w(v) dv \quad (2.6.11)$$

where the function w only depends on v and is positive and uniformly bounded.

L^1 -contraction

Another property that is satisfied by scalar conservation laws is the L^1 -contraction. For the kinetic equation we shall assume that for two given solutions of (2.6.4)-(2.6.5) such that $f - g \in L^1(\mathbb{R} \times V)$ for all $t > 0$

$$\int_V (Q(f) - Q(g)) \text{sign}(f - g) dv \leq 0. \quad (2.6.12)$$

We also assume that the equality holds if and only if $\text{sign}(f - g)$ is constant (independent of v). The L^1 -contraction property now follows from (2.6.12). Subtracting the equations for f and g and multiplying by $\text{sign}(f - g)$ implies that

$$\partial_t \int_V |f - g| dv + \partial_x \int_V v |f - g| dv \leq 0. \quad (2.6.13)$$

Or integrating with respect to x :

$$\frac{d}{dt} \int_{\mathbb{R}} \int_V |f - g| dv dx \leq 0. \quad (2.6.14)$$

Also a comparison principle (that generalizes Proposition 2.6.1) follows easily from (2.6.13) (by the Crandall-Tartar-Lemma [13]). Two solutions f and g with $f - g \in L^1(\mathbb{R}_x \times V)$ for all $t > 0$ clearly satisfy

$$\int_{\mathbb{R}} \int_V (f - g) dv d\xi = \int_{\mathbb{R}} \int_V (f_{init} - g_{init}) dv d\xi \quad \text{for all } t > 0.$$

Then if $f_{init} \geq g_{init}$ then also $f \geq g$ for all $t > 0$.

2.6.1 Existence of kinetic profiles for strong shocks

We now look for travelling wave solutions with speed s connecting different equilibrium states. The travelling wave variable is defined by

$$\xi = x - st. \quad (2.6.15)$$

and we look for functions $f(\xi, v)$ that satisfy

$$(v - s)\partial_\xi f = Q(f) \quad (2.6.16)$$

$$\lim_{\xi \rightarrow -\infty} f(\xi, v) = \mathcal{M}(U_-, v) \quad \lim_{\xi \rightarrow +\infty} f(\xi, v) = \mathcal{M}(U_+, v) \quad (2.6.17)$$

with $U_\pm \in \mathbb{R}$, where the far field conditions hold almost everywhere in v .

Indeed, integrating (2.6.16) with respect to v over V gives

$$\partial_\xi \int_V (v - s)f \, dv = 0$$

and integration with respect to ξ , using (2.6.17), implies that

$$\int_V (v - s)f \, dv = J(U_-) - sU_- = J(U_+) - sU_+ \quad (2.6.18)$$

thus we recover the Rankine-Hugoniot condition.

Observe that if a solution of (2.6.16)-(2.6.17) exists then (2.6.1) implies that $U_- < U_+$, and no solution exists if $U_- > U_+$. This is a consequence of (2.1.6): multiplying (2.6.16) by χ and integration with respect to v gives

$$\partial_\xi \int_V (v - s)H(f, v) \, dv \leq 0$$

(this is the travelling wave version of (2.1.6)). Integration with respect to ξ and (2.6.17) implies

$$(\Psi(U_+) - \Psi(U_-)) - \frac{J(U_+) - J(U_-)}{U_+ - U_-} (\eta(U_+) - \eta(U_-)) \leq 0. \quad (2.6.19)$$

It is now a standard exercise of scalar conservation laws to prove that $U_- < U_+$: since this inequality holds for all convex entropies, we choose to write if for $\eta(U) = U^2/2$, and Ψ now satisfies $\Psi'(U) = UJ'(U)$. Defining

$$L(U) := (\Psi(U) - \Psi(U_-)) - \frac{1}{2}(J(U) - J(U_-))(U + U_-)$$

the inequality (2.6.19) becomes $L(U_+) \leq 0$. We now compute $L(U_-) = 0$ and observe

$$L'(U) = \frac{1}{2}[J'(U)(U - U_-) - (J(U) - J(U_-))] < 0,$$

which holds by the concavity of J , thus $U_- < U_+$.

The general result is the following

Theorem 2.6.1. *If $J''(U) < 0$ and $U_- < U_+$, there exists a travelling wave solution f (unique up to translation in ξ) such that*

$$\lim_{\xi \rightarrow \pm\infty} f(\xi, v) \rightarrow \mathcal{M}(U_{\pm}, v) \quad \text{weakly in } L_v^1(V).$$

Moreover, its macroscopic density is continuous and monotonically increasing.

We describe the steps of the proof in some detail.

Step 1: The slab problem First one constructs profiles on the intervals $[-L, +L]$ for all $L > 0$, by solving the equation

$$(v - s)\partial_{\xi} f^L = Q(f^L), \quad \xi \in (-L, +L), \quad v \in V \quad (2.6.20)$$

subject to the inflow boundary conditions

$$f^L(-L, v) = \mathcal{M}(U_-, v), \quad \text{for } v > s \quad (2.6.21)$$

$$f^L(+L, v) = \mathcal{M}(U_+, v), \quad \text{for } v < s. \quad (2.6.22)$$

The definition of a fixed point map will depend in each case on $Q(f)$, and is similar to the fixed point map defined to prove existence of the evolution equation. In particular, this fixed point map iteration will preserve the maximum and the comparison principles. In general, regularity of the macroscopic slab profiles might need to be proved additionally by means of averaging lemmas, for instance.

A general result can be formulated as follows

Proposition 2.6.2. *With the assumptions (2.6.3) and that the collision operator admits a maximum principle and that (2.6.10) hold, the slab problem (2.6.20)-(2.6.22) has a solution $f^L \in L_{x,v}^1((-L, L) \times V)$ with continuous macroscopic density U^L , and satisfying*

$$\mathcal{M}(U_-, v) \leq f^L(\xi, v) \leq \mathcal{M}(U_+, v) \quad \text{for all } \xi \in \mathbb{R}, \quad v \in V, \quad (2.6.23)$$

then also $U_+ \leq U^L \leq U_-$.

The analogous for the Boltzmann equation is still open. Some results on a slab appear in Arkeryd, Cercignani, and Illner [1], Arkeryd and Nouri [2] and Ukai [38]. A maximum principle is not available here.

Step 2: Centering the profile The limiting problem for $L = \infty$ is translation invariant with respect to ξ . For this reason, before taking the limit $L \rightarrow \infty$, we normalize the shift of the profiles f^L . First we observe that by (2.6.23) and the inflow boundary conditions

$$\begin{aligned} U^L(-L) &= \int_{-\infty}^s f^L(-L, v) dv + \int_s^{+\infty} f^L(-L, v) dv \\ &\leq \int_{-\infty}^s \mathcal{M}(U_+, v) dv + \int_s^{+\infty} \mathcal{M}(U_-, v) dv \leq U^L(L). \end{aligned}$$

Then, for all $L > 0$, by continuity of U^L we can take $\xi^L \in [-L, L]$ such that

$$U^L(\xi^L) = U^* := \int_{-\infty}^s \mathcal{M}(U_+, v) dv + \int_s^{+\infty} \mathcal{M}(U_-, v) dv.$$

Next we shift the point ξ^L to the origin. Before that we first need to extend the ξ domain of f^L to $\mathbb{R}$:

$$f_1^L(\xi, v) := \begin{cases} f^L(-L, v) & \xi < -L \\ f^L(\xi, v) & -L \leq \xi \leq L \\ f^L(L, v) & L < \xi. \end{cases}$$

For a sequence $L_n \rightarrow \infty$ as $n \rightarrow \infty$ we let $\xi_n := \xi^{L_n}$ and $f_n(\xi, v) := f_1^{L_n}(\xi - \xi_n, v)$, and $U_n := \int_V f_n dv$. Clearly now $U_n(0) = U^*$ for all n .

Passing to the limit $L \rightarrow \infty$ in the equation will differ in each case. But, the bound (2.6.23) extends trivially and holds for f_n . Then f_n converges weakly* in $L_{\xi, v}^\infty(\mathbb{R} \times V)$ to some f satisfying (2.6.23). Applying velocity averaging one obtains that $U_n \rightarrow U_f$ uniformly on compact sets and $U_f(0) = U^*$. The weak limit f of f_1^L solves the limit equation in the distributional sense (the limit of the non-linear terms is treated in a similar way as for the existence proof; it depends on the specific form of $Q(f)$). It is yet necessary to prove that the shifted intervals $[-L_n + \xi_n, L_n + \xi_n]$ tend to $\mathbb{R}$.

Proposition 2.6.3. *$L + \xi^L, L - \xi^L \rightarrow \infty$ as $L \rightarrow \infty$. And there exist sequences $\xi_n \rightarrow \infty$ and $\eta_n \rightarrow -\infty$ such that the solution of the limit problem satisfies*

$$f(\xi_n, v) \rightarrow \mathcal{M}(U_+, v), \quad f(\eta_n, v) \rightarrow \mathcal{M}(U_-, v) \quad v - a.e. .$$

In the proof one argues by contradiction, assuming that for a sequence $L_n \rightarrow \infty$, $\xi_n - L_n \rightarrow \xi^* > -\infty$ as $n \rightarrow \infty$. Then, by passing to the limit in the equation in the distributional sense, the limit f of f_n satisfies a half-space problem for $\xi \geq \xi^*$ with equilibrium inflow data:

$$\begin{aligned} (v - s)\partial_\xi f &= Q(f), \quad \text{for } \xi \geq \xi^* \\ f(\xi^*, v) &= \mathcal{M}(U_-, v), \quad \text{for } v > s. \end{aligned}$$

One then proves that $f(\xi^*, v) = \mathcal{M}(U_-, v)$ also holds for $v \leq s$ v -a.e. and actually that $f(\xi, v) = \mathcal{M}(U_-, v)$ for $\xi \geq \xi^*$ v -a.e..

With the aid of (2.6.11) and the continuity of U_f one proves that $U_f \rightarrow U_{+\infty}$ as $\xi \rightarrow \infty$, and that, restricted to a subsequence ξ_n , $f(\xi_n, v) \rightarrow \mathcal{M}(U_{+\infty})$ v -a.e. Using the maximum principle and the inflow boundary condition it can be shown that $U_- \leq U_{+\infty} \leq U_+$ and that

$$\int_{v < s} (v - s)(f(\xi^*, v) - \mathcal{M}(U_-, v)) dv = 0, \quad J(U_{+\infty}) - sU_{+\infty} = J(U_-) - sU_-$$

then, by (2.6.23),

$$f(\xi^*, v) = \mathcal{M}(U_-, v) \quad v - a.e., \quad (2.6.24)$$

as anticipated. It is next shown that

$$\int_V (Q(f) - Q(\mathcal{M}(U_-))) \text{sign}(f - \mathcal{M}(U_-)) dv = 0$$

and (2.6.12) gives that $\text{sign}(f - \mathcal{M}(U_-))$ is constant, actually zero by (2.6.24), thus $f(\xi, v) = \mathcal{M}(U_-, v)$ for all $\xi \geq \xi^*$. This is in contradiction with $U_f(0) = U^*$, then $-L + \xi^L \rightarrow -\infty$.

A similar argument shows that $L + \xi^L \rightarrow \infty$ and that there is a sequence $\xi_k \rightarrow -\infty$ as $k \rightarrow \infty$ such that there exists $U_{-\infty}$ with $f(\xi_k, v) \rightarrow \mathcal{M}(U_{-\infty}, v)$ as $k \rightarrow \infty$ v -a.e. The entropy inequality again implies that $U_{-\infty} \leq U_{+\infty}$. Finally, $J(U_{-\infty}) - sU_{-\infty} = J(U_+) - sU_+$, as before, hence $U_{-\infty} = U_-$ and $U_{+\infty} = U_+$.

Step 3: monotonicity with respect to ξ

Monotonicity of the macroscopic profiles now follows as a consequence of (2.6.12). And in particular implies that the far-field conditions hold in the stronger sense of Theorem 2.6.1.

First we observe that the following holds

Lemma 2.6.1. *Let f and g be two solutions of (2.6.4) such that there exist sequences $\xi_n \rightarrow \infty$ and $\eta_n \rightarrow -\infty$ with*

$$\lim_{n \rightarrow \infty} (f(\xi_n, v) - g(\xi_n, v)) = 0 \quad \lim_{n \rightarrow \infty} (f(\eta_n, v) - g(\eta_n, v)) = 0$$

then $\text{sign}(f - g)$ is independent of v .

This follows by subtracting the equations of f and g , multiplying by $\text{sign}(f - g)$ and integrating with respect to v and ξ (as in (2.6.13)) gives

$$0 = \int_{\mathbb{R}} \partial_{\xi} \int_V (v - s) |f - g| dv d\xi = \int_{\mathbb{R}} \int_V (Q(f) - Q(g)) \text{sign}(f - g) dv d\xi,$$

thus $\text{sign}(f - g)$ is independent of v .

This means that if we consider a travelling wave solution f , any translation of it $\hat{f}(\xi, v) = f(\xi + a, v)$ with $a > 0$, $\hat{f}$ is clearly a travelling wave solution as well, and the above lemma applies, giving that $\text{sign}(f - \hat{f})$ is independent of v . In particular, that f is monotone with respect to ξ holds if an expression of the form $\int w(v) f dv$ for any positive w , is monotone. This step can be performed in the examples and will depend on the form of $Q(f)$.

This completes the proof of Theorem 2.6.1.

2.6.2 Stability of kinetic profiles for strong shocks

We now study dynamic stability of the travelling wave just constructed. We prove that solutions of the Cauchy problem (2.6.4)-(2.6.5) approach a travelling wave solution as $t \rightarrow \infty$ if the initial condition has the same far field behavior of a shock profile.

We let $\varphi(\xi, v)$ be a kinetic shock profile, i.e. a solution to (2.6.16) and (2.6.17). We also let it, by a shift in ξ if necessary, be chosen such that for the initial datum $f_{init}(\xi, v)$

$$\int_{\mathbb{R}} \int_V (f_{init} - \varphi) dv d\xi = 0. \quad (2.6.25)$$

We denote the difference between the solution of the initial value problem and the shock profile by $h(t, \xi, v) := f(t, \xi, v) - \varphi(\xi, v)$, so h satisfies

$$\partial_t h + (v - s) \partial_\xi h = Q(f) - Q(\varphi), \quad (2.6.26)$$

$$h(t = 0) = f_{init} - \varphi, \quad \int_{\mathbb{R}} \int_V h(t, \xi, v) dv d\xi = 0. \quad (2.6.27)$$

Multiplying (2.6.26) by $\text{sign}(h)$ and integrating with respect to v and ξ we get

$$\frac{d}{dt} \int_{\mathbb{R}} \int_V |h| dv d\xi \leq \int_{\mathbb{R}} \int_V (Q(f) - Q(\varphi)) \text{sign}(h) dv d\xi \leq 0.$$

And so $\lim_{t \rightarrow \infty} \|h\|_{L^1_{x,v}} < \infty$, and also

$$\int_{\mathbb{R}} \int_V |h| dv d\xi \leq \int_{\mathbb{R}} \int_V |h_{init}| dv d\xi \quad \text{for all } t > 0.$$

For each $t_n \rightarrow \infty$ we define $h_n(t, \xi, v) := h(t_n + t, \xi, v) \rightarrow h_\infty(t, \xi, v)$. The sequences $\{h_n\}_n$ are bounded in $L^\infty((0, \infty); L^1_{\xi,v} \cap L^\infty_{\xi,v}(\mathbb{R} \times V))$. Then (restricted to a subsequence) $h_n \rightarrow h_\infty$ as $n \rightarrow \infty$ in $L^\infty_{x,v}(\mathbb{R} \times V)$ weak*. Because of the translation invariance in ξ we get equicontinuity in the ξ -direction; by applying the L^1 contractivity to the difference $h(t, \xi + h, v) - h(t, \xi, v)$. Thus we can conclude that there is a subsequence of t_n such that

$$h_n \rightarrow h_\infty \quad \text{as } n \rightarrow \infty \quad \text{in } L^\infty_{\xi,v}(\mathbb{R} \times V) \text{ weak*}.$$

Also since $\int_{\mathbb{R}} |\partial_\xi(U_f - U_g)| d\xi \leq \int_{\mathbb{R}} \int_V |\partial_\xi(f - g)| dv d\xi$, there exists a U_∞ such that

$$\int_V h_n dv = U_{h_n} \rightarrow U_\infty \quad \text{strongly in } L^1_\xi(\mathbb{R}). \quad (2.6.28)$$

We now observe that

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R} \times V} (Q(f_n) - Q(\varphi)) \text{sign}(h_n) dv d\xi dt \\ &= \int_{t_n}^\infty \int_{\mathbb{R} \times V} (Q(f) - Q(\varphi)) \text{sign}(h) dv d\xi dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The term $Q(f_n) - Q(\varphi)$ can be rearranged in the examples and one can take the limit in the weak formulation of the equation satisfied by h_n , using (2.6.28) in the nonlinear term. In addition the above implies that the $\text{sign}(h_\infty)$ does not depend on v , and it is easy to check that $\int \int h_\infty dv d\xi = 0$, since this property is preserved in t . The limit equation is

$$\begin{aligned} \partial_t h_\infty + (v - s) \partial_\xi h_\infty &= \tilde{Q} h_\infty \\ \text{sign}(h_\infty) &= \text{sign}(U_{h_\infty}), \quad \int_{\mathbb{R}} \int_V h_\infty dv d\xi = 0 \end{aligned}$$

and holds in the weak sense. Here $\tilde{Q}$ results from the linearisation of $Q(f_n) - Q(\varphi)$ and taking the limit $n \rightarrow \infty$.

Let us see that $h_\infty \equiv 0$. We argue by contradiction, first we assume that there is a (t_0, ξ_0, v_0) such that $h_\infty(t_0, \xi_0, v_0) > 0$. There must be a (ξ_1, v_1) such that $h_\infty(t_0, \xi_1, v_1) < 0$. This implies $h_\infty(t_0, \xi_0, v) > 0$ and $h_\infty(t_0, \xi_1, v) < 0$ for all v , because $\text{sign}(h_\infty)$ does not depend on v . In fact $\text{sign}(h_\infty)$ does not change along characteristics. So taking $\xi_2 = \xi_0 + (v_0 - s)(t_1 - t_0)$ and $t_1 \neq t_0$, we get $h(t_1, \xi_2, v) > 0$. Now we can choose v_2 such that $\xi_1 = \xi_2 + (v_2 - s)(t_0 - t_1)$, implying $h_\infty(t_0, \xi_1, v) > 0$, a contradiction.

The stability result can now be stated

Theorem 2.6.2. *Let f be a solution of (2.6.4)-(2.6.5), with $\mathcal{M}(U_-, v) \leq f_{init} \leq \mathcal{M}(U_+, v)$, such that $\lim_{x \rightarrow \pm\infty} f_{init}(x, v) = \mathcal{M}(U_\pm, v)$. Let φ be a travelling wave solution such that (2.6.25) holds, then for every sequence $t_n \rightarrow \infty$, $f(t_n + t, \xi, v) \rightarrow \varphi(\xi, v)$ in $L^\infty((0, T) \times \mathbb{R} \times V)$ weak*.*

2.6.3 Examples

BGK-model for scalar conservation laws

The above program has been carried out in [14], [18] for the BGK-model of scalar conservation laws described in Section 2.2.1.

Kinetic entropy inequalities are obtained from (2.6.3) by letting ζ be the inverse of $\mathcal{M}(U, v)$ as we already described in Section 2.2.1. For this model (2.6.11) readily holds with $w(v) = (\sup_{U_- \leq U \leq U_+} \partial_U \mathcal{M}(U, v))^{-1}$.

In several of the arguments that follow a subsequence of distribution functions will converge in $L_{x,v}^\infty(\mathbb{R} \times V)$ weak* as a consequence of the maximum principle. Then either by an averaging lemma or a uniform estimate derived from the equation, the corresponding sequence of macroscopic densities U_f converges strongly in some $L_x^p(\mathbb{R})$ with $1 \leq p < \infty$ or uniformly in $C_x(\mathbb{R})$. Passing to the limit in the equation can be done easily in the weak formulation because the only nonlinear term $\mathcal{M}(U_f, v)$ involves f through U_f .

Let us briefly see why Proposition 2.6.1 holds. The mild formulation (2.6.6) which now reads

$$f(t, x, v) = e^{-t} f_0(x - vt, v) + \int_0^t e^{(s-t)} \mathcal{M}(U_f(s, x - v(t-s)), v) ds,$$

and a standard fixed point argument gives local existence in time. A comparison principle follows easily from the mild formulation too. Let f_{init}^1 and f_{init}^2 be two initial conditions such that $f_{init}^1 \leq f_{init}^2$, and let f^1 and f^2 denote the corresponding solutions, let also $g_{init} := f_{init}^2 - f_{init}^1$ and $g := f^2 - f^1$, then g satisfies

$$\partial_t g + \partial_x g + g = \partial_U \mathcal{M}(\bar{U}, v) U_g,$$

for some $\bar{U}$. Thus by (2.6.3) the positivity of the initial condition is preserved in time. This can be applied to steady solutions $\mathcal{M}(U^*, v)$ where U^* is constant, and so the maximum principle follows.

We observe that in this case the mild formulation gives a self-consistent formulation in terms of U_f :

$$U_f(t, x) = e^{-t} \int_V f_0(x - vt, v) dv + \int_0^t e^{(s-t)} \int_V \mathcal{M}(U_f(s, x - v(t-s)), v) dv ds.$$

A similar formulation is used to solve the slab problem associated to the travelling wave equation, as we shall see.

Let us now check that the L^1 -contraction property holds. We just need to show that (2.6.12) holds for any two solutions of the initial value problem, with different initial conditions, f and g and such that $f - g \in L^1_{x,v}(\mathbb{R} \times V)$. We compute

$$\begin{aligned} & \int_V \{ \mathcal{M}(U_f, v) - \mathcal{M}(U_g, v) - (f - g) \} \text{sign}(f - g) dv = \\ & \int_V \{ | \mathcal{M}(U_f, v) - \mathcal{M}(U_g, v) | - |f - g| \} dv = |U_f - U_g| - \int_V |f - g| dv \leq 0. \end{aligned}$$

where we have used (2.6.3).

We now sketch the proofs of existence and stability of travelling waves. We recall that with the travelling wave variable (2.6.15) we look for solutions of the problem (2.6.16) subject to (2.6.17), where the travelling wave speed s is given by (2.6.18). That $U_- < U_+$ under the assumption (2.6.1) follows from (2.6.19), as before.

Let us turn now to the existence of travelling waves. To solve the slab problem (2.6.20) subject to (2.6.21) and (2.6.22), the following fixed point map can be used

$$\mathcal{T} : U^L \mapsto \int_V f^L dv,$$

where f^L solves (2.6.20), for a given U^L , subject to the boundary conditions (2.6.21) and (2.6.22). So the proof is an application of the Schauder's fixed point theorem. The special form of the collision operator allows to define a fixed point map that is in fact an operator of macroscopic densities and that maps a subspace of the locally continuous functions into itself. The compactness of the operator holds by applying an averaging lemma. The definition of the operator implies that there is a unique f^L that solves the equation and whose macroscopic density is the fixed point, giving the existence.

The passage to the limit $L \rightarrow \infty$ after the profile has been centered can be carried out in the same way as above and the proof of Proposition 2.6.3, in particular, follows similarly.

The only thing left to check in order to conclude that Theorem 2.6.1 holds is the monotonicity of the profiles. By Lemma 2.6.1 we only need to prove that if f is a travelling wave solution which satisfies the far field conditions in the sense of Proposition 2.6.3, then $\int_V w(v)f dv$ is monotone increasing for a positive function w . In this case we can take $w(v) \equiv (v - s)^2$; we let $g = f - \mathcal{M}(U_-, v)$ and multiply the equation satisfied by g by $(v - s)$. Integrating with respect to v yields

$$\partial_\xi \int_V (v - s)^2 g dv = J(U_g + U_-) - J(U_-) - sU_g - \int_V (v - s)g dv. \quad (2.6.29)$$

But integration with respect to v of the equation for g implies that $\int_V (v-s)g \, dv$ is constant, and further, after taking the limit along the sequences ξ_n and η_n , that $\int_V (v-s)g \, dv = 0$. Finally, by (2.6.1) and $U_- \leq U_f \leq U_+$,

$$\partial_\xi \int_V (v-s)^2 g \, dv = J(U_f) - J(U_-) - s(U_f - U_-) > 0.$$

The proof of stability of the shock profiles can be readily adapted and we shall not comment further on it.

Fermions in a background medium

We now consider the kinetic model for fermion-phonon interaction in the presence of a large electric field E described in Section 2.2.4 and review the results from [5]. In one space dimension, numerical computations of $J(U)$ suggest that for $E \neq 0$ then $\text{sign}(J''(U)) = \text{sign}(E)$, see [5] and [19]. Let us assume without loss of generality that $E > 0$ and that $J''(U) > 0$.

Existence and the maximum principle are proved by noticing that $Q_s(f)$ can be split into a linear and a nonlinear part

$$Q_s(f) = \lambda_1(f) + \lambda_2(f)f \quad (2.6.30)$$

where the operators $\lambda_1(f)$ and $\lambda_2(f)$ are linear integral operators. The fixed point iteration is defined by solving at each step the linear equation

$$\varepsilon \partial_t f^{n+1} + \varepsilon v \partial_x f^{n+1} + E \partial_v f^{n+1} - \lambda_2(f^n) f^{n+1} = \lambda_1(f^n), \quad (2.6.31)$$

with

$$f^0 = f_{init} \in L_v^1(\mathbb{R}; W_x^{1,1}(\mathbb{R}))$$

where (2.6.31) can be solved by the method of characteristics or by semigroup theory. Positivity of solutions f^n follows by observing that if $f \geq 0$ then $-\lambda_2(f) \geq 0$ and $\lambda_1(f) \geq 0$. That $f^n \leq 1$ for all n also follows, by writing the equation in terms of $g^n = 1 - f^n$ and using that $Q_s(1 - g^n) = -Q_s(g^n)$ gives the equation (2.6.31) with f^{n+1} and f^n replaced by g^{n+1} and by g^n , respectively. So $g^n \geq 0$ for all n if $g^0 = 1 - f^0 \geq 0$.

Thus, the sequence f^n is uniformly bounded in L^∞ and there is a subsequence that converges weakly* to some $f \in L^\infty$. It has been shown, Mustieles [32], that the sequence f^n converges strongly in $L^\infty([0, T]; L_{x,v}^1(\mathbb{R}^2))$, by deriving L^1 estimates of the form $\|f^{n+1} - f^n\|_{L^1} \leq C \|f^n - f^{n-1}\|_{L^1}$, and the existence holds.

That (2.6.10) and (2.6.11) hold has been shown by Ben Abdallah, Chaker and Schmeiser [5]. We do not go into the proof here, we just remark that (2.6.11) holds for any convex entropy η , not only for $\eta = U^2/2$, namely

$$\int_{\mathbb{R}} (Q_s(f) - E \partial_v f) \chi(\zeta(f, v)) \, dv \leq -C \int_{\mathbb{R}} (f - F(\bar{U}_f))^2 M(v) \, dv$$

holds for any strictly increasing function $\chi \in C^1$, with ζ , as before, defined by (2.6.2) and with $\bar{U}_f := \int \zeta(f, v) M(v) \, dv$. The inequality (2.6.12), and hence L^1 -contraction, was proved by Poupaud [36].

To prove the existence of the slab problem (2.6.20)-(2.6.22) one can proceed as in [4] for the Milne problem. Using the iteration, analogous to (2.6.31),

$$(v - s)\partial_\xi f^{n+1} + E\partial_v f^{n+1} - \lambda_2(f^n)f^{n+1} = \lambda_1(f^n),$$

for $E > 0$ the iteration procedure is started with $f^0 = \mathcal{M}(U_+, v)$, thus clearly $f^1 \leq f^0$, the comparison principle implies that $f^{n+1} \leq f^n$ for all n , thus the sequence defined by the iteration is decreasing, and $f^n \in L^1([-L, L] \times \Omega) \cap L^\infty([-L, L] \times \Omega)$. The existence of fixed points is achieved by passing to the limit $n \rightarrow \infty$; the convergence is strong in L^1 by the monotone convergence theorem, and weak* in L^∞ , this allows passing to the limit in Q . An additional argument that uses the inequality (2.6.12) is needed to prove the uniqueness of the fixed point. For $E < 0$ the iteration procedure is started at $f^0 = \mathcal{M}(U_-, v)$ instead.

The rest of the existence proof now follows as in Section 2.6.1. We only remark that the monotonicity of f with respect to ξ is directly proved for U_f (here $w(v) = 1$), it requires a technical lemma that follows by analyzing the collision term, we refer to [5] for details.

The proof of stability follows the same lines. We only remark that the splitting of the operator (2.6.30) and averaging lemmas are used here to pass to the limit in the h equation (2.6.26).

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Chapter 3

Weak shocks of a BGK kinetic model for isentropic gas dynamics¹²

3.1 Introduction

We analyse the existence and stability of small amplitude travelling wave solutions for the one-dimensional BGK equation

$$\partial_t f + v \partial_x f = M(\rho_f, m_f, v) - f, \quad \text{with } t > 0, x \in \mathbb{R}, v \in \mathbb{R}, \quad (3.1.1)$$

where $f(t, x, v)$ is a density of particles moving with velocity v at the time-space position (t, x) . The functions $\rho_f(t, x)$ and $m_f(t, x)$ denote the macroscopic density and momentum corresponding to the distribution f , i.e. the zeroth and first order moments with respect to the velocity v

$$\rho_f(t, x) = \int f(t, x, v) dv, \quad m_f(t, x) = \int v f(t, x, v) dv.$$

Here and in the following integrations with respect to v are over $\mathbb{R}$.

