universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„On a differential calculus for metric measure spaces“

verfasst von / submitted by

Darius Erös, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Mag. Mag. Dr. Michael Kunzinger

Abstract

The last few decades have seen an increasing interest in the geometric study of spaces seemingly incapable of supporting any sort of differential calculus. Among these are metric measure spaces, which a priori only come equipped with notions of distance and volume. Despite this lack of structure, however, efforts have succeeded in building a framework for analogues of one-forms, vector fields and differentials in this setting by adopting a variational point of view.

The goal of this thesis is to explore such an approach, put forth in large parts by Nicola Gigli, building on Sobolev calculus and the theory of normed modules. To do so, we first spend some time discussing a perspective on weak upper gradients via test plans, providing us with a class of differentiable functions, and then move on to introduce normed modules, allowing us to perform bundle constructions in an appropriate sense. Finally, we establish a first-order differential calculus consisting of tangent and cotangent modules together with a notion of differentials for structure preserving maps. In all of this, a strong emphasis is put on giving full and detailed proofs, as well as on providing motivational examples from smooth differential geometry.

Abriss

Über die letzten Jahre hinweg ist es zu einem stetigen Anstieg des Interesses am geometrischen Studium von Räumen gekommen, die einem scheinbar keine Möglichkeit bieten, auf ihnen Mittel der Differentialrechnung anzuwenden. Zu dieser Klasse an Räumen gehören metrische Maßräume, die lediglich mit Begriffen von Distanz und Volumen ausgestattet sind. Diesem Fehlen an Struktur zum Trotz ist es gelungen, Analogien von 1-Formen, Vektorfeldern und Differentialen im diesem Kontext zu definieren, indem ein variationeller Blickwinkel eingenommen wurde.

Das Ziel dieser Arbeit ist es nun, einen solchen Zugang, der vor allem von Nicola Gigli entwickelt wurde und auf einem Sobolev-Kalkül und der Theorie normierter Moduln aufbaut, näher zu durchleuchten. Dazu werden wir zunächst einige Zeit aufwenden, um eine Perspektive auf schwache Oberhalb-Gradienten über Test Pläne zu diskutieren, die uns eine Klasse an differenzierbaren Funktionen liefert, und anschließend normierte Moduln einführen, die es uns erlauben, gewohnte Bündelkonstruktionen durchzuführen. Schließlich werden wir ein Differentialkalkül erster Ordnung einführen können, das aus Tangential- und Cotangentialmoduln zusammen mit einem Begriff eines Differentials für strukturerhaltende Abbildungen besteht. Bei all dem ist ein starker Fokus auf das Bereitstellen ausführlicher und vollständiger Beweise gelegt worden, sowie auch darauf, motivierende Beispiele aus der glatten Differentialgeometrie zu liefern.

Acknowledgement

Not having been aware of the richness of ideas stemming from metric geometry, I wish to thank my supervisor Michael Kunzinger very much for introducing me to this vast field of astonishing connections between analysis and geometry. Also, I would like to use this opportunity to thank him for giving me the freedom to compose this thesis according to my personal interests and for his sympathetic recognition of my, at times overly ambitious, time schedule (with a record time of proofreading and correcting 8 new pages in under 4 hours upon sending!).

Moreover, I owe a debt of gratitude to Ann Sophie and my parents for their caring support and encouragement while putting together these pages, for their patience and their trust.

Contents

1 Introduction **1**
1.1 Motivation 1
1.2 Overview 2
1.3 Preliminaries 3
1.3.1 Spaces of curves 3
1.3.2 Measure theoretic and functional analytic tools 5
1.3.3 The Bochner integral 7

2 Sobolev spaces **12**
2.1 Test plans 13
2.2 The Sobolev class $S^2(X)$ 17
2.2.1 Weak upper gradients 17
2.2.2 Interlude: Lipschitz functions 19
2.2.3 Alternative characterizations 23
2.3 The Sobolev space $W^{1,2}(X)$ 28
2.3.1 The minimal weak upper gradient $|Df|$ 28
2.3.2 Calculus rules 32

3 Normed Modules **39**
3.1 L^2-normed L^∞-modules 41
3.2 Morphisms and Submodules 43
3.3 Operations on Modules 48
3.3.1 Dualization 48
3.3.2 Pullback 52
3.4 Hilbert Modules 63

4 First-Order Calculus **71**
4.1 The cotangent module 71
4.2 The tangent module 79
4.3 Gradients 81
4.4 Infinitesimal Hilbertianity 87
4.5 Maps of bounded deformation 95

1 Introduction

1.1 Motivation

There is a long history of efforts to extend calculus results for smooth functions on differentiable manifolds to more singular classes of functions, often used as approximations of the regular situation. However, only recently, a new perspective has emerged, in which, instead of considering rough functions on smooth spaces, we lower the regularity of the *spaces* themselves, cf. [17].

In this endeavor, we wish to carve out the essential characteristics needed to define and quantify infinitesimal variations of objects in the smooth setting and use those as an axiomatization of new, more singular spaces. One such approach is based on the concept of a *metric measure space* $(X, \mathrm{d}, \mathfrak{m})$, in which only notions of *distance* and *volume* persist. A possible first definition of a differential in this context, put forth by Heinonen and Koskela in [18], is based on the notion of an *upper gradient*. That is, given a real-valued function u on our space X, any non-negative function g satisfying

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g \,\mathrm{d}s$$

along all rectifiable curves γ in X. Now, in order to determine a *unique* (minimal) upper gradient, we need to allow this relation to fail on *negligible* sets of curves, see [7]. Making this statement precise is a non-trivial task and has led to the development of several different viewpoints on the subject, cf. [2].

The method we are going to follow, proposed by Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré in [1], consists in testing the above inequality against a family of probability measures on the space of curves. Thereby creating a duality between upper gradients and probability measures, we can then pick out certain criteria for our collection of test measures to ensure the existence of a minimal (weak) upper gradient. As we will discover in the subsequent chapters, this fruitful idea leads us to the development of a *Sobolev calculus* on metric measure spaces.

Even more astonishing, however, is the possibility of providing an abstract framework for notions of *one-forms* and *vector fields* on any metric measure space via *normed modules*. Building on the previously developed Sobolev calculus, this concept axiomatizes certain crucial aspects of the space of *integrable sections* of a vector bundle over a Riemannian manifold and was first introduced by Nicola Gigli in [11]. Developing and connecting these ideas will serve as our central ambition for the upcoming sections.

1.2 Overview

Our main objective over the following four chapters is to establish a *first-order differential calculus* on any arbitrary metric measure space. More concretely, we want to show that on any such space we can meaningfully speak of one-forms, vector fields and differentials. The general structuring of the material is based on the book *Lectures on Nonsmooth Differential Geometry* [13] by Nicola Gigli and Enrico Pasqualetto. However, in contrast to their presentation, we put a strong emphasis on giving complete and detailed proofs, provide additional intermediary results and spend some time connecting the theory to its classical, smooth counterpart by way of motivational examples.

To start off, in the remainder of this chapter we collect preliminary results from metric geometry, measure theory and functional analysis. Here, we do no yet give proofs and rather focus on important constructions and expand on definitions needed in the further development of the theory.

Chapter 2 will then introduce, as one of our central concepts, the notion of a *weak upper gradient*, serving as a generalization of the norm of the distributional differential in Euclidean space. Before doing so, however, we establish a few basic facts about the family of *test plans*, which provide us with the necessary tool for a well-defined notion of a *Sobolev class*. We will then spend some time exploring different viewpoints on these objects and on proving the existence of a unique *minimal* weak upper gradient. This will allow us to define a *Banach space of Sobolev functions* and to recreate typical rules of calculation, such as the Leibniz and the chain rule.

In Chapter 3, we lay out a more algebraic treatise of so-called *normed modules*. These objects arise from a possible axiomatization of integrable one-forms on a Riemannian manifold after having removed any traces of a smooth structure by integration. Completing the categorical picture, we will then define morphisms and substructures before moving on to describing operations on these modules, such as dualization and pullback. Finally, as a way of encoding infinitesimally the structure of an inner product space, we introduce Hilbert modules and study the relation to their duals, proving an analogue to the Theorem of Riesz.

Lastly, in Chapter 4, we put together the Sobolev calculus of Chapter 2 and the theory of normed modules developed in Chapter 3 to provide at last an answer to the question:

> How does one define a (co)vector field on an abstract metric measure space?

In our approach, generalizing a bundle viewpoint on the differential structure of a Riemannian manifold, we answer this question by constructing a suitable (co)tangent module. In the spirit of Lebesgue integration, a general one-form will then be the result of a limiting procedure involving *simple* one-forms obtained by piecing together localized differentials of Sobolev functions along a Borel partition of our space. After expanding on some of the properties of these modules, including an alternative description of vector fields as L^2-derivations, we take on the task of characterizing those spaces on which we can define a *unique gradient* of any Sobolev function, leading to the notion of *infinitesimal*

Hilbertianity. Our last endeavor will then consist in proving that any structure preserving map between metric measure spaces induces a *pullback of one-forms* and possesses a *unique differential*, completing our study of a basic first-order differential structure on metric measure spaces.

1.3 Preliminaries

In this subsection, we collect results for later use and establish notation. We begin by introducing concepts related to the space of continuous curves on a metric space, following mostly [13]. Results whose proofs are omitted will be cited more explicitly.

1.3.1 Spaces of curves

Let (X, d) be a metric space. Then we call any map $\gamma\colon [0,1] \to X$ a *curve* in X and denote its value at time $t \in [0,1]$ by the shorthand notation $\gamma_t := \gamma(t)$. To be able to use this time-evaluation as a map of its own, we will also frequently use the notation $e_t(\gamma) := \gamma_t$, where now $e_t\colon \mathcal{C}([0,1], X) \to X$ associates any continuous curve γ to its value at $t \in [0,1]$.

Further, we endow $\mathcal{C}([0,1], X)$ with the metric

$$\mathbf{d}(\gamma, \tilde{\gamma}) := \max_{t\in[0,1]} d(\gamma_t, \tilde{\gamma}_t), \quad \text{for any two curves } \gamma, \tilde{\gamma} \in \mathcal{C}([0,1], X),$$

a distance which is not invariant under reparametrization, allowing us to incorporate a notion of *speed*.

As the next proposition shows, the resulting metric space of curves inherits crucial topological properties from its underlying space (X, d), see Prop 1.2.1. in [13].

Proposition 1.3.1. *Let (X, d) be a complete and separable metric space. Then the space $(\mathcal{C}([0,1], X), \mathbf{d})$ is also complete and separable.*

Although it is impossible to talk about derivatives of curves in X, i.e. of *velocities*, within a certain, more regular family of curves, we can at least recover a notion of speed, namely that of the *metric speed.*

Definition 1.3.2. Let $\gamma \in \mathcal{C}([0,1], X)$. We say that γ is *absolutely continuous*, whenever we can find a function $h \in L^1(0,1)$ such that

$$d(\gamma_t, \gamma_s) \le \int_s^t h(r)\,\mathrm{d}r \tag{1.3.1}$$

for all $s, t \in [0,1], s < t$. Moreover, we denote by $\mathrm{AC}([0,1], X)$ the space of all absolutely continuous curves in X.

Remark 1.3.3. In the special case $X = \mathbb{R}$, the space $\mathrm{AC}([0,1], X)$ coincides with the Sobolev space $W^{1,1}(0,1)$ by the fundamental theorem of calculus for the Lebesgue integral. More specifically, $f \in W^{1,1}(0,1)$ if and only if f is integrable and agrees with an absolutely continuous function almost everywhere. See e.g. Example 9.8.2 in *Real and Functional Analysis* [6] by Bogachev and Smolyanov for details.

Theorem 1.3.4. *Let* $\gamma \in \mathrm{AC}([0,1], X)$*. Then,*

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|}$$

exists for a.e. $t \in [0,1]$*. Further, the map* $t \mapsto |\dot{\gamma}_t|$ *belongs to the space* $L^1(0,1)$ *and is the minimal choice of functions in* (1.3.1) *up to a.e.-equivalence.*

The proof can again be found in [13] under Theorem 1.2.5.

A possibility of selecting curves $\Gamma \subseteq \mathcal{C}([0,1], X)$ of restricted speed is to require higher integrability of the maps $t \mapsto |\dot{\gamma}_t|$ for all $\gamma \in \Gamma$. This we do by way of an energy functional.

Definition 1.3.5. We define the *kinetic energy functional* $\mathrm{E}\colon \mathcal{C}([0,1], X) \to [0, \infty]$ as

$$\mathrm{E}(\gamma) := \begin{cases} \int_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t & \text{if } \gamma \in \mathrm{AC}([0,1], X) \\ \infty & \text{otherwise} \end{cases}$$

We end this subsection by stating a few technical results on calculus rules for functions belonging to the Sobolev space $W^{1,1}(0,1)$.

As already remarked, this space has a close relation to the space of absolutely continuous functions on $(0,1)$. This connection can be exploited, for example, to derive a version of the fundamental theorem of calculus which holds for *almost every point*.

Lemma 1.3.6. *Suppose* $f \in L^1(0,1)$*. Then* $f \in W^{1,1}(0,1)$ *if and only if*

$$f_s - f_t = \int_t^s h_r \, \mathrm{d}r \quad \text{for a.e. } s, t \text{ with } s > t$$

and some function $h \in L^1(0,1)$*.*

By choosing for a given $f \in W^{1,1}(0,1)$ some absolutely continuous representative and applying to it the classical version of the theorem, this can be viewed as a consequence of Theorem 4.3.7 in [6].

Next, we record the following variant of the chain rule for weakly differentiable functions, as well as the Leibniz rule.

Lemma 1.3.7. *Let* $f \in W^{1,1}(0,1)$ *and suppose* $\varphi \in \mathcal{C}^1(\mathbb{R})$*. Then* $\varphi \circ f \in W^{1,1}(0,1)$ *with*

$$(\varphi \circ f)' = (\varphi' \circ f)\, f' \quad \text{a.e. in } (0,1).$$

For a proof, we refer to *Functional Analysis, Sobolev Spaces and Partial Differential Equations* by Haim Brezis [8] Corollary 8.11, noting that we can assume w.l.o.g. that $\varphi(0) = 0$ in our case as we are dealing with a bounded interval (see also the remark in the book).

Finally, we state the Leibniz rule, see also [8] Corollary 8.10.

Proposition 1.3.8. *The space $W^{1,1}(0,1)$ is an algebra whose elements satisfy the Leibniz rule. Explicitly, this means that $f, g \in W^{1,1}(0,1)$ implies $fg \in W^{1,1}(0,1)$ and*

$$(fg)' = f'g + fg'.$$

1.3.2 Measure theoretic and functional analytic tools

In the subsequent chapters we are going to formulate a variety of concepts in a weak or variational sense, meaning that (in)equalities are only required to hold when integrated with respect to a specified family of measures. Therefore, we want to revisit some concepts from measure theory and functional analysis beforehand.

We begin by introducing basic concepts pertaining to the theory of Lebesgue integration and general measure theory. All definitions and results can also be found in *Measure Theory* [5] by Vladimir I. Bogachev.

Definition 1.3.9. Let (X, τ) be a topolgical space. Then, we can equip X with the structure of a measurable space by declaring

$$\sigma(\tau) := \bigcap \{\mathcal{A} \,|\, \mathcal{A} \text{ is a } \sigma\text{-algebra containing } \tau\}$$

to be the collection of all measurable sets on X. We call this collection the *Borel* σ-algebra or the σ-algebra *generated* by τ.

In the setting of metric spaces we of course apply the σ-operator to the topology τ_d induced by the metric.

Definition 1.3.10. Let (X, d) be a metric space. We denote by $\mathcal{P}(X)$ the set of all Borel probability measures of X, i.e. the set of all measures $\mu\colon \sigma(\tau_d) \to [0,1]$ satisfying $\mu(X) = 1$.

Any structure preserving map between two measurable spaces also induces a map between its sets of measures and further allows for the statement of an integral transformation formula. More precisely, let X and Y be two spaces with σ-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively and let $T\colon X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable map. Then we can transport μ to $(Y, \mathcal{B})$ by setting

$$T_*\mu(E) := \mu(T^{-1}(E))$$

for each $E \in \mathcal{B}$. We call $T_*\mu$ the *push-forward* measure or the *image measure* under the map T.

The next proposition, whose proof can be found in [5] Theorem 3.6.1, makes precise our second claim about an integral transformation formula.

Proposition 1.3.11. *Let* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *be two measurable spaces together with a measurable map* $T: X \to Y$ *between them. Given any measure* μ *on* X *and any measurable function* g *on* Y *we have*

$$g \in L^1(T_*\mu) \quad \text{if and only if} \quad g \circ T \in L^1(\mu)$$

and

$$\int_Y g \, \mathrm{d}T_*\mu = \int_X g \circ T \, \mathrm{d}\mu. \tag{1.3.2}$$

Definition 1.3.12. Let (X, d) be a complete and separable metric space. Then we call $(X, d, \mathfrak{m})$ a *metric measure space* provided $\mathfrak{m}: \sigma(\tau_d) \to [0, \infty]$ is a non-trivial Borel measure taking only finite values on bounded sets. We will abbreviate this by saying that $\mathfrak{m} \not\equiv 0$ is *boundedly finite*.

Remark 1.3.13. Let $\mathfrak{m}$ be a measure as in the definition above. Then there exists a *probability measure* $\tilde{\mathfrak{m}}$ that is *equivalent* to $\mathfrak{m}$, meaning that

$$\mathfrak{m}(A) = 0 \quad \text{if and only if} \quad \tilde{\mathfrak{m}}(A) = 0$$

for any Borel set $A \subseteq X$. Indeed, choose any *Borel partition* $(E_n)_{n \geq 1}$ of X, i.e. any sequence of disjoint Borel sets such that $X = \biguplus_{n \geq 1} E_n$, up to $\mathfrak{m}$-null sets (e.g. set $E_n := \overline{B_n(x_0)} \setminus B_{n-1}(x_0)$ for some $x_0 \in X$) and assume w.l.o.g. that $\mathfrak{m}(E_n) > 0$ for all $n \geq 1$. Now set

$$\tilde{\mathfrak{m}}(A) := \sum_{n \geq 1} 2^{-n} \frac{\mathfrak{m}(A \cap E_n)}{\mathfrak{m}(E_n)}$$

where $A \subseteq X$ is any Borel set. Clearly, $\tilde{\mathfrak{m}}(A) \leq \sum_n 2^{-n} = 1$ and $\tilde{\mathfrak{m}}(A) = 0$ precisely when $\mathfrak{m}(A) = \sum_n \mathfrak{m}(A \cap E_n) = 0$, so that $\tilde{\mathfrak{m}}$ possesses all properties claimed.

The class of Borel measures on a metric space X includes *Dirac deltas*, by which we mean probability measures δ_{x_0} with $x_0 \in X$ giving full weight to a set $A \subseteq X$ only if $x_0 \in A$ and setting it to zero otherwise, i.e. $\delta_{x_0}(A) = 1_A(x_0)$. Clearly, these measures display a distinct behavior by focusing exclusively on the relative position of a single point. A generalization of this type of discrete mass concentration is given by the notion of an *atom*.

Definition 1.3.14. Let (X, μ) be a measure space and let $A \subseteq X$ be a measurable subset. Then we call A an *atom* for the measure μ if $\mu(A) > 0$ and for any $B \subseteq A$ with $\mu(B) < \mu(A)$ it automatically follows that $\mu(B) = 0$. If μ does not possess any atoms, we call it *atomless*.

Remark 1.3.15. For a Dirac delta δ_{x_0}, the singleton $\{x_0\}$ trivially comprises an atom since $\delta_{x_0}(\{x_0\}) > 0$ and the only subset having strictly smaller measure is the empty set. Following this reasoning more generally, we see that no atomless measure μ can assign positive weight to a singleton.

Finally, we introduce a notion of regularity for measures, capturing an approximability of our metric measure space by its compact subsets.

Definition 1.3.16. A finite Borel measure μ on a metric space (X, d) is called *inner regular* provided

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E \,\text{compact}\}$$

for any $E \subseteq X$ Borel. We also refer to any such measure as a (finite) *Radon measure.*

Proposition 1.3.17. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. Then any finite Borel measure μ on X is inner regular. In particular, the equivalent probability measure $\tilde{\mathfrak{m}}$, as in Remark 1.3.13, is a Radon measure.*

See Theorem 7.17 in [5] for a proof.

Remark 1.3.18. This result also implies that the push-forward measure $T_*\tilde{\mathfrak{m}}$ is a Radon measure for any measurable $T\colon X \to Y$ into a complete and separable metric space (Y, d_Y). Indeed, we can simply apply the proposition to the metric measure space $(Y, \mathrm{d}_Y, T_*\tilde{\mathfrak{m}})$.

1.3.3 The Bochner integral

As we will encounter the need to talk about integrability of maps taking values in infinite-dimensional Banach spaces, we collect here some results on a possible extension of the Lebesgue integral to this more general setting, namely the *(Lebesgue-)Bochner integral.*

Most definitions are taken from [13] Chapter 1.3, with a number of results from *Topics in Real Analysis* by Gerald Teschl [23] which will be cited more explicitly.

Our basic setup consists of a Banach space $(\mathbb{B}, \|\cdot\|)$ together with a metric measure space (X, d, μ), where μ is a probability measure on X. Unsurprisingly, maps $f\colon X \to \mathbb{B}$ which can be written in the form

$$f = \sum_{i=1}^{n} 1_{E_i} v_i$$

with $\{v_i\}_{i=1}^n \subseteq \mathbb{B}$ and $\{E_i\}_{i=1}^n \subseteq X$ a Borel partition of X, are called *simple.*

A first subtlety arises in defining an appropriate notion of measurability since in infinite dimensions the usual definition does no longer guarantee the existence of approximating simple maps, as shown in the following lemma, a proof of which can be found in [23] Lemma 5.20.

Lemma 1.3.19. *A map $f\colon X \to \mathbb{B}$ is the pointwise limit of simple maps if and only if it is measurable and has separable range.*

We therefore need to introduce a stronger notion of measurability, incorporating a countability assumption on the set of values a given map can take.

Definition 1.3.20. We call a map $f\colon X \to \mathbb{B}$ *strongly measurable* if it is measurable and *essentially separably valued*, meaning that $\mathbb{B}$ contains a separable subset V with $f(x) \in V$ for μ-a.e. $x \in X$.

With this definition at hand, we can proceed to define the integral along the usual lines of Lebesgue integration theory: For a simple map $f = \sum_{i=1}^{n} 1_{E_i} v_i$ and a Borel subset $E \subseteq X$, we set

$$\int_E f \, \mathrm{d}\mu := \sum_{i=1}^{n} \mu(E_i \cap E)\, v_i,$$

calling f *integrable* on X if $\mu(E_i) < \infty$ whenever $v_i \neq 0$.

For a general measurable map $f\colon X \to \mathbb{B}$ we first define *Bochner integrability* as the requirement that there exist simple maps $(f_n)_{n\geq 1}$ satisfying the condition

$$\lim_{n\to\infty} \int \|f_n - f\| \, \mathrm{d}\mu = 0$$

and then set

$$\int_E f \, \mathrm{d}\mu := \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu.$$

Well-definedness of this object hinges upon the straightforward observation that the condition $\lim_{n\to\infty} \int \|f_n - f\| \, \mathrm{d}\mu = 0$ guarantees that the sequence $(\int_E f_n \, \mathrm{d}\mu)_{n\geq 1}$ converges in $\mathbb{B}$ via the triangle inequality for integrals of simple maps. It further assures that its value is independent of the chosen approximating sequence $(f_n)_{n\geq 1}$.

Notice also that a Bochner integrable map f is automatically strongly measurable with $\|f\| \in L^1(\mu)$. Indeed,

$$\int \|f\| \, \mathrm{d}\mu \leq \int \|f_n\| \, \mathrm{d}\mu + \int \|f_n - f\| \, \mathrm{d}\mu \leq \|f_n\| + \epsilon$$

for n large enough. Moreover, a slight generalization of Lemma 1.3.19 (c.f. Lemma 1.3.2 in [13]) ensures strong measurability once f is measurable and $\|f_n - f\| \to 0$ merely μ-a.e. The first condition is true by assumption, while the second is easily seen to again be a consequence of $\lim_{n\to\infty} \int \|f_n - f\| \, \mathrm{d}\mu = 0$.

What is more, we have the following characterization of our integrability-assumption, see [23] Lemma 5.21.

Proposition 1.3.21. *A map $f\colon X \to \mathbb{B}$ is Bochner integrable if and only if it is strongly measurable and $\|f\|$ is integrable.*

As expected, we now define the space $L^1(\mu; \mathbb{B})$ as the quotient set of all Bochner integrable maps $f\colon X \to \mathbb{B}$ under a.e.-equivalence with respect to the measure μ and endow it with the norm

$$\|f\|_{L^1(\mu;\mathbb{B})} := \int \|f\| \, \mathrm{d}\mu.$$

Similarly, we denote by $L^p(\mu;\mathbb{B})$ with $p \geq 1$ the quotient set of strongly measurable maps $f\colon X \to \mathbb{B}$ for which

$$\|f\|_{L^p(\mu;\mathbb{B})} := \left(\int \|f\|^p \,\mathrm{d}\mu\right)^{1/p}$$

is finite. Notice how, by Proposition 1.3.21, both formulations agree for $p = 1$.

Proposition 1.3.22. *The spaces $L^p(\mu;\mathbb{B})$ are Banach spaces for any $1 \leq p \leq \infty$ with the integrable simple functions as a dense subset if $p < \infty$. Moreover, if $\mathbb{B}$ is separable, so is $L^p(\mu;\mathbb{B})$ for $1 \leq p < \infty$.*

For a proof of this result, see [23] Theorem 5.32 and Lemma 5.28.

Example 1.3.23. In our context, the Bochner integral will most often appear when changing perspective on an iterated integral of the form

$$\int_0^1 \int F(\gamma, t) \,\mathrm{d}\pi(\gamma) \,\mathrm{d}t$$

with π a probability measure on $\mathcal{C}([0,1], X)$ for a metric measure space X and with a suitable function $F \in L^1(\pi \otimes \lambda^1)$. Here, λ^1 denotes the restriction of the Lebesgue measure to $[0,1]$.

As it is usually also done for non-classical solutions of parabolic PDEs (cf. the introduction to Chapter 10.1 in [8]), we can now reinterpret F as a *curve into a Banach space*, in our example $L^1(\pi)$. This can be achieved by considering

$$[0,1] \ni t \mapsto F(\,\cdot\,, t) \in L^1(\pi).$$

Then, for example, the integrability condition $F \in L^1(\pi \otimes \lambda^1)$ translates to

$$\int_0^1 \|F(\,\cdot\,, t)\|_{L^1(\pi)} \,\mathrm{d}t < \infty$$

and we are led to wonder how the spaces $L^1(\lambda^1; L^1(\pi))$ and $L^1(\pi \otimes \lambda^1)$ are related. This will be the content of Theorem 1.3.25.

Having introduced all basic notions, we collect some further results reminiscent of the finite-dimensional setting, again taken from [13], specifically Proposition 1.3.9.

Proposition 1.3.24. *Let $\mathbb{V}$ be a further Banach space and let $E \subseteq X$ be a Borel subset of X. Then the following hold true:*

i) For any $f \in L^1(\mu;\mathbb{B})$ we have

$$\left\|\int_E f \,\mathrm{d}\mu\right\| \leq \int_E \|f\| \,\mathrm{d}\mu.$$

In particular, this implies that integration is a linear and continuous operation on $L^1(\mu;\mathbb{B})$.

ii) The space of continuous and bounded maps $\mathcal{C}_b(X, \mathbb{B})$ is densely contained within $L^1(\mu; \mathbb{B})$.

iii) Any linear and continuous map $\ell\colon \mathbb{B} \to \mathbb{V}$ commutes with integration in the sense that $\ell \circ f \in L^1(\mu; \mathbb{V})$ for every $f \in L^1(\mu; \mathbb{B})$ and

$$\ell\left(\int_E f \,\mathrm{d}\mu\right) = \int_E \ell \circ f \,\mathrm{d}\mu.$$

We will now discuss the relation of $L^1(\mu; L^1(\nu))$ and $L^1(\mu \otimes \nu)$ for two metric measure spaces (X, d_X, μ) and (Y, d_Y, ν) with μ and ν both finite, anticipated in Example 1.3.23. For this, we need to distinguish a function $g\colon X \to \mathbb{R}$ from its equivalence class in $L^1(\mu)$ and we will henceforth denote the latter by $[g]_\mu$.

In fact, if $L^1(\nu)$ were to consist simply of individual *functions*, for a given μ-integrable map $f\colon X \to L^1(\nu)$, $x \mapsto f_x$, we could set

$$\tilde{f}(x, y) := f_x(y) \tag{1.3.3}$$

right away to get a measurable function $\tilde{f}\colon X \times Y \to \mathbb{R}$ with

$$\int_{X \times Y} |\tilde{f}| \,\mathrm{d}(\mu \otimes \nu) = \int_X \|f_x\|_{L^1(\nu)} \,\mathrm{d}\mu(x) < \infty \tag{1.3.4}$$

by Fubini's theorem, i.e. an element of $L^1(\mu \otimes \nu)$. Notice that this also establishes the equality $\|f\|_{L^1(\mu; L^1(\nu))} = \|f\|_{L^1(\mu \otimes \nu)}$.

Having to work with equivalence classes, however, simply evaluating at a given $y \in Y$ is not possible any more and the construction becomes a bit less tangible. Nonetheless, the main idea is still the same: Given a μ-integrable map $f\colon X \to L^1(\nu)$, $x \mapsto [f_x]_\nu$, we choose a Borel function $\tilde{f}\colon X \times Y \to \mathbb{R}$ that serves as a representative of f in the sense that

$$[\tilde{f}(x, \cdot\,)]_\nu = [f_x]_\nu \tag{1.3.5}$$

for μ-a.e. $x \in X$ (compare this to (1.3.3)). We then follow computation (1.3.4) to convince ourselves that also in this case $\tilde{f} \in L^1(\mu \otimes \nu)$. Finally, taking its equivalence class, we have found a way of identifying an element of $L^1(\mu; L^1(\nu))$ with one belonging to the space $L^1(\mu \otimes \nu)$.

That the choice of such a function $\tilde{f}$ is indeed possible and, further, that it is unique up to a.e.-equivalence with respect to the product measure $\mu \otimes \nu$, is the content of Proposition 1.3.19 in [13].

Theorem 1.3.25. *The map* $\Phi : L^1(\mu; L^1(\nu)) \to L^1(\mu \otimes \nu)$ *defined by*

$$[f(\,\cdot\,)]_\nu \mapsto [\tilde{f}]_{\mu\otimes\nu},$$

with $\tilde{f}$ *as in* (1.3.5)*, is an isometric isomorphism of Banach spaces. Moreover,*

$$\left(\int [f(x)] \,\mathrm{d}\mu(x)\right)(y) = \int \tilde{f}(x, y) \,\mathrm{d}\mu(x)$$

for ν*-a.e.* $y \in Y$*.*

See Propositions 1.3.20 and 1.3.21 in [13] for a proof.

We end this section with a result similar to Theorem 1.3.4. For its formulation, we again denote by λ^1 the one-dimensional Lebesgue measure restricted to the interval $[0, 1]$ and revert to a less cluttered notation, in which we leave out the brackets whenever referring to elements of some L^1-space.

Proposition 1.3.26. *Let* $f \colon [0, 1] \to L^1(\nu)$ *and suppose that for some given* $g \in L^1(\lambda^1; L^1(\nu))$*,*

$$|f_t(y) - f_s(y)| \leq \int_s^t g_r(y) \,\mathrm{d}r$$

for ν*-a.e.* $y \in Y$ *and every* $s, t \in [0, 1]$ *with* $s < t$*. Then* f *is absolutely continuous and the strong* $L^1(\nu)$*-limit*

$$|f'_t| := \lim_{h\to 0} \frac{f_{t+h} - f_t}{h} \in L^1(\nu)$$

exists for λ^1*-a.e.* $t \in [0, 1]$*. This derivative further satisfies*

$$|f'_t|(y) \leq g_t(y)$$

for $(\lambda^1 \otimes \nu)$*-a.e. pair* $(t, y) \in [0, 1] \times Y$*.*

For a proof, we refer to [13] Proposition 1.3.23.

2 Sobolev spaces

To introduce a notion of differentiability on a general metric measure space $(X, \mathrm{d}, \mathfrak{m})$, we exploit the possibility of using variational techniques in the presence of a regular Borel measure. More precisely, motivated by the idea of defining the metric speed of a curve $\gamma\colon [0,1] \to X$ as the minimal choice of a non-negative function $h \in L^1(0,1)$ such that

$$d(\gamma_t, \gamma_s) \leq \int_s^t h(r)\,\mathrm{d}r \quad \text{for all } s, t \in [0,1], s < t,$$

roughly speaking, we will define the *minimal weak upper gradient* of a function $f\colon X \to \mathbb{R}$ as the minimal L^2-integrable function $G \geq 0$ satisfying

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi(\gamma) \leq \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t$$

for a family of probability measures $\pi \in \Pi$ which exclude a class of *exceptional* curves.

On $\mathbb{R}^n$, we can motivate this definition very explicitly. To do so, consider a function $f \in \mathcal{C}^1(\mathbb{R}^n)$ and denote by $\mathrm{d}f\colon \mathbb{R}^n \to (\mathbb{R}^n)^*$ its (classical) differential and by ∇f its gradient. Then we immediately see that any continuous function $G\colon \mathbb{R}^n \to [0,\infty)$ satisfying

$$G(x) \geq \|\mathrm{d}f(x)\| = \sup_{\|v\| \leq 1} |\mathrm{d}f(x)v| \quad \text{for all } x \in \mathbb{R}^n$$

must also fulfill

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)\|\dot{\gamma}_t\|\,\mathrm{d}t \tag{2.0.1}$$

for every $\gamma \in \mathcal{C}^1([0,1], \mathbb{R}^n)$. In fact, we can simply invoke the fundamental theorem of calculus to compute

$$\begin{aligned} |f(\gamma_1) - f(\gamma_0)| = \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\,\mathrm{d}t \right| &= \left| \int_0^1 \mathrm{d}f(\gamma_t)\dot{\gamma}_t\,\mathrm{d}t \right| \\ &\leq \int_0^1 \|\mathrm{d}f(\gamma_t)\|\|\dot{\gamma}_t\|\,\mathrm{d}t \\ &\leq \int_0^1 G(\gamma_t)\|\dot{\gamma}_t\|\,\mathrm{d}t. \end{aligned}$$

The crucial observation here is that with the given regularity assumption on G, we can also revert the argument. Indeed, suppose indirectly that $G(x_0) < \|\mathrm{d}f(x_0)\| = \|\nabla f(x_0)\|$

for some $x_0 \in \mathbb{R}^n$. Then, by continuity, the same holds within a ball $B_\epsilon(x_0)$ around x_0 of radius $\epsilon > 0$. Now, choose γ in $\mathcal{C}^1([0,1],\mathbb{R}^n)$ with image entirely inside this ball, starting at $\gamma_0 = x_0$ and moving forward with velocity $\dot\gamma_t$ proportional to $\nabla f(\gamma_t)$. This choice is possible due to Peano's local existence theorem for ordinary differential equations with continuous right-hand side (up to reparametrizing the curve as to be defined on the interval $[0,1]$; which is the reason for merely requiring proportionality rather than equality). Along this particular curve γ, we now have

$$\begin{aligned}|f(\gamma_1) - f(\gamma_0)| = \left|\int_0^1 \langle \nabla f(\gamma_t), \dot\gamma_t\rangle \,\mathrm{d}t\right| &= \int_0^1 \|\nabla f(\gamma_t)\|\|\dot\gamma_t\| \,\mathrm{d}t \\ &> \int_0^1 G(\gamma_t)\|\dot\gamma_t\| \,\mathrm{d}t,\end{aligned}$$

completing the argument.

Our aim is therefore, as already hinted at, to characterize the *norm of the differential* $\|\mathrm{d}f\|$ as a minimal element of a suitable class of functions satisfying an inequality of the form (2.0.1), derived from the *fundamental theorem of calculus*. In our approach, we will merely impose that this relation be satisfied in a certain *weak sense*, allowing a weak upper gradient to violate (2.0.1) on a negligible set of curves, see Chapter 1.1. More precisely, we ask for the inequality to hold when integrated against a family of Radon measures on $\mathcal{C}([0,1],X)$ which we will now define. We again closely follow the presentation in [13].

2.1 Test plans

Definition 2.1.1. Let $(X, d, \mathfrak{m})$ be a metric measure space and let π be a Borel probability measure on $(\mathcal{C}([0,1],X), \mathbf{d})$. We call π a *test plan* on X provided

$i)$ there exists a constant $C > 0$ such that $(e_t)_*\pi \leq C\mathfrak{m}$ for each $t \in [0,1]$, (2.1.1)

$ii)$ it holds that $\int \mathrm{E}(\gamma)\,\mathrm{d}\pi(\gamma) < \infty$. (2.1.2)

Further, we denote by $\mathrm{Comp}(\pi)$ the infimum among all constants in item (i) and call it the *compression constant* of π.

Remark 2.1.2. The first condition in this definition describes a compression estimate for π in the following sense:

Consider the set of all curves γ taking values in a Borel subset $A \subseteq X$ at a specified time $t \in [0,1]$. Then π measures the size of this set to be no more than a multiple $C\mathfrak{m}(A)$ of the measure of A itself. Crucially, this bound has to hold *uniformly* in both t and A. Roughly, this means that the test plan does not concentrate too much at any given time, see Example 2.1.3. As a direct consequence of this estimate, the plan does not select any curves taking values in an $\mathfrak{m}$-null set and it transfers properties that hold true in an almost everywhere sense on X to the space $\mathcal{C}([0,1],X)$, see e.g. Proposition 2.1.5.

The second item restricts the space of curves on which π is concentrated to the set $\mathrm{AC}([0,1],X)$ of absolutely continuous curves in X. By the very definition of the kinetic energy functional E, cf. Definition 1.3.5, we therefore also get the equality

$$\int \mathrm{E}(\gamma)\,\mathrm{d}\pi(\gamma) = \int_0^1 \int |\dot{\gamma}_t|^2\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t,$$

upon using Fubini-Tonelli's theorem (the left-hand side is finite and all involved quantities are non-negative). Hence, condition (*ii*) describes a bound on the speed of the curves selected by the plan for, in general, we only have L^1-integrability of the metric speed.

Example 2.1.3. Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space with $\mathfrak{m}$ atomless and suppose at some time $t \in [0,1]$ the push-forward measure $(e_t)_*\pi$ were to be concentrated at some $x_0 \in X$, i.e.

$$(e_t)_*\pi = \delta_{x_0}.$$

Then π is not a test plan on X. Indeed, for any $\epsilon > 0$, we have

$$(e_t)_*\pi(B_\epsilon(x_0)) = \delta_{x_0}(B_\epsilon(x_0)) = 1$$

while

$$\lim_{\epsilon\searrow 0}\mathfrak{m}(B_\epsilon(x_0)) = \mathfrak{m}\Big(\bigcap_{\epsilon>0} B_\epsilon(x_0)\Big) = \mathfrak{m}(\{x_0\}) = 0$$

by continuity from above and the fact that $\mathfrak{m}$ possesses no atoms, see Remark 1.3.15. This clearly makes it impossible to find a constant $C > 0$ such that

$$(e_t)_*\pi(B_\epsilon(x_0)) \leq C\mathfrak{m}(B_\epsilon(x_0)) \quad \text{for all } \epsilon > 0.$$

Definition 2.1.4. Let $s,t \in [0,1]$ and let $\Gamma \subseteq \mathcal{C}([0,1],X)$ be Borel. We define restriction operators on the space of curves $\mathcal{C}([0,1],X)$ and on its space of Borel probability measures $\mathcal{P}(\mathcal{C}([0,1],X))$ as follows:

i) For any $\gamma \in \mathcal{C}([0,1],X)$, we define a continuous curve $\mathrm{Res}_t^s(\gamma) \in \mathcal{C}([0,1],X)$ as

$$\mathrm{Res}_t^s(\gamma)_r := \gamma_{(1-r)t+rs}$$

for each point in time $r \in [0,1]$.

ii) For any $\mu \in \mathcal{P}(\mathcal{C}([0,1],X))$ such that $\mu(\Gamma) > 0$ we define a probability measure $\mathrm{Res}_\Gamma(\mu)$ by setting

$$\mathrm{Res}_\Gamma(\mu) := \frac{1}{\mu(\Gamma)}\mu|_\Gamma.$$

Proposition 2.1.5. *Denote by* Π *the set of all test plans on* X*. Then*

i) Π *is stable under localization in time:* $(Res_t^s)_*\pi \in \Pi$ *for any* $\pi \in \Pi$.

ii) Π *is stable under localization in space:* $Res_\Gamma(\pi) \in \Pi$ *for any* $\pi \in \Pi$ *satisfying* $\pi(\Gamma) > 0$.

iii) Any $\pi \in \Pi$ *preserves pointwise convergence: Suppose a sequence of Borel functions* $(f_n)_{n\geq 1}$ *on* X *converges pointwise a.e. to* $f\colon X \to \mathbb{R}$*. Then* $(f_n(\gamma_t))_{n\geq 1}$ *converges to* $f(\gamma_t)$ *for* π*-a.e.* $\gamma \in \mathcal{C}([0,1],X)$ *and each* $t \in [0,1]$.

