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„Reduction of asymptotically AdS$_3$ spin-N gravity to sl(N) Toda theory“

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Zusammenfassung

Die klassische Reduktion der Gravitation auf asymptotisch AdS_3 Raumzeiten auf ein $\mathsf{SL}(2;\mathbb{R})$ Wess–Zumino–Witten-Modell, weiterführend zu der konformen Liouville-Feldtheorie, wird als Korrespondenz von Phasenräumen formuliert. Wir zeigen die Äquivalenz der Dynamik und der physikalischen Observablen zwischen diesen Theorien. Ausgehend von der Korahmen-Formulierung, wird eine lineare Feldredefinition durchgeführt, um das vorliegende Problem in der Sprache der $\mathsf{SL}(2;\mathbb{R})$ Chern–Simons-Eichfeldtheorie umzuformulieren. Durch die Übersetzung der asymptotischen AdS_3 Randbedingungen in diese Sprache ergeben sich zwei Paare von Randbedingungen für die Chern–Simons Felder. Das erste Paar erlaubt die Reduktion auf das $\mathsf{SL}(2;\mathbb{R})$ Wess–Zumino–Witten-Modell, das zweite Paar erlaubt die Reduktion auf die konforme Liouville-Feldtheorie.

Diese Analyse wird erweitert auf die higher-spin Verallgemeinerung der asymptotisch AdS_3 Gravitation. Ausgehend von einer $\mathsf{SL}(N;\mathbb{R})$ Chern–Simons Formulierung, führt die Umsetzung der gleichen Schritte wie vorher zuerst zum $\mathsf{SL}(N;\mathbb{R})$ Wess–Zumino–Witten-Modell und dann zur konformen $\mathsf{sl}(N;\mathbb{R})$ Toda-Feldtheorie. Explizite Rechnungen werden für den Fall $N = 3$ durchgeführt.

Abstract

The classical reduction of asymptotically AdS_3 general relativity to an $\mathsf{SL}(2;\mathbb{R})$ Wess–Zumino–Witten model, further to Liouville conformal field theory, is phrased as correspondence of phase spaces. We demonstrate the equivalence of dynamics and physical observables between these theories. Starting from the coframe formulation, a linear field redefinition is performed to rephrase the problem at hand in the language of $\mathsf{SL}(2;\mathbb{R})$ Chern–Simons gauge field theory. Translating the asymptotically AdS_3 boundary conditions into this language results in two pairs of boundary conditions for the Chern–Simons fields. The first pair allows for reduction to the $\mathsf{SL}(2;\mathbb{R})$ Wess–Zumino–Witten model, the second pair allows for reduction to Liouville conformal field theory.

This analysis is extended to the higher-spin generalization of asymptotically AdS_3 gravity. Starting from an $\mathsf{SL}(N;\mathbb{R})$ Chern–Simons setting, performing the same steps as before, yields first the $\mathsf{SL}(N;\mathbb{R})$ Wess–Zumino–Witten model, then the $\mathsf{sl}(N;\mathbb{R})$ Toda conformal field theory. Explicit calculations are performed for the $N = 3$ case.

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Contents

Spacetime conventions

Throughout this thesis, natural units $c = 1$, $\hbar = 1$ are used.

The (2+1)-dim. gravitational constant G_3 is kept explicit. Then,

$$L_{\text{Planck}} = G_3$$

The Minkowski metric uses the mostly plus convention:

$$\eta = \text{diag}(-, +, +)$$

The permutation symbol is denoted by ε, the permutation tensor is denoted by ϵ.

Curvature tensor, torsion tensor of a connection Γ are defined via commutators:

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\lambda &= R_{\mu\nu\sigma}{}^\lambda V^\sigma \\
[\nabla_\mu, \nabla_\nu] f &= T_{\mu\nu}{}^\lambda \nabla_\lambda f
\end{aligned}$$

Their associated spacetime 2-forms with values in a framed tangent space are:

$$\begin{aligned}
R^a{}_b &= \frac{1}{2} e^{\sigma a} e_{\lambda b} R_{\mu\nu\sigma}{}^\lambda \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
T^a &= \frac{1}{2} e_\sigma{}^a T_{\mu\nu}{}^\sigma \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu
\end{aligned}$$

For a (2+1)-dimensional coordinate system $\begin{cases} \tau & \text{timelike coordinate,} \\ \rho & \text{radial coordinate,} \\ \theta & \text{angular coordinate,} \end{cases}$
the volume form $\mathrm{d}\tau \wedge \mathrm{d}\rho \wedge \mathrm{d}\theta$ is declared to be positively oriented.

Lightcone coordinates $\{x^+, x^-\}$ and time-space coordinates $\{\tau, \theta\}$ are related as:

$$\begin{aligned}
x^+ &= \tau + \theta \qquad & \tau &= \tfrac{1}{2}(x^+ + x^-) \\
x^- &= \tau - \theta \qquad & \theta &= \tfrac{1}{2}(x^+ - x^-)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\tau \wedge \mathrm{d}\theta &= -\frac{1}{2} \, \mathrm{d}x^+ \wedge \mathrm{d}x^- \\
\mathrm{d}\tau \wedge \mathrm{d}\rho \wedge \mathrm{d}\theta &= \frac{1}{2} \, \mathrm{d}\rho \wedge \mathrm{d}x^+ \wedge \mathrm{d}x^-
\end{aligned}$$

The (1+1)-dimensional exterior derivative splits as $\mathrm{d} = \partial + \bar{\partial}$, where:

$$\partial = \partial_+ \, \mathrm{d}x^+ \qquad \bar{\partial} = \partial_- \, \mathrm{d}x^-$$

List of symbols

$\mathcal{M}$ & $\partial\mathcal{M}$; manifold & its boundary

$C^\infty(\mathcal{M})$; space of smooth functions on $\mathcal{M}$

$F \to \mathcal{M}$; fibre bundle over $\mathcal{M}$

$\Gamma(\mathcal{M}, F)$; space of sections of $F \to \mathcal{M}$

$T\mathcal{M}$ & $T^*\mathcal{M}$; tangent bundle & cotangent bundle

$\mathfrak{X}(\mathcal{M})$; space of vector fields on $\mathcal{M}$

$\Omega^q(\mathcal{M})$; space of differential q-forms on $\mathcal{M}$

$\Omega^q(\mathcal{M}, \mathcal{V})$; space of differential q-forms on $\mathcal{M}$ valued in vector space $\mathcal{V}$

$\mathrm{d}x^\mu$; coordinate differentials

d; exterior derivative; horizontal differential

ι; interior product

$\pounds$; Lie derivative

$[\ ,\ ]$; commutator; Lie bracket

$\mathcal{F}$, $\mathcal{S}$; space of fields, space of on-shell fields

$\delta\varphi^a$; field variations

δ; exterior variation; vertical differential

$\Omega^{p,q}_{\text{local}}(\mathcal{F} \times \mathcal{M})$; space of (p,q)-biforms on $\mathcal{F} \times \mathcal{M}$

$\mathbf{L}$; Lagrangian density

$\boldsymbol{\vartheta}$; (pre-)symplectic potential

$\boldsymbol{\omega}$; (pre-)symplectic current

$\boldsymbol{\Omega}$; (pre-)symplectic form

$\mathfrak{G}$, $\mathfrak{H}$; Lie groups

Ad_G; adjoint action of $G \in \mathfrak{G}$

$\mathfrak{g}$, $\mathfrak{h}$; Lie algebras of Lie groups $\mathfrak{G}$, $\mathfrak{H}$

$\{t_A\}$; generators of Lie algebra $\mathfrak{g}$

ad_t; adjoint action of $t \in \mathfrak{g}$

${f_{AB}}^C$; structure constants of Lie algebra $\mathfrak{g}$

tr; matrix trace; Killing form

γ_{AB}; components of the Killing form

Φ; space of positive roots

Φ_{s}; space of simple roots

α; positive root

α^i; simple root

$\alpha^{\vee i}$; simple coroot

$\langle\ ,\ \rangle$; inner product on root space

A^{ij}; Cartan matrix

G^{ij}; symmetrized Cartan matrix

G_{ij}; quadratic form matrix

k_i; heights of fundamental weights

$\{H, E_+, E_-\}$; Chevalley $\mathsf{sl}(2)$ generators

$\{H^i, E^i_+, E^i_-\}$; Chevalley generators associated to the simple root α^i

E^α_+, E^α_-; raising/lowering generators associated to positive root α

GCT; group of general coordinate transformations

$\mathsf{GL}(N)$; general linear group

$\mathsf{gl}(N)$; Lie algebra of $\mathsf{GL}(N)$; matrix Lie algebra of $N \times N$ matrices

$\mathsf{SL}(N)$; special linear group

$\mathsf{sl}(N)$; Lie algebra of $\mathsf{SL}(N)$; matrix Lie algebra of traceless matrices

$\mathsf{SO}(p,q)$; special orthogonal group of signature (p,q)

$\mathsf{so}(p,q)$; Lie algebra of $\mathsf{SO}(p,q)$; matrix Lie algebra of antisymmetric matrices

$\mathsf{Sp}(2n)$; symplectic group

$\mathsf{sp}(2n)$; Lie algebra of $\mathsf{Sp}(2n)$; matrix Lie algebra of Hamiltonian matrices

$\{j_a\}$; $\mathsf{so}(2,1)$ generators

$\{L_m\}$; spin-2 $\mathsf{sl}(2)$ generators

$\{W^s{}_m\}$; spin-s generators of $\mathsf{sl}(N)$

1. Introduction

General relativity comes in three flavours, determined by the cosmological constant. This is most apparent in 3-dimensional general relativity, where the spacetime spectra differ substantially depending on the cosmological constant, owing to the different geometries allowed by field equations with positive, zero or negative cosmological constant. In essence, the Einstein field equations in 3-dimensions enforce a Riemann curvature tensor of a maximally symmetric spacetime, so every solution has to behave locally like a de Sitter, Minkowski or an anti-de Sitter (AdS) spacetime. However, the curvature tensor is only a local quantity; 3-dimensional general relativistic spacetimes differ in their global properties.

In this thesis, the focus will be on the negative cosmological constant flavour. Only with a negative cosmological constant can 3-dimensional general relativity admit Kerr-like black hole solutions. Furthermore, and more relevant to this thesis, AdS spaces naturally possess physical timelike boundary. 3-dimensional general relativity with negative cosmological constant is therefore not just a good toy model for black holes, it is also a good arena to investigate the so-called AdS/CFT correspondence and the subject of holography in general. The key observation motivating the study of AdS holography is the fact that the isometry group of AdS_n, $\mathsf{SO}(n-1,2)$, is the conformal group of $(n-1)$-dimensional Minkowski spacetime. As symmetries are directly related to physical observables via Noether theorems, one hopes to establish correspondences between $(n-1)$-dim. conformal field theories, living on AdS's boundary, and n-dimensional gravity-like theories on the bulk AdS backgrounds. This has proven to be a fruitful endeavour so far, as (more often than not) easy aspects of one side of the correspondence map to hard aspects of the other side and vice versa. For more general information on the AdS/CFT correspondence and holography, see the textbook [AE15], the standard string theory-adjacent review [AGM+00], and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ reviews [Car05], [Kra08].

While 3-dim. general relativity contains no local degrees of freedom and is therefore seen as being "poorer" than its higher-dimensional counterparts, the opposite is true for 2-dim. conformal field theory. Because the infinitesimal conformal symmetry algebra is infinite-dimensional in 2 dimensions, CFT_2 is seen as being "richer" than its higher-dimensional counterparts. In a sense, the fact that AdS_3 gravity is somehow related to CFT_2 is a subtle hint that the subject of 3-dim. gravitational theories is much richer than it appears at first glance. One example of this wealth is the fact that 3-dim. higher-spin theories of gravity are much better behaved than their higher-dimensional counterparts: results about the usual spin-2 theory lift as straightforwardly as possible to the spin-N case. The subject of this thesis provides an example: the reduction of the 3-dimensional general relativity to Liouville field theory generalizes to Toda field theory.

The reduction of asymptotically AdS_3 general relativity to a Liouville field theory on the boundary is well known [CHvD95], see [Car05] for the standard reference on the subject and [Don16] for a pedagogical introduction. Originally, the search for a boundary theory for asymptotically AdS_3 general relativity was motivated by the Brown–Henneaux discovery [BH86], that the asymptotic symmetry algebra of asymptotically AdS_3 general relativity is the Virasoro algebra. Higher-spin generalization of the Brown–Heneaux result, determining the asymptotic symmetry algebra of asymptotically AdS_3 higher-spin gravity to be the W_N algebra was achieved in [CFPT10].

Knowing that Virasoro is the symmetry algebra of Liouville field theory and W_N is the symmetry algebra of $\mathsf{sl}(N)$ Toda field theory, a higher-spin generalization of Liouville theory, one makes an educated guess that asymptotically AdS_3 higher-spin gravity can be reduced to an $\mathsf{sl}(N)$ Toda field theory on the boundary. This hunch is further supported by the Hamiltonian reduction of the $\mathsf{SL}(2;\mathbb{R})$ WZW model to Liouville field theory generalizing to the Hamiltonian reduction of the $\mathfrak{G}$ WZW model to $\mathfrak{g}$ Toda field theory, where $\mathfrak{g}$ is a normal real form of a simple Lie algebra [FWB+89], [BFO+90]. Showing the $\mathrm{aAdS}_3 \to$ Toda reduction is the first aim of this thesis. These asymptotic symmetry algebras in [BH86], [CFPT10], and follow-up works were derived using the so-called Regge-Teitelboim method [RT74], see [BR16] for a modern review. The second aim of this thesis is to reproduce these results from scratch using the coordinate-free methods of covariant phase space formalism. Lastly, we aim to present these results in an ordered fashion. To achieve the stated goals, two pairs of boundary conditions and two pairs of gauge-fixings have to be implemented; this is done systematically in a top-down way.

This thesis is broken up into four parts: Part I gives an exposition to the methods and language of covariant phase space formalism, used to formulate the results in this thesis. An algebraic description of the variational bicomplex, an extension of the Cartan–de Rham calculus for variational methods in classical field theories, is given. Making use of this variational calculus, generalities for Lagrangian field theories and their associated (pre-)phase spaces are derived. Part II commences with necessary background on simple Lie algebras, leading into a classical analysis of Wess–Zumino–Witten models, Chern–Simons gauge field theories and their well-known correspondence using the methods of covariant phase space formalism. As asymptotically AdS 3-dimensional general relativity, and its higher-spin generalization, are formulated in the language of Chern–Simons field theories, these results constitute preliminary knowledge for the rest of this thesis. Part III describes well-known results about 3-dimensional general relativity and AdS spaces, leading into the Chern–Simons reformulation of 3-dimensional general relativity with negative cosmological constant and its reduction to a conformal field theory on the boundary. This is extended in Part IV to higher-spin AdS_3 gravity. We conclude by summarizing the results and outlining next steps for future research.

Part I.
Covariant phase space formalism

2. Variational bicomplex

2.1. Lie derivative

We use [Fec11] as the reference for basic notions in differential geometry.

$\mathcal{M}$ denotes an n-manifold with coordinates $\{x^\mu\}$. The basis of $T^*\mathcal{M}$ is given by coordinate differentials $\{\mathrm{d}x^\mu\}$, the basis of $T\mathcal{M}$ is given by coordinate partial derivatives $\{\partial_\mu\}$. Vector fields on $\mathcal{M}$ are local sections of the tangent bundle $T\mathcal{M} \to \mathcal{M}$. The space of vector fields is denoted as $\mathfrak{X}(\mathcal{M}) \coloneqq \Gamma(\mathcal{M}, T\mathcal{M})$,

$$\mathfrak{X}(\mathcal{M}) \ni \xi = \xi^\mu(x)\partial_\mu \tag{1}$$

ξ, ζ will denote generic manifold vector fields.

For a vector field ξ, for a point x_0 in $\mathcal{M}$, there is (in a local neighbourhood of x_0) an integral curve γ of ξ passing through x_0, ie. a curve whose tangent field is ξ. Through γ, ξ defines a one parameter family of diffeomorphisms ϕ_t, called the flow of ξ,

$$\begin{aligned} \phi_t : \mathcal{M} &\longrightarrow \mathcal{M} \\ x_0 \equiv \gamma(t_0) &\longmapsto \gamma(t_0 + t) \end{aligned} \tag{2}$$

As ϕ_t is a diffeomorphism, it acts on tensors on $\mathcal{M}$ via pullback. The Lie derivative wrt. ξ is defined as:

$$\pounds_\xi A \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^* A\Big|_{t=0} \tag{3}$$

It immediately follows that $\pounds_\xi$ is the linearization of ϕ_t,

$$\phi^*_{\varepsilon\to 0} A = A + \varepsilon \pounds_\xi A + O(\varepsilon^2) \tag{4}$$

The Lie derivative encodes the notion of "infinitesimal change in the direction of ξ".

Infinitesimal variations in field theory can also be encoded as Lie derivatives in some "field space". For example, compare the usual definition of an infinitesimal symmetry with the definition of a Lie derivative: a transformation $\Delta\varphi$ is a symmetry of a Lagrangian field theory, if:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\varphi + \varepsilon\Delta\varphi)\Big|_{\varepsilon=0} = 0 \tag{5}$$

The variational bicomplex formalism was developed in order to formalize field-theoretic variational calculus in a "coordinate-free" manner, similarly to how Cartan-de Rham calculus formalizes tensor calculus in a coordinate-free manner. The Cartan-de Rham calculus on "total field theory space" is described in the following, in order to formulate the notion of a Lagrangian field theory and its phase space.

2.2. Cartan-de Rham calculus on $\mathcal{M}$

A differential 1-form $\mathbf{A} \in \Gamma(\mathcal{M}, T^*\mathcal{M})$ is a linear map $\mathbf{A} : \mathfrak{X}(\mathcal{M}) \to C^\infty(\mathcal{M})$. A differential q-form is an alternating (totally antisymmetric) multilinear map $\mathbf{A} : \mathfrak{X}(\mathcal{M})^{\otimes q} \to C^\infty(\mathcal{M})$,

$$\begin{aligned} \mathbf{A} : \mathfrak{X}(\mathcal{M}) \otimes \cdots \otimes \mathfrak{X}(\mathcal{M}) &\longrightarrow C^\infty(\mathcal{M}) \\ (\xi_1, \ldots, \xi_q) &\longmapsto \mathbf{A}(\xi_1, \ldots, \xi_q) \end{aligned} \tag{6}$$

$$\mathbf{A}(\ldots, \xi_i, \xi_j, \ldots) = -\mathbf{A}(\ldots, \xi_j, \xi_i, \ldots) \tag{7}$$

The space of differential q-forms is defined as,

$$\Omega^q(\mathcal{M}) \coloneqq \Gamma(\mathcal{M}, \bigwedge^q T^*\mathcal{M}) \tag{8}$$

0-forms are defined to be functions, $\Omega^0(\mathcal{M}) \coloneqq C^\infty(\mathcal{M})$. $\Omega^q(\mathcal{M})$ identically vanish for $q > n$ on grounds of antisymmetry. Bases for the $\Omega^q(\mathcal{M})$'s are generated by exterior products of $\{\mathrm{d}x^\mu\}$,

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \coloneqq \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{9}$$

A differential q-form $\mathbf{A} \in \Omega^p(\mathcal{M})$ then has the component representation,

$$\mathbf{A}^{(q)} = \frac{1}{q!} A_{\mu_1 \ldots \mu_q} \,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q} \tag{10}$$

where $A_{\mu_1 \ldots \mu_q} \in C^\infty(\mathcal{M})$ is totally antisymmetric.

The exterior product $\wedge : \Omega^{q_1}(\mathcal{M}) \otimes \Omega^{q_2}(\mathcal{M}) \to \Omega^{q_1+q_2}(\mathcal{M})$ is given by,

$$\begin{aligned} \mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} &= \frac{1}{(q_1+q_2)!} (\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)})_{\mu_1 \ldots \mu_{q_1} \nu_1 \ldots \nu_{q_2}} \,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{q_1}} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_{q_2}} \\ &\coloneqq \frac{1}{(q_1+q_2)!} \frac{(q_1+q_2)!}{q_1! q_2!} A_{[\mu_1 \ldots \mu_{q_1}} B_{\nu_1 \ldots \nu_{q_2}]} \,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{q_1}} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_{q_2}} \end{aligned} \tag{11}$$

From $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$ it follows that $\wedge$ is a graded commutative product, satisfying:

$$\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} = (-1)^{q_1 q_2} \mathbf{B}^{(q_2)} \wedge \mathbf{A}^{(q_1)} \tag{12}$$

The exterior derivative $\mathrm{d} : \Omega^q(\mathcal{M}) \to \Omega^{q+1}(\mathcal{M})$ is given by,

$$\begin{aligned} \mathrm{d}\mathbf{A}^{(q)} &= \frac{1}{(q+1)!} (\mathrm{d}\mathbf{A}^{(q)})_{\mu\nu_1 \ldots \nu_q} \,\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q} \\ &\coloneqq \frac{1}{(q+1)!} (q+1) \partial_{[\mu} A_{\nu_1 \ldots \nu_q]} \,\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q} \end{aligned} \tag{13}$$

Its component representation is,

$$\mathrm{d} = \partial_\mu \,\mathrm{d}x^\mu \tag{14}$$

From the Leibnitz property of ∂_μ and $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$ it follows that d is a graded derivation of degree +1, satisfying,

$$
\begin{aligned}
\mathrm{d}^2 &= 0 \\
\mathrm{d}\left(\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)}\right) &= \mathrm{d}\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} + (-1)^{q_1}\mathbf{A}^{(q_1)} \wedge \mathrm{d}\mathbf{B}^{(q_2)}
\end{aligned}
\tag{15}
$$

The spaces of differential forms together with the exterior derivative form the de Rham complex on $\mathcal{M}$:

$$
\left(\Omega^\bullet(\mathcal{M}) \coloneqq \bigoplus_{q=0}^{n} \Omega^q(\mathcal{M}), \mathrm{d}\right) \tag{16}
$$

For any vector field $\mathfrak{X}(\mathcal{M}) \ni \xi = \xi^\mu \partial_\mu$, the interior product $\iota_\xi : \Omega^q(\mathcal{M}) \to \Omega^{q-1}(\mathcal{M})$ is given by,

$$
(\iota_\xi \mathbf{A})^{(q-1)} = \mathbf{A}^{(q)}(\xi, \_, \ldots, \_) \tag{17}
$$

The interior product of a function is defined to be zero. It follows that ι is a graded derivation of degree −1, satisfying:

$$
\begin{aligned}
\iota_\xi \iota_\zeta + \iota_\zeta \iota_\xi &= 0 \\
\iota_\xi\left(\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)}\right) &= \iota_\xi \mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} + (-1)^{q_1}\mathbf{A}^{(q_1)} \wedge \iota_\xi \mathbf{B}^{(q_2)}
\end{aligned}
\tag{18}
$$

The Lie derivative, viewed as an operation on differential forms $\pounds_\xi : \Omega^q(\mathcal{M}) \to \Omega^q(\mathcal{M})$, satisfies Cartan's magic formula:

$$
\pounds_\xi = \iota_\xi \mathrm{d} + \mathrm{d}\iota_\xi \tag{19}
$$

It is a derivation of degree 0, satisfying,

$$
\begin{aligned}
\pounds_\xi\left(\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)}\right) &= \pounds_\xi \mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} + \mathbf{A}^{(q_1)} \wedge \pounds_\xi \mathbf{B}^{(q_2)} \\
\mathrm{d}\pounds_\xi - \pounds_\xi \mathrm{d} &= 0 \\
\pounds_\xi \pounds_\zeta - \pounds_\zeta \pounds_\xi &= \pounds_{[\xi,\zeta]} \\
\iota_\xi \pounds_\zeta - \pounds_\zeta \iota_\xi &= \iota_{[\xi,\zeta]}
\end{aligned}
\tag{20}
$$

The operations $\{\mathrm{d}, \iota, \pounds\}$ along with the relations stated above, constitute the Cartan calculus on $\mathcal{M}$.

The action of d on differential forms is determined by its action on functions and generators of the bases, as well as its graded derivation property:

$$
\begin{aligned}
\mathrm{d}f = \partial_\mu f\, \mathrm{d}x^\mu &\quad \mathrm{d}(\mathrm{d}x^\mu) = 0 \\
\mathrm{d}\left(\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)}\right) &= \mathrm{d}\mathbf{A}^{(q_1)} \wedge \mathbf{B}^{(q_2)} + (-1)^{q_1}\mathbf{A}^{(q_1)} \wedge \mathrm{d}\mathbf{B}^{(q_2)}
\end{aligned}
\tag{21}
$$

The action of ι on differential forms is determined by its linear dependence on its argument, its action on functions and generators, as well as its graded derivation property:

$$\begin{gathered}\iota_{\xi+c\zeta}=\iota_\xi+c\iota_\zeta,\text{ in particular }\iota_\xi=\xi^\mu\iota_{\partial_\mu}\\ \iota_\xi f=0\quad \iota_{\partial_\mu}(\mathrm{d}x^\nu)=\mathrm{d}x^\nu\left(\partial_\mu\right)=\delta^\nu{}_\mu\\ \iota_\xi\left(\mathbf{A}^{(q_1)}\wedge\mathbf{B}^{(q_2)}\right)=\iota_\xi\mathbf{A}^{(q_1)}\wedge\mathbf{B}^{(q_2)}+(-1)^{q_1}\mathbf{A}^{(q_1)}\wedge\iota_\xi\mathbf{B}^{(q_2)}\end{gathered}\tag{22}$$

Given d and ι_ξ, the Lie derivative $\pounds_\xi$ (of differential forms) may be defined using Cartan's magic formula (19).

Differential forms valued in a Lie algebra

Differential forms on $\mathcal{M}$ may take value in some vector space $\mathcal{V}$ instead of the real numbers; ie. the components are no longer $C^\infty(\mathcal{M})$, but smooth maps $\mathcal{M}\to\mathcal{V}$

$$\mathbf{A}=\frac{1}{q!}A_{\mu_1\ldots\mu_q}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_q}\,,\ \text{where }A_{\mu_1\ldots\mu_q}:\mathcal{M}\longrightarrow\mathcal{V}\tag{23}$$

The space of differential q-forms valued in $\mathcal{V}$ is $\Omega^q(\mathcal{M},\mathcal{V}):=\Omega^q(\mathcal{M})\otimes\mathcal{V}$. The components of $\mathbf{A}\in\Omega^p(\mathcal{M},\mathcal{V}^{\otimes k})$ in terms of a vector space basis $\{\vec{\mathcal{E}}_a\}$ are:

$$\mathbf{A}=\frac{1}{p!}A_{\mu_1\ldots\mu_p}{}^{a_1\ldots a_k}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\,\vec{\mathcal{E}}_{a_1}\otimes\cdots\otimes\vec{\mathcal{E}}_{a_k},$$

$\wedge$ can be extended to $\wedge:\Omega^{q_1}(\mathcal{M},\mathcal{V}^{\otimes k})\otimes\Omega^{q_2}(\mathcal{M},\mathcal{V}^{\otimes l})\to\Omega^{q_1+q_2}(\mathcal{M},\mathcal{V}^{\otimes k+l})$.

Is $\mathcal{V}=\mathfrak{g}$ a Lie algebra of a Lie group $\mathfrak{G}$, then $\wedge$ can combine with the Lie bracket and Killing form on $\mathfrak{g}$ to produce operations on $\Omega^\bullet(\mathcal{M},\mathfrak{g})\otimes\Omega^\bullet(\mathcal{M},\mathfrak{g})\to\Omega^\bullet(\mathcal{M},\mathfrak{g})$. $\mathfrak{G}$-connections on $\mathcal{M}$ are locally elements of $\Omega^1(\mathcal{M},\mathfrak{g})$. The Lie bracket is,

$$\begin{aligned}[\ ,\ ]:\mathfrak{g}\otimes\mathfrak{g}&\longrightarrow\mathfrak{g}\\ (t,s)&\longmapsto[t,s]\end{aligned}$$

and $\mathfrak{g}$ is assumed to be equipped with a Killing form (trace),

$$\begin{aligned}\mathrm{tr}[\ \ ]:\mathfrak{g}\otimes\mathfrak{g}&\longrightarrow\mathbb{R}\\ (t,s)&\longmapsto\mathrm{tr}[ts]\end{aligned}$$

In this case one may combine the exterior product $\wedge$ into the wedge-bracket,

$$\Omega^p(\mathcal{M},\mathfrak{g})\otimes\Omega^q(\mathcal{M},\mathfrak{g})\xrightarrow{\wedge}\Omega^{p+q}(\mathcal{M},\mathfrak{g}\otimes\mathfrak{g})\xrightarrow{[\ ,\ ]}\Omega^{p+q}(\mathcal{M},\mathfrak{g})\tag{24}$$

and into the wedge-form,

$$\Omega^p(\mathcal{M},\mathfrak{g})\otimes\Omega^q(\mathcal{M},\mathfrak{g})\xrightarrow{\wedge}\Omega^{p+q}(\mathcal{M},\mathfrak{g}\otimes\mathfrak{g})\xrightarrow{\mathrm{tr}}\Omega^{p+q}(\mathcal{M})\tag{25}$$

2.3. Variational bicomplex

Conventions adapted from [Dic03, Chapter 19], [Mne22, Section 3].

The variational bicomplex of local forms will be described here as a set of algebraic rules, extending the Cartan-de Rham calculus on $\mathcal{M}$, to be used to formulate Lagrangian field theory. For the full construction, see [Zuc86], [And92], [Del18], [CS21], [Blo23].

$\{\varphi^a\}$ denotes the collection of some physical fields on spacetime $\mathcal{M}$. Typically, the space of field configurations is the space of sections of a field bundle $F \to \mathcal{M}$, $\Gamma(\mathcal{M}, F) =: \mathcal{F}$. $\mathcal{F}$ is referred to as the "field space"; it is an infinite-dimensional manifold, with a natural set of coordinates given by the jet coordinates $\{\varphi^a, {\varphi^a}_{\mu_1}, {\varphi^a}_{\mu_1\mu_2}, \dots\}$. The components of the r-th jet of φ^a are defined as,

$$
{\varphi^a}_{\mu_1\dots\mu_r} = \partial_{\mu_1}\dots\partial_{\mu_r}\varphi^a \equiv {\varphi^a}_{(\mu)} \tag{26}
$$

where $(\mu) \equiv (\mu_1 \dots \mu_r)$ is a multi-index. The basis elements of $T^*\mathcal{F}$ are called variations,

$$
\{\delta\varphi^a, \delta{\varphi^a}_{\mu_1}, \delta{\varphi^a}_{\mu_1\mu_2}, \dots\} \tag{27}
$$

the basis of $T\mathcal{F}$ are formal partial derivatives,

$$
\{\frac{\partial}{\partial\varphi^a}, \frac{\partial}{\partial{\varphi^a}_{\mu_1}}, \frac{\partial}{\partial{\varphi^a}_{\mu_1\mu_2}}, \dots\}, \quad \frac{\partial{\varphi^a}_{(\mu)}}{\partial{\varphi^b}_{(\nu)}} = {\delta^a}_b{\delta_{(\mu)}}^{(\nu)}, \quad {\partial_a}^{(\mu)} \equiv \frac{\partial}{\partial{\varphi^a}_{(\mu)}} \tag{28}
$$

Vector fields on $\mathcal{F}$ are sections of the tangent bundle $T\mathcal{F} \to \mathcal{F}$. The space of field space vector fields is denoted as $\mathfrak{X}(\mathcal{F}) := \Gamma(\mathcal{F}, T\mathcal{F})$,

$$
\mathfrak{X}(\mathcal{F}) \ni V = V^a_{(\mu)}\left(\varphi^b, {\varphi^b}_{\nu_1}, {\varphi^b}_{\nu_1\nu_2}, \dots\right) {\partial_a}^{(\mu)} \tag{29}
$$

V, W will denote generic field space vector fields.

A variational 1-form $\mathbf{A} \in \Gamma(\mathcal{F}, T^*\mathcal{F})$ is a linear map $\mathbf{A} : \mathfrak{X}(\mathcal{F}) \to C^\infty(\mathcal{F})$. A variational p-form is an alternating multilinear map $\mathbf{A} : \mathfrak{X}(\mathcal{F})^{\otimes p} \to C^\infty(\mathcal{F})$, defined analogously to differential forms. The space of variational p-forms is defined as

$$
\Omega^p(\mathcal{F}) := \Gamma(\mathcal{F}, \bigwedge^p T^*\mathcal{F}) \tag{30}
$$

0-forms are defined to be functions, $\Omega^0(\mathcal{F}) := C^\infty(\mathcal{F})$. Because $\mathcal{F}$ is an infinite-dimensional manifold, $p \in \mathbb{N}$. Bases for the $\Omega^p(\mathcal{F})$'s are generated by exterior products of $\{\delta{\varphi^a}_{(\mu)}\}$,

$$
\delta{\varphi^a}_{(\mu)} \wedge \delta{\varphi^b}_{(\nu)} = -\delta{\varphi^b}_{(\nu)} \wedge \delta{\varphi^a}_{(\mu)} \tag{31}
$$

A variational p-form $\mathbf{A} \in \Omega^p(\mathcal{F})$ then has the component representation,

$$
\mathbf{A}^{(q)} = \sum_{r\geq 0} \frac{1}{p!} A^{(\mu_1)\dots(\mu_p)}_{a_1\ \dots a_p} \delta{\varphi^{a_1}}_{(\mu_1)} \wedge \dots \wedge \delta{\varphi^{a_p}}_{(\mu_p)} \tag{32}
$$

where $(\mu_1) = (\mu_{1_1} \dots \mu_{1_r})$ etc., $A^{(\mu_1)\dots(\mu_p)}_{a_1\ \dots a_p} \in C^\infty(\mathfrak{F})$ is totally antisymmetric.

