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MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Presentation of the affine version of the Jones monoid"

verfasst von / submitted by

Martin Zahradníček

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	ao. Univ.-Prof. i.R. Dr. Karl Auinger

Abstract

This Master's thesis deals with a family of monoids of diagrams drawn by lines on the surface of a cylinder without intersections. The composition is defined by stacking the cylinders, joining the lines and removing contractible loops. This is an affine analogue of the Jones monoid in which the diagrams are drawn on rectangles instead of cylinders. In this thesis, we define these monoids, which we call *affine Jones monoids*, in a short exposition. We then develop some tools to ultimately give a presentation of these affine monoids. Finally, we comment on previous work done towards this presentation.

Zusammenfassung

Diese Master-Arbeit befasst sich mit einer Klasse von Monoiden, deren Elemente Diagramme sind, die aus einander nicht schneidenden Kurven auf der Oberfläche eines Zylinders bestehen. Die binäre Verknüpfung ist durch Zusammenkleben der Zylinder an den Rändern, Verbinden der Kurven und Entfernen kontrahierbarer Kreise definiert. Damit bekommen wir ein affines Analogon zu den Jones Monoiden, in welchen die Diagramme anstatt auf einem Zylinder in einem Rechteck gezeichnet sind. In der vorliegenden Arbeit werden diese Monoide, bezeichnet als *affine Jones Monoide*, kurz definiert. Dann werden einige Hilfsmittel entwickelt, die es schließlich erlauben, eine Präsentation dieser Monoide (durch erzeugende Elemente und definiernde Relationen) herzuleiten. Abschließend kommentieren wir noch einige zu dieser Thematik in der Literatur vorkommende Vorarbeiten.

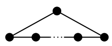
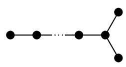
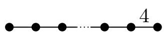
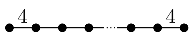
Acknowledgements

Firstly, I would like to express my gratitude to my advisor professor Auinger for his excellent tutorage. I thank him for the large amount of time and effort he must have expended to read through all the versions of the text (and the first versions were barely legible) that he commented on with great detail. I am grateful for all suggested modifications and simplifications for both proofs and definitions which hopefully made this text easier to read.
Secondly, I would like to thank my girlfriend for her patience and support during the creation of this thesis which took far longer than it should have. Děkuju.

Contents

Introduction

Let Γ be a finite Coxeter diagram with all bonds of strength 2 or 3, i. e. Γ is a finite unoriented unlabelled graph. There are some standard classes of Coxeter diagrams we will refer to later:

- $A_n =$ is the open path with n vertices,
- $\widetilde{A_n} =$  is the cycle with $n+1$ vertices.
- $D_n =$  is an open path with n vertices with an added leaf on the penultimate vertex.
- $B_n =$  is an open path n vertices with the last bond being of strength 4 (this does not satisfy our condition but we will comment on it later).
- $\widetilde{C_n} =$  is an open path with $n+1$ vertices with the first and last bond being of strength 4.

The Coxeter group $\mathrm{W}(\Gamma)$ of Γ is the group with generators being the vertices $V(\Gamma)$ of Γ subject to the relations

$$\begin{aligned} v^2 &= 1, & \\ uv &= vu & \text{when } u, v \text{ are not adjacent in } \Gamma, \\ uvu &= vuv & \text{when } u, v \text{ are adjacent in } \Gamma. \end{aligned}$$

Thanks to the first relation one obtains a group even when this presentation is considered to be one of a monoid.
Let $\mathscr{R} := \mathbb{Z}[q, q^{-1}]$. The Hecke algebra $\mathcal{H}(\Gamma)$ is the $\mathscr{R}$-algebra with generators $e_v, v \in \mathrm{V}(\Gamma)$ subject to the relations

$$\begin{aligned} e_v^2 &= (q^2-1)e_v + q^2, & v \in V(\Gamma), \\ e_u e_v &= e_v e_u, & u, v \in V(\Gamma) \text{ are not adjacent in } \Gamma, \\ e_u e_v e_u &= e_v e_u e_v, & u, v \in V(\Gamma) \text{ are adjacent in } \Gamma. \end{aligned}$$

The Hecke algebra then has the basis

$$\{e_w \mid w \in \mathrm{W}(\Gamma)\}$$

where

$$e_w = e_{v_1} \cdots e_{v_l}$$

whenever $w = v_1 \cdots v_l$ is a word of minimal length for w. One then defines for each edge $e = \{u, v\} \in \Gamma$ an element

$$i_e = e_{uvu} + e_{uv} + e_{vu} + e_u + e_v + 1 = \sum_{g \in \langle u,v \rangle} e_g$$

Then the generalised Temperley-Lieb algebra is the quotient $\mathcal{TL}(\Gamma) := \mathcal{H}(\Gamma)/I$ where I is the ideal generated by all the elements i_e for all edges $e \in \Gamma$.
Denote by ϖ the projection $\mathcal{H}(\Gamma) \to \mathcal{TL}(\Gamma)$. Let $\mathrm{W}_c(\Gamma)$ be the elements of $\mathrm{W}(\Gamma)$ whose expressions of minimal length do not contain the subword uvu for adjacent $u, v \in V(\Gamma)$. Define

$$f_v := \varpi(q^{-1}(e_v + 1))$$

and

$$f_w := f_{v_1} \cdots f_{v_l}$$

where $w = v_1 \cdots v_l$ is an expression for $w \in \mathrm{W}_c(\Gamma)$ of minimal length. By [Fan96, section 1] $\mathcal{TL}(\Gamma)$ has basis

$$\{f_w \mid w \in \mathrm{W}_c(\Gamma)\}$$

where $f_w := f_{v_1} \cdots f_{v_l}$ whenever $v_1 \cdots v_l$ is a word of minimal length for w. Moreover as a consequence of [Fan96, lemma 1] $\mathcal{TL}(\Gamma)$ is generated as an $\mathscr{R}$-algebra by $f_v, v \in V(\Gamma)$ subject the relations

$$\begin{aligned} f_v^2 &= \delta f_v, && v \in \mathrm{V}(\Gamma), \\ f_u f_v &= f_v f_u, && u, v \in V(\Gamma) \text{ are not adjacent in } \Gamma, \\ f_u f_v f_u &= f_u, && u, v \in V(\Gamma) \text{ are adjacent in } \Gamma \end{aligned}$$

where $\delta := (q + q^{-1})$. To be precise, the expressions found in [Fan96] are slightly differing to what we have written here but it is equivalent as is explained in [Gre06] where this form of expressions can be found but with different justification as it deals with a more general case of Coxeter diagrams with stronger bonds.
For the case $\Gamma = A_n$ the Tempereley-Lieb algebra first appears (according to [Jon85, p. 104]) in [TL71]. It has been studied extensively by Jones. As a quotient of the Hecke algebra, it is presented by Jones in [Jon87]. The generalized case is studied first in [Fan96].
One can then extract a presentation of a monoid $\mathrm{M}(\Gamma)$ by saying $\mathrm{M}(\Gamma)$ has generators $V(\Gamma)$ subject to the relations

$$\begin{aligned} v^2 &= v, && \\ uv &= vu && \text{when } u, v \text{ are not adjacent in } \Gamma, \\ uvu &= u && \text{when } u, v \text{ are not adjacent in } \Gamma. \end{aligned}$$

Then $\mathcal{TL}(\Gamma)$ is isomorphic to the $\mathscr{R}$-algebra $\mathcal{M}(\Gamma, \Delta)$ with the underlying $\mathscr{R}$-module $\mathscr{R}[\mathrm{M}(\Gamma)]$ and the multiplication defined by

$$e_{m_1}e_{m_2} = \Delta(m_1, m_2)e_{m_1m_2}.$$

The map $\Delta : \mathrm{M}(\Gamma) \times \mathrm{M}(\Gamma) \to \mathscr{R}$ is defined by

$$\begin{aligned}\Delta(v, v) &= (v + v^{-1})\\ \Delta(u, v) &= 1, u \neq v\\ \Delta(\epsilon, \epsilon) &= 1\end{aligned}$$

and by the condition required for the associativity [i]

$$\Delta(p, qr)\Delta(q, r) = \Delta(p, q)\Delta(pq, r).$$

This construction is called the twisted monoid algebra and the map Δ is sometimes called the twisting (e. g. [Wil07]).
A monoid from the Temperley-Lieb algebra was first distilled by Kauffman in [Kau90] for the case $\Gamma = A_n$. He was the first to notice the equivalence between the Temperley-Lieb algebra and the algebra of certain diagrams which naturally form a monoid. The formulation by twisted monoid algebra is due to Jones [Jon94] who directly cites Kauffmann. Wilcox then puts all the diagram algebras in the light of twisted monoid algebras in [Wil07].
As mentioned, for the case $\Gamma = A_n$, $\mathrm{M}(A_n)$ was described as a monoid of diagrams by Kauffman, but a thorough proof of equivalence of the monoids was given more than ten years later in [BDP02] and more recently a different proof was given by East in [Eas21]. One can allow the bonds in Γ to be of strength 4, since such bond still yields a relation with only two monomials as can be seen for example in [Gre06, prop. 2.6]. One can then alter Δ to still get the twisted algebra formulation. As such a diagram treatment was given for cases $\Gamma = B_n$ and $\Gamma = D_n$ in [Gre98a]. recently a case $\Gamma = \widetilde{C_n}$ was given a diagram treatment in two papers [Ern12],[Ern18] by Ernst. All of these are only in the language of algebras.
This thesis then presents $\mathrm{M}(\Gamma)$ as a monoid of diagrams for the case $\Gamma = \widetilde{A_n}$. Viewing $\mathcal{TL}(\widetilde{A_n})$ as an algebra of diagrams on a cylinder was first introduced in [MS93]. There is however no proof given of the isomorphism. That has been addressed (still in the language of algebras) in the paper of Fan and Green [FG99]. We have however found some problems with the proof in this paper we attempt to rectify.
We show that $M(\widetilde{A_n})$ is a certain submonoid of the monoid of certain diagrams drawn on the surface of a cylinder (whereas the Kauffman diagrams are drawn in a rectangle). We then also give a presentation of the whole monoid of the diagrams on the cylinder.
Note that in [GL98] the twisted algebra for this larger monoid is called the affine Temperley-Lieb algebra whereas in [FG99] the algebra $\mathcal{TL}(\widetilde{A_n})$ is called the affine Temperley-Lieb algebra. Thus we choose to avoid the naming affine Temperley-Lieb (as

[i] This amounts to saying that Δ is a 2-cocycle (for the trivial action $\mathrm{M}(\Gamma) \to \mathrm{Aut}(\mathscr{R})$).

has been previously used for the more general case of categories in [AV20]) and we use the name affine Jones monoid.
The affine Jones monoid (or rather its twisted algebra) was presented in a subsequent paper by Green [Gre98b]. We found no problems with this paper, but we restate the statement in the language of monoids and give a more detailed proof.
After presenting our proof we compare our approach to the one given by Fan and Green and point out where we believe there to be the shortcomings.

Chapter 1

Definitions

1.1 Disjoint unions

By $[n]$ we mean the set $\{1, \ldots, n\}$.
For $t \geq 0$ let $S_0, \ldots, S_t$ be sets. By their disjoint union we understand the set

$$S_0 \sqcup \cdots \sqcup S_t := \{(s, i) \mid i \in \{0, \ldots, t\}, s \in S_i\}.$$

In the case $S_0 = S_1 = \cdots = S_t$ this definition reduces to

$$S \sqcup \cdots \sqcup S := S \times \{0, \ldots, t\}.$$

We write the elements (s, i) as

$$\boxed{s|i}$$

and we use the box even just for naming, i. e.

$$\boxed{a} = \boxed{s|i}.$$

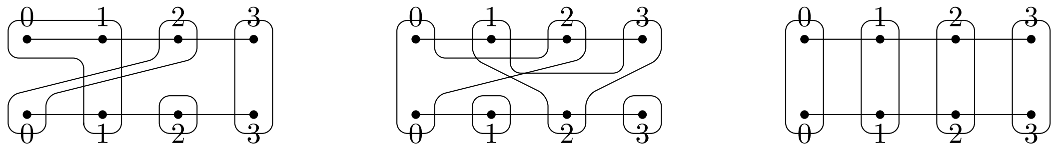
We denote the inclusions $S_i \to S_0 \sqcup \cdots \sqcup S_t$

$$s \mapsto \boxed{s|i}$$

by ι_i for all $i = 0, \ldots, t$.

1.2 Partitions

A **partition** of a set H is an equivalence on H given as the set of equivalence classes. We call the equivalence classes **blocks**.
In the context of partition monoids, the set H is always of the form $S \sqcup S$. We will call partitions of $S \sqcup S$ S**-partitions** (we will omit the letter S when the context is clear).
We call the elements $\boxed{s|0}$ **top nodes** and the elements $\boxed{s|1}$ **bottom nodes**.
Let p_1, p_2 be two S-partitions. Then we can identify the bottom nodes $\iota_1(S)$ of p_1 with the top nodes $\iota_0(S)$ of p_2 and this way obtain a relation q on $S \sqcup S \sqcup S$. Let q^{cl} be the



(a) An example of a $\mathbb{Z}_4$-partition. We draw the elements of each copy of S on a horizontal line. The top nodes ($\boxed{\cdot \mid 0}$) lie on the top line and the bottom nodes ($\boxed{\cdot \mid 1}$) on the bottom line.

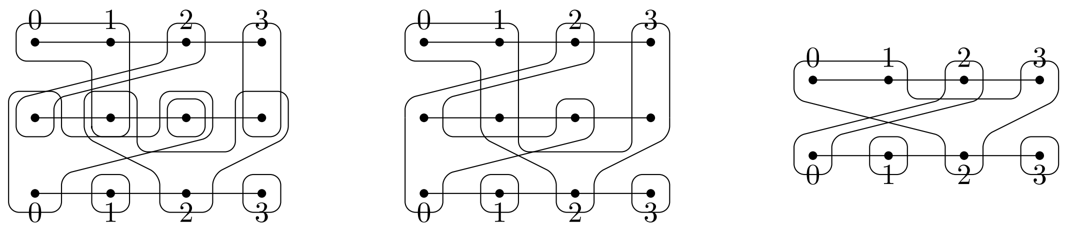
(b) The $\mathbb{Z}_4$-partition for multiplication by 2 modulo 4 obtained by the construction in exmpl. 1.2.1

(c) The partition $\mathrm{Id}_{\mathbb{Z}_4}$

Figure 1.1: Examples of partitions

transitive closure of q, that is, the smallest equivalence containing this relation. Then we define the **composition** $p_1 \circ p_2$ to be the S-partition obtained by the restriction of q to the subset $\mathrm{V}^{\diamond} := \iota_1(S) \cup \iota_3(S) \cong S \sqcup S$ which we call the **outer nodes** of $S \sqcup S \sqcup S$. In an analogous way we can define the composition for any finite sequence $p_1, \dots, p_t$ by identifying the bottom nodes of p_i with the top nodes of p_{i+1} and then show associativity by showing

$$(p_1 \circ p_2) \circ p_3 = p_1 \circ p_2 \circ p_3 = p_1 \circ (p_2 \circ p_3).$$



(a) The relation q.

