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„Applications of the Grothendieck Spectral Sequence to Group and Lie Algebra (Co)Homology"

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Abstract

Spectral sequences were introduced in 1946 by J. Leray and are important computational tools in homological algebra, where they are used to calculate homology groups. In this thesis, we study diverse construction methods of spectral sequences, namely from exact couples, filtered complexes and double complexes, while also discussing convergence properties respectively. Using this we construct the Grothendieck spectral sequence, with which one can calculate the derived functors of a composition of functors form the derived functors of the components. From it we derive several spectral sequences that were originally introduced by G. Hochschild and J.P. Serre in their study of group and Lie algebra (co)homology.

Zusammenfassung

Spektralsequenzen wurden in 1946 von J. Leray eingeführt und sind wichtige Werkzeuge in der homologischen Algebra, wo sie verwendet werden, um Homologiegruppen zu bestimmen. In dieser Arbeit studieren wir diverse Konstruktionsarten von spektralen Sequenzen, nämlich von exakten Paaren, gefilterten Kettenkomplexen und Doppelkomplexen, während wir auch jeweils Konvergenzeigenschaften diskutieren. Damit konstruieren wir die Grothendieck Spektralsequenz, mit der man die abgeleiteten Funktoren einer Komposition von Funtoren aus den Ableitungen der Komponenten berechnen kann. Daraus leiten wir einige Spektralsequenzen ab, die ursprünglich von G. Hochschild und J.P. Serre eingeführt wurden, um (Ko)Homologien von Gruppen und Lie Algebren zu bestimmen.

Contents

1 Preliminaries

1.1 Diagrams and (Co)Limits

Functors are one of the most fundamental notion of categories with many applications. One of them is the formalisation of the intuitive term "diagram". In this first introductory section we shall recite the definitions of diagrams, limits and some related notions and state a few results about them. A reader unfamiliar with category theory or one who is interested in the proof is hereby referred to [17] or any other introductory book. As usual in category theory, there is a dual version of every notion and statement discussed here and as such we will usually focus on one version and just state the other. This principle is best expressed by Francis Borceux's "metatheorem" in [2]:

Theorem 1.1. *Suppose the validity, in every category, of a statement expressing the existence of some objects or morphisms or the equality of some composites. Then the "dual statement" is also valid in every category; this dual statement is obtained by reversing the direction of every arrow and replacing every composite $f \circ g$ by the composite $g \circ f$.*

Definition. Considering a category $\mathcal{C}$ and a small category Γ, a *diagram* D in $\mathcal{C}$ in the shape of Γ is a functor $\Gamma \to \mathcal{C}$.

Definition. Consider two diagrams D and $\tilde{D}$. A *morphism of diagrams* α is a natural transformation $D \to \tilde{D}$, i.e. is a tuple of morphisms

$$(\alpha_X : DX \to \tilde{D}X)_{X \in \Gamma}$$

such that for every morphism $f : X \to Y$ in Γ the following diagram commutes

$$\begin{array}{ccc} DX & \xrightarrow{Df} & DY \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ \tilde{D}X & \xrightarrow[\tilde{D}f]{} & \tilde{D}Y \end{array}$$

Together with the pointwise composition of these morphisms, we form the category of diagrams C^{Γ}.

Proposition 1.2. *For any two small categories Γ and Λ and a category $\mathcal{C}$ there are isomorphisms of categories*

$$(\mathcal{C}^{\Gamma})^{\Lambda} \cong_{\phi} \mathcal{C}^{\Gamma \times \Lambda} \cong_{\psi} (\mathcal{C}^{\Lambda})^{\Gamma}$$

given by

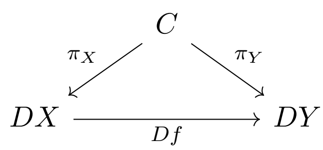
$$\phi^{-1} : D \to \Big(Y \to (X \to D(X,Y))\Big) \quad \textit{and}$$
$$\psi : D \to \Big(X \to (Y \to D(X,Y))\Big)$$

for a diagram $D \in \mathcal{C}^{\Gamma \times \Lambda}$ *and objects* $X \in \Gamma$ *and* $Y \in \Lambda$. *We usually omit these isomorphisms in our notation and write* $D_{X\bullet}$ *instead of* $\psi(D)(X)$ *and* $D_{\bullet Y}$ *instead of* $\phi^{-1}(D)(Y)$.

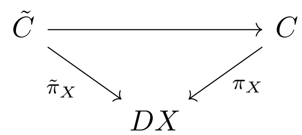
Definition. A *cone* of a diagram $D : \Gamma \to \mathcal{C}$ is an object $C \in \mathcal{C}$ together with a tuple of morphisms

$$\big(\pi_X : C \to DX\big)_{X \in \Gamma}$$

such that for every morphism $f : X \to Y$ in Γ the following triangle commutes.



Definition. A cone $(C, (\pi_X)_{X\in\Gamma})$ of some diagram D is called *universal* or *limit* of D if for every other cone $(\tilde{C}, (\tilde{\pi}_X)_{X\in\Gamma})$ there is a unique morphism $\tilde{C} \to C$ such that the triangle



commutes for all $X \in \Gamma$. In this case we write

$$C = \lim_{X \in \Gamma} DX = \lim_{\Gamma} D$$

This notation focuses on the object $\lim_\Gamma D$ but remember that the morphisms π_X are an integral part of any cone and hence any limit. In the case that the cone is a limit we sometimes call its morphisms *projections*.

Remark. Dually one defines cocones and colimits.

Definition. If Γ is a discrete small category (i.e. a set) then the limits of diagrams in this shape are called *products* and written as

$$\prod_{X \in \Gamma} DX$$

Dually the *colimit* of such diagrams are called coproducts and written as

$$\coprod_{X \in \Gamma} DX$$

Example. If R is an associative ring and Mod_R the category of modules over that ring, then there the product and coproduct are the direct product and the direct sum respectively.

Definition. If Γ is the category

$$\bullet \rightrightarrows \bullet$$

then diagrams in this shape consist of two objects A and B and two parallel morphisms f and g between them. Note that cones of such diagrams can be defined as single morphisms $h : T \to A$ with $fh = gh$. We call the limit of such a diagram the *equalizer* $\mathrm{eq}(f, g)$ and think of the universal cone as a single morphism

$$\mathrm{eq}(f, g) \to A$$

Dually the coequalizer can be thought of as a single morphism

$$B \to \mathrm{coeq}(f, g)$$

Example. In Mod_R the equalizer of two R-linear maps f and g is $\ker(f - g)$ and the coequalizer is $B/\operatorname{im}(f - g)$.

Definition. If Γ is the category

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

then diagrams in this shape consist three objects and two morphism in the configuration

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Limits of such diagrams are called *pullbacks* and are denoted as $A \times_C B$. We think of the universal cone as two morphisms f' and g' with codomain B and A respectively.

$$\begin{array}{ccc} A \times_C B & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Example. In the category Mod_R pullbacks exist and can be described explicitly:

$$A \times_C B = \{(x, y) \in A \oplus B \mid f(x) = g(y)\}$$

Definition. Consider the linear order $\mathbb{Z}$ as a category. Due to functionality it suffices to define a diagram $C \in \mathcal{C}^{\mathbb{Z}}$ only on objects and the covering relations $i+1 \geq i$, i.e. on the directed graph

$$\cdots \longrightarrow i+1 \xrightarrow{\geq} i \xrightarrow{\geq} i-1 \longrightarrow \cdots$$

We call diagrams in this shape *sequences* and denote them as tuples $C = (C_i, d_i)_{i \in \mathbb{Z}}$ or

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots$$

Also the morphisms d_i are called boundary morphisms.

Proposition 1.3. *Limits, if they exist, are unique up to unique isomorphism.*

Proposition 1.4. *If every diagram $D \in C^{\Gamma}$ has a limit then there is an, up to natural isomorphism, unique functor*

$$\lim_{\Gamma} : C^{\Gamma} \to C$$

that sends diagrams to their limits.

Remark. We think of limits as defined only up to isomorphism. Hence if two objects defined as limits of some diagram are isomorphic, we will call them equal.

Definition. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called *left adjoint* if there is a functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism

$$\mathcal{D}(\mathcal{F}(-), -) \cong \mathcal{C}(-, \mathcal{G}(-))$$

Proposition 1.5. *Consider the functor*

$$\Delta : \mathcal{C} \to \mathcal{C}^{\Gamma}$$

that sends objects in $\mathcal{C}$ to the diagrams that are constantly that object. Then this is the left adjoint of the functor lim.

Remark. Dually to lim we can define the functor colim that sends diagrams to their colimits. The above proposition then reads

$$\operatorname*{colim}_{\Gamma} \dashv \Delta \dashv \lim_{\Gamma}$$

This is indeed equivalent to the definitions of the functors lim and colim.

Definition. A category $\mathcal{C}$ is called *complete* if all diagrams $D : \Gamma \to \mathcal{C}$ have limits, for all small categories Γ. A category $\mathcal{C}$ is called *finitely complete* if all diagrams $D : \Gamma \to \mathcal{C}$ have limits, for all finite categories Γ.

Theorem 1.6. *A category is complete if and only if all products and all equalizers exist. Similarly a category is finitely complete if and only if all finite products and all equalizers exist.*

Definition. Consider two categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{F}$ induces a functor $\tilde{\mathcal{F}} : \mathcal{C}^\Gamma \to \mathcal{D}^\Gamma$ by postcomposing, i.e. applying it componentwise to a diagram. We call a functor *continuous* if it preserves all limits, meaning that for every small category Γ there is a natural isomorphism

$$\mathcal{F} \circ \lim_{\Gamma} \cong \lim_{\Gamma} \circ \tilde{\mathcal{F}}$$

Moreover we call a functor *finitely continuous* if it preserves all finite limits.

Lemma 1.7. *Continuous functors preserve monomorphisms.*

Proposition 1.8. *Left adjoint functors are continuous.*

Lemma 1.9. *A functor is continuous if and only if it preserves products and equalizers. Similarly it is finitely continuous if and only if it preserves finite products and equalizers.*

Proposition 1.10. *Consider two small categories Γ and Λ. Then we have a commuting triangle of functors*

$$\begin{array}{ccc} (\mathcal{C}^\Gamma)^\Lambda & \xrightarrow{\ \cong\ } & (\mathcal{C}^\Lambda)^\Gamma \\ & {\scriptstyle \lim_\Lambda} \searrow \quad \swarrow {\scriptstyle \widetilde{\lim}_\Lambda} & \\ & \mathcal{C}^\Gamma & \end{array}$$

where the isomorphism is from Proposition 1.2 and $\tilde{\lim}_\Lambda$ *is the componentwise application of* $\lim_\Lambda : \mathcal{C}^\Lambda \to \mathcal{C}$ *form the definition of continuity above. In fact for any vertex* $X \in \Gamma$ *and a diagram* $D \in \mathcal{C}^{\Gamma\times\Lambda}$ *we have*

$$(\lim_{Y\in\Lambda} D_{\bullet Y})(X) = (\tilde{\lim_{Y\in\Lambda}} D_{\bullet Y})(X) = \lim_{Y\in\Lambda} D_{X\bullet}(Y)$$

This means that we can calculate limits in diagram categories component wise.

Corollary 1.11. *If* $\mathcal{C}$ *is (finitely) complete then so is* $\mathcal{C}^\Gamma$*.*

Corollary 1.12. *The functor* $\lim_\Gamma$ *is continuous for any small category* Γ*. This means that for every small category* Λ *there is a natural isomorphism*

$$\lim_\Gamma \circ \lim_\Lambda \cong \lim_\Lambda \circ \lim_\Gamma$$

1.2 Abelian Categories

In this section we introduce abelian categories and define some basic notions for them. A more detailed discussion can for example be found in [1].

