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Spaces of smooth functions of Denjoy-Carleman-type

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1 Introduction

Function spaces of Denjoy-Carleman-type are special classes of smooth functions. These classes consist of functions which satisfy with all their derivatives certain estimates. For each sequence of positive real numbers $M := (M_p)_p$, which we will call a weight sequence, we can define now a Denjoy-Carleman-class of smooth functions. Therefore we distinguish two different classes: Functions of Romieu-type $\mathcal{E}_{\{M\}}$ and of Beurling-type $\mathcal{E}_{(M)}$.

The aim is now to characterize important and desirable properties of these classes, like closedness under pointwise multiplication, composition and inversion and the closedness under solving ODE's, in terms of the weight sequence. Furthermore we study the injectivity and the surjectivity of the Borel-mapping.

Afterwards we construct concrete examples and apply the proven results to them. So we can illustrate the properties of the function spaces and their dependence on a parameter. Furthermore we prove an interesting decomposition theorem for smooth functions.

In the last chapter we finally introduce an alternative method to define spaces of weighted functions: Instead of the discrete variant which uses the weight sequence M one can consider a continuous one, in particular one considers a weight function ω to define spaces $\mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(\omega)}$. We remark that in [3] a connection is established between the definition via weight functions and decreasing properties of the Fourier-transformation of smooth functions with compact support. We compare both useful ways and prove, in which cases both variants lead to the same function space. In particular we show in which cases $\mathcal{E}_{\{M\}} \cong \mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(M)} \cong \mathcal{E}_{(\omega)}$ is satisfied.

2 Basic Definitions

Notations: With $\log(\cdot)$ we will always denote the natural logarithm. We set $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$ and analogously $\mathbb{R}_{\geq 0}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we will write $\alpha! := \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$, $|\alpha| := \sum_{i=1}^n \alpha_i$ and, if f is a smooth function, then we set $f^{(\alpha)}(x_1, \dots, x_n) := \partial^\alpha f(x_1, \dots, x_n) = \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Furthermore we introduce the binomial coefficient for multiindices:

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ where $\beta \leq \alpha$, which means $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n$, then we put $\binom{\alpha}{\beta} := \prod_{i=1}^n \binom{\alpha_i}{\beta_i} = \prod_{i=1}^n \frac{\alpha_i!}{\beta_i! (\alpha_i - \beta_i)!} = \frac{\alpha!}{\beta! (\alpha - \beta)!}$. If $\beta > \alpha$, then we set $\binom{\alpha}{\beta} = 0$.

With $\|\cdot\|_\infty$ we will always denote the supremum norm.

Let $\mathcal{E}(G)$ be the set of all real-valued smooth functions on an open set G in $\mathbb{R}^n$ and let $\mathcal{E}(K)$ be the set of all smooth real-valued functions on K° with the following property: $f^{(\alpha)}$ can be continuously extended to $K = \bar{K}^\circ$ for all $\alpha \in \mathbb{N}^n$.

An arbitrary sequence of positive real numbers $M := (M_p)_p$ will always be called a *weight sequence*. Let $(M_p)_p$ be now a weight sequence, then we define the space $\mathcal{E}_{M,h}(K)$ for $h > 0$ as follows:

$$\mathcal{E}_{M,h}(K) := \{f \in \mathcal{E}(K) : |f|_{K,h} < \infty\}$$

where we have set

$$|f|_{K,h} := \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot M_{|\alpha|}}.$$

$\mathcal{E}_{M,h}(K)$ is clearly a *Banach-space* and an easy consequence is: If $f \in \mathcal{E}_{M,h}(K)$, then $f^{(\alpha)} \in \mathcal{E}_{M+|\alpha|,h}(K)$, where $M + |\alpha| := M_{p+|\alpha|}$ for all $p \in \mathbb{N}$.

In the following let $G \subseteq \mathbb{R}^n$ be an open connected subset. First we define the space $\mathcal{E}_{\{M\}}(G)$:

$$\mathcal{E}_{\{M\}}(G) := \{f \in \mathcal{E}(G) : \forall K \subset G \text{ compact } \exists h > 0 : |f|_{K,h} < \infty\}.$$

If $f \in \mathcal{E}_{\{M\}}(G)$, then we call f *ultradifferentiable of Romieu-type* on G .

We obtain the locally convex topology on this vectorspace via the representation

$$\mathcal{E}_{\{M\}}(G) = \varprojlim_K \varinjlim_h \mathcal{E}_{M,h}(K). \quad (2.0.1)$$

Sometimes it is useful to put $\mathcal{E}_{\{M\},K} := \varinjlim_h \mathcal{E}_{M,h}(K)$.

Further we define the space $\mathcal{E}_{(M)}(G)$:

$$\mathcal{E}_{(M)}(G) := \{f \in \mathcal{E}(G) : \forall K \subset G \text{ compact } \forall h > 0 : |f|_{K,h} < \infty\}.$$

If $f \in \mathcal{E}_{(M)}(G)$, then we call f *ultradifferentiable of Beurling-type* on G . Here we have the representation

$$\mathcal{E}_{(M)}(G) = \varprojlim_K \varprojlim_h \mathcal{E}_{M,h}(K), \quad (2.0.2)$$

and we set $\mathcal{E}_{(M),K} := \varprojlim_h \mathcal{E}_{M,h}(K)$. By (2.0.2) it follows that $\mathcal{E}_{(M)}(G)$ is a Fréchet-space.

We will often write only $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ if G is not specified. With $\mathcal{O}(G)$ we denote the set of all real analytic functions on G and we have immediately $\mathcal{O} = \mathcal{E}_{\{(p!)_p\}}$.

Now we consider the case $\mathcal{E}_{((p!)_p)}$. We see, that $\mathcal{E}_{((p!)_p)}$ is the space of all real and imaginary parts of entire functions. This holds because we have by definition for $f \in \mathcal{E}_{((p!)_p)}$: $\frac{1}{|f^{(p)}(x)/p!|} \geq \frac{1}{C^{1/p} \cdot h}$ has to be valid for all $p \in \mathbb{N}$ and $h > 0$, thus the radius of convergence for f is infinity.

Note that the condition $|f|_{K,h} < \infty$ is equivalent to $|f|_{K,h} \leq C$ for a constant $C > 0$ and it has the following consequence: The constant h is more important than C , because

2 Basic Definitions

$\lim_{|\alpha| \rightarrow \infty} C^{1/|\alpha|} = 1$, hence

$$|f|_{K,h} < \infty \iff \limsup_{|\alpha| \rightarrow \infty} \left(\frac{\|f^{(\alpha)}|_K\|_\infty}{M_{|\alpha|}} \right)^{1/|\alpha|} \leq h.$$

In the Romieu-case we have: If $f \in \mathcal{E}_{\{M\}}(G)$, then for all $K \subseteq G$ compact there exists a constant $h > 0$ depending on K such that $\limsup_{|\alpha| \rightarrow \infty} \left(\frac{\|f^{(\alpha)}|_K\|_\infty}{M_{|\alpha|}} \right)^{1/|\alpha|} \leq h$.

In the Beurling-case we obtain: If $f \in \mathcal{E}_{(M)}(G)$, then for all $K \subseteq G$ compact

$$\limsup_{|\alpha| \rightarrow \infty} \left(\frac{\|f^{(\alpha)}|_K\|_\infty}{M_{|\alpha|}} \right)^{1/|\alpha|} = 0$$

has to be satisfied because the estimate has to hold for all $h > 0$. Note that in the Romieu-case the constants C and h depend on K , in the Beurling-case C depends on K and h . An immediate consequence by the definition of the spaces is

$$\boxed{\mathcal{E}_{(M)} \subseteq \mathcal{E}_{\{M\}} \text{ for all sequences } (M_p)_p,}$$

because on K compact take a $h > 0$ and $C_h = C$ in the estimate.

We remark that one can always assume $M_0 = 1$, because $\mathcal{E}_{\{M\}} = \mathcal{E}_{\{C \cdot M\}}$ resp. $\mathcal{E}_{(M)} = \mathcal{E}_{(C \cdot M)}$, where $C \cdot M := (C \cdot M_p)_p$ for a constant $C > 0$. Furthermore one has $\mathcal{E}_{\{M\}} = \mathcal{E}_{\{M^D\}}$ resp. $\mathcal{E}_{(M)} = \mathcal{E}_{(M^D)}$, where $M^D = (M_p^D)_p$ and $M_p^D := D^p \cdot M_p$ for an arbitrary constant $D > 0$ and all p . This holds because: Take $h > 0$ arbitrary, then the constant $C_{h \cdot D}$ in the estimate of a function f in $\mathcal{E}_{(M)}$ is the constant C_h in the estimate for f in $\mathcal{E}_{(M^D)}$.

A further consequence is that the spaces $\mathcal{E}_{(M)}$ resp. $\mathcal{E}_{\{M\}}$ are never empty and are infinite dimensional because both cases contain the polynomials:

$$\limsup_{|\alpha| \rightarrow \infty} \left(\frac{\|p^{(\alpha)}|_K\|_\infty}{M_{|\alpha|}} \right)^{1/|\alpha|} = 0$$

is clearly satisfied for all polynomials p and K compact, hence the infinite dimensional vector space of all polynomials is contained in $\mathcal{E}_{(M)}$. We are testing now further "well-known" functions:

The exponential function $\exp : x \rightarrow e^x$ is an element in $\mathcal{E}_{\{M\}}(\mathbb{R})$ if $M_p \geq 1$ for all $p \in \mathbb{N}$: Take $h := 1$ and $C := \max_{x \in K} e^x = e^b$ if $K = [a, b]$. But for $\exp \in \mathcal{E}_{(M)}(\mathbb{R})$ one needs $\limsup_{p \rightarrow \infty} \left(\frac{\|e^x|_K\|_\infty}{M_p} \right)^{1/p} = 0$ for all compact sets K , in particular $\lim_{p \rightarrow \infty} \left(\frac{1}{M_p} \right)^{1/p} = 0$ has to hold, which means: The sequence $(M_p)_p$ has to tend very fast to infinity.

In fact we have the same situation for the trigonometric functions $\sin$ and $\cos$, where we can take in the Romieu-case $C = h = 1$.

In particular for $\exp \in \mathcal{E}_{\{M\}}(\mathbb{R})$ it is sufficient to have $\sup_p \frac{e^b}{M_p} \leq C$, where C is depending on the interval $K = [a, b]$. For $\sin \in \mathcal{E}_{\{M\}}(\mathbb{R})$ it is sufficient to have $\sup_p \frac{1}{M_p} \leq C$.

Summarizing one can say for the Beurling-case: If the sequence $(M_p)_p$ is not or not fast enough increasing, then $\mathcal{E}_{(M)}$ contains only functions with sufficiently small high order derivatives. If $\mathcal{E}_{(M)}$ shall contain functions with no small high order derivatives, like the exponential function, then $(M_p)_p$ has to tend very fast to infinity.

The following useful notations will be used:

$$\boxed{m_p := \frac{M_p}{p!} \quad \mu_p := \frac{M_p}{M_{p-1}} = p \cdot \frac{m_p}{m_{p-1}},}$$

and so we obtain important new sequences $m := (m_p)_p$ and $\mu := (\mu_p)_p$. Remark that μ_p makes only sense for $p \geq 1$, but sometimes the convention $\mu_0 := 1$ is used. If the sequence μ is given

with $M_0 = 1$, one can compute M via

$$M_p = \prod_{i=1}^p \mu_i.$$

In particular we see: The property $\mu_p \geq 1$ for all p is clearly equivalent to the fact that $(M_p)_p$ is an increasing sequence.

Applying this notation we can write

$$|f|_{K,h} = \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot |\alpha|! \cdot m_{|\alpha|}}.$$

We show now that we can replace $|\alpha|!$ by $\alpha!$ in the definition of $|f|_{K,h}$ above without changing the associated function space:

Lemma 2.0.1 *We get the following implications:*

$$\begin{aligned} \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot |\alpha|! \cdot m_{|\alpha|}} < \infty &\Rightarrow \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{(n \cdot h)^{|\alpha|} \cdot \alpha! \cdot m_{|\alpha|}} < \infty \text{ and} \\ \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot \alpha! \cdot m_{|\alpha|}} < \infty &\Rightarrow \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot |\alpha|! \cdot m_{|\alpha|}} < \infty. \end{aligned}$$

Proof. We write $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and conclude: For $n = 1$ we get clearly $\alpha_1! \leq |\alpha_1|!$. For $n - 1 \mapsto n$ we set $A := \sum_{i=1}^{n-1} \alpha_i$ and estimate:

$$\alpha! = \prod_{i=1}^n \alpha_i! \leq \underbrace{A! \cdot \alpha_n!}_{\text{I.H.}} \leq A! \cdot (A+1) \cdots (A+\alpha_n) = (A+\alpha_n)! = |\alpha|!$$

which shows $\alpha! = \prod_{i=1}^n \alpha_i! \leq (\sum_{i=1}^n \alpha_i)! = |\alpha|!$.

On the other side $\binom{|\alpha|}{\alpha_1, \dots, \alpha_n} \leq n^{|\alpha|}$ is satisfied by the *multinomial theorem*, thus $|\alpha|! \leq n^{|\alpha|} \cdot \alpha!$ is satisfied. □

We introduce now the very important notion of *quasi-analyticity*: Let $\mathcal{A}(G)$ be a subspace of $\mathcal{E}(G)$ and put $j_x^\infty : \mathcal{A}(G) \rightarrow \mathbb{R}^{\mathbb{N}^n}$, where

$$j_x^\infty(f) := (f^{(\alpha)}(x))_\alpha, \alpha \in \mathbb{N}^n.$$

The mapping j_x^∞ is called the *Borel mapping* at the point x . We call $\mathcal{A}(G)$ *quasi-analytic*, if j_x^∞ is injective for each $x \in G$. This can be interpreted as follows: If $f \in \mathcal{A}(G)$ is given such that $f^{(\alpha)}(0) = 0$ for all $\alpha \in \mathbb{N}^n$, then we obtain $f = 0$. We will always write j^∞ instead of j_0^∞ .

For a sequence $x := (x_\alpha)_\alpha, x_\alpha \in \mathbb{C}^n$, we put

$$|x|_h := \sup_{\alpha \in \mathbb{N}^n} \frac{|x_\alpha|}{h^{|\alpha|} \cdot M_{|\alpha|}}$$

and $\Lambda_{M,h}^n := \{x = (x_\alpha)_\alpha : x_\alpha \in \mathbb{C}^n, |x|_h < \infty\}$. Furthermore we set

$$\Lambda_{\{M\}}^n := \{x = (x_\alpha)_\alpha : x_\alpha \in \mathbb{C}^n, \exists h > 0 : |x|_h < \infty\},$$

and

$$\Lambda_{(M)}^n := \{x = (x_\alpha)_\alpha : x_\alpha \in \mathbb{C}^n, \forall h > 0 : |x|_h < \infty\}.$$

With these definitions we get immediately $j_x^\infty : \mathcal{E}_{\{M\}}(G) \rightarrow \Lambda_{\{M\}}^n$ resp. $j_x^\infty : \mathcal{E}_{(M)}(G) \rightarrow \Lambda_{(M)}^n$ for all $x \in G$. In the case $n = 1$ we will always write $\Lambda_{\{M\}}$ resp. $\Lambda_{(M)}$.

We point out that in the literature sometimes the following notation is used: Denote by $\mathcal{F}(\mathbb{R}^n)$ the set of all formal power series in $\mathbb{R}^n$, then one can consider the Borel-map $\hat{j}_x^\infty : \mathcal{A}(G) \rightarrow$

$\mathcal{F}(\mathbb{R}^n)$, where $\tilde{j}_x^\infty(f) := \left(\frac{f^{(\alpha)}(x)}{\alpha!} \right)_\alpha$, $\alpha \in \mathbb{N}^n$. Now one can define analogously weighted power series spaces $\mathcal{F}_{\{M\}}(\mathbb{R}^n)$ and $\mathcal{F}_{(M)}(\mathbb{R}^n)$:

$$\mathcal{F}_{\{M\}}(\mathbb{R}^n) := \{F = (F_\alpha)_\alpha \in \mathcal{F}(\mathbb{R}^n) : \exists C, h > 0, \forall \alpha \in \mathbb{N}^n : |F_\alpha| \leq C \cdot h^{|\alpha|} \cdot m_{|\alpha|}\} \text{ and}$$

$$\mathcal{F}_{(M)}(\mathbb{R}^n) := \{F = (F_\alpha)_\alpha \in \mathcal{F}(\mathbb{R}^n) : \forall h > 0 \exists C > 0, \forall \alpha \in \mathbb{N}^n : |F_\alpha| \leq C \cdot h^{|\alpha|} \cdot m_{|\alpha|}\}.$$

Hence we see, using 2.0.1 and this notation, that $\tilde{j}_x^\infty : \mathcal{E}_{\{M\}}(G) \rightarrow \mathcal{F}_{\{M\}}(\mathbb{R}^n)$ and $\tilde{j}_x^\infty : \mathcal{E}_{(M)}(G) \rightarrow \mathcal{F}_{(M)}(\mathbb{R}^n)$ is satisfied for all $x \in G$.

We remark that $\tilde{j}_x^\infty = i \circ j_x^\infty$ holds for all $x \in G$, where $i : \mathbb{R}^{\mathbb{N}^n} \rightarrow \mathcal{F}(\mathbb{R}^n)$ is given by $(x_\alpha)_\alpha \mapsto \left(\frac{x_\alpha}{\alpha!} \right)_\alpha$, $\alpha \in \mathbb{N}^n$, hence i is clearly an isomorphism and both notations are equivalent.

To study properties of the spaces $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ we have to introduce conditions for the weight sequences. The most important condition is the *logarithmic convexity*. A weight sequence $(M_k)_k$ is called *logarithmic convex* if the following holds:

$$2 \cdot \log(M_k) \leq \log(M_{k+1}) + \log(M_{k-1}) \quad \forall k. \quad (2.0.3)$$

We immediately see that (2.0.3) is equivalent to the condition

$$\boxed{M_k^2 \leq M_{k-1} \cdot M_{k+1}, \quad \forall k.}$$

The logarithmic convexity of a weight sequence is a very important condition and it will have a lot of consequences for the associated function spaces. Another interpretation is the following: Let $(M_k)_k$ be a given logarithmic convex weight sequence, then we can consider the sequence of points $(P_k)_k$, where $P_k := (k, \log(M_k))$. If we connect these points with straight lines, we obtain a convex polygon.

Example 2.0.2 Obviously the sequence $(k)_k$ is not log. convex, because the graph of the logarithm is not a convex function, or formally $k^2 \leq (k-1) \cdot (k+1) \Leftrightarrow 0 \leq -1$ is never satisfied. If one takes $(e^p)_p$, then $(e^p)^2 \leq e^{p-1} \cdot e^{p+1} \Leftrightarrow 2p \leq p-1 + p+1 = 2p$ holds clearly, hence $(e^p)_p$ is log. convex.

The sequence $(p!)_p$, which is used to define $\mathcal{O}$ is log. convex, because $(p!)^2 \leq (p-1)! \cdot (p+1)! \Leftrightarrow p \leq p+1$, which is obviously satisfied for all $p \in \mathbb{N}$.

Remark 2.0.3 1. If the sequence $m := (m_k)_k$ satisfies the log. convexity condition, then also the sequence $M := (M_k)_k$, because:

$$\begin{aligned} m_k^2 \leq m_{k-1} \cdot m_{k+1} &\stackrel{\text{Def.}}{\Leftrightarrow} \left(\frac{M_k}{k!} \right)^2 \leq \left(\frac{M_{k-1}}{(k-1)!} \right) \cdot \left(\frac{M_{k+1}}{(k+1)!} \right) \\ &\Leftrightarrow \frac{(k-1)! \cdot (k+1)!}{k!^2} \cdot M_k^2 \leq M_{k-1} \cdot M_{k+1} \\ &\Leftrightarrow \underbrace{\frac{k+1}{k}}_{\geq 1} \cdot M_k^2 \leq M_{k-1} \cdot M_{k+1} \Rightarrow M_k^2 \leq M_{k-1} \cdot M_{k+1}. \end{aligned}$$

Therefore the log. convexity condition of $(m_k)_k$ is often called *strong log. convexity*, (2.0.3) for the sequence $(M_k)_k$ is called *weak log. convexity*.

2. The log. convexity condition for the sequence $(M_k)_k$ is clearly equivalent to the property that the sequence $(\mu_k)_k$ is increasing:

$$\frac{\mu_{k+1}}{\mu_k} = \frac{M_{k+1} \cdot M_{k-1}}{M_k^2} \geq 1 \quad \forall k \Leftrightarrow (2.0.3) \text{ holds for } (M_k)_k.$$

In particular we see: If $(M_k)_k$ is log. convex and $\mu_1 = \frac{M_1}{M_0} \geq 1$ holds, then $\mu_k \geq 1$ for all k , hence $(M_k)_k$ is an increasing sequence, too.

Assuming (2.0.3), we prove now some useful easy consequences.

Lemma 2.0.4 For a given weight sequence $M := (M_k)_k$, where $M_0 = 1$, we have the following property: If M is log. convex. then $(M_k^{\frac{1}{k}})_k$ is an increasing sequence.

Proof. First we remark that the monotonicity condition of $(M_k^{\frac{1}{k}})_k$ is, after applying log, equivalent to

$$\log(M_{k+1}) \geq \frac{k+1}{k} \cdot \log(M_k), \forall k \in \mathbb{N} \setminus \{0\}.$$

Now we use induction on k :

For $k = 1$ we have by (2.0.3):

$$\log(M_2) + \underbrace{\log(M_0)}_{=0} \geq 2 \cdot \log(M_1) \implies \log(M_2) \geq 2 \cdot \log(M_1).$$

And for $k \mapsto k+1$:

$$\begin{aligned} 2 \cdot \log(M_k) &\leq \log(M_{k+1}) + \log(M_{k-1}) \underbrace{\leq}_{I.H.} \log(M_{k+1}) + \frac{k-1}{k} \cdot \log(M_k) \\ \implies \log(M_{k+1}) &\geq \frac{2k-k+1}{k} \cdot \log(M_k) = \frac{k+1}{k} \cdot \log(M_k), \forall k \in \mathbb{N} \setminus \{0\}. \end{aligned}$$

□

Remark 2.0.5 Lemma 2.0.4 implies

1. $\log(M_{k+1}) \geq \frac{k+1}{k} \cdot \log(M_k) \geq \log(M_k)$ for all k , hence $(M_k)_k$ is increasing for all k .
2. For any $p \geq q \geq 1$ we have:

$$\begin{aligned} \log(M_p) &\geq \frac{p}{p-1} \cdot \log(M_{p-1}) \geq \frac{p}{p-1} \cdot \frac{p-1}{p-2} \cdot \log(M_{p-2}) \geq \dots \\ &\geq \frac{p}{p-1} \cdot \frac{p-1}{p-2} \dots \frac{q+1}{q} \cdot \log(M_q) = \frac{p}{q} \cdot \log(M_q). \end{aligned}$$

Lemma 2.0.6 For a given weight sequence $M := (M_k)_k$, where $M_0 = 1$, we have the following property: If M is log. convex then $M_l \cdot M_k \leq M_{l+k}$ for all $l, k \in \mathbb{N}$.

Proof. We use induction on k :

$$M_l \cdot \underbrace{M_0}_{=1} = M_l \leq M_{l+0}, \forall l \in \mathbb{N}.$$

$\frac{M_{k+1}}{M_k}$ is an increasing sequence because M is logarithmic convex, and so:

$$\begin{aligned} M_l \cdot M_k &= M_l \cdot M_{k-1} \cdot \frac{M_k}{M_{k-1}} \underbrace{\leq}_{I.H.} \frac{M_k}{M_{k-1}} \cdot M_{l+k-1} \\ &\leq \frac{M_{k+1}}{M_k} \cdot M_{l+k-1} \leq \dots \leq \frac{M_{k+l}}{M_{l+k-1}} \cdot M_{l+k-1} = M_{k+l}, \forall l \in \mathbb{N}. \end{aligned}$$

□

Lemma 2.0.7 [16, 2.9. Lemma] Let $M := (M_k)_k$ be a log. conv. weight sequence such that $M_0 = 1$. Then we have the following inequality:

$$M_1^k \cdot M_k \geq M_j \cdot M_{\alpha_1} \dots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N} \text{ such that } \sum_{i=1}^j \alpha_i = k.$$

Proof. We prove this statement again with induction on k . We have $M_1^k \cdot M_k \geq M_k \cdot \underbrace{(M_1 \cdots M_1)}_{k\text{-times}}$ for the case $k = j$ (if $k = 0$ then $1 \cdot 1 \geq 1 \cdot 1$). If $j < k$ then we can find an index i such that $\alpha_i \geq 2$ and we set $\alpha'_i := \alpha_i - 1$. For this we get by I.H.:

$$M_j \cdot M_{\alpha_1} \cdots M_{\alpha'_i} \cdots M_{\alpha_j} \leq M_1^{k-1} \cdot M_{k-1}.$$

Because the sequence $(M_k)_k$ is log. conv. $\left(\frac{M_{k+1}}{M_k}\right)_k$ is increasing and so

$$M_j \cdot M_{\alpha_1} \cdots M_{\alpha_j} = \underbrace{M_j \cdot M_{\alpha_1} \cdots M_{\alpha'_i} \cdots M_{\alpha_j}}_{\leq M_1^{k-1} \cdot M_{k-1}} \cdot \underbrace{\frac{M_{\alpha_i}}{M_{\alpha'_i}}}_{\leq \frac{M_k}{M_{k-1}}} \leq M_1^k \cdot M_k.$$

□

We use now 2.0.6 to prove the first important result:

Proposition 2.0.8 *Let $M := (M_k)_k$ be a log. conv. weight sequence with $M_0 = 1$. Then the spaces $\mathcal{E}_{\{M\}}(G)$ and $\mathcal{E}_{(M)}(G)$, $G \subseteq \mathbb{R}^n$ open, are closed under pointwise multiplication of functions, hence commutative rings.*

Proof. We proof the statement for the Romieu-case: Let $f, g \in \mathcal{E}_{\{M\}}(G)$, then by assumption $\exists C_1, C_2, h_1, h_2 > 0$ such that $|f^{(\alpha)}(x)| \leq C_1 \cdot h_1^{|\alpha|} \cdot M_{|\alpha|}$ resp. $|f^{(\beta)}(x)| \leq C_2 \cdot h_2^{|\beta|} \cdot M_{|\beta|}$ holds for all x in a compact set $K \subseteq G$ and all $\alpha, \beta \in \mathbb{N}^n$. To estimate the product $f \cdot g$ we use *Leibnitz formula* and recall that for $\alpha, \beta \in \mathbb{N}^n$ with $\beta_i \leq \alpha_i$ for all i we put $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta! \cdot (\alpha - \beta)!} = \prod_{i=1}^n \binom{\alpha_i}{\beta_i}$ and 0 otherwise. We calculate as follows:

$$\begin{aligned} |(f \cdot g)^{(\alpha)}(x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \cdot |f^{(\beta)}(x)| \cdot |g^{(\alpha - \beta)}(x)| \\ &\leq \underbrace{\sum_{f, g \in \mathcal{E}_{\{M\}}} \binom{\alpha}{\beta}}_{\leq C_3} \cdot \underbrace{C_1 \cdot C_2}_{=: C_3} \cdot h_1^{|\beta|} \cdot \underbrace{h_2^{|\alpha - \beta|}}_{= h_2^{|\alpha| - |\beta|}} \cdot \underbrace{M_{|\beta|} \cdot M_{|\alpha - \beta|}}_{\leq M_{|\alpha|} \text{ by 2.0.6}} \\ &\leq C_3 \cdot M_{|\alpha|} \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \cdot h_1^{|\beta|} \cdot h_2^{|\alpha| - |\beta|} \leq C_3 \cdot M_{|\alpha|} \cdot \prod_{i=1}^n \sum_{\beta_i \leq \alpha_i} \binom{\alpha_i}{\beta_i} \cdot h_1^{\beta_i} \cdot h_2^{\alpha_i - \beta_i} \\ &= C_3 \cdot M_{|\alpha|} \cdot \prod_{i=1}^n h_3^{\alpha_i} = C_3 \cdot h_3^{|\alpha|} \cdot M_{|\alpha|}, \end{aligned}$$

where we have put $h_3 := h_1 + h_2$ and used the *binomial theorem* n -times.

Obviously the proof above holds also for the Beurling case $\mathcal{E}_{(M)}$.

In particular we have shown: If $(M_p)_p$ is a log. convex weight sequence with $M_0 = 1$, then $f \in \mathcal{E}_{M, h_1}(K)$ and $g \in \mathcal{E}_{M, h_2}(K)$ implies $f \cdot g \in \mathcal{E}_{M, h_1 + h_2}(K)$.

□

We finish the first chapter with the following remark:

One can imitate the construction for $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ above using *function germs* in $\mathbb{R}^n$. First we remark that spaces of function germs are defined as inductive limits as follows:

$$\mathcal{E}(\mathbb{R}^n \supseteq A, \mathbb{C}) := \varinjlim_U \mathcal{E}(U, \mathbb{C}),$$

where U runs in the limit above over all open sets with $A \subseteq U \subseteq \mathbb{R}^n$. If one wants to deal with function germs at the origin in $\mathbb{R}^n$ we put $A := \{0\}$. Note that the elements of $\mathcal{E}(\mathbb{R}^n \supseteq A, \mathbb{C})$ are equivalence classes of functions and this space is not Hausdorff.

We denote by $\mathcal{O}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ the ring of all real-analytic complex-valued function germs and by $\mathcal{E}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ the ring of all infinitely differentiable complex-valued function germs at the

origin in $\mathbb{R}^n$. We have also consider in this situation the canonical *Borel map* $j_x^\infty : \mathcal{E}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \longrightarrow \mathbb{R}^{\mathbb{N}^n}$ which is defined in the usual sense by

$$j_x^\infty(f) := (f^{(\alpha)}(x))_\alpha, \alpha \in \mathbb{N}^n.$$

Now we construct weighted function classes in the following way: Let $M := (M_p)_p$ be a weight sequence, then we define the space $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ via

$$\{f \in \mathcal{E}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) : \exists U \text{ open, } U \supseteq \{0\}, \exists C, h > 0, \forall \alpha \in \mathbb{N}^n \forall x \in U : \\ |f^{(\alpha)}(x)| \leq C \cdot h^{|\alpha|} \cdot M_{|\alpha|}\}.$$

Similarly one can define the Beurling-case $\mathcal{E}_{(M)}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ if one changes in the definition above $\exists C, h > 0$ into $\forall h > 0, \exists C > 0$. From our definitions we get immediately for an arbitrary weight sequence $(M_p)_p$ the following inclusions: $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \subseteq \mathcal{E}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ and $\mathcal{E}_{(M)}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \subseteq \mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$. If $m_0 = 1$ and $(m_p)_p$ is increasing, then $\mathcal{O}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \subseteq \mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ holds, too.

From these definitions it follows immediately again that we have for any weight sequence $(M_p)_p$

$$j_x^\infty \left(\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \right) \subseteq \Lambda_{\{M\}}^n \text{ resp. } j_x^\infty \left(\mathcal{E}_{(M)}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) \right) \subseteq \Lambda_{(M)}^n.$$

One has to be careful to apply the results of the spaces $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$, which we will prove in the next chapters, to $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ resp. $\mathcal{E}_{(M)}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$.

3 Stability properties

3.1 Inclusion of spaces

If it is not stated otherwise, we assume in this section that the sequence

$$(m_p)_p \text{ is log. convex and } m_0 = M_0 = 1.$$

Consider two different weight sequences $M := (M_p)_p$ and $M^* := (M_p^*)_p$, then we want to compare the associated function spaces $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{\{M^*\}}$ resp. $\mathcal{E}_{(M)}$ and $\mathcal{E}_{(M^*)}$. For this we introduce now the following notation: Let $m := (m_p)_p$ and $m^* := (m_p^*)_p$, where $m_p = \frac{M_p}{p!}$ and $m_p^* = \frac{M_p^*}{p!}$, then we write $m \preceq m^*$ if

$$\exists D > 0, \forall p \geq 0 : m_p \leq D^p \cdot m_p^*.$$

An equivalent description is:

$$D := \sup_p \left(\frac{m_p}{m_p^*} \right)^{1/p} < \infty. \quad (3.1.1)$$

Note that this supremum doesn't change if one uses the sequences M and M^* because the factorial term cancels, hence $m \preceq m^* \Leftrightarrow M \preceq M^*$. For the sequences $\mu := (\mu_p)_p$, $\mu_p := \frac{M_p}{M_{p-1}}$, and $\mu^* := (\mu_p^*)_p$, $\mu_p^* := \frac{M_p^*}{M_{p-1}^*}$, (3.1.1) has the form

$$\sup_p \prod_{j=1}^p \left(\frac{\mu_j}{\mu_j^*} \right)^{1/p} < \infty.$$

From the definition it's clear that the relation $\preceq$ is reflexive (with $D = 1$) and transitive and $m \preceq m^*$ implies $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{M^*\}}$ resp. $\Lambda_{\{M\}}^n \subseteq \Lambda_{\{M^*\}}^n$.

Example: If $(m_p)_p$ is an arbitrary weight sequence such that $m_p \geq 1$ for all $p \in \mathbb{N}$ holds, then clearly $\mathcal{O} \subseteq \mathcal{E}_{\{M\}}$ is satisfied.

Furthermore we have the same result for the Beurling-case: $m \preceq m^*$ implies $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(M^*)}$ and $\Lambda_{(M)}^n \subseteq \Lambda_{(M^*)}^n$. This holds because $C_h \cdot h^p \cdot M_p \leq C_h \cdot (h \cdot D)^p \cdot M_p^*$, hence the constant C_h in the estimation for the weight sequence $(M_p)_p$ is the constant $C_{h \cdot D}$ in the estimation for $(M_p^*)_p$.

We will write $M \approx M^*$ if and only if $M \preceq M^*$ and $M^* \preceq M$ is satisfied, thus $\approx$ is an equivalence relation on the set of all weight sequences and $M \approx M^* \Rightarrow \mathcal{E}_{\{M\}} = \mathcal{E}_{\{M^*\}}$ resp. $M \approx M^* \Rightarrow \mathcal{E}_{(M)} = \mathcal{E}_{(M^*)}$.

The converse direction is clear for the sequence space case:

If a sequence $c \in \Lambda_{\{M\}}^n$ resp. $c \in \Lambda_{(M)}^n$ belongs to $\Lambda_{\{M^*\}}^n$ resp. $\Lambda_{(M^*)}^n$, then $m \preceq m^*$ holds.

But in the case of function spaces $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ we need another argument! The problem is that the defined Borel-map j^∞ is in general not surjective, so we don't know if for a given sequence $c \in \Lambda_{\{M\}}^n$ resp. $c \in \Lambda_{(M)}^n$, which belongs to $\Lambda_{\{M^*\}}^n$ resp. $\Lambda_{(M^*)}^n$, we can find a function $f \in \mathcal{E}_{\{M^*\}}$ resp. $f \in \mathcal{E}_{(M^*)}$ such that $j^\infty(f) = c$ holds.

Remark 3.1.1 Let $M := (M_p)_p$ be an arbitrary weight sequence and $\mu := (\mu_p)_p$, where we set $\mu_p := \frac{M_p}{M_{p-1}}$. If one assumes that the sequence μ is increasing, one can change μ into a new sequence $\tilde{\mu}$ and M into $\tilde{M} := (\tilde{M}_p)_p$, where $\tilde{M}_p := \prod_{i=0}^p \tilde{\mu}_i$, with the following properties: $\tilde{\mu}$

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is strict increasing and $\mathcal{E}_{\{M\}} = \mathcal{E}_{\{\tilde{M}\}}$ resp. $\mathcal{E}_{(M)} = \mathcal{E}_{(\tilde{M})}$ is satisfied. In particular one has to define $(\tilde{\mu}_p)_p$ in such a way that $\sup_p \prod_{j=0}^p \left(\frac{\mu_p}{\tilde{\mu}_p}\right)^{1/p} < \infty$ and $\sup_p \prod_{j=0}^p \left(\frac{\tilde{\mu}_p}{\mu_p}\right)^{1/p} < \infty$ holds.

If $(\mu_p)_p$ will not become constant, which means there exists no $n_0 \in \mathbb{N}$ such that $\mu_{n_0} = \mu_p$ for all $p \geq n_0$, then one can define the sequence $(\tilde{\mu}_p)_p$ in such a way that the point $(p, \tilde{\mu}_p)$ lie on affine lines.

If $(\mu_p)_p$ will become constant, then one has to choose $(\tilde{\mu}_p)_p$ such that $\lim_{p \rightarrow \infty} \tilde{\mu}_p < \infty$.

A first important result is that in $\mathcal{E}_{\{M\}}$ there are functions with sufficiently large derivatives. We prove this statement now for the one-dimensional case:

Proposition 3.1.2 [24, Theorem 1, p. 3-4] *Let $(M_j)_j$ be a log. convex weight sequence, then there exists a function $\theta \in \mathcal{E}_{\{M\}}$, such that $|\theta^{(j)}(0)| \geq j! \cdot m_j = M_j$ holds for all $j \in \mathbb{N}$.*

Proof. By assumption $(M_j)_j$ is logarithmic convex, hence $(\mu_j)_j$ is an increasing sequence. First we prove the following inequality:

$$\left(\frac{1}{\mu_k}\right)^{k-j} \leq \frac{M_j}{M_k} \quad \forall (j, k) \in \mathbb{N}^2. \quad (3.1.2)$$

For this we distinguish two cases:

$j > k$: We can write the right side in (3.1.2) in the form:

$$\frac{M_j}{M_k} = \frac{M_{k+1}}{M_k} \cdot \frac{M_{k+2}}{M_{k+1}} \cdots \frac{M_j}{M_{j-1}} = \mu_{k+1} \cdots \mu_j. \quad (3.1.3)$$

Hence, because the sequence $(\mu_j)_j$ is increasing, $\mu_j \geq 1$ for all j and the product has $j - k$ factors:

$$\mu_{k+1} \cdots \mu_j \geq (\mu_k)^{j-k},$$

which proves the first case.

$k \geq j$: We write again the right side as

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j+1}} \cdot \frac{M_{j+1}}{M_{j+2}} \cdots \frac{M_{k-1}}{M_k} = \frac{1}{\mu_{j+1}} \cdots \frac{1}{\mu_k}. \quad (3.1.4)$$

Here we have $k - j$ factors and by the increasing property of the sequence $(\mu_j)_j$ we have

$$\frac{1}{\mu_{j+1}} \cdots \frac{1}{\mu_k} \geq \left(\frac{1}{\mu_k}\right)^{k-j},$$

which proves the second case.

Then we define the (complex-valued!) function $\theta : \mathbb{R} \rightarrow \mathbb{C}$ as follows:

$$\theta(x) := \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot \exp(2i\mu_k x).$$

We estimate now θ for all $x \in \mathbb{R}$:

$$|\theta(x)| = \left| \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot \exp(2i\mu_k x) \right| \leq \sum_{k=0}^{\infty} \left| \frac{M_k}{(2\mu_k)^k} \right| \cdot \underbrace{|\exp(2i\mu_k x)|}_{=1}.$$

Then we use the inequality (3.1.2) for the case k arbitrary and $j = 0$ to obtain:

$$\sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \stackrel{(3.1.2)}{\leq} \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot M_k \cdot \frac{M_0}{M_k} = M_0 \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 \cdot M_0 < \infty.$$

From this it follows that the function θ is an absolutely convergent sum in the space of holomorphic functions and so we can interchange summation and differentiation to obtain:

$$\theta^{(j)}(x) = \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot (2i\mu_k)^j \cdot e^{2i\mu_k x}.$$

We can estimate as follows:

$$\begin{aligned} \left| \theta^{(j)}(x) \right| &\leq \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot |(2i\mu_k)^j| = \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k} = \sum_{k=0}^{\infty} M_k \cdot \frac{1}{(2\mu_k)^{k-j}} \\ &\stackrel{(3.1.2)}{\leq} \sum_{k=0}^{\infty} M_k \cdot \frac{1}{2^{k-j}} \cdot \frac{M_j}{M_k} = 2^j \cdot M_j \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = 2^{j+1} \cdot M_j. \end{aligned}$$

This calculation shows: $\theta \in \mathcal{E}_{\{M\}}$, where $C = h = 2$. On the other side we have

$$\left| \theta^{(j)}(0) \right| = \left| \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot (2i\mu_k)^j \right| = \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k} \geq M_j = j! \cdot m_j$$

for all $j \in \mathbb{N}$.

Remark: θ is a complex-valued function! But we can write

$$\theta(x) = \underbrace{\sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot \cos(2\mu_k x)}_{=: \theta_1(x) = (\Re \circ \theta)(x)} + i \cdot \underbrace{\sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot \sin(2\mu_k x)}_{=: \theta_2(x) = (\Im \circ \theta)(x)}$$

and so we get, using (3.1.2): $\theta_1, \theta_2 \in \mathcal{E}_{\{M\}}$ (again with $C = h = 2$). Furthermore we have $\theta_1^{(j)}(0) = \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k}$ for even j and $\theta_1^{(j)}(0) = 0$ if j is odd, resp. $\theta_2^{(j)}(0) = \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k}$ for odd j and $\theta_2^{(j)}(0) = 0$ if j is even. We define now a real-valued function $\tilde{\theta} := \theta_1 + \theta_2$, which has the following properties: We can estimate

$$\left| \tilde{\theta}^{(j)}(x) \right| \leq 2 \cdot \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k} \stackrel{(3.1.2)}{\leq} 2^{j+1} \cdot M_j \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = 2^{j+2} \cdot M_j,$$

which shows $\tilde{\theta} \in \mathcal{E}_{\{M\}}$ with $C = 4$ and $h = 2$. Finally we have $\left| \tilde{\theta}^{(j)}(0) \right| \geq M_j$ for all $j \in \mathbb{N}$, because $\tilde{\theta}^{(j)}(0) = \theta_1^{(j)}(0)$ if j is even and $\tilde{\theta}^{(j)}(0) = \theta_2^{(j)}(0)$ if j is odd.

In particular we see: If $\theta = \Re \circ \theta + i \cdot \Im \circ \theta$ is a complex-valued function, then we obtain $\theta \in \mathcal{E}_{\{M\}} \Leftrightarrow \Re \circ \theta \in \mathcal{E}_{\{M\}}$ and $\Im \circ \theta \in \mathcal{E}_{\{M\}}$ resp. $\theta \in \mathcal{E}_{(M)} \Leftrightarrow \Re \circ \theta \in \mathcal{E}_{(M)}$ and $\Im \circ \theta \in \mathcal{E}_{(M)}$.

Finally we remark that in fact $\theta \in \mathcal{E}_{\{M\}} \setminus \mathcal{E}_{(M)}$, because assume $\theta \in \mathcal{E}_{(M)}$, then:

$$M_j \leq \left| \theta^{(j)}(0) \right| \leq C \cdot h^j \cdot M_j \iff 1 \leq \left| \frac{\theta^{(j)}(0)}{M_j} \right|^{1/j} \leq C^{1/j} \cdot h.$$

Since $\lim_{j \rightarrow \infty} C^{1/j} = 1$ and the inequality has to be valid for all $h > 0$ we get a contradiction! $\square$

Using 3.1.2, we can show now the following theorem:

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Theorem 3.1.3 *Let $(M_j)_j$ be a logarithmic convex weight sequence, then we have:*
 $M \preceq M^* \iff \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{M^*\}}.$

Proof. We have only to show $(\Leftarrow)$: For this let $\theta \in \mathcal{E}_{\{M\}}$ like in 3.1.2 be given. By assumption $\theta \in \mathcal{E}_{\{M^*\}}$ and so we get:

$$j! \cdot m_j \leq \left| \theta^{(j)}(0) \right| \leq C \cdot h^j \cdot j! \cdot m_j^*$$

for $C, h > 0$ constants. This implies $\frac{m_j}{m_j^*} \leq C \cdot h^j$, hence $\left(\frac{m_j}{m_j^*} \right)^{1/j} \leq C^{1/j} \cdot h$ holds for all $j \in \mathbb{N}$. Because $\lim_{j \rightarrow \infty} C^{1/j} = 1$ for the constant $C > 0$ we get (3.1.1). □

3.1.3 has an important consequence: If M and M^* are both log. convex, then $M \approx M^* \iff \mathcal{E}_{\{M\}} = \mathcal{E}_{\{M^*\}}$, hence on the set of all log. convex weight sequences $(M_p)_p$ the equivalence relation $\approx$ characterizes uniquely the equivalence of two function spaces of Romieu-type.

We will apply now 3.1.3 to prove two important and useful corollaries. A desired property of spaces of smooth functions is *closedness under derivation*. Therefore we remark that the first order derivatives of functions in $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ belong to $\mathcal{E}_{\{M'\}}$ resp. $\mathcal{E}_{(M')}$, where we have set $M' := (M'_j)_j$, $M'_j := M_{j+1}$. With this notation we get the following corollary:

Corollary 3.1.4 [24, Corollary 2, p. 4] *Let $(M_j)_j$ be a log. convex weight sequence, then $\mathcal{E}_{\{M\}}$ is stable under differentiation if and only if the sequence $(m_j)_j$ satisfies the property $\sup_j \left(\frac{m_{j+1}}{m_j} \right)^{1/j} < \infty$.*

Proof. $(\Leftarrow)$ Replace the sequence $(m_j)_j$ by $(m'_j)_j$, where $m'_j := m_{j+1}$, and the sequence $(m_j^*)_j$ by $(m_j)_j$ in (3.1.1).

$(\Rightarrow)$ Assume that the supremum above would be unbounded. First choose a function θ like in 3.1.2. The closedness under derivation of $\mathcal{E}_{\{M\}}$ implies $\theta' \in \mathcal{E}_{\{M\}}$ and so the inequality

$$\left| \frac{\theta^{(k+1)}(x)}{k! \cdot m_k} \right| \leq C \cdot h^k \quad (3.1.5)$$

holds for constant $C, h > 0$ and all $k \in \mathbb{N}$, where $x \in K$ and K is a compact set. On the other hand, by 3.1.2, we have the estimation $\theta^{(k+1)}(0) \geq (k+1)! \cdot m_{k+1}$ for all $k \in \mathbb{N}$, thus

$$\begin{aligned} \left| \frac{\theta^{(k+1)}(0)}{k! \cdot m_k} \right| &\geq \frac{(k+1)! \cdot m_{k+1}}{k! \cdot m_k} = (k+1) \cdot \frac{m_{k+1}}{m_k} \quad \forall k \in \mathbb{N} \\ \implies \sup_k \left| \frac{\theta^{(k+1)}(0)}{k! \cdot m_k} \right|^{1/k} &\geq \sup_k (k+1)^{1/k} \cdot \left(\frac{m_{k+1}}{m_k} \right)^{1/k}. \end{aligned}$$

But $\sup_k \left(\frac{m_{k+1}}{m_k} \right)^{1/k}$ is unbounded by assumption and $\lim_{k \rightarrow \infty} (k+1)^{1/k} = 1$, hence $\sup_k \left| \frac{\theta^{(k+1)}(0)}{k! \cdot m_k} \right|^{1/k}$ is unbounded and we obtain a contradiction to (3.1.5). □

Note: The supremum-condition in 3.1.4 doesn't change if we use the sequence $(M_j)_j$, because $\frac{M_{j+1}}{M_j} = \frac{(j+1) \cdot m_{j+1}}{m_j}$ and $\lim_{j \rightarrow \infty} (j+1)^{1/j} = 1$. $(\Leftarrow)$ in 3.1.4 holds for the Beurling-case, too.

We formulate and prove a second corollary:

Corollary 3.1.5 [24, Corollary 1, p. 4] We obtain the following equivalence: $\sup_j m_j^{1/j} < \infty \iff \mathcal{O} = \mathcal{E}_{\{M\}}$.

Proof. ($\Rightarrow$) First note that the weight sequence $(m_j)_j$ for $\mathcal{O}$ is the constant sequence $(1, 1, \dots)$, so we replace in the notation above the sequence $(m_j^*)_j$ by $(1, 1, \dots)$. If $\sup_j m_j^{1/j} < \infty$, then $\mathcal{E}_{\{M\}} \subseteq \mathcal{O}$ holds.

On the other hand, by assumption $m_0 = 1$ and $(m_j)_j$ is log. convex, so we know by 2.0.4, that the sequence $(m_j^{1/j})_j$ is increasing. Thus $\sup_j \left(\frac{1}{m_j}\right)^{1/j} < \infty$ and so we have $\mathcal{O} \subseteq \mathcal{E}_{\{M\}}$, too. In particular, to have $\mathcal{O} \subseteq \mathcal{E}_{\{M\}}$, it would be sufficient to assume $m_j \geq 1$ for all $j \in \mathbb{N}$.

($\Leftarrow$) Holds by 3.1.3.

□

Remark 3.1.6 We have seen that the strong log. convexity and normalization imply by 2.0.4 the fact $\mathcal{O} \subseteq \mathcal{E}_{\{M\}}$. But if we want a strict inclusion $\mathcal{O} \subsetneq \mathcal{E}_{\{M\}}$, by 3.1.5, this is equivalent to condition $\sup_j (m_j^{1/j})_j = \infty$.

If one uses the weight sequence $(M_p)_p$, then the supremum in 3.1.5 changes into $\sup_j \left(\frac{M_j}{j!}\right)^{1/j}$.

Let $N := (N_p)_p$ and $M := (M_p)_p$ be two arbitrary weight sequences. Then we have seen: $N \preceq M \Rightarrow \mathcal{E}_{\{N\}} \subseteq \mathcal{E}_{\{M\}}$ resp. $N \preceq M \Rightarrow \mathcal{E}_{(N)} \subseteq \mathcal{E}_{(M)}$. In this case one can consider now the following short exact sequences of locally convex vector spaces:

$$0 \rightarrow \mathcal{E}_{\{N\}} \hookrightarrow \mathcal{E}_{\{M\}} \twoheadrightarrow \mathcal{E}_{\{M\}}/\mathcal{E}_{\{N\}} \rightarrow 0 \text{ resp.}$$

$$0 \rightarrow \mathcal{E}_{(N)} \hookrightarrow \mathcal{E}_{(M)} \twoheadrightarrow \mathcal{E}_{(M)}/\mathcal{E}_{(N)} \rightarrow 0$$

In the Romieu-case, using (2.0.1), we have for a compact subset K the description

$$\mathcal{E}_{\{M\}}(K)/\mathcal{E}_{\{N\}}(K) = \lim_{\rightarrow h} \mathcal{E}_{M,D \cdot h}(K)/\mathcal{E}_{N,h}(K),$$

where D is the constant appearing in (3.1.1) because inductive limits and quotients commute. Assume now that there exists a constant $D > 0$ such that for all $j \in \mathbb{N}$ we have $N_j = D^j \cdot M_j$, then $\mathcal{E}_{\{M\}}/\mathcal{E}_{\{N\}} = \mathcal{E}_{(M)}/\mathcal{E}_{(N)} = \{0\}$. In this case one can conclude: There doesn't exist a weight sequence L , $L := (L_p)_p$, with $\mathcal{E}_{\{L\}} = \mathcal{E}_{\{M\}}/\mathcal{E}_{\{N\}}$, $\mathcal{E}_{(L)} = \mathcal{E}_{(M)}/\mathcal{E}_{(N)}$, because the polynomials are always contained in $\mathcal{E}_{\{L\}}$ resp. $\mathcal{E}_{(L)}$.

We are going to prove a first characterizing result for the surjectivity of the Borel-map j^∞ . The proof will use 3.1.5, but before formulating the theorem we need some preparations. First we need the theorem of Arzelà-Ascoli:

Lemma 3.1.7 Let X be a compact topological space and F an arbitrary locally convex vector space. Then the subset $E \subseteq C(X, F)$ is precompact if and only if it is equicontinuous and pointwise precompact.

Proof. A proof can be found for example in [15, 6.15 Theorem, p. 104-105].

□

Now we can formulate the second important preparatory result which uses 3.1.7.

Lemma 3.1.8 [11, Proposition 2.2, p. 41] Let $K \subseteq \mathbb{R}^n$ be a compact set, $M := (M_p)_p$ an arbitrary weight sequence and $h < k$, then the inclusion mapping $\iota : \mathcal{E}_{M,h}(K) \hookrightarrow \mathcal{E}_{M,k}(K)$ is compact.

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Proof. Let $I := I_1 \times \cdots \times I_n$, where $I_j \subseteq \mathbb{R}$ is a closed interval for all j . We show the compactness of $\iota : \mathcal{E}_{M,h}(I) \hookrightarrow \mathcal{E}_{M,k}(I)$. Let $\mathcal{B}$ denote the unit ball in the space $\mathcal{E}_{M,h}(I)$ and let $\varepsilon > 0$. Now we choose a number $m \in \mathbb{N}$ so that

$$\left(\frac{h}{k}\right)^m < \frac{\varepsilon}{2} \quad (3.1.6)$$

holds. For $|\alpha| \leq m$ set $\mathcal{B}^\alpha := \{f^{(\alpha)} : f \in \mathcal{B}\}$ and for $x, y \in K$ arbitrary we estimate

$$\begin{aligned} \left|f^{(\alpha)}(x) - f^{(\alpha)}(y)\right| &\leq \int_0^1 \left| \frac{d}{dt} f^{(\alpha)}(y + t \cdot (x - y)) \right| dt \leq \int_0^1 \sum_{j=1}^n \left| f^{(\alpha+e_j)}(y + t(x-y)) \cdot (x_j - y_j) \right| dt \\ &\leq \|f^{(\alpha+1)}\|_\infty \cdot \|x - y\|_1 \leq k, \end{aligned}$$

where we have set $e_j := (0, \dots, \underbrace{1}_j, \dots, 0)$ and $\alpha + 1 := (\alpha_1 + 1, \dots, \alpha_n + 1)$. Thus we have

shown that $\mathcal{B}^\alpha$ is equicontinuous and the fact that $\mathcal{B}^\alpha$ is pointwise precompact is clear.

So, by the theorem of *Arzelà-Ascoli* 3.1.7, we conclude that the set $\mathcal{B}^\alpha$ is relative compact in $\mathcal{C}(I)$. Hence there exist $f_{1,\alpha}, \dots, f_{n_\alpha,\alpha} \in \mathcal{B}$ such that for each $f \in \mathcal{B}$ we can find an index i with $\|f^{(\alpha)} - f_{i,\alpha}^{(\alpha)}\|_\infty \leq \varepsilon \cdot k^{|\alpha|} \cdot M_{|\alpha|}$. There exists a finite number of functions $f_1, f_2, \dots, f_N \in \mathcal{B}$ such that $\{f_1, \dots, f_N\} = \bigcup_{\alpha: |\alpha| \leq m} \{f_{1,\alpha}, \dots, f_{n_\alpha,\alpha}\}$ and for each $f \in \mathcal{B}$ we can find an index j , $1 \leq j \leq N$, for which

$$\|(f - f_j)^{(\alpha)}\|_\infty \leq \varepsilon \cdot k^{|\alpha|} \cdot M_{|\alpha|}$$

holds for all $|\alpha| \leq m$. For $|\alpha| > m$ the inequality above is clear because $f_j \in \mathcal{B}$ and (3.1.6). Hence the set $\{f_1, \dots, f_N\}$ is an ε -net for $\mathcal{B}$ in $\mathcal{E}_{M,k}(I)$ and so $\mathcal{B}$ is precompact.

For arbitrary compact sets $K = \bigcup_j I_j$, where $I_j = I_{j_1} \times \cdots \times I_{j_n}$ and $I_{j_i} \subseteq \mathbb{R}$ is a compact interval for all i , we have $\mathcal{E}_{M,h}(K) = \mathcal{E}_{M,h}(\bigcup_j I_j)$. The compactness of the mapping $\bar{\iota} : \mathcal{E}_{M,h}(K) \hookrightarrow \mathcal{E}_{M,k}(K)$, with $h < k$, follows now: Because we have $\mathcal{E}_{M,k}(\bigcup_j I_j) \hookrightarrow \prod_j \mathcal{E}_{M,k}(I_j) \longrightarrow \mathcal{E}_{M,k}(I_j)$ we can reduce the problem to show the compactness of $\mathcal{E}_{M,h}(\bigcup_j I_j) \longrightarrow \mathcal{E}_{M,k}(I_j)$. So in particular, here it's sufficient to have the compactness for the mapping $\iota : \mathcal{E}_{M,h}(I_j) \hookrightarrow \mathcal{E}_{M,k}(I_j)$ (note that the compact operators form an two-sided-ideal in the ring of all bounded linear operators).

□

Theorem 3.1.9 [24, Theorem 3, p. 4-8] Assume that $\mathcal{E}_{\{M\}}$ is quasi-analytic and that $\mathcal{O} \subsetneq \mathcal{E}_{\{M\}}$. Then the Borel map $j^\infty : \mathcal{E}_{\{M\}} \longrightarrow \Lambda_{\{M\}}^n$ is not surjective.

We will follow the proof in the paper of Thilliez [24], which uses some Hilbert space techniques from functional analysis. The original proof of Carleman is more complicated.

Proof. The proof consists of two parts: The first part deals with a representation formula for functions in quasi-analytic spaces $\mathcal{E}_{\{M\}}$ and the second part follows the original proof of Carleman.

We remark that it is enough to prove this theorem for the case $n = 1$: Therefore not that $\mathcal{E}_{\{M\}}(\mathbb{R}) \hookrightarrow \mathcal{E}_{\{M\}}(\mathbb{R}^n)$ holds by setting the additional $n - 1$ variables constant. Conversely one has the restriction mapping $\mathcal{E}_{\{M\}}(\mathbb{R}^n) \rightarrow \mathcal{E}_{\{M\}}(\mathbb{R})$.

If one assumes now that $j^\infty : \mathcal{E}_{\{M\}}(\mathbb{R}^n) \longrightarrow \Lambda_{\{M\}}^n$ would be surjective, one can restrict j^∞ to $\mathcal{E}_{\{M\}}(\mathbb{R})$ and one would obtain that $j^\infty : \mathcal{E}_{\{M\}}(\mathbb{R}) \longrightarrow \Lambda_{\{M\}}$ is surjective, which is a contradiction to the case $n = 1$.

Now we start with the first part of the proof of 3.1.9:

For any integer $\nu \geq 1$ and any real number $h > 0$, we put $I_\nu = (-1/\nu, 1/\nu)$ and define

$$\mathcal{E}_{M,h}(I_\nu) := \{f \in \mathcal{E}(I_\nu) : |f|_{I_\nu,h} < \infty\}$$

with

$$|f|_{I_\nu, h} := \sup_{p \in \mathbb{N}, x \in I_\nu} \frac{|f^{(p)}(x)|}{h^p \cdot M_p}.$$

From functional analysis it follows that $\mathcal{E}_{M, h}(I_\nu)$ is a Banach space. For $h < h'$ we have the canonical injection $\iota : \mathcal{E}_{M, h}(I_\nu) \hookrightarrow \mathcal{E}_{M, h'}(I_\nu)$, which is a compact map by 3.1.8.

In the following we set $I := I_1 = (-1, 1)$ and denote by $\mathcal{E}(\bar{I})$ the space of functions f which are smooth on $I = (-1, 1)$ and such that all derivatives $f^{(j)}$, $j \geq 0$, extend continuously to the closure $\bar{I}$. For any continuous function f on I we put

$$\|f\|_{L^2(I)} := \left(\int_I |f(x)|^2 dx \right)^{1/2}$$

and

$$\|f\|_{L^\infty(I)} := \sup_{x \in I} |f(x)|.$$

For any $f \in \mathcal{E}(\bar{I})$ and for any integer $j \geq 0$ we show now the following inequalities:

$$\frac{1}{\sqrt{2}} \|f^{(j)}\|_{L^2(I)} \leq \|f^{(j)}\|_{L^\infty(I)} \leq \sqrt{2} \left(\|f^{(j)}\|_{L^2(I)} + \|f^{(j+1)}\|_{L^2(I)} \right). \quad (3.1.7)$$

The right inequality follows since for a given $g \in \mathcal{E}(\bar{I})$ there exists $c \in I$ such that $|g(c)| = \frac{1}{2} \int_I |g(x)| dx$ (by the mean value theorem for integrals). For any $x \in I$ it follows that

$$|g(x)| \leq |g(c)| + \left| \int_c^x |g'(y)| dy \right| \leq \frac{1}{2} \int_I |g(x)| dx + \int_I |g'(x)| dx. \quad (3.1.8)$$

By the *Cauchy-Schwarz-inequality* we have

$$|g(x)| \leq \frac{\sqrt{2}}{2} \left(\int_I |g(x)|^2 dx \right)^{1/2} + \sqrt{2} \left(\int_I |g'(x)|^2 dx \right)^{1/2}. \quad (3.1.9)$$

If we set $g = f^{(j)}$ for arbitrary $j \geq 0$ the right inequality follows and since the left inequality is clear we have shown (3.1.7).

Now put $\|f\|_M^2 := \sum_{j=0}^\infty M_j^{-2} \|f^{(j)}\|_{L^2(I)}^2$ and denote by $\mathcal{H}_M$ the space of all functions $f \in \mathcal{E}(\bar{I})$ such that $\|f\|_M^2 < \infty$ holds. $\mathcal{H}_M$ is a Hilbert space for the norm $\|\cdot\|_M$ because $\|\cdot\|_M$ is given by a ℓ^2 -sum of weighted L^2 -norms and we will denote the associated scalar product of this space by $\langle \cdot, \cdot \rangle$.

Now we set $M' := (M'_j)_j$, where $M'_j := M_{j+1}$, and by inequality (3.1.7) we get for any $\eta \in (0, 1)$ the topological inclusions

$$\mathcal{E}_{M, 1-\eta}(I) \subseteq \mathcal{H}_M \subseteq \mathcal{E}_{M', 1+\eta}(I). \quad (3.1.10)$$

The inclusion $\mathcal{E}_{M, h}(I_\nu^n) \hookrightarrow \mathcal{E}_{M, h'}(I_\nu^n)$ is a compact map for $h < h'$ by 3.1.8. Because the parameter η above is arbitrary and the compact operators form an ideal in the ring of all bounded linear operators it follows that both canonical injections in (3.1.10) are compact, too. We also have that for all integers $i \geq 0$ the map $f \mapsto f^{(i)}(0)$ is a continuous (because bounded) linear functional on the Hilbert space $\mathcal{H}_M$ and so, by the *theorem of Riesz*, there exists $e_i \in \mathcal{H}_M$, such that

$$f^{(i)}(0) = \langle e_i, f \rangle \quad \forall f \in \mathcal{H}_M \quad (3.1.11)$$

holds, where $\langle \cdot, \cdot \rangle := \|\cdot\|_M^2$. For any element $g \in \mathcal{H}_M$ we introduce for $k \in \mathbb{N} \setminus \{0\}$ the following system and study the solution functions $u \in \mathcal{H}_M$:

$$u^{(i)}(0) = g^{(i)}(0) \quad 0 \leq i < k. \quad (3.1.12)$$

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We denote now by $\mathcal{V}_k$ the subspace of $\mathcal{H}_M$ which is spanned by the elements $e_0, e_1, \dots, e_{k-1}$. If $u \in \mathcal{V}_k$ and $u = \sum_{j=0}^{k-1} \xi_j \cdot e_j$, then the system (3.1.12) can be rewritten as a $(k \times k)$ -linear system with unknowns $\xi_0, \xi_1, \dots, \xi_{k-1}$ in the following way:

$$g^{(i)}(0) \underset{(3.1.12)}{=} u^{(i)}(0) \underset{(3.1.11)}{=} \langle e_i | u \rangle = \left\langle e_i \left| \sum_{j=0}^{k-1} \xi_j \cdot e_j \right. \right\rangle = \sum_{j=0}^{k-1} \langle e_i | e_j \rangle \cdot \xi_j \quad 0 \leq i < k. \quad (3.1.13)$$

Claim: The elements e_j are linearly independent in $\mathcal{H}_M$.

If we take scalar products with monomials, then we can see because of the definition of the e_j that $\langle e_j | x^k \rangle = (x^k)^{(j)}(0) = 0$ for $j > k$ and for $j < k$. For $j = k$ we have $\langle e_j | x^k \rangle = k!$, thus for scalars $c_0, \dots, c_{k-1}$ and arbitrary l it follows:

$$\begin{aligned} c_0 \cdot e_0 + \dots + c_{k-1} \cdot e_{k-1} = 0 &\Rightarrow c_0 \cdot \langle e_0 | x^l \rangle + \dots + c_{k-1} \cdot \langle e_{k-1} | x^l \rangle = 0 \\ &\Rightarrow c_l \cdot \langle e_l | x^l \rangle = c_l \cdot l! = 0 \Rightarrow c_l = 0, \end{aligned}$$

which proves the claim.