We consider a class of Maxwellians, which has been introduced by Lions, Perthame and Tadmor in [11]:

$$M(\rho, m, v) = d \left(\frac{1 + 2\alpha}{\alpha} \rho^{2\alpha} - \left(v - \frac{m}{\rho} \right)^2 \right)_+^\beta, \quad (3.1.2)$$

where

$$\beta = \frac{1 - \alpha}{2\alpha}, \quad d = \frac{1}{J_\beta} \left(\frac{1 + 2\alpha}{\alpha} \right)^{-\frac{1}{2\alpha}}, \quad J_\beta = \int_{-1}^1 (1 - z^2)^\beta dz = \frac{\sqrt{\pi} \Gamma(\beta + 1)}{\Gamma(\beta + 3/2)},$$

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and $0 < \alpha < 1$, see e.g. also [3], [1]. The equilibrium distributions satisfy the moment conditions

$$\int M(\rho, m, v) dv = \rho, \quad \int v M(\rho, m, v) dv = m, \quad (3.1.3)$$

$$\int v^2 M(\rho, m, v) dv = \frac{m^2}{\rho} + \rho^{1+2\alpha} =: P(\rho, m), \quad (3.1.4)$$

$$\int v^3 M(\rho, m, v) dv = \frac{m^3}{\rho^2} + 3\rho^{2\alpha} m. \quad (3.1.5)$$

Therefore we obtain for the collision term

$$\int \begin{pmatrix} 1 \\ v \end{pmatrix} (M(\rho_f, m_f, v) - f) dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3.1.6)$$

implying the conservation of mass and momentum:

$$\begin{aligned} \partial_t \rho_f + \partial_x m_f &= 0, \\ \partial_t m_f + \partial_x P_f &= 0, \quad \text{with } P_f := \int v^2 f dv. \end{aligned}$$

With this notation $P_{M(\rho, m)} = P(\rho, m)$. A connection between the kinetic equation and continuum mechanics models can be established by the macroscopic limit, based on the rescaling $(t, x) \rightarrow (t/\epsilon, x/\epsilon)$ with $0 < \epsilon \ll 1$:

$$\epsilon(\partial_t f + v \partial_x f) = M_f - f, \quad (3.1.7)$$

where we use the abbreviation $M_f(t, x, v) := M(\rho_f(t, x), m_f(t, x), v)$. In the macroscopic limit $\epsilon \rightarrow 0$, one formally obtains $f(t, x, v) \rightarrow M(\rho(t, x), m(t, x), v)$ and the system of isentropic gas dynamics

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \partial_x \begin{pmatrix} m \\ P(\rho, m) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.1.8)$$

A correction of this system can be obtained by the Chapman-Enskog procedure, which amounts to an approximation of the error $f - M_f$ by substituting $M(\rho, m, v)$ in the left hand side of (3.1.7) and using (3.1.8) for the evaluation of the time derivatives. This leads to

$$P_f = P(\rho_f, m_f) + P_{f-M_f} \sim P(\rho, m) - \epsilon P_{\partial_t M + v \partial_x M},$$

with

$$\partial_t M + v \partial_x M \sim -\partial_\rho M \partial_x m - \partial_m M \partial_x P(\rho, m) + v \partial_\rho M \partial_x \rho + v \partial_m M \partial_x m.$$

The computation of the second order moment is facilitated by taking the derivatives of (3.1.4) and (3.1.5) with respect to ρ and m and leads to

$$P_{f-M_f} \sim -\epsilon D(\rho) \partial_x \left(\frac{m}{\rho} \right), \quad \text{with } D(\rho) = 2(1 - \alpha)\rho^{1+2\alpha},$$

and, thus, to the Navier-Stokes version of isentropic gas dynamics

$$\partial_t \rho + \partial_x m = 0, \quad (3.1.9)$$

$$\partial_t m + \partial_x P(\rho, m) = \epsilon \partial_x \left[D(\rho) \partial_x \left(\frac{m}{\rho} \right) \right]. \quad (3.1.10)$$

We are interested in the BGK model as a regularisation of hyperbolic conservation laws and in particular in the construction and the dynamic stability of travelling waves. One of the first authors considering such relaxation approximations for discrete velocity spaces was Natalini [15]. Bouchut [1] gave a general framework for BGK models and concentrated together with Berthelin on the relaxation to the isentropic system in [4], [2], [3]. Cuesta and Schmeiser [6] studied small amplitude travelling wave solutions of (3.1.1) for the scalar conservation law. For small amplitude shock profile solutions of the Boltzmann equation existence was proven by Caflisch and Nicolaenko in [5] and stability by Liu and Yu [13]. In this article we aim to extend the results from the scalar case in [6] to the isentropic system and finally also to the isothermal system.

We recall some properties of (3.1.8), which for smooth solutions can be written as

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \mathbf{A}(\rho, m) \cdot \partial_x \begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with

$$\mathbf{A}(\rho, m) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}. \quad (3.1.11)$$

Here and in the following u denotes the macroscopic velocity and c the speed of sound:

$$u(\rho, m) = \frac{m}{\rho}, \quad c(\rho) = \sqrt{1 + 2\alpha \rho^\alpha}.$$

The eigenvalues of $\mathbf{A}$ are given by

$$\lambda_{1/2}(\rho, m) = u(\rho, m) \mp c(\rho).$$

For $\rho > 0$ (away from vacuum) $\lambda_1 < \lambda_2$ holds, and the system is strictly hyperbolic. With the corresponding right eigenvectors

$$r_1(\rho, m) = \begin{pmatrix} 1 \\ \lambda_1(\rho, m) \end{pmatrix}, \quad r_2(\rho, m) = \begin{pmatrix} 1 \\ \lambda_2(\rho, m) \end{pmatrix}$$

one can see that the system is genuinely nonlinear, i.e.:

$$\nabla \lambda_k \cdot r_k \neq 0, \quad k = 1, 2. \quad (3.1.12)$$

We consider shock wave solutions of (3.1.8) of the form

$$(\rho(t, x), m(t, x)) = \begin{cases} (\rho_l, m_l) & \text{for } x < st, \\ (\rho_r, m_r) & \text{for } x > st. \end{cases}$$

Here s is the shock speed and $(\rho_{l,r}, m_{l,r})$ are the constant left and right states, which (by the theory of nonlinear hyperbolic conservation laws, see e.g. [10]) are related by the Rankine-Hugoniot condition

$$s \begin{pmatrix} \rho_r - \rho_l \\ m_r - m_l \end{pmatrix} = \begin{pmatrix} m_r - m_l \\ P_r - P_l \end{pmatrix}, \quad (3.1.13)$$

where we denote $P_{l,r} := P(\rho_{l,r}, m_{l,r})$. For a fixed left state (ρ_l, m_l) the Hugoniot locus is defined as the set of all (ρ_r, m_r) such that (3.1.13) is satisfied for an appropriate s . In a neighbourhood of (ρ_l, m_l) the Hugoniot locus consists of two curves intersecting in (ρ_l, m_l) . At (ρ_l, m_l) the k -th curve is tangent to $r_k(\rho_l, m_l)$ and the shock speed s takes the value $\lambda_k(\rho_l, m_l)$ (see, e.g., [9]). If (ρ_r, m_r) lies on the k -th curve of the Hugoniot locus we refer to $\{\rho_{l,r}, m_{l,r}, s\}$ as a k -shock. The Lax entropy condition

$$\lambda_k(\rho_r, m_r) < s < \lambda_k(\rho_l, m_l) \quad (3.1.14)$$

for k -shocks (see [9]) is a stability condition, which means that the characteristics go into the shock. One can show that the admissibility condition for a 1-shock reduces to

$$\rho_l < \rho_r, \quad (3.1.15)$$

and for a 2-shock to

$$\rho_l > \rho_r.$$

For simplicity we only consider 1-shocks, the procedure for 2-shocks is analogous. The entropy conditions can also be obtained from viscous regularisations, see e.g. [7]. Travelling wave solutions of the viscous system are regularisations of admissible shocks. In this paper the regularisation by the kinetic transport equation (3.1.1) is studied.

A (mathematical) entropy density for the system (3.1.8) is given by the (physical) energy density

$$\eta(\rho, m) = \frac{1}{2} \frac{m^2}{\rho} + \frac{1}{2\alpha} \rho^{1+2\alpha},$$

which, in the limit of vanishing viscosity, satisfies the entropy inequality

$$\partial_t \eta + \partial_x \Psi \leq 0 \quad (3.1.16)$$

in the weak sense. For the entropy flux

$$\Psi(\rho, m) = \frac{1}{2} \frac{m^3}{\rho^2} + \frac{1+2\alpha}{2\alpha} \rho^{2\alpha} m$$

the relation $\nabla \Psi = \nabla \eta \mathbf{A}$ holds. For our choice of Maxwellians (3.1.2), the convex kinetic entropy corresponding to η is given by

$$H(f, v) = \frac{v^2}{2} f + \frac{1}{2d^{1/\beta}} \frac{f^{1+1/\beta}}{1 + \frac{1}{\beta}}. \quad (3.1.17)$$

The kinetic entropy is related to the macroscopic one by

$$\begin{aligned}\eta(\rho, m) &= \int H(M(\rho, m, v), v) dv, \\ \Psi(\rho, m) &= \int v H(M(\rho, m, v), v) dv,\end{aligned}\tag{3.1.18}$$

and satisfies the minimisation principle

$$\eta(\rho, m) = \min_{\substack{\int f dv = \rho \\ \int v f dv = m}} \int H(f, v) dv,\tag{3.1.19}$$

see Bouchut [1]. Multiplying (3.1.1) with $\partial_f H(f, v)$ gives

$$\partial_t H(f, v) + v \partial_x H(f, v) = \partial_f H(f, v) \frac{M_f - f}{\epsilon} \leq \frac{H(M_f, v) - H(f, v)}{\epsilon},$$

where the last inequality comes from the convexity of H . Integration implies

$$\partial_t \int H(f, v) dv + \partial_x \int v H(f, v) dv \leq 0,$$

which is the microscopic version of (3.1.16). For a rigorous proof see Berthelin and Bouchut [3].

It is our aim to study the existence and stability of small amplitude travelling wave solutions of (3.1.1). Our approach is based on the fact that for small amplitude solutions (3.1.1) can be approximated by the Navier-Stokes system (3.1.9), (3.1.10). Similarly to [6] we give a constructive existence proof in Section 3, which shows the asymptotic closeness of viscous and kinetic profiles for small shocks. The main ingredient is a micro-macro decomposition in the sense of Caflisch and Nicolaenko [5] for the Boltzmann equation.

In Section 3.4 we prove the asymptotic dynamic stability of small amplitude travelling wave solutions. We apply a micro-macro projection to the perturbation equation. We treat the macroscopic part similarly to Matsumura and Nishihara [14], where the stability of travelling waves is proven by L^2 -energy estimates for the isentropic system for a compressible viscous gas in Lagrangian coordinates. The microscopic terms are treated as in [6].

In the remainder of this Section, we present the formal construction of small amplitude kinetic shock profiles and also give an idea of the energy method for proving stability of viscous profiles. In Section 2, linearizations of the collision operator are discussed. An essential tool is the H-theorem involving a weighted L^2 -norm, where the weight is an inverse power of the reference equilibrium state. Since equilibrium distributions have compact support, it is essential to construct a reference state with a support large enough to contain the support of the whole shock profile. This technical point is one of the main differences between the present study and earlier work.

Assumptions on the data are also discussed in Section 2. Moment conditions are required for the equilibrium distributions, leading to the restriction $\alpha < 1/13$ or, in other words, the adiabatic exponent $1 + 2\alpha$ has to be smaller than $15/13$.

3.1.1 Formal construction of small amplitude kinetic shock profiles

We look for travelling wave solutions of (3.1.1), depending on x and t through the travelling wave variable $\xi = x - st$, where s denotes the wave speed. The travelling wave version of (3.1.1) is

$$(v - s)\partial_\xi f = M_f - f, \quad \text{for } \xi \in \mathbb{R}, v \in \mathbb{R}, \quad (3.1.20)$$

subject to the far-field conditions

$$f(\pm\infty, v) = M_{r,l}(v), \quad \text{for } v \in \mathbb{R}, \text{ with } M_{r,l}(v) := M(\rho_{r,l}, m_{r,l}, v). \quad (3.1.21)$$

We only consider small amplitude waves, so that

$$\rho_r - \rho_l = \epsilon, \quad 0 < \epsilon \ll 1, \quad (3.1.22)$$

where the positivity of ϵ reflects the entropy condition for a 1-shock (3.1.15). Observe that by computing the zeroth and first order moments in v of (3.1.20) and then integrating with respect to ξ , we recover the Rankine-Hugoniot relations (3.1.13). After the macroscopic scaling $\xi \rightarrow \xi/\epsilon$ the travelling wave equation becomes

$$\epsilon(v - s)\partial_\xi f = M_f - f. \quad (3.1.23)$$

We repeat the Chapman-Enskog procedure and introduce the micro-macro decomposition

$$f = M_f + \epsilon^2 f^\perp, \quad (3.1.24)$$

where, due to (3.1.6),

$$\int f^\perp dv = 0, \quad \int v f^\perp dv = 0.$$

We compute the zeroth and first order moments in v of (3.1.23) and integrate in ξ to obtain equations for ρ_f and m_f

$$m_f - m_l = s(\rho_f - \rho_l), \quad (3.1.25)$$

$$P(\rho_f, m_f) + \epsilon^2 P_{f^\perp} - P_l = s(m_f - m_l). \quad (3.1.26)$$

Since we are considering small amplitude waves, the macroscopic density and momentum are close to their far-field values at $\xi = -\infty$:

$$\rho_f = \rho_l + \epsilon y_1, \quad m_f = m_l + \epsilon y_2, \quad (3.1.27)$$

where (3.1.25) implies the relation

$$y_2 = s y_1. \quad (3.1.28)$$

Using this and comparing $O(\epsilon)$ -terms in (3.1.26) gives the eigenvalue equation $\nabla P_l \cdot (1, s_0)^T = s_0^2$ for the limit s_0 of s as $\epsilon \rightarrow 0$. At this point we decide for 1-shocks and set

$$s = \lambda_{1l} + \epsilon \sigma, \quad \text{with } \lambda_{1l} := \lambda_1(\rho_l, m_l).$$

The final equation determining $y_1(\xi)$ will follow from comparing $O(\epsilon^2)$ -terms in (3.1.26). For this purpose we need to approximate $P_{f^\perp}$. Expanding M_f in (3.1.24) gives

$$f = M_l + \epsilon \mathbf{F}_l \cdot r_{1l} y_1 + O(\epsilon^2), \quad \text{with } \mathbf{F}_l(v) := \nabla M_l(v) \quad \text{and } r_{1l} := r_1(\rho_l, m_l),$$

where here and in the following $\nabla M(\rho, m, v) := \nabla_{(\rho, m)} M(\rho, m, v)$. Substituting this into (3.1.23) we obtain

$$f^\perp = -(v - \lambda_{1l}) \mathbf{F}_l \cdot r_{1l} \partial_\xi y_1 + O(\epsilon). \quad (3.1.29)$$

The computation of $P_{f^\perp}$ is facilitated by computing the gradient with respect to ρ and m of (3.1.5):

$$P_{f^\perp} = D_0 \partial_\xi y_1 \quad (3.1.30)$$

with

$$D_0 := - \int v^2 (v - \lambda_{1l}) \mathbf{F}_l dv \cdot r_{1l} = 2(1 - \alpha) c_l \rho_l^{2\alpha} > 0, \quad \text{where } c_l := c(\rho_l). \quad (3.1.31)$$

Comparing the $O(\epsilon^2)$ -terms in (3.1.13) and (3.1.26) now gives

$$D_0 \partial_\xi y_1 = -2\sigma c_l y_1 (1 - y_1), \quad \text{with } \sigma = -\frac{(\alpha + 1)}{2\rho_l} c_l < 0. \quad (3.1.32)$$

Up to a scaling, this is the travelling wave form of the viscous Burgers equation. Obviously, it has a solution (unique up to shifts) connecting the far-field values $y_1(-\infty) = 0$ and $y_1(\infty) = 1$. Note that (3.1.32) could also have been derived from the Navier-Stokes system (3.1.9), (3.1.10) since

$$u \sim u_l - \epsilon \frac{c_l}{\rho_l} y_1 \quad \text{and} \quad D(\rho) \partial_\xi u \sim -\epsilon D(\rho_l) \frac{c_l}{\rho_l} \partial_\xi y_1 = -\epsilon D_0 \partial_\xi y_1.$$

We make this approximation rigorous in Section 3, where we prove the existence of a small amplitude kinetic shock profile satisfying

$$f(\xi, v) = M_l(v) + \epsilon \mathbf{F}_l(v) \cdot r_{1l} y_1(\xi) + O(\epsilon^2), \quad (3.1.33)$$

where y_1 is the viscous Burgers profile solving (3.1.32).

3.1.2 Stability of Navier-Stokes shock profiles

In this section we shall deal with the stability of travelling waves of a simplified version of the Navier-Stokes system (3.1.9), (3.1.10). The simplification concerns the viscosity term, which will be approximated by its linearization around the left far-field state. Introducing the travelling wave variable $\xi = x - st$ then gives

$$\begin{aligned} \partial_t \rho - s \partial_\xi \rho + \partial_\xi m &= 0, \\ \partial_t m - s \partial_\xi m + \partial_\xi P(\rho, m) &= \epsilon \tilde{D}_0 \partial_\xi^2 (m - u_l \rho), \end{aligned}$$

with $\tilde{D}_0 = D(\rho_l)/\rho_l > 0$. Travelling waves of the Navier-Stokes system are steady states $(\rho_\phi(\xi), m_\phi(\xi))$ of this system satisfying (after integration)

$$\begin{aligned} -s(\rho_\phi - \rho_l) + m_\phi - m_l &= 0, \\ -s(m_\phi - m_l) + P(\rho_\phi, m_\phi) - P(\rho_l, m_l) &= \epsilon \tilde{D}_0 \partial_\xi (m_\phi - u_l \rho_\phi). \end{aligned}$$

Elimination leads to an equation for ρ_ϕ :

$$\epsilon \tilde{D}_0 (u_l - s) \partial_\xi \rho_\phi = s^2 (\rho_\phi - \rho_l) - \frac{1}{\rho_\phi} [m_l + s(\rho_\phi - \rho_l)]^2 + \frac{m_l^2}{\rho_l} - \rho_\phi^{1+2\alpha} + \rho_l^{1+2\alpha}.$$

The entropy condition (3.1.14) for 1-shocks implies the positivity of the coefficient $u_l - s$. The right hand side is easily seen to be a strictly concave function of ρ_ϕ and, thus, positive between its zeroes $\rho_\phi = \rho_l$ and $\rho_\phi = \rho_r$. This shows that viscous profiles exist for all shocks. The density ρ_ϕ and the velocity $u_\phi = m_\phi/\rho_\phi$ are strictly monotone:

$$\partial_\xi \rho_\phi > 0, \quad \partial_\xi u_\phi = \frac{s\rho_\phi - m_\phi}{\rho_\phi^2} \partial_\xi \rho_\phi = -\frac{\rho_l(u_l - s)}{\rho_\phi^2} \partial_\xi \rho_\phi < 0. \quad (3.1.34)$$

The stability of the shock profiles will be investigated by introducing the primitives of the deviations

$$W_\rho(t, \xi) = \int_{-\infty}^{\xi} [\rho(t, \eta) - \rho_\phi(\eta)] d\eta, \quad W_m(t, \xi) = \int_{-\infty}^{\xi} [m(t, \eta) - m_\phi(\eta)] d\eta,$$

and assuming for the initial data

$$W_\rho(0, \pm\infty) = W_m(0, \pm\infty) = 0.$$

This requires not only appropriate decay of $\rho(t=0)$ and $m(t=0)$ to the far-field values of the travelling wave, but also that the total mass and the total momentum of the deviation vanish initially. Whereas one of these conditions can be satisfied by choosing the shift of the travelling wave accordingly, the other one is a 'well-preparedness' assumption on the initial data, whose violation would make the problem of proving convergence to a travelling wave significantly more complicated (see, e.g., [12]).

The linearized equations for the unknowns W_ρ and W_m read

$$\begin{aligned} \partial_t W_\rho - s \partial_\xi W_\rho + \partial_\xi W_m &= 0, \\ \partial_t W_m - s \partial_\xi W_m + \nabla P(\rho_\phi, m_\phi) \cdot (\partial_\xi W_\rho, \partial_\xi W_m) &= \epsilon \tilde{D}_0 \partial_\xi^2 (W_m - u_l W_\rho). \end{aligned}$$

The form of the viscous term suggests to introduce the variable $W_u := W_m - u_l W_\rho$:

$$\partial_t W_\rho + (u_l - s) \partial_\xi W_\rho + \partial_\xi W_u = 0, \quad (3.1.35)$$

$$\partial_t W_u + K_1(\phi) \partial_\xi W_\rho + K_2(\phi) \partial_\xi W_u = \epsilon \tilde{D}_0 \partial_\xi^2 W_u, \quad (3.1.36)$$

where $K_1(\phi) := c_\phi^2 - (u_l - u_\phi)^2$ and $K_2(\phi) := u_l - s + 2(u_\phi - u_l)$. With a smallness assumption on the amplitude of the travelling wave we see that $K_1(\phi)$ and

$K_2(\phi)$ are close to positive constants. Testing (3.1.35) with W_ρ and (3.1.36) with $K_1(\phi)^{-1}W_u$ implies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (W_\rho^2 + K_1(\phi)^{-1}W_u^2) d\xi + \int \left(-\partial_\xi \frac{K_2(\phi)}{K_1(\phi)} \right) \frac{W_u^2}{2} d\xi \\ & + \epsilon \tilde{D}_0 \int K_1(\phi)^{-1} (\partial_\xi W_u)^2 d\xi + \epsilon \tilde{D}_0 \int (\partial_\xi K_1(\phi)^{-1}) W_u \partial_\xi W_u d\xi = 0. \end{aligned} \quad (3.1.37)$$

Since $\partial_\xi u_\phi$ and $\partial_\xi \rho_\phi$ have the same decay rate (see (3.1.34)) we obtain $\partial_\xi K_1(\phi) > 0$ and $\partial_\xi K_2(\phi) < 0$ uniformly in ξ , and hence the second integral in (3.1.37) has the right sign. We apply Young's inequality to the last term

$$\left| \int (\partial_\xi K_1(\phi)^{-1}) W_u \partial_\xi W_u d\xi \right| \leq \frac{1}{\sqrt{\epsilon}} \int (-\partial_\xi K_1(\phi)^{-1}) \frac{W_u^2}{2} d\xi + \sqrt{\epsilon} C_2 \|\partial_\xi W_u\|_{L^2}^2,$$

for a positive constant C_2 . For ϵ small these terms can be controlled by the second and third integral in (3.1.37). Now we are able to write down the final estimate:

$$\frac{1}{2} \frac{d}{dt} \int (W_\rho^2 + K_1(\phi)^{-1}W_u^2) d\xi + \epsilon C_0 \|\partial_\xi W_u\|_{L^2}^2 \leq 0,$$

where C_0 is a positive constant, which implies the global existence of W_ρ and W_u in L^2 . We note that due to the lack of a viscous term in W_ρ we cannot deduce asymptotic stability from this estimate.

Now we briefly explain the ideas how we are going to control the nonlinear terms in Section 4. Since the diffusion only gives a positive term in $\partial_\xi W_u$, we, leaning on [14], artificially produce also a positive term in $\partial_\xi W_\rho$. We will construct a functional J , which controls the H^2 -norm of W_ρ , W_u and derive an estimate of the following form:

$$\frac{d}{dt} J + (1 - \tilde{C}(\|W_\rho\|_\infty + \|W_u\|_\infty + \gamma)) (\|\partial_\xi W_\rho\|_{H^2}^2 + \|\partial_\xi W_u\|_{H^2}^2) \leq 0,$$

where $\tilde{C}$ depends on $\|\partial_\xi W_\rho\|_\infty, \|\partial_\xi W_u\|_\infty$ and $\gamma > 0$ is a constant resulting from producing a linear combination of the integral estimates. By the Sobolev-imbedding we know $H^2(\mathbb{R}) \subset C_b^1(\mathbb{R})$. If $1 > \tilde{C}(\|W_\rho\|_\infty + \|W_u\|_\infty + \gamma)$ at $t = 0$, it remains so for all times t for small enough initial data, since these coefficients in turn are controlled by J . This gives global existence and local stability of W_ρ and W_u in $H^2(\mathbb{R})$. Since also the microscopic terms will be controlled by this estimate, we can deduce the local asymptotic stability of small amplitude travelling wave solutions.

3.2 The linearized collision operator

The fact that equilibrium velocity distributions have compact support requires special care in our analysis, which will be based on linearization around a global (i.e. independent of ξ) equilibrium. It will be important that the support of this equilibrium includes the velocity supports of all other distributions occurring in our

analysis. The velocity support of an equilibrium distribution (3.1.2) is determined by

$$u - \frac{c}{\sqrt{\alpha}} \leq v \leq u + \frac{c}{\sqrt{\alpha}}. \quad (3.2.1)$$

The formal approximation of a shock profile computed in Section 1.1 has the monotonicity properties $\partial_\xi \rho_f > 0$, $\partial_\xi u_f \sim -\frac{c_l}{\rho_l} \partial_\xi \rho_f < 0$, implying that the left hand side of (3.2.1) is strictly decreasing. The same is true for the right hand side of (3.2.1) by

$$\partial_\xi \left(u_f + \frac{c_f}{\sqrt{\alpha}} \right) \sim \frac{c_l}{\rho_l} (\sqrt{\alpha} - 1) \partial_\xi \rho_f < 0.$$

So neither the support of M_l is contained in the support of M_r nor vice versa, excluding both of them as candidates for the required global equilibrium. We shall construct a constant state $(\hat{\rho}, \hat{m})$ such that the support of $\hat{M} := M(\hat{\rho}, \hat{m})$ includes the supports of M_f for the formal approximation of the shock profile for all ξ . Actually, the support of the travelling wave stays a $O(\epsilon)$ distance away from the boundaries of $\text{supp } \hat{M}$. We choose

$$\hat{u} = u_l, \quad \hat{c} = c_r(1 + \epsilon/\rho_r), \quad (3.2.2)$$

which defines $\hat{\rho}$ and $\hat{m}$ uniquely. Then for ϵ small $\hat{M}$ has the desired properties, i.e. there exist positive δ_1, δ_2 such that

$$\hat{u} - \frac{\hat{c}}{\sqrt{\alpha}} = u_r - \frac{c_r}{\sqrt{\alpha}} - \epsilon \delta_1, \quad \hat{u} + \frac{\hat{c}}{\sqrt{\alpha}} = u_l + \frac{c_l}{\sqrt{\alpha}} + \epsilon \delta_2,$$

where asymptotically $\delta_1 \sim \frac{c_l}{\rho_l} \left(\frac{1}{\sqrt{\alpha}} - 1 \right)$, $\delta_2 \sim \frac{c_l}{\rho_l} \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right)$.

From now on we linearize around the Maxwellian $\hat{M}$ with the support

$$\Omega := \left[\hat{u} - \frac{\hat{c}}{\sqrt{\alpha}}, \hat{u} + \frac{\hat{c}}{\sqrt{\alpha}} \right].$$

For the macroscopic wave profiles we use the following alternative expansions to (3.1.27), (3.1.28)

$$\begin{aligned} \rho_f &= \hat{\rho} + \epsilon \hat{y}_1 = \hat{\rho} + \epsilon \left(y_1 + \frac{\rho_l - \hat{\rho}}{\epsilon} \right), \\ m_f &= \hat{m} + \epsilon \hat{y}_2 = \hat{m} + \epsilon \left(s y_1 + \frac{m_l - \hat{m}}{\epsilon} \right). \end{aligned} \quad (3.2.3)$$

Also the wave speed s is expanded around $\hat{\lambda}_1 := \lambda_1(\hat{\rho}, \hat{m})$:

$$s = \lambda_{1l} + \epsilon \sigma = \hat{\lambda}_1 + \epsilon \hat{\sigma}, \quad \text{with } \hat{\sigma} \sim \frac{1+\alpha}{2\hat{\rho}} \hat{c}.$$

We introduce the notation $\mathbf{F} := \nabla \hat{M}$. Clearly $\int \mathbf{F} d\nu = (1, 0)^T$, $\int \nu \mathbf{F} d\nu = (0, 1)^T$. Then the linearized collision operator reads

$$\mathcal{L}f = \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f. \quad (3.2.4)$$

We define the inner product

$$\langle f, g \rangle_v := \frac{1}{2\beta d^{\frac{1}{\beta}}} \int f g \hat{M}^{\frac{1}{\beta}-1} dv, \quad \text{for } \text{supp } f, \text{supp } g \subset \Omega, \quad (3.2.5)$$

where the weight is the second derivative of the kinetic entropy $H''(\hat{M})$. The induced norm and space we denote by $(L_v^2, \|\cdot\|_v)$.

The following relations appear several times:

$$\langle \mathbf{F}, f \rangle_v = \int f \nabla H'(\hat{M}) dv = \frac{1}{\hat{\rho}} \begin{pmatrix} \hat{c}^2 + \hat{u}^2 & -\hat{u} \\ -\hat{u} & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix}, \quad (3.2.6)$$

where $\hat{c} = c(\hat{\rho})$ and $\hat{u} = u(\hat{\rho}, \hat{m})$. We note that the matrix on the right hand side is the Hessian of $\hat{\eta} = \eta(\hat{M})$.

This relation also holds in general for any choice of Maxwellians with twice differentiable kinetic entropies. The inner product in v is defined as above with the weight being the second derivative of H . The minimization principle (3.1.19) has the further consequence that $H'(M(\rho, m))$ is linear in the collision invariants, i.e. there exists a vector $b_{(\rho, m)} \in \mathbb{R}^n$ such that $H'(M(\rho, m)) = b_{(\rho, m)} \cdot (1, v)^T$. If we take the gradient of the first relation in (3.1.18) it turns out that $b_{(\rho, m)} = \nabla \eta(\rho, m)$, such that $H'(M(\rho, m)) = \nabla \eta(\rho, m) \cdot (1, v)^T$. Taking again the gradient of this equality we discover $\langle \mathbf{F}, f \rangle_v = \mathcal{H}(\hat{\eta})(\rho_f, m_f)^T$.

With respect to the above weighted inner product in v the linearized collision operator $\mathcal{L}$ is symmetric and negative semidefinite

$$\langle \mathcal{L}f, f \rangle_v \leq 0.$$

The symmetry follows directly from (3.2.6). To see the semidefiniteness we write

$$\begin{aligned} \langle \mathcal{L}f, f \rangle_v &= \left\langle \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f, f - \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} \right\rangle_v \\ &\quad + \left\langle \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix}, \mathbf{F} \cdot \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} - f \right\rangle_v. \end{aligned}$$

The first term is nonpositive and the second one vanishes because of (3.2.6).