Definition. A category $\mathcal{A}$ is called *preadditive* or *$\mathcal{A}b$-category* if for any three objects $X, Y, Z \in \mathcal{A}$ the set of morphisms $\mathcal{A}(X,Y)$ is equipped with an abelian group structure and the composition

$$\circ : \mathcal{A}(Y,Z) \times \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)$$

is bilinear. Our notation will not distinguish between the operations of these groups as all are written additively. Moreover a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between $\mathcal{A}b$-categories is called *$\mathcal{A}b$-functor* if each map

$$\mathcal{F} : \mathcal{A}(X,Y) \to \mathcal{B}(\mathcal{F}X, \mathcal{F}Y)$$

is a group homomorphism.

Remark. In particular this means that between any two objects there is a distinct morphism called 0 which is the neutral element of the group operation.

Proposition 1.13. *Remember that morphism in diagram categories $\mathcal{A}^\Gamma$ are tupels of morphism in $\mathcal{A}$. If $\mathcal{A}$ is an $\mathcal{A}b$-category, then so is $\mathcal{A}^\Gamma$ with componentwise addition.*

Proposition 1.14. *Let I be a finite set and $\mathcal{A}$ an $\mathcal{A}b$-category. Then, if both sides of the equation exist, there is a natural isomorphism between the coproduct and product functors*

$$\alpha : \coprod_I \to \prod_I$$

such that for any family of objects $X = (X_i)_{i \in I}$ and for any indices j and k the composition

$$X_j \xrightarrow{\iota_k} \coprod_{i \in I} X_i \xrightarrow{\alpha_X} \prod_{i \in I} X_i \xrightarrow{\pi_k} X_k$$

is the identity if $j = k$ and zero otherwise. As discussed in section 1.1 we can in this case think of the product and coproduct as being the same object and denote them both as the direct product

$$\bigoplus_{i \in I} X_i$$

Corollary 1.15. *In every category the empty product is the terminal and the empty coproduct the initial object, if they exist. In an $\mathcal{A}b$-category with finite (co)products there* *is a unique object*

$$0 := \bigoplus_{i \in \emptyset}$$

called zero, which is both initial and terminal. It has the property that zero morphisms $0 : A \to B$ *factor uniquely through it and any morphism factoring through the zero object is already zero.*

Proposition 1.16. *$\mathcal{A}b$-functors preserve finite products and finite coproducts. Conversely, a functor between $\mathcal{A}b$-categories that preserves finite (co)products is an $\mathcal{A}b$-functor.*

Definition. Consider a morphism $f : X \to Y$ in $\mathcal{A}$ and the neutral object 0 of the group operation of $\mathcal{A}(X, Y)$. Then we call the equalizer f and 0 the *kernel* of f. It has the universal property that every morphism $g : Z \to X$ with $f \circ g = 0$ factors uniquely through the kernel.

$$\begin{array}{ccccc} \ker f & \longrightarrow & X & \xrightarrow{f} & Y \\ \uparrow & \nearrow & & & \\ Z & & & & \end{array}$$

Dually the coequalizer of f and 0 is called *cokernel.* If f is monic then we write

$$Y/X := \operatorname{coker}(f)$$

Lemma 1.17. *The kernel and cokernel of the zero morphism* $0 : X \to Y$ *are the identity on* X *and* Y *respectively. Conversely the (co)kernel of an isomorphism is zero.*

Proposition 1.18. *Consider the pullback diagram*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Now if g is monic, then the induced morphism $\ker(f') \to \ker(f)$ *is an isomorphism.*

$$\begin{array}{ccccc} \ker(f') & \longrightarrow & X \times_Z Y & \xrightarrow{f'} & Y \\ {\scriptstyle \cong}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ \ker(f) & \longrightarrow & X & \xrightarrow[f]{} & Z \end{array}$$

Definition. We call functors that preserve kernels *left exact.* Dually functors are called *right exact* if they preserve cokernels and *exact* if they are both left and right exact.

Proposition 1.19. *Left exact $\mathcal{A}b$-functors are finitely continuous.*

Proof. This follows from Corollary 1.9, Proposition 1.16 and the fact that for any two morphisms $f, g : X \to Y$ we have

$$\mathrm{eq}(f, g) = \ker(f - g)$$

$\square$

Definition. Since kernels are thought of simply as morphisms we can take their cokernels to define

$$\begin{aligned} \mathrm{im}(f) &:= \ker(\mathrm{coker}(f)) \quad \text{and} \\ \mathrm{coim}(f) &:= \mathrm{coker}(\ker(f)) \end{aligned}$$

Definition. An $\mathcal{A}b$-category $\mathcal{A}$ is called *abelian* if it is both finitely complete and finitely cocomplete, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel

Example. If R is an associative ring then the category Mod_R is abelian. The usual kernel there is also the categorical kernel and the cokernel of a morphism $f : X \to Y$ is $Y/\mathrm{im}(f)$.

Lemma 1.20. *(Homomorphism Theorem) In abelian categories for every morphism $f : X \to Y$ there is a unique natural isomorphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow & & \uparrow \\ \mathrm{coim}(f) & \xrightarrow[\cong]{} & \mathrm{im}(f) \end{array}$$

As such, we think of the image and coimage as being equal.

Lemma 1.21. *If $\mathcal{A}$ is abelian then so is $\mathcal{A}^\Gamma$ for any small category Γ.*

Proof. Firstly, Corollary 1.11 ensures that $\mathcal{A}^\Gamma$ is both finitely complete and finitely cocomplete. Secondly, note that natural transformations are monic if and only if each of their components is monic. Hence if $\alpha : D \to \tilde{D}$ is a monomorphism of diagrams, then α_X is also monic for every $X \in \Gamma$ and therefore the kernel of its cokernel. But according to Proposition 1.10 (co)limits are calculated component wise hence we have

$$\ker(\mathrm{coker}(\alpha))_X = \ker(\mathrm{coker}(\alpha)_X) = \ker(\mathrm{coker}(\alpha_X)) = \alpha_X$$

□

Definition. Let Γ be directed set, i.e. a preorder where every pair of elements has an upper bound, viewed as a category. Then (co)limits of diagrams $\Gamma \to \mathcal{C}$ are called *filtered*. Dually (co)limits of diagrams $\Gamma^{op} \to \mathcal{C}$ are called *cofiltered*.

Example. Every total order is directed, so in particular (co)limits of sequences $\mathbb{Z} \to \mathcal{C}$ are filtered.

Definition. An abelian category $\mathcal{A}$ is called *AB5* if it is cocomplete and taking filtered colimits is exact. Due to Proposition 1.19 this is equivalent to saying that for any directed set Γ and any finite category Λ we have

$$\operatorname*{colim}_{\Gamma} \circ \lim_{\Lambda} = \lim_{\Lambda} \circ \operatorname*{colim}_{\Gamma}$$

Example. The category Mod_R is AB5 for any associative ring R.

1.3 Chain Complexes and Derived Functors

Continuing from the previous section, this section introduces the basic notions of homological algebra.

Definition. Let $\mathcal{A}$ be an abelian category and $C = (C_i, d_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ a sequence

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots \tag{1.1}$$

We call C a *chain complex* if $d_i \circ d_{i+1} = 0$ for all integers i. Note that the chain complexes in $\mathcal{A}$ together with morphisms of diagrams define a full subcategory of the category of sequences $\mathcal{A}^{\mathbb{Z}}$. We call it the category of chain complexes $\mathrm{Ch}(\mathcal{A})$.

Proposition 1.22. *If $\mathcal{A}$ is an abelian category, then the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes is also abelian.*

Proof. Similar to $\mathcal{A}^{\mathbb{Z}}$, $\mathrm{Ch}(\mathcal{A})$ is an $\mathcal{A}b$-category using componentwise addition. Also as $\mathcal{A}^{\mathbb{Z}}$ is abelian, we show that (co)limits there are chain complexes. Indeed for two chain complexes $C = (C_i, d_i)$ and $C' = (C'_i, d'_i)$ we have

$$(d_i \oplus d'_i) \circ (d_{i+1} \oplus d'_{i+1}) = (d_i \circ d_{i+1}) \oplus (d'_i \circ d'_{i+1}) = 0.$$

and therefore $C \oplus C'$ is a chain complex. Further for a morphism of sequences $f : C \to C'$, its kernel in $\mathcal{A}^{\mathbb{Z}}$ is a sequence, whose boundary morphisms $\tilde{d}_i : \ker(f_i) \to \ker(f_{i-1})$ fit into

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \ker(f_{i+1}) & \xrightarrow{\tilde{d}_{i+1}} & \ker(f_i) & \xrightarrow{\tilde{d}_i} & \ker(f_{i-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota} & & \\
\cdots & \longrightarrow & C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \longrightarrow & C'_{i+1} & \xrightarrow{d'_{i+1}} & C'_i & \xrightarrow{d'_i} & C'_{i-1} & \longrightarrow & \cdots
\end{array}$$

and hence $\tilde{d}_{i-1} \circ \tilde{d}_i = 0$ because C is a chain complex and ι is monic. The dual argument shows that also the cokernel in $\mathcal{A}^{\mathbb{Z}}$ of chain complexes is a chain complex. Finally, we need to check that $\mathrm{Ch}(\mathcal{A})$ does not have more mono- and epimorphisms than $\mathcal{A}^{\mathbb{Z}}$. But we have just shown that the inclusion functor $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}^{\Gamma}$ is exact and therefore preserves epi- and monomorphisms, see Proposition 1.19 and Lemma 1.7. □

Proposition 1.23. *If $C = (C_i, d_i) \in \mathrm{Ch}(\mathcal{A})$ is a chain complex, then for any integer i*

there is a unique natural monomorphism $\operatorname{im}(d_{i+1}) \to \ker(d_i)$ *that fits into the diagram*

$$\begin{array}{ccccc} C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} \\ & \nearrow & & \nwarrow & \\ \operatorname{im}(d_{i+1}) & & \hookrightarrow & & \ker(d_i) \end{array}$$

We then define the i*-th homology of* C *to be*

$$H_i C := \ker(d_i)/\operatorname{im}(d_{i+1})$$

and note that this by definition extends to a functor $H_i : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$*. Also if we do not fix the index* i *we get the total homology functor* $H : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}^{\mathcal{U}\mathbb{Z}}$ *where* $\mathcal{U}\mathbb{Z}$ *is set of integers, i.e. a discrete category.*

Remark. The dual of the above proposition also holds, giving us an epimorphism $\operatorname{coker}(d_i) \to \operatorname{coim}(d_{i+1})$ whose kernel is also $H_i C$.

Definition. A morphism of chain complexes $f : C \to C'$ is said to be a *quasi isomorphism* if its image under the homology functor is an isomorphism

$$Hf : HC \to HC'$$

Definition. We call a sequence $C \in \mathcal{A}^{\mathbb{Z}}$ *exact* if it is a chain complex with

$$H_i C = 0 \quad \text{for all } i \in \mathbb{Z}$$

Lemma 1.24. *Consider an exact sequence* C *whith* $C_i = C_{i+3} = 0$ *for some integer* i*. Then* $d_{i+2} : C_{i+2} \to C_{i+1}$ *is an isomorphism.*

$$0 \longrightarrow C_{i+2} \xrightarrow{\cong} C_{i+1} \longrightarrow 0$$

Definition. An exact sequence is called *short*, if it there are only three neighbouring nonzero entries.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

In this case we have $f = \ker(g)$ and $g = \operatorname{coker}(f)$. Moreover a short exact sequence is called *split* if it is isomorphic (in the category of sequences) to a short exact sequence where the middle object is the direct product of the outer two

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0$$

Definition. Let $\mathcal{A}$ be and abelian category and $P \in \mathcal{A}$ be an object. P is said to be *projective* if the hom-functor $\mathcal{A}(P, -) : \mathcal{A} \to \mathcal{A}b$ is exact. Moreover we say that $\mathcal{A}$ has *enough projectives* if for every object $A \in \mathcal{A}$ there is a projective object $P \in \mathcal{A}$ and an epimorphism $P \to A$. The dual notion of projective is *injective*.