Hence the dimension of $\mathcal{V}_k = k$ and the *Gram matrix* $A := (\langle e_i | e_j \rangle)_{0 \leq i, j \leq k-1}$ is invertible and so the system (3.1.12) has a solution g_k . Because $g_k \in \mathcal{V}_k$, we can write g_k in the form $g_k = \sum_{j=0}^{k-1} \xi_{j,k} \cdot e_j$ for suitable $\xi_{j,k} \in \mathbb{C}$. The scalars $\xi_{j,k}$ depend linearly on $g^{(i)}(0)$, where $0 \leq i \leq k-1$, because:

$$g^{(i)}(0) = g_k^{(i)}(0) = \sum_{j=0}^{k-1} \xi_{j,k} \cdot e_j^{(i)}(0) = \sum_{j=0}^{k-1} \xi_{j,k} \cdot \langle e_i | e_j \rangle. \quad (3.1.14)$$

We want to obtain a family of functions $(u_{j,k})_{0 \leq j \leq k-1}$, $u_{j,k} \in \mathcal{H}_M$ for $0 \leq j \leq k-1$, which depend not on g and such that we have the following formula for g_k :

$$g_k(t) = \sum_{i=0}^{k-1} u_{i,k}(t) \cdot g^{(i)}(0) \quad \forall t \in I. \quad (3.1.15)$$

To show (3.1.15) we put $B := (b_{i,j})_{1 \leq i, j \leq k-1} := A^{-1}$, so we have $\sum_{i=0}^{k-1} b_{i,j} \cdot \langle e_i | e_j \rangle = \delta_{l,j}$. With this notation we obtain now

$$\sum_{i=0}^{k-1} b_{i,j} \cdot g^{(i)}(0) \underset{(3.1.14)}{=} \sum_{i=0}^{k-1} b_{i,j} \cdot \sum_{j=0}^{k-1} \xi_{j,k} \cdot \langle e_i | e_j \rangle = \sum_{j=0}^{k-1} \xi_{j,k} \cdot \underbrace{\sum_{i=0}^{k-1} b_{i,j} \cdot \langle e_i | e_j \rangle}_{\delta_{l,j}} = \xi_{l,k}.$$

Hence

$$g_k = \sum_{l=0}^{k-1} \xi_{l,k} \cdot e_l = \sum_{l=0}^{k-1} \left(\sum_{i=0}^{k-1} b_{i,l} \cdot g^{(i)}(0) \right) \cdot e_l = \sum_{i=0}^{k-1} \underbrace{\left(\sum_{l=0}^{k-1} b_{i,l} \cdot e_l \right)}_{=: u_{i,k}} \cdot g^{(i)}(0),$$

which proves (3.1.15). In particular we have $u_{j,k} \in \mathcal{V}_k$ for $0 \leq j \leq k-1$ and each function $u_{j,k}$ depends only on the Hilbert space $\mathcal{H}_M$, which means on the weight sequence $(M_j)_j$.

Let u be another solution of (3.1.12), so we get: $\langle e_i | g_k - u \rangle = g_k^{(i)}(0) - u^{(i)}(0) = 0$ for $0 \leq i < k$. Hence $g_k - u \in \mathcal{V}_k^\perp$, but $g_k \in \mathcal{V}_k$ by the expansion above. Now we use Pythagoras theorem for Hilbert spaces to obtain that g_k is the minimal solution of (3.1.12) in $\mathcal{H}_M$ with respect to the norm $\|\cdot\|_M$, because $\|g_k\|_M^2 + \|g_k - u\|_M^2 = \|u\|_M^2$. In particular we have $\|g_k\|_M \leq \|g\|_M$, for all $k \geq 1$, and so the sequence $(g_k)_k$ is bounded in $\mathcal{H}_M$.

Next we claim that $g_k \rightarrow g$ in $\mathcal{E}_{M', 1+\eta}(I)$ for $k \rightarrow \infty$.

The inclusion $\mathcal{H}_M \hookrightarrow \mathcal{E}_{M',1+\eta}(I)$ is compact and so it suffices to show that g is the only possible limit for any subsequence of $(g_k)_k$ which is converging in $\mathcal{E}_{M',1+\eta}(I)$. So let h be the limit of such a subsequence. If we take limits in (3.1.12) we get:

$$h^{(i)}(0) = g^{(i)}(0) \quad \forall i \geq 0. \quad (3.1.16)$$

Now an important step in our proof: By assumption $\mathcal{E}_{\{M\}}$ is quasi-analytic and so $\mathcal{E}_{\{M'\}}$ is also quasi-analytic (because we have only an index shift). Thus (3.1.16) shows that $h = g$ holds.

Take $f \in \mathcal{E}_{\{M\}}$ and define for any x the function f_x by $f_x(t) := f(x \cdot t)$. This function belongs to $\mathcal{E}_{M,1-\eta}(I)$ and to $\mathcal{H}_M$ (via (3.1.10)). We apply the expansion formula (3.1.15) and the converging property of $(g_k)_k$ in $\mathcal{E}_{M',1+\eta}(I)$ to $g := f_x$. Set $t = 1$ and define $\omega_{j,k} := j! \cdot u_{j,k}(1)$ to obtain the following important *representation formula* for f :

$$g(1) = f_x(1) = f(x) = \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \omega_{j,k} \cdot \frac{f^{(j)}(0)}{j!} \cdot x^j. \quad (3.1.17)$$

Therefore note: $g^{(j)}(t) = f^{(j)}(t) \cdot x^j$ for all j .

Now we begin with the second part of the proof: If we apply the representation formula (3.1.17) to the monomial $f(x) := x^l$, where $l \geq 0$ is arbitrary, we get $\lim_{k \rightarrow \infty} \omega_{l,k} = 1$. This follows because all terms in the sum on the right hand side of (3.1.17) vanish except the l -th one and so we get

$$x^l = \lim_{k \rightarrow \infty} \omega_{l,k} \cdot x^l.$$

It is possible to select by induction an increasing sequence $(k_p)_{p \geq 0}$ of positive integers such that

$$\sum_{j=0}^{k_p-1} |\omega_{j,k_p} - 1| \cdot m_j \leq 1, \quad \forall p \geq 1. \quad (3.1.18)$$

By assumption $\mathcal{O} \subsetneq \mathcal{E}_{\{M\}}$ and so by 3.1.5 it follows that

$$\lim_{p \rightarrow \infty} (m_{k_p})^{\frac{1}{k_p}} = \infty. \quad (3.1.19)$$

We consider $F := \sum_{j=0}^{\infty} F_j \cdot x^j$ such that $(F_j)_j \in \mathcal{F}_{\{M\}}(\mathbb{R})$, where we have put $F_j := m_j$ for $j \in \{k_p : p \geq 0\}$ and $F_j := 0$ otherwise. Next we claim that this power series does not belong to the image of the Borel map j^∞ . We prove this by contradiction.

Assume that we can find $f \in \mathcal{E}_{\{M\}}$ such that $j^\infty(f) = F$. Now we take a sufficiently small real number $a > 0$ and by the representation formula (3.1.17) we get:

$$f(a) = \lim_{p \rightarrow \infty} \sum_{j=0}^{k_p-1} \omega_{j,k_p} F_j a^j. \quad (3.1.20)$$

By the definition of F , we get

$$\sum_{j=0}^{k_p-1} \omega_{j,k_p} F_j a^j = \sum_{j=0}^{k_p-1} \omega_{j,k_p} F_j a^j, \quad \sum_{j=0}^{k_p-1} F_j a^j = \sum_{q=0}^{p-1} m_{k_q} a^{k_q}.$$

If we use this two equations together with the property $\lim_{k \rightarrow \infty} \omega_{j,k} = 1$ we can write (3.1.20) in the form:

$$f(a) = \lim_{p \rightarrow \infty} \left(\sum_{q=0}^{p-1} m_{k_q} a^{k_q} + \sum_{j=0}^{k_p-1} (\omega_{j,k_p} - 1) F_j a^j \right). \quad (3.1.21)$$

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By (3.1.18) the second sum in the representation formula above is bounded by 1 for $a < 1$ uniformly with respect to the index p and so it follows that the sequence of partial sums of $\sum_{q \geq 0} m_{k_q} a^{k_q}$ is bounded. But this is a contradiction to $\lim_{p \rightarrow \infty} (m_{k_p})^{\frac{1}{k_p}} = \infty$.

□

Remark 3.1.10 *This proof above yields a more precise statement: To any given quasi-analytic ring $\mathcal{E}_{\{M\}}$, such that $\mathcal{O} \subsetneq \mathcal{E}_{\{M\}}$ holds, we can associate an increasing sequence of positive integers $(k_p)_p$ such that the only series $\sum_{p \geq 0} F_{k_p} x^{k_p}$, which belongs to the image $j^\infty(\mathcal{E}_{\{M\}})$, are the locally convergent ones (because of (3.1.19) and the definition of the power series F).*

3.2 Composition theorem

In this section the weight sequence $(m_k)_k$ will play the key role, and we assume throughout that

$$(m_k)_k \text{ is logarithmic convex.}$$

An important property of function spaces is *closedness under composition*. First we give a definition:

Definition 3.2.1 *Let us assume that for all $U \subseteq \mathbb{R}^n$ open one has a $\mathbb{R}$ -subalgebra $\mathcal{C}(U)$ of $\mathcal{E}(U)$. We say that $\mathcal{C}(U)$ is closed under composition if for a given $V \subseteq \mathbb{R}^p$ open and a map $\varphi = (\varphi_1, \dots, \varphi_p) : U \rightarrow V$ such that $\varphi_i \in \mathcal{C}(U) \forall i$ holds, it follows that for all $g \in \mathcal{C}(V)$ we have $g \circ \varphi \in \mathcal{C}(U)$.*

If one uses the function germs we have: Let $\mathcal{C}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ be an arbitrary $\mathbb{R}$ -subalgebra of $\mathcal{E}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$. We say that $\mathcal{C}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ is closed under composition if for $f = (f_1, \dots, f_p) \in (\mathcal{C}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}))^p$ with $f(0) = 0$ and any $g \in \mathcal{C}(\mathbb{R}^p \supseteq \{0\}, \mathbb{C})$ the composition $g \circ f \in \mathcal{C}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$.

Notation: In this section we set $f_\alpha := \frac{f^{(\alpha)}}{\alpha!}$ for a multiindex $\alpha \in \mathbb{N}^n$. We remark that with this notation one obtains

$$|f|_{K,h} = \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} \cdot M_{|\alpha|}} = \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f_\alpha(x)| \cdot \alpha!}{h^{|\alpha|} \cdot |\alpha|! \cdot m_{|\alpha|}},$$

and by 2.0.1 we can replace $|\alpha|!$ by $\alpha!$. In particular we have

$$|f|_{K,h} < \infty \Rightarrow \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f_\alpha(x)|}{(n \cdot h)^{|\alpha|} \cdot m_{|\alpha|}} < \infty \text{ and } \sup_{\alpha \in \mathbb{N}^n, x \in K} \frac{|f_\alpha(x)|}{h^{|\alpha|} \cdot m_{|\alpha|}} < \infty \Rightarrow |f|_{K,h} < \infty.$$

Before we prove the main theorem we have to deal with some formal calculations and inequalities. For this we introduce the following useful notation: Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, then we set:

$$a^\alpha = (a_1, \dots, a_n)^{(\alpha_1, \dots, \alpha_n)} := \prod_{i=1}^n a_i^{\alpha_i}.$$

Now we start with the first lemma:

Lemma 3.2.2 [1, Lemma 4.2., p. 9] *Let $a_i \in \mathbb{R}^p, i = 1, \dots, l$, and let $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$. Then we get*

$$(a_1 + \dots + a_l)^\alpha = \sum_{(k_1, \dots, k_l) \in (\mathbb{N}^p)^l, \alpha = \sum_{i=1}^l k_i} \frac{\alpha!}{k_1! \dots k_l!} a_1^{k_1} \dots a_l^{k_l}. \quad (3.2.1)$$

Remember that with our notation we can write for $a_i = (a_{i,1}, \dots, a_{i,p}) \in \mathbb{R}^p, \alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,p}) \in \mathbb{N}^p$ and all i :

$$a_i^{\alpha_i} = (a_{i,1}, \dots, a_{i,p})^{(\alpha_{i,1}, \dots, \alpha_{i,p})} = \prod_{j=1}^p a_{i,j}^{\alpha_{i,j}}.$$

Proof.

$$\begin{aligned} (a_1 + \dots + a_l)^\alpha &= \left(\sum_{i=1}^l a_{i,1}, \dots, \sum_{i=1}^l a_{i,p} \right)^{(\alpha_1, \dots, \alpha_p)} \\ &= \prod_{j=1}^p (a_{1,j} + \dots + a_{l,j})^{\alpha_j} \\ &= \prod_{j=1}^p \left(\sum_{\alpha_j = \sum_i k_{i,j}, k_{i,j} \in \mathbb{N}} \frac{\alpha_j!}{k_{1,j}! \dots k_{l,j}!} \cdot a_{1,j}^{k_{1,j}} \dots a_{l,j}^{k_{l,j}} \right). \end{aligned}$$

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First we have used the notation above to obtain a product of the components of the vector and then we have used the usual *multinomial theorem*. In the last product each term is a unique product of terms, one from each of the p factors in the product.

□

Now consider a composite function $h = f \circ g : \mathbb{R}^n \rightarrow \mathbb{R}$ for which we have $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ with $g(x) = (g_1(x), \dots, g_p(x))$, $x = (x_1, \dots, x_n)$ and $f : \mathbb{R}^p \rightarrow \mathbb{R}$, $f(y) = f(y_1, \dots, y_p)$. We assume that f and g are smooth functions. If we write $g_\gamma := (g_{1,\gamma}, \dots, g_{p,\gamma})$, $\gamma \in \mathbb{N}^n$, and $h_\gamma(x) := (f \circ g)_\gamma(x)$, $\gamma \in \mathbb{N}^n$, then we define the following formal power series in the indeterminate $u = (u_1, \dots, u_n)$ with coefficients $h_\gamma(x)$:

$$\sum_{\gamma \in \mathbb{N}^n} h_\gamma(x) u^\gamma.$$

We note that the Borel map j^∞ satisfies for all functions f and g , which are smooth in a neighbourhood of 0 and such that $g(0) = g_0 = 0$ holds, the following property: $j^\infty(f \circ g) = j^\infty(f) \circ j^\infty(g)$, where $\circ$ denotes the composition of formal power series.

We are able to prove now the following statement, which is also known under the name *Faà de Bruno formula*.

Proposition 3.2.3 [1, Proposition 4.3., p. 9-10] *For all $\gamma \in \mathbb{N}^n \setminus \{0\}$, it follows that*

$$h_\gamma(x) = \sum \frac{\alpha!}{k_1! \dots k_l!} f_\alpha(g(x)) g_{\delta_1}(x)^{k_1} \dots g_{\delta_l}(x)^{k_l}. \quad (3.2.2)$$

The sum is taken over all $\{\delta_1, \dots, \delta_l\}$ of l distinct elements of $\mathbb{N}^n \setminus \{0\}$ and over all ordered l -tuples $(k_1, \dots, k_l) \in (\mathbb{N}^p \setminus \{0\})^l$ for $l \in \mathbb{N} \setminus \{0\}$ such that $\gamma = \sum_{i=1}^l |k_i| \delta_i$ (so $\gamma \in \mathbb{N}^n \setminus \{0\}$). In the equation above we have set $\alpha = (\alpha_1, \dots, \alpha_p) = \sum_{i=1}^l k_i$.

Proof. By the statements above about the homomorphism property of the Borel map and the power series we obtain

$$\sum_{\gamma \in \mathbb{N}^n} h_\gamma(x) u^\gamma = \sum_{\alpha \in \mathbb{N}^p} f_\alpha(g(x)) \left(\sum_{\delta \in \mathbb{N}^n \setminus \{0\}} g_\delta(x) u^\delta \right)^\alpha.$$

Now we apply the formula (3.2.1) to this situation to obtain the result. Note that by a comparison of the coefficients in the equation there are on the right hand side only finite many summands.

□

To prove the composition theorem we need an important statement:

the *inequality of Childress*, which is connected to the Faà de Bruno formula in one variable. We recall that in the following the sequence $m := (m_k)_k$ is always assumed to be log. convex.

Proposition 3.2.4 [1, Proposition 4.4, p. 10-11] *Let $k_1, \dots, k_n$ be nonnegative integers such that the following condition is satisfied: $\sum_{i=1}^n i \cdot k_i = n$. If we set $\sum_{i=1}^n k_i = k$, then the sequence $(m_k)_k$ has the property*

$$m_k \cdot m_1^{k_1} \dots m_n^{k_n} \leq m_1^k \cdot m_n \quad (3.2.3)$$

Proof. If we have $k_n = 1$ we get by the condition above $k_1 = k_2 = k_3 = \dots = k_{n-1} = 0$ and $k = 1$, and so (3.2.3) turns into $m_1 \cdot m_n^1 \leq m_1^1 \cdot m_n$ which is correct. So we can assume now $k_n = 0$ and we distinguish two cases.

Case 1: $k_1 \neq 0$. Set $k'_1 = k_1 - 1$ and $k' = k - 1$. Then we get $\underbrace{k'}_{k-1} = \underbrace{k'_1}_{k_1-1} + k_2 + \dots + k_{n-1}$ and

$n - 1 = \underbrace{k'_1}_{k_1-1} + 2k_2 + \dots + (n-1)k_{n-1}$. Now we use induction on n : For $n = 0$ we get $k_i = 0$

for all i , and so $k = 0 \Rightarrow m_0 \leq m_0$ in (3.2.3). For $n > 0$ we have

$$m_{k-1} \cdot m_1^{k'_1} \cdot m_2^{k_2} \cdots m_{n-1}^{k_{n-1}} \cdot \underbrace{m_n^{k_n}}_1 \leq m_1^{k'_1} \cdot m_{n-1},$$

hence we write

$$\begin{aligned} m_k \cdot m_1^{k_1} \cdots m_n^{k_n} &= \frac{m_k}{m_{k-1}} \cdot m_{k-1} \cdot m_1 \cdot m_1^{k'_1} \cdot m_2^{k_2} \cdots m_n^{k_n} \\ &\leq \underbrace{m_1 \cdot m_1^{k'_1} \cdot m_{n-1}}_{I.H.} \cdot \frac{m_k}{m_{k-1}} \\ &\leq \underbrace{m_1 \cdot m_1^{k'_1} \cdot \frac{m_n}{m_{n-1}} \cdot m_{n-1}}_{\log. \text{ conv.}, n \geq k} = m_1^k \cdot m_n. \end{aligned}$$

Case 2: $k_1 = 0$. Then we get by the definition of n and k :

$$n - k = k_2 + 2k_3 + \cdots + (n - k)k_{n-k+1} + \underbrace{(n - k + 1)k_{n-k+2} + \cdots}_{> n-k} \underbrace{(n - 2)k_{n-1}}_{> n-k}$$

thus $k_j = 0$ if $j > n - k + 1$ and so $k = k_2 + \cdots + k_{n-k+1}$. Now we apply induction on n again: $m_{k+1} \cdot m_2^{k_2} \cdots m_{n-k+1}^{k_{n-k+1}} \leq m_2^k \cdot m_{n-k+1}$. For this we can also write

$$m_{k+1} \cdot m_1^{k_1} \cdots m_n^{k_n} \leq m_2^k \cdot m_{n-k+1},$$

because we can insert all factors in the product with vanishing exponents. If we multiply this inequality above with m_k and divide it by m_{k+1} we obtain:

$$\begin{aligned} m_k \cdot m_1^{k_1} \cdots m_n^{k_n} &\leq \frac{m_k}{m_{k+1}} \cdot m_2^k \cdot m_{n-k+1} \\ &\leq \frac{m_1}{m_2} \cdot m_2^k \cdot m_{n-k+1} = m_1 \cdot m_2^{k-1} \cdot m_{n-k+1} \\ &\leq m_1^2 \cdot m_2^{k-2} \cdot m_{n-k+2} \\ &\leq \cdots \leq m_1^k \cdot m_n. \end{aligned}$$

Here we have used again the log. conv. of the sequence $(m_k)_k$: first we have $\frac{m_1}{m_2} \geq \frac{m_k}{m_{k+1}}$ and then $\frac{m_1}{m_2} \geq \frac{m_{n-k+j}}{m_{n-k+j+1}}$ for $1 \leq j \leq k - 1$. □

Corollary 3.2.5 [1, Corollary 4.5., p. 11] Let $k_1, \dots, k_l \in \mathbb{N} \setminus \{0\}$ and $\delta_1, \dots, \delta_l \in \mathbb{N} \setminus \{0\}$. Now set $\alpha = k_1 + \cdots + k_l$ and $\gamma = |k_1|\delta_1 + \cdots + |k_l|\delta_l$. In this setting we obtain

$$m_{|\alpha|} \cdot m_{|\delta_1|}^{k_1} \cdots m_{|\delta_l|}^{k_l} \leq m_1^{|\alpha|} \cdot m_{|\gamma|}. \quad (3.2.4)$$

Note: $|\alpha| = |k_1| + \cdots + |k_l|$ and $|\gamma| = |k_1||\delta_1| + \cdots + |k_l||\delta_l|$.

Proof. We can apply the inequality of Childress (3.2.3). Additionally we can assume that all $|\delta_i|$ are distinct. Otherwise, if $|\delta_i| = |\delta_j|$ for i and j with $i \neq j$, then we replace a product $m_{|\delta_i|}^{k_i} \cdot m_{|\delta_j|}^{k_j}$ on the left hand side of (3.2.3) by $m_{|\delta_j|}^{k_i + k_j}$. □

Lemma 3.2.6 [1, Lemma 4.8., p. 12] Let $\lambda > 0$ and set $H(u) := \sum_{\gamma \in \mathbb{N}^n} H_\gamma u^\gamma$, $u = (u_1, \dots, u_n)$, where $H_0 = 1$ and for all $\gamma \in \mathbb{N}^n \setminus \{0\}$ we define

$$H_\gamma := \sum \frac{\alpha!}{k_1! \cdots k_l!} \lambda^{|\alpha|}, \quad (3.2.5)$$

where the summation in (3.2.5) is precisely the same as in (3.2.2). Then it follows that the power series H is locally convergent.

Proof. We define

$$G_j(u_1, \dots, u_n) := \prod_{i=1}^n \left(\frac{1}{1 - u_i} \right) - 1, \text{ for } j = 1, \dots, p$$

and

$$F(z_1, \dots, z_p) := \prod_{j=1}^p \left(\frac{1}{1 - \lambda z_j} \right).$$

Hence

$$G_j(u) = \prod_{i=1}^n \left(\sum_{k=0}^{\infty} u_i^k \right) - 1 = \sum_{\delta \in \mathbb{N}^n \setminus \{0\}} u^\delta, \forall j,$$

and similarly

$$F(z) = \prod_{j=1}^p \left(\sum_{k=1}^{\infty} (\lambda z_j)^k \right) = \sum_{\alpha \in \mathbb{N}^p} \lambda^{|\alpha|} z^\alpha.$$

If we apply (3.2.2) then $H = F \circ G$. Note that in this situation, by the expressions for $G_j(u)$ and $F(z)$ above, we have $f_\alpha(g(x)) = \lambda^{|\alpha|}$ for all α , and $g_\delta(x) = 1$ for all δ .

□

With these preparations we can prove now the closedness under composition:

Theorem 3.2.7 [1, Theorem 4.7., p. 11-12] *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$ be open. Now let $f \in \mathcal{E}_{\{M\}}(V)$ and $g = (g_1, \dots, g_p) : U \rightarrow V$, where $g_j \in \mathcal{E}_{\{M\}}(U)$, $j = 1, \dots, p$. Then also $f \circ g \in \mathcal{E}_{\{M\}}(U)$.*

Proof. Let K be a compact subset of U . Then by the definition of the spaces $\mathcal{E}_{\{M\}}(V)$ resp. $\mathcal{E}_{\{M\}}(U)$ there exist constants $a, b, c, d > 0$ in $\mathbb{R}$, such that

$$|f_\alpha(y)| \leq a \cdot b^{|\alpha|} \cdot m_{|\alpha|}, \forall y \in g(K) \subseteq V, \alpha \in \mathbb{N}^p$$

and

$$|g_{j,\delta}(x)| \leq c \cdot d^{|\delta|} \cdot m_{|\delta|}, \forall x \in K, \delta \in \mathbb{N}^n, j = 1, \dots, p.$$

Now define $h := f \circ g$ and let $\gamma \in \mathbb{N}^n \setminus \{0\}$. By the proven results above we obtain for $x \in K$:

$$\begin{aligned} |h_\gamma(x)| &\stackrel{(3.2.2)}{\leq} a \cdot \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot (b \cdot c)^{|\alpha|} \cdot \delta^{|\gamma|} \cdot m_{|\alpha|} \cdot m_{|\delta_1|}^{[k_1]} \dots m_{|\delta_l|}^{[k_l]} \\ &\stackrel{(3.2.4)}{\leq} a \cdot d^{|\gamma|} \cdot m_{|\gamma|} \cdot \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot (b \cdot c \cdot m_1)^{|\alpha|}. \end{aligned}$$

By the previous lemma 3.2.6 it follows that there exist constants $C, D > 0$, which depend only on $b \cdot c \cdot m_1, n$ and p and such that we can estimate

$$\sum \frac{\alpha!}{k_1! \dots k_l!} \cdot (b \cdot c \cdot m_1)^{|\alpha|} \leq C \cdot D^{|\gamma|}, \forall \gamma \in \mathbb{N}^n \setminus \{0\}. \quad (3.2.6)$$

The summation in the previous sums is always the same as in (3.2.2). If we combine now the estimate for h_γ above with (3.2.6) we get the result:

$$|h_\gamma(x)| \leq a \cdot C \cdot (d \cdot D)^{|\gamma|} \cdot m_{|\gamma|}.$$

Note that this proof is also valid for the Beurling-case $\mathcal{E}_{(M)}$.

□

If one introduces the spaces via function germs then the following consequence holds:

Proposition 3.2.8 [24, Proposition 1, p. 3] *Let $(m_k)_k$ be a log. convex weight sequence such that $m_0 = 1$ and $m_k \geq 1$ holds for all $k \in \mathbb{N}$. Then the space $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ is a local ring, i.e. there exists a maximal ideal. This maximal ideal is given by $\mathcal{M}_{\{M\}} := \{h \in \mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C}) : h(0) = 0\}$.*

Proof. First we remark that by 2.0.8 the space $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ is a commutative ring. If we take a function $h \in \mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$ such that $h(0) \neq 0$, then it has to be invertible in $\mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$. This can be seen as follows: Define $f(x) := h(x) - h(0)$ and $g(t) := (h(0) + t)^{-1}$. Note that now $g \in \mathcal{O}(\mathbb{R} \supseteq \{0\}, \mathbb{C}) \subseteq \underbrace{\mathcal{E}_{\{M\}}(\mathbb{R} \supseteq \{0\}, \mathbb{C})}_{m_k \geq 1, \forall k}$ holds and so, because the ring $\mathcal{E}_{\{M\}}(\mathbb{R} \supseteq \{0\}, \mathbb{C})$ is closed under composition by 3.2.7 and $f(0) = h(0) - h(0) = 0$, we obtain $g \circ f \in \mathcal{E}_{\{M\}}(\mathbb{R}^n \supseteq \{0\}, \mathbb{C})$. Finally we have

$$(g \circ f)(x) = (h(0) + f(x))^{-1} = h(x)^{-1}.$$

□

3.3 Inverse function theorem

We use the techniques introduced in the previous chapter to prove the inverse function theorem for the class $\mathcal{E}_{\{M\}}$. Again we assume in this section that

$$(m_k)_k \text{ is logarithmic convex.}$$

Definition 3.3.1 Assume that for all open $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, we have an $\mathbb{R}$ -subalgebra $\mathcal{C}(U)$ of $\mathcal{E}(U)$. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$ be open, so $\mathcal{C}(U)$ and $\mathcal{C}(V)$ are $\mathbb{R}$ -subalgebras of $\mathcal{E}(U)$ resp. of $\mathcal{E}(V)$. A mapping $f : U \rightarrow V$, $f(x) := (f_1(x), \dots, f_p(x))$, is called a $\mathcal{C}$ -mapping if $g \circ f \in \mathcal{C}(U)$, $\forall g \in \mathcal{C}(V)$.

If we have $\mathcal{C}(U) = \mathcal{E}_{\{M\}}(U)$ then, by the log. convexity for $(m_k)_k$ and 3.2.7, definition 3.3.1 means that $f_i \in \mathcal{E}_{\{M\}}(U)$ for all i . We formulate and prove now the main theorem in this chapter:

Theorem 3.3.2 [1, Theorem 4.10., p. 13-15] Let $U, V \subseteq \mathbb{R}^n$ be open and let $f = (f_1, \dots, f_n) : U \rightarrow V$ be an $\mathcal{E}_{\{M\}}$ -mapping. Let $x_0 \in U$ be an arbitrary point and suppose that the Jacobian matrix $\frac{\partial f}{\partial x}(x_0)$ is invertible. Then there exists neighbourhoods U' of x_0 and V' of $y_0 := f(x_0)$ and an $\mathcal{E}_{\{M\}}$ -mapping $g = (g_1, \dots, g_n) : V' \rightarrow U'$ such that $g(y_0) = x_0$ and $f \circ g = \text{id}_{V'}$.

A generalization of this theorem will be proven in 3.4.5.

Proof. By the usual inverse function theorem we can assume that f has an $\mathcal{E}$ -inverse. Let now K be a compact subset of U . Then, by $f_i \in \mathcal{E}_{\{M\}}(U)$, $i = 1, \dots, n$, there are constants $a, b > 0$ such that for all $\alpha \in \mathbb{N}^n$, $x \in K$ and $i = 1, \dots, n$:

$$|f_{i,\alpha}(x)| \leq a \cdot b^{|\alpha|} \cdot m_{|\alpha|}. \quad (3.3.1)$$

Let $x_0 \in K$ and consider the smooth solution $x = g(y)$ of the equation $f(x_0 + x) = f(x_0) + y$.

We have to show the existence of constants $c, d > 0$ which are *independent* of the point $x_0 \in K$ such that the following inequality holds for all $\beta \in \mathbb{N}^n$ and $j = 1, \dots, n$:

$$|g_{j,\beta}(0)| \leq c \cdot d^{|\beta|} \cdot m_{|\beta|}. \quad (3.3.2)$$

First define $\varphi : U \rightarrow \mathbb{R}^n$ by:

$$\underbrace{f(x_0 + x) - f(x_0)}_y = f'(x_0) \cdot x - f'(x_0) \cdot \varphi(x),$$

which means

$$\varphi(x) = x - f'(x_0)^{-1} \underbrace{(f(x_0 + x) - f(x_0))}_y.$$

Now define $\Theta(x) := (\theta_{i,j}(x))_{1 \leq i,j \leq n} = f'(x)^{-1}$ for $x \in \mathbb{R}^n$ and we obtain

$$\underbrace{g(y)}_x = \Theta(x_0) \cdot y + \varphi \underbrace{(g(y))}_x.$$

From this we get $g(0) = 0$, $\varphi(0) = 0$ and $\varphi'(0) = 0$, because:

$$\varphi'(0)(v) = v - f'(x_0)^{-1}(f'(x_0)(v) - 0) = v - v = 0.$$

For $i = 1, \dots, n$ and $|\alpha| \geq 2$ we have

$$\varphi_{i,\alpha}(0) = - \sum_{j=1}^n \theta_{i,j}(x_0) f_{j,\alpha}(x_0). \quad (3.3.3)$$

Now choose a real number $r > 0$ such that $\underbrace{|\theta_{i,j}(x)|}_{\in \mathbb{R}^n} \leq r$ for all $i, j = 1, \dots, n$ and $x \in K$. If we estimate (3.3.3) it follows that for all i, α there exist $a, b > 0$ such that

$$|\varphi_{i,\alpha}(0)| \leq n \cdot r \cdot a \cdot b^{|\alpha|} \cdot m_{|\alpha|}, \quad (3.3.4)$$

$$\text{because: } |\varphi_{i,\alpha}(0)| \leq \sum_{j=1}^n \underbrace{|\theta_{i,j}(x_0)|}_{\leq r} \cdot \underbrace{|f_{j,\alpha}(x_0)|}_{\leq a \cdot b^{|\alpha|} \cdot m_{|\alpha|}, \forall j}.$$

Then we remark, that $g(y) = \Theta(x_0) \cdot y + (\varphi \circ g)(y)$ and so the first term is linear in the variable y . Hence for $|\gamma| \geq 2$ we obtain $g_\gamma = (\varphi \circ g)_\gamma$. We apply now (3.2.2) for the function g at the point 0 where $|\gamma| \geq 2$:

$$g_\gamma(0) = \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot \varphi_\alpha(0) \cdot g_{\delta_1}(0)^{k_1} \dots g_{\delta_l}(0)^{k_l}.$$

In this sum only the terms with $|\alpha| \geq 2$ are not zero and so only the $g_{\delta_j}(0)$ with $|\delta_j| < |\gamma|$ survives (because of the summation in (3.2.2)).

Let $\Phi(x) := \sum_\alpha \Phi_\alpha x^\alpha$ be the power series which is defined by

$$\Phi(x) := nra \cdot \sum_{|\alpha| \geq 2} (m_1 \cdot b)^{|\alpha|} \cdot x^\alpha.$$

The radius of convergence of Φ is $\frac{1}{m_1 \cdot b} > 0$ and so this power series is *locally convergent*. Consider now the following system of equations for the function $G = (G_1, \dots, G_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$:

$$G_i(y) = \frac{r}{m_1} \cdot (y_1 + \dots + y_n) + \Phi(G(y)), \quad i = 1, \dots, n, \quad (3.3.5)$$

where $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

For this we write $G_i(y) = \sum_\gamma G_{i,\gamma} y^\gamma$ and we use again the following notation for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$:

$$G(y)^\alpha = (G_1(y)^{\alpha_1}, \dots, G_n(y)^{\alpha_n}) = \prod_{i=1}^n G_i(y)^{\alpha_i}.$$

First we see that $G_1 = \dots = G_n$ has to be satisfied since the equation (3.3.5) is independent of i on the right hand side. Thus if we want to solve (3.3.5) we have to consider the following implicit equation:

$$G_1(y) = \frac{r}{m_1} \cdot (y_1 + \dots + y_n) + \underbrace{\Phi(G_1(y), \dots, G_1(y))}_{n\text{-times}}.$$

If we introduce the function $\Phi_1 : \mathbb{R} \rightarrow \mathbb{R}$, where $\Phi_1(x) := \underbrace{\Phi(x, \dots, x)}_{n\text{-times}}$, we can write $((\text{id} - \Phi_1) \circ$

$G_1)(y) = \frac{r}{m_1} \cdot (\sum_{i=1}^n y_i)$ for $y = (y_1, \dots, y_n)$. Because Φ_1 is locally convergent around the point 0 we can take the inverse, hence finally we get

$$G_1(y) = (\text{id} - \Phi_1)^{-1} \left(\frac{r}{m_1} \cdot \sum_{i=1}^n y_i \right).$$

We see that the solution G of (3.3.5) exists and it is locally convergent. Thus for all i the radius of convergence of the power series $G_i(y) = \sum_\gamma G_{i,\gamma} y^\gamma$ is strict positive, which means that $\limsup_{|\gamma| \rightarrow \infty} |G_{i,\gamma}|^{1/|\gamma|}$ is bounded for all i .

So it follows that there exist constants c and d which depend only on the parameters n, m_1, r, a and b , such that we can estimate for all i and γ :

$$|G_{i,\gamma}| \leq c \cdot d^{|\gamma|}. \quad (3.3.6)$$

3 Stability properties

Note that by (3.3.5) and the calculations above we see that all $G_{i,\gamma}$ are nonnegative. We have to prove now for all $|\gamma| \geq 1$:

$$|g_{i,\gamma}(0)| \leq G_{i,\gamma} \cdot m_{|\gamma|}. \quad (3.3.7)$$

From (3.3.6) and (3.3.7) we get (3.3.2) and this finishes the proof. To show (3.3.7) we use induction on $|\gamma|$.

First consider $\gamma = (j) := (0, \dots, \underbrace{1}_j, \dots, 0)$ and so we have:

$$|g_{i,(j)}(0) = |\theta_{i,j}(x_0)| \leq r \underbrace{=}_{(3.3.5)} G_{i,(j)} \cdot m_1.$$

Now we start with the induction step, the summation is always the same as in (3.2.2):

$$\begin{aligned} |g_{i,\gamma}(0)| &\stackrel{(3.2.2)}{\leq} \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot |\varphi_{i,\alpha}(0)| \cdot |g_{\delta_1}(0)^{k_1}| \dots |g_{\delta_l}(0)^{k_l}| \\ &\stackrel{(3.3.4)}{\leq} \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot nrab^{|\alpha|} \cdot m_{|\alpha|} \cdot |g_{\delta_1}(0)^{k_1}| \dots |g_{\delta_l}(0)^{k_l}| \\ &\stackrel{I.H.}{\leq} \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot nrab^{|\alpha|} \cdot m_{|\alpha|} \cdot G_{\delta_1}^{k_1} \dots G_{\delta_l}^{k_l} \cdot m_{|\delta_1|}^{k_1} \dots m_{|\delta_l|}^{k_l} \\ &\stackrel{\text{def. of } \Phi_\alpha}{=} \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot \Phi_\alpha \cdot G_{\delta_1}^{k_1} \dots G_{\delta_l}^{k_l} \cdot \frac{m_{|\alpha|}}{m_1^{|\alpha|}} \cdot m_{|\delta_1|}^{k_1} \dots m_{|\delta_l|}^{k_l} \\ &\stackrel{(3.2.4)}{\leq} m_{|\gamma|} \cdot \sum \frac{\alpha!}{k_1! \dots k_l!} \cdot \Phi_\alpha \cdot G_{\delta_1}^{k_1} \dots G_{\delta_l}^{k_l} \\ &= G_{i,\gamma} \cdot m_{|\gamma|}. \end{aligned}$$

In the last equality we have used the Faà de Bruno formula (3.2.2) for the $G_{i,\gamma}$, which are the coefficients of the G_i in (3.3.5). □

Remark 3.3.3 1. We have also proven automatically that the implicit function theorem is valid in $\mathcal{E}_{\{M\}}$: Therefore we apply 3.3.2 to the inverse of the function $(x, y) \mapsto (x, f(x, y))$.

2. The proof above is not valid for the Beurling-case $\mathcal{E}_{(M)}$ because then for all $b > 0$ there exists $a > 0$ such that (3.3.1) is satisfied for all α, i and $x \in K$. But in this case we would obtain for all $b > 0$ a family Φ_b with radius of convergence $\frac{1}{m_1 \cdot b} > 0$ and a family of systems (3.3.5) with solutions G_b . To show now that the inverse function g is an element in $\mathcal{E}_{(M)}$, too, (3.3.6) has to hold for all $d > 0$, in particular one would need that G_b is an entire function for all $b > 0$ which is not true.

3. 3.3.2 is not true for the Beurling-case, because for the sequence $M_p := p!$, for all $p \in \mathbb{N}$, the associated function class $\mathcal{E}_{(M)}$ are the entire functions and this class is not closed under inversion (take for example $\sinh : \mathbb{R} \rightarrow \mathbb{R}$, which is an entire function, but $\sinh^{-1}$ is not entire any more).

3.4 Closedness under solving ODE's

In this section we are going to prove the property *closedness under solving ODE's* for the spaces $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{(M)}$. In general both types are not closed under solving ODE's, one has to assume special conditions on the weight sequence $M := (M_p)_p$ to prove this property. There exists a proof by Komatsu (see [14]), but here we will follow the proof of Yamanaka (see [26]): In particular we will prove a Banach space version for the theorem of closedness under solving ODE's in the Romieu-case. Afterwards we will use a technique by Komatsu to prove the theorem for the Beurling-case, too. We formulate now our central theorem in this section:

Theorem 3.4.1 *Let $(M_p)_p$ be a weight sequence, such that $M_0 = M_1 = 1$, $M_2 \geq 2$ is satisfied and assume that there exists $H \geq 1$ such that $\left(\frac{M_q}{q!}\right)^{1/(q-1)} \leq H \cdot \left(\frac{M_p}{p!}\right)^{1/(p-1)}$ for all $2 \leq q \leq p$. Let the initial value problem*

$$\begin{cases} x'(t) = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

be given and assume that $f \in \mathcal{E}_{\{M\}}$ holds. Then we obtain for the solution $x \in \mathcal{E}_{\{M\}}$, wherever the smooth solution x exists, too.

Before we start with the proof of 3.4.1 we study and compare some conditions for weight sequences and we have to prove some results.

So let $(M_p)_p$ be a weight sequence, then we consider the following conditions

$$M_0 = M_1 = 1 \tag{3.4.1}$$

$$\left(\frac{M_q}{q!}\right)^{\frac{1}{q-1}} \leq \left(\frac{M_p}{p!}\right)^{\frac{1}{p-1}} \Leftrightarrow m_q^{\frac{1}{q-1}} \leq m_p^{\frac{1}{p-1}}, \quad 2 \leq q \leq p. \tag{3.4.2}$$

Furthermore we recall the log. convexity condition for the sequence $(m_p)_p$

$$\left(\frac{M_p}{p!}\right)^2 \leq \left(\frac{M_{p-1}}{(p-1)!}\right) \cdot \left(\frac{M_{p+1}}{(p+1)!}\right) \Leftrightarrow m_p^2 \leq m_{p-1} \cdot m_{p+1} \quad p \in \mathbb{N}, \tag{3.4.3}$$

and

$$\frac{M_p}{p \cdot M_{p-1}} \longrightarrow \infty, \quad \text{for } p \longrightarrow \infty. \tag{3.4.4}$$

We see immediately: Condition (3.4.2) is equivalent to $\frac{1}{q-1} \cdot \log(m_q) \leq \frac{1}{p-1} \cdot \log(m_p)$ for $2 \leq q \leq p$. Hence the log. convexity condition for $(m_p)_p$ implies (3.4.2).

If one assumes (3.4.1) and the log. convexity for the sequence $(m_p)_p$, then $\log(m_2) \geq 0 \Leftrightarrow m_2 \geq 1 \Leftrightarrow M_2 \geq 2$ has to be satisfied.

$\frac{M_p}{p \cdot M_{p-1}} = \frac{p! \cdot m_p}{p \cdot (p-1)! \cdot m_{p-1}} = \frac{m_p}{m_{p-1}}$, hence (3.4.4) is the property, that the sequence $\left(\frac{m_p}{m_{p-1}}\right)_p$ is unbounded. Another equivalent useful description for (3.4.4) is $\lim_{p \rightarrow \infty} \frac{\mu_p}{p} = \infty$.

Komatsu proved in [14] the closedness under solving ODE's: For the Romieu-case he assumed (3.4.1) and (3.4.2) and for the Beurling-case he assumed additionally (3.4.3) and (3.4.4).

We study now condition (3.4.2) in detail.

For this we remark that the condition that the sequence $(m_p^{1/p})_p$ is increasing can be written in the following form:

$$m_q^{1/q} \leq m_p^{1/p} \text{ for } 1 \leq q \leq p \Leftrightarrow \left(\frac{M_q}{q!}\right)^{1/q} \leq \left(\frac{M_p}{p!}\right)^{1/p} \text{ for } 1 \leq q \leq p, \tag{3.4.5}$$

which looks similar to (3.4.2). If additionally (3.4.1) is satisfied, then (3.4.5) implies again $M_2 \geq 2$. In particular we have: If $M_1 = 1$, then (3.4.5) implies $M_p \geq p!$ for all $p \geq 1$.

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We prove now: If $m_1 = M_1 = 1$ and $m_2 \geq 1 \Leftrightarrow M_2 \geq 2$, then (3.4.2) implies (3.4.5):

$$(3.4.2) \Leftrightarrow \frac{1}{q-1} \cdot \log(m_q) \leq \frac{1}{p-1} \cdot \log(m_p) \Leftrightarrow \log(m_q) \leq \frac{q-1}{p-1} \cdot \log(m_p) \text{ for } 2 \leq q \leq p,$$

and

$$(3.4.5) \Leftrightarrow \frac{1}{q} \cdot \log(m_q) \leq \frac{1}{p} \cdot \log(m_p) \Leftrightarrow \log(m_q) \leq \frac{q}{p} \cdot \log(m_p) \text{ for } 1 \leq q \leq p.$$

To get (3.4.2) $\Rightarrow$ (3.4.5) we distinguish now three cases: For $2 \leq q \leq p$ we are done because $\frac{q-1}{p-1} \leq \frac{q}{p} \Leftrightarrow p \geq q$. If $p = 1$, then $q = 1$ and in this case (3.4.5) is clearly satisfied. If $q = 1$ and $p \geq 2$, then we have by assumption $m_1 = 1 \leq m_2$ and so, by monotony, we are in the first case.

We summarize:

1. If the sequence $(m_p)_p$ is log. convex and $m_0 = 1$, then by 2.0.4 the sequence $(m_p^{\frac{1}{p}})_p$ is increasing.
2. If (3.4.2), $m_1 = 1$ and $m_2 \geq 1$ are satisfied, then $(m_p^{\frac{1}{p}})_p$ is increasing.

Furthermore we remark, that (3.4.2) implies $\log(m_p) \geq \log(m_q)$, for $2 \leq q \leq p$, because:

$$(3.4.2) \Leftrightarrow \frac{1}{q-1} \cdot \log(m_q) \leq \frac{1}{p-1} \cdot \log(m_p) \Leftrightarrow \log(m_p) \geq \frac{p-1}{q-1} \cdot \log(m_q).$$

Now set $q := p-1$ to obtain $\log(m_p) \geq \frac{p-1}{p-2} \cdot \log(m_{p-1})$ for all $p \geq 3$ and we have $\frac{p}{p-1} \leq \frac{p-1}{p-2} \Leftrightarrow p \cdot (p-2) \leq (p-1)^2$ for all $p \geq 3$.

We summarize again:

1. If the sequence $(m_p)_p$ is log. conv. and $m_0 = 1$, then by 2.0.4 $\log(m_p) \geq \frac{p}{p-1} \cdot \log(m_{p-1})$ is satisfied for all $p \geq 2$.
2. (3.4.2) implies $\log(m_p) \geq \frac{p}{p-1} \cdot \log(m_{p-1})$ for all $p \geq 3$.

As we have already mentioned the logarithmic convexity for the sequence $(m_p)_p$ implies (3.4.2). In this way we can say that condition (3.4.2) is closely related to the log. convexity condition for $(m_p)_p$. But (3.4.2) is strict weaker than the log. convexity for $(m_p)_p$: If (3.4.2) is satisfied, then the points $(p, \log(m_p))$ can lie away from $(1, 0)$ on a locally strict concave curve.

We have shown that (3.4.2) implies (3.4.5). We introduce now

$$\left(\frac{M_{q-1}}{q!} \right)^{1/(q-1)} \leq \left(\frac{M_{p-1}}{p!} \right)^{1/(p-1)} \text{ for } 2 \leq q \leq p, \quad (3.4.6)$$

which is condition (3.4.2) for the sequence $(N_p)_p$, where $N_p := M_{p-1}$ for all p . We obtain (3.4.5) $\Rightarrow$ (3.4.6) because

$$(3.4.5) \Leftrightarrow M_q^{1/q} \leq M_p^{1/p} \cdot \frac{q!^{1/q}}{p!^{1/p}} \text{ for } 1 \leq q \leq p$$

resp.

$$(3.4.6) \Leftrightarrow M_q^{1/q} \leq M_p^{1/p} \cdot \frac{q!^{1/q}}{p!^{1/p}} \cdot \frac{(q+1)^{1/q}}{(p+1)^{1/p}} \text{ for } 1 \leq q \leq p$$

and $(q+1)^{1/q} \geq (p+1)^{1/p} \Leftrightarrow \frac{1}{q} \cdot \log(q+1) \geq \frac{1}{p} \cdot \log(p+1)$ is satisfied for $1 \leq q \leq p$.

In this section, if it is not stated otherwise, we will assume always the condition

$$\exists H \geq 1 : \left(\frac{M_q}{q!} \right)^{1/q-1} \leq H \cdot \left(\frac{M_p}{p!} \right)^{1/p-1} \quad \text{for } 2 \leq q \leq p, \quad (3.4.7)$$

which is weaker than (3.4.2), condition (3.4.1) ($\Leftrightarrow M_0 = M_1 = 1$) and additionally $M_2 \geq 2$. We see: (3.4.7) implies

$$\exists H \geq 1 : \left(\frac{M_q}{q!} \right)^{1/q} \leq H \cdot \left(\frac{M_p}{p!} \right)^{1/p} \quad \text{for } 1 \leq q \leq p \quad (3.4.8)$$

and remark that (3.4.8) implies only $M_2 \geq H^{-2} \cdot 2 \cdot M_1$ where $H \geq 1$.

As mentioned above we will prove a Banach-space version, hence for this we have to introduce some further notation: Let E and F be two real Banach-spaces with norms $\|\cdot\|_E$ resp. $\|\cdot\|_F$. Let $U \subseteq E$ be open and $f : E \supseteq U \rightarrow F$. Suppose that f is an infinitely differentiable function in the sense of Fréchet and let $(M_n)_n$ be an arbitrary weight sequence. Then $f \in \mathcal{E}_{\{M\},U}$, if and only if there exist $C, h > 0$ such that

$$\|f^{(n)}(x)\|_{L^n(E;F)} \leq C \cdot h^n \cdot M_n$$

holds for all $x \in U$ and $n \in \mathbb{N}$. In the same way we have $f \in \mathcal{E}_{(M),U}$ if for all $h > 0$ there exists $C > 0$ such that for all $x \in U$ and $n \in \mathbb{N}$ we have

$$\|f^{(n)}(x)\|_{L^n(E;F)} \leq C \cdot h^n \cdot M_n.$$

We remark: The estimates for $\mathcal{E}_{\{M\},U}$ resp. $\mathcal{E}_{(M),U}$ are global, no compact set K is involved in the defining estimates and we use here the operator norm in the defining inequalities. This is an alternative method to introduce function spaces of Romieu- resp. Beurling-type and it is equivalent to the definitions given in the first Chapter.

In this section the differentiability of functions between Banach-spaces has always to be understood in the sense of Fréchet.

With this notation one can easily show that the classes $\mathcal{E}_{\{M\},E}$ resp. $\mathcal{E}_{(M),E}$ are closed under affine linear transformations: Let $g : E \rightarrow E, x \mapsto A(x) + b$, where $A \in L(E, E)$ and $b \in E$, then we have by the chainrule $(f \circ g)'(x) = f'(A(x) + b) \circ A$. Hence $(f \circ g)^{(p)}(x) = f^{(p)}(A(x) + b) \circ A^p$ for all $p \geq 1$ and so we can estimate:

$$\|(f \circ g)^{(p)}(x)\|_{L^p(E;F)} \leq \|f^{(p)}(A(x) + b)\|_{L^p(E;F)} \cdot \|A\|_{L(E;E)}^p.$$

Let E and F be real Banach spaces, $U \subseteq E$ open and $g : E \supseteq U \rightarrow F$ be infinitely differentiable. Furthermore let $V \subseteq \mathbb{R}$ open, $b \in V$, be given and $G : \mathbb{R} \supseteq V \rightarrow \mathbb{R}$ a smooth function. Then we write

$$g \lll G : \Leftrightarrow \left\| g^{(p)}(x) \right\|_{L^p(E;F)} \leq G^{(p)}(b) \quad \text{for } 0 \leq p \leq l, x \in U.$$

If $g' \lll G'$ holds, which is a clearly weaker condition than $g \lll G$, we write $g \ll G$. An easy consequence of the Faà-de-Bruno-Formula (3.2.2) is the following:

Lemma 3.4.2 [26, Lemma 5.1., p. 608] *Let E_1, E_2 and E_3 be three real Banach-spaces, $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ open and $f : E_2 \supseteq U_2 \rightarrow E_3$, $g : E_1 \supseteq U_1 \rightarrow E_2$ be two infinitely differentiable mappings. Let $V_1, V_2 \subseteq \mathbb{R}$ open and $F : V_2 \rightarrow \mathbb{R}$, $G : V_1 \rightarrow V_2$ be two smooth mappings such that $f \lll F$ and $g \ll G$ holds for a point $a \in V_1$ with $V_2 \ni b = G(a)$. Then we get $f \circ g \lll F \circ G$.*

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Proof. Let $x \in U_1$, then we have by assumption $\|(f \circ g)(x)\|_{E_3} \leq (F \circ G)(a)$. For $1 \leq p \leq N$ we put for $q = (q_1, \dots, q_{p-j+1}) \in \mathbb{N}^{p-j+1}$: $|q| := \sum_{i=1}^{p-j+1} q_i$ and $\|q\| := \sum_{i=1}^{p-j+1} i \cdot q_i$. Now we estimate as follows:

$$\begin{aligned} \|(f \circ g)^{(p)}(x)\|_{L^p(E_1; E_3)} &\stackrel{(3.2.2)}{\leq} p! \sum_{j=1}^p \|f^{(j)}(g(x))\|_{L^j(E_2; E_3)} \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \left(\frac{\|g^{(i)}(x)\|_{L^i(E_1; E_2)}}{i!} \right)^{q_i} \\ &\leq p! \cdot \sum_{j=1}^p F^{(j)}(b) \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot \left(\frac{G^{(i)}(a)}{i!} \right)^{q_i} = (F \circ G)^{(p)}(a). \end{aligned}$$

□

Let $E_1, \dots, E_n$ and F be real Banach-spaces, put $E := E_1 \times \dots \times E_n$ and consider an open set $U \subseteq E$. Let $g : E \supseteq U \rightarrow F$ be an infinitely differentiable function and $V \subseteq \mathbb{R}^n$. Now fix a point $a := (a_1, \dots, a_n) \in V$ and consider a smooth mapping $G : \mathbb{R}^n \supseteq V \rightarrow \mathbb{R}$. Take $\alpha \in \mathbb{N}^n$, then we write

$$g \ll_{(U, \alpha, a)} G$$

if and only if $\|g^{(\beta)}(x)\|_{L^\beta(E; F)} \leq G^{(\beta)}(a)$ holds for all $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ with $\beta_i \leq \alpha_i$, $1 \leq i \leq n$, and all $x \in U$, where $L^\beta(E; F) = L^{\beta_1, \dots, \beta_n}(E_1, \dots, E_n; F) \cong \oplus_{i=1}^n L^{\beta_i}(E_i; F)$. (The letter c in the definition is standing for componentwise.) Similarly we write $g \ll_c G$, if

$\|g^{(\beta)}(x)\|_{L^\beta(E; F)} \leq G^{(\beta)}(a)$ holds only for all $\beta \in \mathbb{N}^n \setminus \{0\}$ with $\beta_i \leq \alpha_i$, $1 \leq i \leq n$, and all $x \in U$. With this notation we can formulate the following lemma:

Lemma 3.4.3 [26, Lemma 5.4., p. 610-611] *Let $E_1, \dots, E_n, F$ and G be real Banach-spaces. Set $E := E_1 \times \dots \times E_n$ and let $U_1 \subseteq E$ and $U_2 \subseteq F$ be two open sets. Let $h_1 : U_1 \rightarrow U_2$ and $h_2 : U_2 \rightarrow G$ be infinitely differentiable functions and $V_1 \subseteq \mathbb{R}^n$, $V_2 \subseteq \mathbb{R}$ be two open subsets. Furthermore let $H_1 : \mathbb{R}^n \supseteq V_1 \rightarrow V_2$ and $H_2 : V_2 \rightarrow \mathbb{R}$ be two smooth functions and $a = (a_1, \dots, a_n) \in V_1$ a point such that $b = H_1(a)$ holds. We assume that for $\alpha \in \mathbb{N}^n$ the following holds: $h_2 \ll_{(U_2, |\alpha|, b)} H_2$ and $h_1 \ll_{(U_1, \alpha, a)} H_1$. Then we get:*

$$h_2 \circ h_1 \ll_{(U_1, \alpha, a)} H_2 \circ H_1.$$

Proof. One uses induction on n (the case $n = 1$ is 3.4.2), and again (3.2.2). For details look at [26, p. 610-611].

□

In the proof of the theorem 3.4.1 about the closedness under solving ODE's we will use the following version of the implicit function theorem for Banach-spaces which generalizes 3.3.2. Again (3.2.2) will play the key role in the proof and we will use the *Lagrange inversion theorem*:

Proposition 3.4.4 *Let the equation*

$$y = a + x \cdot \varphi(y) \tag{3.4.9}$$

be given, where φ is analytic around the point $a \in \mathbb{R}$. We denote by $\gamma : x \mapsto y$ the solution of (3.4.9), which is analytic by the classical implicit function theorem. Furthermore let f be a smooth function, which is analytic around a , too, then we obtain the following formula for $n \geq 1$:

$$(f \circ \gamma)^{(n)}(0) = \left(\varphi^n \cdot f' \right)^{(n-1)}(a).$$

Proof. A proof can be found in [5, Chapitre IX., no. 195, p. 481-484].

□

Theorem 3.4.5 [25, Theorem 2, p. 200-202] *Let E and F be two real Banach-spaces, $U \subseteq E$ and $V \subseteq F$ open subsets and $f : U \rightarrow V$ an infinitely differentiable function. We assume that $f \in \mathcal{E}_{\{M\},U}$. Let $x_0 \in U$ and assume that $(f'(x_0))^{-1} \in L(F;E)$ exists. Then there exist open sets U_0 and V_0 , such that $x_0 \in U_0 \subseteq U$ and $f(x_0) \in V_0 \subseteq V$ holds and $f : U_0 \rightarrow V_0$ is a $\mathcal{E}$ -diffeomorphism. Furthermore we have $f^{-1} \in \mathcal{E}_{\{M\},V_0}$.*

Proof. By the usual inverse mapping theorem for smooth mappings we know: There exist U_0, V_0 open subsets in E resp. F such that we have $x_0 \in U_0 \subseteq U$ resp. $f(x_0) \in V_0 \subseteq V$, and $f : U_0 \rightarrow V_0$ is a $\mathcal{E}$ -diffeomorphism. Let $g := f^{-1} : V_0 \rightarrow U_0$ and $a \in U_0$ be fixed. Put $b := f(a)$ and $S := f'(a)$, $T := S^{-1} = g'(b)$. Furthermore we put for $x \in U_0$

$$\phi(x) := x - (T \circ f)(x).$$

Hence, for $y \in V_0$, we have $g(y) = T(y) + g(y) - T(y) = T(y) + g(y) - (T \circ f)(g(y)) = T(y) + (\phi \circ g)(y)$. Then we remark, that $\phi(a) = \text{id} - (T \circ f')(a) = \text{id} - T \circ S = 0$, thus $g'(b) = T + \phi'(g(b)) \cdot g'(b) = T + \phi'(a) \cdot g'(b) = T$. For $p \geq 2$ we can use the Faà-de-Bruno formula (3.2.2) to get a recursion formula for $g^{(p)}(b)$:

$$g^{(p)}(b) = \text{sym} \left(p! \cdot \sum_{j=2}^p \phi^{(j)}(a) \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot \left(\frac{g^{(i)}(b)}{i!} \right)^{q_i} \right),$$

where sym denotes the symmetrization of a multi-linear operator and for $q = (q_1, \dots, q_{p-j+1}) \in \mathbb{N}^{p-j+1}$ we have set again $|q| := q_1 + \dots + q_{p-j+1}$ resp. $\|q\| := q_1 + 2q_2 + \dots + (p-j+1)q_{p-j+1}$. So we get immediately the following estimate:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \leq p! \sum_{j=2}^p \|\phi^{(j)}(a)\|_{L^j(E;F)} \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \left(\frac{\|g^{(i)}(b)\|_{L^i(F;E)}}{i!} \right)^{q_i}. \quad (3.4.10)$$

By assumption $f \in \mathcal{E}_{\{M\},U}$ holds, hence there exist $C, h > 0$ such that for all $x \in U_0$ and $j \geq 1$ we get

$$\|f^{(j)}\|_{L^j(E;F)} \leq C \cdot h^j \cdot M_j,$$

and finally for $j \geq 2$:

$$\|\phi^{(j)}\|_{L^j(E;F)} = \|T \circ f^{(j)}\|_{L^j(E;F)} \leq C \cdot \|T\|_{L(F;E)} \cdot h^j \cdot M_j.$$

In the following let $A \geq \max \left\{ \|g(b)\|_E, \|g'(b)\|_{L(F;E)} \right\}$ be a constant and let $N \in \mathbb{N}$ such that $N \geq 2$, we define for $t \in \mathbb{R}$ the following sum:

$$\psi_N(t) := C \cdot A \cdot \sum_{j=2}^N \frac{h^j \cdot M_j}{j!} \cdot t^j.$$

We consider the equation $s = \frac{t - \psi_N(t)}{A}$ and we want to solve it with respect to the variable t . Therefore we put $g_N(s) := \sum_{j=1}^{\infty} c_j \cdot s^j$ and consider now the equation $s = \frac{g_N(s) - \psi_N(g_N(s))}{A}$ for $s \in \mathbb{R}$ near 0, in particular we have set $g_N(s) = t$ for all small s .

For this we apply 3.4.4 as follows: We put $f := \text{id}$, $x := s$, $y := t$, $a := 0$ and $\varphi(t) := \frac{A}{1 - \psi_N(t)/t} = \frac{t}{s}$ which is analytic near $t = 0$. Then (3.4.9) is satisfied and we obtain for $n \geq 1$:

$$\left. \frac{\partial^n g_N(s)}{\partial s^n} \right|_{s=0} = \left(\frac{d^{n-1}}{dt^{n-1}} \left(\frac{A}{1 - \psi_N(t)/t} \right)^n \right) \Big|_{t=0}.$$

Hence the constants c_i , $i \geq 1$, in the series above can be expressed in the following way:

$$c_i = \frac{1}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\frac{A}{1 - \psi_N(t)/t} \right)^i \right) \Big|_{t=0}. \quad (3.4.11)$$

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On the other side we see that $g_N(s) = A \cdot s + (\psi_N \circ g_N)(s)$, hence for $p \geq 2$ we can use again the Faà de Bruno formula 3.2.2 to obtain:

$$\begin{aligned} g_N^{(p)}(0) &= p! \cdot \sum_{j=2}^p \psi_N^{(j)}(0) \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot \left(\frac{g_N^{(i)}(0)}{i!} \right)^{q_i} \\ &= p! \cdot \sum_{j=2}^p C \cdot A \cdot h^j \cdot M_j \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot c_i^{q_i}. \end{aligned}$$

The second equality above holds, because for all i we get $g_N^{(i)}(0) = i! \cdot c_i$ and for $2 \leq j \leq N$ we have $\psi_N^{(j)}(0) = C \cdot A \cdot h^j \cdot M_j$.

For $2 \leq i \leq N$ we show now for the c_i , which are all positive real numbers, the following estimate:

$$c_i < A(4A(CAh + 1)Hh)^{i-1} \cdot \frac{M_i}{i!}. \quad (3.4.12)$$

In particular we calculate for $2 \leq i \leq N$ in the following way:

$$\begin{aligned} 0 < c_i &\stackrel{(1)}{=} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} \left(CA \cdot \sum_{r=1}^{i-1} \frac{M_{r+1} \cdot h}{(r+1)!} \cdot (ht)^r \right)^s \right) \right) \Big|_{t=0} \\ &\stackrel{(3.4.7)}{\leq} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} h^s \cdot \left(CA \cdot \sum_{r=1}^{\infty} \left(H \cdot \left(\frac{M_i}{i!} \right)^{1/(i-1)} \cdot ht \right)^r \right)^s \right) \right) \Big|_{t=0} \\ &= \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} (CAh)^s \cdot \sum_{r=0}^{\infty} \binom{r+s-1}{r} \cdot \left(H \cdot \left(\frac{M_i}{i!} \right)^{1/(i-1)} \cdot ht \right)^{r+s} \right) \right) \Big|_{t=0} \\ &= \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{r=0}^{\infty} CAh \cdot (CAh + 1)^{r-1} \cdot \left(H \cdot \left(\frac{M_i}{i!} \right)^{1/(i-1)} \cdot ht \right)^r \right) \right) \Big|_{t=0} \\ &\leq \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \sum_{r=0}^{\infty} \binom{i+r-1}{r} \left((CAh + 1) \cdot H \cdot \left(\frac{M_i}{i!} \right)^{1/(i-1)} \cdot ht \right)^r \right) \Big|_{t=0} \\ &\stackrel{(2)}{=} \frac{A^i}{i!} \cdot \binom{2i-2}{i-1} \cdot ((CAh + 1) \cdot (Hh))^{i-1} \cdot (i-1)! \cdot \frac{M_i}{i!} \\ &\leq A^i \cdot 4^{i-1} \cdot ((CAh + 1) \cdot (Hh))^{i-1} \cdot \frac{M_i}{i!}. \end{aligned}$$

Recall that H in the calculation above is the constant appearing in (3.4.7).

For (1) we have used geometric series expansion and (3.4.11). Furthermore we remember: If $2 \leq i \leq N$, then $\psi_N^{(i)}(0) = C \cdot A \cdot h^i \cdot M_i$ holds and because we consider the $i-1$ -derivative on the right hand side above at the point $t = 0$, we can break off the summation at the index $i-1$.

(2) follows, because only the term for $r = i-1$ survives. The last inequality above holds, because $4^{i-1} = 2^{2i-2} = \sum_{k=0}^{2i-2} \binom{2i-2}{k}$.

In the next step we want to compare $\|g^{(p)}(b)\|_{L^p(F;E)}$ with $g_N^{(p)}(0)$ for $1 \leq p \leq N$. First we have $g'_N(0) = c_1 = A$ and so we see:

$$\|g'(b)\|_{L(F;E)} = \|T\|_{L(F;E)} \leq A = g'_N(0). \quad (3.4.13)$$

For $2 \leq p \leq N$ we have $\psi_N^{(p)}(0) = C \cdot A \cdot h^p \cdot M_p$ and so

$$\|\phi^{(p)}\|_{L^p(E;F)} \leq C \cdot \|T\|_{L(F;E)} \cdot h^p \cdot M_p \leq C \cdot A \cdot h^p \cdot M_p = \psi_N^{(p)}(0). \quad (3.4.14)$$

Finally we get by induction the following estimate for $2 \leq p \leq N$:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \underbrace{\leq}_{(3.4.10),(3.4.13),(3.4.14)} g_N^{(p)}(0) = p! \cdot c_p \underbrace{\leq}_{(3.4.12)} A(4A(CAh + 1)Hh)^{p-1} \cdot M_p.$$

But $N \in \mathbb{N}$, $N \geq 2$, was chosen arbitrary, hence for all $p \geq 2$ we obtain:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \leq A(4A(CAh + 1)Hh)^{p-1} \cdot M_p.$$

Because $M_0 = M_1 = 1$ and $\|g(b)\|_E \leq A$ resp. $\|g'(b)\|_{L(F;E)} = \|T\|_{L(F;E)} \leq A$ we have shown: $g \in \mathcal{E}_{\{M\}, V_0}$.