The exterior product $\wedge : \Omega^{p_1}(\mathfrak{F}) \otimes \Omega^{p_2}(\mathfrak{F}) \to \Omega^{p_1+p_2}(\mathfrak{F})$ is given by,

$$\begin{aligned} \mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)} &= \frac{1}{(p_1+p_2)!} (\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)})^{(\mu_1)\dots(\mu_p)(\nu_1)\dots(\nu_p)}_{a_1\ \dots a_p\ b_1\ \dots b_p} \\ &\quad \times \delta\varphi^{a_1}{}_{(\mu_1)} \wedge \dots \wedge \delta\varphi^{a_p}{}_{(\mu_p)} \wedge \delta\varphi^{b_1}{}_{(\nu_1)} \wedge \dots \wedge \delta\varphi^{b_p}{}_{(\nu_p)} \\ &:= \frac{1}{(p_1+p_2)!} \frac{(p_1+p_2)!}{p_1! p_2!} A^{(\mu_1)\dots(\mu_p)}_{[a_1\ \dots a_p} B^{(\nu_1)\dots(\nu_p)}_{b_1\ \dots b_p]} \\ &\quad \times \delta\varphi^{a_1}{}_{(\mu_1)} \wedge \dots \wedge \delta\varphi^{a_p}{}_{(\mu_p)} \wedge \delta\varphi^{b_1}{}_{(\nu_1)} \wedge \dots \wedge \delta\varphi^{b_p}{}_{(\nu_p)} \end{aligned} \tag{33}$$

$\wedge$ is a graded commutative product, satisfying:

$$\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)} = (-1)^{p_1 p_2} \mathbf{B}^{(p_2)} \wedge \mathbf{A}^{(p_1)} \tag{34}$$

The exterior variation $\delta : \Omega^p(\mathfrak{F}) \to \Omega^{p+1}(\mathfrak{F})$ is determined by its action on functions and generators of the bases, as well as its graded derivation (of degree $+1$) property:

$$\begin{gathered} \delta f = \sum_{r \geq 0} \partial_a{}^{(\mu)} f \delta\varphi^a{}_{(\mu)} \quad \delta(\delta\varphi^b{}_{(\nu)}) = 0 \\ \delta^2 = 0 \\ \delta\left(\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)}\right) = \delta\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)} + (-1)^{p_1} \mathbf{A}^{(p_1)} \wedge \delta\mathbf{B}^{(p_2)} \end{gathered} \tag{35}$$

It has the component representation,

$$\delta = \sum_{r \geq 0} \partial_a{}^{(\mu)} \delta\varphi^a{}_{(\mu)} \tag{36}$$

The spaces of variational forms together with the exterior variation form the variational complex on $\mathfrak{F}$,

$$\left(\Omega^\bullet(\mathfrak{F}) := \bigoplus_{p=0}^{\infty} \Omega^p(\mathfrak{F}), \delta\right) \tag{37}$$

For a vector field $V \in \mathfrak{X}(\mathfrak{F})$, the interior product $\iota_V : \Omega^p(\mathfrak{F}) \to \Omega^{p-1}(\mathfrak{F})$ is determined by its linear dependence on its argument, its action on functions and generators, as well as its graded derivation (of degree -1) property:

$$\begin{gathered} \iota_{V+cW} = \iota_W + c\iota_W \\ \iota_V f = 0 \quad \iota_{\partial_a{}^{(\mu)}} \delta\varphi^b{}_{(\nu)} = \delta\varphi^b{}_{(\nu)}(\partial_a{}^{(\mu)}) = \delta^b{}_a \delta_{(\nu)}{}^{(\mu)} \\ \iota_V + \iota_W = 0 \\ \iota_V\left(\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)}\right) = \iota_V \mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)} + (-1)^{p_1} \mathbf{A}^{(p_1)} \wedge \iota_V \mathbf{B}^{(p_2)} \end{gathered} \tag{38}$$

Given δ and ι_V, the Lie derivative $\pounds_V : \Omega^p(\mathcal{F}) \to \Omega^p(\mathcal{F})$ is given via Cartan's magic formula:

$$\pounds_V = \iota_V \delta + \delta \iota_V \tag{39}$$

It is a derivation of degree 0, satisfying,

$$\begin{aligned}
\pounds_V \left(\mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)}\right) &= \pounds_V \mathbf{A}^{(p_1)} \wedge \mathbf{B}^{(p_2)} + \mathbf{A}^{(p_1)} \wedge \pounds_V \mathbf{B}^{(p_2)} \\
\delta \pounds_V - \pounds_V \delta &= 0 \\
\pounds_V \pounds_W - \pounds_W \pounds_V &= \pounds_{[V,W]} \\
\iota_V \pounds_W - \pounds_W \iota_V &= \iota_{[V,W]}
\end{aligned} \tag{40}$$

The variational complex on $\mathcal{F}$ combines with the de Rham complex on $\mathcal{M}$ to form the variational bicomplex on the total field theory space $\mathcal{F} \times \mathcal{M}$. A local (p,q)-biform is a variational p-form on $\mathcal{F}$ valued in differential q-forms on $\mathcal{M}$, such that its value at $x \in \mathcal{M}$ depends only on x and the jet of fields at x,

$$\mathbf{A}^{(p,q)}_{\text{local}} = \mathbf{A}^{(p,q)}(x, \varphi^a(x), {\varphi^a}_{\mu_1}(x), {\varphi^a}_{\mu_1\mu_2}(x), \dots) \tag{41}$$

The space of local (p,q)-biforms is defined as,

$$\Omega^{p,q}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \coloneqq \Omega^p(\mathcal{F}) \wedge \Omega^q(\mathcal{M}) \tag{42}$$

$(0,0)$-forms are defined to be local functions, $\Omega^{(0,0)}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \coloneqq C^\infty_{\text{local}}(\mathcal{F} \times \mathcal{M})$, ie. their values at $x \in \mathcal{M}$ depend on x and jet of fields at x. Bases for the spaces of (p,q)-biforms are generated by exterior products of $\{\mathrm{d}x^\mu\, ,\ \delta{\varphi^a}_{(\mu)}\}$,

$$\begin{aligned}
\delta{\varphi^a}_{(\mu)} \wedge \delta{\varphi^b}_{(\nu)} &= -\delta{\varphi^b}_{(\nu)} \wedge \delta{\varphi^a}_{(\mu)} \\
\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu &= -\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \\
\delta{\varphi^a}_{(\mu)} \wedge \mathrm{d}x^\nu &= -\,\mathrm{d}x^\nu \wedge \delta{\varphi^a}_{(\mu)}
\end{aligned} \tag{43}$$

The component representation of a (p,q)-biform $A \in \Omega^{p.q}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is,

$$\mathbf{A}^{(p,q)} = \sum_{r \geq 0} \frac{1}{(p+q)!} {A^{(\mu_1)\dots(\mu_p)}_{a_1\ \ \dots a_p}}_{\nu_1\dots\nu_q} \delta{\varphi^{a_1}}_{(\mu_1)} \wedge \dots \wedge \delta{\varphi^{a_p}}_{(\mu_p)} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_q} \tag{44}$$

where ${A^{(\mu_1)\dots(\mu_p)}_{a_1\ \ \dots a_p}}_{\nu_1\dots\nu_q} \in C^\infty_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is totally antisymmetric.

The exterior product $\wedge : \Omega^{p_1,q_1}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \otimes \Omega^{p_2,q_2}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \to \Omega^{p_1+p_2,q_1+q_2}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is a graded commutative product,

$$\mathbf{A}^{(p_1,q_1)} \wedge \mathbf{B}^{(p_1,q_2)} = (-1)^{(p_1+q_1)(p_2+q_2)} \mathbf{B}^{(p_2,q_2)} \wedge \mathbf{A}^{(p_1,q_1)} \tag{45}$$

The variational bicomplex has two differentials:

- vertical differential $\delta : \Omega^{p,q}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \to \Omega^{p+1,q}_{\text{local}}(\mathcal{F} \times \mathcal{M})$
- horizontal differential $\mathrm{d} : \Omega^{p,q}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \to \Omega^{p,q+1}_{\text{local}}(\mathcal{F} \times \mathcal{M})$

The local nature of the biforms is encoded in the action of the differentials on functions and generators of the bases,

$$\delta f = \sum_{r\geq 0} \partial_a{}^{(\mu)} f \delta\varphi^a{}_{(\mu)} \quad \mathrm{d}f = \Big(\partial_\nu f + \sum_{r\geq 0} \varphi^a{}_{\nu(\mu)} \partial_a{}^{(\mu)} f\Big)\, \mathrm{d}x^\nu \tag{46}$$

$$\begin{aligned} \delta(\delta\varphi^a{}_{(\mu)}) &= 0 = \mathrm{d}(\mathrm{d}x^\mu) \\ \delta(\mathrm{d}x^\mu) &= 0 \\ \mathrm{d}\big(\delta\varphi^a{}_{(\mu)}\big) &= -\delta \mathrm{d}\varphi^a{}_{(\mu)} = -\delta\varphi^a{}_{\nu(\mu)} \wedge \mathrm{d}x^\nu \end{aligned} \tag{47}$$

The differentials satisfy,

$$\begin{aligned} \delta^2 &= 0 \\ \mathrm{d}^2 &= 0 \\ \mathrm{d}\delta + \delta\mathrm{d} &= 0 \\ \delta\left(\mathbf{A}^{(p_1,q_1)} \wedge \mathbf{B}^{(p_2,q_2)}\right) &= \delta\mathbf{A}^{(p_1,q_1)} \wedge \mathbf{B}^{(p_1,q_2)} + (-1)^{p_1+q_1}\mathbf{A}^{(p_1+q_1)} \wedge \delta\mathbf{B}^{(p_2+q_2)} \\ \mathrm{d}\left(\mathbf{A}^{(p_1,q_1)} \wedge \mathbf{B}^{(p_2,q_2)}\right) &= \mathrm{d}\mathbf{A}^{(p_1,q_1)} \wedge \mathbf{B}^{(p_1,q_2)} + (-1)^{p_1+q_1}\mathbf{A}^{(p_1+q_1)} \wedge \mathrm{d}\mathbf{B}^{(p_2+q_2)} \end{aligned} \tag{48}$$

They have the component representation,

$$\begin{aligned} \delta &= \sum_{r\geq 0} \partial_a{}^{(\mu)} \delta\varphi^a{}_{(\mu)} \\ \mathrm{d} &= \Big(\partial_\nu + \sum_{r\geq 0} \varphi^a{}_{\nu(\mu)} \partial_a{}^{(\mu)}\Big)\, \mathrm{d}x^\nu \end{aligned} \tag{49}$$

The spaces of local biforms together with the two differentials form the variational bicomplex on $\mathcal{F} \times \mathcal{M}$:

$$\Big(\Omega^{\bullet,\bullet}_{\text{local}}(\mathcal{F} \times \mathcal{M}) \coloneqq \bigoplus_{p=0}^{n} \bigoplus_{q=0}^{\infty} \Omega^{p,q}_{\text{local}}(\mathcal{F} \times \mathcal{M}), \delta, \mathrm{d}\Big) \tag{50}$$

On the total space, local vector fields $(\xi, V) \in \mathfrak{X}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ are simply linear combinations of manifold vector fields $\xi \in \mathfrak{X}(\mathcal{M})$ and field space vector fields $V \in \mathfrak{X}(\mathcal{F})$, $\mathfrak{X}_{\text{local}}(\mathcal{F} \times \mathcal{M}) = \mathfrak{X}(\mathcal{F}) \oplus \mathfrak{X}(\mathcal{M})$:

$$(\xi, V) \equiv \sum_{r\geq 0} V^a{}_{(\mu)}\, \partial_a{}^{(\mu)} + \xi^\nu \partial_\nu \tag{51}$$

By its linear dependence on its argument, the interior product ι on $\Omega^{\bullet,\bullet}_{\text{local}}(\mathcal{F}\times\mathcal{M})$ also splits,

$$\iota_{(\xi,V)} = \iota_\xi + \iota_V \tag{52}$$

its action on generators is,

$$\begin{aligned} \iota_{\partial_\mu}\,\mathrm{d}x^\nu &= \delta^\nu{}_\mu \qquad & \iota_{\partial_\mu}\delta\varphi^b{}_{(\nu)} &= 0 \\ \iota_{\partial_a{}^{(\mu)}}\,\mathrm{d}x^\nu &= 0 \qquad & \iota_{\partial_a{}^{(\mu)}}\delta\varphi^b{}_{(\nu)} &= \delta^b{}_a\delta_{(\nu)}{}^{(\mu)} \end{aligned} \tag{53}$$

The Lie derivative similarly splits,

$$\pounds_{(\xi,V)} = \pounds_\xi + \pounds_V \tag{54}$$

The Cartan calculi on $\mathcal{M}$ and $\mathcal{F}$ graded-commute; in total:

$$\mathrm{d}^2 = 0 \quad \delta^2 = 0 \quad \mathrm{d}\delta + \delta\mathrm{d} = 0 \tag{55}$$

$$\begin{aligned} \iota_\xi\iota_\zeta + \iota_\zeta\iota_\xi &= 0 & \iota_V\iota_W + \iota_W\iota_V &= 0 \\ \mathrm{d}\pounds_\xi - \pounds_\xi\mathrm{d} &= 0 & \delta\pounds_V - \pounds_V\delta &= 0 \\ \iota_\xi\mathrm{d} + \mathrm{d}\iota_\xi &= \pounds_\xi & \iota_V\delta + \delta\iota_V &= \pounds_V \\ \pounds_\xi\pounds_\zeta - \pounds_\zeta\pounds_\xi &= \pounds_{[\xi,\zeta]} & \pounds_V\pounds_W - \pounds_W\pounds_V &= \pounds_{[V,W]} \\ \pounds_\xi\iota_\zeta - \iota_\zeta\pounds_\xi &= \iota_{[\xi,\zeta]} & \pounds_V\iota_W - \iota_W\pounds_V &= \iota_{[V,W]} \end{aligned} \tag{56}$$

$$\begin{aligned} \delta\pounds_\xi - \pounds_\xi\delta &= 0 & \delta\iota_\xi + \iota_\xi\delta &= 0 \\ \mathrm{d}\pounds_V - \pounds_V\mathrm{d} &= 0 & \mathrm{d}\iota_V + \iota_V\mathrm{d} &= 0 \\ \pounds_\xi\iota_V - \iota_V\pounds_\xi &= 0 & \pounds_V\iota_\xi - \iota_\xi\pounds_V &= 0 \\ \iota_\xi\iota_V + \iota_V\iota_\xi &= 0 & \pounds_\xi\pounds_V - \pounds_V\pounds_\xi &= 0 \end{aligned} \tag{57}$$

3. Lagrangian field theory, phase space

3.1. Equations of motion

Introducing some nomenclature, an element of $\Omega^{\cdot,n}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is called a density, an element of $\Omega^{\cdot,n-1}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is called a current.

A Lagrangian field theory for a field bundle $F \to \mathcal{M}$ is specified by a choice of Lagrangian density $\mathbf{L} \in \Omega^{0,n}_{\text{local}}(\mathcal{F} \times \mathcal{M})$, in turn specifying the action of the theory,

$$S[\varphi] \coloneqq \int_{\mathcal{M}} \mathbf{L}(\varphi^a, {\varphi^a}_{\mu_1}, {\varphi^a}_{\mu_1\mu_2}, \dots) \tag{58}$$

The variation of the action defines the equations of motion (EoM) of the theory,

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} \delta\varphi^a \wedge \mathrm{EL}_a(\mathbf{L}) - \int_{\partial\mathcal{M}} \boldsymbol{\vartheta} \\ \Longleftrightarrow \delta\mathbf{L} &= \delta\varphi^a \wedge \mathrm{EL}_a(\mathbf{L}) - \mathrm{d}\boldsymbol{\vartheta} \end{aligned} \tag{59}$$

$$\text{EoM:} \quad \mathrm{EL}(\mathbf{L}) = 0 \tag{60}$$

as well as a (pre-)symplectic potential $\boldsymbol{\vartheta} \in \Omega^{1,n-1}_{\text{local}}(\mathcal{F}\times\mathcal{M})$, that, for a choice of maximally inextensible codim.-1 (spacelike) surface $\mathcal{C} \subset \mathcal{M}$, defines a (pre-)symplectic structure on $\mathcal{F}$:

$$\boldsymbol{\omega} \coloneqq \delta\boldsymbol{\vartheta} \tag{61}$$

$$\boldsymbol{\Omega} \coloneqq \int_{\mathcal{C}} \boldsymbol{\omega} \tag{62}$$

$\boldsymbol{\omega} \in \Omega^{2,n-1}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is called the (pre-)symplectic current, $\boldsymbol{\Omega} \in \Omega^{2,0}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ is called the (pre-)symplectic form. The prefix "pre-" signifies that, as defined from $\mathbf{L}$, $\boldsymbol{\Omega}$ may be nondegenerate, ie. there exist some $\mathfrak{X}(\mathcal{F}) \ni V \neq 0$ st. $\iota_V\boldsymbol{\Omega} = 0$.

The EoM carve out a subspace in $\mathcal{F}$, their locus, called the shell,

$$\mathcal{S} \coloneqq \{\varphi \in \mathcal{F} \,|\, \mathrm{EL}(\mathbf{L})(\varphi) = 0\} \tag{63}$$

A field configuration satisfying the EoM is called being on-shell. Equalities that only hold on $\mathcal{S} \subset \mathcal{F}$ will be denoted by $\overset{\text{EoM}}{=}$. The (pre-)symplectic manifold $(\mathcal{S}, \boldsymbol{\Omega})$ is called the (pre-)phase space of the Lagrangian field theory ($F \to \mathcal{M}$, $\mathbf{L}$).

On-shell fields describe the bulk dynamics of the theory. Any $\varphi_s \in \mathcal{S}$, supported away from the boundary, extremizes the action: for any infinitesimal variation $\pounds_\Delta\varphi^a = \Delta\varphi^a$,

$$\pounds_\Delta S[\varphi_s] = \iota_\Delta \delta S[\varphi_s] = \int_{\mathcal{M}} \Delta\varphi^a\, \mathrm{EL}_a(\mathbf{L})(\varphi_s) = 0 + \int_{\partial\mathcal{M}} \dots \tag{64}$$

Meaning of the EL operator

EL is the Euler-Lagrange derivative: to see this, use the fact that the space of local $(1,n)$-forms decomposes into so-called source forms plus d-exact terms,

$$\begin{array}{c} \Omega^{1,n}_{\mathrm{local}} = \Omega^{1,n}_{\mathrm{source}} \oplus \mathrm{d}\Omega^{1,n-1}_{\mathrm{local}} \\ \Cup \\ \mathbf{B}^{(1,n)} = \mathbf{B}_{\mathrm{source}} + \mathrm{d}\boldsymbol{\eta}^{(1,n-1)} \end{array} \tag{65}$$

where a source form contains only 0th order variations $\delta\varphi^a$, ie. $\mathbf{B}_{\mathrm{source}} = \delta\varphi^a \wedge \mathbf{B}^{(0,n)}_a$. This is shown by moving ∂_μ's from the variations onto the $(0,n)$ components, in turn getting a d-exact term [Mne22]; using the identity $\partial_\mu = \mathrm{d}\iota_{\partial_\mu}$,

$$\mathbf{B}^{(1,n)} = \sum_{r\geq 0} \delta{\varphi^a}_{\mu_1\ldots\mu_r} \wedge \mathbf{B}^{\mu_1\ldots\mu_r}_a, \quad {\varphi^a}_{\mu_1\ldots\mu_r} = \partial_{\mu_r}{\varphi^a}_{\mu_1\ldots\mu_{r-1}} = \mathrm{d}\iota_{\partial_{\mu_r}}{\varphi^a}_{\mu_1\ldots\mu_{r-1}}$$

$$\begin{aligned} \delta{\varphi^a}_{\mu_1\ldots\mu_r} \wedge \mathbf{B}^{\mu_1\ldots\mu_r}_a &= -\delta{\varphi^a}_{\mu_1\ldots\mu_{r-1}} \wedge \partial_{\mu_r}\mathbf{B}^{\mu_1\ldots\mu_r}_a + \mathrm{d}\iota_{\partial_{\mu_r}}(\delta{\varphi^a}_{\mu_1\ldots\mu_{r-1}} \wedge \mathbf{B}^{\mu_1\ldots\mu_r}_a) \\ &= \ldots \\ &= (-1)^r\delta\varphi^a \wedge \partial_{\mu_1}\cdots\partial_{\mu_r}\mathbf{B}^{\mu_1\ldots\mu_r}_a + \mathrm{d}(\ldots) \end{aligned} \tag{66}$$

By the linearity of $\wedge$, this suffices. Because d acts on the variations $\delta{\varphi^a}_{(\mu)}$, $\Omega^{1,n}_{\mathrm{source}} \cap \mathrm{d}\Omega^{1,n-1}_{\mathrm{local}} = 0$, this decomposition is unique. Because the spaces $\Omega^{\cdot,q}_{\mathrm{local}}$ identically vanish only for $q > n$, this only works for $(1,n)$-forms. The Euler-Lagrange derivative of $\mathbf{L}$ is the projection of $\delta\mathbf{L}$ onto the source forms, explicitly:

$$\mathrm{EL}_a(\mathbf{L}) = \sum_{r\geq 0}(-1)^r\,\partial_{\mu_1}\cdots\partial_{\mu_r}\frac{\partial\mathbf{L}}{\partial{\varphi^a}_{\mu_1\ldots\mu_r}} \tag{67}$$

The EoM are, in fact, the Euler-Lagrange equations for $\mathbf{L}$.

3.2. Noether's theorem

For a deeper dive into Noether theorems in the variational bicomplex formalism, cf. [Zuc86], [Del18], [CF18, Lecture 1], [Fio21], [Blo23].

A field space vector field $V \in \mathfrak{X}(\mathcal{F})$ is an infinitesimal symmetry of the Lagrangian field theory $(F \to \mathcal{M}, \mathbf{L})$ if,

$$\begin{aligned} \pounds_V S &= \int_{\partial\mathcal{M}} \mathbf{B}_V \\ \iff \pounds_V \mathbf{L} &= d\mathbf{B}_V \end{aligned} \tag{68}$$

for some $\mathbf{B}_V \in \Omega^{0,n-1}_{\text{local}}(\mathcal{F} \times \mathcal{M})$. Such transformations map solutions of the EoM (supported away from the boundary) to other solutions of the EoM,

$$\begin{aligned} \pounds_V S = \iota_V \delta S &= \int_{\mathcal{M}} \pounds_V \varphi^a \, \mathrm{EL}_a(\mathbf{L}) + \int_{\partial\mathcal{M}} \dots \\ &= \int_{\mathcal{M}} \pounds_V(\varphi^a \underbrace{\mathrm{EL}_a(\mathbf{L})}_{\overset{\text{EoM}}{=} 0}) - \int_{\mathcal{M}} \varphi^a \pounds_V \mathrm{EL}_a(\mathbf{L}) + \int_{\partial\mathcal{M}} \dots \\ &\overset{V \text{ symm.}}{=} \int_{\partial\mathcal{M}} \dots \end{aligned} \tag{69}$$

$$\implies \pounds_V \mathrm{EL}(\mathbf{L}) \overset{\text{EoM}}{=} 0 \tag{70}$$

Put another way, infinitesimal symmetries of the theory preserve the shell $\mathcal{S} \subset \mathcal{F}$.

Intoducing some more nomenclature: a conserved current is any element $\mathbf{J} \in \Omega^{0,n-1}_{\text{local}}(\mathcal{F} \times \mathcal{M})$ that is closed on-shell:

$$\mathrm{d}\mathbf{J} \overset{\text{EoM}}{=} 0 \tag{71}$$

There is an equivalence class of conserved currents: $\mathbf{J} \sim \mathbf{J}'$ if they differ on-shell by a d-exact term,

$$\mathbf{J}' \overset{\text{EoM}}{=} \mathbf{J} + \mathrm{d}\mathbf{K} \tag{72}$$

Conserved charges are obtained by integrating conserved currents over $\mathcal{C}$,

$$H := \int_{\mathcal{C}} \mathbf{J} \tag{73}$$

For $\partial\mathcal{M} = \varnothing \implies \partial\mathcal{C} = \varnothing$, the conserved charges depend only on the equivalence class of $\mathbf{J}$. They are conserved in the sense of evaluating the same on two cobordant $\mathcal{C}_1$, $\mathcal{C}_2$, ie. for an $\mathcal{N} \subset \mathcal{M}$ such that $\partial\mathcal{N} = -\mathcal{C}_1 \sqcup \mathcal{C}_2$,

$$H|_{\mathcal{C}_2} - H|_{\mathcal{C}_1} = \int_{\mathcal{C}_2} \mathbf{J} - \int_{\mathcal{C}_1} \mathbf{J} = \int_{\mathcal{N}} \mathrm{d}\mathbf{J} \overset{\text{EoM}}{=} 0 \tag{74}$$

Noether's theorem states the existence of a conserved current $\mathbf{J}_V$ associated to any infinitesimal symmetry $V \in \mathfrak{X}(\mathcal{F})$:

$$\begin{aligned}
\pounds_V \mathbf{L} &= \iota_V \delta \mathbf{L} \overset{\text{EoM}}{=} -\iota_V \mathrm{d}\boldsymbol{\vartheta} = \mathrm{d}\iota_V \boldsymbol{\vartheta} \\
&\overset{V \text{ symm.}}{=} \mathrm{d}\mathbf{B}_V \\
\mathbf{J}_V &\coloneqq \iota_V \boldsymbol{\vartheta} - \mathbf{B}_V \\
\mathrm{d}\mathbf{J}_V &= \mathrm{d}\iota_V \boldsymbol{\vartheta} - \mathrm{d}\mathbf{B}_V \overset{\text{EoM}}{=} 0
\end{aligned} \tag{75}$$

$\mathbf{J}_V$ is called the Noether current associated to V, $H_V \coloneqq \int_{\mathcal{C}} \mathbf{J}_V$ is called the Noether charge associated to V.

For $\partial\mathcal{M} = \varnothing$, a symmetry $V \in \mathfrak{X}(\mathcal{F})$ of $(F \to \mathcal{M}, \mathbf{L})$ corresponds to a Hamiltonian vector field of $\boldsymbol{\Omega}$, with H_V being its Hamiltonian function :

$$\begin{aligned}
\pounds_V \delta \mathbf{L} = \delta \pounds_V \mathbf{L} &\implies \pounds_V \boldsymbol{\vartheta} \overset{\text{EoM}}{=} \delta \mathbf{B}_V + \mathrm{d}(\dots) \\
\iota_V \boldsymbol{\omega} = \pounds_V \boldsymbol{\vartheta} - \delta \iota_V \boldsymbol{\vartheta} &\overset{\text{EoM}}{=} -\delta \mathbf{J}_V + \mathrm{d}(\dots) \\
&\implies \iota_V \boldsymbol{\Omega} \overset{\text{EoM}}{=} -\delta H_V
\end{aligned} \tag{76}$$

Trivial cohomology implies: symplectomorphisms of $\mathcal{S} \overset{1:1}{\longleftrightarrow}$ Hamiltonian vector fields:

$$\pounds_V \boldsymbol{\Omega} = \delta \iota_V \boldsymbol{\Omega} \overset{\text{EoM}}{=} -\delta^2 H_V = 0 \tag{77}$$

Note on boundary terms

One may add to the action a boundary term,

$$\begin{aligned}
S &\longmapsto S + \int_{\partial\mathcal{M}} \mathbf{f} \\
\iff \mathbf{L} &\longmapsto \mathbf{L} + \mathrm{d}\mathbf{f}
\end{aligned} \tag{78}$$

Such a change does not affect the bulk dynamics or the symplectic structure,

$$\begin{aligned}
\delta S &\longmapsto \int_{\mathcal{M}} \delta\varphi^a \wedge \mathrm{EL}_a(\mathbf{L}) - \int_{\partial\mathcal{M}} \boldsymbol{\vartheta} - \int_{\partial\mathcal{M}} \delta \mathbf{f} \\
\iff \delta \mathbf{L} &\longmapsto \varphi^a \wedge \mathrm{EL}_a(\mathbf{L}) - \mathrm{d}\boldsymbol{\vartheta} - \mathrm{d}\delta\mathbf{f}
\end{aligned} \tag{79}$$

$$\begin{aligned}
\implies \boldsymbol{\vartheta} &\longmapsto \boldsymbol{\vartheta} + \delta \mathbf{f} \\
\boldsymbol{\omega} &\longmapsto \boldsymbol{\omega} + \delta^2 \mathbf{f} = \boldsymbol{\omega}
\end{aligned} \tag{80}$$

It also does not affect any Noether currents,

$$\begin{aligned}
\pounds_V \mathbf{L} &\longmapsto \mathrm{d}\mathbf{B}_V + \mathrm{d}\pounds_V \mathbf{f} \\
\hat{=}\ \mathbf{B}_V &\longmapsto \mathbf{B}_V + \pounds_V \mathbf{f} \\
\implies \mathbf{J}_V &\longmapsto \iota_V \boldsymbol{\vartheta} + \iota_V \delta \mathbf{f} - \mathbf{B}_V - \pounds_V \mathbf{f} = \mathbf{J}_V
\end{aligned} \tag{81}$$

3.3. Poisson bracket

Let $V, W \in \mathfrak{X}(\mathcal{F})$ be two symmetries of $(F \to \mathcal{M}, \mathbf{L})$. Then, the (pre-)symplectic form $\mathbf{\Omega}$ on $\mathcal{S}$ carries a Poisson bracket of their Hamiltonian functions:

$$\pounds_V H_W = \iota_V \delta H_W \overset{\text{EoM}}{=} -\iota_V \iota_W \mathbf{\Omega} = \iota_W \iota_V \mathbf{\Omega} \equiv \{H_V, H_W\} \tag{82}$$

Further assume that $[V, W] \in \mathfrak{X}(\mathcal{F})$ is also a symmetry. Using,

$$\pounds_{[V,W]} = \pounds_V \pounds_W - \pounds_W \pounds_V \implies \mathbf{B}_{[V,W]} = \pounds_V \mathbf{B}_W - \pounds_W \mathbf{B}_V + \mathrm{d}(\ldots) \tag{83}$$

the Noether current of the commutator can be expressed in terms of V, W as,

$$\begin{aligned}
\mathbf{J}_{[V,W]} &= \iota_{[V,W]} \boldsymbol{\vartheta} - \mathbf{B}_{[V,W]} \\
&= \iota_V \pounds_W \boldsymbol{\vartheta} - \pounds_W \iota_V \boldsymbol{\vartheta} - \pounds_V \mathbf{B}_W + \pounds_W \mathbf{B}_V + \mathrm{d}(\ldots) \\
&= -\pounds_W \mathbf{J}_V + \iota_V \pounds_W \boldsymbol{\vartheta} \underbrace{-\pounds_V \mathbf{B}_W}_{=-\iota_V \delta \mathbf{B}_W} + \mathrm{d}(\ldots) \\
&\overset{\text{EoM}}{=} -\pounds_W \mathbf{J}_V + \iota_V \pounds_W \boldsymbol{\vartheta} - \iota_V \pounds_W \boldsymbol{\vartheta} + \mathrm{d}(\ldots) \qquad\qquad = -\pounds_W \mathbf{J}_V
\end{aligned}$$

$$\implies \pounds_V \mathbf{J}_W \overset{\text{EoM}}{=} \mathbf{J}_{[V,W]} + \mathrm{d}(\ldots) \tag{84}$$

For $\partial\mathcal{M} = \varnothing$, the Poisson bracket carries a representation of the symmetry commutator algebra:

$$\pounds_V H_W \overset{\text{EoM}}{=} H_{[V,W]} \implies \{H_V, H_W\} = H_{[V,W]} \tag{85}$$

In general, the Poisson bracket carries a centrally-extended representation of the symmetry commutator algebra. Eschewing the above calculation, $H_{[V,W]}$ can be directly obtained from Ω:

$$\begin{aligned}
\iota_{[V,W]} \mathbf{\Omega} &\overset{\text{EoM}}{=} -\delta H_{[V,W]} \\
&= (\iota_V \pounds_W - \pounds_W \iota_V) \mathbf{\Omega} \overset{W \text{ symm.}}{=} -\pounds_W \iota_V \mathbf{\Omega} \overset{\text{EoM}}{=} \pounds_W \delta H_V \\
&= \delta \pounds_W H_V \overset{\text{EoM}}{=} \delta\{H_W, H_V\} = -\delta\{H_V, H_W\}
\end{aligned}$$

$$\implies \{H_V, H_W\} = H_{[V,W]} + \kappa_{V,W}, \quad \delta\kappa_{V,W} = 0 \tag{86}$$

It can be shown that $\kappa_{V,W}$ satisfies the required Lie algebra 2-cocycle properties. Important for the context of this thesis, the above calculation is less ambiguous for $\partial\mathcal{M} \neq \varnothing$.

Part II.
Preliminaries

4. Simple Lie algebras

We use [FS03] as the reference for basic notions in Lie algebras.

The goal of this section is to ultimately construct the so-called spin basis for the Lie algebra $\mathsf{sl}(N;\mathbb{R})$.

A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying:

$$\begin{aligned} [t, s] &= -[s, t] \\ [t, [s, r]] + [s, [r, t]] + [r, [t, s]] &= 0 \end{aligned} \tag{87}$$

In this text, we will mostly consider finite-dimensional real Lie algebras.

The prototypical Lie algebra is some subalgebra of the general linear matrix algebra $\mathsf{gl}(N) = \{N \times N \text{ matrices}\}$ with $[\ ,\ ]$ being the matrix commutator. Two elements are said to commute if their Lie bracket vanishes. If the Lie bracket vanishes, ie. $[\mathfrak{g}, \mathfrak{g}] = 0$, the Lie algebra is called commutative.

Lie algebras may be represented as commutators of linear maps of some vector space $\mathcal{V}$. For the purposes of this thesis, it suffices to consider $\mathcal{V} = \mathbb{R}^N$. A representation of Lie algebra $\mathfrak{g}$ is a map,

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathsf{gl}(N;\mathbb{R}) \\ t &\longmapsto \rho(t) \end{aligned} \tag{88}$$

such that the Lie bracket maps onto the matrix commutator,

$$\rho([t, s]) = \rho(t)\rho(s) - \rho(s)\rho(t) \tag{89}$$

For any element $t \in \mathfrak{g}$, there is an associated derivation of $\mathfrak{g}$, called the adjoint action of t on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{ad}_t : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ s &\longmapsto [t, s] \end{aligned} \tag{90}$$

The associated Lie group $\mathfrak{G}_0 = \exp \mathfrak{g}$ acts on $\mathfrak{g}$ via the adjoint action

$$\begin{aligned} \mathrm{Ad}_G : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ G &\longmapsto GtG^{-1} \end{aligned} \tag{91}$$

Ad and ad are related via the useful identity $\mathrm{Ad}_{e^t} = e^{\mathrm{ad}_t}$.

A subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the Lie bracket, ie.