Proposition 1.25. *The coproduct of two projective objects is also projective.*

Proposition 1.26. *Let P be a projective object, then any short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

Proposition 1.27. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories that is left adjoint to an exact functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$. If $P \in \mathcal{A}$ is projective, then so is $\mathcal{F}P$.*

Proof. We calculate that $\mathcal{B}(\mathcal{F}P, -)$ is the composition of two exact functors and it is therefore exact.

$$\mathcal{B}(\mathcal{F}P, -) \cong \mathcal{A}(P, \mathcal{G}(-)) = \mathcal{A}(P, -) \circ \mathcal{G}$$

□

Proposition 1.28. *Let I be any set and $\mathcal{A}$ a complete abelian category with enough projectives. Then the product functor*

$$\prod_I : \mathcal{A}^I \to \mathcal{A}$$

is exact.

Proposition 1.29. *We can think of every object $A \in \mathcal{A}$ as a chain complex $\tilde{A}$ by setting*

$$\tilde{A}_i = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now if $\mathcal{A}$ has enough projectives then there is an exact sequence P where all P_i are zero for negative i and projective if i is not negative.

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

Moreover there is a quasi isomorphism of chain complexes $f : P \to \tilde{A}$ which we can think *of as a single morphism $P_0 \to A$.*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

In particular we know the homology of P, namely

$$H_i P = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

We call P a projective resolution of A.

Theorem 1.30. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a right exact $\mathcal{A}b$-functor between abelian categories where $\mathcal{A}$ has enough projectives. For any $A \in \mathcal{A}$ and a projective resolution P of A we set*

$$L_i\mathcal{F}(A) := H_i(\mathcal{F}P)$$

then this does, up to isomorphism, not depend on the choice of P. Moreover we can extend all $L_i\mathcal{F}$ to functors $\mathcal{A} \to \mathcal{B}$, which we call the i-th left derived functor of $\mathcal{F}$.

Corollary 1.31. *The zeroth derived functor of $\mathcal{F}$ is $\mathcal{F}$, so $L_i\mathcal{F}$ is only interesting for positive indices i.*

Corollary 1.32. *If $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is right exact and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ is exact, then*

$$L_i(\mathcal{G}\mathcal{F}) = \mathcal{G} \circ L_i\mathcal{F}$$

1.4 Diagram Lemmas

This short section is dedicated to presenting some theorems about larger diagrams that are exact at some points. The snake lemma, being the first here, is essential in the proves of the other three.

Proposition 1.33. *(Snake Lemma) Consider and abelian category $\mathcal{A}$ and the following diagram in $\mathcal{A}$*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

where the rows are exact and all squares commute. Then there is another exact sequence

$$\begin{array}{ccccccc} \ker(f) & \longrightarrow & \ker(g) & \longrightarrow & \ker(h) & & \\ & & & \swarrow & & & \\ & & \operatorname{coker}(f) & \longrightarrow & \operatorname{coker}(g) & \longrightarrow & \operatorname{coker}(h) \end{array}$$

Proposition 1.34. *Since the category of chain complexes in an abelian category $\mathcal{A}$ is again abelian, we can consider short exact sequences in $\mathrm{Ch}(\mathcal{A})$.*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

meaning that all of A, B and C are exact chain complexes and f and g are morphisms of sequences. Now there is another exact sequence

$$\cdots \longrightarrow H_iA \xrightarrow{H_if} H_iB \xrightarrow{H_ig} H_iC \longrightarrow H_{i-1}A \xrightarrow{H_{i-1}f} \cdots$$

between the homologies of the complexes.

Proposition 1.35. *(Nine Lemma) Consider the following commutative diagram, again* *in any abelian category.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

If all rows and the first two columns are exact, then so is the third column.

Proposition 1.36. *(Horseshoe Lemma) Now let $\mathcal{A}$ be abelian with enough projectives and consider the short exact sequence*

$$0 \longrightarrow A'' \longrightarrow A \longrightarrow A' \longrightarrow 0$$

If P'' and P' are projective resolutions of A'' and A' respectively then there is an epimorphism

$$P_0'' \oplus P_0' \to A$$

which makes $P'' \oplus P'$ into a projective resolution of A, in such a way that the following diagram commutes and of course all rows and columns are exact.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1' & \longrightarrow & P_0' & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1'' \oplus P_1' & \longrightarrow & P_0'' \oplus P_0' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

1.5 Graded and Filtered Objects

We end the introductory chapter with a short section about graded and filtered objects, as they will be needed in the construction of spectral sequences in Chapter 2.

Definition. Let G be an abelian group, $\mathcal{U}G$ its underlying set and $\mathcal{C}$ a category. We call objects $C \in \mathcal{C}$ *graded* by G if they are of the form

$$C = \prod_{g \in G} C_g$$

Definition. Then any $g_0 \in G$ defines an endofunctor on the discrete category $\mathcal{U}G$ using the groups addition

$$\tilde{g}_0 : \mathcal{U}G \to \mathcal{U}G, \quad g \to g + g_0$$

and we define the *index shifting functor* $[g_0] : \mathcal{C}^{\mathcal{U}G} \to \mathcal{C}^{\mathcal{U}G}$ as the precomposition with $\tilde{g}_0$. Concretely for a tupel $C = (C_g)_{g \in G}$ we have

$$[g_0]C = (C_{g+g_0})_{g \in G}$$

Notice that, as $[g_0]$ is invertible, it is both continuous and cocontinuous, hence it makes sense to apply it to graded objects. A tupel of morphisms

$$f : C \to [g_0]C' \quad f = (f_g : C_g \to C'_{g+g_0})_{g \in G}$$

is said be a morphism of *degree* g_0.

Lemma 1.37. *Taking the product of a tupel $C = (C_g)_{g \in G}$ is invariant under the index shifting functor. In other words we have*

$$\prod_{g \in G} C_g = \prod_{g \in G} C_{g+g_0}$$

Definition. Let $C = \prod C_g$ and $C' = \prod C'_g$ be graded objects. Now for a tupel of morphisms $(f_g : C_g \to C'_{g+g_0})$, we say that its product

$$\prod_{g \in G} f_g : \prod_{g \in G} C_g \to \prod_{g \in G} C'_g$$

is a *graded morphism* of degree g_0.

Remark. Since in an abelian category taking kernels commutes with taking products, the kernel of a graded morphism will again be a graded object. Since the kernel is defined only up to isomorphism we will always assume that for $f : A \to B$ the degree of the morphism $\ker f \to A$ will be 0. If in the category taking products is also right exact, then we will assume the same for cokernels.

Definition. Let $\mathcal{A}$ be a cocomplete abelian category and $A \in \mathcal{A}$ an object. Then a *filtration* FA of A is a sequence

$$\cdots \hookrightarrow F_{i-1}A \hookrightarrow F_iA \hookrightarrow F_{i+1}A \hookrightarrow \cdots$$

where every morphism is monic,$A = \operatorname{colim}_{i\in\mathbb{Z}} F_iA$ is the colimit of the sequence and the limit $\lim_{i\in\mathbb{Z}} F_iA$ is the zero object. The object A together with a filtration is called a *filtered object.* Moreover a filtration is called *constant* if all morphisms are isomorphism and called *finite* if for small indices it is constantly zero and for large indices it is constantly A.

$$0 = F_iA \hookrightarrow F_{i+1}A \hookrightarrow \cdots \hookrightarrow F_{j-1}A \hookrightarrow F_jA = A$$

Example. The Prüfer group $\mathbb{Z}(p^\infty) = \{c \in \mathbb{C} \mid c^{(p^n)} = 1 \text{ for some } n \geq 0\}$ for some prime number p comes with a canonical filtration

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \cdots \hookrightarrow Z/p^i \hookrightarrow Z/p^{i+1} \hookrightarrow \cdots$$

where every morphism is multiplication by p.

Definition. Every chain complex $C \in \mathrm{Ch}(\mathcal{A})$ has a *canonical filtration* FC by setting some of its points to be zero, namely we define the chain complex F_iC as

$$(F_iC)_j := \begin{cases} C_j & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases} \tag{1.2}$$

with the boundary morphism being either from C or 0. The inclusions $F_iC \hookrightarrow F_{i+1}C$ are natural transformations which are at every point either 0 or the identity.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C_i & \longrightarrow & C_{i-1} & \longrightarrow & \cdots \\ & & \big\downarrow 0 & & \big\downarrow 0 & & \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & C_{i+1} & \longrightarrow & C_i & \longrightarrow & C_{i-1} & \longrightarrow & \cdots \end{array} \tag{1.3}$$

Note that we cannot reverse the inequalities in (1.2) as not all arrows in (1.3) would remain monic.

Definition. For a filtered object FA we define the *associated graded object*

$$GA := (G_i A)_{i\in\mathbb{Z}} \quad \text{where} \quad G_i A := F_i A / F_{i-1} A$$

Proposition 1.38. *A filtration is FA of A is finite if and only if for the associated graded object GA we have*

$$G_i A = 0$$

for all but finitely many indices i.

Proposition 1.39. *If $C \in \mathrm{Ch}(\mathcal{A})$ is a chain complex, FC its canonical filtration and GC the associated graded object, then*

$$(G_i C)_j := \begin{cases} C_j & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

2 General Spectral Sequences

2.1 General Notions

We start the discussion on spectral sequences with their abstract definition and a few notions of about their convergence.

Definition. Let $\mathcal{A}$ be an abelian category. A *spectral sequence* $E = (E^n, d^n)_{n\in\mathbb{N}}$ is a series of endomorphisms

$$d^n : E^n \to E^n$$

in $\mathcal{A}$, such that

- $(d^n)^2 = 0$ for all n, i.e each pair (E^n, d^n) is a chain
- $E^{n+1} = \ker d^n / \operatorname{im} d^n$

We call the chain (E^n, d^n) the n-th page.

Lemma 2.1. *Let $\mathcal{A}$ be complete and have enough projectives and let G be an abelian group. If the starting page of a spectral sequence in $\mathcal{A}$ has the form*

$$E^0 = \prod_{g\in G} E^0_g$$

and the boundary morphism $d^0 = \prod d^0_g$ is of degree $g_0 \in G$ then the first page of the spectral sequence is

$$E^1 = \prod_{g\in G} \ker(d^0_g) / \operatorname{im}(d^0_{g-g_0})$$

One can inductively craft a graded structure on all pages of the spectral sequence, assuming that all d^n are graded. If this is the case we call the spectral sequence graded by G.

Proof. By Proposition 1.28 the product functor $\prod : \mathcal{A}^{\mathcal{U}G} \to \mathcal{A}$ is exact, so it preserves homology. $\square$

One is interested in how the pages of a spectral sequence behave for large n. To be concrete, we need notions for convergence for spectral sequences, the simplest being the following.

Definition. A spectral sequence E is said to *terminate* at the n-th page if the boundary map d^m at the m-th page is zero for all $m \geq n$. This immediately implies $E^m = E^n$.

When one wants to do explicit calculations using spectral sequences, it's important for them to terminate early, preferably at the first or second page. In general however, this will not be the case so we need more general notions of convergence. In order to define the next one, we need the following lemma.