□

The theorem above implies immediately the following Banach-space-version of the *implicit function theorem*:

Theorem 3.4.6 [25, Theorem 3, p. 202] *Let E, F and G be real Banach-spaces, $U \subseteq E$ and $V \subseteq F$ open sets and $f : U \times V \rightarrow G$ an infinitely differentiable function such that $f(a, b) = 0$ holds for a point $(a, b) \in U \times V$. Furthermore we assume that $(f^{(0,1)}(a, b))^{-1} \in L(G; F)$ exists and*

$$\exists C, h > 0 : \|f^{(p,q)}(x, y)\|_{L^{p,q}(E, F; G)} \leq C \cdot h^{p+q} \cdot M_{p+q}$$

holds for $(p, q) \in \mathbb{N}^2$ and all $(x, y) \in U \times V$. Then there exist open sets U_0 and V_0 in E resp. in F such that $a \in U_0 \subseteq U$ and $b \in V_0 \subseteq V$ holds and there exists a unique infinitely differentiable function $g : U_0 \rightarrow V_0$ with the following properties: $g(a) = b$, $f(x, g(x)) = 0$ holds for all $x \in U_0$ and $g \in \mathcal{E}_{\{M\}, U_0}$.

Proof. We apply 3.4.5 to the inverse function of $(x, y) \mapsto (x, f(x, y))$.

□

With these preparations we can start now with the proof of 3.4.1:

First we formulate our central problem. For this let E_1, E_2 be two real Banach-spaces, then we consider the infinitely differentiable function $f : E_1 \times \mathbb{R} \times E_2 \supseteq W \rightarrow E_1$, where we assume that there exist $C, h > 0$ such that

$$\|f^{(i,j,k)}(x, t, \lambda)\|_{L^{i,j,k}(E_1, \mathbb{R}, E_2; E_1)} \leq C \cdot h^{i+j+k} \cdot M_{i+j+k} \quad (3.4.15)$$

holds for all $(i, j, k) \in \mathbb{N}^3$ and $(x, t, \lambda) \in W$. Let the following ODE be given:

$$\begin{cases} x'(t) = f(x(t), t, \lambda) \\ x(0) = x_0, \quad x_0 \in E_1. \end{cases} \quad (3.4.16)$$

We remark that (3.4.16) defines a more general initial value problem than the ODE in 3.4.1 because we have introduced here an additional parameter λ . First we simplify the given initial value problem above: First put $f_1(x(t), t, \lambda, x_0) := f(x(t) + x_0, t, \lambda)$ to get

$$\begin{cases} x'(t) = f_1(x(t), t, \lambda, x_0) \\ x(0) = 0. \end{cases} \quad (3.4.17)$$

Then define $y(t) := (x(t), \xi)$, $\mu := (\lambda, x_0)$ and finally $g(y(t), \mu) := (f_1(x(t), \xi, \lambda, x_0), 1) = (f(x(t) + x_0, \xi, \lambda), 1)$. Thus we obtain the following new ODE:

$$\begin{cases} y'(t) = g(y(t), \mu) \\ y(0) = 0, \end{cases} \quad (3.4.18)$$

3 Stability properties

where $y : \mathbb{R} \rightarrow E_1 \times \mathbb{R} =: E_3$ is the unknown function and $\mu \in E_2 \times E_1 =: E_4$ is a parameter. If f satisfies (3.4.15) with respect to (x, t, λ) , then g satisfies (3.4.15) with respect to (y, μ) . If the solution y of (3.4.18) satisfies (3.4.15) with respect to (t, μ) , then the solution x of (3.4.16) satisfies (3.4.15) with respect to (t, λ, x_0) .

Now let E_3 and E_4 be two real Banach-spaces, U be a neighborhood of the origin in the space E_3 and let V be an open set of E_4 . Let $g : E_3 \times E_4 \supseteq U \times V \rightarrow E_3$, $(y, \mu) \mapsto g(y, \mu)$, be an infinitely differentiable function with the following property: $\exists C, h \geq 1$ such that for all $(j, k) \in \mathbb{N}^2$ and $(y, \mu) \in U \times V$ we have the estimate

$$\left\| g^{(j,k)}(y, \mu) \right\|_{L^{j,k}(E_3, E_4; E_3)} \leq C \cdot h^{j+k} \cdot M_{j+k}. \quad (3.4.19)$$

We assume that the function g in (3.4.18) satisfies (3.4.19) and U is a neighborhood of the origin in E_3 , w.l.o.g. a ball with radius R .

(3.4.19) has the following consequences: The function $y \mapsto g(y, \mu)$ is bounded in the norm $\|\cdot\|_{E_3}$ by the constant C and it is *Lipschitz-continuous* on U with respect to y and uniformly for $\mu \in V$. If $|t| \leq R/C$, then g is bounded by R , so $g(y, \mu) \in U$. Thus for each $\mu \in V$ and $t \in \mathbb{R}$ with $|t| \leq R/C$ the classical existence theorem of ODE's implies the existence of a unique solution y where $y : \mathbb{R} \times E_4 \rightarrow E_3$, $(t, \mu) \mapsto y(t, \mu)$, is defined for $|t| \leq R/C$ and $\mu \in V$. It's well known that y is infinitely differentiable in (t, μ) , and to prove 3.4.1 we have to estimate the derivatives of y .

In particular we have to show: For $(t, \mu) \in I \times V$, where I is an open real interval with $0 \in I$, there exist constants $B, \xi > 0$, such that

$$\left\| y^{(j,k)}(t, \mu) \right\|_{L^{j,k}(\mathbb{R}, E_4; E_3)} \leq B \cdot \xi^{j+k} \cdot M_{j+k} \quad (3.4.20)$$

is satisfied.

We start now with the proof of (3.4.20).

In the first step of the proof we estimate $y^{(0,l)}$, $l \in \mathbb{N}$ arbitrary.

Let I be a compact interval such that $0 \in I$ and let $\mathcal{C}(I, E_3)$ be the space of all continuous functions from I to E_3 . The space $\mathcal{C}(I, E_3)$ with the topology which is induced by the maximum norm becomes a Banach-spaces and we put $\mathcal{C}(I, U) := \{y \in \mathcal{C}(I, E_3) : y(I) \subseteq U\}$. Let $(y, \mu, t) \in \mathcal{C}(I, U) \times V \times I$, then we define the following integral

$$\alpha(y, \mu)(t) := \int_0^t g(y(\tau), \mu) d\tau$$

and obtain a function $\alpha : \mathcal{C}(I, U) \times V \rightarrow \mathcal{C}(I, E_3)$. Let $T \in \mathbb{R}$ be given with $0 < T < R/C$ and put $I_T := [-T, T]$. We define the sets $\mathcal{B}_{E_3, R} := \{y \in E_3 : \|y\|_{E_3} \leq R\}$ and $\mathcal{B}_{E_3, R}^\circ := \{y \in E_3 : \|y\|_{E_3} < R\}$. In the following we want to apply *Banach's fix point theorem* to the mapping α .

Claim: $\alpha : \mathcal{C}(I_T, \mathcal{B}_{E_3, R}) \times V \rightarrow \mathcal{C}(I_T, \mathcal{B}_{E_3, R})$ resp. $\alpha : \mathcal{C}(I_T, \mathcal{B}_{E_3, R}^\circ) \times V \rightarrow \mathcal{C}(I_T, \mathcal{B}_{E_3, R}^\circ)$ holds.

Therefore we estimate

$$\left\| \int_0^t g(y(\tau), \mu) d\tau \right\|_{E_3} \leq \int_0^t \|g(y(\tau), \mu)\|_{E_3} d\tau \stackrel{(3.4.19)}{\leq} C \cdot h^0 \cdot M_0 \cdot T < C \cdot (R/C) = R,$$

hence

$$\|\alpha(y, \mu)(t)\|_{E_3} = \left\| \int_0^t g(y(\tau), \mu) d\tau \right\|_{E_3} \leq R. \quad (3.4.21)$$

Claim: α is a contraction map.

If $(y, z, \mu) \in \mathcal{B}_{E_3, R} \times \mathcal{B}_{E_3, R} \times V$, then we get by (3.4.19):

$$\|g(y, \mu) - g(z, \mu)\|_{E_3} \leq C \cdot h \cdot M_1 \cdot \|y - z\|_{E_3} = C \cdot h \cdot \|y - z\|_{E_3}.$$

Thus for $(y, z, \mu) \in \mathcal{C}(I_T, \mathcal{B}_{E_3, R}) \times \mathcal{C}(I_T, \mathcal{B}_{E_3, R}) \times V$ and $T \in \mathbb{R}$ with $0 < T < (C \cdot h)^{-1}$ we can estimate:

$$\|\alpha(y, \mu) - \alpha(z, \mu)\|_{E_3} \leq C \cdot h \cdot \|y - z\|_{E_3} \cdot T < \|y - z\|_{E_3}, \quad (3.4.22)$$

which proves the claim.

We summarize: By (3.4.22) α is for $T \in \mathbb{R}$ small enough a contraction map and by (3.4.21) it is a self-mapping on $\mathcal{C}(I_T, \mathcal{B}_{E_3, R}) \subseteq \mathcal{C}(I_T, E_3)$, which is a closed subset of a Banach-space. So we can use *Banach's fix point theorem* to obtain for each $\mu \in V$ a unique function $\hat{y}(\mu) \in \mathcal{C}(I_T, \mathcal{B}_{E_3, R})$ with $\alpha(\hat{y}(\mu), \mu) = \hat{y}(\mu)$. This function is the unique solution of (3.4.18), for $t \in I_T$ we have $\hat{y}(\mu)(t) = y(t, \mu)$.

In the next step we estimate the derivatives of α .

Claim: α satisfies (3.4.15) with respect to the variables y and μ .

For this we have to introduce some further notation: Let $\gamma : U \times V \rightarrow L^{j,k}(E_3, E_4; E_3)$ be a continuous mapping and let $y \in \mathcal{C}(I_T, U)$, $y_1, \dots, y_j \in \mathcal{C}(I_T, E_3)$, $\mu \in V$ and $\mu_1, \dots, \mu_k \in E_4$. Then for all $t \in I_T$ we put

$$\rho(y, \mu)(y_1, \dots, y_j; \mu_1, \dots, \mu_k)(t) := \int_0^t \gamma(y(\tau), \mu)(y_1(\tau), \dots, y_j(\tau); \mu_1, \dots, \mu_k) d\tau.$$

The integral on the right side above is well-defined for all $t \in I_T$, furthermore we have $\rho(y, \mu)(y_1, \dots, y_j; \mu_1, \dots, \mu_k) \in \mathcal{C}(I_T, E_3)$ and $\rho(y, \mu) \in L^{j,k}(\mathcal{C}(I_T, E_3), E_4; \mathcal{C}(I_T, E_3)) =: \mathcal{L}^{j,k}$. We apply this notation to $g^{(j,k)}$ and get

$$\alpha^{(j,k)}(y, \mu)(y_1, \dots, y_j; \mu_1, \dots, \mu_k)(t) = \int_0^t g^{(j,k)}(y(\tau), \mu)(y_1(\tau), \dots, y_j(\tau); \mu_1, \dots, \mu_k) d\tau.$$

Now we estimate as follows:

$$\left\| \alpha^{(j,k)}(y, \mu) \right\|_{\mathcal{L}^{j,k}} \leq T \cdot \sup_{a \in U, b \in V} \left\{ \left\| g^{(j,k)}(a, b) \right\|_{L^{j,k}(E_3, E_4; E_3)} \right\} \stackrel{(3.4.19)}{\leq} T \cdot C \cdot h^{j+k} \cdot M_{j+k},$$

which proves the claim.

Claim: The solution y of (3.4.18) satisfies (3.4.15) in the variable μ uniformly in t .

Therefore we define now $\beta(y, \mu) := y - \alpha(y, \mu)$. The fixed point $\hat{y}(\mu)$ of α satisfies clearly $\beta(\hat{y}(\mu), \mu) = 0$, hence $\hat{y}(\mu)$ is determined via the implicit equation $\beta(y, \mu) = 0$. We remark that $\beta^{(1,0)}(y, \mu) \in L(\mathcal{C}(I_T, E_3), \mathcal{C}(I_T, E_3)) =: \mathcal{L}^{1,0}$, so it is a linear operator and $\beta^{(1,0)}(y, \mu) = \text{id}_{\mathcal{L}^{1,0}} - \alpha^{(1,0)}(y, \mu)$. We have chosen $T < (C \cdot h)^{-1}$, hence again by (3.4.19) we get

$$\left\| \alpha^{(1,0)}(y, \mu) \right\|_{\mathcal{L}^{1,0}} \leq T \cdot C \cdot h < 1.$$

This implies $\left\| \beta^{(1,0)}(y, \mu) \right\|_{\mathcal{L}^{1,0}} > 0$ and we can compute the inverse operator $(\beta^{(1,0)}(y, \mu))^{-1}$ in the Banach-algebra $\mathcal{L}^{1,0}$ as follows:

$$\left(\beta^{(1,0)}(y, \mu) \right)^{-1} = \left(\text{id}_{\mathcal{L}^{1,0}} - \alpha^{(1,0)}(y, \mu) \right)^{-1} = \sum_{j=0}^{\infty} \left(\alpha^{(1,0)}(y, \mu) \right)^j.$$

Because α satisfies (3.4.15) we obtain that β satisfies (3.4.15) with respect to y and μ , too. We can apply now 3.4.6, which implies the fact that the fixpoint $\hat{y}(\mu)$ shares also property (3.4.15). If y is the solution of (3.4.18), then $\hat{y}(\mu)(t) = y(t, \mu)$ holds for all $t \in I_T$, which implies that y satisfies (3.4.15) in the variable μ uniformly in t . With other words there exist constants $A, k > 0$ such that for all $l \in \mathbb{N}$ and $(t, \mu) \in I_T \times V$ we have

$$\left\| y^{(0,l)}(t, \mu) \right\|_{L^l(E_4; E_3)} \leq A \cdot \eta^l \cdot M_l. \quad (3.4.23)$$

In the next step of the proof we have to estimate $y^{(j,l)}$ for $j \geq 1$.

For the following computations we assume w.l.o.g. $A \geq \max\{1, C\}$ and $\eta \geq \max\{1, 2h\}$.

Furthermore we put for $l \in \mathbb{N}$, $l \geq 2$, $p_l := H \cdot \left(\frac{M_l}{l!}\right)^{1/(l-1)}$. Then we define the functions $G_l, Y_{l,0} : \mathbb{R} \rightarrow \mathbb{R}$ for $|s|$ small in the following way:

$$G_l(s) := A \cdot \left(\sum_{j=0}^{\infty} (p_l \cdot \eta \cdot s)^j \right) = A \cdot (1 - p_l \cdot \eta \cdot s)^{-1}$$

$$Y_{l,0}(s) := A \cdot \left(1 + \eta \cdot s \cdot \left(\sum_{j=0}^{\infty} (p_l \cdot \eta \cdot s)^j \right) \right) = A \cdot (\eta \cdot s \cdot (1 - p_l \cdot \eta \cdot s)^{-1} + 1).$$

For $i \in \mathbb{N}$ we obtain

$$G_l^{(i)}(s) = A \cdot (p_l \cdot \eta)^i \cdot \left(\sum_{j=i}^{\infty} j \cdot (j-1) \cdots (j-i+1) \cdot (p_l \cdot \eta \cdot s)^{j-i} \right),$$

hence $G_l^{(i)}(0) = A \cdot i! \cdot \eta^i \cdot p_l^i$ holds.

We introduce now the following ODE:

$$\begin{cases} Y_l'(t, \sigma) = G_l(Y(t, \sigma) - A + \sigma) \\ Y_l(0, \sigma) = Y_{l,0}(\sigma), \end{cases} \quad (3.4.24)$$

where σ will be regarded as a complex parameter.

The strategy of the proof will be to compare y , which is the solution of (3.4.18), with the solution Y_l of (3.4.24).

First we are going to solve (3.4.24): We set $\varrho(\sigma) := Y_{l,0}(\sigma) - A + \sigma$, then the solution of (3.4.24) is given by

$$Y_l(t, \sigma) = \frac{1 - ((1 - p_l \cdot \eta \cdot \varrho(\sigma))^2 - 2 \cdot A \cdot p_l \cdot \eta \cdot t)^{1/2}}{p_l \cdot \eta} + A + \sigma.$$

We prove this by direct calculation: First we compute the initial value

$$Y_l(0, \sigma) = \frac{1 - (1 - p_l \cdot \eta \cdot \varrho(\sigma))}{p_l \cdot \eta} + A + \sigma = \underbrace{\frac{\varrho(\sigma)}{Y_{l,0}(\sigma) - A + \sigma}}_{1} + A - \sigma = Y_{l,0}(\sigma).$$

Now we compute the derivative with respect to t :

$$\begin{aligned} Y_l'(t, \sigma) &= \frac{(2 \cdot A \cdot p_l \cdot \eta) \cdot ((1 - p_l \cdot \eta \cdot \varrho(\sigma))^2 - 2 \cdot A \cdot p_l \cdot \eta \cdot t)^{-1/2}}{2 \cdot p_l \cdot \eta} \\ &= \frac{A}{\sqrt{(1 - p_l \cdot \eta \cdot \varrho(\sigma))^2 - 2 \cdot A \cdot p_l \cdot \eta \cdot t}}. \end{aligned}$$

On the other hand we have

$$G_l(Y_l(t, \sigma) - A + \sigma) = \frac{A}{1 - p_l \cdot \eta \cdot (Y_l(t, \sigma) - A + \sigma)} = \frac{A}{\sqrt{(1 - p_l \cdot \eta \cdot \varrho(\sigma))^2 - 2 \cdot A \cdot p_l \cdot \eta \cdot t}}.$$

Claim:

$$\left\| y^{(0,i)}(t, \mu) \right\|_{L^i(E_4; E_3)} \leq Y_{l,0}^{(i)}(0) = Y_l^{(0,i)}(0, 0) \quad (3.4.25)$$

holds for $0 \leq i \leq l$.

First we recall that we assumed condition (3.4.7):

$$\exists H \geq 1 : \left(\frac{M_i}{i!} \right)^{1/(i-1)} \leq H \cdot \left(\frac{M_l}{l!} \right)^{1/(l-1)} \quad \text{for } 2 \leq i \leq l.$$

Hence for $2 \leq i \leq l$ we get

$$m_i = \frac{M_i}{i!} \leq H^{i-1} \cdot \left(\frac{M_l}{l!} \right)^{(i-1)/(l-1)} = p_l^{i-1} = p_l^i \cdot p_l^{-1}. \quad (3.4.26)$$

Obviously (3.4.26) is still valid for $i = 1$ because in this case we have $1 = M_1 \leq p_l^0 = 1$. By assumption we have $M_2 \geq 2 \Leftrightarrow m_2 \geq 1$ and so we get by (3.4.7) for $l \geq 2$:

$$1 \leq \frac{M_2}{2} \leq H \cdot \left(\frac{M_l}{l!} \right)^{1/(l-1)} = p_l. \quad (3.4.27)$$

We summarize:

$$m_i \underbrace{\leq}_{(3.4.26)} p_l^i \cdot p_l^{-1} \underbrace{\leq}_{(3.4.27)} p_l^i \quad (3.4.28)$$

holds for $2 \leq i \leq l$. (3.4.28) is still satisfied for $i \in \{0, 1\}$: If $i = 0$, then $m_0 = 1 = p_l^0$ holds for all l and if $i = 1$, then $m_1 = 1 \leq p_l$, which holds by (3.4.27) for all $l \geq 2$.

By assumption we have $A \geq C$ and $\eta \geq 2h$, thus by (3.4.28):

$$C \cdot (2h)^i \cdot M_i \leq A \cdot \eta^i \cdot p_l^i \cdot i! = G_l^{(i)}(0) \quad \text{for } 0 \leq i \leq l. \quad (3.4.29)$$

For $i \in \mathbb{N}$ consider $Y_{l,0}^{(i)}(s)$ and we get:

$$Y_{l,0}^{(i)}(0) = A \cdot i! \cdot \eta^i \cdot p_l^{i-1} \underbrace{\geq}_{(3.4.26)} A \cdot \eta^i \cdot M_i \quad \text{for } 1 \leq i \leq l.$$

This implies

$$\left\| y^{(0,i)}(t, \mu) \right\|_{L^i(E_4; E_3)} \underbrace{\leq}_{(3.4.23)} A \cdot \eta^i \cdot M_i \leq Y_{l,0}^{(i)}(0) = Y_l^{(0,i)}(0, 0) \quad (3.4.30)$$

for $1 \leq i \leq l$. (3.4.30) is still valid for $i = 0$ because in this case we have $\|y(t, \mu)\|_{E_3} \leq A \cdot \eta^0 \cdot M_0 = A = Y_{l,0}(0) = Y_l(0, 0)$, which proves the claim.

Claim:

$$\left\| y^{(j,k)}(t, \mu) \right\|_{L^{j,k}(\mathbb{R}, E_4; E_3)} \leq Y_l^{(j,k)}(\tau, \sigma)|_{\tau=\sigma=0} \quad (3.4.31)$$

holds for $j + k \leq l$.

First we recall that y is a solution of (3.4.18), hence smooth by the classical existence theorem for ODE's and note that (3.4.25) implies (3.4.31) for $j = 0$ and $k \leq l$. Suppose now that $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_1 < l$ and $\alpha_1 + \alpha_2 = l$ and that (3.4.31) is valid for all (j, k) with $0 \leq j \leq \alpha_1$ and $0 \leq k \leq \alpha_2$. We write now $\alpha := (\alpha_1, \alpha_2)$ and $E_5 := E_3 \times E_4$, hence g can be viewed as a Fréchet-infinitely differentiable function from $U \times V \subseteq E_5$ into the space E_3 . Let $z \in U \times V$, $z := (y, \mu)$, then we denote by $g^{(p)}$ the p -th derivative of g with respect to the variable z , and we get

$$\left\| g^{(q)}(z) \right\|_{L^q(E_5; E_3)} \leq 2^q \cdot \sup_{j+k=q} \left\| g^{(j,k)}(y, \mu) \right\|_{L^{j,k}(E_3, E_4; E_3)}.$$

Hence for $0 \leq q \leq l$ and $z \in U \times V$ we estimate

$$\left\| g^{(q)}(z) \right\|_{L^q(E_5; E_3)} \underbrace{\leq}_{(3.4.19)} 2^q \cdot C \cdot h^q \cdot M_q \underbrace{\leq}_{(3.4.29)} G_l^{(q)}(0).$$

3 Stability properties

In our introduced notation this is equivalent to have $g \lll G_l$. Now put $z(t, \mu) := (y(t, \mu), \mu)$ and $Z(\tau, \sigma) := Y_l(\tau, \sigma) - A + \sigma$. Because $y(t, \mu) \in \mathcal{B}_{E_3, R} \subseteq U$ holds for all $t \in I_T$ and $\mu \in V$, one has $z : I_T \times V \rightarrow U \times V$, and by componentwise differentiation we see

$$\begin{aligned} \|z^{(j,k)}(t, \mu)\|_{L^{j,k}(\mathbb{R}, E_4; E_3, E_4)} &= \max \left\{ \|y^{(j,k)}(t, \mu)\|_{L^{j,k}(\mathbb{R}, E_4; E_3)}, \|\mu^{(j,k)}\|_{E_4} \right\} \\ &\leq \|y^{(j,k)}(t, \mu)\|_{L^{j,k}(\mathbb{R}, E_4; E_3)} + \|\mu^{(j,k)}\|_{E_4}. \end{aligned}$$

If $j + k > 0$, then $Z^{(j,k)}(\tau, \sigma) = Y_l^{(j,k)}(\tau, \sigma) + \sigma^{(j,k)}$. If $0 \leq j \leq \alpha_1$ and $0 \leq k \leq \alpha_2$, then (3.4.31) holds, hence for $(j, k) \in \mathbb{N}^2$ such that $0 \leq j \leq \alpha_1$, $0 \leq k \leq \alpha_2$ and $j + k > 0$ we obtain:

$$\|z^{(j,k)}(t, \mu)\|_{L^{j,k}(\mathbb{R}, E_4; E_3, E_4)} \leq Z^{(j,k)}(\tau, \sigma)|_{\tau=\sigma=0} \quad \text{for } (t, \mu) \in I_T \times V.$$

Using again the introduced notation above this means $z \lll_c Z$, and because we have already shown $g \lll G_l$ we get by 3.4.3: $g \circ z \lll_c G_l \circ Z$. This is equivalent to

$$\|(g \circ z)^{(j,k)}\|_{L^{j,k}(E_3, E_4; E_3)} = \|g^{(j,k)}(y(t, \mu), \mu)\|_{L^{j,k}(E_3, E_4; E_3)} \leq G_l^{(j,k)}(Y_l(\tau, \sigma) - A + \sigma)|_{\tau=\sigma=0}$$

for $(j, k) \in \mathbb{N}^2$ with $0 \leq j \leq \alpha_1$ and $0 \leq k \leq \alpha_2$. We remark: y is a solution of (3.4.18) and Y_l of (3.4.24), so it follows that (3.4.31) holds now for $0 \leq j \leq \alpha_1 + 1$ and $0 \leq k \leq \alpha_2$. Thus we have shown that (3.4.31) holds for $(j, k) \in \mathbb{N}^2$ with $j + k \leq l$.

In the next step of the proof we estimate $Y_l^{(j,k)}(\tau, \sigma)|_{\tau=\sigma=0}$ for $j \geq 1$ and we start to estimate $Y_l^{(j,0)}(\tau, \sigma)|_{\tau=0}$ for $j \geq 1$:

$$\begin{aligned} Y_l^{(j,0)}(\tau, \sigma)|_{\tau=0} &= -(p_l \eta)^{-1} \cdot (-2A p_l \eta)^j \cdot 2^{-1} \cdot (2^{-1} - 1) \cdots (2^{-1} - j + 1) \cdot (1 - p_l \eta \varrho(\sigma))^{1-2j} \\ &= (-1)^{j+1} \cdot (p_l \eta)^{j-1} \cdot (2A)^j \cdot (-1)^j \cdot (-2^{-1}) \cdots (j - 1 - 2^{-1}) \cdot (1 - p_l \eta \varrho(\sigma))^{1-2j} \\ &= (p_l \eta)^{j-1} \cdot (2A)^j \cdot \frac{\Gamma(j - 2^{-1})}{2\pi^{1/2}} \cdot (1 - p_l \eta \varrho(\sigma))^{1-2j}, \end{aligned}$$

where $\Gamma(x) := \int_0^\infty e^{-t} \cdot t^{x-1} dt$ denotes the *Gamma-function*. So the last equality above holds because:

$$\Gamma(j - 2^{-1}) = (j - 1 - 2^{-1}) \cdot \Gamma(j - 1 - 2^{-1}) = \cdots = (j - 1 - 2^{-1}) \cdots (-2^{-1}) \cdot \Gamma(-2^{-1})$$

and $\pi^{1/2} = \Gamma(2^{-1}) = \Gamma(-2^{-1} + 1) = -2^{-1} \cdot \Gamma(-2^{-1})$, thus $\Gamma(-2^{-1}) = -2 \cdot \pi^{1/2}$.

Hence we have shown:

$$Y_l^{(j,0)}(\tau, \sigma)|_{\tau=0} = (p_l \eta)^{j-1} \cdot (2A)^j \cdot \frac{\Gamma(j - 2^{-1})}{2\pi^{1/2}} \cdot (1 - p_l \eta \varrho(\sigma))^{1-2j} \quad (3.4.32)$$

for all $j \in \mathbb{N}$, $j \geq 1$.

We study now the expression $(1 - p_l \cdot \eta \cdot \varrho(\sigma))^{1-2j}$, which appears in (3.4.32), for $j \in \mathbb{N}$, $j \geq 1$, in detail. First we see that

$$\begin{aligned} 1 - p_l \cdot \eta \cdot \varrho(\sigma) &= 1 - p_l \cdot \eta \cdot (Y_{l,0}(\sigma) - A + \sigma) \\ &= 1 - p_l \cdot \eta \cdot ((A \cdot \eta \cdot \sigma \cdot (1 - p_l \cdot \eta \cdot \sigma)^{-1} + A) - A + \sigma) \\ &= 1 - p_l \cdot \eta \cdot \sigma - A \cdot p_l \cdot \eta^2 \cdot \sigma \cdot (1 - p_l \cdot \eta \cdot \sigma)^{-1}. \end{aligned}$$

So for $\sigma \in \mathbb{C}$ with $|\sigma|$ small we can estimate as follows:

$$\begin{aligned} |1 - p_l \cdot \eta \cdot \varrho(\sigma)| &\geq 1 - p_l \cdot \eta \cdot |\sigma| - A \cdot p_l \cdot \eta^2 \cdot |\sigma| \cdot (1 - p_l \cdot \eta \cdot |\sigma|)^{-1} \\ &\geq \underbrace{1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma|}_{A, \eta \geq 1} - A \cdot p_l \cdot \eta^2 \cdot |\sigma| \cdot (1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma|)^{-1} \\ &= 1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma| - (1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma|)^{-1} + 1. \end{aligned}$$

Now we want to find a sufficient condition for σ such that

$$|1 - p_l \cdot \eta \cdot \varrho(\sigma)| \geq \frac{1}{2} \quad (3.4.33)$$

is satisfied. For this we set $x := 1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma|$, then by the calculation above $|1 - p_l \cdot \eta \cdot \varrho(\sigma)| \geq x - x^{-1} + 1$ holds. Because $x - x^{-1} + 1 \geq 1/2 \Rightarrow 2x^2 + x - 2 \geq 0$ we are interested in solving the quadratic equation $2x^2 + x - 2 = 0$. We get the solutions $x_{1,2} = \frac{-1 \pm \sqrt{17}}{4}$, so for $x \geq \frac{-1 + \sqrt{17}}{4}$ (3.4.33) is satisfied. In particular we can take $x \geq 4/5$, which means $1 - A \cdot p_l \cdot \eta^2 \cdot |\sigma| \geq 4/5 \Leftrightarrow |\sigma| \leq (5 \cdot A \cdot p_l \cdot \eta^2)^{-1}$.

Now take $\sigma \in \mathbb{C}$ such that $|\sigma| \leq (5 \cdot A \cdot p_l \cdot \eta^2)^{-1}$ holds and put

$$\omega_{l,j} : \mathbb{C} \rightarrow \mathbb{C}, \quad \omega_{l,j}(\sigma) := (1 - p_l \cdot \eta \cdot \varrho(\sigma))^{1-2j}. \quad (3.4.34)$$

Claim:

$$|\omega_{l,j}^{(i)}(0)| < i! \cdot (2^{2j-1}) \cdot (8 \cdot A \cdot p_l \cdot \eta^2)^i \quad (3.4.35)$$

holds for $i \in \mathbb{N}$:

By (3.4.33) we get $|\omega_{l,j}(\sigma)| \leq 2^{2j-1}$ and we can use *Cauchy's integral formula* to estimate $\omega_{l,j}^{(i)}$ at 0 for $i \in \mathbb{N}$ in the following way:

$$\begin{aligned} |\omega_{l,j}^{(i)}(0)| &\leq \frac{i!}{2\pi} \cdot \int_{\{\sigma \in \mathbb{C} : |\sigma|=r\}} |\omega_{l,j}(\sigma)| \cdot |\sigma|^{-i-1} d\sigma \\ &\leq i! \cdot (2^{2j-1}) \cdot r^i < i! \cdot (2^{2j-1}) \cdot (8 \cdot A \cdot p_l \cdot \eta^2)^i, \end{aligned}$$

where we have set $r := (5 \cdot A \cdot p_l \cdot \eta^2)^{-1}$.

Furthermore we have $0 < \omega_{l,j}^{(i)}(0)$ for $i \in \mathbb{N}$: This holds by the definition of $\varrho(\sigma)$ and the Faà-de-Bruno-formula (3.2.2).

Claim:

$$Y_l^{(j,k)}(0,0) \leq 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^{j+k} \cdot (j+k)! \cdot (p_l)^{k+j-1} \quad (3.4.36)$$

holds for $j \in \mathbb{N}$, $j \geq 1$. For this we estimate as follows:

$$\begin{aligned} 0 < Y_l^{(j,k)}(0,0) &\stackrel{(3.4.32)}{=} \underbrace{(p_l \eta)^{j-1} \cdot (2A)^j \cdot \frac{\Gamma(j-2^{-1})}{2\pi^{1/2}} \cdot \omega_{l,j}^{(k)}(0)}_{(3.4.35)} \\ &\stackrel{(3.4.35)}{\leq} \underbrace{(p_l \eta)^{j-1}}_{\leq p_l \eta^2} \cdot (2A)^j \cdot \frac{\Gamma(j-2^{-1})}{2\pi^{1/2}} \cdot 2^{2j-1} \cdot k! \cdot (8A p_l \eta^2)^k \\ &\leq \underbrace{2^{3(j+k)-2}}_{=8^{j+k} \cdot 2^{-2}} \cdot \pi^{-1/2} \cdot A^{j+k} \cdot \underbrace{\Gamma(j-2^{-1}) \cdot k!}_{\leq j! = \Gamma(j+1)} \cdot (p_l \eta^2)^{k+j-1} \\ &\leq 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^{j+k} \cdot (j+k)! \cdot (p_l)^{k+j-1}, \end{aligned}$$

which proves (3.4.36). The last inequality above holds because $\eta \geq 1$ and we have used $1 \leq \frac{(j+k)!}{j! \cdot k!} = \binom{j+k}{k}$, which holds for all $j, k \in \mathbb{N}$.

With this estimate we can finish the proof: If we put $j+k=l$ in (3.4.36), then we obtain

$$Y_l^{(j,k)}(\tau, \sigma)|_{\tau=\sigma=0} \leq 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^l \cdot l! \cdot p_l^{l-1} = 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2 H)^l \cdot M_l. \quad (3.4.37)$$

3 Stability properties

For the last equality above recall that by definition we have $p_l^{l-1} = \frac{M_l}{l!}$. Hence, by (3.4.31), we have for $j + k = l$ and $j \geq 1$:

$$\left\| y^{(j,k)}(t, \mu) \right\|_{L^{j,k}(\mathbb{R}, E_4; E_3)} \underbrace{\leq}_{(3.4.31)} Y_l^{(j,k)}(\tau, \sigma)|_{\tau=\sigma=0} \underbrace{\leq}_{(3.4.37)} 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2 H)^{j+k} \cdot M_{j+k}. \quad (3.4.38)$$

But since $l \in \mathbb{N}$ is arbitrary and we have already shown (3.4.23) we are done. $\square$

We have shown the closedness under solving ODE's 3.4.1 now for the Romieu-case. For the Beurling-case we have to assume additionally

$$\boxed{\text{conditions (3.4.3) and (3.4.4) for the weight sequence } (M_p)_p.}$$

Then the proof of the closedness under solving ODE's in the Beurling-case can be reduced to the Romieu-case with the following technique which was introduced by Komatsu in [12]:

Let $M := (M_p)_p$ and $L := (L_p)_p$ be two weight sequences, then we write $L \leq M$ if

$$\forall h > 0 \exists C_h > 0 : L_p \leq C_h \cdot h^p \cdot M_p, \quad \text{for } p \in \mathbb{N}. \quad (3.4.39)$$

We remark that $L \leq M$ implies $\lim_{p \rightarrow \infty} \left(\frac{L_p}{M_p} \right)^{1/p} = 0$ and with this notation we can formulate the following very important lemma:

Lemma 3.4.7 [12, Lemma 6, p. 248-249] *Let $M := (M_p)_p$ be a weight sequence, such that $(m_p)_p$ is logarithmic convex and (3.4.4) holds. Then for each weight sequence $L := (L_p)_p$ with $L \leq M$ there exists a weight sequence $N := (N_p)_p$, satisfying also (3.4.3) and (3.4.4), such that*

$$L \leq N \leq M$$

holds ($L \leq N \Leftrightarrow L_p \leq N_p \forall p \in \mathbb{N}$).

Proof. First we define a sequence $\bar{L} := (\bar{L}_p)_p$ with $\bar{L}_p := \inf_{h>0} \{C_h \cdot h^p \cdot M_p\}$, where C_h is the least real number, such that (3.4.39) is satisfied. Hence $L \leq \bar{L} \leq M$ holds and, because $L_p > 0$ for all $p \in \mathbb{N}$, the infimum $\bar{L}_p$ is attained by an $h_p > 0$. We get:

$$\left(\frac{M_p}{\bar{L}_p} \right)^2 = \frac{1}{(C_{h_p} \cdot h_p^p)^2} = \frac{1}{C_{h_p} \cdot h_p^{p-1}} \cdot \frac{1}{C_{h_p} \cdot h_p^{p+1}} \leq \left(\frac{M_{p-1}}{\bar{L}_{p-1}} \right) \cdot \left(\frac{M_{p+1}}{\bar{L}_{p+1}} \right). \quad (3.4.40)$$

From (3.4.40) it follows that the sequence $\frac{M_p}{\bar{L}_p}$ is log. convex and so we obtain that the quotient sequence

$$c_p := \frac{M_p}{\bar{L}_p} \bigg/ \frac{M_{p-1}}{\bar{L}_{p-1}} = \frac{M_p \cdot \bar{L}_{p-1}}{\bar{L}_p \cdot M_{p-1}}$$

is increasing. Since $\bar{L} \leq M$, the sequence $c_p \rightarrow \infty$ for $p \rightarrow \infty$. Then we define $\mu_p := \frac{M_p}{M_{p-1}}$ for all $p \in \mathbb{N}$ and the sequence $(\nu_p)_p$ as follows:

$$\frac{\nu_p}{p} := \max \left\{ \sqrt{\frac{\mu_p}{p}}, \max \left\{ \frac{\mu_q}{q \cdot c_q}; 1 \leq q \leq p \right\} \right\}.$$

From this we define the sequence $N := (N_p)_p$ via

$$N_p := \bar{L}_0 \cdot \prod_{q=1}^p \nu_q.$$

Next we remark

$$\frac{\mu_q}{c_q} = \frac{M_q \cdot \bar{L}_q \cdot M_{q-1}}{M_{q-1} \cdot M_q \cdot \bar{L}_{q-1}} = \frac{\bar{L}_q}{\bar{L}_{q-1}} \implies \bar{L}_p = \bar{L}_0 \cdot \prod_{q=1}^p \frac{\mu_q}{c_q},$$

and by the definition of the ν_q above we see that $\frac{\mu_q}{c_q} \leq \nu_q$ holds for all q . Finally we have $\bar{L}_p \leq N_p$ for all p , hence $\bar{L} \leq N$.

From (3.4.4) and the definition of the sequence $(\nu_p)_p$ we obtain that $\frac{\nu_p}{p}$ is increasing. Because $\nu_p = \frac{N_p}{N_{p-1}}$ is the quotient sequence this implies that $(\frac{N_p}{p!})_p$ is log. conv. and this gives (3.4.3) for the sequence $(N_p)_p$. Note that in our notation (3.4.4) is equivalent to $\frac{\mu_p}{p} \longrightarrow \infty$ for $p \longrightarrow \infty$. Another consequence of the definition is $\frac{\nu_p}{p} \geq \sqrt{\frac{\mu_p}{p}}$ for each p , and so $(N_p)_p$ satisfies also (3.4.4).

It remains to show: $N \trianglelefteq M$. We consider the following quotient:

$$\frac{\nu_p}{\mu_p} = \max \left\{ \sqrt{\frac{p}{\mu_p}}, \max \left\{ \frac{p}{\mu_p} \cdot \frac{\mu_q}{q \cdot c_q}; 1 \leq q \leq p \right\} \right\}.$$

Because the sequence c_q is increasing, for all $\varepsilon > 0$ we can find an index q_ε such that $\frac{1}{c_q} \leq \varepsilon$ for $q > q_\varepsilon$, and so

$$\frac{\nu_p}{\mu_p} \leq \max \left\{ \sqrt{\frac{p}{\mu_p}}, \varepsilon, \frac{p}{\mu_p} \cdot \frac{\mu_q}{q \cdot c_q}; 1 \leq q \leq q_\varepsilon \right\} \leq \varepsilon.$$

The last inequality holds for sufficiently large number p , because $\frac{p}{\mu_p} \longrightarrow 0$ for $p \longrightarrow \infty$.

From this we obtain now $N \trianglelefteq M$: Note that $N_p = \bar{L}_0 \cdot \nu_1 \cdots \nu_p$ and $M_p = \mu_1 \cdots \mu_p$.

□

We apply now 3.4.7 in the following way to prove the closedness under solving ODE's in the Beurling-case:

Corollary 3.4.8 *Let $M := (M_p)_p$ be a weight sequence with the following properties: (3.4.1) and $M_2 \geq 2$ are satisfied and furthermore we have (3.4.3) and (3.4.4). Then the class $\mathcal{E}_{(M)}$ is closed under solving ODE's.*

Proof. First we remark that (3.4.3), which is the strong log. convexity condition, implies (3.4.2) and so (3.4.7). Hence the class $\mathcal{E}_{\{M\}}$ is closed under solving ODE's by 3.4.1.

Let $f \in \mathcal{E}_{(M),U}$ and we set $L_p := \|f^{(p)}\|_{L^p(E;F)}$ to obtain a sequence $L := (L_p)_p$. Then, by definition of the space $\mathcal{E}_{(M),U}$, we get $L \trianglelefteq M$ and so, by 3.4.7, $\exists$ sequence $N := (N_p)_p$ depending on f such that $L \leq N \trianglelefteq M$ holds. Hence $f \in \mathcal{E}_{\{N\},U}$, where $C = h = 1$ are sufficient. Finally we have

$$L \trianglelefteq M \implies \mathcal{E}_{\{L\},U} \subseteq \mathcal{E}_{(M),U} \text{ and } \mathcal{E}_{\{N\},U} \subseteq \mathcal{E}_{(M),U}. \quad (3.4.41)$$

In our situation for $x'(t) = f(x(t), t, \lambda)$ we obtain:

$$f \in \mathcal{E}_{(M),U} \implies f \in \mathcal{E}_{\{N\},U} \xrightarrow[3.4.1]{N \trianglelefteq M} x \in \mathcal{E}_{\{N\},U} \xrightarrow[N \trianglelefteq M]{} x \in \mathcal{E}_{(M),U}.$$

□

We finish this section with the following result: 3.4.7 can also be used to prove the composition theorem 3.2.7 and the inverse function theorem 3.3.2 for the Beurling-case.

Proposition 3.4.9 *Let $M := (M_p)_p$ be a weight sequence such that the sequence $m := (m_p)_p$ is logarithmic convex and (3.4.4) holds. Then the class $\mathcal{E}_{(M)}$ is closed under composition and the inverse function theorem is valid.*

Proof. First we remark that 3.2.7 and 3.3.2 are still valid for the Romieu-case $\mathcal{E}_{\{M\}}$, because the assumption there was only log. convexity for $(m_p)_p$.

We prove now 3.2.7 for the Beurling-case:

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, $f \in \mathcal{E}_{(M)}(V)$ and $g = (g_1, \dots, g_m) : U \rightarrow V$ such that $g_j \in \mathcal{E}_{(M)}(U)$ for all j . Take an arbitrary compact set $K_1 \subseteq U$ and put $K_2 := g(K_1)$. Because g is continuous, K_2 is a compact set in V and we define a weight sequence $L := (L_p)_p$ as follows:

$$L_p := \max \left\{ \|g^{(p)}|_{K_1}\|_\infty, \|f^{(p)}|_{K_2}\|_\infty \right\},$$

where $\|g^{(p)}|_{K_1}\|_\infty := \sup_{x \in K_1} \{\|g^{(p)}(x)\|_{L^p(\mathbb{R}^n, \mathbb{R}^m)}\}$ and similarly for f .

We have $L \trianglelefteq M$ and so, by 3.4.7, we obtain: There exists a weight sequence $N := (N_p)_p$ depending on K_1 , f and g which satisfies (3.4.3) and (3.4.4) and such that $L \leq N \trianglelefteq M$ holds.

We have $f \in \mathcal{E}_{\{L\}, K_2}$ and $g_j \in \mathcal{E}_{\{L\}, K_1}$ for each j , hence $f \in \mathcal{E}_{\{N\}, K_2}$ and $g_j \in \mathcal{E}_{\{N\}, K_1}$ is satisfied for all j and we can apply now the composition theorem 3.2.7 for the Romieu-case $\mathcal{E}_{\{N\}}$ to get

$$f \circ g \in \mathcal{E}_{\{N\}, K_1} \underbrace{\subseteq}_{N \trianglelefteq M} \mathcal{E}_{(M), K_1}.$$

Since K_1 was arbitrary we have $f \circ g \in \mathcal{E}_{(M)}(U)$.

For 3.3.2 we proceed as follows:

Let $U, V \subseteq \mathbb{R}^n$ be open, $f = (f_1, \dots, f_n) : U \rightarrow V$ be a $\mathcal{E}$ -diffeomorphism and assume that $f \in \mathcal{E}_{(M)}(U)$, which means $f_j \in \mathcal{E}_{(M)}(U)$ for all j , then we have to prove that $f^{-1} \in \mathcal{E}_{(M)}(V)$. Let $K \subseteq V$ be an arbitrary compact set, we show now $f^{-1} \in \mathcal{E}_{(M), K}$. The set $f^{-1}(K)$ is a compact subset of $U \subseteq \mathbb{R}^n$ and we choose now U_1 open such that $f^{-1}(K) \subseteq U_1 \subseteq U$ holds and such that U_1 is relatively compact in U , in particular we have $\overline{U_1} \subseteq U$. We define now again the sequence $L := (L_p)_p$ via

$$L_p := \|f^{(p)}|_{\overline{U_1}}\|_\infty.$$

So $L \trianglelefteq M$ and by 3.4.7 we get a weight sequence N with $L \leq N \trianglelefteq M$ and so $f_j \in \mathcal{E}_{\{N\}, K}$, $\forall j$ and $f : U_1 \rightarrow f(U_1)$ is a $\mathcal{E}$ -diffeomorphism. So we can apply 3.3.2 for the class $\mathcal{E}_{\{N\}}$ to obtain: $f^{-1} \in \mathcal{E}_{\{N\}}(f(U_1))$. Because $N \trianglelefteq M$ we have $f^{-1} \in \mathcal{E}_{(M)}(f(U_1))$ and because $K \subseteq f(U_1)$ this implies $f^{-1} \in \mathcal{E}_{(M)}(K)$, too. Since $K \subseteq V$ was chosen arbitrary, we are done.

□

Remark 3.4.10 As we have pointed out in 3.3.3, the Beurling-classes are in general not closed under inversion. For example the entire functions, the defining weight sequence is $M_p := p!$ for all $p \in \mathbb{N}$. Note that in this case condition (3.4.3) is clearly satisfied ($m_p = 1$ for all $p \in \mathbb{N}$ is clearly log. convex), but (3.4.4) is violated because $\lim_{p \rightarrow \infty} \frac{M_p}{p \cdot M_{p-1}} = 1$.

Komatsu (see [13]) has shown the closedness under inversion for the Beurling-case assuming only $M_0 = M_1 = 1$, (3.4.7) and (3.4.4) and he pointed out that condition (3.4.4) is important for the proof.

4 The Denjoy-Carleman-theorem

In this chapter we are going to prove the famous *Denjoy-Carleman-Theorem*, which gives a full characterization for the injectivity of the Borel-map. We will consider two proofs, both for the one-dimensional Romieu-case. The first proof, which was given by Rudin (see [23]), is only valid if the weight sequence $(M_n)_n$ is assumed to be log. convex and if one has $M_0 = 1$. The second version, which is due to Hörmander (see [8]), uses only the normalization $M_0 = 1$, there are no further restrictions on the weight sequence $(M_n)_n$.

4.1 Denjoy-Carleman-theorem 1

Throughout this section, if it is not stated otherwise, we assume that

$$(M_n)_n \text{ is a log. convex weight sequence and } M_0 = 1.$$

For the first version of the Denjoy-Carleman-theorem we need some preparations. We start with a useful proposition, which gives a relation between quasi-analyticity of function spaces and functions with compact support in it.

Proposition 4.1.1 [23, 19.10 Theorem, p. 379] *The class $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ for a weight sequence $M := (M_n)_n$ is quasi-analytic if and only if there is no nontrivial function with compact support in $\mathcal{E}_{\{M\}}$ resp. in $\mathcal{E}_{(M)}$.*

Proof. We prove this statement for the one-dimensional Romieu-case. Let $\mathcal{E}_{\{M\}}$ be quasi-analytic and take a function $f \in \mathcal{E}_{\{M\}}$ such that f has compact support. Then it follows that there exists a point x_0 (which is not in the support of f) such that f and all its derivatives vanish at this point. The quasi-analyticity implies that $f(x) = 0, \forall x$.

For the converse direction assume that $\mathcal{E}_{\{M\}}$ is *not quasi-analytic*. So $\exists f \in \mathcal{E}_{\{M\}}, \exists x_0 \neq 0$, such that f and all its derivatives vanish at the point 0 but $f(x_0) \neq 0$. We can assume that $x_0 > 0$ and then we define a function g as follows: $g(x) = f(x), x \geq 0$, and $g(x) = 0, x < 0$, hence $g \in \mathcal{E}_{\{M\}}$. Now put $h(x) := g(x) \cdot g(2x_0 - x)$. The log. convexity of the weight function implies by 2.0.8 that the space $\mathcal{E}_{\{M\}}$ is a ring and so $h \in \mathcal{E}_{\{M\}}$. Further we have $h(x) = 0$ for $x < 0$ and for $x > 2x_0$. But by assumption $h(x_0) = f^2(x_0) \neq 0$ and so h is a nontrivial element in $\mathcal{E}_{\{M\}}$ with compact support. □

In particular 4.1.1 has the following remarkable consequence: In quasi-analytic function spaces there doesn't exist any bump function, hence we cannot construct a partition of unity.

We put $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and proceed with an important result about functions in the Hardy Space $H^\infty := H^\infty(\mathbb{D}) := \{f : \text{bounded analytic on } \mathbb{D}\}$, which is equipped with the supremum norm. Note that one has $f|_{\mathbb{T}} \in L^\infty(\mathbb{T})$.

Proposition 4.1.2 [23, 15.19 Theorem, p. 300-301] *For $f \in H^\infty$, which is not identically 0, we define*

$$\mu_r(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|f(r \cdot e^{i\theta})|) d\theta \quad (0 < r < 1)$$

and for the radial limit function f^ of f , where $f^*(e^{i\theta}) := \lim_{r \rightarrow 1} f(r \cdot e^{i\theta})$, we set*

$$\mu^*(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|f^*(e^{i\theta})|) d\theta.$$

Then we obtain three properties:

$$\mu_r(f) \leq \mu_s(f) \quad \text{for } 0 < r < s < 1 \quad (4.1.1)$$

$$\mu_r(f) \longrightarrow \log(|f(0)|) \quad \text{as } r \longrightarrow 0 \quad (4.1.2)$$

$$\mu_r(f) \leq \mu^*(f) \quad \text{for } 0 < r < 1 \quad (4.1.3)$$

Proof. For a given function $f \in H^\infty$ there is a $m \in \mathbb{Z}, m \geq 0$, and a function $g \in H^\infty, g(0) \neq 0$, such that $f(z) = z^m \cdot g(z)$. The function g has no zero at the origin and so we can apply *Jensen's formula* (for a prove see [23, 15.18 Theorem, p. 299-300]) to g :

$$e^{\{\mu_r(g) - \log(|g(0)|)\}} = \prod_{n=1}^k \frac{r}{|s_n|}$$

where the s_n are the zeroes of g inside the circle with radius r . We see that the right side above has to increase if r increases, hence $\mu_r(g) \leq \mu_s(g)$ for $r < s$. Because $\log(|f(z)|) = \log(|z^m \cdot g(z)|) = \log(|g(z)|) + \underbrace{m \cdot \log(|z|)}_r$ we obtain

$$\mu_r(f) = \mu_r(g) + m \cdot \log(r)$$

and so we get property (4.1.1).

Now we assume w.l.o.g. that $|f| \leq 1$, and we write $f_r(e^{i\theta})$ instead of $f(r \cdot e^{i\theta})$. So we see that $f_r \longrightarrow f(0)$ for $r \longrightarrow 0$ and $f_r \longrightarrow f^*$ for $r \longrightarrow 1$. Because

$$0 \leq -\log(\underbrace{|f(r \cdot e^{i\theta})|}_{\leq 1, \forall r}) = -\log(|f_r|) = \log\left(\frac{1}{|f_r|}\right) =: g_r$$

we obtain $\frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta) d\theta = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|f(r \cdot e^{i\theta})|) d\theta = -\mu_r(f)$. Hence by *Fatou's lemma* and (4.1.1):

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \liminf_{r \rightarrow 1} g_r(\theta) d\theta &\leq \liminf_{r \rightarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta) d\theta \Rightarrow -\mu^*(f) \leq \liminf_{r \rightarrow 1} (-\mu_r(f)) \\ &\Rightarrow \limsup_{r \rightarrow 1} (\mu_r(f)) \leq \mu^*(f) \Rightarrow (4.1.3). \end{aligned}$$

Further for $r \rightarrow 0$:

$$\begin{aligned} \underbrace{-\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|f(0)|) d\theta}_{= -\log(|f(0)|)} &\leq \liminf_{r \rightarrow 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta) d\theta \Rightarrow -\log(|f(0)|) \leq \liminf_{r \rightarrow 0} (-\mu_r(f)) \\ &\Rightarrow \log(|f(0)|) \geq \limsup_{r \rightarrow 0} (\mu_r(f)). \end{aligned}$$

Because $\log(|f(0)|) \leq \limsup_{r \rightarrow 0} (\mu_r(f))$ holds also, we have shown (4.1.2). □

Theorem 4.1.2 has an important consequence: Choose a radius r , such that $f(z) \neq 0$ if $|z| = r$. This can be done because f is not identically zero. Then obviously $\mu_r(f) < \infty$ and so for $r \rightarrow 1$ also $\mu^*(f) < \infty$ by (4.1.3). So $\log|f^*| \in L^1(\mathbb{T})$ and $f^*(e^{i\theta}) \neq 0$ at almost every point of $\mathbb{T}$.

Theorem 4.1.3 [23, 19.3 Theorem, p. 375-376] Let $A > 0$ and $C > 0$ be two constants, $A, C \in \mathbb{R}$, and let f be an entire function which satisfies the following two estimations:

$$|f(z)| \leq C \cdot e^{A \cdot |z|} \quad \forall z \in \mathbb{C},$$

which means that f is of exponential type and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

which means that $f|_{\mathbb{R}} \in L^2(\mathbb{R})$.

Then $\exists F \in L^2(-A, A)$ such that $\forall z \in \mathbb{C}$:

$$f(z) = \int_{-A}^A F(t) \cdot e^{itz} dt,$$

which means that f is the inverse Fourier-Laplace-transformation of F .

Proof. A proof can be found at [23, p. 375-376]. □

The last preparation for the proof of the *Denjoy-Carleman-Theorem* is the following proposition:

Proposition 4.1.4 Let $(a_n)_n \in \mathbb{C}^{\mathbb{N}}$ and assume that $\sum_{n=1}^{\infty} |a_n| < \infty$. Then $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent.

Proof. First we remark that $\sum_{n=1}^{\infty} |a_n| < \infty$ implies clearly $\lim_{n \rightarrow \infty} a_n = 0$ (*). Now let $\log : \mathbb{C}^- \rightarrow \mathbb{C}$ be a branch of the complex logarithm, where $\mathbb{C}^- := \{z = x + iy \in \mathbb{C} : y = 0, x \leq 0\}$. Because $1 = \log'(1) = \lim_{z \rightarrow 0} \frac{\log(1+z)}{z}$ it follows: $\exists \delta > 0$ such that

$$\left| 1 - \frac{\log(1+z)}{z} \right| \leq \left| 1 - \frac{\log(1+z)}{z} \right| \leq \frac{1}{2}$$

holds for all $|z| \leq \delta$. Hence $\frac{1}{2} \cdot |a_n| \leq |\log(1 + a_n)| \leq \frac{3}{2} \cdot |a_n|$ holds for n large enough, because then $|a_n| \leq \delta$ by (*). So we can estimate:

$$\sum_{n=1}^{\infty} |\log(1 + a_n)| \leq C_{\delta} + \frac{3}{2} \cdot \sum_{n=l_{\delta}}^{\infty} |a_n| < \infty.$$

Put $s_n := \sum_{k=1}^n \log(1 + a_k)$ and $s_{\infty} := \lim_{n \rightarrow \infty} s_n$. So we can write

$$\prod_{k=1}^n (1 + a_k) = \prod_{k=1}^n \exp(\log(1 + a_k)) = \exp\left(\sum_{k=1}^n \log(1 + a_k)\right) = \exp(s_n),$$

and clearly $\exp(s_n) \rightarrow \exp(s_{\infty})$ for $n \rightarrow \infty$. This implies the existence of $\prod_{k=1}^{\infty} (1 + a_k) = \exp(s_{\infty}) < \infty$. □

Now we formulate and prove the *Denjoy-Carleman-Theorem*:

Theorem 4.1.5 [23, 19.11 Theorem, p. 380-383] Let $M := (M_n)_n$ be a weight sequence. We define for $x \in \mathbb{R}, x > 0$, the mappings $Q, q : \mathbb{R}_{>0} \rightarrow \mathbb{R} \cup \{+\infty\}$, where $Q(x) := \sum_{n=0}^{\infty} \frac{x^n}{M_n}$ and $q(x) := \sup_n \frac{x^n}{M_n}$. Then the following are equivalent:

1. $\mathcal{E}_{\{M\}}$ is not quasi-analytic,
2. $\int_0^{\infty} \log(Q(x)) \frac{1}{1+x^2} dx < \infty$,
3. $\int_0^{\infty} \log(q(x)) \frac{1}{1+x^2} dx < \infty$,
4. $\sum_{n=1}^{\infty} \left(\frac{1}{M_n}\right)^{1/n} < \infty$,
5. $\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \sum_{n=1}^{\infty} \frac{1}{\mu_n} < \infty$.

Remarks: If the sequence $M_n \rightarrow \infty$ for $n \rightarrow \infty$ very fast, then the function $Q(x) \rightarrow \infty$ for $x \rightarrow \infty$ very slowly. This holds because $Q(x) \geq \frac{x}{M_1}$ for all x and Q is a monotone function. So the conditions above say that M_n is increasing fast enough. In the case where $Q(x_0) = \infty$ for a $x_0 < \infty$ the integral in condition 2 has the value $+\infty$ and so $\mathcal{E}_{\{M\}}$ is in fact quasi-analytic. Because $Q(x) \geq 1$ and $q(x) \geq 1$ the integrals in condition 2. and 3. are always defined in $\mathbb{R} \cup \{+\infty\}$.

Condition 5. has the consequence, that the sequence $(\mu_n)_n$ has to increase fast enough and $\lim_{n \rightarrow \infty} \mu_n = \infty$.

Proof. (1. $\Rightarrow$ 2.) We assume that $\mathcal{E}_{\{M\}}$ is not quasi-analytic, hence by 4.1.1 above there exists a nontrivial function in $\mathcal{E}_{\{M\}}$ with compact support. Now we use an affine change of the variable (note: $\mathcal{E}_{\{M\}}$ is closed under affine transformations because $f^{(p)}(a \cdot x + b) = a^p \cdot f(a \cdot x + b)$ for $a, b \in \mathbb{R}, a > 0$) to obtain a function $F \in \mathcal{E}_{\{M\}}$ with support in the compact interval $[0, A]$, for some $A > 0$. We can arrange A such that we get for $n \in \mathbb{N}$:

$$\|F^{(n)}\|_{\infty} \leq 2^{-n} \cdot M_n \quad (4.1.4)$$

and such that F is not the zero-function. Remark: In the inequality above we have $C := 1$ and $h := 1/2$. Next we define the following functions:

$$f(z) := \int_0^A F(t) \cdot e^{itz} dt \quad (4.1.5)$$

which is the inverse Fourier-Laplace-transformation of F and then the Möbius-transformation of f :

$$g(w) := f\left(\frac{i - iw}{1 + w}\right).$$

From (4.1.5) we get that f is an entire function and for $z = x + iy$, if $\Im z > 0$, we can estimate the integrand as follows:

$$|F(t) \cdot e^{it(x+iy)}| = |F(t)| \cdot \underbrace{|e^{itx}|}_{=1} \cdot \underbrace{|e^{-ty}|}_{\leq 1} \leq |F(t)|,$$

because $t \in [0, A]$, hence positive. Finally by (4.1.4) $|f(z)| \leq \underbrace{\|F\|_{\infty}}_{\leq M_0} \cdot A \leq A$ for all z in the upper

half plane of the complex numbers and so f is bounded there. For the Möbius-transformation above we get $1 \mapsto 0, -1 \mapsto \infty, i \mapsto 1$ and $-i \mapsto -1$ and therefore we see that g is a bounded holomorphic function on the unitball $\mathbb{D}$. But g is also continuous on $\overline{\mathbb{D}}$, except at the point -1 . By the uniqueness-theorem of the Fourier transformation we have that f cannot be identically zero (because F is not identically zero), thus g is also not the zero-function. Now we apply theorem 4.1.2 to obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|g(e^{i\theta})|) d\theta > -\infty.$$

If we set $x = i \cdot \frac{1-e^{i\theta}}{1+e^{i\theta}} = \tan\left(\frac{\theta}{2}\right)$, then $\frac{dx}{d\theta} = \frac{1}{2 \cdot \cos^2(\theta/2)} = \frac{\sin^2(\theta/2) + \cos^2(\theta/2)}{2 \cdot \cos^2(\theta/2)} = \frac{1}{2} \cdot (1+x^2)$. From the integral above we obtain now with a change of variables:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \log(|f(x)|) \frac{dx}{1+x^2} > -\infty. \quad (4.1.6)$$

On the other hand when we multiply the function f with the term $(iz)^n$ we get by partial integration of (4.1.5) (iterated application):

$$f(z) = (iz)^{-n} \int_0^A F^{(n)}(t) \cdot e^{itz} dt \text{ for } z \neq 0. \quad (4.1.7)$$

To obtain the equality above note that F and all its derivatives vanish at the points 0 and A . We restrict now f to the real axis and use (4.1.4) and (4.1.7) to get for all real x and all $n \in \mathbb{N}$:

$$|x^n \cdot f(x)| \leq 2^{-n} \cdot M_n \cdot A. \quad (4.1.8)$$

This is clear, because we have only to estimate the integral in (4.1.7) and to note that $x \in \mathbb{R}$ and so $|e^{itx}| = 1$.

From (4.1.8) we get for all $x \in \mathbb{R}$, $x \geq 0$ by multiplying with the function $Q(x)$:

$$Q(x) \cdot |f(x)| = \sum_{n=0}^{\infty} \underbrace{\frac{x^n}{M_n}}_{\leq 2^{-n} \cdot A, \forall n} \cdot |f(x)| \leq A \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \cdot A. \quad (4.1.9)$$

Now we use (4.1.9) to see $\log(Q(x)) \leq \log(2 \cdot A) - \log(|f(x)|)$ and (4.1.6) together with the monotony of the integral to finish the first step in this proof.

(2. $\implies$ 3.) Clear, because $q(x) \leq Q(x)$.

(3. $\implies$ 4.) First we set $a_n := M_n^{1/n}$. The log. conv. of the sequence $(M_n)_n$ and $M_0 = 1$ implies by 2.0.4 that $a_n \leq a_{n+1}$ for all $n > 0$. If $x \geq e \cdot a_n$, then we get $\frac{x^n}{M_n} \geq e^n$. So we conclude by remembering the definition of the function $q(x)$ that

$$\log(q(x)) \geq \log\left(\frac{x^n}{M_n}\right) \geq \log(e^n) = n. \quad (4.1.10)$$

Thus we get for every $N \in \mathbb{N}$:

$$\begin{aligned} e \cdot \int_{e \cdot a_1}^{\infty} \log(q(x)) \cdot \frac{dx}{x^2} &\stackrel{(4.1.10)}{\geq} e \cdot \sum_{n=1}^N n \cdot \int_{e \cdot a_n}^{e \cdot a_{n+1}} x^{-2} dx + e \cdot \int_{e \cdot a_{N+1}}^{\infty} (N+1) \cdot x^{-2} dx \\ &\stackrel{\text{integrating}}{=} \sum_{n=1}^N n \cdot \left(\frac{1}{a_n} - \frac{1}{a_{n+1}} \right) + \frac{N+1}{a_{N+1}} = \frac{1}{a_1} - \frac{1}{a_2} + \frac{2}{a_2} - \dots + \frac{N+1}{a_{N+1}} \\ &= \sum_{n=1}^{N+1} \frac{1}{a_n} = \sum_{n=1}^{N+1} \left(\frac{1}{M_n} \right)^{1/n}. \end{aligned}$$

So we finished this part of the proof.