Proof.
i) We need to check that $(\mathrm{Res}_t^s)_*\pi$ is a probability measure on $\mathcal{C}([0,1],X)$ and that is satisfies both the compression estimate and the energy bound. Clearly, $(\mathrm{Res}_t^s)_*\pi \in \mathcal{P}(\mathcal{C}([0,1],X))$ for any $\pi \in \Pi$. Also,

$$(e_r)_*(\mathrm{Res}_t^s)_*\pi = (e_r \circ \mathrm{Res}_t^s)_*\pi = (e_{\rho(r)})_*\pi \leq \mathrm{Comp}(\pi)\,\mathfrak{m}$$

for an arbitrary $r \in [0,1]$, where $\rho(r) := (1-r)t + rs$. To verify the energy bound, notice first that $\mathrm{Res}_t^s(\mathrm{AC}([0,1],X)) \subseteq \mathrm{AC}([0,1],X)$ since for any curve σ of the form $\mathrm{Res}_t^s(\gamma) = \gamma \circ \rho$ we have

$$d(\sigma_r, \sigma_l) = d(\gamma_{\rho(r)}, \gamma_{\rho(l)}) \leq \int_{\rho(r)}^{\rho(l)} h(u)\,\mathrm{d}u = \int_r^l h(\rho(v))(s-t)\,\mathrm{d}v$$

for all $r, l \in [0,1]$, where $h \in L^1(0,1)$ is as in definition 1.3.2. This allows us to compute the metric speed of any such curve as

$$|\dot\sigma_r| = |\rho'(r)||\dot\gamma_{\rho(r)}| = |s-t||\dot\gamma_{\rho(r)}|. \tag{2.1.3}$$

Hence, denoting by $\mathsf{ms}\colon \mathcal{C}([0,1],X) \to L^1(0,1)$ the map associating to each γ the function $t \mapsto |\dot\gamma_t|$ and using the integral transformation formula (1.3.2) we conclude

$$\begin{aligned}\int |\dot\gamma_r|^2\,\mathrm{d}((\mathrm{Res}_t^s)_*\pi)(\gamma) &= \int \mathsf{ms}^2(\gamma)(r)\,\mathrm{d}((\mathrm{Res}_t^s)_*\pi)(\gamma)\\ &\overset{(1.3.2)}{=} \int (\mathsf{ms}^2 \circ \mathrm{Res}_t^s)(\gamma)(r)\,\mathrm{d}\pi(\gamma)\\ &\overset{(2.1.3)}{=} |s-t|^2 \int |\dot\gamma_{\rho(r)}|^2\,\mathrm{d}\pi(\gamma).\end{aligned}$$

Since further

$$\int_0^1 |\dot\gamma_{\rho(r)}|^2\,\mathrm{d}r = \frac{1}{|s-t|}\int_t^s |\dot\gamma_r|^2\,\mathrm{d}r,$$

we finally have

$$\begin{aligned}\int_0^1 \int |\dot\gamma_r|^2 \, \mathrm{d}((\mathrm{Res}_t^s)_*\pi)(\gamma) \, \mathrm{d}r &= |s-t| \int_t^s \int |\dot\gamma_r|^2 \, \mathrm{d}\pi(\gamma) \, \mathrm{d}r \\ &\leq |s-t| \int_0^1 \int |\dot\gamma_r|^2 \, \mathrm{d}\pi(\gamma) \, \mathrm{d}r < \infty.\end{aligned}$$

ii) We again check the two requirements for a test plan separately. The compression estimate for $\mathrm{Res}_\Gamma(\pi)$ with $\pi \in \Pi$ such that $\pi(\Gamma) > 0$ is immediate because of the trivial bound

$$(e_t)_*(\mathrm{Res}_\Gamma(\pi)) = \frac{1}{\pi(\Gamma)}(e_t)_*(\pi|_\Gamma) \leq \frac{1}{\pi(\Gamma)}(e_t)_*\pi.$$

Further,

$$\int \mathrm{E}(\gamma) \, \mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma) = \frac{1}{\pi(\Gamma)} \int_\Gamma \mathrm{E}(\gamma) \, \mathrm{d}\pi(\gamma) < \infty.$$

Hence, $\mathrm{Res}_\Gamma(\pi) \in \mathcal{P}(\mathcal{C}([0,1], X))$ is a test plan.

iii) We want to convince ourselves that the set of all curves at which $(f_n \circ e_t)_{n\geq 1}$ converges has full π-measure for any $t \in [0,1]$ and any $\pi \in \Pi$. Considering instead the measure of its complement, this is a direct consequence of the compression estimate for the test plan π as the following calculation shows:

$$\begin{aligned}\pi(\{\gamma \mid f_n(\gamma_t) \not\to f(\gamma_t)\}) &= \pi((e_t)^{-1}\{x \mid f_n(x) \not\to f(x)\}) \\ &= (e_t)_*\pi(\{x \mid f_n(x) \not\to f(x)\}) \\ &\leq \mathrm{Comp}(\pi) \, \mathfrak{m}(\{x \mid f_n(x) \not\to f(x)\}) = 0.\end{aligned}$$ □

To be able to define an $L^2(\mathfrak{m})$-based Sobolev space on X using the notion of test plans, we prepare the following easy but far-reaching technical result. It is going to ensure *existence* and *uniqueness* of a suitable generalization of the norm of the differential.

Lemma 2.1.6. *Let π be a test plan on X. Then the map*

$$L^2(\mathfrak{m}) \ni G \mapsto \int_0^1 \int G(\gamma_t) |\dot\gamma_t| \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t$$

is linear and continuous. In particular, for any such G, the map $(t,\gamma) \mapsto G(\gamma_t)|\dot\gamma_t|$ belongs to the space $L^1(\pi \otimes \lambda^1)$.

Remark 2.1.7. By abuse of notation, we will always denote by λ^1 the Lebesgue measure on whichever bounded or unbounded interval we are currently considering.

Proof. Linearity is immediate, and we can therefore deduce continuity from boundedness of the map. For this, notice that the two requirements for a test plan together with the integrability assumption on G are exactly what is needed to establish a bound via Hölder's inequality. Indeed, first using the integral transformation formula (1.3.2) to derive the bound

$$\int_0^1 \int G^2 \circ e_t \,\mathrm{d}\pi\,\mathrm{d}t = \int_0^1 \int G^2 \,\mathrm{d}((e_t)_*\pi)\,\mathrm{d}t \leq \mathrm{Comp}(\pi) \int_0^1 \int G^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \mathrm{Comp}(\pi)\,\|G\|^2_{L^2(\mathfrak{m})},$$

we see that

$$\left(\int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}(\pi\otimes\lambda^1)(\gamma,t)\right)^2 \leq \left(\int_0^1 \int G^2\circ e_t\,\mathrm{d}\pi\,\mathrm{d}t\right)\left(\int_0^1 \int |\dot{\gamma}_t|^2\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t\right) \leq \mathrm{Comp}(\pi)\,\|G\|^2_{L^2(\mathfrak{m})} \int \mathrm{E}(\gamma)\,\mathrm{d}\pi(\gamma) < \infty. \qquad \square$$

Remark 2.1.8. Notice how this shows that

$$\Phi\colon L^2(\mathfrak{m}) \to \mathbb{R},\ G \mapsto \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t$$

is even *weakly continuous* as we have proven that $\Phi \in L^2(\mathfrak{m})^*$, allowing us to conclude

$$\Phi(G_n) \to \Phi(G) \quad \text{whenever} \quad G_n \rightharpoonup G.$$

2.2 The Sobolev class $S^2(X)$

2.2.1 Weak upper gradients

Definition 2.2.1. Let $f\colon X \to \mathbb{R}$ be a Borel function. We say that f belongs to the *Sobolev class* $S^2(X)$ if there exists a function $G \in L^2(\mathfrak{m})$, $G \geq 0$ such that

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi(\gamma) \leq \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t \tag{2.2.1}$$

for every test plan π on X. We call any such G a *weak upper gradient* of f.

Remark 2.2.2. For a given test plan π on X,

$$f\circ e_1 - f\circ e_0 \in L^1(\pi) \quad \text{for any } f \in S^2(X)$$

as a direct consequence of Lemma 2.1.6. We keep this in mind for an equivalent but more convenient description of the class $S^2(X)$, namely Theorem 2.2.12, which we will prepare for over the next few pages.

As a first showcase of the utility of the localization properties inherent to the family of test plans, we can immediately recover from this weak formulation an almost everywhere statement along curves.

Proposition 2.2.3. *Let $f \in S^2(X)$ and let G be a weak upper gradient for f. Then for any test plan π on X and π-almost every γ, we have*

$$|f(\gamma_s) - f(\gamma_t)| \leq \int_t^s G(\gamma_r)|\dot{\gamma}_r| \, \mathrm{d}r \tag{2.2.2}$$

for all $s, t \in [0,1]$ with $s > t$.

Proof. We argue by contradiction and hence suppose the existence of a test plan π together with two points $s > t$ in time and a Borel set $\Gamma \subseteq \mathcal{C}([0,1], X)$ with $\pi(\Gamma) > 0$ on which inequality (2.2.2) fails to hold. Integrating this reversed inequality over Γ and averaging leaves us with

$$\frac{1}{\pi(\Gamma)} \int_\Gamma |f(\gamma_s) - f(\gamma_t)| \, \mathrm{d}\pi(\gamma) > \frac{1}{\pi(\Gamma)} \int_\Gamma \int_t^s G(\gamma_r)|\dot{\gamma}_r| \, \mathrm{d}\pi(\gamma) \, \mathrm{d}r. \tag{2.2.3}$$

The idea now is to restrict π, invoking Proposition 2.1.5, and use the defining relation (2.2.1) for the restricted test plan $(\mathrm{Res}_t^s)_* \mathrm{Res}_\Gamma(\pi)$, denoted in the following by $\hat{\pi}$. That is, we want to compare (2.2.3) to

$$\int |f(\sigma_1) - f(\sigma_0)| \, \mathrm{d}\hat{\pi}(\sigma) \leq \int_0^1 \int G(\sigma_r)|\dot{\sigma}_r| \, \mathrm{d}\hat{\pi}(\sigma) \, \mathrm{d}r, \tag{2.2.4}$$

where we have denoted by σ curves in the image of $\mathrm{Res}_t^s \colon \mathcal{C}([0,1], X) \to \mathcal{C}([0,1], X)$.

To prepare, we first transform the two pieces of inequality (2.2.4) separately using the integral transformation formula (1.3.2). For the first piece, this gives

$$\begin{aligned}
\int |f(\sigma_1) - f(\sigma_0)| \, \mathrm{d}\hat{\pi}(\sigma) &= \int |f \circ e_1 - f \circ e_0| \, \mathrm{d}((\mathrm{Res}_t^s)_* \mathrm{Res}_\Gamma(\pi)) \\
&= \int |f \circ e_1 - f \circ e_0| \circ \mathrm{Res}_t^s \, \mathrm{d}(\mathrm{Res}_\Gamma(\pi)) \\
&= \int |f(\gamma_s) - f(\gamma_t)| \, \mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma),
\end{aligned}$$

so that

$$\int |f(\sigma_1) - f(\sigma_0)| \, \mathrm{d}\hat{\pi}(\sigma) = \frac{1}{\pi(\Gamma)} \int_\Gamma |f(\gamma_s) - f(\gamma_t)| \, \mathrm{d}\pi(\gamma). \tag{2.2.5}$$

For the second piece, we begin by recalling from the proof of Proposition 2.1.5 that $\mathsf{ms}(\mathrm{Res}_t^s(\gamma), r) = (s-t)|\dot{\gamma}_{\rho(r)}|$, where $\mathsf{ms}(\gamma, r)$ is the metric speed of the curve γ at time r and $\rho(r) = (1-r)t + rs$. With this at hand, we similarly compute

$$\begin{aligned}
\int_0^1 \int G(\sigma_r)|\dot{\sigma}_r| \, \mathrm{d}\hat{\pi}(\sigma) \, \mathrm{d}r &= \int_0^1 \int (G \circ e_r) \cdot \mathsf{ms}(\,\cdot\,, r) \; \mathrm{d}((\mathrm{Res}_t^s)_* \mathrm{Res}_\Gamma(\pi)) \, \mathrm{d}r \\
&= \int_0^1 \int ((G \circ e_r) \cdot \mathsf{ms}(\,\cdot\,, r)) \circ \mathrm{Res}_t^s \; \mathrm{d}(\mathrm{Res}_\Gamma(\pi)) \, \mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int G(\gamma_{\rho(r)})\, \mathsf{ms}(\mathrm{Res}_t^s(\gamma), r)\ \mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma)\, \mathrm{d}r \\
&= \int_0^1 \int G(\gamma_{\rho(r)})(s-t)|\dot\gamma_{\rho(r)}|\ \mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma)\, \mathrm{d}r.
\end{aligned}
$$

Using Fubini (all quantities are non-negative), we can absorb the factor $(s-t)$ into the second differential by substituting $u = \rho(r)$ so that $\mathrm{d}u = (s-t)\,\mathrm{d}r$. Concretely,

$$
\int_0^1\int G(\gamma_{\rho(r)})(s-t)|\dot\gamma_{\rho(r)}|\,\mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma)\,\mathrm{d}r = \int\int_t^s G(\gamma_u)|\dot\gamma_u|\,\mathrm{d}(\mathrm{Res}_\Gamma(\pi))(\gamma)\,\mathrm{d}u.
$$

Putting this together, we have established

$$
\int_0^1 \int G(\sigma_r)|\dot\sigma_r|\,\mathrm{d}\hat\pi(\sigma)\,\mathrm{d}r = \frac{1}{\pi(\Gamma)}\int_\Gamma\int_t^s G(\gamma_u)|\dot\gamma_u|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}u \tag{2.2.6}
$$

for the second piece.

Summing up, these computations contradict (2.2.3) due to the stability of the family of test plans under restrictions, as previously claimed. Indeed, rewriting both sides in (2.2.4) according to our calculations (2.2.5) and (2.2.6) yields exactly the reversed inequality:

$$
\begin{aligned}
\frac{1}{\pi(\Gamma)}\int_\Gamma |f(\gamma_s)-f(\gamma_t)|\,\mathrm{d}\pi(\gamma) &= \int |f(\sigma_1)-f(\sigma_0)|\,\mathrm{d}\hat\pi(\sigma) \\
&\overset{(2.2.1)}{\leq} \int_0^1\int G(\sigma_r)|\dot\sigma_r|\,\mathrm{d}\hat\pi(\sigma)\,\mathrm{d}r \\
&= \frac{1}{\pi(\Gamma)}\int_\Gamma\int_t^s G(\gamma_u)|\dot\gamma_u|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}u
\end{aligned}
$$ □

2.2.2 Interlude: Lipschitz functions

In the following, a lot of computations will involve estimates for Lipschitz functions on X. We therefore briefly collect a few important notions and results on this topic.

Definition 2.2.4. We say that a function $f\colon X \to \mathbb{R}$ is *Lipschitz* if the quantity

$$
\mathrm{Lip}(f) := \sup_{x,y\in X, x\neq y} \frac{|f(y)-f(x)|}{\mathrm{d}(y,x)} \tag{2.2.7}
$$

is finite. The set of all such functions on X is denoted by $\mathrm{Lip}(X)$.

Further, given $f \in \mathrm{Lip}(X)$ we define a function $\mathrm{lip}(f)\colon X \to [0,\infty)$ by

$$
\mathrm{lip}(f)(x) := \varlimsup_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(y,x)} \tag{2.2.8}
$$

for any accumulation point $x \in X$ and by $\mathrm{lip}(f)(x) = 0$ otherwise. Its value at a given point $x \in X$ is called the *local* Lipschitz constant for f at x.

Lemma 2.2.5. *Let $f \in \mathrm{Lip}(X)$ and $\gamma \in \mathrm{AC}([0,1],X)$. Then the composite function $f \circ \gamma$ is again absolutely continuous with*

$$|(f \circ \gamma)'_t| \leq \mathrm{lip}(f)(\gamma_t)|\dot{\gamma}_t| \tag{2.2.9}$$

for a.e. $t \in [0,1]$.

Proof. To establish absolute continuity of $f \circ \gamma$, take $s, t \in [0,1]$ with $s < t$ and assume for the moment that $\gamma_t \neq \gamma_s$. Then,

$$|f(\gamma_t) - f(\gamma_s)| = \frac{|f(\gamma_t) - f(\gamma_s)|}{\mathrm{d}(\gamma_t, \gamma_s)} \mathrm{d}(\gamma_t, \gamma_s) \leq \mathrm{Lip}(f) \int_s^t |\dot{\gamma}_r| \,\mathrm{d}r$$

with Theorem 1.3.4 guaranteeing the required integrability. By non-negativity of the right hand side, the inequality trivially holds in the remaining case $\gamma_t = \gamma_s$, so that $f \circ \gamma \in \mathrm{AC}([0,1],X)$.

To prove (2.2.9), we use essentially the same trick and thus again distinguish cases: If γ is not constant in any neighborhood of $t \in [0,1]$ (where both $(f \circ \gamma)'_t$ and $|\dot{\gamma}_t|$ exist), we similarly estimate

$$\begin{aligned}
(f \circ \gamma)'_t &= \lim_{h \to 0} \frac{|f(\gamma_{t+h}) - f(\gamma_t)|}{|h|} \\
&\leq \overline{\lim_{h \to 0}} \frac{|f(\gamma_{t+h}) - f(\gamma_t)|}{\mathrm{d}(\gamma_{t+h}, \gamma_t)} \lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|} \\
&\leq \mathrm{lip}(f)(\gamma_t)|\dot{\gamma}_t|.
\end{aligned}$$

Exactly as before, the inequality trivially remains true in the complementary case where γ is constant on some neighborhood of t since this enforces $(f \circ \gamma)'_t = 0$, whereas the right hand side is again non-negative. □

Proposition 2.2.6. *Let $f \in \mathrm{Lip}(X)$ have bounded support. Then f belongs to the class $S^2(X)$.*

Proof. We want to prove that, in this case, the function $\mathrm{lip}(f)$ qualifies as a weak upper gradient for f. Therefore, we start by checking $L^2(\mathfrak{m})$-integrability.

For this, we first observe that $\mathrm{lip}(f)$ vanishes outside the support of f. Indeed, choosing any $x \in X \setminus \mathrm{supp}(f)$, we can find a whole neighborhood $\tilde{U}$ of x with the property that

$$f(y) = 0 \quad \text{for all } y \in \tilde{U}.$$

Thus,

$$\mathrm{lip}(f)(x) = \inf_{U \in \mathcal{U}(x)} \sup_{y \in U \setminus \{x\}} \frac{|f(x) - f(y)|}{\mathrm{d}(y, x)} \leq \sup_{y \in \tilde{U} \setminus \{x\}} \frac{|f(x) - f(y)|}{\mathrm{d}(y, x)} = 0$$

with $\mathcal{U}(x)$ denoting the neighborhood filter for x.

Now, recall that $\mathfrak{m}$ is boundedly finite and note that $\mathrm{lip}(f) \leq \mathrm{Lip}(f) < \infty$. Hence, the estimate

$$\| \mathrm{lip}(f) \|_{L^2(\mathfrak{m})}^2 \leq (\mathrm{Lip}(f))^2\, \mathfrak{m}(\mathrm{supp}(f))$$

implies $\mathrm{lip}(f) \in L^2(\mathfrak{m})$.

Next, invoking Lemma 2.2.5, we see that $f \circ \gamma \in \mathrm{AC}([0,1], X)$ for any absolutely continuous curve γ in X. Moreover, by (2.2.9) we have

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 \mathrm{lip}(f)(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}t,$$

so that integrating with respect to any test plan π on X yields (2.2.1) on the subspace $\mathrm{AC}([0,1], X)$. This finishes the proof by Remark 2.1.2, which implies that it is indeed sufficient to check condition (2.2.1) merely for absolutely continuous curves. □

To be able to use local results in a global setting, we would like to resort to a cut-off argument while, of course, remaining within the class $\mathrm{Lip}(X)$.

Definition 2.2.7. For $x_0 \in X$ and $r > 0$, we define a *Lipschitz cut-off function* $\eta_{x_0,r} \in \mathrm{Lip}(X)$ as

$$\eta_{x_0,r} := (1 - \mathrm{d}(\,\cdot\,, \overline{B_r(x_0)}))^+, \tag{2.2.10}$$

where $(\,\cdot\,)^+ := \max(\,\cdot\,, 0)$.

Remark 2.2.8. The Lipschitz functions $\eta_{x_0,r}$ *cut off* around x_0 in the following sense:

$$\eta_{x_0,r} = 1 \text{ on } \overline{B_r(x_0)} \text{ with } \mathrm{supp}(\eta_{x_0,r}) \subseteq \overline{B_{r+1}(x_0)} \text{ and } \mathrm{lip}(\eta_{x_0,r}) \leq 1 \tag{2.2.11}$$

Indeed, the first claim follows immediately from the definition of $\mathrm{d}(\,\cdot\,, \overline{B_r(x_0)})$. For the second assertion, choose $(y_n)_{n\geq 1}$ with $\eta_{x_0,r}(y_n) > 0$ converging to some given $y \in \mathrm{supp}(\eta_{x_0,r})$. Noticing that

$$\eta_{x_0,r}(y_n) > 0 \quad \text{if and only if} \quad \mathrm{d}(y_n, \overline{B_r(x_0)}) < 1,$$

we conclude

$$\mathrm{d}(x_0, y_n) \leq \mathrm{d}(x_0, x) + \mathrm{d}(x, y_n) < r + 1,$$

having chosen an arbitrary $x \in \overline{B_r(x_0)}$. Now, simply taking limits yields the claim.

Finally, the local Lipschitz-bound again follows easily using the triangle inequality since

$$\mathrm{d}(y, \overline{B_r(x_0)}) - \mathrm{d}(x, \overline{B_r(x_0)}) \leq d(y, x) \tag{2.2.12}$$

for any $x, y \in X$ implies $\eta_{x_0,r} \in \mathrm{Lip}(X)$ and

$$\begin{aligned}
\mathrm{lip}(\eta_{x_0,r})(x) = &\ \varlimsup_{y\to x} \frac{(1-(\mathrm{d}(y,\overline{B_r(x_0)}))^+ - (1-\mathrm{d}(x,\overline{B_r(x_0)}))^+}{\mathrm{d}(y,x)} \\
\leq &\ \varlimsup_{y\to x} \frac{(\mathrm{d}(y,\overline{B_r(x_0)}) - \mathrm{d}(x,\overline{B_r(x_0)}))^+}{\mathrm{d}(y,x)} \\
\overset{(2.2.12)}{\leq} &\ \varlimsup_{y\to x} \frac{\mathrm{d}(y,x)}{\mathrm{d}(y,x)} \\
= &\ 1.
\end{aligned}$$

To shorten notation for what is to come, we introduce two symbols $\vee$ and $\wedge$ to mean

$$f \vee g := \max(f,g) \quad \text{and} \quad f \wedge g := \min(f,g)$$

for any two real-valued functions f and g. For example, given a constant $m \in \mathbb{R}$, the function $(f \vee (-m)) \wedge m$ confines the image of f to the interval $[-m, m]$, extending it at constant height wherever its values would exceed these bounds.

Remark 2.2.9. Using $f \vee g = \frac{1}{2}(f + g + |f-g|)$ for general f and g with a similar formula for $f \wedge g$, we can conclude that once we know $|u| \in W^{1,1}(0,1)$ for $u \in W^{1,1}(0,1)$, we also have

$$f \vee g \in W^{1,1}(0,1) \quad \text{and} \quad f \wedge g \in W^{1,1}(0,1) \tag{2.2.13}$$

for $f, g \in W^{1,1}(0,1)$. A proof of this can be found in Lemma 7.6 in *Elliptic Partial Differential Equations of Second Order* by David Gilbarg and Neil S. Trudinger [14]. This result also provides an explicit expression for these weak derivatives, as for example

$$(f \vee g)' = \frac{1}{2}(f' + g' + \mathrm{sign}(f-g)(f'-g')). \tag{2.2.14}$$

Lemma 2.2.10. *Let $f\colon X \to \mathbb{R}$ be a Borel function and let $\gamma \in \mathrm{AC}([0,1], X)$ such that $f \circ \gamma \in W^{1,1}(0,1)$. Then the restricted functions $f^{mn} \circ \gamma$ also belong to $W^{1,1}(0,1)$, where*

$$f^{mn} := \eta_{x_0,n}((f \vee (-m)) \wedge m) \tag{2.2.15}$$

with $x_0 \in X$ and integers $m, n \geq 1$. Further,

$$|(f^{mn} \circ \gamma)'_t| \leq m|\dot{\gamma}_t| 1_{X\setminus B_n(x_0)}(\gamma_t) + |(f \circ \gamma)'_t| \quad \mathfrak{m}\text{-a.e.} \tag{2.2.16}$$

Proof. We start off by proving $f^{mn} \circ \gamma \in W^{1,1}(0,1)$. Conveniently, Proposition 1.3.8, stating that $W^{1,1}(0,1)$ is an algebra, allows us to investigate the two factors of the product in (2.2.15) separately. This can be done easily:

To ensure $\eta_{x_0,n} \circ \gamma \in W^{1,1}(0,1)$, we can use Lemma 2.2.5, stating that the composition of a function $\eta_{x_0,n} \in \mathrm{Lip}(X)$ and a curve $\gamma \in \mathrm{AC}([0,1], X)$ is absolutely continuous and then recall that $\mathrm{AC}([0,1],\mathbb{R}) \subseteq W^{1,1}(0,1)$, as explained in Remark 1.3.3. For the second factor, it is enough to notice that $W^{1,1}(0,1)$ is stable under both $\vee$ and $\wedge$ by (2.2.13).

For the estimate (2.2.16), we resort to the Leibniz rule for weak derivatives, c.f. Proposition 1.3.8. Estimating the derivative of the first factor $\eta_{x_0,n} \circ \gamma$ and recalling the conclusion of Lemma 2.2.5, we have

$$|(\eta_{x_0,n} \circ \gamma)'_t| \leq \mathrm{lip}(\eta_{x_0,n})(\gamma_t)|\dot{\gamma}_t| \leq 1_{X\setminus B_n(x_0)}(\gamma_t)|\dot{\gamma}_t|$$

using (2.2.9) for the first and (2.2.11) for the second bound, noticing additionally that $\mathrm{lip}(\eta_{x_0,n}) = 0$ on $B_n(x_0)$. Since clearly $|((f \circ \gamma) \vee (-m)) \wedge m| \leq m$, in total, we have a bound on the first summand in the Leibniz rule of the form

$$m\,|\dot{\gamma}_t| 1_{X\setminus B_r(x_0)}(\gamma_t).$$

For the second summand, we observe that $|\eta_{x_0,r}| \leq 1$ and that the magnitude of the derivative of the second factor in (2.2.15) is bounded above by $|(f\circ\gamma)'_t|$ almost everywhere. In fact, by (2.2.14) we have $(f \vee (-m))' = \frac{1}{2}(f' + \mathrm{sign}(f+m)\,f') \leq f'$ with a similar calculation for the outer term. Taken together, this establishes the claimed estimate. $\square$

2.2.3 Alternative characterizations

To connect the newly introduced concept of weak upper gradients to more familiar notions of L^p-derivatives and classical Sobolev spaces, we now prove equivalent characterizations of $S^2(X)$. To do so, however, we need one more technical lemma.

Lemma 2.2.11. *Given any test plan π on X and any $p \in [1,\infty)$, the map*

$$[0,1] \ni t \mapsto f \circ e_t \in L^p(\pi)$$

is continuous, provided $f \in L^p(\mathfrak{m})$.

Proof. We start with the easy case in which $f \in \mathcal{C}_b(X) \cap L^p(\mathfrak{m})$. Here, the claim follows easily from dominated convergence. In fact, continuity ensures

$$|f(\gamma_s) - f(\gamma_t)|^p \to 0 \ \text{ as } \ s \to t$$

for any $\gamma \in \mathcal{C}([0,1],X)$, while the boundedness assumption guarantees

$$|f(\gamma_s) - f(\gamma_t)|^p \leq 2\|f\|^p_{\mathcal{C}_b(X)} \in L^1(\pi).$$

Hence, invoking dominated convergence,

$$\lim_{s\to t} \|f \circ e_s - f \circ e_t\|_{L^p(\pi)} = 0. \tag{2.2.17}$$

The general case now follows from density of $\mathcal{C}_b(X) \cap L^p(\mathfrak{m})$ within $L^p(\mathfrak{m})$, cf. for example item (*ii*) in Proposition 1.3.24. Indeed, by the integral transformation formula (1.3.2)

$$\|f \circ e_t\|_{L^p(\pi)}^p = \int |f|^p \circ e_t \, \mathrm{d}\pi \overset{(1.3.2)}{=} \int |f|^p \, \mathrm{d}((e_t)_* \pi) \le \mathrm{Comp}(\pi) \|f\|_{L^p(\mathfrak{m})}^p, \tag{2.2.18}$$

allowing us to conclude with the usual argumentation: Having fixed some $t \in [0,1]$, we choose a sequence $(g_k)_{k \ge 1} \subseteq \mathcal{C}_b(X) \cap L^p(\mathfrak{m})$ converging in L^p to a given $f \in L^p(\mathfrak{m})$. Then, by the triangle inequality,

$$\begin{aligned} \|f \circ e_s - f \circ e_t\|_{L^p(\pi)} \le \|f \circ e_s - g_k \circ e_s\|_{L^p(\pi)} &+ \|g_k \circ e_s - g_k \circ e_t\|_{L^p(\pi)} \\ &+ \|g_k \circ e_t - f \circ e_t\|_{L^p(\pi)}, \end{aligned}$$

where the outer terms are seen to vanish (independently of s and t) as $k \to \infty$ using the common bound

$$\mathrm{Comp}(\pi)^{1/p} \|f - g_k\|_{L^p(\mathfrak{m})}$$

guaranteed by (2.2.18), while the remaining inner piece vanishes as $s \to t$ due to (2.2.17). ☐

Theorem 2.2.12. *Let $f \colon X \to \mathbb{R}$ be a Borel function and let $G \in L^2(\mathfrak{m})$ with $G \ge 0$. Then the following are equivalent:*

i) The function f belongs to the Sobolev class $S^2(X)$ and G is a weak upper gradient for f.

ii) For any test plan π on X, the map $t \mapsto f \circ e_t - f \circ e_0 \in L^1(\pi)$ is absolutely continuous and possesses an $L^1(\pi)$-derivative for a.e. $t \in [0,1]$. Moreover, this derivative, denoted by $\mathrm{Der}_\pi(f)$, satisfies

$$|\mathrm{Der}_\pi(f)_t|(\gamma) \le G(\gamma_t)|\dot{\gamma}_t|$$

for $\pi \otimes \lambda^1$-a.e. pair (γ, t).

iii) For any test plan π on X and π-almost every γ, the composite function $f \circ \gamma$ belongs to the Sobolev space $W^{1,1}(0,1)$. In addition, its derivative satisfies

$$|(f \circ \gamma)'_t| \le G(\gamma_t)|\dot{\gamma}_t|$$

for a.e. $t \in [0,1]$.

Moreover, we have the equality $\mathrm{Der}_\pi(f)_t(\gamma) = (f \circ \gamma)'_t$ for $\pi \otimes \lambda^1$-a.e. pair (γ, t) if any of the above hold.

Proof.
(i) $\implies$ *(ii)*:
For absolute continuity of the map $t \mapsto f \circ e_t - f \circ e_0$, we check the requirements of Proposition 1.3.26. That is, we argue that there is some $g \in L^1(\lambda^1; L^1(\pi))$ such that

$$|f(\gamma_s) - f(\gamma_t)| \leq \int_t^s g_r(\gamma)\,\mathrm{d}r$$

for π-a.e. γ and any pair $(s,t) \in [0,1]^2$ with $s > t$. This, however, has already been accomplished in Proposition 2.2.2, recalling further that $L^1(\lambda^1; L^1(\pi)) \cong L^1(\lambda^1 \otimes \pi)$ by Theorem 1.3.25 with $(\gamma, t) \mapsto G(\gamma_t)|\dot{\gamma}_t| \in L^1(\pi \otimes \lambda^1)$, as proven in Lemma 2.1.6. Thus, invoking the Theorem of Fubini-Tonelli establishes the claim.

Moreover, Proposition 1.3.26 additionally asserts that under these assumptions, the strong $L^1(\pi)$-limit of

$$\frac{f \circ e_{t+h} - f \circ e_t}{h}$$

exists for a.e. $t \in [0,1]$ as $h \to 0$ and that $|\mathrm{Der}_\pi(f)_t|(\gamma) \leq G(\gamma_t)|\dot{\gamma}_t|$ for a.e. pair (γ, t), having denoted this limit by $\mathrm{Der}_\pi(f)$.
(ii) $\implies$ *(iii)*:
Having fixed a test plan π on X, we need to show that $f \circ \gamma$ possesses a weak derivative on the interval $[0,1]$ for π-a.e. curve γ. Moreover, we also want both $f \circ \gamma$ and $(f \circ \gamma)'$ to be L^1-integrable there.

By assumption, $t \mapsto f \circ e_t - f \circ e_0$ is absolutely continuous with values in the Banach space $L^1(\pi)$ for a.e. $t \in [0,1]$, hence an element of $L^1(\lambda^1; L^1(\pi))$ by item *(ii)* in Proposition 1.3.24. Written out explicitly, this means that for a.e. $t \in [0,1]$,

$$(\,\gamma \mapsto f(\gamma_t) - f(\gamma_0)\,) \in L^1(\pi).$$

Using the theorem of Fubini-Tonelli to conclude $L^1(\lambda^1; L^1(\pi)) \cong L^1(\pi; L^1(\lambda^1))$, we see that, equivalently, for π-a.e. γ, we have

$$(\,t \mapsto f(\gamma_t) - f(\gamma_0)\,) \in L^1(\lambda^1),$$

hence, $f \circ \gamma$ is integrable for π-a.e. γ.

To prove the remaining assertions, we prepare the following identity: For π-a.e. curve γ and alomst every $s, t \in [0,1]$ with $s > t$, we have

$$f(\gamma_s) - f(\gamma_t) = \int_t^s \mathrm{Der}_\pi(f)_r(\gamma)\,\mathrm{d}r. \tag{2.2.19}$$

Indeed, using Fubini's theorem and dominated convergence, we get

$$\begin{aligned}
&\lim_{h\to 0}\left\|\int_t^s \mathrm{Der}_\pi(f)_r - \frac{f\circ e_{r+h} - f\circ e_r}{h}\,\mathrm{d}r\right\|_{L^1(\pi)} \\
&\quad= \int_t^s \lim_{h\to 0}\left\|\mathrm{Der}_\pi(f)_r - \frac{f\circ e_{r+h} - f\circ e_r}{h}\right\|_{L^1(\pi)}\mathrm{d}r = 0,
\end{aligned}$$

by definition of $\mathrm{Der}_\pi(f)$, so that, up to choosing a subsequence, we have the pointwise equality

$$\int_t^s \mathrm{Der}_\pi(f)_r(\gamma)\,\mathrm{d}r = \lim_{h\to 0}\int_t^s \frac{f(\gamma_{r+h}) - f(\gamma_r)}{h}\,\mathrm{d}r$$

for π-a.e. γ. Thus, by the fundamental theorem of calculus and continuity of the map $r\mapsto f\circ e_r$,

$$\begin{aligned}
\int_t^s \mathrm{Der}_\pi(f)_r(\gamma)\,\mathrm{d}r &= \lim_{h\to 0}\left(\frac{1}{h}\int_t^s f(\gamma_{r+h})\,\mathrm{d}r - \frac{1}{h}\int_t^s f(\gamma_r)\,\mathrm{d}r\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_s^{s+h} f(\gamma_r)\,\mathrm{d}r - \frac{1}{h}\int_t^{t+h} f(\gamma_r)\,\mathrm{d}r\right) \\
&= f(\gamma_s) - f(\gamma_t)
\end{aligned}$$

for almost every s,t with $s>t$ and π-a.e. γ.

Now, we are in the position to apply Lemma 1.3.6, which together with the just established equality (2.2.19), allows us to conclude $f\circ\gamma\in W^{1,1}(0,1)$. Further,

$$\int_t^s (f\circ\gamma)'_r\,\mathrm{d}r = f(\gamma_s) - f(\gamma_t) = \int_t^s \mathrm{Der}_\pi(f)_r(\gamma)\,\mathrm{d}r$$

for almost every s,t where $s>t$ so that, $(f\circ\gamma)'_t = \mathrm{Der}_\pi(f)_t(\gamma)$ for a.e. $t\in[0,1]$. This finally implies $|(f\circ\gamma)'_t| = |\mathrm{Der}_\pi(f)_t|(\gamma)\le G(\gamma_t)|\dot\gamma_t|$ for a.e. $t\in[0,1]$.
(iii) $\implies$ *(i)*:
The final implication is a bit subtle. What we would like to do is to conclude that

$$|f(\gamma_1) - f(\gamma_0)| \le \int_0^1 |(f\circ\gamma)'_r|\,\mathrm{d}r \le \int_0^1 G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r, \tag{2.2.20}$$

ensured by the fundamental theorem of calculus, implies our claim when integrated over $\mathcal{C}([0,1],X)$ with respect to a given test plan π. However, this argumentation fails as $f\circ\gamma$ satisfies the fundamental theorem only in *almost every point*, cf. Lemma 1.3.6. In particular, we cannot simply evaluate at the endpoints. To remedy the situation, we approximate f by the functions f^{mn} we constructed in Lemma 2.2.10 with the advantage that these belong to $L^1(\mathfrak{m})$, meaning that the map

$$[0,1]\ni t\mapsto f^{mn}\circ e_t\in L^1(\pi)$$

is continuous by Lemma 2.2.11. Therefore, inequality (2.2.20) indeed holds when applied to f^{mn} instead of f.

From there on, we are left merely with an approximation argument. Since the f^{mn} clearly convergence pointwise almost everywhere to f, Fatou's lemma and dominated convergence, together with the bound (2.2.16) yield

$$\begin{aligned}
\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) &\leq \varliminf_{m\to\infty} \varliminf_{n\to\infty} \int |f^{mn}(\gamma_1) - f^{mn}(\gamma_0)| \,\mathrm{d}\pi(\gamma) \\
&\overset{(2.2.20)}{\leq} \varliminf_{m\to\infty} \varliminf_{n\to\infty} \int \int_0^1 |(f^{mn} \circ \gamma)'_t| \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) \\
&\overset{(2.2.16)}{\leq} \varliminf_{m\to\infty} \varliminf_{n\to\infty} \int \Big[m|\dot\gamma_t| 1_{X\setminus B_n(x_0)}(\gamma_t) + |(f\circ\gamma)'_t| \Big] \,\mathrm{d}\pi(\gamma) \\
&= \varliminf_{m\to\infty} \int |(f\circ\gamma)'_t| \,\mathrm{d}\pi(\gamma) \\
&\leq \int \int_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t \,\mathrm{d}\pi(\gamma),
\end{aligned}$$

where we have used the fundamental theorem for f^{mn} in the second line, as explained above. ☐

Proposition 2.2.13. *The class $S^2(X)$ is stable under the operation $\wedge$. That is, the pointwise minimum $G_1 \wedge G_2$ of two weak upper gradients G_1 and G_2 of $f \in S^2(X)$ is again a weak upper gradient of f.*

Proof. This is an easy consequence of the equivalence of items (*i*) and (*ii*) in the previous theorem. Indeed, since L^2-integrability and non-negativity are clearly preserved, it remains to show that

$$|\mathrm{Der}_\pi(f)_t|(\gamma) \leq (G_1 \wedge G_2)(\gamma_t)|\dot\gamma_t|$$

for π-a.e. curve γ and almost every $t \in [0,1]$. This, however, holds trivially because the left-hand side is a common lower bound for $G_1(\gamma_t)|\dot\gamma_t|$ and $G_2(\gamma_t)|\dot\gamma_t|$. ☐

2.3 The Sobolev space $W^{1,2}(X)$

2.3.1 The minimal weak upper gradient $|Df|$

Having introduced the Sobolev class $S^2(X)$ and having explored some of its properties, we now wish to find a way to associate to a given $f \in S^2(X)$ a *unique* weak upper gradient that will serve as a generalization of the norm of $\mathrm{d}f$ in Euclidean space, as outlined in the introduction. For what is to come, keep in mind how our motivating example also convinced us that such an element might be found through *minimization* within a suitable class of functions.