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \tag{92}$$

is a subalgebra of $\mathfrak{g}$. An ideal is a subalgebra satisfying the stronger property,

$$[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h} \tag{93}$$

Every Lie algebra contains two trivial ideals, the zero element $\{0\}$ and $\mathfrak{g}$ itself. The direct sum of vector spaces,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \tag{94}$$

is a direct sum of Lie algebras if:

$$[\mathfrak{g}_i, \mathfrak{g}_j] = 0 \quad \forall\, i \neq j \in \{1, \ldots, n\} \tag{95}$$

A simple Lie algebra is a noncommutative Lie algebra containing no nontrivial ideals, a semisimple Lie algebra is a direct sum of simple Lie algebras. $\mathsf{sl}(2)$ is the simplest example of simple Lie algebras. A canonical basis for $\mathsf{sl}(2)$ is given by the so-called Chevalley triple $\{H, E_+, E_-\}$, satisfying:

$$[H, E_\pm] = \pm 2E_\pm \quad [E_+, E_-] = H \tag{96}$$

Denote by $\{t_A\}$ a basis of $\mathfrak{g}$. The basis elements $\{t_A\}$ are called generatos, the components of the commutator wrt. to $\{t_A\}$ are called structure constants:

$$[t_A, t_B] = f_{AB}{}^C t_C \tag{97}$$

We will always take the Killing form to correspond to a matrix trace tr[ ] in some representation. Given a basis $\{t_A\}$, the Killing form is represented by the matrix,

$$\mathrm{tr}[t_A t_B] = \gamma_{AB} \tag{98}$$

Data of a simple Lie algebra

Any simple Lie algebra is completely specified by its root system. Here, we name the associated quantities and display the calculation rules. We refer to [FS03] for any background knowledge.

Φ denotes the space of positive roots, Φ_s denotes the space of simple roots. $\langle\ ,\ \rangle$ is the inner product on root space. Positive roots are $\alpha, \beta \in \Phi$, simple roots are $\alpha^i \in \Phi_\mathrm{s}$, coroots are $\alpha^\vee = \frac{2}{|\alpha|^2}\alpha$.

The components of the Cartan matrix, symmetrized Cartan matrix, quadratic form matrix of some root system Φ_{s} are defined as,

$$\begin{aligned} A^{ij} &= \frac{2}{|\alpha^j|^2}\langle \alpha^i, \alpha^j \rangle \\ G^{ij} &= \langle \alpha^{\vee i}, \alpha^{\vee j} \rangle = \frac{2}{|\alpha^i|^2} A^{ij} \\ G_{ij} &= (G^{-1})_{ij} = \langle \Lambda_{(i)}, \Lambda_{(j)} \rangle \end{aligned} \tag{99}$$

These quantities may also be associated to nonsimple roots $\alpha, \beta \in \Phi$,

$$\begin{aligned} A^{\alpha\beta} &= \frac{2}{|\beta|^2}\langle \alpha, \beta \rangle \\ G^{\alpha\beta} &= \langle \alpha^{\vee}, \beta^{\vee} \rangle = \frac{2}{|\alpha|^2} A^{\alpha\beta} \end{aligned} \tag{100}$$

The simple Lie algebra associated to the set of r simple roots $\Phi_{\mathrm{s}} = \{\alpha^i\}$ is uniquely determined as the Lie algebra generated by Chevalley $\mathsf{sl}(2)$ triples $\{H^i, E^i_+, E^i_-\}$ $\forall$ simple root α^i, satisfying:

- $[H^i, H^j] = 0$
- $[H^i, E^j_\pm] = \pm A^{ji} E^j_\pm$
- $[E^i_+, E^j_-] = \delta^{ij} H^i$
- $(\mathrm{ad}_{E^i_\pm})^{1-A^{ji}} E^j_\pm = 0$ (Chevalley-Serre relations)

The list of all simple Lie algebras and the tabulation of their associated quantites can be found in [FS03, Chapter 7].

The components of the Killing form are listed below,

$$\begin{aligned} &\mathrm{tr}\big[H^i H^j\big] = G^{ij} \quad \mathrm{tr}\big[H^i E^j_\pm\big] = 0 \quad \mathrm{tr}\big[E^i_+ E^j_-\big] = \frac{2}{|\alpha^i|^2}\delta^{ij} \quad \mathrm{tr}\big[E^i_\pm E^j_\pm\big] = 0 \\ &\mathrm{tr}\big[H^\alpha H^\beta\big] = G^{\alpha\beta} \quad \mathrm{tr}\Big[H^\alpha E^\beta_\pm\Big] = 0 \quad \mathrm{tr}\Big[E^\alpha_+ E^\beta_-\Big] = \frac{2}{|\alpha|^2}\delta^{\alpha\beta} \quad \mathrm{tr}\Big[E^\alpha_\pm E^\beta_\pm\Big] = 0 \end{aligned} \tag{101}$$

The so-called heights of fundamental weights are defined as

$$k_i = \sum_{j=1}^{N-1} \frac{2}{|\alpha^j|^2} G_{ij} \tag{102}$$

Spin basis for $\mathsf{sl}(N;\mathbb{R})$

The aim is to construct the so-called spin basis for the maximally noncompact real form of a simple Lie algebra $\mathfrak{g}$ out of its root system data. Here, the conventions of [CF23] are used, cf. [RS97] as well.

We will want to work with $\mathsf{sl}(2;\mathbb{R})$ generators $\{L_+, L_0, L_-\}$ satisfying,

$$[L_m, L_n] = (m-n)L_{m+n} \tag{103}$$

Out of any such generators $\{L_+, L_0, L_-\}$, the $\mathsf{so}(2,1) \cong \mathsf{sl}(2;\mathbb{R})$ generators $\{j_0, j_1, j_2\}$ satisfying,

$$[j_a, j_b] = \varepsilon_{abc} j^c \quad \operatorname{tr}[j_a j_b] = \operatorname{tr}[j_2 j_2]\eta_{ab} \quad (\eta_{ab}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{104}$$

can always be constructed as:

$$j_0 = \frac{1}{2}(L_+ + L_-) \quad j_1 = \frac{1}{2}(L_+ - L_-) \quad j_2 = L_0 \tag{105}$$

One thinks of $\{L_m\}$ as the "gravitational $\mathsf{sl}(2;\mathbb{R})$".

The triple $\{L_+, L_0, L_-\}$ can always be constructed as the generators of a principally embedded $\mathsf{sl}(2;\mathbb{R}) \hookrightarrow \mathfrak{g}$ from $\mathfrak{g}$'s root system data as [RS97, Section 3.1]:

$$L_0 = \sum_{i=1}^{r} k_i H^i \quad L_\pm = \pm\sqrt{2}\sum_{i=1}^{r}\sqrt{k_i}E^i_\mp \tag{106}$$

A straightforward calculation shows,

$$\operatorname{tr}[L_0 L_0] = \sum_{i=1}^{r}\frac{2}{|\alpha^i|^2}k_i \quad \operatorname{tr}[L_+ L_-] = -2\operatorname{tr}[L_0 L_0] \tag{107}$$

$\mathsf{sl}(N;\mathbb{R})$ decomposes under the adjoint action of this $\mathsf{sl}(2;\mathbb{R})$ as

$$\mathsf{sl}(N;\mathbb{R}) = \bigoplus_{s=2}^{r}\mathfrak{g}^{(s)}, \tag{108}$$

where $\mathfrak{g}^{(2)} \equiv \mathsf{sl}(2;\mathbb{R})$, $\dim\mathfrak{g}^{(s)} = 2s-1$.

For $\mathsf{sl}(N;\mathbb{R})$, the higher-spin generators $\{W^s{}_m\}$, $s > 2$, are obtained from the $\{L_+, L_0, L_-\}$,

$$W^s{}_m = (-1)^{s-m-1}\frac{(s+m-1)!}{(2s-2)!}\,(\mathrm{ad}_{L_-})^{s-m-1}(L_+)^{s-1} \tag{109}$$

The spin basis $\{W^s{}_m\}$ labelled by a spin number $s \in \{2,\dots,N\}$ and mode number $m \in \{-s+1,\dots,s-1\}$, $W^2{}_m \equiv L_m$, satisfies:

$$[L_m, W^s{}_n] = ((s-1)m - n)W^s{}_{m+n} \tag{110}$$

It follows that ad_{L_0} can be used to check mode numbers, as $[L_0, W^s{}_n] = -nW^s{}_n$. The Jacobi identity then implies that the mode number is additive under the Lie bracket.

$$[L_0, [W^s{}_m, W^r{}_n]] = -(m+n)[W^s{}_m, W^r{}_n] \tag{111}$$

This principal embedding $\mathsf{sl}(2;\mathbb{R}) \hookrightarrow \mathsf{sl}(N;\mathbb{R})$ is in fact a principal gradation of $\mathsf{sl}(N;\mathbb{R})$, see Fig. 1. A $\mathbb{Z}$-graded Lie algebra is a direct sum of vector spaces,

$$\mathfrak{g} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_m \tag{112}$$

such that the Lie bracket is a graded operation,

$$[\mathfrak{g}_m, \mathfrak{g}_s] \subset \mathfrak{g}_{m+s} \tag{113}$$

We can use this m-gradation to define the subspaces of positive and negative modes:

$$\begin{gathered}\mathfrak{g} = \mathfrak{g}_< \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_> \\ \mathfrak{g}_< := \bigoplus_{m<0} \mathfrak{g}_m \quad \mathfrak{g}_> := \bigoplus_{m>0} \mathfrak{g}_m\end{gathered} \tag{114}$$

Straightforward calculation gives that the Cartan generators have mode number 0, Chevalley raising operators E^i_+ have mode number -1, Chevalley lowering operators E^i_- have mode number 1. The Chevalley-Serre relations then imply,

$$E^\alpha_+ \in \mathfrak{g}_< \quad E^\alpha_- \in \mathfrak{g}_> \tag{115}$$

The trace components in the spin basis are antidiagonal in the mode numbers,

$$\mathrm{tr}[W^s{}_m W^r{}_n] = \delta^{sr}\delta_{m+n,0}t^s{}_m \tag{116}$$

where $t^s{}_m \in \mathbb{R}$. The explicit form of the $t^s{}_m$'s is known, see for example [CFP11].

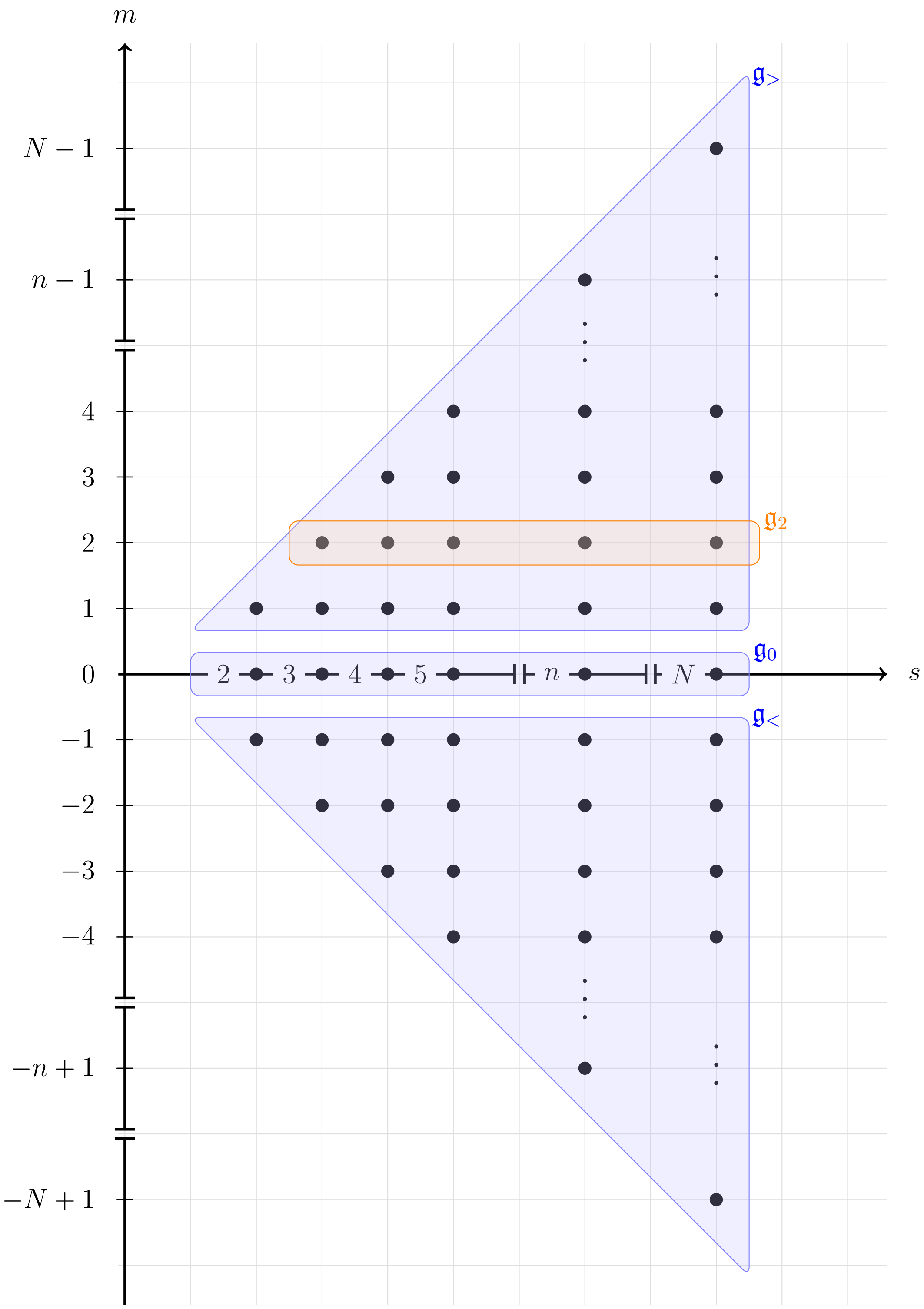


Figure 1: Principal $\mathbb{Z}$-gradation of $\mathsf{sl}(N;\mathbb{R})$.

For easier comparison with the literature, we provide the explicit matrix representations for the $\mathsf{sl}(2)$ and $\mathsf{sl}(3)$ case.

$\mathsf{sl}(2)$

$$K^{ij} = 2 \quad G_{ij} = \frac{1}{2} \quad k_i = \frac{1}{2}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; L_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; L_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$j_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; j_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; j_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$t^2{}_0 = \frac{1}{2} \quad t^2{}_1 = -1$$

$\mathsf{sl}(3)$

$$K^{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad G_{ij} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad k_i = (1,\ 1)$$

$$H^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; E^1_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; E^1_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; E^2_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; E^2_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; L_+ = +\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; L_- = -\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$j_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \; j_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; j_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$W^3{}_m \equiv W_m$

$$W_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; W_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad W_2 = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$W_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; W_{-2} = 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$t^2{}_0 = 2 \quad t^2{}_1 = -4$$

$$t^3{}_0 = \frac{2}{3} \quad t^3{}_1 = -2 \quad t^3{}_2 = 4$$

5. Wess–Zumino–Witten models

Wess–Zumino–Witten (WZW) models provide a field-theoretic description of 2-dim. conformal field theories. As we wish to reduce a 3-dim. gauge field theory to a 2-dim. conformal field theory, some knowledge about WZW models is required.

Witten's paper [Wit84] provides the original context for WZW models, but the conventions here will follow [Gaw97, Lecture 4]; for more details also see [Gaw99], [Gaw02].

Let Σ be a 2-dimensional surface such that the exterior derivative splits $\mathrm{d}|_\Sigma = \partial + \bar{\partial}$,

$$\Omega^\bullet(\Sigma) = \Omega^0(\Sigma) \oplus \big(\Omega^{(1,0)}(\Sigma) \oplus \Omega^{(0,1)}(\Sigma)\big) \oplus \Omega^{(1,1)} \tag{117}$$

In this thesis, Σ will be a Lorentzian cylinder mantle. The Wess–Zumino–Witten (WZW) action for a Lie group $\mathfrak{G}$-valued function $g : \Sigma \to \mathfrak{G}$ is,

$$S_{\mathrm{WZW}}[g] = \frac{k}{4\pi} \int_\Sigma \mathrm{tr}\big[g^{-1}\partial g \wedge g^{-1}\bar{\partial} g\big] - \frac{k}{4\pi} \int_\Sigma g^* \boldsymbol{\beta} \tag{118}$$

composed from a σ-model term,

$$\mathbf{L}_0[g] := \frac{k}{4\pi} \mathrm{tr}\big[g^{-1}\partial g \wedge g^{-1}\bar{\partial} g\big] \tag{119}$$

and a topological term[1]: $\boldsymbol{\beta}$ is a 2-form on $\mathfrak{G}$ such that

$$\frac{1}{4\pi}\mathrm{d}\boldsymbol{\beta}(g) = \frac{1}{12\pi} \mathrm{tr}\big[(g^{-1}\mathrm{d}g)^{\wedge 3}\big] =: \mathbf{L}_{\mathrm{WZ}}[g] \tag{120}$$

Cf. [Gaw97] for details on this construction. k is a coupling constant. For future convenience, compute separately the variation of the σ-model term,

$$\begin{aligned} \delta \mathbf{L}_0 = &\frac{k}{4\pi} \mathrm{tr}\big[-g^{-1}\delta g \wedge \partial(g^{-1}\bar{\partial} g) + g^{-1}\delta g \wedge \bar{\partial}(g^{-1}\partial g)\big] \\ &- \mathrm{d}\frac{k}{4\pi} \mathrm{tr}\big[\delta g \wedge (\partial g^{-1} - \bar{\partial} g^{-1})\big] \end{aligned} \tag{121}$$

$\mathbf{L}_{\mathrm{WZ}}[g]$ is called the Wess–Zumino term. Even though it is a 3-form, variation of $\mathbf{L}_{\mathrm{WZ}}$ is d-exact:

$$\delta \mathbf{L}_{\mathrm{WZ}} = \mathrm{d}\frac{1}{4\pi} \mathrm{tr}\big[g^{-1}\delta g \wedge \partial(g^{-1}\bar{\partial} g) + g^{-1}\delta g \wedge \bar{\partial}(g^{-1}\partial g)\big] \tag{122}$$

The variation of the WZW action is obtained by putting (121) and (122) together, while employing the following computational trick: assume that $\mathcal{M}$ is a 3-manifold with

[1] The σ-model term depends on a choice of metric on Σ, whereas this topological term does not.

boundary Σ, $\partial\mathcal{M} = \Sigma$, futher assume that $g : \Sigma \to \mathfrak{G}$ extends to a map $g : \mathcal{M} \to \mathfrak{G}$. Then,

$$\begin{aligned}\delta S_{\mathrm{WZW}}[g] &= \delta \int_\Sigma \mathbf{L}_0 - \delta \frac{k}{4\pi} \int_\Sigma g^* \boldsymbol{\beta} = \delta \int_\Sigma \mathbf{L}_0 - \delta \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}\boldsymbol{\beta}(g) \\ &= \delta \int_\Sigma \mathbf{L}_0 - \delta k \int_{\mathcal{M}} \mathbf{L}_{\mathrm{WZ}} = \int_\Sigma \delta \mathbf{L}_0 - k \int_{\mathcal{M}} \delta \mathbf{L}_{\mathrm{WZ}} \\ &= \frac{k}{2\pi} \int_\Sigma \mathrm{tr}\big[-g^{-1}\delta g \wedge \partial(g^{-1}\bar{\partial} g)\big] - \frac{k}{4\pi} \int_{\partial\Sigma} \mathrm{tr}\big[\delta g \wedge (\partial g^{-1} - \bar{\partial} g^{-1})\big]\end{aligned}$$

For $\mathfrak{G}$ compact, connected, and simply connected and $k \in \mathbb{Z}$, $S_{\mathrm{WZW}}[g]$ and $\delta S_{\mathrm{WZW}}[g]$ are unambiguously defined. On-shell g's factorize into a right-mover (x^+-mover) and a left-mover (x^--mover),

$$\partial(g^{-1}\bar{\partial} g) \overset{\mathrm{EoM}}{=} 0 \implies g \overset{\mathrm{EoM}}{=} \sigma(x^+)\bar{\sigma}(x^-) \tag{123}$$

the (pre-)symplectic potential of the theory is,

$$\boldsymbol{\vartheta}_{\mathrm{WZW}} = \frac{k}{4\pi} \mathrm{tr}\big[\delta g \wedge (\partial g^{-1} - \bar{\partial} g^{-1})\big] \tag{124}$$

WZW models possess a local $\mathfrak{G}_+ \times \mathfrak{G}_-$ symmetry,

$$g \longmapsto Q^{-1}(x^+) g \bar{Q}(x^-) \tag{125}$$

with the corresponding infinitesimal transformation generated by Lie algebra-valued functions $\lambda, \bar{\lambda} : \Sigma \to \mathfrak{g}$, $Q = \exp\lambda$, $\bar{Q} = \exp\bar{\lambda}$:

$$g \longmapsto g - \lambda(x^+) g + g \bar{\lambda}(x^-) \tag{126}$$

Due to a critical shortage of symbols, the corresponding field space vectors will also be denoted by λ, $\bar{\lambda}$:

$$\pounds_\lambda g \equiv -\lambda g \quad \pounds_{\bar{\lambda}} g \equiv g\bar{\lambda} \quad \pounds_{\lambda,\bar{\lambda}} \equiv -\lambda g + g\bar{\lambda} \tag{127}$$

$$\begin{aligned}\pounds_{\lambda,\bar{\lambda}} S_{\mathrm{WZW}}[g] &= \frac{k}{4\pi} \int_{\partial\Sigma} \mathrm{tr}\big[-\lambda \partial g g^{-1} - \lambda \bar{\partial} g g^{-1} - \bar{\lambda} g^{-1} \partial g - \bar{\lambda} g^{-1} \bar{\partial} g\big] \\ \implies \mathbf{B}_{\mathrm{WZW}}(\lambda,\bar{\lambda}) &= \frac{k}{4\pi} \mathrm{tr}\big[-\lambda \partial g g^{-1} - \lambda \bar{\partial} g g^{-1} - \bar{\lambda} g^{-1} \partial g - \bar{\lambda} g^{-1} \bar{\partial} g\big]\end{aligned} \tag{128}$$

The corresponding Noether currents split into a chiral and antichiral part,

$$\begin{aligned}\mathbf{J}_{\mathrm{WZW}}(\lambda,\bar{\lambda}) &:= \iota_{\lambda,\bar{\lambda}} \boldsymbol{\vartheta}_{\mathrm{WZW}} - \mathbf{B}_{\mathrm{WZW}}(\lambda,\bar{\lambda}) \\ &= -\frac{k}{2\pi} \mathrm{tr}[\lambda \mathbf{J}] + \frac{k}{2\pi} \mathrm{tr}\big[\bar{\lambda} \bar{\mathbf{J}}\big] \\ &\equiv \mathbf{J}(\lambda) + \bar{\mathbf{J}}(\bar{\lambda})\end{aligned} \tag{129}$$

Above, the WZW currents $\mathbf{J} \coloneqq -\partial g g^{-1}$, $\bar{\mathbf{J}} \coloneqq g^{-1}\bar{\partial} g$ were introduced; $\mathbf{J}, \bar{\mathbf{J}} \in \Omega^1(\Sigma, \mathfrak{g})$. The WZW EoM translate directly into current conservation:

$$\bar{\partial}\mathbf{J} \overset{\text{EoM}}{=} 0 \overset{\text{EoM}}{=} \partial\bar{\mathbf{J}} \tag{130}$$

The WZW currents have the infinitesimal transformation behaviour of (chiral) gauge connections:

$$\begin{aligned} &\pounds_{\lambda}\mathbf{J} = \mathrm{d}\lambda + [\mathbf{J}, \lambda] && \pounds_{\bar{\lambda}}\mathbf{J} = 0 \\ &\pounds_{\lambda}\bar{\mathbf{J}} = 0 && \pounds_{\bar{\lambda}}\bar{\mathbf{J}} = \mathrm{d}\bar{\lambda} + [\bar{\mathbf{J}}, \bar{\lambda}] \end{aligned} \tag{131}$$

Denote a maximally inextensible, closed codim.-1 surface on Σ by $\mathcal{S}$; $\mathcal{S}$ is diffeomorphic to a circle. $\mathcal{S}$ is assumed to have an orientation such that $\mathrm{d}x^+|_{\mathcal{S}}$ is positively oriented and $\mathrm{d}x^-|_{\mathcal{S}}$ is negatively oriented. The natural coordinate on $\mathcal{S}$ is $\theta \in [0, 2\pi]$, assumed st. $\int_{\mathcal{S}} \mathrm{d}\theta = \oint_0^{2\pi} \mathrm{d}x^+ = -\oint_0^{2\pi} \mathrm{d}x^-$. Define the chiral and antichiral conserved charges,

$$\begin{aligned} H(\lambda) &\coloneqq \int_{\mathcal{S}} \mathbf{J}(\lambda) = -\frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\lambda\mathbf{J}] \\ \bar{H}(\bar{\lambda}) &\coloneqq \int_{\mathcal{S}} \bar{\mathbf{J}}(\bar{\lambda}) = \quad \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\big[\lambda\bar{\mathbf{J}}\big] \end{aligned} \tag{132}$$

These charges constitute a Poisson charge algebra on $\mathcal{S}$,

$$\begin{aligned} \pounds_{\lambda}H(\mu) = \{H(\lambda), H(\mu)\} &= H([\lambda, \mu]) - \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\mu\partial\lambda] \\ \pounds_{\bar{\lambda}}\bar{H}(\bar{\mu}) = \{\bar{H}(\bar{\lambda}), \bar{H}(\bar{\mu})\} &= \bar{H}([\bar{\lambda}, \bar{\mu}]) + \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\big[\bar{\mu}\bar{\partial}\bar{\lambda}\big] \\ \{H(\lambda), \bar{H}(\bar{\mu})\} &= 0 \end{aligned} \tag{133}$$

Writing $\mathbf{J} = J(x^+)\,\mathrm{d}x^+$, $\bar{\mathbf{J}} = \bar{J}(x^-)\,\mathrm{d}x^-$, $\oint_0^{2\pi} \equiv \oint$, the charges evaluate as,

$$H(\lambda) = -\frac{k}{2\pi}\oint \mathrm{d}x^+\,\mathrm{tr}[\lambda J] \quad \bar{H}(\bar{\lambda}) = -\frac{k}{2\pi}\oint \mathrm{d}x^-\,\mathrm{tr}\big[\bar{\lambda}\bar{J}\big] \tag{134}$$

$$\begin{aligned} \{H(\lambda), H(\mu)\} &= H([\lambda, \mu]) - \frac{k}{2\pi}\oint \mathrm{d}x^+\,\mathrm{tr}[\mu\partial_+\lambda] \\ \{\bar{H}(\bar{\lambda}), \bar{H}(\bar{\mu})\} &= \bar{H}([\bar{\lambda}, \bar{\mu}]) - \frac{k}{2\pi}\oint \mathrm{d}x^-\,\mathrm{tr}\big[\bar{\mu}\partial_-\bar{\lambda}\big] \end{aligned} \tag{135}$$

This charge algebra is, in fact, two copies of the affine Lie algebra $\hat{\mathfrak{g}}_k$. Expand the WZW current into its Lie algebra components,

$$J(x^+) = J^A(x^+)t_A \quad \bar{J}(x^+) = \bar{J}^A(x^+)t_A \tag{136}$$

$$[t_A, t_B] = f_{AB}{}^C t_C \quad \mathrm{tr}[t_A t_B] = \gamma_{AB} \quad \gamma^{AB}\gamma_{BC} = \delta^A{}_C \quad \gamma^{AB} t_B = t^A \tag{137}$$

The components $J^A(x^+)$, $\bar{J}^A(x^+)$ can be represented in terms of chiral, antichiral charges,

$$\begin{aligned} H(t^A\delta(x^+ - y^+)) &= -\frac{k}{2\pi}\oint \mathrm{d}x^+\,\delta(x^+ - y^+)\,\mathrm{tr}\big[t^A J(x^+)\big] \\ &= -\frac{k}{2\pi}\oint \mathrm{d}x^+\,\delta(x^+ - y^+) J^A(x^+)\underbrace{\mathrm{tr}\big[t^A t_B\big]}_{=\gamma^A{}_B} \quad = -\frac{k}{2\pi}J^A(y^+) \end{aligned} \tag{138}$$

$$\implies J^A(y^+) = -\frac{2\pi}{k}H(t^A\delta(x^+ - y^+)) \quad \bar{J}^A(y^-) = -\frac{2\pi}{k}\bar{H}(t^A\delta(x^- - y^-)) \tag{139}$$

It follows,

$$\begin{aligned} \{J^A(x^+), J^B(y^+)\} &= \frac{k}{2\pi}\left(-f^{AB}{}_C J^C(x^+)\delta(x^+ - y^+) - \gamma^{AB}\delta'(x^+ - y^+)\right) \\ \{\bar{J}^A(x^-), \bar{J}^B(y^-)\} &= \frac{k}{2\pi}\left(-f^{AB}{}_C \bar{J}^C(x^-)\delta(x^- - y^-) - \gamma^{AB}\delta'(x^- - y^-)\right) \\ \{J^A(x^+), \bar{J}^B(y^-)\} &= 0 \end{aligned} \tag{140}$$

The currents can also be expanded into modes,

$$J(x^+) = \frac{1}{k}\sum_{m\in\mathbb{Z}} J^A{}_m e^{-imx^+} t_A \quad \bar{J}(x^-) = \frac{1}{k}\sum_{m\in\mathbb{Z}} \bar{J}^A{}_m e^{-imx^-} t_A \tag{141}$$

the modes $J^A{}_m$, $\bar{J}^A{}_m$ can also be represented in terms of the chiral, antichiral charges,

$$\begin{aligned} H(t^A e^{imx^+}) &= -\frac{k}{2\pi}\oint \mathrm{d}x^+\, e^{imx^+}\,\mathrm{tr}\big[t^A J(x^+)\big] \\ &= -\frac{k}{2\pi}\oint \mathrm{d}x^+\, e^{imx^+}\,\mathrm{tr}\left[t^A \frac{1}{k}\sum_{n\in\mathbb{Z}} J^B{}_n e^{-inx^+} t_B\right] \\ &= -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \underbrace{\mathrm{tr}\big[t^A t_B\big] J^B{}_n}_{=J^A{}_n} \underbrace{\oint \mathrm{d}x^+\, e^{i(n-m)x^+}}_{=2\pi\delta_{mn}} \qquad = -J^A{}_m \end{aligned} \tag{142}$$

$$\implies J^A{}_m = -H(t^A e^{imx^+}) \quad \bar{J}^A{}_m = -\bar{H}(t^A e^{imx^-}) \tag{143}$$

It follows,

$$\begin{aligned} \{J^A{}_m, J^B{}_n\} &= -f^{AB}{}_C J^C{}_{m+n} - imk\gamma^{AB}\delta_{m+n,0} \\ \{\bar{J}^A{}_m, \bar{J}^B{}_n\} &= -f^{AB}{}_C \bar{J}^C{}_{m+n} - imk\gamma^{AB}\delta_{m+n,0} \\ \{J^A{}_m, \bar{J}^B{}_n\} &= 0 \end{aligned} \tag{144}$$

WZW model from a 3-dimensional point of view

In the section above, the WZW model was described as inherent to a 2-dim. surface Σ. As a computational trick, one introduces a 3-manifold $\mathcal{M}$ with boundary Σ in order to compute the variation of the topological term. WZW models also appear in contexts of inherently 3-dimensional theories, describing their boundary dynamics. Then, the Lagrangian of the theory is a 3-form instead of a 2-form,

$$\mathbf{L}^{(3)}_{\mathrm{WZW}} = \mathrm{d}\mathbf{L}_0 + \mathbf{L}_{\mathrm{WZ}}, \text{ cf. } \mathbf{L}^{(2)}_{\mathrm{WZW}} = \mathbf{L}_0 - \frac{k}{4\pi} g^* \boldsymbol{\beta} \tag{145}$$

Because of anticommuting differentials $\delta \mathrm{d} = -\mathrm{d}\delta$, the derived covariant phase space quantities will pick up a minus sign:

$$\begin{aligned} \delta\mathbf{L}^{(3)}_{\mathrm{WZW}} &= \delta\mathrm{d}\mathbf{L}_0 + \delta\mathbf{L}_{\mathrm{WZ}} = -\mathrm{d}\delta\mathbf{L}_0 + \delta\mathbf{L}_{\mathrm{WZW}} \\ &= -\mathrm{d}\delta\mathbf{L}^{(2)}_{\mathrm{WZW}} \end{aligned} \tag{146}$$

In particular, resulting Noether currents and Noether charges will pick up a minus sign,

$$\mathbf{J}^{(3)}_{\mathrm{WZW}} = -\mathbf{J}^{(2)}_{\mathrm{WZW}} \quad H^{(3)}_{\mathrm{WZW}} = -H^{(2)}_{\mathrm{WZW}} \tag{147}$$

6. Chern–Simons gauge field theory

We review the basics of the classical Chern–Simons field theory loosely following [Mne17, Section 2].

The Chern–Simons (CS) action on a 3-manifold $\mathcal{M}$ for a $\mathfrak{G}$-connection $\mathbf{A} \in \Omega^1(\mathcal{M}, \mathfrak{g})$ is:

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \operatorname{tr}\left[\mathbf{A} \wedge \mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A}^{\wedge 3}\right] \tag{148}$$

where k is a coupling constant of the theory. The variation of the action gives the equation of motion and (pre-)symplectic potential:

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{2\pi} \int_{\mathcal{M}} \operatorname{tr}[\delta\mathbf{A} \wedge (\mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A})] - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \operatorname{tr}[\delta\mathbf{A} \wedge \mathbf{A}] \tag{149}$$

$$\begin{aligned} &\implies \mathbf{F} \coloneqq \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0 \\ &\implies \boldsymbol{\vartheta}_{\mathrm{CS}} = \frac{k}{4\pi} \operatorname{tr}[\delta\mathbf{A} \wedge \mathbf{A}] \end{aligned} \tag{150}$$

$\mathbf{F}$ is $\mathbf{A}$'s curvature; the equation of motion is a flatness condition for $\mathbf{A}$.[2]

Recall the behaviour of gauge connections under (global) gauge transformations specified by a $Q : \mathcal{M} \to \mathfrak{G}$:

$$\mathbf{A} \longmapsto \mathbf{A}^Q = Q^{-1}\mathrm{d}Q + Q^{-1}\mathbf{A}Q \tag{151}$$

S_{CS} has the following behaviour under gauge transformations:

$$S_{\mathrm{CS}}[A] \longmapsto S_{\mathrm{CS}}[A] + \frac{k}{4\pi} \int_{\partial\mathcal{M}} \operatorname{tr}\left[Q^{-1}\mathbf{A}Q \wedge Q^{-1}\mathrm{d}Q\right] - \frac{k}{12\pi} \int_{\mathcal{M}} \operatorname{tr}\left[(Q^{-1}\mathrm{d}Q)^{\wedge 3}\right] \tag{152}$$

For a manifold $\mathcal{M}$ without boundary, gauge invariance requires just the quantization of the coupling constant: for $k \in \mathbb{Z}$ and $\mathfrak{G}$ a compact Lie group, the quantity $\exp(iS_{\mathrm{CS}}[A])$ is gauge invariant. k is then called the level of the theory.