(4. $\implies$ 5.) First we remember that we have put

$$\mu_n^{-1} := \frac{M_{n-1}}{M_n}$$

and the logarithmic convexity of the sequence $(M_n)_n$ is equivalent to the fact that $(\mu_n)_n$ is an increasing sequence, thus $\mu_n^{-1} \geq \mu_{n+1}^{-1}$ for all $n \in \mathbb{N}$. Now we put as before $a_n := M_n^{1/n}$ and so we get:

$$\underbrace{(a_n \cdot \mu_n^{-1})^n}_{(*)} \leq M_n \cdot \mu_1^{-1} \cdots \mu_n^{-1} = M_n \cdot \underbrace{\frac{M_0}{M_1} \cdot \frac{M_1}{M_2} \cdots \frac{M_{n-1}}{M_n}}_{1/M_1} = 1.$$

(*) above holds because $(\mu_n)_n$ is increasing, hence $\mu_n^{-1} \leq \mu_i^{-1}$, for all $i \in \mathbb{N}$ with $i \leq n$.

From this we get $\mu_n^{-1} \leq \frac{1}{a_n}$ for all $n \in \mathbb{N}$, and the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{1}{M_n}\right)^{1/n} = \sum_{n=1}^{\infty} \frac{1}{a_n}$ implies immediately the convergence of $\sum_{n=1}^{\infty} \frac{1}{\mu_n}$.

(5. $\implies$ 1.) Our assumption here is $\sum_{n=1}^{\infty} \frac{1}{\mu_n} < \infty$. We study the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by:

$$f(z) := \left(\frac{\sin z}{z}\right)^2 \cdot \prod_{n=1}^{\infty} \frac{\sin(\mu_n^{-1} \cdot z)}{z} \cdot \mu_n. \quad (4.1.11)$$

Claim: f is an entire function, which is not identically zero.

First we remark that $\left|1 - \frac{\sin z}{z}\right| \rightarrow 0$ for $|z| \rightarrow 0$. So we can find a constant $B > 0$ such that for $|z| \leq 1$ we have

$$\left|1 - \frac{\sin z}{z}\right| \leq B \cdot |z|. \quad (4.1.12)$$

For $|z| \leq \mu_n$ we get now

$$\left|1 - \frac{\sin(\mu_n^{-1} \cdot z)}{z} \cdot \mu_n\right| \leq \frac{B \cdot |z|}{\mu_n},$$

and so we can estimate as follows:

$$\sum_{n=1}^{\infty} \left|1 - \frac{\sin(\mu_n^{-1} \cdot z)}{\mu_n^{-1} \cdot z}\right| \leq \sum_{n=1}^{\infty} \frac{B \cdot |z|}{\mu_n} = B \cdot |z| \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{\mu_n}}_{< \infty, \text{ by assumpt.}} < \infty. \quad (4.1.13)$$

It follows that the series on the left hand side in (4.1.13) is convergent uniformly on compact sets. Remember that $\mu_n \rightarrow 0$ for $n \rightarrow \infty$, because we have by assumption $\sum_n \frac{1}{\mu_n} < \infty$. We use now 4.1.4 to conclude, that (4.1.11) defines an entire function f which is not identically zero.

Claim: f is an entire function of exponential type which means: There exist constants $A, C > 0$ such that $|f(z)| \leq C \cdot e^{A|z|}$ for all $z \in \mathbb{C}$.

For this we calculate:

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 e^{itz} dt &= \frac{1}{2} \int_{-1}^1 \cos(t \cdot z) dt + \frac{i}{2} \int_{-1}^1 \sin(t \cdot z) dt \\ &= \frac{1}{2} \left[\sin(t \cdot z) \cdot \frac{1}{z} \right]_{-1}^1 + \underbrace{\frac{i}{2} \left[-\cos(t \cdot z) \cdot \frac{1}{z} \right]_{-1}^1}_{=0} = \frac{\sin z}{z}. \end{aligned}$$

This identity shows that for $z = x + iy$ we get $\left|\frac{\sin z}{z}\right| \leq e^{|y|}$, because we can estimate the integrand for $t \in [-1, 1]$ as follows:

$$|e^{itz}| = \underbrace{|e^{itx}|}_{=1} \cdot |e^{-ty}| \leq e^{|y|} \leq e^{|z|}.$$

So we get $|f(z)| \leq e^{A|z|}$ with $A := 2 + \sum_{n=1}^{\infty} \frac{1}{\mu_n}$ and we have shown that f is of exponential type.

Claim: f satisfies for $x \in \mathbb{R}$ and $k \in \mathbb{N}$

$$|x^k \cdot f(x)| \leq M_k \cdot \left(\frac{\sin x}{x} \right)^2. \quad (4.1.14)$$

If $x \in \mathbb{R}$ arbitrary then $|\sin x| \leq |x|$ and clearly $|\sin x| \leq 1$. Hence for each $k \in \mathbb{N}$: we have $\prod_{n=k+1}^{\infty} \frac{\sin(\mu_n^{-1} \cdot x)}{x} \cdot \mu_n \leq 1$ and if we split the infinite product we can estimate:

$$\begin{aligned} |x^k \cdot f(x)| &\leq |x|^k \cdot \left(\frac{\sin x}{x} \right)^2 \cdot \prod_{n=1}^k \underbrace{\left| \frac{\sin(\mu_n^{-1} \cdot x)}{x} \cdot \mu_n \right|}_{\leq \mu_n \cdot |x|^{-1}, \text{ for all } n} \\ &\leq \left(\frac{\sin x}{x} \right)^2 \cdot \mu_1 \cdots \mu_k = \left(\frac{\sin x}{x} \right)^2 \cdot \frac{M_1}{M_0} \cdot \frac{M_2}{M_1} \cdots \frac{M_k}{M_{k-1}} \\ &= \underbrace{\frac{M_k}{M_0}}_{=M_k} \cdot \left(\frac{\sin x}{x} \right)^2, \end{aligned}$$

and this shows (4.1.14).

Next we remark that $\int_0^\infty \left(\frac{\sin(ax)}{x} \right)^2 dx = \frac{\pi}{2} \cdot |a|$ holds (see [4, p. 1083, integral (21.16)]), hence $\int_{-\infty}^\infty \left(\frac{\sin x}{x} \right)^2 dx = \pi$. If we integrate (4.1.14) for the case $k = 0$ we obtain

$$\int_{-\infty}^\infty |f(x)| dx \leq \int_{-\infty}^\infty \left(\frac{\sin x}{x} \right)^2 dx = \pi$$

and so $f \in L^1(\mathbb{R})$. Furthermore $f \in L^2(\mathbb{R})$ because f is a continuous function. Integrating (4.1.14) for arbitrary $k \in \mathbb{N}$ leads to

$$\frac{1}{\pi} \int_{-\infty}^\infty |x^k \cdot f(x)| dx \leq M_k \cdot \frac{1}{\pi} \int_{-\infty}^\infty \left(\frac{\sin x}{x} \right)^2 dx = M_k. \quad (4.1.15)$$

So we have shown that the function f satisfies the hypotheses of theorem 4.1.3 (note: we restrict now f to the real axis) and so we obtain the (classical) Fourier transform of f

$$F(t) := \frac{1}{2\pi} \int_{-\infty}^\infty f(x) \cdot e^{-itx} dx \text{ for } t \in \mathbb{R},$$

which defines a function with compact support. Furthermore F is not identically zero, because f is not identically zero. (4.1.15) shows that $x^k \cdot f(x) \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}$, hence F is smooth and we obtain:

$$F^{(k)}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty (-ix)^k \cdot f(x) \cdot e^{-itx} dx.$$

In particular (4.1.15) shows that $\|F^{(k)}\|_\infty \leq \frac{1}{2} M_k \leq M_k$ holds for all $k \in \mathbb{N}$ and so all derivatives of F exist and are bounded. We have $F \in \mathcal{E}_{\{M\}}$, where $C = h = 1$, and this shows that $\mathcal{E}_{\{M\}}$ contains a nontrivial function with compact support and by theorem 4.1.1 $\mathcal{E}_{\{M\}}$ is a not quasi-analytic space.

□

Remark 4.1.6 1. Note that the proof for the Denjoy-Carleman-Theorem above uses theorems and statements from complex analysis. In several steps we have defined functions on the complex plane, then we have used complex analysis to prove certain properties, and after that we have restricted them to the real axis.

2. Condition (5), which is also called the non quasi-analyticity-condition, is the most popular condition to characterize the quasi-analyticity of function spaces. Note that if one uses the weight sequence $m := (m_k)_k$, then condition (5) for $\mathcal{E}_{\{M\}}$ being not quasi-analytic becomes $\sum_{n=1}^{\infty} \frac{m_{n-1}}{n \cdot m_n} < \infty$.
3. We recall: If $\mathcal{E}_{\{M\}}$ is quasi-analytic then there are no bump-functions, hence no $\mathcal{E}_{\{M\}}$ -partitions of unity. Because $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{\{M\}}$ holds for all weight sequences M , we see by restricting the Borel-map: If $\mathcal{E}_{\{M\}}$ is quasi-analytic, then so is $\mathcal{E}_{(M)}$.

An example of a quasi-analytic space is the space of all real analytic functions $\mathcal{O}$, where we have $(M_p)_p := (p!)_p$, hence $(\mu_p)_p := (p)_p$. $M_0 = 1$ and we have already seen that $(p!)_p$ is log. convex. $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \sum_{p=1}^{\infty} \frac{1}{p} = \infty$ and we can apply 4.1.5 to get the quasi-analyticity for $\mathcal{O}$. So we see:

For any weight sequence $N := (N_p)_p$, $n_p := \frac{N_p}{p!}$, with the property $\sup_p n_p^{1/p} < \infty$, we obtain by (3.1.1) $\mathcal{E}_{\{N\}} \subseteq \mathcal{O}$. So we can restrict the injective Borel-map from $\mathcal{O}$ and obtain the quasi-analyticity for $\mathcal{E}_{\{N\}}$, and because $\mathcal{E}_{(N)} \subseteq \mathcal{E}_{\{N\}}$, for $\mathcal{E}_{(N)}$, too.

The proof of 4.1.5 above has shown the equivalence of (4) and (5). This can be proven directly:

Proposition 4.1.7 *Let $M := (M_n)_n$ be a weight sequence. Then we obtain the following equivalence:*

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} = \infty \iff \sum_{n=1}^{\infty} \left(\frac{1}{M_n} \right)^{1/n} = \infty.$$

Proof. $\implies$: Holds by 4.1.5, (4. $\Rightarrow$ 5.)

$\impliedby$: To prove the inverse direction we assume that $\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty$ and we use the following inequality, which is called *Carleman's inequality*:

$$\sum_{n=1}^{\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} \leq e \cdot \sum_{n=1}^{\infty} a_n, \quad (4.1.16)$$

where $a_i > 0$, $\forall i$, and $\sum_{n=1}^{\infty} a_n < \infty$. If we apply (4.1.16) to $a_i := \frac{M_{i-1}}{M_i}$ we obtain:

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{M_0}{M_1} \cdots \frac{M_{n-1}}{M_n} \right)^{1/n}}_{= \frac{M_0}{M_n} = \frac{1}{M_n}} \leq e \cdot \sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty.$$

□

To complete the proof above we prove now (4.1.16). Remark that in the literature there are a lot of different possibilities and techniques to prove the Carleman inequality.

Proposition 4.1.8 [10, Proof 3, p. 3-4] *Let $(a_n)_n$ be a sequence of positive real numbers and we assume that $\sum_{n=1}^{\infty} a_n < \infty$. Then the following inequality holds:*

$$\sum_{n=1}^{\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} \leq e \cdot \sum_{n=1}^{\infty} a_n.$$

Proof. For the proof we use the known inequality between the arithmetic and geometric mean, which we will call *AG-inequality*, and the following identity:

$$\frac{(k+1)^k}{k!} = \left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right)^2 \cdots \left(1 + \frac{1}{k}\right)^k < e^k. \quad (4.1.17)$$

First we prove (4.1.17). The inequality part is clear from analysis since $e = \sup_k (1 + 1/k)^k$. To prove the equality part we use induction on k :

$$k=1: \frac{(1+1)^1}{1!} = 2 = (1 + \frac{1}{1})^1.$$

$k \mapsto k+1$: The left hand side gives

$$\begin{aligned} \frac{1}{(k+1)!} \cdot ((k+1)+1)^{k+1} &= \frac{1}{(k+1)!} \cdot \sum_{j=0}^{k+1} \binom{k+1}{j} \cdot (k+1)^j \\ &= \frac{1}{k!} \cdot \sum_{j=0}^{k+1} \binom{k+1}{j} \cdot (k+1)^{j-1} \underbrace{=}_{j \mapsto k+1-j} \frac{1}{k!} \cdot \sum_{j=0}^{k+1} \binom{k+1}{j} \cdot (k+1)^{k-j}. \end{aligned}$$

The right hand side gives

$$\begin{aligned} \left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right)^2 \cdots \left(1 + \frac{1}{k}\right)^k \cdot \left(1 + \frac{1}{k+1}\right)^{k+1} &\underbrace{=}_{I.H.} \frac{(k+1)^k}{k!} \cdot \sum_{j=0}^{k+1} \binom{k+1}{j} \cdot \frac{1}{(k+1)^j} \\ &= \frac{1}{k!} \cdot \sum_{j=0}^{k+1} \binom{k+1}{j} \cdot (k+1)^{k-j}. \end{aligned}$$

Now we can prove the Carleman inequality as follows:

$$\begin{aligned} \sum_{i=1}^{\infty} a_i &= \sum_{i=1}^{\infty} i a_i \cdot \underbrace{\left(\sum_{k=i}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right)}_{=\frac{1}{i}} = \sum_{i=1}^{\infty} i a_i \cdot \left(\sum_{k=i}^{\infty} \frac{1}{k \cdot (k+1)} \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{k \cdot (k+1)} \cdot \sum_{i=1}^k i a_i = \sum_{k=1}^{\infty} \frac{a_1 + 2a_2 + \cdots + k a_k}{k \cdot (k+1)} \\ &\underbrace{>}_{A.G.} \sum_{k=1}^{\infty} \frac{1}{k+1} \cdot \left(k! \cdot \prod_{i=1}^k a_i \right)^{\frac{1}{k}} = \sum_{k=1}^{\infty} \left(\frac{k!}{(k+1)^k} \right)^{\frac{1}{k}} \cdot \left(\prod_{i=1}^k a_i \right)^{\frac{1}{k}} \\ &\underbrace{\geq}_{(4.1.17)} \frac{1}{e} \cdot \sum_{k=1}^{\infty} \left(\prod_{i=1}^k a_i \right)^{\frac{1}{k}}. \end{aligned}$$

The inequality in the calculation above is strict because (4.1.17) is strict.

□

4.2 Denjoy-Carleman-theorem 2

We will give now a second proof of the Denjoy-Carleman-theorem, which is due to Hörmander and uses again Carleman's inequality. But, as mentioned at the beginning of this chapter, this version doesn't assume the log. convexity for the weight sequence. In this section let

$$M := (M_j)_j \text{ be an arbitrary weight sequence with } M_0 = 1.$$

First define the sequence $M^i := (M_j^i)_j$ in the following way:

$$M_j^i := \inf_{k \geq j} M_k^{1/k}, \quad (4.2.1)$$

which is by definition the largest increasing minorant of the sequence $(M_j^{1/j})_j$. We remark: If we assume the log. convexity condition for $(M_j)_j$ then, by 2.0.4, the sequence $(M_j^{1/j})_j$ itself is increasing, hence $(M_j^{1/j})_j = (M_j^i)_j$.

Furthermore we introduce now the sequence $M^c := (M_j^c)_j$:

$$M_j^c := \inf \{ M_k^{\frac{l-j}{l-k}} \cdot M_l^{\frac{j-k}{l-k}} : k \leq j \leq l, k \neq l \}, \quad M_0^c = M_0 = 1. \quad (4.2.2)$$

Clearly we have always $M_j^c > 0$ for all j , unless the case where $M_j^c = 0$ for all j . Furthermore, if we set $l = j$ and $k \leq j - 1$, we see that the set on the right hand side contains the element M_j , hence $M_j^c \leq M_j$ holds for all j . We show now: M^c satisfies the log. convexity condition. The sequence M^c is the *largest log. convex minorant* of the sequence M . If M is assumed to be log. convex, then $M = M^c$ is satisfied.

Lemma 4.2.1 *Let $(M_j)_j$ be an arbitrary weight sequence with $M_0 = 1$. For the sequence M^c then we get:*

$$(M_j^c)^2 \leq M_{j-1}^c \cdot M_{j+1}^c$$

is satisfied for all $j \in \mathbb{N}$.

Proof. First we apply log to the sequence M^c and get

$$\log(M_j^c) := \inf \left\{ \frac{l-j}{l-k} \cdot \log(M_k) + \frac{j-k}{l-k} \cdot \log(M_l) : k \leq j \leq l, k \neq l \right\}, \quad \log(M_0^c) = 0.$$

We show now that the function $j \mapsto \log(M_j^c)$ is convex. Therefore put $P_j := (j, \log(M_j))$ and $P_j^c := (j, \log(M_j^c))$ and we denote with σ the line segment from P_{j-}^c to P_{j+}^c , where $j_- < j < j_+$. We assume now that the point P_j^c lies above σ by an $\varepsilon > 0$.

By definition of M^c there exists a line segment σ_- from P_{j--} to P_{j-+} where $j_{--} < j_- < j_{-+}$ and such that the point $(j_-, \log(M_{j_-}^c) + \varepsilon)$ lies above σ_- . One of the endpoints of σ_- , we denote it with P_{n_1} , has to lie below the segment $\sigma + (0, \varepsilon)$ and $n_1 < j$. Otherwise P_j^c would lie below $\sigma + (0, \varepsilon)$, which is a contradiction.

On the other side there exists a line segment σ_+ from P_{j+-} to P_{j++} with $j_{+-} < j_+ < j_{++}$ and such that the point $(j_+, \log(M_{j_+}^c) + \varepsilon)$ lies above the segment σ_+ . One of the endpoints of σ_+ , which will be denoted by P_{n_2} , has to lie below $\sigma + (0, \varepsilon)$ and $n_2 > j$.

But so the point $(j, \log(M_j^c) - \varepsilon)$ has to lie below the segment between P_{n_1} and P_{n_2} which is a contradiction.

□

We set for all $j \in \mathbb{N}$

$$\mu_j^c := \frac{M_j^c}{M_{j-1}^c} \quad (4.2.3)$$

and obtain a sequence $\mu^c := (\mu_j^c)_j$, which is then increasing. If $M_j^c = 0$, then we set $(\mu_j^c)^{-1} := \infty$.

To prove the new version of the Denjoy-Carleman-Theorem we need two preparatory results, which will be used later, too. In the first proposition we construct a bump-function with certain estimates for its derivatives.

For this we define the function $H_a : \mathbb{R} \rightarrow \mathbb{R}$, where $a \in \mathbb{R}$, as follows:

$$H_a(x) := \begin{cases} a^{-1} & \text{for } 0 < x < a \\ 0 & \text{otherwise.} \end{cases} \quad (4.2.4)$$

By definition $\int_{-\infty}^{\infty} H_a(x)dx = 1$ holds and if $u : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function we obtain for the convolution:

$$(u * H_a)(x) = \int_{-\infty}^{\infty} u(x-t) \cdot H_a(t)dt = a^{-1} \cdot \int_0^a u(x-t)dt = a^{-1} \cdot \int_{x-a}^x u(s)ds.$$

Hence $u * H_a$ is $\mathcal{C}^1$ with derivative

$$(u * H_a)'(x) = \frac{u(x) - u(x-a)}{a}. \quad (4.2.5)$$

If u is a $\mathcal{C}^k$ -function, then $(u * H_a)(x)$ is a $\mathcal{C}^{k+1}$ -function.

Proposition 4.2.2 [8, Theorem 1.3.5., p. 19-20] *Let $(a_i)_i$ be a decreasing sequence of positive numbers such that $a := \sum_{j=0}^{\infty} a_j < \infty$ holds and we define $u_k := H_{a_0} * \dots * H_{a_k}$. Then $u_k \in \mathcal{C}^{k-1}(\mathbb{R})$ and $\text{supp}(u_k)$ is contained in $[0, a]$. For $k \rightarrow \infty$ we have $u_k \rightarrow u$, where $u \in \mathcal{E}(\mathbb{R})$ and $\text{supp}(u)$ is contained in $[0, a]$, too. Furthermore $\int_{-\infty}^{\infty} u(x)dx = 1$ holds and we obtain the following inequalities for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$:*

$$|u^{(j)}(x)| \leq \frac{1}{2} \cdot \int_{-\infty}^{\infty} |u^{(j+1)}(x)|dx \leq \frac{2^j}{a_0 \dots a_j}.$$

Proof. By definition $u_1(x) = (H_{a_0} * H_{a_1})(x) = a_1^{-1} \cdot \int_{x-a_1}^x H_{a_0}(t)dt$, hence u_1 vanishes except in the interval $[0, a_0 + a_1]$. Furthermore $u_1(0) = u_1(a_0 + a_1) = 0$ and because $a_1 \leq a_0$: $u_1(a_1) = a_1^{-1} \cdot \int_0^{a_1} H_{a_0}(t)dt = a_0^{-1}$, $u_1(a_0) = a_1^{-1} \cdot \int_{a_0-a_1}^{a_0} H_{a_0}(t)dt = a_0^{-1}$.

From this we get: u_1 is increasing linearly with slope $(a_0 \cdot a_1)^{-1}$ in $[0, a_1]$ to a_0^{-1} , then it is constant a_0^{-1} in $[a_1, a_0]$ and decreasing linearly in $[a_0, a_0 + a_1]$ to 0. Thus u_1 is continuous on $\mathbb{R}$, which implies $u_k \in \mathcal{C}^{k-1}$ by the definition of u_k and the remarks above. We have $\int_{-\infty}^{\infty} u_1(x)dx = 2 \cdot \frac{a_1}{2a_0} + \frac{a_0 - a_1}{a_0} = 1$ and the support of u_k is contained in the interval $[0, \sum_{j=0}^k a_j]$ because $\text{supp}(u_k) = \text{supp}(u_{k-1} * H_{a_k}) \subseteq \text{supp}(u_{k-1}) + \text{supp}(H_{a_k})$.

Let $v, w : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions with compact support such that $\int_{-\infty}^{\infty} v(x)dx = \int_{-\infty}^{\infty} w(x)dx = 1$, then

$$\int_{-\infty}^{\infty} (v * w)(x)dx = 1 \quad (4.2.6)$$

holds. This follows by *Fubini's theorem*:

$$\begin{aligned} \int_{-\infty}^{\infty} (v * w)(x)dx &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(x-y) \cdot w(y)dy \right) dx \\ &= \int_{-\infty}^{\infty} w(y) \cdot \left(\int_{-\infty}^{\infty} v(x-y)dx \right) dy = \int_{-\infty}^{\infty} w(y)dy = 1. \end{aligned}$$

In particular we have $\int_{-\infty}^{\infty} u_k(x)dx = \int_{-\infty}^{\infty} (u_{k-1} * H_{a_k})(x)dx = 1$ for all k and furthermore, because $\int_{-\infty}^{\infty} H_{a_j}(x)dx = 1$ holds for all a_j , we get

$$\int_{-\infty}^{\infty} (H_{a_j} * \dots * H_{a_k})(x)dx = 1 \quad (4.2.7)$$

for all $j, k \in \mathbb{N}$ with $0 \leq j \leq k$. In the following we write $(\tau_a u)(x) := u(x-a)$ for the translation by a . Using this notation we can write for $u_k = H_{a_0} * (H_{a_1} * \cdots * H_{a_k})$ by (4.2.5):

$$u'_k(x) = \frac{(H_{a_1} * \cdots * H_{a_k})(x) - (H_{a_1} * \cdots * H_{a_k})(x - a_0)}{a_0} = \frac{1}{a_0}(\text{id} - \tau_{a_0})(H_{a_1} * \cdots * H_{a_k})(x).$$

Iterating this procedure we obtain for $1 \leq j \leq k-1$ the following expression:

$$u_k^{(j)} = \left(\prod_{i=0}^{j-1} \frac{1}{a_i} (\text{id} - \tau_{a_i}) \right) (H_{a_j} * \cdots * H_{a_k}). \quad (4.2.8)$$

We remark that one can always estimate the convolution $v * w$ by $\sup_{x \in \mathbb{R}} |v(x)| \cdot \int_{-\infty}^{\infty} |w(y)| dy$. Hence for $1 \leq j \leq k-1$ and arbitrary $x \in \mathbb{R}$ we have

$$\begin{aligned} |u_k^{(j)}(x)| &\stackrel{(4.2.8)}{\leq} \frac{2^j}{a_0 \cdots a_{j-1}} \cdot |(H_{a_j} * (H_{a_{j+1}} * \cdots * H_{a_k}))(x)| \\ &\leq \frac{2^j}{a_0 \cdots a_{j-1}} \cdot \sup_{y \in \mathbb{R}} |H_{a_j}(y)| \cdot \int_{-\infty}^{\infty} |(H_{a_{j+1}} * \cdots * H_{a_k})(t)| dt \\ &\stackrel{(4.2.7)}{=} \frac{2^j}{a_0 \cdots a_j} \end{aligned}$$

and

$$\int_{-\infty}^{\infty} |u_k^{(j)}(x)| dx \stackrel{(4.2.8)}{\leq} \frac{2^j}{a_0 \cdots a_{j-1}} \cdot \int_{-\infty}^{\infty} |(H_{a_j} * \cdots * H_{a_k})(x)| dx \stackrel{(4.2.7)}{=} \frac{2^j}{a_0 \cdots a_{j-1}}.$$

We summarize: For $1 \leq j \leq k-1$ we have

$$|u_k^{(j)}(x)| \leq \frac{2^j}{a_0 \cdots a_j} \text{ for all } x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{\infty} |u_k^{(j)}(x)| dx \leq \frac{2^j}{a_0 \cdots a_{j-1}}. \quad (4.2.9)$$

Let $h, \phi : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions with compact support, $\int_{-\infty}^{\infty} \phi(x) dx = 1$ and put $h_\phi := h * \phi$. For $y \in \text{supp}(\phi)$ we have:

$$|(h - h_\phi)(x)| = \left| \int_{-\infty}^{\infty} (h(x) - h(x-y)) \cdot \phi(y) dy \right| \leq \sup_{y \in \text{supp}(\phi)} |h(x) - h(x-y)|. \quad (4.2.10)$$

Now we estimate for all $k \in \mathbb{N}$ and $m \geq 2$:

$$\begin{aligned} |(u_{k+m} - u_m)(x)| &= |(u_m * (H_{a_{m+1}} * \cdots * H_{a_{m+k}}) - u_m)(x)| \\ &\stackrel{(4.2.10)}{\leq} (a_{m+1} + \cdots + a_{m+k}) \cdot \sup_{x \in \mathbb{R}} |u'_m(x)| \stackrel{(4.2.9)}{\leq} \frac{2}{a_0 \cdot a_1} \cdot (a_{m+1} + \cdots + a_{m+k}), \end{aligned}$$

where we have used (4.2.10) for $h := u_m$ and $\phi := H_{a_{m+1}} * \cdots * H_{a_{m+k}}$ with $\text{supp}(\phi) \subseteq [0, a_{m+1} + \cdots + a_{m+k}]$. So $(u_k)_k$ has a uniform limit $u := \lim_{k \rightarrow \infty} u_k$. We can repeat the calculation above for the derivatives $u_k^{(j)}$, $j \leq k-1$, and so $u^{(j)} = \lim_{k \rightarrow \infty} u_k^{(j)}$, hence $u \in \mathcal{E}(\mathbb{R})$. By (4.2.6) we have $\int_{-\infty}^{\infty} u(x) dx = 1$ and (4.2.9) implies the required inequalities for the function u .

□

As mentioned we have always $\sup_{z \in \mathbb{R}} |(u * v)(z)| \leq \sup_{x \in \mathbb{R}} |u(x)| \cdot \int_{-\infty}^{\infty} |v(y)| dy$. So in proposition 4.2.2 we obtain

$$\sup_{z \in \mathbb{R}} |u_{m+k}(z)| \leq \sup_{x \in \mathbb{R}} |u_m(x)| \cdot \int_{-\infty}^{\infty} |(H_{a_{m+1}} * \cdots * H_{a_{m+k}})(y)| dy = \sup_{x \in \mathbb{R}} |u_m(x)|$$

for all $m \in \mathbb{N}$ and $k \in \mathbb{N}$, hence $\sup_{x \in \mathbb{R}} |u(x)| \leq a_0^{-1}$.

The second preparatory result is the following:

Proposition 4.2.3 [8, Lemma 1.3.6., p. 20-22] Let $T > 0$ and $u \in \mathcal{C}^m((-\infty, T])$ a function, which is vanishing on the negative half axis. Furthermore let $(a_j)_j$ be a sequence of positive decreasing numbers such that $T \leq \sum_{j=1}^m a_j$. Then

$$|u(x)| \leq \sum_{j \in J_m} 2^{2j} \cdot a_1 \cdots a_j \cdot \sup_{y \in \mathbb{R}: y < x} |u^{(j)}(y)| \quad (4.2.11)$$

holds for $x \leq T$, where the set J_m is defined as follows: $J_m := \{j \in \mathbb{N} : 1 \leq j \leq m, a_{j+1} < a_j\} \cup \{m\}$.

Proof. With the notations defined above we get $u(x) = (H_a * (a \cdot u'))(x) + (\tau_a u)(x)$ for any a , because the right hand side is equal to

$$a^{-1} \cdot a \cdot \int_{x-a}^x u'(t) dt + u(x-a) = u(x) - u(x-a) + u(x-a).$$

So for $0 \leq j \leq m-1$ we have

$$u^{(j)} = H_{a_{j+1}} * (a_{j+1} \cdot u^{(j+1)}) + \tau_{a_{j+1}} u^{(j)}. \quad (4.2.12)$$

To prove (4.2.11) we start with $j = 0$ and use iterated application of formula (4.2.12) and the property $\tau_a(f * g) = f * (\tau_a g)$.

In the following we will consider terms of the form

$$\tau_{a_{k_1}} \cdots \tau_{a_{k_i}} H_{a_1} * \cdots * H_{a_j} * a_1 \cdots a_j \cdot u^{(j)}; \quad i \leq m, j \leq m \text{ and } a_{k_l} \geq a_l, \forall l, l \leq i. \quad (4.2.13)$$

The condition $a_{k_l} \geq a_l$ for all l with $l \leq i$ has the consequence that $\tau_{a_{k_1}} \cdots \tau_{a_{k_i}}$ is a translation by at least $\sum_{l=1}^i a_l$. We call an expression in (4.2.13) a *legitimate term of type (i, j)* and we will represent it by the point $(i, j) \in \mathbb{N}^2$. Furthermore we introduce the set $Z \subseteq \mathbb{N}^2$ as follows:

$$Z := \{(i, j) \in \mathbb{N}^2 : 0 \leq i < m, 0 \leq j < m, a_{j+1} \geq a_{i+1}\}.$$

If $i, j < m$, then we can apply (4.2.12) to (4.2.13) and obtain two new terms:

$$\begin{aligned} \tau_{a_{k_1}} \cdots \tau_{a_{k_i}} H_{a_1} * \cdots * H_{a_j} * a_1 \cdots a_j \cdot u^{(j)} &= \tau_{a_{k_1}} \cdots \tau_{a_{k_i}} H_{a_1} * \cdots * H_{a_{j+1}} * a_1 \cdots a_{j+1} \cdot u^{(j+1)} \\ &\quad + \tau_{a_{k_1}} \cdots \tau_{a_{k_i}} \tau_{a_{j+1}} H_{a_1} * \cdots * H_{a_j} * a_1 \cdots a_j \cdot u^{(j)}. \end{aligned}$$

The first new term above is clearly a legitimate one of type $(i, j+1)$, the second new term is legitimate of type $(i+1, j)$ if and only if $a_{j+1} \geq a_{i+1}$ is satisfied. Thus we see: If we take a legitimate term of type (i, j) such that additionally $(i, j) \in Z$ is satisfied and apply (4.2.12) to it, then, because $a_{j+1} \geq a_{i+1}$ holds now by assumption, the second new term is legitimate of type $(i+1, j)$, too.

We start with $j = 0$ where we have by (4.2.12): $u = H_{a_1} * a_1 \cdot u' + \tau_{a_1} u$. Thus the first term on the right hand side is clearly legitimate of type $(0, 1)$, and the second of type $(1, 0)$. Additionally the second term of type $(1, 0)$ is always contained in Z , because $a_2 \leq a_1$ holds by assumption. But the first term $(0, 1)$ lies in Z if and only if $a_1 \leq a_2$, in particular if $a_1 = a_2$ holds. Hence either $(0, 1) \in Z$ or $a_2 < a_1$, which means $1 \in J_m$. If $1 \in J_m$, then the term $(0, 1)$ occurs in the sum on the right hand side of (4.2.11). In the following we distinguish two cases:

If $i, j < m$ and $i > j$, then by assumption on the sequence $(a_k)_k$ we have $a_{j+1} \geq a_{i+1}$ and so $(i, j) \in Z$ holds. We apply (4.2.12) and both new terms belong again to Z unless the cases where $i+1 = m$ or $j+1 = m$: $(i, j+1) \in Z$ holds because $j+1 \leq i$ by assumption, so $j+2 \leq i+1$ which implies $a_{j+2} \geq a_{i+1}$. $(i+1, j) \in Z$ is satisfied because $a_{i+2} \leq a_{i+1} \leq a_{j+1}$. If $j+1 = m$, then $j+1 \in J_m$ and the term $(i, j+1)$ occurs in the sum on the right hand side of (4.2.11).

If $i, j < m$ with $i \leq j$ and $(i, j) \in Z$, then the term $(i+1, j)$ lies again in Z unless the case $i+1 = m$: This is clear because $a_{i+2} \leq a_{i+1} \leq a_{j+1}$. For the term $(i, j+1)$ we have to

distinguish: Either $j + 1 \in J_m$, which means $a_{j+2} < a_{j+1}$ or $j + 1 = m$. In this case the term $(i, j + 1)$ occurs in the sum on the right hand side of (4.2.11). Or $j + 1 \notin J_m$, which means $a_{j+2} = a_{j+1}$, but then $a_{j+2} = a_{j+1} \geq a_{i+1}$ holds and so $(i, j + 1) \in Z$.

Hence we see: We can iterate the procedure above and it stops because u is a sum of terms in (4.2.13) with either $i \geq m$ or $j \in J_m$ and $i + 1 \leq j \leq m$. The terms (i, j) for $i \geq m$ vanish because if $i \geq m$ and j arbitrary, then $\tau_{a_{k_1}} \dots \tau_{a_{k_i}}$ is a translation by at least $\sum_{l=1}^m a_l$. If $i + 1 \leq j$ for $j \in J_m$, then the term (i, j) occurs in the sum on the right hand side of (4.2.11).

The number of legitimate terms of type (i, j) is at most 2^{i+j} because each term corresponds to an increasing path from $(0, 0)$ to (i, j) in the lattice $\mathbb{Z}^2$. So one needs $i + j$ steps with at most two alternatives for each of them.

Finally we remark that $\sum_{i \in \mathbb{N}: i < j} 2^{i+j} < 2^{2j}$ holds for all $i, j \in \mathbb{N}$. We show this by induction. If $j = 1$ we have $2 < 2^2$. For $j \mapsto j + 1$ we calculate:

$$\sum_{i: i < j+1} 2^{i+j+1} = 2 \cdot \sum_{i: i < j+1} 2^{i+j} = 2 \cdot \sum_{i: i < j} 2^{i+j} + 2 \cdot 2^{2j} \underbrace{<}_{\text{I.H.}} 2 \cdot 2^{2j} + 2^{2j+1} = 2 \cdot 2^{2j+1} = 2^{2j+2}.$$

This shows the required estimation (4.2.11) for the function u .

□

If we drop the assumption $T \leq \sum_{j=1}^m a_j$, we set $c := \frac{T}{\sum_{j=1}^m a_j}$ and apply the result above to $u(c \cdot x)$ for $x \leq \sum_{j=1}^m a_j \Leftrightarrow c \cdot x \leq T$. Hence we obtain for $x \leq T$ the new estimate

$$|u(x)| \leq \sum_{j \in J_m} (4c)^j \cdot a_1 \cdots a_j \cdot \sup_{y \in \mathbb{R}: y < x} |u^{(j)}(y)|.$$

Now let us assume that

$$|u^{(j)}(x)| \leq \frac{D^j}{a_1 \cdots a_j} \quad \text{for } j \in J_m, \quad x \leq T, \quad (4.2.14)$$

for a constant $D > 0$, then we have $|u(x)| \leq \sum_{j \in J_m} (4cD)^j$ for $x \leq T$. Let $4cD < 8cD < 1$, then

$$|u(x)| \leq 8cD = \frac{8TD}{\sum_{j=1}^m a_j} \quad \text{for } x \leq T \leq \frac{\sum_{j=1}^m a_j}{8D}. \quad (4.2.15)$$

This holds because $|u(x)| \leq \sum_{j \in J_m} (4cD)^j \leq \sum_{j=1}^{\infty} (4cD)^j = \frac{4cD}{1-4cD} < 8cD$: The last inequality is equivalent to $8cD < 1$.

If we assume that u is a smooth function, $\sum_{j=1}^{\infty} a_j = \infty$ holds and (4.2.14) is valid for all $j \in \mathbb{N}$ (or for all j in an infinite subset of $\mathbb{N}$), then by (4.2.15) we see: $u = 0$.

Now we can formulate and prove the new version of the Denjoy-Carleman-Theorem:

Theorem 4.2.4 [8, Theorem 1.3.8., p. 23-24] Let $M := (M_k)_k$ be a weight sequence with $M_0 = 1$, $K \subseteq \mathbb{R}$ an arbitrary closed interval, then the following conditions are equivalent:

- (i) The space $\mathcal{E}_{\{M\}}(K)$ is quasi-analytic.
- (ii) $\sum_{j=0}^{\infty} \frac{1}{M_j^c} = \infty$, where $M_j^c := \inf_{k \geq j} M_k^{1/k}$.
- (iii) $\sum_{j=0}^{\infty} \left(\frac{1}{M_j^c} \right)^{1/j} = \infty$, where $(M_j^c)_j$ is the largest log. convex minorant of $(M_j)_j$.
- (iv) $\sum_{j=1}^{\infty} (\mu_j^c)^{-1} = \infty$, where $(\mu_j^c)^{-1} := \frac{M_{j-1}^c}{M_j^c}$.

Proof. First we consider the case, where the sequence $M^i := (M_j^i)_j$ is bounded: $\exists C > 0$ such that $M_j^i \leq C$ for all $j \in \mathbb{N}$. Thus $\lim_{j \rightarrow \infty} M_{k_j}^{1/k_j} \leq C$ for a subsequence $(k_j)_j$. Now we take in (4.2.2) $k = 0$ and let l run through the subsequence $(k_j)_j$ to see

$$M_j^c \leq \inf\{M_0^{(l-j)/l} \cdot M_l^{j/l}, 0 \leq j \leq l, l \neq 0\} = \inf\{M_l^{j/l}, 0 \leq j \leq l, l \neq 0\},$$

which implies

$$C^j \cdot M_0 \geq M_j^c = M_0 \cdot \mu_1^c \cdots \mu_j^c \geq M_0 \cdot (\mu_1^c \cdots \mu_{i-1}^c) \cdot (\mu_i^c)^{j-i+1} \quad \text{for } i \leq j.$$

Then we fix i , take the j -th root and let $j \rightarrow \infty$. So, by

$$\lim_{j \rightarrow \infty} (\mu_1^c \cdots \mu_{i-1}^c \cdot (\mu_i^c)^{j-i+1})^{1/j} = \mu_i^c,$$

we get $\mu_i^c \leq C$ and we estimate $\sum_{j=1}^{\infty} (\mu_j^c)^{-1} \geq \sum_{j=1}^{\infty} \frac{1}{C} = \infty$, which shows condition (iv).

By assumption we have $\frac{1}{C} \leq \frac{1}{M_j^i}$ for all $j \in \mathbb{N}$, which shows (ii) and furthermore $\frac{1}{C} \leq \left(\frac{1}{M_{k_j}}\right)^{1/k_j} \leq \left(\frac{1}{M_{k_j}^c}\right)^{1/k_j}$, which shows (iii).

Condition (i) is also satisfied: Let $f \in \mathcal{E}_{\{M\}}(K)$, and $f^{(j)}(x_0) = 0$ for a point $x_0 \in K$ and all $j \in \mathbb{N}$. Then we use Taylor's formula to estimate for every $x \in K$ the residue term:

$$|f(x)| \leq D \cdot h^{k_j} \cdot |x - x_0|^{k_j} \cdot \frac{M_{k_j}}{k_j!}. \quad (4.2.16)$$

If we apply the k_j -th root of (4.2.16) then, by *Stirling's formula*, one can see that

$$\left(\frac{1}{k_j!}\right)^{1/k_j} \sim \frac{e}{k_j} \cdot (2\pi k_j)^{-\frac{1}{(2k_j)}},$$

which tends to 0 for $j \rightarrow \infty$. Note that $\lim_{j \rightarrow \infty} M_{k_j}^{1/k_j} \leq C$, so the right hand side in (4.2.16) tends to 0 if $j \rightarrow \infty$, which shows (i).

In the following we assume now that the sequence $M^i := (M_j^i)_j$ is not bounded, thus $\lim_{j \rightarrow \infty} M_j^{1/j} = \infty$.

(iii) $\Leftrightarrow$ (iv): The sequence $M^c := (M_j^c)_j$ is log. convex with $M_0^c = 1$ and so we can use here 4.1.7 for the sequence M^c .

(iv) $\Rightarrow$ (ii): By assumption $\lim_{j \rightarrow \infty} M_j^{1/j} = \infty$ holds, applying the function $\log$ to the sequence $(M_j)_j$ leads to $\lim_{j \rightarrow \infty} \frac{1}{j} \cdot \log(M_j) = \infty$. Thus the points $(j, \log(M_j))$ will lie above lines with arbitrarily high slope for $j \in \mathbb{N}$ large enough, hence by definition all M_j^c are positive and furthermore $\lim_{j \rightarrow \infty} (\mu_j^c)^{-1} = 0$. Now we define the set $\hat{J} := \{j \in \mathbb{N} : \mu_j^c < \mu_{j+1}^c\}$. At points $(j, \log(M_j^c))$, where $j \in \hat{J}$, the function $j \mapsto \log(M_j^c)$ has a corner.

We show now: $M_j^c = M_j$ holds for $j \in \hat{J}$. So let $j \in \hat{J}$ and $k < j < l$, then we estimate

$$\begin{aligned} M_k^{\frac{l-j}{l-k}} \cdot M_l^{\frac{j-k}{l-k}} &\underset{M_j \geq M_j^c, \forall j}{\geq} M_k^{\frac{l-j}{l-k}} \cdot M_l^{\frac{j-k}{l-k}} \underset{(*)}{\geq} M_j^c \cdot (\mu_j^c)^{-\frac{(l-j)(j-k)}{l-k}} \cdot (\mu_{j+1}^c)^{\frac{(l-j)(j-k)}{l-k}} \\ &\underset{(**)}{\geq} M_j^c \cdot \underbrace{\left(\frac{\mu_{j+1}^c}{\mu_j^c}\right)^{1/2}}_{>1} > M_j^c. \end{aligned}$$

(*) holds, because the sequence μ^c is increasing and $(**) \Leftrightarrow 2(l-j)(j-k) \geq (l-k) \Leftrightarrow 2 \geq \frac{1}{l-j} + \frac{1}{j-k}$ is clearly satisfied. So the infimum in the definition of M_j^c is attained at $j = k$ or at $j = l$, hence $M_j = M_j^c$.

Because $(\mu_j^c)_j$ is increasing we have $M_j^c = M_0^c \cdot \mu_1^c \cdots \mu_j^c \leq (\mu_j^c)^j$ for all $j \in \mathbb{N}$ and so we get for $j \in \hat{J}$

$$M_j^i \leq M_j^{1/j} \underbrace{=}_{j \in \hat{J}} (M_j^c)^{1/j} \leq \mu_j^c, \quad (4.2.17)$$

where the first inequality above holds by the definition of the sequence $(M_j^i)_j$. Let $j \in \hat{J}$ be given and assume that for a $k \in \mathbb{N}$ with $k < j$, $k \leq l < j$ implies $l \notin \hat{J}$. Then clearly $\mu_l^c = \mu_j^c$ holds for all l with $k \leq l \leq j$ and it follows for such l :

$$M_l^i \leq M_j^i \underbrace{\leq}_{(4.2.17)} \mu_j^c = \mu_l^c. \quad (4.2.18)$$

Thus (4.2.17) and (4.2.18) together imply $M_j^i \leq \mu_j^c$ for all $j \in \mathbb{N}$.

(ii) $\Rightarrow$ (iii): The function $j \mapsto \log(M_j^c)$ is convex and $M_0^c = 1$. Applying 2.0.4 we obtain: The sequence $((M_j^c)^{1/j})_j$ is increasing. The sequence $(M_j^i)_j$ is by definition the largest increasing minorant of $(M_j^{1/j})_j$ and because $(M_j^c)^{1/j} \leq M_j^{1/j}$ for all j we have $(M_j^c)^{1/j} \leq M_j^i$ for all j .

(i) $\Rightarrow$ (iv): Let us assume that (iv) is not valid, then we set $a_j := (\mu_j^c)^{-1}$ and by 4.2.2 there exists a non-trivial function $u \in \mathcal{E}(K)$ with compact support and

$$|u^{(j)}(x)| \underbrace{\leq}_{(4.2.2)} \frac{2^j}{a_1 \cdots a_j} = 2^j \cdot \mu_1^c \cdots \mu_j^c = 2^j \cdot M_j^c \leq 2^j \cdot M_j,$$

hence $u \in \mathcal{E}_{\{M\}}(K)$ where $C = 1$ and $h = 2$. But $u \neq 0$ and so condition (i) is not satisfied.

(iv) $\Rightarrow$ (i): Again we put $a_j := (\mu_j^c)^{-1}$ and remark: The set J_m which was defined in 4.2.3 is now by definition the set $\{j \in \hat{J} : j \leq m\} \cup \{m\}$ and for $j \in \hat{J}$ we have $M_j = M_j^c = \mu_1^c \cdots \mu_j^c = \frac{1}{a_1 \cdots a_j}$. If $u \in \mathcal{E}_{\{M\}}(K)$ with compact support then we obtain (4.2.14) for $j \in \hat{J}$. By assumption the set $\hat{J}$ is infinite and $\sum_{j=1}^{\infty} a_j = \infty$. So we can apply now (4.2.15) to conclude $u = 0$. □

Remark 4.2.5 Both proofs of the Denjoy-Carleman-theorem are given for dimension $n = 1$. But the higher dimensional case is equivalent to the one-dimensional case:

It's clear by restriction of the Borel-map, that the quasi-analyticity of the higher dimensional case implies the quasi-analyticity of the one-dimensional case.

Conversely assume that $\mathcal{E}_{\{M\}}$ is quasi-analytic for the one-dimensional case and assume that there exists $f \in \mathcal{E}_{\{M\}}$ in the higher-dimensional case such that $j_0^\infty(f) = 0$ and f is a non-trivial function. So there exists a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \neq 0$. But then we can restrict f to the straight line $l := \{t \cdot x_0 : t \in \mathbb{R}\}$, which is passing through the origin and x_0 . So we get $f|_l \neq 0$, which is a contradiction to 4.1.1.

5 Surjectivity of the Borel-map

The goal in this section is to prove a full characterization for the surjectivity of the Borel-map in terms of the weight sequence as given by [22].

First we recall notation: The sequence $\mu := (\mu_p)_p$ is defined by $\mu_p := \frac{M_p}{M_{p-1}} = \frac{p \cdot m_p}{m_{p-1}}$ for $p > 0$ and $\mu_0 := 1$. So, if $M_0 = 1$, then we can write $M_p = \prod_{i=0}^p \mu_i$ to compute the sequence $M := (M_p)_p$. Furthermore we put in this section $\mu^* := (\mu_p^*)_p$, where $\mu_p^* := \frac{\mu_p}{p} = \frac{m_p}{m_{p-1}}$ for $p > 0$ and $\mu_0^* := 1$.

For the proof of the central theorem in this chapter we have to assume that the sequence $(\mu_p)_p$ is strict increasing, $\mu_0 = 1$ and furthermore we assume the non-quasi-analyticity condition for $(M_p)_p$, which means $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$. Hence the sequence $(M_p)_p$ is then log. convex and $\mu_p \nearrow \infty$ holds. Under these assumptions the spaces $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ are rings by 2.0.8 and, by 4.1.5, $\mathcal{E}_{\{M\}}$ is not quasi-analytic.

We recall: If one assumes only that $(\mu_p)_p$ is increasing, then by 3.1.1 we can change the weight sequence into a new one such that the associated function space does not change and the new sequence $(\tilde{\mu}_p)_p$ is strict increasing.

We introduce now several further conditions for the sequence $(\mu_p)_p$:

$$\begin{aligned}
 (\alpha_1) &: \Leftrightarrow 1 = \mu_0^*, \mu_p^* \nearrow \infty \\
 (\beta_1) &: \Leftrightarrow \exists k \in \mathbb{N} : \liminf_{p \rightarrow \infty} \frac{\mu_{kp}^*}{\mu_p^*} > 1 \quad (\beta_1^0) : \Leftrightarrow \inf_{p \geq 1} \frac{\mu_{2p}^*}{\mu_p^*} > 1 \\
 (\beta_2) &: \Leftrightarrow \forall \varepsilon > 0 \exists k \in \mathbb{N}, k > 1 : \limsup_{p \rightarrow \infty} \left(\frac{M_{kp}}{M_p} \right)^{\frac{1}{p \cdot (k-1)}} \cdot \frac{1}{\mu_{kp}} \leq \varepsilon \\
 (\beta_2^0) &: \Leftrightarrow \exists k \in \mathbb{N} : \lim_{p \rightarrow \infty} \frac{\mu_{kp}}{\mu_p} = \infty \\
 (\gamma_1) &: \Leftrightarrow \sup_{p \in \mathbb{N}} \mu_p^* \cdot \sum_{j \geq p} \frac{1}{\mu_j} < \infty \\
 (\gamma_2) &: \Leftrightarrow \exists k \in \mathbb{N} : \lim_{p \rightarrow \infty} \mu_p^* \cdot \sum_{j \geq kp} \frac{1}{\mu_j} = 0
 \end{aligned}$$

First we remark that the following implications are obviously true: (α_1) implies $\mu_0 = 1$ and $\mu_p \nearrow \infty$, furthermore

$$\boxed{(\beta_1^0) \Rightarrow (\beta_1), (\beta_2^0) \Rightarrow (\beta_1) \text{ and } (\gamma_2) \Rightarrow (\gamma_1) \Rightarrow \sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty,}$$

which is the non-quasi-analyticity condition. Another consequence is:

$$(\beta_1) \Leftrightarrow \exists k \in \mathbb{N} : \liminf_{p \rightarrow \infty} \frac{\mu_{kp}}{\mu_p} > k. \quad (5.0.1)$$

It follows immediately by definition: If $\mu_p \nearrow \infty$ and $\mu_0 = M_0 = 1$, then $M_p \nearrow \infty$ and the condition (α_1) implies the log. conv. for the sequence $(m_p)_p$.

We recall: For weight sequences $M := (M_p)_p$ and $N := (N_p)_p$ we write $M \approx N$ if

$$0 < \inf_p \left(\frac{M_p}{N_p} \right)^{1/p} \leq \sup_p \left(\frac{M_p}{N_p} \right)^{1/p} < \infty,$$

in particular if $M \preceq N$ and $N \preceq M$ holds. So $M \approx N \Rightarrow \mathcal{E}_{(M)} = \mathcal{E}_{(N)}$ and, if M and N both are assumed to be logarithmic convex, then we have by 3.1.3: $\mathcal{E}_{\{M\}} = \mathcal{E}_{\{N\}} \Leftrightarrow M \approx N$.

5 Surjectivity of the Borel-map

We can define a new equivalence relation $\approx^*$ for sequences as follows. Put $\nu := (\nu_p)_p$, $\nu_p := \frac{N_p}{N_{p-1}}$, then we say $\nu \approx^* \mu$, if

$$0 < \inf_p \frac{\mu_p}{\nu_p} \leq \sup_p \frac{\mu_p}{\nu_p} < \infty \iff 0 < \inf_p \frac{M_p \cdot N_{p-1}}{N_p \cdot M_{p-1}} \leq \sup_p \frac{M_p \cdot N_{p-1}}{N_p \cdot M_{p-1}} < \infty$$

is satisfied. So it follows that the conditions (β_2) , (β_2^0) , (γ_1) and (γ_2) are stable under $\approx^*$ and $\nu \approx^* \mu \iff \nu^* \approx^* \mu^*$ where $\nu_p^* := \frac{\nu_p}{p}$. We note that the relation $\approx$ is obviously strictly weaker than $\approx^*$.

In this chapter we will consider the one dimensional case.

So let $K \subseteq \mathbb{R}$ be a closed interval, then we recall the definitions of the function spaces associated to a weight sequence $(M_p)_p$:

$$\mathcal{E}_{(M)}(K) := \{f \in \mathcal{E}(K) : |f|_{K,h} < \infty, \forall h > 0\},$$

the Beurling-case, resp.

$$\mathcal{E}_{\{M\}}(K) := \{f \in \mathcal{E}(K), \exists h > 0 : |f|_{K,h} < \infty\},$$

the Romieu-case, where we set

$$|f|_{K,h} := \sup_{p \in \mathbb{N}, y \in K} \frac{|f^{(p)}(y)|}{h^p \cdot M_p}.$$

To study the surjectivity of the Borel-map j^∞ we recall the definition of the introduced sequence spaces: For a sequence $x := (x_p)_p, x_p \in \mathbb{C}$ we put

$$|x|_h := \sup_{p \in \mathbb{N}} \frac{|x_p|}{h^p \cdot M_p},$$

and then we define: $\Lambda_{M,h} := \{x = (x_p)_p : x_p \in \mathbb{C}, |x|_h < \infty\}$. Furthermore we set

$$\Lambda_{(M)} := \{x = (x_p)_p : x_p \in \mathbb{C}, \forall h > 0 : |x|_h < \infty\},$$

and

$$\Lambda_{\{M\}} := \{x = (x_p)_p : x_p \in \mathbb{C}, \exists h > 0 : |x|_h < \infty\}.$$

We remark that these sequence spaces are endowed with a natural projective resp. inductive topology via $\Lambda_{(M)} := \varprojlim_h \Lambda_{M,h}$ and $\Lambda_{\{M\}} := \varinjlim_h \Lambda_{M,h}$. Furthermore they are both rings with respect to convolution $(x \star y)_n := \sum_{k=0}^n x_k \cdot y_{n-k}$, because:

$$\begin{aligned} |(x \star y)_{2h}|_n &= \sup_n \frac{|(x \star y)_n|}{(2h)^n \cdot M_n} \leq \sup_n \frac{\sum_{k=0}^n |x_k| \cdot |y_{n-k}|}{(2h)^n \cdot M_n} \\ &\stackrel{(\diamond)}{\leq} \sup_n \sum_{k=0}^n \frac{1}{2^k} \cdot \underbrace{\left(\frac{|x_k|}{h^k \cdot M_k} \right)}_{\leq |x|_h < \infty} \cdot \underbrace{\left(\frac{|y_{n-k}|}{h^{n-k} \cdot M_{n-k}} \right)}_{\leq |y|_h < \infty} < \infty. \end{aligned}$$

$(\diamond)$ holds because $(M_p)_p$ is log. convex and $M_0 = 1$, so (2.0.6) implies $M_k \cdot M_{n-k} \leq M_n$ for all n, k . Note that the estimate above is valid for the Romieu- and the Beurling-case.

By the definitions above we obtain in this notation

$$j^\infty : \mathcal{E}_{\{M\}}(K) \longrightarrow \Lambda_{\{M\}} \text{ and } j^\infty : \mathcal{E}_{(M)}(K) \longrightarrow \Lambda_{(M)},$$

where $j^\infty(f) := (f^{(p)}(0))_p$.

j^∞ is a linear and bounded operator in both cases, because for all $f \in \mathcal{E}_{\{M\}}(K)$ resp. $f \in \mathcal{E}_{(M)}(K)$ we can estimate:

$$|j^\infty(f)|_h = |(f^{(p)}(0))_p|_h \leq |f|_{K,h}.$$

A mapping $E : \Lambda_{\{M\}} \rightarrow \mathcal{E}_{\{M\}}(K)$ resp. $E : \Lambda_{(M)} \rightarrow \mathcal{E}_{(M)}(K)$ such that $j^\infty \circ E = \text{id}_{\Lambda_{\{M\}}}$ resp. $j^\infty \circ E = \text{id}_{\Lambda_{(M)}}$ holds, is called an *extension operator*. If such an extension operator exists, one can say that the Borel-mapping j^∞ admits a section. In the following we put $K = [-1, 1]$ and formulate now our central theorem in this chapter (see [22, Theorems 2.1., 3.1. 3.4 and 3.5.]):

Theorem 5.0.6

(I) *For the Beurling-case the following are equivalent:*

- (i) *There exists a continuous linear extension operator E .*
- (ii) *The sequence $(\mu_p)_p$ satisfies condition (β_1) .*
- (iii) *The sequence $(\mu_p)_p$ satisfies condition (γ_1) .*
- (iv) *The Borel-mapping j^∞ is surjective.*

(II) *For the Romieu-case we obtain:*

- (a) *We have an equivalence between*
 - (i) *There exists a continuous linear extension operator E .*
 - (ii) *The sequence $(\mu_p)_p$ satisfies condition (β_2) .*
- (b) *Furthermore the following are equivalent:*
 - (i) *The Borel-mapping j^∞ is surjective.*
 - (ii) *The sequence $(\mu_p)_p$ satisfies condition (β_1) .*
 - (iii) *The sequence $(\mu_p)_p$ satisfies condition (γ_1) .*

5.1 Proof of the surjectivity criterion

In this section we will give a proof of 5.0.6, but first we need some preparations and we start with a comparison of the introduced conditions for a fixed sequence $\mu := (\mu_p)_p$. We remark that for these comparison results it's enough to assume that $(\mu_p)_p$ is increasing, $\mu_0 = 1$ and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$.

Lemma 5.1.1 [22, 1.1. Proposition, p. 300-301] *Let $\mu := (\mu_p)_p$ be an increasing sequence such that $\mu_0 = 1$ and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ hold.*

(a) *The following statements are equivalent:*

- (i) μ satisfies condition (γ_1) ,
- (ii) μ satisfies condition (β_1) ,
- (iii) *There exists a sequence $\nu := (\nu_p)_p$, such that $\nu \approx^* \mu$ and ν satisfies (α_1) and (β_1^0) .*

(b) *The conditions (γ_2) and (β_2^0) are equivalent for μ .*

Proof. (b) $(\gamma_2) \Rightarrow (\beta_2^0)$: First we put $A_p^{-1} := \mu_p^* \cdot \sum_{j \geq kp} \frac{1}{\mu_j} < \infty$, then clearly, if k is chosen like in (γ_2) , we have $A_p \rightarrow \infty$ for $p \rightarrow \infty$. So we obtain condition (β_2^0) , if we estimate as follows:

$$\frac{\mu_{2kp}^*}{\mu_p^*} \geq \mu_{2kp}^* \cdot A_p \cdot \sum_{j=kp+1}^{2kp} \frac{1}{\mu_j} \geq \frac{A_p}{2}.$$

The first inequality holds because

$$\frac{1}{\mu_p^*} \geq A_p \cdot \sum_{j=kp+1}^{2kp} \frac{1}{\mu_j} \Leftrightarrow 1 \geq \frac{\sum_{j=kp+1}^{2kp} \frac{1}{\mu_j}}{\sum_{j \geq kp} \frac{1}{\mu_j}},$$

and the second inequality follows by

$$\mu_{2kp}^* \cdot \sum_{j=kp+1}^{2kp} \frac{1}{\mu_j} \underset{\mu \text{ incr.}}{\geq} \frac{\mu_{2kp}}{2kp} \cdot kp \cdot \frac{1}{\mu_{2kp}} = \frac{1}{2}.$$

$(\beta_2^0) \Rightarrow (\gamma_2)$: If k is now given as in (β_2^0) , then for all $q > 1$ there exists $p_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ with $p \geq p_0$

$$\frac{\mu_{kp}^*}{\mu_p^*} = \frac{\mu_{kp}}{k \cdot \mu_p} \geq q \quad (5.1.1)$$

holds. We show (γ_2) with the following estimate:

$$\begin{aligned} \sum_{j \geq kp} \frac{1}{\mu_j} &= \sum_{l=1}^{\infty} \sum_{j=k^l p}^{k^{l+1} p - 1} \frac{1}{\mu_j} \underset{\mu \text{ incr.}}{\leq} \sum_{l=1}^{\infty} k^l p \cdot (k-1) \cdot \frac{1}{\mu_{k^l p}} = (k-1) \cdot \sum_{l=1}^{\infty} \frac{1}{\mu_{k^l p}^*} \\ &\underset{(5.1.1)}{\leq} \frac{k-1}{\mu_p^*} \cdot \sum_{l=1}^{\infty} q^{-l} = \frac{k-1}{\mu_p^*} \cdot \left(\frac{q}{q-1} - 1 \right) = \frac{k-1}{\mu_p^*} \cdot \underbrace{\frac{1}{q-1}}_{\rightarrow 0, \text{ for } q \rightarrow \infty}. \end{aligned}$$

(a) To show $(ii) \Rightarrow (i)$ we can use the estimate of $(\beta_2^0) \Rightarrow (\gamma_2)$ above: Condition (β_1) means that there exist $q > 1, p_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ with $p \geq p_0$ equation (5.1.1) is satisfied for k as in (β_1) and so we get

$$\begin{aligned} \sum_{j \geq p} \frac{1}{\mu_j} &= \sum_{l=0}^{\infty} \sum_{j=k^l p}^{k^{l+1} p - 1} \frac{1}{\mu_j} \underset{\mu \text{ incr.}}{\leq} \sum_{l=0}^{\infty} k^l p \cdot (k-1) \cdot \frac{1}{\mu_{k^l p}} = (k-1) \cdot \sum_{l=0}^{\infty} \frac{1}{\mu_{k^l p}^*} \\ &\underset{(5.1.1)}{\leq} \frac{k-1}{\mu_p^*} \cdot \sum_{l=0}^{\infty} q^{-l} = \frac{k-1}{\mu_p^*} \cdot \frac{q}{q-1}. \end{aligned}$$

(i) $\Rightarrow$ (iii) : First we define a sequence $\tau_p := \frac{1}{\mu_p^*} + \sum_{k \geq p} \frac{1}{\mu_k}$. Because the sequence μ is increasing we see: $\tau_{p+1} - \tau_p = \frac{p+1}{\mu_{p+1}} - \frac{p}{\mu_p} - \frac{1}{\mu_p} \leq 0$, hence $(\tau_p)_p$ is a decreasing sequence. Condition (γ_1) implies that there exists $A \geq 1$, such that

$$\frac{\mu_p^*}{A} \leq \frac{1}{\tau_p} \leq \mu_p^*, \quad (5.1.2)$$

which is equivalent to $0 < \frac{\tau_1}{A} \leq \frac{p \cdot \tau_1}{\tau_p} \cdot \frac{1}{\mu_p} \leq \tau_1 < \infty$ for all p , hence $\left(\frac{p \cdot \tau_1}{\tau_p}\right)_p \approx^* \mu_p$. So we see that $\left(\frac{p \cdot \tau_1}{\tau_p}\right)_p$ satisfies conditions (γ_1) , which is stable under $\approx^*$, and condition (α_1) : $\left(\frac{p \cdot \tau_1}{\tau_p}\right)_p^* = \frac{\tau_1}{\tau_p}$ is increasing because $(\tau_p)_p$ is decreasing. We denote this new sequence $\left(\frac{p \cdot \tau_1}{\tau_p}\right)_p$ in the following by $(\tilde{\mu}_p)_p$ and define a sequence $(\tilde{\tau}_p)_p$ as above, which means $\tilde{\tau}_p = \frac{\tau_p}{\tau_1} + \sum_{k \geq p} \frac{\tau_k}{k \cdot \tau_1}$. Finally we set $\nu_p := \frac{p \cdot \tilde{\tau}_1}{\tilde{\tau}_p}$, where $\nu_0 := 1$. $(\nu_p)_p$ satisfies again the conditions (α_1) and (γ_1) like above but also (β_1^0) . For this we estimate for $p \geq 1$ where we set again $\nu_p^* := \frac{\nu_p}{p} = \frac{\tilde{\tau}_1}{\tilde{\tau}_p}$:

$$\begin{aligned} \frac{\nu_{2p}^*}{\nu_p^*} &= \frac{\tilde{\tau}_p}{\tilde{\tau}_{2p}} = \frac{\frac{1}{\tilde{\mu}_p^*} + \sum_{k \geq p} \frac{1}{\tilde{\mu}_k}}{\frac{1}{\tilde{\mu}_{2p}^*} + \sum_{k \geq 2p} \frac{1}{\tilde{\mu}_k}} \underset{\tilde{\mu}_p^* \nearrow \infty}{\geq} \frac{\frac{1}{\tilde{\mu}_{2p}^*} + \sum_{k \geq p} \frac{1}{\tilde{\mu}_k}}{\frac{1}{\tilde{\mu}_{2p}^*} + \sum_{k \geq 2p} \frac{1}{\tilde{\mu}_k}} = 1 + \frac{\sum_{k=p}^{2p-1} \frac{1}{\tilde{\mu}_k}}{\frac{1}{\tilde{\mu}_{2p}^*} + \sum_{k \geq 2p} \frac{1}{\tilde{\mu}_k}} \\ &\underset{\substack{\geq \\ \tilde{\mu} \text{ incr.}}}{\geq} 1 + \frac{p \cdot \frac{1}{\tilde{\mu}_{2p}}}{\frac{1}{\tilde{\mu}_{2p}^*} + \sum_{k \geq 2p} \frac{1}{\tilde{\mu}_k}} = 1 + \frac{p}{\tau_{2p} \cdot \tilde{\mu}_{2p}} \underset{(5.1.2)}{\geq} 1 + \frac{1}{2A} > 1. \end{aligned}$$

(iii) $\Rightarrow$ (ii) : By assumption there exist $H > 0, q > 1$ such that for all $p \geq 1$ we have

$$\frac{1}{H} \cdot \nu_p^* \leq \mu_p^* \leq H \cdot \nu_p^*, \quad \frac{\nu_{2p}^*}{\nu_p^*} \geq q.$$

If we take $l \in \mathbb{N}$ large enough, then we get (β_1) by

$$\frac{\mu_{2^l p}^*}{\mu_p^*} \geq H^{-2} \cdot \frac{\nu_{2^l p}^*}{\nu_p^*} \geq H^{-2} \cdot q^l > 1.$$

□

Remark 5.1.2 We have shown $(\gamma_1) \Leftrightarrow (\beta_1)$. Hence also condition (β_1) is invariant under relation $\approx^*$ and we remark that the equivalence is very useful because condition (γ_1) is more complicate than (β_1) .