Proposition 2.3.1. *The set of all weak upper gradients of a given $f \in S^2(X)$ is closed and convex in $L^2(\mathfrak{m})$ and hence contains a unique element of minimal norm.*

Proof. Fix any test plan π on X. Invoking Lemma 2.1.6, we directly prove both convexity and closedness. To this end, let G_1 and G_2 be two weak upper gradients for f and let $\lambda \in [0,1]$. Denoting by Φ the *linear and continuous* map $L^2(\mathfrak{m}) \ni H \mapsto \int_0^1 \int H(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t$ with $H \geq 0$ as in Lemma 2.1.6, the straightforward calculation

$$\begin{aligned}\Phi(\lambda G_1 + (1-\lambda)G_2) &= \lambda\Phi(G_1) + (1-\lambda)\Phi(G_2)\\ &\geq (\lambda + (1-\lambda)) \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma)\\ &= \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma)\end{aligned}$$

establishes convexity. Next, suppose a sequence of weak upper gradients $(G_n)_{n\geq 1}$ converges strongly to some $G \in L^2(\mathfrak{m})$. Then,

$$\Phi(G) = \lim_{n\to\infty} \Phi(G_n) \geq \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma)$$

ensuring closedness. Finally, as a non-empty, closed and convex subset of a Hilbert space, the set of weak upper gradients contains a unique minimal element. □

Definition 2.3.2. For a given $f \in S^2(X)$, we call the weak upper gradient of minimal $L^2(\mathfrak{m})$-norm found in Proposition 2.3.1 the *minimal weak upper gradient* of f and denote it by $|Df|$.

Using our theorem on alternative characterizations of $S^2(X)$, we see that $|Df|$ is not only minimal in the L^2-sense but, in fact, even pointwise $\mathfrak{m}$-a.e.

Proposition 2.3.3. *Any weak upper gradient G of some given $f \in S^2(X)$ satisfies*

$$|Df| \leq G \quad \mathfrak{m}\text{-a.e.}$$

Proof. By way of contradiction, suppose we could find some weak upper gradient $G \in L^2(\mathfrak{m})$ such that $\mathfrak{m}(\{G < |Df|\}) > 0$. Invoking Proposition 2.2.13, we see that $G \wedge |Df|$

is again a weak upper gradient. However,

$$\begin{aligned}\| G \wedge |Df| \|_{L^2(\mathfrak{m})}^2 &= \int_{G<|Df|} | G \wedge |Df| |^2 \,\mathrm{d}\mathfrak{m} + \int_{|Df|\le G} | G \wedge |Df| |^2 \,\mathrm{d}\mathfrak{m} \\ &= \int_{G<|Df|} |G|^2 \,\mathrm{d}\mathfrak{m} + \int_{|Df|\le G} |Df|^2 \,\mathrm{d}\mathfrak{m} \\ &< \int_{G<|Df|} |Df|^2 \,\mathrm{d}\mathfrak{m} + \int_{|Df|\le G} |Df|^2 \,\mathrm{d}\mathfrak{m} = \| |Df| \|_{L^2(\mathfrak{m})}^2,\end{aligned}$$

contradicting L^2-minimality of $|Df|$. □

Next, we would like to find a suitable type of convergence for sequences of functions $(f_n)_{n\geq 1}$ in $S^2(X)$. That is, we want to find appropriate notions for which $f_n \to f$ with weak upper gradients $G_n \to G$ ensures $f \in S^2(X)$ with G as a weak upper gradient for f. The most general result for our purposes is given by the following proposition.

Proposition 2.3.4. *Suppose $(f_n)_{n\geq 1} \subseteq S^2(X)$ converges pointwise a.e. to some Borel function $f\colon X \to \mathbb{R}$ with corresponding weak upper gradients $(G_n)_{n\geq 1}$ converging weakly to some G in $L^2(\mathfrak{m})$. Then the limiting function f also belongs to the Sobolev class $S^2(X)$ and G constitutes a weak upper gradient for f.*

Proof. Denoting again by $\Phi : L^2(\mathfrak{m}) \to \mathbb{R}$ the map $H \mapsto \int_0^1 \int H(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t$, we want to prove

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \le \Phi(G).$$

We begin by recalling from item (*iii*) in Proposition 2.1.5 that $f_n \circ e_t \xrightarrow[n\to\infty]{} f \circ e_t$ π-a.e. for any $t \in [0,1]$. Thus, by Fatou's lemma and the definition of the G_n,

$$\begin{aligned}\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) &\le \varliminf_{n\to\infty} \int |f_n(\gamma_1) - f_n(\gamma_0)| \,\mathrm{d}\pi(\gamma) \\ &\le \varliminf_{n\to\infty} \Phi(G_n).\end{aligned}$$

Finally, by Remark 2.1.8 and weak convergence of $(G_n)_{n\geq 1}$, we conclude

$$\lim_{n\to\infty} \Phi(G_n) = \Phi(G),$$

finishing the proof. □

Definition 2.3.5 (Sobolev space)**.** We define the *Sobolev space* $W^{1,2}(X)$ associated to X as the space of $f \in S^2(X)$ additionally satisfying the integrability assumption $f \in L^2(\mathfrak{m})$, in short $W^{1,2}(X) := S^2(X) \cap L^2(\mathfrak{m})$. Further, we define a norm $\| \,.\, \|_{W^{1,2}(X)}$ on this space as

$$\|f\|_{W^{1,2}(X)} := \sqrt{\|f\|_{L^2(\mathfrak{m})}^2 + \| |Df| \|_{L^2(\mathfrak{m})}^2}$$

for any $f \in W^{1,2}(X)$.

Lemma 2.3.6. *The class $S^2(X)$ carries the structure of an $\mathbb{R}$-vector space. More specifically, we have*

$$|D(\lambda f)| = |\lambda||Df| \quad \mathfrak{m}\text{-a.e.} \tag{2.3.1}$$
$$|D(f+g)| \leq |Df| + |Dg| \quad \mathfrak{m}\text{-a.e.} \tag{2.3.2}$$

for all $f, g \in S^2(X)$ and $\lambda \in \mathbb{R}$.

Proof. We start with claim (2.3.2) and show that $|Df| + |Dg|$ constitutes a weak upper gradient for the sum $f + g$. This will then ensure

$$|D(f+g)| \leq |Df| + |Dg| \quad \mathfrak{m}\text{-a.e.}$$

by Proposition 2.3.3. Although we could easily prove the assertion starting with our original definition (2.2.1), we instead check item *(iii)* in Theorem 2.2.12 for reasons of visual clarity. We therefore fix a test plan π on X and estimate

$$\begin{aligned} |((f+g)\circ\gamma)'_t| &\leq |(f\circ\gamma)'_t| + |(g\circ\gamma)'_t| \\ &\leq (\,|Df| + |Dg|\,)(\gamma_t)|\dot{\gamma}_t| \end{aligned}$$

for π-a.e. curve γ and a.e. $t \in [0,1]$ using the triangle inequality. This already establishes our claim, again by Theorem 2.2.12.

We proceed analogously for the claimed relation (2.3.1). Since in this case we even have the equality

$$|((\lambda f)\circ\gamma)'_t| = |\lambda|\,|(f\circ\gamma)'_t|,$$

we can separately bound the right-hand side by $|\lambda||Df|$ to establish $|D(\lambda f)| \leq |\lambda||Df|$ $\mathfrak{m}$-a.e. and then the left-hand side to get $|D(\lambda f)| \geq |\lambda||Df|$ $\mathfrak{m}$-a.e., having used Proposition 2.3.3 in both cases. $\square$

Theorem 2.3.7. *The space $(W^{1,2}(X), \|\,.\,\|_{W^{1,2}(X)})$ is a Banach space.*

Proof. First, we prove that $(W^{1,2}(X), \|\,.\,\|_{W^{1,2}(X)})$ is a normed space and start by noticing that the previous lemma guarantees that $W^{1,2}(X) = S^2(X) \cap L^2(\mathfrak{m})$ is in fact a vector space. Further, the relations (2.3.1) and (2.3.2), also proven in the previous lemma, ensure that $f \mapsto \|\,|Df|\,\|_{L^2(\mathfrak{m})}$ is a seminorm. Since $\|f\|_{W^{1,2}(X)} = 0$ in particular implies $\|f\|_{L^2(\mathfrak{m})} = 0$, we see that $\|\,.\,\|_{W^{1,2}(X)}$ is indeed a norm on the linear space $W^{1,2}(X)$.

It remains to establish completeness. To this end, let $(f_n)_{n\geq 1}$ be any Cauchy sequence in $W^{1,2}(X)$. Then both $(f_n)_{n\geq 1}$ and $(|Df_n|)_{n\geq 1}$ are Cauchy in $L^2(\mathfrak{m})$. Hence, in particular, they are bounded. Moreover, we can find an $f \in L^2(\mathfrak{m})$ such that $f_n \to f$ strongly as $n \to \infty$, so that along some subsequence $(f_{n_k})_{k\geq 1}$ we have

$$f_{n_k} \xrightarrow[k\to\infty]{} f \quad \text{pointwise } \mathfrak{m}\text{-a.e.}$$

Invoking the theorem of Banach-Alaoglu, we conclude

$$|Df_{n_k}| \xrightharpoonup[k\to\infty]{} G$$

for some $G \in L^2(\mathfrak{m})$. In fact, up to the choice of a further subsequence, w.l.o.g. again $(|Df_{n_k}|)_{k\geq 1}$, the above-mentioned boundedness in the Hilbert space $L^2(\mathfrak{m})$ ensures this by a well-known consequence of the theorem of Banach-Alaoglu, see e.g. Theorem 3.18 in [8]. Thus, $f \in W^{1,2}(X)$ with G as a weak upper gradient, by Proposition 2.3.4.

We are left with proving

$$f_n \xrightarrow[k\to\infty]{W^{1,2}(X)} f.$$

This is achieved once we know $f_{n_k} \to f$ in $W^{1,2}(X)$ as $k \to \infty$ since $(f_n)_{n\geq 1}$ is a Cauchy sequence in this space. The L^2-convergence is already established and we therefore check whether

$$|Df_{n_k}| \xrightarrow[k\to\infty]{L^2(\mathfrak{m})} |Df|.$$

Considering instead the shifted sequence $(f_{n_k} - f_{n_m})_{m\geq 1}$, we can find some $H \in L^2(\mathfrak{m})$ with

$$|D(f_{n_k} - f_{n_m})| \xrightharpoonup[m\to\infty]{} H,$$

following the same reasoning as above. By L^2-minimality of $|D(f_{n_k} - f)|$ and weak lower semicontinuity of the $L^2(\mathfrak{m})$-norm, we now have

$$\| \, |D(f_{n_k} - f)| \, \|_{L^2(\mathfrak{m})} \leq \|H\|_{L^2(\mathfrak{m})} \leq \varliminf_{m\to\infty} \| \, |D(f_{n_k} - f_{n_m})| \, \|_{L^2(\mathfrak{m})}.$$

So,

$$\varlimsup_{k\to\infty} \| \, |D(f_{n_k} - f)| \, \|_{L^2(\mathfrak{m})} \leq \varlimsup_{k\to\infty} \varliminf_{m\to\infty} \| \, |D(f_{n_k} - f_{n_m})| \, \|_{L^2(\mathfrak{m})} = 0,$$

again by the fact that $(f_n)_{n\geq 1}$ is $W^{1,2}(X)$-Cauchy. This finally yields the claim. $\square$

We now carve out a general property of minimal weak upper gradients that we have already used in the previous proof, namely the *lower semicontinuity* of the assignment $f \mapsto \| \, |Df| \, \|_{L^2(\mathfrak{m})}$.

Proposition 2.3.8. *Let $f\colon X \to \mathbb{R}$ be a Borel function that is obtained as a pointwise $\mathfrak{m}$-a.e. limit of $(f_n)_{n\geq 1} \subseteq S^2(X)$. Then,*

$$\| \, |Df| \, \|_{L^2(\mathfrak{m})} \leq \varliminf_{n\to\infty} \| \, |Df_n| \, \|_{L^2(\mathfrak{m})} \tag{2.3.3}$$

with $\| \, |Df| \, \|_{L^2(\mathfrak{m})} := \infty$ if f does not belong to $S^2(X)$.

In particular, whenever $g \in L^2(\mathfrak{m})$ is the L^2-limit of some $(g_n)_{n\geq 1} \subseteq W^{1,2}(X)$, we have

$$\| \, |Dg| \, \|_{L^2(\mathfrak{m})} \leq \varliminf_{n\to\infty} \| \, |Dg_n| \, \|_{L^2(\mathfrak{m})}$$

Proof. Up to technical details, we essentially repeat what we have already done in the proof of the previous theorem. Indeed, replacing the sequence by a subsequence if needed, we assume that $(|Df_n|)_{n\geq 1}$ in fact converges in $L^2(\mathfrak{m})$, as there is nothing to prove in the case in which the limit inferior is infinite.

Arguing as above, this now guarantees that $(|Df_n|)_{n\geq 1}$ is bounded in $L^2(\mathfrak{m})$, so that

$$|Df_{n_k}| \xrightharpoonup[k\to\infty]{} G \in L^2(\mathfrak{m})$$

with Proposition 2.3.4 ensuring $f \in S^2(X)$ with G as a weak upper gradient. Hence,

$$\| \, |Df| \, \|_{L^2(\mathfrak{m})} \leq \|G\|_{L^2(\mathfrak{m})} \leq \varliminf_{k\to\infty} \| \, |Df_{n_k}| \, \|_{L^2(\mathfrak{m})} = \varliminf_{n\to\infty} \| \, |Df_n| \, \|_{L^2(\mathfrak{m})}$$

by minimality, weak lower semicontinuity of the norm and finally L^2-convergence of $(|Df_n|)_{n\geq 1}$.

To handle the case in which g is an $L^2(\mathfrak{m})$-limit of $(g_n)_{n\geq 1} \subseteq W^{1,2}(X)$, we pass to a subsequence $(g_{n_k})_{k\geq 1}$ fulfilling

$$g_{n_k} \xrightarrow[k\to\infty]{} g \quad \mathfrak{m}\text{-a.e.}$$

and

$$\lim_{k\to\infty} \| \, |Df_{n_k}| \, \|_{L^2(\mathfrak{m})} = \varliminf_{n\to\infty} \| \, |Df_n| \, \|_{L^2(\mathfrak{m})},$$

where again the right-hand side is assumed finite without loss of generality. Since g is now seen to be a pointwise limit of a sequence in $S^2(X)$, the statement follows by applying the first part, with the final equality enabling a reversion to the initial sequence. □

2.3.2 Calculus rules

Up until now, we do not have gained much control over weak upper gradients of algebraic expressions involving functions in $S^2(X)$. However, having introduced the concept as a possible generalization of the norm of the distributional derivative in Euclidean space, we might hope to be able to state some rules of calculation for minimal weak upper gradients, reminiscent of this classical setting.

To start off, we further develop our calculus for Lipschitz functions from section 2.2.2 as this will provide us with the possibility of resorting to approximation arguments via the lower semicontinuity of $S^2(X) \ni f \mapsto \| \, |Df| \, \|_{L^2(\mathfrak{m})}$ proved in the previous proposition. Our efforts will then culminate in Theorem 2.3.13, where we prove, among other results, versions of the chain rule and the Leibniz rule.

Proposition 2.3.9. *Let $f \in S^2(X)$ and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then $\varphi \circ f \in S^2(X)$ with*

$$|D(\varphi \circ f)| \leq \mathrm{Lip}(\varphi)\, |Df| \quad \mathfrak{m}\text{-a.e.}$$

If, in addition, φ is continuously differentiable, we have

$$|D(\varphi \circ f)| \leq |\varphi'| \circ f\, |Df| \quad \mathfrak{m}\text{-a.e.} \tag{2.3.4}$$

Proof. For the first claim, we simply estimate

$$|(\varphi \circ f)(\gamma_1) - (\varphi \circ f)(\gamma_0)| \leq \mathrm{Lip}(\varphi)|f(\gamma_1) - f(\gamma_0)|,$$

implying that $\mathrm{Lip}(\varphi)|Df|$ is a weak upper gradient for $(\varphi \circ f)$, once we have integrated with respect to any test plan π on X. Thus, $(\varphi \circ f) \in S^2(X)$ with the estimate following from Proposition 2.3.3, ensuring almost everywhere minimality of $|D(\varphi \circ f)|$.

Suppose now that $\varphi \in \mathcal{C}^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ and fix a test plan π on X. Invoking Lemma 1.3.7, we see that

$$(\varphi \circ (f \circ \gamma)) \in W^{1,1}(0,1)$$

for π-a.e. curve γ as Theorem 2.2.12 guarantees $f \circ \gamma \in W^{1,1}(0,1)$. Further, using the estimates provided by the two results stated, we conclude

$$|(\varphi \circ f \circ \gamma)'_t| = |\varphi'|((f \circ \gamma)_t)\, |(f \circ \gamma)'_t| \leq [(|\varphi'| \circ f)|Df|\,](\gamma_t)|\dot{\gamma}_t|,$$

so that $(|\varphi'| \circ f)|Df| \leq \mathrm{Lip}(\varphi)|Df| \in L^2(\mathfrak{m})$ is, in fact, a weak upper gradient for $\varphi \circ f$, again by Theorem 2.2.12. Exactly as above, this implies $|D(\varphi \circ f)| \leq (|\varphi'| \circ f)|Df|$ $\mathfrak{m}$-a.e. $\square$

Proposition 2.3.10. *Let $f \in S^2(X)$ and suppose $N \subseteq \mathbb{R}$ is a null set with respect to λ^1. Then,*

$$|Df| = 0 \quad \mathfrak{m}\text{-a.e. in } f^{-1}(N).$$

Proof. We first argue that it suffices to prove the statement for $K \subseteq \mathbb{R}$ compact. Indeed, let N be any Borel subset of $\mathbb{R}$ and choose a probability measure $\tilde{\mathfrak{m}}$ that is equivalent to $\mathfrak{m}$, cf. Remark 1.3.13. Then, by Remark 1.3.18, the measure $\mu = f_*\tilde{\mathfrak{m}}$ is inner regular, allowing us to find a sequence $(K_n)_{n\geq 1}$ of compact sets with the property that

$$\mu\Big(N \setminus \bigcup_{n\geq 1} K_n\Big) = 0.$$

Written out explicitly, this implies that $f^{-1}(N)$ and $\bigcup_{n\geq 1} f^{-1}(K_n)$ differ only by an $\mathfrak{m}$-null set, establishing our assertion.

Before continuing, we briefly outline the main idea: Recalling lower semicontinuity of $f \mapsto \|\,|Df|\,\|_{L^2(\mathfrak{m})}$, we try to construct a sequence of $\mathcal{C}^1$ Lipschitz functions φ_n

approximating $\mathrm{id}_{\mathbb{R}}$ and possessing the property that their derivates vanish precisely on the null set K (by now chosen to be compact). This will then imply

$$\| \, |Df| \, \|_{L^2(\mathfrak{m})} \leq \varliminf_{n\to\infty} \| \, |D(\varphi_n \circ f)| \, \|_{L^2(\mathfrak{m})} \leq \varliminf_{n\to\infty} \| \, |\varphi_n'| \circ f \, |Df| \, \|_{L^2(\mathfrak{m})}. \tag{2.3.5}$$

Thus,

$$\| \, |Df| \, \|_{L^2(\mathfrak{m})} \leq \| \, 1_{X \setminus f^{-1}(K)} |Df| \, \|_{L^2(\mathfrak{m})} \leq \| \, |Df| \, \|_{L^2(\mathfrak{m})}, \tag{2.3.6}$$

so that, indeed, $|Df|$ has to vanish on $f^{-1}(K)$.

We now start to construct such a sequence. By the above, we may consider $K \subseteq \mathbb{R}$ compact with $\lambda^1(K) = 0$, ensuring

$$\bigcap_{n\geq 1} \{ x \in \mathbb{R} \,|\, \mathrm{d}(x, K) < 1/n \} = K,$$

so that

$$\lim_{n\to\infty} \lambda^1 \left(\{ \mathrm{d}(\,\cdot\,, K) < 1/n \} \right) = \lambda^1(K) = 0 \tag{2.3.7}$$

by continuity from above for the Lebesgue measure and boundedness of K. Since, by the reverse triangle inequality, $\mathrm{d}(\,\cdot\,, K)$ is a *continuous* function on $\mathbb{R}$, we can set $\psi_n := n \, \mathrm{d}(\,\cdot\,, K) \wedge 1$, with $n \geq 1$ an integer, to define a sequence in $\mathcal{C}_b(\mathbb{R})$ with members that become increasingly indistinguishable from the constant function 1 since

$$\lambda^1(\{\psi_n < 1\}) = \lambda^1 \left(\{ \mathrm{d}(\,\cdot\,, K) < 1/n \} \right) \xrightarrow[n\to\infty]{} 0, \tag{2.3.8}$$

due to (2.3.7). Integrating these functions from zero to $t \in \mathbb{R}$, we expect to ever more closely approximate $\mathrm{id}_{\mathbb{R}}$ as $n \to \infty$. In fact, in defining $\varphi_n(t) := \int_0^t \psi_n(s) \, \mathrm{d}t$, we approximate $\mathrm{id}_{\mathbb{R}}$ uniformly by a family of *continuously differentiable Lipschitz functions* with the crucial property that

$$\varphi_n' = \psi_n = 0 \quad \textit{precisely} \text{ on } K. \tag{2.3.9}$$

Indeed, clearly $\varphi_n \in \mathcal{C}^1(\mathbb{R})$ and

$$|\varphi_n(t) - \varphi_n(s)| \leq \int_s^t |\psi_n(r)| \, \mathrm{d}r \leq \|\psi_n\|_{\mathcal{C}_b(\mathbb{R})} |t - s| \leq |t - s|.$$

Furthermore,

$$\sup_{t\in\mathbb{R}} |\varphi_n(t) - t \,| \leq \int_0^\infty |\psi_n(r) - 1| \, \mathrm{d}r \leq \|\psi_n - 1\|_{\mathcal{C}_b(\mathbb{R})} \lambda^1(\{\psi_n < 1\}) \to 0,$$

as a consequence of (2.3.8). In particular, we have

$$\lim_{n\to\infty} (\varphi_n \circ f)(x) = f(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{2.3.10}$$

We can therefore proceed as outlined above, by invoking lower semicontinuity as guaranteed by Proposition 2.3.8 as well as the Lipschitz bound proved in Proposition 2.3.9 to get

$$\int |Df|^2 \,\mathrm{d}\mathfrak{m} \overset{(2.3.3)}{\leq} \varliminf_{n\to\infty} \int |D(\varphi_n \circ f)|^2 \,\mathrm{d}\mathfrak{m} \overset{(2.3.4)}{\leq} \varliminf_{n\to\infty} \int |\varphi_n'|^2 \circ f \, |Df|^2 \mathrm{d}\mathfrak{m} \overset{(2.3.9)}{=} \int 1_{X\setminus f^{-1}(K)} |Df|^2 \mathrm{d}\mathfrak{m}.$$

As explained, cf. inequalities (2.3.5) and (2.3.6), this leads to the conclusion that $|Df|$ vanishes $\mathfrak{m}$-a.e. on $f^{-1}(K)$. □

This result now allows us to relax the technical assumptions posed in Proposition 2.3.9. More concretely, any $\varphi \in \mathrm{Lip}(\mathbb{R})$ is absolutely continuous on $\mathbb{R}$ and thus differentiable almost everywhere by the fundamental theorem of calculus for the Lebesgue integral. We would therefore expect the second conclusion of the proposition to hold without the additional differentiability requirement, at least in an almost everywhere sense. Rigorously proving this involves some calculations with mollifiers, results on which we now briefly collect in the next remark.

Remark 2.3.11. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int \rho \,\mathrm{d}\lambda^n = 1$ (see definition (1.6.2) in *Weakly Differentiable Functions* [24] by William P. Ziemer for an example of such a function). Then for any $\varepsilon > 0$, we further define $\rho_\varepsilon := \varepsilon^{-n}\rho(\frac{\cdot}{\varepsilon}) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so that $\mathrm{supp}(\rho_\varepsilon) \subseteq \overline{B_\varepsilon(0)}$ but still $\int \rho_\varepsilon \,\mathrm{d}\lambda^n = 1$. This family of functions, each member of which we call a *mollifier*, has the crucial property of allowing us to approximate rough functions by sequences of *smooth* functions.

Results of this sort, which we are going to need, are:

i) Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then $(\rho_\varepsilon * f) \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $(\rho_\varepsilon * f) \xrightarrow[\varepsilon\to 0]{} f$ locally in L^1. Moreover,

$$(\rho_\varepsilon * f)(x) \xrightarrow[\varepsilon\to 0]{} f(x) \quad \text{for almost every } x \in \mathbb{R}^d. \tag{2.3.11}$$

ii) If further $f \in \mathcal{C}(\mathbb{R}^d)$, then $(\rho_\varepsilon * f) \xrightarrow[\varepsilon\to 0]{} f$ locally *uniformly*. In particular,

$$(\rho_\varepsilon * f)(x) \xrightarrow[\varepsilon\to 0]{} f(x) \quad \text{for every } x \in \mathbb{R}^d. \tag{2.3.12}$$

See e.g. [24] Theorem 1.6.1.

Proposition 2.3.12. *Let $f \in S^2(X)$ and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then $\varphi \circ f \in S^2(X)$ with*

$$|D(\varphi \circ f)| \leq |\varphi'| \circ f \, |Df| \quad \mathfrak{m}\text{-a.e.} \tag{2.3.13}$$

Proof. As anticipated, we prove this result by regularizing φ with the help of a family of mollifiers $(\rho_\varepsilon)_{\varepsilon>0} \subseteq \mathcal{C}_c^\infty(\mathbb{R})$. This allows us to apply Proposition 2.3.9 to

$$\varphi_\varepsilon := \varphi * \rho_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}),$$

concluding

$$|D(\varphi_\varepsilon \circ f)| \leq |\varphi_\varepsilon'| \circ f\, |Df|. \tag{2.3.14}$$

Our task now is to show that this inequality remains valid in the limit as $\varepsilon \to 0$.

First, notice how $\varphi \in \mathrm{Lip}(\mathbb{R})$ implies that $\varphi \in \mathcal{C}(\mathbb{R})$ and that φ' extends to an element of $L^1_{\mathrm{loc}}(\mathbb{R})$, as briefly mentioned above. Thus,

$$\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \varphi \text{ pointwise} \quad \text{and} \quad \varphi_\varepsilon' \xrightarrow[\varepsilon\to 0]{} \varphi' \text{ a.e.},$$

as stated in Remark 2.3.11. Since

$$|\varphi_\varepsilon'| \circ f\, |Df| \leq \mathrm{Lip}(\varphi)|Df| \in L^2(\mathfrak{m}), \tag{2.3.15}$$

this pointwise convergence implies

$$|\varphi_\varepsilon'| \circ f\, |Df| \xrightarrow[\varepsilon\to 0]{L^2(\mathfrak{m})} |\varphi'| \circ f\, |Df|$$

by the dominated convergence theorem and the observation that $|Df|$ vanishes $\mathfrak{m}$-a.e. in the set of points $x \in \mathbb{R}$ where $\varphi_\varepsilon'(f(x))$ does not converge to $\varphi'(f(x))$ or $\varphi'(f(x))$ does not exist, due to the previous proposition.

The inequalities (2.3.14) and (2.3.15) now permit the final conclusion

$$|D(\varphi \circ f)| \leq |\varphi'| \circ f\, |Df| \quad \mathfrak{m}\text{-a.e.},$$

as they imply that we have found a family of functions $\varphi_\varepsilon \circ f$ in $S^2(X)$ with weak upper gradients $|\varphi_\varepsilon'| \circ f\, |Df|$ converging (weakly) to $|\varphi'| \circ f\, |Df|$ while $\varphi_\varepsilon \circ f \to \varphi \circ f$ pointwise. Hence, Proposition 2.3.4 gives the claim. $\square$

Theorem 2.3.13. *The association of a minimal weak upper gradient to functions $f, g \in S^2(X)$ obeys the following rules of calculation:*

A) Locality. Within the set $\{f = g\}$, we have $|Df| = |Dg|$ $\mathfrak{m}$-a.e.

B) Chain rule.
i) $|Df|$ vanishes wherever f takes values in a λ-null set, i.e.

$$|Df| = 0 \quad \mathfrak{m}\text{-a.e. on } f^{-1}(N) \text{ for any } N \subseteq \mathbb{R} \text{ null.}$$

ii) For any Lipschitz function $\varphi\colon \mathbb{R} \to \mathbb{R}$, we have $\varphi \circ f \in S^2(X)$ and

$$|D(\varphi \circ f)| = |\varphi'| \circ f \, |Df| \quad \mathfrak{m}\text{-a.e.}$$

with $|\varphi'|$ arbitrarily defined wherever φ' does not exist.

C) Leibniz rule. If $f, g \in S^2(X) \cap L^\infty(\mathfrak{m})$, then $fg \in S^2(X) \cap L^\infty(\mathfrak{m})$ and

$$|D(fg)| \leq |f||Dg| + |g||Df| \quad \mathfrak{m}\text{-a.e.}$$

Proof.
A) By instead considering the function $h := f - g \in S^2(X)$, this is easily seen to be a consequence of Proposition 2.3.10 and the triangle inequality (2.3.2) as

$$\big|\,|Df| - |Dg|\,\big| \leq |Dh| = 0$$

$\mathfrak{m}$-a.e. within $h^{-1}(0) = \{f = g\}$.

B) i) This is exactly the content of Proposition 2.3.10.

ii) To prove this claim, we make use of the following observation: Assume $\varphi' \geq 0$ and suppose we could find some $\psi \in \mathrm{Lip}(\mathbb{R})$ such that

$$\psi + \varphi = \mathrm{id}_{\mathbb{R}} \quad \text{with} \quad \psi' \geq 0. \tag{2.3.16}$$

Then, by the triangle inequality (2.3.2) and Proposition 2.3.12,

$$\begin{aligned} |Df| &\overset{(2.3.16)}{\leq} |D(\psi \circ f)| + |D(\varphi \circ f)| \\ &\overset{(2.3.13)}{\leq} (|\psi'| \circ f + |\varphi'| \circ f)|Df| \\ &\overset{(2.3.16)}{=} ((1 - \varphi') \circ f + \varphi' \circ f)|Df| \\ &= |Df|, \end{aligned}$$

implying

$$|D(\varphi \circ f)| = \varphi' \circ f \, |Df| \quad \mathfrak{m}\text{-a.e.}$$

in this special case, as otherwise we could find a Borel set of positive $\mathfrak{m}$-measure on which $|Df| \leq |D(\psi \circ f)| + |D(\varphi \circ f)| < \psi' \circ f \, |Df| + \varphi' \circ f \, |Df| = |Df|$.

We therefore try to identify such a function ψ, generalizing the argument slightly to handle at once also the case of an arbitrary sign of φ'. To this end, we set

$$\psi := \operatorname{sgn}(\varphi')\,\mathrm{id}_{\mathbb{R}} - \varphi \in \operatorname{Lip}(\mathbb{R}) \tag{2.3.17}$$

and assume $\operatorname{Lip}(\varphi) = 1$ (otherwise we replace φ by ${}^{1}/_{\operatorname{Lip}(\varphi)}\,\varphi$). This definition now indeed implies

$$\operatorname{sgn}(\varphi')\psi' = \operatorname{sgn}(\varphi')(\operatorname{sgn}(\varphi') - \varphi') = (1 - |\varphi'|) = (\operatorname{Lip}(\varphi) - |\varphi'|) \geq 0$$

which is consistent with (2.3.16), in the case of positive sign. More generally, we now have

$$|\psi'| + |\varphi'| = \operatorname{sgn}(\varphi')(\psi' + \varphi') = \operatorname{sgn}(\varphi')(\operatorname{sgn}(\varphi') - \varphi' + \varphi') = 1,$$

allowing us to follow the above calculation without restricting to a particular sign of φ' and thereby establishing the full claim.

C) Let $f, g \in S^2(X) \cap L^\infty(\mathfrak{m})$ and fix any test plan π on X. Then we know by Theorem 2.2.12 that for π-a.e. curve γ the composite functions $f \circ \gamma$ and $g \circ \gamma$ belong to the Sobolev space $W^{1,1}(0,1)$ with

$$|(f \circ \gamma)'_t| \leq |Df|(\gamma_t)|\dot{\gamma}_t| \quad \text{and} \quad |(g \circ \gamma)'_t| \leq |Dg|(\gamma_t)|\dot{\gamma}_t| \tag{2.3.18}$$

for a.e. $t \in [0,1]$. Proposition 1.3.8 now permits the conclusion that also $(fg) \circ \gamma \in W^{1,1}(0,1)$ with

$$|((fg) \circ \gamma)'_t| \leq |f|(\gamma_t)|(g \circ \gamma)'_t| + |g|(\gamma_t)|(f \circ \gamma)'_t| \leq (\,|f||Dg| + |g||Df|\,)(\gamma_t)|\dot{\gamma}_t|$$

for a.e. $t \in [0,1]$ by the Leibniz rule, together with (2.3.18). This gives the claim by Theorem 2.2.12 and Proposition 2.3.3 with the boundedness assumption guaranteeing integrability of the right-hand side as $L^\infty(\mathfrak{m}) \cdot L^2(\mathfrak{m}) \subseteq L^2(\mathfrak{m})$. □

Remark 2.3.14. Finally, it is worth mentioning that the definition of the Sobolev space we gave here is consistent with the classical definition on $\mathbb{R}^n$. More precisely, it can be proven that a given Borel function $f\colon \mathbb{R}^n \to \mathbb{R}$ belongs to the Sobolev space $W^{1,2}(\mathbb{R}^n)$ in our sense if and only if both f and its distributional differential $\mathrm{d}f$ are square-integrable. Moreover, in this case, the minimal weak upper gradient $|Df|$ agrees λ^n-almost everywhere with the norm of $\mathrm{d}f$, see Theorem 2.1.37 in [13].

3 Normed Modules

After having found a substitute for the differential of a function on a general metric measure space, we would now like to generalize *one-forms*. The usual definition in smooth differential geometry is via $\mathcal{C}^\infty$-sections of the cotangent bundle. Equivalently, these sections were found to constitute a module over the ring $\mathcal{C}^\infty(M)$, leading to a more algebraic interpretation. Generalizing this approach and removing the need for a differential structure by integration, we arrive at the theory of normed modules.

To start off, we briefly discuss the classical situation of an oriented n-dimensional Riemannian manifold (M, g), which we assume to be second countable and Hausdorff. In this setting, one can check that

$$\mathcal{C}_c(M) \ni f \mapsto \sum_i \int_{\varphi(U_i)} (\chi_i \sqrt{\det(g)} f) \circ \varphi^{-1} \, \mathrm{d}\lambda^n$$

defines a continuous, positive linear functional on $\mathcal{C}_c(M)$, independent of the choice of atlas $(U_i, \varphi_i)_i$ and partition of unity $\{\chi_i\}_i$ subordinate to $\{U_i\}_i$. See e.g. Definition 1.73 in *Nonlinear Analysis on Manifolds. Monge-Ampère Equations* [3] by Thierry Aubin. Hence, by the Riesz representation theorem (cf. Theorem 7.28 in *Measure Theory* [9] by Donald L. Cohn), the action of this functional can be captured precisely as an integral with respect to a (uniquely defined) Radon measure λ_g on M, i.e.

$$\int_M f \, \mathrm{d}\lambda_g = \sum_i \int_{\varphi(U_i)} (\chi_i \sqrt{\det(g)} f) \circ \varphi^{-1} \, \mathrm{d}\lambda^n \tag{3.0.1}$$

for all $f \in \mathcal{C}_c(M)$. This allows us to use the usual measure theoretic definitions for the L^p-spaces, $1 \leq p \leq \infty$, in the context of integration on a Riemannian manifold.

Moreover, using the metric g, we can now also introduce the Banach spaces $L^p(T^*M)$, $1 \leq p < \infty$, of L^p-integrable one-forms. We start by endowing the cotangent bundle (T^*M, M, π) with its natural Borel σ-algebra and defining any measurable map $\omega\colon M \to T^*M$ satisfying $\pi \circ \omega = \mathrm{id}_M$ to be a *measurable one-form.* The quotient space of all such forms under a.e.-equivalence with respect to λ_g will be denoted by $\Gamma(T^*M)$. In defining a product $f \cdot \omega$ by

$$(f \cdot \omega)(p) := f(p)\omega(p), \quad p \in M \tag{3.0.2}$$

for any (representatives of) $f \in L^\infty(\lambda_g)$ and $\omega \in \Gamma(T^*M)$, we introduce the structure of a module over the ring $L^\infty(\lambda_g)$ on the set $\Gamma(T^*M)$.

Finally, we define

$$L^p(T^*M) := \{\omega \in \Gamma(T^*M) \mid |\omega|_g \in L^p(\lambda_g)\},$$

using the notation $|\omega|_g := \sqrt{g(\omega^\#, \omega^\#)}$, where $\omega^\# \in TM$ is the vector field on M uniquely defined by $g(\omega^\#, \cdot\,) = \omega$.

This space is isometrically isomorphic to the space $L^p(M;\mathbb{R}^n)$, making it a Banach space. In fact, this is a consequence of the existence of a *global trivialization* of the cotangent bundle in the measurable sense, which we will construct below.

So as to align with our approach to Sobolev spaces, we focus now on the space of square-integrable one-forms $L^2(T^*M)$. We observe that this space has the following three properties, which we will generalize to our abstract setting of a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ to arrive at the study of so-called $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-modules.*

Lemma 3.0.1. *Let (M, g) be an oriented n-dimensional Riemannian manifold. Then the space of L^2-integrable one-forms $L^2(T^*M)$ satisfies the following:*

i) The vector space $L^2(T^*M)$ equipped with the norm $\|\omega\| = \sqrt{\int_M |\omega|_g^2 \,\mathrm{d}\lambda_g}$ is a Banach space.

ii) $L^2(T^*M)$ can be endowed with the structure of an $L^\infty(\lambda_g)$-module. That is, there exists a bilinear multiplication $\cdot\colon L^\infty(\lambda_g) \times L^2(T^*M) \to L^2(T^*M)$ satisfying

$$\begin{aligned} f \cdot (g \cdot \omega) &= (fg) \cdot \omega \quad && \text{for all } f, g \in L^\infty(\lambda_g) \text{ and } \omega \in L^2(T^*M), \\ \hat{1} \cdot \omega &= \omega && \text{for every } \omega \in L^2(T^*M), \end{aligned}$$

where $\hat{1}$ denotes the equivalence class of the function identically equal to one.

iii) The map $|\cdot|_g\colon L^2(T^*M) \to L^2(\lambda_g)$ fulfills

$$\begin{aligned} |\omega|_g &\geq 0 \ \lambda_g\text{-a.e.} && \text{for every } \omega \in L^2(T^*M), \\ |f \cdot \omega|_g &= |f|\, |\omega|_g \ \lambda_g\text{-a.e.} && \text{for every } f \in L^\infty(\lambda_g) \text{ and } \omega \in L^2(T^*M), \\ \|\omega\| &= \|\, |\omega|_g \|_{L^2(\lambda_g)} && \text{for every } \omega \in L^2(T^*M). \end{aligned}$$

Proof.
(i) Starting from a local trivialization $\{(U_i, \varphi_i)\}_{i\geq 1}$ of the cotangent bundle (T^*M, M, π), we construct a *global* trivialization in the measurable sense. For this, we assume that each $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ is a fibrewise isometry, when we equip $\pi^{-1}(p)$ with the inner product induced by the metric g and $\mathbb{R}^n$ with the standard inner product (e.g. choosing an orthonormal frame $\omega_1, \ldots, \omega_n \in \Gamma^\infty(U_i, T^*M)$, we map $(q, \sum_i \lambda_i(q)\omega_i(q)) \in T^*M|_q \subseteq \pi^{-1}(U_i)$ onto $(q, \lambda_1(q), \ldots, \lambda_n(q)) \in U_i \times \mathbb{R}^n$).

To patch together the maps φ_i, we first define $E_k := U_k \setminus \bigcup_{i=1}^{k-1} U_i$ to partition M by a collection of Borel sets $(E_k)_{k\geq 1}$ and then set

$$\varphi := \sum_{k\geq 1} 1_{\pi^{-1}(E_k)} \varphi_k.$$

By continuity of $\pi\colon T^*M \to M$, the sets $\pi^{-1}(E_k) = \pi^{-1}(U_k) \setminus \bigcup_{i=1}^{k-1} \pi^{-1}(U_i)$ form a Borel partition of T^*M (cf. Remark 1.3.13). Hence, $\varphi\colon T^*M \to M \times \mathbb{R}^n$ is measurable with measurable inverse (mapping $(p, v) \in M \times \mathbb{R}^n$ back to T^*M using φ_k, whenever $p \in E_k$).

Now, let $\omega\colon M \to T^*M$ be any measurable one-form. Composing it with φ and projecting onto the second factor, we obtain a unique measurable map $f := \mathrm{pr}_2 \circ \varphi \circ \omega\colon M \to \mathbb{R}^n$ with $\|f(p)\| = |\omega(p)|_{g(p)}$ for any $p \in M$ since $\mathrm{pr}_2 \circ \varphi$ is an isometric isomorphism on each fibre. Thus, $L^2(M; \mathbb{R}^n)$ contains an isometric image of $L^2(T^*M)$. Moreover, as mapping any measurable $f\colon M \to \mathbb{R}^n$ onto

$$M \ni p \mapsto \varphi^{-1}(p, f(p)) \in T^*M$$

is clearly an inverse to this (one-to-one) identification of measurable one-forms with measurable $\mathbb{R}^n$-valued maps, we conclude $L^2(T^*M) \cong L^2(M; \mathbb{R}^n)$ isometrically.

See also Remark I.17 in *Covariant Schrödinger Semigroups on Riemannian Manifolds* [15] by Batu Güneysu for a similar construction using orthonormal frames.