(b) The equivalence q^{cl}.

(c) The composition partition.

Figure 1.2: Composition of the $\mathbb{Z}_4$-parititions from example fig. 1.1.

Example 1.2.1. Let $\mathrm{f} : S \to S$ be any map. Then corresponding to it is an S-partition p_{f} with one block for each $s \in S$

$$b_s = \{\boxed{s \mid 1}\} \cup \iota_0(\mathrm{f}^{-1}(s)).$$

The composition of partitions then corresponds to the composition of maps, so that,

$$p_{\mathrm{f}} \circ p_{\mathrm{g}} = p_{\mathrm{g} \circ \mathrm{f}}.$$

$\diamondsuit$

We also have an S-partition $\mathrm{Id}_S := \{\{\boxed{s|0}, \boxed{s|1}\} \mid s \in S\}$. This is the S-partition p_{Id_S} from the example above. This partition satisfies

$$p \circ \mathrm{Id}_S = p = \mathrm{Id}_S \circ p$$

for any S-partition p, and so we have a monoid $(\mathfrak{P}_S, \circ, \mathrm{Id}_S)$ called the **partition monoid on** S. This was introduced by Martin in [Mar96]. He generalised this notion from some submonoids the oldest of which is the Brauer monoid introduced by Brauer in [Bra37]. By a **Brauer** S-partition we understand an S-partition where all blocks are of size 2. This allows us to view a Brauer S-partition as a graph on the vertices $S \sqcup S$ where each edge corresponds to a block. In this way, we get all graphs with the degrees of all vertices equal to 1, i. e. all perfect matchings. Thus we call the blocks of Brauer partitions **edges**. We denote the edge $e = \{\boxed{a}, \boxed{b}\}$ by

$$\boxed{a}-\boxed{b}$$

or by

$$\underset{e}{\boxed{a}-\boxed{b}}$$

should we need the name e to be included. When we know, for example, that $\boxed{a}$ is a top node, we will emphasize it by the notation

$$\boxed{a}\,\boxed{b}$$

or again

$$\underset{e}{\boxed{a}\,\boxed{b}}.$$

This allows us to implicitly pose some conditions, e. g.

$$\boxed{s_1|x_1}\,\boxed{s_1|x_2}$$

implies $x_2 = x_1$ or $x_2 = x_1 - 1$ (which will be important later for the product graphs). We call the edges $\boxed{a|0}\,\boxed{b|0}$ **top horizontal**, $\boxed{a|1}\,\boxed{b|1}$ **bottom horizontal** and the remaining edges **vertical**.

Example 1.2.2. A class of examples of Brauer partitions is provided by p_σ for any bijection $\sigma : S \to S$ from the construction in the example exmpl. 1.2.1. One can obtain in this way precisely all the Brauer partitions without horizontal edges. $\lozenge$

For Brauer partitions a very important notion is the **product graph** introduced in [Aui14] and [Eas14] (in the second reference it is introduced for an arbitrary set S and also non-Brauer partitions). Let $P = (p_1, \ldots, p_t)$ be a sequence of Brauer S-partitions. The product graph $\mathcal{G}(P)$ is a (labeled multi) graph on vertices $\underbrace{S \sqcup \cdots \sqcup S}_{t+1 \text{ times}}$ with edges

$$\mathrm{f}_1(p_1) \sqcup \cdots \sqcup \mathrm{f}_t(p_t)$$

where f_i is the map induced by the map on vertices

$$\mathrm{f}_i(\boxed{s|0}) = \iota_{i-1}(s)$$
$$\mathrm{f}_i(\boxed{s|1}) = \iota_i(s).$$

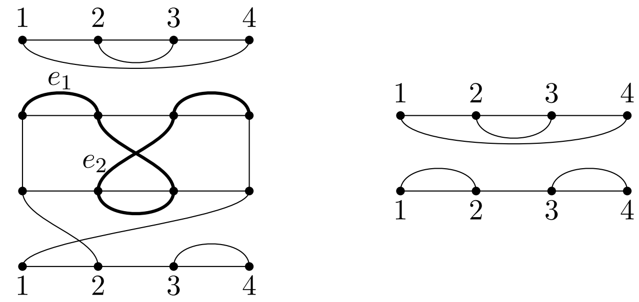
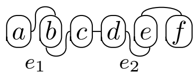


Figure 1.3: Example of a product graph of three Brauer [4]-partitions. The middle partition can be obtained from the construction in exmpl. 1.2.2 for the transposition $(2,3)$. The thick path can be described by the example string (1.1) when $\boxed{a} = \boxed{1|1}, \boxed{b} = \boxed{2|1}, \boxed{c} = \boxed{3|2}, \boxed{d} = \boxed{2|2}, \boxed{e} = \boxed{3|1}, \boxed{f} = \boxed{4|1}$.
The notation $\boxed{x}\ \boxed{2|1}$ uniquely describes the edge $\boxed{1|1}\ \boxed{2|1}$ since $\boxed{x}$ has to be a node $\boxed{i|0}$ or $\boxed{i|1}$ and so the edge $\boxed{3|2}\ \boxed{2|1}$ does not fit.
On the right is the resulting composition (which is just the first partition).

In other words, it is the graph $p_1 \sqcup \cdots \sqcup p_t$ after identifying the bottom vertices of p_i with top vertices of p_{i+1}. Note that this gives the edges labels and so we know, that an edge (e,i) is an image $\mathrm{f}_i(e)$, $e \in p_i$. Hence, we will not distinguish between $(e,i) \in \mathcal{G}(P)$ and $e \in p_i$.
The product graph can be identified with the relation q from the definition of $p_1 \circ \cdots \circ p_t$ which is now also a graph since the partitions p_i are Brauer.[i] The vertices of the connected components of $\mathcal{G}(P)$ are precisely the blocks of q^{cl} and therefore their restrictions to the outer nodes $\mathrm{V}^\diamond = \iota_1(S) \cup \iota_{t+1}(S)$ give the edges of $p_1 \circ \cdots \circ p_t$. We call the connected components of $\mathcal{G}(P)$ **strings**. Note that not all strings correspond to edges in the composition as they might not contain any outer node.
Also note that the vertices $\mathrm{V}^\diamond$ are of degree 1 and all other vertices are of degree 2. Consequently, any connected component of $\mathcal{G}(P)$ is an (open simple) path (possibly infinite in one or both directions) or a cycle and the open paths end only in the outer

[i]That is the reason for calling it product graph, since one might call the composition of partitions their product.

nodes $V^\diamond$. Therefore, all blocks of $p_1 \circ \cdots \circ p_t$ are of size 1 or 2 and the blocks of size 1 correspond to paths infinite in one direction.
As a result, the Brauer $[n]$-partitions, denoted $\mathfrak{B}_n$, form a submonoid of $\mathfrak{P}_{[n]}$, as there can be no infinite paths in a finite graph.
In the rest of the thesis, a **path** in a graph will always mean an open simple path.
Finally, note that the product graph might technically be a multigraph, but the parallel edges always form a cycle of size 2. Keeping this in mind, we will usually give walks through product graphs only by a sequence of vertices with the notation for edges in between as introduced earlier. The string

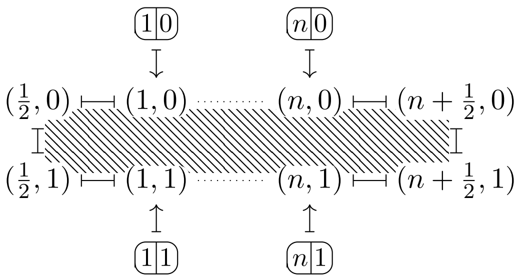


(1.1)

is an example of our notation for paths.

1.3 Jones partitions

We shall understand the y-axis of the Euclidean plane being downward directed (deviating from the usual convention) as the diagrams are usually composed from top to bottom. Also, for the rest of this thesis $n \geq 3$ is a fixed natural number.
We can define an even smaller submonoid of Brauer $[n]$-partitions by giving a topological restriction. By a **Jones** $[n]$-partition we understand a Brauer $[n]$-partition that has a planar (with pairwise non-intersecting edges) drawing in the interior of the rectangle $\square$:



with the vertices placed as noted.

Example 1.3.1. The construction from exmpl. 1.2.2 produces a Jones $[n]$-partition only for the identity. $\diamondsuit$

If p_1, p_2 are Jones $[n]$-partitions then by stacking their planar drawings on top of each other we get a planar drawing for their product graph. The vertices are thus placed in the plane by

$$\boxed{i|k} \mapsto (i, k), \quad i \in [n], k = 0, 1, 2.$$

We can remove the (drawings for the) components without end-vertices from this drawing and "squish" to get a planar drawing of $p_1 \circ p_2$. Also, $\mathrm{Id}_{[n]}$ is obviously Jones and so the **Jones monoid** $\mathfrak{J}_n$ (the set of Jones $[n]$-partitions) is a submonoid of $\mathfrak{B}_n$.

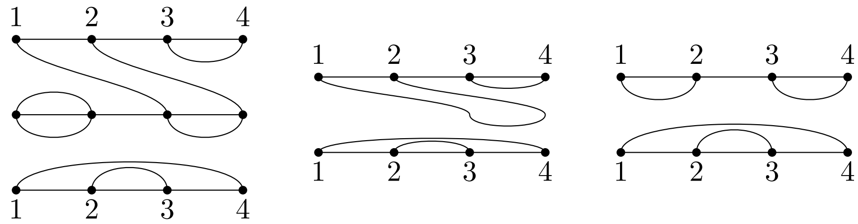


Figure 1.4: Example of a composition of two Jones [4]-partitions. Removing cycles and squishing of the stacked planar drawings produces a planar drawing of the composition partition, which can be seen in the middle. After some straightening, we get the drawing on the right.

As mentioned in the introduction these diagrams were invented by Kauffmann in [Kau90]. But the monoid he defined was slightly larger since he allowed for contractible loops to be present alongside the $[n]$-partition. He essentially added a generator o in the centre whose power counts these loops. This allowed him to define the Temperley-Lieb algebra as a quotient of the (untwisted) monoid algebra for the Kauffmann monoid. The twisted formulation producing the smaller monoid was given by Jones and so this smaller monoid is called Jones first named as such in [LF06]. One might then call the Kauffmann monoid the deformed Jones monoid using the terminology from e. g. [AV20].
Note, that combinatorially we can describe the Jones $[n]$-partitions by defining which edges intersect. Consider $[n] \sqcup [n]$ ordered by

$$1|0 \lhd \cdots \lhd n|0 \lhd n|1 \lhd \cdots \lhd 1|1,$$

i. e. the order is given by circumventing the perimeter of the rectangle clockwise. Then we say that the edges a-b, c-d, $a \lhd b$, $c \lhd d$ **intersect** if and only if one of the following holds

$$a \lhd c \lhd b \lhd d,$$
$$c \lhd a \lhd d \lhd b.$$

Then the partition is Jones if and only if it does not contain intersecting edges.

1.4 Annular partitions

By glueing the sides $\begin{array}{c}(\frac{1}{2},1)\\ | \\ (\frac{1}{2},0)\end{array}$ and $\begin{array}{c}(n+\frac{1}{2},1)\\ | \\ (n+\frac{1}{2},0)\end{array}$ of the rectangle ▭ we create a cylinder ⌭ and with it we obtain a slightly larger class of $[n]$-partitions called the **annular** $[n]$-partitions: those $[n]$-partitions which admit a planar (pairwise non-intersecting edges) drawing on the cylinder. Again by stacking cylinders, we see that annular partitions form another submonoid called the **annular monoid** $\mathfrak{A}_n$

$$\mathfrak{J}_n \subseteq \mathfrak{A}_n \subseteq \mathfrak{B}_n.$$

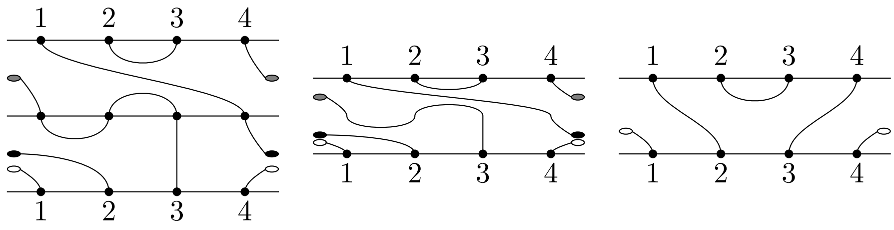


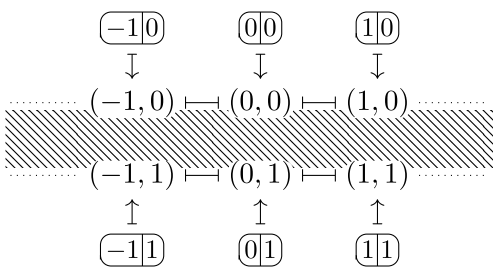
Figure 1.5: Example of a composition of two annular [4]-partitions. We use different marks to stress which parts of the edges that go through the glueing belong together. Note that both partitions are only annular and not Jones.
In the middle is the drawing on the cylinder of the composition obtained from squishing the stacked cylinders. On the right is the same partition with prettier edges. Note that the middle and right drawings have non-homotopic edges.