The standard norms and spaces of functions of ξ we denote with $(L_\xi^2, \|\cdot\|_\xi)$, $(H_\xi^k, \|\cdot\|_{H_\xi^k})$, $(L_\xi^\infty, \|\cdot\|_\infty)$. The Hilbert space $L_{\xi, v}^2$ is then naturally defined by the scalar product

$$\langle f, g \rangle_{\xi, v} = \int_{\mathbb{R}} \langle f, g \rangle_v d\xi, \quad \text{where } \text{supp } f, \text{supp } g \subset \Omega,$$

with the induced norm $\|\cdot\|_{\xi, v}$. Similarly the spaces $H_\xi^k(L_v^2)$ of functions, whose derivatives in ξ up to order k are in L_v^2 , are defined by

$$\|f\|_{H_\xi^k(L_v^2)} = \left(\|f\|_{\xi, v}^2 + \cdots + \|\partial_\xi^k f\|_{\xi, v}^2 \right)^{1/2}.$$

For the existence and stability proofs we need the following property for Maxwellians $M(\rho, m)$ with $\text{supp } M(\rho, m) \subset \Omega$:

$$\sup_{\xi} \left| \int \left(\partial_{\rho}^j \partial_m^k M(\rho, m, v) \right)^2 \hat{M}^{\frac{1}{\beta}-1} dv \right| < \infty, \quad (3.2.7)$$

for $j + k = 0, \dots, 4$. In order to guarantee that this holds, we have to make a technical assumption and restrict in the following α to values

$$0 < \alpha < \frac{1}{13}.$$

To see that this implies (3.2.7) we first observe that it is sufficient to show the uniform boundedness of

$$\int_{\text{supp } M(\rho, m)} |p(v)| \left(\frac{c^2}{\alpha} - (v - u)^2 \right)^{2(\beta-n)} \left(\frac{\hat{c}^2}{\alpha} - (v - \hat{u})^2 \right)^{1-\beta} dv, \quad (3.2.8)$$

for $n = 0, \dots, 4$. Here $p(v)$ is a polynomial in v , which can also be neglected since the integration is over a bounded domain. The assumption $\text{supp } M(\rho, m) \subset \Omega$ implies

$$\left(\frac{c}{\sqrt{\alpha}} + u - v \right) \left(\frac{c}{\sqrt{\alpha}} - u + v \right) \leq \left(\frac{\hat{c}}{\sqrt{\alpha}} + \hat{u} - v \right) \left(\frac{\hat{c}}{\sqrt{\alpha}} - \hat{u} + v \right),$$

for all $v \in \text{supp } M(\rho, m)$ and $\xi \in \mathbb{R}$. Hence, assuming for the moment $\beta > 1$, the integral in (3.2.8) is bounded by

$$\int \left(\frac{c^2}{\alpha} - (v - u)^2 \right)_+^{\beta+1-2n} dv.$$

A transformation of variable leads to the Beta function and hence (3.2.7) is valid only if $\beta + 1 - 2n > -1$, i.e. $\beta > 6$ or equivalently $0 < \alpha < 1/13$.

Moreover, if $\text{supp } f \subset \Omega$, then

$$\left| \int v^k f dv \right| \leq C \|f\|_v, \quad \text{for } k \leq 3, \quad (3.2.9)$$

and clearly also

$$\|\rho_f\|_{H_{\xi}^k} \leq C \|f\|_{H_{\xi}^k(L_v^2)}, \quad \|m_f\|_{H_{\xi}^k} \leq C \|f\|_{H_{\xi}^k(L_v^2)}. \quad (3.2.10)$$

3.3 Existence of small amplitude travelling waves

As in the formal asymptotics we want to expand f in powers of ϵ . Similarly to [6], we construct a formal asymptotic approximation

$$f_{as} := M(\rho, m) + \epsilon^2 f^{\perp}[\rho, m], \quad (3.3.1)$$

where ρ, m are yet undetermined and the leading term of $f^\perp$ is chosen as in (3.1.29) with a correction term of $O(\epsilon)$

$$f^\perp[\rho, m] := -\frac{1}{\epsilon}(v-s)\partial_\xi M(\rho, m) + \epsilon \left(\frac{1}{\epsilon^2} \partial_\xi (P(\rho, m) - sm) \partial_m \hat{M} \right). \quad (3.3.2)$$

This choice gives $\rho^\perp = -\frac{1}{\epsilon} \partial_\xi (m - s\rho)$ and $m^\perp = 0$. The residual is

$$\epsilon^3 h := \epsilon(v-s)\partial_\xi f_{as} - M_{f_{as}} + f_{as}. \quad (3.3.3)$$

We again pose the far-field conditions

$$f_{as}(\pm\infty, v) = M_{r,l}(v), \quad (3.3.4)$$

and require for ρ, m the relation

$$m - m_l = s(\rho - \rho_l), \quad (3.3.5)$$

which implies $\rho^\perp = 0$ and also $\int h dv = 0$. Therefore

$$\rho_{as} = \rho, \quad m_{as} = m.$$

Moreover we determine ρ_{as}, m_{as} such that $\int v h dv = 0$. This condition is equivalent to the differential equation

$$\begin{aligned} \frac{1}{\epsilon} \partial_\xi \left(\int v(v-s)^2 M_{f_{as}} dv - (2\hat{u}-s)(P(\rho_{as}, m_{as}) - sm_{as}) \right) \\ = \frac{1}{\epsilon^2} (P(\rho_{as}, m_{as}) - P_l - s(m_{as} - m_l)). \end{aligned} \quad (3.3.6)$$

With the ansatz

$$\rho_{as} = \rho_l + \epsilon y_1, \quad m_{as} = m_l + \epsilon y_2, \quad (3.3.7)$$

equation (3.3.5) implies $y_2 = s y_1$ and therefore (3.3.6) becomes an ODE for y_1 (which is at leading order the viscous Burgers equation (3.1.32)):

$$(D_0 + \epsilon \mathcal{N}_1(y_1)) \partial_\xi y_1 = -2\sigma c_l y_1 (1 - y_1) + \epsilon \mathcal{N}_2(y_1),$$

where $\mathcal{N}_1(y_1)$ and $\mathcal{N}_2(y_1)$ are bounded if y_1 is bounded and $\mathcal{N}_2(0) = \mathcal{N}_2(1) = 0$. Hence for small ϵ a bounded smooth monotone solution y_1 connecting the values $y_1(-\infty) = 0$ to $y_1(\infty) = 1$ exists, which is made unique by the initial condition

$$\rho_{as}(0) = \frac{\rho_l + \rho_r}{2}. \quad (3.3.8)$$

Since y_1 is uniformly bounded in $\xi \in \mathbb{R}$, the same holds for $\partial_\xi y_1$. Differentiation shows that $\partial_\xi^k y_1$ for $k = 2, \dots, 4$ are bounded and therefore clearly also $\rho_{as}, m_{as}, \partial_\xi^k \rho_{as}/\epsilon, \partial_\xi^k m_{as}/\epsilon$, where $k = 1, \dots, 4$. Moreover the decay of all these terms is exponential as $\xi \rightarrow \pm\infty$.

As in Section 3.2 the monotonicity of ρ_{as} and m_{as} imply $\text{supp } M(\rho_{as}, m_{as}) \subset \Omega$, from which we deduce $\text{supp } f_{as} \subset \Omega$.

Lemma 3.3.1. *The asymptotic profile f_{as} satisfies the far-field conditions (3.3.4), and the travelling wave equation (3.1.23) up to the residual $\epsilon^3 h$, where h is bounded uniformly in $H_\xi^2(L_v^2)$ and fulfills the moment conditions*

$$\int h dv = 0, \quad \int v h dv = 0. \quad (3.3.9)$$

Proof. The far-field conditions and (3.3.9) result directly from the construction of f_{as} . The boundedness of h in $H_\xi^2(L_v^2)$ we obtain from (3.2.7) and from the exponential decay of all terms, which allows us to integrate in ξ . $\square$

3.3.1 The micro-macro decomposition of the correction term

We introduce the correction term

$$\epsilon^2 g = f - f_{as}$$

with

$$\epsilon^2 \rho_g = \rho_f - \rho_{as}, \quad \epsilon^2 m_g = m_f - m_{as}.$$

Observe that due to (3.1.25) and (3.3.5) the relation

$$m_g = s \rho_g \quad (3.3.10)$$

holds. The travelling wave problem for the correction term g can be written as follows

$$\begin{aligned} \epsilon(v-s)\partial_\xi g - \mathcal{L}g &= (\mathbf{F}_{f_{as}} - \mathbf{F}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_g \\ &+ \frac{1}{\epsilon^2} \left(M_{f_{as} + \epsilon^2 g} - M_{f_{as}} - \epsilon^2 \mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_g \right) + \epsilon h, \end{aligned} \quad (3.3.11)$$

subject to

$$g(\pm\infty, v) = 0 \quad \text{for all } v \in \Omega. \quad (3.3.12)$$

Here we denoted $\mathbf{F}_{f_{as}} := \nabla M_{f_{as}}$. The left hand side is the linearization of the travelling wave equation (3.1.23). On the right hand side we have a linear term of order $O(\epsilon)$, a nonlinear term of order $O(\epsilon^2)$ and the residual term. Lemma 3.3.1, (3.3.12) and integration with respect to ξ give

$$\int (v-s)g dv = 0, \quad \int (v-s)^2 g dv = 0,$$

where the first relation is just (3.3.10). We introduce a decomposition of g into a macroscopic and into a microscopic part, which was done by Caflisch and Nicolaenko in [5] for the Boltzmann equation

$$g(\xi, v) = z(\xi)\Phi(v) + \epsilon w(\xi, v), \quad (3.3.13)$$

where Φ is chosen such that to leading order $\mathcal{L}\Phi = \epsilon(\nu - s)\tau\Phi$ for a constant τ and moreover

$$\int (\nu - s)\Phi d\nu = 0, \quad \int (\nu - s)^2\Phi d\nu = 0. \quad (3.3.14)$$

The choice

$$\Phi(\nu) = \mathbf{F} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} + \epsilon \frac{1}{\hat{D}_0} \hat{\sigma}(2\hat{c} - \epsilon\hat{\sigma})(\nu - \hat{\lambda}_1) \mathbf{F} \cdot \begin{pmatrix} 1 \\ \hat{\lambda}_1 \end{pmatrix} \quad (3.3.15)$$

with $\hat{D}_0 := 2(1 - \alpha)\hat{\rho}^{2\alpha}\hat{c}$ is sufficient for all required properties. We denote

$$\hat{D} := - \int (\nu - s)^3 \Phi d\nu = \hat{D}_0 + O(\epsilon).$$

Since

$$\langle (\nu - s)\Phi, \Phi \rangle_\nu = \epsilon \frac{\hat{D}}{\hat{D}_0} \frac{\hat{\sigma}\hat{c}}{\hat{\rho}} (2\hat{c} - \epsilon\hat{\sigma}) \neq 0,$$

the decomposition of g is made unique by the orthogonality condition

$$\langle (\nu - s)\Phi, w \rangle_\nu = 0.$$

Moreover by computing the zeroth and first order moments of (3.3.15) we see that the macroscopic density and momentum of Φ are the constant values

$$\rho_\Phi = 1, \quad m_\Phi = s,$$

which together with (3.3.10) give

$$s\rho_w = m_w. \quad (3.3.16)$$

The definition of the micro-macro decomposition implies the following properties of z and w :

Lemma 3.3.2. *If g satisfies (3.3.11) and (3.3.12), then*

$$w(\pm\infty, \nu) \equiv 0, \quad z(\pm\infty) = 0 \quad (3.3.17)$$

and

$$\int (\nu - s)^k w(\xi, \nu) d\nu = 0, \quad \text{for all } \xi \in \mathbb{R} \text{ and } k = 1, 2, 3. \quad (3.3.18)$$

Substitution of (3.3.13) into (3.3.11) and division by ϵ gives

$$(\nu - s)\Phi \partial_\xi z - \Lambda z + \epsilon(\nu - s)\partial_\xi w - \mathcal{L}w = \epsilon\Gamma\rho_w + \epsilon R(\rho_g) + h, \quad (3.3.19)$$

where

$$\begin{aligned} \Lambda &= \frac{1}{\epsilon} \left(\mathbf{F}_{fas} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} - \Phi \right), & \Gamma &= \frac{1}{\epsilon} (\mathbf{F}_{fas} - \mathbf{F}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}, \\ R(\rho_g) &= \frac{1}{\epsilon^4} \left(M_{fas+\epsilon^2 g} - M_{fas} - \epsilon^2 \mathbf{F}_{fas} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_g \right). \end{aligned}$$

These terms are formally $O(1)$, such that the ϵ -powers in (3.3.19) show the expected orders of magnitude. Moreover

$$\int \varphi dv = 0, \quad \int v\varphi dv = 0 \quad \text{for } \varphi \in \{R, \Gamma, \Lambda, h\}.$$

First we derive an equation for z by multiplying the equation with $-(v-s)^2$ and integrating with respect to v :

$$\hat{D}\partial_\xi z + \psi(\xi)z = - \int (v-s)^2 (\epsilon \hat{\Gamma} \rho_w + \epsilon R(\rho_g) + h) dv, \quad (3.3.20)$$

where

$$\begin{aligned} \psi(\xi) &= \int (v-s)^2 \Lambda dv = \frac{1}{\epsilon} \left(\nabla P(\rho_{as}, m_{as}) \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} - s^2 \right), \\ \hat{\Gamma} &= \frac{1}{\epsilon} \mathbf{F}_{f_{as}} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}. \end{aligned}$$

Here we have used (3.3.14), Lemma 3.3.1 and Lemma 3.3.2. Equation (3.3.20) does not contain derivatives of w and becomes independent of w as $\epsilon \rightarrow 0$. Expanding $\psi(\xi)$ shows that $\psi \sim 2c_l \sigma(1-2y_1)$, where y_1 is the profile of ρ_{as} (3.3.7). Thus to leading order equation (3.3.20) is the inhomogenous linearized viscous Burgers equation. In particular there exist constants $\gamma, \bar{\xi} > 0$ such that

$$\psi(\xi) \leq -\gamma \text{ for } \xi \leq -\bar{\xi} \quad \text{and} \quad \psi(\xi) \geq \gamma \text{ for } \xi \geq \bar{\xi}. \quad (3.3.21)$$

To derive an equation for the microscopic term w we substitute (3.3.20) into (3.3.19), which is the same as applying the projection

$$\Pi f := f + \frac{(v-s)\Phi}{\hat{D}} \int (v-s)^2 f dv$$

to (3.3.19), giving

$$\epsilon(v-s)\partial_\xi w - \mathcal{L}w = \Pi \Lambda z + \epsilon \tilde{\Gamma} \rho_w + \epsilon \Pi R + \Pi h, \quad (3.3.22)$$

where

$$\tilde{\Gamma} = \Pi \Gamma + \frac{(v-s)\Phi}{\hat{D}} \frac{1}{\epsilon} \int (v-s)^2 \mathbf{F} dv \cdot \begin{pmatrix} 1 \\ s \end{pmatrix}.$$

The idea of the following manipulation of the equation for w is again from Caflisch and Nicolaenko [5] and was also used in [6] for the scalar conservation law. Since $\mathcal{L}$ is only negative semidefinite, we introduce a new operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions satisfying the moment conditions in (3.3.18):

$$\mathcal{M}w := \mathcal{L}w - v(v-s)\mathbf{F} \cdot \langle v(v-s)\mathbf{F}, w \rangle_v. \quad (3.3.23)$$

Using (3.2.6) we see that

$$\langle v(v-s)\mathbf{F}, w \rangle_v = \frac{1}{\hat{\rho}} \begin{pmatrix} \hat{c}^2 + \hat{u}^2 & -\hat{u} \\ -\hat{u} & 1 \end{pmatrix} \int \begin{pmatrix} 1 \\ v \end{pmatrix} v(v-s)w dv.$$

Lemma 3.3.3. *On the set of functions w with $m_w = s\rho_w$, the operator $\mathcal{M}$ satisfies the following properties:*

(i) $\mathcal{M}$ is symmetric with respect to $\langle \cdot, \cdot \rangle_v$.

(ii) $\mathcal{M}$ coincides with $\mathcal{L}$ on the set of functions w with

$$\int (v-s)^k w dv \equiv 0, \quad \text{for } k = 1, 2, 3.$$

(iii) $\mathcal{M}$ is negative definite in L_v^2 . There exists a constant $\kappa > 0$, such that

$$-\langle \mathcal{M}w, w \rangle_v \geq \kappa \|w\|_v^2. \quad (3.3.24)$$

Proof. We proceed as in [5] and [6]. Since $\mathcal{L}$ is symmetric, the same holds for $\mathcal{M}$. The second property is obvious.

Now it remains to show the estimate (3.3.24). We decompose

$$w = \mathbf{F} \cdot \begin{pmatrix} 1 \\ s \end{pmatrix} \rho_w + w^\perp.$$

This implies $\mathcal{L}w = -w^\perp$, $\langle \mathcal{L}w, w \rangle_v = -\|w^\perp\|_v^2$ and $\|w\|_v^2 = \|\mathbf{F} \cdot (1, s)^T \rho_w\|_v^2 + \|w^\perp\|_v^2$. With the relations in (3.2.6) and (3.1.5) we see that $\langle v(v-s)\partial_m \tilde{M}, F \cdot (1, s)^T \rangle_v = -\hat{D}_0/\hat{\rho} + O(\epsilon) =: -\hat{D}_1/\hat{\rho} < 0$ and obtain

$$\begin{aligned} -\langle \mathcal{M}w, w \rangle_v &= \|w^\perp\|_v^2 + |\langle v(v-s)\mathbf{F}, w \rangle_v|^2 \\ &\geq \|w^\perp\|_v^2 + \frac{1}{\hat{\rho}^2} \left(\hat{D}_1 \rho_w + \hat{u} \int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv \right)^2 \\ &= \|w^\perp\|_v^2 + \gamma \frac{\hat{D}_1^2}{\hat{\rho}^2} \rho_w^2 - \frac{\gamma}{1-\gamma} \frac{1}{\hat{\rho}^2} \left(\hat{u} \int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv \right)^2 \\ &\quad + \frac{(1-\gamma)}{\hat{\rho}^2} \left[\hat{D}_1 \rho_w + \frac{1}{1-\gamma} \left(\hat{u} \int v(v-s)w^\perp dv - \int v^2(v-s)w^\perp dv \right) \right]^2 \end{aligned}$$

According to (3.2.9) there exists a $C > 0$ such that

$$-\langle \mathcal{M}w, w \rangle_v \geq \gamma \frac{\hat{D}_1^2}{\hat{\rho}^2} \rho_w^2 + \|w^\perp\|_v^2 \left(1 - \frac{\gamma}{1-\gamma} C \right) \geq \kappa \|w\|_v^2$$

for a $\kappa > 0$ and $\gamma \in (0, 1)$ sufficiently small. $\square$

Instead of solving (3.3.20), (3.3.22) subject to (3.3.17), we now replace the operator $\mathcal{L}$ by $\mathcal{M}$

$$\epsilon(v-s)\partial_\xi w - \mathcal{M}w = \Pi \Lambda z + \epsilon \tilde{\Gamma} \rho_w + \epsilon \Pi R + \Pi h \quad (3.3.25)$$

and show the existence of a solution of (3.3.20), (3.3.25) together with (3.3.17). The equivalence of these problems is not obvious.

Lemma 3.3.4. *The function $g = z\Phi + \epsilon w$ is a solution of (3.3.11) and (3.3.12), iff z and w solve (3.3.20), (3.3.25) subject to (3.3.17).*

Proof. Let g be a solution of (3.3.11), (3.3.12). According to Lemma 3.3.2 and Lemma 3.3.3 (ii) we obtain $\mathcal{L}w = \mathcal{M}w$, and therefore (z, w) solves (3.3.20), (3.3.25), (3.3.17).

Conversely let now (z, w) be a solution of (3.3.20), (3.3.25), (3.3.17). We have to show $\int (v-s)^k w dv \equiv 0$ for $k = 1, 2, 3$, such that $\mathcal{L}$ and $\mathcal{M}$ coincide. Note that

$$\int v^i \varphi dv = 0, \quad i = 0, 1 \quad \Rightarrow \quad \int (v-s)^j \Pi \varphi dv = 0, \quad j = 0, 1, 2,$$

and thus these relations hold for $\varphi \in \{\Lambda, \Gamma, R, h\}$. Calculating the corresponding moments of (3.3.25) we obtain the following linear system of ordinary differential equations

$$\epsilon \begin{pmatrix} \partial_\xi \int (v-s)w dv \\ \partial_\xi \int v(v-s)w dv \\ \partial_\xi \int v^2(v-s)w dv \end{pmatrix} = \mathbf{B} \cdot \begin{pmatrix} \int (v-s)w dv \\ \int v(v-s)w dv \\ \int v^2(v-s)w dv \end{pmatrix},$$

where $\mathbf{B}$ is a matrix with constant coefficients. We know that $\int v^k (v-s)w(\pm\infty, v) dv = 0$ for $k = 0, 1, 2$. Hence the only possible solution is

$$\int v^k (v-s)w(\xi, v) dv \equiv 0 \quad \text{for } k = 0, 1, 2.$$

□

3.3.2 Existence

We now show the existence of a solution of the problem (3.3.20), (3.3.25), (3.3.17) by first solving the linear and finally the full nonlinear system of differential equations for the decomposition of g . The solvability of a similar problem was already shown in [6]. Here the results are just repeated. We start with the linear problem and regard the right hand sides of equations (3.3.20), (3.3.25) as given inhomogenities

$$\partial_\xi z + \psi(\xi)z = h_z, \quad \text{with } h_z \in H_\xi^1, \quad (3.3.26)$$

$$\epsilon(v-s)\partial_\xi w - \mathcal{M}w = h_w, \quad \text{with } h_w \in H_\xi^2(L_v^2). \quad (3.3.27)$$

To prove the stability result in Section 4, we need L_ξ^∞ -bounds for the macroscopic profiles of the travelling wave and for their first derivatives. Hence we look for solutions in the spaces H_ξ^2 and $H_\xi^2(L_v^2)$. This requires homogenous far-field conditions and already provides uniqueness for the solution of (3.3.27). Equation (3.3.26) has a one parameter set of solutions, which is due to the arbitrary shift of travelling wave solutions. We pose the initial condition

$$z(0) = z_0. \quad (3.3.28)$$

The following result was shown in [6]:

Lemma 3.3.5. *The unique solution z of (3.3.26), (3.3.28) is bounded by*

$$\|z\|_{H_\xi^2} \leq C(|z_0| + \|h_z\|_{H_\xi^1}).$$

The variation of constant formula gives the mild formulation of the unique solution of (3.3.26), (3.3.28). From the properties of ψ given in (3.3.21) one can deduce the L^2 -continuous dependence of z on z_0 and h_z . From (3.3.26) we get the same estimate for $\partial_\xi z$. Using the uniform boundedness of $\partial_\xi \psi$, differentiation of (3.3.26) finally gives the bound for $\partial_\xi^2 z$.

Lemma 3.3.6. *There exists a unique solution $w \in H_\xi^2(L_v^2)$ of (3.3.27). For this solution the bound*

$$\|\partial_\xi^k w\|_{\xi, v} \leq \frac{1}{\kappa} \|\partial_\xi^k h_w\|_{\xi, v}, \quad \text{for } k = 0, 1, 2,$$

holds, where κ is the same as in (3.3.24).

Sketch of the proof. We can apply the proof given in [6], which is based on a discretisation of the velocity variable. The latter yields a system of ordinary differential equations. Since the discretised version of the operator $\mathcal{M}$ is again symmetric and negative definite we can deduce the existence and uniqueness of a bounded solution which converges to zero as $\xi \rightarrow \pm\infty$. Choosing the quadrature formula appropriate the solution of the discretised problem is bounded by $\frac{1}{\kappa} \|h_w\|_\xi$, implying weak convergence. Then we can pass to the limit. Also the estimate carries over to the solution w of (3.3.27). The estimates for the derivatives are then derived from the differentiated equation in the same way. $\square$

We now apply a fix-point argument to solve the nonlinear equations (3.3.20), (3.3.25) subject to the initial condition $z(0) = z_0$, which is related to the initial condition for the original unknown by

$$\langle (v-s)\Phi, f - f_{as} \rangle_v (\xi = 0) = \epsilon^2 \langle (v-s)\Phi, \Phi \rangle_v z_0. \quad (3.3.29)$$

For the contraction argument we need bounds for the right hand sides of (3.3.20), (3.3.25).

Lemma 3.3.7. (i) *The operator $\Pi : L_v^2 \rightarrow L_v^2$ is bounded.*

(ii) *There exists a constant $C > 0$, such that*

$$\begin{aligned} \|\Pi \Lambda z\|_{H_\xi^2(L_v^2)} &\leq C \|z\|_{H_\xi^2}, & \|\tilde{\Gamma} \rho_w\|_{H_\xi^2(L_v^2)} &\leq C \|w\|_{H_\xi^2(L_v^2)}, \\ \left\| \int (v-s)^2 \hat{\Gamma} dv \rho_w \right\|_{H_\xi^2} &\leq C \|\rho_w\|_{H_\xi^2}. \end{aligned}$$

(iii) *There exists a constant $K > 0$, such that for all ρ_1, ρ_2 with $\|\rho_1\|_{H_\xi^2}, \|\rho_2\|_{H_\xi^2} \leq \frac{K}{\epsilon}$ we obtain for the nonlinearity*

$$\|R(\rho_1) - R(\rho_2)\|_{H_\xi^2(L_v^2)} \leq C(\|\rho_1\|_{H_\xi^2} + \|\rho_2\|_{H_\xi^2}) \|\rho_1 - \rho_2\|_{H_\xi^2}. \quad (3.3.30)$$

Proof. The proof of (i) is straightforward using (3.2.9).

For the estimates on the linear terms in (ii) we use (i), inequality (3.2.10), the boundedness of $\|\hat{y}_1\|_\infty, \|\hat{y}_2\|_\infty$, (see (3.2.3)), and of $\|\partial_\xi^k \rho_{as}/\epsilon\|_\infty, \|\partial_\xi^k m_{as}/\epsilon\|_\infty$ for $k = 1, 2$.

We expand the nonlinearities

$$R(\rho_1) - R(\rho_2) = (1, s) \cdot \mathcal{H}(M_1)(1, s)^T (\rho_2 + \vartheta_1(\rho_1 - \rho_2))(\rho_1 - \rho_2),$$

where $M_1 = M((\rho_{as}, m_{as}) + \epsilon^2 \vartheta_2(\rho_2 + \vartheta_1(\rho_1 - \rho_2))(1, s))$ and $0 \leq \vartheta_1, \vartheta_2 \leq 1$. Due to the construction of Ω there exists a constant $K > 0$ such that $\text{supp } M_1 \subset \Omega$ for $\|\rho_1\|_\infty, \|\rho_2\|_\infty \leq K/\epsilon$. If now ρ_1, ρ_2 are bounded in H_ξ^2 by a constant of order $O(\epsilon^{-2})$, which is guaranteed by our assumption, we obtain the given estimate by differentiation, using (3.2.7) and the one-dimensional Sobolev imbedding. $\square$

We define the following norm in $H_\xi^2(L_v^2)$:

$$\|g\| := \|z\|_{H_\xi^2} + \epsilon \|w\|_{H_\xi^2(L_v^2)},$$

and observe that $\|g\|_{H_\xi^2(L_v^2)} \leq C \|g\|$ for a positive constant C .

Theorem 3.3.1. *For every $z_0 \in \mathbb{R}$ and for ϵ small enough, there exists a solution f of (3.1.23) satisfying (3.3.29), unique in a ball in $(H_\xi^2(L_v^2), \|\cdot\|)$ with center f_{as} and an $O(\epsilon)$ radius. It satisfies*

$$\|f - M(\rho_{as}, m_{as})\|_{H_\xi^2(L_v^2)} = O(\epsilon^2),$$

where (ρ_{as}, m_{as}) is the solution of (3.3.5), (3.3.6) and (3.3.8). More precisely

$$f = M(\rho_{as}, m_{as}) + \epsilon^2 f^\perp[\rho_{as}, m_{as}] + \epsilon^2 z \Phi + \epsilon^3 w,$$

where $\|z\|_{H_\xi^2}$ and $\|w\|_{H_\xi^2(L_v^2)}$ are uniformly bounded as $\epsilon \rightarrow 0$.

Proof. We proceed as in [6]. As a consequence of Lemma 3.3.7 (ii) we can extend the results from Lemma 3.3.5 and Lemma 3.3.6 to the full linear problem

$$\begin{aligned} \hat{D} \partial_\xi z + \psi(\xi) z &= -\epsilon \int (\nu - s)^2 \hat{\Gamma} \rho_w d\nu + h_z, \\ \epsilon(\nu - s) \partial_\xi w - \mathcal{M} w &= \Pi \Lambda z + \epsilon \tilde{\Gamma} \rho_w + h_w, \end{aligned}$$

with inhomogenities h_z, h_w and $z(0) = z_0$. Applying the solution operator to the nonlinearities and residual terms in (3.3.20) and (3.3.25) gives a fixed point problem $(z, w) = \mathcal{G}(z, w)$. Due to Lemma 3.3.7 (iii) the fixed point operator is bounded by $\|\mathcal{G}(z, w)\| \leq c(1 + \epsilon\|(z, w)\|^2)$ if $\|(z, w)\| \leq C/\epsilon$ for some $C > 0$. This implies that for ϵ small enough $\mathcal{G}$ maps both the ball with radius $2c$ and the ball with radius $\epsilon^{-1} \min\{1/(2c), C\}$ into themselves. Also $\mathcal{G}$ is a contraction on the ball with an $O(\epsilon^{-1})$ radius. We conclude that for ϵ small the fixed point problem has a solution with $\|(z, w)\| \leq 2c$, which is unique in a ball with an $O(\epsilon^{-1})$ radius. Knowing this and returning to the fixed point equation for w , also the boundedness of $\|w\|_{H_\xi^2(L_v^2)}$ follows. $\square$

The monotonicity of ρ_f can be deduced in the same way as it was done in [6]:

Lemma 3.3.8. *Let the assumptions of (the existence) Theorem 3.3.1 hold and let f be the solution of (3.1.23) with initial condition (3.3.29). Then $\rho_f(\xi)$ is strictly increasing.*

The proof relies on the fact that the map $z_0 \mapsto \rho_f(0)$ is invertible for ϵ small, meaning that the travelling wave can also be made locally unique by prescribing the value of $\rho_f(0)$ instead of z_0 . This argument can of course be repeated for every $\xi_0 \in \mathbb{R}$ instead of the origin. Now assuming ρ_f is not strictly monotone would lead to the periodicity of f as a consequence of the uniqueness result, which contradicts the far-field conditions.