The infinitesimal gauge transformation corresponding to $Q = \exp\lambda$ is specified by a $\lambda : \mathcal{M} \to \mathfrak{g}$; once again, we also denote the corresponding field space vector by λ:

$$\pounds_\lambda \mathbf{A} \equiv \mathrm{d}\lambda + [\mathbf{A}, \lambda] \tag{153}$$

$$\begin{aligned} \pounds_\lambda S_{\mathrm{CS}}[A] &= \frac{k}{4\pi} \int_{\partial\mathcal{M}} \operatorname{tr}[-\mathrm{d}\lambda \wedge \mathbf{A}] \\ \implies \mathbf{B}_{\mathrm{CS}}[\lambda] &= \frac{k}{4\pi} \operatorname{tr}[-\mathrm{d}\lambda \wedge \mathbf{A}] \end{aligned} \tag{154}$$

[2] Note: $\boldsymbol{\Omega}_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{C}} \operatorname{tr}[\delta\mathbf{A} \wedge \delta\mathbf{A}]$ is the Atiyah–Bott (pre-)symplectic form.

The associated Noether current is on-shell d-exact:

$$
\begin{aligned}
\mathbf{J}_{\mathrm{CS}}(\lambda) &:= \iota_\lambda \boldsymbol{\vartheta}_{\mathrm{CS}} - \mathbf{B}_{\mathrm{CS}}(\lambda) \\
&= \frac{k}{2\pi}\operatorname{tr}[\mathrm{d}\lambda \wedge \mathbf{A} - 2\lambda \mathbf{A}\wedge\mathbf{A}] \\
&\overset{\text{EoM}}{=} \frac{k}{2\pi}\operatorname{tr}[\mathrm{d}\lambda \wedge \mathbf{A} + \lambda \mathrm{d}\mathbf{A}] \qquad = \mathrm{d}\frac{k}{2\pi}\operatorname{tr}[\lambda\mathbf{A}]
\end{aligned}
\tag{155}
$$

The associated conserved charge is therefore nonzero only for manifolds $\mathcal{M}$ with boundary $\partial\mathcal{M} \neq \varnothing$,

$$
H_{\mathrm{CS}}(\lambda) := \int_{\mathcal{C}} \mathbf{J}_{\mathrm{CS}} \overset{\text{EoM}}{=} \frac{k}{2\pi}\int_{\mathcal{C}} \mathrm{d}\operatorname{tr}[\lambda\mathbf{A}] = \frac{k}{2\pi}\int_{\partial\mathcal{C}} \operatorname{tr}[\lambda\mathbf{A}] \tag{156}
$$

In that case, the Poisson bracket carries a centrally extended representation of the symmetry algebra:

$$
\pounds_\lambda H_{\mathrm{CS}}(\mu) = \{H_{\mathrm{CS}}(\lambda), H_{\mathrm{CS}}(\mu)\} = H_{\mathrm{CS}}([\lambda,\mu]) + \frac{k}{2\pi}\int_{\partial\mathcal{C}} \operatorname{tr}[\mu \mathrm{d}\lambda] \tag{157}
$$

If $\partial\mathcal{M} \neq \varnothing$, the variational principle is not well-defined: a non-zero boundary term means that the action is not extremized on-shell, ie. the boundary dynamics of the theory are not well-defined. For the field theory to have well-defined dynamics on the entirety of $\mathcal{M}$, the CS action needs to be improved by a boundary term and boundary conditions for $\mathbf{A}$ have to be specified, so that the variation of the action vanishes on-shell. Furthermore, allowed gauge transformations have to be restricted to those that preserve these boundary conditions.

Such counterterms and boundary conditions typically require the specification of additional structure on $\partial\mathcal{M}$, such as a metric or a (para-)complex strucutre. To that end, for cylindrical manifolds $\mathcal{M}$, a standard procedure is to first equip $\partial\mathcal{M}$ with a Lorentzian metric, so that $\mathrm{d}|_{\partial\mathcal{M}} = \partial + \bar{\partial}$, $\mathbf{A}|_{\partial\mathcal{M}} = \mathbf{A}_+ + \mathbf{A}_-$, and then fix either the chiral or the antichiral component. Upon improving the action by adding a suitable boundary term, in the sense of (78), the variation of the improved action vanishes on-shell.

An appropriate boundary term is:

$$
S_\partial[A] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}[\mathbf{A}_+ \wedge \mathbf{A}_-] \qquad
\begin{cases}
\delta(S_{\mathrm{CS}} - S_\partial)[A] \overset{\text{EoM}}{=} -\dfrac{k}{2\pi}\displaystyle\int_{\partial\mathcal{M}} \operatorname{tr}[\delta\mathbf{A}_- \wedge \mathbf{A}_+] \\
\delta(S_{\mathrm{CS}} + S_\partial)[A] \overset{\text{EoM}}{=} -\dfrac{k}{2\pi}\displaystyle\int_{\partial\mathcal{M}} \operatorname{tr}[\delta\mathbf{A}_+ \wedge \mathbf{A}_-]
\end{cases}
$$

A well-defined variational principle may now be obtained by imposing one of two boundary conditions:

1. Chiral boundary conditions

$$
S^+_{\mathrm{CS}}[A] := S_{\mathrm{CS}}[A] - S_\partial[A] \longrightarrow \mathbf{A}_-\big|_{\partial\mathcal{M}} \overset{\text{BC}}{=} 0 \tag{158}
$$

2. Antichiral boundary conditions

$$S^-_{\rm CS}[A] \coloneqq S_{\rm CS}[A] + S_\partial[A] \longrightarrow \mathbf{A}_+\big|_{\partial\mathcal{M}} \overset{\rm BC}{=} 0 \tag{159}$$

BC-preserving gauge transformations match the chosen chirality of $\mathbf{A}$:

1.

$$\mathbf{A}_-^Q = Q^{-1}\bar\partial Q + Q^{-1}\mathbf{A}_- Q \overset{!}{=} \big|_{\partial\mathcal{M}}\, 0 \implies \bar\partial Q\big|_{\partial\mathcal{M}} \overset{!}{=} 0 \tag{160}$$

2.

$$\mathbf{A}_+^Q = Q^{-1}\partial Q + Q^{-1}\mathbf{A}_+ Q \overset{!}{=} \big|_{\partial\mathcal{M}}\, 0 \implies \partial Q\big|_{\partial\mathcal{M}} \overset{!}{=} 0 \tag{161}$$

Infinitesimally, this translates to $\bar\partial\lambda = 0$ ($\partial\lambda = 0$). The associated Noether charges adapt the chosen chirality of $\mathbf{A}$:

1.

$$H_{\rm CS}(\lambda) = \frac{k}{2\pi}\int_{\partial\mathcal{C}} \mathrm{tr}[\lambda\mathbf{A}] \longrightarrow H(\lambda) = \frac{k}{2\pi}\int_{\partial\mathcal{C}} \mathrm{tr}\big[\lambda(x^+)\mathbf{A}_+\big] \tag{162}$$

2.

$$H_{\rm CS}(\lambda) = \frac{k}{2\pi}\int_{\partial\mathcal{C}} \mathrm{tr}[\lambda\mathbf{A}] \longrightarrow \bar H(\lambda) = \frac{k}{2\pi}\int_{\partial\mathcal{C}} \mathrm{tr}\big[\lambda(x^-)\mathbf{A}_-\big] \tag{163}$$

Restricting to BC-preserving gauge transformations also restores gauge invariance on the entirety of $\mathcal{M}$:

1.

$$S^+_{\rm CS}[A^Q] = S^+_{\rm CS}[A] + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[2Q^{-1}\mathbf{A}_- Q\wedge Q^{-1}\partial Q - Q^{-1}\partial Q\wedge Q^{-1}\bar\partial Q\big] \overset{\rm BC}{=} 0 \tag{164}$$

2.

$$S^-_{\rm CS}[A^Q] = S^-_{\rm CS}[A] + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[2Q^{-1}\mathbf{A}_+ Q\wedge Q^{-1}\bar\partial Q + Q^{-1}\partial Q\wedge Q^{-1}\bar\partial Q\big] \overset{\rm BC}{=} 0 \tag{165}$$

A note on asymptotic symmetries

Typically, Noether charges associated to gauge transformations vanish. However, as demonstrated above, on manifolds with boundary, this is not guaranteed. The set of (BC-preserving) infinitesimal gauge transformations with nonvanishing associated Noether charges are called the asymptotic symmetries of the theory. The rest are called trivial gauge symmetries of the theory. Trivial gauge symmetries map a solution to itself, while asymptotic symmetries map solutions to other solutions.

The covariant phase space formalism produces a (pre-)symplectic manifold $(\mathbf{\Omega}, \mathcal{S})$, called the (pre-)phase space, out of a Lagrangian field theory $(F \to \mathcal{M}, \mathbf{L})$; asymptotic symmetries may be interpreted as the nondegenrate directions in $\mathcal{S}$. Here, this is shown using the example of CS gauge field theory.

Recall the CS (pre-)symplectic form:

$$\mathbf{\Omega}_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{C}} \mathrm{tr}[\delta\mathbf{A} \wedge \delta\mathbf{A}] \tag{166}$$

Contracting this with an infinitesimal symmetry λ gives,

$$\iota_\lambda \mathbf{\Omega}_{\mathrm{CS}} \overset{\mathrm{EoM}}{=} \frac{k}{2\pi} \int_{\partial\mathcal{C}} \mathrm{tr}[\lambda\delta\mathbf{A}] \tag{167}$$

This is generally zero if $\lambda|_{\partial\mathcal{M}} = 0$. Such a λ will have a vanishing charge, therefore such a λ is a trivial gauge symmetry. In this context, trivial gauge symmetries correspond to degenerate directions wrt. to the (pre-)symplectic form on $\mathcal{S}$. On the other hand, if $\lambda|_{\partial\mathcal{M}} \neq 0$, then $\iota_\lambda \mathbf{\Omega}_{\mathrm{CS}} \neq 0$; such a λ will have a nonvanishing charge, therefore will be an asymptotic symmetry.

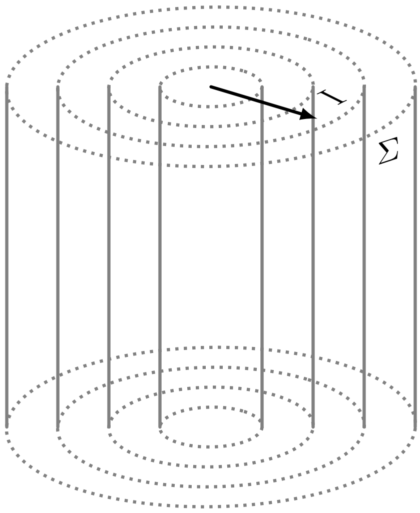
Restricting $\mathcal{S}$ to only have tangent directions corresponding to asymptotic symmetries produces a nondegenerate $\mathbf{\Omega}_{\mathrm{CS}}$; this subset may be called the physical phase space of the theory.

7. The CS/WZW correspondence

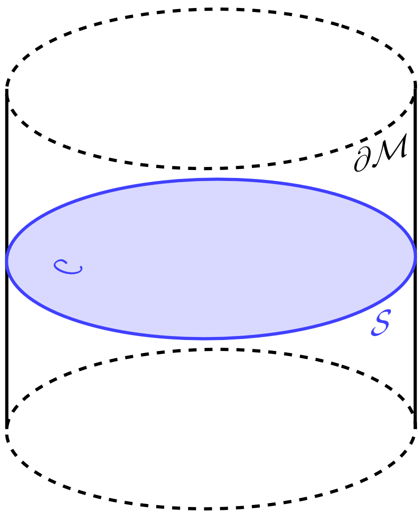
The reductions of 3-dim. gauge field theory to 2-dim. conformal field theory described in this thesis are closely related to the well-known correspondence between CS field theory and WZW models, the classical aspects of which are sketched out in the next few paragraphs.

Content in this section is adapted from [EMSS89] to fit the setting in this text.

Consider a CS theory on a cylindrical manifold $\mathcal{M}$, such that $\mathcal{M} = I \times \Sigma$, $\partial\mathcal{M} = \Sigma$:[3]



Furthermore, for the purposes of the correspondence, assume that the maximally inextensible codim.-1 surface $\mathcal{C}$ is chosen such that $\partial\mathcal{C} = \mathcal{C} \cap \partial\mathcal{M} \equiv \mathcal{S}$,



[3]Typically, for $\mathcal{M} = I \times \Sigma$, $\partial_{\mathcal{M}} = -\partial\Sigma_{\text{in}} + \partial\Sigma_{\text{out}}$; here, Σ_{in} degenerates.

Conservation of charges means that such a choice can be always be made.

The discussion about CS theories on manifolds with boundary will be extended here, showing that the boundary dynamics are described by a WZW model on Σ. Note that as part of the boundary conditions, Σ has been made into a Lorentzian cylinder mantle.

The CS EoM $\Leftrightarrow$ flatness constraint on $\mathbf{A}$ is solved by a "pure gauge"[4] connection,

$$\mathbf{F} = 0 \implies \mathbf{A} = U^{-1}\mathrm{d}U \tag{168}$$

Plugging a pure gauge connection into the improved CS action results in a WZW action for the field U:

$$\begin{aligned} S_{\mathrm{CS}}[A = U^{-1}\mathrm{d}U] &= -\frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{tr}\big[(U^{-1}\mathrm{d}U)^{\wedge 3}\big] = -kS_{\mathrm{WZ}}[U] \\ S_{\partial}[A = U^{-1}\mathrm{d}U] &= \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[U^{-1}\partial U \wedge U^{-1}\bar{\partial}U\big] = S_0[U] \end{aligned} \tag{169}$$

$$\begin{aligned} S^{+}_{\mathrm{CS}}[A = U^{-1}\mathrm{d}U] &= -kS_{\mathrm{WZ}}[U] - S_0[U] = -S_{\mathrm{WZW}}[U] \\ S^{-}_{\mathrm{CS}}[A = U^{-1}\mathrm{d}U] &= -kS_{\mathrm{WZ}}[U] + S_0[U] = S_{\mathrm{WZW}}[U^{-1}] \end{aligned} \tag{170}$$

The CS level k corresponds directly to the WZW level k. Also note that the resulting WZW actions for the chiral and antichiral boundary conditions are parity conjugate.

This shows that the boundary dynamics of a CS gauge field theory is determined by a WZW model.

While a general solution to the WZW EoM (123) has a chiral and an antichiral part, the CS boundary conditions select only one sector:

1.
$$\mathbf{A}_{-} \overset{\mathrm{EoM}}{=} U^{-1}\bar{\partial}U \overset{!}{=} \big|_{\Sigma}\, 0 \implies U \overset{\mathrm{EoM}}{=} U(x^{+}) \tag{171}$$

2.
$$\mathbf{A}_{+} \overset{\mathrm{EoM}}{=} U^{-1}\partial U \overset{!}{=} \big|_{\Sigma}\, 0 \implies U \overset{\mathrm{EoM}}{=} U(x^{-}) \tag{172}$$

As a consequence, even though the boundary conditions only set $\mathbf{A}_{-}$ ($\mathbf{A}_{+}$) to zero on the boundary, on-shell it is zero on the whole of $\mathcal{M}$.

[4] $0^U = U^{-1}\mathrm{d}U$, hence a connection of this form can be thought of as a "gauge transformed nothing".

The CS (pre-)symplectic potential translates into a chiral WZW (pre-)symplectic potential,

1.
$$\boldsymbol{\vartheta}_{\mathrm{CS}} = \frac{k}{4\pi}\,\mathrm{tr}[\delta\mathbf{A}\wedge\mathbf{A}] \longrightarrow \boldsymbol{\vartheta}_{\mathrm{WZW}} = \mathrm{d}\frac{k}{4\pi}\,\mathrm{tr}\big[\delta U\wedge\partial U^{-1}\big] \tag{173}$$

2.
$$\boldsymbol{\vartheta}_{\mathrm{CS}} = \frac{k}{4\pi}\,\mathrm{tr}[\delta\mathbf{A}\wedge\mathbf{A}] \longrightarrow \boldsymbol{\vartheta}_{\mathrm{WZW}} = \mathrm{d}\frac{k}{4\pi}\,\mathrm{tr}\big[\delta U\wedge\bar{\partial} U^{-1}\big] \tag{174}$$

The same expressions get derived directly from $-S_{\mathrm{WZW}}[U]$ ($S_{\mathrm{WZW}}[U^{-1}]$), subject to the chiral (antichiral) boundary conditions (158), (159), keeping in mind the issue of the minus sign (146).

The infinitesimal gauge transformation of $\mathbf{A}$ by $\lambda : \mathcal{M} \to \mathfrak{g}$, $\pounds_\lambda\mathbf{A} = \mathrm{d}\lambda + [\mathbf{A},\lambda]$, translates into the following transformation of the WZW field,

$$\pounds_\lambda U \equiv U\lambda \tag{175}$$

As before, λ adapts the chirality chosen by the boundary conditions. Computing the associated WZW Noether currents and charges gives,

1.
$$\mathbf{J}_{\mathrm{WZW}}(\lambda) = \frac{k}{2\pi}\,\mathrm{tr}\big[\lambda U^{-1}\partial U\big] \implies H_{\mathrm{WZW}}(\lambda) = \frac{k}{2\pi}\int_{\mathcal{S}}\mathrm{tr}\big[\lambda U^{-1}\partial U\big] \tag{176}$$

2.
$$\mathbf{J}_{\mathrm{WZW}}(\lambda) = \frac{k}{2\pi}\,\mathrm{tr}\big[\lambda U^{-1}\bar{\partial} U\big] \implies H_{\mathrm{WZW}}(\lambda) = \frac{k}{2\pi}\int_{\mathcal{S}}\mathrm{tr}\big[\lambda U^{-1}\bar{\partial} U\big] \tag{177}$$

exact correspondence with (162), (163). The nonzero component of $\mathbf{A}$ on Σ plays the role of the WZW current. This shows that the Poisson charge algebra of CS gauge field theory is one copy of the affine Lie algebra $\hat{\mathfrak{g}}_k$, corresponding to one sector of a WZW model's charge algebra.

One may make the following statement of CS/WZW correspondence:

The (classical) phase space of a single CS gauge field theory on a cylindrical manifold $\mathcal{M}$ with boundary Σ, subject to chiral (antichiral) boundary conditions, reduces to the chiral (antichiral) sector of a WZW phase space on Σ.

Equivalence of semiclassical path integrals

The CS action can be reduced to the WZW action by using the geometry of $\mathcal{M} = I \times \Sigma$: there is a splitting into transverse (I) and longitudinal (Σ) parts,

$$\int_{\mathcal{M}} = \int_I \int_\Sigma \qquad \mathrm{d} = \mathrm{d}_I + \mathrm{d}_\Sigma \qquad \mathbf{A} = \mathbf{A}_I + \mathbf{A}_\Sigma \tag{178}$$

Assuming $\partial\Sigma = \varnothing$, the CS action takes on a canonical form,

$$S_{\mathrm{CS}}[A = A_I + A_\Sigma] = \frac{k}{4\pi} \int_I \int_\Sigma \mathrm{tr}[\mathbf{A}_\Sigma \wedge \mathrm{d}_I \mathbf{A}_\Sigma + 2\mathbf{A}_I \wedge \mathbf{F}_\Sigma] \tag{179}$$

where $\mathbf{A}_\Sigma$ becomes the dynamical field and $\mathbf{A}_I$ takes on the role of a Lagrange multiplier, enforcing the flatness constraint $\mathbf{F}_\Sigma = \mathrm{d}_\Sigma \mathbf{A}_\Sigma + \mathbf{A}_\Sigma \wedge \mathbf{A}_\Sigma = 0$. This constraint is solved by $\mathbf{A}_\Sigma = U^{-1}\mathrm{d}_\Sigma U$. Upon solving the flatness constrain, the CS action reduces to the Wess-Zumino term:

$$S_{\mathrm{CS}}[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] = -kS_{\mathrm{WZ}}[U] \tag{180}$$

The boundary counterterm evaluated on a pure gauge connection reduces to the σ-model term:

$$S_\partial[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] = \frac{k}{4\pi} \int_\Sigma \mathrm{tr}\big[U^{-1}\partial U \wedge U^{-1}\bar{\partial} U\big] = S_0[U] \tag{181}$$

Therefore, by solving the canonical constraint, the improved CS actions are dynamically equivalent to WZW models on the boundary:

$$\begin{aligned} S^+_{\mathrm{CS}}[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] &= -k\Gamma[U] - S_0[U] = -S_{\mathrm{WZW}}[U] \\ S^-_{\mathrm{CS}}[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] &= -k\Gamma[U] + S_0[U] = S_{\mathrm{WZW}}[U^{-1}] \end{aligned} \tag{182}$$

This classical equivalence between the CS action on $\mathcal{M} = I \times \Sigma$ and a WZW action on Σ translates directly into an equivalence of semiclassical partition functions:

$$\begin{aligned} \mathcal{Z}_{\mathrm{CS}} &= \int [\mathrm{D}A]\; e^{iS^+_{\mathrm{CS}}[A]} \\ &= \int [\mathrm{D}A_I][\mathrm{D}A_\Sigma] \exp\left(i\frac{k}{4\pi}\int_I\int_\Sigma \mathrm{tr}[\mathbf{A}_\Sigma \wedge \mathrm{d}_I \mathbf{A}_\Sigma] - i\frac{k}{4\pi}\int_\Sigma \mathrm{tr}[\mathbf{A}_+ \wedge \mathbf{A}_-]\right) \\ &\qquad \times \exp\left(i\frac{k}{2\pi}\int_I\int_\Sigma \mathrm{tr}[\mathbf{A}_I \wedge \mathbf{F}_\Sigma]\right) \end{aligned}$$

The Lagrange multiplier can be integrated out using the path integral δ-function identity $\int [\mathrm{D}A_I] \exp\big(i\frac{k}{2\pi}\int_I\int_\Sigma \mathrm{tr}[\mathbf{A}_I \wedge \mathbf{F}_\Sigma]\big) = \delta(\mathbf{F}_\Sigma)$,

$$= \int [\mathrm{D}A_\Sigma]\delta(\mathbf{F}_\Sigma) e^{-iS_{\mathrm{WZW}}[U]}$$

It can be shown that $[\mathrm{D}\mathbf{A}_\Sigma]\delta(\mathbf{F}_\Sigma) = [\mathrm{D}U]$, where $[\mathrm{D}U]$ is the Haar measure on $\mathfrak{G}$,

$$= \int [\mathrm{D}U]\; e^{-iS_{\mathrm{WZW}}[U]} = \mathcal{Z}_{\mathrm{WZW}}$$

See [EMSS89, Section 2.1], and the footnote on p. 111, for the argument why $[\mathrm{D}A_\Sigma]\delta(\mathbf{F}_\Sigma)$ is the Haar measure on $\mathfrak{G}$.

8. More on Wess–Zumino–Witten models

Gauged WZW models

For future computational needs, we recall how a gauge-invariant extension of the WZW action is derived, following [Wit92], [Gaw97, Lecture 4].

An important property satisfied by the WZW action is the Polyakov–Wiegmann (P–W) identity:

$$\begin{aligned} S_{\mathrm{WZW}}[gh] &= S_{\mathrm{WZW}}[g] + S_{\mathrm{WZW}}[h] + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\big[g^{-1}\bar\partial g \wedge h\partial h^{-1}\big] \\ S_{\mathrm{WZW}}[gh^{-1}] &= S_{\mathrm{WZW}}[g] + S_{\mathrm{WZW}}[h^{-1}] + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\big[g^{-1}\bar\partial g \wedge h^{-1}\partial h\big] \end{aligned} \tag{183}$$

Using the P–W identities, one constructs an extension of the WZW action that is gauge invariant under the action of an (anomaly-free) subgroup $\mathfrak{H} \subset \mathfrak{G}$.

$$\begin{aligned} &S_{\mathrm{WZW}}[hg\tilde h^{-1}] = S_{\mathrm{WZW}}[g] + S_{\mathrm{WZW}}[h\tilde h^{-1}] \\ &\quad + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\Big[-h^{-1}\bar\partial h \wedge \partial g g^{-1} + g^{-1}\bar\partial g \wedge \tilde h^{-1}\partial\tilde h + h^{-1}\bar\partial h \wedge g\tilde h^{-1}\partial\tilde h g^{-1} - h^{-1}\bar\partial h \wedge \tilde h^{-1}\partial\tilde h\Big] \end{aligned}$$

For $h, \tilde h : \mathfrak{H} \to \Sigma$, identifying the components of a $\mathfrak{H}$-connection $\mathbf{B}$ as $\begin{cases} \mathbf{B}_- \equiv h^{-1}\bar\partial h, \\ \mathbf{B}_+ \equiv \tilde h^{-1}\partial\tilde h, \end{cases}$ the $\mathfrak{H}$-invariant extension of the WZW action is:

$$\begin{aligned} &S_{\mathrm{WZW}}[g, B] := S_{\mathrm{WZW}}[hg\tilde h^{-1}] - S_{\mathrm{WZW}}[h\tilde h^{-1}] \\ &= S_{\mathrm{WZW}}[g] + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\big[-\mathbf{B}_- \wedge \partial g g^{-1} + g^{-1}\bar\partial g \wedge \mathbf{B}_+ + \mathbf{B}_- \wedge g\mathbf{B}_+ g^{-1} - \mathbf{B}_- \wedge \mathbf{B}_+\big] \end{aligned}$$

Gauge transformations $U : \mathfrak{H} \to \Sigma$ act as $\begin{cases} g \mapsto UgU^{-1}, \\ \mathbf{B} \mapsto \mathbf{B}^U = U^{-1}\mathrm{d}U + U^{-1}\mathbf{B}U. \end{cases}$ $S_{\mathrm{WZW}}[g, B]$ is called the gauged WZW action.

Note that this extension makes sense even for $\mathfrak{H} = \mathfrak{G}$.

Reduction of $\mathsf{SL}(N;\mathbb{R})$ WZW model to $\mathsf{sl}(N;\mathbb{R})$ Toda field theory

In [FWB+89], [BFO+90] it was shown how constraining WZW currents may reduce the $\mathsf{SL}(2;\mathbb{R})$ WZW model to a Liouville field theory, or more generally, for a simple Lie group $\mathfrak{G}$, reduce the $\mathfrak{G}$ WZW model to a $\mathfrak{g}$ Toda field theory.

$\mathsf{SL}(2;\mathbb{R})$ admits a Gauß decomposition[5]:

$$\mathsf{SL}(2;\mathbb{R}) \ni g = \underbrace{\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}}_{a} \underbrace{\begin{pmatrix} e^{\Phi/2} & 0 \\ 0 & e^{-\Phi/2} \end{pmatrix}}_{b} \underbrace{\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}}_{c} \tag{184}$$

$\{H, E_+, E_-\}$ is the standard Chevalley basis of $\mathsf{sl}(2)$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{185}$$

Then,

$$g = e^{XE_+} e^{\frac{1}{2}\Phi H} e^{YE_-} \tag{186}$$

The $\mathsf{SL}(2;\mathbb{R})$ WZW action may be written in terms of the variables X, Y, Φ:

$$\begin{aligned} S_{\text{WZW}}[abc] &= S_{\text{WZW}}[a] + S_{\text{WZW}}[b] + S_{\text{WZW}}[c] \\ &\quad + \frac{k}{2\pi}\int_\Sigma \operatorname{tr}\left[a^{-1}\bar\partial a \wedge b\partial b^{-1} + b^{-1}\bar\partial b \wedge c\partial c^{-1} + a^{-1}\bar\partial a \wedge bc\partial c^{-1}b^{-1}\right] \\ &= \frac{k}{4\pi}\int_\Sigma \left(\frac{1}{2}\partial\Phi \wedge \bar\partial\Phi + 2\partial Y \wedge \bar\partial X \, e^{-\Phi}\right) \end{aligned} \tag{187}$$

The main observation of [FWB+89], [BFO+90] is that the current constraints:

$$\begin{aligned} \mathbf{J}[E_+] &\overset{!}{=} \frac{k}{2\pi}\mu \, \mathrm{d}x^+ \;\hat{=}\; \partial Y \overset{!}{=} -\mu e^{\Phi} \, \mathrm{d}x^+ \\ \bar{\mathbf{J}}[E_-] &\overset{!}{=} \frac{k}{2\pi}\nu \, \mathrm{d}x^- \;\hat{=}\; \bar\partial X \overset{!}{=} -\nu e^{\Phi} \, \mathrm{d}x^- \end{aligned} \tag{188}$$

reduce the $\mathsf{SL}(2;\mathbb{R})$ WZW action to the Liouville action:

$$S_{\mathrm{L}} = \frac{k}{4\pi}\int_\Sigma \mathrm{d}x^+ \wedge \mathrm{d}x^- \left(\frac{1}{2}\partial_+\Phi\partial_-\Phi + 2\mu\nu e^{\Phi}\right) \tag{189}$$

[5] $\mathsf{SL}(2;\mathbb{R})_0$, the component connected to the identity, admits a Gauß decomposition.

In fact, this results generalizes to any locally uniquely Gauß-decomposable Lie group. These are the simply connected Lie groups associated to the normal real forms[6] of simple Lie algebras $A_r \cong \mathsf{sl}(r+1,\mathbb{R})$, $B_r \cong \mathsf{so}(r+1,r)$, $C_r \cong \mathsf{sp}(2r,\mathbb{R})$, $D_r \cong \mathsf{so}(r,r)$.

For such a simple Lie group $\mathfrak{G}$, one parameterizes the Gauß decomposition of its elements with the simple Lie algebra data: simple roots $\alpha^i \in \Phi_s$ with their associated Chevalley triple $\{H^i, E^i_+, E^i_-\}$, positive roots $\alpha \in \Phi$ with their associated ladder operators $\{E^\alpha_+, E^\alpha_-\}$. The $\mathsf{sl}(2)$ Gauß decomposition (186) generalizes to:

$$\mathfrak{G} \ni g = \underbrace{\exp\left(\sum_{\alpha\in\Phi} X_\alpha E^\alpha_+\right)}_{a} \underbrace{\exp\left(\sum_i \frac{1}{2}\Phi_i H^i\right)}_{b} \underbrace{\exp\left(\sum_{\alpha\in\Phi} Y_\alpha E^\alpha_-\right)}_{c} \tag{190}$$

Using $\mathrm{Ad}_{e^t} = e^{\mathrm{ad}_t}$ and the Chevalley basis relations, the $\mathfrak{G}$ WZW action may be written in terms of the variables X_i, Y_i, Φ_i: [7]

$$\begin{aligned} S_{\mathrm{WZW}}[abc] &= S_{\mathrm{WZW}}[a] + S_{\mathrm{WZW}}[b] + S_{\mathrm{WZW}}[c] \\ &\quad + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\left[a^{-1}\bar\partial a \wedge b\partial b^{-1} + b^{-1}\bar\partial b \wedge c\partial c^{-1} + a^{-1}\bar\partial a \wedge bc\partial c^{-1}b^{-1}\right] \\ &= \frac{k}{4\pi}\int_\Sigma \left(\sum_{ij}\frac{1}{4}G^{ij}\partial\Phi_i \wedge \bar\partial\Phi_j + \sum_i \frac{2}{|\alpha^i|^2}\partial Y_i \wedge \bar\partial X_i e^{-\sum_j \frac{1}{2}A^{ij}\Phi_j}\right) \end{aligned} \tag{191}$$

[FWB+89], [BFO+90] provide the natural generalization of the $\mathsf{SL}(2;\mathbb{R})$ current constraints:

$$\begin{aligned} &\mathbf{J}[E^i_+] \overset{!}{=} \frac{k}{2\pi}\mu_i\,\mathrm{d}x^+ && \hat{=}\ \partial Y_i \overset{!}{=} -\frac{|\alpha^i|^2}{2}\mu_i e^{\sum_j \frac{1}{2}A^{ij}\Phi_j}\,\mathrm{d}x^+ \\ &\bar{\mathbf{J}}[E^i_-] \overset{!}{=} \frac{k}{2\pi}\nu_i\,\mathrm{d}x^- && \hat{=}\ \bar\partial X_i \overset{!}{=} -\frac{|\alpha^i|^2}{2}\nu_i e^{\sum_j \frac{1}{2}A^{ij}\Phi_j}\,\mathrm{d}x^- \\ &\mathbf{J}[E^\alpha_+] = 0 = \bar{\mathbf{J}}[E^\alpha_-], \\ &\alpha \in \Phi \setminus \Phi_s \end{aligned} \tag{192}$$

These reduce the $\mathfrak{G}$ WZW action to the $\mathfrak{g}$ Toda action:

$$S_T[\Phi_i] = \frac{k}{4\pi}\int_\Sigma \mathrm{d}x^+ \wedge \mathrm{d}x^- \left(\sum_{ij}\frac{1}{4}G^{ij}\partial_+\Phi_i \wedge \partial_-\Phi_j + \sum_i \frac{1}{2}|\alpha^i|^2\mu_i\nu_i e^{\sum_j \frac{1}{2}A^{ij}\Phi_j}\right) \tag{193}$$

[6] Also called the maximally noncompact real forms.
[7] The components associated to nonsimple roots fall out.

Part III.
Spin-2

9. 3-dimensional general relativity

Main reference for standard lore about 3-dim. general relativity used here is [CF18, Lecture 2], see also the textbook [Car03].