We prove now the following corollary:

Corollary 5.1.3 [22, 1.3. Corollary, p. 301] Let $\mu = (\mu_p)_p$ be an increasing sequence and $\mu_0 = 1$, $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$, then we obtain:

- (a) If a weight sequence $(\mu_p)_p$ satisfies condition (γ_1) , then there exists $\varepsilon > 0$ and a sequence $(\nu_p)_p$, such that $(\nu_p)_p \approx^* (\mu_p)_p$ and the sequence $(\nu_p \cdot p^{-\varepsilon})_p$ satisfies (α_1) and (γ_1) .
- (b) If $(\mu_p)_p$ satisfies (γ_2) then the conclusion of (a) holds for arbitrarily large ε .

Proof. (a) We assume (γ_1) , then 5.1.1 (a) shows: $(\gamma_1) \Leftrightarrow (\beta_1)$, hence there exists $k \in \mathbb{N}$, such that $\lim_{p \rightarrow \infty} \inf \frac{\mu_{kp}^*}{\mu_p^*} = q > 1$. If ε is chosen in such a way that $q \cdot k^{-\varepsilon} > 1$, then the sequence $(\mu_p \cdot p^{-\varepsilon})_p$ satisfies (β_1) , too. So, by 5.1.1 (a), the conclusion follows.

(b) This statement follows analogously as above: We have $(\gamma_2) \Rightarrow (\gamma_1)$ and by 5.1.1 (b) the ε can here be chosen arbitrarily large. □

The next lemma gives equivalent conditions for (β_2) :

Lemma 5.1.4 [22, 1.5. Lemma, p. 302-303] *Let $\mu = (\mu_p)_p$ be an increasing sequence, then:*

(a) *For $0 \leq j < p$ we obtain: The mapping $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$, $(p, j) \mapsto \left(\frac{M_p}{M_j}\right)^{1/(p-j)}$, is an increasing function in both variables.*

(b) *Furthermore the following statements are equivalent:*

(i) $(\mu_p)_p$ satisfies condition (β_2) ,

(ii) $\exists l : \forall \varepsilon > 0 \exists 0 < \beta < 1 \exists p_0 : \forall p \geq p_0 \exists j : \beta \cdot p \leq j < p : \left(\frac{M_p}{M_j}\right)^{1/(p-j)} \leq \varepsilon \cdot \mu_{lp}$,

(iii) $\forall \varepsilon > 0 \exists 0 < \beta < 1 \exists p_0 : \forall p \geq p_0 : \max_{j \leq \beta p} \frac{M_p}{M_j} \cdot \frac{1}{\mu_p^{p-j}} \leq \varepsilon^p$.

Proof. (a) By assumption $(\mu_p)_p$ is increasing, with other words the sequence $(M_p)_p$ is log. convex and so we can use 2.0.4 to obtain the result. More precisely: Fix j , then define a sequence $(N_k)_k$ with $N_k := \frac{M_{j+k}}{M_j} = \mu_{j+1} \cdots \mu_{j+k}$ and $N_0 := 1$. Thus $(N_k)_k$ is again log. convex and so by 2.0.4 we see that $N_k^{1/k} = \left(\frac{M_p}{M_j}\right)^{1/(p-j)}$ is an increasing sequence, where $p := k + j$.

(b) (i) $\Rightarrow$ (ii) : We set $\beta := \frac{1}{k}$, where $k = k(\varepsilon/2)$, and $\varepsilon > 0$ is given by condition (β_2) . Then we use (a) to estimate for large p , where $p \leq kj < 2p$:

$$\left(\frac{M_p}{M_j}\right)^{1/(p-j)} \cdot \frac{1}{\mu_{2p}} \underbrace{\leq}_{(a), \mu \text{ incr.}} \left(\frac{M_{kj}}{M_j}\right)^{1/(j(k-1))} \cdot \frac{1}{\mu_{kj}} \underbrace{\leq}_{(\beta_2)} \varepsilon.$$

(ii) $\Rightarrow$ (iii) : We apply (ii) for $\varepsilon^{3l} < 1$ and because of (a) we can assume $j \leq \tilde{\beta}p$ and $\tilde{\beta} < \frac{1}{2l}$. We put $q := \lfloor p/l \rfloor$ and estimate for p large enough:

$$\left(\frac{M_p}{M_j}\right)^{1/(p-j)} \cdot \frac{1}{\mu_p} = \left(\frac{M_p}{M_j} \cdot \frac{1}{\mu_p^{p-j}}\right)^{1/(p-j)} \underbrace{\leq}_{(\star)} \left(\frac{M_q}{M_j} \cdot \frac{1}{\mu_{lq}^{q-j}}\right)^{1/(p-j)} \underbrace{\leq}_{(ii)} \varepsilon^{3l \cdot \frac{q-j}{p-j}} \leq \varepsilon^{p/(p-j)}.$$

($\star$) holds because $M_p \cdot \mu_{lq}^{q-j} \leq M_q \cdot \mu_p^{p-j}$. The last inequality is valid because $\varepsilon < 1$ and:

$$\begin{aligned} 3l \cdot (q - j) &= 3lq - 3lj > 3lq - 3lp/(2l) = 3lq - 3p/2 \\ &> 3 \cdot (p - l) - 3p/2 \geq p \Leftrightarrow p/2 \geq 3l, \end{aligned}$$

which is true for large p and given l . To obtain the result we can decrease j by (a), because (ii) holds for large j .

(iii) $\Rightarrow$ (i) : First we choose $\frac{1}{k} \leq \beta$. Hence for $p = kj$ we have $j \leq \beta p$ and so we see for large j and $\varepsilon \leq 1$:

$$\left(\frac{M_{kj}}{M_j} \right)^{1/j(k-1)} \cdot \underbrace{\frac{1}{\mu_{kj}}}_{(iii)} \leq \varepsilon^{\frac{kj}{j(k-1)}} = \varepsilon^{k/(k-1)} \leq \varepsilon.$$

□

Remark 5.1.5 A further easy implication for increasing μ is $(\beta_2^0) \Rightarrow (\beta_2)$, because first we take $k \in \mathbb{N}$, which is given by (β_2^0) , and then we estimate:

$$\underbrace{\left(\frac{M_{2kp}}{M_p} \right)^{1/p(2k-1)}}_{\mu_{p+1} \cdots \mu_{2kp}} \cdot \frac{1}{\mu_{2kp}} \underbrace{\leq}_{\mu \text{ incr.}} \left(\mu_{2p}^p \cdot \mu_{2kp}^{p(2k-2)} \right)^{1/p(2k-1)} \cdot \frac{1}{\mu_{2kp}} = \left(\frac{\mu_{2p}}{\mu_{2kp}} \right)^{1/(2k-1)},$$

and by (β_2^0) the right side tends to 0 for $p \rightarrow \infty$.

For the first important characterizing result we need the following lemma, which is related to 4.2.2.

Lemma 5.1.6 [22, 2.2. Theorem, p. 306] Let $(M_p)_p$ be a log. convex weight sequence with $M_0 = 1$ and assume that $a := \sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$. Then there exists a smooth function φ which support is contained in $[-a, a]$, such that $0 \leq \varphi(x) \leq 1$ for all $x \in [-a, a]$, and $\varphi^{(p)}(0) = \delta_{p,0}$. Furthermore we have $\|\varphi^{(p)}\|_{\infty} \leq 2^p \cdot M_p$ for all $p \in \mathbb{N}$.

In particular one can say: φ is a non-trivial function ($\varphi(0) = 1$) with compact support and $\varphi \in \mathcal{E}_{\{M\}}$ (take $C = 1$ and $h = 2$). Thus, by 4.1.1, the space $\mathcal{E}_{\{M\}}$ is not quasi-analytic.

Proof. To prove this lemma we change the proof of 4.2.2 in the following way: We set $\tilde{H}_a(x) := a^{-1}$ for $-a/2 < x < a/2$ and $\tilde{H}_a(x) := 0$ otherwise, so $\int \tilde{H}_a(x) dx = 1$. If u is a continuous function we obtain for the convolution

$$(u * \tilde{H}_a)(x) = a^{-1} \cdot \int_{-a/2}^{a/2} u(x-t) dt = a^{-1} \cdot \int_{x-a/2}^{x+a/2} u(t) dt$$

and $(u * \tilde{H}_a)'(x) = \frac{u(x+a/2) - u(x-a/2)}{a}$ which is a continuous function.

Now we put $a_j := \mu_j^{-1}$, hence $\mu_0 = a_0 = 1$ and the sequence $(a_p)_p$ is decreasing. Furthermore $a = \sum_{j=1}^{\infty} a_j < \infty$ holds. Then we define the functions $\varphi_k := \tilde{H}_{a_0} * \cdots * \tilde{H}_{a_k}$ and $\varphi := \lim_{k \rightarrow \infty} \varphi_k$. Thus $\int_{-\infty}^{\infty} \varphi_k(x) dx = 1$ holds for all k and φ_k is a $\mathcal{C}^{k-1}$ -function. We follow now the proof of 4.2.2: First analyze the function $\varphi_1(x) = a_1^{-1} \cdot \int_{x-a_1/2}^{x+a_1/2} \tilde{H}_{a_0}(t) dt$. It is identically zero in $(-\infty, -a_0/2 - a_1/2]$ and in $[a_0/2 + a_1/2, \infty)$, identically $a_0^{-1} = 1$ in $[-a_0/2 + a_1/2, a_0/2 - a_1/2]$. Furthermore it is linearly increasing with slope $\frac{1}{a_0 \cdot a_1}$ in $[-a_0/2 - a_1/2, -a_0/2 + a_1/2]$ and linearly decreasing with slope $-\frac{1}{a_0 \cdot a_1}$ in $[a_0/2 - a_1/2, a_0/2 + a_1/2]$. Thus we see that φ_1 is a continuous function with support in $[-a_0/2 - a_1/2, a_0/2 + a_1/2]$ and so φ is by definition a smooth function which support is contained in $[-a, a]$. (4.2.8) in 4.2.2 changes into the following form, where $1 \leq j \leq k-1$:

$$\varphi_k^{(j)} = \left(\prod_{i=0}^{j-1} \frac{1}{a_i} \cdot (\tau_{-a_i/2} - \tau_{a_i/2}) \right) (\tilde{H}_{a_j} * \cdots * \tilde{H}_{a_k}). \quad (5.1.3)$$

The other conclusions in the proof of 4.2.2 are valid and because of the definition of the sequence $(a_j)_j$ we have $\frac{2^p}{a_0 \cdots a_p} = 2^p \cdot \mu_0 \cdots \mu_p = 2^p \cdot M_p$ for all p , hence the required estimations are satisfied. Now we prove the property $\varphi^{(p)}(0) = \delta_{p,0}$:

First we see that $\varphi(0) = a_0^{-1} = 1$. All the $\tilde{H}_{a_k}$ are by definition even functions, which means $\tilde{H}_{a_k}(x) = \tilde{H}_{a_k}(-x)$ for all x and k . Let f, g be two even functions, then the convolution $f * g$, if it exists, is even, too:

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{\infty} f(x-y) \cdot g(y) dy = \int_{-\infty}^{\infty} f(-(x-y)) \cdot g(-y) dy \\ &= \int_{-\infty}^{\infty} f(-x+y) \cdot g(-y) dy \underset{y' := -y}{=} - \int_{\infty}^{-\infty} f(-x-y') \cdot g(y') dy' = (f * g)(-x). \end{aligned}$$

Hence for all j and k the functions $\tilde{H}_{a_j} * \dots * \tilde{H}_{a_k}$ used in (5.1.3) are even, which implies $\varphi_k^{(j)}(0) = 0$ for $1 \leq j \leq k-1$ and so $\varphi^{(j)}(0) = \delta_{j,0}$.

□

With these preparations we can prove now the first important proposition.

Proposition 5.1.7 [22, 2.1. Theorem, p. 305-308]

- (a) Let $\mu := (\mu_p)_p$, $\mu_0 = 1$, be an increasing sequence such that condition (γ_1) ($\Rightarrow \sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$) is satisfied, then
 - (i) there exists a continuous linear extension operator $E : \Lambda_{(M)} \longrightarrow \mathcal{E}_{(M)}([-1, 1])$, in particular the Borel-map $j^{\infty} : \mathcal{E}_{(M)}([-1, 1]) \rightarrow \Lambda_{(M)}$ is onto
 - (ii) $j^{\infty} : \mathcal{E}_{\{M\}}([-1, 1]) \rightarrow \Lambda_{\{M\}}$ is onto.
- (b) Let $\mu := (\mu_p)_p$, $\mu_0 = 1$, be an increasing sequence such that $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ and assume additionally condition (β_2) . Then there exists a cont. linear extension operator $E : \Lambda_{\{M\}} \longrightarrow \mathcal{E}_{\{M\}}([-1, 1])$.

Proof. To prove (a)(i) we will construct a family $\chi := (\chi_p)_p$ of functions such that $\chi_p \in \mathcal{E}_{(M)}$ for all $p \in \mathbb{N}$ and such that all χ_p have compact support. Furthermore we will show that $\chi_p^{(j)}(0) = \delta_{j,p}$ and $|\chi_p|_{\mathbb{R},h} \leq C_h \cdot \frac{H_h^p}{M_p}$ holds for all small $h > 0$ and constants C_h and H_h depending not on p .

Because then we can define for a sequence $x = (x_p)_p \in \Lambda_{(M)}$

$$E(x) := \sum_{p \geq 0} x_p \cdot \chi_p,$$

and with this definition we can show now that $E(x) \in \mathcal{E}_{(M)}([-1, 1])$ for all $x \in \Lambda_{(M)}$ and the continuity of E which can be stated as follows: For all $h > 0$ there exist $K, k > 0$ such that $|E(x)|_{[-1,1],h} \leq K \cdot |x|_k$ holds for all $x \in \Lambda_{(M)}$.

Therefore we take $h > 0$ arbitrary, put $k := \frac{1}{2H_h}$ and estimate for $x \in \Lambda_{(M)}$ and $y \in [-1, 1]$:

$$|E^{(l)}(x)(y)| \leq \sum_{p \geq 0} \underbrace{|x_p|}_{\leq |x|_k \cdot k^p \cdot M_p} \cdot \underbrace{|\chi_p^{(l)}(y)|}_{\leq h^l \cdot M_l \cdot C_h \cdot \frac{H_h^p}{M_p}} \leq |x|_k \cdot C_h \cdot h^l \cdot M_l \cdot \sum_{p \geq 0} (k \cdot H_h)^p,$$

and so we can write $|E(x)|_{[-1,1],h} \leq |x|_k \cdot C_h \cdot \sum_{p \geq 0} \underbrace{(k \cdot H_h)^p}_{1/2} = |x|_k \cdot C_h \cdot 2 < \infty$, hence we

have $E(x) \in \mathcal{E}_{(M)}([-1, 1])$ and $|E(x)|_{[-1,1],h} \leq \underbrace{2 \cdot C_h}_K \cdot |x|_k$ which implies the continuity of E .

Finally we have immediately $E^{(l)}(x)(0) = x$ for all $l \in \mathbb{N}$ and $x \in \Lambda_{(M)}$, and so $j^{\infty} \circ E = \text{id}_{\Lambda_{(M)}}$.

We start now with the construction of $\chi = (\chi_p)_p$: By 5.1.3(a) we can assume condition (γ_1) for the sequence $(\mu_p \cdot p^{-\varepsilon})_p$ and $\mu_p \cdot p^{-\varepsilon} \nearrow \infty$, for a suitable chosen $\varepsilon > 0$. Hence the sequence

$$\underbrace{\mu_p, \dots, \mu_p}_{p\text{-times}}, \mu_{p+1} \cdot \left(\frac{p}{p+1}\right)^{\varepsilon}, \mu_{p+2} \cdot \left(\frac{p}{p+2}\right)^{\varepsilon}, \dots,$$

is increasing. We denote this sequence with $(\tilde{\mu}_p)_p$ and define $(\tilde{M}_p)_p$, where $\tilde{M}_p := \prod_{j=0}^p \tilde{\mu}_j$. By (γ_1) for the sequence $(\tilde{\mu}_p)_p$ we have: There exists $A > 0$ such that for all $p \in \mathbb{N}$

$$\underbrace{\frac{p}{\mu_p}}_{=\sum_{j=1}^p 1/\mu_p} + \sum_{k>p} \frac{1}{\mu_k} \cdot \left(\frac{k}{p}\right)^\varepsilon \leq \frac{A}{\mu_p^*}.$$

We apply now (5.1.6): There exist smooth functions ϱ_p , such that $\text{supp}(\varrho_p)$ is contained in $\left[-\frac{A}{\mu_p^*}, \frac{A}{\mu_p^*}\right]$, $\varrho_p^{(j)}(0) = \delta_{j,0}$ and $\|\varrho_p^{(j)}\|_\infty \leq 2^j \cdot \tilde{M}_j, \forall p, j$. Then for $0 \leq j \leq p$ we have:

$$2^j \cdot \tilde{M}_j = 2^j \cdot \tilde{\mu}_0 \cdot \tilde{\mu}_1 \cdots \tilde{\mu}_j = 2^j \cdot \mu_p^j.$$

If $j > p$, then we obtain:

$$2^j \cdot \tilde{M}_j = 2^j \cdot \mu_p^p \cdot \tilde{\mu}_{p+1} \cdots \tilde{\mu}_j = 2^j \cdot \mu_p^p \cdot \underbrace{\mu_{p+1} \cdots \mu_j}_{=M_j/M_p} \cdot \left(\frac{p^{j-p} \cdot p!}{j!}\right)^\varepsilon.$$

We summarize:

$$\|\varrho_p^{(j)}\|_\infty \leq \begin{cases} 2^j \cdot \mu_p^j, & \text{for } 0 \leq j \leq p \\ 2^j \cdot \mu_p^p \cdot \frac{M_j}{M_p} \cdot \left(\frac{p^{j-p} \cdot p!}{j!}\right)^\varepsilon, & \text{for } j > p. \end{cases} \quad (5.1.4)$$

After that we define now:

$$\chi_p(t) := \varrho_p(t) \cdot \frac{t^p}{p!}, \text{ for } t \in \mathbb{R}.$$

First we obtain that all χ_p have compact support, because all ϱ_p have compact support and we see by differentiating that $\chi_p^{(p)}(0) = \varrho_p(0) = \varrho_p^{(0)}(0) = \delta_{0,0} = 1$ and $\chi_p^{(j)}(0) = 0$ for $j \neq p$. To estimate $\chi_p^{(j)}$ we have to distinguish three cases:

1. $0 \leq j \leq p$:

$$\begin{aligned} \|\chi_p^{(j)}\|_\infty &\stackrel{(5.1.4)}{\leq} \sum_{l=0}^j \binom{j}{l} \cdot 2^l \cdot \mu_p^l \cdot \left(\frac{A}{\mu_p}\right)^{p+l-j} \cdot \frac{p^{p+l-j}}{(p+l-j)!} \\ &\stackrel{(\circ)}{\leq} \underbrace{\frac{M_j}{M_p}}_{=\frac{1}{\mu_{j+1} \cdots \mu_p}} \cdot A^p \cdot \underbrace{\sum_{l=0}^j \binom{j}{l} \cdot 2^l \cdot \left(\frac{1}{A}\right)^{j-l}}_{=(2+\frac{1}{A})^j} \cdot \frac{\mu_{j+1} \cdots \mu_p}{\mu_p^{p-j}} \cdot \frac{p^p}{p!} \\ &\stackrel{(\circ\circ)}{\leq} \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A}\right)^j \cdot \underbrace{\frac{\mu_{j+1} \cdots \mu_p}{\mu_p^{p-j}}}_{\leq 1, \mu \text{ incr.}} \\ &\leq \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A}\right)^j. \end{aligned}$$

($\circ$) holds, because: $1 \leq \frac{p^{j-l}}{(p+l-j+1) \cdots p}, 0 \leq l \leq j$, and ($\circ\circ$) holds, because $\frac{p^p}{p!} \leq e^p$ for all $p \in \mathbb{N}$.

2. $j \geq 2p$: We estimate as before to obtain:

$$\begin{aligned}
 \|\chi_p^{(j)}\|_\infty &\stackrel{(5.1.4)}{\leq} \sum_{l=j-p}^j \binom{j}{l} \cdot 2^l \cdot \mu_p^p \cdot \frac{M_l}{M_p} \cdot \left(\frac{p^{l-p} \cdot p!}{l!} \right)^\varepsilon \cdot \left(\frac{A}{\mu_p} \right)^{p+l-j} \cdot \frac{p^{p+l-j}}{(p+l-j)!} \\
 &\leq \underbrace{\frac{M_j}{M_p}}_{=\mu_{p+1} \cdots \mu_j} (A \cdot e)^p \cdot \left(\frac{p^{j-2p} \cdot p!}{(j-p)!} \right)^\varepsilon \cdot \underbrace{\sum_{l=j-p}^j \binom{j}{l} 2^l \left(\frac{1}{A} \right)^{j-l}}_{=(2+\frac{1}{A})^j} \cdot \underbrace{\frac{\mu_p^{j-l} \cdot \overbrace{\mu_{p+1} \cdots \mu_l}^{=M_l/M_p}}{\mu_{p+1} \cdots \mu_j}}_{\leq 1, \forall l} \\
 &\stackrel{(\circ \circ \circ)}{\leq} \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A} \right)^j \cdot \left(\frac{p^{j-2p}}{(j-2p)!} \right)^\varepsilon.
 \end{aligned}$$

$(\circ \circ \circ)$ holds, because for $j \geq 2p$ we have: $p! \leq (j-2p+1) \cdots (j-p)$.

3. $p < j < 2p$: We use again (5.1.4) and estimate as above. For this case $\frac{p^{j-2p}}{(j-2p)!} < 1$, hence:

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A} \right)^j.$$

Now take $h > 0$ and define $H = H_h := A \cdot e \cdot \left(2 + \frac{1}{A} \right)^2 \cdot h^{-2} \cdot \exp\left(\varepsilon \cdot \left(h^{-1} \cdot \left(2 + \frac{1}{A} \right) \right)^{1/\varepsilon}\right)$. We remark, that for $B > 0$ and $j \geq 2p$ we have

$$\frac{(p \cdot B)^{j-2p}}{(j-2p)!} \leq e^{p \cdot B}, \quad (5.1.5)$$

which follows by the power series expansion of $\exp$.

Then, using the second case above, we get:

$$\begin{aligned}
 \|\chi_p^{(j)}\|_\infty &\stackrel{\text{case 2.}}{\leq} \frac{M_j \cdot h^j}{M_p} \cdot \left(A \cdot e \cdot \left(2 + \frac{1}{A} \right)^2 \cdot h^{-2} \right)^p \cdot \underbrace{\left(\frac{p^{j-2p}}{(j-2p)!} \cdot \left(h^{-1} \cdot \left(2 + \frac{1}{A} \right) \right)^{\frac{j-2p}{\varepsilon}} \right)^\varepsilon}_{\text{by (5.1.5)} \leq \exp\left(p \cdot \varepsilon \cdot \left(h^{-1} \cdot \left(2 + \frac{1}{A} \right) \right)^{1/\varepsilon}\right)} \\
 &\stackrel{\text{Def. of } H}{\leq} \frac{M_j \cdot h^j}{M_p} \cdot H^p.
 \end{aligned}$$

On the other hand, if $j < 2p$ and $0 < h \leq 2 + \frac{1}{A}$, then

$$\left(2 + \frac{1}{A} \right)^j \leq \frac{h^j}{h^{2p}} \cdot \left(2 + \frac{1}{A} \right)^{2p} \quad (5.1.6)$$

and

$$\exp\left(\varepsilon \cdot \underbrace{\left(h^{-1} \cdot \left(2 + \frac{1}{A} \right) \right)^{1/\varepsilon}}_{\geq 1}\right) \geq \exp(\varepsilon) \geq \exp(0) = 1. \quad (5.1.7)$$

We use the cases 1. resp. 3. to obtain:

$$\|\chi_p^{(j)}\|_\infty \stackrel{(5.1.6), \text{ cases 1.,3.}}{\leq} \frac{M_j \cdot h^j}{M_p} \cdot (A \cdot e)^p \cdot \left(\left(2 + \frac{1}{A} \right) \cdot h^{-1} \right)^{2p} \stackrel{(5.1.7), \text{ def. of } H}{\leq} \frac{M_j \cdot h^j}{M_p} \cdot H^p.$$

Finally we see: $|\chi_p|_{[-1,1],h} \leq \frac{H^p}{M_p} < \infty$ holds for all $p \in \mathbb{N}$ and all $h > 0$ (small).

Now we prove (a)(ii):

The result in this case is much weaker. For a sequence $x := (x_p)_p \in \Lambda_{\{M\}}$ we can find $H, C > 0$ (H large), such that $|x_p| \leq C \cdot M_p \cdot H^p$ for all p . For an $h > 0$ we set now $\mu_j \mapsto h \cdot \mu_j =: \tilde{\mu}_j$, for all j , in the construction of ϱ_p above. So we see that ε is not necessary here and:

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot \left(\frac{A \cdot e}{h}\right)^p \cdot \left(h \cdot \left(2 + \frac{1}{A}\right)\right)^j \quad \forall j, p$$

hence $|\chi_p|_{\mathbb{R}, h \cdot (2+1/A)} \leq \frac{1}{M_p} \cdot \left(\frac{A \cdot e}{h}\right)^p$ for all p . Choose h large enough, such that $\frac{A \cdot e}{h} < \frac{1}{H}$, and so

$$\frac{1}{M_p} \cdot \left(\frac{A \cdot e}{h}\right)^p < \frac{1}{M_p \cdot H^p}, \quad \forall p.$$

We define again $\chi := \sum_{p \geq 0} x_p \cdot \chi_p$. So $\chi \in \mathcal{E}_{\{M\}}([-1, 1])$, because

$$\begin{aligned} |\chi|_{h \cdot (2+1/A)} &\leq \sum_{p \geq 0} |x_p \cdot \chi_p|_{h \cdot (2+1/A)} \leq \sum_{p \geq 0} C \cdot M_p \cdot H^p \cdot \frac{1}{M_p} \cdot \left(\frac{A \cdot e}{h}\right)^p \\ &= C \cdot \sum_{p \geq 0} \left(\underbrace{H \cdot \frac{A \cdot e}{h}}_{< 1 \Leftrightarrow (A \cdot e)/h < 1/H} \right)^p < \infty, \end{aligned}$$

and has the property

$$j^\infty(\chi) = (\chi^{(p)}(0))_p = \left(\sum_{l \geq 0} x_l \cdot \underbrace{\chi_l^{(p)}(0)}_{= \delta_{p,l}} \right)_p = (x_p)_p.$$

Finally we consider the case (b):

We imitate the procedure of (a)(i): Define $E(x) := \sum_{p \geq 0} x_p \cdot \chi_p$ for $x := (x_p)_p \in \Lambda_{\{M\}}$ and the functions χ_p have to satisfy: For all $H > 0$ (H small) there exist $h, C > 0$ (both large), such that $|\chi_p|_{\mathbb{R}, h} \leq C \cdot \frac{H^p}{M_p}$ holds.

First we show again the continuity of E : Let $x \in \Lambda_{\{M\}}$, then there exists $k > 0$ such that $x \in \Lambda_{M,k}$. Put $H := \frac{1}{2k}$, then we can estimate for all $y \in [-1, 1]$ and $l \in \mathbb{N}$:

$$\begin{aligned} |E^{(l)}(x)(y)| &\leq \sum_{p \geq 0} \underbrace{|x_p|}_{\leq |x|_k \cdot k^p \cdot M_p} \cdot \underbrace{|\chi_p^{(l)}(y)|}_{\leq h^l \cdot M_l \cdot C \cdot \frac{H^p}{M_p}} \leq |x|_k \cdot C \cdot h^l \cdot M_l \cdot \sum_{p \geq 0} \underbrace{(k \cdot H)^p}_{1/2} \\ &= 2 \cdot |x|_k \cdot C \cdot h^l \cdot M_l < \infty. \end{aligned}$$

Hence we have shown $|E(x)|_{[-1, 1], h} \leq 2 \cdot C \cdot |x|_k$, which means $E(x) \in \mathcal{E}_{M, h}([-1, 1])$ for all $x \in \Lambda_{M, k}$, and so the continuity of E is proven.

We show now the estimates for the functions χ_p . First we apply now 5.1.6 for all p to the following sequence:

$$\underbrace{\mu_p, \dots, \mu_p}_{p\text{-times}}, \mu_{p+1}, \mu_{p+2}, \dots$$

and we can define now χ_p as in (a)(i). Then for a given $\varepsilon > 0$ we choose a real number β with $0 < \beta < 1$ like in 5.1.4 (b)(iii). If we estimate $\|\chi_p^{(j)}\|_\infty$ as in (a)(i), we see:

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A}\right)^j, \quad \forall j, p.$$

5 Surjectivity of the Borel-map

For $j > p\beta$ and $h \geq 1$ we have $1 = h^{-j} \cdot h^j \leq h^{-p\beta} \cdot h^j$ and so

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot (A \cdot e \cdot h^{-\beta})^p \cdot \left(h \cdot \left(2 + \frac{1}{A}\right)\right)^j. \quad (5.1.8)$$

If $j \leq p\beta$, then $0 \leq j \leq p$ and we can use the first case in (a)(i) to see:

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot (A \cdot e)^p \cdot \left(2 + \frac{1}{A}\right)^j \cdot \underbrace{\frac{M_p}{M_j}}_{=\mu_{j+1} \cdots \mu_p} \cdot \frac{1}{\mu_p^{p-j}}.$$

Applying 5.1.4 (b)(iii) we obtain for p large enough $\max_{0 \leq j \leq p\beta} \frac{M_p}{M_j} \cdot \frac{1}{\mu_p^{p-j}} \leq \varepsilon^p$, hence

$$\|\chi_p^{(j)}\|_\infty \leq \frac{M_j}{M_p} \cdot (A \cdot e \cdot \varepsilon)^p \cdot \left(2 + \frac{1}{A}\right)^j. \quad (5.1.9)$$

The inequalities (5.1.8) and (5.1.9) together show for sufficiently large p and arbitrary $h \geq 1$:

$$|\chi_p|_{\mathbb{R}, h \cdot (2+1/A)} \leq \frac{1}{M_p} \cdot \left(A \cdot e \cdot \max\{\varepsilon, h^{-\beta}\}\right)^p.$$

So we see: We can take $A > 0$ such that $A \cdot e \geq h > 0$ and define $\varepsilon := \frac{H}{A \cdot e}$ and $h := \left(\frac{A \cdot e}{H}\right)^{1/\beta}$. Then $h^{-\beta} = \left(\frac{A \cdot e}{H}\right)^{-1} = \frac{H}{A \cdot e} = \varepsilon$ and so finally $\left(A \cdot e \cdot \max\{\varepsilon, h^{-\beta}\}\right)^p = (A \cdot e \cdot \varepsilon)^p = H^p$. $\square$

5.1.7 has a further important consequence: In step (a)(i) we have constructed with 5.1.6 a non-trivial function with compact support lying in $\mathcal{E}_{(M)}$. In particular we have shown that there exists a sequence $(\chi_p)_p$ of non-trivial functions with compact support and such that $\chi_p \in \mathcal{E}_{(M)}$ for all $p \in \mathbb{N}$. So we can formulate the following corollary:

Corollary 5.1.8 *Let $(M_p)_p$ be a log. convex weight sequence such that $M_0 = 1$ and condition (γ_1) is satisfied for the sequence $(\mu_p)_p$. Then the space $\mathcal{E}_{(M)}$ is not quasi-analytic.*

Proof. By 5.1.7 (a)(i) it follows that there exists $f \in \mathcal{E}_{(M)}$ such that f is non-trivial with compact support. Hence, by 4.1.1, the space $\mathcal{E}_{(M)}$ is not quasi-analytic. $\square$

With these preparations we can start now with the proof of 5.0.6. From now on we assume that

$$\mu_0 = 1, (\mu_p)_p \text{ is strict increasing and } \sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty \text{ is satisfied.}$$

Proof of 5.0.6.

First we obtain by 5.1.1 that $(\beta_1) \Leftrightarrow (\gamma_1)$ is satisfied.

We prove now $(II)(a)(i) \Leftrightarrow (II)(a)(ii)$: $(II)(a)(ii) \Rightarrow (II)(a)(i)$ holds by 5.1.7 (b).

$(II)(a)(i) \Rightarrow (II)(a)(ii)$:

We define $\chi_p := E(e_p)$, where $e_p = (0, \dots, \underbrace{1}_{p\text{-th}}, \dots)$ is the p -th unit vector. For $x := (x_p)_p \in$

$\Lambda_{\{M\}}$ we have $E(x) = \sum_{p \geq 0} x_p \cdot E(e_p) = \sum_{p \geq 0} x_p \cdot \chi_p$ and because $j^\infty(E(x)) = (\sum_{p \geq 0} x_p \cdot \chi_p^{(j)}(0))_j = x_p \forall p$, we obtain $\chi_p^{(j)}(0) = \delta_{p,j}$. By assumption E is continuous and because both

spaces are of type (LF) , we can apply *Grothendieck's factorization theorem* (for a proof see [6, Théorème A, p. 16] or [21, 24.33, p. 271-272]). We have $|e_p|_{1/H} = \frac{H^p}{M_p}$ and so for all $H > 0$ ($H \leq 1$) there exist $C, h > 0$ ($C, h \geq 1$), such that

$$|E(e_p)|_{[-1,1],h} = |\chi_p|_{[-1,1],h} \leq C \cdot \frac{H^p}{M_p} \quad \forall p. \quad (5.1.10)$$

Hence, by Taylors's formula, for all $j \in \mathbb{N}$ and $0 \leq x \leq 1$:

$$|\chi_p^{(p)}(x) - 1| \leq \frac{x^j}{j!} \cdot \|\chi_p^{(p+j)}\|_{[-1,1]} \leq \underbrace{C \cdot \frac{(h \cdot x)^j}{j!} \cdot (H \cdot h)^p \cdot \underbrace{\mu_{p+1} \cdots \mu_{p+j}}_{=M_{p+j}/M_p}}_{(5.1.10)}.$$

Then we set $j = p$ and let us assume that

$$0 \leq x \leq B \cdot \frac{p^{1/p}}{\mu_{2p}} \quad (5.1.11)$$

for a constant $B > 0$. From the estimate above and $\frac{M_{p+j}}{M_p} \cdot \frac{1}{\mu_{2p}^p} \leq 1$ (note: $(\mu_p)_p$ is increasing), it follows that

$$|\chi_p^{(p)}(x) - 1| \leq C \cdot (B \cdot h^2 \cdot H)^p.$$

For $H \leq 1$ we choose $B < h^{-2} < 1$. So $C \cdot (B \cdot h^2 \cdot H)^p < C$, with $C \geq 1$, and so for large p and x as in (5.1.11) we have $\chi_p^{(p)}(x) \geq \frac{1}{2}$. After integrating $p - j$ times, for $0 \leq j \leq p$, we obtain

$$\chi_p^{(j)}(x) \geq \frac{1}{2} \cdot \frac{x^{p-j}}{(p-j)!}.$$

Hence,

$$\frac{1}{2} \cdot \frac{1}{(p-j)!} \cdot \left(B \cdot \frac{p^{1/p}}{\mu_{2p}} \right)^{p-j} \leq \|\chi_p^{(j)}\|_{[-1,1]} \leq C \cdot \frac{H^p}{M_p} \cdot h^j \cdot M_j,$$

where the second inequality above is exactly (5.1.10). Because $p^{1/p} \geq (p-j)!^{1/(p-j)}$ we get now for $0 \leq j < p$ and p large enough:

$$\left(\frac{M_p}{M_j} \right)^{\frac{1}{p-j}} \cdot \frac{1}{\mu_{2p}} \leq (2C)^{\frac{1}{p-j}} \cdot B^{-1} \cdot (H^p \cdot h^j)^{\frac{1}{p-j}}.$$

For $0 < \beta < \frac{1}{2}$ small we put j to be the smallest integer which is $\geq \beta \cdot p$. Then for p large enough $\beta_p := \beta + \frac{1}{p} \leq \frac{1}{2}$ and we have $2j \leq p \Leftrightarrow p \leq 2p - 2j \Leftrightarrow \frac{1}{p-j} \leq \frac{2}{p}$ and $p \cdot \beta_p = p \cdot \beta + 1 \geq j$. Hence, because $h \geq 1$,

$$\left(\frac{M_p}{M_j} \right)^{\frac{1}{p-j}} \cdot \frac{1}{\mu_{2p}} \leq (2C)^{2/p} \cdot B^{-1} \cdot (H \cdot h^{\beta_p})^{\frac{p}{p-j}}.$$

Then for a given ε' with $0 < \varepsilon' \leq B^{-1}$ we choose $0 < H < \varepsilon' \cdot B/2$ and then β small to reach $h^{\beta_p} \leq 2$, for all p which are large enough. So $H \cdot h^{\beta_p} < \varepsilon' \cdot B \leq 1$, and

$$\left(\frac{M_p}{M_j} \right)^{\frac{1}{p-j}} \cdot \frac{1}{\mu_{2p}} \leq (2C)^{2/p} \cdot B^{-1} \cdot (\varepsilon' \cdot B)^1 \leq \varepsilon, \quad \forall \varepsilon > 0,$$

because $(2C)^{2/p} \rightarrow 1$ for $p \rightarrow \infty$. So we have proven condition 5.1.4 (b)(ii), where here we have $l = 2$, hence condition (β_2) .

Now we prove $(I)(i) \Leftrightarrow (I)(ii)$:

$(I)(ii) \Rightarrow (I)(i)$ holds by 5.1.7 (a)(i).

$(I)(i) \Rightarrow (I)(ii)$:

By the continuity of E we have: For all $h > 0$ ($h \leq 1$) we can find $C, H > 0$ ($C, H \geq 1$) such that (5.1.10) holds. Now we imitate the proof in (a) and choose for $j = kp, k \in \mathbb{N}$. Then we assume

$$0 \leq x \leq B \cdot \frac{(kp)!^{1/(kp)}}{\mu_{(k+1)p}},$$

where $B > H_1^2$ and H_1 is the constant which appears in (5.1.10) for $h = 1$. Then we choose $0 < h < B^{-1}$ and $k \in \mathbb{N}$ large enough such that $\underbrace{(B \cdot h)^k}_{<1} \cdot H \cdot h < 1$. Like in the case above we get for p large enough (here: $2p$ and after p -times integrating):

$$\frac{1}{2p!} \cdot \left(B \cdot \frac{(k2p)!^{1/(2kp)}}{\mu_{2(k+1)p}} \right)^p \leq \|\chi_{2p}^{(p)}|_{[-1,1]}\|_\infty \leq C \cdot \frac{H_1^{2p}}{\mu_p^p}. \quad (5.1.12)$$

From (5.1.12) we can conclude condition (β_1) as follows:

$$(5.1.12) \Leftrightarrow \underbrace{\frac{1}{(2 \cdot C)^{1/p}}}_{\rightarrow 1, \text{ for } p \rightarrow \infty} \cdot \underbrace{\frac{B}{H_1^2}}_{1 <} \cdot \underbrace{\frac{(2kp)!^{1/2kp}}{p^{1/p}}}_{(*)} \leq \frac{\mu_{2(k+1)p}}{\mu_p}. \quad (5.1.13)$$

For the expression $(*)$ we use *Stirling's formula*:

$$(*) \sim \frac{(2k) \cdot (p/e) \cdot (4\pi kp)^{1/(4kp)}}{(p/e) \cdot (2\pi p)^{1/(2p)}} \rightarrow 2k, \text{ for } p \rightarrow \infty.$$

Finally we divide both sides in (5.1.13) through $2(k+1)$ to obtain condition (β_1) .

With this part of the proof of 5.0.6 we can formulate now the following corollary:

Corollary 5.1.9 *The conditions (β_1) , (β_2) and (γ_1) are invariant under the equivalence relation $\approx$.*

Proof. This follows immediately from above because, as we have already seen, the spaces $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{(M)}$ are clearly stable under the relation $\approx$. □

We continue with the proof of 5.0.6: $(I)(i) \Rightarrow (I)(iv)$ is clearly satisfied.

We show now $(I)(iv) \Rightarrow (I)(ii)$ to finish the first part of the theorem:

First we remark that j^∞ is an open mapping between two Fréchet-spaces by the *open mapping theorem* because it is surjective. So there exist $H, C > 0$ and functions $\chi_p \in \mathcal{E}_{(M)}([-1, 1])$, such that

$$|\chi_p|_{[-1,1],1} \leq C \cdot \frac{H^p}{M_p} \quad \forall p \quad \text{and} \quad \chi_p^{(j)}(0) = \delta_{j,p}, \quad (5.1.14)$$

which means $j^\infty(\chi_p) = e_p$.

Then define for all p the number $\tau_p := \inf\{x \in [0, 1] : \chi_p^{(p)}(x) < 1/2\}$ and so $\chi_p^{(p)}(x) \geq 1/2$ on the closed interval $[0, \tau_p]$. For $p' := 2p$ we integrate this inequality p -times and obtain:

$$\chi_{2p}^{(p)}(x) \geq \frac{1}{2} \cdot \frac{x^p}{p!} \quad \text{on } [0, \tau_{2p}].$$

By (5.1.14) we see now $\frac{1}{2} \cdot \frac{\tau_{2p}^p}{p!} \leq C \cdot \frac{H^{2p} \cdot M_p}{M_{2p}}$ and because $(\mu_p)_p$ is an increasing sequence we have for all p : $M_p/M_{2p} = (\mu_{p+1} \cdots \mu_{2p})^{-1} \leq \mu_p^{-p}$. In general $p^{1/p} \leq p \Leftrightarrow p! \leq p^p$ holds, hence

$$\left(\frac{M_p \cdot p!}{M_{2p}} \right)^{1/p} \leq \frac{p}{\mu_p} \quad \forall p,$$

which implies

$$\tau_{2p} \leq \frac{(2C)^{1/p} \cdot H^2}{\mu_p^*}. \quad (5.1.15)$$

Now we want to show that there exists $c > 0$ such that for all p which are sufficiently large

$$\tau_p \geq \sum_{k \geq 2p} \frac{c}{\mu_k} \quad (5.1.16)$$

holds. Then we could prove condition (γ_1) , because:

$$\sum_{k \geq p} \frac{1}{\mu_k} \underbrace{\leq}_{(\mu_p)_p \text{ incr.}} \underbrace{\frac{3}{\mu_p^*}}_{=3p/\mu_p} + \sum_{k \geq 4p} \frac{1}{\mu_k} \underbrace{\leq}_{(5.1.16)} \frac{3}{\mu_p^*} + \frac{\tau_{2p}}{c} \underbrace{\leq}_{(5.1.15)} \frac{A}{\mu_p^*},$$

for a constant $A > 0$. Note that $(2C)^{1/p}$ is for large p close to 1, hence bounded.

To prove (5.1.16) we first define the set $P := \{p \in \mathbb{N} : \tau_p < \sum_{k \geq 2p} \frac{c}{\mu_k}\}$, where $c < \min\{\frac{1}{4}, \frac{1}{(4H)}\}$.

Claim: The set P is finite. We assume in the following that $p \in P$ and we consider the function

$$\varrho_p(x) := \begin{cases} 0 & \text{for } x < 0 \\ \chi_p^{(p)}(x) - 1 & \text{for } x \geq 0. \end{cases} \quad (5.1.17)$$

We want to apply now 4.2.3 to this situation. For this we need the property that $(\mu_j)_j$ is *strict increasing*.

We proceed as follows: First we define the sequence $(a_k)_k$ appearing in 4.2.3. Set

$$a_k := \begin{cases} \frac{c}{\mu_{2p}} & \text{for } k \leq p \\ \frac{c}{\mu_{k+p}} & \text{for } k > p \end{cases} \quad (5.1.18)$$

and finally $T := \tau_p$. So $(a_k)_k$ is a decreasing sequence and *strict decreasing* for $k \geq p$. Using the notation of 4.2.3 (where $m = \infty$) we obtain for the set J_∞ : $J_\infty = \{k \in \mathbb{N} : k \geq p\}$. Furthermore

$$\sum_{k \geq 1} a_k = p \cdot \frac{c}{\mu_{2p}} + \sum_{k > 2p} \frac{c}{\mu_k} \geq \sum_{k \geq 2p} \frac{c}{\mu_k} > \tau_p = T$$

holds by definition of the set P . Finally we estimate as follows:

$$\frac{1}{2} \leq \sum_{k \geq p} 4^k \cdot \frac{c^k}{\mu_{2p}^p \cdot \mu_{2p+1} \cdots \mu_{p+k}} \cdot C \cdot H^p \cdot \frac{M_{p+k}}{M_p} \leq \frac{C}{1-4c} \cdot \underbrace{(4 \cdot c \cdot H)^p}_{<1}.$$

The first inequality holds because we use 4.2.3 where we estimate for $T = \tau_p$ and by (5.1.14).

The second inequality follows by the fact that $\frac{M_{p+k}}{M_p} \cdot (\mu_{2p}^p \cdot \mu_{2p+1} \cdots \mu_{p+k})^{-1} \leq 1$ (again by the increasing property of $(\mu_p)_p$). Furthermore $(4 \cdot h) < 1$ holds and so we can make a geometric series expansion.

Finally we conclude that P can only consist of finitely many indices because $(4 \cdot c \cdot H)^p \rightarrow 0$ for $p \rightarrow \infty$.

We finish the proof of 5.0.6 showing the following equivalence: $(II)(b)(i) \Leftrightarrow (II)(b)(ii)$.

$(II)(b)(ii) \Rightarrow (II)(b)(i)$ holds by 5.1.7 (a)(ii).

$(II)(b)(i) \Rightarrow (II)(b)(ii)$:

Applying 5.1.10 we obtain: The set $\{\Lambda_{\{M\}} \ni x := (x_p)_p : |x|_1 \leq 1\}$ is contained in the set $\{j^\infty(\chi) : |\chi|_{[-1,1],h} \leq C\}$ for some suitable $h, C > 0$. So we can find functions $\chi_p \in \mathcal{E}_{\{M\}}([-1,1])$, such that:

$$|\chi_p|_{[-1,1],h} \leq C \cdot \frac{1}{M_p}, \quad \text{and} \quad \chi_p^{(j)}(0) = \delta_{j,p}, \quad \forall j, p.$$

Now we can proceed as in the proof of $(I)(iv) \Rightarrow (I)(ii)$, where $H = 1$.

□

To complete the proof of 5.0.6 we have to prove the following result, which is due to Grothendieck:

Proposition 5.1.10 [6, Théorème B (1.), p. 17-18] *Let E and F be two separable locally convex vector spaces and assume that F is (LB) and E is (LF) or endowed with a coarser topology. Let $u : E \longrightarrow F$ be a surjective continuous linear mapping, then it is already a topological homomorphism (=open mapping on its image).*

Proof. By assumption of the space E we have a sequence of Fréchet-spaces $(E_i)_i$ and continuous linear mappings $f_i : E_i \longrightarrow E$ such that $E = \bigcup_i f_i(E_i)$. So it is sufficient to prove the case where E is a (LF) -space. We can factor out the kernel and $E_i / \ker(f_i)$ is still a Fréchet-space for all i , thus we can assume w.l.o.g. that the mappings f_i are all injective. Furthermore we can assume that u is injective, otherwise we can factor out the kernel of u , because the quotient space $E / \ker(u)$ is still of type (LF) . F is a (LB) -space and so we conclude: To prove the continuity of the mapping u^{-1} it is sufficient to prove the continuity for $u^{-1} \circ v : G \longrightarrow E$ for all continuous linear mappings $v : G \longrightarrow F$, where G is an arbitrary Banach-space.

Let $u_i := u \circ f_i$ for all i , then the statement follows by applying *Grothendieck's factorization theorem* (see [6, Théorème A, p. 16] or [21, 24.33, p. 271-272]):

Because $v(G) \subseteq \bigcup_i u_i(E_i)$ there exists an index i_0 such that $v(G) \subseteq u_{i_0}(E_{i_0})$ and because the mapping u_{i_0} is injective there exists a continuous linear mapping $w : G \rightarrow E_{i_0}$ such that $v = u_{i_0} \circ w$. Thus $u^{-1} \circ v = u^{-1} \circ u_{i_0} \circ w = u^{-1} \circ u \circ f_{i_0} \circ w = f_{i_0} \circ w$ is a continuous linear mapping.

□

5.2 Important consequences of the surjectivity

We summarize: In fact we have shown that in the case, where $\mu_0 = 1$, $(\mu_p)_p$ is strict increasing and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ holds, condition (β_1) for the sequence $(\mu_p)_p$ characterizes the surjectivity of the Borel-map in the Romieu- and in the Beurling-case. Recall that for $(\mu_p)_p$ increasing and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ we have $(\beta_1) \Leftrightarrow (\gamma_1)$ whereas (γ_1) implies always $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$.

Therefore we introduce another condition, which is often used in the literature:

$$\exists C > 0 : \sum_{j=k}^{\infty} \frac{m_j}{(j+1) \cdot m_{j+1}} \leq C \cdot \frac{m_k}{m_{k+1}}, \forall k \geq 0. \quad (5.2.1)$$

This condition is called *strong non-quasi-analyticity* for the weight sequence $(m_j)_j$. It is equivalent to

$$\exists C > 0 : \sum_{j=k}^{\infty} \frac{M_j}{M_{j+1}} \leq C \cdot \frac{(k+1) \cdot M_k}{M_{k+1}}, \forall k \geq 0,$$

if we use the sequence $(M_j)_j$.

First we see that $(\gamma_1) \Leftrightarrow (5.2.1)$ holds in general, because:

$$\begin{aligned} (5.2.1) &\Leftrightarrow \sum_{j=k}^{\infty} \frac{j! \cdot m_j}{j! \cdot (j+1) \cdot m_{j+1}} \leq C \cdot \frac{k! \cdot m_k}{k! \cdot m_{k+1}} \Leftrightarrow \sum_{j=k}^{\infty} \frac{M_j}{M_{j+1}} \leq C \cdot \frac{(k+1) \cdot M_k}{M_{k+1}} \\ &\Leftrightarrow \sum_{j=k}^{\infty} \frac{1}{\mu_{j+1}} \leq C \cdot \underbrace{\frac{k+1}{\mu_{k+1}}}_{=1/\mu_{k+1}^*} \Leftrightarrow (\gamma_1). \end{aligned}$$

If the sequence $(\mu_j)_j$ is increasing and $\sum_{j=1}^{\infty} \frac{1}{\mu_j} < \infty$ holds, then condition (β_1) is equivalent to (5.2.1) because $(\beta_1) \Leftrightarrow (\gamma_1)$ holds by 5.1.1.

In the previous section we have shown a full characterization of the Borel-map for the spaces $\mathcal{E}_{\{M\}}([-1, 1])$ resp. $\mathcal{E}_{(M)}([-1, 1])$. Remember that for $G \subseteq \mathbb{R}$ we have $\mathcal{E}_{\{M\}}(G) = \lim_{\leftarrow K} \mathcal{E}_{\{M\}}(K)$ resp. $\mathcal{E}_{(M)}(G) = \lim_{\leftarrow K} \mathcal{E}_{(M)}(K)$, where $K \subseteq G$ compact. We extend now the characterization for the surjectivity of the Borel-map to $\mathcal{E}_{\{M\}}(G)$ resp. $\mathcal{E}_{(M)}(G)$ in the following way:

We assume that $\mu_0 = 1$, $(\mu_p)_p$ is increasing and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ is satisfied. Then $\mathcal{E}_{\{M\}}$ is not quasi-analytic by 4.1.5. If additionally (β_1) holds, which is then equivalent to (γ_1) by 5.1.1, we have that $\mathcal{E}_{(M)}$ is not quasi-analytic by 5.1.8. So we can use 4.1.1 to obtain the existence of bump-functions in $\mathcal{E}_{\{M\}}(G)$ resp. $\mathcal{E}_{(M)}(G)$ and so we can restrict the problem to the interval $[-1, 1]$. Note that the assumptions on $(\mu_p)_p$ in the proof of 5.0.6 were stronger.

We summarize: If we assume $\mu_0 = 1$, $(\mu_p)_p$ is strict increasing and condition (γ_1) for $(\mu_p)_p$, then $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{(M)}$ are not quasi-analytic, and the Borel-mapping is surjective in both cases. If we assume $\mu_0 = 1$, $(\mu_p)_p$ is strict increasing and $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$, then in the Romieu- and the Beurling-case the surjectivity of the Borel-mapping implies condition (γ_1) .

The surjectivity of the Borel-map has another important consequence for the Beurling-case. First we remark that $\Lambda_{(M)} \cong \lambda^{\infty}(A)$ where $\lambda^{\infty}(A)$ is the sequence space associated to the *Köthe-Matrix* $A := (a_{j,k})_{j,k}$, with $a_{j,k} := \frac{k^j}{M_j}$. The fact that A defines a Köthe-Matrix follows immediately, because $a_{j,k} > 0$ holds for all $j, k \in \mathbb{N}$ and $a_{j,k} \leq a_{j,k+1} \Leftrightarrow \frac{k^j}{M_j} \leq \frac{(k+1)^j}{M_j}$ is clearly satisfied for all $j, k \in \mathbb{N}$.

First we apply this isomorphism in the following way: By [21, Lemma 27.4(2), p. 308], we see that if $j^{\infty} : \mathcal{E}_{(M)} \rightarrow \Lambda_{(M)}$ is surjective, then there exists $f \in \mathcal{E}_{(M)}$ such that $f^{(j)}(0) > 0$ for all $j \in \mathbb{N}$.

Furthermore we have: For all infinite $I \subseteq \mathbb{N}$ and for all $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $\inf_{j \in I} \frac{a_{j,n}}{a_{j,k}} = \inf_{j \in I} \frac{n^j}{k^j} = 0$, because we can take $k > n$. Therefore condition (6) in [21, 27.9 Theorem, p. 310-312], is valid, hence by condition (3) we get $\lambda^\infty(A) = c_0(A)$.

Let $M := (M_p)_p$ and $N := (N_p)_p$ be two arbitrary weight sequences and assume that the Borel-mapping $j^\infty : \mathcal{E}_{(M)} \rightarrow \Lambda_{(M)}$ is surjective. Furthermore we assume that $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(N)}$ holds, then the surjectivity of j^∞ for $\mathcal{E}_{(M)}$ implies $\Lambda_{(M)} \subseteq \Lambda_{(N)}$. Hence $\lambda^\infty(A) \subseteq \lambda^\infty(B)$ and so $c_0(A) \subseteq c_0(B)$, where $A := (a_{j,k})_{j,k}$ and $B := (b_{j,k})_{j,k}$ with $a_{j,k} := \frac{k^j}{M_j}$ and $b_{j,k} := \frac{k^j}{N_j}$. With this preparation we are going to prove now the following comparison result:

Proposition 5.2.1 *Let $M := (M_p)_p$ and $N := (N_p)_p$ be arbitrary weight sequences and we assume that $j^\infty : \mathcal{E}_{(M)} \rightarrow \Lambda_{(M)}$ is surjective and $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(N)}$ holds. Then we obtain $M \preceq N$.*

Proof. By assumption we have $\lambda^\infty(A) \subseteq \lambda^\infty(B)$, so $\sup_{j \in \mathbb{N}} |x_j| \cdot \frac{k^j}{M_j} < \infty$ for all $k \in \mathbb{N}$ implies $\sup_{j \in \mathbb{N}} |x_j| \cdot \frac{k^j}{N_j} < \infty$ for all $k \in \mathbb{N}$, where $(x_j)_j \in \mathbb{C}^\mathbb{N}$. We put $y_j := \frac{x_j}{M_j}$, then $\sup_{j \in \mathbb{N}} |y_j| \cdot k^j < \infty$ for all $k \in \mathbb{N}$ implies $\sup_{j \in \mathbb{N}} |y_j| \cdot k^j \cdot Q_j < \infty$ for all $k \in \mathbb{N}$ where we have set $Q_j := \frac{M_j}{N_j}$. Define for the Köthe-matrix $\tilde{A} := (a_{j,k})_{j,k}$, where $a_{j,k} := k^j$, the associated sequence space

$$\lambda^\infty(\tilde{A}) := \{y = (y_j)_j \in \mathbb{C}^\mathbb{N} : \sup_{j \in \mathbb{N}} |y_j| \cdot k^j < \infty, \forall k \in \mathbb{N}\}.$$

$\lambda^\infty(\tilde{A})$ is a Fréchet-space, equipped with the (countable many) seminorms $\|y\|_k := \sup\{|y_j| \cdot k^j : j \in \mathbb{N}\}$. We see that the mapping $Q : \lambda^\infty(\tilde{A}) \rightarrow \mathbb{C}$, $Q : y \mapsto \sum_{j=0}^\infty y_j \cdot Q_j$ is a bounded linear functional. Thus there exist $k_0 \in \mathbb{N}$ and a constant $D > 0$ such that for all $y = (y_j)_j \in \lambda^\infty(\tilde{A})$

$$|Q(y)| = \left| \sum_{j=0}^\infty y_j \cdot Q_j \right| \leq D \cdot \|y\|_{k_0} \quad (5.2.2)$$

is satisfied. Let $k_0 \in \mathbb{N}$ such that (5.2.2) holds, then we put $y_j := \frac{1}{k_0^j}$ for all $j \in \mathbb{N}$ which implies $\|y\|_{k_0} = 1$. Hence, by (5.2.2), we get

$$\sup_j \frac{Q_j}{k_0^j} \leq D \implies \sup_j \left(\frac{M_j}{N_j} \right)^{1/j} \leq D^{1/j} \cdot k_0 < \infty,$$

and this is exactly (3.1.1), so $M \preceq N$ is satisfied. □

With 5.2.1 we can formulate now the following corollary, which is similar to 3.1.3:

Corollary 5.2.2 *Let $\mu_0 = 1$, $(\mu_p)_p$ be strict increasing, and assume that condition (γ_1) ($\implies \sum_{p=1}^\infty \frac{1}{\mu_p} < \infty$) holds for $(\mu_p)_p$. Let $N := (N_p)_p$ be an arbitrary weight sequence. Then $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(N)}$ implies $M \preceq N$.*

If $\nu := (\nu_p)_p$, where $\nu_p := \frac{N_p}{N_{p-1}}$ satisfies $\nu_0 = 1$, $(\nu_p)_p$ strict increasing, and condition (γ_1) , too, then we have $M \approx N \iff \mathcal{E}_{(M)} = \mathcal{E}_{(N)}$.

Proof. 5.0.6 implies the surjectivity for $j^\infty : \mathcal{E}_{(M)} \rightarrow \Lambda_{(M)}$, and because $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(N)}$ holds, 5.2.1 implies $M \preceq N$. □

We finish this section with the following comparison:

2.0.3 shows, that the log. conv. condition for the sequence $m := (m_p)_p$ is stronger than for $M := (M_p)_p$. Now we want to obtain analog results for the introduced conditions above. For this we write $(\cdot)_\mu$ resp. $(\cdot)_{\bar{\mu}}$, where $\bar{\mu}_p := p! \cdot \mu_p = \frac{p! \cdot M_p}{M_{p-1}}$. We set $\bar{\mu}_p^* := \frac{\bar{\mu}_p}{p} = (p-1)! \cdot \mu_p = \frac{p! \cdot m_p}{m_{p-1}} = \frac{M_p}{m_{p-1}} = p! \cdot \mu_p^*$, and $\bar{\mu}_0^* = \bar{\mu}_0$. Now define another sequence $\widehat{M} := (\widehat{M}_p)_p$, where $\widehat{M}_p := \prod_{j=0}^p \bar{\mu}_j = \prod_{j=0}^p j! \cdot M_p$. We show that $(\cdot)_\mu \Rightarrow (\cdot)_{\bar{\mu}}$ holds for all conditions:

First we see that $\mu_p \nearrow \infty$ and $\mu_0 = 1$ imply $\bar{\mu}_p \nearrow \infty$ and $\bar{\mu}_0 = 1$:

$\bar{\mu}_0 = 0! \cdot \mu_0 = 1$ and $\bar{\mu}_p \nearrow \infty$, because $\frac{(p+1)!}{p!} = p+1 \geq 1$.

A further easy consequence is the following: $\frac{1}{\bar{\mu}_p} < \frac{1}{\mu_p}$ for all p , thus $\sum_{p=1}^{\infty} \frac{1}{\bar{\mu}_p} \leq \sum_{p=1}^{\infty} \frac{1}{\mu_p}$.

$(\alpha_1)_\mu \Rightarrow (\alpha_1)_{\bar{\mu}}$: $1 = \mu_0^* = \bar{\mu}_0^*$ and $\bar{\mu}_p^* \nearrow \infty$ is obvious.

$(\beta_1)_\mu \Rightarrow (\beta_1)_{\bar{\mu}}$: Holds because $\frac{\bar{\mu}_{kp}^*}{\bar{\mu}_p^*} = \frac{(kp)! \cdot \mu_{kp}^*}{p! \cdot \mu_p^*} = (kp) \cdots (p+1) \cdot \frac{\mu_{kp}^*}{\mu_p^*}$ for all $k \in \mathbb{N}$, $k > 1$.

$(\beta_1^0)_\mu \Rightarrow (\beta_1^0)_{\bar{\mu}}$: $\frac{\bar{\mu}_{2p}^*}{\bar{\mu}_p^*} = (2p) \cdots (p+1) \cdot \frac{\mu_{2p}^*}{\mu_p^*}$.

$(\beta_2)_\mu \Rightarrow (\beta_2)_{\bar{\mu}}$: For any $k \in \mathbb{N}$, $k > 1$, we have

$$\left(\frac{\widehat{M}_{kp}}{\widehat{M}_p} \right)^{1/p(k-1)} \cdot \frac{1}{\bar{\mu}_{kp}} = \left((p+1)! \cdots (kp)! \cdot \frac{M_{kp}}{M_p} \right)^{1/p(k-1)} \cdot \frac{1}{(kp)! \cdot \mu_{kp}} \text{ and}$$

$$((p+1)! \cdots (kp)!)^{1/p(k-1)} \cdot \frac{1}{(kp)! \cdot \mu_{kp}} \leq \left(((kp)!)^{p(k-1)} \right)^{1/p(k-1)} \frac{1}{(kp)! \cdot \mu_{kp}} = \frac{1}{\mu_{kp}}.$$

$(\beta_2^0)_\mu \Rightarrow (\beta_2^0)_{\bar{\mu}}$: Trivial, because: $\frac{\bar{\mu}_{kp}}{\bar{\mu}_p} \geq \frac{\mu_{kp}}{\mu_p}$ for any $k \in \mathbb{N}$.

$(\gamma_1)_\mu \Rightarrow (\gamma_1)_{\bar{\mu}}$: We have

$$\bar{\mu}_p^* \cdot \sum_{j \geq p} \frac{1}{\bar{\mu}_j} = (p-1)! \cdot \mu_p \cdot \underbrace{\sum_{j \geq p} \frac{1}{j! \cdot \mu_j}}_{\leq \frac{1}{p!} \cdot \sum_{j \geq p} \frac{1}{\mu_j}} \leq \mu_p^* \cdot \sum_{j \geq p} \frac{1}{\mu_j}, \forall p.$$

$(\gamma_2)_\mu \Rightarrow (\gamma_2)_{\bar{\mu}}$: Similarly we get

$$\bar{\mu}_p^* \cdot \sum_{j \geq kp} \frac{1}{\bar{\mu}_j} \leq \underbrace{\frac{(p-1)!}{(kp)!}}_{= \frac{1}{(kp) \cdots p} \leq \frac{1}{p}} \cdot \mu_p \cdot \sum_{j \geq kp} \frac{1}{\mu_j} \leq \mu_p^* \cdot \sum_{j \geq kp} \frac{1}{\mu_j}, \forall k, p \in \mathbb{N}.$$

6 Important examples

6.1 Gevrey-spaces

In this chapter we study a certain family of function spaces: The *Gevrey-spaces*. The defining weight sequence has the form $M_p := p!^s$, where $s \in \mathbb{R}$. Hence $m_p := p!^{s-1}$ and $\mu_p = p^s$. For a certain parameter $s \in \mathbb{R}$ we distinguish as usual: In the Romieu-case we write $\mathcal{G}_{\{s\}}$, in the Beurling-case $\mathcal{G}_{(s)}$ for the associated function spaces. Note that $\mathcal{G}_{\{1\}} = \mathcal{O}$ and $\mathcal{G}_{(1)}$ is the space of (the real parts of) entire functions. Usually Gevrey-spaces are defined only for $s > 1$, but here we will study the more general case. Therefore we check for the different properties:

Logarithmic convexity: First we see that the strong log. convexity condition is satisfied if

$$(p!^{s-1})^2 \leq (p+1!)^{s-1} \cdot (p-1)!^{s-1} \Leftrightarrow p^{s-1} \leq (p+1)^{s-1}$$

holds, thus exactly for all $s \geq 1$. For the weak log. convexity condition we have

$$(p!^s)^2 \leq (p+1!)^s \cdot (p-1)!^s \Leftrightarrow p^s \leq (p+1)^s,$$

which holds exactly for all $s \geq 0$.