This is the monoid introduced in [Jon94]

Example 1.4.1. The construction from exmpl. 1.2.2 produces an annular $[n]$-partition precisely for the powers of the rotation $(1, \ldots, n)$. ♢

1.5 Bare diagrams

Whereas the strip is simply connected, the cylinder is not. So two drawings of the same edge are not necessarily homotopic. To understand the larger and more detailed structure where we distinguish drawings with non-homotopic edges we turn to the universal cover (which is always simply connected).
For the case of the cylinder, the universal cover is the infinite strip



with the covering map $\pi :$ $\to$

$$(kn + x, y) \mapsto (x, y), \quad k \in \mathbb{Z}, x \in \left[\frac{1}{2}, n + \frac{1}{2}\right), y \in [0, 1]$$

where we understand $\mathbb{Z} \sqcup \mathbb{Z} \subseteq \square$ as noted. With this identification we have $\pi^{-1}([n] \sqcup [n]) = \mathbb{Z} \sqcup \mathbb{Z}$ and the covering map then restricts to the map $\pi : \mathbb{Z} \sqcup \mathbb{Z} \to [n] \sqcup [n]$

$$\boxed{kn+i|y} \mapsto \boxed{i|y}, \quad k \in \mathbb{Z}, i \in [n], y = 0, 1.$$

The covering map comes with the automorphism group $\mathrm{Aut}(\pi)$ generated by the translation $\tau : \square \to \square$

$$(x, y) \mapsto (x + n, y)$$

which restricts to an automorphism of $\mathbb{Z} \sqcup \mathbb{Z} \subseteq \square$ which we will also denote τ.
We say a $\mathbb{Z}$-partition p is

- n**-periodic** if and only if it is invariant under the translation τ, meaning,

$$\tau(p) = p$$

  where we extend the translation τ onto edges by translating their end-nodes;

- **planar** when it is Brauer and has a drawing in the strip with pairwise non-intersecting edges. Combinatorially we can describe this notion again with the order

$$\cdots \lhd \boxed{-1|0} \lhd \boxed{0|0} \lhd \boxed{1|0} \lhd \cdots\cdots \boxed{1|1} \lhd \boxed{0|1} \lhd \boxed{-1|1} \lhd \cdots .$$

  and defining that two edges

$$\boxed{a}\!-\!\boxed{b}, \boxed{c}\!-\!\boxed{d}, \quad \boxed{a} \lhd \boxed{b}, \boxed{c} \lhd \boxed{d}$$

  intersect when either

$$\boxed{a} \lhd \boxed{c} \lhd \boxed{b} \lhd \boxed{d},$$

  or

$$\boxed{c} \lhd \boxed{a} \lhd \boxed{d} \lhd \boxed{b}.$$

  which is just the natural order of the top nodes followed by the reversed natural order of the bottom nodes. One might again think of it as the order given by the perimeter of the infinite rectangle which is the strip.

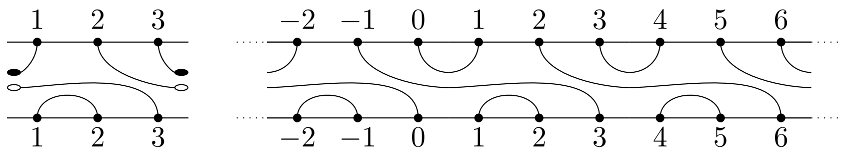


Figure 1.6: A planar drawing of an annular [3]-partition on the left and its lift by the covering to a planar drawing of a 3-periodic $\mathbb{Z}$-partition.

Let $c : [0,1] \to \square$ be a lift of a planar drawing of an edge of some annular $[n]$-partition. Then $\tau \circ c$ is a lift of the same drawing. So any $\mathbb{Z}$-partition obtained by endpoints of all lifts of a drawing is n-periodic.
Also, non-intersecting curves have non-intersecting lifts so any such $\mathbb{Z}$-partition is planar. On the other hand, any n-periodic planar Brauer $\mathbb{Z}$-partition has a drawing by non-intersecting edges that maps by the covering map to a planar drawing on the cylinder of some annular $[n]$-partition (for example by straight lines for vertical edges and squished half-circles for the horizontal edges).
Notice that any n-periodic Brauer $\mathbb{Z}$-partition is composed of n τ-orbits of edges. These orbits are precisely the $\mathbb{Z}$-edges whose drawings map onto a single homotopy class of drawings on the cylinder of single $[n]$-edge.
This translates onto the product graphs. These are also n-periodic and composed of some finite number of τ-orbits of strings. We can study the implications in more detail.

Discussion 1.5.1. Let

$$\boxed{i_1|x_1} - \cdots - \boxed{i_l|x_l}$$

be a path[ii] through a product graph of two or more n-periodic Brauer $\mathbb{Z}$-partitions. Let there be $a < b$ such that $x_a = x_b$ and

$$i_a \equiv i_b \pmod{n}.$$

This means that $\boxed{i_a|x_a}$ and $\boxed{i_b|x_b}$ are from the same orbit. Choose a, b such that $b - a$ is minimal. Let $i_b = i_a + mn$. Then

$$\boxed{i_a|x_a} - \cdots - \boxed{i_b = i_a{+}mn|x_b} - \boxed{i_{a+1}{+}mn|x_{a+1}} - \cdots - \boxed{i_b{+}mn|x_b}$$

is a path through the graph. That is because from the sequence

$$i_a, \ldots, i_b = i_a + mn, i_{a+1} + mn, \ldots, i_b + mn$$

only i_j and $i_j + mn$ are in the same class modulo n (for any j) but they differ by mn. Hence

$$\begin{gathered}\cdots - \boxed{i_a{-}mn|x_a} - \cdots - \boxed{i_{b-1}{-}mn|x_{b-1}} - \\ - \boxed{i_b{-}mn = i_a|x_a} - \cdots - \boxed{i_b = i_a{+}mn|x_b} - \\ - \boxed{i_{a+1}{+}mn|x_{a+1}} - \cdots - \boxed{i_b{+}mn|x_b} - \cdots\end{gathered}$$

is an infinite path through the product graph. Such a path then traverses the whole string it is part of.
One can see fig. 1.7 for illustration.
To sum it up, whenever a string contains two different nodes in the same τ-orbit this string is necessarily a path infinite in both directions and can be described as repeating translations of the finite path between the two nodes. $\Diamond$

[ii] As mentioned before, we always understand an open simple path, i. e. no vertex is visited more than once.

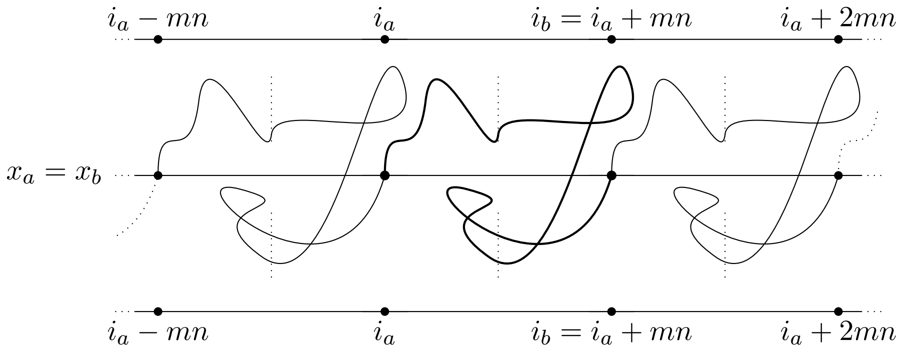


Figure 1.7: Whenever there appears the thick path, i. e. a path between two different nodes of the same τ-orbit ($\boxed{i_a|x_a}$ and $\boxed{i_a|x_b}$), we can translate this path in both directions and join the translations to the thick path. Thus such a path can appear only on a string infinite in both directions.

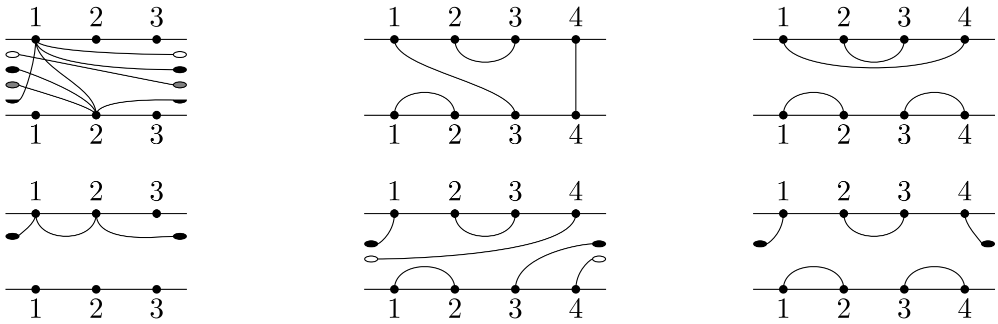
As a first consequence, there can be no infinite string containing an outer node. Hence any composition of n-periodic Brauer $\mathbb{Z}$-partitions is n-periodic Brauer.
Moreover, by stacking planar drawings of n-periodic planar $\mathbb{Z}$-partitions, we get a planar drawing of their product graph. Again by removing the components without end-vertices and by squishing, we get a planar drawing of their composition. And so the composition is not only n-periodic Brauer by the above but also planar. By a **bare diagram**, we understand an n-periodic planar Brauer $\mathbb{Z}$-partition that is planar and n-periodic. The bare diagrams thus form a submonoid of $\mathfrak{P}_{\mathbb{Z}}$ which we call the **bare affine Jones monoid** and denote it $\mathfrak{a}\mathfrak{J}_n^{\mathrm{b}}$.
This exposition of the diagrams is analogous to the approach in [GL98] where the (in our opinion less readable) language of planar involutions is used. But some key elements like the total order $\triangleleft$ or having the infinite $\mathbb{Z}$-partition as the basic view come from this paper. One can also confront our discussion with [GL98, lemmma 0.8].
The surjective homomorphism $\mathfrak{a}\mathfrak{J}_n^{\mathrm{b}} \twoheadrightarrow \mathfrak{A}_n$ is far from being a bijection. For any annular $[n]$-bipartition p the preimage

- is finite only if p does not contain any vertical edge;
- consists of $|\mathbb{Z}|$ different bare diagrams which differ by translating only the bottom (or only the top) nodes by a power of τ whenever p contains a vertical edge.

See fig. 1.8 for examples of different bare diagrams mapping onto the same annular partition.
For an edge $e = \boxed{i|x}\!-\!\boxed{j|y}$ of a bare diagram we call

$$l(e) = |i - j|$$

the **length of the edge** e. Then $l(\tau(e)) = l(e)$ and thereupon, length is an invariant of



(a) Different (non-homotopic) drawings of two edges that can appear in a [3]-partition. Every horizontal edge possesses $|\mathbb{Z}|$, whereas any vertical edge has only two non-homotopic drawings that do not self-intersect.

(b) Two of $\mathbb{Z}$ different non-equivalent planar drawings of the exact same annular [4]-partition.

(c) Two out of the nine non-equivalent planar drawings of the same [4]-partition without any vertical edges.

Figure 1.8

an orbit. Hence it makes sense to define the **length** $l(D)$ **of a bare diagram** D to be the sum of the lengths of its orbits.

Example 1.5.2. Recall the lift of the annular [3]-partition from fig. 1.6. In this lift the length of the vertical edges (all in the same orbit) is 4 while all horizontal edges (in two different orbits) are of length 1. $\Diamond$

Also, note that horizontal edges have a length of at most $n-1$, as otherwise, they intersect their own translation.
Although we defined the bare diagrams in terms of $\mathbb{Z}$-partitions it is useful to keep in mind the representation as a planar drawing of an annular $[n]$-partition. We will be at times also drawing the bare diagram (and diagrams) this way.

1.6 Diagrams

Either by looking at the cylinder or considering the implications of n-periodicity, we see that the product graph of bare diagrams D_1, D_2 contains strings of 3 types (another consequence of disc. 1.5.1):

- Finite strings with two end-vertices. These correspond bijectively to the edges of $D_1 D_2$. Each edge on such a string is from a different edge-orbit as each vertex is from a different vertex-orbit by disc. 1.5.1.
- Finite strings without end-vertices (cycles). By the covering map these map to the contractible loops on the cylinder. Again the orbits of the edges and vertices are

distinct by disc. 1.5.1.

- Infinite strings invariant under τ. These can occur only for even n and if there are no vertical strings present. These correspond by the covering map to non-contractible loops wrapping around the cylinder (which wrap around it once[iii]).

  This wrapping once then means that we can specialise disc. 1.5.1 in this case as the number m from this discussion always satisfies $m = \pm 1$.

  Discussion 1.6.1. Let s be an infinite string in some product graph of d bare diagrams. Then there are integers $p \geq 1$[iv] and

  $$\ldots, x_{-1}, x_0, x_1, \ldots$$

  and

  $$\ldots, r_{-1}, r_0, r_1, \ldots$$

  $(0 < r_p < d)$ such that for all $i \in \mathbb{Z}$ we have $x_{i+p} = x_i + n$ and $r_{i+p} = r_i$ and s is the infinite path

  $$\cdots \boxed{x_{-1} | r_{-1}} - \boxed{x_0 | r_0} - \boxed{x_1 | r_1} \cdots .$$

  The whole string is then covered by translations by powers of τ of any of the finite subpaths

  $$\boxed{x_i | r_i} - \cdots - \boxed{x_{i+p} | r_{i+p}}$$

  and each edge-orbit appearing on this string is represented exactly once on this subpath.

  These subpaths then correspond to one full path around the cylinder.

  ◊

These infinite components will turn out to be of significant importance later.
By a **diagram**, we understand a bare diagram together with a finite set of what we call **non-contractible circular edges** which set is empty if the bare diagram contains a vertical edge. Formally it is a pair (D, k) where D is a bare diagram and $k = 0$ if D contains a vertical edge.
When e is an abstract non-contractible circular edge (not necessarily as an edge of a diagram), we will say that e **intersects** any vertical edge and no horizontal or non-contractible circular edge.

Example 1.6.2. Let $n = 4$ and D be the diagram

$$\{\boxed{1|0}\,\boxed{4|0}, \boxed{2|0}\,\boxed{3|0}, \boxed{1|1}\,\boxed{4|1}, \boxed{2|1}\,\boxed{3|1}, e\}$$

where e is non-contractible circular. Then a vertical edge $\boxed{2|0}\,\boxed{1|1}$ intersects the edges $\boxed{1|0}\,\boxed{4|0}$ and e of D. ◊

[iii] curve wrapping around the cylinder more times must self-intersect as can be seen e. g. by the intermediate value theorem

[iv] p for period

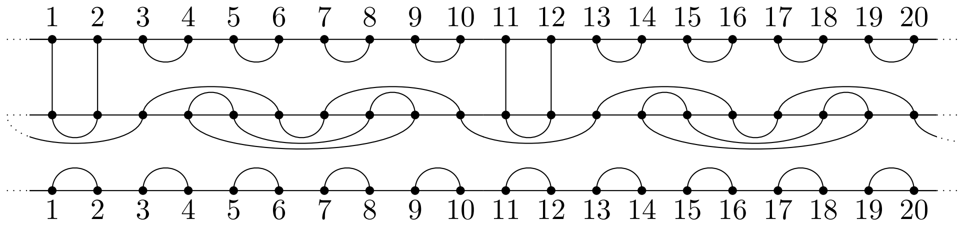


Figure 1.9: Example of a product graph of two bare diagrams for $n = 10$. We see ten τ-orbits of finite strings. An example of such a string is $\boxed{1|0}\ \boxed{1|1}\ \boxed{1|2}\ \boxed{2|0}$. Then we can also see one orbit of infinitely many cycles a representative of which is $\boxed{4|1}\ \boxed{5|1}\ \boxed{8|1}\ \boxed{9|1}\ \boxed{4|1}$. Finally we have one orbit of an infinite string. This orbit necessarily (thanks to planarity) contains only one string. This string is covered by translations of the path $\boxed{3|1}\ \boxed{6|1}\ \boxed{7|1}\ \boxed{10|1}\ \boxed{13|1}$. Each edge of this string appears on exactly one of the translations of this path.