9.1. No local degrees of freedom

The Einstein-Hilbert (EH) action for a smooth Lorentzian 3-manifold $(\mathcal{M}, g)$,

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G_3} \int_{\mathcal{M}} \mathrm{d}^3x \sqrt{-g}\,(R - 2\Lambda)\,, \tag{194}$$

has as its equations of motion the Einstein field equations (EFE),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{195}$$

so an on-shell Ricci tensor in 3-dim. general relativity (GR_3) is proportional to the metric[8]:

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}. \tag{196}$$

By counting the independent components of the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R^{\sigma}{}_{\mu\nu\alpha}$ of a (pseudo-)Riemannian n-manifold, one sees that in 3 dimensions they possess the same amount of degrees of freedom (doF):

$$\left.\begin{aligned} \mathrm{doF}(R_{\mu\nu}) &= \frac{n(n+1)}{2} \\ \mathrm{doF}(R_{\alpha\beta\mu\nu}) &= \frac{n^2(n^2-1)}{12} \end{aligned}\right\} \implies \mathrm{doF}(R_{\mu\nu}) \overset{n=3}{=} \mathrm{doF}(R_{\alpha\beta\mu\nu}) \tag{197}$$

Therefore, in three spacetime dimensions, the EFE completely determine the curvature tensor:

$$R_{\mu\nu} \overset{\mathrm{EoM}}{=} 2\Lambda g_{\mu\nu} \implies R_{\alpha\beta\mu\nu} \overset{\mathrm{EoM}}{=} \Lambda(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \tag{198}$$

This means that there are no local doF in GR_3. In fact, the on-shell Riemann tensor (198) is that of a spacetime of constant curvature.

The maximally symmetric solutions of the EFE (195) are called

$$\left.\begin{aligned} \Lambda > 0 :&\quad \text{de Sitter (dS)} \\ \Lambda = 0 :&\quad \text{Minkowski (Mink)} \\ \Lambda < 0 :&\quad \text{anti-de Sitter (AdS)} \end{aligned}\right\} \text{spacetimes.}$$

These solutions are the spacetimes of globally constant curvature. While all solutions of GR_3 are forced by the EFE to be locally $\mathrm{dS}_3/\mathrm{Mink}_3/\mathrm{AdS}_3$, this does not necessarily fix the global geometry.

[8] In n dimensions: $R_{\mu\nu} = \frac{2}{n-2}\Lambda g_{\mu\nu}$

9.2. 3-dimensional anti-de Sitter space

AdS spaces are the maximally symmetric solutions to the EFE with $\Lambda < 0$. A characteristic global length scale is introduced, the AdS radius ℓ:

$$\Lambda = -\frac{1}{\ell^2} \tag{199}$$

Here, the properties of AdS_3 will be surveyed, following [CF18, Lecture 2], [Don16, Section 5], and [Car03, Section 4.4]. Note that everything said here about AdS_3 is also valid for AdS_n, $n > 3$, with the only difference being the number of angular coordinates[9].

AdS_3 is a static spacetime

In global coordinate systems, any complete geodesic may be completely extended.

In terms of physical global coordinates (t, r, θ), $\begin{cases} t \in \mathbb{R} & \text{timelike coordinate,} \\ r \in \mathbb{R}_{\geq 0} & \text{luminosity distance,} \\ \theta \in [0, 2\pi] & \text{angular coordinate,} \end{cases}$

the AdS_3 line element reads:

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = -\left(1 + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \tag{200}$$

Its short-range behaviour, for $r \ll \ell$, is that of a Minkowski spacetime,

$$\overset{r \ll \ell}{\sim} -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 = \mathrm{d}s^2_{\mathrm{Mink}3}\,, \tag{201}$$

while its long-range behaviour, for $r \gg \ell$ (equivalently, for $r \to \infty$), shows that its conformal boundary is a cylinder mantle:

$$\underset{r \to \infty}{\overset{r \gg \ell,}{\sim}} \frac{\ell^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{\ell^2}\big(\underbrace{-\,\mathrm{d}t^2 + \ell^2\,\mathrm{d}\theta^2}_{\text{cylinder metric}}\big) \tag{202}$$

AdS_3 is clearly a static spacetime: ∂_t is an obvious Killing vector of the AdS_3 metric, that is in addition orthogonal to $t = \text{const.}$ slices ($g_{ti} = 0$, $i \in \{r, \theta\}$).

[9] ie. to obtain properties of AdS_n, take $\theta \longrightarrow \{\theta_i\}$ appropriately.

AdS_3 is a cylinder spacetime

Long-range behaviour of $\mathrm{d}s^2_{\mathrm{AdS}_3}$ already shows that its conformal boundary at $r \to \infty$ is a cylinder mantle. It is fruitful to further examine the cylindrical nature of AdS_3.

First, change to dimensionless global coordinates (τ, ρ, θ), $\begin{cases} \tau = t/\ell, \\ \ell \sinh \rho = r, \end{cases}$

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \ell^2 \left(-\cosh^2 \rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \, \mathrm{d}\theta^2 \right) \tag{203}$$

Now, the radial direction may be compactified by introducing a conformal radial coordinate q as

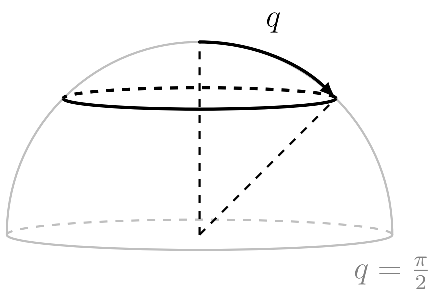
$$\cos q = \frac{1}{\cosh \rho} \implies \mathrm{d}\rho = \frac{\mathrm{d}q}{\cos q} \tag{204}$$

$$\rho \in \mathbb{R}_{\geq 0} \implies \cosh \rho \in [1, \infty) \implies q \in [0, \tfrac{\pi}{2}) \tag{205}$$

In terms of conformal coordinates (τ, q, θ),

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{AdS}_3} &= \frac{\ell^2}{\cos^2 q} \left(-\mathrm{d}\tau^2 + \mathrm{d}q^2 + \sin^2 q \, \mathrm{d}\theta^2 \right) && (206) \\ &= \frac{\ell^2}{\cos^2 q} \left(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_2{}^2 \right) && (207) \end{aligned}$$

Taking into account the coordinate range $q \in [0, \frac{\pi}{2})$, $\tau =$ const. slices are topologically disks:



Furthermore, while the physical distance between $q = 0$ and $q = \frac{\pi}{2}$ is infinite, a light signal starting from $q = 0$ reaches $q = \frac{\pi}{2}$ in a finite physical time coordinate interval $\Delta\tau = \frac{\pi}{2}$, $\hat{=}\ \Delta t = \ell\pi/2$. Therefore, the spatial boundary of AdS_3 is physical; the coordinate range of q can be extended to $q \in [0, \frac{\pi}{2}]$. Then, the line element in terms of the conformal coordinates (206) states that the whole of AdS_3 is conformally equivalent to an infinite cylinder $\mathbb{R} \times D^2$.

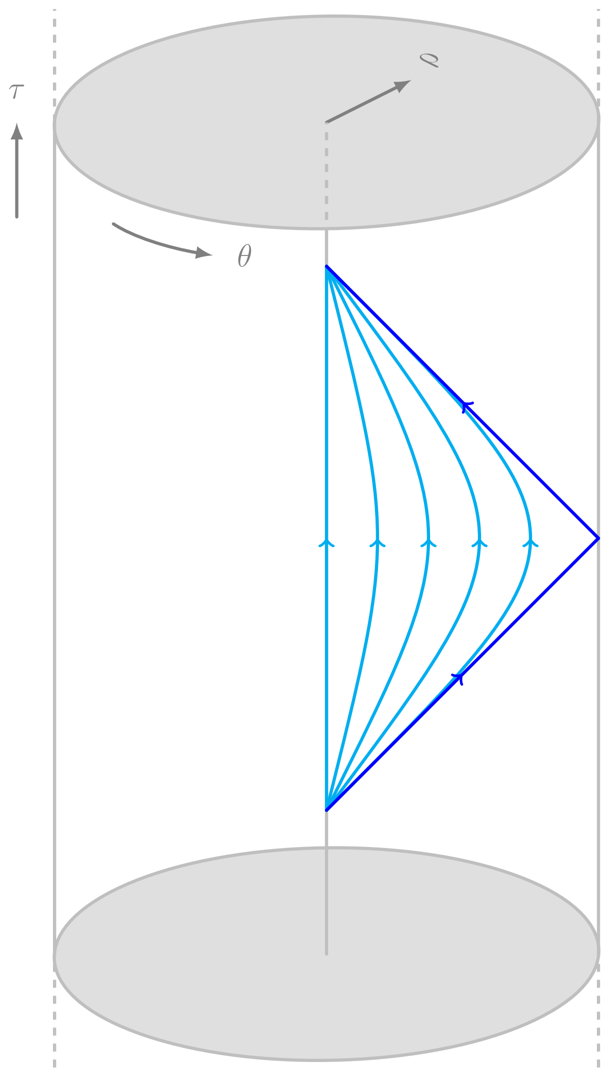


Figure 2: AdS_3 as an infinite cylinder. Cyan curves depict lightlike radial geodesics, blue curves depict null radial geodesics.

AdS$_3$'s isometry group is $\mathsf{SO}(2,2)$

AdS$_3$ may be viewed as the universal covering of the following hyperboloid inside $\mathbb{R}^{2,2}$:

$$-(y^{-1})^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 = -\ell^2 \tag{208}$$

The AdS$_3$ line element in global unitless coordinates (τ, ρ, θ) (203) is induced from the flat $\mathbb{R}^{2,2}$ line element, by parameterizing $\{y^{-1}, y^0, y^1, y^2\}$ satisfying (208) as

$$\begin{aligned} y^{-1} &= \ell \cosh\rho \cos\tau \\ y^0 &= \ell \cosh\rho \sin\tau \\ y^1 &= \ell \sinh\rho \cos\theta \\ y^2 &= \ell \sinh\rho \sin\theta \end{aligned} \tag{209}$$

In the above parameterization, τ and θ are 2π-periodic. AdS$_3$, the universal covering of this hyperboloid, is obtained by decompatifying τ, ie. taking $\tau \in \mathbb{R}$ instead of $\tau \in [0, 2\pi]$.

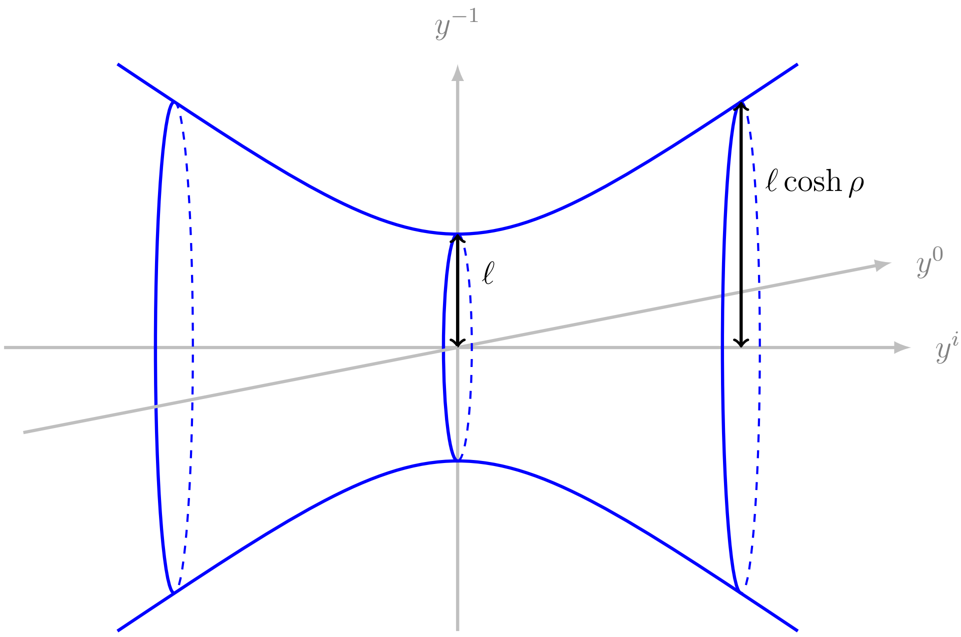


Figure 3: The hyperboloid immersed in $\mathbb{R}^{2,2}$, angular direction θ is surpressed. The hyperboloid contains closed timelike curves; in fact, it may be viewed as AdS$_3$ with its τ-axis wrapped infinitely often around the origin in the $y^{-1}\, y^0$-plane.

This representation of AdS$_3$ makes it clear that its isometry group is $\mathsf{SO}(2,2)$.
(The isometry group of AdS$_n$ is $\mathsf{SO}(n-1,2)$.)

Asymptotic radial coordinates

For the analysis of asymptotically AdS_3 spacetimes, different radial coordinates r, ρ are more suitable.

A more convenient form of the AdS_3 metric is obtained by shifting the radius:

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \left(1 + \frac{r^2}{\ell^2}\right)^{-1} \mathrm{d}r^2 - \left(1 + \frac{r^2}{\ell^2}\right) \mathrm{d}t^2 + r^2\,\mathrm{d}\theta^2 \tag{210}$$

$$\Big\downarrow \; r \mapsto \frac{1}{2}\left(r + \sqrt{r^2 + \ell^2}\right) \tag{211}$$

$$= \underbrace{\frac{\ell^2}{r^2}\,\mathrm{d}r^2}_{\text{cf. } r \to \infty \text{ behaviour of } \mathrm{d}s^2_{\mathrm{AdS}_3}} - \left(\frac{r}{\ell} + \frac{\ell}{4r}\right)^2 \mathrm{d}\tau^2 + \left(r - \frac{\ell^2}{4r}\right)^2 \mathrm{d}\theta^2 \tag{212}$$

Note the coordinate range shift: $r \in [0, \infty) \mapsto r \in [\ell/2, \infty)$.

A more convenient dimensionless radial coordinate ρ is given by:

$$\ell e^\rho = r \implies \mathrm{d}r = \ell e^\rho\,\mathrm{d}\rho \implies \mathrm{d}\rho = \frac{\mathrm{d}r}{r} \tag{213}$$

The AdS_3 line element in terms of (τ, ρ, θ) is then:

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \ell^2\left(\mathrm{d}\rho^2 - \left(e^\rho + \frac{1}{4}e^{-\rho}\right)^2 \mathrm{d}\tau^2 - \left(e^\rho - \frac{1}{4}e^{-\rho}\right)^2 \mathrm{d}\theta^2\right) \tag{214}$$

Asymptotically AdS_3 spacetimes

An asymptotically AdS_3 spacetime may be thought of as any spacetime, whose line element $\mathrm{d}s^2$ does not deviate from the AdS_3 line element too much for $r \to \infty$:

$$\mathrm{d}s^2 - \mathrm{d}s^2_{\mathrm{AdS}_3} \overset{r\to\infty}{\sim} O(1) \tag{215}$$

In general, any metric admitting the following expansion near the $r \to \infty$ boundary,

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\ell^2}{r^2}\,\mathrm{d}r^2 + r^2 g^\partial_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j + O(1) \\ &= \ell^2\,\mathrm{d}\rho^2 + \ell^2 e^{2\rho} g^\partial_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j + O(1) \end{aligned} \tag{216}$$

is referred to as "asymptotically AdS_3". This form of the metric is called the Fefferman-Graham gauge, it is expressed in the Fefferman-Graham coordinates (r, x^i) or (ρ, x^i). Note, such coordinate systems (τ, r, θ), (τ, ρ, θ) were obtained in the section 9.2. Here, g^∂_{ij} is the boundary (r-independent) metric, specifying the (leading terms) geometry on the outer cylinder mantle. Fixing g^∂ amounts to specifying the boundary conditions; distinct solutions for given boundary conditions differ by subleading terms only.

On closer inspection, the intuitive definition of an asymptotically AdS_3 spacetime (215) already fixes g^∂ to be the flat 2-dim. Minkowski metric η: using the long-range behaviour of the AdS_3 metric (202), any line element satisfying (215) can be written in the neighbourhood of the boundary as

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\,\mathrm{d}r^2 + r^2(-\,\mathrm{d}\tau^2 + \mathrm{d}\theta^2) + O(1) \tag{217}$$

Setting $g^\partial = \eta$ is known as Brown-Henneaux boundary conditions.

10. Coframe formulation of general relativity

10.1. General relativity with torsion

The standard reference on GR with torsion is [HvdHKN76].

The Einstein–Hilbert metric formulation is the standard way to talk about general relativity. While it undoubtedly excels at describing macroscopic spacetime behaviour, it contains room for well-motivated generalization. The underlying geometric language is that of Riemannian manifolds, ie. manifolds equipped with a nondegenerate metric tensor that fully determines a connection and a curvature tensor. However, this can be generalized to allow a metric-compatible connection that is not fully determined the metric, containing extra degrees of freedom, called torsion.

Note that the metricity condition for a connection Γ,

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{218}$$

does not fully determine Γ as a function of g; its antisymmetric part, torsion $T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$, is allowed by the metricity condition to be arbitrary.

General metric-compatible connections decompose into the metric-dependent part, given by Christoffel symbols $\left\{ {\lambda \atop \mu\nu} \right\}$, and the torsion part, given by contortion $K_{\mu\nu}{}^{\lambda}$:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} - K_{\mu\nu}{}^{\lambda}, \text{ where } \begin{cases} \left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \\ K_{\mu\nu}{}^{\lambda} = \frac{1}{2} (-T_{\mu\nu}{}^{\lambda} + T_{\nu}{}^{\lambda}{}_{\mu} - T^{\lambda}{}_{\mu\nu}) \end{cases} \tag{219}$$

The Einstein-Hilbert-Palatini (EHP) action is a simple generalization of the Einstein-Hilbert action, obtained by promoting the connection Γ to an independent variable:

$$S_{\mathrm{EHP}}[g, \Gamma] = \frac{1}{16\pi G_3} \int_{\mathcal{M}} \mathrm{d}^3 x \sqrt{-g} \, (R - 2\Lambda) \tag{220}$$

Then, the equations of motion are:

$$\begin{cases} \delta S / \delta g \stackrel{!}{=} 0 & \Longrightarrow \; R_{\mu\nu} = 2\Lambda g_{\mu\nu} : \quad \text{constant curvature} \\ \delta S / \delta \Gamma \stackrel{!}{=} 0 & \Longrightarrow \; T_{\mu\nu}{}^{\lambda} = 0 : \quad \text{no torsion} \end{cases} \tag{221}$$

Hence, imposing (in the absence of sources) an a priori torsion-free condition on Γ is not required, as the EHP action is on-shell equivalent to the EH action. The EHP field equations are equivalent to the EFE – therefore, this is an equivalent formulation of GR.

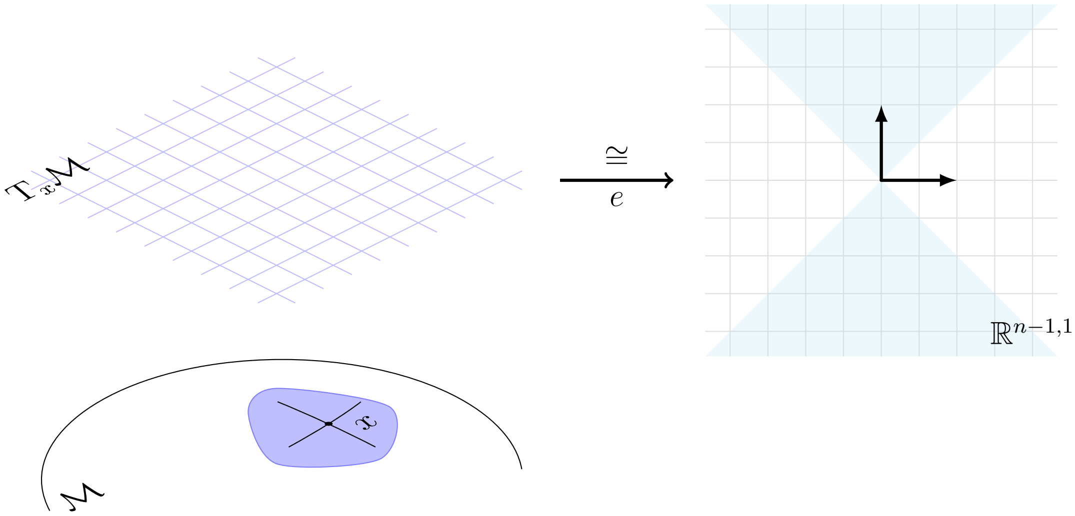
10.2. Coframe description of spacetimes

The exposition in this section broadly follows [Kra20, Chapter 3] as well as [HZ16, Chapter 3]. See also [Tec19].

In the coframe formulation of general relativity, one distinguishes between the spacetime manifold as the base of a fibre bundle, and the local tangent space where Lorentz-invariant physical quantities "live". Cartan's insight is that the equivalence principle,

> Locally, the gravitational force may be nullified by switching to a freely falling inertial frame, in which special relativity holds.

should be interpreted as the existence of a coframe isomorphism $e : T_x\mathcal{M} \xrightarrow{\cong} \mathbb{R}^{n-1,1}$ at every point $x \in \mathcal{M}$ adapted to $\mathbb{R}^{n-1,1}$'s geometric structure.[10]



In terms of local $\mathcal{M}$ coordinates $\{x^\mu\}$ and $\mathbb{R}^{n-1,1}$ coordinates $\{z^a\}$, this isomorphism is given by the coframe field:

$$\mathrm{d}z^a = e_\mu{}^a(x)\,\mathrm{d}x^\mu \tag{222}$$

The inverse of the coframe field $e_\mu{}^a$ is the frame field $e^\mu{}_a$, satisfying $\begin{cases} e_\mu{}^a e^\mu{}_b = \delta^a{}_b \\ e_\mu{}^a e^\nu{}_a = \delta_\mu{}^\nu. \end{cases}$

Note that while tensors in $\mathcal{M}$ are tensors under GCT transformations, tensors in $\mathbb{R}^{n-1,1}$ are invariant under the Lorentz transformation group $\mathsf{SO}(n-1,1)$. $\mathbb{R}^{n-1,1}$ contains two canonical $\mathsf{SO}(n-1,1)$ tensors, the flat metric η_{ab} and the permutation symbol $\varepsilon_{a_1\ldots a_n}$. The coframe then defines a metric $g_{\mu\nu}$ and a permutation tensor $\epsilon_{\mu_1\ldots\mu_n}$ on $\mathcal{M}$ via pullback:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{223}$$

$$\epsilon_{\mu_1\ldots\mu_n} = e_{\mu_1}{}^{a_1}\cdots e_{\mu_n}{}^{a_n}\varepsilon_{a_1\ldots a_n} \tag{224}$$

[10] In other words, $T\mathcal{M}$ should be isomorphic to a principal $\mathsf{SO}(n-1,1)$–bundle.

Under a Lorentz transformation $Q \in \mathsf{SO}(n-1,1)$, the coframe transforms as a vector:

$$e_\mu{}^a \mapsto (Q^{-1})^a{}_b e_\mu{}^b \tag{225}$$

Therefore, under local Lorentz transformations, the metric (223) and permutation tensor (224) remain unchanged.

The coframe can convert GCT vector fields $\mathfrak{X}(\mathcal{M}) \ni \xi = \xi^\mu \partial_\mu$ into $\mathsf{SO}(n-1,1)$ vector-valued fields,

$$\xi^a(x) = e_\mu{}^a(x)\xi^\mu(x) \tag{226}$$

While ξ^a transforms as a vector under local Lorentz transformations, $\partial_\mu \xi^a$ does not. To rectify this, introduce a Lorentz covariant derivative by specifying a Lorentz connection,

$$D_\mu \xi^a \coloneqq \partial_\mu \xi^A + w_\mu{}^a{}_b \xi^b \tag{227}$$

with the usual generalization to the covariant derivative of $\mathsf{SO}(n-1,1)$ covectors and higher valence tensors. w transforms as required under local Lorentz transformations,

$$w_\mu{}^a{}_b \longmapsto (Q^{-1})^a{}_d w_\mu{}^d{}_c Q^c{}_b - (Q^{-1})^a{}_c \partial_\mu Q^c{}_b \tag{228}$$

While the flat metric η_{ab} is a constant and invariant $\mathsf{SO}(n-1,1)$ tensor in each individual $T_x\mathcal{M}$, it should be in addition the same invariant tensors across all tangent spaces. w defines a notion of parallel transport on $T\mathcal{M}$; therefore, one demands η_{ab} to be covariantly constant,

$$D_\mu \eta_{ab} \overset{!}{=} 0 \tag{229}$$

As a consequence, the Lorentz connection w has to be antisymmetric[11],

$$w_\mu{}^{ab} \overset{!}{=} -w_\mu{}^{ba} \tag{230}$$

Such a connection w is said to be compatible with $\mathbb{R}^{n-1,1}$'s geometric structure.

w defines an affine connection on $\mathcal{M}$ via e's pullback,

$$\begin{aligned} \nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + \Gamma_{\mu\lambda}{}^\nu \xi^\lambda \\ &\coloneqq e^\nu{}_a D_\mu \xi^a \end{aligned} \tag{231}$$

with analogous definition for GCT covectors and higher valence tensors. Consistency now requires the analogue of the metricity condition (218) for the coframe, the vanishing of its total covariant derivative:

$$\mathcal{D}_\mu e_\lambda{}^a \coloneqq \partial_\mu e_\lambda{}^a + w_\mu{}^a{}_b e_\lambda{}^b - \Gamma^\sigma{}_{\mu\lambda} e_\sigma{}^a = 0 \tag{232}$$

By its definition as an isomorphism, the coframe field is nondegenerate. Together with the vanishing of its total covariant derivative, g, as defined by the relation (223), is indeed a (pseudo-)metric tensor.

[11]Tangent space indices are raised and lowered with η.

Notice that the coframe e and Lorentz connection w are spacetime 1-forms:

$$e^a = e_\mu{}^a \, \mathrm{d}x^\mu \tag{233}$$
$$w^a{}_b = w_\mu{}^a{}_b \, \mathrm{d}x^\mu \tag{234}$$

By the previous discussion, $\mathcal{M}$'s geometric quantities may be re-expressed wlog. in terms of these 1-forms. As exterior forms do not carry coordinate indices, any further study of GR using e and w is manifestly coordinate independent.

Curvature and torsion in this language are spacetime 2-forms with values in $\mathbb{R}^{n-1,1}$,

$$R^a{}_b = \mathrm{d}w^a{}_b + w^a{}_c \wedge w^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{235}$$
$$T^a = De^a := \mathrm{d}e^a + w^a{}_b \wedge e^b = \frac{1}{2} T_{\mu\nu}{}^a \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{236}$$

10.3. Einstein-Cartan action

The Einstein-Hilbert-Palatini action (220) expressed in the coframe language is called the Einstein-Cartan (EC) action; in $n > 3$ dimensions, it has the form:

$$S_{\mathrm{EC}}[e,w] = \frac{1}{16\pi G_n} \int_{\mathcal{M}} \varepsilon_{a\ldots bcd} \left(\frac{1}{n-2} e^a \wedge \cdots \wedge e^b \wedge R^{cd} - \frac{\Lambda}{n} e^a \wedge \cdots \wedge e^b \wedge e^c \wedge e^d \right)$$

The equations of motion are:

$$\begin{cases} \delta S/\delta e \overset{!}{=} 0 & \implies \varepsilon_{ab\ldots cde} e^b \wedge \cdots \wedge e^c \wedge R^{de} = \Lambda \varepsilon_{ab\ldots cde} e^b \wedge \cdots \wedge e^c \wedge e^d \wedge e^e \\ \delta S/\delta w \overset{!}{=} 0 & \implies T^a = 0 \end{cases}$$

These are the requirements that $\mathcal{M}$ be torsion-free and an Einstein manifold, ie. $R \propto g_{\mu\nu}$, written in the coframe language.

The EC action generalizes straightforwardly to $n = 3$:

$$S_{\mathrm{EC}}[e,w] = \frac{1}{16\pi G_3} \int_{\mathcal{M}} \varepsilon_{abc} \left(e^a \wedge R^{bc} - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c \right) \tag{237}$$

For the next part, cf. [Kra20, Section 4]. In 3 dimensions, something special occurs: ω and R can be dualized to $\mathsf{SO}(2,1)$ vectors:

$$w_a := \frac{1}{2} \varepsilon_{abc} w^{bc} \tag{238}$$
$$R_a := \frac{1}{2} \varepsilon_{abc} R^{bc} = \mathrm{d}\omega_a + \frac{1}{2} \varepsilon_{abc} w^b \wedge w^c \tag{239}$$

In terms of the dualized variables, the 3-dimensional EC action reads

$$\begin{aligned} S_{\mathrm{EC}}[e,\omega] &= \frac{1}{8\pi G_3}\int_{\mathcal{M}}\left(\eta_{ab}\, e^a\wedge R^b - \frac{\Lambda}{6}\varepsilon_{abc}\, e^a\wedge e^b\wedge e^c\right) \\ &= \frac{1}{8\pi G_3}\int_{\mathcal{M}}\left(\eta_{ab}\, e^a\wedge \mathrm{d}w^b + \frac{1}{2}\varepsilon_{abc} e^a\wedge w^b\wedge w^c - \frac{\Lambda}{6}\varepsilon_{abc}\, e^a\wedge e^b\wedge e^c\right), \end{aligned} \tag{240}$$

its equations of motion are equivalent to the EHP equations of motion (221):

$$\begin{cases} \delta S/\delta e \stackrel{!}{=} 0 & \Longrightarrow\ R_a = \frac{\Lambda}{2}\varepsilon_{abc} e^b\wedge e^c : \quad \text{constant curvature} \\ \delta S/\delta w \stackrel{!}{=} 0 & \Longrightarrow\ T^a = 0 : \quad \text{no torsion} \end{cases} \tag{241}$$

Observe that GR_3 may be completely described using $\mathsf{SO}(2,1)$ vectors e^a, w^a, R^a, T^a; using the isomorphism $\mathbb{R}^{2,1}\cong\mathsf{so}(2,1)$, they can be interpreted as $\mathsf{so}(2,1)$-valued forms. Interpreting $\mathsf{so}(2,1)$ generators $\{j_a\}_{a\in\{0\hat{=}\tau,1\hat{=}\theta,2\hat{=}\rho\}}$,

$$j_0 = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ j_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ j_2 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{242}$$

satisfying

$$\begin{aligned} [j_a, j_b] = \varepsilon_{abc} j^c \quad \mathrm{tr}[j_a j_b] = \frac{1}{2}\eta_{ab} \\ \Longrightarrow\quad \eta_{ab} = 2\,\mathrm{tr}[j_a j_b], \quad \varepsilon_{abc} = 2\,\mathrm{tr}[j_a[j_b, j_c]] \end{aligned} \tag{243}$$

as the basis vectors in $\mathbb{R}^{2,1}$, define the $\mathsf{so}(2,1)$-valued 1-forms:

$$\mathbf{e} \coloneqq e^a j_a \quad \mathbf{w} \coloneqq w^a j_a \tag{244}$$

Plugging the identification (243) between $\mathsf{so}(2,1)$ and $\mathbb{R}^{2,1}$ into the 3-dimensional EC action (240) leads to the reformulation of GR_3 as an $\mathsf{SO}(2,1)$ gauge theory:

$$S_{\mathrm{EC}}[e,w] = \frac{1}{4\pi G_3}\int_{\mathcal{M}} \mathrm{tr}\left[\mathbf{e}\wedge \mathrm{d}\mathbf{w} + \mathbf{e}\wedge\mathbf{w}\wedge\mathbf{w} - \frac{\Lambda}{3}\mathbf{e}^{\wedge 3}\right] \tag{245}$$

From the variation of the action,

$$\begin{aligned}\delta S_{\mathrm{EC}}[e,w] = \frac{1}{4\pi G_3}\int_{\mathcal{M}} \mathrm{tr}[\delta\mathbf{e}\wedge(\mathrm{d}\mathbf{w}+\mathbf{w}\wedge\mathbf{w}-\Lambda\,\mathbf{e}\wedge\mathbf{e})+\delta\mathbf{w}\wedge(\mathrm{d}\mathbf{e}+\mathbf{w}\wedge\mathbf{e}+\mathbf{e}\wedge\mathbf{w})] \\ -\frac{1}{4\pi G_3}\int_{\partial\mathcal{M}}\mathrm{tr}[\mathbf{e}\wedge\delta\mathbf{w}]\end{aligned} \tag{246}$$

the equations of motion (241) can be read off:

$$R^a j_a =: \mathbf{R} = \mathrm{d}\mathbf{w}+\mathbf{w}\wedge\mathbf{w} = \Lambda\,\mathbf{e}\wedge\mathbf{e} \tag{247}$$

$$T^a j_a =: \mathbf{T} = \mathrm{d}\mathbf{e}+\mathbf{w}\wedge\mathbf{e}+\mathbf{e}\wedge\mathbf{w} = 0 \tag{248}$$

Once again, the resulting field equations are conditions for constant curvature, torsion-free spacetimes, cf. the EHP equations of motion (221). If $\partial\mathcal{M}\neq\varnothing$, the well-definedness of the variational principle requires an implementation of boundary conditions.

Under a local Lorentz transformation, specified by a $Q:\mathcal{M}\to\mathsf{SO}(2,1)$, $\mathbf{e}$ and $\mathbf{w}$ admit the following transformation behaviour:

$$\begin{aligned}\mathbf{e} &\longmapsto Q^{-1}\mathbf{e}Q \\ \mathbf{w} &\longmapsto Q^{-1}\mathrm{d}Q + Q^{-1}\mathbf{w}Q = \mathbf{w}^Q\end{aligned} \tag{249}$$

$\mathbf{w}$'s behaviour is typical of a gauge connection. However, $\mathbf{e}\in\Omega^1(\mathcal{M},\mathsf{so}(2,1))$ carries the requirement of being nondegenerate, and cannot therefore be treated as a second gauge connection.

11. $\Lambda < 0$ GR$_3$ in Chern–Simons variables

See also [CFPT10], [CFP11], [Don16], [Oli18], [CF23].