Lemma 2.2. *For every spectral sequence E there are two filtrations*

$$0 \hookrightarrow B^0 \hookrightarrow B^1 \hookrightarrow \cdots \quad \textit{and} \quad \cdots \hookrightarrow Z^1 \hookrightarrow Z^0 \hookrightarrow E^0$$

such that $B^n \hookrightarrow Z^n$ and $E^{n+1} = Z^n/B^n$ for all non negative integers n. We call the B^n the n-almost boundaries and Z^n the n-almost cycles.

Proof. We start by setting Z^0 and B^0 to be the kernel and image of d^0 respectively. Now define Z^n and B^n inductively as the pullbacks

$$Z^n := Z^{n-1} \times_{E^n} \ker(d^n) \quad \text{and} \quad B^n := Z^n \times_{\ker(d^n)} \operatorname{im}(d^n)$$

which fit into the following pullback diagram

$$\begin{array}{ccccc} B^n & \hookrightarrow & Z^n & \hookrightarrow & Z^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{im}(d^n) & \hookrightarrow & \ker(d^n) & \hookrightarrow & E^n \end{array} \tag{2.1}$$

It remains to show that $E^{n+1} = Z^n/B^n$ and for that we may assume that $E^n = Z^{n-1}/B^{n-1}$, i.e. B^{n-1} is the kernel of the right most vertical arrow in 2.1. But by Proposition 1.18, B^{n-1} must be the kernel of all three downwards facing arrows in (2.1). We arrive at the commuting diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^{n-1} & \xrightarrow{\text{id}} & B^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^n & \longrightarrow & Z^n & \longrightarrow & Z^n/B^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im}(d^n) & \longrightarrow & \ker(d^n) & \longrightarrow & E^{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We just discussed that the first two columns are exact and also all rows are exact by their respective definitions. Hence by the Nine Lemma 1.35 the last column is also exact which implies $E^{n+1} = Z^n/B^n$ by Lemma 1.24. □

Corollary 2.3. *If $\mathcal{A}$ is complete, products are exact and the spectral sequence E is graded, then its n-almost cycles and boundaries are also graded, meaning that they can be written as*

$$B^n = \prod_{g \in G} B^n_g \qquad \text{and}$$
$$Z^n = \prod_{g \in G} Z^n_g$$

Proof. The product functor is continuous and exact (Propositions 1.10 and 1.28) meaning that is preserves pullbacks, kernels and images, hence the constructions of the almost cycles and boundaries can be done component wise. □

Definition. Let E be a spectral sequence and Z^n and B^n the corresponding n-almost cycles and boundaries, then we set

$$B^\infty := \operatorname{colim}_n B^n \qquad \text{and} \qquad Z^\infty := \lim_n Z^n$$

One can show that there is a canonical monomorphism $B^\infty \to Z^\infty$, so we define the *limit page* of E as

$$E^\infty := Z^\infty / B^\infty$$

Proposition 2.4. *If a spectral sequence E terminates at the n-th page, then the limit page exists and $E^\infty = E^n$.*

Proof. For any $m \geq n$ we have $d^m = 0$, so the the pullback diagram (2.1) yields

$$Z^m = Z^{m-1} \times_{E^m} E^m = Z^{m-1}$$
$$B^m = Z^{m-1} \times_{E^m} 0 = \ker(Z^{m-1} \to E^m) = B^{m-1}$$

In particular, the filtrations Z and B are finite, hence their (co)limits exist and are Z^{n-1} and B^{n-1} respectively. So we have

$$E^\infty = Z^\infty / B^\infty = Z^{n-1}/B^{n-1} = E^n$$

□

Definition. Let $\mathcal{A}$ be complete with enough projectives and let E be a spectral sequence in $\mathcal{A}$ graded by some group G. We say that E *abuts* if for every index $g \in G$ there is some natural number n_g such that $Z_g^n = Z_g^{n_g}$ and $B_g^n = B_g^{n_g}$ whenever $n \geq n_g$. Of course every terminating spectral sequence abuts.

Corollary 2.5. *If $\mathcal{A}$ is $AB5$, i.e. a cocomplete abelian category where taking filtered colimits is exact, and the spectral sequence E is graded and abuts, then its limit page E^∞ exists, is graded by some group G and for every index $g \in G$ there is some number n_g such that*

$$E_g^\infty = E_g^{n_g}$$

Proof. By assumption, taking the (co)limits of the filtrations commutes with the product and the product is also exact, so the proof of 2.4 can be done component wise. □

Definition. Let E be a spectral sequence graded by $\mathbb{Z}^2$ and H an object graded by $\mathbb{Z}$. Then E is said to *converge* to H if there is a filtration FH of H such that

$$E_{ij}^\infty = G_j H_i$$

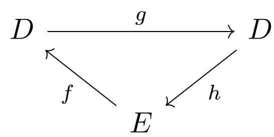
for all indices $i, j \in \mathbb{Z}$, where GH is the graded object associated to FH. We write this as

$$E_{ij}^n \implies H_i$$

2.2 Spectral Sequences of Exact Couples

There are several ways of constructing spectral sequences. We start very general, using exact couples.

Definition. An *exact couple* is a diagram



in some abelian category $\mathcal{A}$, which is exact at every vertex, i.e.

$$\ker g = \operatorname{im} f \quad \ker h = \operatorname{im} g \quad \ker f = \operatorname{im} h$$

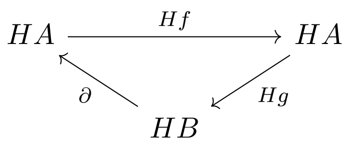
Example. Consider a short exact sequence of chain complexes

$$0 \longrightarrow A \xrightarrow{f} A \xrightarrow{g} B \longrightarrow 0$$

in an abelian category where products are exact. Then Proposition 1.34 then gives us a long exact sequence

$$\cdots \longrightarrow H_iA \xrightarrow{H_if} H_iA \xrightarrow{H_ig} H_iB \xrightarrow{\partial_i} H_{i-1}A \longrightarrow \cdots$$

between the homologies of the chain complexes which describes an exact couple in total homology



Construction 2.6. (*spectral sequence of an exact couple*) We start out our spectral sequence with $E^1 := E$ and $d^1 := h \circ f$. Note that $fh = 0$ and therefore also $d^1 \circ d^1 = hfhf = 0$, hence d^1 indeed defines a chain. Now derive the first page by constructing another exact couple. Let

$$D^2 := \operatorname{im} g \quad \text{and}$$
$$E^2 := H(E^0, d^0) = \ker(hf)/\operatorname{im}(hf)$$

To construct the morphisms we first note that $h \circ f|_{\ker(hf)}$ and $f|_{\operatorname{im}(hf)}$ are both zero and hence f factors to

$$f^2 : \ker(hf)/\operatorname{im}(hf) \to \ker h = \operatorname{im} g \tag{2.2}$$

Next we restrict g to its image.

$$g^2 : \operatorname{im} g \to \operatorname{im} g \tag{2.3}$$

Finally note that h has values in $\ker(hf)$, hence we can define

$$h^2 : \operatorname{im} g \xrightarrow{g^{-1}} D/\operatorname{im} f \xrightarrow{h} \ker(hf)/\operatorname{im}(hf) \tag{2.4}$$

since g induces an isomorphism $D/\ker g \to \operatorname{im} g$. One easily checks that these three morphisms define another exact couple

$$\begin{array}{ccc} \operatorname{im} g & \xrightarrow{\quad g^2 \quad} & \operatorname{im} g \\ & {}_{f^2}\nwarrow \qquad \swarrow_{h^2} & \\ & H(E^1, d^1) & \end{array}$$

This procedure is iterated to construct the entire spectral sequence.

Proposition 2.7. *Let G be an abelian group and assume that the product functor $\prod_{\mathcal{U}G}$ is exact. If the three objects and the three morphisms of an exact couple are graded by G, then so are the objects and morphisms of its derived exact couple. Induction also shows that the resulting spectral sequence is also graded. Moreover the morphisms of the n-th derived exact couple have the following degrees:*

$$\begin{aligned} \deg f^n &= \deg f^1 \\ \deg g^n &= \deg g^1 \\ \deg h^n &= \deg h^1 - (n-1)\deg g^1 \\ \deg d^n &= \deg f^n + \deg h^n \end{aligned}$$

Proof. Derived exact couples are defined using only kernels and cokernels, all of which commute with taking products so the derived exact couples are also graded. From constructions of the derived morphisms (2.2), (2.3) and (2.4) we see that their degrees are

$$\deg f^{n+1} = \deg f^n \quad \deg g^{n+1} = \deg g^n \quad \deg h^{n+1} = \deg h^n - \deg g^n$$

The proposition follows via induction. □

2.3 Spectral Sequences of Filtered Complexes

From now on let $\mathcal{A}$ be a complete $AB5$-category with enough projectives. This ensures that taking products is finitely cocontinuous and taking filtered colimits is continuous.

Definition. Consider a chain complex $C \in \mathrm{Ch}(\mathcal{A})$, then we call a filtration FC of C a *filtered complex*. Note that every filtration step F_jC is a chain complex and we call its i-th object F_jC_i, resulting in the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \hookrightarrow & F_{j-1}C_{i+1} & \hookrightarrow & F_jC_{i+1} & \hookrightarrow & F_{j+1}C_{i+1} & \hookrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \hookrightarrow & F_{j-1}C_{i} & \hookrightarrow & F_jC_{i} & \hookrightarrow & F_{j+1}C_{i} & \hookrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \hookrightarrow & F_{j-1}C_{i-1} & \hookrightarrow & F_jC_{i-1} & \hookrightarrow & F_{j+1}C_{i-1} & \hookrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{2.5}$$

Here all horizontal morphisms are monic, i.e the rows are filtrations of the objects C_i in $\mathcal{A}$ and we denote them as FC_i

Remark. As in section 1.5, for a filtered complex FC, we can define the associated graded object

$$GC := \prod_{j\in\mathbb{Z}} G_j \quad \text{where} \quad G_jC := F_jC/F_{j-1}C$$

This is obviously a chain complex in $\mathcal{A}$, as FC is a filtered object in $\mathrm{Ch}(\mathcal{A})$. We denote the i-th object in this chain as G_jC_i.

Construction 2.8. (*spectral sequence of a filtered complex*) We construct a spectral sequence from a filtered complex by reducing to the case of exact couples. At first start with a filtered complex FC and consider the graded object GC. Choose this to be zeroth page E^0 with the boundary map d^0 just being the sum of boundary maps in the chains G_jC. This gives $E^1_{ij} = H_i(G_jC)$ for the first page, from which we construct an exact couple by setting

$$D^1 := \prod_{i,j\in\mathbb{Z}} H_i(F_jC) \quad \text{and} \quad E^1 := \prod_{i,j\in\mathbb{Z}} H_i(G_jC)$$

To define the morphisms of the exact couple we fix an index j and look at the short exact sequence of chain complexes

$$0 \longrightarrow F_{j-1}C \longrightarrow F_jC \longrightarrow G_jC \longrightarrow 0$$

from which Proposition 1.34 gives us a long exact sequence

$$\cdots \longrightarrow H_i(F_{j-1}C) \xrightarrow{\iota_{i,j-1}} H_i(F_jC) \xrightarrow{\pi_{ij}} H_i(G_jC) \xrightarrow{\partial_{ij}} H_{i-1}(F_{j-1}C) \longrightarrow \cdots$$

Then we use these morphisms and the fact that both D^1 and E^1 are graded by $\mathbb{Z}^2$ to define the exact couple

$$\begin{array}{ccc} D^1 & \xrightarrow{\quad\iota\quad} & D^1 \\ & {}_{\partial}\nwarrow \quad \swarrow_{\pi} & \\ & E^1 & \end{array}$$

Now apply construction 2.6 to construct the rest of the spectral sequence.

