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DISSERTATION

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Long-Time Asymptotics for the Asymptotically Periodic Korteweg–de Vries Equation

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Alice Mikikits-Leitner

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Abstract

We will consider one of the most famous examples of a completely integrable nonlinear wave equation, the Korteweg–de Vries (KdV) equation. The goal of this thesis is to investigate the long-time asymptotic behavior of solutions of the KdV equation which are short-range perturbations of (quasi-)periodic finite-gap KdV solutions.

It is well-known that in the classical case with constant background in the limit for long times the following picture appears: the perturbed solution splits up into a number of solitons (solitary waves) generated by the eigenvalues of the Lax operator. Apart from that there exists a decaying radiation part corresponding to the continuous spectrum. In other words, the solitons constitute the stable part of the solution.

We will show that, if the constant background is replaced by a (quasi-)periodic one, in the long-time limit one can observe solitons traveling on a limiting solution, which is not the background solution. In the remaining regions the perturbed solution also does not approach the (quasi-)periodic background solution but a modulated solution.

The method we will make use of is to formulate the inverse spectral problem as a Riemann–Hilbert problem set on the underlying hyperelliptic Riemann surface. We will then use the method of nonlinear steepest descent/stationary phase. For large times the original Riemann–Hilbert problem can be reduced to a simpler one, which is localized at the stationary phase points and can be explicitly solved.

Zusammenfassung

Wir betrachten eines der bekanntesten Beispiele einer vollständig integrablen nichtlinearen Wellengleichung, die Korteweg–de Vries (KdV) Gleichung. Das Ziel dieser Arbeit ist es das Langzeitverhalten von Lösungen der KdV Gleichung zu untersuchen, die kleine Störungen von (quasi-)periodischen KdV Lösungen darstellen.

Es ist bekannt, dass im klassischen Fall mit konstantem Hintergrund im Langzeitlimes das Folgende passiert: die gestörte Lösung spaltet auf in eine Reihe von Solitonen (solitären Wellen), die den Eigenwerten des Laxoperators entsprechen. Darüber hinaus gibt es einen oszillierenden Anteil, der vom kontinuierlichen Spektrum herrührt. Mit anderen Worten, die Solitonen bilden den stabilen Anteil der gestörten Lösung.

Wir zeigen, dass, wenn der konstante Hintergrund durch einen (quasi-)periodischen ersetzt wird, für große Zeiten Solitonen beobachtet werden können, die auf einer Lösung wandern, die nicht die Hintergrundlösung darstellt. In den übrigen Bereichen nähert sich die gestörte Lösung nicht der (quasi-)periodischen Hintergrundlösung an, sondern einer modulierten Lösung.

Die Methode von der wir Gebrauch machen ist das inverse Spektralproblem als Riemann–Hilbert Problem auf der zugrundeliegenden hyperelliptischen Riemannfläche umzuschreiben. Dann werden wir die Methode der stationären Phase anwenden. Das ursprüngliche Riemann–Hilbert Problem kann für große Zeiten auf ein einfacheres, explizit lösbares reduziert werden, welches nun bei den stationären Phasepunkten lokalisiert ist.

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Zu guter Letzt sage ich meinen Eltern Danke dafür, dass sie immer an mich geglaubt haben und Respekt für meine Arbeit zeigen, obwohl sie wahrscheinlich nie viel davon verstehen werden.

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1. Introduction

In this thesis we will consider one of the most famous examples of a nonlinear wave equation, namely the so-called Korteweg–de Vries (KdV) equation

$$V_t(x,t) = 6V(x,t)V_x(x,t) - V_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where the subscripts denote differentiation with respect to the corresponding variables.

One can see that the KdV equation consists of two terms, namely a nonlinear term given by $6V(x,t)V_x(x,t)$ and a dispersive term $V_{xxx}(x,t)$. If we consider the linearized version of the KdV equation, i.e.,

$$V_t(x,t) = -V_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R},$$

and an initial condition $V(x,0) = V_0(x) \in \mathcal{S}(\mathbb{R})$ this equation can be solved by means of Fourier analysis for linear partial differential equations. Concerning the long-time behavior of the solutions one sees that an initial wave packet spreads for large times t, a phenomenon called dispersion. On the other hand, neglecting the linear term one arrives at the nonlinear version of the KdV equation, i.e.,

$$V_t(x,t) = 6V(x,t)V_x(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

This equation is known as Burgers' equation and can be solved by the method of characteristics. The characteristics $V(x,t) = c$ are straight lines with the slope $6V(x_0,0) = 6c$. An initial wave packet will steepen until it breaks, thus there exist no global solutions as in the linearized case.

The KdV equation (1.1) contains both terms, linear and nonlinear. Thus it can happen that both nonlinearity and dispersion also occur together, which may result in so-called solitary wave solutions. These are waves that do not change their shape for all times traveling at a constant speed. The appearance of solitary waves, also known as "solitons", as special solutions of the KdV equation explains why this equation has become so interesting, not only mathematically but also in a practical sense.

The solitary wave, i.e., a spatially localized wave, was first discovered by John Scott Russell in 1834. He followed a wave in the Edinburgh–Glasgow canal and made the observation that the wave did not change its shape. He wrote his discovery down in his "Report on Waves" 1844 reporting on, in his words, "the great wave of translation". Later on Russell also performed experiments in his laboratory to study the astonishing phenomenon he had been observing.

After their discovery solitary waves have been studied theoretically by Stokes, Boussinesq, and Lord Rayleigh in the 1870's. However, the equation, which provides the mathematical theory to describe solitons was finally named after Korteweg and de Vries, who derived it in 1895 [Korteweg95].

In 1955 Fermi, Pasta, and Ulam numerically examined a model, which is related to the discretization of the KdV equation, with the first computers at Los Alamos Science Laboratories [Fermi55]. More precisely, they studied a lattice of particles coupled by an anharmonic potential and made the observation that the energy of one mode injected initially is not equipartitioned among the other modes.

Later in 1965 Zabusky and Kruskal considered the initial value problem for the KdV equation by numerical simulations [Zabusky65]. They showed that for large times an imposed periodic initial condition may split up into a number of well-defined and separated waves. These waves emerging from the initial condition showed to not change their shapes and velocities. Because of the particle-like behavior of these waves Zabusky and Kruskal chose the word "soliton".

Solitons have been and still are an interesting and active research topic, not only for mathematicians but also for physicists. From the mathematical point of view this may be because of the remarkable properties of integrable systems. From the viewpoint of physicists the major interest in solitons is due to the fact that they describe a wide range of various physical phenomena. Phenomena like the propagation of hydrodynamic waves, propagation of signals in optical fibers, localized waves in astrophysical plasmas, as well as localized modes in magnetic crystals, to name just a few. This explains also the practical and technical interest in soliton theory. However, one has to be aware of the fact that the mathematical theory of solitons is only able to provide a model and thus to describe such physical systems approximately. In [Dauxois06] a wide range of applications of solitons in physics is discussed.

Apart from the KdV equation there are also many other soliton equations, which are of physical importance, such as the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, the Toda lattice, the Boussinesq equation, and the Camassa–Holm equation.

Concerning the existence and uniqueness of solutions of the KdV equation Gardner, Greene, Kruskal, and Miura [Gardner74] have developed the so-called inverse scattering transform in 1974, which provides a systematic method to obtain the solutions of an integrable system, i.e., a system with infinitely many conservation laws. The inverse scattering method can be used to study the evolution of a rapidly decaying initial condition of the KdV equation. Further it can be used to construct so-called N-soliton solutions corresponding to reflectionless potentials and N bound states. Moreover, the inverse scattering transform ensures that the KdV equation corresponds to a completely integrable system.

The beauty of this approach is that the solution of a nonlinear equation can be established via a number of linear steps. The inverse scattering method can be viewed as a nonlinear analogue of the Fourier transform which is used to solve linear partial differential equations. Like the Fourier modes appearing from the study of a linear equation, solitons arise from nonlinear equations and hence play the role of 'nonlinear normal modes' [Ablowitz74].

Another decisive step towards understanding integrable wave equations was taken by Lax [Lax68]. The idea of the Lax method is to find a Lax pair, i.e., a pair of two matrices or operators $L(t)$, $P(t)$ defined on a Hilbert space such that the Lax equation

$$L_t = [P, L],$$

is equivalent to the original nonlinear differential equation. Here the bracket denotes the commutator of the operators, i.e., $[P, L] = PL - LP$. Moreover, the Lax method can be used to derive a full hierarchy of equations.

One generalization scheme which goes beyond the Lax method has been made in 1972 to solve the NLS equation [Zakharov72]. In 1974 Ablowitz, Kaup, Newell, and Segur [Ablowitz74] proposed a scheme, the so-called AKNS method, which provides the ability to solve many completely integrable systems, such as the KdV, sine-Gordon, the NLS equation, etc. by an inverse scattering transform.

For a detailed study and description of the inverse scattering, Lax, and AKNS method we refer to [Drazin90], which in general can be considered a very good introductory literature to the theory of solitons.

Concerning the long-time asymptotic behavior of solutions of the KdV equation it is well-known that an arbitrary short-range solution of the KdV equation may split into a number of solitons traveling on the constant background and a decaying radiation part, cf. Figure 1.1, which is taken from [Grunert09]. Numerical evidence for this separation into solitons was first found by Zabusky and Kruskal [Zabusky65], as already mentioned above.

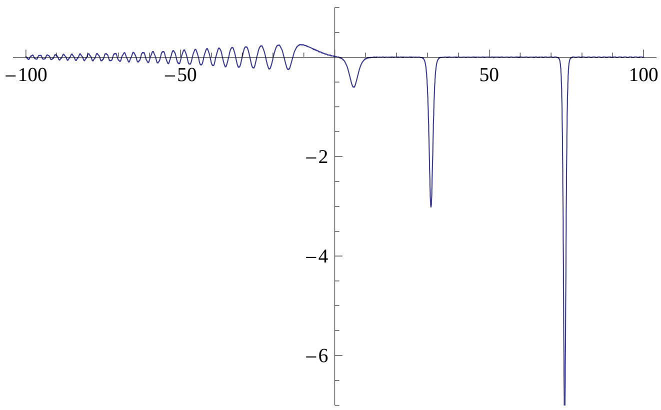

Figure 1.1.: Solution of the KdV equation $V(x,t)$ at time $t = 5$ with initial condition $V(x,0) = \text{sech}(x+3) - 5\text{sech}(x-1)$ computed numerically. The solution is seen to split up into a number of solitons traveling to the right and a decaying radiation tail on the left. [Grunert09]

First mathematical results were given by Ablowitz and Newell [Ablowitz73], Manakov [Manakov74], and Šabat [Šabat73]. Rigorous results for the KdV equation were proved by Šabat [Šabat73] and Tanaka [Tanaka75]. A detailed study of the radiation part concerning its asymptotic behavior were derived by Zakharov and Manakov [Zakharov76], by Ablowitz and Segur [Ablowitz77, Segur81], by Buslaev [Buslaev81, Budylin96], and later on justified and extended to all orders by Buslaev and Sukhanov [Buslaev86].

A detailed rigorous proof using the method of nonlinear steepest descent for oscillatory Riemann–Hilbert problems (RHP) was given by Deift and Zhou [Deift93] based on earlier work of Manakov [Manakov74] and Its [Its81, Its86, Its87, Its82]. The method of nonlinear steepest descent (also known as nonlinear stationary phase method) which they applied to gain the asymptotics of solutions of a Riemann–Hilbert factorization problem can be viewed as an analogue to the steepest descent method to approximate oscillatory integrals, a well-known principle of asymptotic analysis. The idea is that in the asymptotic limit for large times the solution of the RHP reduces to a simpler RHP localized at the stationary phase points, which is exactly solvable.

Recently Grunert and Teschl [Grunert09] used exactly this method to derive the long-time asymptotics of solutions of the KdV equation involving the case of solitons. They followed the work done by Krüger and Teschl [Krüger09a, Krüger09b] concerning the long-time asymptotic behavior of solutions of the Toda lattice, a discrete version of the KdV equation.