We summarize: All Gevrey-sequences with $s \geq 1$ are strong log. convex and so the closedness under composition theorem 3.2.7 and the inverse function theorem 3.3.2 are valid for $\mathcal{G}_{\{s\}}$, $s \geq 1$. If $s \geq 0$, then the sequences are weak log. convex. Thus for $s \in [0, 1)$ the sequences are weak but not strong log. convex. Finally, if $s < 0$, then the sequences are neither weak nor strong log. convex.

Furthermore we have: $M_0 = 0!^s = 1$ and $M_1 = 1!^s = 1$ hold automatically for all $s \in \mathbb{R}$, the sequences $(M_p)_p$ and $(\mu_p)_p$ are strict increasing for $s > 0$ and $\lim_{p \rightarrow \infty} M_p = \lim_{p \rightarrow \infty} \mu_p = \infty$ is clearly satisfied for all parameters $s > 0$. We remark: $M_p \geq 1$ for all p holds if $s \geq 0$ and $m_p \geq 1$ for all p if $s \geq 1$.

Quasi-analyticity-property: We calculate: $\sum_{p=1}^{\infty} \frac{(p-1)!^s}{p!^s} = \sum_{p=1}^{\infty} \frac{1}{p^s} < \infty$, if and only if $s > 1$. By 4.1.5 we obtain the fact that for $s > 1$ the space $\mathcal{G}_{\{s\}}$ is not quasi-analytic. If $s \in [0, 1]$, then the weight sequences $(p!^s)_p$ are log. convex but the sum above is divergent, hence by 4.1.5: $\mathcal{G}_{\{s\}}$ and so $\mathcal{G}_{(s)}$ are quasi-analytic.

Closedness under solving ODE's: Condition (3.4.7) has the form

$$\left(\frac{q!^s}{q!}\right)^{1/q-1} \leq H \cdot \left(\frac{p!^s}{p!}\right)^{1/p-1} \Leftrightarrow q!^{\frac{s-1}{q-1}} \leq H \cdot p!^{\frac{s-1}{p-1}} \quad \text{for } 2 \leq q \leq p,$$

where $H \geq 1$. We have already seen that $(p!^s)_p$ is strong log. convex, which is condition (3.4.3), for $s \geq 1$. Hence condition (3.4.7) is satisfied in this case (in particular (3.4.2) is satisfied where $H = 1$). Furthermore, as we have mentioned, $M_0 = M_1 = 1$ holds for all parameters $s \in \mathbb{R}$ and $M_2 = 2^s \geq 2$ is satisfied for all $s \geq 1$. But (3.4.4) holds only for $s > 1$: $\lim_{p \rightarrow \infty} \frac{\mu_p}{p} = \lim_{p \rightarrow \infty} p^{s-1} = \infty \Leftrightarrow s > 1$. Thus $\mathcal{G}_{\{s\}}$ is closed under solving ODE's for all parameters $s \geq 1$ by 3.4.1 and $\mathcal{G}_{(s)}$ for $s > 1$ by 3.4.7 and 3.4.8.

Furthermore we have that for $s > 1$ the Beurling-classes $\mathcal{G}_{(s)}$ are closed under composition and under inversion: 3.2.7 resp. 3.3.2 are valid for $\mathcal{G}_{(s)}$, $s > 1$, because of 3.4.9.

Inclusion of spaces: To simplify the calculations we will often use *Stirling's formula*: For sequences $a := (a_p)_p$ and $b := (b_p)_p$ we write $a \sim b$ if $\lim_{p \rightarrow \infty} \frac{a_p}{b_p} = 1$ is satisfied. So Stirling's

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formula has the form $p!^{s/p} \sim \binom{p}{e}^s \cdot \underbrace{(2\pi p)^{s/2p}}_{\rightarrow 1, \text{ for } p \rightarrow \infty}$.

First we see, that $\lim_{p \rightarrow \infty} ((p!)^{s-1})^{1/p} < \infty \Leftrightarrow \lim_{p \rightarrow \infty} p^{s-1} < \infty$, which holds if and only if $s \leq 1$ and $\lim_{p \rightarrow \infty} p^{s-1} = \infty$ for all $s > 1$. Hence, by 3.1.5, we obtain: $\mathcal{O} \subsetneq \mathcal{G}_{\{s\}}$ for all $s > 1$ and for $s = 1$ clearly $\mathcal{G}_{\{s\}} = \mathcal{O}$. If $s < 1$, then $\mathcal{G}_{\{s\}} \subseteq \mathcal{O}$.

Furthermore we see:

$$\sup_p \left(\frac{p!^s}{p!^t} \right)^{1/p} < \infty \Leftrightarrow \sup_p p^{s-t} < \infty \Leftrightarrow s \leq t.$$

Hence $s \leq t$ implies $\mathcal{G}_{\{s\}} \subseteq \mathcal{G}_{\{t\}}$, resp. $\mathcal{G}_{(s)} \subseteq \mathcal{G}_{(t)}$ for all $s, t \in \mathbb{R}$. On the other side let $s \geq 0$, then the sequence $(p!^s)_p$ is log. convex, thus by 3.1.3 $\mathcal{G}_{\{s\}} \subseteq \mathcal{G}_{\{t\}}$ has the consequence that the supremum is bounded, hence $s \leq t$. We summarize: For all $s, t \geq 0$ we get $s = t \Leftrightarrow \mathcal{G}_{\{s\}} = \mathcal{G}_{\{t\}}$.

Next we want to compare the Gevrey-space with a general function-space of Romieu- resp. Beurling-type with weight sequence $(N_p)_p$. First we consider

$$\sup_p \left(\frac{p!^s}{N_p} \right)^{1/p} < \infty \Leftrightarrow \frac{1}{e^s} \cdot \sup_p \frac{p^s}{N_p^{1/p}} < \infty.$$

So the sequence $(N_p)_p$ has to grow at least as fast as $(p^{sp})_p$ to have $\mathcal{G}_{\{s\}} \subseteq \mathcal{E}_{\{N\}}$ resp. $\mathcal{G}_{(s)} \subseteq \mathcal{E}_{(N)}$. If $s \geq 0$, then we have for the Romieu-case by 3.1.3: $\mathcal{G}_{\{s\}} \subseteq \mathcal{E}_{\{N\}}$ implies the fact that $(N_p)_p$ has to grow at least as fast as $(p^{sp})_p$.

Similarly we obtain:

$$\sup_p \left(\frac{N_p}{p!^s} \right)^{1/p} < \infty \Leftrightarrow e^s \cdot \sup_p \frac{N_p^{1/p}}{p^s} < \infty.$$

Here the sequence $(p^{sp})_p$ has to grow at least as fast as $(N_p)_p$ to get $\mathcal{E}_{\{N\}} \subseteq \mathcal{G}_{\{s\}}$ resp. $\mathcal{E}_{(N)} \subseteq \mathcal{G}_{(s)}$.

Let $(N_p)_p$ be log. convex, then we have by 3.1.3: $\mathcal{E}_{\{N\}} \subseteq \mathcal{G}_{\{s\}}$ has the consequence that $\sup_p \frac{N_p^{1/p}}{p^s} < \infty$, hence $(p^{sp})_p$ has to grow at least as fast as $(N_p)_p$.

Stability under derivation: We have that $\sup_p \left(\frac{(p+1)!^s}{p!^s} \right)^{1/p} < \infty \Leftrightarrow \sup_p (p+1)^{s/p} < \infty$ holds for all $s \in \mathbb{R}$, so by (3.1.1) we see that $\mathcal{G}_{\{s\}}$ and $\mathcal{G}_{(s)}$ are closed under differentiation for $s \in \mathbb{R}$ and for $s \geq 0$ we see by 2.0.8: $\mathcal{G}_{\{s\}}$ and $\mathcal{G}_{(s)}$ are differential algebras.

Moderate Growth: A weight sequence $m := (m_p)_p$ satisfies this condition, if

$$\sup_{j,k} \left(\frac{m_{j+k}}{m_j \cdot m_k} \right)^{1/j+k} < \infty. \quad (6.1.1)$$

It has an important interpretation: For $\mathbb{R}^n$ we can define the function space $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ in two different ways: First via the usual semi-norms $|\cdot|_{K,h}$ and second via iterated partial derivatives in the following way: For $x \in \mathbb{R}^n$ we write $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^l$, $k+l=n$, and we set for $\gamma \in \mathbb{N}^n$ $\gamma = (\alpha, \beta)$ where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^l$. Then we can define new seminorms on a compact set K in $\mathbb{R}^n$ and $h_1, h_2 > 0$:

$$|f|_{K, h_1, h_2} := \sup_{\alpha, \beta, x \in K} \frac{|(f^{(\alpha, 0)})^{(0, \beta)}(x_1, x_2)|}{h_1^{|\alpha|} \cdot |\alpha|! \cdot m_{|\alpha|} \cdot h_2^{|\beta|} \cdot |\beta|! \cdot m_{|\beta|}}.$$

So one can put $\mathcal{E}_{M, h_1, h_2}(K) := \{f \in \mathcal{E}(K) : |f|_{K, h_1, h_2} < \infty\}$ and obtain the following representations for $G \subseteq \mathbb{R}^n$ open:

$$\widetilde{\mathcal{E}_{\{M\}}(G)} := \lim_{\leftarrow K} \lim_{\rightarrow h_1} \lim_{\rightarrow h_2} \mathcal{E}_{M, h_1, h_2}(K) \text{ resp. } \widetilde{\mathcal{E}_{(M)}(G)} := \lim_{\leftarrow K} \lim_{\leftarrow h_1} \lim_{\leftarrow h_2} \mathcal{E}_{M, h_1, h_2}(K).$$

It is a very natural assumption, that both definitions shall lead to the same function space which means $\mathcal{E}_{\{M\}}(G) = \widetilde{\mathcal{E}_{\{M\}}}(G)$ resp. $\mathcal{E}_{(M)}(G) = \widetilde{\mathcal{E}_{(M)}}(G)$ for all open subsets G . For this, if we assume normalization ($M_0 = m_0 = 1$) and log. conv. for the sequence $(M_p)_p$, then we can use (3.1.1): The inclusions $\widetilde{\mathcal{E}_{\{M\}}}(G) \subseteq \mathcal{E}_{\{M\}}(G)$ resp. $\widetilde{\mathcal{E}_{(M)}}(G) \subseteq \mathcal{E}_{(M)}(G)$ hold by 2.0.6 (in fact the supremum in (3.1.1) is bounded by 1). The inverse inclusions are satisfied if condition (6.1.1) holds. Hence only for normalized log. conv. weight sequences with condition (6.1.1) the two different definitions coincide. Note, that for the sequence $(M_p)_p$ the property (6.1.1) looks the same, because:

$$\begin{aligned} \sup_{j,k} \left(\frac{M_{j+k}}{M_j \cdot M_k} \right)^{1/j+k} < \infty &\Leftrightarrow \sup_{j,k} \left(\underbrace{\frac{(j+k)!}{j! \cdot k!}}_{1 \leq \binom{j+k}{j} \leq 2^{j+k}} \cdot \frac{m_{j+k}}{m_j \cdot m_k} \right)^{1/j+k} < \infty \\ &\Leftrightarrow \sup_{j,k} \left(\frac{m_{j+k}}{m_j \cdot m_k} \right)^{1/j+k} < \infty. \end{aligned}$$

We have

$$1 \leq \binom{j+k}{j} \leq 2^{j+k} \quad \forall j, k \implies 1 \leq \sup_{j,k} \left(\frac{(j+k)!}{j! \cdot k!} \right)^{1/j+k} \leq 2.$$

So for the Gevrey-sequences $(p!)^s$, $s \geq 0$, we can estimate $1 \leq \sup_{j,k} \left(\frac{(j+k)!^s}{j!^s \cdot k!^s} \right)^{1/j+k} \leq 2^s < \infty$

and for $s < 0$ we get $1 \geq \sup_{j,k} \left(\frac{(j+k)!^s}{j!^s \cdot k!^s} \right)^{1/j+k} \geq 2^s$. Thus for all $s \in \mathbb{R}$ the Gevrey-sequences satisfy (6.1.1).

If a weight sequence satisfies (5.2.1) and (6.1.1) then we call it *strong regular*.

Stability under ultradifferential operators:

For given function spaces $\mathcal{E}_{(M)}$ resp. $\mathcal{E}_{\{M\}}$ we call a linear PDO with constant coefficients of the form $P(\partial) := \sum_{|\alpha|=0}^{\infty} a_{\alpha} \cdot \partial^{\alpha}$, $a_{\alpha} \in \mathbb{C}$, an *ultradifferentiable operator of class $(M_p)_p$* , if

$$\boxed{\exists L, C > 0 : |a_{\alpha}| \leq \frac{C \cdot L^{|\alpha|}}{M_{|\alpha|}}, \forall |\alpha|, \text{ resp. } \forall L > 0 \exists C > 0 : |a_{\alpha}| \leq \frac{C \cdot L^{|\alpha|}}{M_{|\alpha|}}, \forall |\alpha|}.$$

We show now: If the weight sequence $(M_p)_p$ satisfies

$$\exists A, H > 0 : M_p \leq A \cdot H^p \cdot \min_{0 \leq q \leq p} M_q \cdot M_{p-q} \quad \forall p \in \mathbb{N}, \quad (6.1.2)$$

then the associated function space $\mathcal{E}_{(M)}$ resp. $\mathcal{E}_{\{M\}}$ is stable under $P(\partial)$. We estimate the α -term in the sum of $P(\partial)$ on a compact set $K \subseteq \mathbb{R}^n$ as follows:

$$\begin{aligned} \sup_{x \in K} |(a_{\alpha} \cdot f^{(\alpha)})^{(\beta)}(x)| &\leq \sup_{x \in K} |a_{\alpha}| \cdot |f^{(\alpha+\beta)}(x)| \leq \frac{C_1 \cdot L^{|\alpha|}}{M_{|\alpha|}} \cdot C_2 \cdot h^{|\alpha+\beta|} \cdot M_{|\alpha+\beta|} \\ &\stackrel{(6.1.2)}{\leq} \underbrace{\frac{C_1 \cdot L^{|\alpha|}}{M_{|\alpha|}} \cdot C_2 \cdot h^{|\alpha+\beta|} \cdot A \cdot H^{|\alpha+\beta|}}_{\substack{\leq M_{|\alpha|} \cdot M_{|\beta|}, \text{ for } q=|\alpha|}} \cdot \underbrace{\min_{0 \leq q \leq |\alpha+\beta|} M_q \cdot M_{|\alpha+\beta|-q}}_{\substack{\leq M_{|\alpha|} \cdot M_{|\beta|}, \text{ for } q=|\alpha|}} \\ &\leq C_1 \cdot C_2 \cdot A \cdot (h \cdot H)^{|\beta|} \cdot M_{|\beta|} \cdot \underbrace{(L \cdot h \cdot H)^{|\alpha|}}_{(*)}. \end{aligned}$$

Note that by the definitions above one can reach that $(*)$ is small enough: Take $h > 0$ small in the Beurling-case and $L > 0$ small in the Romieu-case. Hence the sum over the multiindex α is convergent.

Finally we note that (6.1.2) is equivalent to condition (6.1.1).

We show now that the Gevrey-sequences satisfies (6.1.2) for $s \in \mathbb{R}$: This holds because (6.1.1) is equivalent to (6.1.2) or by direct calculation:

$$(6.1.2) \Leftrightarrow A \cdot H^p \geq \frac{p!^s}{\min_{0 \leq q \leq p} \{q!^s \cdot (p-q)!^s\}} = \max_{0 \leq q \leq p} \left(\frac{p}{q} \right)^s,$$

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and for this it is sufficient to set $A := 1, H := 2^s$.

Condition (β_2) : It is satisfied for no $s \in \mathbb{R}$, because first we choose in 5.1.4 (b)(iii) $j = 0$, hence

$$(\beta_2) \Rightarrow \lim_{p \rightarrow \infty} \frac{M_p^{1/p}}{\mu_p} = 0. \quad (6.1.3)$$

But if we use again Stirling's formula we get a contradiction:

$$\lim_{p \rightarrow \infty} \frac{M_p^{1/p}}{\mu_p} = \lim_{p \rightarrow \infty} \left(\frac{p!^{1/p}}{p} \right)^s = \lim_{p \rightarrow \infty} \left(\frac{(p/e) \cdot (2\pi p)^{1/2p}}{p} \right)^s = \frac{1}{e^s} \cdot \lim_{p \rightarrow \infty} (2\pi p)^{\frac{s}{2p}} = \frac{1}{e^s} > 0.$$

Condition (β_0^1) : Is only satisfied for $s > 1$ because $\inf_p \frac{(2p)^s/(2p)}{p^s/p} = 2^{s-1} > 1$.

Condition (β_1) : Holds for $s > 1$, because $(\beta_0^1) \Rightarrow (\beta_1)$ or by direct calculation:

$$\liminf_{p \rightarrow \infty} \frac{(kp)^s/(kp)}{p^s/p} = k^{s-1} > 1, \text{ where } k \in \mathbb{N}.$$

We have already seen: The spaces $\mathcal{G}_{\{s\}}$, where $s > 1$, are not quasi-analytic by 4.1.5, because $\sum_{p=1}^{\infty} \frac{1}{p^s} < \infty$. But for $s > 1$ we can use 5.1.1 to obtain $(\beta_1) \Leftrightarrow (\gamma_1)$, hence by 5.1.8 we can conclude that the spaces $\mathcal{G}_{(s)}$ are not quasi-analytic, too.

Furthermore, by 5.2.2, we see: If $s > 1$ and $\mathcal{G}_{(s)} \subseteq \mathcal{G}_{(t)}$ holds, then $(p!^s)_p \preceq (p!^t)_p$ and so $\sup_p p!^{(s-t)/p} < \infty \Leftrightarrow \sup_p p^{s-t} < \infty \Leftrightarrow s \leq t$. Finally, if $s, t > 1$, then $\mathcal{G}_{(s)} = \mathcal{G}_{(t)} \Leftrightarrow s = t$.

Condition (β_0^2) : Use 5.1.5 or by direct calculation: $\lim_{p \rightarrow \infty} \frac{(kp)^s}{p^s} = k^s < \infty$ holds for all $k \in \mathbb{N}$ and $s \in \mathbb{R}$, hence (β_0^2) doesn't hold for all $s \in \mathbb{R}$.

Condition (γ_1) : Holds for $s > 1$, because 5.1.1 (a) is valid. So (5.2.1) is satisfied, hence for $s > 1$ the spaces are strong regular.

Condition (γ_2) : Is not satisfied by 5.1.1 (b) for any $s \in \mathbb{R}$.

We summarize: For parameters $s > 1$ we obtain: For $\mathcal{G}_{\{s\}}$ and $\mathcal{G}_{(s)}$ the Borel-map is *surjective*, but it is *not injective*. We can observe a big difference here between the Romieu- and the Beurling-case for parameters $s > 1$: $\mathcal{G}_{(s)}$ admits always the existence of an Extension-operator, but $\mathcal{G}_{\{s\}}$ doesn't share this property for any s .

Parameters	inj.	surj.	E for $\{\}$	E for $()$	mod. growth	strong reg.
$s < 0$	—	—	—	—	yes	—
$0 \leq s \leq 1$	yes	—	—	—	yes	no
$s > 1$	no	yes	no	yes	yes	yes

6.2 Further examples

Next we want to vary the Gevrey-sequences, to study the remarks above and their stability properties in detail. For this we introduce the following notation. If $M := (M_p)_p$ is a weight sequence with associated function spaces $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ then we construct new weight sequences $M' := (M'_p)_p$ in the following way: Set $M'_p := M_p \cdot f(p)$, where $f : \mathbb{N} \rightarrow \mathbb{R}_{>0}$, $p \mapsto f(p)$. We put $\mathcal{E}_{\{M \cdot f\}} := \mathcal{E}_{\{M'\}}$ resp. $\mathcal{E}_{(M \cdot f)} := \mathcal{E}_{(M')}$ and remark: The function f can be used to introduce (additional) parameters and one can study the dependence of the properties of the function spaces on the parameter(s).

If $f(p) = 1$ holds for all $p \in \mathbb{N}$, then we have clearly $(M'_p)_p = (M_p)_p$. Furthermore we obtain: $(M_p \cdot f(p))_p$ is log. conv. if and only if $(\mu_p \cdot \frac{f(p)}{f(p-1)})_p$ is an increasing sequence, hence if $(M_p)_p$ and $(f(p))_p$ are both log. convex, then $(M_p \cdot f(p))_p$, too.

We remark: If one has two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}_{>0}$, with $\|f - g\|_\infty \leq \varepsilon$, then we estimate:

$$\sup_p |M_p \cdot f(p) - M_p \cdot g(p)| \leq \sup_p |M_p| \cdot \|f - g\|_\infty \leq \varepsilon \cdot \sup_p |M_p|.$$

This shows: If the functions f and g are very close, in particular if ε is very small, one can not conclude, that the sequences $(M_p \cdot f(p))_p$ and $(M_p \cdot g(p))_p$ are close.

But if $f := (f(p))_p$ and $g := (g(p))_p$ and $f \preceq g$, then $\mathcal{E}_{\{M \cdot f\}} \subseteq \mathcal{E}_{\{M \cdot g\}}$ resp. $\mathcal{E}_{(M \cdot f)} \subseteq \mathcal{E}_{(M \cdot g)}$. Furthermore we have:

If $\sup_p \frac{1}{f(p)^{1/p}} < \infty$, then $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{M \cdot f\}}$ resp. $\mathcal{E}_{(M)} \subseteq \mathcal{E}_{(M \cdot f)}$ and if $\sup_p f(p)^{1/p} < \infty$, then $\mathcal{E}_{\{M \cdot f\}} \subseteq \mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M \cdot f)} \subseteq \mathcal{E}_{(M)}$.

We consider now some examples:

1. Let $f(p) := \frac{1}{p!}$, then $(M_p \cdot f(p))_p = m_p$.
2. Let $f(p) := \frac{1}{\mu_p} = \frac{M_{p-1}}{M_p}$, then $(M_p \cdot f(p))_p = (M_{p-1})_p$. Hence, if the sequence $(M_p)_p$ is logarithmic convex, then we can use 3.1.4 to obtain: $\mathcal{E}_{\{M\}}$ is stable under differentiation if and only if $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{M \cdot f\}}$.
3. Set $f(p) := \frac{1}{m_p}$, then $(M_p \cdot f(p))_p = \left(\frac{p! \cdot m_p}{m_p}\right)_p = (p!)_p$. Hence $\mathcal{E}_{\{M \cdot f\}} = \mathcal{O}$.
4. Set $f(p) := \mu_p$, then $(M_p \cdot f(p))_p = \left(\frac{M_p^2}{M_{p-1}}\right)_p$. So $((M_p \cdot f(p))_p)$ is log. conv. if and only if

$$\frac{M_{p+1}^4}{M_p^2} \leq \frac{M_p^2}{M_{p-1}} \cdot \frac{M_{p+2}^2}{M_{p+1}} \Leftrightarrow \frac{\mu_{p+1}}{\mu_p} \leq \frac{\mu_{p+2}^2}{\mu_{p+1}^2} \quad \forall p.$$

One can iterate to obtain the following "sequence of sequences": $(M_p)_p, (M_p \cdot f(p))_p, (M_p \cdot f(p)^2)_p, \dots$. So we can consider the orbit of a weight sequence with respect to a function f : $\{(M_p \cdot f(p)^n)_p : n \in \mathbb{Z}\}$. For $f(p) := \frac{1}{\mu_p}$ and $n \in \mathbb{N}$ we get shifted sequences: $(M_p)_p, (M_{p-1})_p, (M_{p-2})_p, \dots$

If $\mathcal{E}_{\{M \cdot f\}} \subseteq \mathcal{E}_{\{M\}}$ and $(M_p \cdot f(p))_p$ is log. convex, then, by 3.1.3, we have $\sup_p f(p)^{1/p} < \infty$.

Because $\sup_p (f(p)^n)^{1/p} = (\sup_p f(p)^{1/p})^n < \infty$ for all $n \in \mathbb{N}$ we get:

$$\dots \subseteq \mathcal{E}_{\{M \cdot f^n\}} \subseteq \mathcal{E}_{\{M \cdot f^{n-1}\}} \dots \subseteq \mathcal{E}_{\{M \cdot f\}} \subseteq \mathcal{E}_{\{M\}}.$$

If we assume that $\mathcal{E}_{\{M\}}$ is quasi-analytic, then, by restricting the Borel-map, we see now that all spaces $\mathcal{E}_{\{M \cdot f^n\}}$, $n \in \mathbb{N}$, are quasi-analytic.

For the following we assume that $(M_p)_p$ and $(f(p))_p$ are both log. conv. Then $\{(M_p \cdot f(p)^n)_p : n \in \mathbb{N}\}$ consists only of log. conv. sequences, because:

$$\begin{aligned} (M_p)_p \text{ log.conv.} &\Leftrightarrow 1 \leq \frac{M_{p-1} \cdot M_{p+1}}{M_p^2} \quad \forall p \\ &\Rightarrow \left(\frac{f(p)^2}{f(p-1) \cdot f(p+1)} \right)^n \leq \frac{M_{p-1} \cdot M_{p+1}}{M_p^2} \quad \forall p, \end{aligned}$$

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where $n \in \mathbb{N}$ arbitrary.

To transfer quasi-analyticity-properties we can use 4.1.5 in the following way: First we normalize all $(M_p \cdot f(p)^n)_p$, in particular set $M_0 = f(0) = 1$.

We additionally assume now that $(f(p))_p$ is decreasing. If $\mathcal{E}_{\{M\}}$ is quasi-analytic, then

$$\infty = \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} \leq \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} \cdot \left(\frac{f(p-1)}{f(p)} \right)^n, \forall n \in \mathbb{N},$$

which shows, that all spaces $\mathcal{E}_{\{M \cdot f^n\}}$, $n \in \mathbb{N}$, are quasi-analytic.

But if one assumes that $(f(p))_p$ is an increasing sequence, then we obtain: If $\mathcal{E}_{\{M\}}$ is not quasi-analytic, then the spaces $\mathcal{E}_{\{M \cdot f^n\}}$, $n \in \mathbb{N}$, are not quasi-analytic because:

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} \cdot \left(\frac{f(p-1)}{f(p)} \right)^n \leq \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty, \forall n \in \mathbb{N}.$$

Now we assume that the sequences $(M_p)_p$ and $(f(p))_p$ both satisfy condition (6.1.2) then the sequence $(M_p \cdot f(p))_p$ shares this property, too:

$$\begin{aligned} M_p \cdot f(p) &\leq (A \cdot H^p \cdot \min_{0 \leq q \leq p} \{M_{p-q} \cdot M_q\}) \cdot f(p) \\ &\leq (A \cdot H^p \cdot \min_{0 \leq q \leq p} \{M_{p-q} \cdot M_q\}) \cdot (B \cdot K^p \cdot \min_{0 \leq q \leq p} \{f(p-q) \cdot f(q)\}) \\ &\leq A \cdot B \cdot (H \cdot K)^p \cdot \min_{0 \leq q \leq p} \{(M_{p-q} \cdot f(p-q)) \cdot (M_q \cdot f(q))\}. \end{aligned}$$

Hence all sequences in the set $\{(M_p \cdot f(p)^n)_p : n \in \mathbb{N}\}$ have property (6.1.2). Assume additionally $f(p) \geq 1$ for all p then the sequence $(f(p)^{-1})_p$ satisfies (6.1.2) automatically, because

$$\begin{aligned} \frac{1}{f(p)} &\leq f(p) \leq A \cdot H^p \cdot \min_{0 \leq q \leq p} f(p-q) \cdot f(q) \leq A \cdot H^p \cdot \max_{0 \leq q \leq p} f(p-q) \cdot f(q) \\ &= A \cdot H^p \cdot \min_{0 \leq q \leq p} \frac{1}{f(p-q)} \cdot \frac{1}{f(q)}. \end{aligned}$$

So we obtain: In this case all sequences $\{(M_p \cdot f(p)^n)_p : n \in \mathbb{Z}\}$ have property (6.1.2).

Now let us return to the Gevrey-spaces $\mathcal{G}_{\{s\}}$ resp. $\mathcal{G}_{(s)}$, defined via the weight sequence $(p!)^s$, $s \in \mathbb{R}$. Then we set $\mu_p := p^s \cdot f(p)$ for $p > 0$ and $\mu_0 := f(0) := 1$, hence $M_p := p!^s \cdot \prod_{i=0}^p f(i)$ resp. $m_p := p!^{s-1} \cdot \prod_{i=0}^p f(i)$ and we denote the associated function spaces with $\mathcal{G}_{\{s, f\}}$ resp. with $\mathcal{G}_{(s, f)}$. If $f(p) = 1, \forall p \in \mathbb{N}$, then we obtain the "usual" Gevrey-spaces, which we have discussed above. The characterizing conditions for the weight sequence depend now essentially on the function f . First we see that $M_0 = 1$ holds automatically for all $s \in \mathbb{R}$ and we have $M_1 = f(0) \cdot f(1)$, so $M_1 = 1 \Leftrightarrow f(1) = 1$. Thus we see: (3.4.1) is satisfied, if and only if $f(0) = f(1) = 1$.

Condition $M_2 \geq 2$ has the form $2^s \cdot \underbrace{f(0)}_1 \cdot f(1) \cdot f(2) \geq 2 \Leftrightarrow 2^{s-1} \geq \frac{1}{f(1) \cdot f(2)}$.

The most important condition, if we want to apply the theorems above, is of course logarithmic convexity: For the strong log. convexity condition we get

$$\begin{aligned} m_p^2 &\leq m_{p-1} \cdot m_{p+1} \Leftrightarrow p!^{2s-2} \cdot \prod_{i=0}^p (f(i))^2 \leq \left((p-1)!^{s-1} \cdot \prod_{i=0}^{p-1} f(i) \right) \cdot \left((p+1)!^{s-1} \cdot \prod_{i=0}^{p+1} f(i) \right) \\ &\Leftrightarrow \frac{p!^{2s}}{(p+1)!^s \cdot (p-1)!^s} \cdot \frac{p+1}{p} \leq \frac{f(p+1)}{f(p)} \quad \forall p \\ &\Leftrightarrow \frac{p^{s-1}}{(p+1)^{s-1}} \leq \frac{f(p+1)}{f(p)} \quad \forall p \end{aligned}$$

and for the weak log. convexity we have

$$\begin{aligned} M_p^2 \leq M_{p-1} \cdot M_{p+1} &\Leftrightarrow p!^{2s} \cdot \prod_{i=0}^p (f(i))^2 \leq \left((p-1)!^s \cdot \prod_{i=0}^{p-1} f(i) \right) \cdot \left((p+1)!^s \cdot \prod_{i=0}^{p+1} f(i) \right) \\ &\Leftrightarrow \frac{p^s}{(p+1)^s} \leq \frac{f(p+1)}{f(p)} \quad \forall p. \end{aligned}$$

For (3.4.7) we calculate:

$$\begin{aligned} (3.4.7) &\Leftrightarrow \exists H \geq 1 : \left(q!^{s-1} \cdot \prod_{i=0}^q f(i) \right)^{1/(q-1)} \leq \left(p!^{s-1} \cdot \prod_{i=0}^p f(i) \right)^{1/(p-1)} \quad \text{for } 2 \leq q \leq p \\ &\Leftrightarrow \exists H \geq 1 : \left(\frac{q!^{q-1}}{p!^{p-1}} \right)^{s-1} \leq \frac{\prod_{i=0}^p f(i)^{1/(p-1)}}{\prod_{i=0}^q f(i)^{1/(q-1)}} \quad \text{for } 2 \leq q \leq p. \end{aligned}$$

Finally (3.4.4) holds if $\lim_{p \rightarrow \infty} p^{s-1} \cdot f(p) = \infty$.

If we want to compare the new spaces $\mathcal{G}_{\{s,f\}}$ resp. $\mathcal{G}_{(s,f)}$, we have by (3.1.1):

$$\begin{aligned} \sup_p \left(p!^{s-t} \cdot \frac{\prod_{i=0}^p f(i)}{\prod_{i=0}^p g(i)} \right)^{1/p} < \infty &\Leftrightarrow \sup_p p^{s-t} \cdot \prod_{i=0}^p \left(\frac{f(i)}{g(i)} \right)^{1/p} < \infty \\ &\Rightarrow \mathcal{G}_{\{s,f\}} \subseteq \mathcal{G}_{\{t,g\}} \quad \text{resp.} \quad \mathcal{G}_{(s,f)} \subseteq \mathcal{G}_{(t,g)}. \end{aligned}$$

If $(p!^s \cdot \prod_{i=0}^p f(i))_p$ is log. convex then, by 3.1.3, also $\Leftarrow$ holds above in the Romieu-case.

Furthermore we have: If

$$\begin{aligned} \sup_p \left(\frac{(p+1)!^s \cdot \prod_{i=0}^{p+1} f(i)}{p!^s \cdot \prod_{i=0}^p f(i)} \right)^{1/p} < \infty &\Leftrightarrow \sup_p \left((p+1)^s \cdot f(p+1) \right)^{1/p} < \infty \\ &\Leftrightarrow \sup_p f(p+1)^{1/p} < \infty \end{aligned}$$

is satisfied, then $\mathcal{G}_{\{s,f\}}$ resp. $\mathcal{G}_{(s,f)}$ is closed under differentiation. By 3.1.4 we see: If $(p!^s \cdot \prod_{i=0}^p f(i))_p$ is log. convex, then the closedness under differentiation of $\mathcal{G}_{\{s,f\}}$ implies $\sup_p f(p+1)^{1/p} < \infty$.

Finally, if $m_p = p!^{s-1} \cdot \prod_{i=0}^p f(i) \geq 1$ for all p and $(p!^s \cdot \prod_{i=0}^p f(i))_p$ is log. convex, then we get by 3.1.5 that $\mathcal{O} = \mathcal{G}_{\{s,f\}}$ if and only if

$$\sup_p \left((p!)^{s-1} \cdot \prod_{i=0}^p f(i) \right)^{1/p} < \infty \quad \Leftrightarrow \quad \underbrace{\sup_p p^{s-1} \cdot \left(\prod_{i=0}^p f(i) \right)^{1/p}}_{\text{Stirling's formula}} < \infty.$$

Now we look at the important characterizing conditions for the Borel-map:

$$\begin{aligned} (\beta_1) &\Leftrightarrow \exists k \in \mathbb{N} : \liminf_{p \rightarrow \infty} \frac{p \cdot (kp)^s \cdot f(kp)}{p^s \cdot (kp) \cdot f(p)} = k^{s-1} \cdot \liminf_{p \rightarrow \infty} \frac{f(kp)}{f(p)} > 1 \\ &\Leftrightarrow \exists k \in \mathbb{N} : \liminf_{p \rightarrow \infty} \frac{f(kp)}{f(p)} > \frac{1}{k^{s-1}}. \end{aligned}$$

The condition for the quasi-analyticity of a function space is, by 4.1.5:

$$\sum_{p=1}^{\infty} \frac{(p-1)!^s \cdot \prod_{i=0}^{p-1} f(i)}{p!^s \cdot \prod_{i=0}^p f(i)} = \sum_{p=1}^{\infty} \frac{1}{p^s \cdot f(p)} = \infty.$$

Another important condition is (β_2^0) , because $(\beta_2^0) \Rightarrow (\beta_2)$ by 5.1.5 and $(\beta_2^0) \Rightarrow (\beta_1)$:

$$(\beta_2^0) \Leftrightarrow \exists k \in \mathbb{N} : \lim_{p \rightarrow \infty} \frac{(kp)^s \cdot f(kp)}{p^s \cdot f(p)} = k^s \cdot \lim_{p \rightarrow \infty} \frac{f(kp)}{f(p)} = \infty \Leftrightarrow \exists k \in \mathbb{N} : \lim_{p \rightarrow \infty} \frac{f(kp)}{f(p)} = \infty.$$

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Applying once again Stirling's formula, (6.1.3) leads to a further possibility to check (β_2) :

$$\lim_{p \rightarrow \infty} \frac{M_p^{1/p}}{\mu_p} = 0 \Leftrightarrow \frac{1}{e^s} \cdot \lim_{p \rightarrow \infty} (2\pi p)^{s/2p} \cdot \frac{\left(\prod_{i=0}^p f(i)\right)^{1/p}}{f(p)} = 0 \Leftrightarrow \lim_{p \rightarrow \infty} \frac{\left(\prod_{i=0}^p f(i)\right)^{1/p}}{f(p)} = 0.$$

We remark: The important conditions (β_1) and (β_0^2) depend only on the function f , while the condition to check the quasi-analyticity of a function space is not independent of the "original weight sequence". But this statement is not true in general, it's a special property of the Gevrey-sequences.

Example Set $f_\alpha(p) := \alpha^{p-1}$ for $p > 0$ and $f_\alpha(0) := 1$, where $\alpha \in \mathbb{R}_{>0}$ arbitrary. Then $\mu_p := p^s \cdot \alpha^{p-1}$ for $p > 0$ and $\mu_0 := 1$, hence $M_p := p!^s \cdot \alpha^{p(p-1)/2}$ and we have $M_0 = M_1 = \alpha^0 = 1$. So we have introduced a new parameter α and we write $\mathcal{G}_{\{s, f_\alpha\}}$ resp. $\mathcal{G}_{(s, f_\alpha)}$ for the associated function spaces. Clearly we have $\mathcal{G}_{\{s, f_1\}} = \mathcal{G}_{\{s\}}$ resp. $\mathcal{G}_{(s, f_1)} = \mathcal{G}_{(s)}$, so we will study the cases $0 < \alpha < 1$ and $1 < \alpha$. For this example we will restrict the parameter s to $(1, \infty)$. Then we have: $\alpha \geq 1$ implies that $(\mu_p)_p = (p^s \cdot \alpha^{p-1})_p$ is a strict increasing sequence and $\lim_{p \rightarrow \infty} \mu_p = \lim_{p \rightarrow \infty} p^s \cdot \alpha^{p-1} = \infty$. Furthermore for $\alpha \geq 1$ condition (3.4.4) is satisfied, because $\lim_{p \rightarrow \infty} \frac{\mu_p}{p} = p^{s-1} \cdot \alpha^{p-1} = \infty$.

First we check (6.1.1):

$$\begin{aligned} (6.1.1) &\Leftrightarrow \sup_{j,k} \left(\frac{\alpha^{(j+k)(j+k-1)/2}}{\alpha^{j(j-1)/2} \cdot \alpha^{k(k-1)/2}} \right)^{1/j+k} < \infty \Leftrightarrow \sup_{j,k} (\alpha^{2kj})^{1/2(j+k)} < \infty \\ &\Leftrightarrow \sup_{j,k} \alpha^{\frac{jk}{j+k}} < \infty. \end{aligned}$$

The first equivalence holds, because we have already shown that $\sup_{j,k} \left(\frac{(j+k)!^s}{j!^s \cdot k!^s} \right)^{1/(j+k)} < \infty$ for any s . Hence the modified weight sequences satisfy (6.1.1) $\Leftrightarrow 0 < \alpha \leq 1$. For $\alpha > 1$ we have $\sup_{j,k} \alpha^{\frac{jk}{j+k}} \geq \sup_k \alpha^{\frac{k^2}{2k}} = \infty$.

Now we characterize *log. convexity*:

Let $\alpha \geq 1$, since $\sup_p \left(\frac{p}{p+1} \right)^{s-1} = 1 \leq \frac{f_\alpha(p+1)}{f_\alpha(p)} = \alpha$ holds, we obtain the strong log. convexity for all parameters $s > 1$. So the closedness under composition 3.2.7 and the inverse function theorem 3.3.2 are valid for $1 \leq \alpha$ and $s > 1$ arbitrary for the Romieu-case and the Beurling-case: Therefore note that (3.4.4) is satisfied and so we can use 3.4.9 in the Beurling-case.

But if $\alpha < 1$, then we have $\frac{f_\alpha(p+1)}{f_\alpha(p)} = \alpha < 1$. Thus in this case the sequence doesn't satisfy the weak log. convexity condition for any parameter $s > 1$.

closedness under solving ODE's:

As we have seen $M_0 = M_1 = 1$ and furthermore we have $M_2 \geq 2$ if and only if $2^{s-1} \cdot \alpha \geq 1$ which is for example satisfied for any $s > 1$ and $\alpha \geq 1$. Furthermore for all $s > 1$ and $\alpha \geq 1$ the weight sequence is strong log. convex, hence (3.4.2) and so (3.4.7) are satisfied. Thus for $s > 1$ and $\alpha \geq 1$ the classes $\mathcal{G}_{\{s, f_\alpha\}}$ are closed under solving ODE's by 3.4.1 and because for $s > 1$ and $\alpha \geq 1$ (3.4.4) is satisfied we have by 3.4.8 the closedness under solving ODE's for $\mathcal{G}_{(s, f_\alpha)}$, too.

We want to compare the new spaces $\mathcal{G}_{\{s, f_\alpha\}}$ and $\mathcal{G}_{\{t, f_\alpha\}}$ resp. $\mathcal{G}_{(s, f_\alpha)}$ and $\mathcal{G}_{(t, f_\alpha)}$. Therefore we have:

$$\sup_p \underbrace{\left(p!^{s-t} \right)^{1/p}}_{\sim p^{s-t}} \cdot \left(\frac{\alpha}{\beta} \right)^{(p-1)/2} < \infty \implies \mathcal{G}_{\{s, f_\alpha\}} \subseteq \mathcal{G}_{\{t, f_\beta\}} \text{ resp. } \mathcal{G}_{(s, f_\alpha)} \subseteq \mathcal{G}_{(t, f_\beta)}$$

The supremum on the left hand side above is bounded if and only if $\beta > \alpha$ and $t, s > 1$ arbitrary or if $\beta = \alpha$ and $t \geq s$ is satisfied. If the sequence $(p!^s \cdot \alpha^{p(p-1)/2})_p$ is log. convex, in particular if $\alpha \geq 1$, then we obtain by 3.1.3 in the Romieu-case also $\Leftarrow$. We summarize: $\mathcal{G}_{\{s, f_\alpha\}} = \mathcal{G}_{\{t, f_\beta\}} \Leftrightarrow 1 \leq \alpha < \beta, t, s > 1, \text{ or } 1 \leq \alpha = \beta, t \geq s$.

Using (3.1.1) we get, that $\mathcal{G}_{\{s, f_\alpha\}}$ resp. $\mathcal{G}_{(s, f_\alpha)}$ is closed under differentiation for all $s > 1$ and $\alpha \in \mathbb{R}_{>0}$, because:

$$\sup_p \left((p+1)^s \cdot f_\alpha(p+1) \right)^{1/p} < \infty \Leftrightarrow \sup_p \left((p+1)^s \cdot \alpha^p \right)^{1/p} < \infty \Leftrightarrow \alpha < \infty.$$

Furthermore we see by 2.0.8: $\mathcal{G}_{\{s, f_\alpha\}}$ and $\mathcal{G}_{(s, f_\alpha)}$ are differential algebras for $s > 1$ and $\alpha \geq 1$.

Because

$$\sup_p m_p^{1/p} < \infty \Leftrightarrow \sup_p \left(p^{s-1} \cdot \alpha^{(p-1)/2} \right) < \infty \Leftrightarrow \alpha < 1$$

we see by (3.1.1): $\mathcal{G}_{\{s, f_\alpha\}} \subseteq \mathcal{O}$ if $\alpha < 1$. On the other side one has

$$\sup_p \left(\frac{1}{m_p} \right)^{1/p} < \infty \Leftrightarrow \sup_p \left(p^{1-s} \cdot \alpha^{-(p-1)/2} \right) < \infty \Leftrightarrow \alpha \geq 1,$$

hence $\mathcal{O} \subseteq \mathcal{G}_{\{s, f_\alpha\}}$ if $\alpha \geq 1$. Note that in this case the weight sequences are strong log. convex and so by 3.1.5 we get for $\alpha \geq 1$: $\mathcal{O} \subsetneq \mathcal{G}_{\{s, f_\alpha\}}$.

In the following we want to study the dependence of the characterizing properties of the Borel-map on the function f . We distinguish now two cases for the parameter α : $0 < \alpha < 1$ and $1 < \alpha$.

Case 1: $1 < \alpha$:

As we have seen, in this case the weight sequences are strong log. convex for all $s > 1$. First we have

$$\sum_{p=1}^{\infty} \frac{1}{p^s \cdot f_\alpha(p)} = \sum_{p=1}^{\infty} \frac{1}{p^s \cdot \alpha^{p-1}} \leq \sum_{p=1}^{\infty} \frac{1}{p^s} < \infty, \forall s > 1.$$

Furthermore

$$(\beta_2^0) : \lim_{p \rightarrow \infty} \frac{f_\alpha(kp)}{f_\alpha(p)} = \lim_{p \rightarrow \infty} \underbrace{\frac{\alpha^{kp-1}}{\alpha^{p-1}}}_{\alpha^{p(k-1)}} = \infty, \text{ for } k \in \mathbb{N}, k > 1$$

holds, hence condition (β_2^0) is satisfied and so (β_1) and (β_2) .

We summarize: The spaces $\mathcal{G}_{\{s, f_\alpha\}}$, where $1 < \alpha$ and $s > 1$ arbitrary, are not quasi-analytic. In particular the strong non-quasi-analyticity condition is satisfied, because by 5.1.1 $(\beta_1) \Leftrightarrow (\gamma_1)$ holds. Hence, by 5.1.8, the spaces $\mathcal{G}_{(s, f_\alpha)}$ are not quasi-analytic for arbitrary $s > 1$ and $1 < \alpha$, too.

Furthermore we see for $\alpha > 1$ and $s > 1$: If $\mathcal{G}_{(s, f_\alpha)} \subseteq \mathcal{G}_{(t, f_\beta)}$ holds, then by 5.2.2, $\sup_p p^{s-t} \cdot (\frac{\alpha}{\beta})^{(p-1)/2} < \infty$ has to be satisfied.

Case 2: $0 < \alpha < 1$:

As stated above, the weight sequence is *not weak logarithmic convex*, so we cannot apply any of our theorems and we loose a lot of information! But since we have shown $\mathcal{G}_{\{s, f_\alpha\}} \subseteq \mathcal{O}$ in the case where $\alpha < 1$ we obtain the quasi-analyticity for $\mathcal{G}_{\{s, f_\alpha\}}$ (hence for $\mathcal{G}_{(s, f_\alpha)}$) for $0 < \alpha < 1$ and all $s > 1$: The Borel-map is injective for $\mathcal{O}$, hence we can restrict it.

The following tabular shall summarize and illustrate our results for the function spaces $\mathcal{G}_{\{s, f_\alpha\}}$ resp. $\mathcal{G}_{(s, f_\alpha)}$, where $\alpha \in \mathbb{R}_{>0}$ is the important parameter and $s > 1$ arbitrary. E denotes the Extension-operator, injectivity and surjectivity are related to the Borel-map, defined on $\mathcal{G}_{\{s, f_\alpha\}}$.

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Parameter	inj.	surj.	E for $\{\}$	E for $()$	mod. growth	strong reg.
$0 < \alpha < 1$	yes	—	—	—	yes	—
$\alpha = 1$	no	yes	no	yes	yes	yes
$\alpha > 1$	no	yes	yes	yes	no	no

Finally we can say: If we want to apply our strong characterizing results we have to study the important properties of weight sequences in detail because they depend essentially on given parameters.

7 A decomposition theorem

In this chapter we are going to prove a decomposition theorem for smooth functions on the compact interval $[-1, 1]$ which is due to S. Mandelbrojt (see [18]):

Theorem 7.0.1 *Let $f \in \mathcal{E}([-1, 1])$ be given, then there exist weight sequences $M^1 := (M_p^1)_p$ and $M^2 := (M_p^2)_p$ such that the spaces $\mathcal{E}_{\{M^1\}}([-1, 1])$ and $\mathcal{E}_{\{M^2\}}([-1, 1])$ are both quasi-analytic and such that $f = g + h$ holds for certain $g \in \mathcal{E}_{\{M^1\}}([-1, 1])$ and $h \in \mathcal{E}_{\{M^2\}}([-1, 1])$, where the sequences M^1 and M^2 depend on the function f .*

Proof. Before starting the proof of the theorem, we have to study the *Tchebychev-polynomials*, which form a basis for $\mathcal{E}([-1, 1])$, in detail: We denote with

$$T_n(x) := \cos(n \cdot \arccos(x))$$

the n -th *Tchebychev-polynomial*, which is a polynomial of degree n and $|T_n(x)| \leq 1$, for all $x \in [-1, 1]$ and all $n \in \mathbb{N}$. Each $f \in \mathcal{E}([-1, 1])$ can be written uniquely in the form:

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot T_n(x), \quad (7.0.1)$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ (see [21, 29.5 Beispiel(4), p. 343-344]). Note that this series and all its derivatives are absolutely convergent on $[-1, 1]$. The strategy of the proof of 7.0.1 is that we will decompose the sequence $(a_n)_n$ appearing in (7.0.1).

First, for $r \geq 1$, we define the following function:

$$S(r) := \max_{1 \leq n \leq r} \frac{r^n}{\|f^{(n)}\|_{\infty}}.$$

Therefore we remark: If there exists $N \in \mathbb{N}$ such that $\|f^{(n)}\|_{\infty} = 0$ holds for all $n \geq N$, then f is a polynomial!

Claim: We show now that there exists a constant $A > 2$ such that for all integers $n > A$ we have

$$|a_n| < \frac{1}{S(n/A)}. \quad (7.0.2)$$

If we take $x = \cos(\theta)$ we obtain:

$$\varphi(\theta) := f(\cos(\theta)) = \sum_{n=0}^{\infty} a_n \cdot \cos(n\theta) \quad \text{and} \quad \varphi_p(\theta) := f^{(p)}(\cos(\theta)) = \sum_{n=0}^{\infty} a_n^{(p)} \cdot \cos(n\theta),$$

where $f^{(p)}$ denotes the p -th derivative. From this we see

$$\varphi_p'(\theta) = -\sin(\theta) \cdot f^{(p+1)}(\cos(\theta)) = -\sin(\theta) \cdot \varphi_{p+1}(\theta). \quad (7.0.3)$$

Hence

$$\begin{aligned} \underbrace{\sum_{n=1}^{\infty} a_n^{(p)} \cdot n \cdot \sin(n\theta)}_{-\varphi_p'(\theta)} &= \sin(\theta) \cdot \underbrace{\sum_{n=0}^{\infty} a_n^{(p+1)} \cdot \cos(n\theta)}_{\varphi_{(p+1)}(\theta)} \\ &\stackrel{(*)}{=} \frac{1}{2} \cdot \sum_{n=1}^{\infty} \left(a_{n-1}^{(p+1)} - a_{n+1}^{(p+1)} \right) \cdot \sin(n\theta) + \frac{1}{2} \cdot a_0^{(p+1)} \cdot \sin(\theta). \end{aligned}$$

(*) holds, because

$$-\frac{1}{2} \cdot \sin((n-1)\theta) + \frac{1}{2} \cdot \sin((n+1)\theta) = \sin(\theta) \cdot \cos(n\theta).$$

From this we get for all $n \geq 2$:

$$a_n^{(p)} = \frac{a_{n-1}^{(p+1)} - a_{n+1}^{(p+1)}}{2n}. \quad (7.0.4)$$

To obtain the $a_n^{(p)}$ we can also calculate as follows:

$$\begin{aligned} \frac{1}{\pi} \cdot \int_0^{2\pi} \varphi_p(\theta) \cdot \cos(n\theta) d\theta &= \frac{1}{\pi} \cdot \int_0^{2\pi} \left(\sum_{l=0}^{\infty} a_l^{(p)} \cdot \cos(l\theta) \right) \cdot \cos(n\theta) d\theta \\ &= \sum_{l=0}^{\infty} a_l^{(p)} \cdot \underbrace{\left(\frac{1}{\pi} \cdot \int_0^{2\pi} \cos(l\theta) \cdot \cos(n\theta) d\theta \right)}_{=\delta_{l,n}} = a_n^{(p)}. \end{aligned}$$

On the other side we can use integration by parts and (7.0.3) to obtain:

$$\begin{aligned} a_n^{(p)} &= \frac{1}{\pi} \cdot \int_0^{2\pi} \varphi_p(\theta) \cdot \cos(n\theta) d\theta = \underbrace{\left[\varphi_p(\theta) \cdot \frac{1}{n} \cdot \sin(n\theta) \right]_0^{2\pi}}_{=0} - \frac{1}{\pi n} \cdot \int_0^{2\pi} \varphi_p'(\theta) \cdot \sin(n\theta) d\theta \\ &= \frac{1}{\pi n} \cdot \int_0^{2\pi} \sin(\theta) \cdot f^{(p+1)}(\cos(\theta)) \cdot \sin(n\theta) d\theta, \end{aligned}$$

for all $n \geq 1$. Now we can estimate the coefficients as follows:

$$\begin{aligned} |a_n^{(p)}| &= \frac{1}{\pi n} \cdot \left| \int_0^{2\pi} \sin(\theta) \cdot f^{(p+1)}(\cos(\theta)) \cdot \sin(n\theta) d\theta \right| \leq \frac{1}{\pi n} \cdot \int_0^{2\pi} \|f^{(p+1)}\|_{\infty} d\theta \\ &= \frac{2 \cdot \|f^{(p+1)}\|_{\infty}}{n}. \end{aligned}$$

We summarize:

$$|a_n^{(p)}| \leq \frac{2 \cdot \|f^{(p+1)}\|_{\infty}}{n}, \quad \forall n \geq 1. \quad (7.0.5)$$

Hence, if $n > 2$ and $1 \leq p \leq n-1$:

$$\begin{aligned} |a_{n-p+1}^{(p-1)}| &\stackrel{(7.0.4)}{\leq} \frac{|a_{n-p}^{(p)}| + |a_{n-p+2}^{(p)}|}{2 \cdot (n-p+1)} \stackrel{(7.0.5)}{\leq} \frac{\|f^{(p+1)}\|_{\infty}}{n-p+1} \cdot \underbrace{\left(\frac{1}{n-p} + \frac{1}{n-p+2} \right)}_{\frac{2(n-p+1)}{(n-p)(n-p+2)}} \\ &\leq \frac{2 \cdot \|f^{(p+1)}\|_{\infty}}{(n-p) \cdot (n-p+1)}. \end{aligned}$$

The last inequality holds because $\frac{1}{n-p+2} < \frac{1}{n-p+1}$. If we iterate this estimation and use always (7.0.4), finally we obtain:

$$\begin{aligned} |a_n| &\leq 2 \cdot \|f^{(p+1)}\|_{\infty} \cdot \frac{1}{(n-p) \cdot (n-p+1) \cdots n} = 2 \cdot \|f^{(p+1)}\|_{\infty} \cdot \frac{(n-p-1)!}{n!} \\ &\stackrel{(**)}{\leq} \frac{2 \cdot B \cdot e^{p+1}}{n^{p+1}} \cdot \|f^{(p+1)}\|_{\infty} \leq \frac{A^{p+1} \cdot \|f^{(p+1)}\|_{\infty}}{n^{p+1}} \quad \text{for } 1 \leq p \leq n-1, \end{aligned}$$

for a constant A , which depends on the constant B . (**) above holds for a constant B , because by *Stirling's formula* we know, that $n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$, hence

$$\frac{(n-p-1)!}{n!} \approx \underbrace{\frac{\sqrt{n-p-1}}{\sqrt{n}}}_{<1} \cdot \underbrace{\frac{(n-p-1)^{n-p-1}}{n^n}}_{< n^{n-p-1}/n^n} \cdot e^{p+1} < \frac{e^{p+1}}{n^{p+1}}.$$

The estimate above shows in fact $|a_n| \leq \min_{2 \leq p \leq n} \frac{A^p \cdot \|f^{(p)}\|_\infty}{n^p}$. For $p = 1$ we obtain $|a_n| \leq \frac{2 \cdot \|f'\|_\infty}{n} \leq \frac{A \cdot \|f'\|_\infty}{n}$, because

$$|a_n| = \frac{1}{\pi n} \cdot \left| \int_0^{2\pi} \sin(\theta) \cdot f'(\cos(\theta)) \cdot \sin(n\theta) d\theta \right| \leq \frac{1}{\pi n} \cdot \|f'\|_\infty \cdot 2\pi.$$

Now we are able to prove (7.0.2):

$$|a_n| \leq \min_{1 \leq p \leq n} \frac{A^p \cdot \|f^{(p)}\|_\infty}{n^p} \leq \min_{1 \leq p \leq (n/A)} \frac{A^p \cdot \|f^{(p)}\|_\infty}{n^p} \underbrace{=}_{\text{def. of } S} \frac{1}{S(n/A)}, \quad (7.0.6)$$

thus the claim is proven.

In the following we study the properties of the function $S : r \mapsto S(r)$. First we remark that one can generalize in the definition of S the $\|f^{(n)}\|_\infty$ to an arbitrary sequence $(M_n)_n$, where $M_n > 0$, $n \in \mathbb{N}$:

$$S(r) := \max_{1 \leq n \leq r} \frac{r^n}{M_n}.$$

We set $n(r) \in \mathbb{N}$ to be the greatest integer $n \leq r$, such that $S(r) = \frac{r^n}{M_n}$ holds. It follows that $S(r) = \frac{r^{n(r)}}{M_{n(r)}}$, and $n(r) \leq r$, and S is an increasing function in r . The function $n : r \mapsto n(r)$ is also increasing in r , because suppose for some $r_1 > r$, we would have $n(r_1) < n(r)$, then:

$$r_1^{n(r)-n(r_1)} > r^{n(r)-n(r_1)} \geq \frac{M_{n(r)}}{M_{n(r_1)}}. \quad (***)$$

(**) holds because of the definition of $S(r)$ and $n(r)$ and because $n(r_1) < n(r)$. Hence

$$\frac{r_1^{n(r)}}{M_{n(r)}} > \frac{r_1^{n(r_1)}}{M_{n(r_1)}} = S(r_1), \quad \text{where } n(r) \leq r < r_1,$$

and this is a contradiction to the definition of $S(r_1)$.

For $r \rightarrow \infty$ it follows that $n(r) \rightarrow \infty$: We take $r > \max\left(\frac{M_2}{M_1}, \frac{M_3}{M_2}, \dots, \frac{M_p}{M_{p-1}}, p\right)$, so $r > \frac{M_i}{M_{i-1}}$ for $2 \leq i \leq p$. Note that for all $p \in \mathbb{N}$, $p > n$, we have

$$\frac{M_p}{M_n} = \underbrace{\frac{M_{n+1}}{M_n}}_{< r} \dots \underbrace{\frac{M_p}{M_{p-1}}}_{< r} \leq r^{p-n},$$

which implies $\frac{r^n}{M_n} \leq \frac{r^p}{M_p}$. So, because $p < r$ holds, we obtain by definition $n(r) \geq p$.

In the following we put $s(r) := \log(S(r)) = n(r) \cdot \log(r) - \log(M_{n(r)})$. The image of the function n consists of positive integers, which are increasing in r . So we see: In a given closed interval $[1, R]$, $R > 1$, the function s is increasing and there are only a finite number of points β_i , $i = 1, \dots, l_R$, where s is not continuous, in particular where it is not differentiable. But s is continuous from the right at each point, which means $s(r) = s^+(r) := \lim_{t \searrow 0} s(r+t)$ for all r and it has a derivative from the right at each point, too: $s'(r) := \lim_{t \searrow 0} \frac{s(r+t) - s(r)}{t} = \frac{n(r)}{r}$.

Let $1 \leq r_0 < r_1 < \beta_1$, then s is $\mathcal{C}^1$ on $[r_0, r_1]$ and $s(r_1) - s(r_0) = \int_{r_0}^{r_1} s'(\tau) d\tau$ holds. We set $s^-(r) := \lim_{t \searrow 0} s(r-t)$, then $s^-(\beta_1) - s(r_0) = \int_{r_0}^{\beta_1} s'(\tau) d\tau$, hence

$$s(\beta_1) - s(r_0) = \int_{r_0}^{\beta_1} s'(\tau) d\tau + \underbrace{s(\beta_1) - s^-(\beta_1)}_{c_1}.$$

Note that s is continuous from the right at each point, so $s^+(\beta_1) = s(\beta_1)$, and $c_1 > 0$ denotes the height of the jump at the point β_1 . Similarly we calculate for all points β_i to obtain for $1 \leq r_0 < r$

$$s(r) - s(r_0) = \int_{r_0}^r \frac{n(t)}{t} dt + \sum_{i=1}^{l_r} c_i, \quad (7.0.7)$$

where $c_i > 0$ denotes the height of the jump of s at β_i and the sum is taken over all i such that $\beta_i \in [r_0, r]$.

Proof of 7.0.1.

Now we can start with the proof of the theorem! First we choose $r_0 \in \mathbb{R}$, $r_0 \geq 1$, such that $n(r_0) \geq 2$ and $s(r_0) > 0$. We set $m(r) := n(r) - 2 \geq 0$ for $r \geq r_0$ and $m(r) = 0$ for $r < r_0$. Hence for $r \geq r_0$ one obtains

$$\begin{aligned} s(r) &\stackrel{(7.0.7)}{\geq} s(r_0) + \int_{r_0}^r \frac{n(t)}{t} dt \stackrel{\text{definition of } m(t)}{=} s(r_0) + \int_{r_0}^r \frac{2}{t} dt + \int_0^r \frac{m(t)}{t} dt \\ &\stackrel{s(r_0) > 0}{\geq} 2 \cdot \log\left(\frac{r}{r_0}\right) + \int_0^r \frac{m(t)}{t} dt \stackrel{t \mapsto t^{1/2}}{=} 2 \cdot \log\left(\frac{r}{r_0}\right) + \int_0^{r^2} \frac{m(t^{1/2})}{2t} dt. \end{aligned}$$

We define $N(t) := \left\lfloor \frac{m(t^{1/2})}{2} \right\rfloor$, where $[x]$ is defined to be the largest integer, which is less or equal to x . For $r \geq r_0$:

$$s(r) \geq 2 \cdot \log\left(\frac{r}{r_0}\right) + \int_0^{r^2} \frac{N(t)}{t} dt, \quad (7.0.8)$$

because $[x] \leq x$ for all $x \in \mathbb{R}$. The image of the function $N : t \mapsto N(t)$ consists of positive integers, which are increasing with t . Furthermore we have that N vanishes locally around the point zero, because $m(r) = 0$ for $r < r_0$, and $N(t) \rightarrow \infty$ for $t \rightarrow \infty$, because $n(t) \rightarrow \infty$ for $t \rightarrow \infty$, hence $m(t) \rightarrow \infty$ for $t \rightarrow \infty$.

In the next step we consider a sequence $(t_q)_q$ of increasing real numbers with the following properties, where $a > 1$ is a constant:

$$t_0 = 0, \quad t_q > a \cdot t_{q-1} \quad \text{and} \quad \left\lfloor \frac{\int_{t_{q-1}}^{t_q} \frac{N(t)}{t} dt}{\log(t_q) - \log(t_{q-1})} \right\rfloor \geq N(t_{q-1}) \quad \forall q \geq 1. \quad (7.0.9)$$

Note that this choice is possible, because the function N is increasing and $N(t) \rightarrow \infty$ for $t \rightarrow \infty$. More precisely we have: Let φ be a positive increasing function such that $\varphi(x) \rightarrow \infty$ for $x \rightarrow \infty$. Then we choose a sufficient large λ and $\frac{\log(\lambda)}{\log(\beta)}$ sufficient small to obtain for $\beta > \lambda \cdot \alpha > 0$:

$$\frac{\int_{\alpha}^{\beta} \frac{\varphi(t)}{t} dt}{\log(\beta) - \log(\alpha)} > \frac{\int_{\lambda \cdot \alpha}^{\beta} \frac{\varphi(t)}{t} dt}{\log(\beta) - \log(\alpha)} \stackrel{\varphi \text{ incr.}}{\geq} \varphi(\lambda \cdot \alpha) \cdot \underbrace{\frac{\log(\beta) - \log(\alpha) - \log(\lambda)}{\log(\beta) - \log(\alpha)}}_{=1 - \frac{\log(\lambda)}{\log(\beta) - \log(\alpha)}}.$$

Now let $(N_q)_q$ be defined by:

$$N_q := \left\lfloor \frac{\int_{t_{q-1}}^{t_q} \frac{N(t)}{t} dt}{\log(t_q) - \log(t_{q-1})} \right\rfloor \quad (7.0.10)$$

and so we get by definition:

$$N(t_{q-1}) \leq N_q \leq N(t_q) \quad \text{for all } q \geq 1. \quad (7.0.11)$$

Using the introduced sequences $(t_q)_q$ and $(N_q)_q$, we can define two functions N_1 and N_2 in the following way:

$$N_1(r) := \begin{cases} N(r) & \text{for } t_{2q} \leq r < t_{2q+1} \\ N_{2(q+1)} & \text{for } t_{2q+1} \leq r < t_{2(q+1)} \end{cases} \quad \text{for all } q \geq 0, \quad (7.0.12)$$

and

$$N_2(r) := \begin{cases} N(r) & \text{for } t_{2q+1} \leq r < t_{2(q+1)} \text{ for all } q \geq 0 \text{ and } t_0 = 0 \leq r < t_1 \\ N_{2q+1} & \text{for } t_{2q} \leq r < t_{2q+1} \text{ for all } q \geq 1. \end{cases} \quad (7.0.13)$$

Finally we define the following three important functions T, T_1^* and T_2^* :

$$\begin{cases} \log(T(r)) := \int_0^r \frac{N(t)}{t} dt \\ \log(T_1^*(r)) := \int_0^r \frac{N_1(t)}{t} dt \\ \log(T_2^*(r)) := \int_0^r \frac{N_2(t)}{t} dt. \end{cases} \quad (7.0.14)$$

By the definitions above and (7.0.11) we remark: $N_1(t)$ and $N_2(t)$ are increasing in t , tending to infinity for $t \rightarrow \infty$. The images of both functions consist of integers with ≥ 0 . Furthermore we remark that all three integrals defined in (7.0.14) are finite for $r < \infty$: Recall that the function N vanishes near zero, hence N_1 and N_2 by construction, too.