Proposition 2.9. *The pages E^n of the spectral sequence constructed from a filtered complex are graded by $\mathbb{Z}^2$ and the boundary morphisms d^n have degree $(-1,-n)$. In other words we can write*

$$d^n = \prod_{i,j\in\mathbb{Z}} d^n_{ij} \qquad \text{where} \qquad d^n_{ij} : E^n_{ij} \to E^n_{i-1,j-n}$$

Proof. For the zeroth page it is obvious that d^0 has degree $(-1,0)$. In construction 2.8 we saw that ι, π and ∂ evidently have degrees $(0,1)$, $(0,0)$ and $(-1,-1)$ respectively and therefore $d^1 = \pi \circ \partial$ has degree $(-1,-1)$. The rest follows from Proposition 2.7. $\square$

Since the pages of the spectral sequence induced by a filtered complex are graded by the index set $\mathbb{Z}^2$, we can talk about their convergence. This section contains two important theorems. The first gives an easy but sufficient condition for abutting, while the second will allows us to calculate the limit page of an abutting spectral sequence induced by a filtered complex.

Definition. Consider a filtered complex FC and remember the diagram (2.5) where the columns FC_i are filtrations of C_i for each index i. We call FC *bounded*, if FC_i is finite and also

$$\partial_{i,j+1} = \partial_{ij} : F_jC_i = F_{j+1}C_i \to F_jC_{i-1} = F_{j+1}C_{i-1}$$

for the boundary morphisms in FC and large enough indices j. Note that this a strictly weaker notion than FC itself being finite.

Theorem 2.10. *If a filtered complex FC is bounded, then the associated spectral sequence abuts.*

Proof. First we fix some indices i and j and show that d^n_{ij} is zero for some large n. Note that the filtered objects FC_{i-1} and FC_{i+1} are finite, hence we find N, such that that

$$\begin{aligned} G_{j-n}C_{i-1} &= 0 \quad \text{and} \\ G_{j+n}C_{i+1} &= 0 \quad \text{for} \quad n \geq N \end{aligned}$$

In particular, the corresponding homologies

$$\begin{aligned} E^1_{i-1,j-n} &= H_{i-1}(G_{j-n}C) \quad \text{and} \\ E^1_{i+1,j+n} &= H_{i+1}(G_{j+n}C) \end{aligned}$$

are already zero. Since the inclusions in Proposition 2.2 preserve the grading (by definition), we know that also $E^n_{i-1,j-n}$ and $E^n_{i+1,j+n}$ need to be zero. But d^n has degree $(-1,-n)$ so we get

$$0 = E^n_{i+1,j+n} \xrightarrow{d^n_{i+1,j+n}} E^n_{ij} \xrightarrow{d^n_{ij}} E^n_{i-1,j-n} = 0$$

To conclude the proof just note that by definition we have

$$E^{n+1}_{ij} = \ker d^n_{ij} / \operatorname{im} d^n_{i+1,j+n} = E^n_{ij}$$

for large n, so inductively we get $E^n_{ij} = E^N_{ij} = E^\infty_{ij}$.

□

Corollary 2.11. *If the filtered complex FC is not only bounded but finite, then the corresponding spectral sequence terminates.*

Proof. In this case, N in the prove above can be chosen independently of the indices i and j, hence $E^N = E^\infty$

□

To prove the next theorem we need a filtration on HC.

Lemma 2.12. *Let FC be a bounded filtered complex in an $AB5$-category. and let HC be the total homology of C. Then by setting*

$$F_jHC := \mathrm{im}(HF_jC \to HC)$$

for each index j, we can define a filtration of HC.

Proof. Think of C as a sequence of chain complexes that is constantly C. Then by assumption we are given a morphism of sequences $FC \to C$ to which we apply the total homology functor and are left with a morphism of sequences

$$HFC \to HC$$

We take the image of this morphism and the resulting sequence is indeed a filtration as its morphisms are monic since they fit into

$$\begin{array}{ccc} F_jHC & \longrightarrow & F_{j+1}HC \\ \downarrow & & \downarrow \\ HC & \xrightarrow[\mathrm{id}]{} & HC \end{array}$$

Taking the colimit of a sequence is (co)continuous in $AB5$-categories, so we get that

$$\begin{aligned} \operatorname{colim} FHC &= \operatorname{colim}(\mathrm{im}(HFC \to HC)) = \\ &= \mathrm{im}(\operatorname{colim} HFC \to \operatorname{colim} HC) = \\ &= \mathrm{im}(H(\operatorname{colim} FC) \to HC) = HC \end{aligned}$$

□

Theorem 2.13. *The spectral sequence of a bounded filtered complex FC converges to the homology of the complex HC. i.e*

$$E_{ij}^n \implies H_iC$$

Proof. Consider the commuting diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & HFC & \xrightarrow{\mathrm{id}} & HFC & \xrightarrow{\mathrm{id}} & HFC & \longrightarrow & \cdots \\ & & \downarrow \iota^{n-1} & & \downarrow \iota^{n} & & \downarrow \iota^{n+1} & & \\ \cdots & \longrightarrow & HFC & \xrightarrow{\iota} & HFC & \xrightarrow{\iota} & HFC & \longrightarrow & \cdots \end{array}$$

which is a morphism of sequences between the constant sequence HF_jC and HFC, whose image will again be a sequence.

$$\cdots \longrightarrow \operatorname{im}(\iota^{n-1}) \longrightarrow \operatorname{im}(\iota^{n}) \longrightarrow \operatorname{im}(\iota^{n+1}) \longrightarrow \cdots$$

Now remember construction 2.8, where we called the objects of that sequence D^n. Fixing the two indices i and j we in particular have

$$D^n_{ij} = \operatorname{im}\left((\iota^n)_{i,j-n} : H_iF_{j-n}C \to H_iF_jC\right)$$

and get the sequence

$$\cdots \longrightarrow D^{n-1}_{i,j+n-1} \longrightarrow D^{n}_{i,j+n} \longrightarrow D^{n+1}_{i,j+n+1} \longrightarrow \cdots$$

Again because the filtered colimits are exact, can easily calculate the colimit of this sequence, namely

$$\begin{aligned}\operatorname*{colim}_n D^n_{i,j+n} &= \operatorname*{colim}_n \operatorname{im}(H_iF_jC \to H_iF_{j+n}C) = \\ &= \operatorname{im}(\operatorname*{colim}_n H_iF_jC \to \operatorname*{colim}_n H_iF_{j+n}C) = \\ &= \operatorname{im}(H_iF_jC \to H_iFC) = F_jH_iC\end{aligned}$$

where F_jH_iC is the filtration of H_iC from Lemma 2.12 above. Now remember that every derived exact couple is indeed exact, so in particular for every positive n there is an exact sequence

$$D^n_{i,j+n-1} \longrightarrow D^n_{i,j+n} \longrightarrow E^n_{ij} \longrightarrow D^n_{i-1,j}$$

But we assumed that the filtered complex is bounded, so can see that $D^n_{i-1,j}$ will be zero and $E^n_{ij} = E^{n_0}_{ij}$ will be constant due to Theorem 2.13 for n larger then some n_0. This gives us a sequence of exact sequences

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
D^n_{i,j+n-1} & \longrightarrow & D^n_{i,j+n} & \longrightarrow & E^{n_0}_{ij} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
D^{n+1}_{i,j+n} & \longrightarrow & D^{n+1}_{i,j+n+1} & \longrightarrow & E^{n_0}_{ij} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

whose colimit is

$$F_{j-1}H_iC \longrightarrow F_jH_iC \longrightarrow E^\infty_{ij} \longrightarrow 0$$

which is exact, again because taking this filtered colimit is exact. Finally in Lemma 2.12 we saw that FHC is a filtration of HC, so the spectral sequence converges to HC and in particular we have

$$E^\infty_{ij} = G_jH_iC$$

□

2.4 Spectral Sequences of Double Complexes

Another important source of spectral sequences are double complexes, i.e. chain complexes of chain complexes. Indeed they each yield two spectral sequences and much information can be gained by comparing them, as we shall later see.

Definition. A *double complex* is a commuting diagram in the shape of the partial order $\mathbb{Z} \times \mathbb{Z}$ in some abelian category $\mathcal{A}$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{i+1,j+1} & \xrightarrow{\delta_{i+1,j+1}} & C_{i+1,j} & \xrightarrow{\delta_{i+1,j}} & C_{i+1,j-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \partial_{i+1,j+1}} & & \downarrow{\scriptstyle \partial_{i+1,j}} & & \downarrow{\scriptstyle \partial_{i+1,j-1}} & & \\
\cdots & \longrightarrow & C_{i,j+1} & \xrightarrow{\delta_{i,j+1}} & C_{i,j} & \xrightarrow{\delta_{i,j}} & C_{i,j-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \partial_{i,j+1}} & & \downarrow{\scriptstyle \partial_{i,j}} & & \downarrow{\scriptstyle \partial_{i,j-1}} & & \\
\cdots & \longrightarrow & C_{i-1,j+1} & \xrightarrow{\delta_{i-1,j+1}} & C_{i-1,j} & \xrightarrow{\delta_{i-1,j}} & C_{i-1,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

such that for all i and j we have

$$\partial_{i-1,j} \circ \partial_{ij} = 0 \quad \text{and} \quad \delta_{i,j-1} \circ \delta_{ij} = 0$$

This gives the category of double complexes over $\mathcal{A}$ as a full subcategory of $\mathcal{A}^{\mathbb{Z}\times\mathbb{Z}}$. Alternatively we think of double complexes as objects $C = \prod_{i,j\in\mathbb{Z}} C_{ij}$ graded by $\mathbb{Z} \times \mathbb{Z}$ together with two graded morphisms ∂ and δ of degree $(-1, 0)$ and $(0, -1)$ respectively, such that

$$\partial^2 = 0 \quad \delta^2 = 0 \quad \partial\delta = \delta d$$

Remark. Note that each column and each row of a double complex is itself a chain complex. We denote them as $(C_{\bullet j}, \partial_{\bullet j})$ and $(C_{i\bullet}, \delta_{i\bullet})$ respectively.

Lemma 2.14. *As of Proposition 1.22 we can define* $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$*, the category of chain complexes of chain complexes. This is isomorphic to the category of double complexes.*

Proof. The isomorphism $\mathcal{A}^{\mathbb{Z}\times\mathbb{Z}} \cong (\mathcal{A}^{\mathbb{Z}})^{\mathbb{Z}}$ descends to an isomorphism of the respective subcategories, which we denote as >. □

Remark. Mirroring a double complex along its anti diagonal, i.e precomposing with it with $\mathbb{Z}^2 \to \mathbb{Z}^2$, $(i,j) \to (j,i)$, is an automorphism of categories. We define $\vee$ to be the composition of this automorphism and $>$. Explicitly, for a double complex $C = (C, \partial, \delta)$ this yields

$$C^{>} = ((C_{\bullet j}, \partial_{\bullet j})_j, \delta) \quad \text{and} \quad C^{\vee} = ((C_{i\bullet}, \delta_{i\bullet})_i, \partial)$$

Definition. Given a double complex C in $\mathcal{A}$ we can define its *total complex* $|C| \in \mathrm{Ch}(\mathcal{A})$ as follows.

$$\cdots \longrightarrow \prod_{i+j=n} C_{ij} \xrightarrow{d_n} \prod_{i+j=n-1} C_{ij} \longrightarrow \cdots \tag{2.6}$$

where

$$d_n = \prod_{i+j=n} \partial_{ij} + \prod_{i+j=n} (-1)^i \delta_{ij}$$

Note that this extends to a functor $|\bullet| : \mathrm{Ch}(\mathrm{Ch}(\mathcal{A})) \to \mathrm{Ch}(\mathcal{A})$.