Generally speaking, the long-time behavior of KdV solutions on constant background has been studied for a long time and is well-known. However, an interesting task to be investigated in this thesis is the long-time asymptotics of KdV solutions on (quasi-)periodic background, i.e., of KdV solutions which are short-range perturbations of (quasi-)periodic KdV solutions. Kamvissis and Teschl have been recently studying this question for solutions of the Toda lattice [Kamvissis07b, Kamvissis07a], also concerning higher order asymptotics [Kamvissis08], by the method of nonlinear steepest descent for Riemann–Hilbert problems. Here the RHP is not set on the complex plane but on a hyperelliptic Riemann surface. The generalization to involve the case where solitons are present was later made by Krüger and Teschl [Krüger09c], who made use of the ideas presented in [Deift96] how to treat solitons in the context of Riemann–Hilbert problems.

All in all, concerning the question of the long-time asymptotics of solutions of the periodic Toda lattice it turned out that the radiation tail does not decay as in the constant case, rather it approaches a modulated lattice instead of the background solution. Apart from that solitons can be observed traveling on a limiting lattice.

As mentioned above, the task of this thesis is to determine the long-time asymptotic behavior of solutions of the periodic KdV equation.

The content of the thesis at hand will be structured in the following way:

Chapter 2 deals with hyperelliptic curves of the KdV-type, i.e., curves that have a branchpoint at infinity. The way how to construct a hyperelliptic Riemann surface $\mathcal{K}_g$ of genus $g \in \mathbb{N}_0$ associated with this curve is described in detail.

Chapter 3 presents the so-called Baker–Akhiezer function. Then via the Lax approach the Its–Matveev formula for quasi-periodic finite-gap KdV solutions, from which class we will choose our background solution, is obtained.

Chapter 4 establishes the asymptotic behavior of the Jost solutions, i.e., solutions which asymptotically look like the Baker–Akhiezer functions. Apart from that, the main results from scattering theory will be presented, from which we will be able to derive the vector Riemann–Hilbert problem set on the hyperelliptic Riemann surface $\mathcal{K}_g$. Finally we obtain a formula for the one-soliton solution corresponding to a vanishing reflection coefficient and one bound state.

Chapter 5 is devoted to the method of stationary phase. That means we will conjugate our Riemann–Hilbert problem and deform the jump contours of it in such a way that the problem reduces to a much simpler, explicitly solvable Riemann–Hilbert problem, which has jumps on small crosses centered at the stationary phase points.
Moreover, we introduce the so-called oscillatory and soliton region and derive the long-time asymptotic behavior of our perturbed KdV solution in the latter.

Chapter 6 presents the long-time asymptotics inside the oscillatory region. For that purpose a decoupling argument will be used.

Chapter 7 describes how to establish an analytic approximation of the reflection coefficient. This is necessary since it is not guaranteed that, while performing the deformation step, it has an analytic extension in the corresponding regions.

Chapter 8 finally summarizes the results obtained in the scope of this thesis.

Appendix A deals with the connection between singular integral equations and Riemann–Hilbert problems, which is used when deriving the long-time asymptotics of the perturbed KdV solution.

Appendix B presents the solution of a Riemann–Hilbert problem with a jump on a small cross. The problem is here set in the complex plane. Nevertheless, we are able to use this result to gain the asymptotics inside the oscillatory region.

2. Hyperelliptic Curves of the KdV-Type

In this thesis we will investigate the long-time asymptotic behavior of solutions of the KdV equation which are short-range perturbations of quasi-periodic KdV solutions. The latter will be chosen from a class of solutions of the KdV equation, which can be constructed by means of algebraic geometry. To derive such algebro-geometric KdV solutions we will need an underlying algebraic curve of the KdV-type.

Hence this chapter, which is heavily based on [Gesztesy03, Appendix B] and [Teschl00, Section A.7] summarizes the basic facts on such hyperelliptic KdV-type curves, i.e. curves branched at infinity. Since we only take a look at the self-adjoint case, where all E_j are real and distinct points, we refer to [Gesztesy03, Appendix B] for further information and a detailed study of the general case.

Let $g \in \mathbb{N}_0$. In this chapter we are going to describe how to construct the hyperelliptic Riemann surface $\mathcal{K}_g$ of genus g associated with the KdV-type curve

$$R_{2g+1}(z) = \prod_{j=0}^{2g} (z - E_j), \quad \{E_j\}_{j=0,\dots,2g}. \tag{2.1}$$

Let $\{E_j\}_{j=0,\dots,2g}$ be real and distinct fixed points, i.e.

$$\{E_j\}_{j=0,\dots,2g} \subset \mathbb{R}, \qquad E_0 < E_1 < \cdots < E_{2g}, \quad g \in \mathbb{N}_0. \tag{2.2}$$

Define the cut plane

$$\Pi = \mathbb{C} \backslash (\bigcup_{j=0}^{g-1} [E_{2j}, E_{2j+1}] \cup [E_{2g}, \infty)), \tag{2.3}$$

which is obviously a domain, that is an open and connected subset of $\mathbb{C}$. Hence we may declare a holomorphic function on Π

$$\begin{array}{rccc} R_{2g+1}^{1/2}(.): & \Pi & \rightarrow & \mathbb{C} \\ & z & \mapsto & \big(\prod\limits_{j=0}^{2g} (z - E_j) \big)^{1/2} \end{array} . \tag{2.4}$$

The square root branch in (2.4) is chosen such that

$$R_{2g+1}^{1/2}(z) = \lim_{\varepsilon \downarrow 0} R_{2g+1}^{1/2}(z + \mathrm{i}\varepsilon), \quad \text{for } z \in \mathbb{C} \backslash \Pi \tag{2.5}$$

and

$$R_{2g+1}^{1/2}(z) = |R_{2g+1}^{1/2}(z)| \cdot \\ \cdot \begin{cases} (-1)^g \mathrm{i} & \text{for } z \in (-\infty, E_0), \\ (-1)^{g+j}\mathrm{i} & \text{for } z \in (E_{2j-1}, E_{2j}),\ j = 1, \dots, g, \\ (-1)^{g+j} & \text{for } z \in (E_{2j}, E_{2j+1}),\ j = 0, \dots, g-1, \\ 1 & \text{for } z \in (E_{2g}, \infty). \end{cases} \tag{2.6}$$

Next, define the following set

$$\mathcal{M}_g = \{(z, \sigma R_{2g+1}(z)^{1/2}) | z \in \mathbb{C},\ \sigma \in \{-1, +1\}\} \cup \{p_\infty\}, \tag{2.7}$$

where $p_\infty = (\infty, \infty)$ is the point at infinity. To describe charts on $\mathcal{M}_g$ we need to introduce more notation. Let $p_0 \in \mathcal{M}_g$, $U_{p_0} \subset \mathcal{M}_g$ a neighborhood of p_0, $\zeta_{p_0} : U_{p_0} \to V_{p_0} \subset \mathbb{C}$ a homeomorphism defined below, and write

$$\begin{aligned} p_0 &= (z_0, \sigma_0 R_{2g+1}^{1/2}(z_0)) \text{ or } p_0 = p_\infty \\ p &= (z, \sigma R_{2g+1}^{1/2}(z)) \in U_{p_0} \subset \mathcal{M}_g, \qquad V_{p_0} = \zeta_{p_0}(U_{p_0}) \subset \mathbb{C}. \end{aligned} \tag{2.8}$$

The set

$$\mathcal{B} = \{(E_j, 0)\}_{0 \le j \le 2g} \cup \{p_\infty\} \tag{2.9}$$

is called the set of branch points on $\mathcal{M}_g$. For introducing charts on $\mathcal{M}_g$ one has to distinguish three different cases; (i) $p_0 \in \mathcal{M}_g \backslash \mathcal{B}$, (ii) $p_0 = (\infty, \infty)$, and (iii) $p_0 = (E_i, 0)$ for some $i = 0, \dots, 2g$.

(i) $p_0 \in \mathcal{M}_g \setminus \mathcal{B}$: Then one defines

$$U_{p_0} = \{p \in \mathcal{M}_g |\ |z - z_0| < C_0 \text{ and } \sigma R_{2g+1}^{1/2}(z) \xrightarrow{\gamma} \sigma_0 R_{2g+1}^{1/2}(z_0)\}, \tag{2.10}$$

where

$$C_0 = \min_{j=0,\dots,2g} |z_0 - E_j| > 0 \tag{2.11}$$

and $\sigma R_{2g+1}^{1/2}(z) \xrightarrow{\gamma} \sigma_0 R_{2g+1}^{1/2}(z_0)$ means that $\sigma R_{2g+1}^{1/2}(z)$ is the branch reached by analytic continuation along γ, the straight line from z to z_0,

$$V_{p_0} = \{\zeta \in \mathbb{C} |\ |\zeta| < C_0\}, \tag{2.12}$$

and

$$\begin{aligned} \zeta_{p_0} : U_{p_0} &\to V_{p_0} \\ p &\mapsto (z - z_0) \end{aligned} \tag{2.13}$$

with inverse

$$\begin{aligned} \zeta_{p_0}^{-1} : V_{p_0} &\to U_{p_0}, \\ \zeta &\mapsto (z_0 + \zeta, \sigma R_{2g+1}^{1/2}(\zeta + z_0)). \end{aligned} \tag{2.14}$$

(ii) $p_0 = p_\infty$: Here one introduces

$$U_{p_\infty} = \{p \in \mathcal{M}_g |\ |z| > C_\infty\}, \quad C_\infty = \max_{j=0,\dots,2g} |E_j| < \infty, \tag{2.15}$$

$$V_{p_\infty} = \{\zeta \in \mathbb{C} |\ |\zeta| < C_\infty^{-1/2}\} \tag{2.16}$$

and

$$\begin{aligned} \zeta_{P_\infty} : U_{P_\infty} &\to V_{P_\infty}, \\ p &\mapsto \sigma/z^{1/2}, \\ p_\infty &\mapsto 0 \end{aligned} \tag{2.17}$$

with inverse

$$\begin{aligned} \zeta_{p_\infty}^{-1} : V_{p_\infty} &\to U_{p_\infty}, \\ \zeta &\mapsto \Big(\zeta^{-2}, \Big(\prod_{j=0}^{2g}(1-\zeta^2 E_j)\Big)^{1/2}\zeta^{-2g-1}\Big) \\ 0 &\mapsto p_\infty, \end{aligned} \tag{2.18}$$

The former square root is defined as

$$z^{1/2} = |z^{1/2}| \mathrm{e}^{\mathrm{i}\arg(z)/2}, \quad 0 \le \arg(z) < 2\pi, \tag{2.19}$$

and the latter root is holomorphic on U_{p_∞} with the sign fixed by

$$\Big(\prod_{j=0}^{2g}(1-\zeta^2 E_j)\Big)^{1/2} = 1 - \frac{1}{2}\Big(\sum_{n=0}^{2g} E_n\Big)\zeta^2 + O(\zeta^4). \tag{2.20}$$

(iii) $p_0 = (E_i, 0)$: Define

$$\begin{aligned} U_{p_0} &= \{p \in \mathcal{M}_g |\ |z - E_i| < C_i\}, && (2.21) \\ C_i &= \begin{cases} \min\limits_{\substack{j=0,\dots,2g \\ j\neq i}} |E_j - E_i| > 0, & g \in \mathbb{N} \\ \infty, & g = 0, \end{cases} \\ V_{p_0} &= \{\zeta \in \mathbb{C} |\ |\zeta| < C_i^{1/2}\}, && (2.22) \end{aligned}$$

and

$$\begin{aligned} \zeta_{p_0} : U_{p_0} &\to V_{p_0} \\ p &\mapsto \sigma(z - E_i)^{1/2}, \end{aligned} \tag{2.23}$$

with inverse

$$\begin{aligned} \zeta_{p_0} : V_{p_0} &\to U_{p_0} \\ \zeta &\mapsto \Big(E_i + \zeta^2, \Big(-\prod_{\substack{j=0 \\ j\neq i}}^{2g}(E_i - E_j - \zeta^2)\Big)^{1/2}\zeta\Big), \end{aligned} \tag{2.24}$$

where the first root is defined as

$$(z-E_i)^{1/2} = |z-E_i|^{1/2}\exp\big(\mathrm{i}\arg(z-E_i)/2\big)$$
$$\arg(z-E_i)\in\begin{cases}[0,2\pi) & \text{if } i \text{ is even,}\\ (-\pi,\pi] & \text{if } i \text{ is odd,}\end{cases} \tag{2.25}$$

and the second root is holomorphic on U_{p_0} with the sign fixed by

$$\Big(\prod_{\substack{j=0\\ j\neq i}}^{2g}(E_i-E_j-\zeta^2)\Big)^{1/2} = \Big(\prod_{\substack{j=0\\ j\neq i}}^{2g}(E_i-E_j)\Big)^{1/2}\Big(\prod_{\substack{j=0\\ j\neq i}}^{2g}\left(1-(E_m-E_n)^{-1}\zeta^2\right)\Big)^{1/2}$$
$$= (-1)^g\mathrm{i}^{-i}\Big|\Big(\prod_{\substack{j=0\\ j\neq i}}^{2g}(E_i-E_j)\Big)^{1/2}\Big|\Big(1-\frac{1}{2}\big(\sum_{\substack{j=0\\ j\neq i}}^{2g}(E_i-E_j)^{-1}\big)\zeta^2+O(\zeta^4)\Big) \tag{2.26}$$

The set $\mathcal{M}_g$ defined in (2.7) together with the charts in (2.10)–(2.26) yields a compact Riemann surface of genus g, which we will denote by $\mathcal{K}_g$.