11.1. $\Lambda < 0$ GR$_3$ action in terms of $\mathsf{SO}(2,1)$ Chern–Simons connections

Following Achúcarro&Townsend [AT86], introduce $\mathsf{SO}(2,1)$ CS connections $\mathbf{A}$, $\bar{\mathbf{A}}$ as linear field redefinitions of $\mathbf{e}$, $\mathbf{w}$:

$$\left.\begin{aligned} \mathbf{A} &= \mathbf{w} + \frac{1}{\ell}\,\mathbf{e} \\ \bar{\mathbf{A}} &= \mathbf{w} - \frac{1}{\ell}\,\mathbf{e} \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} \mathbf{e} &= \frac{\ell}{2}\left(\mathbf{A} - \bar{\mathbf{A}}\right) \\ \mathbf{w} &= \frac{1}{2}\left(\mathbf{A} + \bar{\mathbf{A}}\right) \end{aligned}\right. \tag{250}$$

Compute the individual tr terms in the gauge field theory form of the EC action (245),

$$\begin{aligned} \operatorname{tr}[\mathbf{e}\wedge \mathrm{d}\mathbf{w}] &= \frac{\ell}{4}\operatorname{tr}\big[\mathbf{A}\wedge\mathrm{d}\mathbf{A} - \bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}} - \mathrm{d}\big(\mathbf{A}\wedge\bar{\mathbf{A}}\big)\big], \\ \operatorname{tr}[\mathbf{e}\wedge\mathbf{w}\wedge\mathbf{w}] &= \frac{\ell}{8}\operatorname{tr}\big[\mathbf{A}^{\wedge 3} - \bar{\mathbf{A}}^{\wedge 3} + \mathbf{A}\wedge\bar{\mathbf{A}}\wedge\mathbf{A} - \bar{\mathbf{A}}\wedge\mathbf{A}\wedge\bar{\mathbf{A}}\big], \\ \operatorname{tr}\left[\frac{1}{3\ell^2}\mathbf{e}^{\wedge 3}\right] &= \frac{\ell}{8}\operatorname{tr}\left[\frac{1}{3}\mathbf{A}^{\wedge 3} - \frac{1}{3}\bar{\mathbf{A}}^{\wedge 3} - \mathbf{A}\wedge\mathbf{A}\wedge\bar{\mathbf{A}} + \bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\mathbf{A}\right], \end{aligned}$$

define the GR$_3$ CS coupling constant as

$$k := \frac{\ell}{4G_3},$$

plug the individual tr terms into the EC action (245) to obtain:

$$\begin{aligned} S_{\mathrm{EC}}[e,w] &= \frac{k}{4\pi}\int_{\mathcal{M}}\operatorname{tr}\left[\mathbf{A}\wedge\mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A}^{\wedge 3} - \bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}} - \frac{2}{3}\bar{\mathbf{A}}^{\wedge 3}\right] \\ &\quad - \frac{k}{4\pi}\int_{\partial\mathcal{M}}\operatorname{tr}\big[\mathbf{A}\wedge\bar{\mathbf{A}}\big] \\ \Longrightarrow\ S_{\mathrm{EC}}[e,w] &= S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] - \frac{k}{4\pi}\int_{\partial\mathcal{M}}\operatorname{tr}\big[\mathbf{A}\wedge\bar{\mathbf{A}}\big] \end{aligned} \tag{251}$$

The flatness constraints on $\mathbf{A}$ and $\bar{\mathbf{A}}$ are equivalent to (247)&(248), the equations of motion for $\mathbf{e}$ and $\mathbf{w}$:

$$\left.\begin{aligned} \mathbf{F} &= \mathrm{d}\mathbf{A} + \mathbf{A}\wedge\mathbf{A} = 0 \\ \bar{\mathbf{F}} &= \mathrm{d}\bar{\mathbf{A}} + \bar{\mathbf{A}}\wedge\bar{\mathbf{A}} = 0 \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} &\mathbf{F} + \bar{\mathbf{F}} = 0 \to \mathrm{d}\mathbf{w} + \mathbf{w}\wedge\mathbf{w} + \frac{1}{\ell^2}\mathbf{e}\wedge\mathbf{e} = 0 \\ &\mathbf{F} - \bar{\mathbf{F}} = 0 \to \mathrm{d}\mathbf{e} + \mathbf{w}\wedge\mathbf{e} + \mathbf{e}\wedge\mathbf{w} = 0 \end{aligned}\right. \tag{252}$$

11.2. Infinitesimal gauge symmetries of the GR_3 action

$S_{\mathrm{EC}}[A,\bar{A}]$ differs from $S_{\mathrm{CS}}[A]-S_{\mathrm{CS}}[\bar{A}]$ only by a boundary term. In the covariant phase space formalism, boundary terms do not change the infinitesimal symmetries/conserved quantities of the theory. Using that fact that, the infinitesimal symmetries/conserved quantities of $\Lambda<0$ GR_3 can be inferred to be equal to two copies of $\mathsf{SO}(2,1)$ CS infinitesimal symmetries,

$$\pounds_{\Lambda}\mathbf{A}=\mathrm{d}\Lambda+[\mathbf{A},\Lambda]\quad \pounds_{\bar{\Lambda}}\bar{\mathbf{A}}=\mathrm{d}\bar{\Lambda}+[\bar{\mathbf{A}},\bar{\Lambda}] \tag{253}$$

where $(\Lambda,\bar{\Lambda})\in\mathsf{so}(2,1)\times\mathsf{so}(2,1)$.

$$\begin{aligned}\boldsymbol{\vartheta}_{\mathrm{EC}}[A,\bar{A}]&=\boldsymbol{\vartheta}_{\mathrm{CS}}[A]-\boldsymbol{\vartheta}_{\mathrm{CS}}[\bar{A}]\\ \mathbf{B}(\Lambda,\bar{\Lambda})&=\mathbf{B}_{\mathrm{CS}}(\Lambda)[A]-\mathbf{B}_{\mathrm{CS}}(\bar{\Lambda})[\bar{A}]\end{aligned} \tag{254}$$

$$\implies \mathbf{J}_{\mathrm{EC}}(\Lambda,\bar{\Lambda})=\mathbf{J}_{\mathrm{CS}}(\Lambda)-\mathbf{J}_{\mathrm{CS}}(\bar{\Lambda})\equiv\mathbf{J}(\Lambda)+\bar{\mathbf{J}}(\bar{\Lambda}) \tag{255}$$

The Noether current and charge split into an unbarred and barred conserved current and charge,

$$\begin{aligned}\mathbf{J}(\Lambda)&=\frac{k}{2\pi}\,\mathrm{tr}[\Lambda\mathbf{A}] & \bar{\mathbf{J}}(\bar{\Lambda})&=-\frac{k}{2\pi}\,\mathrm{tr}\big[\bar{\Lambda}\bar{\mathbf{A}}\big]\\ H(\Lambda)&=\frac{k}{2\pi}\int_{\mathcal{S}}\mathrm{tr}[\Lambda\mathbf{A}] & \bar{H}(\bar{\Lambda})&=-\frac{k}{2\pi}\int_{\mathcal{S}}\mathrm{tr}\big[\bar{\Lambda}\bar{\mathbf{A}}\big]\end{aligned} \tag{256}$$

GR admits two symmmetries: local Lorentz invariance and diffeomorphism invariance. Their infinitesimal transformations are contained in the $\mathsf{SO}(2,1)\times\mathsf{SO}(2,1)$ infinitesimal symmetry of the $S_{\mathrm{CS}}[A]-S_{\mathrm{CS}}[\bar{A}]$.

Lorentz invariance

The Lorentz transformation behaviour of $\mathbf{e}$, $\mathbf{w}$ translates into a diagonal transformation behaviour on $\mathbf{A}$, $\bar{\mathbf{A}}$:

$$\left.\begin{aligned}&\mathbf{e}\mapsto Q^{-1}\mathbf{e}Q\\ &\mathbf{w}\mapsto Q^{-1}\mathrm{d}Q+Q^{-1}\mathbf{w}Q\end{aligned}\right\}\implies\left\{\begin{aligned}&\mathbf{A}\mapsto Q^{-1}\mathrm{d}Q+Q^{-1}\mathbf{A}Q=\mathbf{A}^{Q}\\ &\bar{\mathbf{A}}\mapsto Q^{-1}\mathrm{d}Q+Q^{-1}\bar{\mathbf{A}}Q=\bar{\mathbf{A}}^{Q}\end{aligned}\right. \tag{257}$$

Recall the behaviour of S_{CS} under a general gauge transformation:

$$S_{\mathrm{CS}}[A]\longmapsto S_{\mathrm{CS}}[A^{Q}]=S_{\mathrm{CS}}[A]+\frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{tr}\big[Q^{-1}\mathbf{A}Q\wedge Q^{-1}\mathrm{d}Q\big]-\frac{k}{12\pi}\int_{\mathcal{M}}\mathrm{tr}\big[(Q^{-1}\mathrm{d}Q)^{\wedge 3}\big]$$

The EC action is in contrast manifestly invariant under $\mathsf{SO}(2,1)$ gauge transformations acting diagonally on $\mathbf{A}$, $\bar{\mathbf{A}}$:

$$
\begin{aligned}
S_{\mathrm{EC}}[A,\bar{A}] &= S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}\big[\mathbf{A}\wedge\bar{\mathbf{A}}\big] \\
&\big\downarrow \mathbf{A}\mapsto\mathbf{A}^Q,\ \bar{\mathbf{A}}\mapsto\bar{\mathbf{A}}^Q \\
&S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}\big[\mathbf{A}\wedge\bar{\mathbf{A}}\big] \\
&+\frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}\big[Q^{-1}\mathbf{A}Q\wedge Q^{-1}\mathrm{d}Q\big] - \frac{k}{12\pi}\int_{\mathcal{M}} \operatorname{tr}\big[(Q^{-1}\mathrm{d}Q)^{\wedge 3}\big] \\
&-\frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}\big[Q^{-1}\bar{\mathbf{A}}Q\wedge Q^{-1}\mathrm{d}Q\big] + \frac{k}{12\pi}\int_{\mathcal{M}} \operatorname{tr}\big[(Q^{-1}\mathrm{d}Q)^{\wedge 3}\big] \\
&-\frac{k}{4\pi}\int_{\partial\mathcal{M}} \operatorname{tr}\big[Q^{-1}\mathbf{A}Q\wedge Q^{-1}\mathrm{d}Q - Q^{-1}\bar{\mathbf{A}}Q\wedge Q^{-1}\mathrm{d}Q\big] \\
&= S_{\mathrm{EC}}[A,\bar{A}]
\end{aligned}
$$

Infinitesimally, for $Q=\exp\Lambda$, infinitesimal Lorentz transformation are a diagonal subset (Λ,Λ) of $(\Lambda,\bar{\Lambda})\in\mathsf{so}(2,1)\times\mathsf{so}(2,1)$.

Diffeomorphism invariance

GR, in addition to local Lorentz gauge transformations, is also invariant under diffeomorphisms, specified infinitesimally by vector fields $\xi\in\mathfrak{X}(\mathcal{M})$, acting via Lie derivatives:

$$
\left.\begin{aligned}\Delta_\xi\mathbf{e} &= \pounds_\xi\mathbf{e}\\ \Delta_\xi\mathbf{w} &= \pounds_\xi\mathbf{w}\end{aligned}\right\} \implies \left\{\begin{aligned}\Delta_\xi\mathbf{A} &= \pounds_\xi\mathbf{A}\\ \Delta_\xi\bar{\mathbf{A}} &= \pounds_\xi\bar{\mathbf{A}}\end{aligned}\right. \tag{258}
$$

Using Cartan's formula (for 1-forms), $\pounds_\xi = \mathrm{d}\iota_\xi + \iota_\xi\mathrm{d}$, where the interior product acts on 1-forms as $\iota_\xi\mathbf{A}^{(1)} = A(\xi)$, $\iota_\xi(\mathbf{A}^{(1)}\wedge\mathbf{B}^{(1)}) = A(\xi)\mathbf{B} - \mathbf{A}B(\xi)$, compute the behaviour of a flat connection ($\mathrm{d}\mathbf{A}+\mathbf{A}\wedge\mathbf{A}=0$) under a diffeomorphism transformation:

$$
\begin{aligned}
\pounds_\xi\mathbf{A} &= \mathrm{d}\iota_\xi\mathbf{A} + \iota_\xi\mathrm{d}\mathbf{A} \\
&\overset{\text{flat}}{=} \mathrm{d}\iota_\xi\mathbf{A} - \iota_\xi(\mathbf{A}\wedge\mathbf{A}) \\
&= \mathrm{d}A(\xi) - A(\xi)\mathbf{A} + \mathbf{A}A(\xi) = \mathrm{d}A(\xi) + [\mathbf{A}, A(\xi)] \\
&\hat{=}\ \pounds_{A(\xi)}\mathbf{A}
\end{aligned}
$$

As a result, diffeomorphisms generated by $\xi\in\mathfrak{X}(\mathcal{M})$ translate on-shell to a subset $(A(\xi),\bar{A}(\xi))$ of $(\Lambda,\bar{\Lambda})\in\mathsf{so}(2,1)\times\mathsf{so}(2,1)$.

11.3. AdS_3 Chern-Simons connections

Recall the AdS_3 line element in asymptotic unitless coordinates (τ, ρ, θ):

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \ell^2 \left(\mathrm{d}\rho^2 - \left(e^\rho + \frac{1}{4} e^{-\rho} \right)^2 \mathrm{d}\tau^2 - \left(e^\rho - \frac{1}{4} e^{-\rho} \right)^2 \mathrm{d}\theta^2 \right) \tag{259}$$

Utilizing the relations (223) and (243), the $\mathsf{SO}(2,1)$ connection $\mathbf{e}$ creates a line element via

$$\mathrm{d}s^2 = 2\,\mathrm{tr}[\mathbf{ee}] = -(e^0)^2 + (e^1)^2 + (e^2)^2 \tag{260}$$

An appropriate AdS_3 coframe is then $\begin{cases} e^0_{\mathrm{AdS}_3} = \ell \left(e^\rho + \frac{1}{4} e^{-\rho} \right) \mathrm{d}\tau\,, \\ e^1_{\mathrm{AdS}_3} = \ell \left(e^\rho - \frac{1}{4} e^{-\rho} \right) \mathrm{d}\theta\,, \\ e^2_{\mathrm{AdS}_3} = \ell\, \mathrm{d}\rho\,, \end{cases}$

contracted with the $\mathsf{so}(2,1)$ basis (242) gives the AdS_3 $\mathsf{so}(2,1)$ frame:

$$\mathbf{e}_{\mathrm{AdS}_3} = \ell \left(\left(e^\rho + \frac{1}{4} e^{-\rho} \right) j_0\, \mathrm{d}\tau + \left(e^\rho - \frac{1}{4} e^{-\rho} \right) j_1\, \mathrm{d}\theta + j_2\, \mathrm{d}\rho \right) \tag{261}$$

For future convenience, it is useful to switch to an $\mathsf{sl}(2;\mathbb{R}) \cong \mathsf{so}(2,1)$ basis $\{L_m\}_{m\in\{+,0,-\}}$, obtained from $\{j_a\}$ as[12]

$$L_0 = j_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \; L_+ = j_0 + j_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; L_- = j_0 - j_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{262}$$

$$[L_m, L_n] = (m-n) L_{m+n} \tag{263}$$

Then, the AdS_3 $\mathsf{so}(2,1) \cong \mathsf{sl}(2;\mathbb{R})$ frame in lightcone coordinates $x^\pm = \tau \pm \theta$ is:

$$\mathbf{e}_{\mathrm{AdS}_3} = \ell \left(\frac{1}{2} \left(e^\rho L_+ + \frac{1}{4} e^{-\rho} L_- \right) \mathrm{d}x^+ + \frac{1}{2} \left(\frac{1}{2} e^{-\rho} L_+ + e^\rho L_- \right) \mathrm{d}x^- + L_0\, \mathrm{d}\rho \right) \tag{264}$$

The no torsion condition (248) determines a compatible AdS_3 Lorentz connection:

$$\mathbf{w}_{\mathrm{AdS}_3} = \frac{1}{2} \left(e^\rho L_+ + \frac{1}{4} e^{-\rho} L_- \right) \mathrm{d}x^+ - \frac{1}{2} \left(\frac{1}{4} e^{-\rho} L_+ + e^\rho L_- \right) \mathrm{d}x^- \tag{265}$$

The $\mathsf{SL}(2;\mathbb{R})$ CS connections corresponding to global AdS_3 are obtained from (250):

$$\begin{aligned} \mathbf{A}_{\mathrm{AdS}_3} &= \mathbf{e}_{\mathrm{AdS}_3} + \frac{1}{\ell} \mathbf{w}_{\mathrm{AdS}_3} = \quad L_0\, \mathrm{d}\rho + \left(e^\rho L_+ + \frac{1}{4} e^{-\rho} L_- \right) \mathrm{d}x^+ \\ \bar{\mathbf{A}}_{\mathrm{AdS}_3} &= \mathbf{e}_{\mathrm{AdS}_3} - \frac{1}{\ell} \mathbf{w}_{\mathrm{AdS}_3} = -L_0\, \mathrm{d}\rho - \left(\frac{1}{4} e^{-\rho} L_+ + e^\rho L_- \right) \mathrm{d}x^- \end{aligned} \tag{266}$$

Note that $\mathbf{A}^{\mathrm{AdS}_3}_- = 0 = \bar{\mathbf{A}}^{\mathrm{AdS}_3}_+$.

[12] One thinks of j_0, j_1 as the generators corresponding to τ, θ; then, L_+, L_- correspond to x^+, x^-.

11.4. Asymptotically AdS$_3$ Chern–Simons connections

A pair of $\mathsf{SL}(2;\mathbb{R})$ CS connections defines a 3-dim. line element via: (cf. (260))

$$\mathrm{d}s^2 = \frac{\ell^2}{2}\,\mathrm{tr}\big[(\mathbf{A}-\bar{\mathbf{A}})(\mathbf{A}-\bar{\mathbf{A}})\big] \tag{267}$$

It therefore makes sense to define an asymptotically AdS$_3$ pair of $\mathsf{SL}(2;\mathbb{R})$ CS connections (subject to Brown-Henneaux boundary conditions) analogously to the definition of an asymptotically AdS$_3$ metric (215). These aAdS$_3$ boundary conditions for $\mathbf{A}$, $\bar{\mathbf{A}}$ come in two pairs:

(i.) Chiral behaviour near $\partial\mathcal{M}$:

$$\mathbf{A}_-|_{\partial\mathcal{M}} = 0 \quad \bar{\mathbf{A}}_+|_{\partial\mathcal{M}} = 0 \tag{268}$$

(ii.) Asymptotically AdS$_3$ behaviour:

$$\mathbf{A}_+ - \mathbf{A}_+^{\mathrm{AdS}_3} \overset{r\to\infty}{\sim} O(1) \quad \bar{\mathbf{A}}_- - \bar{\mathbf{A}}_-^{\mathrm{AdS}_3} \overset{r\to\infty}{\sim} O(1) \tag{269}$$

By employing the EoM flatness constraint and using up gauge freedom, the most general form of an aAdS$_3$ $\mathsf{SL}(2;\mathbb{R})$ CS connection pair may be obtained.

First, gauge freedom may be used to factor out ρ-dependence from both connections,

$$\begin{aligned}\mathbf{A} &= b(\rho)^{-1}\mathrm{d}b(\rho) + b(\rho)^{-1}\mathbf{a}b(\rho)\\ \bar{\mathbf{A}} &= b(\rho)\mathrm{d}b(\rho)^{-1} + b(\rho)\bar{\mathbf{a}}b(\rho)^{-1}\end{aligned} \tag{270}$$

with $b(\rho) = e^{\rho L_0}$. The gauge transformation that brings $\mathbf{A}, \bar{\mathbf{A}}$ into the above form can always be found[13] [CFPT10]. For flat connections in such form, subject to (268), this implies

$$\begin{aligned}\mathbf{A}_- &\overset{\mathrm{EoM}}{=} 0 \quad \mathbf{A} \overset{\mathrm{EoM}}{=} b(\rho)^{-1}\mathrm{d}b(\rho) + b(\rho)^{-1}\mathbf{a}(x^+)b(\rho)\\ \bar{\mathbf{A}}_+ &\overset{\mathrm{EoM}}{=} 0 \quad \bar{\mathbf{A}} \overset{\mathrm{EoM}}{=} b(\rho)\mathrm{d}b(\rho)^{-1} + b(\rho)\bar{\mathbf{a}}(x^-)b(\rho)^{-1}\end{aligned} \tag{271}$$

This form will be called ”radial gauge”. Written in this radial gauge, the AdS$_3$ CS connections are:

$$\begin{aligned}\mathbf{A}_{\mathrm{AdS}_3} &= b(\rho)^{-1}\mathrm{d}b(\rho) + b(\rho)^{-1}\left(\ L_+ - \frac{1}{4}L_-\right)\mathrm{d}x^+\, b(\rho)\\ \bar{\mathbf{A}}_{\mathrm{AdS}_3} &= b(\rho)\mathrm{d}b(\rho)^{-1} + b(\rho)\quad\left(-\frac{1}{4}L_+ + L_-\right)\mathrm{d}x^-\, b(\rho)^{-1}\end{aligned} \tag{272}$$

[13]Under the assumption of analycity.

The fall-off conditions (269) constrain the reduced gauge connections $\mathbf{a}, \bar{\mathbf{a}}$ to,

$$\begin{aligned} \mathbf{a} &= (a^{(0)} L_0 + L_+ + a^{(-1)} L_-)\, \mathrm{d}x^+ \\ \bar{\mathbf{a}} &= (\bar{a}^{(0)} L_0 + \bar{a}^{(+1)} L_+ + L_-)\, \mathrm{d}x^- \end{aligned} \tag{273}$$

Using an infinitesimal gauge tranformation $\lambda \propto L_-$, the L_0 components of the reduced gauge connections can be set to zero. For later convenience, the remaining free components of $\mathbf{a}$, $\bar{\mathbf{a}}$ will be relabelled as $\begin{cases} a^{(-1)}(x^+) \equiv -\mathcal{L}(x^+) \\ \bar{a}^{(+1)}(x^-) \equiv -\bar{\mathcal{L}}(x^-) \end{cases}$

Then,

$$\begin{aligned} \mathbf{a} &= (\quad L_+ - \mathcal{L}(x^+) L_-)\, \mathrm{d}x^+ \\ \bar{\mathbf{a}} &= (-\bar{\mathcal{L}}(x^-) L_+ + L_-)\, \mathrm{d}x^- \end{aligned} \tag{274}$$

Such form of CS connections will be called "highest-weight gauge" (for $\mathbf{a}$)/"lowest-weight gauge" (for $\bar{\mathbf{a}}$), owing to representation-theoretic properties of L_-, L_+, cf. [FS03, Section 5.7] for more details.

Therefore, we obtain the most general pair of aAdS$_3$ $\mathsf{SL}(2;\mathbb{R})$ CS connections:

$$\begin{aligned} \mathbf{A}_{\mathrm{aAdS}_3} &= \quad L_0\, \mathrm{d}\rho + \left(\quad e^{\rho} L_+ - e^{-\rho} \mathcal{L}(x^+) L_-\right) \mathrm{d}x^+ \\ \bar{\mathbf{A}}_{\mathrm{aAdS}_3} &= -L_0\, \mathrm{d}\rho + \left(-e^{-\rho} \bar{\mathcal{L}}(x^-) L_+ + e^{\rho} L_-\right) \mathrm{d}x^- \end{aligned} \tag{275}$$

After applying boundary conditions and gauge-fixings, we see that the reduced physical phase space of the theory is parameterized by two functions:

$$\mathcal{S}_{\mathrm{phys}} = \{\mathbf{A}_{\mathrm{aAdS}_3}(\mathcal{L}(x^+)), \bar{\mathbf{A}}_{\mathrm{aAdS}_3}(\bar{\mathcal{L}}(x^-))\} \tag{276}$$

11.5. Global symmetries

A symmetry of a field theory preserves the shell $\mathcal{S}$. After gauge-fixing, one has to restrict the allowed symmetry transformations, such that $\mathcal{S}_{\text{phys}}$ is preserved. The procedure of determining the residual symmetries here follows [CFPT10], [CFP11], [CF23].

Recall the behaviour of a gauge connection under an infinitesimal gauge transformation specified by $\Lambda : \mathcal{M} \to \mathfrak{g}$:

$$\pounds_\Lambda \mathbf{A} = \mathrm{d}\Lambda + [\mathbf{A}, \Lambda] \tag{277}$$

The set of such transformation that preserve the form of the aAdS_3 CS connections (271), (274) are called the global symmetries. These will map physical solutions of the theory to other physical solutions.

Radial gauge (and chirality) preserving Λ's have to satisfy:

$$\pounds_\Lambda(b^{-1}\partial_\rho b) \stackrel{!}{=} b^{-1}\partial_\rho b, \quad \pounds_\Lambda(b^{-1}\mathbf{a}(x^+)b) \stackrel{!}{=} b^{-1}\widetilde{\mathbf{a}}(x^+)b \tag{278}$$

The infinitesimal transformation parameters Λ, $\bar{\Lambda}$ that preserve chirality and the radial gauge (271) are of the form:

$$\Lambda = b^{-1}\lambda(x^+)b \quad \bar{\Lambda} = b\bar{\lambda}(x^-)b^{-1} \tag{279}$$

In fact, the reduced transformation parameters λ, $\bar{\lambda}$ may be viewed as acting on the reduced connections $\mathbf{a}$, $\bar{\mathbf{a}}$:

$$\pounds_\lambda \mathbf{a} \equiv \partial\lambda + [\mathbf{a}, \lambda] \quad \pounds_{\bar{\lambda}} \bar{\mathbf{a}} \equiv \bar{\partial}\bar{\lambda} + [\bar{\mathbf{a}}, \bar{\lambda}] \tag{280}$$

As $\mathbf{a}$, $\bar{\mathbf{a}}$ contain all of the dynamical content and different $\mathbf{a}$'s, $\bar{\mathbf{a}}$'s correspond to different physical solutions, λ, $\bar{\lambda}$ that do not vanish at $\partial\mathcal{M}$ generate global symmetries of the theory.

Highest-weight (lowest-weight) gauge conditions (274) may be used to further reduce doF contained in λ ($\bar{\lambda}$). Gauge-preserving λ's have to satisfy:

$$\mathbf{a} = a(x^+)\,\mathrm{d}x^+ = (L_+ - \mathcal{L}(x^+)L_-)\,\mathrm{d}x^+$$
$$a \longmapsto a + \pounds_\lambda a \stackrel{!}{=} L_+ - \widetilde{\mathcal{L}}(x^+)L_-$$
$$\implies \pounds_\lambda a \stackrel{!}{=} -\Delta_\lambda \mathcal{L} L_-, \quad \Delta\mathcal{L} \coloneqq \widetilde{\mathcal{L}} - \mathcal{L}$$

In other words, global transformations of aAdS_3 connections fully reduce to transformations of the functions $\mathcal{L}$, $\bar{\mathcal{L}}$ that distuinguish one aAdS_3 solution from another. Expanding λ in the $\mathsf{sl}(2;\mathbb{R})$ basis as $\lambda = \lambda^0 L_0 + \lambda^+ L_+ + \lambda^- L_-$, the above constraints reduce allowed λ's to a single doF, λ^+, acting on $\mathcal{L}$ as:

$$\Delta_\lambda \mathcal{L} = -\frac{1}{2}\partial_+^3\lambda^+ + \partial_+\mathcal{L}\lambda^+ + 2\mathcal{L}\partial_+\lambda^+ \tag{281}$$

11.6. Global charges

Evaluating the conserved charges in the radial gauge gives:

$$H(\lambda) = \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\lambda \mathbf{a}] \quad \bar{H}(\bar{\lambda}) = -\frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\left[\bar{\lambda}\bar{\mathbf{a}}\right] \tag{282}$$

$$\begin{aligned} \pounds_\lambda H(\mu) &= \{H(\lambda), H(\mu)\} = H([\lambda,\mu]) + \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\mu \partial \lambda] \\ \pounds_{\bar{\lambda}} \bar{H}(\bar{\mu}) &= \{\bar{H}(\bar{\lambda}), \bar{H}(\bar{\mu})\} = \bar{H}([\bar{\lambda},\bar{\mu}]) + \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\left[\bar{\mu}\bar{\partial}\bar{\lambda}\right] \\ & \{H(\lambda), \bar{H}(\bar{\mu})\} = 0 \end{aligned} \tag{283}$$

The AdS_3 charge algebra reveals itself to be two copies of the $\hat{\mathfrak{sl}}_2$ affine Lie algebra; expanding into Fourier modes, with $\mathbf{a} = a(x^+)\,\mathrm{d}x^+$, $\bar{\mathbf{a}} = \bar{a}(x^-)\,\mathrm{d}x^-$, yields the familiar form

$$\begin{aligned} a(x^+) = \frac{1}{k}\sum_{m\in\mathbb{Z}} a^A{}_m e^{-imx^+} t_A \quad \bar{a}(x^-) = \frac{1}{k}\sum_{m\in\mathbb{Z}} \bar{a}^A{}_m e^{-imx^-} t_A \\ \implies a^A{}_m = H(t^A e^{imx^+}) \quad \bar{a}^A{}_m = H(t^A e^{imx^-}) \end{aligned} \tag{284}$$

$$\begin{aligned} &\{a^A{}_m, a^B{}_n\} = f^{AB}{}_C\, a^C{}_{m+n} + imk\gamma^{AB}\delta_{m+n,0} \\ &\{\bar{a}^A{}_m, \bar{a}^B{}_n\} = f^{AB}{}_C\, \bar{a}^C{}_{m+n} + imk\gamma^{AB}\delta_{m+n,0} \\ &\{a^A{}_m, \bar{a}^B{}_n\} = 0 \end{aligned} \tag{285}$$

Compared to CS theory's charge algebra containing one copy of the $\hat{\mathfrak{g}}$ affine Lie algebra, the AdS_3 charge algebra contains two copies of the $\hat{\mathfrak{sl}}_2$ affine Lie algebra. This is a hint that the boundary dynamics of the theory are controlled by a full WZW model; in section 12 this will be shown. In Section 12, it will also be shown that highest-weight gauge conditions further reduce this $\hat{\mathfrak{sl}}_2$ affine Lie algebra to a Virasoro algebra.

11.7. Aside: the aAdS$_3$ spectrum

Plugging in the general pair of asymptotically AdS$_3$ CS connections (275) into the line element formula (267), one obtains the Bañados metric:

$$\mathrm{d}s^2 = \ell^2 \left(\mathrm{d}\rho^2 + \mathcal{L}(x^+)(\mathrm{d}x^+)^2 + \bar{\mathcal{L}}(x^-)(\mathrm{d}x^-)^2 - (e^{2\rho} + e^{-2\rho}\mathcal{L}(x^+)\bar{\mathcal{L}}(x^-))\,\mathrm{d}x^+\,\mathrm{d}x^-\right) \tag{286}$$

The Bañados metric is the most general aAdS$_3$ metric satisfying the Brown-Henneaux boundary conditions; the whole aAdS$_3$ spectrum is parameterized by two functions, $\mathcal{L}(x^+)$ and $\bar{\mathcal{L}}(x^-)$.

BTZ subspectrum

The Bañados metric (286) contains a static subspectrum specified by two constants, $\mathcal{L}$ and $\bar{\mathcal{L}}$:

$$\partial_\tau\,\mathrm{d}s^2 \stackrel{!}{=} 0 \implies \begin{array}{l} \partial_\tau\mathcal{L}(x^+) = \partial_+\mathcal{L}(x^+) \stackrel{!}{=} 0 \\ \partial_\tau\bar{\mathcal{L}}(x^-) = \partial_-\bar{\mathcal{L}}(x^-) \stackrel{!}{=} 0 \end{array} \implies \begin{array}{l} \mathcal{L}(x^+) = \mathcal{L} = \text{const.} \\ \bar{\mathcal{L}}(x^-) = \bar{\mathcal{L}} = \text{const.} \end{array} \tag{287}$$

Note that, because the Bañados metric depends on a temporal coordinate only through lightcone coordinates, all stationary asymptotically AdS$_3$ spacetimes have to be static.

A Bañados metric with $\mathcal{L}(x^+) = \text{const.} = \bar{\mathcal{L}}(x^-)$ gives the BTZ metric:

$$\mathrm{d}s^2_{\mathrm{BTZ}} = \ell^2 \left(\mathrm{d}\rho^2 + \mathcal{L}(\mathrm{d}x^+)^2 + \bar{\mathcal{L}}(\mathrm{d}x^-)^2 - (e^{2\rho} + e^{-2\rho}\mathcal{L}\bar{\mathcal{L}})\,\mathrm{d}x^+\,\mathrm{d}x^-\right) \tag{288}$$

- The AdS$_3$ solution is recovered for $\mathcal{L} = -\frac{1}{4} = \bar{\mathcal{L}}$.

- For other $\mathcal{L} < 0$ and $\bar{\mathcal{L}} < 0$, the metric describes either a conical defect or a conical excess, corresponding to single particle solutions.

- For $\mathcal{L} \geq 0$ and $\bar{\mathcal{L}} \geq 0$, the metric describes a (2+1)-dim. Kerr-like black hole with
  $\begin{cases} \text{mass } M & \ell M = k(\mathcal{L} + \bar{\mathcal{L}}), \\ \text{spin } J & J = k(\mathcal{L} - \bar{\mathcal{L}}). \end{cases}$
  Its event horizon is located at $\rho = \rho+$, its inner horizon is located at $\rho = \rho_-$,

  $$\rho_\pm = \frac{1}{2}\log\left(\frac{k}{\ell}M\left(1 \pm \sqrt{1 - \left(\frac{J}{\ell M}\right)^2}\right)\right) \tag{289}$$

  For $\mathcal{L} = 0$ and/or $\bar{\mathcal{L}} = 0$ (equivalently: for $|J| = \ell M$), the black hole is extremal.

- For the remaining cases, the spacetime is pathological, containing closed timelike curves.