Remark. The total complex functor, up to a sign of the boundary morphism, is invariant under the automorphism $C^{>} \to C^{\vee}$.

Remark. There is another notion of the total complex, using coproducts instead of products. If the (co)products are finite, the two notions are of course equivalent by Proposition 1.14.

Lemma 2.15. *The total complex functor $|\bullet|$ is left exact. If products are exact in the underlying category $\mathcal{A}$ then the total complex functor is also right exact.*

Proof. This follows directly from Propositions 1.10 and 1.28. □

We now give two ways of constructing a filtration of the total complex of a double complex.

Construction 2.16. *(spectral sequences of a double complex)* As both $C^{>}$ and $C^{\vee}$ are chain complexes we can use the canonical filtration to get filtered complexes $FC^{>}$ and $FC^{\vee}$ whose objects are double complexes that are bounded below in i and j respectively.

$$F_jC^{>}: \quad \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C_{i+1,j} & \xrightarrow{\delta} & C_{i+1,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C_{i,j} & \xrightarrow{\delta} & C_{i,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C_{i-1,j} & \xrightarrow{\delta} & C_{i-1,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$F_iC^{\vee}: \quad \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{i,j+1} & \xrightarrow{\delta} & C_{i,j} & \xrightarrow{\delta} & C_{i,j-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
\cdots & \longrightarrow & C_{i-1,j+1} & \xrightarrow{\delta} & C_{i-1,j} & \xrightarrow{\delta} & C_{i-1,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We can apply the total complex functor to these filtrations to get chain complexes $|F_iC^{>}|$ and $|F_iC^{\vee}|$, where the objects are

$$|F_jC^{>}|_i = \prod_{\substack{k+l=i \\ l\leq j}} C_{kl} \quad \text{and} \quad |F_iC^{\vee}|_j = \prod_{\substack{k+l=i \\ k\leq j}} C_{kl}$$

with the corresponding boundary morphisms. If again all those coproducts are finite, then by lemma 2.15 both of them are filtrations of the total complex $|C|$ i.e. filtered complexes. We denote them as $F^{>}C$ and $F^{\vee}C$ respectively. Now we simply apply Construction 2.8 to get two spectral sequences ${}^{>}E$ and ${}^{\vee}E$.

Proposition 2.17. *Consider the associated graded objects of $F^{>}C$ and $F^{\vee}C$ which we call $G^{>}C$ and $G^{\vee}C$ respectively. Then every $G^{>}_{j}C$ is a chain complex with*

$$G^{>}_{j}C_i = C_{i-j,j}$$

and the boundary morphism just being ∂ from the the double complex C. Similarly the $G^{\vee}_{j}C$ are chain chain complexes, where the boundary morphism is δ from the double complex and

$$G^{\vee}_{j}C_i = C_{j,i-j}$$

Proof. By construction 2.16, the filtration steps $F^{>}_{q}C$ are chain complexes, so $G^{>}C$ is a graded object in $\mathrm{Ch}(\mathcal{A})$, meaning that the $G^{>}_{q}C$ are chain complexes. Also we have

$$G^{>}_{j}C_i = \coprod_{\substack{k+l=i\\ l\leq j}} C_{kl} \Big/ \coprod_{\substack{k+l=i\\ l\leq j-1}} C_{kl} = C_{j-i,j}$$

□

Proposition 2.18. *We can calculate the zeroth and first page of ${}^{>}E$ and ${}^{\vee}E$ directly from their construction, namely*

$$\begin{aligned}
{}^{>}E^{0}_{ij} &= G^{>}_{j}C_i = C_{i-j,j}\\
{}^{\vee}E^{0}_{ij} &= G^{\vee}_{j}C_i = C_{j,i-j}\\
{}^{>}E^{1}_{ij} &= H_iG^{>}_{j}C = H_{i-j}(C_{\bullet j}, \partial_{\bullet j})\\
{}^{\vee}E^{1}_{ij} &= H_iG^{\vee}_{j}C = H_{i-j}(C_{j\bullet}, \delta_{j\bullet})
\end{aligned}$$

Remark. Many sources use different indices, namely $p = i - j$ and $q = j$, in order to align the spectral sequence with the double complex.

$${}^{>}E^{0}_{pq} = C_{pq}$$

We will not do this in order to keep the focus on the filtrations of the total complex.

Proposition 2.19. *For a double complex C, both of its spectral sequences ${}^{>}E$ and ${}^{\vee}E$ both converge to the homology of the total complex $|C|$.*

$${}^{>}E^{n}_{ij}, {}^{\vee}E^{n}_{ij} \implies H_i|C|$$

Proof. This immediately follows from Theorem 2.13. □

Proposition 2.20. *Since we can consider* $\delta_{\bullet j}$ *as a morphism of chain complexes* $(C_{\bullet j}, \partial_{\bullet j}) \to (C_{\bullet j-1}, \partial_{\bullet j-1})$*, it induces a morphism in homology*

$$H_{i-j}\delta_{\bullet j} : H_{i-j}(C_{\bullet j}, \partial_{\bullet j}) \to H_{i-j}(C_{\bullet,j-1}, \partial_{\bullet,j-1})$$

This is the boundary morphism of the first page of ${}^{>}E$*. The analogous statement holds for* ${}^{\vee}E$*.*

Proof. Consider the morphisms $\delta_{i-j,j} : C_{i-j,j} \to C_{i-j,j-1}$, all which factor through $F^{>}_{j-1}C$, giving us a commuting diagram of chain complexes

$$\begin{array}{ccc} & F^{>}_{j-1}C & \\ \nearrow & & \searrow \\ G^{>}_{j}C & \xrightarrow[\delta_{\bullet j}]{\qquad\qquad} & G^{>}_{j-1}C \end{array}$$

To see the compatibility with the respective boundary morphisms, just remember that $d \circ \partial = \delta \circ \partial$. Applying homology we get another commuting triangle

$$\begin{array}{ccc} & H_{i-j}F^{>}_{j-1}C & \\ \overset{\Phi}{\nearrow} & & \overset{\pi}{\searrow} \\ {}^{>}E^{1}_{ij} & \xrightarrow[H_{i-j}\delta_{\bullet j}]{\qquad\qquad} & {}^{>}E^{1}_{i-1,j-1} \end{array}$$

Now one can check that this Φ is the connecting morphism in the exact sequence

$$\cdots \longrightarrow H_i F^{>}_{j} C \xrightarrow{\pi} H_i G^{>}_{j} C \xrightarrow{\Phi} H_{i-1} F^{>}_{j-1} C \longrightarrow \cdots$$

which proves the claim. $\square$

2.5 Grothendieck Spectral Sequence

Remember that if an abelian category $\mathcal{A}$ has enough projectives, then so does $\mathrm{Ch}(\mathcal{A})$. With that we are going to construct double complexes using special projective resolutions of chain complexes, yielding the Grothendieck spectral sequence.

Proposition 2.21. *(Cartan-Eilenberg resolution) Let $(C_i, d_i) \in \mathrm{Ch}(\mathcal{A})$ be a bounded below chain complex, meaning $C_i = 0$ for negative i. Then there is a projective resolution $P \in \mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ of C such that both P_{ij} and $H_i P_{\bullet j}$ are projective for all i, j. Also the induced morphisms*

$$H_i P_{\bullet j} \to H_i P_{\bullet j-1}$$

form a projective resolution of $H_i C$. We call such a resolution a Cartan-Eilenberg resolution.

Proof. Consider the two short exact sequences

$$0 \longrightarrow \operatorname{im} d_i \longrightarrow C_{i-1} \longrightarrow \operatorname{coker} d_i \longrightarrow 0$$

$$0 \longrightarrow H_i C \longrightarrow \operatorname{coker} d_{i+1} \longrightarrow \operatorname{im} d_i \longrightarrow 0$$

Given a projective resolution $P''_{1\bullet}$ of $\operatorname{coker} d_1$ we choose two more projective resolutions $P'_{i\bullet}$ and $P^H_{i\bullet}$ of $\operatorname{im} d_i$ and $H_i C$ respectively. Proposition 1.36 now gives us two projective resolutions $P_{i-1\bullet}$ and $P''_{i+1\bullet}$, of C_{i-1} and $\operatorname{coker} d_{i+1}$ respectively, that fit into the short exact sequences of complexes

$$0 \longrightarrow P'_{i\bullet} \longrightarrow P_{i-1\bullet} \longrightarrow P''_{i\bullet} \longrightarrow 0$$

$$0 \longrightarrow P^H_{i\bullet} \longrightarrow P''_{i+1\bullet} \longrightarrow P'_{i\bullet} \longrightarrow 0$$

Starting out with an arbitrary choice for $P''_{1\bullet}$ we iteratively construct these resolutions for all non negative indices i. Now define $d_{i\bullet} : P_{i\bullet} \to P_{i-1\bullet}$ to be the composition

$$P_{i\bullet} \longrightarrow P''_{i+1\bullet} \longrightarrow P'_{i\bullet} \longrightarrow P_{i-1\bullet}$$

which indeed is a projective resolution of the chain complex (C_i, d_i), where all P_{ij} are projective and so are the $H_i P_{\bullet j} = P^H_{ij}$. □

Lemma 2.22. *If $C = (C_i, d_i)$ is a chain complex in an abelian category $\mathcal{A}$ such that all* $\operatorname{im} d_i$ *and $H_i(C)$ are projective, and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a right exact functor, then we have*

$$\mathcal{F}H(C) = H(\mathcal{F}C)$$

Proof. By Proposition 1.23 we have a short exact sequence

$$0 \longrightarrow H_iC \longrightarrow \operatorname{coker} d_{i+1} \longrightarrow \operatorname{im} d_i \longrightarrow 0$$

which splits because the image is projective by assumption, see Proposition 1.26. This implies that the cokernel in the center of the above sequence is a direct sum of projective objects and hence is projective by Proposition 1.25. But therefore the sequence

$$0 \longrightarrow \operatorname{im} d_i \longrightarrow C_{i-1} \longrightarrow \operatorname{coker} d_i \longrightarrow 0$$

must also split, so the exactness of both sequences is preserved by $\mathcal{F}$. Because of the latter one we calculate that $\mathcal{F}$ preserves the images of the d_i.

$$\begin{aligned} \mathcal{F} \operatorname{im} d_i &= \ker(\mathcal{F} \operatorname{coker} d_i) \\ &= \ker \operatorname{coker}(\mathcal{F}d_i) = \operatorname{im}(\mathcal{F}d_i) \end{aligned}$$

where the second equality uses the right exactness of $\mathcal{F}$. A similar calculation using the first exact sequence yields the result.

$$\begin{aligned} \mathcal{F}H_i(C) &= \ker \mathcal{F}(\operatorname{coker} d_i \to \operatorname{im} d_{i-1}) \\ &= \ker(\operatorname{coker} \mathcal{F}d_i \to \operatorname{im} \mathcal{F}d_{i-1}) = H(\mathcal{F}C) \end{aligned}$$

□

Definition. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories with enough projectives. An object $A \in \mathcal{A}$ is called *$\mathcal{F}$-acyclic* if the total left derived functor $L\mathcal{F}$ maps A to zero, i.e. $L_i\mathcal{F}(A) = 0$ for all $i \geq 1$.