(ii) We have already noted that $\Gamma(T^*M)$ is an $L^\infty(\lambda_g)$-module, allowing $L^2(T^*M)$ to inherit this structure once we have established

$$\int_M |f \cdot \omega|_g^2 \,\mathrm{d}\lambda_g \overset{(3.0.2)}{=} \int_M |f|^2 \,|\omega|_g^2 \,\mathrm{d}\lambda_g < \infty$$

for all $f \in L^\infty(\lambda_g)$ and $\omega \in L^2(T^*M)$. This, however, follows by a simple application of Hölder's inequality.

(iii) The first property is a consequence of the positivity of the metric g, the second follows from (3.0.2) together with $\mathbb{R}$-linearity while, finally, the third is true by definition. □

3.1 L^2-normed L^∞-modules

Mimicking the above statement, we now introduce the general definition of a normed module, removing any traces of a smooth manifold. Most results and definitions of this and the subsequent sections are again taken from [13] Chapter 3.

Definition 3.1.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. An $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module on X is a tuple $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \,\cdot\,, |\cdot|)$ consisting of the following data:

i) A Banach space $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$.

ii) A bilinear multiplication $\cdot : L^\infty(\mathfrak{m}) \times \mathcal{M} \to \mathcal{M}$ satisfying

$$\begin{aligned} f \cdot (g \cdot v) = (fg) \cdot v \quad & \text{for all } f, g \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathcal{M}, \\ \hat{1} \cdot v = v \quad & \text{for every } v \in \mathcal{M}, \end{aligned}$$

where $\hat{1}$ denotes the equivalence class of the function identically equal to one.

iii) A *pointwise* norm $|\cdot| : \mathcal{M} \to L^2(\mathfrak{m})$ fulfilling

$$\begin{aligned} &a)\ |v| \geq 0 \ \mathfrak{m}\text{-a.e.} && \text{for every } v \in \mathcal{M}, && (3.1.1) \\ &b)\ |f \cdot v| = |f||v| \ \mathfrak{m}\text{-a.e.} && \text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathcal{M}, && (3.1.2) \\ &c)\ \|v\|_{\mathcal{M}} = \|\,|v|\,\|_{L^2(\mathfrak{m})} && \text{for every } v \in \mathcal{M} && (3.1.3) \end{aligned}$$

with $|f|$ denoting the absolute value for $f \in L^\infty(\mathfrak{m})$.

Proposition 3.1.2. *Any normed module $\mathcal{M}$ further possesses the following properties:*

i) The multiplication by $L^\infty(\mathfrak{m})$-functions is continuous with

$$\|f \cdot v\|_{\mathcal{M}} \leq \|f\|_{L^\infty(\mathfrak{m})} \|v\|_{\mathcal{M}}. \tag{3.1.4}$$

ii) The $\mathbb{R}$-multiplication and the $L^\infty(\mathfrak{m})$-multiplication agree on (almost everywhere) constant functions, that is

$$\lambda v = \hat{\lambda} \cdot v \quad \text{for any } \lambda \in \mathbb{R}, v \in \mathcal{M} \tag{3.1.5}$$

with $\hat{\cdot}$ again denoting the respective equivalence class.

iii) The pointwise norm is indeed a norm. Explicitly,

$$|v + w| \leq |v| + |w|, \tag{3.1.6}$$
$$|\lambda v| = |\lambda||v|, \tag{3.1.7}$$
$$|v| = 0 \text{ only if } v = 0 \tag{3.1.8}$$

for any $v, w \in \mathcal{M}$ and $\lambda \in \mathbb{R}$ with the (in)equalities holding $\mathfrak{m}$-a.e. Moreover,

$$\|\,|v| - |w|\,\|_{L^2(\mathfrak{m})} \leq \|v - w\|_{\mathcal{M}}, \tag{3.1.9}$$

rendering the assignment $\mathcal{M} \ni v \mapsto |v| \in L^2(\mathfrak{m})$ continuous.

Proof. Notice for (i) that

$$\|f \cdot v\|_{\mathcal{M}} = \|\,|f||v|\,\|_{L^2(\mathfrak{m})} \leq \|f\|_{L^\infty(\mathfrak{m})} \,\|\,|v|\,\|_{L^2(\mathfrak{m})} = \|f\|_{L^\infty(\mathfrak{m})} \,\|v\|_{\mathcal{M}}$$

by the definition of the pointwise norm and Hölder's inequality.

Item (ii) follows immediately from the definition as a module over $L^\infty(\mathfrak{m})$ since this implies

$$\hat{\lambda} \cdot v = (\lambda \hat{1}) \cdot v = \lambda(\hat{1} \cdot v) = \lambda v$$

due to bilinearity for the second equality and $\hat{1} \cdot v = v$ for the last one.

For *(iii)*, by way of contradiction, suppose the existence of $v, w \in \mathcal{M}$ and of an $\mathfrak{m}$-positive Borel set $E \subseteq X$ with

$$|v + w| > |v| + |w| \text{ on } E.$$

To make calculations easier, we will squeeze in constants $a, b, c \in \mathbb{R}$ with $c > a + b$ such that $a \geq |v|$, $b \geq |w|$ and $|v + w| \geq c$. This leads to

$$\| 1_E(v + w) \|_{\mathcal{M}} \overset{(3.1.3)}{=} \sqrt{\int_E |v + w|^2 \, \mathrm{d}\mathfrak{m}} \geq c \sqrt{\mathfrak{m}(E)},$$

whereas, similarly,

$$\| 1_E v \|_{\mathcal{M}} + \| 1_E w \|_{\mathcal{M}} \leq (a + b) \sqrt{\mathfrak{m}(E)} < c \sqrt{\mathfrak{m}(E)},$$

implying

$$\| 1_E(v + w) \|_{\mathcal{M}} > \| 1_E v \|_{\mathcal{M}} + \| 1_E w \|_{\mathcal{M}},$$

contradicting the fact that $\| \cdot \|_{\mathcal{M}}$ is subadditive.

The second claim, equality (3.1.7), is a consequence of item *(ii)*, as this implies $|\lambda v| = |\hat{\lambda} \cdot v| = |\hat{\lambda}||v| = |\lambda||v|$. Concerning the third claim (3.1.8), we notice that $|v| = 0$ enforces $\|v\|_{\mathcal{M}} = \| \, |v| \, \|_{L^2(\mathfrak{m})} = 0$ and thus $v = 0$. Finally, to establish continuity, we invoke the reverse triangle inequality for the pointwise norm to conclude

$$\| |v| - |w| \|_{\mathcal{M}}^2 \overset{(3.1.3)}{=} \int (|v| - |w|)^2 \, \mathrm{d}\mathfrak{m} \overset{(3.1.6)}{\leq} \int |v - w|^2 \, \mathrm{d}\mathfrak{m} = \|v - w\|_{\mathcal{M}}^2 \qquad \square$$

Remark 3.1.3. Given some $v \in \mathcal{M}$ and a Borel set $E \subseteq X$, we say that v *vanishes* on E provided $1_E v = 0$. Again using the defining properties of the pointwise norm, this is easily seen to be equivalent to the requirement $|v| = 0$ $\mathfrak{m}$-a.e. on E. In fact, $1_E v = 0$ precisely when $\int_E |v|^2 \, \mathrm{d}\mathfrak{m} = \|1_E v\|_{\mathcal{M}}^2 = 0$, which is the case if and only if $|v|$ vanishes $\mathfrak{m}$-a.e. on the set E.

3.2 Morphisms and Submodules

To get a clearer grasp of the implications of our somewhat interwoven axiomatization of a normed module, we now introduce notions of structure preserving maps and define substructures of modules.

Definition 3.2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules. Then any map $\Phi\colon \mathcal{M} \to \mathcal{N}$ is called a *module homomorphism* if it is $L^\infty(\mathfrak{m})$-linear and bounded with respect to the pointwise norms, meaning $|\Phi(v)| \le C|v|$ for all $v \in \mathcal{M}$ and some universal constant $C > 0$.

Remark 3.2.2. Due to the interplay between the pointwise norm and the Banach space norm, we could have equivalently demanded that $\Phi\colon \mathcal{M} \to \mathcal{N}$ be bounded in the Banach space sense. In fact, Φ is seen to be continuous with respect to the Banach space norms by estimating, for a given $v \in \mathcal{M}$,

$$\|\Phi(v)\|_{\mathcal{N}} \overset{(3.1.3)}{=} \||\Phi(v)|\|_{L^2(\mathfrak{m})} \le C\||v|\|_{L^2(\mathfrak{m})} = \|v\|_{\mathcal{M}}.$$

Conversely, by way of contradiction, suppose Φ satisfied the bound

$$\|\Phi(v)\|_{\mathcal{N}} \le C\|v\|_{\mathcal{M}} \quad \text{for all } v \in \mathcal{M} \tag{3.2.1}$$

and some universal constant $C > 0$, while being unbounded with respect to the pointwise norms. Then, for this constant C, we could find some $w \in \mathcal{M}$ and a Borel set $E \subseteq X$ with $\mathfrak{m}(E) > 0$ such that

$$|\Phi(w)| > C|w| \text{ on } E. \tag{3.2.2}$$

By $L^\infty(\mathfrak{m})$-linearity of Φ, however, this would imply

$$\begin{aligned}\|\Phi(1_E w)\|_{\mathcal{N}}^2 \overset{(3.1.3)}{=} \int |\Phi(1_E w)|^2 \,\mathrm{d}\mathfrak{m} \overset{(3.1.2)}{=} \int 1_E |\Phi(w)|^2 \,\mathrm{d}\mathfrak{m} &\overset{(3.2.2)}{>} C^2 \int 1_E |w|^2 \,\mathrm{d}\mathfrak{m} \\ &\overset{(3.1.3)}{=} C^2 \|1_E w\|_{\mathcal{M}}^2,\end{aligned}$$

contradicting (3.2.1).

Thus, using (3.1.5) to deduce $\mathbb{R}$-linearity of Φ, we conclude that any module homomorphism is continuous. Moreover, as the next proposition shows, another possible definition could have been: Any $\mathbb{R}$-*linear* and continuous map $L\colon \mathcal{M} \to \mathcal{N}$ is called a module homomorphism.

Proposition 3.2.3. *Let $\Phi\colon \mathcal{M} \to \mathcal{N}$ be $\mathbb{R}$-linear and bounded w.r.t. the pointwise norms of $\mathcal{M}$ and $\mathcal{N}$. Then Φ is $L^\infty(\mathfrak{m})$-linear and hence a module homomorphism.*

Proof. We show first that it suffices to prove homogeneity w.r.t. indicators, i.e.

$$\Phi(1_E v) = 1_E \Phi(v) \quad \text{for all } v \in \mathcal{M} \text{ and } E \subseteq X \text{ Borel.} \tag{3.2.3}$$

To this end, let $(f_n)_{n \ge 1} \subseteq L^\infty(\mathfrak{m})$ be a sequence of simple functions converging to some given $f \in L^\infty(\mathfrak{m})$ and let $v \in \mathcal{M}$. By $\mathbb{R}$-linearity combined with assumption (3.2.3), we see that $\Phi(f_n v) = f_n \Phi(v)$ for all $n \ge 1$. Continuity of the $L^\infty(\mathfrak{m})$-multiplications on $\mathcal{M}$ and $\mathcal{N}$, respectively, further implies $f_n v \to f v$ as well as $f_n \Phi(v) \to f\Phi(v)$. Hence, continuity of Φ (see Remark 3.2.2) finally establishes $\Phi(fv) = f\Phi(v)$.

We now start to prove (3.2.3). Noticing that additivity of Φ grants

$$\Phi(1_E v) + \Phi(1_{X\setminus E} v) = \Phi(v) = 1_E\Phi(v) + 1_{X\setminus E}\Phi(v), \tag{3.2.4}$$

we will establish the fact that $\Phi(1_{X\setminus E} v)$ vanishes on E (as discussed in Remark 3.1.3) due to boundedness w.r.t. the pointwise norms. Multiplying (3.2.4) by 1_E, this will imply $1_E\Phi(1_E v) = 1_E\Phi(v)$. Thus,

$$\Phi(1_E v) = 1_E\Phi(1_E v) + 1_{X\setminus E}\Phi(1_E v) = 1_E\Phi(v),$$

as, by analogy, it will follow that $1_{X\setminus E}\Phi(1_E v) = 0$. This remaining claim is now easily established using absolute homogeneity of the pointwise norm (cf. Remark 3.1.3):

$$|\Phi(1_{X\setminus E} v)| \leq C\,|1_{X\setminus E} v| \overset{(3.1.2)}{=} C\,1_{X\setminus E}|v| = 0 \ \mathfrak{m}\text{-a.e. on } E \tag{3.2.5}$$

□

Definition 3.2.4. Let $\mathcal{M}$ be a module and let $E \subseteq X$ be a Borel subset of X. Then we call

$$\mathcal{M}|_E := \{1_E v \mid v \in \mathcal{M}\}$$

the *localization* of $\mathcal{M}$ onto E.

Remark 3.2.5. Given a linear and continuous map $\Phi\colon \mathcal{M} \to \mathcal{N}$, the condition (3.2.3), which is equivalent to

$$\Phi(fv) = f\Phi(v) \quad \text{for all } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathcal{M}$$

in this context, is also referred to as the *locality condition*, see e.g. Definition 1.2.1 in *Nonsmooth differential geometry - An approach tailored for spaces with Ricci curvature bounded from below* [11] by Nicola Gigli. It ensures that Φ respects restrictions in the sense that $\Phi(\mathcal{M}|_E) \subseteq \mathcal{N}|_E$. As a consequence, $\Phi(v)$ will vanish $\mathfrak{m}$-a.e. within $\{v = 0\} = \{|v| = 0\}$.

Definition 3.2.6. Let $\mathcal{P} \subseteq \mathcal{M}$ be a closed vector subspace of $\mathcal{M}$. Then we call $\mathcal{P}$ a *submodule* if it is closed under the multiplication by elements of $L^\infty(\mathfrak{m})$.

Lemma 3.2.7. *The localization $\mathcal{M}|_E$ of $\mathcal{M}$ onto E is a submodule of $\mathcal{M}$.*

Proof. Clearly, $\mathcal{M}|_E$ is a vector subspace of $\mathcal{M}$ and it is closed under the $L^\infty(\mathfrak{m})$-multiplication by commutativity of the ring $L^\infty(\mathfrak{m})$: $f(1_E v) = (f1_E)v = (1_E f)v = 1_E(fv)$. Now, to establish closedness, let $1_E v_n \xrightarrow[n\to\infty]{} w \in \mathcal{M}$. Then also

$$1_E|v_n| \xrightarrow[n\to\infty]{} |w| \in L^2(\mathfrak{m})$$

by the continuity of the pointwise norm, see item (*iii*) in Proposition 3.1.2, and its homogeneity, as presupposed in (3.1.2). Thus,

$$1_{X\setminus E}|w| = \lim_{n\to\infty} 1_{X\setminus E}(1_E|v_n|) = 0 \ \mathfrak{m}\text{-a.e.}$$

and so $w = 1_{X\setminus E}w + 1_E w = 1_E w \in \mathcal{M}|_E$. □

Remark 3.2.8. It is easily verified that the kernel and the closure of the image of any module homomorphism $\Phi\colon \mathcal{M} \to \mathcal{N}$ are submodules of $\mathcal{M}$. Indeed, both are clearly closed under multiplication by elements of $L^\infty(\mathfrak{m})$ and $\ker\Phi = \Phi^{-1}(\{0\})$ is (topologically) closed by continuity of Φ, whereas $\overline{\operatorname{im}\Phi}$ is by definition.

Finally, given an arbitrary subset $S \subseteq \mathcal{M}$, we will be interested in the smallest submodule of $\mathcal{M}$ that contains it. In particular, we want to find a criterion ensuring that S is rich enough to be able to *generate* $\mathcal{M}$ in an appropriate sense.

Proposition 3.2.9. *For any $S \subseteq \mathcal{M}$, the set $\mathcal{M}(S)$, defined as the closure of the set $\mathcal{S}$ of all*

$$\sum_{i=1}^n f_i v_i \quad \text{with } (f_i)_{i=1}^n \subseteq L^\infty(\mathfrak{m}),\ (v_i)_{i=1}^n \subseteq S,\ n \in \mathbb{N},$$

is a module. Moreover, it is the smallest submodule containing S.

Proof. Clearly, $\mathcal{M}(S)$ constitutes a vector subspace of $\mathcal{M}$ which is closed by definition. To check that it is closed under $L^\infty(\mathfrak{m})$-multiplication, simply notice that if $v \in \mathcal{M}(S)$ and $f \in L^\infty(\mathfrak{m})$, then for any $\varepsilon > 0$, there exists $\sum_i f_i v_i \in \mathcal{S}$ with $\|v - \sum_i f_i v_i\|_{\mathcal{M}} < \varepsilon/\|f\|_{L^\infty(\mathfrak{m})}$ and thus $\|fv - \sum_i (ff_i)v_i\|_{\mathcal{M}} \le \|f\|_{L^\infty(\mathfrak{m})}\|v - \sum_i f_i v_i\|_{\mathcal{M}} < \varepsilon$ by inequality (3.1.4), i.e. $fv \in \mathcal{M}(S)$.

Suppose now that $\mathcal{M}'$ is another submodule containing the set S. Being a vector subspace closed under the $L^\infty(\mathfrak{m})$-multiplication, it must also contain $\mathcal{S}$. Hence, $\mathcal{M}(S) = \overline{\mathcal{S}} \subseteq \overline{\mathcal{M}'} = \mathcal{M}'$. □

Definition 3.2.10. Let $S \subseteq \mathcal{M}$ be an arbitrary subset of a module $\mathcal{M}$. Then we call $\mathcal{M}(S)$ the *module generated by* S. If $\mathcal{M}|_E = \mathcal{M}(S)|_E$ for some Borel subset $E \subseteq X$, we also say that S *generates* $\mathcal{M}$ *on* E.

Example 3.2.11. The space $L^2(\mathfrak{m})$, viewed as a module, is generated by any singleton $\{g\}$, where $g \in L^2(\mathfrak{m})$ is non-zero $\mathfrak{m}$-a.e. Indeed, denote by $\mathcal{M}$ the submodule generated by $\{g\}$ and choose a sequence $(f_n)_{n\ge 1} \subseteq L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$ converging to some given $f \in L^2(\mathfrak{m})$. Now, introducing $E_m := \{|g| \ge 1/m\}$, we have

$$1_{E_m} f_n = \left(1_{E_m}\frac{f_n}{g}\right)\cdot g \in L^\infty(\mathfrak{m})\cdot g \subseteq \mathcal{M} \quad \text{for all } n, m \ge 1,$$

as $|\frac{f_n}{g}| \le m\|f_n\|_{L^\infty(\mathfrak{m})}$ on E_m. Moreover, since $\bigcup_{m\ge 1} E_m = X$ up to null-sets,

$$\begin{aligned}\varlimsup_{n\to\infty}\varlimsup_{m\to\infty}\|1_{E_m}f_n - f\|_{L^2(\mathfrak{m})} &\le \varlimsup_{n\to\infty}\varlimsup_{m\to\infty}(\|1_{E_m}f_n - f_n\|_{L^2(\mathfrak{m})} + \|f_n - f\|_{L^2(\mathfrak{m})}) \\ &= \varlimsup_{n\to\infty}\|f_n - f\|_{L^2(\mathfrak{m})} = 0,\end{aligned}$$

i.e. $L^2(\mathfrak{m}) \subseteq \overline{\mathcal{M}} = \mathcal{M}$, giving the claim.

Proposition 3.2.12. *Let V be a vector subspace of a module $\mathcal{M}$. Then $\mathcal{M}(V)$ coincides with the $\mathcal{M}$-closure of the set $\mathcal{V}$ of all*

$$\sum_{i=1}^n 1_{E_i}v_i \quad \text{with } (E_i)_{i=1}^n \text{ a Borel partition of } X,\ (v_i)_{i=1}^n \subseteq V,\ n\in\mathbb{N}. \tag{3.2.6}$$

Proof. Evidently, $\overline{\mathcal{V}} \subseteq \mathcal{M}(V)$. For the converse, we would need to show $\sum_i f_i v_i \in \overline{\mathcal{V}}$ for $(f_i)_i \subseteq L^\infty(\mathfrak{m})$, $(v_i)_i \subseteq V$. Using the vector space structure of $\overline{\mathcal{V}}$, it suffices, however, to prove that any element of the form fv with $f \in L^\infty(\mathfrak{m})$ and $v \in V\setminus\{0\}$ belongs to the closure of $\mathcal{V}$. Now, recalling that $\|fv - gv\| \le \|f-g\|_{L^\infty(\mathfrak{m})}\|v\|$ due to (3.1.4), this can easily be achieved by approximating f by a simple function $g = \sum 1_{E_i}\alpha_i$ such that for some given $\varepsilon > 0$ we have $\|f-g\|_{L^\infty(\mathfrak{m})} < \varepsilon/\|v\|$ with $gv = \sum 1_{E_i}(\alpha_i v) \in \mathcal{V}$. □

3.3 Operations on Modules

3.3.1 Dualization

As any module $\mathcal{M}$ possesses the structure of an algebraic module over the ring $L^\infty(\mathfrak{m})$, we might be tempted to introduce the dual of $\mathcal{M}$ as the set of all $L^\infty(\mathfrak{m})$-linear and continuous maps into $L^\infty(\mathfrak{m})$. This, however, does certainly not lead to a desirable definition, as the next example demonstrates.

Example 3.3.1. Let $L\colon \mathcal{M} \to L^\infty(\mathfrak{m})$ be $L^\infty(\mathfrak{m})$-linear and continuous with $\mathfrak{m}$ atomless. By adapting and expanding on Example 1.2.8 in [11], we show that this enforces $L = 0$.

In fact, assuming otherwise, we can find an element $v \in \mathcal{M}$ and a constant $b > 0$ such that $|L(v)| \geq b$ on some Borel set $E \subseteq X$ with $\mathfrak{m}(E) > 0$. Since $|v| \in L^2(\mathfrak{m})$, we can further assume $|v| \leq a$ on E for some $a > 0$. Indeed, if $\mathfrak{m}(E \cap \{|v| \leq a\}) = 0$ for any choice of constant $a > 0$, we would have

$$\||v|\|^2_{L^2(\mathfrak{m})} \geq \int_E |v|^2 \,\mathrm{d}\mathfrak{m} > a^2 \mathfrak{m}(E)$$

for any $a > 0$, contradicting $\||v|\|^2_{L^2(\mathfrak{m})} < \infty$. We can therefore replace E by the Borel set $\tilde{E} := E \cap \{|v| \leq a\}$ with $a > 0$ suitably chosen so that $\mathfrak{m}(\tilde{E}) > 0$. On this set, we now clearly have $|v| \leq a$ as well as $|L(v)| \geq b$.

Now, choose $f \in L^2(\mathfrak{m}) \setminus L^\infty(\mathfrak{m})$ non-negative, vanishing outside of E (we will prove below that such an element exists) and set $f_n := f \wedge n \in L^\infty(\mathfrak{m})$, so that $\|f_n\|_{L^\infty(\mathfrak{m})} = n$ for all $n \geq 1$, due to the unboundedness of f. Then,

$$\|f_n v - f v\|_{\mathcal{M}} \overset{(3.1.3)}{=} \|\,|f_n - f||v|\,\|_{L^2(\mathfrak{m})} \leq a \,\|f_n - f\|_{L^2(\mathfrak{m})} \xrightarrow[n\to\infty]{} 0,$$

so that, by the assumed continuity of L,

$$\|L(f_n v) - L(f v)\|_{L^\infty(\mathfrak{m})} \xrightarrow[n\to\infty]{} 0.$$

However, as $|L(v)| \geq b$ on E and $\|f_n\|_{L^\infty(\mathfrak{m})} = n$, we also have, by $L^\infty(\mathfrak{m})$-linearity, that

$$\|L(f_n v)\|_{L^\infty(\mathfrak{m})} = \|f_n L(v)\|_{L^\infty(\mathfrak{m})} \geq b\|f_n\|_{L^\infty(\mathfrak{m})} = bn \xrightarrow[n\to\infty]{} \infty,$$

leading to a contradiction.

We now supply a proof of the claim that there exists a square-integrable but unbounded function f that vanishes outside a Borel set E of positive measure if we require $\mathfrak{m}$ to be atomless.

For this, we first assume w.l.o.g. that E is of finite $\mathfrak{m}$-measure by otherwise replacing it by any of its bounded subsets of positive measure (e.g. by $E \cap B_m(x_0)$ with $x_0 \in E$ for $m \geq 1$ large enough, arguing via continuity from below and recalling that $\mathfrak{m}$ is boundedly finite). Then we make use of the fact that, within E, we can find a sequence of sets

$(E_n)_{n\geq 1}$ such that

$$\mathfrak{m}(E_n) = \frac{1}{n^2} \quad \text{with} \quad E_{n+1} \subseteq E_n \quad \text{for all } n \geq 1, \tag{3.3.1}$$

see for example Lemma 2.5 in chapter 2 of *Interpolation of Operators* by Colin Bennett and Robert C. Sharpley [4]. Now, we set

$$f := \sqrt{\sum_{n\geq 1} 1_{E_n}},$$

wherever the sum is finite, and compute

$$\|f\|^2_{L^2(\mathfrak{m})} = \int |f|^2 \,\mathrm{d}\mathfrak{m}|_E = \sum_{n\geq 1} \mathfrak{m}(E_n) \overset{(3.3.1)}{=} \sum_{n\geq 1} \frac{1}{n^2} < \infty.$$

Hence, f is square-integrable, while being unbounded due to our choice of $(E_n)_{n\geq 1}$, as can be seen by observing that for any $m \geq 1$, we have

$$\sum_{n=1}^{m} 1_{E_n} = m \quad \text{on} \quad \bigcap_{n=1}^{m} E_n, \;\text{ where } \mathfrak{m}(\bigcap_{n=1}^{m} E_n) \overset{(3.3.1)}{\geq} \mathfrak{m}(E_{m+1}) > 0.$$

Returning to our task of defining a dual module, we note that, conceptually, the right kind of dual should (again) consist of *suitable integrands*, in the sense that the equality

$$\sup_{\|v\|\leq 1} \|L(v)\| = \|L\| = \|\, |L| \,\|_{L^2(\mathfrak{m})}$$

must hold true for an appropriate norm on the target space of the linear map L. Choosing the pointwise norm for L in such a way that $|L(v)| \leq |L||v|$, an attempt at this is to require that $L(v) \in L^1(\mathfrak{m})$ as Hölder's inequality would then ensure

$$\|L(v)\|_{L^1(\mathfrak{m})} \leq \||L||v|\|_{L^1(\mathfrak{m})} \leq \|\,|L|\,\|_{L^2(\mathfrak{m})}\|\,|v|\,\|_{L^2(\mathfrak{m})} = \|\,|L|\,\|_{L^2(\mathfrak{m})}\|v\|,$$

so that $\|L\|_{L^1(\mathfrak{m})} \leq \|\,|L|\,\|_{L^2(\mathfrak{m})}$. In fact, this approach will yield a meaningful definition of $\mathcal{M}^*$ as an $L^2(\mathfrak{m})$-normed module. We again closely follow the presentation given in [13] Chapter 3.2.1.

Definition 3.3.2. Let $(\mathcal{M}, \|\cdot\|, \cdot, |\cdot|)$ be a module. Then we define by

$$\mathcal{M}^* := \{\, L\colon \mathcal{M} \to L^1(\mathfrak{m}) \mid L \text{ is } L^\infty(\mathfrak{m})\text{-linear and continuous} \,\} \tag{3.3.2}$$

its *dual module* and equip it with

$$\|L\|_* := \sup_{\|v\|\leq 1} \|L(v)\|_{L^1(\mathfrak{m})} \quad \text{and} \quad |L|_* := \operatorname*{ess\,sup}_{|v|\leq 1\ \mathfrak{m}\text{-a.e.}} |L(v)|, \tag{3.3.3}$$

together with pointwise multiplication, i.e. $(f\cdot L)(v) := fL(v)$ for $f \in L^\infty(\mathfrak{m})$, $L \in \mathcal{M}^*$.

Lemma 3.3.3. *For any $v \in \mathcal{M}$ and $L \in \mathcal{M}^*$, we have*

$$|L(v)| \le |L|_* |v| \ \mathfrak{m}\text{-a.e.} \tag{3.3.4}$$

Proof. Given some $v \in \mathcal{M}$, we would like to conclude, as usual,

$$|L(v)| = |v||L(|v|^{-1}v)| \le |v||L|_* \ \mathfrak{m}\text{-a.e.}$$

but this will only work if $|v|, |v|^{-1} \in L^\infty(\mathfrak{m})$. Using locality of L and a decomposition (up to null sets) of our underlying space X as $X = \{|v| = 0\} \cup \bigcup_{n \ge 1} S_n$, where $S_n := \{1/n \le |v| \le n\}$ for $n \ge 1$ an integer, we can however reduce the general situation to this special case. In fact, $1_{S_n} v$ clearly has the required properties for any $n \ge 1$ so that

$$1_{S_n}|L(v)| = |L(1_{S_n} v)| \le |1_{S_n} v||L|_* = 1_{S_n}|v||L|_* \ \mathfrak{m}\text{-a.e.},$$

whereas $|L(v)| = 0$ $\mathfrak{m}$-a.e. on $\{|v| = 0\}$ (cf. Remark 3.2.5), giving the claim on all of X. □

Proposition 3.3.4. *The space $(\mathcal{M}^*, \|\cdot\|_*, \cdot\,, |\cdot|_*)$ is indeed an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.*

Proof. It is a general fact that the set of all linear and continuous maps between two Banach spaces again forms a Banach space. Similarly, the set of all module homomorphisms (in the usual algebraic sense) between two modules over a *commutative* ring is again a module over the same ring. As clearly $|L|_* \ge 0$, as well as $|fL|_* = |f||L|_*$, we are left with proving

$$\|L\|_* = \|\, |L|_*\|_{L^2(\mathfrak{m})}, \tag{3.3.5}$$

which will establish well-definedness of the pointwise norm as a map $|\cdot|_*\colon \mathcal{M}^* \to L^2(\mathfrak{m})$, as well as the missing relation (3.1.3).

For this, we first invoke Lemma 3.3.3 and Hölder's inequality to conclude

$$\|L(v)\|_{L^1(\mathfrak{m})} \overset{(3.3.3)}{\le} \||L|_*|v|\|_{L^1(\mathfrak{m})} \le \|\,|L|_*\|_{L^2(\mathfrak{m})}\|\,|v|\,\|_{L^2(\mathfrak{m})} \overset{(3.1.3)}{=} \|\,|L|_*\|_{L^2(\mathfrak{m})}\|v\|.$$

Hence, $\|L\|_* \le \|\,|L|_*\|_{L^2(\mathfrak{m})}$. To prove the converse inequality, we use a duality argument. Concretely, by the Theorem of Hahn-Banach and reflexivity of $L^2(\mathfrak{m})$, we may equivalently establish the bound on

$$\sup_{f \in L^2(\mathfrak{m}), \|f\| \le 1} \left| \int f|L|_* \,\mathrm{d}\mathfrak{m} \right| = \|\,|L|_*\|_{L^2(\mathfrak{m})}.$$

By a cut-off argument, we see that this supremum is unchanged if we further assume $f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$. Moreover, as $|L|_* \ge 0$, we have

$$\int f|L|_* \,\mathrm{d}\mathfrak{m} = \int f^+|L|_* \,\mathrm{d}\mathfrak{m} - \int f^-|L|_* \,\mathrm{d}\mathfrak{m} \le \int f^+|L|_* \,\mathrm{d}\mathfrak{m},$$

where we have used the decomposition $f = f^+ - f^-$ into positive and negative parts f^+ and f^-, respectively. Thus, may also suppose that f is non-negative

Choosing a sequence $(v_n)_{n\geq 1} \subseteq \mathcal{M}$ with $|v_n| \leq 1$ $\mathfrak{m}$-a.e. such that $|L|_* = \sup_{n\geq 1}|L(v_n)|$, this now allows us to conclude

$$\int f L(v_n)\,\mathrm{d}\mathfrak{m} = \int L(f v_n)\,\mathrm{d}\mathfrak{m} \leq \|L\|_*\|f v_n\| \leq \|L\|_*\|f\|_{L^2(\mathfrak{m})}$$

by $L^\infty(\mathfrak{m})$-linearity of L. Modifying the sequence $(v_n)_{n\geq 1}$ so that $(L(v_n))_{n\geq 1}$ increases monotonically to $|L|_*$, we thus obtain the statement by monotone convergence and continuity of the multiplication. This modification can be obtained as follows: recursively define $(\tilde{v}_n)_{n\geq 1}$ via

$$\tilde{v}_{n+1} := 1_{\{|L(v_{n+1})|\geq|L(\tilde{v}_n)|\}} v_{n+1} + 1_{\{|L(v_{n+1})|<|L(\tilde{v}_n)|\}} \tilde{v}_n$$

with $\tilde{v}_0 := v_0$. Then $|L(\tilde{v}_n)| = \sup_{k\leq n}|L(v_k)|$, implying $|L|_* = \sup_{n\geq 1}|L(\tilde{v}_n)|$, while, clearly, $|\tilde{v}_n| \leq 1$ $\mathfrak{m}$-a.e. $\square$

Lemma 3.3.5. *Let $L\colon \mathcal{M} \to L^1(\mathfrak{m})$ be an $\mathbb{R}$-linear and continuous map. Then the following are equivalent:*

i) $L(1_E v) = 1_E L(v)$ *for all* $v \in \mathcal{M}$ *and* $E \subseteq X$ *Borel*

ii) $|L(v)| \leq f|v|$ *for all* $v \in \mathcal{M}$ *and some* $f \in L^2(\mathfrak{m})$

iii) $L \in \mathcal{M}^*$

Proof. Clearly, statement (*iii*) includes (*i*) as a special case, and it implies (*ii*) by Lemma 3.3.3 together with equality (3.3.5). Finally, to see that (*iii*) follows from either (*i*) or (*ii*), we simply repeat the argumentation of the proof of Proposition 3.2.3, as in each case we only need to establish homogeneity over $L^\infty(\mathfrak{m})$. More specifically, if L is homogeneous with respect to indicators, i.e. if (*i*) holds, we argue as we did below (3.2.3) to deduce $L^\infty(\mathfrak{m})$-linearity of L. Finally, if $|L(v)| \leq f|v|$ for all $v \in \mathcal{M}$ and some $f \in L^2(\mathfrak{m})$, for a given $E \subseteq X$ Borel, we replace estimate (3.2.5) by

$$|L(1_{X\setminus E} v)| \leq f|1_{X\setminus E} v| = f 1_{X\setminus E}|v| = 0 \text{ } \mathfrak{m}\text{-a.e.on } E$$

and conclude $L(1_E v) = 1_E L(v)$ as we did there. $\square$

Proposition 3.3.6. *Let V be a generating vector subspace of $\mathcal{M}$ and suppose $\ell\colon V \to L^1(\mathfrak{m})$ is an $\mathbb{R}$-linear map satisfying*

$$|\ell(v)| \leq g|v| \text{ } \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in V \text{ and some } g \in L^2(\mathfrak{m}). \tag{3.3.6}$$

Then there exists a unique extension $L \in \mathcal{M}^*$ of the map ℓ such that $|L|_* \leq g$ $\mathfrak{m}$-a.e.

Proof. By assumption, the set $\mathcal{V}$ consisting of all finite sums $\sum_i 1_{E_i} v_i$ for $(v_i)_i \subseteq V$ and $(E_i)_i$ a Borel partition of X is dense within $\mathcal{M}$, see Proposition 3.2.12. Linearity and locality force us to define

$$L(v) := \sum_i 1_{E_i} \ell(v_i) \quad \text{for } v = \sum_i 1_{E_i} v_i \in \mathcal{V}.$$

This is indeed a proper definition as, for any other representation $v = \sum_j 1_{F_j} w_j \in \mathcal{V}$,

$$\Big| \sum_i 1_{E_i} \ell(v_i) - \sum_j 1_{F_j} \ell(w_j) \Big| = \sum_{i,j} 1_{E_i \cap F_j} |\ell(v_i) - \ell(w_j)| \overset{(3.3.6)}{\leq} g \sum_{i,j} 1_{E_i \cap F_j} |v_i - w_j| = 0.$$

Since we similarly have $|L(v)| \leq g|v|$ $\mathfrak{m}$-a.e. for all $v \in \mathcal{V}$, it follows that $\|L(v)\|_{L^1(\mathfrak{m})} \leq \|g\|_{L^2(\mathfrak{m})} \|v\|$. This, in turn, guarantees that the map $L\colon \mathcal{V} \to L^1(\mathfrak{m})$ extends linearly and continuously to all of $\mathcal{M}$.

Now, by approximating any $v \in \mathcal{M}$ by a sequence $(v_n)_{n \geq 1}$ within $\mathcal{V}$ and using continuity of the pointwise norm and L, we can extend the bound $|L(v_n)| \leq g|v_n|$ $\mathfrak{m}$-a.e. to the whole of $\mathcal{M}$, giving $|L|_* \leq g$. To see this, we first estimate, using Hölder's inequality,

$$\| g|v_n| - g|v| \|_{L^1(\mathfrak{m})} \leq \|g\|_{L^2(\mathfrak{m})} \| |v_n| - |v| \|_{L^2(\mathfrak{m})} \overset{(3.1.9)}{\leq} \|g\|_{L^2(\mathfrak{m})} \|v_n - v\|_{\mathcal{M}},$$

so that $g|v_n| \xrightarrow[n \to \infty]{L^1(\mathfrak{m})} g|v|$. Since by continuity of $L\colon \mathcal{M} \to L^1(\mathfrak{m})$, also $L(v_n) \xrightarrow[n \to \infty]{L^1(\mathfrak{m})} L(v)$, along some subsequence $(v_{n_k})_{k \geq 1}$, it must hold that

$$|L(v)| - g|v| = \lim_{k \to \infty} \left(|L(v_{n_k})| - g|v_{n_k}| \right) \leq 0 \ \mathfrak{m}\text{-a.e.} \tag{3.3.7}$$

Finally, this implies $L \in \mathcal{M}^*$ as a consequence of Lemma 3.3.5. □

3.3.2 Pullback

Our next task will be to generalize the notion of a *pullback bundle* to the setting of normed modules. Loosely speaking, using a map $\varphi\colon Y \to X$ between two metric measure spaces X and Y, we want to be able to pull back elements of a normed module over X to define a new module over Y. To motivate this, we briefly recall the situation for manifolds and vector bundles.

In smooth differential geometry, given a diffeomorphism $f\colon M \to N$ between two manifolds M and N, we can always push forward a smooth vector field $X \in \Gamma^\infty(TM)$ to a unique vector field $f_* X \in \Gamma^\infty(TN)$ by demanding the following diagram to commute:

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ X \big\uparrow & & \big\uparrow f_* X \\ M & \xrightarrow{f} & N \end{array}$$

Since f is a diffeomorphism, we thus have $f_*X = Tf \circ X \circ f^{-1} \in \Gamma^\infty(TN)$, so that $(f_*X)_{f(p)} = T_pf(X_p)$, see also Proposition 8.19 in *Introduction to Smooth Manifolds* [20] by John M. Lee.

In the general case, where $f\colon M \to N$ is merely assumed to be smooth, this construction obviously fails (e.g. if f is not surjective, there is no natural way of extending f_*X beyond the image of f). However, despite this inability to identify the push forward with a vector field on N, we can still make sense of it as a vector field *along* f. By this we mean any smooth map $Y\colon M \to TN$ which satisfies $\pi \circ Y = f$, or as a diagram,

$$\begin{tikzcd} & TN \arrow[d, "\pi"] \\ M \arrow[ur, "Y"] \arrow[r, "f"] & N \end{tikzcd}$$

Now clearly, $Tf \circ X\colon M \to TN$ is an example of a vector field along f, as it satisfies $(Tf \circ X)(p) = T_pf(X_p) \in T_{f(p)}N$.

Turning this condition, that Y may only take values in $T_{f(p)}N$ for $p \in M$, into the defining feature for the smooth sections of some bundle $f^*(TN)$ over M, we introduce the *pullback bundle*.

Such a bundle can be obtained as follows: Given a smooth map $f\colon M \to N$ and any vector bundle (E, N, π) over N, loosely speaking, we want to match up the fibers above $f(p) \in N$ with all points $p \in M$. Starting on a set-theoretic level, this can be achieved by defining the total space f^*E as the pullback (in the sense of category theory; see e.g. Definition 3.1.15 in *Category Theory in Context* [22] by Emily Riehl) of the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Concretely, $f^*E = M \times_N E = \{(p, e) \in M \times E \mid f(p) = \pi(e)\}$, so that indeed $(p, e) \in f^*E$ precisely when $e \in E$ lies in the fiber above $f(p)$. Together with the bundle projection

$$f^*\pi\colon f^*E \to M, (p, e) \mapsto p \quad \text{and the map} \quad \tilde{f}\colon f^*E \to E, (p, e) \mapsto e,$$

we have the following commutative diagram, with respect to which $(f^*E, f^*\pi, \tilde{f})$ is *universal*,

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Written out, this means that for any other triple (F, p, q) consisting of a set F together with two maps $p\colon F \to M$ and $q\colon F \to E$ making the above diagram commutative, i.e.