Claim: On every interval $[t_{2q}, t_{2q+1})$ where $q \geq 0$ we have $T_1^*(r) \leq T(r)$ for all $r \in [t_{2q}, t_{2q+1})$ and on $[t_{2q+1}, t_{2(q+1)})$, $q \geq 0$, we have $T_2^*(r) \leq T(r)$ for all $r \in [t_{2q+1}, t_{2(q+1)})$. We prove this for the function T_1^* . If $t_{2q} \leq r < t_{2q+1}$, then:

$$\begin{aligned} \log(T_1^*(r)) &= \int_0^r \frac{N_1(t)}{t} dt \stackrel{(7.0.12)}{=} \sum_{j=0}^{q-1} \int_{t_{2j}}^{t_{2j+1}} \frac{N(t)}{t} dt + \underbrace{\sum_{j=0}^{q-1} \int_{t_{2j+1}}^{t_{2(j+1)}} \frac{N_{2(j+1)}}{t} dt}_{= N_{2(j+1)} \cdot \log\left(\frac{t_{2(j+1)}}{t_{2j+1}}\right)} + \int_{t_{2q}}^r \frac{N(t)}{t} dt. \end{aligned}$$

By (7.0.10) we have: $N_{2p} \cdot \log\left(\frac{t_{2p}}{t_{2p-1}}\right) \leq \int_{t_{2p-1}}^{t_{2p}} \frac{N(t)}{t} dt$, for $1 \leq p \leq q$. Hence

$$\log(T_1^*(r)) \leq \int_0^r \frac{N(t)}{t} dt = \log(T(r)) \implies T_1^*(r) \leq T(r).$$

For T_2^* we can imitate the proof above and the claim follows.

The next step in the proof is to show

$$\int_1^\infty \frac{\log(T_1^*(r))}{r^2} dr = \int_1^\infty \frac{\log(T_2^*(r))}{r^2} dr = \infty. \quad (7.0.15)$$

We restrict to the first integral. Let $t_{2q-1} \leq r < t_{2q}$, then:

$$\begin{aligned} \log(T_1^*(r)) &= \int_0^r \frac{N_1(t)}{t} dt \underset{N_1(t) \geq 0}{\geq} \int_{t_{2q-1}}^r \frac{N_1(t)}{t} dt \underset{N_1 \text{ increasing}}{\geq} N_1(t_{2q-1}) \cdot \log\left(\frac{r}{t_{2q-1}}\right) \\ &\stackrel{(7.0.12)}{=} N_{2q} \cdot \log\left(\frac{r}{t_{2q-1}}\right). \end{aligned}$$

We set $\tau := \frac{r}{t_{2q-1}}$ and use the estimate above to obtain:

$$\int_{t_{2q-1}}^{t_{2q}} \frac{\log(T_1^*(r))}{r^2} dr \geq N_{2q} \cdot \int_{t_{2q-1}}^{t_{2q}} \frac{\log\left(\frac{r}{t_{2q-1}}\right)}{r^2} dr = \frac{N_{2q}}{t_{2q-1}} \cdot \int_1^{\frac{t_{2q}}{t_{2q-1}}} \frac{\log(\tau)}{\tau^2} d\tau \stackrel{(7.0.9)}{=} b > 0,$$

where b is a constant. Again we can imitate the proof for T_2^* .

This is the crucial point in this proof: We define now two smooth functions g and h , which will decompose $f = \sum_{n=0}^{\infty} a_n \cdot T_n$. Using the sequence $(t_q)_q$, which was constructed in (7.0.9), let $(b_n)_n$ and $(c_n)_n$ be the two sequences defined by:

$$b_n := \begin{cases} a_n & \text{for } A \cdot \sqrt{t_{2q}} \leq n < A \cdot \sqrt{t_{2q+1}} \\ 0 & \text{for } A \cdot \sqrt{t_{2q+1}} \leq n < A \cdot \sqrt{t_{2(q+1)}} \end{cases} \quad \text{for all } q \geq 0, \quad (7.0.16)$$

$$c_n := \begin{cases} a_n & \text{for } A \cdot \sqrt{t_{2q+1}} \leq n < A \cdot \sqrt{t_{2(q+1)}} \\ 0 & \text{for } A \cdot \sqrt{t_{2q}} \leq n < A \cdot \sqrt{t_{2q+1}} \end{cases} \quad \text{for all } q \geq 0, \quad (7.0.17)$$

Then we define the functions g and h using the introduced sequences above:

$$g(x) := \sum_{n=0}^{\infty} b_n \cdot T_n(x), \quad h(x) := \sum_{n=0}^{\infty} c_n \cdot T_n(x) \quad (7.0.18)$$

It follows immediately that $f(x) = g(x) + h(x)$ for all $x \in [-1, 1]$.

Claim: For $i \in \{1, 2\}$ there exist weight sequences $M^i := (M_p^i)_p$ such that $g \in \mathcal{E}_{M^1}$, $h \in \mathcal{E}_{M^2}$ and $\mathcal{E}_{M^i}$ are quasi-analytic for $i \in \{1, 2\}$. We prove this for g :

First we remember that on (t_{2q}, t_{2q+1}) we have $T_1^*(r) \leq T(r)$ and remark that we obtain by (7.0.16):

$$b_n = a_n \Leftrightarrow \sqrt{t_{2q}} \leq \frac{n}{A} < \sqrt{t_{2q+1}} \Rightarrow t_{2q} \leq \frac{n^2}{A^2} < t_{2q+1}. \quad (7.0.19)$$

We can estimate the b_n for $n > A$ as follows:

$$|b_n| \stackrel{(7.0.2)}{\leq} \frac{1}{S(n/A)} \stackrel{(7.0.8), (7.0.14)}{\leq} \left(\frac{A \cdot r_0}{n} \right)^2 \cdot \frac{1}{T(n^2/A^2)} \stackrel{(7.0.19)}{\leq} \left(\frac{A \cdot r_0}{n} \right)^2 \cdot \frac{1}{T_1^*(n^2/A^2)},$$

thus for $n > A$ we have

$$|b_n| \leq \left(\frac{A \cdot r_0}{n} \right)^2 \cdot \frac{1}{T_1^*(n^2/A^2)}. \quad (7.0.20)$$

Furthermore we use the following property of the Tchebychev-polynomials: By [19, formula (6.2.4), p. 206] we have $T_n^{(p)}(1) \leq \frac{1}{2} \cdot \left(\frac{2}{p} \right)^p \cdot n^{2p}$ for $1 \leq p \leq n$ and by [19, formula (6.4.1), p. 217] we have $|T_n^{(p)}(x)| \leq 2^p \cdot T_n^{(p)}(1)$ for all $x \in [-1, 1]$ and $0 \leq p \leq n$. Thus on $[-1, 1]$ and for $1 \leq p \leq n$ we can estimate the Tchebychev-polynomials as follows:

$$|T_n^{(p)}(x)| \leq \frac{1}{2} \cdot \left(\frac{4}{p} \right)^p \cdot n^{2p}. \quad (7.0.21)$$

So for $1 \leq p \leq n$ and $A < n$ we obtain the following estimate:

$$\begin{aligned} |g^{(p)}(x)| &\leq \sum_{n \geq p} |b_n| \cdot |T_n^{(p)}(x)| \stackrel{(7.0.20), (7.0.21)}{\leq} (A \cdot r_0)^2 \cdot \frac{1}{2} \cdot \left(\frac{4}{p} \right)^p \cdot \sum_{n \geq p} \frac{n^{2p}}{n^2 \cdot T_1^*(n^2/A^2)} \\ &\leq (A \cdot r_0)^2 \cdot \frac{1}{2} \cdot \left(\frac{4}{p} \right)^p \cdot \left(\sum_{n \geq p} \frac{1}{n^2} \right) \cdot \sup_{n \geq p} \frac{n^{2p}}{T_1^*(n^2/A^2)} \leq C^p \cdot \sup_{n \geq p} \frac{(n/A)^{2p}}{T_1^*((n/A)^2)} \\ &\leq C^p \cdot \sup_{r \geq 0} \frac{r^p}{T_1^*(r)} =: C^p \cdot M_p^1, \end{aligned}$$

where $C > 0$ is a constant. The last inequality is valid because for $r \geq 0$ we can insert in the last supremum more values. Note that this estimate above in fact shows $|g^{(p)}(x)| \leq C^p \cdot M_p^1$

and it is only valid for $p > A$: Because for $n \in \mathbb{N}$, such that $n \geq p$, we have used (7.0.2). But if we choose a bigger C it is also true for $1 \leq p \leq A$ (this is only a finite number of possible p 's).

Claim:

$$M_p^1 := \sup_{r \geq 0} \frac{r^p}{T_1^*(r)} \quad \forall p \geq 1 \quad (7.0.22)$$

implies

$$T_1^*(r) = \sup_{p \geq 1} \frac{r^p}{M_p^1} \quad (7.0.23)$$

First remark that $\sup_{q \geq 1} \frac{r^q}{M_q^1} \leq T_1^*(r)$ follows from (7.0.22), because this equation holds for all $p \geq 1$. Because $N_1(r) \in \mathbb{N}$ we can write:

$$\sup_{q \geq 1} \frac{r^q}{M_q^1} = \sup_{q \geq 1} \frac{r^q}{\sup_{\varrho \geq 0} \frac{\varrho^q}{T_1^*(\varrho)}} \geq \frac{r^{N_1(r)}}{\sup_{\varrho \geq 0} \frac{\varrho^{N_1(r)}}{T_1^*(\varrho)}}. \quad (7.0.24)$$

For $\varrho_1 < \varrho_2$ we have

$$\begin{aligned} \underbrace{\log\left(\frac{\varrho_2^{N_1(r)}}{T_1^*(\varrho_2)}\right) - \log\left(\frac{\varrho_1^{N_1(r)}}{T_1^*(\varrho_1)}\right)}_{=:\log(\varrho_1, \varrho_2)} &\stackrel{(7.0.14)}{=} N_1(r) \cdot \log(\varrho_2) - \int_0^{\varrho_2} \frac{N_1(t)}{t} dt \\ &\quad - \left(N_1(r) \cdot \log(\varrho_1) - \int_0^{\varrho_1} \frac{N_1(t)}{t} dt \right) \\ &= N_1(r) \cdot \log\left(\frac{\varrho_2}{\varrho_1}\right) - \int_{\varrho_1}^{\varrho_2} \frac{N_1(t)}{t} dt. \end{aligned}$$

We estimate the last term above ($N_1(r)$ is positive and increasing):

$$N_1(\varrho_1) \cdot \log\left(\frac{\varrho_2}{\varrho_1}\right) = N_1(\varrho_1) \cdot \int_{\varrho_1}^{\varrho_2} \frac{1}{t} dt \leq \int_{\varrho_1}^{\varrho_2} \frac{N_1(t)}{t} dt \leq N_1(\varrho_2) \cdot \int_{\varrho_1}^{\varrho_2} \frac{1}{t} dt = N_1(\varrho_2) \cdot \log\left(\frac{\varrho_2}{\varrho_1}\right).$$

Hence

$$\left(N_1(r) - N_1(\varrho_2) \right) \cdot \log\left(\frac{\varrho_2}{\varrho_1}\right) \leq \log(\varrho_1, \varrho_2) \leq \left(N_1(r) - N_1(\varrho_1) \right) \cdot \log\left(\frac{\varrho_2}{\varrho_1}\right).$$

So we can say that $\frac{\varrho^{N_1(r)}}{T_1^*(\varrho)}$ and both expressions above stay constant for $N_1(\varrho) = N_1(r)$, they increase for $N_1(\varrho) < N_1(r)$ and decrease for $N_1(\varrho) > N_1(r)$. Hence

$$\sup_{\varrho \geq 0} \frac{\varrho^{N_1(r)}}{T_1^*(\varrho)} = \frac{r^{N_1(r)}}{T_1^*(r)}, \quad (7.0.25)$$

and together with (7.0.24) we obtain:

$$\sup_{q \geq 1} \frac{r^q}{M_q^1} \geq \frac{r^{N_1(r)}}{\frac{r^{N_1(r)}}{T_1^*(r)}} = T_1^*(r),$$

which shows (7.0.23) and the claim is proven.

Claim: For $i \in \{1, 2\}$ the weight sequences $(M_p^i)_p$ are quasi-analytic. We show this for M^1 : Let $p > 1$ and if $r_2 > r_1 > 0$ then we have

$$(p - N_1(r_2)) \cdot \log\left(\frac{r_2}{r_1}\right) \leq \log\left(\frac{r_2^p}{T_1^*(r_2)}\right) - \log\left(\frac{r_1^p}{T_1^*(r_1)}\right) \leq (p - N_1(r_1)) \cdot \log\left(\frac{r_2}{r_1}\right).$$

Now we order the values under the function N_1 , such that they are increasing: $n_1 < n_2 < n_3 < \dots$. If $n_i \leq p < n_{i+1}$, then

$$M_p^1 = \frac{\tilde{r}^p}{T_1^*(\tilde{r})},$$

where $\tilde{r}$ is the smallest r such that $N_1(r) = p$ (because of (7.0.25)). Hence, for $n_i \leq p < p+1 \leq n_{i+1}$:

$$\log(M_{p+1}^1) - \log(M_p^1) = \log(\tilde{r}) \quad (7.0.26)$$

If p is increasing, then clearly $\tilde{r}$ is also increasing, so the difference on the left hand side above increases, which shows that $\log(M_p^1)$ is a convex function in p . We summarize:

1. $T_1^*(r) = \sup_{p \geq 1} \frac{r^p}{M_p^1}$ by (7.0.23)

2. $\int_1^\infty \frac{\log(T_1^*(r))}{r^2} dr = \infty$ by (7.0.15)

3. The sequence $(M_p^1)_p$ is log. conv. by (7.0.26)

1) and 2) together correspond to the third condition in 4.1.5, because:

$$\infty = \int_1^\infty \frac{\log(T_1^*(r))}{r^2} dr \leq \int_0^\infty \frac{\log(T_1^*(r))}{r^2} dr \quad \text{and} \quad \frac{1}{r^2} \sim \frac{1}{1+r^2} \quad \left(\Leftrightarrow \lim_{r \rightarrow \infty} \frac{\frac{1}{r^2}}{\frac{1}{1+r^2}} = 1 \right).$$

Hence we have shown that g belongs to a quasi-analytic space. The proof for h is the same and we summarize: We can write $f(x) = g(x) + h(x)$ for all $x \in [-1, 1]$, where g and h are functions in the quasi-analytic spaces $\mathcal{E}_{\{M^1\}}([-1, 1])$ resp. $\mathcal{E}_{\{M^2\}}([-1, 1])$ with $M^1 := (M_p^1)_p$ resp. $M^2 := (M_p^2)_p$.

In fact we have shown the decomposition on $[-1, 1]$, because for $\mathcal{E}([-1, 1])$ one has the representation by the *Tchebychev-polynomials*. If we apply an affine linear transformation $x \mapsto a \cdot x + b$, $a, b \in \mathbb{R}, a > 0$, we obtain the general case. Note that the function space $\mathcal{E}_{\{M^1\}}$ resp. $\mathcal{E}_{\{M^2\}}$ is closed under such transformations.

□

Remark 7.0.2 From the theorem above we obtain also the following property: If $F = \log(G)$ for functions $F, G \in \mathcal{E}$ (in particular if $G(x) > 0, \forall x$), then $F = F_1 + F_2$ by the theorem above and so

$$G = e^F = e^{F_1 + F_2} = e^{F_1} \cdot e^{F_2},$$

where $F_1 \in \mathcal{E}_{\{M^1\}}$ resp. $F_2 \in \mathcal{E}_{\{M^2\}}$. If for $i \in \{1, 2\}$ the weight sequences $M^i := (M_p^i)_p$ satisfies $M_p^i \geq 1$ for all $p \in \mathbb{N}$, then $\exp \in \mathcal{E}_{\{M^i\}}$ (take $C := \max_{x \in K} e^x, h := 1, \forall K \subseteq \mathbb{R}$ compact).

If we additionally assume that the sequences $(m_p^1)_p$ and $(m_p^2)_p$ are both logarithmic convex, where $m_p^1 := \frac{M_p^1}{p!}$ resp. $m_p^2 := \frac{M_p^2}{p!}$ for all $p \in \mathbb{N}$, then we can use 3.2.7 to obtain:

$$F_1 \in \mathcal{E}_{\{M^1\}} \implies e^{F_1} \in \mathcal{E}_{\{M^1\}}, \quad F_2 \in \mathcal{E}_{\{M^2\}} \implies e^{F_2} \in \mathcal{E}_{\{M^2\}}.$$

So we have decomposed G into a product of two functions, which both belong to a quasi-analytic space.

8 A comparison with weight functions

The goal of this chapter is to compare two different definitions of weighted function spaces. Up to now we have worked with weight sequences $M := (M_p)_p$. Such a sequence M is nothing else but a mapping $M : \mathbb{N} \rightarrow \mathbb{R}_{>0}$. The new strategy in this chapter will be to define weighted function spaces of Romieu- resp. Beurling $\mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(\omega)}$, where $\omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ will be called a *weight function*. Therefore we want to compare the new spaces with the "old ones". Here again we have to introduce and compare certain properties for M and ω .

A natural question arises: In which cases do we obtain $\mathcal{E}_{\{M\}} \cong \mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(M)} \cong \mathcal{E}_{(\omega)}$ as locally convex vector spaces? We will give a complete answer to this question and also construct a counterexample.

Another important item will be: Is there a correspondence $M \leftrightarrow \omega$, which means: Given a weight sequence M , can one associate to M a weight function ω ? And, if ω is a given weight function, can one "build" a weight sequence M which is related to ω ?

8.1 Regularization of weight sequences

At the beginning we discuss the technique of the *regularization of sequences*, which was introduced by Mandelbrojt (see [19, Chapitre 1]). In the following let $(a_n)_n$ be an arbitrary sequence of real numbers such that $a_n > -\infty$ holds for all $n \in \mathbb{N}$. We allow also $a_n = \infty$ for finitely many n and so we get a sequence of points $P_n := (n, a_n)$ in $\mathbb{R} \times (\mathbb{R} \cup \{+\infty\})$.

Let $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \cup \{+\infty\}$ be an increasing continuous function with $\phi(0) \geq 1$. Such a function ϕ will be called a *regularizing function*. If $\phi(t_0) = \infty$ for some $t_0 \in \mathbb{R}_{\geq 0}$, then $\phi(t) = \infty$ for all $t \geq t_0$. In this case we assume the following condition on the sequence $(a_n)_n$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \infty. \quad (8.1.1)$$

We discuss now the geometric process of regularizing the sequence $a := (a_n)_n$ with respect to a regularizing function ϕ . The regularized sequence will be denoted with $a^\phi := (a_n^\phi)_n$.

Set

$$B_t := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \phi(t)\},$$

for all $t \in \mathbb{R}_{\geq 0}$ we define a set D_t satisfying the following property:

D_t consists of the intersection of the minimal line with slope t through a point $p_n \in B_t$ with the stripe B_t .

The mapping $t \mapsto D_t$ is well-defined: If $\phi(t) < \infty$ for all t this is clear because by assumption $\phi(t) \geq 1$ holds for all t . If $\phi = \infty$ on $[t_0, \infty)$ then we can use (8.1.1). Note that in the second case the segments D_t , where $t \geq t_0$, are half-lines.

Now we consider the subsequence $(P_{n_i})_i$ of $(P_n)_n$ of points lying on a segment D_t , where $t \in \mathbb{R}_{\geq 0}$. Then $(P_{n_i})_i$ consists of infinitely many points: If there exists an index $i_0 \in \mathbb{N}$ such that $P_{n_{i_0}} \notin D_t$ for any t , then $a_n = \infty$ for $n \geq n_{i_0}$. But this is a contradiction to $a_n = \infty$ for only finitely many indices. We call the sequence $(n_i)_i$, $i \in \mathbb{N}$, the *principal indices sequence*.

If we take a principal index n_i and assume that $P_{n_i} \in D_{t_1}$ and $P_{n_i} \in D_{t_2}$ where $t_1 < t_2$, then immediately $P_{n_i} \in D_t$ holds for $t \in [t_1, t_2]$. So for all principal indices n_i we can associate an interval I_i such that $P_{n_i} \in D_t$ for $t \in I_i$. Such an interval is in general half-open and of the form $[t_i, \tau_i)$. $t_i \in I_i$ holds always because ϕ is a continuous function. In the cases where we

have $\tau_i \notin I_i$ we conclude $\phi(\tau_i) = n_{i+1}$. The indices n_{i+1} with this property are called *indices of discontinuity*.

For all $i \in \mathbb{N}$ we have clearly $\tau_i = t_{i+1}$ and we remark: If $t_{i+1} = t_{i+2} = \dots = t_{i+h}$ occur for an $h > 1$, then the points $P_{n_{i+1}}, \dots, P_{n_{i+h}}$ are lying on a line, hence the indices $n_{i+1}, \dots, n_{i+h}$ are indices of discontinuity.

Now we define the line segments L_i as follows: L_i is the segment of the line with slope $\tau_i = t_{i+1}$ which is passing through the point $P_{n_i} = (n_i, a_{n_i})$ and such that $\text{pr}_1(L_i) = [n_i, n_{i+1})$ holds for all i . If n_{i+1} is not an index of disc. then we have $L_i \subseteq D_{t_{i+1}}$, in the opposite case L_i lies over $D_{t_{i+1}}$ because $\phi(t_{i+1}) = \phi(\tau_i) = n_{i+1}$. The slopes of L_i are increasing for $i \rightarrow \infty$.

We put $\mathcal{B} := \bigcup_{i \in \mathbb{N}} L_i$, where L_0 is defined to be the horizontal line between the points $(0, a_{n_1})$ and P_{n_1} . Thus the set $\mathcal{B}$ consists of a family of convex polygons, which are closed on the left side and open on the right side. In particular between two indices of disc. n_i and n_{i+1} we obtain a certain half-open convex polygon. The cusps of the polygons, except the extremal points on the right side, are exactly the points P_{n_i} , where n_i is a principal index.

We define the regularization mapping r^ϕ as follows: $P_n = (n, a_n) \mapsto P_n^\phi = (n, a_n^\phi)$, where P_n^ϕ denotes the projection of the point P_n onto $\mathcal{B}$. The regularized sequence $a^\phi := (a_n^\phi)_n$ satisfies $a_n^\phi \leq a_n$ for all n with equality for the principal indices.

Furthermore we remark: Given two regularizing functions ϕ_1 and ϕ_2 with $\phi_1(t) \leq \phi_2(t)$ for all t we obtain immediately $a_n^{\phi_2} \leq a_n^{\phi_1}$ for all n and the number of the indices of disc. decreases. If $(\phi_i)_i$ is an increasing sequence of regularizing functions, then the sequences $a^{\phi_1}, a^{\phi_2}, \dots$ become more and more regular, but $|a_n - a_n^{\phi_i}|$ is increasing for all n if $i \rightarrow \infty$. An important example is $\phi = \infty$. In this case the set $\mathcal{B}$ coincides with the Newton polygon, hence $\mathcal{B}$ is connected and the set of indices of disc. is empty.

For $t \geq 0$ we set $m(t) := \sup\{n \in \mathbb{N} : P_n \in D_t\}$. The function $m : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}$ is well-defined because there are only finitely many points P_n lying on a segment D_t . This is clear if $\phi(t) < \infty$ for all t and if $\phi(t_0) = \infty$ for a $t_0 > 0$, then this follows by (8.1.1). We show now that m is an increasing function and $\lim_{t \rightarrow \infty} m(t) = \infty$. Let $t_1 < t_2$, then we distinguish two cases: If $P_{m(t_1)} \in D_{t_2}$ then obviously $m(t_1) \leq m(t_2)$ holds. If $P_{m(t_1)}$ lies above the segment D_{t_2} , then all points P_n , where $n \leq m(t_1)$ holds, lie above D_{t_2} . Thus we obtain in this case $m(t_1) < m(t_2)$. Suppose that there exists $k \in \mathbb{N}$ such that $m(t) \leq k$ for all t , then $a_n = \infty$ for all $n > k$ which is a contradiction to our assumption.

Claim: The function m is continuous from the right. For the segments D_t we have $D_t = D_t^+$, where $D_t^+ := \lim_{\varepsilon \searrow 0} D_{t+\varepsilon}$, hence they are continuous from the right. If n_i is not an index of disc., then we obtain $D_t = D_t^-$, where $D_t^- := \lim_{\varepsilon \searrow 0} D_{t-\varepsilon}$, too. Analogously we define $m(t)^+$ and with this notation we have to show $m(t) = m(t)^+$ for all t . Because $m(t) \in \mathbb{N}$ for all $t \geq 0$ we have $m(t)^+ \in \mathbb{N}$, thus there exists $\varepsilon_0 > 0$ such that $m(t + \varepsilon) = m(t)^+$ for all ε with $0 < \varepsilon \leq \varepsilon_0$. Then $P_{m(t)^+} \in D_{t+\varepsilon}$ holds for all those ε , hence $P_{m(t)^+} \in D_t$ because $D_t = D_t^+$. Finally $m(t)^+ \leq m(t)$ implies the desired property.

We give now the important definition of the *trace function* $A : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ of a sequence $(a_n)_n$ with respect to a regularizing function ϕ :

Let $-A(t)$ be the value of the trace of the segment D_t on the line $\{(0, y) : y \in \mathbb{R}\}$.

By definition A is a strict increasing function and $\lim_{t \rightarrow \infty} A(t) = \infty$ because otherwise we would obtain $a_n = \infty$ for infinitely many $n \in \mathbb{N}$. If we take an arbitrary line l with slope t which is passing through a point $P_n = (n, a_n)$ with $n \leq \phi(t)$, then the ordinate of l at the origin is equal to $a_n - n \cdot t$. By definition of D_t we see: $a_n - n \cdot t \leq -A(t) \Leftrightarrow A(t) \leq n \cdot t - a_n$ and equality holds for $n = m(t)$. We summarize:

$$A(t) = \sup\{n \cdot t - a_n : n \in \mathbb{N}, n \leq \phi(t)\}. \quad (8.1.2)$$

A is also continuous from the right, it is discontinuous for $t = t_i$ where n_i is an index of disc. If we regularize with respect to $\phi = \infty$, then A is continuous. The correspondence between the functions m and A is the following: For the segment D_t , with slope t , by definition of the function m we have $P_{m(t)} \in D_t$. By $P_{m(t)} = (m(t), a_{m(t)})$ we see $A(t) = t \cdot m(t) - a_{m(t)}$ and this implies the fact that m is nothing else but the derivative from the right of A .

To each sequence $(a_n)_n$ like above we have constructed a trace function but there are different sequences which lead to the same trace function. First we observe that for a and a^ϕ one obtains the same trace function with respect to ϕ : The cusps of the set $\mathcal{B}$ in B_t are lying by definition above the segment D_t , except the principal points which are lying on D_t . Thus all points P_n^ϕ with $n \leq \phi(t)$ are lying below the segment D_t , except the points $P_{n_i} = P_{n_i}^\phi$ where n_i is a principal index. We see: For both sequences a and a^ϕ we can take the same segments D_t for all $t \geq 0$, hence we obtain the same trace function A .

The next result characterizes the relation between a sequence $(a_n)_n$ and the trace function A which can be considered as a *Galois-correspondence* between $(a_n)_n$ and A :

Lemma 8.1.1 [19, 1.4.I., p. 6-7] *In the set of all sequences $(a_n)_n$ which lead to the same trace function A with respect to a regularizing function ϕ we can find a "most regularized" one. This sequence $(\alpha_n)_n$ is given by*

$$\alpha_n = \sup\{n \cdot t - A(t) : t \in \mathbb{R}_{\geq 0}, n \leq \phi(t)\}. \quad (8.1.3)$$

Proof. By the previous statements it is enough to show that

$$a_n^\phi = \sup\{n \cdot t - A(t) : t \in \mathbb{R}_{\geq 0}, n \leq \phi(t)\} \quad (8.1.4)$$

holds, in particular $a_n^\phi = \alpha_n$ for all $n \in \mathbb{N}$. We consider the line $l_n := \{(x, y) \in \mathbb{R}^2 : x = n\}$ for an arbitrary $n \in \mathbb{N}$ and for all $t \geq 0$ with $n \leq \phi(t)$ the segment D_t . Hence D_t intersects l_n in $(n, n \cdot t - A(t))$, but the point $P_n^\phi = (n, a_n^\phi)$ is not lying below D_t which implies $n \cdot t - A(t) \leq a_n^\phi$. By construction of a^ϕ we see that there exists i such that $P_n^\phi \in L_i$. We distinguish now two cases: If the index n_{i+1} is not an index of discontinuity we have $L_i \subseteq D_{t_{i+1}}$ and because $\phi(t_{i+1}) = \phi(\tau_i) \geq n_{i+1}$ holds we are done because the supremum in (8.1.4) is reached for $t = t_{i+1}$. If n_{i+1} is an index of disc. then $L_i \subseteq D_{t_{i+1}}^-$ where $D_{t_{i+1}}^- := \lim_{\varepsilon \searrow 0} D_{t_{i+1}-\varepsilon}$ and the supremum in (8.1.4) is not reached for finite t .

□

Remark 8.1.2 By (8.1.2) and (8.1.4) one can now define a regularized sequence a^ϕ and a trace function A in an abstract way without geometric arguments, calculations and restrictions on the sequence $(a_n)_n$. Sometimes it's useful to extend the trace function to an even function on $\mathbb{R}$, where we put $A(t) := A(-t)$ for $t \in \mathbb{R}$, $t < 0$. Finally we remark that (8.1.1) implies $A(t) = \sup\{n \cdot t - a_n^\phi : n \in \mathbb{N}, n \leq \phi(t)\}$.

8.2 Associated function

We return now to an arbitrary weight sequences $(M_n)_n$ and apply the regularization procedure above as follows: We regularize for $\phi \equiv \infty$ the sequence $a := (a_n)_n$, where $a_n := \log(M_n)$. As we already noticed the set $\mathcal{B}$ coincides in this case with the Newton polygon relative to the points $P_n = (n, a_n)$ and it is a continuous convex polygon. We point out that condition (8.1.1) is in this notation $\lim_{n \rightarrow \infty} M_n^{1/n} = \infty$. Recall that for $M_0 = 1$ and $(M_n)_n$ log. convex by 2.0.4 the sequence $(M_n^{1/n})_n$ is increasing. Furthermore we remark that $\mathcal{B}$ is the maximal element of the set of convex lines which lie not above the points P_n .

After regularization we set $M_n^\phi := \exp(a_n^\phi)$. By construction the sequence $(M_n^\phi)_n$ coincides with the largest convex minorant of $(M_n)_n$, thus $M_n^\phi = M_n^c$ for all n . We put

$$T(\varrho) := \sup_{p \in \mathbb{N}} \frac{\varrho^p}{M_p} \quad (8.2.1)$$

and $A(\varrho) := \log(T(\exp(\varrho))) = \sup_{p \in \mathbb{N}} (p \cdot \varrho - \log(M_p))$, so A is the *trace function* of the sequence $(a_n)_n = (\log(M_n))_n$. A and T are both continuous functions and by (8.1.4) we can write now

$$M_n^c = M_n^\phi = \sup_{r \in \mathbb{R}_{\geq 0}} \frac{r^n}{T(r)}.$$

Remark that (8.1.1) implies for all $n \geq 1$ that $\lim_{r \rightarrow \infty} \frac{r^n}{T(r)} = 0$ holds.

Let $(M_p)_p$ be an arbitrary sequence of positive numbers. We define now the *associated function* $\omega_M : \mathbb{R} \longrightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ via

$$\omega_M(\varrho) := \sup_{p \in \mathbb{N}} \log \left(\frac{|\varrho|^p \cdot M_0}{M_p} \right) \quad (8.2.2)$$

for $\varrho \neq 0$ and $\omega_M(0) := 0$. One can see: For $M_0 = 1$ we get $\omega_M(\varrho) = A(\log(\varrho)) = \log(T(\varrho))$, if $M_0 \neq 1$ we have $\omega_M(\varrho) = A(\log(\varrho)) + \log(M_0) = \log(T(\varrho)) + \log(M_0)$.

Clearly ω_M is an even function and often we restrict ω_M to $\mathbb{R}_{\geq 0}$. We remark: If we assume that the sequence $\left(\frac{M_p}{M_0}\right)^{1/p}$ is bounded below by a positive constant, then ω_M is an increasing convex function in the variable $\log(\varrho)$. Furthermore ω_M vanishes for sufficiently small $\varrho > 0$ and it is increasing faster than $\log(\varrho^p)$ for any p as $\varrho \longrightarrow \infty$. If $(M_p)_p$ is log. convex then by 2.0.4 this assumption is satisfied.

Furthermore we introduce the function $\mu : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{N} \cup \{+\infty\}$ defined by:

$$\mu(\lambda) := |\{p \in \mathbb{N} : \mu_p \leq \lambda\}| = |\{p \in \mathbb{N} : M_p \leq \lambda \cdot M_{p-1}\}|. \quad (8.2.3)$$

We see: If $(M_p)_p$ is log. convex, then $(\mu_p)_p$ is increasing and so the function μ is increasing. If $(\mu_p)_p$ is increasing and additionally $\lim_{p \rightarrow \infty} \mu_p = \infty$, then $\mu(\lambda)$ is a finite value for all $\lambda < \infty$.

If we define the sequence $M^{\omega_M} := (M_p^{\omega_M})_p$ in the following way

$$M_p^{\omega_M} := M_0 \cdot \sup_{\varrho \in \mathbb{R}} \frac{|\varrho|^p}{\exp(\omega_M(\varrho))}, \quad \text{for } p \in \mathbb{N},$$

then we obtain immediately:

Lemma 8.2.1 *The sequence $(M_p^{\omega_M})_p$ coincides with the largest log. convex minorant of the sequence $(M_p)_p$, which means $M_p^{\omega_M} = M_p^c$ for all p .*

Proof. Follows by Lemma 8.1.1, the definition of the associated function and the remarks above.

□

An easy consequence is now the following result:

Proposition 8.2.2 [11, Proposition 3.2., p. 49] *A weight sequence $(M_p)_p$ is logarithmic convex if and only if*

$$M_p = M_0 \cdot \sup_{\varrho \in \mathbb{R}} \frac{|\varrho|^p}{\exp(\omega_M(\varrho))} \quad (8.2.4)$$

is satisfied for all p .

Proof. For a log. convex sequence $(M_p)_p$ we have $M_p = M_p^c$ for all p and so we are done by 8.2.1. □

Remark 8.2.3 In [19, 1.7., p. 14-16] it is pointed out that another important regularizing function is $\phi := \exp$, where we restrict $\exp$ to $\mathbb{R}_{\geq 0}$. $\exp$ is clearly continuous and increasing and $\exp(0) = 1$, hence we can apply the regularizing procedure. Let $(M_n)_n$ be an arbitrary weight sequence, we put again $a_n := \log(M_n)$ for all n and regularize the sequence $(a_n)_n$ with respect to $\exp$ to obtain $(a_n^{\exp})_n$. Then we define $M_n^o := \exp(a_n^{\exp})$ for all n . The function S , where we put

$$S(r) := \max_{n \in \mathbb{N}: n \leq r} \frac{r^n}{M_n},$$

is well defined for $r \geq 0$ (in particular $S(r) = M_0^{-1}$ for $r \in [0, 1)$) and we have $\log(S(r)) = \max_{n \leq r} (n \cdot \log(r) - \log(M_n))$. Hence $A(t) := \log(S(\exp(t)))$ is nothing else but the trace function of the sequence $(a_n)_n = (\log(M_n))_n$ with respect to the regularizing function $\exp$. Finally 8.1.1 leads to

$$M_n^o = \sup_{r \in \mathbb{R}: r \geq n} \frac{r^n}{S(r)} \quad \text{and} \quad S(r) = \max_{n \leq r} \frac{r^n}{M_n^o}.$$

Note that S is the important function, which we used in the decomposition theorem 7.0.1. In particular in 7.0.1 we have taken the maximum over all $n \in \mathbb{N}$ with $1 \leq n \leq r$. But if one assumes $M_1 \leq M_0$, then $\max_{n \in \mathbb{N}: n \leq r} \frac{r^n}{M_n} = \max_{n \in \mathbb{N}: 1 \leq n \leq r} \frac{r^n}{M_n}$ for $r \geq 1$.

From now on we assume always that

$$\boxed{M_0 \leq M_1 \text{ is satisfied.}}$$

In the next step we want to derive an explicit formula for the associated function. To do so we first prove the following result:

Lemma 8.2.4 [19, 1.8.III., p. 18-19] *Let $(a_n)_n$ and $(b_n)_n$ be two increasing sequences of positive real numbers. We define a function $N : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ in the following way: $N(x) := 0$ for $0 < x \leq a_1$ and $N(x) := b_n$ for $a_n < x \leq a_{n+1}$ where $n \geq 1$. Furthermore put $b_0 := 0$ and $N_n := \sum_{i=1}^n (b_i - b_{i-1}) \cdot a_i \stackrel{b_0=0}{=} \sum_{i=1}^{n-1} b_i \cdot (a_i - a_{i+1}) + a_n \cdot b_n$. So $N_0 = 0$, $N_1 = a_1 \cdot b_1$ and we obtain for $a_m \leq x \leq a_{m+1}$, $m \geq 1$:*

$$\int_0^x N(t) dt = b_m \cdot x - N_m = \sup_{k \geq 1} (b_k \cdot x - N_k) = \sup_{k \geq 0} (b_k \cdot x - N_k).$$

Proof. First we remark that $b_k \cdot x - N_k \geq b_k \cdot a_k - N_k = \sum_{i=1}^{k-1} b_i \cdot (a_{i+1} - a_i) \geq 0$ for all $k \geq 1$ and $b_0 \cdot x - N_0 = 0$, hence $\sup_{k \geq 1} (b_k \cdot x - N_k) = \sup_{k \geq 0} (b_k \cdot x - N_k)$.

Let $x \in [a_m, a_{m+1}]$, then we get immediately by definition of N_n and the function N :

$$\int_0^x N(t) dt = \sum_{k=1}^{m-1} b_k \cdot (a_{k+1} - a_k) + b_m \cdot (x - a_m) = b_m \cdot x - N_m.$$

On the other hand we can estimate for $0 < p < m < q$:

$$\begin{aligned}
b_p \cdot x - N_p &= \sum_{k=1}^{p-1} (b_k \cdot (a_{k+1} - a_k)) + b_p \cdot (x - a_p) \\
&= \sum_{k=1}^{p-1} (b_k \cdot (a_{k+1} - a_k)) + b_p \cdot \underbrace{\sum_{k=p}^{m-1} (a_{k+1} - a_k)}_{=a_m - a_p} + b_p \cdot (x - a_m) \\
&\stackrel{(*)}{\leq} \sum_{k=1}^{m-1} (b_k \cdot (a_{k+1} - a_k)) + b_m \cdot (x - a_m) = b_m \cdot x - N_m.
\end{aligned}$$

(*) holds because $b_p \leq b_k$ for $p \leq k \leq m$ and $a_m \leq x$. Furthermore we estimate

$$b_m \cdot x - N_m \stackrel{(**)}{\geq} b_m \cdot x - N_m - \sum_{k=m+1}^q ((b_k - b_{k-1}) \cdot a_k) + (b_q - b_m) \cdot x = b_q \cdot x - N_q.$$

(**) holds because $x \leq a_k$ for $m+1 \leq k \leq q$. Hence we see: For $0 < p < m < q$ we have $b_q \cdot x - N_q \leq b_m \cdot x - N_m$ and $b_p \cdot x - N_p \leq b_m \cdot x - N_m$. □

Note that this result is still valid for $a_1 = 0$, and in this case we have $\int_0^x N(t)dt = b_1 \cdot (x - a_1) = b_1 \cdot x$ for $x \in [0, a_2]$. Applying the lemma above we can prove an important and very useful integral representation formula for the associated function, which will be often used:

Proposition 8.2.5 [19, 1.8.V., p. 20-21] *If $(M_p)_p$ is a logarithmic convex weight sequence, then we obtain for $\varrho > 0$ the following expression for the associated function:*

$$\omega_M(\varrho) = \int_0^\varrho \frac{\mu(\lambda)}{\lambda} d\lambda = \int_{\mu_1}^\varrho \frac{\mu(\lambda)}{\lambda} d\lambda,$$

where $\mu(\lambda) := |\{p \in \mathbb{N} : \mu_p \leq \lambda\}|$ is defined as in 8.2.3.

Proof. We apply 8.2.4 in the following way: Put $a_k := \log(\mu_k)$ and $b_k := k$, then both sequences are increasing because of the log. convexity of $(M_p)_p$ and a_k is positive, too: Since $M_0 \leq M_1$ we get $\mu_1 = \frac{M_1}{M_0} \geq 1$, hence $a_1 = \log(\mu_1) \geq 0$. First we remark that for the function N we have by definition $N(\log(x)) = \mu(x)$ for all $x \in \mathbb{R}$, $x \geq 1$. For N_n we compute:

$$N_n = \sum_{i=1}^n (b_i - b_{i-1}) \cdot a_i = \sum_{i=1}^n a_i = \sum_{i=1}^n \log(\mu_i) = \log(M_n) - \log(M_0).$$

Then we obtain for $x \in [a_n, a_{n+1}]$ and the trace function A :

$$\int_0^x N(t)dt \stackrel{8.2.4}{=} \sup_{k \geq 0} \{b_k \cdot x - N_k\} = \sup_{k \geq 0} \{k \cdot x + \log(M_0) - \log(M_k)\} = A(x) + \log(M_0).$$

Finally we have for $\log(y) = x$, where $y \geq \mu_1$:

$$\omega_M(y) = A(\log(y)) + \log(M_0) = \int_0^x N(t)dt \stackrel{t=\log(\lambda)}{=} \int_1^y \frac{N(\log(\lambda))}{\lambda} d\lambda = \int_1^y \frac{\mu(\lambda)}{\lambda} d\lambda.$$

Note that $\mu(\lambda) = 0$ on $[0, \mu_1)$ by definition, hence on $[0, 1)$, and so we are done. □

Remark 8.2.6 If $M_1 \leq M_0$ holds, then $\sup_{p \in \mathbb{N}} \log \left(\frac{|\varrho|^p \cdot M_0}{M_p} \right) = \sup_{p \in \mathbb{N}; p \geq 1} \log \left(\frac{|\varrho|^p \cdot M_0}{M_p} \right)$ for $|\varrho| \geq 1$: Assume that the supremum is attained for $p = 0$, so $0 \geq p \cdot \log(|\varrho|) + \log(M_0) - \log(M_p)$ has to be satisfied for all $p \geq 1$, which is equivalent to $\frac{M_p}{M_0} \geq |\varrho|^p$ for all $p \geq 1$. But for the case $p = 1$ we need $\frac{M_1}{M_0} \geq |\varrho| \geq 1$.

Hence we see by 8.2.5: If $M_0 = M_1$ holds, then $\int_1^\varrho \frac{\mu(\lambda)}{\lambda} d\lambda = \omega_M(\varrho) = \sup_{p \in \mathbb{N}; p \geq 1} \log \left(\frac{\varrho^p \cdot M_0}{M_p} \right)$ for all $\varrho \geq 1$.

In the following we define a further property of weight sequences:

$$\exists A, H > 0 : \forall p \in \mathbb{N} : M_{p+1} \leq A \cdot H^p \cdot M_p. \quad (8.2.5)$$

We call a sequence $(M_p)_p$ *stable under differential operators* (with constant coefficients) if (8.2.5) is satisfied. It's clear, that (8.2.5) is equivalent to $\sup_p \left(\frac{M_{p+1}}{M_p} \right)^{1/p} < \infty$, which implies closedness under derivation for $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$. If $(M_p)_p$ satisfies (8.2.5), then we can estimate for the operator $a \cdot \frac{d}{dx_i}$, $a \in \mathbb{C}$, on a fixed compact set K :

$$\begin{aligned} \sup_{x \in K} |(a \cdot f^{(e_i)})^{(\alpha)}(x)| &\leq \sup_{x \in K} |a| \cdot |f^{(\alpha+e_i)}(x)| \leq |a| \cdot C \cdot h^{|\alpha|+1} \cdot M_{|\alpha|+1} \\ &\leq \underbrace{(|a| \cdot C \cdot h \cdot A)}_{(8.2.5)} \cdot (h \cdot H)^{|\alpha|} \cdot M_{|\alpha|}. \end{aligned}$$

Here we have set $e_i := (0, \dots, \underbrace{1}_i, \dots, 0)$.

The next proposition gives a full characterization of property (8.2.5) for log. convex sequences $(M_p)_p$ in terms of the function μ .

Proposition 8.2.7 [11, Proposition 3.4., p. 50-51] A log. convex weight sequence $(M_p)_p$ satisfies (8.2.5) if and only if there exist constants $A, H > 1$, such that

$$\mu(\lambda) \geq \frac{\log(\lambda/A)}{\log(H)}, \quad \forall \lambda > 0. \quad (8.2.6)$$

Under these equivalent conditions we have the following inequality for all $k, \varrho > 0$:

$$\omega_M(k \cdot \varrho) - \omega_M(\varrho) \geq \frac{\log(\varrho/A) \cdot \log(k)}{\log(H)}. \quad (8.2.7)$$

Proof. (8.2.5) can be viewed as $\mu_{p+1} \leq A \cdot H^p$ for all p and $A, H > 0$. Because $\lambda < \mu_{p+1}$ holds for $p := \mu(\lambda)$ we have

$$\lambda < \mu_{p+1} \leq A \cdot H^{\mu(\lambda)} \Leftrightarrow \log(\lambda) - \log(A) \leq \mu(\lambda) \cdot \log(H) \Leftrightarrow \frac{\log(\lambda/A)}{\log(H)} \leq \mu(\lambda).$$

Conversely, by the log. convexity of $(M_p)_p$, let $\mu_{p_0} < \mu_{p_0+1} = \dots = \mu_{p+1} \leq \dots$ and if $\lambda \nearrow \mu_{p_0+1}$ in (8.2.6), we see

$$p_0 \geq \frac{\log(\mu_{p+1}/A)}{\log(H)} \Leftrightarrow H^{p_0} \cdot A \geq \mu_{p+1} = \mu_{p_0+1} \quad \forall p_0,$$

which implies (8.2.5). If now (8.2.6) is satisfied, we use 8.2.5 to calculate:

$$\begin{aligned} \omega_M(k \cdot \varrho) - \omega_M(\varrho) &\geq \int_\varrho^{k \cdot \varrho} \frac{\log(\lambda/A)}{\log(H) \cdot \lambda} d\lambda = \frac{1}{\log(H)} \cdot \int_\varrho^{k \cdot \varrho} \frac{\log(\lambda)}{\lambda} d\lambda - \frac{\log(A)}{\log(H)} \cdot \int_\varrho^{k \cdot \varrho} \frac{1}{\lambda} d\lambda \\ &= \frac{1}{\log(H)} \cdot \left[\frac{(\log(\lambda))^2}{2} \right]_\varrho^{k \cdot \varrho} - \frac{\log(A)}{\log(H)} \cdot \log(k) \\ &= \frac{(2 \log(\varrho/A) + \log(k)) \cdot \log(k)}{2 \log(H)} \geq \frac{\log(\varrho/A) \cdot \log(k)}{\log(H)}. \end{aligned}$$

The last inequality holds, because $H > 1$, and for the integration we have used $\int \frac{\log(\lambda)}{\lambda} d\lambda = (\log(\lambda))^2 - \int \frac{\log(\lambda)}{\lambda} d\lambda$, which follows by integration by parts, hence $\int \frac{\log(\lambda)}{\lambda} d\lambda = \frac{(\log(\lambda))^2}{2}$.

□

Lemma 8.2.8 [11, Lemma 3.5., p. 51] Let $(M_p)_p$ and $(M'_p)_p$ be two log. convex weight sequences, then we denote with ω_M resp. $\omega_{M'}$ their associated functions and we write μ resp. μ' for the functions, which we have defined in (8.2.3). Then the sequence

$$N_p := \min_{0 \leq q \leq p} M_q \cdot M'_{p-q}$$

is also logarithmic convex and the associated function ω_N resp. the function $\nu : \lambda \mapsto \nu(\lambda)$, $\nu(\lambda) := |\{p \in \mathbb{N} : \frac{N_p}{N_{p-1}} \leq \lambda\}|$, can be computed via

$$\nu(\lambda) = \mu(\lambda) + \mu'(\lambda) \quad \omega_N(\varrho) = \omega_M(\varrho) + \omega_{M'}(\varrho).$$

Proof. We have

$$N_p = \min_{0 \leq q \leq p} M_q \cdot M'_{p-q} = \underbrace{N_0}_{=M_0 \cdot M'_0} \cdot \min_{0 \leq q \leq p} \mu_1 \cdots \mu_q \cdot \mu'_1 \cdots \mu'_{p-q},$$

hence the sequence $\nu_p := \frac{N_p}{N_{p-1}}$ is obtained if we rearrange the set $\{\mu_1, \mu_2, \dots, \mu'_1, \mu'_2, \dots\}$ in the order of growth. So we see, that $(\nu_p)_p$ is an increasing sequence, thus $(N_p)_p$ is log. convex. Because

$$\nu_p \leq \lambda \Leftrightarrow N_p \leq \lambda \cdot N_{p-1} \Leftrightarrow \min_{0 \leq q \leq p} \mu_1 \cdots \mu_q \cdot \mu'_1 \cdots \mu'_{p-q} \leq \lambda \cdot \min_{0 \leq q \leq p-1} \mu_1 \cdots \mu_q \cdot \mu'_1 \cdots \mu'_{p-q-1}$$

we see that $\nu(\lambda) = \mu(\lambda) + \mu'(\lambda)$ is clearly satisfied. Now we use 8.2.5 to finish the proof.

□

With the lemma above we can prove a full characterization of weight sequences satisfying the moderate growth condition via the associated function:

Proposition 8.2.9 [11, Proposition 3.6., p. 51-52] A logarithmic convex weight sequence $(M_p)_p$ satisfies moderate growth if and only if

$$2 \cdot \omega_M(\varrho) \leq \omega_M(H \cdot \varrho) + \log(A \cdot M_0) \quad \forall \varrho \quad (8.2.8)$$

is satisfied, where A and H are the constants appearing in (6.1.2).

Proof. Applying 8.2.8 we see: $2 \cdot \omega_M$ is the associated function of the sequence $N_p := \min_{0 \leq q \leq p} M_q \cdot M_{p-q}$. If $(M_p)_p$ satisfies condition (6.1.2), then:

$$\begin{aligned} 2 \cdot \omega_M(\varrho) &= \sup_{p \in \mathbb{N}} \log \left(\frac{|\varrho|^p \cdot N_0}{N_p} \right) = \sup_{p \in \mathbb{N}} \log \left(\frac{|\varrho|^p \cdot M_0^2}{\min_{0 \leq q \leq p} M_q \cdot M_{p-q}} \right) \\ &\stackrel{(6.1.2)}{\leq} \sup_{p \in \mathbb{N}} \log \left(\frac{(H \cdot |\varrho|)^p \cdot M_0 \cdot A \cdot M_0}{M_p} \right) = \omega_M(H \cdot \varrho) + \log(A \cdot M_0). \end{aligned}$$

Conversely we remark that $(N_p)_p$ is a log. convex sequence, because $(M_p)_p$ is log. convex and 8.2.8. Hence

$$N_p \stackrel{8.2.2}{=} N_0 \cdot \sup_{p \in \mathbb{N}} \left(\frac{|\varrho|^p}{\exp(2\omega_M(\varrho))} \right) \stackrel{(6.1.2)}{\geq} M_0^2 \cdot \sup_{p \in \mathbb{N}} \left(\frac{|\varrho|^p}{\exp(\omega_M(H \cdot \varrho)) \cdot A \cdot M_0} \right) \stackrel{8.2.2}{=} \frac{M_p}{A \cdot H^p}.$$

□

Finally in the next two results we will give a characterization of the important relations $\preceq$ and $\trianglelefteq$ on the set of all sequences via the associated functions.

Proposition 8.2.10 [11, Lemma 3.8., p. 52] *For two log. convex weight sequences $(M_p)_p$ and $(N_p)_p$ we have: $M \preceq N$ if and only if there exist constants $L, C > 0$ such that*

$$\omega_N(\varrho) \leq \omega_M(L \cdot \varrho) + \log(C) \quad (8.2.9)$$

for all $\varrho \in \mathbb{R}$.

Proof. If $M \preceq N$, then we calculate

$$\begin{aligned} \omega_N(\varrho) &= \sup_{p \in \mathbb{N}} \log \left(\frac{|\varrho|^p \cdot N_0}{N_p} \right) \leq \sup_{p \in \mathbb{N}} \log \left(\frac{(L \cdot |\varrho|)^p \cdot M_0}{M_p} \right) + \log \left(\frac{C \cdot N_0}{M_0} \right) \\ &= \omega_M(L \cdot \varrho) + \log \left(\frac{C \cdot N_0}{M_0} \right). \end{aligned}$$

Conversely we obtain

$$M_p \underset{8.2.2}{=} M_0 \cdot \sup_{\varrho \in \mathbb{R}} \frac{|\varrho|^p}{\exp(\omega_M(\varrho))} \leq M_0 \cdot C \cdot \sup_{\varrho \in \mathbb{R}} \frac{|\varrho|^p}{\exp(\omega_N(\varrho/L))} \underset{8.2.2}{=} \frac{M_0 \cdot C}{N_0} \cdot L^p \cdot N_p.$$

□

For the following proposition we restrict the associated function ω_M to $\mathbb{R}_{\geq 0}$.

Proposition 8.2.11 [11, Lemma 3.10., p. 53] *Given two log. convex weight sequences $(M_p)_p$ and $(N_p)_p$, then the following are equivalent:*

- (a) $M \trianglelefteq N$
- (b) For all $L > 0$ there exists a constant $C > 0$ such that equation (8.2.9) is satisfied
- (c) There exists a continuous increasing function ε on $\mathbb{R}_{\geq 0}$ which satisfies
 - (i) $\varepsilon(0) = 0$,
 - (ii) $\varepsilon(\varrho)/\varrho \rightarrow 0$ for $\varrho \rightarrow \infty$,
 - (iii) $\omega_N = \omega_M \circ \varepsilon$.

Proof. (a) $\Leftrightarrow$ (b) follows with the same calculation as in 8.2.10 and the definition of $\trianglelefteq$.

(b) $\Rightarrow$ (c) We define $\varepsilon(\varrho)$ implicit via $\omega_N(\varrho) = \omega_M(\varepsilon(\varrho))$ for $\varrho > \nu_1$ and for $\varrho \leq \nu_1$ by $\frac{\mu_1 \cdot \varrho}{\nu_1} \leq \mu_1$. Using 8.2.5 the properties (c)(i) and (c)(iii) are satisfied. We prove (c)(ii) with contradiction: Suppose that there exists $L > 0$ and a sequence $\varrho_j \rightarrow \infty$ such that $\varepsilon(\varrho_j) > 2L \cdot \varrho_j$. Then $\omega_N(\varrho_j) = \omega_M(\varepsilon(\varrho_j)) > \omega_M(2L \cdot \varrho_j)$, and we choose now a constant C according to L , such that (8.2.9) is satisfied. So we obtain

$$\omega_M(2L \cdot \varrho_j) < \omega_N(\varrho_j) \underset{(8.2.9)}{\leq} \omega_M(L \cdot \varrho_j) + \log(C).$$

But this is a contradiction to the fact that the function $\exp(\omega_M(\varrho))$ increases faster than any power of ϱ , if $\varrho \rightarrow \infty$.

(c) $\Rightarrow$ (b): For any $L > 0$ we can find a ϱ_L , such that

$$\omega_N(\varrho) = \omega_M(\varepsilon(\varrho)) \leq \omega_M(L \cdot \varrho), \quad \text{for } \varrho \geq \varrho_L.$$

This is possible because $\varepsilon(\varrho) \sim L \cdot \varrho$ would be a contradiction to property (c)(ii). So we get (8.2.9) by putting $C := \exp(\omega_N(\varrho_L))$, because $\omega_N(\varrho) \leq \omega_N(\varrho_L)$ for $\varrho < \varrho_L$.

□

8.3 Periodical functions in $\mathcal{E}_{\{M\}}$, $\mathcal{E}_{(M)}$

In this section we study properties of $\mathcal{E}_{\{M\}}^{2\pi}$ resp. $\mathcal{E}_{(M)}^{2\pi}$, which denote the spaces of all 2π -periodic functions in $\mathcal{E}_{\{M\}}$ resp. in $\mathcal{E}_{(M)}$. In the proofs we will deal again with the associated function and we assume throughout this section that

$$M_0 \leq M_1.$$

First we will prove the next proposition, which gives important isomorphisms between $\mathcal{E}_{\{M\}}^{2\pi}$, $\mathcal{E}_{(M)}^{2\pi}$ and sequence spaces:

Proposition 8.3.1 [17, 1.2. Lemma, p. 111-112] *Let $M := (M_p)_p$ be a logarithmic convex weight sequence, which satisfies condition (8.2.5). We set for $c = (c_j)_j \in \mathbb{C}^{\mathbb{N}}$ the norm $\|c\|_k := \left(\sum_{j=1}^{\infty} |c_j|^2 \cdot e^{2\omega_M(j \cdot k)} \right)^{1/2}$, where ω_M denotes the associated function of M . Then we obtain the following isomorphisms:*

- (a) $\mathcal{E}_{(M)}^{2\pi} \cong \Lambda((M)) := \{c = (c_j)_j \in \mathbb{C}^{\mathbb{N}} : \forall k \in \mathbb{N} : \|c\|_k < \infty\}$.
- (b) $\mathcal{E}_{\{M\}}^{2\pi} \cong \Lambda(\{M\}) := \{c = (c_j)_j \in \mathbb{C}^{\mathbb{N}} : \exists k \in \mathbb{N} : \|c\|_{1/k} < \infty\}$.
- (c) $\Lambda((M)) \cong \Lambda(\mu) := \{c = (c_j)_j \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} |c_j|^2 \cdot e^{2k \cdot \mu(j \cdot k)} < \infty, \forall k \in \mathbb{N}\}$, where μ is the function defined in (8.2.3).

Proof. To prove the statements above we have to work with *Fourier-coefficients*: $\hat{f}_j := \frac{1}{2\pi} \cdot \int_0^{2\pi} f(t) \cdot e^{-ijt} dt$ for $j \in \mathbb{Z}$. So it's convenient for our calculations to extend the index set above from $\mathbb{N} \setminus \{0\}$ to $\mathbb{Z}$ and so we get two new sequence spaces $\tilde{\Lambda}((M))$ resp. $\tilde{\Lambda}(\{M\})$. First we show now that $\Lambda((M)) \cong \tilde{\Lambda}((M))$ resp. $\Lambda(\{M\}) \cong \tilde{\Lambda}(\{M\})$ holds:

Therefore let $\tilde{c} = (\tilde{c}_j)_j \in \mathbb{C}^{\mathbb{Z}}$, we define the mapping

$$\begin{cases} \tilde{c}_{-j} \mapsto c_{2j+1} & \text{for } j \geq 0 \\ \tilde{c}_j \mapsto c_{2j} & \text{for } j \geq 1. \end{cases} \quad (8.3.1)$$

Now we estimate as follows:

$$\begin{aligned} \|c\|_k^2 &= \sum_{j=1}^{\infty} |c_j|^2 \cdot e^{2\omega_M(j \cdot k)} = \sum_{j=1}^{\infty} |c_{2j}|^2 \cdot e^{2\omega_M((2j) \cdot k)} + \sum_{j=0}^{\infty} |c_{2j+1}|^2 \cdot e^{2\omega_M((2j+1) \cdot k)} \\ &= \sum_{j=1}^{\infty} |\tilde{c}_j|^2 \cdot e^{2\omega_M((2j) \cdot k)} + \sum_{j=0}^{\infty} |\tilde{c}_{-j}|^2 \cdot e^{2\omega_M((2j+1) \cdot k)} \leq \underbrace{\sum_{j \in \mathbb{Z}} |\tilde{c}_j|^2 \cdot e^{2\omega_M(j \cdot k')}}_{(\diamond)}. \end{aligned}$$

$(\diamond)$ is satisfied for $k' := 4k$ because ω_M is an even function which increases on $\mathbb{R}_{\geq 0}$. In particular we have $\omega_M((2j) \cdot k) \leq \omega_M(j \cdot 4k)$ and $\omega_M((2j+1) \cdot k) \leq \omega_M(j \cdot 4k)$ because $k \leq 2kj$ for all k and j .

On the other side we get

$$\begin{aligned} \|\tilde{c}\|_k^2 &= \sum_{j \in \mathbb{Z}} |\tilde{c}_j|^2 \cdot e^{2\omega_M(j \cdot k)} = \sum_{j=1}^{\infty} |\tilde{c}_j|^2 \cdot e^{2\omega_M(j \cdot k)} + \sum_{j=0}^{-\infty} |\tilde{c}_j|^2 \cdot e^{2\omega_M(j \cdot k)} \\ &\stackrel{(\diamond \diamond)}{=} \underbrace{\sum_{j=1}^{\infty} |c_{2j}|^2 \cdot e^{2\omega_M(j \cdot k)}}_{(\diamond \diamond)} + \sum_{j=0}^{\infty} |c_{2j+1}|^2 \cdot e^{2\omega_M(j \cdot k)} \leq \sum_{j=1}^{\infty} |c_j|^2 \cdot e^{2\omega_M(j \cdot k)}. \end{aligned}$$

$(\diamond \diamond)$ holds because ω_M is an even function. The last inequality above is valid because $\omega_M(j \cdot k) \leq \omega_M((2j) \cdot k)$ and $\omega_M(j \cdot k) \leq \omega_M((2j+1) \cdot k)$ for all k and j .

(a) Let $f \in \mathcal{E}_{(M)}^{2\pi}$ and denote by $(\hat{f}_j)_j$ the sequence of all Fourier-coefficients, then we estimate:

$$\begin{aligned} \frac{1}{M_0^2} \cdot \|(\hat{f}_j)_j\|_k^2 &\stackrel{(\star)}{\leq} \sum_{i \geq 0, j \in \mathbb{Z}} |\hat{f}_j|^2 \cdot |j|^{2i} \cdot \frac{k^{2i}}{M_i^2} = \sum_{i \geq 0} \left(\sum_{j \in \mathbb{Z}} |\widehat{f^{(i)}}_j|^2 \right) \cdot \frac{k^{2i}}{M_i^2} \stackrel{(\star\star)}{=} \frac{1}{2\pi} \cdot \sum_{i \geq 0} \|f^{(i)}\|_{L^2}^2 \cdot \frac{k^{2i}}{M_i^2} \\ &\leq \sum_{i \geq 0} \sup_{t \in [0, 2\pi]} |f^{(i)}(t)|^2 \cdot \frac{k^{2i}}{M_i^2} \leq \sum_{i \geq 0} |f|_{[0, 2\pi], 1/(2k)}^2 \cdot \frac{M_i^2}{(2k)^{2i}} \cdot \frac{k^{2i}}{M_i^2} \\ &= |f|_{[0, 2\pi], 1/(2k)}^2 \cdot \sum_{i \geq 0} \frac{1}{4^i} = \frac{4}{3} \cdot |f|_{[0, 2\pi], 1/(2k)}^2 < \infty. \end{aligned}$$

($\star$) holds because of the definitions of the norm $\|\cdot\|_k$ and of the associated function: $e^{2\omega_M(j \cdot k)} = \exp(2 \cdot \sup_p \log((|j| \cdot k)^p \cdot M_0/M_p)) \leq \sum_{i \geq 0} |j|^{2i} \cdot \frac{k^{2i} \cdot M_0^2}{M_i^2}$.

($\star\star$) holds because the Fourier-transformation is an isometry.