Topologically, $\mathcal{K}_g$ can be constructed as indicated in Figure 2.1: Take two copies $\Pi_\pm$ of the cut plane Π defined as in (2.32) and glue the rims of the cuts on Π_+ together crosswise with the rims of the corresponding cuts on Π_-, i.e. the $+$-rims on Π_+ with the $-$-rims on Π_- and vice versa. This leads to the compact hyperelliptic Riemann surface $\mathcal{K}_g$ of genus g (corresponding to $\mathcal{M}_g$).

Next we will introduce the representatives $\{a_j,b_j\}_{j=1}^g$ of a canonical homology basis for $\mathcal{K}_g$ as it is sketched in Figure 2.2. Set $E_{2g+1}=\infty$, then for b_j we start at a point in the interval (E_{2j-2},E_{2j-1}) on Π_+ (i.e. on the upper rim of the cut $[E_{2j-2},E_{2j-1}]$), surround E_{2g} (i.e. b_j hits a point on the cut $[E_{2g},\infty)$) thereby changing to Π_- and return to our starting point encircling E_{2j-1}, again changing sheets. For a_j we choose a cycle surrounding the cut $[E_{2j-2},E_{2j-1}]$, $j=1,\dots,g$ clockwise (once) on Π_+. The cycles are chosen so that their intersection matrix reads

$$a_i\circ b_k=\delta_{ik}. \tag{2.27}$$

We will now define three maps on $\mathcal{K}_g$. The holomorphic sheet exchange map (involution) is defined as

$$\begin{array}{rlll} *: & \mathcal{K}_g & \to & \mathcal{K}_g \\ & (z,R_{2g+1}^{1/2}(z)) & \mapsto & (z,\sigma R_{2g+1}^{1/2}(z))^* = (z,-\sigma R_{2g+1}^{1/2}(z)) \\ & p_\infty & \mapsto & p_\infty \end{array}. \tag{2.28}$$

Moreover, we define two meromorphic projections:

$$\begin{array}{rlll} \pi: & \mathcal{K}_g & \to & \mathbb{C}\cup\{\infty\} \\ & (z,\sigma R_{2g+1}^{1/2}(z)) & \mapsto & z \\ & p_\infty & \mapsto & \infty \end{array}, \tag{2.29}$$

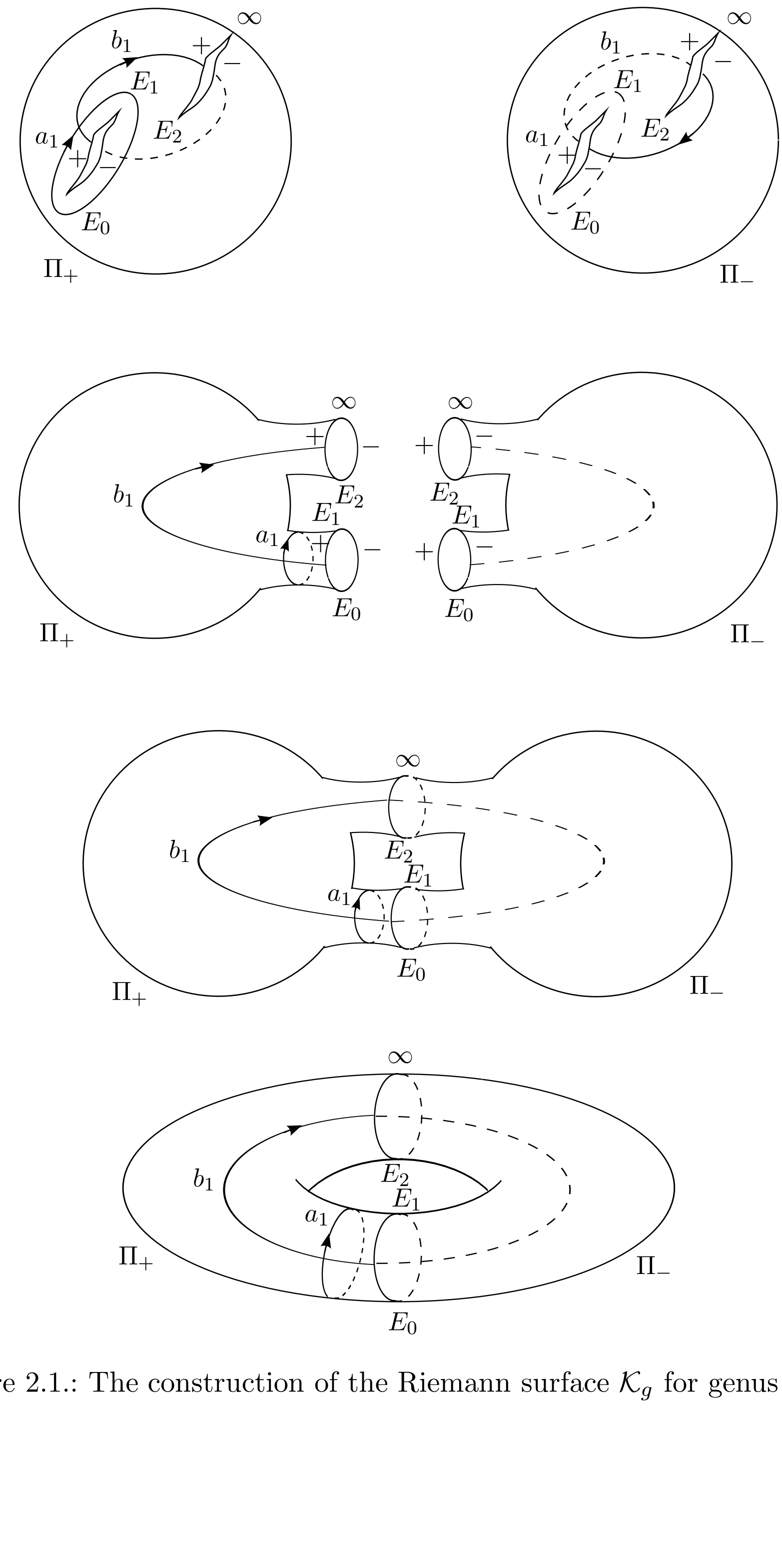

Figure 2.1.: The construction of the Riemann surface $\mathcal{K}_g$ for genus $g = 1$.

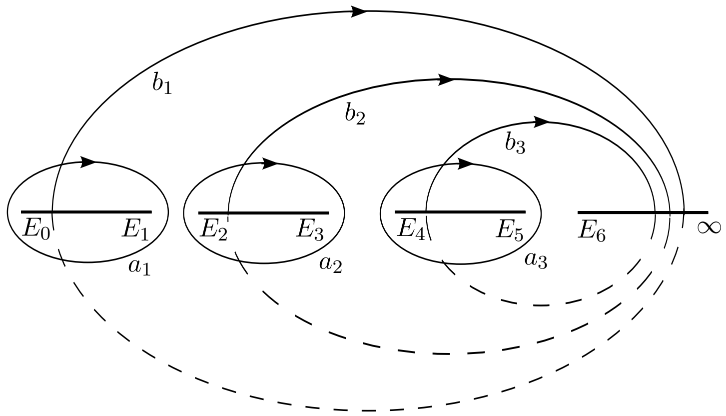

Figure 2.2.: The homology basis $\{a_j, b_j\}_{j=1}^{g}$ on $\mathcal{K}_g$ for genus $g = 2$. Here the solid lines indicate the parts on Π_+ and the doted ones the parts on Π_-.

$$F: \begin{array}{lcl} \mathcal{K}_g & \to & \mathbb{C} \cup \{\infty\} \\ (z, \sigma R_{2g+1}^{1/2}(z)) & \mapsto & \sigma R_{2g+1}^{1/2}(z) \\ p_\infty & \mapsto & \infty \end{array} . \tag{2.30}$$

π has a pole of order 2 at p_∞ and two simple zeros at $(0, \pm R_{2g+1}^{1/2}(0))$ if $R_{2g+1}^{1/2}(0) \neq 0$, respectively one double zero at $(0,0)$ if $R_{2g+1}^{1/2}(0) = 0$, and F has a pole of order $2g+1$ at p_∞ and $2g+1$ simple zeros at $(E_j, 0)$. Notice that

$$\pi(p^*) = \pi(p), \quad F(p^*) = -F(p), \quad p \in \mathcal{K}_g. \tag{2.31}$$

Therefore we can make the following conclusions: $\mathcal{K}_g$ is a two–sheeted, ramified covering of the Riemann sphere (i.e. $\mathcal{K}_g \cong \mathbb{C} \cup \{\infty\}$). Moreover, $\mathcal{K}_g$ is compact, since π is open and $\mathbb{C} \cup \{\infty\}$ is compact. Finally, we have that $\mathcal{K}_g$ is hyperelliptic, since it admits a function with a single pole of order 2.

Let us take two subsets

$$\Pi_\pm = \{(z, \pm R_{2g+1}^{1/2}(z)) | z \in \Pi\} \subset \mathcal{M}_g, \tag{2.32}$$

which will be called upper and lower sheet, respectively, and define two more quite useful charts

$$\zeta_\pm: \begin{array}{lcl} \Pi_\pm & \to & \Pi \\ p & \mapsto & z. \end{array}$$

The charts (2.10)–(2.26) are chosen such that there are compatible with the charts $\zeta_\pm$ wherever they overlap. Now consider

$$\frac{d\pi}{F}. \tag{2.33}$$

Again using local charts we see that $d\pi/F$ is holomorphic everywhere and has a zero of order $2g-2$ at p_∞. So we may conclude that

$$\eta_j = \frac{\pi^{j-1} d\pi}{F}, \qquad 1 \le j \le g, \tag{2.34}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_g$.

If we introduce the constants $\underline{c}(.)$ via

$$\begin{aligned}
\underline{c}(k) &= \big(c_1(k), \dots, c_g(k)\big), \quad c_j(k) = (C^{-1})_{j,k}, \quad j,k = 1, \dots, g, \\
C &= (C_{j,k})_{j,k=1,\dots,g}, \quad C_{j,k} = \int_{a_k} \eta_j = 2 \int_{E_{2k-2}}^{E_{2k-1}} \frac{z^{j-1} dz}{R_{2g+1}^{1/2}(z)} \in \mathbb{R},
\end{aligned}$$

(C is invertible since otherwise there would be a nonzero differential with vanishing a-periods.) the differentials ζ_j

$$\zeta_j = \sum_{k=1}^{g} c_j(k) \eta_k, \quad j = 1, \dots, g, \tag{2.35}$$

fulfill

$$\int_{a_k} \zeta_j = \delta_{j,k}, \quad j = 1, \dots, g, \tag{2.36}$$

and are thus a basis of the required form. The matrix of b-periods

$$\tau_{jk} = \int_{b_k} \zeta_j = \sum_{j=1}^{g} c_j(m) \int_{b_k} \eta_j = 2 \sum_{j=1}^{g} c_j(m) \sum_{\ell=k}^{g} \int_{E_{2\ell-1}}^{E_{2\ell}} \frac{z^{j-1} dz}{R_{2g+1}^{1/2}(z)} \in \mathrm{i}\mathbb{R} \tag{2.37}$$

fulfills

$$\underline{\tau} = \mathrm{i}\underline{T}, \quad \underline{T} > 0.$$

Moreover, $\underline{\tau}$ is symmetric, that is

$$\tau_{jk} = \tau_{kj}.$$

3. Algebro-geometric solutions of the KdV-equation

We want to choose our background solution, denoted by $V_q(x,t)$, from the class of algebro-geometric quasi-periodic solutions of the KdV equation corresponding to g–gap initial values, i.e., from the finite-gap class of stationary solutions of the KdV hierarchy.