3.4 Local stability of small amplitude travelling waves

In this section we prove the asymptotic, dynamic stability of small amplitude travelling waves constructed before. The methods we apply are commonly used for conservation laws with diffusion terms. This motivates to introduce the parabolic scaling $\xi \rightarrow \frac{\xi}{\epsilon}, t \rightarrow \frac{t}{\epsilon^2}$ in equation (3.1.20). Let f be the solution of

$$\epsilon^2 \partial_t f + \epsilon(v - s) \partial_\xi f = M_f - f \quad (3.4.1)$$

with the far-field conditions

$$f(t, \xi = \pm\infty, v) = M_{r,l}(v).$$

Let ϕ be the travelling wave solution as in Theorem 3.3.1. For given initial data $f_0(\xi, v) = f(0, \xi, v)$ we fix the shift in the travelling wave ϕ such that

$$\int_{\mathbb{R}} [\rho_{f_0}(\xi) - \rho_\phi(\xi)] d\xi = 0. \quad (3.4.2)$$

In addition we restrict ourselves to initial data satisfying

$$\int_{\mathbb{R}} [m_{f_0}(\xi) - m_\phi(\xi)] d\xi = 0. \quad (3.4.3)$$

This way we guarantee

$$\int_{\mathbb{R}} [\rho_f(t, \xi) - \rho_\phi(\xi)] d\xi = \int_{\mathbb{R}} [m_f(t, \xi) - m_\phi(\xi)] d\xi = 0, \quad \forall t \geq 0. \quad (3.4.4)$$

Introducing the perturbation G by

$$\epsilon G = f - \phi, \quad \rho := \rho_G, \quad m := m_G,$$

we obtain

$$\epsilon \partial_t G + (v - s) \partial_\xi G = \frac{1}{\epsilon^2} [M_{\phi + \epsilon G} - M_\phi] - \frac{1}{\epsilon} G. \quad (3.4.5)$$

As in [6] we apply a micro-macro decomposition to the deviation G

$$G = \mathbf{F} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \epsilon g. \quad (3.4.6)$$

Then the norm of G satisfies

$$\|\partial_\xi^k G\|_{\xi, \nu}^2 = \frac{1}{\hat{\rho}} \left[\hat{c}^2 \|\partial_\xi^k \rho\|_\xi^2 + \left\| \partial_\xi^k (m - \hat{u} \rho) \right\|_\xi^2 \right] + \epsilon^2 \|\partial_\xi^k g\|_{\xi, \nu}^2. \quad (3.4.7)$$

Macroscopic equations for ρ and m are obtained by computing the zeroth and first order moments of equation (3.4.5)

$$\epsilon \partial_t \rho + \partial_\xi (m - s \rho) = 0, \quad (3.4.8)$$

$$\epsilon \partial_t m + \partial_\xi \left(\nabla \hat{P} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} - s m \right) + \epsilon \partial_\xi P_g = 0, \quad (3.4.9)$$

where $\hat{P} := P_{\hat{M}} = P(\hat{\rho}, \hat{m})$ and as before $P_g = \int v^2 g \, dv$. Next we apply the microscopic projection $-\mathcal{L}G = \epsilon g$ to (3.4.5) to get an equation for g

$$\epsilon^2 \partial_t g - \partial_\xi \left(\mathbf{F} \cdot \left[(\mathbf{A} - \nu \mathbf{I}) \begin{pmatrix} \rho \\ m \end{pmatrix} \right] \right) - \epsilon \partial_\xi \mathcal{L}((v - s)g) = R_2(\rho, m) - g \quad (3.4.10)$$

with the Jacobian $\mathbf{A} := \mathbf{A}(\hat{\rho}, \hat{m})$ given in (3.1.11) and the nonlinearity

$$\begin{aligned} R_2(\rho, m) &= \frac{\mathbf{F}_\phi - \mathbf{F}}{\epsilon} \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} + \tilde{R}_2(\rho, m), \\ \tilde{R}_2(\rho, m) &= \frac{1}{\epsilon^2} \left[M_{\phi + \epsilon G} - M_\phi - \epsilon \mathbf{F}_\phi \cdot \begin{pmatrix} \rho \\ m \end{pmatrix} \right], \end{aligned}$$

split into its linear and purely quadratic part.

As in the Chapman-Enskog approximation we compute the last term in (3.4.9) using (3.4.10)

$$P_g = q(\rho, m) - \epsilon^2 \partial_t P_g + \epsilon \partial_\xi P_{\mathcal{L}((v-s)g)} - D \partial_\xi (m - \hat{u} \rho), \quad (3.4.11)$$

with the constant $D := 2(1 - \alpha)\hat{\rho}^{2\alpha} > 0$ and the nonlinearity $q(\rho, m) := P_{R_2(\rho, m)}$. According to (3.4.7) and the diffusion term in (3.4.11) it is convenient (as already mentioned in the introduction) to define

$$W_\rho(t, \xi) = \int_{-\infty}^{\xi} \rho(t, \xi) \, d\xi, \quad W_u(t, \xi) = \int_{-\infty}^{\xi} [m(t, \xi) - \hat{u} \rho(t, \xi)] \, d\xi.$$

Integrating (3.4.8), (3.4.9) with respect to ξ gives the macroscopic equations

$$\partial_t W_\rho + \frac{1}{\epsilon} \left[(\hat{c} - \epsilon \hat{\sigma}) \partial_\xi W_\rho + \partial_\xi W_u \right] = 0, \quad (3.4.12)$$

$$\partial_t W_u + \frac{1}{\epsilon} \left[\hat{c}^2 \partial_\xi W_\rho + (\hat{c} - \epsilon \hat{\sigma}) \partial_\xi W_u \right] + P_g = 0. \quad (3.4.13)$$

Observe that the second equation is obtained by a linear combination of (3.4.8), (3.4.9). Substituting (3.4.11) we get the equivalent equation for (3.4.13)

$$\begin{aligned} \partial_t W_u + \frac{1}{\epsilon} \left[\hat{c}^2 \partial_\xi W_\rho + (\hat{c} - \epsilon \hat{\sigma}) \partial_\xi W_u \right] + q - D \partial_\xi^2 W_u \\ = \epsilon^2 \partial_t P_g - \epsilon \partial_\xi P_{\mathcal{L}((v-s)g)}. \end{aligned} \quad (3.4.14)$$

Splitting q as R_2 into its linear and purely quadratic term, the equation can again be reformulated to

$$\begin{aligned} \partial_t W_u + \frac{1}{\epsilon} \left[K_1(\phi) \partial_\xi W_\rho + K_2(\phi) \partial_\xi W_u \right] + \tilde{q} - D \partial_\xi^2 W_u \\ = \epsilon^2 \partial_t P_g - \epsilon \partial_\xi P_{\mathcal{L}((v-s)g)}, \end{aligned} \quad (3.4.15)$$

where $\tilde{q} = P_{\tilde{R}_2}$ and

$$K_1(\phi) := c_\phi^2 - (u_\phi - \hat{u})^2, \quad K_2(\phi) := \hat{c} - \epsilon \hat{\sigma} + 2(u_\phi - \hat{u}). \quad (3.4.16)$$

In the following we will switch between the three different representations of the second macroscopic equation. We will need the signs of K_1, K_2 and of their derivatives. From Lemma 3.3.8 we know that the density of the travelling wave is strictly increasing, which also implies $\partial_\xi u_\phi < 0$, see (3.1.34). Since all components have the same decay rate we get (for ϵ small)

$$\frac{\hat{c}^2}{2} < K_1(\phi) < 2\hat{c}^2, \quad \partial_\xi K_1(\phi) > 0, \quad \partial_\xi (K_1(\phi)^{-1}) < 0, \quad (3.4.17)$$

$$\frac{\hat{c}}{2} < K_2(\phi) < 2\hat{c}, \quad \partial_\xi K_2(\phi) < 0. \quad (3.4.18)$$

We recall the uniform boundedness of $\partial_\xi K_1(\phi), \partial_\xi K_2(\phi)$ by a constant of $O(\epsilon)$ resulting from Theorem 3.3.1.

We start with deriving estimates for the macroscopic part of the system. For controlling the nonlinear terms, L_ξ^∞ -bounds of ρ, m are needed, which we shall control in H_ξ^1 . This means we need to control the H_ξ^2 -norm of W_1, W_2 and therefore we give integral estimates for their derivatives up to second order in the following.

Taylor expansion of $\tilde{R}_2$ gives

$$\tilde{R}_2(\rho, m) = (\rho, m) \cdot \mathcal{H}(M_{\phi+\epsilon\vartheta G}) \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad \text{for a } \vartheta \in (0, 1). \quad (3.4.19)$$

For $\|R_2\|_\nu$ to be well defined we have to guarantee $\text{supp } \tilde{R}_2 \subset \Omega$. Due to the construction of Ω this is only true for sufficiently small $\|\rho\|_\infty, \|m\|_\infty$. We make this smallness assumption for the moment and prove it in the stability result at the end of this section. The nonlinear terms satisfy the estimate

$$\begin{aligned} \|R_2\|_{H_\xi^k(L_\nu^2)}^2 + \|\tilde{R}_2\|_{H_\xi^k(L_\nu^2)}^2 + \|q\|_{H_\xi^k}^2 + \|\tilde{q}\|_{H_\xi^k}^2 \\ \leq \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right] \text{ for } k = 0, 1. \end{aligned} \quad (3.4.20)$$

Here and in the following $\tilde{C}$ depends on $\|\rho\|_\infty, \|m\|_\infty$. By differentiating (3.4.19), using (3.2.7), the Sobolev imbedding and the smoothness of the travelling wave, the estimate for $\tilde{R}_2$ is obtained. Then the bound for R_2 is deduced easily, and finally the estimates for $q, \tilde{q}$ are an immediate consequence of (3.2.9).

We now test (3.4.12) with W_ρ and (3.4.15) with $K_1^{-1}W_u$, which amounts to a symmetrization of the system and yields the cancellation of the mixed terms containing $W_\rho \partial_\xi W_u$ and $W_u \partial_\xi W_\rho$:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|W_\rho\|_\xi^2 + \|K_1^{-1/2}W_u\|_\xi^2] + \frac{1}{2\epsilon} \int \partial_\xi (-K_2 K_1^{-1}) W_u^2 d\xi + \int K_1^{-1}W_u \tilde{q} d\xi \\ & \quad + D \|K_1^{-1/2} \partial_\xi W_u\|_\xi^2 + D \int \partial_\xi (K_1^{-1}) W_u \partial_\xi W_u d\xi \\ & = \epsilon^2 \partial_t \int K_1^{-1} W_u P_g d\xi + \epsilon \int K_1^{-1} [\hat{c}^2 \partial_\xi W_\rho + (\hat{c} - \epsilon \hat{\sigma}) \partial_\xi W_u + \epsilon P_g] P_g d\xi \\ & \quad + \epsilon \int (\partial_\xi K_1^{-1} W_u + K_1^{-1} \partial_\xi W_u) P_{\mathcal{A}((v-s)_g)} d\xi. \end{aligned}$$

Here we have just used integration by parts and equation (3.4.13) for the substitution of $\partial_t W_u$ on the right hand side. We note that $\|K_1^{-1/2} \partial_\xi^k W\|_\xi^2 \geq 1/(2\hat{c}^2) \|\partial_\xi^k W\|_\xi^2$ and the second term has the favourable sign since $\partial_\xi (K_2 K_1^{-1}) \leq 0$ for all ξ . On the left hand side we estimate the purely quadratic nonlinearity by

$$\left| \int K_1^{-1} W_u \tilde{q} d\xi \right| \leq C \|W_u\|_\infty \int |\tilde{q}| d\xi \leq \tilde{C} \|W_u\|_\infty \left[\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2 \right]. \quad (3.4.21)$$

The triangle inequality is used for

$$\left| \int \partial_\xi (K_1^{-1}) W_u \partial_\xi W_u d\xi \right| \leq \int |\partial_\xi (K_1^{-1})| W_u^2 d\xi + \epsilon C \|\partial_\xi W_u\|_\xi^2,$$

where we recall that $\|\partial_\xi \rho_\phi\|_\infty, \|\partial_\xi m_\phi\|_\infty \leq \epsilon C$. We now turn to the right hand side. Recalling (3.2.9), with the Cauchy-Schwarz inequality we can bound the last two integrals by

$$\epsilon C (\|g\|_{\xi,v}^2 + \|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_m\|_\xi^2) + \frac{\epsilon}{2} \int |\partial_\xi (K_1^{-1})| W_u^2 d\xi.$$

Summarizing, we obtain for ϵ small enough

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|W_\rho\|_\xi^2 + \|K_1^{-1/2}W_u\|_\xi^2 - \epsilon^2 \int K_1^{-1} W_u P_g d\xi \right] + \int \kappa W_u^2 d\xi \quad (3.4.22) \\ & + \left(\frac{D}{4\hat{c}^2} - \tilde{C} \|W_u\|_\infty \right) \|\partial_\xi W_u\|_\xi^2 \leq (\tilde{C} \|W_u\|_\infty + \epsilon C) \|\partial_\xi W_\rho\|_\xi^2 + \epsilon C \|g\|_{\xi,v}^2 \end{aligned}$$

where $\kappa(\phi) = \frac{1}{2\epsilon} (\partial_\xi (-K_2 K_1^{-1}) + \epsilon(D + \epsilon) \partial_\xi K_1^{-1}) \geq 0$ for all ξ . The lack of a diffusion term in W_ρ will be overcome with another combination of the macroscopic

equations. We observe that

$$\begin{aligned} \frac{d}{dt} \int W_u \partial_\xi W_\rho d\xi &= \int [\partial_t(W_u \partial_\xi W_\rho) - \partial_\xi(W_u \partial_t W_\rho)] d\xi \\ &= \int [\partial_t W_u \partial_\xi W_\rho - \partial_t W_\rho \partial_\xi W_u] d\xi. \end{aligned}$$

Corresponding to the right hand side we now combine the equations (3.4.12), (3.4.13) yielding

$$\begin{aligned} \epsilon \frac{d}{dt} \int W_u \partial_\xi W_\rho d\xi + \hat{c}^2 \|\partial_\xi W_\rho\|_\xi^2 - \|\partial_\xi W_u\|_\xi^2 \\ = -\epsilon \int \partial_\xi W_\rho P_g d\xi \leq \epsilon (\|\partial_\xi W_\rho\|_\xi^2 + C \|g\|_{\xi, \nu}^2). \end{aligned} \quad (3.4.23)$$

A linear combination of (3.4.22) and (3.4.23) implies the following lemma:

Lemma 3.4.1. *Let W_ρ, W_u be the solution of the system (3.4.12), (3.4.15) and ϵ be small enough. Then there exist constants C and $\tilde{C}$ such that, for any $\alpha_0 > 0$,*

$$\begin{aligned} \frac{d}{dt} J_0 + \left(\frac{\alpha_0 \hat{c}^2}{2} - \tilde{C} \|W_u\|_\infty \right) \|\partial_\xi W_\rho\|_\xi^2 \\ + \left(\frac{D}{4\hat{c}^2} - \alpha_0 - \tilde{C} \|W_u\|_\infty \right) \|\partial_\xi W_u\|_\xi^2 \leq \epsilon C \|g\|_{\xi, \nu}^2 \end{aligned} \quad (3.4.24)$$

with $J_0 = \frac{1}{2} \left[\|W_\rho\|_\xi^2 + \|K_1^{-1/2} W_u\|_\xi^2 - \epsilon^2 \int K_1^{-1} W_u P_g d\xi + 2\epsilon \alpha_0 \int W_u \partial_\xi W_\rho d\xi \right]$.

We now turn to the estimate for the first order derivatives. It is derived by testing the derivatives of (3.4.12) and (3.4.14) with $\hat{c}^2 \partial_\xi W_\rho$ and $\partial_\xi W_u$ respectively. Since there is no contribution of the travelling wave terms in the integration by parts, the derivation is even simpler. The term with the nonlinearity we treat in a different way:

$$\left| \int \partial_\xi q \partial_\xi W_u d\xi \right| = \left| \int q \partial_\xi^2 W_u d\xi \right| \leq \frac{D}{4} \|\partial_\xi^2 W_u\|_\xi^2 + \tilde{C} (\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2).$$

Following the procedure above, where now the differentiated versions of the equations (3.4.12), (3.4.13) are used, we arrive at

Lemma 3.4.2. *Let W_ρ, W_u be the solution of the system (3.4.12), (3.4.14) and ϵ be small enough. Then there exist constants C and $\tilde{C}$ such that, for any $\alpha_1 > 0$,*

$$\begin{aligned} \frac{d}{dt} J_1 + \frac{\alpha_1 \hat{c}^2}{2} \|\partial_\xi^2 W_\rho\|_\xi^2 + \left(\frac{D}{2} - \alpha_1 \right) \|\partial_\xi^2 W_u\|_\xi^2 \\ \leq \tilde{C} (\|\partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2) + \epsilon C \|\partial_\xi g\|_{\xi, \nu}^2 \end{aligned} \quad (3.4.25)$$

with $J_1 = \frac{1}{2} \left[\|\hat{c}^2 \partial_\xi W_\rho\|_\xi^2 + \|\partial_\xi W_u\|_\xi^2 - \epsilon^2 \int \partial_\xi W_u P_{\partial_\xi g} d\xi + 2\epsilon \alpha_1 \int \partial_\xi W_u \partial_\xi^2 W_\rho d\xi \right]$.

For controlling the microscopic terms we use the full kinetic perturbation equation.

Lemma 3.4.3. *Let G , decomposed as in (3.4.6), be the solution of (3.4.5). Then there exists a $\tilde{C}$ such that, for $k = 0, 1$,*

$$\frac{d}{dt} \|\partial_\xi^k G\|_{\xi, \nu}^2 + \|\partial_\xi^k g\|_{\xi, \nu}^2 \leq \tilde{C} \left[\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_u\|_{H_\xi^k}^2 \right].$$

Proof. We take the inner product in ξ and ν of the k th derivative of (3.4.5) with $\partial_\xi^k G$ and divide by ϵ

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_\xi^k G\|_{\xi, \nu}^2 &= \frac{1}{\epsilon} \langle \partial_\xi^k (R_2 - g), \partial_\xi^k G \rangle_{\xi, \nu} = \langle \partial_\xi^k R_2, \partial_\xi^k g \rangle_{\xi, \nu} - \|\partial_\xi^k g\|_{\xi, \nu}^2 \\ &\leq \tilde{C} (\|\partial_\xi W_\rho\|_{H_\xi^k}^2 + \|\partial_\xi W_m\|_{H_\xi^k}^2) - \frac{1}{2} \|\partial_\xi^k g\|_{\xi}^2. \end{aligned}$$

□

Now we are able to prove the main result of this section.

Theorem 3.4.1. *Let the assumptions of Theorem 3.3.1 hold and let ϕ be the travelling wave solution. Let $f_0(\xi, \nu)$ be the initial datum for (3.4.1) and let*

$$\begin{aligned} W_{\rho,0}(\xi) &= \frac{1}{\epsilon} \int_{-\infty}^{\xi} [\rho_{f_0}(\eta) - \rho_\phi(\eta)] d\eta, \\ W_{u,0}(\xi) &= \frac{1}{\epsilon} \int_{-\infty}^{\xi} [(m_{f_0}(\eta) - m_\phi(\eta)) - \hat{u}(\rho_{f_0}(\eta) - \rho_\phi(\eta))] d\eta. \end{aligned}$$

Moreover let $f_0 - \phi \in H_\xi^2(L_\nu^2)$, (implying $f_0(\pm\infty, \nu) = \phi(\pm\infty, \nu)$), and $W_{\rho,0}, W_{u,0} \in L_\xi^2$, which ensures assumption (3.4.2) and (3.4.3). Let

$$\|W_{\rho,0}\|_{L_\xi^2} + \|W_{u,0}\|_{L_\xi^2} + \frac{1}{\epsilon} \|f_0 - \phi\|_{H_\xi^1(L_\nu^2)} \leq \delta \quad (3.4.26)$$

for a δ small enough, which is independent from ϵ . Then for ϵ small enough equation (3.4.1) with initial data f_0 has a unique global solution. In particular, small amplitude travelling waves are locally stable in the sense that

$$\lim_{t \rightarrow \infty} \int_t^\infty \|f(\tau, \cdot) - \phi(\cdot)\|_{H_\xi^1(L_\nu^2)}^2 d\tau = 0.$$

Proof. The main idea is to construct a Lyapunov functional, which is decaying in time. Recall (3.4.1), (3.4.2) and define

$$J := J_0 + \gamma_1 J_1 + \gamma_2 \|G\|_{H_\xi^1(L_\nu^2)}^2,$$

where the ϵ -independent constants $\gamma_1, \gamma_2 > 0$ and $\alpha_0, \alpha_1 > 0$ will be chosen below. For any positive choice of the constants the functional J is bounded from above and below by

$$\|W_\rho\|_{H_\xi^2}^2 + \|W_u\|_{H_\xi^2}^2 + \epsilon^2 \|g\|_{H_\xi^1(L_\nu^2)}^2. \quad (3.4.27)$$

Hence by Sobolev imbedding there exists a constant C_0 such that $\|W_\rho\|_\infty + \|W_u\|_\infty + \|\rho\|_\infty + \|m\|_\infty \leq C_0 \sqrt{J}$. We now combine the estimates from Lemma 3.4.1, Lemma 3.4.2, and Lemma 3.4.3 to get

$$\begin{aligned} & \frac{d}{dt}J + \frac{\gamma_2}{2} \|g\|_{H_\xi^1(L_v^2)}^2 + \left(\frac{\alpha_0 \hat{c}^2}{2} - \tilde{C}(\|W_u\|_\infty + \gamma_1 + \gamma_2) \right) \|\partial_\xi W_\rho\|_\xi^2 \\ & \quad + \left(\frac{D}{4\hat{c}^2} - \alpha_0 - \tilde{C}(\|W_u\|_\infty + \gamma_1 + \gamma_2) \right) \|\partial_\xi W_u\|_\xi^2 \\ & + \left(\gamma_1 \frac{\alpha_1 \hat{c}^2}{2} - \gamma_2 \tilde{C} \right) \|\partial_\xi^2 W_\rho\|_\xi^2 + \left(\gamma_1 \frac{D}{2} - \gamma_1 \alpha_1 - \gamma_2 \tilde{C} \right) \|\partial_\xi^2 W_u\|_\xi^2 \leq 0 \end{aligned}$$

We choose $L = C_0 \sqrt{J(0)}$ small enough, such that there exist constants $\alpha_0, \alpha_1, \gamma_1, \gamma_2 > 0$ satisfying

$$\frac{\alpha_0 \hat{c}^2}{2} > \tilde{C}(L)(L + \gamma_1 + \gamma_2), \quad \gamma_1 \frac{\alpha_1 \hat{c}^2}{2} > \gamma_2 \tilde{C}(L), \quad (3.4.28)$$

$$\frac{D}{4\hat{c}^2} > \alpha_0 + \tilde{C}(L)(L + \gamma_1 + \gamma_2), \quad \gamma_1 \frac{D}{2} > \gamma_1 \alpha_1 + \gamma_2 \tilde{C}(L). \quad (3.4.29)$$

Then the coefficients of $\|\partial_\xi^{k+1} W_\rho\|_\xi^2$, $\|\partial_\xi^{k+1} W_u\|_\xi^2$, $k = 0, 1$, are positive initially. Since J controls the L_ξ^∞ -norms of W_ρ, W_u and their derivatives, these coefficients stay positive. Hence there is a constant $C_1 > 0$ such that

$$\frac{d}{dt}J \leq -C_1 \|G\|_{H_\xi^1(L_v^2)}^2, \quad \text{for all } t \geq 0,$$

showing that J is a Lyapunov functional. The proof is now completed by integration with respect to t . $\square$

Appendix. Isothermal Case

In this appendix we show, how the existence and stability results can be extended to the isothermal system

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} + \partial_x \begin{pmatrix} m \\ \frac{m^2}{\rho} + \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.4.30)$$

A Maxwellian satisfying (3.1.3)-(3.1.4) for $\alpha = 0$ and therefore leading to the isothermal system is given by

$$M(\rho, m, v) = \frac{\rho}{\sqrt{2\pi}} e^{-(v-u)^2/2}, \quad v \in \mathbb{R},$$

see Bouchut [1] and references therein. Observe that in this case the speed of sound is constant, $c(\rho) \equiv 1$. Kinetic and macroscopic entropies are given by

$$H(f, v) = \frac{v^2}{2} f + f \ln f, \quad \eta(\rho, m) = \frac{1}{2} \frac{m^2}{\rho} + (\rho \ln \rho - \rho \ln \sqrt{2\pi}),$$

see again [1]. Now the equilibrium distributions do not have compact support, which simplifies the procedure and allows to linearize around the asymptotic state at $\xi = -\infty$. Again the quadratic approximation of the kinetic entropy close to equilibrium gives the weight for the inner product

$$\langle f, g \rangle_v = \int_{\mathbb{R}} f g \frac{1}{M_l} dv.$$

Then one can show that all conditions from Section 2 are satisfied and therefore the proofs for existence and stability carry over to this case. Only one difference in Section 4 is important to mention. Since the sound speed is constant, the derivative of $K_1(\phi)$ corresponding to (3.4.16) is now of $O(\epsilon^2)$ and has a different sign

$$\partial_\xi K_1(\phi) = -2(u_\phi - u_l)\partial_\xi u_\phi < 0.$$

Therefore the first macroscopic estimate corresponding to (3.4.24) has to be derived differently. We test equation (3.4.13) with $K_1(\phi)W_\rho$ and (3.4.15) with W_u to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|K_1^{1/2} W_\rho\|_\xi^2 + \|W_u\|_\xi^2] + \int \tilde{q} W_u d\xi + D \|\partial_\xi W_u\|_\xi^2 \\ & + \frac{1}{2\epsilon} \int \left[2(-\partial_\xi K_1) W_\rho W_u + (1 - \epsilon\sigma)(-\partial_\xi K_1) W_\rho^2 + (-\partial_\xi K_2) W_u^2 \right] d\xi \\ & = \epsilon^2 \frac{d}{dt} \int W_u P_g d\xi + \epsilon \int [\partial_\xi W_\rho + (1 - \epsilon\sigma)\partial_\xi W_u + \epsilon P_g] P_g d\xi \\ & + \epsilon \int \partial_\xi W_u P_{\mathcal{L}((v-s)g)} d\xi \end{aligned}$$

The difference to the isentropic case appears only in the second line. Using the Young inequality for

$$\left| \int 2(-\partial_\xi K_1) W_\rho W_u d\xi \right| \leq (1 - \epsilon\sigma) \int (-\partial_\xi K_1) W_\rho^2 d\xi + \frac{1}{1 - \epsilon\sigma} \int (-\partial_\xi K_1) W_u^2 d\xi,$$

the whole second line can be bounded from below by $\int \tilde{\kappa}(\phi) W_u^2 d\xi$, where $\tilde{\kappa}(\phi) = \frac{1}{\epsilon} [1 - |u_\phi - u_l|/(1 - \epsilon\sigma)] |\partial_\xi u_\phi| \geq 0$ for all ξ . The remaining estimates are analogous to the isentropic case.

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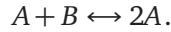
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Chapter 4

Travelling waves of a kinetic transport model for the KPP-Fisher equation¹²

4.1 Introduction

When the chemical reaction



takes place in a setting, where the density of species B can be assumed as constant and species A is subject to one-dimensional diffusion, then the dynamics of the density $\rho(t, x)$ of species A can be described (after nondimensionalization) by the KPP-Fisher equation

$$\partial_t \rho = D \partial_x^2 \rho + \rho(\bar{\rho} - \rho), \quad (4.1.1)$$

with the constant equilibrium states $\rho = 0$, $\rho = \bar{\rho} > 0$ and with the diffusion coefficient $D > 0$. Equation (4.1.1) was introduced by Fisher [5] as a model for a favourable gene in 1937. At the same time also Kolmogoroff, Petrovsky and Piscounoff [8] investigated (4.1.1) with a more general nonlinearity.

In a kinetic transport model, diffusion can be replaced by collisions with a (nonmoving) background medium, which randomize the direction of movement. A kinetic equation for the phase space density $f(t, x, v)$ of particles of species A can be written in the form

$$\varepsilon^2 \partial_t f + \varepsilon v \partial_x f = \mathcal{L} f + \varepsilon^2 Q(f), \quad (4.1.2)$$

with time $t > 0$, position $x \in \mathbb{R}$, and velocity $v \in V \subset \mathbb{R}$. The left hand side describes the free streaming of particles, and the terms on the right hand side are the models for collisions and chemical reactions. The dimensionless parameter ε is assumed positive and small: $0 < \varepsilon \ll 1$. Considering its occurrence on the

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right hand side of (4.1.2), this means that collisions are much more frequent than reactions. The powers of ε on the left hand side can be achieved by appropriate scalings for time and position.

Collisions are described as instantaneous velocity jumps with an equilibrium distribution $M(v)$, satisfying the moment conditions

$$\int_V M dv = 1, \quad \int_V vM dv = 0, \quad \int_V v^2 M dv = D > 0, \quad \int_V v^3 M dv = 0.$$

A typical example is the Maxwellian $M(v) = (2\pi D)^{-1/2} e^{-v^2/(2D)}$, $V = \mathbb{R}$. The simplest collision model then is the relaxation operator

$$\mathcal{L}f = \int_V [M(v)f(v') - M(v')f(v)] dv' = M\rho_f - f,$$

with the macroscopic density $\rho_f(t, x) = \int_V f(t, x, v) dv$. The process obviously conserves mass: $\int_V \mathcal{L}f dv = 0$. For the chemical reactions, it is assumed that they produce particles with the same equilibrium velocity distribution:

$$Q(f) = \int_V [\bar{\rho}M(v)f(v') - f(v)f(v')] dv' = \rho_f(M\bar{\rho} - f).$$

We obtain the kinetic reaction model

$$\varepsilon^2 \partial_t f + \varepsilon v \partial_x f = M\rho_f - f + \varepsilon^2 \rho_f(M\bar{\rho} - f). \quad (4.1.3)$$

A connection between (4.1.3) and the KPP-Fisher equation can be established by the macroscopic limit $\varepsilon \rightarrow 0$. Substitution of the Chapman-Enskog ansatz $f = M\rho_f + \varepsilon f^\perp$ into (4.1.3) and integration with respect to v leads to the macroscopic equation

$$\partial_t \rho_f + \partial_x \int_V v f^\perp dv = \rho_f(\bar{\rho} - \rho_f).$$

On the other hand, (4.1.3) implies

$$f^\perp = -vM\partial_x \rho_f + O(\varepsilon).$$

Hence, in the formal limit $\varepsilon \rightarrow 0$, ρ_f solves the KPP-Fisher equation (4.1.1).