As before we can still draw a diagram by non-intersecting curves in the strip where we draw the infinite edges by curves horizontally unbounded in both directions. These curves are ordered by their vertical position therefore we can assign uniquely the elements of $[k]$ to those curves (by convention 1 will be assigned to the top curve). Thus a non-circular edge of a diagram (D, k) is formally an element of $[k]$ and when we consider a drawing of (D, k), $i \in [k]$ will be the ith unbounded curve from top to bottom.
We compose the diagrams as follows

$$(D, k)(D', k') = (D \circ D', k + k' + \gamma(D, D'))$$

where $\gamma(D, D')$ is the number of infinite strings of the product graph of the bare diagrams $\mathcal{G}(D, D')$.

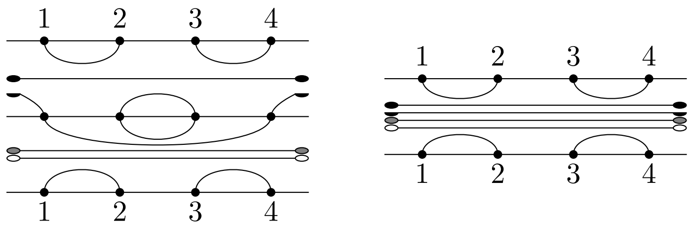


Figure 1.10: Example of a composition of two diagrams. On the left is the product graph, and on the right is the composition. One new non-contractible circular edge is formed.

The monoid of the diagrams is then called the **affine Jones monoid** and is denoted $\mathfrak{a}\mathfrak{J}_n$. The product graph of diagrams is then just a product graph of the bare diagrams with the infinite edges added. So a string is either a string of the product graph of the bare

diagrams or a single non-contractible circular edge of one of the diagrams. This way we can associate a string to any edge of the composition.
By edge, we will from now on mean an edge of some diagram.
For intuition it will be sometimes quite useful to imagine the non-contractible circular edges as a repeated finite (i. e. bounded) curve, each instance with endpoints $\tau^i(x), \tau^{i+1}(x)$ for some $x \in \square$. This is indeed often the correct way since it is consistent with the way we think about the other edges in terms of the covering map.
This justifies the definition of the **length of a non-contractible circular edge** to be n as it is the distance of the endpoints in this sense[v]. Also, the **length** $l(D)$ **of a diagram** (D_b, k) is the length of the bare diagram D_b plus the length of all the non-contractible circular edges, that is,

$$l(D) := l(D_b) + nk.$$

1.7 Some important diagrams

By a **singular** diagram, we understand a non-invertible one. By the **rank** of a diagram, we understand the number of different τ-orbits of vertical edges.

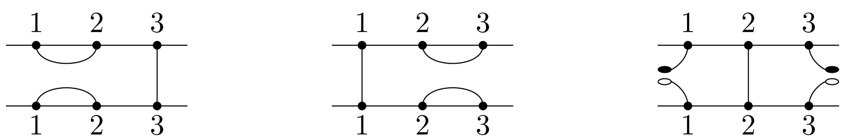


Figure 1.11: All the diapsides for $n = 3$. From left to right $\mathsf{DA}_1, \mathsf{DA}_2, \mathsf{DA}_3$.

For any $i \in [n]$, the **diapsis** is the diagram

$$\mathsf{DA}_i = \left(\mathrm{Id}_{[n]} \setminus \langle\tau\rangle \left\{ \boxed{i|0}\,\boxed{i|1}, \boxed{i{+}1|0}\,\boxed{i{+}1|1} \right\}\right) \cup \langle\tau\rangle \left\{ \boxed{i|0}\,\boxed{i{+}1|0}, \boxed{i|1}\,\boxed{i{+}1|1} \right\}.$$

See fig. 1.11 for pictures. These can be described as singular idempotent diagrams of maximal rank. We call the edges $\boxed{i|0}\,\boxed{i{+}1|0}$ and $\boxed{i|1}\,\boxed{i{+}1|1}$ the **top** and the **bottom apsides**. The term *diapsis* comes from the paper [BDP02][vi]. The original term used by Kauffmann is *hook* and it is still mainly used in the literature.

Figure 1.12: The twist И (left) and the countertwist N (right).

[v]We will see later a cleaner justification
[vi]There the authors explain that *apsis* means *arc* in Greek and so *diapsis* means *double arc*.

The **twist** is the diagram

$$\text{И} = \left\{ (i|0)\,(i{+}1|1) \mid i \in \mathbb{Z} \right\}.$$

It can be thought of as twisting the bottom bounding circle of the cylinder by $\frac{2\pi}{n}$ to the right. The term *twist* originates from [GL98].
The **countertwist** is its inverse

$$\mathsf{N} = \left\{ (i|0)\,(i{-}1|1) \mid i \in \mathbb{Z} \right\}.$$

One can see drawings of both on fig. 1.12.
We chose this notation for the diapsides and the twists since a composition resembles its product graph after rotating by 90 degrees. That is illustrated on fig. 1.13.

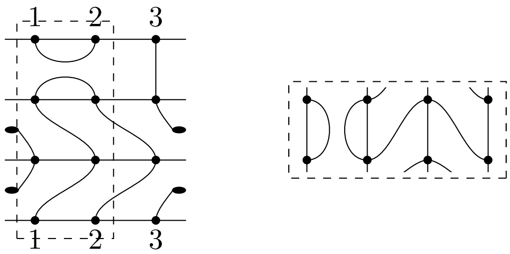


Figure 1.13: The composition Dꓷ$_1$ИN. One can just turn the notation by 90 degrees to get an idea about the product graph.

Chapter 2

Presentation

2.1 Generating diagrams

In this part, we will show that all diagrams are generated by the diapsides and the two twists by first showing that a natural submonoid of $\mathfrak{a}\mathfrak{J}_n$ is generated by the diapsides. This will turn out to be a monoid isomorphic to $\mathrm{M}(\widetilde{A_n})$ from the introduction.
Before the main result of this part is presented in section 2.1.4 we need to observe some properties of the diagrams.

2.1.1 Parity

Consider the edge $\boxed{i|0}\boxed{i+1|1}$. This edge will obviously never appear in a composition of diapsides as any vertical edge in such composition must have an even number of apsides appearing on its string and so has an even length.
To characterise such obstacles, the notion of parity must be introduced.

Definition 2.1.1. Let e be an edge of a diagram. We say that e **intersects the line** ${}^{k}|^{k+1}$ if and only if either

- $e = \boxed{i|x}\!-\!\boxed{j|y}$, $i \leq j$ is non-circular such that $i \leq k$ and $k+1 \leq j$ or
- e is non-contractible circular.

$\Diamond$

Notice that e intersects the line ${}^{k}|^{k+1}$ if and only if all drawings of e intersect the line $x = k + \frac{1}{2}$. One could also use the order definition by adding the pair of nodes $\boxed{k+{}^1\!/\!_2|0}$, $\boxed{k+{}^1\!/\!_2|1}$ with

$$\boxed{k|0} \triangleleft \boxed{k+{}^1\!/\!_2|0} \triangleleft \boxed{k+1|0},$$

$$\boxed{k+1|1} \triangleleft \boxed{k+{}^1\!/\!_2|1} \triangleleft \boxed{k|1}.$$

Then an edge e intersects the line ${}^{k}|^{k+1}$ if and only if e intersects the edge $\boxed{k+{}^1\!/\!_2|0}\boxed{k+{}^1\!/\!_2|1}$ in the combinatorial sense.

Important to realise is that for any diagram, there are only finitely many edges intersecting the line ${}^{k}|^{k+1}$. That is because from any τ-orbit of some e there can be at most $\left\lceil \frac{l(e)}{n} \right\rceil$ edges intersecting this line.

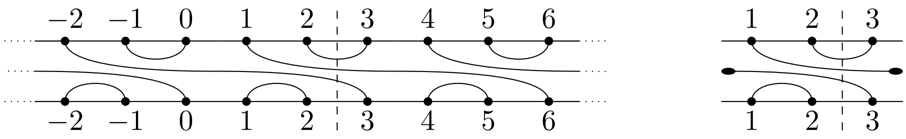


Figure 2.1: This diagram for $n = 3$ intersects the line ${}^{k}|^{k+1}$ 3 times. There are two vertical edges from the same orbit intersecting the line and one horizontal edge. On the cylinder, this means that any drawing of the vertical edge (of the correct homotopy class) intersects the drawn line at least twice.

Definition 2.1.2. By saying a diagram D **intersects the line** ${}^{k}|^{k+1}$ c **times** we mean that there are c edges of D that intersect the line ${}^{k}|^{k+1}$.
We denote this number by $\mathrm{X}({}^{k}|^{k+1}, D)$. $\Diamond$

Lemma 2.1.3. For any k, k', $\mathrm{X}({}^{k}|^{k+1}, D)$ and $\mathrm{X}({}^{k'}|^{k'+1}, D)$ have the same parity, i. e.

$$\mathrm{X}({}^{k}|^{k+1}, D) \equiv \mathrm{X}({}^{k'}|^{k'+1}, D) \pmod 2.$$

$\Diamond$

Proof. We will prove that the parity is the same for the lines ${}^{k}|^{k+1}$ and ${}^{k+1}|^{k+2}$.
We will focus on the two nodes $\boxed{k+1|0}$, $\boxed{k+1|1}$. Note that any edge that does not end in these two nodes intersects either both or none of the lines ${}^{k}|^{k+1}$ and ${}^{k+1}|^{k+2}$.
So only the edges ending in those two nodes can make a difference between the two lines.
Now we have to consider two cases:

- $\boxed{k+1|0}\boxed{k+1|1} \in D$ is an edge of D. Then this edge does not intersect any of the two lines. And so both lines are intersected by the same number of edges (actually by no edges):
$$\mathrm{X}({}^{k}|^{k+1}, D) = \mathrm{X}({}^{k'}|^{k'+1}, D)\ (= 0).$$
- There are two distinct edges e_1, e_2 with end-nodes $\boxed{k+1|0}$, $\boxed{k+1|1}$. Then both edges intersect exactly one of the lines. Thus either both intersect the same line in which case the number of edges is larger by two for one of the lines (but has the same parity):
$$\mathrm{X}({}^{k}|^{k+1}, D) = \mathrm{X}({}^{k'}|^{k'+1}, D) \pm 2.$$
  Otherwise, each edge intersects one of the lines and the number of intersecting edges is the same for both lines.
$$\mathrm{X}({}^{k}|^{k+1}, D) = \mathrm{X}({}^{k'}|^{k'+1}, D).$$

□

The parity being constant then allows us to make the following definition.

Definition 2.1.4. Let D be a diagram. We say that D is **even/odd** if and only if $\mathrm{X}({}^{k}|^{k+1}, D)$ is even/odd for every $k \in \mathbb{Z}$. ◊

Lemma 2.1.5. Let s be a string of $\mathcal{G}(D, D')$ for some diagrams D, D'. If s is a cycle then the number of edges on s intersecting the line ${}^{k}|^{k+1}$ is even. If s is an open path then the number of edges on s intersecting the line ${}^{k}|^{k+1}$ is odd if and only if the associated edge e of DD' intersects the line ${}^{k}|^{k+1}$. ◊

Proof. Consider any path $p = \boxed{i_1|r_1}-\boxed{i_2|r_2}\cdots\boxed{i_{l-1}|r_{l-1}}-\boxed{i_l|r_l}$ on s. Suppose that $i_1 \leq k$. Then by induction on l it is clear that $i_l \leq k$ if the number of edges of p intersecting the line ${}^{k}|^{k+1}$ is even and $i_l \geq k+1$ if that number of edges is odd. Analogously for $i_1 \geq k+1$. From this, the claim follows by considering the maximal path (possibly closed) on s when s is finite (cycle or associated to a non-circular edge).
For infinite strings s we have to use the discussion 1.6.1. So s is an infinite path

$$\cdots\boxed{x_{-1}|1}\;\boxed{x_0|1}\;\boxed{x_1|1}\cdots.$$

such that for some $p \geq 1$ we have $x_{i+p} = x_i + n$ for all $i \in \mathbb{Z}$.
Thus the sequences

$$\ldots, x_{i-p}, x_i, x_{i+p}, \ldots$$

are increasing arithmetic sequences for all $i \in [p]$. Hence there are $a < b \in \mathbb{Z}$ such that for any $j \leq a$

$$x_j \leq k$$

and for any $j \geq b$

$$x_j \geq k+1$$

Then just consider the subpath

$$\boxed{x_a|1}-\cdots-\boxed{x_b|1}$$

which then has $x_a \leq k, x_b \geq k+1$ and so, by the inductive statement, must contain an odd number of edges intersecting the ${}^{k}|^{k+1}$. But also no other edges on s intersect this line (all are too far left or right).
See fig. 2.2 for an example (with three diagrams, since that makes it more instructive as the subpath can then contain multiple edges from the same τ-orbit). □

This combinatorial behaviour of the parity for the individual edges then translates onto the diagrams.

Corollary 2.1.6. The composition of two even or two odd diagrams is even. The composition of an odd and an even diagram is odd. ◊

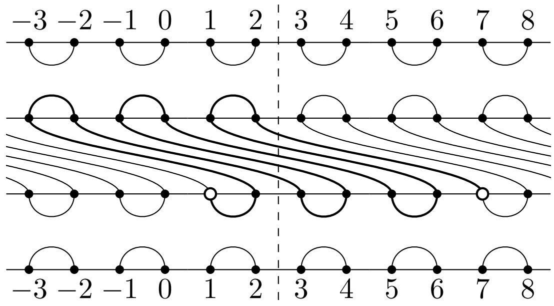


Figure 2.2: In this example for $n = 4$ the infinite string is split into three parts separated by nodes $\boxed{1|2}$ and $\boxed{7|2}$. The finite path between these two nodes contains all the edges on this string which intersect the line ${}^{2}|^{3}$. These two nodes are always on opposite sides of this line as the remaining two half-infinite paths are.

Proof. Let S be the set of all edges of $\mathcal{G}(D, D')$ that intersect the line ${}^{k}|^{k+1}$. Then these edges are partitioned into blocks where a block is defined by being on the same string of $\mathcal{G}(D, D')$. Let B_{odd} be the set of blocks of odd size. Then

$$|B_{\text{odd}}| = \mathrm{X}({}^{k}|^{k+1}, DD')$$

by lem. 2.1.5. But also

$$|B_{\text{odd}}| \equiv |S| \pmod 2$$

and $|S| = \mathrm{X}({}^{k}|^{k+1}, D) + \mathrm{X}({}^{k}|^{k+1}, D')$ is odd if and only if precisely one of D, D' is odd. □

The following lemma explains why the notion of parity is so important for presenting $\mathfrak{a}\mathfrak{J}_n$.