$f \circ p = \pi \circ q$, there exists a unique map $h\colon F \to f^*E$ such that

$$\begin{tikzcd}
F \arrow[drr, bend left, "q"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "p"'] & & \\
& f^*E \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\
& M \arrow[r, "f"] & N
\end{tikzcd}$$

commutes (cf. diagram (3.1.17) in [22]).

For a proof that this construction indeed yields a vector bundle over M and that the universal property remains true in this setting, see Proposition 3.1 in *Fibre Bundles* [19] by Dale Husemoller.

Finally, we remark a (contravariant) functoriality of the assignment $E \mapsto f^*E$, up to isomorphism. Concretely, let (E, N, π) again be a vector bundle over N and suppose we are given two smooth maps $f\colon M \to N$ and $g\colon P \to M$. Then

$$(f \circ g)^*E \cong g^*f^*E \text{ as well as } \operatorname{id}_N^* E \cong E \tag{3.3.8}$$

as vector bundles, cf. Theorem 3.2 in [19].

As we now return to the setting of metric measure spaces, we want to carry over the category theoretical aspects of the pullback bundle to our notion of a pullback module. More explicitly, we want to preserve the universal property and the functoriality described above, see Remark 3.3.12 and Theorem 3.3.13. All results and definitions are borrowed from [13] Chapter 3.2.3 with remarks pertaining to category theory taken from [11] Chapter 1.6.

Definition 3.3.7. Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces and let $\varphi\colon Y \to X$ be a Borel map. Then we say φ is *of bounded compression* if there exists a constant $C > 0$ such that $\varphi_*\mathfrak{m}_Y \leq C\mathfrak{m}_X$. We call the infimum over all such constants the *compression constant* and denote it by $\operatorname{Comp}(\varphi)$.

Theorem 3.3.8. *Let $\varphi\colon Y \to X$ be a map of bounded compression between two metric measure spaces $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$. Suppose further that $\mathcal{M}$ is a normed module over X. Then there exists an essentially unique module $\varphi^*\mathcal{M}$ over Y together with a module homomorphism $\varphi^*\colon \mathcal{M} \to \varphi^*\mathcal{M}$ such that*

i) $|\varphi^*v| = |v| \circ \varphi$ $\mathfrak{m}_Y$-a.e. for every $v \in \mathcal{M}$ (3.3.9)

ii) $\varphi^*\mathcal{M}$ is generated by the set $\{\varphi^*v \mid v \in \mathcal{M}\}$. (3.3.10)

Remark 3.3.9. Here, by *essential uniqueness*, we mean uniqueness up to a unique module-isomorphism, i.e., given another pair $((\varphi^*\mathcal{M})', (\varphi^*)')$ satisfying the conclusion of the theorem, there is a unique module isomorphism $\Phi\colon \varphi^*\mathcal{M} \to (\varphi^*\mathcal{M})'$ so that $\Phi \circ (\varphi^*)' = \varphi^*$.

Proof. We first establish uniqueness. Suppose, therefore, that $((\varphi^*\mathcal{M})', (\varphi^*)')$ is another such pair consisting of a normed module and a homomorphism satisfying the above stated requirements. In this case, any isomorphism $\Psi\colon \varphi^*\mathcal{M} \to (\varphi^*\mathcal{M})'$ would have to satisfy $\Psi(\sum_i 1_{E_i}\varphi^* v_i) = \sum_i 1_{E_i}(\varphi^*)' v_i$ on the generating vector subspace

$$V = \left\{ \sum_{i=1}^n 1_{E_i}\varphi^* v_i \,\middle|\, (E_i)_i \text{ is a Borel partition of } Y, (v_i)_i \subseteq \mathcal{M}, n \in \mathbb{N} \right\} \subseteq \varphi^*\mathcal{M}.$$

Our goal therefore is to take this as the definition of a map $\Phi\colon V \to (\varphi^*\mathcal{M})'$ and extend it linearly and continuously to all of $\varphi^*\mathcal{M}$. This is possible as $|(\varphi^*)' v_i| = |v|\circ\varphi = |\varphi^* v_i|$ $\mathfrak{m}_Y$-a.e. implies

$$\left|\sum_i 1_{E_i}(\varphi^*)' v_i\right| = \sum_i 1_{E_i}|v_i| \circ \varphi = \left|\sum_i 1_{E_i}\varphi^* v_i\right| \ \mathfrak{m}_Y\text{-a.e.}$$

Hence, $\|\Phi(v)\| = \| |\Phi(v)| \|_{L^2(\mathfrak{m})} = \| |v| \|_{L^2(\mathfrak{m})} = \|v\|$ for all $v \in V$, allowing us to extend Φ to a linear and continuous map on $\varphi^*\mathcal{M}$, retaining $|\Phi(w)| = |w|$ $\mathfrak{m}$-a.e. for all $w \in \varphi^*\mathcal{M}$ by arguing as in (3.3.7). This finally ensures the applicability of Proposition 3.2.3, which gives the full claim.

Inspecting the structure of the set V, we notice that two representations $\sum_i 1_{E_i}\varphi^* v_i$ and $\sum_j 1_{F_j}\varphi^* w_j$ of the same element agree if and only if $\sum_{i,j} 1_{E_i\cap F_j}\varphi^*(v_i - w_j)$ vanishes, i.e. precisely when $|v_i - w_j| \circ \varphi = 0$ $\mathfrak{m}_Y$-a.e. on $E_i \cap F_j$ for all i, j. For the existence part of the proof, we are hence led to consider the set

$$\left\{ (E_i, v_i)_{i=1}^n \mid (E_i)_i \text{ is a Borel partition of } Y, (v_i)_i \subseteq \mathcal{M}, n \in \mathbb{N} \right\},$$

on which we define the equivalence relation $(E_i, v_i)_i \sim (F_j, w_j)_j$ whenever $|v_i - w_j| \circ \varphi = 0$ $\mathfrak{m}_Y$-a.e. on $E_i \cap F_j$ for all i, j. The set of equivalence classes $\{[E_i, v_i]_i\}$ will be denoted by Ppb, an abbreviation for *pre-pullback module*. We now want to endow this set with the structure of a normed module. The vector space operations are defined simply as

$$\begin{aligned} [E_i, v_i]_i + [F_j, w_j]_j &= [E_i \cap F_j, v_i + w_j]_{i,j} \\ \lambda[E_i, v_i]_i &= [E_i, \lambda v_i]_i. \end{aligned}$$

By either merely regrouping terms or by using (absolute) homogeneity of the pointwise norm, we see that this is in fact a proper definition. To define an $L^\infty(\mathfrak{m}_Y)$-multiplication, we set

$$\left(\sum_j 1_{F_j}\alpha_j\right) \cdot [E_i, v_i]_i = [E_i \cap F_j, \alpha_j v_i]_{i,j}$$

for simple functions and extend it appropriately later on. The two norms are given by

$$|[E_i, v_i]_i| = \sum_i 1_{E_i}|v_i| \circ \varphi \ \text{ and } \ \|[E_i, v_i]_i\| = \left(\sum_i \int_{E_i} |v_i|^2 \circ \varphi \,\mathrm{d}\mathfrak{m}_Y\right)^{1/2}. \tag{3.3.11}$$

We now begin to prove that this construction will yield a normed module, as claimed.

To this end, we first notice that $\|[E_i, \lambda v_i]_i\| = |\lambda|\|[E_i, v_i]_i\|$ for any $\lambda \in \mathbb{R}$ ensures that the map $\|\cdot\|\colon \mathsf{Ppb} \to [0,\infty)$ is absolutely homogeneous. As

$$\begin{aligned} |[E_i, v_i]_i + [F_j, w_j]| &= \sum_{i,j} 1_{E_i \cap F_j} |v_i + w_j| \circ \varphi \\ &\le \sum_{i,j} 1_{E_i \cap F_j}(|v_i| \circ \varphi + |w_j| \circ \varphi) = |[E_i, v_i]_i| + |[F_j, w_j]_j|, \end{aligned} \tag{3.3.12}$$

we also have

$$\begin{aligned} \|[E_i, v_i]_i + [F_j, w_j]_j\| &\overset{(3.3.11)}{=} \||[E_i, v_i]_i + [F_j, w_j]_j|\|_{L^2(\mathfrak{m})} \\ &\overset{(3.3.12)}{\le} \||[E_i, v_i]_i|\|_{L^2(\mathfrak{m})} + \||[F_j, w_j]_j|\|_{L^2(\mathfrak{m})} = \|[E_i, v_i]_i\| + \|[F_j, w_j]_j\|, \end{aligned}$$

implying subbadditivity of $\|\cdot\|$. Further, $\|[E_i, v_i]_i\| \le \sqrt{\mathrm{Comp}(\varphi)}(\sum_i \int_{E_i} |v_i|^2 \,\mathrm{d}\mathfrak{m}_X)^{1/2}$, by (1.3.2), implies positive definiteness, so that $(\mathsf{Ppb}, \|\cdot\|)$ is seen to constitute a normed space.

Hence, we can define $(\varphi^*\mathcal{M}, \|\cdot\|)$ as the (Banach space) completion of $(\mathsf{Ppb}, \|\cdot\|)$. The multiplication by simple functions then extends to an $L^\infty(\mathfrak{m}_Y)$-multiplication since

$$\begin{aligned} \left\|\left(\sum_j 1_{F_j}\alpha_j\right)\cdot[E_i, v_i]_i\right\| &= \left\|\sum_{i,j} 1_{E_i \cap F_j}|\alpha_j v_i| \circ \varphi\right\|_{L^2(\mathfrak{m}_Y)} \\ &\le \max_j |\alpha_j| \left\|\sum_i 1_{E_i}|v_i| \circ \varphi\right\|_{L^2(\mathfrak{m}_Y)} = \left\|\sum_j 1_{F_j}\alpha_j\right\|_{L^\infty(\mathfrak{m}_Y)} \|[E_i, v_i]\|. \end{aligned}$$

Finally, the reverse triangle inequality

$$\begin{aligned} \||[E_i, v_i]_i| - |[F_j, w_j]_j|\|_{L^2(\mathfrak{m}_Y)} &\overset{(3.3.12)}{\le} \||[E_i, v_i]_i - [F_j, w_j]_j|\|_{L^2(\mathfrak{m}_Y)} \\ &\overset{(3.3.11)}{=} \|[E_i, v_i]_i - [F_j, w_j]_j\| \end{aligned}$$

ensures that the pointwise norm also extends continuously to $\varphi^*\mathcal{M}$, so that the tuple $(\varphi^*\mathcal{M}, \|\cdot\|, \cdot, |\cdot|)$ in fact constitutes an $L^2(\mathfrak{m}_Y)$-normed $L^\infty(\mathfrak{m}_Y)$-module.

Lastly, to define the map $\varphi^*\colon \mathcal{M} \to \varphi^*\mathcal{M}$, we simply put

$$\varphi^* v := [Y, v] \in \mathsf{Ppb} \subseteq \varphi^*\mathcal{M},$$

so that $|\varphi^* v| = |[Y, v]| = |v| \circ \varphi$. To show that the image of φ^* generates $\varphi^*\mathcal{M}$, we recall Proposition 3.2.12 and notice that any $[E_i, v_i]_i \in \mathsf{Ppb}$ can be written as

$$[E_i, v_i]_i = \sum_i 1_{E_i}[Y, v_i]_i = \sum_i 1_{E_i}\varphi^* v_i \qquad \square$$

Example 3.3.10. Viewing $L^2(\mathfrak{m}_X)$ as a module over X, a map of bounded compression $\varphi: Y \to X$ induces a well-defined module homomorphism

$$\varphi^*: L^2(\mathfrak{m}_X) \to L^2(\mathfrak{m}_Y),\ f \mapsto f \circ \varphi,$$

as $\|f \circ \varphi\|_{L^2(\mathfrak{m}_Y)} \leq \sqrt{\mathrm{Comp}(\varphi)}\|f\|_{L^2(\mathfrak{m}_X)}$ by the integral transformation formula (1.3.2). Moreover, $\{\varphi^* f \mid f \in L^2(\mathfrak{m}_X)\}$ generates $L^2(\mathfrak{m}_Y)$ as a module by Example 3.2.11. In fact, given any $f \in L^2(\mathfrak{m}_X)$ with $f \neq 0$ $\mathfrak{m}_X$-a.e., it follows that

$$\mathfrak{m}_Y(\{f \circ \varphi = 0\}) = \varphi_* \mathfrak{m}_X(\{f = 0\}) \leq \mathrm{Comp}(\varphi)\mathfrak{m}_X(\{f = 0\}) = 0,$$

so that $\varphi^* f = f \circ \varphi \neq 0$ $\mathfrak{m}_Y$-a.e. Thus, in total, $\varphi^* L^2(\mathfrak{m}_X) = L^2(\mathfrak{m}_Y)$.

Theorem 3.3.11. *Let $\varphi: Y \to X$ be a map of bounded compression and suppose $T: \mathcal{M} \to \mathcal{N}$ is any linear map between a module $\mathcal{M}$ over X and a module $\mathcal{N}$ over Y that satisfies*

$$|T(v)| \leq C|v| \circ \varphi \ \mathfrak{m}_Y\text{-a.e.} \quad \text{for every } v \in \mathcal{M} \tag{3.3.13}$$

and some universal constant $C > 0$. Then there exists a unique module homomorphism $\hat{T}: \varphi^*\mathcal{M} \to \mathcal{N}$ with $|\hat{T}(w)| \leq C|w|$ $\mathfrak{m}_Y$-a.e. for all $w \in \varphi^*\mathcal{M}$ that makes the following diagram commute:

$$\begin{tikzcd} & \varphi^*\mathcal{M} \arrow[d, "\hat{T}"] \\ \mathcal{M} \arrow[ur, "\varphi^*"] \arrow[r, "T"] & \mathcal{N} \end{tikzcd}$$

Proof. The bound (3.3.13) guarantees that the linear map $S: \varphi^* v \mapsto T(v)$, defined on the generating vector subspace $\{\varphi^* v \mid v \in \mathcal{M}\}$, is well-defined, as it implies

$$|S(\varphi^* v)| = |T(v)| \leq C|v| \circ \varphi \overset{(3.3.9)}{=} C|\varphi^* v| \ \mathfrak{m}_Y\text{-a.e.} \quad \text{for every } v \in \mathcal{M}$$

Thus, $\|S(\varphi^* v)\| \overset{(3.1.3)}{\leq} C\|\varphi^* v\|$, allowing us to extend the map S uniquely to a linear and continuous map $\hat{T}: \varphi^*\mathcal{M} \to \mathcal{N}$ satisfying $|\hat{T}(w)| \leq C|w|$ $\mathfrak{m}_Y$-a.e. by continuity of the pointwise norms, cf. (3.3.7). Hence, Proposition 3.2.3 gives the claim. □

Remark 3.3.12. By a slight generalization of the above statement (see Proposition 1.6.3 in [11]), once can prove the following *universal property* of the pullback module:

Denote by Mod the category whose objects are couples $(X, \mathcal{M})$, consisting of a metric measure space $(X, \mathrm{d}_X, \mathfrak{m}_X)$ together with an $L^2(\mathfrak{m}_X)$-normed $L^\infty(\mathfrak{m}_X)$-module $\mathcal{M}$, and whose morphisms are pairs of maps $(\varphi, T): (Y, \mathcal{N}) \to (X, \mathcal{M})$, where $\varphi: Y \to X$ is a map of bounded compression and $T: \mathcal{M} \to \mathcal{N}$ is a linear map satisfying $|T(v)| \leq C|v| \circ \varphi$ $\mathfrak{m}_Y$-a.e. for every $v \in \mathcal{M}$. Then the pair $(Y, \varphi^*\mathcal{M})$ together with the two morphisms (φ, φ^*) and $(\mathrm{id}_Y, 0)$ is universal (in the sense described in the introduction) with respect to the following commutative diagram:

$$\begin{tikzcd}
(Y,\varphi^*\mathcal{M}) \arrow[r, "(\varphi,\varphi^*)"] \arrow[d, "(\mathrm{id}_Y\,,0)"'] & (X,\mathcal{M}) \arrow[d, "(\mathrm{id}_X\,,0)"] \\
(Y,0) \arrow[r, "(\varphi,0)"] & (X,0)
\end{tikzcd}$$

Thus, the pullback module is indeed a *categorical pullback* in the category Mod, just as the pullback bundle was in its appropriate setting. For more details, see Remark 1.6.4 in [11].

Theorem 3.3.13. *Let* $(X, \mathrm{d}_X, \mathfrak{m}_X), (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ *and* $(Z, \mathrm{d}_Z, \mathfrak{m}_Z)$ *be three metric measure spaces with maps of bounded compression* $\varphi\colon Y \to X$ *and* $\psi\colon Z \to Y$ *between them. Then the composite function* $\varphi \circ \psi\colon Z \to X$ *is also of bounded compression, and we have*

$$(\psi^*(\varphi^*\mathcal{M}), \psi^* \circ \varphi^*) \cong ((\varphi \circ \psi)^*\mathcal{M}, (\varphi \circ \psi)^*)$$

in the sense described in Remark 3.3.9.

Proof. The composite map $\varphi \circ \psi\colon Z \to X$ is easily seen to be of bounded compression, recalling that the push-forward satisfies $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. Thus, starting from a normed module $\mathcal{M}$ over X, Theorem 3.3.8 ensures the existence of an $L^2(\mathfrak{m}_Z)$-normed module $(\varphi \circ \psi)^*\mathcal{M}$ together with a module homomorphism $(\varphi \circ \psi)^*\colon \mathcal{M} \to (\varphi \circ \psi)^*\mathcal{M}$.

We now make use of the previous theorem, lifting the linear map

$$T\colon \mathcal{M} \to \psi^*(\varphi^*\mathcal{M}), v \mapsto \psi^*\varphi^*v$$

to the pullback module $(\varphi \circ \psi)^*\mathcal{M}$. This is possible as

$$|T(v)| = |\psi^*(\varphi^*v)| \overset{(3.3.9)}{=} |\varphi^*v| \circ \psi \overset{(3.3.9)}{=} |v| \circ \varphi \circ \psi \ \mathfrak{m}_X\text{-a.e.} \tag{3.3.14}$$

Thus, we obtain a unique module homomorphism $\hat{T}\colon (\varphi \circ \psi)^*\mathcal{M} \to \psi^*(\varphi^*\mathcal{M})$, mapping $(\varphi \circ \psi)^*v$ onto $\psi^*\varphi^*v$ by commutativity of the diagram in Theorem 3.3.11. To prove that $\hat{T}$ is in fact an isomorphism, we first define the linear map $S\colon \psi^*\varphi^*v \mapsto (\varphi \circ \psi)^*v$ on the set $\{\psi^*\varphi^*v \mid v \in \mathcal{M}\}$. Once we have proved that this set generates $\psi^*(\varphi^*\mathcal{M})$ as a module, equation (3.3.14) ensures that

$$|S(\psi^*\varphi^*v)| = |v| \circ \varphi \circ \psi = |\psi^*(\varphi^*v)| \ \mathfrak{m}_X\text{-a.e.}$$

As in the proof of Theorem 3.3.11, this implies the existence of a unique extension to a module homomorphism $\hat{S}\colon \psi^*(\varphi^*\mathcal{M}) \to (\varphi \circ \psi)^*\mathcal{M}$ with $\psi^*\varphi^*v \mapsto (\varphi \circ \psi)^*v$. Finally, as $\hat{S}$ is the inverse to $\hat{T}$ when both are restricted to generating subspaces, by continuity and L^∞-linearity, this remains true on the whole module. Thus, $(\varphi \circ \psi)^*\mathcal{M}$ and $\psi^*(\varphi^*\mathcal{M})$ are isomorphic with $\psi^* \circ \varphi^*$ getting identified with $(\varphi \circ \psi)^*$.

In remains to prove that the vector space $\{\psi^*\varphi^* v \mid v \in \mathcal{M}\}$ generates the module $\psi^*(\varphi^*\mathcal{M})$ (as compared to $\{\psi^* w \mid w \in \varphi^*\mathcal{M}\}$). For this, we recall Proposition 3.2.12 and show that any element of the form $\sum_{j=1}^m 1_{F_j}\psi^* w_j$ with $(F_j)_j$ a Borel partition and $(w_j)_j \subseteq \varphi^*\mathcal{M}$ can be approximated arbitrarily well by elements of the form $\sum_{i=1}^n 1_{E_i}\psi^*(\varphi^* v_i)$ with $(E_i)_i$ again a Borel partition and $(v_i)_i \subseteq \mathcal{M}$.

We start by noticing that $\sum_j 1_{F_j}\psi^* w_j = \psi^*(\sum_j 1_{F_j} w_j)$ with $\sum_j 1_{F_j} w_j \in \varphi^*\mathcal{M}$. Given any $\varepsilon > 0$, it thus suffices to choose an element $\sum_i 1_{E_i}\varphi^* v_i$ with $\|\sum_j 1_{F_j} w_j - \sum_i 1_{E_i}\varphi^* v_i\| < {}^{\varepsilon}/_{\|\psi^*\|}$ to guarantee that

$$\left\|\sum_j 1_{F_j}\psi^* w_j - \sum_i 1_{E_i}\psi^*(\varphi^* v_i)\right\| \leq \|\psi^*\| \left\|\sum_j 1_{F_j} w_j - \sum_i 1_{E_i}\varphi^* v_i\right\| < \varepsilon. \qquad \square$$

We would now like to understand the relationship between the dual of the pullback and the pullback of the dual, and we will see that, in general, these two notions do not coincide.

To begin with, we observe that the usual dual pairing

$$\mathcal{M} \times \mathcal{M}^* \to L^1(\mathfrak{m}), \ (v, L) \mapsto L(v) \tag{3.3.15}$$

provides us with a natural way of pairing up elements $\varphi^* v \in \varphi^*\mathcal{M}$ with elements of the form $\varphi^* L \in \varphi^*\mathcal{M}^*$. In fact, simply pulling back (3.3.15) along φ, we can define a mapping

$$(\varphi^* v, \varphi^* L) \mapsto L(v) \circ \varphi,$$

as we have the bound

$$|L(v) \circ \varphi| = |L(v)| \circ \varphi \overset{(3.3.4)}{\leq} (|L|_* \circ \varphi)(|v| \circ \varphi) \overset{(3.3.9)}{=} |\varphi^* L||\varphi^* v|, \tag{3.3.16}$$

where we have used the pointwise norms on $\varphi^*\mathcal{M}^*$ and $\varphi^*\mathcal{M}$ to establish the last equality. Extending suitably, as we will do in the next proposition, we then discover that

$$\varphi^*\mathcal{M}^* \hookrightarrow (\varphi^*\mathcal{M})^*,$$

while it will, in general, fail to be a surjection, see Example 3.3.16.

Proposition 3.3.14. *There exists a unique $L^\infty(\mathfrak{m}_Y)$-bilinear and continuous map $B\colon \varphi^*\mathcal{M} \times \varphi^*\mathcal{M}^* \to L^1(\mathfrak{m}_Y)$ such that*

$$B(\varphi^* v, \varphi^* L) = L(v) \circ \varphi \ \mathfrak{m}_Y\text{-a.e.} \tag{3.3.17}$$

for every $v \in \mathcal{M}$ and $L \in \mathcal{M}^*$. Moreover, we have a bound on the absolute value $|B(V, W)| \leq |V||W|$ $\mathfrak{m}_Y$-a.e. for all $V \in \varphi^*\mathcal{M}$ and $W \in \varphi^*\mathcal{M}^*$ with regard to the product of the respective pointwise norms.

Proof. We again define this map as the extension of a suitable map on generating subspaces. For this, using homogeneity of the pointwise norm (3.1.2), we can calculate as in (3.3.16) to get

$$\begin{aligned}\Big|\sum_{i,j} 1_{E_i\cap F_j} L_j(v_i)\circ\varphi\Big| &\overset{(3.1.2)}{=} \sum_{i,j} 1_{E_i\cap F_j}|L_j(v_i)\circ\varphi| \\ &\overset{(3.3.16)}{\leq} \sum_{i,j} 1_{E_i\cap F_j}|\varphi^*L_j|\,|\varphi^*v_i| \overset{(3.1.2)}{=} \Big|\sum_i 1_{E_i}\varphi^*v_i\Big|\Big|\sum_j 1_{F_j}\varphi^*L_j\Big| \ \mathfrak{m}_Y\text{-a.e.}\end{aligned}$$

This allows us to extend the (densely defined) linear map

$$\Big(\sum 1_{E_i}\varphi^*v_i, \sum_j 1_{F_j}\varphi^*L_j\Big) \mapsto \sum_j 1_{F_j}L_j\Big(\sum_i 1_{E_i}v_i\Big)\circ\varphi = \sum_{i,j} 1_{E_i\cap F_j}L_j(v_i)\circ\varphi$$

as in the proof of Theorem 3.3.11 by taking norms on both sides to see that we can extend (linearly and) continuously, while preserving

$$|B(V,W)| \leq |V||W| \ \mathfrak{m}_Y\text{-a.e.}$$

for all $V \in \varphi^*\mathcal{M}$ and $W \in \varphi^*\mathcal{M}^*$, so that Proposition 3.3.6 guarantees $L^\infty(\mathfrak{m})$-bilinearity. □

Proposition 3.3.15. *Let $\varphi\colon Y \to X$ be a map of bounded compression and denote by $B\colon \varphi^*\mathcal{M} \times \varphi^*\mathcal{M}^* \to L^1(\mathfrak{m}_Y)$ the pairing defined in Proposition 3.3.14. Then,*

$$I\colon \varphi^*\mathcal{M}^* \to (\varphi^*\mathcal{M})^*,\ W \mapsto B(\,\cdot\,,W) \tag{3.3.18}$$

is an isometric embedding (of modules).

Proof. Proposition 3.3.14 guarantees that $I(W) = B(\,\cdot\,,W)\colon \varphi^*\mathcal{M} \to L^1(\mathfrak{m}_Y)$ is $L^\infty(\mathfrak{m}_Y)$-linear and continuous, i.e. an element of $(\varphi^*\mathcal{M})^*$. Further, as $|B(V,W)| \leq |W|$ for all $V \in \mathcal{B}_{\varphi^*\mathcal{M}}$, where $\mathcal{B}_{\varphi^*\mathcal{M}}$ is the closed unit ball of $\varphi^*\mathcal{M}$ w.r.t. the pointwise norm, we have $|I(W)|_* \leq |W|$ $\mathfrak{m}_Y$-a.e. Hence, by taking norms on both sides and invoking (3.1.3) for the respective modules, we see that $W \mapsto I(W)$ is continuous. Its $L^\infty(\mathfrak{m}_Y)$-linearity is inherited from the map B.

To prove the missing converse inequality $|I(W)| \geq |W|$ $\mathfrak{m}_Y$-a.e., by density and continuity (recall condition (3.3.10) and then argue as in (3.3.7)), it suffices to establish

$$\operatorname{ess\,sup}|I(W)(V)| \geq |W| \ \mathfrak{m}_Y\text{-a.e.},$$

where the supremum is taken over all $V \in \mathcal{B}_{\varphi^*\mathcal{M}}$ of the form $\sum_i 1_{E_i}\varphi^*v_i$, for all elements $W = \sum_j 1_{F_j}\varphi^*L_j$, where we can assume w.l.o.g. that the partition $(F_j)_{j=1}^m$ agrees with

$(E_i)_{i=1}^n$ by taking common refinements. Since

$$|\sum_i 1_{E_i}\varphi^* v_i| \overset{(3.1.2)}{=} \sum_i 1_{E_i}|\varphi^* v_i| \overset{(3.3.9)}{=} (\sum_i 1_{E_i}|v_i|)\circ\varphi,$$

this amounts to proving

$$\operatorname*{ess\,sup}_{\{v_i\}_{i=1}^n\subseteq\mathcal{B}_{\mathcal{M}}} \left| I(W)(\sum_i 1_{E_i}\varphi^* v_i)\right| \geq |W| \ \mathfrak{m}_Y\text{-a.e.} \tag{3.3.19}$$

with $W=\sum_i 1_{E_i}\varphi^* L_i$. Making use of equality (3.3.17) and the definition of the map I, we conclude that the left-hand side of (3.3.19) equals

$$\sum_i 1_{E_i}\operatorname*{ess\,sup}_{v_i\in\mathcal{B}_{\mathcal{M}}} |L_i(v_i)\circ\varphi| \overset{(3.3.3)}{=} \sum_i 1_{E_i}|L_i|_*\circ\varphi \overset{(3.3.9)}{=} \sum_i 1_{E_i}|\varphi^* L_i| \overset{(3.1.2)}{=} |W| \ \mathfrak{m}_Y\text{-a.e.} \quad \square$$

Example 3.3.16. We now demonstrate that, in the general case, $\varphi^*\mathcal{M}^*$ can only be realized as a *proper* subspace of $(\varphi^*\mathcal{M})^*$. In fact, this already occurs for L^2-Bochner spaces, which naturally carry the structure of a normed module. A thorough treatment of this phenomenon, can be found in Chapter IV of *Vector Measures* [10] by Joseph Diestel and John J. Uhl.

We start by demonstrating how any given Banach space $(\mathbb{B},\|\cdot\|_{\mathbb{B}})$ can be viewed as a normed module over the *singleton space* $(X,\mathrm{d},\mathfrak{m})=(\{*\},0,\delta_*)$ with the zero metric and the Dirac delta δ_* on its only point. For this, we use the identification

$$L^\infty(\delta_*) \overset{\cong}{\to} \mathbb{R},\ f\mapsto f(*) \tag{3.3.20}$$

and thereby associate the vector space structure with the module structure. Further, recalling $L^2(\delta_*)=L^\infty(\delta_*)$, we define

$$|\cdot|\colon \mathbb{B}\to L^2(\delta_*),\ |v|(x)=\|v\|_B. \tag{3.3.21}$$

We also endow the Bochner space $L^2([0,1],\mathbb{B})$ with the structure of a normed module over $[0,1]$ with the Euclidean distance and the Lebesgue measure by similarly defining

$$|f|(x)=\|f(x)\|_{\mathbb{B}}\in L^2([0,1]) \quad \text{for all } f\in L^2([0,1],\mathbb{B}) \tag{3.3.22}$$

up to a.e.-equivalence.

Now, we claim that for the pullback of this module under the constant map $\varphi\colon[0,1]\to\{*\}$, we have

$$(\varphi^*\mathbb{B},\varphi^*)\cong(L^2([0,1],\mathbb{B}),\mathrm{Const}),$$

where $\mathrm{Const}\colon\mathbb{B}\to L^2([0,1],\mathbb{B})$ maps any $v\in\mathbb{B}$ to $\mathrm{Const}(v)\colon[0,1]\to\mathbb{B}$, $t\mapsto v$.

To show this, by Theorem 3.3.8, it suffices to prove

$$|\mathrm{Const}(v)| = |v| \circ \varphi \quad \lambda^1\text{-a.e.} \tag{3.3.23}$$

for the respective pointwise norms, and to argue that the set $\{\sum_i 1_{E_i} \mathrm{Const}(v_i)\}$ with $(E_i)_i$ a Borel partition of $[0,1]$ and $(v_i)_i \subseteq \mathbb{B}$, is dense in $L^2([0,1],\mathbb{B})$. Since the latter follows immediately from the density of simple functions in $L^2([0,1],\mathbb{B})$, see Proposition 1.3.22, we focus on equality (3.3.23).

This, however, is merely a task in unwinding definitions: The left-hand side is given as the equivalence class of the function

$$t \mapsto |\mathrm{Const}(v)|(t) \overset{(3.3.22)}{=} \|\mathrm{Const}(v)_t\|_{\mathbb{B}} = \|v\|_{\mathbb{B}},$$

while the right-hand side evaluates to

$$(|v| \circ \varphi)(t) = |v|(*) \overset{(3.3.21)}{=} \|v\|_{\mathbb{B}}.$$

Applying this construction to $\mathbb{B}$ as well as its dual $\mathbb{B}'$, we therefore have

$$(\varphi^*\mathbb{B})^* \cong (L^2([0,1],\mathbb{B}))' \quad \text{and} \quad \varphi^*\mathbb{B}^* \cong L^2([0,1],\mathbb{B}'),$$

where we have also already used Theorem 3.4.1 to identify $L^2([0,1],\mathbb{B})'$ with the module dual of $L^2([0,1],\mathbb{B})$. Now, it is easy to see that any $\ell \in L^2([0,1],\mathbb{B}')$ can be viewed as an element of $(L^2([0,1],\mathbb{B}))'$ by defining its action on $v \in L^2([0,1],\mathbb{B})$ as

$$\ell(v) := \int_0^1 \ell_t(v_t)\,\mathrm{d}t \in \mathbb{R} \tag{3.3.24}$$

Moreover, it can be proved that this mapping in fact provides an isometric embedding of $L^2([0,1],\mathbb{B}')$ into $(L^2([0,1],\mathbb{B}))'$, cf. the discussion before Theorem 1 in Chapter IV of [10]. To see that it fails to be surjective, let $\mathbb{B} = L^1(0,1)$ and define

$$T\colon L^2([0,1],L^1(0,1)) \to \mathbb{R},\ f \mapsto \int_0^1 \int_0^1 f_t(x) 1_{[0,t]}(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Suppose now that the operator T were given as the action of some $g \in (L^1(0,1))'$. Using (3.3.24) as well as the fact that $(L^1(0,1))' \cong L^\infty(0,1)$ via the identification of $g \in L^\infty(0,1)$ with $h \mapsto \int h(x)g(x)\,\mathrm{d}x$, we see that

$$T(f) \overset{(3.3.24)}{=} \int_0^1 g_t(f_t)\,\mathrm{d}t = \int_0^1 \int_0^1 f_t(x) g_t(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence $g_t(x) = 1_{[0,t]}(x)$, which is not essentially separably valued, contradicting our assumption by Proposition 1.3.21.

3.4 Hilbert Modules

For any normed module $\mathcal{M}$, we have at our disposal two different notions of a dual: $\mathcal{M}'$ as the set of $\mathbb{R}$-linear and continuous functionals $\ell\colon \mathcal{M} \to \mathbb{R}$, and $\mathcal{M}^*$ as the set of $L^\infty(\mathfrak{m})$-linear and continuous maps $L\colon \mathcal{M} \to L^1(\mathfrak{m})$. Since the operation of integrating over the space X provides a natural linear map $\int\colon L^1(\mathfrak{m}) \to \mathbb{R}$ with

$$\left|\int L(v)\,\mathrm{d}\mathfrak{m}\right| \leq \|L\|_*\|v\|,$$

cf. (3.3.3), we immediately see that

$$(\mathcal{M} \ni v \mapsto \int L(v)\,\mathrm{d}\mathfrak{m}) \in \mathcal{M}'$$

Now, we would like to address the converse: Is any continuous linear functional ℓ on $\mathcal{M}$ of the form $\ell(v) = \int L(v)\,\mathrm{d}\mathfrak{m}$ with $L \in \mathcal{M}^*$? As we will prove in the next theorem, this is indeed the case, and we can obtain $L(v)$ as the Radon-Nikodým derivative of the countably additive set-function $X \supseteq E \mapsto \ell(1_E v)$ with respect to $\mathfrak{m}$.

We again closely follow the presentation given in Chapter 3.2.1 in [13] with a result from [11], which will be cited explicitly.

Theorem 3.4.1. *For any normed module $\mathcal{M}$, the map*

$$\mathrm{Int}\colon \mathcal{M}^* \to \mathcal{M}',\ \mathrm{Int}(L)(v) := \int L(v)\,\mathrm{d}\mathfrak{m} \tag{3.4.1}$$

constitutes an isometric (Banach space) isomorphism.

Proof. We first show that the map Int is an isometry, i.e. that $\|\mathrm{Int}(L)\| = \|L\|_*$ for all $L \in \mathcal{M}^*$, or more explicitly, we prove (cf. (3.3.3))

$$\sup_{\|v\|\leq 1}\left|\int L(v)\,\mathrm{d}\mathfrak{m}\right| = \sup_{\|v\|\leq 1}\|L(v)\|_{L^1(\mathfrak{m})}.$$

Clearly, the right-hand side provides a bound for the left-hand side. For the converse, we choose a sequence $(v_n)_{n\geq 1}$ in the unit ball of $\mathcal{M}$ such that $\sup_{n\geq 1}\|L(v_n)\|_{L^1(\mathfrak{m})} = \|L\|_*$. Switching to the modified sequence $(\tilde{v}_n)_{n\geq 1}$, where

$$\tilde{v}_n := 1_{\{L(v_n)\geq 0\}}v_n + 1_{\{L(v_n)<0\}}(-v_n) \tag{3.4.2}$$

so that $L(\tilde{v}_n) = |L(v_n)|$ $\mathfrak{m}$-a.e. while $|\tilde{v}_n| = |v_n|$, we see that indeed also $\|L\|_* = \sup_{n\geq 1}\int|L(v_n)|\,\mathrm{d}\mathfrak{m} = \sup_{n\geq 1}\int L(\tilde{v}_n)\,\mathrm{d}\mathfrak{m} \leq \sup_{\|v\|\leq 1}|\int L(v)\,\mathrm{d}\mathfrak{m}|$.

This further immediately implies that Int is isometric, injective and continuous. Once we have shown that the map is also surjective, we know $\|\mathrm{Int}^{-1}(\ell)\|_* = \|\mathrm{Int}(\mathrm{Int}^{-1}(\ell))\| = \|\ell\|$, implying continuity of the inverse Int^{-1} and thereby concluding the proof.

To establish surjectivity, we proceed as in the standard proof of the assertion $(L^p(\mathfrak{m}))' \cong L^q(\mathfrak{m})$, as for example given in [9] Theorem 4.5.1. That is, fixing $\ell \in \mathcal{M}'$ and $v \in \mathcal{M}$, we obtain $L(v)$ as the Radon-Nikodým derivative $\frac{\mathrm{d}\mu_v}{\mathrm{d}\mathfrak{m}}$ of the finite (signed) Borel measure μ_v w.r.t. $\mathfrak{m}$, where $\mu_v(E) := \ell(1_E v)$ for all $E \subseteq X$ Borel. In particular,

$$\ell(v) = \ell(1_X v) = \mu_v(X) = \int L(v)\,\mathrm{d}\mathfrak{m},$$

which is exactly what we claimed. We now check that this is technically feasible.

To this end, let $\{A_n\}_{n\geq 1}$ be a sequence of disjoint Borel sets in X and denote by $A \subseteq X$ its union. Then, by continuity of the multiplication, $1_A v = \sum_{n\geq 1} 1_{A_n} v$ and so $\mu_v(A) = \ell(1_A v) = \sum_{n\geq 1} \ell(1_{A_n} v) = \sum_{n\geq 1} \mu_v(A_n)$ since $\ell \in \mathcal{M}'$. As $\mathfrak{m}(B) = 0$ clearly implies $\mu_v(B) = 0$, the set function $X \supseteq E \mapsto \mu_v(E)$ constitutes a finite Borel measure that is absolutely continuous with respect to $\mathfrak{m}$. The Theorem of Radon-Nikodým thus grants the existence of an element $\frac{\mathrm{d}\mu_v}{\mathrm{d}\mathfrak{m}} \in L^1(\mathfrak{m})$ with the property that $\mu_v(E) = \int_E \frac{\mathrm{d}\mu_v}{\mathrm{d}\mathfrak{m}}\,\mathrm{d}\mathfrak{m}$ for all $E \subseteq X$ Borel. Denoting this element by $L(v)$, we check that $v \mapsto L(v)$ is linear and satisfies

$$L(1_F v) = 1_F L(v) \text{ for all } F \subseteq X \text{ Borel.}$$

First, we observe that $\mu_{\lambda v + w} = \lambda \mu_v + \mu_w$ due to linearity of ℓ and $\mu_{1_F v}(E) = \ell(1_E 1_F v) = \mu_v(E \cap F)$ for any Borel subset $F \subseteq X$. Hence, fixing some $E \subseteq X$ Borel,

$$\int_E L(\lambda v + w)\,\mathrm{d}\mathfrak{m} = \mu_{\lambda v + w}(E) = \lambda \mu_v(E) + \mu_w(E) = \int_E \lambda L(v) + L(w)\,\mathrm{d}\mathfrak{m}$$

as well as

$$\int_E L(1_F v)\,\mathrm{d}\mathfrak{m} = \mu_{1_F v}(E) = \mu_v(E \cap F) = \int_E 1_F L(v)\,\mathrm{d}\mathfrak{m},$$

implying the assertion by arbitrariness of the set E.

Finally, choosing $\tilde{v} \in \mathcal{M}$ adapted to a given $v \in \mathcal{M}$ as in (3.4.2), we have $\|L(v)\|_{L^1(\mathfrak{m})} = \int L(\tilde{v})\,\mathrm{d}\mathfrak{m} = \ell(\tilde{v}) \leq \|\ell\|\|v\|$, so that $L\colon \mathcal{M} \to L^1(\mathfrak{m})$ is seen to be continuous and hence an element of $\mathcal{M}^*$ by Lemma 3.3.5. $\square$

With this at hand, we can derive the following *dual* characterization of the pointwise norm on $\mathcal{M}$, reminiscent (and in fact a consequence of) the analogous result for Banach spaces.