This is an example of the derivation of reaction-diffusion equations from kinetic models. Formal asymptotics of this kind for much more general cases, in particular also systems, has been carried out already by other authors (see, e.g., [1], [11]). However, a rigorous justification is only known for linear models [1].

We recall some of the results on travelling wave solutions of the KPP-Fisher equation, which have been studied extensively. In terms of the travelling wave variable $\xi = x - st$ with the wave speed s , the travelling wave version of (4.1.1) is given by

$$D\rho'' + s\rho' + \rho(\bar{\rho} - \rho) = 0. \quad (4.1.4)$$

We assume $s \geq 0$, which is no restriction because of the reflection symmetry $s \rightarrow -s$, $\xi \rightarrow -\xi$. The following existence result goes back to Kolmogoroff, Petrovsky, and Piscounoff [8]:

Theorem 4.1.1. *For $s \geq s_0 := 2\sqrt{D\bar{\rho}}$ there exists a solution of (4.1.4), subject to the far field conditions $\rho(-\infty) = \bar{\rho}$ to $\rho(\infty) = 0$, which is positive, unique up to a shift, and strictly decreasing.*

The proof can be found in many books (e.g. Grindrod [6]) and relies on a phase plane analysis as follows: The corresponding system for (ρ, ρ') of (4.1.4) has the critical points $(0, 0)$ and $(\bar{\rho}, 0)$. Linearisation gives the eigenvalues:

- $(0, 0)$: $\lambda_{\pm} = -\frac{s}{2D} \left(1 \mp \sqrt{1 - \frac{s_0^2}{s^2}} \right) \Rightarrow \begin{cases} s < s_0 & \text{stable spiral} \\ s \geq s_0 & \text{stable node} \end{cases}$
- $(\bar{\rho}, 0)$: $\lambda_{\pm} = \frac{s}{2D} \left(-1 \pm \sqrt{1 + \frac{s_0^2}{s^2}} \right) \Rightarrow \text{saddle point}$

We see that a positive solution can only exist for $s \geq s_0$, because otherwise the solution spirals around the origin. Since a trajectory starting from $(\bar{\rho}, 0)$ cannot leave the triangle

$$0 \leq \rho \leq \bar{\rho}, \quad 0 \geq \rho' \geq -\frac{s}{2D}\rho.$$

it must connect $(\bar{\rho}, 0)$ to $(0, 0)$. Moreover we can deduce the following asymptotic behaviour of ρ :

$$\begin{aligned} \bar{\rho} - \rho &\sim e^{\alpha_- \xi} \quad \text{as } \xi \rightarrow -\infty, \quad \text{with } \alpha_- = \frac{s}{2D} \left(\sqrt{1 + \frac{s_0^2}{s^2}} - 1 \right) > 0, \\ s > s_0 : \quad \rho &\sim e^{-\alpha_+ \xi} \quad \text{as } \xi \rightarrow \infty, \quad \text{with } \alpha_+ = \frac{s}{2D} \left(1 - \sqrt{1 - \frac{s_0^2}{s^2}} \right) > 0, \\ s = s_0 : \quad \rho &\sim \xi e^{-s\xi/(2D)} \quad \text{as } \xi \rightarrow \infty. \end{aligned}$$

It is our aim to study the existence and stability of travelling wave solutions of (4.1.3). As a preliminary result, we prove global existence of solutions of the initial value problem for (4.1.3) for initial data bounded by a global equilibrium in Section 4.2.

Our approach for the analysis of travelling waves is based on the fact that for ε small, (4.1.3) can be approximated by the KPP-Fisher equation (4.1.1). In section 3 we present a constructive existence proof for travelling wave solutions of the kinetic equation (4.1.3), which shows the asymptotic closeness of the kinetic profiles to the solutions of (4.1.4). We follow the approach of [4] for travelling waves of kinetic BGK models for scalar conservation laws by first constructing a formal asymptotic approximation, and then showing solvability of the problem for the correction term. For the latter we adapt the micro-macro decomposition introduced by Caflisch and Nicolaenko [2] for the Boltzmann equation. The major difficulty in this work is caused by the fact that, in contrary to [4] and [2], the macroscopic problem is not a conservation law. The existence result can be extended to also give strict monotonicity, and therefore positivity for the macroscopic density of the kinetic profile.

In section 4 we show the asymptotic stability of kinetic profiles with non-critical speed, i.e. $s > s_0$, under perturbations in suitable spaces. Travelling waves for the KPP-Fisher equation are stable under perturbations, which decay faster than (or at least as fast as) the waves. A corresponding result is proven here. The required decay properties are built into an appropriately weighted L^2 space. This has the consequence that we can control the macroscopic terms in a similar way as for the KPP-Fisher equation. Concerning the control of the microscopic terms, we have only been successful under the additional assumption of a bounded set of velocities V .

We shall now turn to the travelling wave version of (4.1.3). In the remainder of this section we repeat the formal asymptotics in terms of the travelling wave variable and also show how the stability of wave profiles of the KPP-Fisher equation can be shown by using energy estimates.

4.1.1 Formal approximation of travelling wave solutions

We shall be looking for travelling wave solutions $f = f(\xi, v)$, $\xi = x - st$, of (4.1.3) satisfying

$$\varepsilon(v - \varepsilon s)\partial_\xi f = M\rho_f - f + \varepsilon^2\rho_f(M\bar{\rho} - f), \quad (4.1.5)$$

subject to the far-field conditions

$$f(-\infty, v) = \bar{\rho}M(v) \quad \text{and} \quad f(\infty, v) = 0 \quad \text{for all } v \in V. \quad (4.1.6)$$

Similar to above we make the ansatz $f(\xi, v) = \rho_f(\xi)M(v) + \varepsilon f^\perp(\xi, v)$. Integration of (4.1.5) then leads to

$$-s\partial_\xi \rho_f + \partial_\xi \int_V v f^\perp dv = \rho_f(\bar{\rho} - \rho_f). \quad (4.1.7)$$

From (4.1.5) we also determine the leading order terms of $f^\perp$:

$$f^\perp = -vM\partial_\xi \rho_f + \varepsilon[sM\partial_\xi \rho_f - v\partial_\xi f^\perp + M\rho_f(\bar{\rho} - \rho_f)] + O(\varepsilon^2) \quad (4.1.8)$$

$$= -vM\partial_\xi \rho_f + \varepsilon(v^2 - D)M\partial_\xi^2 \rho_f + O(\varepsilon^2), \quad (4.1.9)$$

where in the last step we have used (4.1.8) and (4.1.7). Substitution of (4.1.9) into (4.1.7) shows that ρ_f formally solves (4.1.4) even up to $O(\varepsilon^2)$ -terms.

4.1.2 Stability of travelling waves of the Fisher equation

Let ρ_ϕ be the travelling wave solution of (4.1.4) with speed $s \geq s_0$. We shall change to the moving coordinates t and $\xi = x - st$ in (4.1.1):

$$\partial_t \rho_f - D\partial_\xi^2 \rho_f - s\partial_\xi \rho_f - \rho_f(\bar{\rho} - \rho_f) = 0 \quad (4.1.10)$$

and look for solutions, which are small perturbations of ρ_ϕ , i.e. $\rho_f = \rho_\phi + \rho$. The equation for the perturbation ρ reads

$$\partial_t \rho - D\partial_\xi^2 \rho - s\partial_\xi \rho + \rho(2\rho_\phi - \bar{\rho} + \rho) = 0. \quad (4.1.11)$$

It is well known that travelling wave solutions of the Fisher equation are in general unstable to perturbations in the far-field, see e.g. Canosa [3]. Here one first analyses the spectrum of the operator associated to the linearised equation, which also contains the eigenvalue 0 with the eigenfunction $\partial_\xi \rho_\phi$. We note that this eigenfunction is the one generated by perturbations equivalent to small translations in the travelling wave. However, this problem is overcome in the literature by introducing norms with appropriate weights, which push $\partial_\xi \rho_\phi$ out of the space. Sattinger treats a more general case by using L^∞ -based norms [9]. In view of the microscopic terms later we shall rather use an L^2 -based framework. We define the weight function

$$W(\xi) = e^{\frac{s}{2D}\xi},$$

and introduce the Hilbert spaces $L_\xi^2 = L^2(\mathbb{R})$, $H_\xi^1 = H^1(\mathbb{R})$, L_W^2 , and H_W^1 of functions of ξ with the respective norms

$$\begin{aligned} \|\rho\|_\xi^2 &= \int_{\mathbb{R}} \rho^2 d\xi, \quad \|\rho\|_{H_\xi^1}^2 = \|\rho\|_\xi^2 + \|\partial_\xi \rho\|_\xi^2, \\ \|\rho\|_W &= \|\rho W\|_\xi, \quad \|\rho\|_{H_W^1}^2 = \|\rho\|_W^2 + \|\partial_\xi \rho\|_W^2. \end{aligned}$$

Local existence of solutions of the perturbation equation (4.1.11) in $H_\xi^1 \cap H_W^1$ (which means the weight acts only as $\xi \rightarrow \infty$) follows by a standard contraction argument. Hence if we can show the decay of the solution in $H_\xi^1 \cap H_W^1$ as time evolves, global existence follows by the continuation principle.

We assume $s > s_0$ and $\rho_f = \rho_\phi + \rho \geq 0$, which is a consequence of the maximum principle, if it holds initially. The shift in the travelling wave is adjusted such that $\rho_\phi(0) = 3\bar{\rho}/4$, implying

$$\rho_\phi(\xi) \geq \frac{3\bar{\rho}}{4} \quad \text{for } \xi \leq 0. \quad (4.1.12)$$

Multiplication of (4.1.11) with W gives

$$\partial_t(\rho W) - D\partial_\xi^2(\rho W) + (\kappa + 2\rho_\phi + \rho)\rho W = 0, \quad (4.1.13)$$

with $\kappa = s^2/(4D) - \bar{\rho} > 0$ by $s > s_0$. Testing (4.1.11) with ρ and (4.1.13) with $\alpha\rho W$ (with $\alpha > 0$) and adding the resulting equations leads to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\rho\|_\xi^2 + \alpha \|\rho\|_W^2 \right) + D \left(\|\partial_\xi \rho\|_\xi^2 + \alpha \|\partial_\xi(\rho W)\|_\xi^2 \right) \\ & + \frac{1}{2} \int_{\mathbb{R}} \rho^2 \rho_\phi (1 + \alpha W^2) d\xi + \alpha \int_{\mathbb{R}} \rho^2 W^2 (\kappa + \rho_\phi + \rho) d\xi \\ & + \frac{\alpha}{2} \int_{\mathbb{R}} \rho^2 W^2 \rho_\phi d\xi + \int_{\mathbb{R}} \rho^2 (3\rho_\phi/2 - \bar{\rho} + \rho) d\xi = 0. \end{aligned} \quad (4.1.14)$$

All terms after the time derivative are obviously nonnegative, except the last one. Its contribution for $\xi < 0$ can be estimated using (4.1.12):

$$\int_{-\infty}^0 \rho^2 (3\rho_\phi/2 - \bar{\rho} + \rho) d\xi \geq \left(\frac{\bar{\rho}}{8} - \|\rho\|_\infty \right) \int_{-\infty}^0 \rho^2 d\xi.$$

On the other hand, the contributions of the last two terms for $\xi > 0$ can be estimated by

$$\frac{\alpha}{2} \int_0^\infty \rho^2 W^2 \rho_\phi d\xi + \int_0^\infty \rho^2 (3\rho_\phi/2 - \bar{\rho} + \rho) d\xi \geq \int_0^\infty \rho^2 (\alpha W^2 \rho_\phi/2 - \bar{\rho}) d\xi.$$

Since W^2 increases faster than ρ_ϕ decreases, α can be chosen such that

$$\frac{\alpha W^2 \rho_\phi}{2} - \bar{\rho} \geq \frac{\bar{\rho}}{16}$$

holds. If we succeed below in proving $\|\rho\|_\infty \leq \bar{\rho}/16$, then the inequality

$$\frac{1}{2} \frac{d}{dt} \left(\|\rho\|_\xi^2 + \alpha \|\rho\|_W^2 \right) \leq -\frac{\bar{\rho}}{16} \|\rho\|_\xi^2 - \kappa \alpha \|\rho\|_W^2 \leq -\min \left\{ \frac{\bar{\rho}}{16}, \kappa \right\} \left(\|\rho\|_\xi^2 + \alpha \|\rho\|_W^2 \right)$$

is satisfied, implying exponential decay of the perturbation ρ . For proving the required L^∞ -bound, we shall derive a H^1 -bound and use Sobolev imbedding. In terms of $r = \partial_\xi \rho$, the derivative with respect to ξ of (4.1.11) and its product with W read

$$\begin{aligned} \partial_t r - D \partial_\xi^2 r - s \partial_\xi r + r(2\rho_\phi - \bar{\rho} + 2\rho) + 2\rho \partial_\xi \rho_\phi &= 0, \\ \partial_t(rW) - D \partial_\xi^2(rW) + rW(\kappa + 2\rho_\phi + 2\rho) + 2W\rho \partial_\xi \rho_\phi &= 0. \end{aligned}$$

We shall proceed with this system analogously to (4.1.11), (4.1.13), note however the small differences: In the parantheses, 2ρ replaces ρ , and the added last terms in both equations. When testing by r and, respectively, by αrW , the identity $2r\rho = \partial_\xi(\rho^2)$ is used in these terms:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|r\|_\xi^2 + \alpha \|r\|_W^2 \right) + D \left(\|\partial_\xi r\|_\xi^2 + \alpha \|\partial_\xi(rW)\|_\xi^2 \right) + \int_{\mathbb{R}} r^2 (2\rho_\phi - \bar{\rho} + 2\rho) d\xi \\ + \alpha \int_{\mathbb{R}} r^2 W^2 (\kappa + 2\rho_\phi + 2\rho) d\xi = \int_{\mathbb{R}} \rho^2 \partial_\xi ((1 + \alpha W^2) \partial_\xi \rho_\phi) d\xi. \end{aligned} \quad (4.1.15)$$

For the right hand side we observe that

$$|\partial_\xi((1 + \alpha W^2) \partial_\xi \rho_\phi)| \leq c(1 + \alpha W^2) \rho_\phi,$$

because of the exponential behaviour of W^2 and ρ_ϕ as $\xi \rightarrow \pm\infty$. Therefore the right hand side of (4.1.15) can be dominated by a multiple of the last term in the first line of (4.1.14). Similarly, the other problematic term in (4.1.15), $-\bar{\rho} \int_{\mathbb{R}} r^2 d\xi$, can be controlled by a multiple of the term $D \|\partial_\xi \rho\|_\xi^2$ in (4.1.14). Therefore the functional

$$J[\rho] := \|\rho\|_\xi^2 + \alpha \|\rho\|_W^2 + \beta \left(\|\partial_\xi \rho\|_\xi^2 + \alpha \|\partial_\xi \rho\|_W^2 \right),$$

with a small enough positive constant β is nonincreasing in time, as long as

$$\|\rho\|_\infty \leq \frac{\bar{\rho}}{16} \quad (4.1.16)$$

holds. Since, by Sobolev imbedding, $\|\rho\|_\infty^2 \leq cJ[\rho]$, (4.1.16) can be guaranteed for all time under the smallness assumption

$$J[\rho(t=0)] \leq \frac{1}{c} \left(\frac{\bar{\rho}}{16} \right)^2,$$

on the initial data. As shown above, this has the consequence

$$\|\rho\|_\xi^2 + \alpha \|\rho\|_W^2 \leq \left(\|\rho(t=0)\|_\xi^2 + \alpha \|\rho(t=0)\|_W^2 \right) \exp \left(-t\bar{\rho} \min \left\{ \frac{1}{8}, 2 \left(\frac{s^2}{s_0^2} - 1 \right) \right\} \right)$$

Obviously in the case $\kappa = 0$ we cannot deduce exponential convergence using this procedure. The difficulty of proving stability in the case of the critical wave speed was discussed in several works. Sattinger [9], [10] left this case open. Kirchgässner [7] later treated the long time asymptotics also for $s = s_0$. Since we will not adapt this procedure on the kinetic level, we do not go into more detail.

4.1.3 Linear collision operator and norms

We define the weighted inner product in v by

$$\langle f, g \rangle_v = \int_V \frac{f g}{M} dv$$

and denote the induced space and norm with $(L_v^2, \|\cdot\|_v)$. With respect to $\langle \cdot, \cdot \rangle_v$ the linear collision operator $\mathcal{L}f = M\rho_f - f$ is symmetric and negative semidefinite as a consequence of

$$\langle \mathcal{L}f, g \rangle_v = -\langle \mathcal{L}f, \mathcal{L}g \rangle_v.$$

The standard norms and spaces of functions of ξ are denoted by $(L_\xi^2, \|\cdot\|_\xi)$, $(H_\xi^k, \|\cdot\|_{H_\xi^k})$, and $(C_\xi^b, \|\cdot\|_\infty)$. The Hilbert space $(L_{\xi,v}^2, \|\cdot\|_{\xi,v})$ is then naturally defined by the scalar product

$$\langle f, g \rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g \rangle_v d\xi.$$

For $k \in \mathbb{N}_0$, the space $H_\xi^k(L_v^2)$ of functions, whose derivatives up to order k with respect to ξ are in $L_{\xi,v}^2$ is equipped with the norm

$$\|f\|_{H_\xi^k(L_v^2)} = \left(\|f\|_{\xi,v}^2 + \cdots + \|\partial_\xi^k f\|_{\xi,v}^2 \right)^{1/2}.$$

In a similar way $C_\xi^b(L_v^2)$ is defined by

$$\|f\|_{\infty,v} = \sup_{\xi \in \mathbb{R}} \|f\|_v.$$

For later reference we note that the Cauchy-Schwarz inequality implies

$$\|\rho_f\|_\xi \leq \|f\|_{\xi,v}. \quad (4.1.17)$$

4.2 Global existence for the initial value problem

Theorem 4.2.1. *Let $0 \leq f_0(x, v) \leq \bar{\rho}M(v)$ hold. Then the kinetic equation (4.1.3) subject to the initial condition $f(t = 0) = f_0$ has a unique solution $f \in C([0, \infty); L^\infty(\mathbb{R} \times V))$, satisfying*

$$0 \leq f(t, x, v) \leq \max\{\bar{\rho}, \hat{\rho}\}M(v), \quad \forall (t, x, v) \in [0, \infty) \times \mathbb{R} \times V. \quad (4.2.1)$$

Proof. For proving a local existence result, we introduce $g(t, y, v) = f(t, y + vt/\varepsilon, v)$, satisfying

$$\varepsilon^2 \partial_t g = \rho_g M(1 + \varepsilon^2 \bar{\rho}) - g(1 + \varepsilon^2 \rho_g), \quad g(t = 0) = f_0,$$

or, equivalently,

$$\begin{aligned} g(t, y, v) = & f_0(y, v) + M(v) \left(\frac{1}{\varepsilon^2} + \bar{\rho} \right) \int_0^t \rho_g(\tau, y) d\tau \\ & - \int_0^t \left(\frac{1}{\varepsilon^2} + \rho_g(\tau, y) \right) g(\tau, y, v) d\tau. \end{aligned} \quad (4.2.2)$$

For $T > 0$, we introduce the Banach space

$$\begin{aligned} \mathcal{C}_T = & \left\{ g \in C([0, T]; L^\infty(\mathbb{R} \times V)) : \|g\|_{\mathcal{C}_T} < \infty \right\}, \\ \|g\|_{\mathcal{C}_T} = & \sup_{(t, y, v) \in [0, T] \times \mathbb{R} \times V} \frac{|g(t, y, v)|}{M(v)}. \end{aligned}$$

Using the property $|\rho_g| \leq \|g\|_{\mathcal{C}_T}$, it is straightforward to uniquely solve (4.2.2) in $\mathcal{C}_T$ for small enough T by Picard iteration. Global existence will follow from the following a priori estimates. The nonnegativity of f is an obvious consequence of the maximum principle for kinetic equations, after writing (4.1.3) in the form

$$\varepsilon^2 \partial_t f + \varepsilon v \partial_x f + f(1 + \varepsilon^2 \rho_f) = \rho_f M(1 + \varepsilon^2 \bar{\rho}),$$

and solving by a fixed point iteration, where ρ_f is considered as given. The function $h(t, x, v) = \max\{\bar{\rho}, \hat{\rho}\}M(v) - f(t, x, v)$ satisfies

$$\varepsilon^2 \partial_t h + \varepsilon v \partial_x h + h(1 + \varepsilon^2 \rho_f) = \rho_h M + \varepsilon^2 \rho_f M(\hat{\rho} - \bar{\rho})_+, \quad h(t = 0) \geq 0,$$

proving $h \geq 0$ and, thus, (4.2.1). $\square$

4.3 Existence of travelling wave solutions

4.3.1 Asymptotic approximation

Inspired by the formal asymptotics in Section 1.1, we define the asymptotic approximation

$$f_{as} = M\rho_{as} + \varepsilon f^\perp[\rho_{as}],$$

where ρ_{as} is chosen as the travelling wave solution of the Fisher equation:

$$D\rho_{as}'' + s\rho_{as}' + \rho_{as}(\bar{\rho} - \rho_{as}) = 0, \quad \rho_{as}(0) = \frac{\bar{\rho}}{2}, \quad (4.3.1)$$

and, recalling (4.1.9), we set

$$f^\perp[\rho] = -vM\rho' + \varepsilon(v^2 - D)M\rho''.$$

Integration shows that $\int_V f^\perp dv = 0$, implying $\rho_{as} = \rho_{f_{as}}$. The formal approximation f_{as} satisfies the far-field conditions (4.1.6) and the travelling wave equation (4.1.5) up to the residual

$$\begin{aligned} \varepsilon^3 h &= \varepsilon(v - \varepsilon s)\partial_\xi f_{as} - M\rho_{as} + f_{as} - \varepsilon^2 \rho_{as}(M\bar{\rho} - f_{as}) \\ &= \varepsilon^3(s\nu M\rho_{as}'' + (v - \varepsilon s)(v^2 - D)M\rho_{as}''' + \rho_{as}f^\perp). \end{aligned}$$

We see that

$$\int_V h dv = 0, \quad \text{and} \quad \|h\|_{H_\xi^k(L_v^2)} \leq C_k \quad \text{for any } k \in \mathbb{N}.$$

4.3.2 Correction term

The travelling wave equation for the correction term $\varepsilon^2 g = f - f_{as}$ reads

$$\varepsilon(v - \varepsilon s)\partial_\xi g = \mathcal{L}g + \varepsilon^2 Bg + \varepsilon^4 Rg - \varepsilon h, \quad (4.3.2)$$

where

$$Bg = \rho_g(M\bar{\rho} - f_{as}) - \rho_{as}g, \quad Rg = -\rho_g g.$$

The right hand side of (4.3.2) contains the linear collision operator, a linear term of order $O(\varepsilon^2)$, a nonlinear term of order $O(\varepsilon^4)$ and the residual. We look for a solution g of (4.3.2), subject to the far-field conditions

$$g(\pm\infty, v) = 0 \quad \text{for all } v \in V. \quad (4.3.3)$$

Integration shows that necessarily

$$\partial_\xi \int_V (v - \varepsilon s)g dv = \varepsilon \int_V (Bg + \varepsilon^2 Rg)dv = \varepsilon \rho_g(\bar{\rho} - 2\rho_{as}) - \varepsilon^3 \rho_g^2. \quad (4.3.4)$$

The following micro-macro decomposition is motivated by the work of Caflisch and Nicolaenko on the Boltzmann equation [2]. We split g into a leading term with separated variables plus a small term of order ε :

$$g(\xi, v) = \Phi(v)z(\xi) + \varepsilon w(\xi, v),$$

where Φ is chosen such that $\mathcal{L}\Phi = -\varepsilon\tau(v - \varepsilon s)\Phi + O(\varepsilon^2)$ for a constant τ . Hence we make the ansatz

$$\Phi(v) = (1 + \varepsilon\tau(v - \varepsilon s))M(v),$$

and determine the constant τ such that

$$\int_V (v - \varepsilon s) \Phi dv = 0, \quad \text{giving} \quad \tau = \frac{s}{D + \varepsilon^2 s^2}. \quad (4.3.5)$$

Integration shows $\rho_\Phi = 1 - \varepsilon^2 \tau s$ and

$$D_1 := \int_V (v - \varepsilon s)^2 \Phi dv = D + O(\varepsilon^2) > 0.$$

Thus we can make the decomposition unique by requiring

$$\int_V (v - \varepsilon s)^2 w dv = 0. \quad (4.3.6)$$

We also point out that due to (4.3.5), relation (4.3.4) is equivalent to

$$\partial_\xi \int_V (v - \varepsilon s) w dv = \int_V (Bg + \varepsilon^2 Rg) dv = \rho_g (\bar{\rho} - 2\rho_{as}) - \varepsilon^2 \rho_g^2. \quad (4.3.7)$$

We repeat the travelling wave equation for g , but now written in terms of the decomposition:

$$(v - \varepsilon s) \Phi \partial_\xi z + \varepsilon (v - \varepsilon s) \partial_\xi w = \frac{1}{\varepsilon} \mathcal{L} \Phi z + \mathcal{L} w + \varepsilon Bg + \varepsilon^3 Rg - h, \quad (4.3.8)$$

where we have divided by ε . Like g , its micro- and macro-components z and w have to satisfy the homogenous far-field conditions

$$w(\pm\infty, v) \equiv 0, \quad z(\pm\infty) = 0. \quad (4.3.9)$$

As a next step we define another set of macroscopic and microscopic projections, which produce two equations with separated derivatives of z and w . Applying

$$Pf := \int_V (v - \varepsilon s) f dv \quad (4.3.10)$$

to (4.3.8) we obtain, due to (4.3.6),

$$D_1 z' + s \rho_\Phi z = - \int_V vw dv + \varepsilon PBg + \varepsilon^3 PRg - Ph. \quad (4.3.11)$$

We differentiate (4.3.11) and use the moment relation (4.3.7). After multiplying the resulting equation by $D/D_1 = 1 + O(\varepsilon^2)$ and collecting the linear and nonlinear terms we arrive at

$$Dz'' + sz' + z(\bar{\rho} - 2\rho_{as}) = \varepsilon B^z(z, z', w, \partial_\xi w) + \varepsilon^2 R^z(g, \partial_\xi g) - \tilde{h}, \quad (4.3.12)$$

where

$$\begin{aligned} B^z(z, z', w, \partial_\xi w) &= \frac{D}{D_1} \left[-\rho_w (\bar{\rho} - 2\rho_{as}) - s \rho'_w + \partial_\xi PBg \right] \\ &\quad + \frac{1}{\varepsilon} \left(1 - \frac{\rho_\Phi D}{D_1} \right) (sz' + z(\bar{\rho} - 2\rho_{as})), \\ R^z(g, \partial_\xi g) &= \frac{D}{D_1} \left[\rho_g^2 + \varepsilon \partial_\xi PRg \right], \quad \tilde{h} = -\frac{D}{D_1} \partial_\xi Ph. \end{aligned}$$

At leading order, (4.3.12) is the inhomogenous linearized Fisher equation, independent of w .

The microscopic projection

$$\Pi f := f - \frac{(v - \varepsilon s)\Phi}{D_1} \int_V (v - \varepsilon s)f \, dv = f - \frac{(v - \varepsilon s)\Phi}{D_1} Pf \quad (4.3.13)$$

has the properties $\Pi(v - \varepsilon s)\Phi = 0$ and $\Pi(v - \varepsilon s)w = (v - \varepsilon s)w$, due to (4.3.6). Applying Π to (4.3.8) we get the following equation for w :

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{L}w = \frac{(v - \varepsilon s)\Phi}{D_1} \int_V vw \, dv + \varepsilon \Lambda z + \varepsilon \Pi Bg + \varepsilon^3 \Pi Rg - \Pi h, \quad (4.3.14)$$

where $\Lambda = \frac{1}{\varepsilon^2} \Pi \mathcal{L} \Phi$ is (formally) $O(1)$.