Lemma 2.1.7. Let D be a product of some diapsides. Then D is even. ◊

Proof. Diapsides are even. □

This also resolves the motivating example as any diagram containing the edge $\boxed{i|0}\boxed{i{+}1|1}$ is necessarily odd (it is the only edge intersecting the line ${}^{i}|^{i+1}$).
Thus we cannot hope to create odd diagrams as compositions of diapsides. But we will see that this is the only obstacle (for singular diagrams) as the reverse statement cor. 2.1.17 holds.

2.1.2 Length

To prove the reverse statement of lem. 2.1.7 we need to better understand the behaviour of the length of a diagram.

For $r_1 \leq r_2$ we write $\boxed{i \mid r_1}\,\boxed{i \mid r_2}$ for the path

$$\boxed{i \mid r_1}\,\boxed{i \mid r_1{+}1} \cdots \boxed{i \mid r_2{-}1}\,\boxed{i \mid r_2}$$

and analogously $\boxed{i \mid r_1}\,\boxed{i \mid r_2}$ for $r_1 \geq r_2$. We even allow $r_1 = r_2$, i. e. to describe a path of length 0.
By using this notation, we implicitly assume $r_1 \leq r_2$ ($r_1 \geq r_2$) similarly to the notation for the edges.

Definition 2.1.8. By a **forward-backward move**, we understand a path in some product graph of the form

$$\boxed{i \mid r_1}\,\boxed{j \mid r_2}\,\boxed{j \mid r_3}\,\boxed{k \mid r_4}$$

with $i < j > k$ or $i > j < k$. We will understand by a forward-backward move such path even in the other direction, i. e.

$$\boxed{k \mid r_4}\,\boxed{j \mid r_3}\,\boxed{j \mid r_2}\,\boxed{i \mid r_1}.$$

$\Diamond$

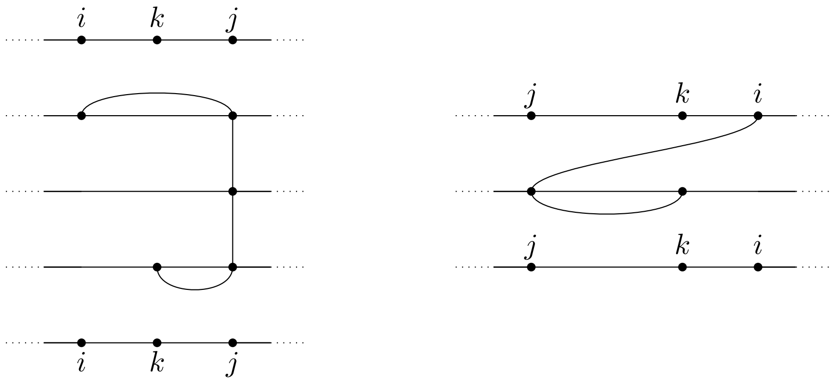


Figure 2.3: Examples of forward-backward moves.

Lemma 2.1.9. Let $D_1, \ldots, D_l$ be diagrams, e be an edge of $D_1 \cdots D_l$ and s be the string of $\mathcal{G}(D_1, \ldots, D_l)$ associated to e. Denote by O the set of distinct edge-orbits appearing on s and R a set of representatives of O. Then $l(e) = \sum_{f \in R} l(f)$ if and only if s contains no forward-backward move.
In particular, if s contains a forward-backward move then $l(e) \leq \sum_{f \in R} l(f) - 2$. $\Diamond$

Proof. For a non-circular edge e, s is a path whose edges form already a set of representatives R.
For a non-contractible circular edge e we will once again use disc. 1.6.1. Consider any path on s

$$s_{\text{cov}} = \boxed{x_i \mid r_i} - \cdots - \boxed{x_{i+p} \mid r_{i+p}}$$

from the discussion. Then again the set of edges appearing on it form a set of representatives R. If s contains a forward-backward move q, we can choose i such that q is a subpath of s_{cov}.
Either way we have a path (s itself or s_{cov}) containing all the edge-orbits of s exactly once and that contains a forward-backward move if s contains one. Let us call this path

$$r = \boxed{a_1} \cdots \boxed{a_l}.$$

Then

$$l(e) = |c_1 l(\boxed{a_1}\boxed{a_2}) + \cdots + c_{l-1} l(\boxed{a_{l-1}}\boxed{a_l})| \tag{2.1}$$

where we put

$$c_i = \begin{cases} 1 & x_i < x_{i+1}, \\ -1 & x_i > x_{i+1}, \\ 0 & x_i = x_{i+1} \end{cases}$$

where $\boxed{a_i} = \boxed{x_i | r_i}$. That means that $c_i = 0$ for vertical edges of length 0 and c_i is 1 or -1 depending on whether p goes left or right through the edge $\boxed{a_i}\boxed{a_{i+1}}$. An example can be observed in fig. 2.4.

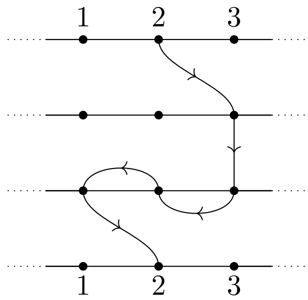


Figure 2.4: Illustration for proof of lem. 2.1.9. When traversing the string in the direction of the arrows (i. e. $\boxed{a_1} = \boxed{2|0}$) we assign to the edges the sequence of c_i's $1, 0, -1, -1, 1$. This reveals that this string contains two forward-backward moves, namely $\boxed{2|0}\boxed{3|1}\boxed{3|2}\boxed{2|2}$ corresponding to the subsequence $1, 0, -1$ and $\boxed{2|2}\boxed{1|2}\boxed{2|3}$ corresponding to the subsequence $-1, 1$.

Now it remains to realise, that $c_i, c_{i+1}, \ldots, c_j$ form a sequence

$$1, 0, \ldots, 0, -1$$

or

$$-1, 0, \ldots, 0, 1$$

(with possibly no zeros between the ones) if and only if

$$a_i - a_{i+1} - \cdots - a_j$$

is a forward-backward move.
Hence

$$l(e) = |c_1|\, l(a_1 - a_2) + \cdots + |c_{l-1}|\, l(a_{l-1} - a_l)$$

if and only if the coefficients c_i are either all non-negative or non-positive which, by the paragraph above, is the case if and only if r (and hence s) contains a forward-backward move. $\square$

Notice that this observation (crucially the equality (2.1)) is a justification for $l(e) = n$ for non-contractible circular edges as it is the only consistent choice.

Corollary 2.1.10. Let $D_1, \ldots, D_l$ be a sequence of diagrams. Then

- $l(D_1 \cdots D_l) \leq l(D_1) + \cdots + l(D_l)$,
- $l(D_1 \cdots D_l) = l(D_1) + \cdots + l(D_l)$ if and only if the product graph contains no forward-backward move.

$\lozenge$

Proof. The first statement is a straightforward corollary accounting for all edges of D and D'.
For the second statement, it is enough to realise, that any string of $\mathcal{G}(D_1, \ldots, D_l)$, that is a cycle, necessarily contains a forward-backward move. $\square$

2.1.3 Neighbours

The last ingredient that will be crucial for the uniqueness of the decomposition of diagrams is the notion of two edges of a diagram being neighbours. A notion characterising when two edges can be replaced by two different edges with the same end-nodes without intersecting other edges in the diagram.

Definition 2.1.11. We say that two edges of a diagram D **are neighbours** if and only if for any drawing of D their drawings can be connected by a curve that does not intersect a drawing of another edge of D. $\lozenge$

Observation 2.1.12. This can be made combinatorial. Let $a - b$, $c - d$ be different edges of D. Then they are neighbours if and only if $a - c$ does not intersect any edge of D. We could have actually picked any of $a - d$, $b - c$, $b - d$ instead of $a - c$.
When e is a topmost non-contractible circular edge, then e is a neighbour of precisely those horizontal edges $a|0 - b|0$, $a < b$ such that there are no horizontal edges $c|0 - d|0$, $c < d$ with $c < a < b < d$. This means that $a|0 - b|0$ is a neighbour of the top most non-contractible circular edge e if and only if $b|0 - a{+}n|0$ does not intersect any edge of D. In other words, $a|0 - b|0$ and e are neighbours if and only if $a|0 - b|0$ and $a{+}n|0 - b{+}n|0$ are neighbours.

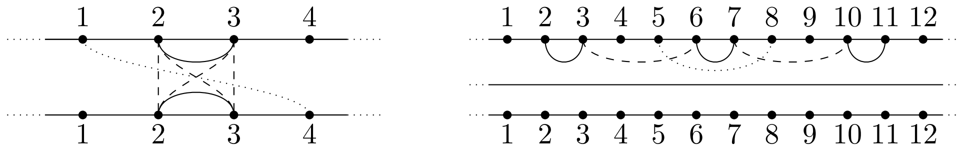


Figure 2.5: For both examples $n = 4$. The two solid edges are neighbours if and only if there are no edges intersecting the dashed edges, e. g. the dotted edge. In the second example one could think about using other edges for the criterium, for example, $(2|0)\,(6|0)$ or $(2|0)\,(7|0)$ but those are not edges in any diagram as they are too long for horizontal edges.

Similarly for bottom most non-contractible circular edges and bottom horizontal edges. See fig. 2.5 for illustration. $\Diamond$

Definition 2.1.13. We say an edge e **separates** two points x, y (that are not the end-nodes of e) in the border of the strip $\square$ if and only if any drawing of e intersects any curve in $\square$ with x, y as endpoints. This notion can again be made combinatorial by adding all the points of the border to the total order $\triangleleft$, i. e.

$$\mathbb{Z} \sqcup \mathbb{Z} \subseteq \mathbb{R} \sqcup \mathbb{R}$$

in the natural way and then we would just use e intersecting $(x)\!-\!(y)$, $(x), (y) \in \mathbb{R} \sqcup \mathbb{R}$ in the combinatorial sense as a definition. $\Diamond$

Observation 2.1.14. Let e be any edge. Then e intersects the line ${}^{i}|^{i+1}$ if and only if it separates the points $(i + \frac{1}{2}, 0)$ and $(i + \frac{1}{2}, 1)$. $\Diamond$

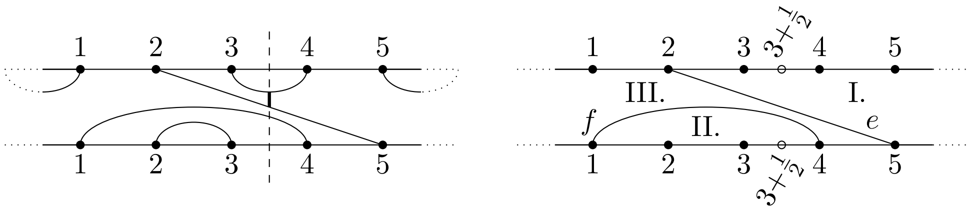
Lemma 2.1.15. Let D be a diagram that contains the top apsis $(i|0)\,(i{+}1|0)$ and a different edge that intersects the line ${}^{i}|^{i+1}$. Then there is a unique edge e that intersects the line ${}^{i}|^{i+1}$ and is a neighbour of $(i|0)\,(i{+}1|0)$. $\Diamond$

Proof. The existence can be proven combinatorially but can be seen easily by drawing the diagram with straight lines and squished half-circles. Then the straight line from $(i + \frac{1}{2}, 0)$ to $(i + \frac{1}{2}, 1)$ first intersects $(i|0)\,(i{+}1|0)$ and then another edge which is then a neighbour of $(i|0)\,(i{+}1|0)$. This edge must intersect the line ${}^{i}|^{i+1}$ since it was drawn by a straight line or a half-circle.
For uniqueness imagine two edges e, f of D intersecting the line ${}^{i}|^{i+1}$. Consider any drawing of D. Then e and f split the strip into three connected components:

I. one bordered only by e,

II. one bordered only by f,

III. one bordered by both e and f.



(a) Example for $n = 5$. When one draws the horizontal edges with half-circles and vertical with straight lines, one obtains the curve connecting the edges $\boxed{3|0}\boxed{4|0}$ and $\boxed{2|0}\boxed{5|1}$ as a segment of the straight line $x = 3 + \frac{1}{2}$. Those edges are thus neighbours and both intersect the line ${}^3|^4$.

(b) Both $\boxed{2|0}\boxed{5|1}$ and $\boxed{1|1}\boxed{4|1}$ intersect the line ${}^3|^4$. However in any drawing of the diagram, the point $(3 + \frac{1}{2}, 0)$ is always in the connected component bordered only by $\boxed{2|0}\boxed{5|1}$. Thus $\boxed{1|1}\boxed{4|1}$ can never be connected to $\boxed{3|0}\boxed{4|0}$ without intersecting $\boxed{2|0}\boxed{5|1}$ and thus $\boxed{1|1}\boxed{4|1}$ and $\boxed{3|0}\boxed{4|0}$ are not neighbours.

Figure 2.6

Since both e and f separate the points $(i + \frac{1}{2}, 0)$ and $(i + \frac{1}{2}, 1)$, none of those points can be in the third component and the points have to be in different components. So one point is in component I. and the other one is in II. Without loss of generality say that $(i + \frac{1}{2}, 0)$ is in the component I. bordered by e. Then also $\boxed{i|0}, \boxed{i{+}1|0}$ are in the component I. and so f is not a neighbour of $\boxed{i|0}\boxed{i{+}1|0}$. This proves the uniqueness.
Look at fig. 2.6 for better understanding. □

2.1.4 Generating diagrams

Since we have done all the preparatory work, we are now ready to formulate and prove the main lemma.