Thus this chapter will be devoted to the construction of such quasi-periodic finite-gap KdV solutions represented in terms of theta functions. The basic Riemann surface will be the hyperelliptic curve $\mathcal{K}_g$, $g \in \mathbb{N}_0$, introduced in Chapter 2.

In the whole chapter we will follow closely the book by Gesztesy and Holden [Gesztesy03, Chapter 1] and the lecture notes by Gesztesy [Gesztesy89].

3.1. The Baker-Akhiezer function

Definition 3.1 (Baker–Akhiezer function)**.** *Let $\mathcal{D}_{p_1,\dots,p_g}$ be a positive divisor of degree g on $\mathcal{K}_g$, $g \in \mathbb{N}_0$, $\zeta = z^{-1/2}$ the local coordinate near $p_\infty = (\infty,\infty) \in \mathcal{K}_g$, and $q : \mathbb{C} \to \mathbb{C}$ a polynomial. Then the* Baker-Akhiezer function

$$\psi_{y,s} : \mathcal{K}_g \setminus \{p_\infty\} \to \mathbb{C} \cup \{\infty\}, \quad y,s \in \mathbb{C}, \tag{3.1}$$

associated with $\mathcal{K}_g$, q, ζ, and $\mathcal{D}_{p_1,\dots,p_g}$ is defined as follows:
(i) $\psi_{y,s}$ is meromorphic on $\mathcal{K}_g \setminus \{p_\infty\}$ and the divisor $\mathcal{D}_{\psi_{y,s}}$ of poles of $\psi_{y,s}\big|_{\mathcal{K}_g \setminus \{p_\infty\}}$ satisfies

$$\mathcal{D}_{\psi_{y,s}} \ge -\mathcal{D}_{p_1,\dots,p_g}. \tag{3.2}$$

(ii) The product

$$\big(\psi_{y,s} \circ z^{-1}\big)(\zeta) \exp\Big(- \mathrm{i} y q(\zeta^{-1}) - \mathrm{i} s q(\zeta^{-3})\Big) = c + O(\zeta), \quad c \in \mathbb{C}, \tag{3.3}$$

is holomorphic near $\zeta = 0$.

For the rest of this chapter we shall confine ourselves to the KdV equation and hence choose

$$q(z) = z, \quad z \in \mathbb{C}.$$

Higher order polynomials q are used for the equations of the KdV hierarchy.

Pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j(x,t))_{j=1}^g = (\mu_j(x,t), \sigma_j)_{j=1}^g \tag{3.4}$$

whose projections lie in the spectral gaps, i.e.,

$$\mu_j(x,t) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, g. \tag{3.5}$$

Associated with these numbers is the divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(x,t)}(p) = \begin{cases} 1 & p = \hat{\mu}_j(x,t),\ j = 1, \dots, g, \\ 0 & \text{else.} \end{cases} \tag{3.6}$$

We introduce

$$\underline{z}(p, x, x_0, t, t_0) = \underline{\hat{\Xi}}_{E_0} - \underline{\hat{A}}_{E_0}(p) + \underline{\hat{\alpha}}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) \in \mathbb{C}^g, \tag{3.7}$$

where $\underline{\hat{\Xi}}_{E_0}$ is the vector of Riemann constants

$$\hat{\Xi}_{E_0,j} = \frac{1 - \sum_{k=1}^{g} \tau_{j,k}}{2}, \quad j = 1, \dots, g \tag{3.8}$$

and $\underline{A}_{E_0}$ ($\underline{\alpha}_{E_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathcal{K}}_g$ (the fundamental polygon associated with $\mathcal{K}_g$ by cutting along the a and b cycles) to $\mathbb{C}^g$.

Moreover, we have the linearizing property of the Abel map ([Gesztesy03, Theorem 1.44])

$$\underline{\hat{\alpha}}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) = \underline{\hat{\alpha}}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \frac{(x - x_0)}{2\pi}\underline{U}_0 + 12\frac{(t - t_0)}{2\pi}\underline{U}_2, \tag{3.9}$$

where $\mathcal{D}_{\underline{\hat{\mu}}} = \mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}$ and $\underline{U}_0$ and $\underline{U}_2$ are the b-periods of the Abelian differential $\omega_{p_\infty,0}$ and $\omega_{p_\infty,2}$, respectively, defined below, i.e.

$$\underline{U}_0 = \big(U_{0,1}, \dots, U_{0,g}\big) \quad \text{where } U_{0,j} = \int_{b_j} \omega_{p_\infty,0},\ j = 1, \dots, g, \tag{3.10}$$

$$\underline{U}_2 = \big(U_{2,1}, \dots, U_{2,g}\big) \quad \text{where } U_{2,j} = \int_{b_j} \omega_{p_\infty,2},\ j = 1, \dots, g. \tag{3.11}$$

By $\theta(\underline{z})$ we denote the Riemann theta function associated with $\mathcal{K}_g$ defined by

$$\theta(\underline{z}) = \sum_{\underline{m} \in \mathbb{Z}^g} \exp 2\pi \mathrm{i} \left(\langle \underline{m}, \underline{z} \rangle + \frac{\langle \underline{m}, \underline{\tau}\, \underline{m} \rangle}{2} \right), \qquad \underline{z} \in \mathbb{C}^g. \tag{3.12}$$

The Riemann theta function fulfills the quasi-periodic property

$$\theta(\underline{z} + \underline{m} + \underline{\tau}\, \underline{n}) = \exp\Big\{2\pi \mathrm{i}\Big(- \langle \underline{n}, \underline{z} \rangle - \frac{\langle \underline{n}, \underline{\tau}\, \underline{n} \rangle}{2}\Big)\Big\}\theta(\underline{z}), \quad \underline{n}, \underline{m} \in \mathbb{Z}^g, \tag{3.13}$$

where $\underline{\tau}$ is the matrix of b-periods defined in (2.37) and $\langle ., . \rangle$ denotes the scalar product in $\mathbb{R}^g$ (cf., e.g. [Farkas92] or [Teschl00, App. A]). We recall that the function $\theta(\underline{z}(p, x, x_0, t, t_0))$ has precisely g zeros $\hat{\mu}_j(x,t)$ (with $\hat{\mu}_j(x_0, t_0) = \hat{\mu}_j$). This follows from Riemann's vanishing theorem (cf. [Teschl00, Theorem A.13]).

Theorem 3.1 (The time dependent Baker-Akhiezer function)**.** *The function*

$$\begin{aligned}\psi_q(p,x,x_0,t,t_0) =& \frac{\theta\big(\underline{z}(p,x,x_0,t,t_0)\big)}{\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)} \frac{\theta\big(\underline{z}(p_\infty,x_0,x_0,t_0,t_0)\big)}{\theta\big(\underline{z}(p,x_0,x_0,t_0,t_0)\big)} \cdot \\ & \cdot \exp\Big(-\mathrm{i}(x-x_0)\int_{E_0}^{p} \omega_{p_\infty,0} - 12\mathrm{i}(t-t_0)\int_{E_0}^{p}\omega_{p_\infty,2}\Big), \end{aligned} \tag{3.14}$$

is the uniquely determined time dependent Baker-Akhiezer function associated with $\mathcal{K}_g$, p_∞, $q = \mathrm{id}$, ζ and the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$, where $\omega_{p_\infty,0}$ and $\omega_{p_\infty,2}$ are normalized Abelian differentials of the second kind with a single pole of multiplicity two at p_∞ and principal part $\zeta^{-2}d\zeta$ and $\zeta^{-4}d\zeta$, respectively.

Proof. The function defined in (3.14) is single-valued on $\mathcal{K}_g$. Indeed, changing the path from E_0 to p in (3.14) results in the changes

$$\begin{aligned} \int_{E_0}^{p}\omega_{p_\infty,0} &\longrightarrow \int_{E_0}^{p}\omega_{p_\infty,0} + \int_\gamma \omega_{p_\infty,0}, \\ \int_{E_0}^{p}\omega_{p_\infty,2} &\longrightarrow \int_{E_0}^{p}\omega_{p_\infty,2} + \int_\gamma \omega_{p_\infty,2}, \\ \underline{A}_{E_0}(p) = \int_{E_0}^{p}\underline{\zeta} &\longrightarrow \underline{A}_{E_0}(p) + \int_\gamma \underline{\zeta}, \quad \underline{\zeta} := \{\zeta_j\}_{j=1}^{g} \end{aligned}$$

for some closed cycle γ

$$\gamma = \sum_{j=1}^{g}(m_j a_j + n_j b_j), \quad m_j, n_j \in \mathbb{Z},\ j = 1,\dots,g. \tag{3.15}$$

Since the Abelian differentials $\omega_{p_\infty,0}$ and $\omega_{p_\infty,2}$ have to be normalized to have vanishing a_j-periods we have

$$\begin{aligned} \int_\gamma \omega_{p_\infty,0} &= \sum_{j=1}^{g} n_j \int_{b_j}\omega_{p_\infty,0} = \underline{n}\,\underline{U}_0, \\ \int_\gamma \omega_{p_\infty,2} &= \sum_{j=1}^{g} n_j \int_{b_j}\omega_{p_\infty,2} = \underline{n}\,\underline{U}_2, \\ \int_\gamma \underline{\zeta} &= \sum_{j=1}^{g}\Big(m_j\int_{a_j}\underline{\zeta} + n_j\int_{b_j}\underline{\zeta}\Big) = \underline{m} + \underline{\tau}\,\underline{n}, \end{aligned}$$

where $\underline{n} := \{n_j\}_{j=1}^{g}$ and $\underline{m} := \{m_j\}_{j=1}^{g}$. Thus the quasi-periodic property of the

Riemann theta function (3.13) implies

$$\begin{aligned}
&\psi_q(p,x,x_0,t,t_0) \to \\
&\to \frac{\theta\big(\underline{z}(p,x,x_0,t,t_0)-\underline{m}-(\underline{\tau}\,\underline{n})\big)}{\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)}\frac{\theta\big(\underline{z}(p_\infty,x_0,x_0,t_0,t_0)\big)}{\theta\big(\underline{z}(p,x_0,x_0,t_0,t_0)-\underline{m}-(\underline{\tau}\,\underline{n})\big)}\cdot \\
&\quad\cdot \exp\Big(-\mathrm{i}(x-x_0)\big(\int_{E_0}^p \omega_{p_\infty,0}+\underline{n}\,\underline{U}_0\big)-12\mathrm{i}(t-t_0)\big(\int_{E_0}^p \omega_{p_\infty,2}+\underline{n}\,\underline{U}_2\big)\Big) \\
&= \psi_q(p,x,x_0,t,t_0)\exp\Big(-\mathrm{i}(x-x_0)\underline{n}\,\underline{U}_0-12\mathrm{i}(t-t_0)\underline{n}\,\underline{U}_2\Big)\cdot \\
&\quad\cdot\exp\Big(2\pi\mathrm{i}\underline{n}\,\underline{U}_0\frac{(x-x_0)}{2\pi}+2\pi\mathrm{i}\underline{n}\,\underline{U}_2 12\frac{(t-t_0)}{2\pi}\Big)
\end{aligned}$$

and hence

$$\psi_q(p,x,x_0,t,t_0) \to \psi_q(p,x,x_0,t,t_0),$$

that is ψ is independent of the path chosen to connect E_0 and p and hence single-valued.

The function defined in (3.14) satisfies (i) and (ii) of Definition 3.1. In fact, the function $\theta\big(\underline{z}(p,x_0,x_0,t_0,t_0)\big)$ has precisely g zeros $\hat{\mu}_j(x_0,t_0)$ and the corresponding divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ is positive. Thus ψ_q satisfies (i). Setting $y=(x-x_0)$ and $s=4(t-t_0)$ we will see later (cf. proof of Theorem 3.6, (3.31)) that (ii) also holds. Note that we have $c=1$, since the function (3.14) is already normalized.