Based on ideas of Caflisch and Nicolaenko in [2], we make another manipulation on this equation. Since the symmetric operator $\mathcal{L}$ is only negative semidefinite, we introduce a new symmetric operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions w satisfying (4.3.6):

$$\mathcal{M}w := \mathcal{L}w - (v - \varepsilon s)^2 M \int_V (v - \varepsilon s)^2 w \, dv.$$

Lemma 4.3.1. *The operator $\mathcal{M}$ is symmetric and negative definite with respect to $\langle \cdot, \cdot \rangle_v$. There exists a constant $\kappa > 0$, such that*

$$-\langle \mathcal{M}w, w \rangle_v \geq \kappa \|w\|_v^2 \quad \text{for all } w \in L_v^2. \quad (4.3.15)$$

Proof. We follow the proof in [4]. The symmetry of $\mathcal{M}$ is an obvious consequence of its definition and of the symmetry of $\mathcal{L}$. To prove the coercivity estimate we decompose $w = M\rho_w + w^\perp$ and observe that $\mathcal{L}w = -w^\perp$:

$$\begin{aligned} -\langle \mathcal{M}w, w \rangle_v &= \|w^\perp\|_v^2 + \left((D + \varepsilon^2 s^2)\rho_w + \int_V (v - \varepsilon s)^2 w^\perp \, dv \right)^2 \\ &= \|w^\perp\|_v^2 + \gamma (D + \varepsilon^2 s^2)^2 \rho_w^2 - \frac{\gamma}{1 - \gamma} \left(\int_V (v - \varepsilon s)^2 w^\perp \, dv \right)^2 \\ &\quad + (1 - \gamma) \left((D + \varepsilon^2 s^2)\rho_w + \frac{1}{1 - \gamma} \int_V (v - \varepsilon s)^2 w^\perp \, dv \right)^2 \\ &\geq \|w^\perp\|_v^2 + \gamma (D + \varepsilon^2 s^2)^2 \rho_w^2 - \frac{\gamma}{1 - \gamma} C \|w^\perp\|_v^2, \end{aligned}$$

for any $\gamma \in (0, 1)$ and a positive constant C . Hence choosing $\gamma \in (0, 1)$ small enough gives (4.3.15). $\square$

We now replace $\mathcal{L}$ in (4.3.14) by the operator $\mathcal{M}$:

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = \frac{(v - \varepsilon s)\Phi}{D_1} \int_V vw \, dv + \varepsilon \Lambda z + \varepsilon \Pi Bg + \varepsilon^3 \Pi Rg - \Pi h. \quad (4.3.16)$$

The equivalence to the original problem is not obvious:

Lemma 4.3.2. *The function $g = \Phi z + \varepsilon w$ is a solution of (4.3.2), (4.3.3) iff z and w solve (4.3.12), (4.3.16) subject to (4.3.9).*

Proof. We follow the proofs in [2] and [4]. The problem (4.3.12), (4.3.16) (4.3.9) has been derived from (4.3.2), (4.3.3) using the properties (4.3.7), (4.3.6) of solutions of the latter. In particular (4.3.6) is not a necessary condition for existence. Hence we have to check that (4.3.6) also holds for solutions of (4.3.12), (4.3.16), (4.3.9), without requiring it as a side condition. Using

$$\int_V \Pi f \, dv = \int_V f \, dv, \quad \int_V (v - \varepsilon s) \Pi f \, dv = 0,$$

integration of (4.3.16) implies

$$\begin{aligned} \varepsilon \partial_\xi \int_V (v - \varepsilon s) w \, dv &= -(D + \varepsilon^2 s^2) \int_V (v - \varepsilon s)^2 w \, dv + \varepsilon \int_V (Bg + \varepsilon^2 Rg) \, dv, \\ \varepsilon \partial_\xi \int_V (v - \varepsilon s)^2 w \, dv &= 2\varepsilon s D \int_V (v - \varepsilon s)^2 w \, dv. \end{aligned}$$

The second equation is a linear constant coefficient ODE for $\int_V (v - \varepsilon s)^2 w \, dv$. Since $w(\pm\infty, v) = 0$, the only possible solution is

$$\int_V (v - \varepsilon s)^2 w \, dv = 0.$$

Knowing this and returning to the first differential equation we also recover (4.3.7). $\square$

We now eliminate the first term on the right hand side in (4.3.16) by substituting (4.3.11):

$$\varepsilon(v - \varepsilon s) \partial_\xi w - \mathcal{M}w = A(z, z') + \varepsilon Bg + \varepsilon^3 Rg - h, \quad (4.3.17)$$

where

$$A(z, z') = -\frac{(v - \varepsilon s)\Phi}{D_1}(D_1 z' + s\rho_\Phi z) + \varepsilon \Lambda z.$$

Thus we have arrived at our final differential problem (4.3.12), (4.3.17), subject to (4.3.9). In the following subsections we will show the solvability based on a fix-point argument.

4.3.3 Linear Problem

We first analyse the leading system of (4.3.12), (4.3.17), where we consider the higher order terms as given inhomogenities. In particular, we prove the solvability of

$$Dz'' + sz' + z(\bar{\rho} - 2\rho_{as}) = h_z, \quad \text{with } h_z \in H_\xi^1, \quad (4.3.18)$$

$$\varepsilon(v - \varepsilon s) \partial_\xi w - \mathcal{M}w = A(z, z') + h_w, \quad \text{with } h_w \in H_\xi^2(L_v^2). \quad (4.3.19)$$

We shall look for solutions in the same spaces as the inhomogenities. This will in the following replace the homogenous far-field conditions, and already provides

uniqueness for the solution of (4.3.19). Moreover this requirement only permits a one-parameter set of solutions of (4.3.18), which reflects the arbitrary shift in the wave. Here uniqueness will be guaranteed by posing the initial condition

$$z(0) = z_0, \quad (4.3.20)$$

for a $z_0 \in \mathbb{R}$. Since (4.3.18) is independent of w , this equation is solved first. Then $A(z, z')$ is considered as a given inhomogeneity in (4.3.19).

Lemma 4.3.3. *Let $h_z \in H_\xi^k$, $k \geq 0$. Then the problem (4.3.18), (4.3.20) possesses a unique solution $z \in H_\xi^{k+2}$, satisfying (with $C > 0$ independent from z_0 and h_z)*

$$\|z\|_{H_\xi^{k+2}} \leq C(|z_0| + \|h_z\|_{H_\xi^k}).$$

Proof. Since (4.3.18) is the linearization of (4.3.1) at its solution ρ_{as} , the derivative ρ'_{as} is a solution of the homogeneous equation. The standard order reduction procedure then allows to rewrite (4.3.18) as the first order system

$$z' = \frac{\rho''_{as}}{\rho'_{as}} z + z_1, \quad z'_1 = -\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) z_1 + \frac{h_z}{D}. \quad (4.3.21)$$

Starting with the second equation, (4.3.1), $\rho'_{as} < 0$, and $0 < \rho_{as} < \bar{\rho}$ imply

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) = \frac{\rho_{as}(\bar{\rho} - \rho_{as})}{D\rho'_{as}} < 0.$$

Since, by the asymptotic behaviour of ρ_{as} , this coefficient converges to negative values as $\xi \rightarrow \pm\infty$, the stronger statement

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) \leq -\gamma < 0,$$

holds. By standard ODE methods, a unique decaying solution z_1 of the second equation in (4.3.21) exists for decaying h_z (using the 'boundary condition' $z_1(-\infty) = 0$). It can be estimated by testing the equation with z_1 , giving

$$\|z_1\|_\xi \leq \frac{1}{\gamma D} \|h_z\|_\xi.$$

Turning to the first equation in (4.3.21), we observe that

$$\lim_{\xi \rightarrow \infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} < 0, \quad \lim_{\xi \rightarrow -\infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} > 0.$$

This is the situation covered in Lemma 3.5 of [4], implying the existence of a unique solution satisfying

$$\|z\|_\xi \leq C'(|z_0| + \|z_1\|_\xi) \leq C' \left(|z_0| + \frac{1}{\gamma D} \|h_z\|_\xi \right).$$

Testing (4.3.18) with z and with z'' we obtain estimates for the first and second derivatives, implying $\|z\|_{H_\xi^2} \leq C(|z_0| + \|h_z\|_{L_\nu^2})$. Finally, the same procedure can be applied to differentiated versions of (4.3.18), completing the proof. $\square$

Now $A(z, z')$ can be considered as a given inhomogeneity in (4.3.19), and the following result from [4] can be used:

Proposition 4.3.1. *Let $\tilde{h}_w \in H_\xi^k(L_\nu^2)$, $k \geq 0$. Then there exists a unique solution $w \in H_\xi^k(L_\nu^2)$ of*

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = \tilde{h}_w,$$

satisfying

$$\|w\|_{H_\xi^k(L_\nu^2)} \leq \frac{1}{\kappa} \|\tilde{h}_w\|_{H_\xi^k(L_\nu^2)},$$

with κ as in Lemma 4.3.1.

Sketch of the proof. Uniqueness and the stability estimate are obtained by testing the equation with w and the k -th derivative of the equation with $\partial_\xi^k w$. Existence can be proven in several ways, one of which is the approximation by a discrete velocity system with a finite number of discrete velocities. This reduces the problem to an ODE system. Care has to be taken in order not to destroy the definiteness of $\mathcal{M}$ by the approximation. $\square$

The final result on the linear problem can now be easily proven.

Lemma 4.3.4. *Let $h_z \in H_\xi^k$ and $h_w \in H_\xi^l(L_\nu^2)$, then there exists a unique solution $(z, w) \in H_\xi^{k+2} \times H_\xi^m(L_\nu^2)$, $m = \min\{k+1, l\}$, of (4.3.18), (4.3.19), (4.3.20), satisfying*

$$\|z\|_{H_\xi^{k+2}(L_\nu^2)} \leq C(|z_0| + \|h_z\|_{H_\xi^k}), \quad \|w\|_{H_\xi^m(L_\nu^2)} \leq C(|z_0| + \|h_z\|_{H_\xi^k} + \|h_w\|_{H_\xi^l(L_\nu^2)}).$$

Proof. The only thing left to note is the estimate

$$\|A(z, z')\|_{H_\xi^{k+1}(L_\nu^2)} \leq \|z\|_{H_\xi^{k+2}},$$

whose proof is straightforward by the definition of A . $\square$

4.3.4 Nonlinear Problem

In this section we prove existence and uniqueness of solutions of the nonlinear problem (4.3.17), (4.3.12) subject $z(0) = z_0$ in the spaces H_ξ^3 and $H_\xi^2(L_\nu^2)$, respectively. After the preparations in the previous subsections, the proof is a straightforward contraction argument. We need, however, estimates for the right hand sides of (4.3.17) and (4.3.12). In the following, C denotes (possibly different) ε -independent constants.

Lemma 4.3.5. (i) *The linear terms B and B^z satisfy the estimate*

$$\|B(\Phi z + \varepsilon w)\|_{H_\xi^2(L_\nu^2)} + \|B^z(z, z', w, \partial_\xi w)\|_{H_\xi^1} \leq C(\|z\|_{H_\xi^2} + \|w\|_{H_\xi^2(L_\nu^2)}).$$

(ii) The nonlinearities R and R^z are quadratic: Let $g_1, g_2 \in H_\xi^2(L_v^2)$, then

$$\begin{aligned} & \|R(g_1) - R(g_2)\|_{H_\xi^2(L_v^2)} + \|R^z(g_1, \partial_\xi g_1) - R^z(g_2, \partial_\xi g_2)\|_{H_\xi^1} \\ & \leq C \left(\|g_1\|_{H_\xi^2(L_v^2)} + \|g_2\|_{H_\xi^2(L_v^2)} \right) \|g_1 - g_2\|_{H_\xi^2(L_v^2)}. \end{aligned}$$

Proof. The proof is straightforward. All that is needed for (ii) is the one-dimensional Sobolev imbedding $H_\xi^1 \subset C_\xi^b$ and (4.1.17). $\square$

According to the spaces of the solutions and inhomogenities of the linear problem we define the norms

$$\|(z, w)\|_* := \|z\|_{H_\xi^3} + \varepsilon \|w\|_{H_\xi^2(L_v^2)}, \quad \|(h_z, h_w)\|_{**} := \|h_z\|_{H_\xi^1} + \varepsilon \|h_w\|_{H_\xi^2(L_v^2)}, \quad (4.3.22)$$

weighted corresponding to the decomposition $g = \Phi z + \varepsilon w$. Obviously $\|g\|_{H_\xi^2(L_v^2)}$ is bounded by $\|(z, w)\|_*$.

Before stating the existence result for travelling waves we note that in terms of the original unknown $f = f_{as} + \varepsilon^2 g$, the condition $z(0) = z_0$ reads

$$\int_V (v - \varepsilon s)^2 (f(0, v) - f_{as}(0, v)) dv = \varepsilon^2 D_1 z_0. \quad (4.3.23)$$

Theorem 4.3.1. *Let the wave speed satisfy $s \geq s_0$. For every $z_0 \in \mathbb{R}$ and for ε small enough, there exists a solution f of (4.1.5) satisfying (4.3.23), which is unique in a ball $\{f : \|f - f_{as}\|_* \leq \delta\}$, where the radius δ is independent of ε . It satisfies*

$$\|f - f_{as}\|_{H_\xi^2(L_v^2)} = O(\varepsilon^2),$$

or, more precisely,

$$f = f_{as} + \varepsilon^2 \Phi z + \varepsilon^3 w = M \rho_{as} - \varepsilon v M \rho'_{as} + \varepsilon^2 (v^2 - D) M \rho''_{as} + \varepsilon^2 \Phi z + \varepsilon^3 w, \quad (4.3.24)$$

where ρ_{as} is the travelling wave solution of the Fisher equation (4.3.1) and $\|z\|_{H_\xi^3}$ and $\|w\|_{H_\xi^2(L_v^2)}$ are uniformly bounded as $\varepsilon \rightarrow 0$.

Proof. Let ε be small enough. Then as a consequence of Lemma 4.3.5 (i), the solvability results for the above linear problem (4.3.18), (4.3.19) can be extended to the full linear problem

$$\begin{aligned} D z'' + s z' + z(\bar{\rho} - 2\rho) &= \varepsilon B^z(z, z', w, \partial_\xi w) + h_z, \\ \varepsilon(v - \varepsilon s) \partial_\xi w - \mathcal{M} w &= A(z, z') + \varepsilon B(z, w) + h_w, \end{aligned}$$

with inhomogenities h_z, h_w and $z(0) = z_0$. Applying the solution operator to the nonlinear problem (4.3.12), (4.3.17), we obtain a fixed point problem $(z, w) = \mathcal{G}(z, w)$, where the fix point operator is bounded by

$$\|\mathcal{G}(z, w)\|_* \leq C_0(1 + \varepsilon^2 \|(z, w)\|_*^2).$$

The constant C_0 bounds the initial condition and the residual terms, and the nonlinear terms are of order ε^2 . We see that for ε small enough, $\mathcal{G}$ maps both the ball with radius $2C_0$ and the ball with radius $1/(2\varepsilon^2 C_0)$ into themselves. Also, with the property of the nonlinearity, the fixed point operator $\mathcal{G}$ is a contraction on a ball with radius of order $O(\varepsilon^{-2})$.

We can conclude that for ε small enough, the fixed point problem has a solution (z, w) with $\|(z, w)\|_* \leq 2C_0$, which is unique in a ball with an $O(\varepsilon^{-2})$ -radius. Knowing this and returning to the fixed point problem, the boundedness of $\|w\|_{H_\xi^2(L_v^2)}$ follows. $\square$

Remark 4.3.1. *Due to Lemma 4.3.4 the contraction argument could also be carried out in $H_\xi^k(L_v^2)$ for any $k \in \mathbb{N}$. As a consequence the existence result holds in $H_\xi^k(L_v^2)$ for any $k \in \mathbb{N}$.*

As in [4] we can now deduce the monotonicity of the density ρ_f of the travelling wave:

Lemma 4.3.6. *Let the assumptions of the Existence-Theorem 4.3.1 hold and let f be the solution of (4.1.5) with initial condition (4.3.23). Then ρ_f is strictly decreasing, implying together with the far-field values also the positivity of ρ_f .*

The proof relies on the fact that the map $z_0 \mapsto \rho_f(0)$ is invertible for ε small, meaning that the travelling wave can also be made locally unique by prescribing the value of $\rho_f(0)$ instead of z_0 . This argument can of course be repeated for every $\xi_0 \in \mathbb{R}$ instead of the origin. Now assuming ρ_f is not strictly monotone would lead to the periodicity of f as a consequence of the uniqueness result, which contradicts the far-field conditions.

4.4 Dynamic stability of travelling waves

In this section we prove the local asymptotic stability of travelling waves with speed $s > s_0$ as constructed in Theorem 4.3.1. For this purpose it will be necessary to make the assumption

$$\text{The set } V \text{ of velocities is bounded, i.e. } |v| \leq v_{\max}, \quad \forall v \in V. \quad (4.4.1)$$

As for the macroscopic equation in Section 4.1.2, we restrict our attention to non-negative solutions. This is a consequence of nonnegativity of initial data and of the maximum principle (see Theorem 4.2.1). Replacing x by the travelling wave variable $\xi = x - st$,

$$\varepsilon^2 \partial_t f + \varepsilon(v - \varepsilon s) \partial_\xi f = M \rho_f - f + \varepsilon^2 \rho_f (M \bar{\rho} - f), \quad (4.4.2)$$

the travelling wave constructed in Theorem 4.3.1 and now denoted by $\phi(v, \xi)$ becomes a stationary solution. We fix the shift such that

$$\rho_\phi(0) = \frac{3}{4} \bar{\rho}, \quad (4.4.3)$$

implying $\rho_\phi(\xi) \geq 3\bar{\rho}/4$ for $\xi \leq 0$ by the monotonicity of ρ_ϕ . The initial datum $G_0(v, \xi)$ of the perturbation

$$G(t, v, \xi) = f(t, v, \xi) - \phi(v, \xi), \quad \rho(t, \xi) := \rho_G(t, \xi),$$

is assumed to satisfy $G_0 + \phi \geq 0$ guaranteeing $G(t, \cdot, \cdot) + \phi \geq 0$ for all $t \geq 0$ and, in particular, $\rho + \rho_\phi \geq 0$. The perturbation G satisfies

$$\varepsilon^2 \partial_t G + \varepsilon(v - \varepsilon s) \partial_\xi G = M\rho - G + \varepsilon^2(M\rho\bar{\rho} - (\rho_\phi + \rho)G - \rho\phi). \quad (4.4.4)$$

We apply a micro-macro decomposition to the perturbation. The kinetic equation (4.4.4) motivates to split G into

$$G = M\rho + \varepsilon g, \quad \text{implying} \quad \|G\|_v^2 = \rho^2 + \varepsilon^2 \|g\|_v^2.$$

The scalar product of (4.4.4) with G gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|G\|_{\xi, v}^2 + \|g\|_{\xi, v}^2 + \int_{\mathbb{R}} (2\rho_\phi + \rho - \bar{\rho}) \rho^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|g\|_v^2 d\xi \\ = -\varepsilon \int_{\mathbb{R}} \rho \langle \phi, g \rangle_v d\xi \leq \varepsilon^2 C(\|\rho\|_\xi^2 + \|g\|_{\xi, v}^2), \end{aligned}$$

with an ε -independent constant C , where we have used $\phi = M\rho_\phi + O(\varepsilon)$. As in the purely macroscopic case the integrand in the third term is negative as $\xi \rightarrow \infty$. We recall from Section 4.1.2 the weight function $W(\xi) = e^{\xi s/(2D)}$, and extend the definition of the corresponding norms to functions of (v, ξ) :

$$\begin{aligned} \|\rho\|_W &= \|\rho W\|_\xi, \quad \|\rho\|_{H_W^1}^2 = \|\rho\|_W^2 + \|\partial_\xi \rho\|_W^2, \\ \|G\|_{W, v} &= \|GW\|_{\xi, v}, \quad \|G\|_{H_W^1(L_v^2)}^2 = \|G\|_{W, v}^2 + \|\partial_\xi G\|_{W, v}^2. \end{aligned}$$

We rewrite (4.4.4) in terms of GW :

$$\begin{aligned} \partial_t(GW) + \frac{1}{\varepsilon}(v - \varepsilon s) \partial_\xi(GW) - \frac{1}{\varepsilon} \frac{s}{2D}(v - \varepsilon s)GW \\ = -\frac{1}{\varepsilon} gW + (M\bar{\rho}\rho - (\rho_\phi + \rho)G - \rho\phi)W, \end{aligned}$$

which gives the estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|G\|_{W, v}^2 + \|g\|_{W, v}^2 + \int_{\mathbb{R}} (\kappa + 2\rho_\phi + \rho) (\rho W)^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) W^2 \|g\|_v^2 d\xi \\ \leq -\frac{s^2}{4D} \|\rho\|_W^2 + \frac{s}{D} \int_{\mathbb{R}} \rho W^2 \int_V v g dv d\xi + \varepsilon \frac{s}{2D} \int_{\mathbb{R}} \langle (v - \varepsilon s)g, g \rangle_v W^2 d\xi \\ - \varepsilon \int_{\mathbb{R}} \rho \langle \phi, g \rangle_v W^2 d\xi \\ \leq \frac{\kappa}{2} \|\rho\|_W^2 + (1 - \gamma) \|g\|_{W, v}^2 + \varepsilon \frac{s}{2D} v_{\max} \|g\|_{W, v}^2 + \varepsilon^2 C(\|\rho\|_W^2 + \|g\|_{W, v}^2), \end{aligned}$$

for a $\gamma = \gamma(\kappa) > 0$, where we have used the Young inequality and $\left(\int_V v g \, dv\right)^2 \leq D \|g\|_v^2$. We point out the necessity of $\kappa > 0$ and of the boundedness of V .

By the representation (4.3.24) of the travelling wave, $\rho_\phi(\xi) \geq \rho_{as}(\xi) - C\varepsilon^2$ holds, which will be used in the third term of the estimate above. We shall use the constant α introduced in Section 4.1.2, and recall that it satisfies

$$\alpha \frac{\rho_{as}(\xi)}{2} W^2(\xi) - \bar{\rho} \geq \frac{\bar{\rho}}{16} \quad \text{for } \xi \geq 0. \quad (4.4.5)$$

We now combine both estimates and bound the macroscopic terms in a similar way as in the introduction:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|G\|_{\xi,v}^2 + \alpha \|G\|_{W,v}^2) + (1 - \varepsilon^2 C) \|g\|_{\xi,v}^2 + \alpha (\gamma - \varepsilon C) \|g\|_{W,v}^2 \\ & + \int_{\mathbb{R}} \left(\frac{3\rho_\phi}{2} - \bar{\rho} + \rho - \varepsilon^2 C \right) \rho^2 d\xi + \frac{\alpha}{2} \int_{\mathbb{R}} \rho_\phi \rho^2 W^2 d\xi + \frac{1}{2} \int_{\mathbb{R}} \rho^2 \rho_\phi (1 + \alpha W^2) d\xi \\ & + \alpha \int_{\mathbb{R}} \left(\frac{\kappa}{2} - \varepsilon^2 C + (\rho_\phi + \rho) \right) \rho^2 W^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|g\|_v^2 (1 + \alpha W^2) d\xi \leq 0. \end{aligned}$$

Due to (4.4.3) we can bound

$$\int_{-\infty}^0 \left(\frac{3\rho_\phi}{2} - \bar{\rho} + \rho - \varepsilon^2 C \right) \rho^2 d\xi \geq \left(\frac{\bar{\rho}}{8} - \varepsilon^2 C - \|\rho\|_\infty \right) \int_{-\infty}^0 \rho^2 d\xi. \quad (4.4.6)$$

Moreover, α was chosen such that

$$\int_0^\infty \left(\frac{\alpha}{2} \rho_\phi W^2 - \bar{\rho} - \varepsilon^2 C \right) \rho^2 d\xi \geq \left(\frac{\bar{\rho}}{16} - \varepsilon^2 C \right) \int_0^\infty \rho^2 d\xi - \alpha \varepsilon^2 C \int_0^\infty \rho^2 W^2 d\xi. \quad (4.4.7)$$

We summarize the estimate in a lemma.

Lemma 4.4.1. *Let (4.4.1) hold, let $\varepsilon > 0$ be small enough, and let ϕ be a travelling wave solution as constructed in Theorem 4.3.1. Let G be a solution of (4.4.4) for initial data G_0 with $G_0 + \phi \geq 0$. Let $\alpha > 0$ satisfy (4.4.5). Then*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|G\|_{\xi,v}^2 + \alpha \|G\|_{W,v}^2) + (1 - \varepsilon^2 C) \|g\|_{\xi,v}^2 + \alpha (\gamma - \varepsilon C) \|g\|_{W,v}^2 \\ & + \left(\frac{\bar{\rho}}{8} - \varepsilon^2 C - \|\rho\|_\infty \right) \int_{-\infty}^0 \rho^2 d\xi + \left(\frac{\bar{\rho}}{16} - \varepsilon^2 C \right) \int_0^\infty \rho^2 d\xi \\ & + \frac{1}{2} \int_{\mathbb{R}} \rho^2 \rho_\phi (1 + \alpha W^2) d\xi + \alpha \int_{\mathbb{R}} \left(\frac{\kappa}{2} - \varepsilon^2 C + (\rho_\phi + \rho) \right) \rho^2 W^2 d\xi \\ & + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|g\|_v^2 (1 + \alpha W^2) d\xi \leq 0. \end{aligned}$$

For control of the nonlinear term we have to bound ρ in $L^\infty(\mathbb{R})$. By Sobolev imbedding this can be done by controlling the $H^1(\mathbb{R})$ -norm. Thus we also derive estimates for the derivative in a similar procedure as above and denote

$$H = \partial_\xi G, \quad r = \partial_\xi \rho, \quad h = \partial_\xi g.$$

We start by differentiating (4.4.4):

$$\partial_t H + \frac{1}{\varepsilon}(v - \varepsilon s)\partial_\xi H = -\frac{1}{\varepsilon}h + M\bar{\rho}r - (\rho_\phi + \rho)H - \phi r - (\rho'_\phi + r)G - \rho\partial_\xi \phi,$$

which gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|H\|_{\xi, v}^2 + \|h\|_{\xi, v}^2 + \int_{\mathbb{R}} (2(\rho_\phi + \rho) - \bar{\rho})r^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|h\|_v^2 d\xi \\ &= -\varepsilon^2 \int_{\mathbb{R}} (\rho'_\phi + r) \langle g, h \rangle_v d\xi - \varepsilon \int_{\mathbb{R}} \langle \phi, h \rangle_v r d\xi - \varepsilon \int_{\mathbb{R}} \langle \partial_\xi \phi, h \rangle_v \rho d\xi + \int_{\mathbb{R}} \rho_\phi'' \rho^2 d\xi \\ &\leq \varepsilon^2 C \left(\|\rho\|_{H_\xi^1}^2 + \|g\|_{H_\xi^1(L_v^2)}^2 \right) + \varepsilon^2 \|g\|_{\infty, v} \left(\|r\|_\xi^2 + \|h\|_{\xi, v}^2 \right) + \frac{s^2}{2D^2} \int_{\mathbb{R}} \rho_\phi \rho^2 d\xi. \end{aligned} \quad (4.4.8)$$

Here we have used the Cauchy-Schwarz inequality, $\phi = M\rho_\phi + O(\varepsilon)$, and the fact that $\rho_\phi - \rho_{as} = O(\varepsilon^2)$, implying $D\rho_\phi'' = -s\rho'_\phi - \rho_\phi(\bar{\rho} - \rho_\phi) + O(\varepsilon^2) \leq \frac{s^2}{2D}\rho_\phi + O(\varepsilon^2)$. As above we rewrite the equation in terms of HW

$$\begin{aligned} & \partial_t(HW) + \frac{1}{\varepsilon}(v - \varepsilon s)\partial_\xi(HW) - \frac{1}{\varepsilon} \frac{s}{2D}(v - \varepsilon s)HW + \frac{1}{\varepsilon}hW \\ &= (M\bar{\rho}r - (\rho_\phi + \rho)H - \phi r - (\rho'_\phi + r)G - \rho\partial_\xi \phi)W, \end{aligned}$$

and compute the scalar product with HW . Treating the terms on the right hand side similarly to (4.4.8) and the transport terms analogously to (4.4.5), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|H\|_{W, v}^2 + \left(\gamma - \varepsilon \frac{s}{2D} v_{max} \right) \|h\|_{W, v}^2 + \int_{\mathbb{R}} \left(\frac{\kappa}{2} + 2(\rho_\phi + \rho) \right) r^2 W^2 d\xi \\ & \quad + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|h\|_v^2 W^2 d\xi \\ & \leq \varepsilon^2 C \left(\|\rho\|_{H_W^1}^2 + \|g\|_{H_W^1(L_v^2)}^2 \right) + \varepsilon^2 \|g\|_{\infty, v} \left(\|r\|_W^2 + \|h\|_{W, v}^2 \right) + \frac{s^2}{2D^2} \int_{\mathbb{R}} \rho_\phi \rho^2 W^2 d\xi, \end{aligned}$$

where we have also used the monotonicity of ρ_ϕ .

In the purely macroscopic case we used the diffusion to control $-\bar{\rho}\|r\|_\xi^2$, which is not directly available from the transport term in the kinetic equation. One can now proceed in two different ways: For the shorter but much more restrictive version one can use the positivity of κ instead. One then chooses $\alpha_2 = \bar{\rho}/(4\kappa)$, such that $\alpha_2 \kappa W^2/4 \geq \bar{\rho}$ for $\xi \geq 0$. This would however result in requiring that the perturbation is initially bounded by κC , which is very restrictive for κ small. We note that in the estimates above even a κ of order ε would be allowed. Therefore we rather perform a Chapman-Enskog approximation, from which we recover the diffusion in the equation for ρ .

Before proceeding with deriving the equation for ρ we summarize the estimates for H and HW as follows.

Lemma 4.4.2. *Let the assumptions of Lemma 4.4.1 hold and let $\alpha > 0$. Then*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|H\|_{\xi,v}^2 + \alpha \|H\|_{W,v}^2 \right) + \left(1 - \varepsilon^2 (C + \|g\|_{\infty,v}) \right) \|h\|_{\xi,v}^2 \\
& + \alpha \left(\gamma - \varepsilon (C + \varepsilon \|g\|_{\infty,v}) \right) \|h\|_{W,v}^2 + \varepsilon^2 \int_{\mathbb{R}} (\rho_\phi + \rho) \|h\|_v^2 (1 + \alpha W^2) d\xi \\
& + 2 \int_{\mathbb{R}} (\rho_\phi + \rho) r^2 (1 + \alpha W^2) d\xi - \left(\bar{\rho} + \varepsilon^2 (C + \|g\|_{\infty,v}) \right) \|r\|_{\xi}^2 \\
& + \alpha \left(\frac{\kappa}{2} - \varepsilon^2 (C + \|g\|_{\infty,v}) \right) \|r\|_W^2 \\
& \leq \frac{s^2}{2D^2} \int_{\mathbb{R}} \rho_\phi \rho^2 (1 + \alpha W^2) d\xi + \varepsilon^2 C \left(\|\rho\|_{\xi,v}^2 + \alpha \|\rho\|_{W,v}^2 + \|g\|_{\xi,v}^2 + \alpha \|g\|_{W,v}^2 \right).
\end{aligned}$$

We still need to control $-\bar{\rho} \|r\|_{\xi}^2$. We split (4.4.4) into two equations according to the splitting of G in ρ and g . Integrating (4.4.4) we obtain

$$\partial_t \rho - sr + (2\rho_\phi + \rho - \bar{\rho})\rho = - \int_V v h dv. \quad (4.4.9)$$

Applying $-\mathcal{L}$ to (4.4.4) gives the equation for g :

$$\varepsilon^2 \partial_t g + v M r - \varepsilon M \int_V v h dv + \varepsilon(v - \varepsilon s)h + g = \varepsilon \rho \mathcal{L} \phi - \varepsilon^2 (\rho_\phi + \rho) g.$$

We compute

$$- \int_V v h dv = D \partial_\xi r + \varepsilon^2 \partial_t \int_V v h dv + \varepsilon \partial_\xi \mathcal{S}[g],$$

where we denote

$$\mathcal{S}[g] = \int_V v(v - \varepsilon s)h dv + \varepsilon \left(\frac{\rho}{\varepsilon} \int_V v \phi dv + (\rho_\phi + \rho) \int_V v g dv \right).$$

In the macroscopic equation we now recover the diffusion:

$$\partial_t \rho - D \partial_\xi r - sr + (2\rho_\phi + \rho - \bar{\rho})\rho = \varepsilon^2 \partial_t \int_V v h dv + \varepsilon \partial_\xi \mathcal{S}[g]. \quad (4.4.10)$$

Testing this equation with ρ gives

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\rho\|_{\xi}^2 + D \|r\|_{\xi}^2 + \int_{\mathbb{R}} (2\rho_\phi + \rho - \bar{\rho}) \rho^2 d\xi = \varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}} \rho \int_V v h dv d\xi \\
& - \varepsilon^2 \int_{\mathbb{R}} \left(sr - (2\rho_\phi + \rho - \bar{\rho})\rho - \int_V v h dv \right) \int_V v h dv d\xi - \varepsilon \int_{\mathbb{R}} \mathcal{S}[g] r d\xi,
\end{aligned}$$

where we have substituted $\partial_t \rho$ according to (4.4.9) and used integration by parts.