Lemma 2.1.16. Let D be an even diagram that contains the apsis $\boxed{i|0}\boxed{i{+}1|0}$. Then there exists a unique diagram D^{red} such that $D = \mathrm{DC}_i D^{\mathrm{red}}$ and $l(D) = l(D^{\mathrm{red}}) + 2$. ◊

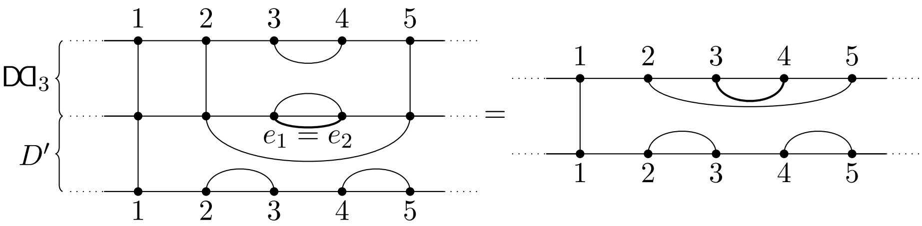
Proof. Consider the sequence $Q = (\mathrm{DC}_i, D')$ for any diagram D'. Let $e_1 = \boxed{a|x}\boxed{i|0} \in D'$ be the edge with one end-node $\boxed{i|0}$ and $e_2 = \boxed{i{+}1|0}\boxed{b|y} \in D'$ the edge with one end-node $\boxed{i{+}1|0}$. If $e_1 = e_2$ then $e_1 = e_2 = \boxed{i|0}\boxed{i{+}1|0}$ and so $\mathrm{DC}_i D' = D'$. Thence D' cannot be the desired D^{red}. So we can assume $e_1 \neq e_2$.
Then $\mathrm{DC}_i D'$ has almost the same edges as D'. In $\mathrm{DC}_i D'$ the edges e_1, e_2 (and their translations) are replaced by the top apsis $\boxed{i|0}\boxed{i{+}1|0}$ and a new edge e_{new} that comes from the composition of e_1, e_2 and the bottom apsis $\boxed{i|1}\boxed{i{+}1|1} \in \mathrm{DC}_i$ (and their translations). More precisely

$$\mathrm{DC}_i D' = (D' \setminus \langle\tau\rangle\{e_1, e_2\}) \cup \langle\tau\rangle\{e_{\mathrm{new}}, \boxed{i|0}\boxed{i{+}1|0}\} \tag{DC}$$

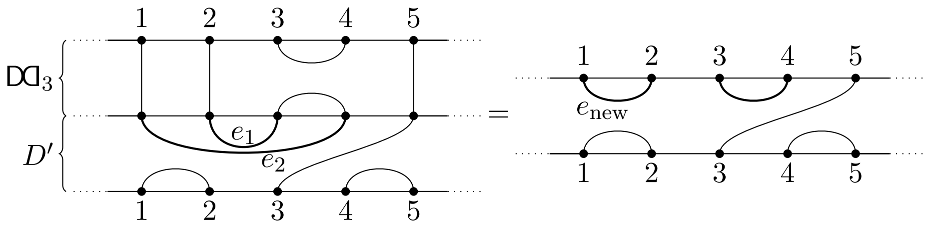
where e_{new} is the edge associated to the string of $\mathcal{G}(Q)$ containing the path

$$\boxed{a|x{+}1} \underset{e_1}{\frown} \boxed{i|1} \frown \boxed{i{+}1|1} \underset{e_2}{\frown} \boxed{b|y{+}1}$$

in $\mathcal{G}(Q)$. Observe that e_{new} and $\boxed{i|0}\ \boxed{i{+}1|0}$ in $\mathsf{DQ}_i D'$ are always neighbours as $\boxed{i|0}\ \boxed{i{+}1|0}$ and $\boxed{i|1}\ \boxed{i{+}1|1}$ are neighbours in DQ_i.



(a) Whenever D' contains the top apsis $\boxed{3|0}\ \boxed{4|0}$ then $\mathsf{DQ}_3 D' = D'$.



(b) Otherwise $\mathsf{DQ}_3 D'$ is D' with $\boxed{2|0}\ \boxed{3|0}$ (e_1) and $\boxed{1|0}\ \boxed{4|0}$ (e_2) replaced by $\boxed{1|0}\ \boxed{2|0}$ (e_{new}) and the top apsis $\boxed{3|0}\ \boxed{4|0}$. The two new edges are neighbours since e_{new} (its string) contains the bottom apsis from DQ_3 and $\boxed{3|0}\ \boxed{4|0}$ comes from the string with one edge being the top apsis from DQ_3 and the apsides are neighbours in DQ_3.
Note that in this case, the string corresponding to e_{new} contains a forward-backward move since e_2 goes left from $\boxed{4|0}$. Thus e_{new} is shorter than $3 + 1 + 1$ (the sum of the lengths of edges on the string).

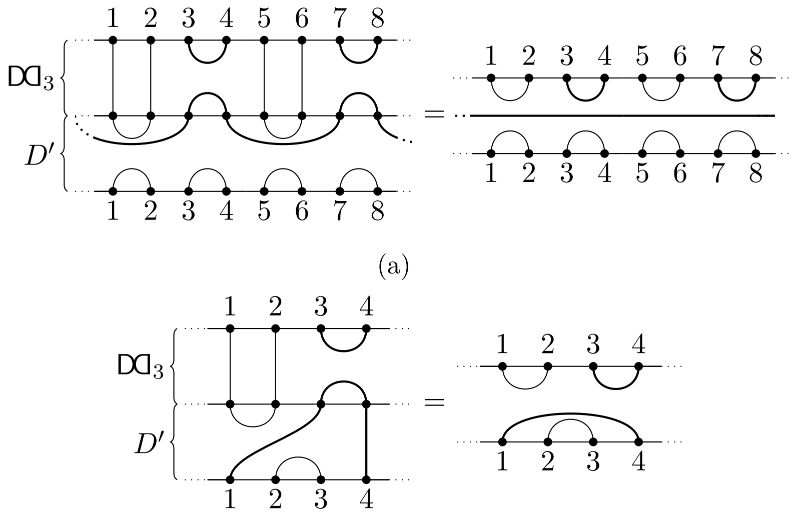
Figure 2.7

Now if we additionally require that $l(\mathsf{DQ}_i D') = l(D') + 2$ then by cor. 2.1.10 $a \leq i$ and $i + 1 \leq b$ so that the product graph does not contain a forward-backward move. Hence there are two cases to be considered.

- $b = a + 2n - 1, x = y = 0$. This is the only case in which e_1 and e_2 belong to the same τ-orbit. Here e_{new} is the top most non-contractible circular edge of $\mathsf{DQ}_i D'$.
- Otherwise, e_1 and e_2 are in different orbits and so $\boxed{a|x}$) and $\boxed{b|y}$ are in different vertex-orbits. Hence $e_{\text{new}} = \boxed{a|x}\!-\!\boxed{b|y}$ is a non-circular edge and it intersects the line ${}^{i}|^{i+1}$.

Notice that in both cases from knowing $\mathsf{DQ}_i D'$ and $e_{\text{new}} \in \mathsf{DQ}_i D'$ one can uniquely reconstruct D':

- e_{new} is non-contractible circular $\implies$ $e_1 = \boxed{i{-}n{+}1|0}\,\boxed{i|0}$, $e_2 = \boxed{i{+}1|0}\,\boxed{i{+}n|0}$,
- $e_{\text{new}} = \boxed{a|x}\boxed{b|y}$, $a \le i, i+1 \le b$ $\implies$ $e_1 = \boxed{a|x}\boxed{i|0}$, $e_2 = \boxed{i{+}1|0}\boxed{b|y}$.



(a)

(b)

Figure 2.8: Examples ($n = 4$) for the two cases in the proof of lem. 2.1.16.

Now let us add the final condition $D = \mathrm{DQ}_i D'$ for the given diagram D. By the decomposition (DC) we need to find $e_{\text{pot}} \in D$ such that

- e_{pot} is a neighbour of $\boxed{i|0}\,\boxed{i{+}1|0}$ and
- e_{pot} is either
  - a non-circular edge intersecting the line ${}^{i}|^{i+1}$ or
  - the top most non-contractible circular edge.

For such an edge e_{pot} we can define

$$D' = \left(D \setminus \langle\tau\rangle \left\{\boxed{i|0}\,\boxed{i{+}1|0}, e_{\text{pot}}\right\}\right) \cup \langle\tau\rangle \{e_1, e_2\}$$

where e_1, e_2 are the unique edges as described above. Such D' is a diagram by obs. 2.1.12, since $\boxed{i|0}\,\boxed{i{+}1|0}$ and e_{pot} are neighbours (and e_1 and e_2 do not intersect). Then by the above this D' is the unique diagrams such that $D = \mathrm{DQ}_i D'$, $l(D) = l(D') + 2$ and $e_{\text{pot}} = e_{\text{new}}$ as described above. An example can be found in fig. 2.9
The claim then follows from the statement lem. 2.1.15 which claims precisely that such $e_{\text{pot}} \in D$ is unique. To apply that statement one just needs to realise, that apart from $\boxed{i|0}\,\boxed{i{+}1|0}$, there has to be at least one other edge in D intersecting the line ${}^{i}|^{i+1}$ since D is even. $\square$

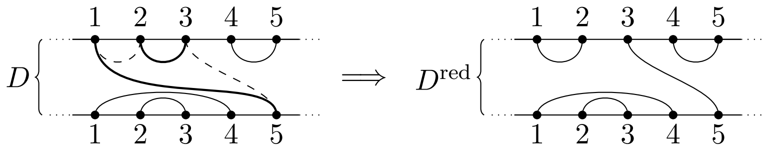


Figure 2.9: In this example, $e_{\text{pot}} = \boxed{1|0}\boxed{5|1}$ is the unique edge neighbouring the top apsis $\boxed{2|0}\boxed{3|0}$ and intersecting the line $^2|^3$. Thus the edges $e_1 = \boxed{1|0}\boxed{2|0}$ and $e_2 = \boxed{3|0}\boxed{5|1}$ do not intersect any edge in D. But also since $1 \leq 2$ and $3 \leq 5$ it is guaranteed that e_1 does not intersect e_2 and so we can replace the top apsis and e_{pot} with e_1 and e_2 to obtain the diagram D^{red} which is the unique diagram from lem. 2.1.16. Indeed the length of D is 10 whereas the length of D^{red} is 8.

For any drawing of the diagram and any curve connecting the points $(i + \frac{1}{2}, 0), (i + \frac{1}{2}, 1)$ the edge e_{pot} will be the first edge intersecting the line $^i|^{i+1}$.

Corollary 2.1.17. The diapsides $\mathsf{DC}_1, \ldots, \mathsf{DC}_n$ generate the singular even part of $\mathfrak{a}\mathfrak{J}_n$. ♢

Now we need to generate the rest of the elements in $\mathfrak{a}\mathfrak{J}_n$. By cor. 2.1.6 we know that the composition of two odd diagrams is even. So since $\mathsf{И}$ (and N) are odd (every line is intersected by just a single edge) we can use them to obtain even diagrams from odd ones.

Observation 2.1.18. Both the twist $\mathsf{И}$ and the counter twist N are odd.
Let D be an odd singular diagram. Then $\mathsf{N}D$ is even singular (by cor. 2.1.6) and thus is generated by the diapsides. ♢

Corollary 2.1.19. Any odd singular diagram D is of the form $\mathsf{И}D$, where D is singular even and thus is generated by the diapsides. ♢

Observation 2.1.20. Any invertible element in $\mathfrak{a}\mathfrak{J}_n$ is of the form $\mathsf{И}^k, k \in \mathbb{Z}$. ♢

Hence we are able to generate all diagrams.

Theorem 2.1.21. The monoid $\mathfrak{a}\mathfrak{J}_n$ is generated by the diapsides $\mathsf{DC}_i, i \in [n]$, the twist $\mathsf{И}$ and the counter-twist N. ♢

2.2 Presentation

2.2.1 Singular Even $\mathfrak{a}\mathfrak{J}_n$

The approach of this section is fairly standard. First, we make each word in the generators as short as possible with the rewriting rules given by the presentation. Then, we in essence just show that two reduced (as short as possible) words are the same when they produce the same diagram.

In the non-annular case, the approach for this part is somewhat simplified by the existence of a unique normal form to work with.[i] See [BDP02] or [Eas21] for details.
In our case, we will instead need to define what we call a normal *clan* which will be of a family of words of which none is special. Then two reduced words producing the same diagram might not be exactly the same, but they will be part of the same normal clan.
We will again first concentrate only on the singular even part of $\mathfrak{a}\mathfrak{J}_n$ which, as we now know, is generated by the diapsides.

Definition 2.2.1. We call Γ the alphabet with one symbol

$$\mathtt{D}_i, i \in [n]$$

for each diapsis D_i.
We will denote $|w|$ the length of a word.
We define the following relations on the words of the free monoid Γ^* generated by Γ:[ii] [iii]

$$\mathtt{D}_i^2 = \mathtt{D}_i, \quad \forall i, \tag{Ip}$$
$$\mathtt{D}_i\mathtt{D}_j = \mathtt{D}_j\mathtt{D}_i, \forall i \notin \{j-1, j, j+1\}, \tag{Cm}$$
$$\mathtt{D}_i\mathtt{D}_{i\pm1}\mathtt{D}_i = \mathtt{D}_i, \quad \forall i. \tag{St}$$

We use addition/subtraction modulo n in (Cm) and (St). We will continue to use this convention in the remainder of this thesis for all indices of diapsides and their symbols without further mentioning it.
Let $\mathcal{S}$ be a set of relations. We denote by $\sim_{\mathcal{S}}$ the congruence relation on Γ^* induced by the relations in $\mathcal{S}$. ♢

We have a natural surjective homomorphism

$$\phi : \Gamma^* \twoheadrightarrow \langle \mathsf{D}_1, \ldots, \mathsf{D}_n \rangle \subseteq \mathfrak{a}\mathfrak{J}_n$$

defined by

$$\mathtt{D}_i \mapsto \mathsf{D}_i.$$

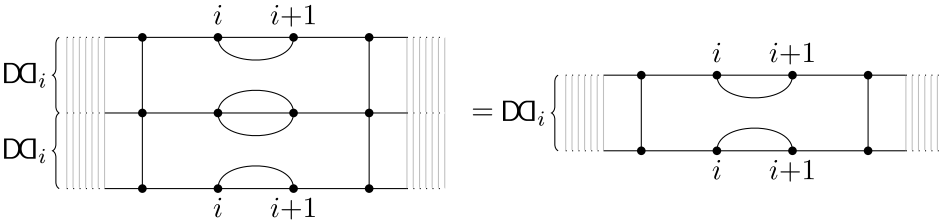
We say a binary relation $\heartsuit$ on Γ^* is **admissible** (with respect to $\mathfrak{a}\mathfrak{J}_n$) if and only if $u \heartsuit v$ implies $\phi(u) = \phi(v)$.