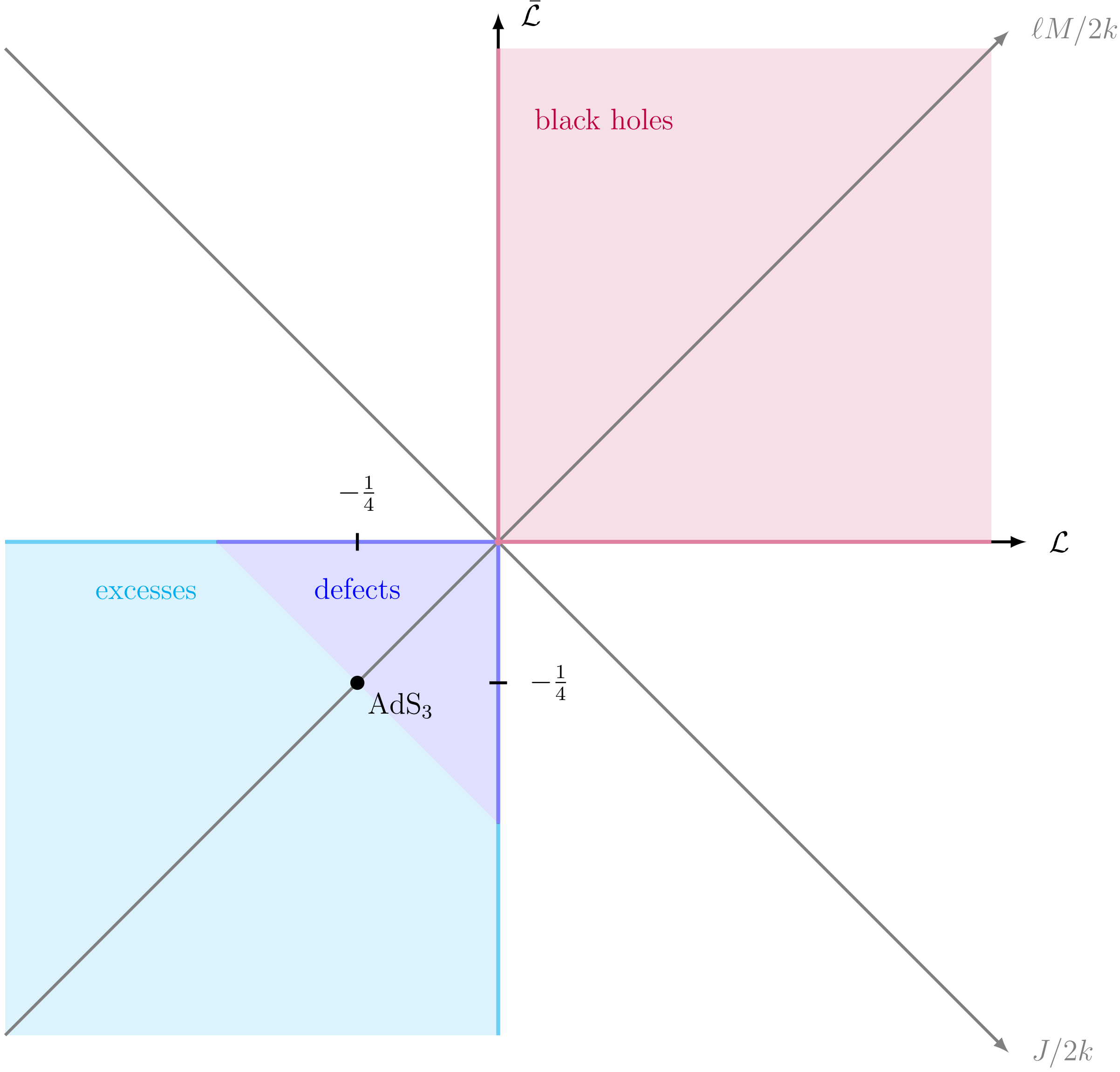


Figure 4: Full spectrum of BTZ metrics parameterized by $\mathcal{L}, \bar{\mathcal{L}}$.
Purple region corresponds to black hole solutions, blue region corresponds to conical defect solutions, cyan region corresponds to conical excess solutions. Unfilled regions correspond to pathological spacetimes containing closed time-like curves.

12. Reduction of $\Lambda < 0$ GR$_3$ to a conformal field theory

One is interested in describing the boundary dynamics of aAdS$_3$ spacetimes $\mathcal{M}$; such spacetimes share the topology of AdS$_3$, namely $\mathcal{M} = I \times \Sigma$ st. $\partial\mathcal{M} = \Sigma$, where Σ is the cylinder mantle located at spatial infinity, see Fig. 2. As part of the Brown-Henneaux boundary conditions, Σ comes naturally equipped with a Minkowski metric η st. $\mathrm{d}|_\Sigma = \partial + \bar{\partial}$. The relation between CS gauge field theory and WZW models may be exploited to reduce $\Lambda < 0$ GR$_3$ to a conformal field theory on the boundary.

12.1. Improved variational principle

The variation of the EC action contains an on-shell nonvanishing boundary term:

$$\begin{aligned} \delta S_{\mathrm{EC}}[A,\bar{A}] &= \delta S_{\mathrm{CS}}[A] - \delta S_{\mathrm{CS}}[\bar{A}] - \delta\left(\frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[\mathbf{A}\wedge\bar{\mathbf{A}}\big]\right) \\ &\overset{\mathrm{EoM}}{=} -\frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[\delta\mathbf{A}\wedge(\mathbf{A}-\bar{\mathbf{A}}) + \delta\bar{\mathbf{A}}\wedge(\mathbf{A}-\bar{\mathbf{A}})\big] \end{aligned} \tag{290}$$

Analogously to standard CS procedures, in order to implement boundary conditions, $\partial\mathcal{M}$ is equipped with a Minkowski metric, such that $\mathrm{d}|_{\partial\mathcal{M}} = \partial + \bar{\partial}$. Following [ABO03], [Kra03] an appropriate (Lorentz invariant) counterterm for GR$_3$ is:

$$S_{\mathrm{ct}}[A,\bar{A}] := \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[(\mathbf{A}_+ - \bar{\mathbf{A}}_+)\wedge(\mathbf{A}_- - \bar{\mathbf{A}}_-)\big] = \frac{1}{4\pi G_3}\int_{\partial\mathcal{M}} \mathrm{tr}\left[\frac{1}{\ell}\mathbf{e}_+\wedge\mathbf{e}_-\right] \tag{291}$$

$$\begin{aligned} \delta(S_{\mathrm{EC}} - S_{\mathrm{ct}})[A,\bar{A}] &\overset{\mathrm{EoM}}{=} -\frac{k}{2\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[\delta\mathbf{A}_-\wedge(\mathbf{A}_+ - \bar{\mathbf{A}}_+) + \delta\bar{\mathbf{A}}_+\wedge(\mathbf{A}_- - \bar{\mathbf{A}}_-)\big] \\ \delta(S_{\mathrm{EC}} + S_{\mathrm{ct}})[A,\bar{A}] &\overset{\mathrm{EoM}}{=} -\frac{k}{2\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[\delta\mathbf{A}_+\wedge(\mathbf{A}_- - \bar{\mathbf{A}}_-) + \delta\bar{\mathbf{A}}_-\wedge(\mathbf{A}_+ - \bar{\mathbf{A}}_+)\big] \end{aligned} \tag{292}$$

A well-defined variational principle may be obtained by imposing one of two possible boundary conditions: $\begin{cases} S_{\mathrm{EC}} - S_{\mathrm{ct}} \longrightarrow \mathbf{A}_-|_{\partial\mathcal{M}} \overset{!}{=} 0 \overset{!}{=} \bar{\mathbf{A}}_+|_{\partial\mathcal{M}} \\ S_{\mathrm{EC}} + S_{\mathrm{ct}} \longrightarrow \mathbf{A}_+|_{\partial\mathcal{M}} \overset{!}{=} 0 \overset{!}{=} \bar{\mathbf{A}}_-|_{\partial\mathcal{M}} \end{cases}$

Since for aAdS$_3$ connections, $\mathbf{A}_+$ and $\bar{\mathbf{A}}_-$ are the dynamical parts, and $\mathbf{A}_-^{\mathrm{AdS}_3} = 0 = \bar{\mathbf{A}}_+^{\mathrm{AdS}_3}$, one wants to fix $\mathbf{A}_-|_{\partial\mathcal{M}}$ and $\bar{\mathbf{A}}_+|_{\partial\mathcal{M}}$. Therefore, for describing aAdS$_3$ spacetimes, the appropriate improved action and boundary condition is:

$$S[A,\bar{A}] := S_{\mathrm{EC}}[A,\bar{A}] - S_{\mathrm{ct}}[A,\bar{A}] \qquad \mathbf{A}_-|_{\partial\mathcal{M}} \overset{!}{=} 0 \overset{!}{=} \bar{\mathbf{A}}_+|_{\partial\mathcal{M}} \tag{293}$$

12.2. Reduction to an $\mathsf{SL}(2;\mathbb{R})$ WZW model

The improved action (293) may be rewritten as,

$$S[A,\bar{A}] = S_{\mathrm{CS}}[A] - S_{\partial}[A] - S_{\mathrm{CS}}[\bar{A}] - S_{\partial}[\bar{A}] - \frac{k}{2\pi}\int_{\Sigma}\mathrm{tr}\big[\mathbf{A}_{-}\wedge\bar{\mathbf{A}}_{+}\big] \tag{294}$$

The equations of motion, a flatness constraint for $\mathbf{A}$ and $\bar{\mathbf{A}}$ are solved by pure gauge connections:

$$\begin{aligned}\mathbf{F}=0 \implies \mathbf{A} = U^{-1}\mathrm{d}U\\ \bar{\mathbf{F}}=0 \implies \bar{\mathbf{A}} = \bar{U}^{-1}\mathrm{d}\bar{U}\end{aligned} \tag{295}$$

Plugging a pure gauge connection into the improved EC action results in a WZW action for a composite group element:

$$\begin{aligned}S_{\mathrm{CS}}[A=U^{-1}\mathrm{d}U] &= -\frac{k}{12\pi}\int_{\mathcal{M}}\mathrm{tr}\big[(U^{-1}\mathrm{d}U)^{\wedge 3}\big] = -kS_{\mathrm{WZ}}[U]\\ S_{\partial}[A=U^{-1}\mathrm{d}U] &= \frac{k}{4\pi}\int_{\partial\mathcal{M}}\mathrm{tr}\big[U^{-1}\partial U\wedge U^{-1}\bar{\partial}U\big] = S_0[U]\end{aligned} \tag{296}$$

$$\begin{aligned}&S[A=U^{-1}\mathrm{d}U,\bar{A}=\bar{U}\mathrm{d}\bar{U}]\\ &= -kS_{\mathrm{WZ}}[U] - S_0[U] + kS_{\mathrm{WZ}}[\bar{U}] - S_0[\bar{U}] - \frac{k}{2\pi}\int_{\Sigma}[U^{-1}\bar{\partial}U\wedge\bar{U}^{-1}\partial\bar{U}]\\ &= -S_{\mathrm{WZW}}[U] - S_{\mathrm{WZW}}[\bar{U}^{-1}] - \frac{k}{2\pi}\int_{\Sigma}[U^{-1}\bar{\partial}U\wedge\bar{U}^{-1}\partial\bar{U}]\\ &\overset{\mathrm{P-W}}{=} -S_{\mathrm{WZW}}[U\bar{U}^{-1}]\end{aligned} \tag{297}$$

The derivation of the $\Lambda<0$ $\mathrm{GR}_3 \to \mathsf{SL}(2;\mathbb{R})$ WZW reduction seems done. However, at this point, the first set of aAdS_3 boundary conditions for the CS connections may be applied. Recall the identity used to define gauged WZW models from Sec. 8,

$$\begin{aligned}&S_{\mathrm{WZW}}[hg\tilde{h}^{-1}] = S_{\mathrm{WZW}}[g] + S_{\mathrm{WZW}}[h\tilde{h}^{-1}]\\ &\quad+\frac{k}{2\pi}\int_{\Sigma}\mathrm{tr}\Big[-h^{-1}\bar{\partial}h\wedge\partial gg^{-1} + g^{-1}\bar{\partial}g\wedge\tilde{h}^{-1}\partial\tilde{h} + h^{-1}\bar{\partial}h\wedge g\tilde{h}^{-1}\partial\tilde{h}g^{-1} - h^{-1}\bar{\partial}h\wedge\tilde{h}^{-1}\partial\tilde{h}\Big]\end{aligned}$$

Define $G \coloneqq U^{-1}\bar{U}$, set $\begin{cases} h=U, & \mathbf{A}_{-}=U^{-1}\bar{\partial}U\\ g=G & \\ \tilde{h}=\bar{U}, & \bar{\mathbf{A}}_{+}=\bar{U}^{-1}\partial\bar{U}\end{cases}$; using $S_{\mathrm{WZW}}[UG\bar{U}^{-1}=\mathbb{1}]=0$:

$$\begin{aligned}&-S_{\mathrm{WZW}}[U\bar{U}^{-1}] = S_{\mathrm{WZW}}[G]\\ &\qquad+\frac{k}{2\pi}\int_{\Sigma}\mathrm{tr}\big[-\mathbf{A}_{-}\wedge\partial GG^{-1} + G^{-1}\bar{\partial}G\wedge\bar{\mathbf{A}}_{+} + \mathbf{A}_{-}\wedge G\bar{\mathbf{A}}_{+}G^{-1} - \mathbf{A}_{-}\wedge\bar{\mathbf{A}}_{+}\big]\end{aligned}$$

Recalling the first set of aAdS$_3$ boundary conditions (268), $\begin{cases} \mathbf{A}_-|_{\partial\mathcal{M}} = 0 \\ \bar{\mathbf{A}}_+|_{\partial\mathcal{M}} = 0 \end{cases}$

$$S[A = U^{-1}\mathrm{d}U, \bar{A} = \bar{U}\mathrm{d}\bar{U}] \overset{\mathrm{BC}}{=} S_{\mathrm{WZW}}[G = U^{-1}\bar{U}] \tag{298}$$

The radial gauge (271) for $\mathbf{A}$, $\bar{\mathbf{A}}$ translates into the following:

$$U = ub,\ \mathbf{a} = u^{-1}\partial u \qquad \bar{U} = \bar{u}b^{-1},\ \bar{\mathbf{a}} = \bar{\partial}\mathrm{d}_\Sigma\bar{u} \tag{299}$$

$$G = b^{-1}gb^{-1},\ g = u^{-1}\bar{u} \tag{300}$$

$$\begin{aligned} S_0[G] &= \frac{k}{4\pi}\int_\Sigma \mathrm{tr}\big[G^{-1}\partial G \wedge G^{-1}\bar{\partial}G\big] \\ &= \frac{k}{4\pi}\int_\Sigma \mathrm{tr}\big[bg^{-1}b\partial(b^{-1}gb^{-1}) \wedge bg^{-1}b\bar{\partial}(b^{-1}gb^{-1})\big] \\ &\overset{\partial b=0=\bar{\partial}b}{=} \frac{k}{4\pi}\int_\Sigma \mathrm{tr}\big[g^{-1}\partial g \wedge g^{-1}\bar{\partial}g\big] = S_0[g] \end{aligned} \tag{301}$$

Viewing $G = b^{-1}gb^{-1} : \mathcal{M} \to \mathsf{SL}(2;\mathbb{R})$ as an extension of the dynamical $\mathsf{SL}(2;\mathbb{R})$-valued field g on Σ, one has:

$$S[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U, \bar{A}_\Sigma = \bar{U}\mathrm{d}_\Sigma\bar{U}] = S_{\mathrm{WZW}}[g] \tag{302}$$

Solving the WZW equations of motion gives results compatible with on-shell aAdS$_3$ CS connections:

$$g(x^+, x^-) \overset{\mathrm{EoM}}{=} u^{-1}(x^+)\bar{u}(x^-) \tag{303}$$

$$\begin{aligned} \mathbf{a}_+ &\overset{\mathrm{EoM}}{=} \mathbf{a}_+(x^+) & \mathbf{a}_- &\overset{\mathrm{EoM}}{=} 0 \\ \bar{\mathbf{a}}_+ &\overset{\mathrm{EoM}}{=} 0 & \bar{\mathbf{a}}_- &\overset{\mathrm{EoM}}{=} \bar{\mathbf{a}}_-(x^-) \end{aligned} \tag{304}$$

The reduced CS connections $\mathbf{a}$, $\bar{\mathbf{a}}$ translate on-shell into WZW currents J, $\bar{J}$:

$$\begin{aligned} \mathbf{J} &:= -\partial g g^{-1} = u^{-1}\partial u - g\bar{u}^{-1}\partial\bar{u}g^{-1} \overset{\mathrm{EoM}}{=} \mathbf{a} \\ \bar{\mathbf{J}} &:= g^{-1}\bar{\partial}g = -g^{-1}u^{-1}\bar{\partial}ug + \bar{u}^{-1}\bar{\partial}\bar{u} \overset{\mathrm{EoM}}{=} \bar{\mathbf{a}} \end{aligned} \tag{305}$$

The gravitational global charges translate on-shell into (3-dim.) WZW charges:

$$\begin{aligned} H(\lambda) &= \frac{k}{2\pi}\int_S \mathrm{tr}[\lambda\mathbf{a}] \overset{\mathrm{EoM}}{=} \frac{k}{2\pi}\int_S \mathrm{tr}[\lambda\mathbf{J}] \\ \bar{H}(\bar{\lambda}) &= -\frac{k}{2\pi}\int_S \mathrm{tr}\big[\bar{\lambda}\bar{\mathbf{a}}\big] \overset{\mathrm{EoM}}{=} -\frac{k}{2\pi}\int_S \mathrm{tr}\big[\bar{\lambda}\bar{\mathbf{J}}\big] \end{aligned} \tag{306}$$

Note on a semiclassical path-integral reduction

Analogously to the CS/WZW correspondence, the EC action can be reduced to the WZW action by using the geometry of $\mathcal{M} = I \times \Sigma$: split into transverse (I) and longitudinal parts (Σ),

$$\int_{\mathcal{M}} = \int_I \int_\Sigma \qquad \mathrm{d} = \mathrm{d}_I + \mathrm{d}_\Sigma \qquad \mathbf{A} = \mathbf{A}_I + \mathbf{A}_\Sigma \tag{307}$$

For aAdS$_3$ spacetimes, $\partial\Sigma = \varnothing$, therefore the CS action takes on a canonical form,

$$S_{\mathrm{CS}}[A = A_I + A_\Sigma] = \frac{k}{4\pi}\int_I\int_\Sigma \mathrm{tr}[\mathbf{A}_\Sigma \wedge \mathrm{d}_I \mathbf{A}_\Sigma + 2\mathbf{A}_I \wedge \mathbf{F}_\Sigma] \tag{308}$$

Once again, $\mathbf{A}_I$ enforces a flatness constraint for $\mathbf{A}_\Sigma$,

$$S_{\mathrm{CS}}[A = A_I + A_\Sigma] = \frac{k}{4\pi}\int_I\int_\Sigma \mathrm{tr}[\mathbf{A}_\Sigma \wedge \mathrm{d}_I \mathbf{A}_\Sigma + 2\mathbf{A}_I \wedge \mathbf{F}_\Sigma], \qquad \begin{cases} \mathbf{A}_\Sigma \overset{\text{flat}}{=} U^{-1}\mathrm{d}_\Sigma U \\ \bar{\mathbf{A}}_\Sigma \overset{\text{flat}}{=} \bar{U}^{-1}\mathrm{d}_\Sigma \bar{U} \end{cases}$$

upon its solution, we obtain the familiar terms,

$$\begin{aligned} S_{\mathrm{CS}}[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] &= -kS_{\mathrm{WZ}}[U] \\ S_\partial[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U] &= \frac{k}{4\pi}\int_\Sigma \mathrm{tr}\big[U^{-1}\partial U \wedge U^{-1}\bar{\partial} U\big] = S_0[U] \end{aligned}$$

Putting the pieces together gives,

$$\begin{aligned} &S[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U, \bar{A}_\Sigma = \bar{U}\mathrm{d}_\Sigma \bar{U}] \\ &= -kS_{\mathrm{WZ}}[U] - S_0[U] + kS_{\mathrm{WZ}}[\bar{U}] - S_0[\bar{U}] - \frac{k}{2\pi}\int_\Sigma [U^{-1}\bar{\partial} U \wedge \bar{U}^{-1}\partial \bar{U}] \\ &= -S_{\mathrm{WZW}}[U] - S_{\mathrm{WZW}}[\bar{U}^{-1}] - \frac{k}{2\pi}\int_\Sigma [U^{-1}\bar{\partial} U \wedge \bar{U}^{-1}\partial\bar{U}] \\ &\overset{\text{P–W}}{=} -S_{\mathrm{WZW}}[U\bar{U}^{-1}] \end{aligned} \tag{309}$$

Now perform the same P–W moves as before, arriving at the desired form of the action,

$$\begin{aligned} -S_{\mathrm{WZW}}[U\bar{U}^{-1}] &= S_{\mathrm{WZW}}[G] \\ &\quad + \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\big[-\mathbf{A}_- \wedge \partial G G^{-1} + G^{-1}\bar{\partial} G \wedge \bar{\mathbf{A}}_+ + \mathbf{A}_- \wedge G\bar{\mathbf{A}}_+ G^{-1} - \mathbf{A}_- \wedge \bar{\mathbf{A}}_+\big] \end{aligned}$$

On a classical level, boundary conditions and gauge-fixing may be implemented etc., giving the same results as before.

Since $(\mathbf{e}, \mathbf{w}) \leftrightarrow (\mathbf{A}, \bar{\mathbf{A}})$ is a linear field redefinition, we may consider the semiclassical path integral:

$$\mathcal{Z}_{\mathrm{EC}} = \int [\mathrm{D}A][\mathrm{D}\bar{A}] e^{iS[A,\bar{A}]} \tag{310}$$

The Lagrange multipliers $\mathbf{A}_I$, $\bar{\mathbf{A}}_I$ can be integrated out at the same time, giving

$$\begin{aligned}
\mathcal{Z}_{\mathrm{EC}} &= \int [\mathrm{D}A][\mathrm{D}\bar{A}] e^{iS[A,\bar{A}]} \\
&= \int [\mathrm{D}A_I][\mathrm{D}A_\Sigma] \exp\left(i\frac{k}{4\pi} \int_I \int_\Sigma \mathrm{tr}[\mathbf{A}_\Sigma \wedge \mathrm{d}_I \mathbf{A}_\Sigma] - i\frac{k}{4\pi} \int_\Sigma \mathrm{tr}[\mathbf{A}_+ \wedge \mathbf{A}_-] \right) \\
&\qquad\qquad \times \exp\left(i\frac{k}{2\pi} \int_I \int_\Sigma \mathrm{tr}[\mathbf{A}_I \wedge \mathbf{F}_\Sigma] \right) \\
&\quad \int [\mathrm{D}\bar{A}_I][\mathrm{D}\bar{A}_\Sigma] \exp\left(-i\frac{k}{4\pi} \int_I \int_\Sigma \mathrm{tr}\big[\bar{\mathbf{A}}_\Sigma \wedge \mathrm{d}_I \bar{\mathbf{A}}_\Sigma\big] - i\frac{k}{4\pi} \int_\Sigma \mathrm{tr}\big[\bar{\mathbf{A}}_+ \wedge \bar{\mathbf{A}}_-\big] \right) \\
&\qquad\qquad \times \exp\left(-i\frac{k}{2\pi} \int_I \int_\Sigma \mathrm{tr}\big[\bar{\mathbf{A}}_I \wedge \bar{\mathbf{F}}_\Sigma\big] \right) \\
&\qquad\qquad \times \exp\left(-i\frac{k}{2\pi} \int_\Sigma \mathrm{tr}\big[\mathbf{A}_- \wedge \bar{\mathbf{A}}_+\big] \right) \\
&= \int [\mathrm{D}A_\Sigma] \delta(\mathbf{F}_\Sigma) [\mathrm{D}\bar{A}_\Sigma] \delta(\bar{\mathbf{F}}_\Sigma) e^{-iS_{\mathrm{WZW}}[U] - iS_{\mathrm{WZW}}[\bar{U}^{-1}] - i\frac{k}{2\pi}\int_\Sigma [U^{-1}\bar{\partial}U \wedge \bar{U}^{-1}\partial\bar{U}]}
\end{aligned}$$

The P–W identites hold inside path integrals, [Gaw97]

$$\begin{aligned}
&\overset{\text{P–W}}{=} \int [\mathrm{D}U][\mathrm{D}\bar{U}] - S_{\mathrm{WZW}}[U\bar{U}^{-1}] \\
&\overset{\text{P–W}}{=} \int [\mathrm{D}G][\mathrm{D}A_-][\mathrm{D}\bar{A}_+] \exp\left(iS_{\mathrm{WZW}}[G]\right) \\
&\qquad \times \exp\left(i\frac{k}{2\pi} \int_\Sigma \mathrm{tr}\big[-\mathbf{A}_- \wedge \partial G G^{-1} + G^{-1}\bar{\partial}G \wedge \bar{\mathbf{A}}_+ + \mathbf{A}_- \wedge G\bar{\mathbf{A}}_+ G^{-1} - \mathbf{A}_- \wedge \bar{\mathbf{A}}_+\big] \right)
\end{aligned}$$

Now, however, we may not simply set $\mathbf{A}_-|_\Sigma = 0 = \bar{\mathbf{A}}_+|_\Sigma$. Without further manipulation, we have arrived at the so-called $\mathfrak{G}/\mathfrak{G}$ partition function [Wit92]. The nondynamical componennts $\mathbf{A}_-$ and $\bar{\mathbf{A}}_+$, may be used to implement the reduction to Liouville theory by adapting the procedure outlined in [OV98].

12.3. Further reduction to Liouville theory

The aAdS$_3$ boundary conditions, as well as the gauge-fixing, came in two pairs. So far, only the first pair has been used up to reduce $\Lambda < 0$ GR$_3$ to $\mathsf{SL}(2;\mathbb{R})$ WZW. The second set of boundary conditions (together with the highest-weight gauge) (269), (274) translates straightforwardly into [FWB+89]-type WZW current constraint that reduces the $\mathsf{SL}(2;\mathbb{R})$ WZW model to a (classical) Liouville conformal field theory.

The gravitational boundary conditions are expressed using the $\mathsf{sl}(2;\mathbb{R})$ spin basis $\{L_+, L_0, L_-\}$, whereas the reduction to Liouville is done using the Chevalley basis $\{H, E_+, E_-\}$,

$$L_+ = E_- \quad L_0 = \frac{1}{2}H \quad L_- = -E_+ \quad \mathrm{tr}[L_m L_n] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{311}$$

Using these relations, we translate the highest-weight gauge-fixing into current constraints:

$$\begin{aligned} \mathbf{a}^+ &= -\frac{2\pi}{k}\mathbf{J}(L_-), \quad \mathbf{a}^+ \stackrel{!}{=} \mathrm{d}x^+ \\ \bar{\mathbf{a}}^- &= \frac{2\pi}{k}\bar{\mathbf{J}}(L_+), \quad \bar{\mathbf{a}}^- \stackrel{!}{=} \mathrm{d}x^- \end{aligned} \tag{312}$$

$$\begin{aligned} -\frac{2\pi}{k}\mathbf{J}(L_-) &= \frac{2\pi}{k}\mathbf{J}(E_+) \stackrel{!}{=} \mathrm{d}x^+ \\ \frac{2\pi}{k}\bar{\mathbf{J}}(L_+) &= \frac{2\pi}{k}\bar{\mathbf{J}}(E_-) \stackrel{!}{=} \mathrm{d}x^- \end{aligned} \tag{313}$$

Recognizing (188) with $\mu = -1$, $\nu = -1$; the WZW action reduces to the following Liouville action:

$$S_{\mathrm{WZW}}[g = e^{XE_+}e^{\frac{1}{2}\Phi H}e^{YE_-}] \stackrel{\text{constr.}}{=} \frac{k}{4\pi}\int_\Sigma \mathrm{d}x^+ \wedge \mathrm{d}x^- \left(\frac{1}{2}\partial_+\Phi\partial_+\Phi - e^{\Phi}\right) \tag{314}$$

Comparing with the standard form of the Liouville action,

$$S_L[\Phi] = \frac{c_L}{24\pi}\int_\Sigma \mathrm{d}x^+ \wedge \mathrm{d}x^- \left(\frac{1}{2}\partial_+\Phi\partial_-\Phi + Me^{\Phi}\right) \tag{315}$$

aAdS$_3$ general relativity is classically reduces to a Liouville theory with:

$$\text{central charge } c = 6k = \frac{3\ell}{2G_3} \quad \text{"cosmological constant" } M = 1 \tag{316}$$

$c = \frac{3\ell}{2G_3}$ is called the Brown-Henneaux central charge [BH86].

12.4. Virasoro algebra

$\Lambda < 0$ GR_3 reduces under (269), (274) to a Liouville theory. The charge algebra reduces to a Virasoro algebra, the symmetry algebra of Liouville field theory. We show this fact only for the unbarred sector; the calculation is analogous for the barred sector.

Recall the allowed transformation (281):

$$\begin{aligned}
\{H(\lambda), H(\mu)\} &= \pounds_\lambda H(\mu) \\
&= \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\mu \pounds_\lambda \mathbf{a}] = \frac{k}{2\pi}\oint \mathrm{d}x^+\,\mathrm{tr}[\mu \pounds_\lambda a] \\
&= -\frac{k}{2\pi}\oint \mathrm{d}x^+\,\mathrm{tr}\big[(\mu^0 L_0 + \mu^+ L_+ + \mu^- L_-)\Delta_\lambda \mathcal{L} L_-\big] \\
&= \frac{k}{2\pi}\oint \mathrm{d}x^+\,\mathrm{tr}\big[\mu^+ \Delta_\lambda \mathcal{L}\big] \\
&= \frac{k}{2\pi}\oint \mathrm{d}x^+\,\mu^+\Big(-\frac{1}{2}\partial_+^3\lambda^+ + \partial_+\mathcal{L}\lambda^+ + 2\mathcal{L}\partial_+\lambda^+\Big) \\
&= \frac{k}{2\pi}\oint \mathrm{d}x^+\,(\mu^+\partial_+\lambda^+ - \lambda^+\partial_+\mu^+)\mathcal{L} - \frac{k}{4\pi}\oint \mathrm{d}x^+\,\mu^+\partial_+^3\lambda^+
\end{aligned}$$

Above is written the Poisson algebra of Virasoro charges,

$$H^{\mathsf{Vir}}(\lambda^+) \coloneqq \frac{k}{2\pi}\oint \mathrm{d}x^+\,\lambda^+\mathcal{L} \tag{317}$$

viewing $\Delta_\lambda \mathcal{L} \equiv \pounds_{\lambda^+}\mathcal{L}$ as the transformation of $\mathcal{L}$ by λ^+,

$$\begin{aligned}
\{H^{\mathsf{Vir}}(\lambda^+), H^{\mathsf{Vir}}(\mu^+)\} \coloneqq \pounds_{\lambda^+} H^{\mathsf{Vir}}(\mu^+) &= \frac{k}{2\pi}\oint \mathrm{d}x^+\,\mu^+\pounds_{\lambda^+}\mathcal{L} \\
&= H^{\mathsf{Vir}}(\mu^+\partial_+\lambda^+ - \lambda^+\partial_+\mu^+) - \frac{k}{4\pi}\oint \mathrm{d}x^+\,\mu^+\partial_+^3\lambda^+
\end{aligned} \tag{318}$$

Expanding $\mathcal{L}(x^+)$ into modes yields the usual form of the Virasoro algebra:[14]

$$\begin{gathered}
\mathcal{L}(x^+) = \frac{1}{k}\sum_n \mathcal{L}_n e^{-inx^+} \qquad \mathcal{L}_m = H^{\mathsf{Vir}}(e^{imx^+}) \\
\{\mathcal{L}_m, \mathcal{L}_n\} = i(m-n)\mathcal{L}_{m+n} + im^3\frac{k}{2}\delta_{m+n,0}
\end{gathered} \tag{319}$$

where once again, the central charge of the theory is $c = 6k = \frac{3\ell}{2G_3}$.

[14] The ”wrong” sign, i instead of $-i$, may be attributed to the inherent 3-dimensionality of the theory, see (146).

Part IV.
Spin-N

13. Higher-spin AdS$_3$ gravity

13.1. Higher-spin action

Beginning with the coframe formulation of 3-dimensional general relativity, a linear redefinition of fields and a Lie algebra isomorphism $\mathsf{so}(2,1) \cong \mathsf{sl}(2;\mathbb{R})$ leads to the Chern–Simons reformulation. For the higher-spin generalization of $\Lambda < 0$ GR$_3$, the derivation from Part III is repeated with an arbitrary representation of $\{L_m\}$, satisfying:

$$\operatorname{tr}[j_2 j_2] = \operatorname{tr}[L_0 L_0] = {t^2}_0 \quad \eta_{ab} = \frac{1}{{t^2}_0} \operatorname{tr}[j_a j_b] \tag{320}$$

$$\begin{aligned} S_{\mathrm{EC}}[e,w] &= \frac{1}{8\pi G_3} \int_{\mathcal{M}} \left(\eta_{ab}\, e^a \wedge \mathrm{d}w^b + \frac{1}{2} \varepsilon_{abc} e^a \wedge w^b \wedge w^c - \frac{\Lambda}{6} \varepsilon_{abc}\, e^a \wedge e^b \wedge e^c \right) \\ &= \frac{1}{{t^2}_0} \frac{1}{8\pi G_3} \int_{\mathcal{M}} \operatorname{tr}\left[\mathbf{e} \wedge \mathrm{d}\mathbf{w} + \mathbf{e} \wedge \mathbf{w} \wedge \mathbf{w} - \frac{\Lambda}{3} \mathbf{e}^{\wedge 3} \right] \\ &\Big\downarrow \mathbf{e} = \frac{\ell}{2} \left(\mathbf{A} - \bar{\mathbf{A}} \right) \quad \mathbf{w} = \frac{1}{2} \left(\mathbf{A} + \bar{\mathbf{A}} \right) \\ &= S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \operatorname{tr}\left[\mathbf{A} \wedge \bar{\mathbf{A}} \right] \end{aligned}$$

The only difference lies in introducing a gravitational Chern–Simons coupling constant for arbitrary representation of $\{L_m\}$,

$$k := \frac{1}{{t^2}_0} \frac{\ell}{8G_3}$$

Now, for some simple Lie algebra $\mathfrak{g}$, take $\mathbf{A}, \bar{\mathbf{A}} \in \Omega^1(\mathcal{M}, \mathfrak{g})$. By choosing some principal embedding $\mathsf{sl}(2;\mathbb{R}) \hookrightarrow \mathfrak{g}$, we may define the higher-spin AdS$_3$ gravitational action as:

$$S_{\mathrm{hs}}[A, \bar{A}] := S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \operatorname{tr}\left[\mathbf{A} \wedge \bar{\mathbf{A}} \right] \tag{321}$$

For $\mathfrak{g} = \mathsf{sl}(N;\mathbb{R})$, the CS EoM $\Leftrightarrow$ flatness constraints,

$$\mathbf{F} = 0 = \bar{\mathbf{F}} \tag{322}$$

linearize to the spin-N Fronsdal equations in an AdS$_3$ background, see [CF23, Section 2] for the derivation and further context.

The principal embedding $\mathsf{sl}(2;\mathbb{R}) \hookrightarrow \mathsf{sl}(N;\mathbb{R})$ defines the basis elements $\{L_m\}$, corresponding to the familiar gravitational spin-2 sector, as well as the higher-spin basis elements $\{{W^s}_m\}$ described in Section 4, corresponding to higher-spin sectors for $s \in 3, \ldots, N$.