Applying the Young inequality we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\rho\|_{\xi}^2 - \varepsilon^2 \int_{\mathbb{R}} \rho \int_V v h dv d\xi \right) + D \|r\|_{\xi}^2 + \int_{\mathbb{R}} (2\rho_{\phi} + \rho - \bar{\rho}) \rho^2 d\xi \\ \leq \frac{D}{4} \|r\|_{\xi}^2 + \varepsilon^2 C \left(\|g\|_{H_{\xi}^1(L_v^2)}^2 + \|\rho\|_{H_{\xi}^1}^2 \right) \\ + \varepsilon \int_{\mathbb{R}} (\rho_{\phi} + \rho) \left(\rho^2 + r^2 + \varepsilon^2 C (\|g\|_v^2 + \|h\|_v^2) \right) d\xi \end{aligned}$$

As before we rewrite (4.4.10) in terms of ρW :

$$\partial_t(\rho W) - D \partial_{\xi}^2(\rho W) + (\kappa + 2\rho_{\phi} + \rho) \rho W = \varepsilon^2 \partial_t \int_V v h W dv + \varepsilon \partial_{\xi} \mathcal{S}[g] W.$$

Due to the weight, integration by parts now produces more terms on the right hand side:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\rho\|_W^2 - \varepsilon^2 \int_{\mathbb{R}} \rho W^2 \int_V v h dv d\xi \right) + \frac{3D}{4} \|\partial_{\xi}(\rho W)\|_{\xi}^2 \\ + \int_{\mathbb{R}} (\kappa + 2\rho_{\phi} + \rho) \rho^2 W^2 d\xi \\ \leq \varepsilon C (\|h\|_{W,v}^2 + \|\rho\|_W^2) + \varepsilon^2 C \left(\|g\|_{H_W^1(L_v^2)}^2 + \|\rho\|_{H_W^1}^2 \right) \\ + \varepsilon \int_{\mathbb{R}} (\rho_{\phi} + \rho) \left(\rho^2 + r^2 + \varepsilon^2 C (\|g\|_v^2 + \|h\|_v^2) \right) W^2 d\xi. \end{aligned}$$

We combine the estimates, treat the macroscopic part as in (4.4.6), (4.4.7), and obtain the following Lemma.

Lemma 4.4.3. *Let the assumptions of Lemma 4.4.1 hold, let α satisfy (4.4.5), and let ε be small. Then*

$$\begin{aligned} \frac{d}{dt} I + \left(\frac{3D}{4} - \varepsilon^2 C \right) (\|r\|_{\xi}^2 + \alpha \|\partial_{\xi}(\rho W)\|_{\xi}^2) + \left(\frac{\bar{\rho}}{8} - \varepsilon C - \|\rho\|_{\infty} \right) \int_{-\infty}^0 \rho^2 d\xi \\ + \left(\frac{\bar{\rho}}{16} - \varepsilon C \right) \int_0^{\infty} \rho^2 d\xi + \alpha (\kappa - \varepsilon C) \|\rho\|_W^2 \\ \leq \varepsilon C \|h\|_{W,v}^2 + \varepsilon^2 C \left(\|g\|_{H_{\xi}^1(L_v^2)}^2 + \alpha \|g\|_{H_{\xi}^1(L_v^2)}^2 \right) \\ + \varepsilon \int_{\mathbb{R}} (\rho_{\phi} + \rho) \left(r^2 + \varepsilon^2 C (\|g\|_v^2 + \|h\|_v^2) \right) (1 + \alpha W^2) d\xi, \end{aligned}$$

where

$$I = \frac{1}{2} \left(\|\rho\|_{\xi}^2 + \alpha \|\rho\|_W^2 - 2\varepsilon^2 \int_{\mathbb{R}} \rho (1 + \alpha W^2) \int_V v h dv d\xi \right).$$

We are now ready to prove the main result of this section.

Theorem 4.4.1. *Let (4.4.1) hold, let ϕ be the travelling wave solution from Theorem 4.3.1 with speed $s > s_0$ made unique by (4.4.3), and let ε be small. Let $f_0(v, \xi)$ satisfy*

$$0 \leq f_0 \leq \hat{\rho} M, \quad \text{and} \quad \|f_0 - \phi\|_{H_\xi^1(L_v^2)} + \|f_0 - \phi\|_{H_W^1(L_v^2)} \leq \delta,$$

for a $\delta > 0$ small enough, but independently of ε , and $\hat{\rho} > 0$.

Then the solution of (4.4.2) with initial datum f_0 satisfies

$$\|f - \phi\|_{H_\xi^1(L_v^2)}^2(t) + \|f - \phi\|_{H_W^1(L_v^2)}^2(t) \leq C e^{-\lambda t} (\|f_0 - \phi\|_{H_\xi^1(L_v^2)}^2 + \|f_0 - \phi\|_{H_W^1(L_v^2)}^2),$$

with an exponential decay rate $\lambda > 0$.

Proof. Applying the standard contraction argument, one can show the well posedness of (4.4.4) in $H_\xi^1(L_v^2) \cap H_W^1(L_v^2)$ for initial data $G_0 \in H_\xi^1(L_v^2) \cap H_W^1(L_v^2)$. Hence it only remains to derive the a priori estimate.

We construct a Lyapunov functional by combining the above estimates. We introduce

$$J(t) = I(t) + \frac{1}{2} \left(\|G\|_{\xi, v}^2 + \alpha \|G\|_{W, v}^2 + \beta (\|H\|_{\xi, v}^2 + \alpha \|H\|_{W, v}^2) \right),$$

where $\beta > 0$ is determined below and $\alpha > 0$ satisfies (4.4.5). The functional J is bounded from above and below by

$$\alpha_* \left(\|G\|_{H_\xi^1(L_v^2)}^2 + \|G\|_{H_W^1(L_v^2)}^2 \right) \leq J \leq \alpha^* \left(\|G\|_{H_\xi^1(L_v^2)}^2 + \|G\|_{H_W^1(L_v^2)}^2 \right),$$

where $\alpha_*, \alpha^* > 0$ are independent of κ and ε , if the same is true for β . For ε small enough, the estimate for J reads

$$\begin{aligned} & \frac{d}{dt} J + 2 \left(\frac{\bar{\rho}}{16} - \|\rho\|_\infty \right) \int_{-\infty}^0 \rho^2 d\xi + \frac{\bar{\rho}}{16} \int_0^\infty \rho^2 d\xi + \frac{\alpha \kappa}{4} \|\rho\|_W^2 \\ & + \left(\frac{1}{2} - \frac{\beta s^2}{2D^2} \right) \int_{\mathbb{R}} \rho_\phi \rho^2 (1 + \alpha W^2) d\xi + \left(\frac{D}{2} - \beta \bar{\rho} - \varepsilon^2 \beta \|g\|_{\infty, v} \right) \|r\|_\xi^2 \\ & \quad + \alpha \beta \left(\frac{\kappa}{4} - \varepsilon^2 \|g\|_{\infty, v} \right) \|r\|_W^2 + \frac{1}{2} \|g\|_{\xi, v}^2 + \alpha \frac{\gamma}{2} \|g\|_{W, v}^2 \\ & + \left(\frac{\beta}{2} - \varepsilon^2 \|g\|_{\infty, v} \right) \|h\|_{\xi, v}^2 + \alpha \beta \left(\frac{\gamma}{2} - \varepsilon^2 \|g\|_{\infty, v} \right) \|h\|_{W, v}^2 \leq 0. \end{aligned}$$

We choose $\beta = \min\{\frac{D}{4\bar{\rho}}, \frac{D^2}{2s^2}\}$. By Sobolev imbedding,

$$\|G\|_{\infty, v} \leq \|G\|_{H_\xi^1(L_v^2)} \leq \sqrt{\frac{J}{\alpha_*}},$$

such that in particular $\|\rho\|_\infty$ and $\varepsilon \|g\|_{\infty, v}$ are bounded by $\sqrt{J/\alpha_*}$. We now denote $M = \sqrt{J(0)/\alpha_*}$ and let the initial data be small enough such that

$$M < \frac{\bar{\rho}}{16},$$

which also implies $\varepsilon M < \min\{\kappa/4, \beta/2, \gamma/2, D/(4\beta)\}$ for ε small enough. Then all coefficients above are positive initially and therefore J is decreasing at $t = 0$. Since in turn J controls the coefficients, the functional J decreases for all times. This in particular means that there exists a constant $\lambda > 0$ such that

$$\frac{d}{dt}J \leq -\lambda J,$$

and we get the exponential decay of J , which is equivalent to $\|G\|_{H_\xi^1(L_v^2)}^2 + \|G\|_{H_w^1(L_v^2)}^2$.

□

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Part II

Travelling waves for a nonlinear nonlocal diffusion equation

Chapter 5

On nonlinear conservation laws with a nonlocal diffusion term¹²

5.1 Introduction

We consider one-dimensional conservation laws for the density $u(t, x)$, $t > 0$, $x \in \mathbb{R}$, of the form

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u, \quad (5.1.1)$$

where $\mathcal{D}^\alpha$ is the non-local operator

$$(\mathcal{D}^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{u'(y)}{(x-y)^\alpha} dy, \quad (5.1.2)$$

with $0 < \alpha < 1$. The flux function $f(u)$ is smooth and satisfies $f(0) = 0$.

We shall analyze the local and global solvability of the Cauchy problem for (5.1.1), as well as the existence and stability of travelling wave solutions. In particular, we shall show that smooth travelling wave solutions exist, which are asymptotically stable. These waves are shock profiles satisfying the standard entropy conditions as those derived from the standard parabolic regularization with $\mathcal{D}^\alpha$ replaced by ∂_x .

Since $\mathcal{D}^\alpha u$ can be written as the convolution of the derivative u' with $\Gamma(1-\alpha)^{-1} \theta(x) x^{-\alpha}$ (with the Heavyside function θ), $\mathcal{D}^\alpha$ is a pseudo-differential operator with symbol

$$\frac{ik}{\Gamma(1-\alpha)} \mathcal{F} \left(\frac{\theta(x)}{x^\alpha} \right) (k) = ik (a_\alpha - ib_\alpha \operatorname{sgn}(k)) |k|^{\alpha-1} = (b_\alpha + ia_\alpha \operatorname{sgn}(k)) |k|^\alpha,$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F} \varphi(k) = \widehat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \varphi(x) dx,$$

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and

$$a_\alpha = \sin\left(\frac{\alpha\pi}{2}\right) > 0, \quad b_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) > 0,$$

(see [2] for the details of the computation). Obviously, the operator on the right hand side of (5.1.1) also is a pseudo-differential operator with symbol

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha) = -(a_\alpha - ib_\alpha \operatorname{sgn}(k)) |k|^{\alpha+1}. \quad (5.1.3)$$

Due to the negativity of its real part, it is dissipative.

Remark 5.1.1. For $s \in \mathbb{R}$, we use the Sobolev space

$$H^s := \{u : \|u\|_{H^s} < \infty\}, \quad \|u\|_{H^s} := \|(1 + |k|)^s \hat{u}\|_{L^2(\mathbb{R})},$$

and the corresponding homogeneous norm

$$\|u\|_{\dot{H}^s} := \| |k|^s \hat{u} \|_{L^2(\mathbb{R})}.$$

The fact $\|\mathcal{D}^\alpha u\|_{\dot{H}^s} = \sqrt{a_\alpha^2 + b_\alpha^2} \|u\|_{\dot{H}^{s+\alpha}}$ justifies to interpret $\mathcal{D}^\alpha$ as a differentiation operator of order α . It is bounded as a map from H^s to $H^{s-\alpha}$.

Denoting by C_b^m , $m \geq 0$, the set of functions, whose derivatives up to order m are continuous and bounded on $\mathbb{R}$, $\mathcal{D}^\alpha u : C_b^1 \rightarrow C_b$ is bounded. This can be easily seen by splitting the domain of integration in (5.1.2) into $(-\infty, x - \delta]$ and $[x - \delta, x]$ for some positive $\delta > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha u$.

The operator $\partial_x \mathcal{D}^{1/3}$ occurs in applications. It has been derived as the physically correct viscosity term in two layer shallow water flows by performing formal asymptotic expansions associated to the triple-deck regularization used in fluid mechanics (see, e.g., [16]). Moreover $\mathcal{D}^{1/3}$ appears in the work of Fowler [11] in an equation for dune formation:

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u. \quad (5.1.4)$$

Here the fractional derivative appears with the negative sign, but this instability is regularized by the second order derivative. Alibaud et al. showed the well posedness of (5.1.4) in L^2 as well as the violation of the maximum principle, which is intuitive in the context of the application due to underlying erosions [1]. Travelling wave solutions of (5.1.4) have been analyzed by Alvarez-Samaniego and Azerad in [2].

Fractal conservation laws of the form

$$\partial_t u + \partial_x f(u) = D^{\alpha+1} u, \quad (5.1.5)$$

where $D^{\alpha+1}$ is the pseudo-differential operator with symbol $-|k|^{\alpha+1}$ (meaning $D^{\alpha+1} u = \mathcal{F}^{-1}(|k|^{\alpha+1} \hat{u})$) have been investigated in several works, see e.g. Biler et al. [5] and Droniuou et al. [7].

This work is organized as follows. In Section 5.2, we analyse the Cauchy problem for (5.1.1). A crucial property of (5.1.1) is the decay of the L^2 -norm. Testing (5.1.1) with u and assuming vanishing far-field values of u , the flux term vanishes

$$\int_{\mathbb{R}} u \partial_x f(u) dx = \int_{\mathbb{R}} u f'(u) \partial_x u dx = \int_{\mathbb{R}} \partial_x G(u) dx = 0, \quad G(u) = \int_0^u v f'(v) dv,$$

since G is smooth and $G(0) = 0$. We obtain the L^2 -estimate:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 dx = -a_\alpha \int_{\mathbb{R}} |k|^{1+\alpha} |\hat{u}|^2 dk \leq 0.$$

Here we have used the Plancherel theorem together with $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$, implying

$$\int_{\mathbb{R}} \operatorname{sgn} k |k|^j |\hat{u}(k, t)|^2 dk = 0.$$

This relation shows that in an L^2 -framework the operator $\partial_x \mathcal{D}^\alpha$ behaves similarly to $D^{\alpha+1}$. Due to the decay of the L^2 -norm of the solution to (5.1.1), one would hope for well posedness of the Cauchy problem for initial data in L^2 allowing us to deduce the global existence. Using a contraction argument similar to the one by Dix for the classical viscous Burgers equation, we can show the well posedness in L^2 for the quadratic flux $f(u) = u^2$ in the case $\alpha > 1/2$. This critical value was already mentioned by Biler, Funaki and Woyczynski [5] for (5.1.5). For the general flux and $\alpha \in (0, 1)$ we have to require higher regularity of the initial data: $u_0 \in H^1$. To deduce global existence of solutions in H^1 , we derive a Lyapunov functional, where we improve the estimates given in [5] for (5.1.5).

An alternative L^∞ -based existence theory uses the nonnegativity of the semigroup generated by $\partial_x \mathcal{D}^\alpha$, which is a consequence of its interpretation as a Riesz-Feller derivative [10], [12]. This allows to prove a maximum principle for solutions of (5.1.1) as in [9].

Section 5.3 is devoted to the analysis of travelling wave solutions connecting different far-field values. Such wave profiles are typically smooth and clearly not in L^2 . Therefore, in contrary to the Cauchy problem, we work with the original representation (5.1.2) of $\mathcal{D}^\alpha$ and obtain a nonlinear Volterra integral equation as the travelling wave version of (5.1.1). Assuming (even a bit less than) convexity of the flux function we can show the existence and uniqueness of monotone solutions satisfying the entropy condition for classical shock waves of the conservation law corresponding to (5.1.1). This essentially requires to extend the well known results for the existence of viscous shock profiles, which solve (local) ordinary differential equations.

Biler et al. [5] showed that no travelling wave solutions of (5.1.5) can exist for $\alpha \in (-1, 0]$. For the case $\alpha \in (0, 1)$ however also no existence result is available.

To show the asymptotic stability of the travelling waves, we use the antiderivative method typically applied in the case of the classical viscous regularisation and derive a Lyapunov functional decaying for all times. This allows to deduce the decay of initially small perturbations.

5.2 The Cauchy Problem

In this section we analyse the Cauchy problem

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u, \quad u(0, x) = u_0(x), \quad (5.2.1)$$

where we use the Fourier-representation of $\partial_x \mathcal{D}^\alpha u$ given in (5.1.3). In the following we show the global existence of a solution to the Cauchy problem in L^2 for $\alpha > \frac{1}{2}$ and $f(u) = u^2$. Assuming the flux to be quadratic allows us to estimate the nonlinearity in the fashion of Dix [7]. For more general flux functions we have to require $u_0 \in H^1$. This existence result resembles the one in [5] for equation (5.1.5). We can achieve an improvement concerning the admissible orders of the derivatives. Then we verify the applicability of the work of Droniou et al [9] for the Cauchy problem of (5.1.5) in L^∞ to (5.2.1), which in particular yields a maximum principle.

Applying the Fouriertransform to the linear evolution equation $\partial_t u = \partial_x \mathcal{D}^\alpha u$ we see that the semigroup generated by the fractional derivative is formally given by the convolution with the kernel

$$K(t, x) = \mathcal{F}^{-1} e^{-\Lambda(k)t}(x), \quad \text{where } \Lambda(k) = (a_\alpha - i b_\alpha \operatorname{sgn}(k)) |k|^{\alpha+1}. \quad (5.2.2)$$

To analyse the well posedness we use the mild formulation of the Cauchy problem

$$u(t, x) = K(t, \cdot) * u_0(x) - \int_0^t K(t - \tau, \cdot) * \partial_x f(\tau, \cdot)(x) d\tau. \quad (5.2.3)$$

Before proceeding with the contraction arguments we shall note that for any $u_0 \in H^s$ we have $K(t, \cdot) * u_0 \rightarrow u_0$ as $t \rightarrow 0$ in H^s . Since the integral

$$\|K(t, \cdot) * u_0 - u_0\|_{H^s}^2 = \int (1 + |k|)^{2s} (e^{-\Lambda(k)t} - 1)^2 |\hat{u}_0(k)|^2 dk$$

is bounded by $4\|u_0\|_{H^s}^2$ we can apply the Dominated Convergence Theorem and pass to the limit under the integral sign.

Proposition 5.2.1 (Global existence in L^2). *Let $f(u) = u^2$ and $\alpha > \frac{1}{2}$. Then for any $u_0 \in L^2$ the Cauchy problem (5.2.1) has a unique solution $u \in L^2$ for all times $t > 0$.*

Proof. Due to the decay of the L^2 norm it only remains to show the well-posedness. Denoting the right hand side of (5.2.3) with $\mathcal{G}u$ the mild formulation gives a fixed point problem $u = \mathcal{G}u$. We briefly explain how to carry out the contraction argument. We shall estimate

$$\mathcal{G}u(t, x) = K(t, \cdot) * u_0(x) - \int_0^t K(t - \tau, \cdot) * \partial_x u^2(\tau, \cdot)(x) d\tau \quad (5.2.4)$$

in L^2 for initial data $u_0 \in L^2$. Using Cauchy-Schwarz inequality it is easy to see that $\|(gh)^\wedge\|_\infty \leq \|g\|_{L^2} \|h\|_{L^2}$. Let $t_0 > 0$ and denote $\|u\|_X = \sup_{t \in [0, t_0]} \|u\|_{L^2}$. Applying Plancherel's Theorem we can bound the L^2 norm of the nonlinear term in $\mathcal{G}$ by

$$C \|u\|_X^2 \int_0^t \|e^{-a_\alpha |k|^{\alpha+1}(t-\tau)} |k|\|_{L^2} d\tau \leq C t^{1-\frac{3}{2(\alpha+1)}} \|u\|_X^2 \int_0^1 (1-\sigma)^{-\frac{3}{2(\alpha+1)}} d\sigma,$$

where we have performed the substitution $k \mapsto k(t - \tau)^{\frac{1}{\alpha+1}}$. We see that the critical value for the exponent is $\alpha = \frac{1}{2}$. For $\alpha > 1/2$ the operator $\mathcal{G}$ is a contraction for small times: There exists a constant $C_0 > 0$, such that

$$\|\mathcal{G}u\|_X \leq C_0 \left(1 + t_0^{1 - \frac{3}{2(\alpha+1)}} \|u\|_X^2 \right), \quad \|\mathcal{G}u_1 - \mathcal{G}u_2\|_X \leq C_0 \delta t_0^{1 - \frac{3}{2(\alpha+1)}} \|u_1 - u_2\|_{L^2},$$

for all $u_1, u_2 \in B_\delta = \{u \in C([0, t_0], L^2) : \|u\|_X \leq \delta\}$. Hence for t_0 small enough, $\mathcal{G}$ maps the ball with radius $\delta = 2C_0$ into itself. Also $\mathcal{G}$ is a contraction on B_δ . With Banachs fixed point argument we can conclude the existence of a solution $u \in B_\delta$ of (5.2.3), which is therefore the solution of (5.2.1) on $(0, t_0)$. The uniqueness result is only local in B_δ . Hence let us now assume $u, v \in C([0, T], L^2)$ are two solutions of (5.2.3) and let $M = \max\{\|u\|_X, \|v\|_X\}$. Then due to the above contraction argument there exists a $T_0 > 0$ such that the solution is unique in B_M on $[0, T_0]$, i.e. $u = v$ on $[0, T_0]$. This argument can of course be repeated. $\square$

Proposition 5.2.2 (Well posedness in H^1). *Let $u_0 \in H^1$. Then there exists a $t_0 > 0$ such the Cauchy problem (5.2.1) has a unique solution $u \in H^1$ for $t \in (0, t_0)$.*

Proof. We consider the fix point operator associated to the right hand side of (5.2.3) and proceed as in Theorem 5.2.1. We shall only show how to estimate the nonlinear term in H^1 . We let $t_0 > 0$ and denote $\|u\|_X = \sup_{t \in [0, t_0]} \|u\|_{H^1}$ and $\|u\|_{t_0, \infty} = \sup_{t \in [0, t_0]} \|u\|_\infty$. Then

$$\begin{aligned} & \int_0^t \left\| (1 + |k|) e^{-a_\alpha |k|^{\alpha+1}(t-\tau)} |(\partial_x f(u))'| \right\|_{L^2} d\tau \\ & \leq \int_0^t \left(1 + \frac{\|y\| e^{-a_\alpha |y|^{\alpha+1}}}{(t-\tau)^{\frac{1}{\alpha+1}}} \right) \|\partial_x f(u)\|_{L^2} d\tau \\ & \leq C \|\partial_x f(u)\|_{t_0, \infty} \left(t_0 + t_0^{1 - \frac{1}{1+\alpha}} \right) \|u\|_X \leq C(\|u\|_X) \left(t_0 + t_0^{1 - \frac{1}{1+\alpha}} \right) \|u\|_X \end{aligned}$$

by Sobolev Imbedding $H^1 \subset C_b$, where $C(\|u\|_X)$ is bounded if $\|u\|_X$ is bounded. As above for t_0 small we can deduce the existence and uniqueness of a solution in H^1 on $(0, t_0)$. $\square$

Proposition 5.2.3 (Global existence in H^1). *Let $u_0 \in H^1$ with $\|u_0\|_{H^1}$ small enough. Then the Cauchy problem (5.2.1) has a unique solution $u \in H^1$ for all times $t > 0$.*

Proof. The main idea is to construct a Lyapunov functional decaying in time. As we have seen in the introduction the L^2 norm is decaying

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = -a_\alpha \|u\|_{\dot{H}^{\frac{1+\alpha}{2}}}^2. \quad (5.2.5)$$

Testing the differentiated evolution equation with $\partial_x u$ gives

$$\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2}^2 + \int \frac{\partial_u^2 f(u)}{2} (\partial_x u)^3 dx = -a_\alpha \|u\|_{\dot{H}^{\frac{3+\alpha}{2}}}^2 \quad (5.2.6)$$

where we used partial integration twice. The flux term we estimate as follows

$$\left| \int \frac{\partial_u^2 f(u)}{2} (\partial_x u)^3 dx \right| \leq \|\partial_u^2 f(u)\|_\infty \|\partial_x u\|_{L^3}^3 \leq C(\|u\|_{H^1}) \|\partial_x u\|_{L^3}^3,$$

where as before we have used $H^1 \subset C_b$. We now apply a generalisation of the celebrated Gagliardo-Nirenberg inequalities (see e.g. [13]) to Sobolev spaces with fractional order, which was proven by Amann [4] (Proposition 4.1):

$$\begin{aligned} \|\partial_x u\|_{L^3}^3 &\leq C \|\partial_x u\|_{H^{\frac{\alpha+1}{4}}}^2 \|\partial_x u\|_{L^2} \leq C \|u\|_{H^1} \left(\|u\|_{H^1}^2 + \|u\|_{H^{\frac{\alpha+5}{4}}}^2 \right) \\ &\leq C \|u\|_{H^1} \left(\|u\|_{H^{\frac{1+\alpha}{2}}}^2 + \|u\|_{H^{\frac{3+\alpha}{2}}}^2 \right) \end{aligned} \quad (5.2.7)$$

The complete estimate reads

$$\frac{d}{dt} \|u\|_{H^1}^2 + 2(a_\alpha - C(\|u\|_{H^1})\|u\|_{H^1}) \left(\|u\|_{H^{\frac{1+\alpha}{2}}}^2 + \|u\|_{H^{\frac{3+\alpha}{2}}}^2 \right) \leq 0$$

Let $M = \|u_0\|_{H^1}$. If M is small enough such that $C(M)M < a_\alpha$, then $\|u\|_{H^1}^2$ is decreasing at $t = 0$. But this in turn implies that for small times the coefficient $a_\alpha - C(\|u\|_{H^1})\|u\|_{H^1}$ stays positive. Repeating this argument we get the decay of $\|u\|_{H^1}^2$ for all times

$$\frac{d}{dt} \|u\|_{H^1}^2 \leq 0, \quad \text{for all } t \geq 0.$$

□

Remark 5.2.1. The stability results for travelling waves in section 5.3.2 are as well global existence results for the Cauchy problem for initial values, which are small perturbations to travelling waves.

Remark 5.2.2. We shall note that the operator $\partial_x \mathcal{D}^\alpha : H^{1+\alpha} \rightarrow L^2$ is the generator of an analytic semigroup. Hence one could also argue by standard existence results, see e.g. Henry [13] and Pazy [15], to deduce the well-posedness of the Cauchy problem in $H^{\alpha+1}$. Then an energy estimate for the $H^{1+\alpha}$ -norm is required to get global existence. For polynomial fluxes such a Lyapunov functional can be derived as follows. We use the decay of the L^2 -norm. Then we multiply the Fouriertransform of the equation with $|k|^{1+\alpha}$ and test it with $|k|^{1+\alpha} \hat{u}(k)$. For the nonlinearity we use the estimate given by Dix [7] (Theorem 3.2): Let $m > 0$, $u \in H^{m+\frac{1}{2}}$ and let $q \geq 2$ be an integer. Then there exists a $C > 0$ such that

$$|\langle D^m \partial_x u^q, D^m u \rangle_{L^2}| \leq C \|u\|_\infty^{q-1} \|u\|_{H^{m+\frac{1}{2}}}^2. \quad (5.2.8)$$

For small enough initial data in $H^{1+\alpha}$ we can derive the decay of u in $H^{1+\alpha}$ in a similar way as in the proof of Theorem 5.2.3.

We now turn to the Cauchy problem (5.2.1) in L^∞ . Droniou et al. [9] showed the solvability of the Cauchy problem in L^∞ for (5.1.5) and in particular derived a maximum principle. The crucial property there was the non-negativity of the

kernel associated to the semigroup generated by $D^{\alpha+1}$. Therefore to carry over the results to the operator $\partial_x \mathcal{D}^\alpha$ we need to investigate the properties of the kernel K given in (5.2.2). As we have already mentioned, the operator $\partial_x \mathcal{D}^\alpha$ is a Riesz-Feller differential operator. Due to Feller [11] the symbol of $\partial_x \mathcal{D}^\alpha$ is therefore the logarithm of the characteristic function of a Levy stable distribution, see e.g. also Gorenflo and Mainardi [12]. Hence the kernel K is non-negative:

$$K(t, x) \geq 0, \quad \text{for all } t > 0, x \in \mathbb{R}.$$

The kernel K satisfies additionally the properties:

$$\forall t > 0, x \in \mathbb{R}, \quad K(t, x) = \frac{1}{t^{1/(1+\alpha)}} K\left(1, \frac{x}{t^{1/(1+\alpha)}}\right) \quad (5.2.9)$$

$$\forall t > 0, \quad \|K(t, \cdot)\|_{L^1(\mathbb{R})} = 1 \quad (5.2.10)$$

K is C^∞ on $(0, \infty) \times \mathbb{R}$ and for all $m \geq 0$ there exists a B_m such that

$$\forall (t, x) \in (0, \infty) \times \mathbb{R}, \quad |\partial_x^m K(t, x)| \leq \frac{1}{t^{(1+m)/(1+\alpha)}} \frac{B_m}{(1 + t^{-2/(1+\alpha)} |x|^2)}. \quad (5.2.11)$$

In particular there exists a C_0 such that for all $t > 0$: $\|\partial_x K(t, \cdot)\|_{L^1(\mathbb{R})} = \frac{C_0}{t^{1/(1+\alpha)}}$.

Relation (5.2.9) follows from the change of variable $\eta = t^{-1/(1+\alpha)}$ under the integral sign. Since K is non-negative we have $\|K(1, \cdot)\|_{L^1(\mathbb{R})} = \int K(1, x) dx = \mathcal{F}(K(1, \cdot))(0) = 1$, which together with (5.2.9) implies (5.2.10). To show (5.2.11) we write $\partial_x^m K(1, x) = \int (ik)^m e^{ikx} e^{-\Lambda(k)} dk$. Since $\alpha > 0$ we can perform integration by parts twice and obtain $\partial_x^m K(1, x) = O(x^2)$. Together with the boundedness of $\partial_x^m K(1, x)$ we get the estimate for $t = 1$. A substitution as above leads to (5.2.11). To see the last relation one just uses (5.2.9) and (5.2.11).