Corollary 3.4.2. *For any $v \in \mathcal{M}$, we have*

$$|v| = \operatorname*{ess\,sup}_{|L|_* \leq 1} |L(v)| \quad \mathfrak{m}\text{-a.e.} \tag{3.4.3}$$

Proof. By Lemma 3.3.3, the right-hand side is clearly bounded by $|v|$ for any $v \in \mathcal{M}$. We now claim that there exists an $L \in \mathcal{M}^*$ with

$$L(v) = |v|^2 = |L|_*^2 \ \mathfrak{m}\text{-a.e.} \tag{3.4.4}$$

To see this, invoking the Theorem of Hahn-Banach, we choose an $\ell \in \mathcal{M}'$ such that

$$\ell(v) = \|v\|^2 = \|\ell\|^2 \tag{3.4.5}$$

and set $L := \mathrm{Int}^{-1}(\ell)$. Then, using Lemma 3.3.3 and Hölder's inequality, we have

$$\int |L(v)| \,\mathrm{d}\mathfrak{m} \overset{(3.3.4)}{\leq} \int |L|_* |v| \,\mathrm{d}\mathfrak{m} \leq \| \, |L|_* \|_{L^2(\mathfrak{m})} \| \, |v| \, \|_{L^2(\mathfrak{m})} \overset{(3.3.5)}{=} \|L\|_* \|v\| \tag{3.4.6}$$

Hence, employing (3.4.5) and recalling that $\|L\|_* = \|\ell\|$, as guaranteed by Theorem 3.4.1, we obtain

$$\|v\|^2 = \ell(v) \overset{(3.4.1)}{=} \int L(v) \,\mathrm{d}\mathfrak{m} \overset{(3.4.6)}{\leq} \|L\|_* \|v\| = \|\ell\| \|v\| = \|v\|^2,$$

Thus, all involved inequalities are in fact equalities and so

$$\int L(v) \,\mathrm{d}\mathfrak{m} = \int |L(v)| \,\mathrm{d}\mathfrak{m} = \int |L|_* |v| \,\mathrm{d}\mathfrak{m}.$$

Moreover, as $L(v) \leq |L(v)| \overset{(3.3.4)}{\leq} |L|_* |v|$, we can conclude that

$$L(v) = |L(v)| = |L|_* |v| \ \mathfrak{m}\text{-a.e.}$$

This is already sufficient to obtain $\operatorname*{ess\,sup}_{|L|_* \leq 1} |L(v)| \geq |v|$ $\mathfrak{m}$-a.e.

Additionally, to see that we even have (3.4.4), we recall the equality case in Hölder's inequality to deduce that

$$|L|_*^2 = t|v|^2 \quad \text{for some scalar } t \geq 0.$$

By integrating this equality over the whole space X and again using $\|L\|_* = \|\ell\|$, we now see that $t = 1$ (unless $v = 0$) as

$$\|v\|^2 \overset{(3.4.6)}{=} \|\ell\|^2 = \|L\|_*^2 \overset{(3.3.5)}{=} \int |L|_*^2 \,\mathrm{d}\mathfrak{m} = t \int |v|^2 \,\mathrm{d}\mathfrak{m} \overset{(3.1.3)}{=} t\|v\|^2. \qquad \square$$

The previous result and the idea of its proof are taken from [11], Corollary 1.2.16.

Corollary 3.4.3. *Any normed module $\mathcal{M}$ isometrically embeds into its double dual $\mathcal{M}^{**}$ via the map*

$$\mathrm{ev}\colon \mathcal{M} \to \mathcal{M}^{**},\ \mathrm{ev}(v)(L) = L(v). \tag{3.4.7}$$

Proof. $L^\infty(\mathfrak{m})$-linearity is easily established as

$$\mathrm{ev}(fv)(L) = L(fv) = fL(v) = f\,\mathrm{ev}(v)(L)$$

for all $f \in L^\infty(\mathfrak{m})$, any $v \in \mathcal{M}$ and $L \in \mathcal{M}^*$. Now, by Corollary 3.4.2

$$|\mathrm{ev}(v)| \overset{(3.3.3)}{=} \operatorname*{ess\,sup}_{|L|_* \leq 1} |\mathrm{ev}(v)(L)| \overset{(3.4.7)}{=} \operatorname*{ess\,sup}_{|L|_* \leq 1} |L(v)| \overset{(3.4.3)}{=} |v| \ \mathfrak{m}\text{-a.e.}$$

Thus, $\mathrm{ev}\colon \mathcal{M} \to \mathcal{M}^{**}$ is an injective module homomorphism preserving the (pointwise) norm. The module $\mathcal{M}$ can therefore be identified isometrically with the submodule $\mathrm{ev}(\mathcal{M}) \subseteq \mathcal{M}^{**}$ (recall Remark 3.2.8). □

As we also do in Banach space theory, we distinguish spaces for which this canonical embedding is a bijection. In fact, using the integration operators defined in Theorem 3.4.1, we will now show that this notion coincides with the usual notion of reflexivity for Banach spaces.

Definition 3.4.4. A normed module $\mathcal{M}$ is said to be *reflexive* if the natural embedding $\mathrm{ev}\colon \mathcal{M} \to \mathcal{M}^{**}$ is surjective.

Proposition 3.4.5. *Let $\mathcal{M}$ be any normed module. Then $\mathcal{M}$ is reflexive as a module if and only if it is reflexive as a Banach space.*

Proof. Denote by $i\colon \mathcal{M} \to \mathcal{M}''$ the canonical embedding of $\mathcal{M}$ as a Banach space into its double dual via

$$\mathcal{M} \ni v \mapsto i(v) \quad \text{with} \quad i(v)(\ell) = \ell(v) \ \text{ for all } \ell \in \mathcal{M}' \tag{3.4.8}$$

and by $\mathrm{Int}_{\mathcal{M}^*}\colon \mathcal{M}^{**} \to (\mathcal{M}^*)'$ the integration operator as defined in (3.4.1). Then both $\mathrm{Int}_{\mathcal{M}^*}$ and $\mathrm{Int}'_{\mathcal{M}}\colon \mathcal{M}'' \to (\mathcal{M}^*)'$, defined as the transpose of $\mathrm{Int}_{\mathcal{M}}$, i.e.

$$\mathrm{Int}'_{\mathcal{M}}(\varphi) = \varphi \circ \mathrm{Int}_{\mathcal{M}} \ \text{ for any } \varphi \in \mathcal{M}'', \tag{3.4.9}$$

are bijective. Moreover, we have

$$\mathrm{Int}_{\mathcal{M}^*}(\mathrm{ev}(v))(L) = \int \mathrm{ev}(v)(L)\,\mathrm{d}\mathfrak{m} \overset{(3.4.7)}{=} \mathrm{Int}_{\mathcal{M}}(L)(v) \overset{(3.4.8)}{=} i(v)(\mathrm{Int}_{\mathcal{M}}(L)) \overset{(3.4.9)}{=} \mathrm{Int}'_{M}(i(v))(L)$$

Hence, the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathrm{ev}"] \arrow[d, "i"'] & \mathcal{M}^{**} \arrow[d, "\mathrm{Int}_{\mathcal{M}^*}", "\cong"'] \\ M'' \arrow[r, "\cong", "\mathrm{Int}'_{\mathcal{M}}"'] & (\mathcal{M}^*)' \end{tikzcd}$$

commutes, implying that the map ev is surjective if and only if the map i is. □

We now turn our attention to a special class of reflexive modules, in which the pointwise norms can be polarized to produce *pointwise scalar products*. Relating this property to the underlying Banach spaces, the duals of such modules will be seen to be isometric to the modules themselves via a pointwise analogue of the Riesz Theorem. Chapter 3.2.2 in [13] will serve as a guideline for this.

Finally, also in the context of defining a differential calculus for metric measure spaces, this class of modules will play a prominent role, as they will serve as a generalization of bundles on a *Riemannian* manifold.

Definition 3.4.6. A normed module $\mathcal{M}$ is said to be a *Hilbert module* if the underlying Banach space $(\mathcal{M}, \|\cdot\|)$ possesses the structure of a Hilbert space. To distinguish notations, we will denote a generic Hilbert module by a calligraphic $\mathcal{H}$ and its scalar product by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$.

As an immediate consequence of the last proposition, we obtain the following result, relating any Hilbert module with its double dual.

Corollary 3.4.7. *Any Hilbert module $\mathcal{H}$ is reflexive.*

We now prove our main result on Hilbert modules, in which we show that any such module admits a *pointwise scalar product* $\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathcal{H}\times\mathcal{H}\to L^1(\mathfrak{m})$ defined via polarization as

$$\langle v, w\rangle := \frac{1}{2}(|v+w|^2 - |v|^2 - |w|^2) \quad \text{for any } v, w \in \mathcal{H}. \tag{3.4.10}$$

Further, this map will be shown to possess the usual properties of a scalar product.

Theorem 3.4.8. *Let $\mathcal{H}$ be a Hilbert module. Then the map defined in* (3.4.10) *is $L^\infty(\mathfrak{m})$-bilinear, symmetric, positive semi-definite (in the almost-everywhere sense) and non-degenerate. Moreover, we can relate* (3.4.10) *to the pointwise norm via*

$$|\langle v, w\rangle| \leq |v||w| \ \mathfrak{m}\text{-a.e.} \quad \text{and} \quad \langle v, v\rangle = |v|^2 \tag{3.4.11}$$

for any $v, w \in \mathcal{H}$. Finally, the pointwise parallelogram law *holds:*

$$2(|v|^2 + |w|^2) = |v+w|^2 + |v-w|^2 \ \mathfrak{m}\text{-a.e.} \tag{3.4.12}$$

Proof. First, by simply inserting into (3.4.10), we get $\langle v, v\rangle = |v|^2 \geq 0$ $\mathfrak{m}$-a.e., as well as $\langle v, w\rangle = \langle w, v\rangle$ for any $v, w \in \mathcal{H}$, establishing positive semi-definiteness and symmetry. By integrating (3.4.10) over the space X and using (3.1.3), we get

$$\langle v, w\rangle_{\mathcal{H}} = \int \langle v, w\rangle \,\mathrm{d}\mathfrak{m} \tag{3.4.13}$$

by the parallelogram law for $\|\cdot\|_{\mathcal{H}}$. Further, integrating the expression $\langle \lambda v, w\rangle$ for $\lambda \in \mathbb{R}$

over any Borel set $E \subseteq X$, we see

$$\begin{aligned}\int_E \langle \lambda v, w\rangle \,\mathrm{d}\mathfrak{m} &\overset{(3.4.10)}{=} \int_E \frac{1}{2}(|\lambda v + w|^2 - |\lambda v|^2 - |w|^2)\,\mathrm{d}\mathfrak{m} \\ &\overset{(3.1.3)}{=} \frac{1}{2}(\|\lambda 1_E v + 1_E w\|_{\mathcal{H}}^2 - \|\lambda 1_E v\|_{\mathcal{H}}^2 - \|1_E w\|_{\mathcal{H}}^2) \\ &= \langle \lambda 1_E v, 1_E w\rangle_{\mathcal{H}} = \lambda \langle 1_E v, 1_E w\rangle_{\mathcal{H}} \overset{(3.4.13)}{=} \int_E \lambda \langle v, w\rangle \,\mathrm{d}\mathfrak{m},\end{aligned}$$

having used the parallelogram law for $\|\cdot\|_{\mathcal{H}}$ and homogeneity of $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$ in the last line. Hence, by arbitrariness of $E \subseteq X$, we get $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$ and with a similar argumentation $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for any $u, v, w \in \mathcal{H}$. Taken together, this implies that $(v, w) \mapsto \langle v, w\rangle$ is $\mathbb{R}$-bilinear, symmetric and positive semi-definite. Since clearly

$$|\langle v, w\rangle| \overset{(3.1.6)}{\leq} \frac{1}{2}((|v| + |w|)^2 - |v|^2 - |w|^2) = |v||w| \;\mathfrak{m}\text{-a.e.},$$

with $|v|$ and $|w|$ both in $L^2(\mathfrak{m})$, Hölder's inequality guarantees that $\langle v, w\rangle \in L^1(\mathfrak{m})$, while Lemma 3.3.5 ensures $L^\infty(\mathfrak{m})$-bilinearity.

It remains to establish non-degeneracy as well as the pointwise parallelogram law. Suppose therefore that $\langle v, w\rangle = 0$ $\mathfrak{m}$-a.e. for all $w \in \mathcal{H}$. Then by (3.4.13), we see that $\langle v, w\rangle_{\mathcal{H}} = \int \langle v, w\rangle \,\mathrm{d}\mathfrak{m} = 0$ for all $w \in \mathcal{H}$, so that $v = 0$ by non-degeneracy of the scalar product on $\mathcal{H}$. Finally, the pointwise parallelogram law is obtained, exactly as above, by integrating over an arbitrary Borel set $E \subseteq X$ and using the parallelogram law for $\|\cdot\|_{\mathcal{H}}$ when applied to the elements $1_E v$ and $1_E w$. □

Corollary 3.4.9. *Let $\mathcal{M}$ be a normed module. Then $\mathcal{M}$ is a Hilbert module if and only if the pointwise norm $|\cdot|$ satisfies the parallelogram law.*

Proof. The previous theorem implies that $|\cdot|$ satisfies the parallelogram law if $\|\cdot\|_{\mathcal{M}}$ does. Moreover, as $\int |v|^2 \,\mathrm{d}\mathfrak{m} = \|v\|_{\mathcal{M}}^2$ for all $v \in \mathcal{M}$, the converse also holds. Hence, the pointwise norm satisfies the parallelogram law if and only if $\|\cdot\|_{\mathcal{M}}$ does, which is the case precisely when $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a Hilbert space by the Jordan-von Neumann Theorem. □

We now discuss the relation of a Hilbert module $\mathcal{H}$ with its dual by proving an analogous result to the Theorem of Riesz for Hilbert spaces. This will imply, as we will see, that the operations of pulling back the module and dualizing it commute (up to unique isomorphism), in contrast to the situation for general normed modules, cf. Example 3.3.16.

Theorem 3.4.10. *Let $\mathcal{H}$ be a Hilbert module. Then any $L \in \mathcal{H}^*$ is of the form*

$$L = \langle v, \,\cdot\,\rangle \quad \textit{for a unique } v \in \mathcal{H} \tag{3.4.14}$$

with $|v| = |L|_*$ $\mathfrak{m}$-a.e.

Proof. By locality of the scalar product, this is an immediate consequence of the classical Theorem of Riesz and Theorem 3.4.1. Indeed, these results imply that $\mathrm{Int}(L) \in \mathcal{H}'$ must be of the form $\mathrm{Int}(L) = \langle v, \cdot \rangle_{\mathcal{H}}$ for a unique $v \in \mathcal{H}$, so that

$$\int_E \langle v, w\rangle \,\mathrm{d}\mathfrak{m} \overset{(3.4.13)}{=} \langle v, 1_E w\rangle_{\mathcal{H}} = \mathrm{Int}(L)(1_E w) \overset{(3.4.1)}{=} \int_E L(w)\,\mathrm{d}\mathfrak{m}$$

for all $w \in \mathcal{H}$ and any $E \subseteq X$ Borel. We must therefore have $L = \langle v, \cdot\rangle$.

To see $|v| = |L|_*$ $\mathfrak{m}$-a.e., we establish

$$|v| = \operatorname*{ess\,sup}_{|w|\leq 1 \ \mathfrak{m}\text{-a.e.}} |\langle v, w\rangle|.$$

By (3.4.11), the pointwise norm $|v|$ is a bound for the essential supremum. We now construct an element $\tilde{w} \in \mathcal{H}$ with $|\tilde{w}| \leq 1$ for which equality is attained. To do so, we first define the set $E := \{|v| = 0\}$ and, invoking locality of the scalar product, we observe that

$$|\langle 1_E v, w\rangle| = 1_E|\langle v, w\rangle| \overset{(3.4.11)}{\leq} (1_E|v|)|w| = 0 \ \mathfrak{m}\text{-a.e.} \tag{3.4.15}$$

for any $w \in \mathcal{H}$. Thus, introducing $\tilde{w} := 1_{X\setminus E}\frac{v}{|v|} = 1_{\{|v|\neq 0\}}\frac{v}{|v|}$, so that

$$1_{X\setminus E} v = |v|\,\tilde{w} \quad \text{and} \quad |\tilde{w}| = 1_{X\setminus E}, \tag{3.4.16}$$

we have

$$\langle v, \tilde{w}\rangle \overset{(3.4.16)}{=} \langle |v|\tilde{w} + 1_E v, \tilde{w}\rangle \overset{(3.4.15)}{=} \langle |v|\tilde{w}, \tilde{w}\rangle \overset{(3.4.11)}{=} |v||\tilde{w}|^2 \overset{(3.4.16)}{=} 1_{X\setminus E}|v| = |v| \ \mathfrak{m}\text{-a.e.} \qquad \square$$

Proposition 3.4.11. *Let $\varphi\colon Y \to X$ be a map of bounded compression and let $\mathcal{H}$ be a Hilbert module on X. Then, as normed modules,*

$$\varphi^*\mathcal{H}^* \cong (\varphi^*\mathcal{H})^*.$$

Proof. We first prove that $\varphi^*\mathcal{H}$ is a Hilbert module on Y. For this, by Corollary 3.4.9 and continuity of the pointwise norm, it suffices to check the pointwise parallelogram law for elements of the form $\sum_{i=1}^n 1_{E_i}\varphi^* v_i$ for $(E_i)_i$ a Borel partition of Y and $(v_i)_i \subseteq \mathcal{H}$ (cf. (3.3.10) and argue as in (3.3.7)). Now, using item (i) in Theorem 3.3.8, we see that

$$\Big|\sum_i 1_{E_i}\varphi^* v_i\Big| \overset{(3.1.2)}{=} \sum_i 1_{E_i}|\varphi^* v_i| \overset{(3.3.9)}{=} \sum_i 1_{E_i}|v_i|\circ\varphi = \Big|\sum_i 1_{E_i} v_i\Big|\circ\varphi,$$

giving the claim by writing out the parallelogram law for the elements $\sum_i 1_{E_i} v_i \in \mathcal{H}$.

To prove the isomorphism, let $I\colon \varphi^*\mathcal{H}^* \to (\varphi^*\mathcal{H})^*$ denote the map $W \mapsto B(\cdot\,, W)$ as in Proposition 3.3.15. We already know that I is an isometric embedding, leaving us with the task of establishing surjectivity. This we do by constructing a right inverse $J\colon (\varphi^*\mathcal{H})^* \to \varphi^*\mathcal{H}^*$. For this, let $R_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}^*$ and $R_{\varphi^*\mathcal{H}}\colon \varphi^*\mathcal{H} \to (\varphi^*\mathcal{H})^*$ denote the respective Riesz isomorphisms as in Theorem 3.4.10. Since, recalling that $|R_{\mathcal{H}}(v)|_* = |v|$,

$$|(\varphi^*(R_{\mathcal{H}}(v))| \overset{(3.3.9)}{=} |R_{\mathcal{H}}(v)|_* \circ \varphi = |v| \circ \varphi \ \mathfrak{m}_Y\text{-a.e.},$$

Theorem 3.3.11 guarantees the existence of a unique module homomorphism $S\colon \varphi^*\mathcal{H} \to \varphi^*\mathcal{H}^*$ with

$$S(\varphi^* v) = \varphi^*(R_{\mathcal{H}}(v)) \quad \text{for all } v \in \mathcal{H}. \tag{3.4.17}$$

We now define the module homomorphism

$$J := S \circ R^{-1}_{\varphi^*\mathcal{H}}\colon (\varphi^*\mathcal{H})^* \to \varphi^*\mathcal{H}^* \tag{3.4.18}$$

and check that $I \circ J = \mathrm{id}_{(\varphi^*\mathcal{H})^*}$ on the generating subspace $\{R_{\varphi^*\mathcal{H}}(\varphi^* v) \mid v \in \mathcal{H}\}$. To this end, we first note that

$$R_{\varphi^*\mathcal{H}}(\varphi^* v)(\varphi^* w) \overset{(3.4.14)}{=} \langle \varphi^* v, \varphi^* w\rangle \overset{(3.3.9)}{=} \langle v, w\rangle \circ \varphi.$$

Hence, the equality

$$\begin{aligned}(I \circ J)(R_{\varphi^*\mathcal{H}}(\varphi^* v))(\varphi^* w) &\overset{(3.4.18)}{=} I(S(\varphi^* v))(\varphi^* w) \overset{(3.4.17)}{=} I(\varphi^*(R_{\mathcal{H}} v))(\varphi^* w) \\ &\overset{(3.3.18)}{=} B(\varphi^*(R_{\mathcal{H}}(v)), \varphi^* w) \overset{(3.3.17)}{=} R_{\mathcal{H}}(v)(w) \circ \varphi \overset{(3.4.14)}{=} \langle v, w\rangle \circ \varphi\end{aligned}$$

establishes the claim and concludes the proof, as it shows that the two module homomorphisms $I \circ J$ and $\mathrm{id}_{(\varphi^*\mathcal{H})^*}$ agree on a generating subspace. □

4 First-Order Calculus

We now turn to the task of building a first-order differential calculus for an arbitrary metric measure space $(X, \mathrm{d}, \mathfrak{m})$. To do so, we first generalize integrable sections of the tangent and cotangent bundles via the theory of normed modules. Thus, in our approach, one forms will be introduced as elements of a *cotangent module* $L^2(T^*X)$, which we equip with a linear differential $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$ built upon the minimal weak upper gradient $|Df|$. We then show that $L^2(T^*X)$ is in fact generated by the differentials of Sobolev functions, so that a generic one form can be obtained as the limit of simple one-forms

$$\omega = \sum_{i=1}^{n} 1_{E_i} \, \mathrm{d}f_i, \tag{4.0.1}$$

where, as usual, $(E_i)_i$ constitutes a Borel partition of X and f_i belongs to the Sobolev class $S^2(X)$ for each $1 \le i \le n$.

Dualizing this module, we then obtain the *tangent module* $L^2(TX)$, whose elements we call *vector fields* on X. In analogy with the classical situation, we will also prove a one-to-one correspondence between these vector fields on X and L^2-derivations on $S^2(X)$.

4.1 The cotangent module

Theorem 4.1.1. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space. Then there exists an essentially unique* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module* $L^2(T^*X)$*, called the* cotangent module*, and an* $\mathbb{R}$*-linear map* $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$*, called the* differential*, such that*

i) $|\mathrm{d}f| = |Df|$ $\mathfrak{m}$-a.e. for every $f \in S^2(X)$, (4.1.1)

ii) $L^2(T^*X)$ is generated by $\{\mathrm{d}f \mid f \in S^2(X)\}$. (4.1.2)

Remark 4.1.2. Again, essential uniqueness is to be understood as in Remark 3.3.9. Thus, we assert that for any other pair $(\widetilde{\mathcal{M}}, \tilde{\mathrm{d}})$ as above, there exists a unique module isomorphism $\Phi\colon L^2(T^*X) \to \widetilde{\mathcal{M}}$ with $\Phi \circ \mathrm{d} = \tilde{\mathrm{d}}$.

Proof. Following an argumentation similiar to the proof of Theorem 3.3.8, we first establish uniqueness and then define the relevant objects in accordance. For this, let $\widetilde{M}$ be another module with differential $\tilde{\mathrm{d}}$ satisfying both conditions stated, and suppose we had already found a module isomorphism $\Psi\colon L^2(T^*X) \to \widetilde{\mathcal{M}}$ satisfying $\Psi \circ \mathrm{d} = \tilde{\mathrm{d}}$. Then, necessarily,

$$\Psi\Big(\sum_{i=1}^{n} 1_{E_i}\,\mathrm{d}f_i\Big) = \sum_{i=1}^{n} 1_{E_i}\Psi(\mathrm{d}f_i) = \sum_{i=1}^{n} 1_{E_i}\tilde{\mathrm{d}}f_i \tag{4.1.3}$$

for any Borel partition $(E_i)_{i=1}^n$ of X and $(f_i)_{i=1}^n \subseteq S^2(X)$. Therefore, we use (4.1.3) to define a map on

$$V := \{\sum_{i=1}^{n} 1_{E_i}\,\mathrm{d}f_i \mid (E_i)_{i=1}^n \text{ is a Borel partition of } X,\ (f_i)_{i=1}^n \subseteq S^2(X)\}, \tag{4.1.4}$$

and extend it to $L^2(T^*X)$. This is possible as property (i) guarantees that

$$\Big|\sum_i 1_{E_i}\,\mathrm{d}f_i\Big| \overset{(3.1.2)}{=} \sum_i 1_{E_i}|\mathrm{d}f_i| \overset{(i)}{=} \sum_i 1_{E_i}|Df_i| = \sum_i 1_{E_i}|\tilde{\mathrm{d}}f_i| = \Big|\sum_i 1_{E_i}\tilde{\mathrm{d}}f_i\Big|, \tag{4.1.5}$$

while V is seen to carry the structure of a vector space due to the linearity of the differential d:

$$\alpha\sum_{i=1}^{n} 1_{E_i}\,\mathrm{d}f_i + \sum_{j=1}^{m} 1_{F_j}\,\mathrm{d}g_j = \sum_{i,j} 1_{E_i\cap F_j}\,\mathrm{d}(\alpha f_i + g_j) \in V, \tag{4.1.6}$$

where the quantities are defined as in (4.1.4) with $\alpha \in \mathbb{R}$. Thus, we are in the position to invoke Proposition 3.2.12 and conclude, by item (ii), that V is dense in $L^2(T^*X)$. Defining $\widetilde{V}$ accordingly for $\tilde{d}$, we can draw the same conclusion for $\widetilde{V}$ as a subspace of $\widetilde{M}$.

Hence, the linear map defined by (4.1.3) has dense domain and codomain. Since it preserves the pointwise norm, as we have seen in the calculation (4.1.5), we can extend it to a module isomorphism using Proposition 3.2.3 (recall that the image of any isometry is closed and notice that $\Phi(V) = \widetilde{V}$ is dense in $\widetilde{M}$, whence $\Phi(V) = \widetilde{M}$).

We now establish existence. To do so, we notice that two representations $\sum_i 1_{E_i}\,\mathrm{d}f_i$ and $\sum_j 1_{F_j}\,\mathrm{d}g_j$ of the same element in V, defined in (4.1.4), coincide precisely when the pointwise norm of their difference vanishes, or equivalently, if and only if

$$|D(f_i - g_j)| = |\mathrm{d}f_i - \mathrm{d}g_j| = 0 \ \mathfrak{m}\text{-a.e. on } E_i \cap F_j \text{ for all } i, j.$$

We are therefore led to consider the quotient set of

$$\Big\{\{(E_i, f_i)\}_{i=1}^n \mid n \geq 1, (E_i)_{i=1}^n \text{ is a Borel partition of } X,\ (f_i)_{i=1}^n \subseteq S^2(X)\Big\}$$

under the equivalence relation $(E_i, f_i)_i \sim (F_j, g_j)_j$ if and only if

$$|D(f_i - g_j)| = 0 \ \mathfrak{m}\text{-a.e. on } E_i \cap F_j \text{ for all } i, j,$$

having used the abbreviation $(E_i, f_i)_i \equiv \{(E_i, f_i)\}_{i=1}^n$. We call this set of equivalence classes the *pre-cotangent module*, denoted Pcm.

Inspired by the calculation (4.1.6), we equip Pcm with a linear structure by setting

$$\begin{aligned} [E_i, f_i]_i + [F_j, g_j]_j &:= [E_i \cap F_j, f_i + g_j]_{i,j} \\ \alpha[E_i, f_i]_i &:= [E_i, \alpha f_i]_i \qquad\qquad \text{for } \alpha \in \mathbb{R}, \end{aligned} \tag{4.1.7}$$

with square brackets indicating equivalence classes. To see that these definitions give rise to well-defined notions of addition and scalar-multiplication, notice for instance that whenever $(E_i, f_i)_i \sim (\tilde{E}_k, \tilde{f}_k)_k$ and $(F_j, g_j)_j \sim (\tilde{F}_l, \tilde{g}_l)_l$, we have

$$|D((f_i + g_j) - (\tilde{f}_k + \tilde{g}_l))| \overset{(2.3.2)}{\leq} |D(f_i - \tilde{f}_k)| + |D(g_j - \tilde{g}_l)| = 0 \ \mathfrak{m}\text{-a.e.}$$

on the intersection of $E_i \cap \tilde{E}_k$ and $F_j \cap \tilde{F}_l$, implying at once $(E_i \cap F_j, f_i + g_j)_{i,j} \sim (\tilde{E}_k \cap \tilde{F}_l, \tilde{f}_k + \tilde{g}_l)_{k,l}$. Using absolute homogeneity of the minimal weak upper gradient, we get the second claim with an even shorter calculation.

Next, we endow the newly entitled vector space Pcm with a multiplication by $L^\infty(\mathfrak{m})$-functions, starting with simple functions for the moment:

$$\left(\sum_j \alpha_j 1_{F_j}\right) \cdot [E_i, f_i]_i := [E_i \cap F_j, \alpha_j f_i]_{i,j}. \tag{4.1.8}$$

For the pointwise norm, we set

$$|\,[E_i, f_i]_i| := \sum_i 1_{E_i} |Df_i|, \tag{4.1.9}$$

where the right-hand side is understood to hold up to almost everywhere equivalence. Again, well-definedness is a consequence of the (reverse) triangle inequality, allowing us to conclude

$$\sum_i 1_{E_i} |Df_i| = \sum_{i,j} 1_{E_i \cap F_j} |Df_i| = \sum_{i,j} 1_{F_j \cap E_i} |Dg_j| = \sum_j 1_{F_j} |Dg_j|, \tag{4.1.10}$$

whenever $|D(f_i - g_j)| = 0$ $\mathfrak{m}$-a.e. on $E_i \cap F_j$ for all i, j, that is whenever $(E_i, f_i)_i$ and $(F_j, g_j)_j$ are equivalent. Finally, setting

$$\|\,[E_i, f_i]_i\| := \|\,|\,[E_i, f_i]_i|\,\|_{L^2(\mathfrak{m})} = \left(\sum_i \int_{E_i} |Df_i|^2 \,\mathrm{d}\mathfrak{m}\right)^{1/2}, \tag{4.1.11}$$

we endow Pcm with its last piece of data so as to become a module (upon completion), namely a norm compatible with (4.1.9). Indeed, $\|\,[E_i, f_i]_i\| = 0$ $\mathfrak{m}$-a.e. immediately implies $|Df_i| = 0$ $\mathfrak{m}$-a.e. on E_i for all i, so that $[E_i, f_i]_i = [X, 0]$. Further, $\|\alpha[E_i, f_i]_i\| = |\alpha|\,\|\,[E_i, f_i]_i\|$ by absolute homogeneity of the weak upper gradient. Lastly, the triangle inequality follows as in (4.1.10), again by using a common refinement of two partitions and the triangle inequality for the weak upper gradient:

$$\begin{aligned}\| [E_i, f_i]_i + [F_j, g_j]_j\| &= \|\sum_{i,j} 1_{E_i\cap F_j}|D(f_i+g_j)|\,\|_{L^2(\mathfrak{m})}\\ &\le \|\sum_{i,j} 1_{E_i\cap F_j}|Df_i|\,\|_{L^2(\mathfrak{m})} + \|\sum_{i,j} 1_{E_i\cap F_j}|Dg_j|\,\|_{L^2(\mathfrak{m})}\\ &= \|\,[E_i, f_i]_i\| + \|\,[F_j, g_j]_j\|.\end{aligned}$$

Finalizing, we denote by $(L^2(T^*X), \|\cdot\|_{L^2(T^*X)})$ the Banach completion of Pcm together with the norm $\|\cdot\|$ defined in (4.1.11). We now check that we can carry the two norms to the completion space. First, by the reverse triangle inequality for (4.1.9), obtained via a refinement from the corresponding reverse triangle inequality for the weak upper gradient (2.3.2), the map $|\cdot|\colon \mathsf{Pcm} \to L^2(\mathfrak{m})$ is uniformly continuous and hence extends to a unique map defined on the whole of $L^2(T^*X)$. Secondly, as

$$\begin{aligned}\left\|\Big(\sum_j 1_{F_j}\alpha_j\Big)\cdot[E_i, v_i]_i\right\| &\overset{(4.1.8)}{=} \left\|\sum_{i,j} 1_{E_i\cap F_j}|D(\alpha_j f_i)|\,\right\|_{L^2(\mathfrak{m})}\\ &\le \max_j|\alpha_j|\left\|\sum_i 1_{E_i}|Df_i|\,\right\|_{L^2(\mathfrak{m})} \overset{(4.1.11)}{=} \left\|\sum_j 1_{F_j}\alpha_j\right\|_{L^\infty(\mathfrak{m})}\|[E_i, v_i]\|,\end{aligned}$$

we see that the multiplication by simple functions extends to a bilinear multiplication $L^\infty(\mathfrak{m}) \times L^2(T^*X) \to L^2(T^*X)$. Thus, $(L^2(T^*X), \|\cdot\|_{L^2(T^*X)}, \cdot\,, |\cdot|)$, in fact, constitutes an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

It remains to define the differential $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$, which we can now easily do by setting

$$\mathrm{d}f := [X, f], \tag{4.1.12}$$

so that the differentials of two Sobolev functions f and g coincide precisely when $|D(f-g)| = 0$ $\mathfrak{m}$-a.e. Then, clearly,

$$\mathrm{d}(\alpha f + g) \overset{(4.1.12)}{=} [X, \alpha f + g] \overset{(4.1.7)}{=} \alpha[X, f] + [X, g] = \alpha\,\mathrm{d}f + \mathrm{d}g$$

for all $f, g \in S^2(X)$ and $\alpha \in \mathbb{R}$ by (4.1.7), guaranteeing $\mathbb{R}$-linearity. Since $|\mathrm{d}f| = |[X, f]| \overset{(4.1.9)}{=} |Df|$ $\mathfrak{m}$-a.e., conclusion (*i*) holds. For the second item, we observe that

$$\sum_{i=1}^n 1_{E_i}\,\mathrm{d}f_i \overset{(4.1.12)}{=} \sum_{i=1}^n 1_{E_i}[X, f_i] \overset{(4.1.8)}{=} [E_i, f_i]_i,$$

which provides the required density of $\{\mathrm{d}f \mid f \in S^2(X)\}$ within $L^2(T^*X)$ by Proposition 3.2.12. $\square$

Remark 4.1.3. As for the Sobolev calculus, one can prove the following consistency result: Consider the space of all square-integrable one-forms on $\mathbb{R}^n$, equipped with the Lebesgue measure, together with the map assigning to each Sobolev function its distributional differential. Then, taking a module viewpoint, this couple is isomorphic to the cotangent module $(L^2(T^*\mathbb{R}^n), \mathrm{d})$ via a unique module isomorphism, see Proposition 4.1.7 in [13].

We now investigate how the rules of calculation for minimal weak upper gradients carry over to the linear differential $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$. Before doing so, however, we clarify upon some functional analytic properties of this unbounded operator. Specifically, we will establish the *closedness* of the differential as a map from $W^{1,2}(X)$ into $L^2(\mathfrak{m})$ with respect to the corresponding weak topologies.

Theorem 4.1.4. *If a sequence of Sobolev functions $(f_n)_{n\geq 1} \subseteq S^2(X)$ converges pointwise $\mathfrak{m}$-almost everywhere to a function $f\colon X \to \mathbb{R}$ in such a way that the sequence of differentials $(\mathrm{d}f_n)_{n\geq 1}$ converges weakly to some $\omega \in L^2(T^*X)$, then $f \in S^2(X)$ with $\mathrm{d}f = \omega$.*

Moreover, the same conclusion holds if $(f_n)_{n\geq 1} \subseteq W^{1,2}(X)$ with $f_n \rightharpoonup f$ and $\mathrm{d}f_n \rightharpoonup \omega$.

Proof. Applying Mazur's lemma (see Corollary 3.8 in [8]) to the weakly convergent sequence of differentials $(\mathrm{d}f_n)_{n\geq 1}$, we can find convex combinations

$$\mathrm{d}f_n' := \sum_{i=n}^{N_n} \lambda_{n,i}\, \mathrm{d}f_i,\ N_n \geq n \quad \text{with} \quad \sum_{i=n}^{N_n} \lambda_{n,i} = 1$$

such that

$$\mathrm{d}f_n' \xrightarrow[n\to\infty]{L^2(T^*X)} \omega \quad \text{in the strong topology.}$$

We now claim that for the sequence $(f_n')_{n\geq 1}$, where $f_n' = \sum_{i=n}^{N_n} \lambda_{n,i} f_i$, we further have

$$f_n' \xrightarrow[n\to\infty]{} f \ \mathfrak{m}\text{-a.e.}$$

To see this, denote by $E \subseteq X$ the null-set of all points where $(f_n)_{n\geq 1}$ does not converge to f. Now, for a given $\varepsilon > 0$ and $x \in X \setminus E$, choose $M \geq 1$ so that $|f_i(x) - f(x)| < \varepsilon$ for all $i \geq M$. Then, using $\sum_{i=m}^{N_m} \lambda_{m,i} = 1$, we have

$$|f_m'(x) - f(x)| = \left|\sum_{i=m}^{N_m} \lambda_{m,i}(f_i(x) - f(x))\right| \leq \sum_{i=m}^{N_m} \lambda_{m,i}|f_i(x) - f(x)| < \varepsilon \sum_{i=m}^{N_m} \lambda_{m,i} = \varepsilon$$

for all $m \geq M$.

Hence, by otherwise replacing our original sequence by the modified $(f_n')_{n\geq 1}$, we can assume w.l.o.g. that $\mathrm{d}f_n \xrightarrow[n\to\infty]{} \omega$ in the strong topology of $L^2(T^*X)$. This implies

$$|Df_n| \overset{(4.1.1)}{=} |\mathrm{d}f_n| \xrightarrow[n\to\infty]{L^2(\mathfrak{m})} |\omega|,$$

by the continuity of the pointwise norm as stated in (3.1.9). Proposition 2.3.4 thus guarantees that $f \in S^2(X)$. To see that $\mathrm{d}f = \omega$, we first fix some $n \geq 1$ and invoke Proposition 2.3.8 to conclude

$$\begin{aligned}\|\mathrm{d}f - \mathrm{d}f_n\|_{L^2(T^*X)} \overset{(4.1.1)}{=} \||D(f-f_n)|\|_{L^2(\mathfrak{m})} &\overset{(2.3.3)}{\leq} \varliminf_{k\to\infty} \||D(f_k-f_n)|\|_{L^2(\mathfrak{m})} \\ &= \varliminf_{k\to\infty} \|\mathrm{d}f_k - \mathrm{d}f_n\|_{L^2(\mathfrak{m})}\end{aligned}$$

Together with the fact that $(\mathrm{d}f_n)_{n\geq 1}$ is a Cauchy sequence in $L^2(T^*X)$, this yields

$$\varlimsup_{n\to\infty} \|\mathrm{d}f - \mathrm{d}f_n\|_{L^2(T^*X)} \leq \varlimsup_{n\to\infty} \varliminf_{k\to\infty} \|\mathrm{d}f_k - \mathrm{d}f_n\|_{L^2(T^*X)} = 0,$$

and so, indeed, $\mathrm{d}f = \omega$.

Finally, to get the same conclusion if $(f_n)_{n\geq 1} \subseteq W^{1,2}(X)$ with $f_n \rightharpoonup f$ and $\mathrm{d}f_n \rightharpoonup \omega$, we apply Mazur's lemma first to the sequence $(\mathrm{d}f_n)_{n\geq 1}$ to obtain $\mathrm{d}f_n' \to \omega$ with $f_n' \rightharpoonup f$ and then to $(f_n')_{n\geq 1}$ to get $\mathrm{d}f_n'' \to \omega$ with $f_n'' \to f$. Now, by choosing an almost-everywhere converging subsequence of $(f_n'')_{n\geq 1}$, we can apply the first part of the theorem. □

Theorem 4.1.5. *The differential* $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$ *obeys the following rules of calculation:*

A) Locality. Within the set $\{f = g\}$, *we have* $\mathrm{d}f = \mathrm{d}g$ $\mathfrak{m}$-a.e.

B) Chain rule.

i) $\mathrm{d}f$ *vanishes wherever* f *takes values in a* λ*-null set, i.e.*

$$\mathrm{d}f = 0 \quad \mathfrak{m}\text{-a.e. on } f^{-1}(N) \text{ for any } N \subseteq \mathbb{R} \text{ null.} \tag{4.1.13}$$

ii) For any Lipschitz function $\varphi\colon I \to \mathbb{R}$, *we have* $\varphi \circ f \in S^2(X)$ *and*

$$\mathrm{d}(\varphi \circ f) = \varphi' \circ f \,\mathrm{d}f \ \mathfrak{m}\text{-a.e.} \tag{4.1.14}$$

with $\varphi' \circ f$ *arbitrarily defined wherever* φ' *does not exist.*

C) Leibniz rule. If $f, g \in S^2(X) \cap L^\infty(\mathfrak{m})$, *then* $fg \in S^2(X) \cap L^\infty(\mathfrak{m})$ *and*

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \ \mathfrak{m}\text{-a.e.} \tag{4.1.15}$$

Proof. Locality and item (i) of the chain rule follow immediately from Theorem 2.3.13, as

$$|\mathrm{d}f - \mathrm{d}g| \overset{(4.1.1)}{=} |D(f-g)| = 0 \ \mathfrak{m}\text{-a.e. in } \{f = g\},$$

by item (A) in Theorem 2.3.13 and

$$|\mathrm{d}f| \overset{(4.1.1)}{=} |Df| = 0 \ \mathfrak{m}\text{-a.e. in } f^{-1}(N),$$

by item (B), so that $\mathrm{d}f = 0$ there, as a consequence of Remark 3.1.3.