To show uniqueness let $\tilde{\psi}_q$ be another Baker-Akhiezer function. Since $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ is the divisor of zeros of ψ_q we get that $\tilde{\psi}_q/\psi_q$ is meromorphic on $\mathcal{K}_g$ with poles in $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$. Since the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ is nonspecial, $\tilde{\psi}_q/\psi_q$ must be equal a constant by the Riemann-Roch theorem. □

Remark 3.2. *The differentials $\omega_{p_\infty,2k}$, $k=0,1,2,\dots$ are Abelian differentials of the second kind with a single pole of multiplicity two at p_∞ and principal part $\zeta^{-2k-2}d\zeta$. They are explicitly given by*

$$\omega_{p_\infty,2k} = \left(\frac{\pi^{g+k}+P_k(\pi)}{-2R_{2g+1}^{1/2}}\right)d\pi. \tag{3.16}$$

Here $P_k(\pi)$ is a monic polynomial of degree $g+k$ whose coefficients have to be determined from the behavior at p_∞ (principal part $\zeta^{-2k-2}d\zeta$) and the normalization, i.e., the differential has vanishing a-periods.

Next, we derive explicit expressions for $\omega_{p_\infty,0}$ and $\omega_{p_\infty,2}$.

Lemma 3.3. *We have*

$$\omega_{p_\infty,0} = -\frac{1}{2}\frac{\prod_{j=1}^g(\pi-\lambda_j)}{R_{2g+1}^{1/2}}d\pi. \tag{3.17}$$

Proof. Clearly (3.17) is an Abelian differential of the second kind. Moreover, introducing $\zeta = z^{-1/2}$ we obtain that near p_∞ the following is valid:

$$\begin{aligned}
\omega_{p_\infty,0} =& -\frac{1}{2}\prod_{j=1}^{g}(\zeta^{-2}-\lambda_j)R_{2g+1}^{-1/2}(\zeta^{-2})(-2\zeta^{-3}d\zeta) \\
=&\zeta^{-2g}\zeta^{-3}\prod_{j=1}^{g}(1-\zeta^2\lambda_j)\zeta^{2g+1}\Big(\prod_{j=0}^{2g}(1-\zeta^2 E_j)^{1/2}\Big)^{-1}d\zeta \\
=&\zeta^{-2}\Big(1-(\sum_{j=1}^{g}\lambda_j)\zeta^2+O(\zeta^4)\Big)\Big(1+\frac{1}{2}(\sum_{j=0}^{2g}E_j)\zeta^2+O(\zeta^4)\Big)d\zeta \\
=&\zeta^{-2}\Big(1+\frac{1}{2}(E_0+\sum_{j=1}^{g}(E_{2j-1}+E_{2j}-2\lambda_j))\zeta^2+O(\zeta^4)\Big)d\zeta.
\end{aligned}$$

Hence,

$$\omega_{p_\infty,0} = \big(\zeta^{-2}+O(1)\big)d\zeta \quad \text{near } p_\infty. \tag{3.18}$$

Furthermore, $\omega_{p_\infty,0}$ is normalized to have vanishing a_j-periods, that is,

$$\int_{a_j}\omega_{p_\infty,0} = 0, \quad j=1,\dots,g. \tag{3.19}$$

□

Lemma 3.4. *We have*

$$\omega_{p_\infty,2} = -\frac{1}{2}\frac{\prod_{j=0}^{g}(\pi-\tilde\lambda_j)}{R_{2g+1}^{1/2}}d\pi, \quad \sum_{j=0}^{g}\tilde\lambda_j = \frac{1}{2}\sum_{j=0}^{2g}E_j. \tag{3.20}$$

Proof. By inspection (3.20) is an Abelian differential of the second kind. Moreover, introducing $\zeta = z^{-1/2}$ we obtain that the following behavior near p_∞ is valid:

$$\begin{aligned}
\omega_{p_\infty,2} =& -\frac{1}{2}\prod_{j=0}^{g}(\zeta^{-2}-\tilde\lambda_j)R_{2g+1}^{-1/2}(\zeta^{-2})(-2\zeta^{-3}d\zeta) \\
=&\zeta^{-2g-2}\zeta^{-3}\prod_{j=0}^{g}(1-\zeta^2\tilde\lambda_j)\zeta^{2g+1}\Big(\prod_{j=0}^{2g}(1-\zeta^2 E_j)^{1/2}\Big)^{-1}d\zeta \\
=&\zeta^{-4}\Big(1-(\sum_{j=0}^{g}\tilde\lambda_j)\zeta^2+O(\zeta^4)\Big)\Big(1+\frac{1}{2}(\sum_{j=0}^{2g}E_j)\zeta^2+O(\zeta^4)\Big)d\zeta \\
=&\zeta^{-4}\Big(1+\Big(\frac{1}{2}(\sum_{j=0}^{2g}E_j)-(\sum_{j=0}^{g}\tilde\lambda_j)\Big)\zeta^2+O(\zeta^4)\Big)d\zeta.
\end{aligned}$$

Since we have chosen $\tilde{\lambda}_j$, $j = 0, \ldots, g$ such that they have to satisfy

$$\sum_{j=0}^{g} \tilde{\lambda}_j = \frac{1}{2} \sum_{j=0}^{2g} E_j, \tag{3.21}$$

we finally deduce

$$\omega_{p_\infty,2} = \big(\zeta^{-4} + O(1)\big) d\zeta \quad \text{near } p_\infty. \tag{3.22}$$

Furthermore, $\omega_{p_\infty,2}$ is uniquely determined by its normalization, i.e.

$$\int_{a_j} \omega_{p_\infty,2} = 0, \quad j = 1, \ldots, g. \tag{3.23}$$

□

Remark 3.5. *In general, the following asymptotic expansions hold*

$$\int_{E_0}^{p} \omega_{p_\infty,2k} \underset{z\to\infty}{=} -(2k+1)^{-1} z^{k+1/2} + c_k + O(z^{-1/2}),$$

where c_k are some constants. Here $\omega_{p_\infty,2k}$, $k = 0, 1, 2, \ldots$ are the Abelian differentials of the second kind given by (3.16). In particular,

$$\int_{E_0}^{p} \omega_{p_\infty,0} \underset{z\to\infty}{=} -z^{1/2} + c_0 + O(z^{-1/2}), \tag{3.24}$$

$$\int_{E_0}^{p} \omega_{p_\infty,2} \underset{z\to\infty}{=} -\frac{1}{3} z^{3/2} + c_2 + O(z^{-1/2}). \tag{3.25}$$

3.2. Quasi-periodic finite-gap KdV solutions

The next theorem establishes the well-known Its-Matveev formula for quasi periodic, finite-gap KdV solutions in terms of theta functions (cf. [Its75a, Its75b]). We will show that the Baker–Akhiezer function defined in (3.14) is a solution of the Schrödinger equation corresponding to the Schrödinger operator H_q with the potential V_q.

Theorem 3.6 (Its–Matveev formula)**.** *Let $p = (z, \sigma R_{2g+1}(z)^{1/2}) \in \mathcal{K}_g \setminus \{p_\infty\}$, $(z, x, x_0, t, t_0) \in \mathbb{C} \times \mathbb{R}^4$. Then the Baker-Akhiezer function $\psi_q(p, x, x_0, t, t_0)$ defined as in (3.14) satisfies the Schrödinger equation*

$$H_q(t)\psi_q(p, x, x_0, t, t_0) = z\psi_q(p, x, x_0, t, t_0), \tag{3.26}$$

where H_q is the operator from the Lax pair of the KdV equation, i.e.,

$$H_q(t) = -\partial_x^2 + V_q(., t), \quad \text{on } H^2(\mathbb{R}) \tag{3.27}$$

and V_q is the quasi-periodic finite-gap solution

$$V_q(x, t) = E_0 + \sum_{j=1}^{g} (E_{2j-1} + E_{2j} - 2\lambda_j) - 2\partial_x^2 \ln \theta\big(\underline{z}(p_\infty, x, x_0, t, t_0)\big). \tag{3.28}$$

Proof. Expanding $\omega_{p_\infty,0}$ in (3.17) near p_∞ we get (cf. proof of Lemma 3.3)

$$\omega_{p_\infty,0} = \big(\zeta^{-2} + C + O(\zeta^2)\big)d\zeta, \tag{3.29}$$

where we have set

$$C = \frac{1}{2}\big(E_0 + \sum_{j=1}^{g}(E_{2j-1} + E_{2j} - 2\lambda_j)\big). \tag{3.30}$$

Then from (3.29) and (3.22) one infers that near p_∞

$$\begin{aligned}\psi_q(p,x,x_0,t,t_0) =& \big(1 + c_1(x,t)\zeta + c_2(x,t)\zeta^2 + c_3(x,t)\zeta^3 + O(\zeta^4)\big)\cdot\\ &\cdot \exp\Big(\mathrm{i}(x-x_0)\big(\zeta^{-1} - C\zeta + O(\zeta^3)\big) + 4\mathrm{i}(t-t_0)\big(\zeta^{-3} + O(\zeta)\big)\Big),\end{aligned} \tag{3.31}$$

where c_j, $j = 1,\dots,3$ denote the coefficients in the expansion of the ratio of theta functions. Note that there are no $O(1)$ terms in the exponents in (3.31) since both integrals over $\omega_{p_\infty,0}$ and $\omega_{p_\infty,2}$ vanish at E_0. From (3.31) one computes

$$\begin{aligned}\psi_{q,xx}(p,x,x_0,t,t_0) =& \big(-\zeta^{-2} - c_1(x,t)\zeta^{-1} + 2C - c_2(x,t) + 2\mathrm{i}c_{1,x}(x,t) + O(\zeta)\big)\cdot\\ &\cdot \exp\Big(\mathrm{i}(x-x_0)\big(\zeta^{-1} - C\zeta + O(\zeta^3)\big) + 4\mathrm{i}(t-t_0)\big(\zeta^{-3} + O(\zeta)\big)\Big)\\ =& \big(-\zeta^{-2} + 2C + 2\mathrm{i}c_{1,x}(x,t) + O(\zeta)\big)\psi_q(p,x,x_0,t,t_0).\end{aligned}$$

Since

$$-\psi_{q,xx} + \big(2C + 2\mathrm{i}c_{1,x}(x,t) - \zeta^{-2}\big)\psi_q = O(\zeta)\psi_q$$

is also a Baker-Akhiezer function with the same essential singularity at p_∞ and the same pole divisor as ψ_q, uniqueness of the Baker-Akhiezer function proves

$$-\psi_{q,xx} + \big(2C + 2\mathrm{i}c_{1,x}(x,t) - \zeta^{-2}\big)\psi_q = 0. \tag{3.32}$$