Observation 2.2.2. (Ip), (Cm) and (St) are all admissible. That can be seen in the pictures in fig. 2.10. ♢

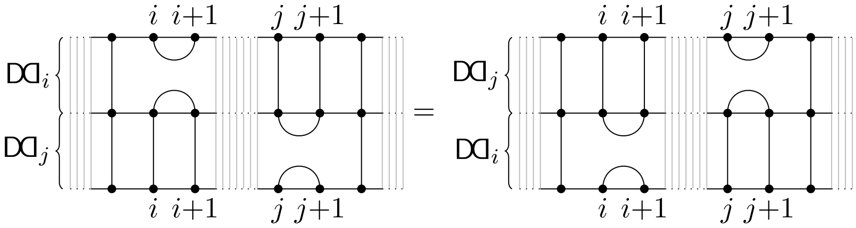
We define a set of equations

$$\mathcal{E} := (\text{Ip}) + (\text{Cm}) + (\text{St}).$$

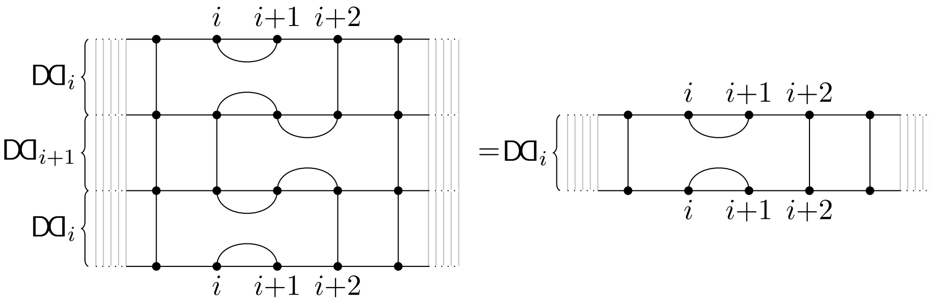
[i] There are many choices for a particular normal form even in this case, but any choice produces only one unique word.
[ii] We use abbreviations instead of numbers as these are in my opinion easier to memorise.
[iii] Mnemotechnics: (Ip) – **I**dem**p**otents, (Cm) – **C**om**m**utes, (St) – **S**hor**t**en.



(a) Admissibility of (Ip).



(b) Admissibility of (Cm).



(c) Admissibility of (St).

Figure 2.10: Picture proof of obs. 2.2.2

We thus have a surjective homomorphism

$\mathcal{E}$ at p. 33

$$\Gamma^*/\sim_{\mathcal{E}} \twoheadrightarrow \langle \bowtie_1, \ldots, \bowtie_n \rangle.$$

We call shortening a word by (Ip) or (St) a **reduction**. Since (Cm) preserves the length of words, the only way to shorten a word is to reduce it.
We call a word **reducible** if and only if it can be reduced.

Definition 2.2.3. Let w be a word. We call a set of all words equivalent to w modulo (Cm) its **clan** and denote it $\langle w \rangle$. We call $\langle w \rangle$ a **normal clan** if none of the words in $\langle w \rangle$ is reducible. When $\langle w \rangle$ is a normal clan, we call w **reduced**. $\Diamond$

Remark. Let w be a word in Γ not containing $\bowtie_n$.
Let $w' \in \langle w \rangle$ be such that there is no subword

$$\bowtie_i \bowtie_j, i > j + 1.$$

It is easy to realise that w' has to be of the form

$$v_1 \cdots v_c$$

where each v_i is a word composed of diapsis symbols with the indices decreasing by 0 or 1, e. g.:

$$\mathtt{DC}_3\mathtt{DC}_2\mathtt{DC}_2\mathtt{DC}_1,$$

and the last symbol of v_i has lower index than the fist symbol of v_{i+1} for all i.
One can show that there is always a unique such $w' \in \langle w \rangle$.[iv] Then w' (and hence w) is reduced if and only if w' is non-reducible.[v] This is the simplification of the non-annular case. $\diamondsuit$

Observation 2.2.4. For any word v in Γ^*, there exists a reduced word $w \in \Gamma^*$ such that $v \sim_{\mathcal{E}} w$. $\diamondsuit$

$\mathcal{E}$ at p. 33

We now characterise the reducedness of a word by means which tie nicely into the associated diagram.
Let $w = d_1 \cdots d_l \in \Gamma^*$. We will associate the product graph

$$\mathcal{G}(w) := \mathcal{G}((\phi(d_1), \ldots, \phi(d_l))).$$

to w.

Lemma 2.2.5. For a word $w \in \Gamma^*$ the following are equivalent

1. w is not reduced,
2. $l(\phi(w)) < 2|w|$,
3. the product graph $\mathcal{G}(w)$ contains a forward-backward move,
4. w contains a subword $\mathtt{DC}_i u \mathtt{DC}_i$ such that
   - u does not contain $\mathtt{DC}_i$ and
   - u does not contain both $\mathtt{DC}_{i-1}$ and $\mathtt{DC}_{i+1}$.

$\diamondsuit$

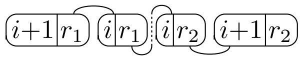
Proof. ($1 \implies 2$) Let $w' \sim_{\mathcal{E}} w$ be a shorter word (which exists by 1). By cor. 2.1.10 $l(\phi(w')) \leq 2|w'| < 2|w|$.

($2 \implies 3$) By cor. 2.1.10.

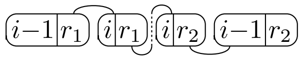
[iv] For example think of the first symbol in w'. It has to be a symbol $\mathtt{DC}_i$ in w such that there is no $\mathtt{DC}_{i-1}$ nor $\mathtt{DC}_{i+1}$ before it in w. And among such symbols, it has to be the one with the lowest index.

[v] Even more precisely, w' is non-reducible if and only if the end symbols of v_i's have strictly increasing indices.

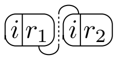
(3 $\implies$ 4) Let $w = d_1 \cdots d_l$ be such that we have a forward-backward move in $\mathcal{G}(w)$. Then, since the diagrams $\phi(d_i)$ are diapsides, the forward-backward path is of the form



or

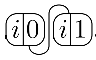


for some $r_1 \leq r_2$. Thus $d_{r_1} = d_{r_2+1} = \mathtt{DQ}_i$ or $d_{r_1} = d_{r_2+1} = \mathtt{DQ}_{i-1}$. Moreover, the subpath (of both cases)



is evidence that for every c with $r_1 < c < r_2 + 1$

$$d_c \neq \mathtt{DQ}_i, \mathtt{DQ}_{i-1}$$

since $\mathtt{DQ}_i, \mathtt{DQ}_{i-1}$ do not contain the edge . Thus

$$d_{r_1} \underbrace{d_{r_1+1} \cdots d_{r_2}}_{u} d_{r_2+1}$$

is the desired subword.

(4 $\implies$ 1) Suppose there is such a subword and pick the one with the minimal length. Denote it by $v = \mathtt{DQ}_i u \mathtt{DQ}_i$ Consider two cases.

- u contains neither one of $\mathtt{DQ}_{i-1}, \mathtt{DQ}_{i+1}$. Here we can repeatedly apply (Cm) to rewrite $\mathtt{DQ}_i u \mathtt{DQ}_i$ to $\mathtt{DQ}_i \mathtt{DQ}_i u$ and thus obtain $w' \in \langle w \rangle$ with a subword $\mathtt{DQ}_i \mathtt{DQ}_i$ which is reducible by (Ip).
- u contains $\mathtt{DQ}_{i-1}$ or $\mathtt{DQ}_{i+1}$. Say it contains $\mathtt{DQ}_{i+1}$. Then if $\mathtt{DQ}_{i+1}$ appears only once in u, we can again use (Cm) to move the two symbols $\mathtt{DQ}_i$ right next to it so that $\mathtt{DQ}_i \mathtt{DQ}_{i+1} \mathtt{DQ}_i$ is a subword of some $w' \in \langle w \rangle$ which is then reducible by (St).

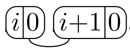
  Otherwise, we have a subword $\mathtt{DQ}_{i+1} t \mathtt{DQ}_{i+1}$ of u. Again pick the shortest one. Since we have picked the shortest one $\mathtt{DQ}_{i+1}$ does not appear in t. But since t is also a subword of u we know that $\mathtt{DQ}_i$ does not appear in t either. Hence $\mathtt{DQ}_{i+1} t \mathtt{DQ}_{i+1}$ is a shorter prohibited subword which is a contradiction with the choice of v.

  Symmetrically when u contains $\mathtt{DQ}_{i-1}$.

□

Recalling cor. 2.1.10 we then state the contrapositive.

Corollary 2.2.6. A word w is reduced if and only if $l(\phi(w)) = 2|w|$. ♢

A consequence of this lemma is the final piece of the puzzle which characterises the reduced words producing diagrams containing the apsis .

Lemma 2.2.7. Let w be a reduced word such that $\bowtie_i \phi(w) = \phi(w)$. Then there is $w' \in \langle w \rangle$ that starts with the symbol $\bowtie_i$, i. e.

$$w' = \bowtie_i w'_{\text{end}}$$

for some word w'_{end}. $\Diamond$

Proof. Let w be such a reduced word. Then $\bowtie_i w$ is not reduced as

$$2|\bowtie_i w| > 2|w| = l(\phi(w)) = l(\bowtie_i \phi(w)) = l(\phi(\bowtie_i w)).$$

Then, since w is reduced but $\bowtie_i w$ is not, there must be (by lem. 2.2.5 (1 $\iff$ 4)) an occurrence of $\bowtie_i$ in w

$$w = v \bowtie_i w_{\text{end}}$$

such that v contains no symbols $\bowtie_i$ and does not contain both $\bowtie_{i-1}, \bowtie_{i+1}$.
Suppose that $\bowtie_{i+1}$ occurs in v. Then consider its first occurrence i. e.

$$v = u \bowtie_{i+1} v_{\text{end}}$$

so that $\bowtie_i, \bowtie_{i+1}$ do not occur in u. Look at the graph $\mathcal{G}(w)$.
Consider a maximal path starting in $\boxed{i{+}1|0}$. Then, since $\bowtie_i, \bowtie_{i+1}$ do not occur in u, we know that the first apsis on this path will be the top apsis of $\bowtie_{i+1}$. But this means, that the apsis $\boxed{i|0}\,\boxed{i{+}1|0}$ cannot be an edge of $\phi(w)$, since w is reduced and therefore $\mathcal{G}(w)$ does not contain a forward-backward move. But $\phi(w) = \bowtie_i \phi(w)$ contains the apsis $\boxed{i|0}\,\boxed{i{+}1|0}$. Thus $\bowtie_{i+1}$ does not appear in u. See fig. 2.11.

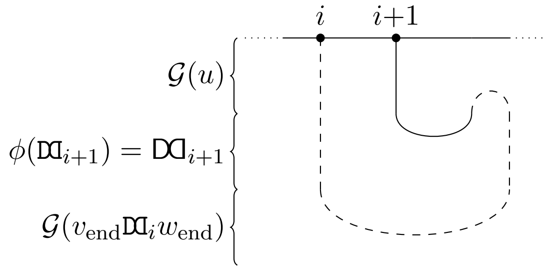


Figure 2.11: The dashed path would need to contain a forward-backward move

Symmetrically $\bowtie_{i-1}$ does not appear in u. Hence

$$w = v \bowtie_i w_{\text{end}}$$

and v contains none of the symbols $\bowtie_{i-1}, \bowtie_i, \bowtie_{i+1}$. So we can use (Cm) repeatedly to move v behind the symbol $\bowtie_i$ and obtain the desired word $w' = \bowtie_i v w_{\text{end}} \in \langle w \rangle$. $\square$

$\mathcal{E}$ at p. 33 **Theorem 2.2.8.** $(\Gamma : \mathcal{E})$ is a presentation for the singular even part of $\mathfrak{a}\mathfrak{J}_n$ under the generator assignment

$$\mathbb{DC}_i \mapsto \mathsf{DC}_i.$$

$\diamondsuit$

Proof. We have already proven in cor. 2.1.17 that the diapsides generate the singular even part. Also by obs. 2.2.2, we know, that the relations are admissible. Thus it remains to show that the relations are sufficient.
Take two reduced words u, v such that $D := \phi(u) = \phi(v)$. Then by cor. 2.2.6 u and v have the same length that being $\frac{l(D)}{2}$. We will prove by induction on their length that $u \sim_{(\mathrm{Cm})} v$.
For u, v empty $u = v$ trivially and so $u \sim_{(\mathrm{Cm})} v$
Now assume u, v are nonempty. Let $\mathbb{DC}_i$ be the first symbol of u so that

$$u = \mathbb{DC}_i w_u.$$

Then $\mathsf{DC}_i D = D$ since $\mathbb{DC}_i u \sim_{(\mathrm{Ip})} u$. But hence $\mathsf{DC}_i \phi(v) = \phi(v)$ and so by lem. 2.2.7 we know that there is $v' \in \langle v \rangle$ such that

$$v' = \mathbb{DC}_i w_{v'}.$$

Then

$$D = \mathsf{DC}_i \phi(w_u) = \mathsf{DC}_i \phi(w_{v'})$$

from which

$$\begin{aligned} l(\phi(w_u)) &\geq l(D) - l(\mathsf{DC}_i) = l(D) - 2 \\ l(\phi(w_v)) &\geq l(D) - l(\mathsf{DC}_i) = l(D) - 2 \end{aligned}$$

as $l(\mathsf{DC}_i) = 2$. But also,

$$\begin{aligned} l(\phi(w_u)) &\leq 2|w_u| = 2|u| - 2 = l(D) - 2 \\ l(\phi(w_v)) &\leq 2|w_v| = 2|v| - 2 = l(D) - 2 \end{aligned}$$

hence first of all w_u, w_v are reduced since

$$l(\phi(w_u)) = 2|w_u| = l(D) - 2 = 2|w_{v'}| = l(\phi(w_v))$$

(one might also realise that suffixes of reduced words are reduced). But thence also both $\phi(w_u), \phi(w_{v'})$ must be the unique diagram D^{red} for D and the apsis $\boxed{i|0}\,\boxed{i{+}1|0} \in D$ from lem. 2.1.16[vi] and so

$$\phi(w_u) = \phi(w_v').$$

But $w_u, w_{v'}$ are shorter than u, v and thus by inductive hypothesis $w_u \sim_{(\mathrm{Cm})} w_{v'}$. From here finally

$$u \sim_{(\mathrm{Cm})} v' \sim_{(\mathrm{Cm})} v.$$

$\square$

[vi] This is a crucial part missing in [FG99], since their version of lem. 2.1.16 is missing this uniqueness

2.2.2 Affine Jones monoid

To present the whole $\mathfrak{a}\mathfrak{J}_n$ is now easy in light of obs. 2.1.18.

Definition 2.2.9. We call Π the alphabet Γ with added symbols

$$\texttt{И}, \quad \texttt{N}.$$

We define additional relations on the words of the free monoid Π^* [vii]

$$\begin{aligned}
\texttt{И}\texttt{Dᗡ}_i &= \texttt{Dᗡ}_{i-1}\texttt{И} \qquad \forall i && \text{(QC)}\\
\texttt{ИИ}\texttt{Dᗡ}_1 &= \texttt{Dᗡ}_{n-1}\cdots\texttt{Dᗡ}_1 && \text{(UT)}\\
\texttt{NИ} &= 1 && \text{(VI)}\\
\texttt{ИN} &= 1 && \text{(AI)}
\end{aligned}$$

Justified by (AI) and (VI), we shall use

$$\texttt{И}^k$$

to mean

$$\texttt{N}^{-k}.$$

Define the relation set

$$\mathcal{G} := \text{(QC)} + \text{(UT)} + \text{(AI)} + \text{(VI)}.$$

$\Diamond$

We will still denote the natural surjective homomorphism $\Pi^* \twoheadrightarrow \mathfrak{a}\mathfrak{J}_n$ by ϕ which makes sense since Γ^* is a submonoid of Π^* and $\phi|_{\Gamma^*}$ is the natural homomorphism from Γ^*.