To prove item (ii), we first observe that

$$\mathrm{d}h = 0 \quad \text{for all } h\colon X \to \mathbb{R} \text{ constant} \tag{4.1.16}$$

by (4.1.13). Now, assume for the moment that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is affine, i.e. $\varphi(t) = at + b$ for $a, b \in \mathbb{R}$ (allowing $a = 0$). Then,

$$\mathrm{d}(\varphi \circ f) = \mathrm{d}(af + b) = a\,\mathrm{d}f + \mathrm{d}b \overset{(4.1.16)}{=} a\,\mathrm{d}f = \varphi' \circ f\,\mathrm{d}f. \tag{4.1.17}$$

In the general case, we first extend $\varphi\colon I \to \mathbb{R}$ to a Lipschitz function $\varphi\colon \mathbb{R} \to \mathbb{R}$ using the Whitney-McShane extension theorem (see Theorem 2.3 in *Lectures on Lipschitz Analysis* [16] by Juha Heinonen). Now, one can prove that for any such function, there exists an approximating sequence of piecewise affine functions $(\varphi_n)_{n\geq 1} \subseteq \mathrm{Lip}(\mathbb{R})$ with the property that

$$\varphi_n \xrightarrow[n\to\infty]{} \varphi \text{ pointwise} \quad \text{and} \quad \varphi_n' \xrightarrow[n\to\infty]{} \varphi' \text{ a.e.,} \tag{4.1.18}$$

so that

$$\sup_{n\geq 1} \mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\varphi), \tag{4.1.19}$$

see claim (4.5) in [13]. This now allows us to conclude using Theorem 4.1.4. Indeed, $(\varphi_n' \circ f)_{n\geq 1}$ converges pointwise to $\varphi' \circ f$ outside $f^{-1}(N)$, where $N \subseteq \mathbb{R}$ is the null-set of all points where φ_n' does not exist for some $n \geq 1$ or the sequence of $(\varphi_n')_{n\geq 1}$ does not converge to φ'. Thus, item (i) guarantees that $\mathrm{d}f$ vanishes $\mathfrak{m}$-a.e. on $f^{-1}(N)$ and (4.1.19) ensures $|\varphi_n' - \varphi'| \circ f\,|\mathrm{d}f| \leq 2\,\mathrm{Lip}(\varphi)|\mathrm{d}f| \in L^2(\mathfrak{m})$. Therefore, by dominated convergence,

$$\|\varphi_n' \circ f\,\mathrm{d}f - \varphi' \circ f\,\mathrm{d}f\|^2_{L^2(T^*X)} \overset{(3.1.3)}{=} \int |\varphi_n' - \varphi'|^2 \circ f\,|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m} \xrightarrow[n\to\infty]{} 0. \tag{4.1.20}$$

Hence, (4.1.18) and (4.1.20) give

$$\varphi_n \circ f \xrightarrow[n\to\infty]{} \varphi \circ f \ \mathfrak{m}\text{-a.e.} \quad \text{and} \quad \mathrm{d}(\varphi_n \circ f) \overset{(4.1.17)}{=} \varphi_n' \circ f\ \mathrm{d}f \xrightarrow[n\to\infty]{L^2(T^*X)} \varphi' \circ f\ \mathrm{d}f \ \mathfrak{m}\text{-a.e.}$$

So, by Theorem 4.1.4, $\varphi \circ f \in S^2(X)$ with $\mathrm{d}(\varphi \circ f) = \varphi' \circ f\ \mathrm{d}f$.

Finally, to prove the Leibniz rule, assume first that $f, g \geq 1$. Then, by the chain rule with $\varphi = \log\colon [1,\infty) \to \mathbb{R}$, we obtain

$$\frac{\mathrm{d}(fg)}{fg} \overset{(4.1.14)}{=} \mathrm{d}(\log(fg)) = \mathrm{d}(\log(f) + \log(g)) \overset{(4.1.14)}{=} \frac{\mathrm{d}f}{f} + \frac{\mathrm{d}g}{g},$$

i.e. $\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$. In the general case, we shift f and g by a constant $c > 0$ so that $f + c$ and $g + c$ are both bounded below by 1, e.g. $c = \max\{\|f\|_{L^\infty(\mathfrak{m})}, \|g\|_{L^\infty(\mathfrak{m})}\} + 1$.

Using the first part, we can now conclude, on the one hand, that

$$\begin{aligned}\mathrm{d}((f+c)(g+c)) &= (f+c)\,\mathrm{d}(g+c) + (g+c)\,\mathrm{d}(f+c)\\ &\overset{(4.1.16)}{=} (f+c)\,\mathrm{d}g + (g+c)\,\mathrm{d}f = f\,\mathrm{d}g + g\,\mathrm{d}f + c\,\mathrm{d}(f+g),\end{aligned}$$

while, on the other hand,

$$\mathrm{d}((f+c)(g+c)) = \mathrm{d}(fg + c(f+g) + c^2) = \mathrm{d}(fg) + c\,\mathrm{d}(f+g).$$

Comparing the two results, we see that we must have $\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$. □

Finally, as a technical result, we note that it suffices to consider the images of L^2-integrable Sobolev functions $f \in W^{1,2}(X)$ under the differential $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$ to generate the whole cotangent module.

Proposition 4.1.6. *The set $\{\mathrm{d}f \mid f \in W^{1,2}(X)\}$ generates the cotangent module $L^2(T^*X)$.*

Proof. The image of the differential $\mathrm{d}\colon S^2(X) \to L^2(T^*X)$ generates $L^2(T^*X)$ by construction, see (4.1.2), and so it suffices to prove $\{\mathrm{d}f \mid f \in S^2(X)\} \subseteq \mathcal{M}$, where $\mathcal{M}$ is the module generated by $\{\mathrm{d}f \mid f \in W^{1,2}(X)\}$, as this would then imply

$$L^2(T^*X) \overset{(3.2.6)}{=} \overline{\left\{\sum_i 1_{E_i}\,\mathrm{d}f_i \,\middle|\, f_i \in S^2(X)\right\}} \subseteq \mathcal{M} \subseteq L^2(T^*X).$$

We do this by way of a cut-off argument similar to Lemma 2.2.10. Concretely, recall from (2.2.15) the definition

$$f^{mn} = \eta_{x_0,n}(f \vee (-m)) \wedge m \in L^2(\mathfrak{m}),$$

where $\eta_{x_0,n}$ is the Lipschitz cut-off function of the ball of radius n around some point $x_0 \in X$ introduced in (2.2.10). Defining $\varphi \in \mathrm{Lip}(\mathbb{R})$ by

$$\varphi(t) = (t \vee (-m)) \wedge m \quad \text{so that} \quad \varphi'(t) = 1_{\{|t| \le m\}}(t), \tag{4.1.21}$$

we see that $f^{mn} = \eta_{x_0,n}\,\varphi \circ f$ with $\varphi \circ f \in S^2(X) \cap L^\infty(\mathfrak{m})$ by the chain rule proved in the preceding theorem. Since Proposition 2.2.6 guarantees that also $\eta_{x_0,n} \in S^2(X) \cap L^\infty(\mathfrak{m})$, the Leibniz rule finally ensures $f^{mn} \in S^2(X) \cap L^2(\mathfrak{m}) = W^{1,2}(X)$.

We now prove that, upon localizing, the differentials of these cut-offs converge in $L^2(T^*X)$ to $\mathrm{d}f$. For this, we use the fact that on $B_n(x_0)$ the function $\eta_{x_0,n}$ satisfies

$$\eta_{x_0,n} = 1 \quad \text{with} \quad \mathrm{d}\eta_{x_0,n} = 0 \tag{4.1.22}$$

by (4.1.16). Therefore, by the Leibniz rule,

$$\begin{aligned}1_{B_n(x_0)}\,\mathrm{d}f^{mn} &\overset{(4.1.15)}{=} 1_{B_n(x_0)}(\eta_{x_0,n}\,\mathrm{d}(\varphi\circ f)+\varphi\circ f\,\mathrm{d}\eta_{x_0,n})\\ &\overset{(4.1.22)}{=} 1_{B_n(x_0)}\,\varphi'\circ f\,\mathrm{d}f \overset{(4.1.21)}{=} 1_{B_n(x_0)}1_{\{|f|\le m\}}\,\mathrm{d}f,\end{aligned}$$

Hence, defining the set $A_{mn} := f^{-1}([-m,m])\cap B_n(x_0)$, we have

$$1_{A_{mn}}\,\mathrm{d}f^{mn} \xrightarrow[n,m\to\infty]{L^2(T^*X)} \mathrm{d}f$$

by the dominated convergence theorem, as $\bigcup_{m,n\ge 1} A_{mn}$ clearly exhausts X. This finally gives the claim since $1_{A_{mn}}\,\mathrm{d}f^{mn}\in\mathcal{M}$ and $\mathcal{M}\subseteq L^2(T^*X)$ is closed. □

4.2 The tangent module

After having defined one-forms on a general metric measure space, we now introduce vector fields by duality.

Definition 4.2.1. Given a metric measures space $(X,\mathrm{d},\mathfrak{m})$, the (dual) module $L^2(TX) := (L^2(T^*X))^*$ is called the *tangent module* of X and any element $v\in L^2(TX)$ is called a *vector field* on X.

Remark 4.2.2. As usual, we denote the action of $v\in L^2(TX)$ on $\omega\in L^2(T^*X)$ by $\omega(v)$. Note that expressions like $f\omega(v)$ for $f\in L^\infty(\mathfrak{m})$ are unambiguous as $(f\omega)(v)=f(\omega(v))$ by $L^\infty(\mathfrak{m})$-linearity of v. Moreover, when referring to the pointwise norm of an element $v\in L^2(TX)$, we drop the additional star in the notation and simply write $|v|$.

Now, in smooth differential geometry, we have already encountered an equivalent characterization of the space of vector fields on a manifold M via derivations on the algebra $\mathcal{C}^\infty(M)$. More concretely, the map taking any vector field v to the derivation

$$\mathcal{C}^\infty(M)\ni f\mapsto \mathrm{d}f(v) \tag{4.2.1}$$

was found to be bijective.

Our goal in this section is to establish an analogous result for $v\in L^2(TX)$ and $f\in S^2(X)$, thereby defining an appropriate notion of a derivation in this context. To begin with, we notice that the counterpart to (4.2.1), that is, the linear map

$$S^2(X)\ni f\mapsto \mathrm{d}f(v)\in L^1(\mathfrak{m}),$$

satisfies

$$|\mathrm{d}f(v)| \overset{(3.3.3)}{\le} |v||\mathrm{d}f| \overset{(4.1.1)}{=} |v||Df| \ \mathfrak{m}\text{-a.e.} \tag{4.2.2}$$

with $|v| \in L^2(\mathfrak{m})$. Therefore, any linear map $L: S^2(X) \to L^1(\mathfrak{m})$ for which

$$L(f) = \mathrm{d}f(v) \quad \text{for some } v \in L^2(TX) \tag{4.2.3}$$

would necessarily also have to satisfy $|L(f)| \le |v||Df|$ $\mathfrak{m}$-a.e. With the help of Proposition 3.3.6, we will see that this bound, in fact, already characterizes all linear maps taking the form (4.2.3).

Definition 4.2.3. We call any $\mathbb{R}$-linear map $L: S^2(X) \to L^1(\mathfrak{m})$ an L^2-*derivation* provided

$$|L(f)| \le \ell |Df| \ \mathfrak{m}\text{-a.e.} \qquad \text{for some } \ell \in L^2(\mathfrak{m}). \tag{4.2.4}$$

The set of all derivations on $S^2(X)$ is denoted by $\mathrm{Der}(S^2(X))$.

Theorem 4.2.4. *There is a one-to-one correspondence between the set of vector fields on X and the set of derivations on $S^2(X)$. More concretely, the map*

$$L^2(TX) \ni v \mapsto v \circ \mathrm{d} \in \mathrm{Der}(S^2(X))$$

is bijective.

Proof. We have already established the claim that $f \mapsto \mathrm{d}f(v)$ is a derivation in (4.2.2). For the converse, we invoke Proposition 3.3.6, as already hinted at. Specifically, for a given derivation $L: S^2(X) \to L^1(\mathfrak{m})$, we define the *linear* map

$$T: V \to L^1(\mathfrak{m}), \mathrm{d}f \mapsto L(f)$$

on the generating vector subspace $\{\, \mathrm{d}f \mid f \in S^2(X)\} =: V \subseteq L^2(T^*X)$. This is possible as

$$|T(\mathrm{d}f)| = |L(f)| \overset{(4.2.4)}{\le} \ell\, |Df| \overset{(4.1.1)}{=} \ell\, |\mathrm{d}f| \tag{4.2.5}$$

Thus, Proposition 3.3.6 guarantees the existence of a unique vector field $v \in L^2(TX)$ for which $\mathrm{d}f(v) = T(\mathrm{d}f) = L(f)$ holds for every $f \in S^2(X)$, concluding the proof. □

Corollary 4.2.5. *Any derivation $L: S^2(X) \to L^1(\mathfrak{m})$ satisfies the Leibniz rule, i.e.*

$$L(fg) = gL(f) + fL(g) \quad \text{for all } f, g \in S^2(X) \cap L^\infty(\mathfrak{m}).$$

Proof. By Theorem 4.2.4, we can find a vector field $v \in L^2(TX)$ such that $L = v \circ \mathrm{d}$. Hence, by the Leibniz rule for the differential $\mathrm{d}: S^2(X) \to L^2(T^*X)$, we have

$$L(fg) = \mathrm{d}(fg)(v) \overset{(4.1.15)}{=} (g\, \mathrm{d}f + f\, \mathrm{d}g)(v) = g\, \mathrm{d}f(v) + f\, \mathrm{d}g(v) = gL(f) + fL(g),$$

having used $L^\infty(\mathfrak{m})$-linearity of $v \in L^2(TX)$ □

4.3 Gradients

Recall that on $\mathbb{R}^n$, equipped with the Euclidean inner product, the gradient ∇f of a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined in duality to its differential $\mathrm{d}f$ via the relation

$$\langle \nabla f|_p, v\rangle = \mathrm{d}f|_p(v) \quad \text{for all } v \in \mathbb{R}^n.$$

Hence, ∇f is the *unique* vector field on $\mathbb{R}^n$, whose (pointwise) image under the Riesz isomorphism $\mathbb{R}^n \overset{\cong}{\to} (\mathbb{R}^n)^*$ is the differential $\mathrm{d}f$.

Now, as our approach to Sobolev functions only captures the *magnitude* of the differential, we try to find an alternative description only involving norms. For this, we exploit the fact that the Riesz isomorphism

$$\mathbb{R}^n \to (\mathbb{R}^n)^*,\ v \mapsto \langle v, \cdot\,\rangle$$

maps any $v \in \mathbb{R}^n$ to a *unique norm-attaining* linear functional $\alpha := \langle v, \cdot\,\rangle$, by which we mean that

$$\alpha(v) = \|v\|^2 = \|\alpha\|^2.$$

The Riesz isomorphism therefore establishes a one-to-one correspondence

$$\{v \in \mathbb{R}^n\} \longleftrightarrow \{\alpha \in (\mathbb{R}^n)^* \mid \alpha \text{ is norm-attaining}\}.$$

In fact, one can show that linearly assigning a unique norm-attaining functional α to any $v \in \mathbb{R}^n$ is possible *only if* the norm is induced by a scalar product, see the solution to Exercise 4.2.11 in [13].

Moreover, denoting by $\mathrm{Dual}(v)$ the set of all $\alpha \in (\mathbb{R}^n)^*$ for which $\alpha(v) = \|v\|^2 = \|\alpha\|^2$, we have the following result, whose proof can be found as Propositions 5.4.24 and 5.4.25 together with Theorem 5.4.17 in *An Introduction to Banach Space Theory* [21] by Robert E. Megginson.

Proposition 4.3.1. *The assignment $v \mapsto \mathrm{Dual}(v)$ is singleton-valued if and only if the norm is (Gateaux) differentiable. Further, $\mathrm{Dual}(v) \cap \mathrm{Dual}(w) = \varnothing$ for $v \neq w$ precisely when the norm is strictly convex.*

Returning to our original problem, we can consequently, for a general norm, only hope to define a gradient $v|_p \in \mathbb{R}^n$ of $g \in \mathcal{C}^1(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ via the requirement that $\mathrm{Dual}(v|_p)$ contain $\mathrm{d}g|_p$ (as opposed to being able to set $\nabla g|_p := \mathrm{Dual}^{-1}(\mathrm{d}g|_p)$ with ∇g uniquely defined). Thus, we introduce

$$\mathrm{Grad}(g) := \{v\colon \mathbb{R}^n \to \mathbb{R}^n \mid \mathrm{d}g(v) = \|\mathrm{d}g\|^2 = \|v\|^2\}$$

with the understanding of pointwise equalities at each $p \in \mathbb{R}^n$.

Now, a way of deciding whether this set contains just a single element, based only on the notion of a differential, is to check (again pointwise) if

$$\max_{v \in \mathrm{Grad}(g)} \mathrm{d}f(v) = \min_{v \in \mathrm{Grad}(g)} \mathrm{d}f(v) \quad \text{for all } f \in \mathcal{C}^1(\mathbb{R}^n). \tag{4.3.1}$$

Indeed, if two gradients v and w of g disagree, say $v_i < w_i$ at some point $p \in \mathbb{R}^n$, we can choose the (linear) projection $f_i := x \mapsto x_i$ to conclude $\min_{v \in \mathrm{Grad}(g)} \mathrm{d}f_i(v) \leq \mathrm{d}f_i(v_i) = v_i < w_i = \mathrm{d}f_i(w) \leq \max_{v \in \mathrm{Grad}(g)} \mathrm{d}f_i(v)$ at this point p.

Example 4.3.2. To see explicitly that gradients may be non-unique, consider $(\mathbb{R}^2, \|\cdot\|_\infty)$ and the function $f\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x$. Then $\mathrm{d}f|_p(x,y) = x$ for any $p \in \mathbb{R}^2$ and so

$$\begin{aligned} \mathrm{Grad}(f)|_p &:= \{(x,y) \in \mathbb{R}^2 \mid x = 1 = \|(x,y)\|_\infty\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x = 1, |y| \leq 1\}. \end{aligned}$$

This effect is of course due to the lack of strict convexity.

Now, as on general spaces, this might be hard to check explicitly, we rephrase both quantities purely in variational terms:

$$\begin{aligned} \max_{v \in \mathrm{Grad}(g)} \mathrm{d}f(v) &= \inf_{\varepsilon > 0} \frac{\|\mathrm{d}(g + \varepsilon f)\|^2 - \|\mathrm{d}g\|^2}{2\varepsilon} \\ \min_{v \in \mathrm{Grad}(g)} \mathrm{d}f(v) &= \sup_{\varepsilon < 0} \frac{\|\mathrm{d}(g + \varepsilon f)\|^2 - \|\mathrm{d}g\|^2}{2\varepsilon} \end{aligned} \tag{4.3.2}$$

This equivalent formulation (see Proposition 4.3.6, where we give a full proof of this equivalence), which only requires knowledge on the magnitude of the differentials, will serve us as a characterization in the abstract setting of metric measure spaces, a treatment of which we now begin to develop.

A similar motivational introduction, on which our presentation is based, can be found in *On the differential structure of metric measure spaces and applications* [12] by Nicola Gigli.

Definition 4.3.3. Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and let $f \in S^2(X)$. We say that $v \in L^2(TX)$ is *a gradient* of f if

$$\mathrm{d}f(v) = |\mathrm{d}f|^2 = |v|^2 \ \mathfrak{m}\text{-a.e.} \tag{4.3.3}$$

The set of all gradients of f is denoted by $\mathrm{Grad}(f)$.

Lemma 4.3.4. *Given $f \in S^2(X)$, the set $\mathrm{Grad}(f)$ is non-empty and is comprised exactly of those $v \in L^2(TX)$ for which*

$$\mathrm{d}f(v) \geq \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|v|^2 \ \mathfrak{m}\text{-a.e.} \tag{4.3.4}$$

Proof. In the proof of Corollary 3.4.2 we have convinced ourselves that for any normed module $\mathcal{M}$ and any $w \in \mathcal{M}$, there exists an $L \in \mathcal{M}^*$ with $L(w) = |w|^2 = |L|_*^2$ $\mathfrak{m}$-a.e., cf. claim (3.4.4). Applying this to $\mathcal{M} = L^2(T^*X)$ and $w = \mathrm{d}f$, this implies the existence of some $v \in L^2(TX) = (L^2(T^*X))^*$ such that

$$\mathrm{d}f(v) = \frac{1}{2}\,\mathrm{d}f(v) + \frac{1}{2}\,\mathrm{d}f(v) = \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|v|^2 \text{ } \mathfrak{m}\text{-a.e.} \tag{4.3.5}$$

Hence, $\mathrm{Grad}(f) \neq \varnothing$. To prove the alternative characterization, notice that for any $v \in \mathrm{Grad}(f)$, we can proceed exactly as in (4.3.5) to verify the inequality (4.3.4). Conversely, suppose that (4.3.4) holds. Then, by Young's inequality,

$$\frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|v|^2 \overset{(4.3.4)}{\leq} \mathrm{d}f(v) \overset{(3.3.4)}{\leq} |\mathrm{d}f||v| \leq \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|v|^2 \text{ } \mathfrak{m}\text{-a.e.}$$

Hence, $\mathrm{d}f(v) = |\mathrm{d}f||v| = \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|v|^2$ $\mathfrak{m}$-a.e., implying $v \in \mathrm{Grad}(f)$ as $\frac{1}{2}(|\mathrm{d}f| - |v|)^2 = \frac{1}{2}|\mathrm{d}f|^2 - |\mathrm{d}f||v| + \frac{1}{2}|v|^2 = 0$ $\mathfrak{m}$-a.e., which enforces $|\mathrm{d}f| = |v|$ $\mathfrak{m}$-a.e. □

Being defined in duality with differentials, it is not surprising that gradients obey similar rules of calculation. However, the analogue of the Leibniz rule in Theorem 4.1.5 is not true for gradients in general. In fact, it will turn out that for (unique) gradients it is equivalent to their linearity and holds only if the tangent module is a Hilbert module.

Theorem 4.3.5. *Let $f \in S^2(X)$. Then the assignment of a gradient of f satisfies the following rules of calculation:*

A) Locality. If $f = g$ on a Borel set $E \subseteq X$ for some $g \in S^2(X)$, then for any choice of gradient $v \in \mathrm{Grad}(f)$, there exists some $w \in \mathrm{Grad}(g)$ for which $v = w$ on E.

B) Chain rule.
i) Any $v \in \mathrm{Grad}(f)$ vanishes wherever f takes values in a λ-null set, i.e.

$$v = 0 \quad \mathfrak{m}\text{-a.e. on } f^{-1}(N) \text{ for any } N \subseteq \mathbb{R} \text{ null.} \tag{4.3.6}$$

ii) For any Lipschitz function $\varphi\colon \mathbb{R} \to \mathbb{R}$, we have

$$(\varphi' \circ f)v \in \mathrm{Grad}(\varphi \circ f) \text{ } \mathfrak{m}\text{-a.e.} \tag{4.3.7}$$

with $(\varphi' \circ f)(x)$ arbitrarily defined whenever φ' does not exist at $f(x)$.

Proof. To prove locality, for a given $v \in \mathrm{Grad}(f)$, we begin by choosing any $\tilde{w} \in \mathrm{Grad}(g)$ and set $w := 1_E v + 1_{X\setminus E}\tilde{w}$. Then, as

$$\mathrm{d}f = \mathrm{d}g \text{ } \mathfrak{m}\text{-a.e. on } E \tag{4.3.8}$$

by Theorem 4.1.5, we have

$$|w|^2 \overset{(3.1.2)}{=} 1_E|v|^2 + 1_{X\setminus E}|\tilde{w}|^2 \overset{(4.3.3)}{=} 1_E|\mathrm{d}f|^2 + 1_{X\setminus E}|\mathrm{d}g|^2$$
$$\overset{(4.3.8)}{=} 1_E|\mathrm{d}g|^2 + 1_{X\setminus E}|\mathrm{d}g|^2 = |\mathrm{d}g|^2 \ \mathfrak{m}\text{-a.e.}$$

Moreover,

$$\begin{aligned}\mathrm{d}g(w) = 1_E\,\mathrm{d}g(v) + 1_{X\setminus E}\,\mathrm{d}g(\tilde{w}) &\overset{(4.3.3)}{=} 1_E\,\mathrm{d}g(v) + 1_{X\setminus E}|\tilde{w}|^2\\ &\overset{(4.3.8)}{=} 1_E\,\mathrm{d}f(v) + 1_{X\setminus E}|\tilde{w}|^2\\ &\overset{(4.3.3)}{=} 1_E|v|^2 + 1_{X\setminus E}|\tilde{w}|^2 = |w|^2\end{aligned}$$

in the $\mathfrak{m}$-almost everywhere sense. Hence, $w \in \mathrm{Grad}(g)$ with $v = w$ on E.

For the chain rule, first observe that $\mathrm{d}f = 0$ $\mathfrak{m}$-a.e. on $f^{-1}(N)$ if $N \subseteq \mathbb{R}$ is λ-null by Theorem 4.1.5. Thus, $|v| = |\mathrm{d}f| = 0$ $\mathfrak{m}$-a.e. and so $v = 0$. Now, let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz. We need to prove

$$\mathrm{d}(\varphi\circ f)((\varphi'\circ f)v) = |\varphi'\circ f|^2|v|^2 = |\mathrm{d}(\varphi\circ f)|^2 \ \mathfrak{m}\text{-a.e.}$$

Recalling that

$$\mathrm{d}(\varphi\circ f) = \varphi'\circ f\,\mathrm{d}f \ \mathfrak{m}\text{-a.e.} \tag{4.3.9}$$

by Theorem 4.1.5, we have

$$\mathrm{d}(\varphi\circ f)((\varphi'\circ f)v) = \varphi'\circ f\,\mathrm{d}(\varphi\circ f)(v) \overset{(4.3.9)}{=} |\varphi'\circ f|^2\,\mathrm{d}f(v) \overset{(4.3.3)}{=} |\varphi'\circ f|^2|v|^2$$

$\mathfrak{m}$-almost everywhere. This also immediately implies

$$\mathrm{d}(\varphi\circ f)((\varphi'\circ f)v) = |\varphi'\circ f|^2|v|^2 \overset{(4.3.3)}{=} |\varphi'\circ f|^2|\mathrm{d}f|^2 \overset{(4.3.9)}{=} |\mathrm{d}(\varphi\circ f)|^2 \ \mathfrak{m}\text{-a.e.},$$

giving the claim. □

We now proceed to prove, in our abstract setting, the equivalence of the aforementioned characterizations (4.3.1) and its variational counterpart (4.3.2). To do so, we introduce the quantity

$$H_{f,g}(\varepsilon) := \frac{1}{2}|D(g+\varepsilon f)|^2 \in L^1(\mathfrak{m}), \tag{4.3.10}$$

where $f, g \in S^2(X)$ and $\varepsilon \in \mathbb{R}$. As we know by Lemma 2.3.6 that the Sobolev class $S^2(X)$ constitutes a real vector space and that $S^2(X) \ni f \mapsto |Df|$ is convex, we can conclude that also $H_{f,g}$ is $\mathfrak{m}$-a.e. convex for all choices of Sobolev functions f and g (as the restriction of the convex minimal weak upper gradient to the segment connecting these two functions in $S^2(X)$). Hence, the difference quotients of $H_{f,g}$ are monotonically non-decreasing and so

$$L^1(\mathfrak{m})\text{-}\lim_{\varepsilon\to 0^+} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} = \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} \tag{4.3.11}$$

exists (cf. Lemma 3.2 in [23]).

Proposition 4.3.6. *Let $f, g \in S^2(X)$ and let $v \in \mathrm{Grad}(g)$. Then,*

$$\operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} \le \mathrm{d}f(v) \le \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} \quad \mathfrak{m}\text{-a.e.} \tag{4.3.12}$$

Moreover, there exist choices of $v_{f,+}$ and $v_{f,-}$ in $\mathrm{Grad}(g)$ for which the right-hand side and the left-hand side, respectively, attain equality.

Proof. For the first part, we invoke Lemma 4.3.4 and Young's inequality to get

$$\varepsilon\,\mathrm{d}f(v) = \mathrm{d}(g+\varepsilon f)(v) - \mathrm{d}g(v) \le \frac{1}{2}|\mathrm{d}(g+\varepsilon f)|^2 + \frac{1}{2}|v|^2 - \frac{1}{2}|dg|^2 - \frac{1}{2}|v|^2 = H_{f,g}(\varepsilon) - H_{f,g}(0).$$

Upon dividing by $\varepsilon > 0$, respectively $\varepsilon < 0$, this gives the claim.

For the second assertion, we focus on the existence of $v_{f,+}$ as $v_{f,-}$ can be obtained analogously. We will argue by compactness and start by picking, for any $\varepsilon \in (0,1)$, some $v_\varepsilon \in \mathrm{Grad}(g+\varepsilon f)$. We then have, by the triangle inequality,

$$\begin{aligned} \|v_\varepsilon\|_{L^2(TX)} &\overset{(3.1.3)}{=} \||v_\varepsilon|\|_{L^2(\mathfrak{m})} \overset{(4.3.3)}{=} \||\mathrm{d}(g+\varepsilon f)|\|_{L^2(\mathfrak{m})} \\ &= \|\mathrm{d}(g+\varepsilon f)\|_{L^2(T^*X)} \le \|\mathrm{d}g\|_{L^2(T^*X)} + \|\mathrm{d}f\|_{L^2(T^*X)} \end{aligned}$$

for this entire family of gradients v_ε. Hence, by the theorem of Banach-Alaoglu, for any $\varepsilon' \in (0,1)$, the set $\{v_\varepsilon \mid \varepsilon \in (0,\varepsilon')\}$ is precompact in the weak* topology. The finite intersection property of compact sets thus implies

$$\bigcap_{0<\varepsilon'<1} \overline{\{v_\varepsilon \mid \varepsilon \in (0,\varepsilon')\}}^{w*} \ne \varnothing.$$

Let $v_{f,+}$ be some element of this set. Defining

$$F_E \colon L^2(TX) \to \mathbb{R},\ v \mapsto \int_E \frac{1}{2}|v|^2 + \frac{1}{2}|\mathrm{d}g|^2 - \mathrm{d}g(v)\,\mathrm{d}\mathfrak{m}$$

for $E \subseteq X$ Borel, we claim that

$$F_E(v_{f,+}) \le \varliminf_{\varepsilon\to 0} \int_E \frac{1}{2}|v_\varepsilon|^2 + \frac{1}{2}|\mathrm{d}g|^2 - \mathrm{d}g(v_\varepsilon)\,\mathrm{d}\mathfrak{m}. \tag{4.3.13}$$

In fact, first note that we can rewrite F_E so that

$$F_E(v) = \frac{1}{2}\|1_E v\|^2_{L^2(TX)} + \frac{1}{2}\|1_E\,\mathrm{d}g\|^2_{L^2(T^*X)} - \int 1_E\,\mathrm{d}g(v)\,\mathrm{d}\mathfrak{m}.$$

Now, if a net $(v_\alpha)_{\alpha\in A} \subseteq L^2(TX)$ converges to some $v \in L^2(TX)$ in the weak* topology, i.e.

$$\int v_\alpha(\omega)\,\mathrm{d}\mathfrak{m} = \mathrm{Int}(v_\alpha)(\omega) \underset{\alpha}{\to} \mathrm{Int}(v)(\omega) = \int v(\omega)\,\mathrm{d}\mathfrak{m} \quad \text{for all } \omega \in L^2(T^*X),$$

cf. also Theorem 3.4.1, then clearly $\int_E \mathrm{d}g(v_\alpha)\,\mathrm{d}\mathfrak{m} = \mathrm{Int}(v_\alpha)(1_E\,\mathrm{d}g) \to \mathrm{Int}(v)(1_E\,\mathrm{d}g) = \int_E \mathrm{d}g(v)\,\mathrm{d}\mathfrak{m}$. Further, the assignment $(L^2(T^*X))' \ni \ell \mapsto \|\ell\|$ is weak* lower semicontinuous, as can be seen by choosing any net $(\ell_\alpha)_\alpha \subseteq (L^2(T^*X))'$ weak* converging to ℓ and observing that

$$|\ell(w)| = \lim_\alpha |\ell_\alpha(\omega)| \leq \varliminf_\alpha \|\ell_\alpha\|$$

for any $\omega \in L^2(T^*X)$ with $\|\omega\| \leq 1$, and so, $\|\ell\| \leq \varliminf_\alpha \|\ell_\alpha\|$. Putting these two facts together, we see that $L^2(TX) \ni v \mapsto F_E(v)$ is weak* lower semicontinuous, implying (4.3.13).

Next, observe that we can rearrange

$$\mathrm{d}g(v_\varepsilon) + \varepsilon\,\mathrm{d}f(v_\varepsilon) \geq \frac{1}{2}|\mathrm{d}(g+\varepsilon f)|^2 + \frac{1}{2}|v_\varepsilon|^2 \ \mathfrak{m}\text{-a.e.}, \tag{4.3.14}$$

guaranteed by Lemma 4.3.4, to have the form

$$\frac{1}{2}|v_\varepsilon|^2 - \mathrm{d}g(v_\varepsilon) \leq \varepsilon\,\mathrm{d}f(v_\varepsilon) - \frac{1}{2}|\mathrm{d}g + \varepsilon\,\mathrm{d}f|^2 \ \mathfrak{m}\text{-a.e.}$$

Using $|\mathrm{d}g + \varepsilon\,\mathrm{d}f| - |\mathrm{d}g| \geq -\varepsilon|\mathrm{d}f|$, as a consequence of the reverse triangle inequality, to conclude

$$|\mathrm{d}g + \varepsilon\,\mathrm{d}f|^2 \geq |\mathrm{d}g|^2 - 2\varepsilon|\mathrm{d}g||\mathrm{d}f| + \varepsilon^2|\mathrm{d}f|^2, \tag{4.3.15}$$

we therefore have

$$\frac{1}{2}|v_\varepsilon|^2 - \mathrm{d}g(v_\varepsilon) \overset{(4.3.15)}{\leq} \varepsilon\,\mathrm{d}f(v_\varepsilon) - \frac{1}{2}|\mathrm{d}g|^2 + \varepsilon|\mathrm{d}g||\mathrm{d}f| - \frac{1}{2}\varepsilon^2|\mathrm{d}f|^2.$$

Introducing $G_\varepsilon := \varepsilon(\mathrm{d}f(v_\varepsilon) + |\mathrm{d}g||\mathrm{d}f| - \frac{1}{2}\varepsilon|\mathrm{d}f|^2)$, we thus finally see that

$$\frac{1}{2}|v_\varepsilon|^2 + \frac{1}{2}|\mathrm{d}g|^2 - \mathrm{d}g(v_\varepsilon) \leq G_\varepsilon \ \mathfrak{m}\text{-a.e.}$$

with $G_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^1(\mathfrak{m})} 0$. Hence, (4.3.13) ensures

$$\int_E \frac{1}{2}|v_{f,+}|^2 + \frac{1}{2}|\mathrm{d}g|^2 - \mathrm{d}g(v_{f,+})\,\mathrm{d}\mathfrak{m} = F_E(v_{f,+}) \leq \varliminf_{\varepsilon\to 0^+} \|G_\varepsilon\|_{L^1(\mathfrak{m})} = 0$$

for any $E \subseteq X$ Borel and so $v_{f,+} \in \mathrm{Grad}(g)$, again by Lemma 4.3.4.

Finally, let $(v_\varepsilon)_{\varepsilon\in(0,1)}$ be a net converging to $v_{f,+} \in \mathrm{Grad}(g)$ in the weak* sense such that $v_\varepsilon \in \mathrm{Grad}(g+\varepsilon f)$ for all $\varepsilon \in (0,1)$. Then, along this net, we have

$$\begin{aligned}\varepsilon\,\mathrm{d}f(v_\varepsilon) = \mathrm{d}(g+\varepsilon f)(v_\varepsilon) - \mathrm{d}g(v_\varepsilon) &\geq \frac{1}{2}|\mathrm{d}(g+\varepsilon f)|^2 + \frac{1}{2}|v_\varepsilon|^2 - \frac{1}{2}|\mathrm{d}g|^2 - \frac{1}{2}|v_\varepsilon|^2 \\ &= H_{f,g}(\varepsilon) - H_{f,g}(0),\end{aligned}$$

using Lemma 4.3.4 to bound $\mathrm{d}(g+\varepsilon f)(v_\varepsilon)$ and Young's inequality for $\mathrm{d}g(v_\varepsilon)$, cf. the proof of Lemma 4.3.4. Therefore,

$$\mathrm{d}f(v_\varepsilon) \geq \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \geq \operatorname*{ess\,inf}_{\varepsilon'>0} \frac{H_{f,g}(\varepsilon') - H_{f,g}(0)}{\varepsilon'} \ \mathfrak{m}\text{-a.e.} \tag{4.3.16}$$

and so

$$\int_E \mathrm{d}f(v_{f,+})\,\mathrm{d}\mathfrak{m} = \mathrm{Int}(v_{f,+})(1_E\,\mathrm{d}f) = \lim_\varepsilon \mathrm{Int}(v_\varepsilon)(1_E\,\mathrm{d}f) \overset{(4.3.16)}{\geq} \int_E \operatorname*{ess\,inf}_{\varepsilon'>0} \frac{H_{f,g}(\varepsilon') - H_{f,g}(0)}{\varepsilon'}$$

for any Borel subset $E \subseteq X$, giving the claim by the first part of the proposition, where we have shown that the essential infimum always gives an upper bound for $\mathrm{d}f(v_{f,+})$, see (4.3.12). □

4.4 Infinitesimal Hilbertianity

Proposition 4.4.1. *Let $g \in S^2(X)$. Then g possesses a unique gradient, meaning* $\mathrm{Grad}(g)$ *is a singleton, if and only if*

$$\operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \geq \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \ \mathfrak{m}\text{-a.e.} \tag{4.4.1}$$

for every $f \in S^2(X)$. By the relation (4.3.12)*, this is the case precisely when both quantities coincide for each $f \in S^2(X)$.*

Proof. If $\mathrm{Grad}(g)$ is a singleton, the claimed inequality follows from Proposition 4.3.6. Conversely, if (4.4.1) is satisfied, then for any pair $v, w \in \mathrm{Grad}(g)$, we have

$$\mathrm{d}f(v) = \mathrm{d}f(w)\ \mathfrak{m}\text{-a.e.} \quad \text{for all } f \in S^2(X). \tag{4.4.2}$$

Indeed, otherwise we could find a Borel set $E \subseteq X$ with $\mathfrak{m}(E) > 0$ on which $\mathrm{d}f(v) < \mathrm{d}f(w)$ for $v, w \in \mathrm{Grad}(g)$. Then, however,

$$\operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \overset{(4.3.12)}{\leq} \mathrm{d}f(v) < \mathrm{d}f(w) \overset{(4.3.12)}{\leq} \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \ \mathfrak{m}\text{-a.e. on } E$$

by Proposition 4.3.6, contradicting (4.4.1).

Switching notation for clarity of the argument (from what we described in Remark 4.2.2), this now implies that $v(\sum 1_{E_i}\,\mathrm{d}f_i) = w(\sum 1_{E_i}\,\mathrm{d}f_i)$ for any Borel partition $(E_i)_i$ of X and any (finite) sequence $(f_i)_i \subseteq S^2(X)$ by $L^\infty(\mathfrak{m})$-linearity. Hence, we have $v(\omega) = w(\omega)$ for all ω in a dense subset of $L^2(T^*X)$. By continuity, this implies $v = w \in \mathrm{Grad}(g)$, proving the claim. $\square$

Definition 4.4.2. Let (X, d, m) be a metric measure space. If every $g \in S^2(X)$ possesses a unique gradient (in the sense of Proposition 4.4.1), then we call X *infinitesimally strictly convex.* Further, for any such $g \in S^2(X)$, we denote this unique gradient by ∇g.

Remark 4.4.3. By definition, the element ∇g is therefore the unique vector field on X satisfying

$$\mathrm{d}g(\nabla g) = |\mathrm{d}g|^2 = |\nabla g|^2 \ \mathfrak{m}\text{-a.e.}$$

The terminology is of course borrowed from Banach space theory, cf. Proposition 4.3.1, expressing the fact that the pointwise norm, as a generalization of the norm on each tangent space (hence the adjective *infinitesimal*), is strictly convex. Moreover, by Proposition 4.3.6, we have

$$\mathrm{d}f(\nabla g) = \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} = \operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \ \mathfrak{m}\text{-a.e.} \tag{4.4.3}$$

for any $f \in S^2(X)$.