Next, we will compute c_1 and c_2 directly. We have

$$\begin{aligned}&\theta\big(\underline{\hat{\Xi}}_{E_0} - \underline{\hat{A}}_{E_0}(p)\big) =\\ &= \theta\big(\underline{\hat{\Xi}}_{E_0} - \underline{\hat{A}}_{E_0}(p_\infty)\big) - \zeta\sum_{j=1}^{g}\frac{U_{0,j}}{2\pi\mathrm{i}}\Big(\frac{\partial}{\partial z_j}\theta(\underline{z})\Big)\Big|_{\underline{z}=\underline{\hat{\Xi}}_{E_0}-\underline{\hat{A}}_{E_0}(p_\infty)}\\ &\quad - \frac{1}{4\pi^2}\frac{1}{2}\zeta^2\sum_{j,l=1}^{g}U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j\partial z_l}\theta(\underline{z})\Big)\Big|_{\underline{z}=\underline{\hat{\Xi}}_{E_0}-\underline{\hat{A}}_{E_0}(p_\infty)},\end{aligned} \tag{3.33}$$

where $U_{0,j}$, $j = 1,\dots,g$ are defined in (3.10). Therefore, expanding the ratios of theta functions near p_∞ yields

$$\begin{aligned}
&\frac{\theta\big(\underline{z}(p,x,x_0,t,t_0)\big)}{\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)} = 1 - \Big[\frac{1}{2\pi \mathrm{i}}\zeta \sum_{j=1}^{g} U_{0,j}\Big(\frac{\partial}{\partial z_j}\theta(\underline{z})\Big) \\
&\quad + \frac{1}{4\pi^2}\frac{\zeta^2}{2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big)\Big]\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x,x_0,t,t_0)} + O(\zeta^3) \\
&= 1 + \mathrm{i}\zeta\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big) \\
&\quad - \frac{1}{4\pi^2}\frac{\zeta^2}{2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big)\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x,x_0,t,t_0)} + O(\zeta^3)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\theta\big(\underline{z}(p_\infty,x_0,x_0,t_0,t_0)\big)}{\theta\big(\underline{z}(p,x_0,x_0,t_0,t_0)\big)} = \\
&= 1 + \Big[\frac{1}{2\pi \mathrm{i}}\zeta \sum_{j=1}^{g} U_{0,j}\Big(\frac{\partial}{\partial z_j}\theta(\underline{z})\Big) + \frac{1}{4\pi^2}\frac{\zeta^2}{2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big) \\
&\quad - \frac{1}{4\pi^2}\zeta^2\Big(\sum_{j=1}^{g} U_{0,j}\Big(\frac{\partial}{\partial z_j}\theta(\underline{z})\Big)\Big)^2\Big]\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x_0,x_0,t_0,t_0)} + O(\zeta^3) \\
&= 1 - \mathrm{i}\zeta\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0} \\
&\quad + \frac{1}{4\pi^2}\frac{\zeta^2}{2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big)\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x_0,x_0,t_0,t_0)} \\
&\quad - \zeta^2\Big(\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0}\Big)^2 + O(\zeta^3).
\end{aligned}$$

Finally, one obtains

$$\begin{aligned}
&\frac{\theta\big(\underline{z}(p,x,x_0,t,t_0)\big)}{\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)}\frac{\theta\big(\underline{z}(p_\infty,x_0,x_0,t_0,t_0)\big)}{\theta\big(\underline{z}(p,x_0,x_0,t_0,t_0)\big)} = \\
&= 1 + \mathrm{i}\zeta\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big) - \mathrm{i}\zeta\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0} \\
&\quad - \frac{\zeta^2}{8\pi^2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big)\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x,x_0,t,t_0)} \\
&\quad + \frac{\zeta^2}{8\pi^2}\sum_{j,l=1}^{g} U_{0,j}U_{0,l}\Big(\frac{\partial^2}{\partial z_j \partial z_l}\theta(\underline{z})\Big)\theta(\underline{z})^{-1}\Big|_{\underline{z}=\underline{z}(p_\infty,x_0,x_0,t_0,t_0)} \\
&\quad + \zeta^2\Big(\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\Big)\Big(\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0}\Big) \\
&\quad - \zeta^2\Big(\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0}\Big)^2 + O(\zeta^3) \\
&= 1 + c_1(x,t)\zeta + c_2(x,t)\zeta^2 + O(\zeta^3),
\end{aligned}$$

and thus

$$\begin{aligned} c_1(x,t) =& \mathrm{i}\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big) - \mathrm{i}\partial_x \ln\theta\big(\underline{z}(p_\infty,x,x_0,t_0,t_0)\big)\Big|_{x=x_0}, \\ c_{1,x}(x,t) =& \mathrm{i}\partial_x^2 \ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big). \end{aligned}$$

Hence

$$2\mathrm{i}c_{1,x}(x,t) = V_q(x,t) - 2C. \tag{3.34}$$

Then (3.32) and (3.34) prove (3.26) and (3.28). □

The next theorem shows that the Baker–Akhiezer function defined in (3.14) solves the equation $\partial_t\psi_q = P_{q,2}\psi_q$, where $P_{q,2}$ denotes the operator from the Lax pair.

Theorem 3.7. *Let $p = (z, \sigma R_{2g+1}(z)^{1/2}) \in \mathcal{K}_g \setminus \{p_\infty\}$, $(z,x,x_0,t,t_0) \in \mathbb{C}\times\mathbb{R}^4$. Then the Baker-Akhiezer function $\psi_q(p,x,x_0,t,t_0)$ defined as in (3.14) satisfies*

$$\psi_{q,t}(p,x,x_0,t,t_0) = P_{q,2}\psi_q(p,x,x_0,t,t_0), \tag{3.35}$$

where $P_{q,2}$ is the operator from the Lax pair of the KdV equation, i.e.,

$$P_{q,2}(t) = -4\partial_x^3 + 6V_q(.,t)\partial_x + 3V_{q,x}(.,t), \quad \text{on } H^3(\mathbb{R}) \tag{3.36}$$

with V_q given by (3.28).

Proof. First, we make use of (3.26) to get

$$\begin{aligned} &\psi_{q,t}(p,x,x_0,t,t_0) - \big(P_{q,2}(t)\psi_q\big)(p,x,x_0,t,t_0) = \\ &\quad= \psi_{q,t}(p,x,x_0,t,t_0) - 2\big(V_q(x,t) + 2\zeta^{-2}\big)\psi_{q,x}(p,x,x_0,t,t_0) \\ &\qquad + V_{q,x}(x,t)\psi(p,x,x_0,t,t_0). \end{aligned} \tag{3.37}$$

Second, (3.31) implies

$$\begin{aligned} \psi_{q,t}(p,x,x_0,t,t_0) =& 4\mathrm{i}\big(\zeta^{-3} + c_1(x,t)\zeta^{-2} + c_2(x,t)\zeta^{-1} + c_3(x,t)\big)\cdot \\ &\cdot\exp\Big(\mathrm{i}(x-x_0)\big(\zeta^{-1} - C\zeta + O(\zeta^3)\big) + 4\mathrm{i}(t-t_0)\big(\zeta^{-3} + O(\zeta)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} \psi_{q,x}(p,x,x_0,t,t_0) =& \Big(\mathrm{i}\zeta^{-1} + \mathrm{i}c_1(x,t) + \big(c_{1,x}(x,t) - \mathrm{i}C + \mathrm{i}c_2(x,t)\big)\zeta \\ &+ \big(c_{2,x}(x,t) - \mathrm{i}Cc_1(x,t) + \mathrm{i}c_3(x,t)\big)\zeta^2\Big)\cdot \\ &\cdot\exp\Big(\mathrm{i}(x-x_0)\big(\zeta^{-1} - C\zeta + O(\zeta^3)\big) + 4\mathrm{i}(t-t_0)\big(\zeta^{-3} + O(\zeta)\big)\Big). \end{aligned}$$

Inserting these expressions into (3.37) yields

$$\begin{aligned} &\psi_{q,t}(p,x,x_0,t,t_0) - \big(P_{q,2}(t)\psi_q\big)(p,x,x_0,t,t_0) = \\ &= \big(2\mathrm{i}c_{1,xx}(x,t) - 4c_{2,x}(x,t) + 4c_1(x,t)c_{1,x}(x,t) + O(\zeta)\big)\psi_q(p,x,x_0,t,t_0). \end{aligned}$$

Note that in the above equation the terms containing the coefficient c_3 cancel, such that there are only terms left which contain c_1, $c_{1,x}$, $c_{1,xx}$ and $c_{2,x}$. Uniqueness of the Baker-Akhiezer function then implies as in (3.32) that

$$\begin{aligned}&\psi_{q,t}(p,x,x_0,t,t_0)-\big(P_{q,2}(t)\psi_q\big)(p,x,x_0,t,t_0)=\\&=\big(2\mathrm{i}c_{1,xx}(x,t)-4c_{2,x}(x,t)+4c_1(x,t)c_{1,x}(x,t)\big)\psi_q(p,x,x_0,t,t_0).\end{aligned}\tag{3.38}$$

The expansion of the ratio of theta functions near p_∞ in the proof of Theorem 3.6 implies

$$\begin{aligned}c_{2,x}(x,t)=&-\frac{1}{2}\partial_x^3\ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\\&-\Big(\partial_x\ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\Big)\Big(\partial_x^2\ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\Big)\\&+\Big(\partial_x^2\ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\Big)\Big(\partial_x\ln\theta\big(\underline{z}(p_\infty,x,x_0,t,t_0)\big)\Big|_{x=x_0}\Big)\end{aligned}$$

and hence

$$2\mathrm{i}c_{1,xx}(x,t)+4c_1(x,t)c_{1,x}(x,t)-4c_{2,x}(x,t)=0.$$

This yields (3.35) by (3.38). □

Combining the last two theorems we easily check that the potential V_q defined as in (3.28) is a solution of the KdV equation, since the operators H_q and $P_{q,2}$ constitute the Lax pair for the KdV equation. More precisely, we have the following result:

Theorem 3.8 (Quasi-periodic finite-gap KdV solutions)**.** *Let $p=(z,\sigma R_{2g+1}(z)^{1/2})\in\mathcal{K}_g\setminus\{p_\infty\}$, $(z,x,x_0,t,t_0)\in\mathbb{C}\times\mathbb{R}^4$. Then $V_q(x,t)$ given by (3.28) satisfies the KdV equation*

$$\mathrm{KdV}(V_q):=V_{q,t}-6V_qV_{q,x}+V_{q,xxx}=0.\tag{3.39}$$

Proof. From (3.26) one gets

$$\begin{aligned}H_q(t)\psi_{q,t}(p,x,x_0,t,t_0)&=-\partial_x^2\psi_{q,t}(p,x,x_0,t,t_0)+V_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&=\partial_t\big(-\partial_x^2\psi_q(p,x,x_0,t,t_0)\big)+V_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&=\partial_t\big(z\psi_q(p,x,x_0,t,t_0)-V_q(x,t)\psi_q(p,x,x_0,t,t_0)\big)+V_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&=z\psi_{q,t}(p,x,x_0,t,t_0)-V_{q,t}\psi_q(p,x,x_0,t,t_0)-V_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&\quad+V_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&=z\psi_{q,t}(p,x,x_0,t,t_0)-V_{q,t}\psi_q(p,x,x_0,t,t_0).\end{aligned}\tag{3.40}$$

One can easily verify that $H_q(t)$ and $P_{q,2}(t)$ satisfy the Lax equation

$$\partial_tH_q(t)-\big[P_{q,2}(t),H_q(t)\big]=\mathrm{KdV}(V_q).\tag{3.41}$$

From (3.35), (3.40), and (3.41) one deduces

$$\begin{aligned}\mathrm{KdV}(V_q)\psi_q(p,x,x_0,t,t_0)&=\big(V_{q,t}(x,t)-zP_{q,2}(t)+H(t)P_{q,2}(t)\big)\psi_q(p,x,x_0,t,t_0)\\&=V_{q,t}(x,t)\psi_q(p,x,x_0,t,t_0)-z\psi_{q,t}(p,x,x_0,t,t_0)+H_q(x,t)\psi_{q,t}(p,x,x_0,t,t_0)\\&=V_{q,t}(x,t)\psi_q(p,x,x_0,t,t_0)-z\psi_{q,t}(p,x,x_0,t,t_0)\\&\quad+\big(z\psi_{q,t}(p,x,x_0,t,t_0)-V_{q,t}(x,t)\psi_q(p,x,x_0,t,t_0)\big)=0,\end{aligned}$$

that is,

$$\mathrm{KdV}(V_q) = 0.$$

□

The two branches of the Baker-Akhiezer function are denoted by

$$\psi_{q,\pm}(z, x, x_0, t, t_0) = \psi_q(p, x, x_0, t, t_0), \qquad p = (z, \pm). \tag{3.42}$$

It is well-known that the continuous spectrum of the quasi-periodic Schrödinger operator

$$H_q(t) = -\partial_x^2 + V_q(., t) \quad \text{on } H^2(\mathbb{R}), \tag{3.43}$$

is time independent and consists of $g + 1$ spectral bands

$$\sigma(H_q(t)) = \bigcup_{j=1}^{g} [E_{2j-2}, E_{2j-1}] \cup [E_{2g}, \infty), \quad t \in \mathbb{R}. \tag{3.44}$$

4. Scattering theory and the Riemann–Hilbert problem

In this chapter we use the same notation as in Chapter 3, but choose $x_0 = t_0 = 0$ and suppress the dependence on x_0, t_0 in the argument of the functions.