Observation 2.2.10. (QC), (UT), (VI) and (AI) are all admissible. (QC) and (UT) can be seen in the pictures in fig. 2.12. (VI) and (AI) are omitted and considered obvious.

$\Diamond$

Let us infer some additional relations

Lemma 2.2.11. The following relations lie in the congruence $\sim_{\mathcal{G}}$:[viii]

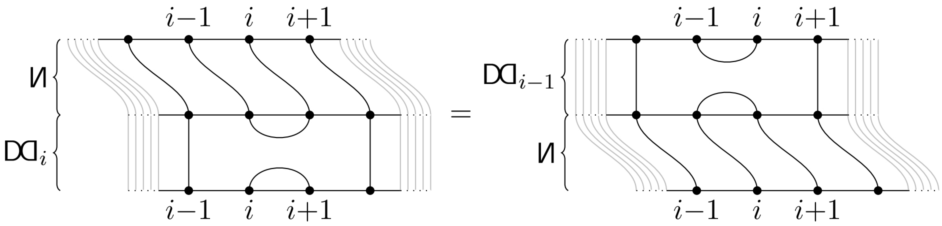
$\mathcal{G}$ at p. 39

$$\begin{aligned}
\texttt{Dᗡ}_i &= \texttt{И}\texttt{Dᗡ}_{i+1}\texttt{N} \qquad \forall i && \text{(Cg)}\\
\texttt{Dᗡ}_i &= \texttt{N}\texttt{Dᗡ}_{i-1}\texttt{И} \qquad \forall i && \text{(CgR)}\\
\texttt{Dᗡ}_i\texttt{N} &= \texttt{N}\texttt{Dᗡ}_{i-1} \qquad \forall i && \text{(QCR)}\\
\texttt{NN}\texttt{Dᗡ}_{n-1} &= \texttt{Dᗡ}_1\cdots\texttt{Dᗡ}_{n-1}. && \text{(UTR)}\\
\texttt{ИИ}\texttt{Dᗡ}_i &= \texttt{Dᗡ}_{i-1}\texttt{Dᗡ}_{i-2}\cdots\texttt{Dᗡ}_{i-n+1} \ \forall i && \text{(UTG)}\\
\texttt{N}\texttt{Dᗡ}_i &= \texttt{И}\texttt{Dᗡ}_{i+2}\cdots\texttt{Dᗡ}_{n+i+1} \qquad \forall i && \text{(Rp)}
\end{aligned}$$

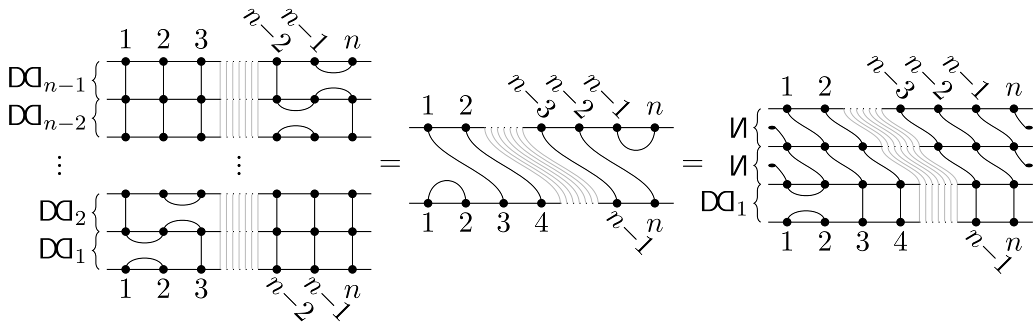
$\Diamond$

[vii] Mnemotechnics: (QC) – **Q**uasi**C**ommutes, (UT) – **U**n**T**wist, (VI)/(AI) – $\texttt{NИ}/\texttt{ИN}$ **I**nverse (notice the shape the diagonal lines make).

[viii] Mnemotechnics (-R) – **R**everse, (-G) – **G**eneralized, (Rp) – **Rep**lace.



(a) Admissibility of (QC).



(b) Admissibility of (UT).

Figure 2.12: Picture proof of obs. 2.2.2

Proof. (Cg), (CgR) and (QCR) are just (QC)N, N(QC) and N(QC)N modulo (AI) and (VI).
For (UTR) we just gradually rewrite (UT).

$$\begin{array}{rcllr}
\text{ИИ}\bowtie_1 & = & \bowtie_{n-1}\cdots\bowtie_1 & \text{(UT)} & \\
\bowtie_1 & = & \mathtt{NN}\bowtie_{n-1}\cdots\bowtie_2\bowtie_1 & |\mathtt{NN}\cdot & \text{mod (VI)} \\
\bowtie_1\bowtie_2 & = & \mathtt{NN}\bowtie_{n-1}\cdots\bowtie_3\bowtie_2 & |\cdot\bowtie_2 & \text{mod (St)} \\
 & \vdots & & \vdots & \\
\bowtie_1\cdots\bowtie_{n-1} & = & \mathtt{NN}\bowtie_{n-1} & |\cdot\bowtie_{n-1} & \text{mod (St).}
\end{array}$$

(UTG) is inferred using (Cg):

$$\begin{aligned}
\text{ИИ}\bowtie_i &\sim_{\text{(Cg)}} \text{ИИИ}^{n-i}\bowtie_1\mathtt{N}^{n-i} \\
&\sim_{\text{(UT)}} \text{И}^{n-i}\bowtie_{n-1}\cdots\bowtie_1\mathtt{N}^{n-i}
\end{aligned}$$

which can be further rewritten as in cor. 2.2.12 to

$$\begin{aligned}
&\sim_{\text{(QC)+(AI)+(VI)}} \bowtie_{n-1-(n-i)}\cdots\bowtie_{1-(n-i)} \\
&= \bowtie_{i-1}\bowtie_{i-2}\cdots\bowtie_{i-n+1}
\end{aligned}$$

.

(Rp) is quite similar to the inference for (UTG)

$$
\begin{aligned}
\mathtt{N}\mathtt{D}\reflectbox{$\mathtt{D}$}_i &\sim_{\text{(CgR)}} \mathtt{N}\mathtt{N}^{i+1}\mathtt{D}\reflectbox{$\mathtt{D}$}_{n-1}\reflectbox{$\mathtt{N}$}^{i+1} \\
&\sim_{\text{(UTR)}} \mathtt{N}^{i}\mathtt{D}\reflectbox{$\mathtt{D}$}_1 \cdots \mathtt{D}\reflectbox{$\mathtt{D}$}_{n-1}\reflectbox{$\mathtt{N}$}^{i+1}
\end{aligned}
$$

which can be further rewritten as in cor. 2.2.12 to

$$
\sim_{\text{(QC)+(AI)+(VI)}} \reflectbox{$\mathtt{N}$}\mathtt{D}\reflectbox{$\mathtt{D}$}_{1+i+1} \cdots \mathtt{D}\reflectbox{$\mathtt{D}$}_{n+i+1}.
$$

□

Corollary 2.2.12. Define the relation set

$$
\mathcal{F} := \text{(QC)} + \text{(AI)} + \text{(VI)}.
$$

Let w be a word in Π^*. Then there exists $v \in \Gamma^*$ and $k \in \mathbb{Z}$ such that

$$
w \sim_{\mathcal{F}} \reflectbox{$\mathtt{N}$}^k v.
$$

◊

Proof. We just move all $\reflectbox{$\mathtt{N}$}$'s and $\mathtt{N}$'s left of all the diapsis symbols using (QC) and (QCR). Then we just annihilate the $\reflectbox{$\mathtt{N}$}$'s and $\mathtt{N}$'s on the left with (VI) and (AI). □

Corollary 2.2.13. Let w be a word in Π^* that contains a symbol from Γ. Then there exists $v \in \Gamma^*$ and $k \in \{0, 1\}$ such that

$$
w \sim_{\mathcal{G}} \reflectbox{$\mathtt{N}$}^k v.
$$

$\mathcal{G}$ at p. 39

◊

Proof. Let us write

$$
w \sim_{\mathcal{F}} \reflectbox{$\mathtt{N}$}^k v, v \in \Gamma^*
$$

$\mathcal{F}$ at p. 41

from cor. 2.2.12 and consider two cases

- $k \geq 0$. Let us proceed by induction on k.

  When $k = 0, 1$, there is nothing to do, thus the basis of induction is established.

  Suppose $k \geq 2$. Then we can use (UTG) (v is nonempty) to obtain

  $$
  \reflectbox{$\mathtt{N}$}^k v \sim_{\mathcal{G}} \reflectbox{$\mathtt{N}$}^{k-2} v'
  $$

  for some word $v' \in \Gamma^*$. We can then apply the inductive hypothesis on the right-hand side.

- $k < 0$, i. e.

$\mathcal{F}$ at p. 41

$$w \sim_{\mathcal{F}} \mathtt{N}^k v, k > 0, v \in \Gamma^*.$$

Then, in the presence of any symbol for a diapsis, all occurrences of $\mathtt{N}$ can be replaced with $\mathtt{И}$ using (Rp).[ix] Then we can proceed by the first part of the proof.

□

Theorem 2.2.14. We define the set of equations

$\mathcal{E}$ at p. 33
$\mathcal{G}$ at p. 39

$$\mathcal{H} := \mathcal{E} + \mathcal{G}.$$

$(\Pi : \mathcal{H})$ is a presentation for $\mathfrak{a}\mathfrak{J}_n$ under the mapping of the generators

$$\mathtt{Dᗡ}_i \mapsto \mathsf{Dᗡ}_i,$$
$$\mathtt{И} \mapsto \mathsf{И},$$
$$\mathtt{N} \mapsto \mathsf{N}.$$

◊

Proof. That the generators generate $\mathfrak{a}\mathfrak{J}_n$ was already seen in thm. 2.1.21.
Let now D be a diagram and u, v be two words s. t. $\phi(u) = \phi(v) = D$. We may assume

$$u = \mathtt{И}^{k_u} w_u, \quad \text{for some } w_u \in \Gamma^* \text{ and } k_u \in \mathbb{Z},$$
$$v = \mathtt{И}^{k_v} w_v, \quad \text{for some } w_v \in \Gamma^* \text{ and } k_v \in \mathbb{Z}$$

by cor. 2.2.12. Then if D is invertible w_u, w_v must be empty. Then for any $k \in \mathbb{Z}$ $\mathtt{И}^k$ can be recognized by having the edge $\boxed{0|0}\boxed{k|1}$ and thus $k_u = k_v$ and so $u = v$.
So let us assume that D is singular. Then w_u, w_v are nonempty. Hence by cor. 2.2.13

$$u \sim_{\mathcal{G}} \mathtt{И}^{m_u} t_u, \quad \text{for some } t_u \in \Gamma^* \text{ and } m_u \in \{0, 1\},$$
$$v \sim_{\mathcal{G}} \mathtt{И}^{m_v} t_v, \quad \text{for some } t_v \in \Gamma^* \text{ and } m_v \in \{0, 1\}.$$

Since D is either odd or even and since the product of diapsides is even we know $m_u = m_v$.
Since $\mathsf{И}$ is invertible we know even for $m_u = m_v = 1$

$$\phi(t_u) = \phi(t_v)$$

from which by thm. 2.2.8

$$t_u \sim_{\mathcal{E}} t_v.$$

From here we get (by multiplying both sides with $\mathtt{И}$ when $m_u = m_v = 1$)

$$\mathtt{И}^{m_u} t_u \sim_{\mathcal{E}} \mathtt{И}^{m_v} t_v$$

and by transitivity

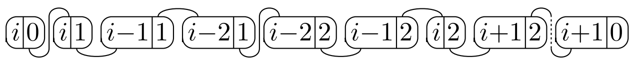
$$u \sim_{\mathcal{G}} v.$$

□

[ix] This should not really surprise us being aware of obs. 2.1.18.

2.3 Comparison

We would like to reflect briefly on the previous work dealing with this topic, namely [FG99, chapter 4].
The definition of composition of diagrams is very unclear since they do not distinguish contractible and non-contractible loops and so the reader is left guessing.
Also, throughout the chapter the term *edge of minimal length* is used several times without ever defining the length of an edge.
We find a problem already with the proof of lem. 4.3.2. In our language the authors are claiming, that for a reduced word the associated product graph does not contain strings that are a cycle. In our opinion their argument does not justify this statement at all.
This is underlined by the lem. 4.3.5(b). This is essentially our lem. 2.2.5 (1 $\iff$ 3).
This lemma is proven much more rigorously than lem. 4.3.2. in the paper. However, our suspicion comes from the fact that both statements have the same underlying argument, that being the absence of forward-backward moves. Indeed that is the approach of proving lem. 4.3.5 which, however, relies on lem 4.3.2 to already have been proven.
The next problem is in lem. 4.3.4. This is again very similar to our lem. 2.2.7. However this again follows only after the lem. 4.3.5(b). Until then it is not clear that a path like



must not occur in the product graph for any reduced word. Thus in our opinion, the authors prove only the less important direction (the reverse direction of lem. 2.2.7).
The lem. 4.3.4 is actually used in the unproven direction already in the proof of 4.3.5(a). to claim that when $w'\mathtt{DC}_i$ is reduced then $\mathcal{G}(w'\mathtt{DC}_i)$ does not contain a cyclic string. But that is (in our opinion wrongly) claimed in lem. 4.3.2 so it is very dubious.
Finally, the most serious problem in our eyes is with the lem. 4.4.8. This can be very much compared to our lem. 2.1.16. However in the third and fourth case the lem. 4.4.8 does not claim uniqueness. Indeed the uniqueness does not follow from the proof and the authors were probably aware of it since they do claim the uniqueness in the first and second case.
The problem is then in the proof of thm. 4.5.2. There they claim that for reduced words $\mathtt{DC}_i a'$ and $\mathtt{DC}_i b'$ with $\phi(\mathtt{DC}_i a') = \phi(\mathtt{DC}_i b')(=D$ in the paper) it follows that $\phi(a') = \phi(b')(= D')$. But for that one does need the uniqueness of the decomposition from lem. 2.1.16.
We are aware that an analogue of the decomposition lem. 2.1.16 can be found as [GL98, prop. 1.8]. The uniqueness is not claimed here either and a lot of work is left to the reader. But this statement could be, with some work, rewritten to include the uniqueness.

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