Theorem 2.23. *(Grothendieck spectral sequence) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $AB5$-categories, where $\mathcal{A}$ and $\mathcal{B}$ have enough projectives, and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : B \to \mathcal{C}$ right exact functors, such that $\mathcal{F}$ maps projective objects in $\mathcal{A}$ to $\mathcal{G}$-acyclic objects in $\mathcal{B}$. Then for every $A \in \mathcal{A}$ there is a convergent spectral sequence E with*

$$E^2_{ij} = L_j\mathcal{G} \circ L_{i-j}\mathcal{F}(A) \implies L_i(\mathcal{G}\mathcal{F})(A)$$

Proof. Let P be a projective resolution of A. Since $\mathcal{B}$ has enough projectives we can find a Cartan-Eilenberg resolution Q of $\mathcal{F}P$ in $\mathrm{Ch}(\mathcal{B})$. We define $C := \mathcal{G}(Q)$, then C_{ij} can only be non zero if i and j are both non negative.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{G}Q_{2,2} & \longrightarrow & \mathcal{G}Q_{2,1} & \longrightarrow & \mathcal{G}Q_{2,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{G}Q_{1,2} & \longrightarrow & \mathcal{G}Q_{1,1} & \longrightarrow & \mathcal{G}Q_{1,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{G}\mathcal{F}G_{0,2} & \longrightarrow & \mathcal{G}\mathcal{F}Q_{0,1} & \longrightarrow & \mathcal{G}\mathcal{F}Q_{0,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now let ${}^{\vee}E$ and ${}^{>}E$ be the corresponding spectral sequences. We saw in Proposition 2.19 that both of them converge to filtrations of $H|C|$, the homology of the total complex of C.

Step 1: We start by computing ${}^{>}E$. Note that by definition, for each j we have that $Q_{\bullet j}$ is a projective resolution of $\mathcal{F}P_j$. Remember that Proposition 2.18 allows us to calculate the first page of ${}^{>}E$, so we get

$${}^{>}E^1_{ij} = H_{i-j}(C_{\bullet j}) = H_{i-j}(\mathcal{G}Q_{\bullet j}) = L_{i-j}\mathcal{G}(\mathcal{F}P_j)$$

But each P_j is projective, so by assumption $\mathcal{F}P_j$ is $\mathcal{G}$-acyclic. This just means that

$${}^{>}E^1_{ij} = L_{i-j}\mathcal{G}(\mathcal{F}P_j) = \begin{cases} \mathcal{G}\mathcal{F}P_j & \text{if } i = j \\ 0 & \text{if otherwise} \end{cases}$$

From Proposition 2.9 we know that the boundary morphism on the n-th page has degree $(-1, -n)$ so for $n \geq 2$ it must be zero. This means that ${}^{>}E$ terminates at the second page, so ${}^{>}E^\infty = {}^{>}E^2$ and we must only calculate this second page. By Proposition 2.20 the differential on the first page ${}^{>}d^1$ is induced by the horizontal boundary morphism of the double complex. In particular ${}^{>}d^1_{qq} : \mathcal{G}\mathcal{F}P_q \to \mathcal{G}\mathcal{F}P_{q-1}$ is just $\mathcal{G}\mathcal{F}$ applied to the boundary morphism of the resolution P. This gives us that

$$^{>}E^{\infty}_{ij} = {}^{>}E^2_{ij} = \begin{cases} H_j\mathcal{G}\mathcal{F}P = L_j(\mathcal{G}\mathcal{F})(A) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

From this we can calculate the homology of the total complex. We fix some index i, then due to Theorem 2.19 the objects $^{>}E^{\infty}_{ij}$ are the quotients F_j/F_{j-1} for some filtration F of $H_i|C|$. This immediately implies

$$F_j = \begin{cases} L_i(\mathcal{G}\mathcal{F})(A) & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

so

$$H_i|C| = L_i(\mathcal{G}\mathcal{F})(A)$$

which means that $^{\vee}E$ does indeed converge to $L_i(\mathcal{G}\mathcal{F})$.
Step 2: We now compute $^{\vee}E$. Remember that we chose Q to be a Cartan-Eilenberg resolution, which means that all $H_{i-j}Q_{j\bullet}$ as well as all images of the horizontal boundary morphisms $\delta_{j,i-j}$ are projective objects. Therefore we can apply Lemma 2.22 to the chains $(Q_{j\bullet}, \delta_{j\bullet})$ to get

$$^{\vee}E^1_{ij} = H_{i-j}(\mathcal{G}Q_{j\bullet}) = \mathcal{G}H_{i-j}Q_{j\bullet}$$

using the fact that $\mathcal{G}$ is right exact and Proposition 2.20. Analogously to step 1, the first page boundary morphisms $^{\vee}d^1_{ij}$ are induced by the vertical boundary morphism of the double complex C. But remember that the $H_{i-j}Q_{j\bullet}$ together with the induced morphisms ∂ form a projective resolution of $H_{i-j}\mathcal{F}P_j$, which means

$$^{\vee}E^2_{ij} = H_j\mathcal{G}(H_{i-j}Q_{i\bullet})_i = L_j\mathcal{G}(H_{i-j}\mathcal{F}P) = L_j\mathcal{G}(L_{i-j}\mathcal{F}(A))$$

□

Corollary 2.24. *If $\mathcal{F}$ is exact in the above setup, then $L_i(\mathcal{G}\mathcal{F}) = L_i\mathcal{G} \circ \mathcal{F}$.*

Proof. If $\mathcal{F}$ is exact then its i-th derived functor is zero for any positive index i. In particular the second page of the Grothendieck specral sequence reads

$$E^2_{ij} = \begin{cases} L_i\mathcal{G}(\mathcal{F}(A)) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We immediately see that the sequence degenerates at this page, so

$$E^2 = E^\infty$$

$\square$

3 Hochschild Serre Spectral Sequences

In this chapter we discuss special cases of the Grothendieck spectral sequence, namely the Hochschild Serre spectral sequences for group and Lie algebra (co)homology.

Proposition 3.1. *A homomorphism of rings $f : S \to R$ induces a functor $\mathcal{F} : \mathrm{Mod}_R \to \mathrm{Mod}_S$ called restriction, by defining an action of S on an R-module A*

$$S \times A \to \mathcal{A}, \quad (s, a) \to f(s)a$$

This functor is exact, since it has both a left and a right adjoint, namely

$$R \otimes_S \dashv \mathcal{F} \dashv \mathrm{Mod}_S(R, -)$$

In other words, there are two natural isomorphisms

$$\mathrm{Mod}_R(R \otimes_S (-), -) \cong \mathrm{Mod}_S(-, F(-))$$
$$\mathrm{Mod}_S(F(-), -) \cong \mathrm{Mod}_R(-, \mathrm{Mod}_S(R, -))$$

Example. If R is a unital ring then the unique unit preserving homomorphism $\mathbb{Z} \to R$ induces the forgetful functor

$$\mathcal{U} : \mathrm{Mod}_R \to \mathcal{A}b$$

which sends R-modules to their underlying abelian group. In particular we can do this if R is a group ring $\mathbb{Z}[G]$. Similarly if R is a unital $\mathbb{K}$-algebra, for example the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, from $\mathbb{K} \to R$ we get the forgetful functor

$$\mathcal{U} : \mathrm{Mod}_R \to \mathrm{Vect}_{\mathbb{K}}$$

Example. The group homomorphism $G \to 1$ induces a homomorphism of group rings $\mathbb{Z}[G] \to \mathbb{Z}$, which in turn induces a functor that equips abelian groups with the trivial action of G.

$$\mathcal{T} : \mathcal{A}b \to \mathrm{Mod}_G$$

Example. The homomorphism of $\mathbb{K}$-Lie algebras $\mathfrak{g} \to 0$ induces a homomorphism of their respective universal enveloping algebras $\mathcal{U}\mathfrak{g} \to \mathbb{K}$. The corresponding functor equips vector spaces with the trivial action of $\mathfrak{g}$.

$$\mathcal{T} : \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Mod}_{\mathfrak{g}}$$

3.1 Group (Co)Homology

Definition. Let G be a group and $A \in \mathrm{Mod}_G$ a G-module, then we call $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ the *G-coinvariants* of A, where $\mathbb{Z}[G]$ is the group ring associated to G and $\mathbb{Z}$ is the trivial G-module. This is of course a right exact functor

$$\mathbb{Z}\otimes_{\mathbb{Z}[G]} : \mathrm{Mod}_G \to \mathcal{A}b$$

Moreover we call the i-th left derived functor of $\mathbb{Z}\otimes_{\mathbb{Z}[G]}$ the i-th *homology*

$$H_i(G, -) := L_i(\mathbb{Z}\otimes_{\mathbb{Z}[G]})$$

Lemma 3.2. *Given a group extension $Q = G/N$ and the restriction functors $\mathcal{U} : \mathrm{Mod}_Q \to \mathcal{A}b$ and $\mathcal{U}' : \mathrm{Mod}_G \to \mathrm{Mod}_N$ there is a natural isomorphism*

$$\mathcal{U} \circ (\mathbb{Z}[Q]\otimes_{\mathbb{Z}[G]}) \cong (\mathbb{Z}\otimes_{\mathbb{Z}[N]}) \circ \mathcal{U}'$$

This, in combination with Corollaries 1.32 and 2.24 the fact that $\mathcal{U}$ and $\mathcal{U}'$ are exact, gives us a natural isomorphism

$$\mathcal{U} \circ L_i(\mathbb{Z}[Q]\otimes_{\mathbb{Z}[G]}) \cong L_i(\mathbb{Z}\otimes_{\mathbb{Z}[N]}) \circ \mathcal{U}'$$

In particular we have a Q-module structure on $H_i(N, A)$ whenever A is a G-module.

Theorem 3.3. *(Hochschild-Serre for group homology) Again given a group extension $Q = G/N$ and a G-module A, there is a convergent spectral sequence*

$$E^2_{ij} = H_j(Q, H_{i-j}(N, A)) \implies H_i(G, A)$$

Proof. There is a well known natural isomorphism

$$\begin{aligned} \mathbb{Z}\otimes_{\mathbb{Z}[G]} &= (\mathbb{Z} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q])\otimes_{\mathbb{Z}[G]} = \\ &= (\mathbb{Z}\otimes_{\mathbb{Z}[Q]}) \circ (\mathbb{Z}[Q]\otimes_{\mathbb{Z}[G]}) \end{aligned}$$

Now by Proposition 3.1 the functor $\mathbb{Z}[Q]\otimes_{\mathbb{Z}[G]}$ is left adjoint to the extension functor $\mathrm{Mod}_Q \to \mathrm{Mod}_G$, meaning that it is right exact and preserves projective objects by Propositions 1.8 and 1.27. Finally $\mathbb{Z}\otimes_{\mathbb{Z}[Q]}$ is also right exact and hence Theorem 2.23 applies, giving us the spectral sequence as an instance of the Grothendieck spectral sequence. □

Definition. Again let G be a group and A a G-module then we call $\mathrm{Mod}_G(\mathbb{Z}, A)$ the G-invariants of A where again $\mathbb{Z}$ is considered as a trivial G-module. This defines the left exact functor

$$\mathrm{Mod}_G(\mathbb{Z}, -) : \mathrm{Mod}_G \to \mathcal{A}b$$

whose i-th right derived functor we call the i-th *cohomology*

$$H^i(G, -) := R^i(\mathrm{Mod}_G(\mathbb{Z}, -))$$

Lemma 3.4. *Given a group extension $Q = G/N$ and the restriction functors $\mathcal{U} : \mathrm{Mod}_Q \to \mathcal{A}b$ and $\mathcal{U}' : \mathrm{Mod}_G \to \mathrm{Mod}_N$ there is a natural isomorphism*

$$\mathcal{U} \circ \mathrm{Mod}_G(\mathbb{Z}[Q], -) \cong \mathrm{Mod}_N(\mathbb{Z}, -) \circ \mathcal{U}'$$

This, in combination with Corollaries 1.32 and 2.24 the fact that $\mathcal{U}$ and $\mathcal{U}'$ are exact, gives us a natural isomorphism

$$\mathcal{U} \circ R^i(\mathrm{Mod}_G(\mathbb{Z}[Q], -)) \cong R^i(\mathrm{Mod}_N(\mathbb{Z}, -)) \circ \mathcal{U}' = H^i(N, -) \circ \mathcal{U}'$$

In particular we have a Q-module structure on $H^i(N, A)$ whenever A is a G-module.