Hence the kernel associated to $\partial_x \mathcal{D}^\alpha$ satisfies the same properties as the one for $D^{\alpha+1}$ required in the work of Droniou et al. [9]. Thus their proof can exactly be carried over to our equation:

Theorem 5.2.1. *Let $u_0 \in L^\infty$. Then there exists a unique solution $u \in L^\infty$ of (5.2.1) for all times, which satisfies the maximum principle*

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \forall t > 0.$$

To motivate the well-posedness we observe that due to the properties of K we can estimate the terms in (5.2.3) for $t > 0$ as follows: $|K(t, \cdot) * u_0(x)| \leq \|u_0\|_\infty$ and

$$\left| \int_0^t \partial_x K(t-s, \cdot) * f(u(s, \cdot)) ds \right| \leq C \|f(u)\|_{L^\infty((0,t) \times \mathbb{R})} t^{1-\frac{1}{1+\alpha}}$$

Due to the Lipschitz continuity of f we get a contraction for small times t_0 on $L^\infty((0, t_0) \times \mathbb{R})$ and therefore the well-posedness.

To show the global existence as well as the maximum principle Droniou et al. [9] constructed an approximate solution by a splitting method and used a compactness argument to pass to the limit.

Moreover it was shown in [9], that under the properties of K given above the differential operator has a regularising effect, such that the solution is smooth for all times $t > 0$.

5.3 Travelling wave solutions

5.3.1 Existence of travelling wave solutions

We introduce the travelling wave variable $\xi = x - st$ with the wave speed s and look for solutions $u(x, t) = u(\xi)$ of (5.1.1), which are connecting the different far-field values u_- and u_+ . A straightforward calculation shows that if u depends on x and t only through the travelling wave variable ξ , then so does $\mathcal{D}^\alpha u$, and we arrive at

$$-su' + f(u)' = (\mathcal{D}^\alpha u)', \quad u(-\infty) = u_-, \quad u(\infty) = u_+,$$

where the prime denotes differentiation with respect to ξ . Integration gives the travelling wave equation

$$h(u) := -s(u - u_-) + f(u) - f(u_-) = \mathcal{D}^\alpha u = d_\alpha \int_0^\infty \frac{u'(\xi - y)}{y^\alpha} dy, \quad (5.3.1)$$

with $d_\alpha = 1/\Gamma(1 - \alpha)$. If u' decays to zero fast enough as $\xi \rightarrow \pm\infty$, we obtain, at least formally, the Rankine-Hugoniot conditions corresponding to shock solutions of the hyperbolic conservation law, relating the far-field values and the wave speed by

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \quad (5.3.2)$$

If the flux function $f(u)$ is convex between the far-field values u_- and u_+ , then the left hand side $h(u)$ of (5.3.1) is negative between its zeroes u_- and u_+ . If $u(\xi)$ is monotone, the right hand side in (5.3.1) has the same sign as u' . Therefore if a monotone solution exists, it has to be nonincreasing, leading to the standard entropy condition

$$u_- > u_+,$$

derived by replacing $\mathcal{D}^\alpha u$ by u' . Under this assumption, the existence of a smooth monotone travelling wave will be proved. The precise assumptions on the flux function will be formulated in terms of $h(u)$: We require

$$\begin{aligned} h \in C^\infty([u_+, u_-]), \quad h(u_+) = h(u_-) = 0, \quad h < 0 \text{ in } (u_+, u_-), \\ \exists u_m \in (u_+, u_-) \text{ such that } h' < 0 \text{ in } (u_+, u_m), \quad h' > 0 \text{ in } (u_m, u_-). \end{aligned} \quad (5.3.3)$$

Note that this is a little less than asking for convexity of f , and it allows for the slightly weakened form $f'(u_+) \leq s < f'(u_-)$ of the Lax entropy condition.

The integral operator

$$\mathcal{D}^\alpha u(\xi) = d_\alpha \int_{-\infty}^\xi \frac{u'(y)}{(\xi - y)^\alpha} dy$$

in the travelling wave problem

$$h(u) = \mathcal{D}^\alpha u, \quad u(-\infty) = u_-, \quad u(\infty) = u_+, \quad (5.3.4)$$

is of the Abel type. It is well known that it can be inverted by multiplying (5.3.4) with $(z - \xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z . This leads to

$$u(\xi) - u_- = \mathcal{D}^{-\alpha}(h(u))(\xi) := d_{1-\alpha} \int_{-\infty}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} dy. \quad (5.3.5)$$

Equations (5.3.4) and (5.3.5) are equivalent if $u \in C_b^1(\mathbb{R})$ and $u' \in L^1(\mathbb{R}_-)$, hence in particular if $u \in C_b^1(\mathbb{R})$ is monotone. We will use both formulations to deduce the existence result. An important property of both integral equations is their translation invariance, which will be used several times below.

The equation (5.3.5) is a nonlinear Volterra integral equation with an integrable kernel, where a well developed theory exists for problems on bounded intervals. Therefore we shall start our investigations by proving a 'local' existence result around $\xi = -\infty$. The subsequent monotonicity and boundedness results will lead to global existence for $\xi \in \mathbb{R}$.

The local existence result is based on linearization at $\xi = -\infty$ (or, equivalently, at $u = u_-$). This can be done for either (5.3.4) or (5.3.5) with the same result. As could be expected for ordinary differential equations, the linearizations

$$h'(u_-)v = \mathcal{D}^\alpha v, \quad v = h'(u_-)\mathcal{D}^{-\alpha}v,$$

have solutions of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where a straightforward computation gives $\lambda = h'(u_-)^{1/\alpha}$. We shall need that these are the only solutions and therefore make the assumption

$$\mathcal{N}(\mathcal{D}^\alpha - h'(u_-)) = \text{span}\{e^{\lambda\xi}\}, \quad \lambda = h'(u_-)^{1/\alpha}, \quad (5.3.6)$$

which seems plausible, see e.g. [6]. The main result of this section is the following.

Theorem 5.3.1. *Let (5.3.3) and (5.3.6) hold. Then there exists a decreasing solution $u \in C_b^1(\mathbb{R})$ of the travelling wave problem (5.3.4). It is unique among all $u \in u_- + H^2((-\infty, 0)) \cap C_b^1(\mathbb{R})$.*

The following local existence result shows that the nonlinear problem has, up to translations, only two nontrivial solutions, which can be approximated by $u_- \pm e^{\lambda\xi}$ for large negative ξ . The choice 1 of the modulus of the coefficient of the exponential is irrelevant due to the translation invariance of the solution.

Lemma 5.3.1. *(Local existence) Let (5.3.6) hold. Then, for every small enough $\varepsilon > 0$, the equation (5.3.5) has solutions $u_{up}, u_{down} \in u_- + H^2(I_\varepsilon)$, $I_\varepsilon = (-\infty, \xi_\varepsilon]$, $\xi_\varepsilon = \log \varepsilon / \lambda$, satisfying*

$$u_{up}(\xi_\varepsilon) = u_- + \varepsilon, \quad u_{down}(\xi_\varepsilon) = u_- - \varepsilon. \quad (5.3.7)$$

These are unique among all functions u satisfying $\|u - u_-\|_{H^2(I_\varepsilon)} \leq \delta$, with δ small enough, but independently from ε . They satisfy (with an ε -independent constant C)

$$\|u_{up} - u_- - e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \leq C\varepsilon^2, \quad \|u_{down} - u_- + e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \leq C\varepsilon^2.$$

Proof. The proof will only be given for existence and uniqueness of u_{down} , which will be of greater interest below, but the proof for u_{up} is analogous.

We start by writing (5.3.4) and the initial condition (5.3.7) in terms of the perturbation $\bar{u}(\xi) = u_{down}(\xi) - u_- + e^{\lambda\xi}$:

$$(\mathcal{D}^\alpha - h'(u_-))\bar{u} = h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u}), \quad \bar{u}(\xi_\varepsilon) = 0. \quad (5.3.8)$$

The idea is to write this as a fixed point problem considering the right hand side as given. Since we shall use the Fourier transform for constructing a particular solution, we need a smooth enough extension to $\xi \in \mathbb{R}$, although we are only interested in $\xi < \xi_\varepsilon$. For $f \in H^2(I_\varepsilon)$, let the extension $\mathcal{E}(f) \in H^2(\mathbb{R})$ satisfy

$$\mathcal{E}(f) \Big|_{I_\varepsilon} = f, \quad \|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \leq \gamma \|f\|_{H^2(I_\varepsilon)}.$$

The bounded solution of the equation

$$(\mathcal{D}^\alpha - h'(u_-))u_{part} = \mathcal{E}(f),$$

and of its derivatives with respect to ξ can be written as

$$u_{part}^{(m)} = \mathcal{F}^{-1} \left[(b_\alpha |k|^\alpha - h'(u_-) + i a_\alpha \operatorname{sgn}(k) |k|^\alpha)^{-1} \mathcal{F} \mathcal{E}(f)^{(m)} \right], \quad m = 0, 1, 2.$$

The coefficient can easily be seen to be bounded uniformly in k , leading to the estimate

$$\|u_{part}\|_{H^2(I_\varepsilon)} \leq \|u_{part}\|_{H^2(\mathbb{R})} \leq C \|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \leq C\gamma \|f\|_{H^2(I_\varepsilon)}.$$

By the assumption (5.3.6), $U[f](\xi) = u_{part}(\xi) - u_{part}(\xi_\varepsilon)e^{\lambda(\xi-\xi_\varepsilon)}$ is the unique solution of

$$(\mathcal{D}^\alpha - h'(u_-))U = f \quad \text{in } I_\varepsilon, \quad U(\xi_\varepsilon) = 0.$$

This allows to write (5.3.8) as a fixed point problem:

$$\bar{u} = U \left[h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u}) \right].$$

The right hand side of (5.3.8) can be written as

$$\frac{h''(\bar{u})}{2} (e^{\lambda\xi} - \bar{u})^2 = \frac{h''(\bar{u})}{2} \left(e^{2\lambda(\xi-\xi_\varepsilon)} - 2e^{\lambda(\xi-\xi_\varepsilon)}\bar{u} + \bar{u}^2 \right).$$

Using the continuous imbedding of $H^2(I_\varepsilon)$ in $C_b(I_\varepsilon)$, it can easily be shown that

$$\begin{aligned} & \left\| h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u}) \right\|_{H^2(I_\varepsilon)} \\ & \leq K(\|\bar{u}\|_{H^2(I_\varepsilon)}) \left(\varepsilon^2 + \varepsilon \|\bar{u}\|_{H^2(I_\varepsilon)} + \|\bar{u}\|_{H^2(I_\varepsilon)}^2 \right), \end{aligned}$$

where K is a positive nondecreasing function. Now it is easily seen that the fixed point map is a contraction in (independently of ε) small enough balls and that it maps a ball with an $O(\varepsilon^2)$ radius into itself. $\square$

Lemma 5.3.2. (*Local monotonicity*) Let the assumptions of Lemma 5.3.1 hold. Then, in I_ε ,

$$u_{up} > u_-, \quad u'_{up} > 0, \quad u_{down} < u_-, \quad u'_{down} < 0.$$

Proof. Again we restrict our attention to u_{down} and skip the analogous proof for u_{up} . As a consequence of Lemma 5.3.1 and of Sobolev imbedding

$$|u_{down}(\xi) - u_- + e^{\lambda\xi}| \leq C\varepsilon^2, \quad \xi \leq \xi_\varepsilon.$$

Thus, there exists ξ^* satisfying

$$u_{down}(\xi^*) = u_- - 2C\varepsilon^2, \quad \xi_{C\varepsilon^2} \leq \xi^* \leq \xi_{3C\varepsilon^2}.$$

Since $u_{down}(\xi) < u_-$ for $\xi \geq \xi^*$, we may restrict our attention in the following to $\xi \leq \xi^*$. Thus, we eliminated a subinterval of length $d_1 \geq \xi_\varepsilon - \xi_{3C\varepsilon^2}$. Now we set $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 2C\varepsilon_1^2$, and, by a shift in ξ , replace ξ^* by ξ_{ε_2} . This means that the shifted solution becomes the unique u_{down} from Lemma 5.3.1, where ε_1 has been replaced by ε_2 . Of course, the argument can be iterated to produce a sequence $\{\varepsilon_n\}$, determined by $\varepsilon_{n+1} = 2C\varepsilon_n^2$, and in each step a subinterval of length $d_n \geq \xi_{\varepsilon_n} - \xi_{3C\varepsilon_n^2}$ can be eliminated, where $u_{down} < u_-$ holds. It is easily seen that, for $\varepsilon_1 = \varepsilon$ small enough, $\sum_{n=1}^\infty d_n = \infty$ completing the proof of $u_{down} < u_-$ in I_ε .

The proof of the second property of u_{down} is completely analogous noting that, again by Sobolev imbedding,

$$|u'_{down}(\xi) + \lambda e^{\lambda\xi}| \leq C\varepsilon^2 \quad \text{for } \xi \leq \xi_\varepsilon.$$

□

Remark 5.3.1. Together with $u_{up} - u_-, u_{down} - u_- \in L^2(I_\varepsilon)$, the result of the lemma implies

$$\lim_{\xi \rightarrow -\infty} u_{up}(\xi) = \lim_{\xi \rightarrow -\infty} u_{down}(\xi) = u_-.$$

Together the two solutions constitute the 'unstable manifold' of the point u_- .

Lemma 5.3.3. (*Continuation principle*) Let $u \in C_b^1((-\infty, \xi_0])$ be a (continuation of a) solution of (5.3.5) as constructed in Lemma 5.3.1. Then there exists a $\delta > 0$, such that it can be extended uniquely to $C_b^1((-\infty, \xi_0 + \delta))$.

Proof. Defining

$$f(\xi) = u_- + d_{1-\alpha} \int_{-\infty}^{\xi_0} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} dy,$$

which can be considered as given and smooth by the assumptions, (5.3.5) can be written as

$$u(\xi) = f(\xi) + d_{1-\alpha} \int_{\xi_0}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} dy.$$

Local existence of a smooth solution for ξ close to ξ_0 is a standard result for Volterra integral equations. □

It is now obvious that, as for ordinary differential equations, boundedness will be enough for global existence.

Lemma 5.3.4. (*Global uniqueness*) Let $u \in u_- + H^2((-\infty, \xi_0))$ be a solution of (5.3.5). Then, up to a shift in ξ , it is the continuation of u_{up} or of u_{down} , or $u \equiv u_-$.

Proof. For every $\delta > 0$ there exists a $\xi^* \leq \xi_0$, such that $\|u - u_-\|_{H^2((-\infty, \xi^*))} < \delta$, and therefore, by Sobolev imbedding, also $|u(\xi^*) - u_-| < \delta$. Choosing δ small enough, there are only the options $u(\xi^*) = u_-$ (implying $u \equiv u_-$) or $u(\xi^*) \neq u_-$ whence, by local uniqueness, u is either equal to u_{up} or to u_{down} , depending on the sign of $u(\xi^*) - u_-$. $\square$

This result already implies the uniqueness of the travelling wave, if it exists.

Lemma 5.3.5. (*Global monotonicity*) Let $u \in C_b^1(-\infty, \xi_0]$ be (a continuation of) the solution u_{down} of (5.3.5) as constructed in Lemma 5.3.1. Then u is nonincreasing.

Proof. We recall the properties of h given in (5.3.3). We shall use both formulations (5.3.4) and (5.3.5). First we prove that the derivative of u remains negative as long as $u \geq u_m$. Assume to the contrary that

$$u(\xi_*) \geq u_m, \quad u'(\xi_*) = 0, \quad u' < 0 \text{ in } (-\infty, \xi_*).$$

Then we obtain from the derivative of (5.3.5), evaluated at $\xi = \xi_*$, the contradiction

$$0 = u'(\xi_*) = d_{1-\alpha} \int_{-\infty}^{\xi_*} \frac{h'(u(y))u'(y)}{(\xi_* - y)^{1-\alpha}} dy < 0.$$

Now we show that u cannot become increasing for $u < u_m$. Again, assume the contrary

$$u(\xi_*) < u_m, \quad u' > 0 \text{ in } (\xi_*, \xi_* + \delta), \quad u' \leq 0 \text{ in } (-\infty, \xi_*],$$

where we assume additionally that δ is small enough such that $u(\xi_* + \delta) < u_m$. This implies

$$\begin{aligned} \int_{-\infty}^{\xi_* + \delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy &= \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy + \int_{\xi_*}^{\xi_* + \delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy \\ &> \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy. \end{aligned}$$

But on the other hand we know

$$0 > h(u(\xi_* + \delta)) - h(u(\xi_*)) = \int_{-\infty}^{\xi_* + \delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy - \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy > 0,$$

leading again to a contradiction. Therefore u' cannot get positive. $\square$

Lemma 5.3.6. (*Boundedness*) Let $u \in C_b^1(-\infty, \xi_0]$ be (a continuation of) the solution u_{down} of (5.3.5) as constructed in Lemma 5.3.1. Then $u_+ < u < u_-$.

Proof. Suppose the solution would reach the value u_+ in finite time, i.e. there exists a ξ_* , such that $u(\xi_*) = u_+$. Since u is nonincreasing and, by Lemma 5.3.2, strictly decreasing at least close to $\xi = -\infty$, we obtain the contradiction

$$0 = h(u_+) = \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy < 0.$$

□

The proof of Theorem 5.3.1 is completed by proving $\lim_{\xi \rightarrow \infty} u(\xi) = u_+$. Assuming to the contrary $\lim_{\xi \rightarrow \infty} u(\xi) > u_+$, would imply $\lim_{\xi \rightarrow \infty} h(u(\xi)) < 0$. Then, however, $-\mathcal{D}^{-\alpha} h(u) = u_- - u$ would increase above all bounds, which is impossible by Lemma 5.3.6.

5.3.2 Asymptotic stability of travelling waves for convex fluxes

We change to the moving coordinate $\xi = x - st$ in (5.1.1),

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u, \quad (5.3.9)$$

and look for solutions of (5.3.9), which are small perturbations of travelling wave solutions and in particular sharing the same far-field values. Let $u_0(\xi)$ be an initial datum and $\phi(\xi)$ a travelling wave solution as constructed in the previous section, with the shift chosen such that

$$\int_{\mathbb{R}} (u_0(\xi) - \phi(\xi)) d\xi = 0. \quad (5.3.10)$$

Due to the conservation property of the equation we see that (formally)

$$\int_{\mathbb{R}} (u(t, \xi) - \phi(\xi)) d\xi = 0, \quad \text{for all } t \geq 0.$$

The flux function will be assumed to be convex between the far-field values of the travelling wave, i.e.

$$f''(\phi(\xi)) \geq 0, \quad \text{for all } \xi \in \mathbb{R}.$$

The perturbation $U = u - \phi$ satisfies the equation

$$\partial_t U + \partial_\xi ((f'(\phi) - s)U) + \frac{1}{2} \partial_\xi (f''(\phi + \vartheta U) U^2) = \partial_\xi \mathcal{D}^\alpha U, \quad (5.3.11)$$

for some $\vartheta \in (0, 1)$. The aim is to show local stability of travelling waves, i.e. the decay of U for small initial perturbations $U_0 = u_0 - \phi$. Testing (5.3.11) with U , we get

$$\frac{1}{2} \frac{d}{dt} \|U\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}} f''(\phi) \phi' U^2 d\xi - \frac{1}{2} \int_{\mathbb{R}} f''(\phi + \vartheta U) U^2 \partial_\xi U d\xi = -a_\alpha \|U\|_{\dot{H}^{(1+\alpha)/2}}^2, \quad (5.3.12)$$

where several integrations by parts have been carried out. Recalling $\phi' \leq 0$, we see that the second term has the unfavourable sign. As one would do for the conservation law with the classical viscous regularisation, we introduce the primitive of the perturbation:

$$W(t, \xi) = \int_{-\infty}^{\xi} U(t, \eta) d\eta, \quad W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta) d\eta.$$

Integration of (5.3.11) gives the equation for W ,

$$\partial_t W + (f'(\phi) - s) \partial_{\xi} W + \frac{1}{2} f''(\phi + \vartheta U) (\partial_{\xi} W)^2 = \partial_{\xi} \mathcal{D}^{\alpha} W, \quad (5.3.13)$$

which we test with W to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}} f''(\phi) \phi' W^2 d\xi + \frac{1}{2} \int_{\mathbb{R}} f''(\phi + \vartheta U) (\partial_{\xi} W)^2 W d\xi \\ = -a_{\alpha} \|W\|_{\dot{H}^{(1+\alpha)/2}}^2. \end{aligned} \quad (5.3.14)$$

This equation has the crucial property that the quadratic terms have the favourable sign. From the cubic term (arising from the nonlinearity) we pull out the L^{∞} -norm of W (and of U if f'' is not constant), which we shall control by Sobolev imbedding.

A contraction argument similar to the one from Theorem 5.2.2 shows that the Cauchy problem for (5.3.13) with $W_0 \in H^2$ has a unique local solution in H^2 . Analogously to Theorem 5.2.1, the case $f(u) = u^2$ together with $\alpha > 1/2$ allows to require less regularity on the initial data. Here the Cauchy problem for (5.3.13) is well posed for an initial datum $W_0 \in H^1$. Global existence will be the consequence of the existence of a Lyapunov functional, which also allows to deduce the asymptotic stability of travelling waves. The Lyapunov functional is also easier to derive in the case of the Burgers flux. Mainly for pedagogical reasons we first derive the result in this simplified situation and then proceed with the stability for the general convex flux function.

Stability of travelling waves for the quadratic flux

Assuming $f(u) = u^2$ and $\alpha > 1/2$, the Cauchy problem for (5.3.13) is well posed in H^1 . Since $f'' = 2$, the nonlinear term in (5.3.12) cancels. Therefore to derive the global existence as well as asymptotic stability it suffices to construct a Lyapunov-functional controlling the H^1 -norm of W .

Theorem 5.3.2. *Let $f(u) = u^2$ and $\alpha > 1/2$. Let ϕ be a travelling wave solution as in Theorem 5.3.1, and let $u_0(\xi)$ be an initial datum for (5.3.9), such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \phi(\eta)) d\eta$ satisfies $W_0 \in H^1$. If $\|W_0\|_{H^1}$ is small enough, equation (5.3.9) with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t \rightarrow \infty} \int_t^{\infty} \|u(\tau, \cdot) - \phi\|_{L^2} d\tau = 0.$$

Remark 5.3.2. Note that the condition (5.3.10), which can be translated to $W_0(\pm\infty) = 0$, is incorporated in the condition $W_0 \in H^1$.

Proof. Equations (5.3.12) and (5.3.14) imply the estimates

$$\frac{1}{2} \frac{d}{dt} \|U\|_{L^2}^2 - C_0 \|U\|_{L^2}^2 \leq -a_\alpha \|U\|_{\dot{H}^{(1+\alpha)/2}}^2, \quad (5.3.15)$$

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 - \|W\|_{L^\infty} \|\partial_\xi W\|_{L^2}^2 \leq -a_\alpha \|W\|_{\dot{H}^{(1+\alpha)/2}}^2, \quad (5.3.16)$$

with $C_0 = \|\phi'\|_{L^\infty}$. We shall construct a Lyapunov functional by a linear combination of these estimates. For $\gamma > 0$, we denote $\gamma_* = \min\{1, \gamma\}$ and $\gamma^* = \max\{1, \gamma\}$. Then

$$J(t) = \frac{1}{2} (\|W\|_{L^2}^2 + \gamma \|U\|_{L^2}^2)$$

is bounded from above and below by

$$\frac{\gamma_*}{2} \|W\|_{H^1}^2 \leq J \leq \frac{\gamma^*}{2} \|W\|_{H^1}^2. \quad (5.3.17)$$

The combined estimate reads

$$\frac{dJ}{dt} - (\gamma C_0 + \|W\|_{L^\infty}) \|W\|_{\dot{H}^1}^2 + a_\alpha (\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma \|W\|_{\dot{H}^{(3+\alpha)/2}}^2) \leq 0.$$

The idea is to control the second term by the third, which seems plausible, since the interpolation inequality

$$\|W\|_{\dot{H}^1}^2 \leq \|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \|W\|_{\dot{H}^{(3+\alpha)/2}}^2, \quad (5.3.18)$$

holds as a consequence of $k^2 \leq |k|^{1+\alpha} + |k|^{3+\alpha}$, $k \in \mathbb{R}$. The same inequality with k replaced by $k(a_\alpha/(2C_0))^{1/(1+\alpha)}$ implies

$$\gamma C_0 \|W\|_{\dot{H}^1}^2 \leq \frac{a_\alpha}{2} (\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma \|W\|_{\dot{H}^{(3+\alpha)/2}}^2),$$

with $\gamma = (a_\alpha/(2C_0))^{2/(1+\alpha)}$. For the term arising from the nonlinearity we use the consequence $\|W\|_{\dot{H}^1}^2 \leq \frac{1}{\gamma_*} (\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma \|W\|_{\dot{H}^{(3+\alpha)/2}}^2)$ of (5.3.18), which leads to

$$\frac{dJ}{dt} + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*} \|W\|_{L^\infty} \right) (\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma \|W\|_{\dot{H}^{(3+\alpha)/2}}^2) \leq 0.$$

By Sobolev imbedding and (5.3.17) we have

$$\|W\|_{L^\infty} \leq \|W\|_{H^1} \leq \sqrt{\frac{2}{\gamma_*} J}.$$

We now let the initial data be small enough such that $J(0) < (\gamma_*)^3 a_\alpha^2/8$. This immediately implies the existence of a $\lambda > 0$, such that

$$\frac{dJ}{dt} \leq -\lambda (\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma \|W\|_{\dot{H}^{(3+\alpha)/2}}^2) \leq -\lambda \gamma_* \|U\|_{L^2}^2, \quad \text{for all } t > 0.$$

Integration with respect to time concludes the proof. $\square$

Stability for a general convex flux function

In contrary to the quadratic flux, now the nonlinearity in estimate (5.3.12) does not cancel:

$$\frac{1}{2} \frac{d}{dt} \|U\|_{L^2}^2 - C_0 \|U\|_{L^2}^2 - K(\|U\|_{L^\infty}) \|U\|_{L^\infty} \|U\|_{H_\xi^1}^2 \leq -a_\alpha \|U\|_{\dot{H}^{(1+\alpha)/2}}^2, \quad (5.3.19)$$

with a positive nondecreasing function K and, similarly to above, $C_0 = \|f''(\phi)\phi'\|_{L^\infty}/2$. The estimate for W reads

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 - K(\|U\|_{L^\infty}) \|W\|_{L^\infty} \|\partial_\xi W\|_{L^2}^2 \leq -a_\alpha \|W\|_{\dot{H}^{(1+\alpha)/2}}^2, \quad (5.3.20)$$

We see that now we have to control U and W in $H^1 \subset L^\infty$, and therefore also need to derive an estimate for $\partial_\xi U$. As we have mentioned above, the Cauchy problem for (5.3.13) is well posed in H^2 . Hence the decay of W in H^2 is needed to repeat the local existence as well as to control the nonlinearities. We differentiate (5.3.11) and test it with $\partial_\xi U$. After several integrations by parts, we can estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_\xi U\|_{L^2}^2 - C_1 \|U\|_{H^1}^2 - K(\|U\|_{L^\infty}) (\|U\|_{L^\infty} \|\partial_\xi U\|_{L^2}^2 + \|\partial_\xi U\|_{L^3}^3) \\ \leq -a_\alpha \|\partial_\xi U\|_{\dot{H}^{(1+\alpha)/2}}^2, \end{aligned} \quad (5.3.21)$$

where C_1 depends on the travelling wave and its derivatives up to order 2. The term $\|\partial_\xi U\|_{L^3}^3$ we bound as in (5.2.7) using the Gagliardo-Nirenberg estimates. We are now ready to prove a stability result similar to Theorem 5.3.2 for the general convex flux function:

Theorem 5.3.3. *Let (5.3.3) hold and let ϕ be a travelling wave solution as in Theorem 5.3.1. Let u_0 be an initial datum for (5.3.9) such that $W_0(\xi) = \int_{-\infty}^{\xi} (u_0(\eta) - \phi(\eta)) d\eta$ satisfies $W_0 \in H^2$. If $\|W_0\|_{H^2}$ is small enough, equation (5.3.9) with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t \rightarrow \infty} \int_t^\infty \|u(\tau, \cdot) - \phi\|_{H^1} d\tau = 0.$$

Proof. We proceed similarly to above and define

$$J(t) = \frac{1}{2} (\|W\|_{L^2}^2 + \gamma_1 \|U\|_{L^2}^2 + \gamma_2 \|\partial_\xi U\|_{L^2}^2),$$

with positive constants $\gamma_1, \gamma_2 > 0$. We denote $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and $\gamma^* = \max\{1, \gamma_1, \gamma_2\}$. Then, as a functional of W , J is equivalent to the square of the H^2 -norm. Combining (5.3.20), (5.3.19) and (5.3.21) together with (5.2.7) gives the complete estimate

$$\begin{aligned} \frac{d}{dt} J + a_\alpha \left(\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot{H}^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot{H}^{(5+\alpha)/2}}^2 \right) \\ - \gamma_1 C_0 \|U\|_{L^2}^2 - \gamma_2 C_1 \|U\|_{H^1}^2 - K(\|W\|_{H^2}) \|W\|_{H^2} \|u\|_{\dot{H}^{(5+\alpha)/4}}^2 \leq 0. \end{aligned}$$

Similarly to above we now choose $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 C_0 \|u\|_{L^2}^2 + \gamma_2 C_1 \|u\|_{H^1}^2 \leq \frac{a_\alpha}{2} \left(\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot{H}^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot{H}^{(5+\alpha)/2}}^2 \right),$$

and get the final estimate

$$\frac{d}{dt} J + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*} K(\|W\|_{H^2}) \|W\|_{H^2} \right) \left(\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \gamma_1 \|W\|_{\dot{H}^{(3+\alpha)/2}}^2 + \gamma_2 \|W\|_{\dot{H}^{(5+\alpha)/2}}^2 \right) \leq 0.$$

Letting again the initial data be such that $J(0)$ is small enough, we can deduce that J is nonincreasing for all times and moreover

$$\int_0^\infty \|U(t, \cdot)\|_{H^1}^2 dt < \infty.$$

□

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