Let $\psi_{q,\pm}(z,x,t)$ be the branches of the Baker-Akhiezer functions defined in Chapter 3. We determine two Jost functions $\psi_\pm(z,x,t)$ for the perturbed problem as solutions of the equation

$$\big(-\frac{d}{dx^2} + V\big)\psi_\pm = z\psi_\pm, \quad z \in \mathbb{C}, \tag{4.1}$$

where V is the perturbed potential satisfying the short-range assumption

$$\int_{-\infty}^{+\infty} (1+|x|)\big(|V(x,t) - V_q(x,t)|\big)dx < \infty. \tag{4.2}$$

Next we establish the existence of such Jost solutions, that is, solutions of the perturbed operator H which asymptotically look like the Baker-Akhiezer functions.

4.1. The asymptotics of the Jost functions

This section is based on [BoutetdeMonvel08] and [Mikikits-Leitner09].

From the definition (3.14) we obtain that the branches of the Baker-Akhiezer function have the following form

$$\psi_{q,\pm}(z,x,t) = \theta_{q,\pm}(z,x,t)\exp(\pm \mathrm{i}xk(z) \mp 12\mathrm{i}t\int_{E_0}^{p}\omega_{p_\infty,2}), \tag{4.3}$$

where $\theta_{q,\pm}(z,x,t)$ is quasi-periodic with respect to x and

$$k(z) = -\int_{E_0}^{p}\omega_{p_\infty,0}, \qquad p = (z,+), \tag{4.4}$$

denotes the quasi-momentum map.

Theorem 4.1. *Assume* (4.2). *For every* $z \in \mathbb{C}\backslash\{E_j\}_{j=0}^{2g}$ *there exist (weak) solutions* $\psi_\pm(z,.,t)$ *of* $H\psi = z\psi$ *satisfying*

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp \mathrm{i}xk(z)}\big(\psi_\pm(z,x,t) - \psi_{q,\pm}(z,x,t)\big) = 0, \tag{4.5}$$

where $\psi_{q,\pm}(z,.,t)$ are the Baker–Akhiezer functions. Moreover, $\psi_\pm(z,.,t)$ are continuous (resp. holomorphic) with respect to z whenever $\psi_{q,\pm}(z,.,t)$ are and

$$\left|\mathrm{e}^{\mp \mathrm{i}xk(z)}\big(\psi_\pm(z,x,t)-\psi_{q,\pm}(z,x,t)\big)\right| \le C(z), \tag{4.6}$$

where $C(z)$ denotes some constant depending only on z.

Proof. Finding solutions of (4.1) is equivalent to solving the equation

$$(H_q - z)\psi = -\hat{V}\psi, \quad z\in\mathbb{C}, \tag{4.7}$$

where H_q is the operator defined in (3.27) and we have set

$$\hat{V} = V - V_q. \tag{4.8}$$

By Theorem 3.6 the Baker-Akhiezer function ψ_q defined in (3.14) is the solution of the homogeneous equation, that is,

$$(H_q - z)\psi_q = 0, \quad z\in\mathbb{C}.$$

Thus by using the variation of constants formula (cf. [Teschl08, Section 5.3]) we obtain the Volterra integral equations for the two Jost functions

$$\begin{aligned}\psi_\pm(z,x,t) = \psi_{q,\pm}(z,x,t) - \frac{1}{W\big(\psi_{q,+},\psi_{q,-}\big)}\int_x^{\pm\infty} \Big(\psi_{q,-}(z,x,t)\psi_{q,+}(z,y,t)\\ - \psi_{q,-}(z,y,t)\psi_{q,+}(z,x,t)\Big)\hat{V}(y,t)\psi_\pm(z,y,t)dy.\end{aligned} \tag{4.9}$$

Moreover, introducing $\tilde{\psi}_\pm(z,x) = \mathrm{e}^{\pm\mathrm{i}xk(z)}\psi_\pm(z,x)$ the resulting integral equation can be solved using the method of successive iterations. This proves the claims. □

Now we are ready to prove the asymptotic behavior of the Jost functions.

Theorem 4.2. *Assume (4.2). The asymptotic behavior of the two Jost functions are given by*

$$\psi_\pm(z,x,t) = \psi_{q,\pm}(z,x,t)\Big(1 + \tilde{V}_\pm(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} + o\big(\frac{1}{\sqrt{z}}\big)\Big), \tag{4.10}$$

as $z\to\infty$, where

$$\tilde{V}_\pm(x,t) = \mp\int_x^{\pm\infty}(V - V_q)(y,t)dy. \tag{4.11}$$

Proof. Again abbreviating $\hat{V} = V - V_q$ and invoking (4.9) one deduces

$$\begin{aligned}\frac{\psi_\pm(z,x,t)}{\psi_{q,\pm}(z,x,t)} &=\\ &= 1 - \frac{1}{W\big(\psi_{q,+},\psi_{q,-}\big)}\int_x^{\pm\infty}\Big(\psi_{q,-}(z,x,t)\psi_{q,+}(z,y,t)\frac{\psi_{q,\pm}(z,y,t)}{\psi_{q,\pm}(z,x,t)}\\ &\quad - \int_x^{\pm\infty}\psi_{q,-}(z,y,t)\psi_{q,+}(z,x,t)\frac{\psi_{q,\pm}(z,y,t)}{\psi_{q,\pm}(z,x,t)}\Big)\hat{V}(y,t)\frac{\psi_\pm(z,y,t)}{\psi_{q,\pm}(z,y,t)}dy\\ &= 1 \mp \int_x^{\pm\infty}\Big(G_q(z,x,x)\frac{\psi_{q,\pm}(z,y,t)^2}{\psi_{q,\pm}(z,x,t)^2} - G_q(z,y,y)\Big)\hat{V}(y,t)\frac{\psi_\pm(z,y,t)}{\psi_{q,\pm}(z,y,t)}dy,\end{aligned} \tag{4.12}$$

where

$$G_q(z,x,y) = \frac{1}{W\big(\psi_{q,+},\psi_{q,-}\big)} \begin{cases} \psi_{q,+}(z,x,t)\psi_{q,-}(z,y,t), & x \ge y, \\ \psi_{q,+}(z,y,t)\psi_{q,-}(z,x,t), & x \le y \end{cases} \tag{4.13}$$

is the Green function of H_q. We have

$$G_q(z,x,x) = \frac{\psi_{q,+}(z,x,t)\psi_{q,-}(z,x,t)}{W\big(\psi_{q,+},\psi_{q,-}\big)} = \frac{\mathrm{i}\prod_{j=1}^{g}\big(z-\mu_j(x,t)\big)}{2R_{2g+1}^{1/2}(z)}. \tag{4.14}$$

Hence for z near ∞ the Green function has the following asymptotic behavior

$$G(z,x,x) = \frac{\mathrm{i}}{2\sqrt{z}}\Big(1+\frac{1}{2}V_q(x,t)\frac{1}{z}+O\big(\frac{1}{z^2}\big)\Big), \tag{4.15}$$

where we made use of the fact that the quasi-periodic potential V_q defined in (3.28) can be written as

$$V_q(x,t) = E_0 + \sum_{j=1}^{g}\big(E_{2j-1}+E_{2j}-2\mu_j(x,t)\big) \tag{4.16}$$

for $p=(z,+)$ near p_∞. Next we insert (4.15) into (4.12) such that iteration implies

$$\frac{\psi_\pm(z,x,t)}{\psi_{q,\pm}(z,x,t)} = 1 \mp \frac{\mathrm{i}}{2\sqrt{z}}\Big(\int_x^{\pm\infty}\frac{\psi_{q,\pm}(z,y,t)^2}{\psi_{q,\pm}(z,x,t)^2}\hat{V}(y,t)dy - \int_x^{\pm\infty}\hat{V}(y,t)dy\Big) + O\big(\frac{1}{z}\big).$$

Next we will show that the first integral vanishes as $\sqrt{z}\to\infty$. We begin with the case $\mathrm{Im}(\sqrt{z})\to\infty$. Making use of Remark 3.5 we compute

$$\begin{aligned} \Big|\int_x^{\pm\infty}\frac{\psi_{q,\pm}(z,y,t)^2}{\psi_{q,\pm}(z,x,t)^2}\hat{V}(y,t)dy\Big| &\le C\int_x^{\pm\infty}\exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\big|\hat{V}(y,t)\big|dy \\ &\le C\int_x^{x+\varepsilon}\big|\hat{V}(y,t)\big|dy + C\cdot\exp\big(\mp 2\mathrm{Im}(\sqrt{z})\varepsilon\big)\int_{x+\varepsilon}^{\pm\infty}\big|\hat{V}(y,t)\big|dy, \end{aligned}$$

such that the first integral can be made arbitrary small if $\varepsilon>0$ is small and the second integral vanishes as $\mathrm{Im}(\sqrt{z})\to\infty$.

Otherwise, if $\mathrm{Re}(\sqrt{z})\to\infty$, we use (4.3) to rewrite the integral as

$$\int_x^{\pm\infty}\Bigg(\frac{\theta_{q,\pm}(z,y,t)^2}{\theta_{q,\pm}(z,x,t)^2}\hat{V}(y,t)\exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\Bigg)\exp\big(\pm 2\mathrm{i}\mathrm{Re}(\sqrt{z})(y-x)\big)dy.$$

Since

$$\left|\frac{\theta_{q,\pm}(z,y,t)^2}{\theta_{q,\pm}(z,x,t)^2}\hat{V}(y,t)\exp\big(\mp 2\mathrm{Im}(\sqrt{z})(y-x)\big)\right| \le |\hat{V}(y,t)|$$

the integral vanishes as $\mathrm{Re}(\sqrt{z})\to\infty$ by a slight variation of the Riemann–Lebesgue lemma.

Hence we finally have

$$\frac{\psi_\pm(z,x,t)}{\psi_{q,\pm}(z,x,t)} = 1 \pm \frac{\mathrm{i}}{2\sqrt{z}}\int_x^{\pm\infty}\hat{V}(y,t)dy + o\big(\frac{1}{\sqrt{z}}\big) \tag{4.17}$$

as $z\to\infty$. □

Moreover, we have the following result which concerns the asymptotics of the Jost solutions at the other side.

Lemma 4.3. *([Mikikits-Leitner09]) Assume* (4.2)*. Then the Jost solutions* $\psi_\pm(z,.,t)$*,* $z\in\mathbb{C}\backslash\sigma(H)$*, satisfy*

$$\lim_{x\to\mp\infty}\left|\mathrm{e}^{\mp\mathrm{i}xk(z)}\big(\psi_\pm(z,x,t)-\alpha(z)\psi_{q,\pm}(z,x,t)\big)\right|=0, \tag{4.18}$$

where

$$\alpha(z)=\frac{W(\psi_-(z),\psi_+(z))}{W(\psi_{q,-}(z),\psi_{q,+}(z))}=\frac{\prod_{j=1}^{g}(z-\mu_j)}{2\mathrm{i}R_{2g+1}^{1/2}(z)}W(\psi_-(z),\psi_+(z)). \tag{4.19}$$

Proof. The proof can be looked up in [Mikikits-Leitner09], proof of Lemma 3.5. □

4.2. The Riemann-Hilbert problem

Now we are ready to deduce the Riemann–Hilbert problem from scattering theory.