Theorem 3.5. *(Hochschild-Serre for group cohomology) Consider again a group extension $Q = G/N$ and a G-module A, then there is a convergent spectral sequence*

$$E^2_{ij} = H^j(Q, H^{i-j}(N, A)) \implies H^i(G, A)$$

Proof. We begin with a natural isomorphism

$$\begin{aligned} \mathrm{Mod}_G(\mathbb{Z}, -) &= \mathrm{Mod}_G(\mathbb{Z} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q], -) = \\ &= \mathrm{Mod}_Q(\mathbb{Z}, \mathrm{Mod}_G(\mathbb{Z}[Q], -)) \end{aligned}$$

Then again by Proposition 3.1 the functor $\mathrm{Mod}_G(\mathbb{Z}[Q], -)$ is right adjoint to the extension functor $\mathrm{Mod}_Q \to \mathrm{Mod}_G$, meaning that it is left exact and preserves injective objects. Since $\mathrm{Mod}_Q(\mathbb{Z}, -)$ is also left exact the dual of Theorem 2.23 applies, so we get the spectral sequence as an instance of the Grothendieck spectral sequence. ☐

Example. As an application we may calculate the cohomology of the lattice $\mathbb{Z}^n$, namely

$$H^i(\mathbb{Z}^n) = \mathbb{Z}^{\binom{n}{i}}$$

via induction over n. It is well known that $H^0(\mathbb{Z}) = H^1(\mathbb{Z}) = \mathbb{Z}$, so consider the group extension

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0$$

The spectral sequence from Theorem 3.5 then yields

$$E^2_{ij} = H^j(\mathbb{Z}, H^{i-j}(\mathbb{Z}^n)) \implies H^i(\mathbb{Z}^{n+1})$$

By the induction hypothesis and the fact that the homology functor is additive, we see that the second page has the form

$$E^2_{ij} = H^j(\mathbb{Z}, \mathbb{Z}^{\binom{n}{i-j}}) = \begin{cases} \mathbb{Z}^{\binom{n}{i}} & \text{if } j = 0 \\ \mathbb{Z}^{\binom{n}{i-1}} & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and hence, because of its degree, the second page boundary morphism d^2 must be zero, hence $E^2 = E^\infty$. Now for a fixed i we have a filtration F of $H^i(\mathbb{Z}^{n+1})$ with $F_j/F_{j-1} = E^2_{ij}$, so

$$H^i(\mathbb{Z}^{n+1}) = \mathbb{Z}^{\binom{n}{i-1}} \oplus \mathbb{Z}^{\binom{n}{i}} = \mathbb{Z}^{\binom{n+1}{i}}$$

Example. We can calculate the cohomology of the integer Heisenberg group $\mathcal{H}_3$ of rank 3. Note that its center is isomorphic to $\mathbb{Z}$, so it fits into the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_3 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

The resulting spectral sequence is

$$E^2_{ij} = H^j(\mathbb{Z}^2, H^{i-j}(Z))$$

This can only not be zero if $i = j$ or $i = j + 1$, in which case

$$E^2_{ij} = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z}^2 & \text{if } j = 1 \\ \mathbb{Z} & \text{if } j = 2 \end{cases}$$

The boundary morphism d^2 has degree $(1, 2)$, so the only non zero component can be

$$d^2_{1,0} : E^2_{1,0} = \mathbb{Z} \to E^2_{2,2} = \mathbb{Z}$$

One can show that this is an isomorphism, resulting in

$$E_{ij}^{\infty} = E_{ij}^{3} = \begin{cases} \mathbb{Z} & \text{if } (i,j) = (0,0) \text{ or } (3,2) \\ \mathbb{Z}^2 & \text{if } (i,j) = (1,1) \text{ or } (2,1) \\ 0 & \text{otherwise} \end{cases}$$

In particular, all the filtrations of $H^i(\mathcal{H}_3)$ only have one step, giving us

$$\begin{aligned} H^0(\mathcal{H}_3) &= \mathbb{Z} \\ H^1(\mathcal{H}_3) &= \mathbb{Z}^2 \\ H^2(\mathcal{H}_3) &= \mathbb{Z}^2 \\ H^3(\mathcal{H}_3) &= \mathbb{Z} \end{aligned}$$

3.2 Lie Algebra (Co)Homology

Definition. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let A a $\mathfrak{g}$-module. We define the i-th *homology* of A to be

$$H_i(\mathfrak{g}, A) := L_i(\mathbb{K} \otimes_{\mathcal{U}\mathfrak{g}})(A)$$

Here $\mathbb{K}$ is considered as the trivial $\mathfrak{g}$-module.

Lemma 3.6. *Considering an extension of Lie algebras $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ and the restriction functors $\mathcal{R} : \mathrm{Mod}_{\mathfrak{q}} \to \mathrm{Vect}_{\mathbb{K}}$ and $\mathcal{R}' : \mathrm{Mod}_{\mathfrak{g}} \to \mathrm{Mod}_{\mathfrak{n}}$ there is a natural isomorphism*

$$\mathcal{R} \circ (\mathcal{U}\mathfrak{q} \otimes_{\mathcal{U}\mathfrak{g}}) \cong (\mathbb{K} \otimes_{\mathfrak{n}}) \circ \mathcal{R}'$$

This, in combination with corollaries 1.32 and 2.24 the fact that $\mathcal{R}$ and $\mathcal{R}'$ are exact, gives us a natural isomorphism

$$\mathcal{R} \circ L_i(\mathcal{U}\mathfrak{q} \otimes_{\mathcal{U}\mathfrak{g}}) \cong L_i(\mathbb{K} \otimes_{\mathfrak{n}}) \circ \mathcal{R}'$$

In particular we have a $\mathfrak{q}$-module structure on $H_i(\mathfrak{n}, A)$ whenever A is a $\mathfrak{g}$-module.

Theorem 3.7. *(Hochschild-Serre for Lie algebra homology) Given an extension $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ of Lie algebras and a G-module A, there is a convergent spectral sequence*

$$E^2_{ij} = H_j(\mathfrak{q}, H_{i-j}(\mathfrak{n}, A)) \implies H_i(\mathfrak{g}, A)$$

Proof. There is a well known natural isomorphism

$$\begin{aligned} \mathbb{K} \otimes_{\mathcal{U}\mathfrak{g}} &= (\mathbb{K} \otimes_{\mathcal{U}\mathfrak{q}} \mathcal{U}\mathfrak{q}) \otimes_{\mathcal{U}\mathfrak{g}} = \\ &= (\mathbb{K} \otimes_{\mathcal{U}\mathfrak{q}}) \circ (\mathcal{U}\mathfrak{q} \otimes_{\mathcal{U}\mathfrak{g}}) \end{aligned}$$

Now by Proposition 3.1 the functor $\mathcal{U}\mathfrak{q} \otimes_{\mathcal{U}\mathfrak{g}}$ is left adjoint to the extension functor $\mathrm{Mod}_{\mathfrak{q}} \to \mathrm{Mod}_{\mathfrak{g}}$, meaning that it is right exact and preserves projective objects by Propositions 1.8 and 1.27. Finally $\mathbb{K} \otimes_{\mathcal{U}\mathfrak{q}}$ is also right exact and hence Theorem 2.23 applies, giving us the spectral sequence as an instance of the Grothendieck spectral sequence. $\square$

Definition. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let A a $\mathfrak{g}$-module. We define the i-th *cohomology* of A to be

$$H^i(\mathfrak{g}, A) := R^i(\mathrm{Mod}_{\mathfrak{g}}(\mathbb{K}, -))(A)$$

Here $\mathbb{K}$ is considered as the trivial $\mathfrak{g}$-module.

Lemma 3.8. *Given an extension $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ of Lie algebras and the restriction functors $\mathcal{T} : \mathrm{Mod}_{\mathfrak{q}} \to \mathrm{Vect}_{\mathbb{K}}$ and $\mathcal{T}' : \mathrm{Mod}_{\mathfrak{g}} \to \mathrm{Mod}_{\mathfrak{n}}$ there is a natural isomorphism*

$$\mathcal{T} \circ \mathrm{Mod}_{\mathfrak{g}}(\mathcal{U}\mathfrak{q}, -) \cong \mathrm{Mod}_{\mathfrak{n}}(\mathbb{K}, -) \circ \mathcal{T}'$$

This, in combination with corollaries 1.32 and 2.24 the fact that $\mathcal{R}$ and $\mathcal{R}'$ are exact, gives us a natural isomorphism

$$\mathcal{R} \circ R^i(\mathrm{Mod}_{\mathfrak{g}}(\mathcal{U}\mathfrak{q}, -)) \cong R^i(\mathrm{Mod}_{\mathfrak{n}}(\mathbb{K}, -)) \circ \mathcal{R}' = H^i(\mathfrak{n}, -) \circ \mathcal{R}'$$

In particular we have a $\mathfrak{q}$-module structure on $H^i(\mathfrak{n}, A)$ whenever A is a $\mathfrak{g}$-module.

Theorem 3.9. *(Hochschild-Serre for Lie algebra cohomology) Consider again an extension $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ and a $\mathfrak{g}$-module A, then there is a convergent spectral sequence*

$$E_{ij}^2 = H^j(\mathfrak{q}, H^{i-j}(\mathfrak{n}, A)) \implies H^i(\mathfrak{g}, A)$$

Proof. We once more begin with a natural isomorphism

$$\begin{aligned} \mathrm{Mod}_{\mathfrak{g}}(\mathbb{K}, -) &= \mathrm{Mod}_{\mathfrak{g}}(\mathbb{K} \otimes_{\mathcal{U}\mathfrak{q}} \mathcal{U}\mathfrak{q}, -) = \\ &= \mathrm{Mod}_{\mathfrak{q}}(\mathbb{K}, \mathrm{Mod}_{\mathfrak{g}}(\mathcal{U}\mathfrak{q}, -)) \end{aligned}$$

Then again by Proposition 3.1 the functor $\mathrm{Mod}_{\mathfrak{g}}(\mathcal{U}\mathfrak{q}, -)$ is right adjoint to the extension functor $\mathrm{Mod}_{\mathfrak{q}} \to \mathrm{Mod}_{\mathfrak{g}}$, meaning that it is left exact and preserves injective objects. Since $\mathrm{Mod}_{\mathfrak{q}}(\mathbb{K}, -)$ is also left exact the dual of Theorem 2.23 applies, giving us the spectral sequence as an instance of the Grothendieck spectral sequence. □

Example. Similar to the cohomology of the lattice $\mathbb{Z}^n$ one can show that the cohomology of the abelian Lie algebra $\mathbb{K}^n$ is

$$H^i(\mathbb{K}^n) = \mathbb{K}^{\binom{n}{i}}$$

Example. We can calculate the cohomology of the three dimensional Heisenberg algebra $\mathfrak{h}_3$ over a field $\mathbb{K}$. Note that its derived subalgebra is one dimensional, so it fits into the short exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{h}_3 \longrightarrow \mathbb{K}^2 \longrightarrow 0$$

Similar to the integer Heisenberg group, from this one can show that

$$\begin{aligned} H^0(\mathfrak{h}_3) &= \mathbb{K} \\ H^1(\mathfrak{h}_3) &= \mathbb{K}^2 \\ H^2(\mathfrak{h}_3) &= \mathbb{K}^2 \\ H^3(\mathfrak{h}_3) &= \mathbb{K} \end{aligned}$$

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