13.2. Infinitesimal symmetries of the hs-action

As in Section 11.2, the infinitesimal symmetries/conserved quantites of higher-spin AdS_3 gravity can be inferred to be equal to two copies of $\mathsf{SL}(N;\mathbb{R})$ CS infinitesimal symmetries,

$$\pounds_{\Lambda}\mathbf{A} = \mathrm{d}\Lambda + [\mathbf{A}, \Lambda] \quad \pounds_{\bar{\Lambda}}\bar{\mathbf{A}} = \mathrm{d}\bar{\Lambda} + [\bar{\mathbf{A}}, \bar{\Lambda}] \tag{323}$$

where $(\Lambda, \bar{\Lambda}) \in \mathsf{sl}(N;\mathbb{R}) \times \mathsf{sl}(N;\mathbb{R})$.

$$\begin{aligned} \boldsymbol{\vartheta}_{\text{hs}}[A, \bar{A}] &= \boldsymbol{\vartheta}_{\text{CS}}[A] - \boldsymbol{\vartheta}_{\text{CS}}[\bar{A}] \\ \mathbf{B}(\Lambda, \bar{\Lambda}) &= \mathbf{B}_{\text{CS}}(\Lambda)[A] - \mathbf{B}_{\text{CS}}(\bar{\Lambda})[\bar{A}] \end{aligned} \tag{324}$$

$$\implies \mathbf{J}_{\text{hs}}(\Lambda, \bar{\Lambda}) = \mathbf{J}_{\text{CS}}(\Lambda) - \mathbf{J}_{\text{CS}}(\bar{\Lambda}) \equiv \mathbf{J}(\Lambda) + \bar{\mathbf{J}}(\bar{\Lambda}) \tag{325}$$

The Noether current and charge again split into an unbarred and barred conserved current and charge,

$$\begin{aligned} \mathbf{J}(\Lambda) &= \frac{k}{2\pi}\operatorname{tr}[\Lambda\mathbf{A}] & \bar{\mathbf{J}}(\bar{\Lambda}) &= -\frac{k}{2\pi}\operatorname{tr}\left[\bar{\Lambda}\bar{\mathbf{A}}\right] \\ H(\Lambda) &= \frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}[\Lambda\mathbf{A}] & \bar{H}(\bar{\Lambda}) &= -\frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}\left[\bar{\Lambda}\bar{\mathbf{A}}\right] \end{aligned} \tag{326}$$

Eventually, a radial gauge will be imposed,

$$\begin{aligned} \mathbf{A} &= b(\rho)^{-1}\mathrm{d}b(\rho) + b(\rho)^{-1}\mathbf{a}b(\rho) \\ \bar{\mathbf{A}} &= b(\rho)\mathrm{d}b(\rho)^{-1} + b(\rho)\bar{\mathbf{a}}b(\rho)^{-1} \end{aligned} \tag{327}$$

Chirality and radial gauge-preserving gauge transformations also have a factorized ρ-dependence,

$$\Lambda = b^{-1}\lambda(x^{+})b \quad \bar{\Lambda} = b\bar{\lambda}(x^{-})b^{-1} \tag{328}$$

Evaluating the conserved charges in the radial gauge gives:

$$H(\lambda) = \frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}[\lambda\mathbf{a}] \quad \bar{H}(\bar{\lambda}) = -\frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}\left[\bar{\lambda}\bar{\mathbf{a}}\right] \tag{329}$$

$$\begin{aligned} \pounds_{\lambda}H(\mu) = \{H(\lambda), H(\mu)\} &= H([\lambda, \mu]) + \frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}[\mu\partial\lambda] \\ \pounds_{\bar{\lambda}}\bar{H}(\bar{\mu}) = \{\bar{H}(\bar{\lambda}), \bar{H}(\bar{\mu})\} &= \bar{H}([\bar{\lambda}, \bar{\mu}]) + \frac{k}{2\pi}\int_{\mathcal{S}}\operatorname{tr}\left[\bar{\mu}\bar{\partial}\bar{\lambda}\right] \\ \{H(\lambda), \bar{H}(\bar{\mu})\} &= 0 \end{aligned} \tag{330}$$

The higher-spin aAdS_3 charge algebra reveals itself to be the $\hat{\mathsf{sl}}_N$ affine Lie algebra; in terms of the Fourier modes, with $\mathbf{a} = a(x^{+})\,\mathrm{d}x^{+}$, $\bar{\mathbf{a}} = \bar{a}(x^{-})\,\mathrm{d}x^{-}$,

$$\begin{aligned} &\{a^{A}{}_{m}, a^{B}{}_{n}\} = f^{AB}{}_{C}\,a^{C}{}_{m+n} + imk\gamma^{AB}\delta_{m+n,0} \\ &\{\bar{a}^{A}{}_{m}, \bar{a}^{B}{}_{n}\} = f^{AB}{}_{C}\,\bar{a}^{C}{}_{m+n} + imk\gamma^{AB}\delta_{m+n,0} \\ &\{a^{A}{}_{m}, \bar{a}^{B}{}_{n}\} = 0 \end{aligned} \tag{331}$$

13.3. Asymptotically AdS$_3$ boundary conditions

Without requiring too much thought, the aAdS$_3$ boundary conditions (268), (269) generalize to the higher-spin case:

(i.) Chiral behaviour near $\partial\mathcal{M}$:

$$\mathbf{A}_-|_{\partial\mathcal{M}} = 0 \quad \bar{\mathbf{A}}_+|_{\partial\mathcal{M}} = 0 \tag{332}$$

(ii.) Asymptotically AdS$_3$ behaviour:

$$\mathbf{A}_+ - \mathbf{A}_+^{\mathrm{AdS}_3} \overset{r\to\infty}{\sim} O(1) \quad \bar{\mathbf{A}}_- - \bar{\mathbf{A}}_-^{\mathrm{AdS}_3} \overset{r\to\infty}{\sim} O(1) \tag{333}$$

Above, the AdS$_3$ CS connections are now viewed as distinguished $\mathsf{SL}(N;\mathbb{R})$ connections,

$$\begin{aligned} \mathbf{A}_{\mathrm{AdS}_3} &= \quad L_0\,\mathrm{d}\rho + \left(e^{\rho}L_+ + \frac{1}{4}e^{-\rho}L_-\right)\mathrm{d}x^+ \\ \bar{\mathbf{A}}_{\mathrm{AdS}_3} &= -L_0\,\mathrm{d}\rho - \left(\frac{1}{4}e^{-\rho}L_+ + e^{\rho}L_-\right)\mathrm{d}x^- \end{aligned} \tag{334}$$

Anticipating the on-shell nonvanishing boundary term in $\delta S_{\mathrm{hs}}[A,\bar{A}]$, a suitable counterterm is:

$$S_{\mathrm{ct}}[A,\bar{A}] \coloneqq \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{tr}\big[(\mathbf{A}_+ - \bar{\mathbf{A}}_+)\wedge(\mathbf{A}_- - \bar{\mathbf{A}}_-)\big] \tag{335}$$

Then, the total action of the higher-spin AdS$_3$ gravity theory is,

$$S[A,\bar{A}] \coloneqq S_{\mathrm{hs}}[A,\bar{A}] - S_{\mathrm{ct}}[A,\bar{A}] \tag{336}$$

Chiral boundary conditions ensure a well-defined variational principle, $\delta S[A,\bar{A}] \overset{\mathrm{EoM}}{=} 0$.

14. Reduction of higher-spin AdS$_3$ gravity to a conformal field theory

14.1. Reduction to an $\mathsf{SL}(N;\mathbb{R})$ WZW model

Starting from the total higher-spin action, $S[A,\bar{A}] := S_{\text{hs}}[A,\bar{A}] - S_{\text{ct}}[A,\bar{A}]$, one may repeat the exact steps from Section 12.2 to reduce it to an $\mathsf{SL}(N;\mathbb{R})$ WZW action:

- Transform to the radial gauge:

$$\begin{aligned}\mathbf{A} &= b(\rho)^{-1}\mathrm{d}b(\rho) + b(\rho)^{-1}\mathbf{a}b(\rho)\\ \bar{\mathbf{A}} &= b(\rho)\mathrm{d}b(\rho)^{-1} + b(\rho)\bar{\mathbf{a}}b(\rho)^{-1}\end{aligned} \tag{337}$$

  Such a gauge transformation can always be found.

- The first set of boundary conditions (332) implies

$$\mathbf{a} \overset{\text{EoM}}{=} a(x^+)\,\mathrm{d}x^+ \qquad \bar{\mathbf{a}} \overset{\text{EoM}}{=} \bar{a}(x^-)\,\mathrm{d}x^- \tag{338}$$

- Solve the flatness constraint with pure gauge connections $\mathbf{A} = U^{-1}\mathrm{d}U$, $\bar{\mathbf{A}} = \bar{U}^{-1}\mathrm{d}\bar{U}$. Plugging them into the action and employing the chiral BC yields a WZW action:

$$\begin{aligned}
&S[A = U^{-1}\mathrm{d}U, \bar{A} = \bar{U}\mathrm{d}\bar{U}]\\
&\overset{\text{P–W}}{=} -S_{\text{WZW}}[U\bar{U}^{-1}]\\
&= S_{\text{WZW}}[G]\\
&+ \frac{k}{2\pi}\int_\Sigma \mathrm{tr}\big[-\mathbf{A}_- \wedge \partial G G^{-1} + G^{-1}\bar{\partial} G \wedge \bar{\mathbf{A}}_+ + \mathbf{A}_- \wedge G\bar{\mathbf{A}}_+ G^{-1} - \mathbf{A}_- \wedge \bar{\mathbf{A}}_+\big]\\
&\overset{\text{BC}}{=} S_{\text{WZW}}[G]
\end{aligned}$$

- In the radial gauge, express the boundary dynamics of the theory in terms of an $\mathsf{SL}(N;\mathbb{R})$-valued field $g = u^{-1}\bar{u}$ on Σ, viewing $G = b^{-1}gb^{-1} : \mathcal{M} \to \mathsf{SL}(N;\mathbb{R})$ as an extension of g to $\mathcal{M}$.

$$S[A_\Sigma = U^{-1}\mathrm{d}_\Sigma U, \bar{A}_\Sigma = \bar{U}\mathrm{d}_\Sigma\bar{U}] = S_{\text{WZW}}[g] \tag{339}$$

14.2. Further reduction to $\mathfrak{sl}(N;\mathbb{R})$ Toda theory

Writing,

$$\mathbf{a} = \sum_{s=2}^{N} \sum_{m=-s+1}^{s-1} \mathbf{a}_s{}^m W^s{}_m \tag{340}$$

the most general pair of aAdS$_3$ $\mathsf{SL}(N;\mathbb{R})$ is obtained as in Section 11.4:

$$\begin{aligned} a &= \sum_m \mathbf{a}_2{}^m L_m + \sum_{s=3}^{N} \sum_{m=-s+1}^{s-1} \mathbf{a}_s{}^m W^s{}_m \\ &\Big\downarrow \text{aAdS}_3 \text{ boundary conditions} \\ &= L_+ \,\mathrm{d}x^+ + \sum_{m=-,0} \mathbf{a}_2{}^m L_m + \sum_{s=3}^{N} \sum_{m=-s+1}^{0} \mathbf{a}_s{}^m W^s{}_m \\ &\Big\downarrow \text{highest-weight gauge} \\ &= L_+ \,\mathrm{d}x^+ + \mathbf{a}_2{}^- L_- + \sum_{s=3}^{N} \mathbf{a}_s{}^{s-1} W^s{}_{s-1} \end{aligned}$$

Using infinitesimal gauge tranformations $\mathfrak{g}_< \ni \lambda \propto L_-, W^3{}_{-2}, W^3{}_{-1}, \ldots,$ the highest-weight gauge can always be achieved.

Put together, aAdS$_3$ boundary conditions and highest-weight gauge-fixing constitute two sets of constraints on $\mathbf{a}$'s components:

$$\begin{array}{ll} \mathbf{a}_2{}^{+1} \stackrel{!}{=} \mathrm{d}x^+ & \mathbf{a}_2{}^0 \stackrel{!}{=} 0 \\ \mathbf{a}_3{}^{+1} \stackrel{!}{=} 0 & \mathbf{a}_3{}^{-1}, \mathbf{a}_3{}^0, \mathbf{a}_3{}^{+2} \stackrel{!}{=} 0 \\ \quad\vdots & \\ \mathbf{a}_N{}^{+1} \stackrel{!}{=} 0 & \mathbf{a}_N{}^{-N+2}, \mathbf{a}_N{}^{-N+3}, \ldots, \mathbf{a}_N{}^0, a_N{}^{+2}, \ldots, \mathbf{a}_N{}^{+N-1} \stackrel{!}{=} 0 \end{array} \tag{341}$$

Due to the antidiagonal nature of tr (116), the first column of constraints can be expressed in terms of unbarred Noether currents of mode number -1 basis elements, the second column can be expressed in terms of unbarred Noether currents of mode number smaller than -1.

As the reduced unbarred CS connection is equal on-shell to the chiral WZW current $\mathbf{a} \stackrel{\text{EoM}}{=} \mathbf{J}$, these two sets of boundary conditions translate into [FWB+89]-type current constraints for an $\mathsf{SL}(N;\mathbb{R})$ WZW model. Recall that the E^i_+ have mode number -1, E^α_+ have mode number smaller than -1 for $\alpha \in \Phi \setminus \Phi_\text{s}$.

Therefore, one will obtain the current constraints,

$$\begin{aligned} \mathbf{J}(E_+^i) &\overset{!}{=} -\frac{k}{2\pi}\mu^i \,\mathrm{d}x^+ && \alpha^i \in \Phi_\mathrm{s} \\ \mathbf{J}(E_+^\alpha) &\overset{!}{=} 0 && \alpha \in \Phi \setminus \Phi_\mathrm{s} \end{aligned} \tag{342}$$

for some μ^i. Similarly, the aAdS$_3$ boundary conditions and lowest-weight gauge-fixing for the reduced barred CS connection $\bar{\mathbf{a}} \overset{\text{EoM}}{=} \bar{\mathbf{J}}$ translate into the complementary current constraints:

$$\begin{aligned} \bar{\mathbf{J}}(E_-^i) &\overset{!}{=} -\frac{k}{2\pi}\nu^i \,\mathrm{d}x^- && \alpha^i \in \Phi_\mathrm{s} \\ \bar{\mathbf{J}}(E_-^\alpha) &\overset{!}{=} 0 && \alpha \in \Phi \setminus \Phi_\mathrm{s} \end{aligned} \tag{343}$$

for some ν^i. Put together, the aAdS$_3$ boundary conditions together with the highest-weight/lowest-weight gauge implement the necessary current constraints to reduce the $\mathsf{SL}(N;\mathbb{R})$ WZW model to $\mathsf{sl}(N;\mathbb{R})$ Toda theory.

The $\mathsf{sl}(3;\mathbb{R})$ case will be explicitely verified here. Writing $W_m \equiv {W^3}_m$, the spin-3 basis can be expressed in terms of the Chevalley basis as:

$$\begin{aligned} & & W_{+2} &= -2[E_-^1, E_-^2] \\ L_+ &= \sqrt{2}(E_-^1 + E_-^2) & W_{+1} &= \frac{1}{\sqrt{2}}(E_-^1 - E_-^2) \\ L_0 &= H^1 + H^2 & W_0 &= \frac{1}{3}(H^1 - H^2) \\ L_- &= -\sqrt{2}(E_+^1 + E_+^2) & W_{-1} &= -\frac{1}{\sqrt{2}}(E_+^1 - E_+^2) \\ & & W_{-2} &= 2[E_+^1, E_+^2] \end{aligned} \tag{344}$$

$$\begin{aligned} {\mathbf{a}_2}^{+1} \overset{!}{=} \mathrm{d}x^+ &\implies \frac{1}{{t^2}_1}\frac{2\pi}{k}\mathbf{J}(L_-) \overset{!}{=} \mathrm{d}x^+ \\ {\mathbf{a}_3}^{+1} \overset{!}{=} 0 &\implies \mathbf{J}(W_{-1}) \overset{!}{=} 0 \end{aligned} \tag{345}$$

$$\implies \left.\begin{aligned} \mathbf{J}(E_+^1) + \mathbf{J}(E_+^2) &\overset{!}{=} 2\sqrt{2}\frac{k}{2\pi}\,\mathrm{d}x^+ \\ \mathbf{J}(E_+^1) - \mathbf{J}(E_+^2) &\overset{!}{=} 0 \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} \mathbf{J}(E_+^1) &\overset{!}{=} \sqrt{2}\frac{k}{2\pi}\,\mathrm{d}x^+ \\ \mathbf{J}(E_+^2) &\overset{!}{=} \sqrt{2}\frac{k}{2\pi}\,\mathrm{d}x^+ \end{aligned}\right. \tag{346}$$

The other $\mathbf{J}(E_+^\alpha)$ are obviously set to zero. Similarly for the barred sector, one obtains

$$\mathbf{J}(E_-^1) \overset{!}{=} \sqrt{2}\frac{k}{2\pi}\,\mathrm{d}x^+ \quad \mathbf{J}(E_-^2) \overset{!}{=} \sqrt{2}\frac{k}{2\pi}\,\mathrm{d}x^+ \tag{347}$$

Plugging in the above constraints $\mu^i = -\sqrt{2} = \nu^i$ into a Gauß-decomposed $\mathsf{SL}(3;\mathbb{R})$ WZW action yields the $\mathsf{sl}(3;\mathbb{R})$ Toda action:

$$S_T[\Phi_i] = \frac{k}{4\pi}\int_\Sigma \mathrm{d}x^+ \wedge \mathrm{d}x^- \left(\sum_{ij}\frac{1}{4}G^{ij}\partial_+\Phi_i \wedge \partial_-\Phi_j + \sum_i \left|\alpha^i\right|^2 e^{\sum_j \frac{1}{2}A^{ij}\Phi_j}\right) \tag{348}$$

14.3. W-algebra

It was shown above that higher-spin AdS_3 gravity reduces under the aAdS_3 boundary conditions and gauge-fixings to $\mathsf{sl}(N;\mathbb{R})$ Toda theory. The symmetry algebra of $\mathsf{sl}(N;\mathbb{R})$ Toda field theory is the so-called W_N algebra. As we will see, the charge algebra will split into various sectors of the W_N algebra. Once again, we show this only for the unbarred sector.

For convenience, relabel the remaining free components of the highest-weight/lowest-weight gauged $\mathbf{a}$, $\bar{\mathbf{a}}$ analogously to (274),

$$\begin{aligned} \mathbf{a} &= (L_+ - \mathcal{W}^{(2)} L_- + \mathcal{W}^{(3)} {W^3}_{-2} + \cdots + (-1)^{N-1} \mathcal{W}^{(N)} {W^N}_{-N+1}\,)\, \mathrm{d}x^+ \\ \bar{\mathbf{a}} &= (L_- - \bar{\mathcal{W}}^{(2)} L_+ + \bar{\mathcal{W}}^{(3)} {W^3}_{+2} + \cdots + (-1)^{N-1} \bar{\mathcal{W}}^{(N)} {W^N}_{N-1}\,)\, \mathrm{d}x^+ \end{aligned} \tag{349}$$

where $\mathcal{W}^{(s)} = \mathcal{W}^{(s)}(x^+)$, $\bar{\mathcal{W}}^{(s)} = \bar{\mathcal{W}}^{(s)}(x^-)$. Global symmetries have to satisfy:

$$\pounds_\lambda a \overset{!}{=} -\Delta_\lambda \mathcal{W}^{(2)} + \Delta_\lambda \mathcal{W}^{(3)} + \cdots + (-1)^{N-1} \Delta_\lambda \mathcal{W}^{(N)} \tag{350}$$

Expanding λ in the $\mathsf{sl}(N;\mathbb{R})$ basis as,

$$\lambda = \sum_{s=2}^{N} \sum_{m=-s+1}^{s-1} \lambda_s^m(x^+) {W^s}_m \tag{351}$$

we infer that the above constraints reduce allowed λ's to $N-1$ doF, $\lambda_2^+, \ldots \lambda_N^{+N-1}$. This follows from, among other things, the mode number-additivity property of the Lie bracket [CFP11]. Therefore, $\Delta_\lambda \mathcal{W}$ is a sum of actions of the components $\lambda^{+s-1} \equiv \lambda_s^{+s-1}$ on $\mathcal{W}$,

$$\Delta_\lambda \mathcal{W} = \pounds_{\lambda^+} \mathcal{W} + \cdots + \pounds_{\lambda^{+N-1}} \mathcal{W} \tag{352}$$

Then, the Noether charge Poisson bracket decomposes into Poisson brackets of individual spin-s charges,

$$\begin{aligned} \{H(\lambda), H(\mu)\} \equiv &\{H^{(2)}(\lambda^+), H^{(2)}(\mu^+)\} + \cdots + \{H^{(N)}(\lambda^{+N-1}), H^{(2)}(\mu^+)\} \\ &\vdots \\ &+ \{H^{(2)}(\lambda^+), H^{(N)}(\mu^{+N-1})\} + \cdots + \{H^{(N)}(\lambda^{+N-1}), H^{(N)}(\mu^{+N-1})\} \end{aligned} \tag{353}$$

where the spin-s charge is defined analogously to the Virasoro charges (317):

$$H^{(s)}(\lambda^{+s-1}) := (-1)^{s-1} {t^s}_{s+1} \frac{k}{2\pi} \oint \mathrm{d}x^+\, \mathcal{W}^{(s)} \tag{354}$$

$$\begin{aligned} \{H^{(s)}(\lambda^{+s-1}), H^{(r)}(\mu^{+r-1})\} &:= \pounds_{\lambda^{+s-1}} H^{(r)}(\mu^{+r-1}) \\ &= (-1)^{r-1} {t^r}_{r-1} \frac{k}{2\pi} \oint \mathrm{d}x^+\, \mu^{+r-1} \pounds_{\lambda^{+s-1}} \mathcal{W}^{(r)} \end{aligned} \tag{355}$$

The charge algebra decomposition (353) contains a nontrivial statement, namely:

$$\{H^{(s)}(\lambda^{+s-1}), H^{(r)}(\mu^{+r-1})\} = -\{H^{(r)}(\lambda^{+r-1}), H^{(s)}(\mu^{+s-1})\}$$
$$\Downarrow$$
$$(-1)^{r-1}t^{r}{}_{r-1}\frac{k}{2\pi}\oint \mathrm{d}x^{+}\,\mu^{+r-1}\pounds_{\lambda^{+s-1}}\mathcal{W}^{(r)} = -(-1)^{s-1}t^{s}{}_{s-1}\frac{k}{2\pi}\oint \mathrm{d}x^{+}\,\mu^{+s-1}\pounds_{\lambda^{+r-1}}\mathcal{W}^{(s)} \tag{356}$$

The standard form of W_N algebra is obtained by computing the bracket of the Fourier modes,

$$\begin{aligned} \mathcal{W}^{(s)} &=: (-1)^{s-1}\frac{1}{t^{s}{}_{s-1}k}\sum_{n}\mathcal{W}^{(s)}{}_{n}e^{-inx^{+}} \\ \mathcal{W}^{(s)}{}_{m} &= H^{(s)}(e^{imx^{+}}) \\ \{\mathcal{W}^{(s)}{}_{m},\mathcal{W}^{(r)}{}_{n}\} &= \{H^{(s)}(e^{imx^{+}}), H^{(r)}(e^{inx^{+}})\} \end{aligned} \tag{357}$$

Here, the W_3 case will be explicitly computed.

$$\lambda = \lambda_2^{+}L_{+} + \lambda_2^{0}L_0 + \lambda_2^{-}L_{-} + \lambda_3^{+2}W_{+2} + \lambda_3^{+1}W_{+1} + \lambda_3^{0}W_0 + \lambda_3^{-1}W_{-1} + \lambda_3^{-2}W_{-2} \tag{358}$$

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} \\ [L_m, W_n] &= (2m-n)W_{m+n} \\ [W_m, W_n] &= -\frac{1}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \end{aligned} \tag{359}$$

$$t^{2}{}_{0} = 2 \quad t^{2}{}_{1} = -4 \quad t^{3}{}_{2} = 4 \tag{360}$$

Writing $\lambda^{+} \equiv \lambda_2^{+}$, $\lambda^{+2} \equiv \lambda_3^{+2}$, obtain the allowed actions of λ on $\mathcal{L}$, $\mathcal{W}$:

$$\begin{aligned} \Delta_\lambda \mathcal{L} &= -\frac{1}{2}\partial_+^3\lambda^{+} + \partial_+\mathcal{L}\lambda^{+} + 2\mathcal{L}\partial_+\lambda^{+} + 2\partial_+\mathcal{W}\lambda^{+2} + 3\mathcal{W}\partial_+\lambda^{+2} \\ &\equiv \pounds_{\lambda^{+}}\mathcal{L} + \pounds_{\lambda^{+2}}\mathcal{L} \end{aligned} \tag{361}$$

$$\begin{aligned} \Delta_\lambda \mathcal{W} &= \partial_+\mathcal{W}\lambda^{+} + 3\mathcal{W}\partial_+\lambda^{+} \\ &\quad + \frac{1}{24}\partial_+^5\lambda^{+2} - \frac{1}{6}\partial_+^3\mathcal{L}\lambda^{+2} - \frac{9}{12}\partial_+^2\mathcal{L}\partial_+\lambda^{+2} - \frac{5}{4}\partial_+\mathcal{L}\partial_+^2\lambda^{+2} \\ &\quad - \frac{5}{6}\mathcal{L}\partial_+^3\lambda^{+2} + \frac{8}{3}\mathcal{L}\partial_+\mathcal{L}\lambda^{+2} + \frac{8}{3}\mathcal{L}^2\partial_+\lambda^{+2} \\ &\equiv \pounds_{\lambda^{+}}\mathcal{W} + \pounds_{\lambda^{+2}}\mathcal{W} \end{aligned} \tag{362}$$

The decomposition of the Noether charge bracket gives:

$$\begin{aligned}
\{H(\lambda), H(\mu)\} &= \pounds_\lambda H(\mu) \\
&= \frac{k}{2\pi}\int_{\mathcal{S}} \mathrm{tr}[\mu \pounds_\lambda \mathbf{a}] = \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mathrm{tr}[\mu \pounds_\lambda a] \\
&= \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mathrm{tr}\big[(\cdots + \mu^+ L_+ + \mu^{+2} W_{+2})(-\Delta_\lambda \mathcal{L} L_- + \Delta_\lambda \mathcal{W} W_{-2})\big] \\
&= -{t^2}_1 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^+ \pounds_{\lambda^+}\mathcal{L} - {t^2}_1 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^+ \pounds_{\lambda^{+2}}\mathcal{L} \\
&\quad + {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \pounds_{\lambda^+}\mathcal{W} + {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \pounds_{\lambda^{+2}}\mathcal{W} \\
&\equiv \{H^{(2)}(\lambda^+), H^{(2)}(\mu^+)\} + \{H^{(3)}(\lambda^{+2}), H^{(2)}(\mu^+)\} \\
&\quad + \{H^{(2)}(\lambda^+), H^{(3)}(\mu^{+2})\} + \{H^{(3)}(\lambda^{+2}), H^{(3)}(\mu^{+2})\}
\end{aligned} \tag{363}$$

where the spin-2 and spin-3 charges are defined in accordance with (354),

$$H^{(2)}(\lambda^+) \coloneqq -{t^2}_1 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \lambda^+ \mathcal{L} \quad H^{(3)}(\lambda^{+2}) \coloneqq {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \lambda^{+2} \mathcal{W} \tag{364}$$

The individual Poisson brackets evaluate to:

$$\begin{aligned}
\{H^{(2)}(\lambda^+), H^{(2)}(\mu^+)\} \coloneqq \pounds_{\lambda^+} H^{(2)}(\mu^+) &= -{t^2}_1 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^+ \pounds_{\lambda^+}\mathcal{L} \\
&= -{t^2}_1 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^+ \Big(-\frac{1}{2}\partial_+^3 \lambda^+ + \partial_+ \mathcal{L}\lambda^+ + 2\mathcal{L}\partial_+\lambda^+\Big) \\
&= H^{(2)}(\mu^+\partial_+\lambda^+ - \lambda^+\partial_+\mu^+) + {t^2}_1 \frac{k}{4\pi}\oint \mathrm{d}x^+\, \mu^+ \partial_+^3 \lambda^+
\end{aligned}$$

$$\begin{aligned}
\{H^{(2)}(\lambda^+), H^{(3)}(\mu^{+2})\} \coloneqq \pounds_{\lambda^+} H^{(3)}(\mu^{+2}) &= {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \pounds_{\lambda^+}\mathcal{W} \\
&= {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \Big(\partial_+ \mathcal{W}\lambda^+ + 3\mathcal{W}\partial_+\lambda^+\Big) \\
&= H^{(3)}(2\mu^{+2}\partial_+\lambda^+ - \lambda^+\partial_+\mu^{+2})
\end{aligned}$$

$$\begin{aligned}
\{H^{(3)}(\lambda^{+2}), H^{(3)}(\mu^{+2})\} \coloneqq \pounds_{\lambda^{+2}} H^{(3)}(\mu^{+2}) &= {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \pounds_{\lambda^{+2}}\mathcal{W} \\
&= {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \frac{1}{24}\partial_+^5 \lambda^{+2} \\
&\quad + {t^3}_2 \frac{k}{2\pi}\oint \mathrm{d}x^+\, \mu^{+2} \Big(-\frac{1}{6}\partial_+^3 \mathcal{L}\lambda^{+2} - \frac{9}{12}\partial_+^2 \mathcal{L}\partial_+\lambda^{+2} - \frac{5}{4}\partial_+\mathcal{L}\partial_+^2\lambda^{+2} \\
&\qquad - \frac{5}{6}\mathcal{L}\partial_+^3\lambda^{+2} + \frac{8}{3}\mathcal{L}\partial_+\mathcal{L}\lambda^{+2} + \frac{8}{3}\mathcal{L}^2\partial_+\lambda^{+2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= {t^3}_2 \frac{k}{2\pi} \oint \mathrm{d}x^+ \, \mu^{+2} \frac{1}{24} \partial_+^5 \lambda^{+2} \\
&+ \frac{1}{12} H^{(2)} (2\mu^{+2}\partial_+^3\lambda^{+2} - 2\lambda^{+2}\partial_+^3\mu^{+2} - 3\partial_+\mu^{+2}\partial_+^2\lambda^{+2} + 3\partial_+^2\mu^{+2}\partial_+\lambda^{+2}) \\
&- \frac{4}{3} {t^2}_1 \frac{k}{2\pi} \oint \mathrm{d}x^+ \, (\mu^{+2}\partial_+\lambda^{+2} - \lambda^{+2}\partial_+\mu^{+2}) \mathcal{L}^2
\end{aligned}
$$

Verifying the (single occurence of) nontrivial statement (356) lets us check our work:

$$
\begin{aligned}
\{H^{(3)}(\lambda^{+2}), H^{(2)}(\mu^+)\} &= -{t^2}_1 \frac{k}{2\pi} \oint \mathrm{d}x^+ \, \mu^+ \pounds_{\lambda^{+2}} \mathcal{L} \\
&= -{t^2}_1 \frac{k}{2\pi} \oint \mathrm{d}x^+ \, \mu^+ \Big(2\partial_+ \mathcal{W} \lambda^{+2} + 3\mathcal{W} \partial_+ \lambda^{+2} \Big) \\
&= {t^3}_2 \frac{k}{2\pi} \oint \mathrm{d}x^+ \, \Big(-2\lambda^{+2}\partial_+\mu^+ + \mu^+\partial_+\lambda^{+2} \Big) \mathcal{W} \\
&= -\{H^{(2)}(\mu^+), H^{(3)}(\lambda^{+2})\}
\end{aligned}
\tag{365}
$$

Expanding $\mathcal{L}$, $\mathcal{W}$ into Fourier modes yields the W_3 algebra in its standard form,

$$
\begin{gathered}
\mathcal{L} =: -\frac{1}{{t^2}_1 k} \sum_n \mathcal{L}_n e^{-inx^+} \quad \mathcal{W} =: \frac{1}{{t^3}_2 k} \sum_n \mathcal{W}_n e^{-inx^+} \\
\mathcal{L}_m = H^{(2)}(e^{imx^+}) \quad \mathcal{W}_m = H^{(3)}(e^{imx^+})
\end{gathered}
\tag{366}
$$

$$
\begin{aligned}
\{\mathcal{L}_m, \mathcal{L}_n\} &= i(m-n)\mathcal{L}_{m+n} + im^3 \frac{1}{12} \frac{3\ell}{2G_3} \delta_{m+n,0} \\
\{\mathcal{L}_m, \mathcal{W}_n\} &= i(2m-n)\mathcal{W}_{m+n} \\
\{\mathcal{W}_m, \mathcal{W}_n\} &= \frac{1}{12} \Big(im^5 \frac{1}{12} \frac{3\ell}{2G_3} \delta_{m+n,0} - i(m-n)(2m^2+2n^2-mn)\mathcal{L}_{m+n} \\
&\qquad + i(m-n) 96 \frac{2G_3}{3\ell} \sum_q \mathcal{L}_{m+n-q} \mathcal{L}_q \Big)
\end{aligned}
\tag{367}
$$

The resulting central charge of the theory is once again $c = \frac{3\ell}{2G_3}$.

Part V. Discussion

Summary of results

We have successfully demonstrated the classical correspondence between asymptotically AdS_3 general relativity, $\mathsf{SL}(2;\mathbb{R})$ Wess–Zumino–Witten models, and Liouville field theory as an equivalence of classical phase spaces. We have shown that the action of the bulk theory becomes the action of the boundary theory in the presence of suitable boundary conditions and gauge-fixing, and that the gravitational algebra of physical observables becomes the WZW $\hat{\mathsf{sl}}(2)$ affine Lie algebra and further reduces to the Virasoro algebra of the Liouville field theory.

The analysis was then extended to asymptotically AdS_3 spin-N gravity, $\mathsf{SL}(N;\mathbb{R})$ Wess–Zumino–Witten models, and $\mathsf{sl}(N;\mathbb{R})$ Toda field theory, with explicit calculations for $N = 3$. The reduction of $\hat{\mathsf{sl}}(3)$ affine Lie algebra to the W_3 algebra was performed explicitly.

Future directions

An immediate follow-up would be to explicitly perform all calculations for arbitrary N. We have already sketched parts of it; however, more work is needed to explicitly determine all necessary quantities for the $\mathsf{sl}(N;\mathbb{R})$ spin-basis $\{W^s{}_m\}$ in terms of the A_{N-1} simple Lie algebra data. Further follow-up is to work out an analogue reduction to Toda field theory for the other families of simple Lie algebra B_m, C_m, D_m, as well as for the exceptional Lie algebras, starting from the appropriate $\mathfrak{G} \times \mathfrak{G}$ hs-action. This would hint at different types of higher-spin theories, differing from the common type-A higher-spin theory that was the subject of this thesis.

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