Now we show the inverse direction: Let $(c_j)_j \in \tilde{\Lambda}((M))$, then, by definition of the associated function, we have

$$\frac{(k \cdot |j|)^i \cdot M_0}{M_i} \leq e^{\omega_M(k \cdot j)} \Leftrightarrow \omega_M(k \cdot j) \geq \log\left(\frac{(k \cdot |j|)^i \cdot M_0}{M_i}\right) \text{ for all } i \in \mathbb{N}. \quad (8.3.2)$$

Thus $(c_j)_j \in s$, where s denotes the Fréchet-space of all rapidly decreasing sequences:

$$s := \{(c_j)_j \in \mathbb{C}^{\mathbb{Z}} : \lim_{|j| \rightarrow \infty} |c_j| \cdot |j|^i = 0, \forall i \in \mathbb{N}\}.$$

Hence $f(t) := \sum_{j \in \mathbb{Z}} c_j \cdot e^{ijt}$ defines a periodical $\mathcal{E}$ -function with period 2π (see [21, 29.5 Example(1), p. 340-341]). We estimate now for all $k \in \mathbb{R}_{>0}$, $i \in \mathbb{N}$ and for all $t \in I$, $I \subseteq [0, 2\pi]$, compact:

$$\begin{aligned} M_0 \cdot |f^{(i)}(t)| \cdot \frac{k^i}{M_i} &\leq M_0 \cdot \sum_{j \in \mathbb{Z}} |c_j| \cdot \frac{(k \cdot |j|)^i}{M_i} \stackrel{(8.3.2)}{\leq} \sum_{j \in \mathbb{Z}} |c_j| \cdot e^{\omega_M(j \cdot k)} \\ &\stackrel{(\Delta)}{\leq} C \cdot \left(\sum_{j \in \mathbb{Z}} |c_j|^2 \cdot e^{2\omega_M(j \cdot l)} \right)^{1/2} < \infty \text{ for } l = \alpha \cdot k, \end{aligned}$$

with $\alpha > H$ and $H \geq 1$ the constant appearing in (8.2.5). For (Δ) we have used *Cauchy-Schwartz-inequality* and 8.2.7:

In particular we get $e^{\omega_M(j \cdot l) - \omega_M(j \cdot k)} \geq \left(\frac{j \cdot k}{A}\right)^{\log(\alpha)/\log(H)}$ for all k, j and $\frac{\log(\alpha)}{\log(H)} > 1$, where $A, H \geq 1$ are the constants appearing in (8.2.5).

We point out that $\Lambda((M))$ is the Fréchet-space $\lambda^2(A_{\omega_M})$, for the symmetric *Köthe-Matrix* $A_{\omega_M} := (\exp(\omega_M(j \cdot k)))_{j,k}$. Therefore note that A_{ω_M} is a Köthe-Matrix, because ω_M is an increasing positive function by assumption on the weight sequence.

(b) The proof here is analogously as for (a) and we remark that $\Lambda(\{M\})$ is a (LF) -space.

(c) We use now 8.2.5 and the fact that μ and ω_M are increasing functions, because the weight sequence is log. convex. Hence for arbitrary $k, j > 0$ we estimate:

$$\begin{aligned} \omega_M(k \cdot j) &= \omega_M(k \cdot j) - \omega_M(j) + \omega_M(j) = \int_j^{kj} \frac{\mu(\lambda)}{\lambda} d\lambda + \omega_M(j) \\ &\leq \mu(k \cdot j) \cdot \int_j^{kj} \frac{1}{\lambda} d\lambda + \omega_M(j) = \mu(k \cdot j) \cdot \log(k) + \omega_M(j) \end{aligned}$$

and

$$k \cdot \mu(k \cdot j) \leq \omega_M(k \cdot e^{k \cdot j}) - \omega_M(k \cdot j) \leq \omega_M(k \cdot e^{k \cdot j}) - \omega_M(j).$$

It follows that the systems $\{k \cdot \mu(k) + \omega_M(\cdot) : k \in \mathbb{N}\}$ and $\{\omega_M(k) : k \in \mathbb{N}\}$ define the same sequence space and (c) follows from the diagonal transformation $(c_j)_j \mapsto (c_j \cdot e^{-\omega_M(j)})_j$.

The space $\Lambda(\mu)$ is the Köthe-sequence space $\lambda^2(A_\mu)$, where $A_\mu := (\exp(k \cdot \mu(j \cdot k)))_{j,k}$ is again a Köthe-matrix by the assumptions on the weight sequence. In particular we have shown $\lambda^2(A_{\omega_M}) \cong \lambda^2(A_\mu)$.

Finally we remark that we can define the sequence spaces above also with the corresponding l^1 -norms instead of the l^2 -norms. This holds because the l^2 -norm is always dominated by the l^1 norm and by inequality (Δ) in (a), which follows from 8.2.7, we obtain the converse estimate, too.

□

The following important result gives an important and useful characterization for weight sequence function spaces of Beurling-type.

Proposition 8.3.2 [17, 3.1. Theorem, p. 116-117] *Let $M := (M_p)_p$ be a weight sequence, which is logarithmic convex and such that condition (8.2.5) holds. Then the following statements are equivalent:*

- (a) $\mathcal{E}_{(M)}$ or $\mathcal{E}_{(M)}^{2\pi}$ is isomorphic to a power series space.
- (b) $\mathcal{E}_{(M)}$ or $\mathcal{E}_{(M)}^{2\pi}$ has the property (Ω) .
- (c) $\exists C \in \mathbb{N}$ such that $2\mu_p \leq \mu_{C \cdot p}$ holds for $p \in \mathbb{N}$ large enough.
- (d) $\mathcal{E}_{(M)}$ and $\mathcal{E}_{(M)}^{2\pi}$ are isomorphic to the power series space of infinite type $\Lambda_\infty(\mu) := \{c = (c_j)_j \in \mathbb{C}^\mathbb{N} : \|c\|_t^2 := \sum_{j=1}^\infty |c_j|^2 \cdot e^{2t \cdot \mu(j)} < \infty, \forall t < \infty\}$.

Proof. (d) $\Rightarrow$ (a): is trivial.

(a) $\Rightarrow$ (b): Holds, because the power series spaces $\Lambda_r(\cdot)$ satisfy condition (Ω) for all $r \in \mathbb{R} \cup \{+\infty\}$ (see [21, 29.11 Lemma (3), p. 347-348]).

(b) $\Rightarrow$ (c): First we claim: If $\mathcal{E}_{(M)}$ satisfies (Ω) then also $\mathcal{E}_{(M)}^{2\pi}$: By [17, 2.1. Lemma, p. 114], $\mathcal{E}_{(M)}$ contains $\mathcal{E}_{(M)}^{2\pi}$ as a complemented subspace, hence there exists a subspace F in $\mathcal{E}_{(M)}$ such that $\mathcal{E}_{(M)}/F \cong \mathcal{E}_{(M)}^{2\pi}$. By [21, 29.11 Lemma (1) and (2), p. 347], the claim follows.

We see that $\mathcal{E}_{(M)}^{2\pi}$ satisfies always condition (Ω) , hence by 8.3.1(c) the space $\Lambda(\mu)$ has the property (Ω) , too. For $c \in \Lambda(\mu)$, $c := (c_j)_j$, the dual norms have the form $\|c\|_k^* = (\sum_{j \in \mathbb{N}} |c_j|^2 \cdot e^{-2k\mu(j \cdot k)})^{1/2}$ and because of the property (Ω) we have:

$$\forall p \in \mathbb{N} \exists q \in \mathbb{N} \forall k \in \mathbb{N} \exists \nu \exists C : \|\cdot\|_q^{*1+\nu} \leq C \cdot \|\cdot\|_k^* \cdot \|\cdot\|_p^{*\nu}.$$

If we set $p = 1$ and $k = 2q$ and apply the inequality above to the unit vectors $e_l, l \in \mathbb{N}$, we see for arbitrary $l \in \mathbb{N}$:

$$2q \cdot \mu(2ql) + \nu \cdot \mu(l) \leq (1 + \nu) \cdot q \cdot \mu(ql) + C.$$

Note that the function μ is increasing and so $q \cdot \mu(ql) \leq q \cdot \mu(2ql)$ holds for all l , thus we obtain the following estimate:

$$q \cdot \mu(2ql) \leq \nu \cdot (q \cdot \mu(ql) - \mu(l)) + C \leq \nu \cdot q \cdot \mu(ql) + C \text{ for } l \in \mathbb{N}.$$

This shows for $t \in \mathbb{R}$ large enough

$$\mu(2t) \leq C \cdot \mu(t) \text{ for a constant } C. \quad (8.3.3)$$

Finally (8.3.3) $\Leftrightarrow$ (c) holds by the definition of the function μ .

(c) $\Rightarrow$ (d): Because μ is an increasing function we have always $\mu(t) \leq \mu(2t)$ for all t and by 8.3.1(c) and inequality (8.3.3) we have now

$$\mathcal{E}_{(M)}^{2\pi} \underbrace{\cong}_{8.3.1(a)} \Lambda((M)) \underbrace{\cong}_{8.3.1(c)} \Lambda(\mu) \underbrace{\cong}_{(8.3.3)} \Lambda_{\infty}(\mu). \quad (8.3.4)$$

In general we have the following short exact sequence of locally convex spaces (see [17, 2.2 Proposition, p. 114-115]):

$$0 \longrightarrow \mathcal{E}_{(M)}^{2\pi} \xrightarrow{I} \mathcal{E}_{(M)} \xrightarrow{T} \Gamma_{(M)} \longrightarrow 0$$

where $\Gamma_{(M)} := \{c = (c_j)_j \in \mathbb{C}^{\mathbb{N}} : \sup_{j \geq 0} |c_j| \cdot \frac{k^j}{M_j} < \infty, \forall k \geq 1\}$ and $T(f) := (f^{(j)}(2\pi) - f^{(j)}(0))_j$. Obviously $\text{im}(I) = \ker(T)$ holds and T is also bounded, hence continuous, because of the definition of the spaces.

Furthermore we have $\Gamma_{(M)} \cong \Lambda_{\infty}(n)$ (by [17, p. 115]) and so we see that in the short exact sequence above the left and the right space are power series spaces of infinite type, hence Fréchet-Hilbert-spaces, and in the middle we have a Fréchet-space, which is nuclear, hence Fréchet-Hilbert, too (by [21, 28.1 Lemma, p. 325]). A power series space of infinite type Λ_{∞} has property (DN) by [21, 29.2 Lemma (3), p. 339], and property (Ω) by [21, 29.11 Lemma (3), p. 347-348]. So we can apply the *splitting-theorem* [21, 30.1 Theorem, p. 357-368] to obtain for some sequence $\beta := (\beta_n)_n$:

$$\mathcal{E}_{(M)} \cong \Lambda_{\infty}(\mu) \oplus \Lambda_{\infty}(n) \cong \Lambda_{\infty}(\beta) \cong \Lambda_{\infty}(\beta)''. \quad (8.3.5)$$

For this note that power-series spaces $\Lambda_r(\cdot)$ are reflexive for $r \in \mathbb{R} \cup \{+\infty\}$ (see [21, p. 337]). Finally we have the following equalities:

$$\Lambda_{\infty}(\beta)' \underbrace{=}_{(\diamond)} \Delta(\Lambda_{\infty}(\beta)) \underbrace{=}_{(8.3.5)} \Delta(\mathcal{E}_{(M)}) \underbrace{=}_{(\diamond\diamond)} \Delta(\Lambda((M))) \underbrace{=}_{(8.3.4)} \Delta(\Lambda_{\infty}(\mu)) \underbrace{=}_{(\diamond)} \Lambda_{\infty}(\mu)'.$$

$\Delta(\cdot)$ denotes here the *diametral dimension* of power series spaces (see [9, p. 212]): $\Delta(\Lambda_{\infty}(\cdot)) = (\Lambda_{\infty}(\cdot))'$. $(\diamond)$ holds by [17, formula (2.2), p. 115] and $(\diamond\diamond)$ is satisfied by [17, 2.3. Corollary (a), p. 116].

Hence we see by reflexivity: $\mathcal{E}_{(M)} \cong \Lambda_{\infty}(\mu)$.

□

8.4 Comparison theorem

We want to use the introduced techniques to prove a comparison theorem. First we define now special real valued functions, which will play the crucial role in this section. We call a mapping $\omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ a *weight function*, if it satisfies the following conditions: ω is continuous, increasing on $\mathbb{R}_{\geq 0}$, $\omega(0) = 0$ and

$$(\omega_1): \omega(2t) = O(\omega(t)) \text{ as } t \rightarrow \infty, \text{ i.e. } \limsup_{t \rightarrow \infty} \frac{\omega(2t)}{\omega(t)} < \infty.$$

$$(\omega_2): \omega(t) = O(t) \text{ as } t \rightarrow \infty, \text{ i.e. } \limsup_{t \rightarrow \infty} \frac{\omega(t)}{t} < \infty.$$

$$(\omega_3): \log(t) = o(\omega(t)) \text{ as } t \rightarrow \infty, \text{ i.e. } \lim_{t \rightarrow \infty} \frac{\log(t)}{\omega(t)} = 0.$$

$$(\omega_4): \varphi_\omega : t \mapsto \omega(e^t) \text{ is a convex function on } \mathbb{R}.$$

If ω also satisfies condition

$$(\omega_Q) : \int_1^\infty \frac{\omega(t)}{t^2} dt = \infty,$$

then ω is called a *quasi-analytic weight*. If ω doesn't satisfy (ω_Q) , then ω is called a *non quasi-analytic weight*. Sometimes ω is extended to an even function on $\mathbb{R}$, by $\omega(-x) := \omega(x)$ for $x \in \mathbb{R}_{\geq 0}$. For the function φ_ω in condition (ω_4) we remark: It is an increasing function by definition and by (ω_4) convex on $\mathbb{R}_{\geq 0}$. Further $0 = \lim_{t \rightarrow \infty} \frac{\log(t)}{\omega(t)} = \lim_{x \rightarrow \infty} \frac{x}{\varphi_\omega(x)}$ holds by (ω_3) . Now define the *Legendre-Fenchel-conjugate* φ_ω^* in the following way:

$$\varphi_\omega^*(x) := \sup_{y \geq 0} \{x \cdot y - \varphi_\omega(y)\}, \text{ for } x \geq 0.$$

With these notations we can define function spaces of Beurling- resp. Romieu-case via a weight function in the following way: Let $K \subseteq \mathbb{R}^n$ a compact subset and $m \in \mathbb{N}$. For $f \in \mathcal{E}(K)$ we set

$$|f|_{K,m} := \sup_{x \in K, \alpha \in \mathbb{N}^n} |f^{(\alpha)}(x)| \cdot \exp\left(-\frac{1}{m} \cdot \varphi_\omega^*(m \cdot |\alpha|)\right)$$

and $\mathcal{E}_{\omega,m}(K) := \{f \in \mathcal{E}(K) : |f|_{K,m} < \infty\}$. For $G \subseteq \mathbb{R}^n$ open we define the Romieu-type function space $\mathcal{E}_{\{\omega\}}(G)$ as follows:

$$\mathcal{E}_{\{\omega\}}(G) := \{f \in \mathcal{E}(G) : \forall K \subset G \text{ compact } \exists m \in \mathbb{N} : |f|_{K,m} < \infty\}.$$

Hence we obtain the locally convex topology via the representation

$$\mathcal{E}_{\{\omega\}}(G) = \lim_{\leftarrow K} \lim_{\rightarrow m} \mathcal{E}_{\omega,m}(K). \quad (8.4.1)$$

Similarly we define the Beurling-type function space on G :

$$\mathcal{E}_{(\omega)}(G) := \{f \in \mathcal{E}(G) : \forall K \subset G \text{ compact } \forall m \in \mathbb{N} : |f|_{K,1/m} < \infty\}.$$

Here we have the representation

$$\mathcal{E}_{(\omega)}(G) = \lim_{\leftarrow K} \lim_{\leftarrow m} \mathcal{E}_{\omega,m}(K), \quad (8.4.2)$$

thus $\mathcal{E}_{(\omega)}(G)$ is a Fréchet-space.

In this section, if it is not stated otherwise, we assume for a weight sequence $M := (M_p)_p$ the following conditions:

$$1 = M_0 \leq M_1, \text{ log. convexity, (8.2.5) for } A, H > 1$$

and furthermore the following new property

$$\exists c > 0 \forall p \in \mathbb{N} : (c \cdot (p+1))^p \leq M_p. \quad (8.4.3)$$

We remark that (8.4.3) has an important consequence:

$$(8.4.3) \iff \exists c > 0 : c \leq \frac{M_p^{1/p}}{p+1} \quad \forall p \in \mathbb{N} \implies \lim_{p \rightarrow \infty} M_p^{1/p} = \infty.$$

Now we introduce a further important property of the sequence $(\mu_p)_p$:

$$(\beta_3) : \Leftrightarrow \exists Q \in \mathbb{N} : \liminf_{j \rightarrow \infty} \frac{\mu_{Q \cdot j}}{\mu_j} > 1.$$

So we see that $(\beta_1) \Rightarrow (\beta_3)$ holds by definition.

The first step for the proof of the comparison theorem is the following lemma:

Lemma 8.4.1 [2, 12. Lemma] *Let $(M_p)_p$ a weight sequence and consider the assertions:*

- (i) *There exists a weight function ω such that $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) \cong \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ as locally convex vector spaces.*
- (ii) *The sequence $(\mu_p)_p$ satisfies condition (β_3) .*
- (iii) *There exists $C > 1$ and $A > 0$ such that for all $t \geq 0$ we have $\mu(2t) \leq C \cdot \mu(t) + A$.*
- (iv) *The associated function ω_M of the sequence $(M_p)_p$ satisfies $\omega_M(2t) = O(\omega_M(t))$ as $t \rightarrow \infty$.*
- (v) *The associated function ω_M of the sequence $(M_p)_p$ is a weight function.*

Then the following implications hold: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$.

Proof. $(i) \Rightarrow (ii)$: By assumption we have the following isomorphisms:

$$\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) \underbrace{\cong}_{(i)} \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R}) \underbrace{\cong}_{[20, 3.8 \text{ Corollary, p. 79}]} \lambda^1(A_\omega),$$

where the second isomorphism above is given by the Fourier-mapping $\mathcal{F} : f \mapsto (\hat{f}_j)_j$, $j \in \mathbb{Z}$, the sequence space $\lambda^1(A_\omega) := \{c := (c_j)_j \in \mathbb{C}^{\mathbb{Z}} : \|c\|_k := \sum_{j \in \mathbb{Z}} |c_j| \cdot e^{k \cdot \omega(j)} < \infty, \forall k \in \mathbb{N}\}$ is a power series space of infinite type and $A_\omega := (\exp(k \cdot \omega(j)))_{j,k}$ defines clearly a Köthe-Matrix. Thus $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R})$ is isomorphic to a power series space of infinite type and so we can apply 8.3.2 (d) $\Leftrightarrow$ (c) to obtain (ii), because $\liminf_{j \rightarrow \infty} \frac{\mu_{Q \cdot j}}{\mu_j} \geq 2 > 1$.

$(ii) \Rightarrow (iii)$: By assumption we have: There exist $\varepsilon > 0$, $p_0 \in \mathbb{N}$ such that $\mu_{Q \cdot j} \geq (1 + \varepsilon) \cdot \mu_j$ holds for all $j \geq p_0$. By iterating this argument we see $\mu_{Q^2 \cdot j} \geq (1 + \varepsilon)^2 \cdot \mu_j$ for all $j \geq p_0$ and so there exists $C := Q^l$ for a $l \in \mathbb{N}$ such that $2\mu_p \leq \mu_{C \cdot p}$ for all $p \geq p_0$. Now we choose $t_0 > 0$ such that

$$C \cdot (p_0 + 1) < \mu(t_0). \quad (8.4.4)$$

Fix $t \in \mathbb{R}$, such that $t \geq t_0$ and let $p_1 \in \mathbb{N}$ be the largest integer such that $\mu_{p_1} \leq 2t$ holds. Finally choose $q \in \mathbb{N}$ such that

$$q \cdot C \leq p_1 < (q + 1) \cdot C \quad (8.4.5)$$

holds. Now we estimate:

$$\mu_{C \cdot (p_0 + 1)} \underbrace{\leq}_{(8.4.4)} \mu_{\mu(t_0)} \leq t_0 \leq t < 2t,$$

and so $C \cdot (p_0 + 1) \leq p_1$ because of the choice of p_1 . By the choice of q and (8.4.5) we have $q \geq p_0$ and so

$$2\mu_q \leq \mu_{C \cdot q} \underbrace{\leq}_{(8.4.5)} \mu_{p_1} \leq 2t.$$

From this we get $\mu_q \leq t$, thus $q \leq \mu(t)$. Finally, by the choice of q and the maximality of p_1 , we can estimate as follows:

$$\mu(2t) = p_1 < (q + 1) \cdot C \leq C \cdot \mu(t) + C, \text{ for } t \geq t_0.$$

This estimate above holds also for $0 \leq t < t_0$, because μ is an increasing function.

(iii) $\Rightarrow$ (iv): First we see that by definition of the associated function for $t > 0$ the following holds:

$$\omega_M(t) \geq \log\left(\frac{t}{M_1}\right) = \log(t) - \log(M_1). \quad (8.4.6)$$

So we can calculate:

$$\begin{aligned} \omega_M(2t) &\stackrel{8.2.5}{=} \int_{\mu_1}^{2t} \frac{\mu(\lambda)}{\lambda} d\lambda = \int_{\mu_1/2}^t \frac{\mu(2s)}{s} ds \stackrel{\text{assumption}}{\leq} \int_{\mu_1/2}^t \frac{C \cdot \mu(s) + A}{s} ds \\ &= C \cdot \int_{\mu_1/2}^t \frac{\mu(s)}{s} ds + A \cdot (\log(t) - \log(\mu_1/2)) \\ &\stackrel{(8.4.6)}{\leq} C \cdot \omega_M(t) + A \cdot (\omega_M(t) + \log(M_1) + \log(2/M_1)) \leq (C + A) \cdot \omega_M(t) + A \cdot \log(2). \end{aligned}$$

(iv) $\Rightarrow$ (v): ω_M is an even and increasing function on $\mathbb{R}_{\geq 0}$ and $\omega_M(0) = 0$. If $t \in [-m, m]$ for $|m| < \infty$, then the supremum in the definition of $\omega_M(t)$ is a maximum and so we obtain the continuity of the associated function. Condition (ω_1) is exactly assumption (iv). To prove (ω_4) we remark that for arbitrary $t \in \mathbb{R}$ by definition

$$\omega_M(e^t) = \sup_{p \in \mathbb{N}} \log\left(\frac{e^{t \cdot p}}{M_p}\right) = \sup_{p \in \mathbb{N}} (t \cdot p - \log(M_p))$$

holds, so $\varphi_{\omega_M} : t \mapsto \omega_M(e^t)$ is a convex function on $\mathbb{R}$. For condition (ω_2) we set

$$\sigma(t) := \sup_{p \in \mathbb{N}} p \cdot \log\left(\frac{|t|}{p+1}\right), \text{ where } t \in \mathbb{R}.$$

So there exists $D \geq 1$ such that $\sigma(t) \leq D \cdot t + D$ and we estimate for $t > 0$, using the new condition (8.4.3):

$$\omega_M(t) = \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{M_p}\right) \stackrel{(8.4.3)}{\leq} \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{(c \cdot (p+1))^p}\right) = \sigma\left(\frac{t}{c}\right) \leq \frac{D}{c} \cdot t + D.$$

This estimate implies (ω_2) and shows $\omega_M(\mathbb{R}) \subseteq \mathbb{R}$, too. Condition (ω_3) is clear by the definition of the associated function because

$$\lim_{t \rightarrow \infty} \frac{\log(t)}{\omega_M(t)} = \lim_{t \rightarrow \infty} \frac{\log(t)}{\sup_{p \in \mathbb{N}} (p \cdot \log(t) - \log(M_p))} = 0.$$

Thus we see: The associated function ω_M is a weight function. □

Proposition 8.4.2 [2, 13. Proposition] *Let $(M_p)_p$ be a weight sequence, then the following statements are equivalent:*

- (i) *There exists a weight function ω such that $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) = \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ as vector spaces.*
- (ii) *There exists a weight function ω such that $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) = \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ as locally convex vector spaces.*
- (iii) *$(M_p)_p$ satisfies conditions (6.1.2) and (β_3) .*
- (iv) *$(M_p)_p$ satisfies condition (6.1.2), the associated function ω_M is a weight function and (i) holds with $\omega = \omega_M$.*

Moreover for every ω satisfying (i) the following holds:

(v) $\exists A \geq 1, B > 0$, such that $\frac{1}{A} \cdot \omega_M(t) - B \leq \omega(t) \leq A \cdot \omega_M(t) + B$ holds for all $t \geq 0$.

Proof. (i) $\Rightarrow$ (ii): For functions in $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R})$ resp. in $\mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ we can restrict them on the compact interval $[0, 2\pi]$, and so by (2.0.2) and (8.4.2), in both cases we have Fréchet-spaces. The identity map $\text{id} : \mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) \rightarrow \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ has closed graph, because the point evaluations are continuous for both topologies. So we can apply now the open mapping theorem for Fréchet-spaces to obtain (ii).

(ii) $\Rightarrow$ (iii): The existence of such a $Q \in \mathbb{N}$ follows by 8.4.1 (i) $\Rightarrow$ (ii). Now we prove that condition (6.1.2) is satisfied: By assumption and by 8.3.1 (a) and [20, 3.8 Corollary, p. 79] we have

$$\lambda^1(A_{\omega_M}) \underbrace{\cong}_{8.3.1(a)} \mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) \underbrace{=}_{(ii)} \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R}) \underbrace{\cong}_{[20, 3.8 \text{ Corollary, p. 79}]} \lambda^1(A_{\omega}).$$

Both isomorphisms above are given by the Fourier-mapping $\mathcal{F} : f \mapsto (\hat{f}_j)_j$, $j \in \mathbb{Z}$, thus we conclude $\lambda^1(A_{\omega_M}) = \lambda^1(A_{\omega})$.

Hence there exist $D > 0, k \in \mathbb{N}$ resp. $E > 0, l \in \mathbb{N}$, such that for all $j \in \mathbb{N}$ we have:

$$\exp(\omega_M(j)) \leq D \cdot \exp(k \cdot \omega(j)) \quad \text{and} \quad \exp(2k \cdot \omega(j)) \leq E \cdot \exp(\omega_M(l \cdot j)).$$

Thus we obtain

$$2\omega_M(j) \leq \omega_M(l \cdot j) + \log(D^2 \cdot E) \quad (8.4.7)$$

for all $j \in \mathbb{N}$. In this case, if $t \in [j, j+1]$, we can estimate:

$$\begin{aligned} 2\omega_M(t) &\leq 2\omega_M(j+1) \underbrace{\leq}_{(8.4.7)} \omega_M(l \cdot (j+1)) + \log(D^2 \cdot E) \leq \omega_M(2l \cdot j) + \omega_M(l) + \log(D^2 \cdot E) \\ &\leq \omega_M(2l \cdot t) + \omega_M(l) + \log(D^2 \cdot E). \end{aligned}$$

Applying 8.2.9 we obtain condition (6.1.2) for the sequence $(M_p)_p$.

(iii) $\Rightarrow$ (iv): The associated function ω_M of the weight sequence $(M_p)_p$ is a weight function by 8.4.1(ii) $\Rightarrow$ (v). By assumption we have (6.1.2) for the weight sequence and so we can apply again 8.2.9. W.l.o.g. we can assume $H \in \mathbb{N}$ and $A \geq 1$ for the two constants appearing in (6.1.2) and we obtain

$$2^l \cdot \omega_M(j) \leq \omega_M(H^l \cdot j) + 2^l \cdot \log(A), \quad (8.4.8)$$

where $l, j \in \mathbb{N}$ by iterating the estimate in 8.2.9. On the other hand the associated function is a weight function, hence by (ω_1) we conclude: There exists $K \in \mathbb{N}$, such that for $t > 0$ $\omega_M(2t) \leq K \cdot \omega_M(t) + K$ holds. Again we iterate this estimate to get for $l, j \in \mathbb{N}$:

$$\omega_M(2^l \cdot j) \leq K^l \cdot \omega_M(j) + l \cdot K^l. \quad (8.4.9)$$

(8.4.8) and (8.4.9) together have the consequence that the *Köthe-matrix* $A_{\omega_M} := (\exp(\omega_M(k \cdot j)))_{k \in \mathbb{N}, j \in \mathbb{Z}}$ defines the same sequence space as $\mathcal{M} := (\exp(k \cdot \omega_M(j)))_{k \in \mathbb{N}, j \in \mathbb{Z}}$, which we denote by $\lambda^1(\mathcal{M})$. We summarize:

$$\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) \underbrace{\cong}_{8.3.1(a)} \lambda^1(A_{\omega_M}) = \lambda^1(\mathcal{M}) \underbrace{\cong}_{[20, 3.8 \text{ Corollary, p. 79}]} \mathcal{E}_{(\omega_M)}^{2\pi}(\mathbb{R}),$$

where both isomorphisms above are given again by the Fourier-mapping $\mathcal{F}$.

Finally the implication (iv) $\Rightarrow$ (i) is trivial.

(v): Here we use the arguments and estimates in (ii) $\Rightarrow$ (iii) and furthermore, because assumption 8.4.1(i) is satisfied, we use 8.4.1(iv) to conclude: There exist $A_1 \geq 1, A_2 \geq 1$ such that for all $j \in \mathbb{N}$ we obtain

$$\omega_M(j) \leq A_1 \cdot \omega(j) + A_1 \quad \text{and} \quad \omega(j) \leq A_2 \cdot \omega_M(j) + A_2.$$

To prove (v) for all $t \geq 0$ we remark that by 8.4.1(v) the associated function ω_M of the sequence $(M_p)_p$ is a weight function.

□

To prove a first version of the comparison theorem we have to deal with increasing properties of the *Legendre-Fenchel-conjugate*:

Proposition 8.4.3 *Let $M := (M_p)_p$ be a weight sequence, if ω_M is a weight function, then for $l \leq k$ and each $p \in \mathbb{N}$ we have: $k \cdot \varphi_{\omega_M}^*\left(\frac{p}{k}\right) \leq l \cdot \varphi_{\omega_M}^*\left(\frac{p}{l}\right)$.*

Proof. ω_M is a weight function, hence it is an increasing function and by (ω_4) it follows that φ_{ω_M} is positive and convex on $\mathbb{R}_{\geq 0}$. Further $\varphi_{\omega_M}(0) = \omega_M(1) = 0$, φ_{ω_M} is increasing by definition and $\lim_{t \rightarrow \infty} \frac{\log(t)}{\omega_M(t)} = \lim_{x \rightarrow \infty} \frac{x}{\varphi_{\omega_M}(x)} = 0$ holds by (ω_3) . Thus the Legendre-Fenchel-conjugate $\varphi_{\omega_M}^* : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is increasing and satisfies $\varphi_{\omega_M}^*(0) = 0$, because $\varphi_{\omega_M}(0) = 0$ and φ_{ω_M} is increasing. $\varphi_{\omega_M}^*$ is convex: Since the function $x \mapsto x \cdot y - f(y)$ is continuous and linear the supremum $\varphi_{\omega_M}^*(x)$ is lower semi-continuous and convex.

It follows now that the function $\frac{\varphi_{\omega_M}^*(x)}{x}$ is increasing on $\mathbb{R}_{\geq 0}$, because we can estimate for $\varphi_{\omega_M}^*$ and $x < y$

$$\varphi_{\omega_M}^*(x) \leq x \cdot \frac{\varphi_{\omega_M}^*(y)}{y} + \left(1 - \frac{x}{y}\right) \cdot \underbrace{\varphi_{\omega_M}^*(0)}_{=0},$$

which is equivalent to $\frac{\varphi_{\omega_M}^*(x)}{x} < \frac{\varphi_{\omega_M}^*(y)}{y}$.

□

Furthermore we will have to use another important result by Grothendieck, which is a special version of the *closed graph theorem*:

Proposition 8.4.4 [6, Théorème B (2), p. 17-18] *Let F be a (LB) and E a (LF)-space and $u : F \rightarrow E$ a linear mapping. Then we get: If the graph of u is closed in $F \times E$, then u is already a continuous linear mapping. In fact, the following is sufficient for the continuity of u : There is no sequence $(x_i)_i$ in F with $x_i \rightarrow 0$ for $i \rightarrow \infty$ such that $u(x_i) \rightarrow \alpha$ in a Fréchet-space E_n , where $\alpha \neq 0$ and $E := \varinjlim E_n$.*

Proof. First we remark that we can reduce the proof to the case, where F is a Banach-space. We denote with H the graph of u which is a closed subspace in $F \times E$ by hypotheses, hence (LF). If we set $H_i := H \cap (F \times E_i)$, then the second hypotheses, which is weaker than the first one, implies the fact that H_i is a closed subspace in $F \times E_i$. The spaces H_i are endowed with a Fréchet-topology which is finer than the topology induced by H and further we can write $H = \bigcup_i H_i$. Applying now 5.1.10 to the mapping $\text{pr}_1 : H \rightarrow F$ yields to the fact that pr_1 is an open mapping on $\text{pr}_1(H) = F$. This implies the continuity of $v := \text{pr}_1^{-1}$ and because $u = \text{pr}_2 \circ v$, where $\text{pr}_2 : H \rightarrow E$, or equivalently $v = (\text{id}_F, u)$, we obtain the continuity of the mapping u .

□

For the proof of the comparison result we will need some further properties of the space $\mathcal{E}_{\{M\}}^{2\pi}$:

Proposition 8.4.5 [17, 4.1. Proposition, p. 119-120] *Let $M := (M_p)_p$ be a log. convex weight sequence such that (8.2.5) holds. Then the following are equivalent:*

- (a) $(\mathcal{E}_{\{M\}}^{2\pi})'_b \in (\underline{DN})$.
- (b) Condition (8.3.3) is satisfied.

Remark that as we have mentioned in (8.3.1)(b), $\Lambda(\{M\})$ is a (LF) -space, its strong dual is a Fréchet-space and so property (DN) makes sense in this situation (therefore see [21, Definition (a), p. 348]).

Proof. (a) $\Rightarrow$ (b): Assume $(\mathcal{E}_{\{M\}}^{2\pi})'_b \in (DN)$, then using again [21, 29.12 Lemma (1), p. 348] and (8.3.1)(b), we see: $(\Lambda(\{M\}))'_b \in (DN)$. Thus we get by definition of property (DN) :

$$\exists r \in \mathbb{N} \forall p \in \mathbb{N} \exists C > 0, n > 1 \exists q \in \mathbb{N} : \|\cdot\|_{1/p}^* \leq C \|\cdot\|_{1/r}^{*^{1/n}} \|\cdot\|_{1/q}^{*^{1-1/n}},$$

where we have set for the dual norms $\|c\|_{1/k}^{*^2} := \sum_{j=1}^{\infty} |c_j|^2 \cdot e^{-2 \cdot \omega_M(j/k)}$, for $c := (c_j)_j \in (\Lambda(\{M\}))'_b$. This is equivalent to

$$-n \cdot \omega_M\left(\frac{t}{p}\right) \leq -\omega_M\left(\frac{t}{r}\right) - (n-1) \cdot \omega_M\left(\frac{t}{q}\right) + \log(C) \text{ for } t \in \mathbb{R},$$

which means

$$\omega_M\left(\frac{t}{r}\right) - \omega_M\left(\frac{t}{p}\right) \leq (n-1) \cdot \left(\omega_M\left(\frac{t}{p}\right) - \omega_M\left(\frac{t}{q}\right)\right) + \log(C) \text{ for } t \in \mathbb{R}. \quad (8.4.10)$$

Note that $(M_p)_p$ is log. convex, thus the functions μ and ω_M are increasing and we can use again 8.2.5. Hence for $p = 4r$ we have:

$$\begin{aligned} \log(2) \cdot \mu\left(\frac{t}{2r}\right) &= \mu\left(\frac{t}{2r}\right) \cdot \int_{t/2r}^{t/r} \frac{1}{\lambda} d\lambda \leq \int_{t/2r}^{t/r} \frac{\mu(\lambda)}{\lambda} d\lambda \underbrace{=}_{8.2.5} \omega_M\left(\frac{t}{r}\right) - \omega_M\left(\frac{t}{2r}\right) \\ &\leq \omega_M\left(\frac{t}{r}\right) - \omega_M\left(\frac{t}{4r}\right) \underbrace{\leq}_{(8.4.10)} (n-1) \cdot \left(\omega_M\left(\frac{t}{4r}\right) - \omega_M\left(\frac{t}{q}\right)\right) + \log(C) \\ &\leq D \cdot \mu\left(\frac{t}{4r}\right) \end{aligned}$$

for a constant $D > 0$, which implies (8.3.3). The last inequality above holds because $\|\cdot\|_{1/q}^*$ is increasing in q and so one can choose w.l.o.g. $q \in \mathbb{N}$ arbitrary large. In particular we take $q \geq p = 4r$ to obtain for $t \in \mathbb{R}$ large enough $\log(C) \leq (n-1) \cdot \omega_M\left(\frac{t}{q}\right)$. Note that t is depending here on q .

(b) $\Rightarrow$ (a): In this case we can estimate in the following way:

$$\begin{aligned} \omega_M\left(\frac{t}{r}\right) - \omega_M\left(\frac{t}{p}\right) &\underbrace{=}_{8.2.5} \int_{t/p}^{t/r} \frac{\mu(\lambda)}{\lambda} d\lambda \leq \log\left(\frac{p}{r}\right) \cdot \mu\left(\frac{t}{r}\right) \underbrace{\leq}_{(8.3.3)} D_1 \cdot \mu\left(\frac{t}{p \cdot r}\right) \\ &\leq D_2 \cdot \int_{t/pr}^{t/p} \frac{\mu(\lambda)}{\lambda} d\lambda \underbrace{=}_{8.2.5} D_2 \cdot \left(\omega_M\left(\frac{t}{p}\right) - \omega_M\left(\frac{t}{p \cdot r}\right)\right). \end{aligned}$$

This implies (8.4.10) which is equivalent to (DN) .

□

Using 8.4.5 we can prove now

Proposition 8.4.6 [17, 4.3. Theorem, p. 121-122] Let $M := (M_p)_p$ be a log. convex weight sequence such that (8.2.5) holds. Consider the following assertions:

- (a) The space $\mathcal{E}_{\{M\}}^{2\pi}$ is isomorphic to the strong dual of a power series space of finite type.
- (b) $(\mathcal{E}_{\{M\}}^{2\pi})'_b \in (\bar{\Omega}) \cap (DN)$.

(c) $(M_p)_p$ satisfies the conditions (6.1.2) and (8.3.3).

Then we get the following implications: (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

For a definition of property $(\bar{\Omega})$ see [21, Definition (b), p. 348].

Proof. (a) $\Rightarrow$ (b): Holds by [21, 29.12 Lemma (4) and (5), p. 348] and reflexivity.

(b) $\Rightarrow$ (c): If $(\mathcal{E}_{\{M\}}^{2\pi})'_b \in (\bar{\Omega})$, then by [21, 29.12 Lemma (1), p. 348] and 8.3.1(b) we obtain $(\Lambda(\{M\}))'_b \in (\bar{\Omega})$, too. So the norms $\|\cdot\|_{1/k}$ in $\Lambda(\{M\})$ are now the "dual norms" and we get the following estimate by the definition of $(\bar{\Omega})$:

$$\forall d > 0 \forall p \in \mathbb{N} \exists q \in \mathbb{N} \forall k \in \mathbb{N} \exists C_k > 0 : \|\cdot\|_{1/q}^{1+d} \leq C_k \|\cdot\|_{1/k} \|\cdot\|_{1/p}^d.$$

In particular for $d = 1$:

$$\forall p \in \mathbb{N} \exists q \in \mathbb{N} \forall k \in \mathbb{N} \exists C_k > 0 : \|\cdot\|_{1/q}^2 \leq C_k \|\cdot\|_{1/k} \|\cdot\|_{1/p}.$$

Now we apply this inequality above to the unit vectors $e_l \in \Lambda(\{M\})$ for all $l \in \mathbb{N}$ and get for the associated function ω_M :

$$2 \cdot \omega_M\left(\frac{l}{q}\right) \leq \omega_M\left(\frac{l}{k}\right) + \omega_M\left(\frac{l}{p}\right) + \log(C_k). \quad (8.4.11)$$

Now we set $p = 1$ and $k = q^3$. Note that by the log. convexity of $(M_p)_p$ the function μ is by definition in (8.2.3) increasing and by (8.4.11) and 8.2.5 we can estimate as follows:

$$\begin{aligned} 2 \cdot \log(q) \cdot \mu\left(\frac{l}{q^3}\right) &\leq \int_{l/(q^3)}^{l/q} \frac{\mu(\lambda)}{\lambda} d\lambda \underbrace{=}_{8.2.5} \omega_M\left(\frac{l}{q}\right) - \omega_M\left(\frac{l}{q^3}\right) \\ &\underbrace{\leq}_{(8.4.11)} \omega_M(l) - \omega_M\left(\frac{l}{q}\right) + \log(C_k) \underbrace{=}_{8.2.5} \int_{l/q}^l \frac{\mu(\lambda)}{\lambda} d\lambda \\ &\leq \log(q) \cdot \mu(l) + \log(C_k). \end{aligned}$$

So we have shown

$$2 \cdot \log(q) \cdot \mu\left(\frac{l}{q^3}\right) \leq \log(q) \cdot \mu(l) + \log(C_k) \quad (8.4.12)$$

and (8.4.12) implies (8.5.2), because $\mu(t) \rightarrow \infty$ for $t \rightarrow \infty$, and by the same argumentation as in 8.5.1 we obtain property (6.1.2). Finally (8.3.3) follows by 8.4.5.

□

Now we formulate and prove the first comparison theorem:

Theorem 8.4.7 [2, 14. Theorem] Let $(M_p)_p$ be a weight sequence, then the following are equivalent:

- (i) There exists a weight function ω such that for all $n \in \mathbb{N}$, $n > 0$, and all $G \subseteq \mathbb{R}^n$ open we have: $\mathcal{E}_{(M)}(G) = \mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G) = \mathcal{E}_{\{\omega\}}(G)$ as vector spaces and/or as locally convex vector spaces.
- (ii) There exists a weight function ω and $n \in \mathbb{N}$, $n > 0$, $G \subseteq \mathbb{R}^n$ open, such that $\mathcal{E}_{(M)}(G) = \mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G) = \mathcal{E}_{\{\omega\}}(G)$ as vector spaces.
- (iii) The weight sequence $(M_p)_p$ satisfies conditions (6.1.2) and (β_3) .
- (iv) $(M_p)_p$ satisfies condition (6.1.2), the associated function ω_M is a weight function and (i) holds with $\omega = \omega_M$.

Proof. First we consider the Beurling case. The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are clearly satisfied.

(ii) $\Rightarrow$ (iii): By assumption we have $\mathcal{E}_{(M)}(G+x) = \mathcal{E}_{(\omega)}(G+x)$ for all $x \in \mathbb{R}^n$ and so

$$\prod_{x \in \mathbb{R}^n} \mathcal{E}_{(M)}(G+x) = \prod_{x \in \mathbb{R}^n} \mathcal{E}_{(\omega)}(G+x).$$

$\mathcal{E}_{(M)}$ and $\mathcal{E}_{(\omega)}$ are sheaves on $\mathbb{R}^n$ and so we get $\mathcal{E}_{(M)}(\mathbb{R}^n) = \mathcal{E}_{(\omega)}(\mathbb{R}^n)$. Of course we have the inclusions $\mathcal{E}_{(M)}(\mathbb{R}) \subseteq \mathcal{E}_{(M)}(\mathbb{R}^n)$ resp. $\mathcal{E}_{(\omega)}(\mathbb{R}) \subseteq \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ by restriction on the first variable and so by assumption $\mathcal{E}_{(M)}(\mathbb{R}) = \mathcal{E}_{(\omega)}(\mathbb{R})$ as vector spaces. Finally, this implies $\mathcal{E}_{(M)}^{2\pi}(\mathbb{R}) = \mathcal{E}_{(\omega)}^{2\pi}(\mathbb{R})$ as vector spaces, and we can use now 8.4.2 (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv): Property (6.1.2) in (iv) follows by assumption and the associated function ω_M is a weight function by 8.4.2 (iii) $\Rightarrow$ (iv). For the third assertion we note, that by the log. convexity of the weight sequence $(M_p)_p$ and by 8.2.2 we have for all $p \in \mathbb{N}$:

$$M_p = \sup_{t>0} \frac{t^p}{\exp(\omega_M(t))}. \quad (8.4.13)$$

$(M_p)_p$ satisfies (6.1.2) and so we can use 8.2.9 to conclude for all $t > 0$:

$$2 \cdot \omega_M(t) \leq \omega_M(H \cdot t) + \log(A).$$

In the following it is enough to assume $H \in \mathbb{N}$. To prove the result we need to show the following two inequalities:

$$\forall h \in (0, 1) \exists k \in \mathbb{N} \exists C > 0 \forall p \in \mathbb{N} : \exp(k \cdot \varphi_{\omega_M}^*(p/k)) \leq C \cdot h^p \cdot M_p \quad (8.4.14)$$

and

$$\forall m \in \mathbb{N} \exists h \in (0, 1) \exists D > 0 \forall p \in \mathbb{N} : h^p \cdot M_p \leq D \cdot \exp(m \cdot \varphi_{\omega_M}^*(p/m)). \quad (8.4.15)$$

(8.4.14) and (8.4.15) together imply (i) with $\omega = \omega_M$.

First we prove (8.4.14): We fix $0 < h < 1$ and choose $m \in \mathbb{N}$ such that

$$\frac{1}{2^m} \leq h. \quad (8.4.16)$$

ω_M is a weight function and so by (ω_1) and formula (8.4.9) in 8.4.2 there exists a $K \in \mathbb{N}$ such that

$$\omega_M(2^m \cdot t) \leq K^m \cdot \omega_M(t) + m \cdot K^m \Leftrightarrow -\omega_M(2^m \cdot t) \geq -K^m \cdot \omega_M(t) - m \cdot K^m. \quad (8.4.17)$$

We estimate as follows:

$$\begin{aligned} \log\left(\frac{1}{2^{mp}} \cdot M_p\right) &\stackrel{(8.4.13)}{=} \sup_{t>0} \left(p \cdot \log\left(\frac{t}{2^m}\right) - \omega_M(t)\right) \stackrel{t \mapsto 2^m \cdot \tau}{=} \sup_{\tau>0} (p \cdot \log(\tau) - \omega_M(2^m \cdot \tau)) \\ &\stackrel{(8.4.17)}{\geq} \sup_{\tau>0} (p \cdot \log(\tau) - K^m \cdot \omega_M(\tau) - m \cdot K^m) \\ &\stackrel{x:=\log(\tau)}{\geq} \sup_{x>0} \left(p \cdot x - K^m \cdot \underbrace{\varphi_{\omega_M}(x)}_{\omega_M(\exp(x))}\right) - m \cdot K^m = K^m \cdot \varphi_{\omega_M}^*\left(\frac{p}{K^m}\right) - m \cdot K^m. \end{aligned}$$

From this we get for all $p \in \mathbb{N}$:

$$\exp\left(K^m \cdot \varphi_{\omega_M}^*\left(\frac{p}{K^m}\right)\right) \leq \exp(m \cdot K^m) \cdot \frac{1}{2^{mp}} \cdot M_p \stackrel{(8.4.16)}{\leq} \exp(m \cdot K^m) \cdot h^p \cdot M_p,$$

which implies (8.4.14) with $k := K^m$ and $C := \exp(m \cdot K^m)$.

To prove (8.4.15) first we use 8.4.3: It suffices to take $m = 2^k, k \in \mathbb{N}$. We fix $k \in \mathbb{N}$ and put $h := \frac{1}{H^k}$, where H is the constant appearing in (6.1.2). So we have, like formula (8.4.8) in 8.4.2:

$$2^k \cdot \omega_M(t) \leq \omega_M(H^k \cdot t) + 2^k \cdot \log(A) \Leftrightarrow -\omega_M(H^k \cdot t) \leq -2^k \cdot \omega_M(t) + 2^k \cdot \log(A). \quad (8.4.18)$$

We estimate:

$$\begin{aligned} \log\left(\frac{M_p}{H^{kp}}\right) &\stackrel{(8.4.13)}{=} \sup_{t>0} \left(p \cdot \log\left(\frac{t}{H^k}\right) - \omega_M(t)\right) \stackrel{t \mapsto H^k \cdot \tau}{=} \sup_{\tau>0} (p \cdot \log(\tau) - \omega_M(H^k \cdot \tau)) \\ &\stackrel{(8.4.18)}{\leq} \sup_{\tau>0} (p \cdot \log(\tau) - 2^k \cdot \omega_M(\tau)) + 2^k \cdot \log(A) \\ &\stackrel{x := \log(\tau)}{=} \sup_{x \in \mathbb{R}} (p \cdot x - 2^k \cdot \varphi_{\omega_M}(x)) + 2^k \cdot \log(A) \\ &\stackrel{(*)}{=} \sup_{x>0} (p \cdot x - 2^k \cdot \varphi_{\omega_M}(x)) + 2^k \cdot \log(A) = 2^k \cdot \varphi_{\omega_M}^*\left(\frac{p}{2^k}\right) + 2^k \cdot \log(A). \end{aligned}$$

(*) holds because by definition we have $\varphi_{\omega_M} \geq 0$ and $\varphi_{\omega_M}(0) = \omega_M(1) = 0$. Furthermore $x \cdot y - f(y) \leq 0$ for $y \leq 0$ and $x \cdot 0 - f(0) = 0$, hence $\sup_{y \in \mathbb{R}} \{x \cdot y - f(y)\} = \sup_{y \geq 0} \{x \cdot y - f(y)\}$. From the estimate above we get for all $p \in \mathbb{N}$:

$$h^p \cdot M_p = \frac{1}{H^{kp}} \cdot M_p \leq A^{2^k} \cdot \exp\left(2^k \cdot \varphi_{\omega_M}^*\left(\frac{p}{2^k}\right)\right),$$

which proves (8.4.15) for $m = 2^k$.

Now we consider the Romieu-case. Again the implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are clear.

(ii) $\Rightarrow$ (iii): We can imitate the sheaf-arguments like in the Beurling-case to conclude $\mathcal{E}_{\{M\}}^{2\pi}(\mathbb{R}) = \mathcal{E}_{\{\omega\}}^{2\pi}(\mathbb{R})$ as vector spaces: Because of the periodicity of the functions in these spaces we can restrict them on a compact set K . Hence, by (2.0.1) and (8.4.1), we obtain two (LB)-spaces. Applying 8.4.4 of Grothendieck to the identity map we obtain equality as locally convex vector spaces. Note that the identity map has closed graph because the point evaluations are continuous linear mappings and point separating.

So $\mathcal{E}_{\{M\}}^{2\pi}(\mathbb{R}) = \mathcal{E}_{\{\omega\}}^{2\pi}(\mathbb{R})$ and, again by [20, 3.8 Corollary, p. 79] it follows now that $\mathcal{E}_{\{M\}}^{2\pi}(\mathbb{R})$ is isomorphic to the strong dual space of a power series space of finite type. Thus by 8.4.6 the weight sequence $(M_p)_p$ satisfies the conditions (6.1.2) and (iii) in 8.4.1. Hence by (iii) $\Rightarrow$ (v) in 8.4.1 we can use (iii) $\Leftrightarrow$ (iv) in 8.4.2.

(iii) $\Rightarrow$ (iv): Property (6.1.2) is clear and the associated function ω_M of $(M_p)_p$ is a weight function via implication (ii) $\Rightarrow$ (v) in 8.4.1. To prove the third statement in (iv) we can use the same calculation as in the Beurling-case.

□

Finally we prove the second important comparison theorem:

Theorem 8.4.8 [2, 16. Corollary] *Let ω be an arbitrary weight function, then the following are equivalent:*

- (i) *There exists a weight sequence $(M_p)_p$ such that for all $n \in \mathbb{N}$, $n > 0$, and $G \subseteq \mathbb{R}^n$ open the spaces $\mathcal{E}_{(M)}(G)$ and $\mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G)$ and $\mathcal{E}_{\{\omega\}}(G)$ are equal as vector spaces and/or as locally convex vector spaces.*
- (ii) *There exists a weight sequence $(M_p)_p$, $n \in \mathbb{N}$, $n > 0$, and $G \subseteq \mathbb{R}^n$ open such that $\mathcal{E}_{(M)}(G)$ and $\mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G)$ and $\mathcal{E}_{\{\omega\}}(G)$ are equal as vector spaces.*

(iii) There exists $H \geq 1$ such that for all $t \geq 0$ we have:

$$2 \cdot \omega(t) \leq \omega(H \cdot t) + H. \quad (8.4.19)$$

Furthermore the sequence $(M_p)_p$ defined by $M_p := \varphi_\omega(p)$ is a weight sequence for which condition (i) holds.

Proof. We concentrate on the Beurling-case: The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) hold clearly.

(ii) $\Rightarrow$ (iii): By assumption condition (ii) in 8.4.7 is satisfied for the weight sequence $(M_p)_p$, hence condition (iv) in 8.4.7 holds, too. So the sequence $(M_p)_p$ satisfies (6.1.2) and by 8.4.2 also condition (v) there. Applying once again 8.2.9 we obtain: There exist $H \geq 1, C > 0, D > 0$ and $B > 0, A \geq 1$ such that for all $t \geq 0$:

$$2\omega(t) \underbrace{\leq}_{(v)} 2A \cdot \omega_M(t) + 2B \underbrace{\leq}_{(6.1.2)} \frac{1}{A} \cdot \omega_M(H \cdot t) + D + 2B \underbrace{\leq}_{(v)} \omega(H \cdot t) + C + DB.$$

This implies (8.4.19) in (iii). Furthermore we remark that condition (v) in 8.4.2 implies for all $y \geq 0$:

$$\varphi_\omega^*(y) \leq \frac{1}{A} \cdot \varphi_{\omega_M}^*(A \cdot y) + B \quad \text{and} \quad \varphi_\omega^*(y) \geq A \cdot \varphi_{\omega_M}^*\left(\frac{y}{A}\right) - B.$$

Hence $\mathcal{E}_{(\omega)}(G) = \mathcal{E}_{(\omega_M)}(G)$ for all open subsets G in $\mathbb{R}^n$. The weight sequence satisfies (6.1.2) and its associated function ω_M is a weight function by (iv) in 8.4.7. So by 8.4.7 we obtain $\mathcal{E}_{(\omega_M)}(G) = \mathcal{E}_{(M)}(G)$ for all open subsets G in $\mathbb{R}^n$.

The proof above for the Beurling-case still holds for the Romieu-case, because 8.4.7 is valid for both cases.

□

8.5 Additional results

In this section we construct a not quasi-analytic weight sequence $M := (M_p)_p$ which satisfies condition (6.1.2) and with the following property: For all weight functions ω , $n \in \mathbb{N}$, $n > 0$, and $G \subseteq \mathbb{R}^n$ open we get: $\mathcal{E}_{(M)}(G) \neq \mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G) \neq \mathcal{E}_{\{\omega\}}(G)$. Furthermore we use the introduced notation to prove a new version of the Denjoy-Carleman-theorem.

To create the weight sequence for the counterexample we prove first the following proposition:

Proposition 8.5.1 [17, 3.3. Example, p. 118-119] *There exists a sequence $M := (M_p)_p$ of positive numbers with $M_0 = M_1 = 1$, which is log. convex, not quasi-analytic and satisfies condition (6.1.2), but not property (8.3.3).*

Proof. Put $c_1 := 1$ and define the sequences of positive integers $(c_n)_n$ and $(d_n)_n$ inductively in the following way: $d_n := \lfloor c_n^{3/2} \rfloor + 1$ and $c_{n+1} := d_n^2 + 1$. Then we define the sequence $(\mu_p)_p$:

$$\mu_p := \begin{cases} c_n^3 & \text{for } c_n \leq p \leq \lfloor c_n^{3/2} \rfloor = d_{n-1} \\ \frac{p^4}{d_n^2} & \text{for } d_n \leq p \leq d_n^2 = c_{n+1} \end{cases} \quad (8.5.1)$$

From the definitions above we conclude some properties of $(\mu_p)_p$. First $(\mu_p)_p$ is an increasing sequence: If $p = d_n$, then $\mu_p = d_n^2 \geq c_n^3 = \mu_{p-1}$ and if $p = c_{n+1}$, then $\mu_p = c_{n+1}^3 \geq d_n^6 = \mu_{p-1}$. Furthermore $\sum_{p=1}^{\infty} \frac{1}{\mu_p} < \infty$ because $\mu_p \geq p^2$ for all p . This holds, because for $c_n \leq p < d_n$ we have $p^2 \leq c_n^3 = \mu_p$ and for $d_n \leq p < c_{n+1}$ we have $d_n^2 \leq p^2$, hence $p^2 \leq \frac{p^4}{d_n^2} = \mu_p$.

Now set $M_p := \prod_{i=1}^p \mu_i$. Hence $M_0 = 1$, $M_1 = \mu_1 = c_1 = 1$ and the sequence $M := (M_p)_p$ is log. convex. By 4.1.5 $(M_p)_p$ is not quasi-analytic and we prove now (6.1.2):

Let $c_n \leq p < d_n$ and so, because $2 \leq d_n$ for all n , we estimate

$$\mu_{2p} \leq \mu_{2d_n} \underbrace{\leq}_{d_n < 2d_n \leq d_n^2} \frac{(2d_n)^4}{d_n^2} = 16d_n^2 \underbrace{\leq}_{(\diamond)} 17 \cdot (d_n - 1)^2 \leq 17c_n^3 = 17\mu_p,$$

where $(\diamond)$ holds for all $p \geq 33$. If $d_n \leq p < c_{n+1}$, then:

$$\mu_{2p} \leq \frac{(2p)^4}{d_n^2} = 16 \cdot \frac{p^4}{d_n^2} = 16\mu_p.$$

Thus $\mu_{2p} \leq 17\mu_p$ holds in both cases for p large enough, which is equivalent to

$$2 \cdot \mu(t) \leq \mu(C \cdot t) \text{ for } t \text{ large enough.} \quad (8.5.2)$$

So, by 8.2.5, we obtain for the associated function ω_M of the sequence $(M_p)_p$:

$$2 \cdot \omega_M(t) = \int_0^t \frac{2 \cdot \mu(\lambda)}{\lambda} d\lambda \leq \int_0^t \frac{\mu(C \cdot \lambda)}{\lambda} d\lambda + B = M(C \cdot t) + B.$$

Applying 8.2.9 we conclude property (6.1.2).

But the sequence $(M_p)_p$ doesn't satisfy condition (8.3.3), which is equivalent to (c) in 8.3.2, because $\mu_{C \cdot c_n} = \mu_{c_n}$ holds for any $C \in \mathbb{N}$ and n large enough (which depends on C).

□

Let $M := (M_p)_p$ be the sequence defined in 8.5.1. Then, because $\mu_p \geq p^2$ for all p , we get $M_p \geq p!^2$ for all p and so this sequence satisfies condition (8.4.3), too. Because M satisfies (6.1.2), also (8.2.5) holds. Hence M satisfies the assumptions for the comparison results and we can apply them to conclude:

Assume for a given weight function ω , $n \in \mathbb{N}$ and $G \subseteq \mathbb{R}^n$ open we would have $\mathcal{E}_{(M)}(G) = \mathcal{E}_{(\omega)}(G)$ resp. $\mathcal{E}_{\{M\}}(G) = \mathcal{E}_{\{\omega\}}(G)$. Then, by 8.4.7 (iii), condition (β_3) has to be satisfied, so

$\liminf_{j \rightarrow \infty} \frac{\mu_{Q \cdot j}}{\mu_j} > 1$ for a $Q \in \mathbb{N}$. By the same argument as in the proof of 8.4.1 (ii) $\Rightarrow$ (iii) there exists a $C \in \mathbb{N}$ ($C := Q^l$) such that

$$2 \cdot \mu_p \leq \mu_{C \cdot p}$$

holds for all large $p \in \mathbb{N}$. But this property is exactly condition (c) in 8.3.2 and by 8.5.1 we obtain a contradiction.

As mentioned, we prove now a new version of the Denjoy-Carleman-theorem, using the functions μ and ω_M :

Theorem 8.5.2 [11, Lemma 4.1., p. 55-56] *Let $M := (M_p)_p$ be a log. convex weight sequence such that (8.4.3) is satisfied and $1 = M_0 \leq M_1$. We denote again by ω_M the associated function of M and μ is the mapping defined in (8.2.3), then the following are equivalent:*

- (i) $\mathcal{E}_{\{M\}}$ is not quasi-analytic,
- (ii) $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$,
- (iii) $\sum_{p=1}^{\infty} \left(\frac{1}{M_p}\right)^{1/p} < \infty$,
- (iv) $\int_0^{\infty} \frac{\mu(\lambda)}{\lambda^2} d\lambda < \infty$,
- (v) $\int_0^{\infty} \frac{\omega_M(t)}{t^2} dt < \infty$.

In particular we see: $\mathcal{E}_{\{M\}}$ is not quasi-analytic if and only if the associated function ω_M doesn't satisfy condition (ω_Q).

Proof. (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) holds by 4.1.5 resp. 4.2.4.

First we remark that we can write each partial sum in (ii) as a *Riemann-Stieltjes-integral* in the following way:

$$\sum_{M_p/M_{p-1} \leq t} \frac{M_{p-1}}{M_p} = \sum_{\mu_p \leq t} \frac{1}{\mu_p} \stackrel{[7, 92.4 \text{ Satz, p. 501}]}{=} \int_0^t \frac{1}{\lambda} d\mu.$$

Then we use integration by parts for Riemann-Stieltjes-integrals [7, 90.2 Satz, p. 491-492] to obtain

$$\int_0^t \frac{1}{\lambda} d\mu = \frac{\mu(t)}{t} - \underbrace{\int_0^t \mu(\lambda) d\left(\frac{1}{\lambda}\right)}_{-1/\lambda^2 d\lambda},$$

where $t > 0$. Note that $\mu(\lambda) = 0$ for $\lambda < \mu_1$.

Now let $t \rightarrow \infty$ in the formula above, then (ii) $\Leftrightarrow$ (iv) follows immediately.

We use 8.2.5 and Fubini to see for $x > 0$:

$$\begin{aligned} \int_0^x \frac{\omega_M(t)}{t^2} dt &\stackrel{8.2.5}{=} \int_0^x \left(\int_0^t \frac{\mu(\lambda)}{\lambda} d\lambda \right) \frac{1}{t^2} dt = \int_0^x \frac{\mu(\lambda)}{\lambda} \cdot \underbrace{\left(\int_{\lambda}^x \frac{1}{t^2} dt \right)}_{=1/\lambda - 1/x} d\lambda \\ &= \int_0^x \frac{\mu(\lambda)}{\lambda^2} d\lambda - \frac{1}{x} \cdot \int_0^x \frac{\mu(\lambda)}{\lambda} d\lambda \stackrel{8.2.5}{=} \int_0^x \frac{\mu(\lambda)}{\lambda^2} d\lambda - \frac{\omega_M(x)}{x}. \end{aligned}$$

We have seen: (8.4.3) implies the fact that there exist $c > 0, D \geq 1$ such that $\omega_M(x) \leq \frac{D}{c} \cdot x + D$, hence $\frac{\omega_M(x)}{x}$ is bounded for $x > 0$. Finally we let $x \rightarrow \infty$ in the formula above to obtain (iv) $\Leftrightarrow$ (v).

□

9 Appendix

9.1 Zusammenfassung

Funktionenräume vom Denjoy-Carleman-Typ sind spezielle Klassen glatter Funktionen, welche mit all ihren Ableitungen gewisse Größenabschätzungen erfüllen. Für jede Folge positiver reeller Zahlen $M := (M_p)_p$, die wir als Gewichtsfolge bezeichnen, können wir nun eine Denjoy-Carleman-Klasse von Funktionen definieren. Dabei unterscheiden wir zwei Klassen: Funktionen vom Romieu-Typ $\mathcal{E}_{\{M\}}$ und vom Beurling-Typ $\mathcal{E}_{(M)}$.

Das Ziel ist nun interessante und wünschenswerte Eigenschaften dieser erhaltenen Klassen, wie etwa Abgeschlossenheit bezüglich punktweiser Multiplikation, Komposition und Inversion sowie die Abgeschlossenheit bezüglich Lösungen gewöhnlicher Differentialgleichungen, durch Eigenschaften der Gewichtsfolge zu charakterisieren. Weiters studieren wir die Injektivität und die Surjektivität der Borel-Abbildung.

Anschließend konstruieren wir einige konkrete Beispiele und wenden die bewiesenen Resultate darauf an. Damit können wir die Eigenschaften der Funktionenräume und deren Abhängigkeit von einem Parameter illustrieren. Weiters beweisen wir einen interessanten Zerlegungssatz für glatte Funktionen.

Im letzten Kapitel stellen wir schließlich eine alternative Methode vor, um Räume von gewichteten Funktionen zu definieren: Man kann anstatt der diskreten Variante der Gewichtsfolge M auch eine kontinuierliche, das heißt eine Gewichtsfunktion ω benutzen, um Räume $\mathcal{E}_{\{\omega\}}$ bzw. $\mathcal{E}_{(\omega)}$ zu erhalten. Wir merken an, dass in [3] ein Zusammenhang zwischen der Definition via Gewichtsfunktionen und dem Abfallverhalten der Fouriertransformierten glatter Funktionen mit kompaktem Träger hergestellt wird. Wir vergleichen beide nützliche Wege und beweisen, wann die beiden Varianten zum gleichen Funktionenraum führen, das heißt, wann $\mathcal{E}_{\{M\}} \cong \mathcal{E}_{\{\omega\}}$ bzw. $\mathcal{E}_{(M)} \cong \mathcal{E}_{(\omega)}$ gilt.

9.2 Lebenslauf

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