Now, an important question in our considerations of infinitesimally strictly convex spaces is whether the *duality relation*

$$\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f) \ \mathfrak{m}\text{-a.e.} \quad \text{for all } f, g \in S^2(X), \tag{4.4.4}$$

holds, as it would imply that the map $(f, g) \mapsto \mathrm{d}f(\nabla g)$ is bilinear and symmetric, turning $f \mapsto \mathrm{d}f(\nabla f) = |\mathrm{d}f|^2$ into a quadratic form. This fact is, as it will turn out, sufficient to conclude that the Sobolev space $W^{1,2}(X)$ is a Hilbert space, in which case X is called *infinitesimally Hilbertian.*

Lemma 4.4.4. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an infinitesimally strictly convex metric measure space and suppose $W^{1,2}(X) \ni f \mapsto \mathrm{E}(f) := \frac{1}{2}\int |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m}$ is a quadratic form. Then the identity*

$$\int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m} \tag{4.4.5}$$

holds for any pair $f, g \in W^{1,2}(X)$.

Proof. By assumption, given $f, g \in W^{1,2}(X)$, the function $\mathbb{R}^2 \ni (t, s) \mapsto \mathrm{E}(tf + sg)$ is a quadratic polynomial as

$$\mathrm{E}(tf + sg) = t^2\mathrm{E}(f) + s^2\mathrm{E}(g) + 2st\mathrm{B}_{\mathrm{E}}(f, g), \tag{4.4.6}$$

where $\mathrm{B_E}$ is the (bilinear) polar form of E defined by this very relation. So, in particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{E}(tf+sg)=\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{E}(tf+sg). \tag{4.4.7}$$

Evaluating the inner derivative of the left-hand side, by (4.3.11), we get

$$\begin{aligned}\left(\lim_{\varepsilon\to 0}\frac{1}{2}\int\frac{|D(tf+\varepsilon g)|^2-|D(tf)|^2}{\varepsilon}\,\mathrm{d}\mathfrak{m}\right)&=\int\operatorname*{ess\,inf}_{\varepsilon>0}\frac{H_{g,tf}(\varepsilon)-H_{g,tf}(0)}{\varepsilon}\,\mathrm{d}\mathfrak{m}\\&=\int\mathrm{d}g(\nabla(tf))\,\mathrm{d}\mathfrak{m}=\left(t\int\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m}\right),\end{aligned}$$

where we have used Proposition 4.3.6 to pass from the first to the second line and the chain rule for the gradient (putting $\varphi=t\,\mathrm{id}_{\mathbb{R}}$ in item (B) of Theorem 4.3.5).

Thus, the left-hand side of (4.4.7) equals $\int\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m}$ and by interchanging the roles of tf and εg in this calculation, we see that the right-hand side reduces to $\int\mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m}$. □

Theorem 4.4.5. *The Sobolev space $W^{1,2}(X)$ is a Hilbert space precisely when*

$$2(|\mathrm{d}f|^2+|\mathrm{d}g|^2)=|\mathrm{d}(f+g)|^2+|\mathrm{d}(f-g)|^2\ \mathfrak{m}\text{-a.e.} \tag{4.4.8}$$

holds for every $f,g\in W^{1,2}(X)$. Moreover, this is equivalent to the assertion that $L^2(T^*X)$ and $L^2(TX)$ are Hilbert modules.

Proof. Suppose first that $W^{1,2}(X)$ is a Hilbert space. This is the case if and only if $\|\cdot\|_{W^{1,2}(X)}$ satisfies the parallelogram law by the Theorem of Jordan-von Neumann and as $\|f\|^2_{W^{1,2}(X)}=\|f\|^2_{L^2(\mathfrak{m})}+\||\mathrm{d}f|\|^2_{L^2(\mathfrak{m})}$, we see that $W^{1,2}(X)\ni f\mapsto\||\mathrm{d}f|\|_{L^2(\mathfrak{m})}$ must satisfy the parallelogram law. Hence, defining

$$W^{1,2}(X)\ni f\mapsto\mathrm{E}(f):=\frac{1}{2}\int|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m} \tag{4.4.9}$$

we have $\mathrm{E}(f+\varepsilon g)+\mathrm{E}(f-\varepsilon g)=2\mathrm{E}(f)+2\varepsilon^2\mathrm{E}(g)$ for all $f,g\in W^{1,2}(X),\varepsilon\neq 0$. Rewriting this, we obtain

$$\frac{\mathrm{E}(f+\varepsilon g)-\mathrm{E}(f)}{\varepsilon}+\frac{\mathrm{E}(f-\varepsilon g)-\mathrm{E}(f)}{\varepsilon}=2\varepsilon\mathrm{E}(g). \tag{4.4.10}$$

Therefore,

$$\begin{aligned}\int \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{g,f}(\varepsilon)-H_{g,f}(0)}{\varepsilon}\,\mathrm{d}\mathfrak{m} &\overset{(4.3.11)}{=} \lim_{\varepsilon\to 0^+}\int \frac{H_{g,f}(\varepsilon)-H_{g,f}(0)}{\varepsilon}\,\mathrm{d}\mathfrak{m}\\ &\overset{(4.3.10)}{=} \lim_{\varepsilon\to 0^+} \frac{\mathrm{E}(f+\varepsilon g)-\mathrm{E}(f)}{\varepsilon}\\ &\overset{(4.4.10)}{=} \lim_{\varepsilon\to 0^-} \frac{\mathrm{E}(f+\varepsilon g)-\mathrm{E}(f)}{\varepsilon}\\ &= \int \operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{g,f}(\varepsilon)-H_{g,f}(0)}{\varepsilon}\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

As Proposition 4.3.6 guarantees that the essential supremum appearing in the above calculation is always bounded from above by the essential infimum, we have thus established

$$\operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{g,f}(\varepsilon)-H_{g,f}(0)}{\varepsilon} = \operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{g,f}(\varepsilon)-H_{g,f}(0)}{\varepsilon}, \tag{4.4.11}$$

or equivalently, by Proposition 4.4.1, that $(X,\mathrm{d},\mathfrak{m})$ is infinitesimally strictly convex.

To get (4.4.8) for f, g (essentially) bounded, we claim that it suffices to show the parallelogram law for all maps

$$W^{1,2}(X)\cap L^\infty(\mathfrak{m}) \ni f \mapsto \int h|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m} \tag{4.4.12}$$

with $h \in \mathrm{Lip}(X)\cap L^\infty(\mathfrak{m})$. In fact, using the Borel partition $(E_n)_{n\geq 1}$ of X given by $E_n := \overline{B_n(x_0)}\setminus B_{n-1}(x_0)$ for some $x_0\in X$ and defining the Lipschitz cut-off functions

$$h_n := (1-\mathrm{d}(\,\cdot\,,E_n))^+ \in \mathrm{Lip}(X)\cap L^\infty(\mathfrak{m}), \tag{4.4.13}$$

this would imply

$$\int h_n F\,\mathrm{d}\mathfrak{m} = 0 \quad \text{for all } n\geq 1, \tag{4.4.14}$$

where $F := 2|\mathrm{d}f|^2+2|\mathrm{d}g|^2-|\mathrm{d}(f+g)|^2-|\mathrm{d}(f-g)|^2 \in L^1(\mathfrak{m})$. Suppose now $F\geq c>0$ on some Borel set $E\subseteq X$ with $\mathfrak{m}(E)>0$. Then we could find some $N\geq 1$ such that $\mathfrak{m}(E\cap E_n)>0$, as otherwise $\mathfrak{m}(E)=\sum_{n\geq 1}\mathfrak{m}(E\cap E_n)=0$. Therefore, as $h_n\equiv 1$ on E_n,

$$\int h_N F\,\mathrm{d}\mathfrak{m} \geq \int_{E\cap E_N} h_N F\,\mathrm{d}\mathfrak{m} = \int_{E\cap E_N} F\,\mathrm{d}\mathfrak{m} \geq c\mathfrak{m}(E\cap E_N) > 0$$

contradicting (4.4.14). Proceeding analogously for $F\leq c$ gives $F\equiv 0$, which is the claim.

To prove the parallelogram law for the maps in (4.4.12), we make use of the Leibniz rule for the differential (cf. Theorem 4.1.5), as well as the chain rule for gradients (cf. Theorem 4.3.5) to obtain

$$\int h|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m} \overset{(4.3.3)}{=} \int h\,\mathrm{d}f(\nabla f)\,\mathrm{d}\mathfrak{m} \overset{(4.1.15)}{=} \int \mathrm{d}(fh)(\nabla f) - f\,\mathrm{d}h(\nabla f)\,\mathrm{d}\mathfrak{m}$$
$$\overset{(4.3.7)}{=} \int \mathrm{d}(fh)(\nabla f) - \mathrm{d}h(\nabla(\frac{1}{2}f^2))\,\mathrm{d}\mathfrak{m}$$

Now, Lemma 4.4.4 guarantees that $\int \mathrm{d}h(\nabla(\frac{1}{2}f^2))\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}(\frac{1}{2}f^2)(\nabla h)\,\mathrm{d}\mathfrak{m}$, which is clearly a quadratic form in f. Finally, again by Lemma 4.4.4, we have $\int \mathrm{d}(fh)(\nabla g)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}g(\nabla(fh))\,\mathrm{d}\mathfrak{m}$. Thus, linearity of the differential renders the assignment $(f,g) \mapsto \int \mathrm{d}(fh)(\nabla g)\,\mathrm{d}\mathfrak{m}$ bilinear, thereby ensuring that $f \mapsto \int \mathrm{d}(fh)(\nabla f)\,\mathrm{d}\mathfrak{m}$ is quadratic. Hence, $f \mapsto \int h|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}$ is the difference of two quadratic forms and so itself quadratic, implying the parallelogram law.

To get rid of the additional boundedness requirement, we invoke locality of the differential (cf. Theorem 4.1.5). This ensures that for $f_n := 1_{\{|f|\leq n\}}f$ we have

$$\mathrm{d}f = \mathrm{d}f_n \ \mathfrak{m}\text{-a.e. on } \{|f| \leq n\}$$

and similarly for $g_n := 1_{\{|g|\leq n\}}g$. Given that the parallelogram law (proved above) holds for f_n, g_n with $n \geq 1$ arbitrary, we get

$$2|\mathrm{d}f|^2 + 2|\mathrm{d}g|^2 = |\mathrm{d}(f+g)|^2 + |\mathrm{d}(f-g)|^2 \ \mathfrak{m}\text{-a.e. on } \bigcup_{n\geq 1}\{|f| \leq n\} \cap \{|g| \leq n\}.$$

Now, as $f, g \in L^2(\mathfrak{m})$, this union has full measure, i.e.

$$\mathfrak{m}(X \setminus \bigcup_{n\geq 1}\{|f| \leq n\} \cap \{|g| \leq n\}) \leq \mathfrak{m}(\bigcap_{n\geq 1}\{|f| > n\}) + \mathfrak{m}(\bigcap_{n\geq 1}\{|g| > n\}) = 0$$

which gives (4.4.8) for all $f, g \in W^{1,2}(X)$.

For the converse assertion, we simply integrate (4.4.8) over all of X to get the parallelogram law for $f \mapsto \mathrm{E}(f)$. As already remarked, this is equivalent to $W^{1,2}(X)$ being a Hilbert space.

Lastly, to see the equivalence of (4.4.8) to the fact that both $L^2(T^*X)$ and $L^2(TX)$ are Hilbert modules, we argue by density. More specifically, suppose the parallelogram law holds for all differentials of Sobolev functions. Then, by $L^\infty(\mathfrak{m})$-linearity of the pointwise norm, we get

$$|\omega + \eta|^2 + |\omega - \eta|^2 = 2|\omega|^2 + 2|\eta|^2 \ \mathfrak{m}\text{-a.e.} \tag{4.4.15}$$

for all simple one-forms $\omega = \sum_i 1_{E_i}\,\mathrm{d}f_i, \eta = \sum_j 1_{F_j}\,\mathrm{d}g_j$. By density of these forms in $L^2(T^*X)$ and by continuity of the pointwise norm, we obtain (4.4.15) for all $\omega, \eta \in L^2(T^*X)$, giving the claim for $L^2(T^*X)$ by Corollary 3.4.9. Moreover, since we have $L^2(TX) \cong (L^2(T^*X))' \cong L^2(T^*X)$ as normed spaces by Theorem 3.4.1, we see that also the tangent module is a Hilbert module. □

Theorem 4.4.6. *On any metric measure space* $(X, \mathrm{d}, \mathfrak{m})$*, the modules* $L^2(T^*X)$ *and* $L^2(TX)$ *are Hilbert modules if and only if* $(X, \mathrm{d}, \mathfrak{m})$ *is infinitesimally strictly convex and for all* $f, g \in W^{1,2}(X)$ *any of the following equivalent conditions hold true:*

i) $\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f)$ $\mathfrak{m}$-a.e.

ii) $\nabla(f+g) = \nabla f + \nabla g$ $\mathfrak{m}$-a.e.

iii) $\nabla(fg) = f\nabla g + g\nabla f$ $\mathfrak{m}$-a.e. *if, additionally,* $f, g \in L^\infty(\mathfrak{m})$.

Proof. Suppose the tangent and cotangent modules are Hilbert modules. By Theorem 4.4.5 this implies that $W^{1,2}(X)$ is a Hilbert space, and so we know that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally strictly convex, following the argumentation preceeding (4.4.11). Moreover, we also have that

$$W^{1,2}(X) \cap L^\infty(\mathfrak{m}) \ni f \mapsto \mathrm{E}_h(f) := \frac{1}{2}\int h|\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m}$$

satisfies the parallelogram law for any $h \in \mathrm{Lip}(X) \cap L^\infty(\mathfrak{m})$ by (4.4.12). This guarantees that $\mathbb{R}^2 \ni (t,s) \mapsto \mathrm{E}_h(tf+sg)$ are quadratic polynomial for all h (cf. equality (4.4.6)) and we can proceed exactly as in the proof of Lemma 4.4.4 to get

$$\int h\,\mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} = \int h\,\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m} \quad \text{for all } f, g \in W^{1,2}(X) \cap L^\infty(\mathfrak{m}).$$

This allows us to conclude

$$\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f)\ \mathfrak{m}\text{-a.e.} \quad \text{for all (essentially) bounded } f, g \in W^{1,2}(X),$$

by choosing appropriate localizing Lipschitz functions $(h_n)_{n\geq 1}$ as in (4.4.13), repeating the argumentation we gave there verbatim. Finally, we have already convinced ourselves that this gives the full claim by a cut-off argument.

Suppose now that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally strictly convex and that (i) holds. We show, conversely, that $W^{1,2}(X)$ is a Hilbert space by verifying the parallelogram law for $f \mapsto \mathrm{E}(f)$, defined in (4.4.9). For this, we note that $[0,1] \ni t \mapsto \mathrm{E}(f+tg)$ is Lipschitz as it is the restriction of the convex function $\mathbb{R} \ni t \mapsto \int H_{g,f}(t)\,\mathrm{d}\mathfrak{m}$ to the compact interval $[0,1]$. Computing its derivative, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(f+tg) = \lim_{\varepsilon\to 0} \frac{\mathrm{E}((f+tg)+\varepsilon g) - \mathrm{E}(f+tg)}{\varepsilon} \tag{4.4.16}$$

$$\overset{(4.3.10)}{=} \lim_{\varepsilon\to 0} \int \frac{H_{g,f+tg}(\varepsilon) - H_{g,f+tg}(0)}{2\varepsilon} \overset{(4.4.3)}{=} \int \mathrm{d}g(\nabla(f+tg))\,\mathrm{d}\mathfrak{m} \tag{4.4.17}$$

Therefore, invoking Lemma 4.4.4, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(f+tg) &\overset{(4.4.16)}{=} \int \mathrm{d}g(\nabla(f+tg))\,\mathrm{d}\mathfrak{m} \overset{(4.4.5)}{=} \int \mathrm{d}(f+tg)(\nabla g)\,\mathrm{d}\mathfrak{m}\\ &\overset{(4.3.3)}{=} \int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + t\int |\mathrm{d}g|^2\,\mathrm{d}\mathfrak{m},\end{aligned}$$

so that integrating over $[0,1]$ yields

$$\mathrm{E}(f+g)-\mathrm{E}(f) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(f+tg)\,\mathrm{d}t = \int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + \mathrm{E}(g). \tag{4.4.18}$$

Replacing g by $-g$, we obtain

$$\mathrm{E}(f-g)-\mathrm{E}(f) = -\int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + \mathrm{E}(g), \tag{4.4.19}$$

giving the claim by adding (4.4.18) and (4.4.19).

We now show the equivalence of conditions (*ii*) and (*iii*). To this end, suppose that $f, g \in W^{1,2}(X)\cap L^\infty(\mathfrak{m})$ and define $f' := \exp(f)$, $g' := \exp(g)$. These functions are seen to belong to $W^{1,2}(X)\cap L^\infty(\mathfrak{m})$ by the chain rule for the differential, as the exponential function restricted to the respective bounded images of f and g is Lipschitz. Using the chain rule for gradients (cf. Theorem 4.3.5), we get the equalities

$$\nabla(f'g') = \nabla(\exp(f+g)) \overset{(4.3.7)}{=} \exp(f+g)\nabla(f+g) = f'g'\nabla(f+g)$$

and

$$g'\nabla f' + f'\nabla g' \overset{(4.3.7)}{=} g'f'\nabla f + f'g'\nabla g = f'g'(\nabla f + \nabla g).$$

Hence, the Leibniz rule (*iii*) applied to f' and g' immediately implies $\nabla(f+g) = \nabla f + \nabla g$. Arguing conversely, assuming (*ii*), we thereby also immediately get the Leibniz rule for exponentials of Sobolev functions, as now the right-hand sides of the above equations are equal by assumption. To recover the functions themselves, we extract (strict) positive and negative parts, that is, we write

$$f = f^+ - f^- \quad \text{and} \quad g = g^+ - g^-, \tag{4.4.20}$$

wherever f and g are non-zero, using the notation $f^\pm = \max(\pm f, 0)$. Then, by locality of ∇ and item (*ii*),

$$\begin{aligned}\nabla(fg) &\overset{(4.4.20)}{=} \nabla((f^+-f^-)(g^+-g^-))\\ &\overset{(ii)}{=} \nabla(f^+g^+) - \nabla(f^+g^-) - \nabla(f^-g^+) + \nabla(f^-g^-)\end{aligned}$$

outside $\{f=0\}\cup\{g=0\}$ in the $\mathfrak{m}$-almost everywhere sense. Now, we can apply the above argumentation piece by piece, as for example $f^\pm = \exp(\log(f^\pm))$. Collecting terms and recalling that both ∇f and ∇g vanish $\mathfrak{m}$-almost everywhere on $\{f=0\}\cup\{g=0\}$

by Theorem 4.3.5, this establishes the claim.

Finally, we show that $W^{1,2}(X)$ is a Hilbert space precisely when $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally strictly convex and $f \mapsto \nabla f$ is linear. By Theorem 4.4.5 and the above, this will ensure the equivalence of the items (i) and (ii), hence also of (i) and (iii). Again invoking Theorem 4.4.5, we see that it suffices to show the pointwise parallelogram law for elements of the form $\mathrm{d}f$ with $f \in W^{1,2}(X)$ to prove that $W^{1,2}(X)$ is a Hilbert space. Now, if $f \mapsto \nabla f$ is linear, the map $(f, g) \mapsto \mathrm{d}f(\nabla g)$ is bilinear and so

$$f \mapsto \mathrm{d}f(\nabla f) = |\mathrm{d}f|^2 \ \mathfrak{m}\text{-a.e.}$$

is a quadratic form, immediately implying the claim.

For the converse, assuming that $W^{1,2}(X)$ is a Hilbert space, or equivalently, that $L^2(T^*X)$ is a Hilbert module, Theorem 3.4.10 ensures the existence of a unique $\omega_v \in L^2(T^*X)$ with

$$v = \langle \omega_v, \cdot \rangle \quad \text{and} \quad |\omega_v| = |v| \overset{(4.3.3)}{=} |\mathrm{d}f| \tag{4.4.21}$$

for each $v \in \mathrm{Grad}(f)$. By Proposition 3.4.8, we further have

$$|\omega_v - \mathrm{d}f|^2 \overset{(3.4.10)}{=} |\omega_v|^2 + |\mathrm{d}f|^2 - 2\langle \omega_v, \mathrm{d}f \rangle \overset{(4.4.21)}{=} 2|\mathrm{d}f|^2 - 2\,\mathrm{d}f(v) \overset{(4.3.3)}{=} 0 \ \mathfrak{m}\text{-a.e.}$$

Thus, $\omega_v = \mathrm{d}f$ for all $v \in \mathrm{Grad}(f)$, implying infinitesimal strict convexity of $(X, \mathrm{d}, \mathfrak{m})$, as well as linearity of $f \mapsto \nabla f = \langle \mathrm{d}f, \cdot \rangle$. □

4.5 Maps of bounded deformation

In this final section, we introduce a class of maps preserving the notion of a test plan. By duality, as we will see, this ensures that along such maps we can pull back arbitrary Sobolev functions. Moreover, these maps induce a pullback of one-forms and, under the additional assumption of infinitesimal strict convexity of their codomain, posses a *unique differential*, completing our picture of a basic first-order calculus for metric measure spaces.

Lemma 4.5.1. *Let $\varphi\colon Y \to X$ be a continuous map between two metric measure spaces, let π be a test plan on Y and denote by $\hat{\varphi}\colon \mathcal{C}([0,1], Y) \to \mathcal{C}([0,1], X)$ the induced map $\gamma \mapsto \varphi \circ \gamma$. Then $\hat{\varphi}_*\pi$ is a test plan on X if we assume φ to be Lipschitz and of bounded compression.*

Proof. We need to establish the compression estimate as well as the energy condition for $\hat{\varphi}_*\pi$. For the first, notice that

$$(e_t)_*(\hat{\varphi}_*\pi) = \varphi_*((e_t)_*\pi) \overset{(2.1.1)}{\leq} \mathrm{Comp}(\pi)\varphi_*\mathfrak{m}_Y,$$

giving the desired estimate if $\varphi_*\mathfrak{m}_Y \leq C\mathfrak{m}_X$ for some $C > 0$, that is, if φ is of bounded compression. For the second, we observe that $\hat{\varphi}(\gamma) = \varphi \circ \gamma$ is an absolutely continuous curve in X for any absolutely continuous $\gamma\colon I \to Y$, as

$$\mathrm{d}_X((\varphi \circ \gamma)_s, (\varphi \circ \gamma)_t) \leq \mathrm{Lip}(\varphi)\,\mathrm{d}_X(\gamma_s, \gamma_t) \leq \mathrm{Lip}(\varphi) \int_s^t |\dot{\gamma}_r|\,\mathrm{d}r.$$

By Theorem 1.3.4, this immediately further implies

$$\mathsf{ms}(\varphi \circ \gamma)(t) \leq \mathrm{Lip}(\varphi)\mathsf{ms}(\gamma)(t) \quad \text{at any time } t \in [0,1] \tag{4.5.1}$$

for the respective metric speeds of these curves. Thus, invoking the integral transformation formula (1.3.2), we get

$$\begin{aligned}
\int_0^1\!\!\int |\dot{\gamma}_t|^2\,\mathrm{d}\hat{\varphi}_*\pi(\gamma)\,\mathrm{d}t = \int_0^1\!\!\int \mathsf{ms}^2(\gamma)(t)\,\mathrm{d}\hat{\varphi}_*\pi(\gamma)\,\mathrm{d}t &\overset{(1.3.2)}{=} \int_0^1\!\!\int \mathsf{ms}^2(\varphi\circ\gamma)(t)\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t \\
&\overset{(4.5.1)}{\leq} \mathrm{Lip}(\varphi)^2 \int_0^1\!\!\int \mathsf{ms}^2(\gamma)(t)\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t \\
&= \mathrm{Lip}(\varphi)^2 \int_0^1\!\!\int |\dot{\gamma}_t|^2\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t < \infty.
\end{aligned}$$

□

Definition 4.5.2. Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces and let $\varphi\colon Y \to X$ be a map between them. Then φ is said to be *of bounded deformation* if it is Lipschitz and of bounded compression.

Proposition 4.5.3. *Let $\varphi\colon Y \to X$ be a map of bounded deformation. Then along φ, any Sobolev function on X pulls back to a Sobolev function on Y. More specifically, $f \circ \varphi \in S^2(Y)$ for all $f \in S^2(X)$ and*

$$|D(f \circ \varphi)| \leq \mathrm{Lip}(\varphi)|Df| \circ \varphi \quad \mathfrak{m}_Y\text{-a.e.}$$

Proof. Recalling Proposition 2.3.3, it suffices to show that $\mathrm{Lip}(\varphi)|Df| \circ \varphi$ constitutes a weak upper gradient for $f \circ \varphi$. Since $\||Df| \circ \varphi\|_{L^2(\mathfrak{m}_X)} \leq \sqrt{\mathrm{Comp}(\varphi)}\||Df|\|_{L^2(\mathfrak{m}_Y)}$, cf. Example 3.3.10, it only remains to establish

$$\int |(f \circ \varphi)(\gamma_1) - (f \circ \varphi)(\gamma_0)| \,\mathrm{d}\pi(\gamma) \leq \mathrm{Lip}(\varphi) \int_0^1 \int (|Df| \circ \varphi)(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t.$$

This is achieved in duality to Lemma 4.5.1 by noticing that the left-hand side of this inequality (using again the integral transformation formula (1.3.2)) is exactly the function $|f \circ e_1 - f \circ e_0|$ integrated against the test plan $\hat{\varphi}_*\pi$. Now, by the definition of the Sobolev class,

$$\int |f \circ e_1 - f \circ e_0| \,\mathrm{d}\hat{\varphi}_*\pi \leq \int_0^1 \int |Df|(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\hat{\varphi}_*\pi(\gamma)\,\mathrm{d}t.$$

Undoing the integral transformation, as in the proof of Lemma 4.5.1, the right-hand side is finally seen to equal

$$\int_0^1 \int |Df|((\varphi \circ \gamma)_t)\mathsf{ms}(\varphi \circ \gamma)(t) \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t \overset{(4.5.1)}{\leq} \mathrm{Lip}(\varphi) \int_0^1 \int (|Df| \circ \varphi)(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t,$$

guaranteeing that $\mathrm{Lip}(\varphi)|Df| \circ \varphi \in L^2(\mathfrak{m})$ is a weak upper gradient for $f \circ \varphi$. □

The ability to pull back Sobolev functions further allows us to define a *pullback of one-forms*, as we know it from smooth differential geometry.

Theorem 4.5.4. *Any map $\varphi\colon Y \to X$ of bounded deformation induces a unique linear and continuous map $\varphi^*\colon L^2(T^*X) \to L^2(T^*Y)$ such that*

i) $\varphi^*\,\mathrm{d}f = \mathrm{d}(f \circ \varphi)$ *for any* $f \in S^2(X)$

ii) $\varphi^*(g\omega) = (g \circ \varphi)\,\varphi^*\omega$ *for all* $g \in L^\infty(\mathfrak{m}_X)$ *and* $\omega \in L^2(T^*X)$,

and which additionally satisfies

$$|\varphi^*\omega| \leq \mathrm{Lip}(\varphi)|\omega| \circ \varphi \;\mathfrak{m}_Y\text{-a.e.} \tag{4.5.2}$$

for every $\omega \in L^2(T^*X)$.

Proof. Using linearity together with items (i) and (ii), we see that such a map φ^* would have to satisfy

$$\varphi^*(\sum_i 1_{E_i}\,\mathrm{d}f_i) = \sum_i \varphi^*(1_{E_i}\,\mathrm{d}f_i) \overset{(ii)}{=} \sum_i (1_{E_i} \circ \varphi)\,\varphi^*\,\mathrm{d}f_i \overset{(i)}{=} \sum_i 1_{E_i} \circ \varphi\ \mathrm{d}(f_i \circ \varphi).$$

In order to define the map φ^*, we therefore (have to) set

$$\varphi^*\Big(\sum_i 1_{E_i}\,\mathrm{d}f_i\Big) := \sum_i 1_{E_i}\circ\varphi\;\mathrm{d}(f_i\circ\varphi). \tag{4.5.3}$$

Now, recalling that $1_{E_i}\circ\varphi = 1_{\varphi^{-1}(E_i)}$ and using Proposition 4.5.3, we can estimate

$$\begin{aligned}\Big|\sum_i 1_{E_i}\circ\varphi\;\mathrm{d}(f_i\circ\varphi)\Big| &= \sum_i 1_{\varphi^{-1}(E_i)}|\mathrm{d}(f_i\circ\varphi)|\\ &\le \mathrm{Lip}(\varphi)\sum_i 1_{\varphi^{-1}(E_i)}|\mathrm{d}f_i|\circ\varphi = \mathrm{Lip}(\varphi)\Big|\sum_i 1_{E_i}\,\mathrm{d}f_i\Big|\circ\varphi.\end{aligned}$$

This implies that the assignment (4.5.3) specifies a well-defined map, which can be extended linearly and continuously to all of $L^2(T^*X)$. By continuity of the multiplication and the pointwise norm, we now get $|\varphi^*\omega| \le \mathrm{Lip}(\varphi)|\omega|\circ\varphi$ for all $\omega\in L^2(T^*X)$ in the $\mathfrak{m}_Y$-almost everywhere sense, as in (3.3.7). To establish item (*ii*), suppose first that $g=\sum\alpha_j 1_{F_j}$ is a simple function and compute, for a simple one-form $\omega=\sum 1_{E_i}\,\mathrm{d}f_i$,

$$\varphi^*(g\omega) = \varphi^*\Big(\sum_{i,j} 1_{E_i\cap F_j}\,\mathrm{d}(\alpha_j f_i)\Big) \overset{(4.5.3)}{=} \sum_j \alpha_j 1_{F_j}\circ\varphi\sum_i 1_{E_i}\circ\varphi\,\mathrm{d}(f_i\circ\varphi) = (g\circ\varphi)\varphi^*\omega.$$

In the general case, approximate $g\in L^\infty(\mathfrak{m}_X)$ by simple functions and $\omega\in L^2(T^*X)$ by simple one-forms and then use continuity of the multiplication and of φ^*. $\square$

Thus, we hereby have at our disposal two different notions of a pullback of one-forms. To distinguish notations, we will henceforth denote the pullback map of the module $L^2(T^*X)$, as defined in Theorem 3.3.8, by $\omega\mapsto[\varphi^*\omega]\in\varphi^*L^2(T^*X)$.

Now, in smooth differential geometry, given any smooth map $\varphi\colon M\to N$ between two manifolds, we can characterize its differential $\mathrm{d}\varphi$ as the unique section of the pullback bundle φ^*TN satisfying

$$[\varphi^*\omega](\mathrm{d}\varphi(X)) = (\varphi^*\omega)(X) \tag{4.5.4}$$

for all vector fields X on M and all one-forms ω on N. This definition via bundles will serve us as inspiration for the notion of a differential in our setting of metric measure spaces.

Theorem 4.5.5. *Any map $\varphi: Y \to X$ of bounded deformation into an infinitesimally strictly convex metric measure space $(X, \mathrm{d}_X, \mathfrak{m}_X)$ possesses a unique differential $\mathrm{d}\varphi$. That is, for any such φ, there exists a unique module homomorphism $\mathrm{d}\varphi: L^2(TY) \to \varphi^* L^2(TX)$ such that*

$$[\varphi^*\omega](\mathrm{d}\varphi(v)) = (\varphi^*\omega)(v) \quad \text{for every } v \in L^2(TY), \omega \in L^2(T^*X). \tag{4.5.5}$$

Moreover, the map $\mathrm{d}\varphi$ satisfies

$$|\mathrm{d}\varphi(v)| \leq \mathrm{Lip}(\varphi)|v| \ \mathfrak{m}_Y\text{-a.e.}$$

for every $v \in L^2(TY)$.

Remark 4.5.6. Our choice of notation in (4.5.5) is motivated by a desire to emphasize the similarity to the classical situation, cf. (4.5.4). However, the assertion should be stated more explicitly as

$$I(\mathrm{d}\varphi(v))([\varphi^*\omega]) = \varphi^*\omega(v) \quad \text{for every } v \in L^2(TY), \omega \in L^2(T^*X), \tag{4.5.6}$$

where $I: \varphi^* L^2(TX) \to (\varphi^* L^2(T^*X))^*$ is the map defined in (3.3.18). Thus, the action of $\mathrm{d}\varphi(v)$ is only specified implicitly via the embedding $\varphi^* L^2(TX) \hookrightarrow (\varphi^* L^2(T^*X))^*$. To be able to identify this differential uniquely, we therefore need a right inverse to this embedding, which is the reason for our additional assumption of infinitesimal strict convexity.

Proof. We begin by arguing that the map $[\varphi^*\omega] \mapsto \varphi^*\omega(v)$ for some fixed $v \in L^2(TY)$ is well-defined and that it can be extended from the generating subspace $\{[\varphi^*\omega] \mid \omega \in L^2(T^*X)\}$ to the whole pullback module. For this, we use the bound (4.5.2) and item (*ii*) in Theorem 3.3.8 to deduce

$$|\varphi^*\omega(v)| \overset{(3.3.4)}{\leq} |\varphi^*\omega||v| \overset{(4.5.2)}{\leq} \mathrm{Lip}(\varphi)|\omega| \circ \varphi|v| \overset{(3.3.9)}{=} \mathrm{Lip}(\varphi)|v||[\varphi^*\omega]| \ \mathfrak{m}_Y\text{-a.e.} \tag{4.5.7}$$

This establishes well-definedness and guarantees continuity (by taking norms on both sides). Together with linearity, these conditions allow us to apply Proposition 3.3.6, yielding an extension $L_v: \varphi^* L^2(T^*X) \to L^1(\mathfrak{m}_Y)$ of our map $[\varphi^*\omega] \mapsto \varphi^*\omega(v)$. As, by assumption, $(X, \mathrm{d}_X, \mathfrak{m}_X)$ is infinitesimally strictly convex, $L^2(T^*X)$ is a Hilbert module (see Theorem 4.4.5). Hence, Proposition 3.4.11 implies the existence of a unique element $\mathrm{d}\varphi(v) \in \varphi^* L^2(TX)$ with the property that

$$[\varphi^*\omega](\mathrm{d}\varphi(v)) = L_v([\varphi^*\omega]) = \varphi^*\omega(v),$$

since, denoting by $J: (\varphi^* L^2(T^*X))^* \to \varphi^* L^2(TX)$ the right inverse of the embedding $I: \varphi^* L^2(TX) \to (\varphi^* L^2(T^*X))^*$ as in (3.4.18), we have

$$[\varphi^*\omega](\mathrm{d}\varphi(v)) \overset{(4.5.6)}{=} I(\mathrm{d}\varphi(v))([\varphi^*\omega]) = I(J(L_v))([\varphi^*\omega]) = L_v([\varphi^*\omega]).$$

Moreover, since J is the right inverse of an isometric embedding, we know that

$$|\mathrm{d}\varphi(v)| = |L_v|_* \overset{(4.5.7)}{\leq} \mathrm{Lip}(\varphi)|v|$$

Therefore, Proposition 3.2.3 finally implies that the $\mathbb{R}$-linear map $v \mapsto \mathrm{d}\varphi(v)$ is, in fact, a module homomorphism. □

As is also the case in smooth differential geometry, this differential $\mathrm{d}\varphi$ can be viewed as a map between the spaces of vector fields $L^2(TY)$ and $L^2(TX)$ themselves if we assume $\varphi\colon Y \to X$ to be an isomorphism. In fact, as we already know that $L^2(TX) \cong (\varphi^{-1})^*(\varphi^* L^2(TX))$ by functoriality, cf. Theorem 3.3.13, in this case we can identify $L^2(TX)$ with $\varphi^* L^2(TX)$ by relating the rings $L^\infty(\mathfrak{m}_X)$ and $L^\infty(\mathfrak{m}_Y)$ via $f \mapsto f \circ \varphi$,

Theorem 4.5.7. *Let $\varphi\colon Y \to X$ be an invertible map of bounded compression such that $\varphi^{-1}\colon X \to Y$ is also of bounded compression. Then its differential can equivalently be expressed as an $\mathbb{R}$-linear and continuous map $\mathrm{d}\varphi\colon L^2(TY) \to L^2(TX)$ such that*

$$\omega(\mathrm{d}\varphi(v)) = (\varphi^*\omega(v)) \circ \varphi^{-1} \ \mathfrak{m}_X\text{-a.e. for every } v \in L^2(TY), \omega \in L^2(T^*X).$$

This map further satisfies the bound

$$|\mathrm{d}\varphi(v)| \leq \mathrm{Lip}(\varphi)|v| \circ \varphi^{-1} \ \mathfrak{m}_X\text{-a.e.} \tag{4.5.8}$$

for any $v \in L^2(TY)$.

Proof. For $v \in L^2(TY)$, we let $\mathrm{d}\varphi(v)\colon L^2(T^*X) \to L^1(\mathfrak{m}_X)$, $\omega \mapsto (\varphi^*\omega(v)) \circ \varphi^{-1}$. Then, since

$$\begin{aligned} |\omega(\mathrm{d}\varphi)(v)| = |\varphi^*\omega(v)| \circ \varphi^{-1} \overset{(3.3.4)}{\leq} (|\varphi^*\omega||v|) \circ \varphi^{-1} &\overset{(4.5.7)}{\leq} (\mathrm{Lip}(\varphi)|\omega| \circ \varphi|v|) \circ \varphi^{-1} \\ &= \ \mathrm{Lip}(\varphi)|\omega|(|v| \circ \varphi^{-1}), \end{aligned}$$

we can again apply Proposition 3.2.3 to conclude that $\mathrm{d}\varphi(v)$ is $L^\infty(\mathfrak{m}_X)$-linear and hence $\mathrm{d}\varphi(v) \in L^2(TX)$. Taking the supremum over all $\omega \in L^2(T^*X)$ with $|\omega| \leq 1$ in the above estimate also gives the bound (4.5.8). □

Bibliography

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below.* 2011. URL: `https://arxiv.org/abs/1106.2090`.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces.* 2011. URL: `https://arxiv.org/abs/1111.3730`.

[3] Thierry Aubin. *Nonlinear Analysis on Manifolds. Monge-Ampère Equations.* Springer New York, NY, 2012. ISBN: 978-1-4612-5734-9.

[4] Colin Bennett and Robert C. Sharpley. *Interpolation of Operators.* Academic Press, 1988, 1988. ISBN: 978-0-12-088730-9.

[5] Vladimir I. Bogachev. *Measure Theory.* Springer Berlin, Heidelberg, 2007. ISBN: 978-3-540-34514-5.

[6] Vladimir I. Bogachev and Oleg G. Smolyanov. *Real and Functional Analysis.* Springer Cham, 2020. ISBN: 978-3-030-38219-3.

[7] Mario Bonk, Luca Capogna, Piotr Hajłasz, Nageswari Shanmugalingam, and Jeremy Tyson. *Analysis on metric spaces.* 2019. URL: `https://arxiv.org/pdf/1911.00430.pdf`.

[8] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer New York, NY, 2010. ISBN: 978-0-387-70913-0.

[9] Donald L. Cohn. *Measure Theory.* Birkhäuser New York, NY, 2013. ISBN: 978-1-4614-6956-8.

[10] Joseph Diestel and John Jerry Uhl. *Vector Measures.* American Mathematical Society 1977, 1977. ISBN: 978-0821815151.

[11] Nicola Gigli. *Nonsmooth differential geometry - An approach tailored for spaces with Ricci curvature bounded from below.* 2014. URL: `https://arxiv.org/abs/1407.0809`.

[12] Nicola Gigli. *On the differential structure of metric measure spaces and applications.* 2014. URL: `https://arxiv.org/abs/1205.6622`.

[13] Nicola Gigli and Enrico Pasqualetto. *Lectures on Nonsmooth Differential Geometry.* Springer Cham, 2020. ISBN: 978-3-030-38613-9.

[14] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer Berlin, Heidelberg, 2015. ISBN: 978-3-642-61798-0.

[15] Batu Güneysu. *Covariant Schrödinger Semigroups on Riemannian Manifolds*. Birkhäuser Cham, 2019. ISBN: 978-3-319-68903-6.

[16] Juha Heinonen. *Lectures on Lipschitz Analysis*. 2005. URL: `http://www.math.jyu.fi/research/reports/rep100.pdf`.

[17] Juha Heinonen. "Nonsmooth Calculus". In: *Bulletin of the American Mathematical Society* 44 (2) (2007), pp. 163–232. URL: `https://community.ams.org/journals/bull/2007-44-02/S0273-0979-07-01140-8/S0273-0979-07-01140-8.pdf`.

[18] Juha Heinonen and Pekka Koskela. "Quasiconformal maps in metric spaces with controlled geometry". In: *Acta Math.* 181(1) (1998), pp. 1–61. DOI: `10.1007/BF02392747`.

[19] Dale Husemoller. *Fibre Bundles*. Springer-Verlag New York 1994, 2013. ISBN: 978-1-4757-2261-1.

[20] John M. Lee. *Introduction to Smooth Manifolds*. Springer Science+Business Media New York 2012, 2012. ISBN: 978-1-4419-9982-5.

[21] Robert E. Megginson. *An Introduction to Banach Space Theory*. Springer Science+Business Media New York 1998, 2012. ISBN: 978-1-4612-0603-3.

[22] Emily Riehl. *Category Theory in Context*. Cambridge University Press, 2014.

[23] Gerald Teschl. *Topics in Real Analysis*. American Mathematical Society, 2020.

[24] William P. Ziemer. *Weakly Differentiable Functions*. Springer New York, NY, 2012. ISBN: 978-1-4612-1015-3.