Lemma 4.4 (Scattering relations)**.** *One has the* scattering relations

$$T(z)\psi_\pm(z,x,t)=\overline{\psi_\mp(z,x,t)}+R_\mp(z)\psi_\mp(z,x,t),\qquad z\in\sigma(H_q), \tag{4.20}$$

where $T(z)$*,* $R_\pm(z)$ *are the transmission respectively the reflection coefficient defined as*

$$T(z)=\frac{W(\overline{\psi_+(z)},\psi_+(z))}{W(\psi_-(z),\psi_+(z))},\qquad R_\pm(z):=-\frac{W(\psi_\mp(z),\overline{\psi_\pm(z)})}{W(\psi_\mp(z),\psi_\pm(z))},\quad z\in\sigma(H_q). \tag{4.21}$$

Here $\psi_\pm(z,x,t)$ *is defined such that* $\psi_\pm(z,x,t)=\lim_{\varepsilon\downarrow0}\psi_\pm(z+\mathrm{i}\varepsilon,x,t)$*,* $z\in\sigma(H_q)$*. If we take the limit from the other side we have* $\overline{\psi_\pm(z,x,t)}=\lim_{\varepsilon\downarrow0}\psi_\pm(z-\mathrm{i}\varepsilon,x,t)$*. Moreover, the transmission* $T(z)$ *and reflection* $R_\pm(z)$ *coefficients satisfy*

$$T(z)\overline{R_+(z)}+\overline{T(z)}R_-(z)=0,\qquad |T(z)|^2+|R_\pm(z)|^2=1. \tag{4.22}$$

In particular one reflection coefficient, say $R(z)=R_+(z)$*, suffices.*

Proof. To see that (4.20) indeed holds note that apart from $\psi_\pm(z,x,t)$ also $\overline{\psi_\pm(z,x,t)}$ is a solution of

$$\Big(-\frac{d^2}{dx^2}+V\Big)\psi=z\psi,\quad z\in\sigma(H_q). \tag{4.23}$$

Therefore we have found four solutions of a second order linear differential equation, hence they have to be linearly dependent, that is,

$$\psi_\pm(z,x,t)=\alpha_\mp(z)\overline{\psi_\mp(z,x,t)}+\beta_\mp(z)\psi_\mp(z,x,t).$$

Thus we have

$$W(\psi_\pm(z),\psi_\mp(z))=\alpha_\mp(z)W(\overline{\psi_\mp(z)},\psi_\mp(z)),$$
$$W(\psi_\pm(z),\overline{\psi_\mp(z)})=\beta_\mp(z)W(\psi_\mp(z),\overline{\psi_\mp(z)}),$$

and hence

$$\alpha_\pm(z) = \frac{W(\psi_\mp(z), \psi_\pm(z))}{W(\overline{\psi_\pm}(z), \psi_\pm(z))} \quad \text{and} \quad \beta_\pm(z) = \frac{W(\psi_\mp(z), \overline{\psi_\pm(z)})}{W(\psi_\pm(z), \overline{\psi_\pm(z)})}. \tag{4.24}$$

Defining

$$T(z) := \alpha(z)^{-1} = \alpha_+(z)^{-1} = \alpha_-(z)^{-1}$$

and

$$R_\pm(z) := \beta_\pm(z)\alpha(z)^{-1}$$

one gets the scattering relations (4.20). The properties (4.22) can be verified directly by straight-forward calculations. □

Lemma 4.5. *The transmission coefficient $T(z)$ has a meromorphic extension to $\mathbb{C}\backslash\sigma(H_q(t))$ with simple poles at the eigenvalues ρ_j. The residues are given by*

$$\mathrm{Res}_{\rho_j} T(z) = \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^{g}(\rho_j - \mu_k)} \frac{\gamma_{\pm,j}}{c_j^{\pm 1}}, \tag{4.25}$$

where

$$\gamma_{\pm,j}^{-1} = \int_{-\infty}^{\infty} |\psi_\pm(\rho_j, y, t)|^2 dy \tag{4.26}$$

are referred to as norming constants and $\psi_-(\rho_j, x, t) = c_j \psi_+(\rho_j, x, t)$.

Proof. For notational simplicity let us suppress the dependence on x and t in the scope of this proof. To see that (4.25) indeed holds we note that

$$\mathrm{Res}_{\rho_j} T(z) = \frac{W(\overline{\psi_+(\rho_j)}, \psi_+(\rho_j))}{\left(\frac{d}{dz} W(\psi_-(z), \psi_+(z))\right)\big|_{z=\rho_j}}, \tag{4.27}$$

since $W(\overline{\psi_+(z)}, \psi_+(z))$ has no zeros at ρ_j but $W(\psi_-(z), \psi_+(z))$ has. Let us denote the differentiation with respect to z with a dot, then we have

$$\frac{d}{dz} W(\psi_-(z), \psi_+(z)) = W(\dot{\psi}_-(z), \psi_+(z)) + W(\psi_-(z), \dot{\psi}_+(z)). \tag{4.28}$$

Moreover, using (4.1) one obtains

$$\begin{aligned} \frac{\partial}{\partial x} W(\dot{\psi}_-(z), \psi_+(z)) &= \psi_-(z)\psi_+(z), \\ \frac{\partial}{\partial x} W(\psi_-(z), \dot{\psi}_+(z)) &= -\psi_-(z)\psi_+(z). \end{aligned} \tag{4.29}$$

From [Mikikits-Leitner09], proof of Lemma 4.1, we know that

$$\begin{aligned} W(\dot{\psi}_-(z), \psi_+(z)) &\longrightarrow \alpha(z) W(\dot{\psi}_{q,-}(z), \psi_{q,+}(z)) \quad \text{as } x \to -\infty, \\ W(\psi_-(z), \dot{\psi}_+(z)) &\longrightarrow \alpha(z) W(\psi_{q,-}(z), \dot{\psi}_{q,+}(z)) \quad \text{as } x \to +\infty, \end{aligned} \tag{4.30}$$

where $\alpha(z)$ is given by (4.19). Since ρ_j is a zero of $W(\psi_-(z),\psi_+(z))$ one has $\alpha(\rho_j)=0$ and thus

$$\begin{aligned} W(\dot\psi_-(\rho_j),\psi_+(\rho_j)) &\longrightarrow 0 \quad \text{as } x\to-\infty,\\ W(\psi_-(\rho_j),\dot\psi_+(\rho_j)) &\longrightarrow 0 \quad \text{as } x\to+\infty. \end{aligned} \tag{4.31}$$

Combining (4.28), (4.30), and (4.31) yields

$$\frac{d}{dz}W(\psi_-(z),\psi_+(z))\big|_{z=\rho_j} = \tag{4.32}$$

$$= \int_{-\infty}^{x}\frac{\partial}{\partial y}W(\dot\psi_-(\rho_j),\psi_+(\rho_j))dy - \int_{x}^{+\infty}\frac{\partial}{\partial y}W(\psi_-(\rho_j),\dot\psi_+(\rho_j))dy =$$

$$= \int_{-\infty}^{\infty}\psi_-(\rho_j)\psi_+(\rho_j)dy, \tag{4.33}$$

where the prime denotes differentiation with respect to z. Hence

$$\big(\frac{d}{dz}W(\psi_-(z),\psi_+(z))\big)\big|_{z=\rho_j} = \gamma_{\pm,j}^{-1}c_j^{\pm1}, \tag{4.34}$$

where $\gamma_{\pm,j}^{-1}=\|\psi_\pm(\rho_j,.,t)\|_2^2$ and $\psi_-(\rho_j,x,t)=c_j\psi_+(\rho_j,x,t)$. Moreover, for $z\in\sigma(H_q)$ we have (cf. [Gesztesy03, Equ. (1.87)])

$$W(\overline{\psi_\pm(z)},\psi_\pm(z)) = W(\psi_{q,\mp}(z),\psi_{q,\pm}(z)) = \pm\frac{2\mathrm{i}R_{2g+1}^{1/2}(z)}{\prod_{k=1}^{g}(z-\mu_k)}, \tag{4.35}$$

which finishes the proof. □

Note that one reflection coefficient, say $R(z)=R_+(z)$ and one set of norming constants, say $\gamma_j=\gamma_{+,j}$ suffices.

We will define a Riemann–Hilbert problem on the Riemann surface $\mathcal{K}_g$ as follows:

$$m(p,x,t) = \begin{cases} \begin{pmatrix} T(z)\frac{\psi_-(z,x,t)}{\psi_{q,-}(z,x,t)} & \frac{\psi_+(z,x,t)}{\psi_{q,+}(z,x,t)} \end{pmatrix}, & p=(z,+)\\ \begin{pmatrix} \frac{\psi_+(z,x,t)}{\psi_{q,+}(z,x,t)} & T(z)\frac{\psi_-(z,x,t)}{\psi_{q,-}(z,x,t)} \end{pmatrix}, & p=(z,-) \end{cases} \tag{4.36}$$

We are interested in the jump condition of $m(p,x,t)$ on Σ, the boundary of $\Pi_\pm$ (oriented counterclockwise when viewed from top sheet Π_+). It consists of two copies $\Sigma_\pm$ of $\sigma(H_q)$ which correspond to non-tangential limits from $p=(z,+)$ with $\pm\mathrm{Im}(z)>0$, respectively to non-tangential limits from $p=(z,-)$ with $\mp\mathrm{Im}(z)>0$.

To formulate our jump condition we use the following convention: When representing functions on Σ, the lower subscript denotes the non-tangential limit from Π_+ or Π_-, respectively,

$$m_\pm(p_0,x,t) = \lim_{\Pi_\pm\ni p\to p_0} m(p,x,t), \qquad p_0\in\Sigma. \tag{4.37}$$

Using the notation above implicitly assumes that these limits exist in the sense that $m(p,x,t)$ extends to a continuous function on the boundary away from the band edges.

Moreover, we will also use symmetries with respect to the sheet exchange map

$$p^* = \begin{cases} (z, \mp) & \text{for } p = (z, \pm), \\ \infty_\mp & \text{for } p = \infty_\pm, \end{cases} \tag{4.38}$$

and complex conjugation

$$\overline{p} = \begin{cases} (\overline{z}, \pm) & \text{for } p = (z, \pm) \not\in \Sigma, \\ (z, \mp) & \text{for } p = (z, \pm) \in \Sigma, \\ \infty_\pm & \text{for } p = \infty_\pm. \end{cases} \tag{4.39}$$

In particular, we have $\overline{p} = p^*$ for $p \in \Sigma$.

Note that we have $\tilde{m}_\pm(p, x, t) = m_\mp(p^*, x, t)$ for $\tilde{m}(p, x, t) = m(p^*, x, t)$ (since $*$ reverses the orientation of Σ) and $\tilde{m}_\pm(p, x, t) = \overline{m_\pm(p^*, x, t)}$ for $\tilde{m}(p, x, t) = \overline{m(\overline{p}, x, t)}$.

Theorem 4.6 (Vector Riemann–Hilbert problem)**.** *Let $\mathcal{S}_+(H(0)) = \{R(z),\ z \in \sigma(H_q);\ (\rho_j, \gamma_j),\ 1 \le j \le N\}$ be the right scattering data of the operator $H(0)$. Then $m(p) = m(p, x, t)$ defined in (4.36) is meromorphic away from Σ and satisfies:*

1. *The jump condition*

$$m_+(p) = m_-(p) J(p), \quad \text{for } p \in \Sigma, \tag{4.40}$$

where the jump matrix is given by

$$J(p) = \begin{pmatrix} 1 - |R(p)|^2 & -\overline{R(p)\Theta(p, x, t)} \mathrm{e}^{-t\phi(p)} \\ R(p)\Theta(p, x, t)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}, \tag{4.41}$$

2. *the divisor conditions*

$$(m_1) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*} - \mathcal{D}_{\underline{\rho}}, \qquad (m_2) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)} - \mathcal{D}_{\underline{\rho}^*}, \tag{4.42}$$

and pole conditions

$$\begin{aligned} &\Big(m_1(p) + \frac{2\mathrm{i} R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^g (\rho_j - \mu_k)} \frac{\gamma_j}{\pi(p) - \rho_j} \frac{\psi_q(p, x, t)}{\psi_q(p^*, x, t)} m_2(p)\Big) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*}, \ \text{near } \rho_j, \\ &\Big(\frac{2\mathrm{i} R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^g (\rho_j - \mu_k)} \frac{\gamma_j}{\pi(p) - \rho_j} \frac{\psi_q(p, x, t)}{\psi_q(p^*, x, t)} m_1(p) + m_2(p)\Big) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)}, \ \text{near } \rho_j^*, \end{aligned} \tag{4.43}$$

3. *the symmetry condition*

$$m(p^*) = m(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.44}$$

4. *and the normalization*

$$m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \tag{4.45}$$

Here

$$\mathcal{D}_{\underline{\rho}} = \sum_j \mathcal{D}_{\rho_j}, \qquad \mathcal{D}_{\underline{\rho}^*} = \sum_j \mathcal{D}_{\rho_j^*}. \tag{4.46}$$

denotes the divisor of poles, and the phase is given by

$$\phi(p, \frac{x}{t}) = -24\mathrm{i} \int_{E_0}^{p} \omega_{p_\infty,2} - 2\mathrm{i}\frac{x}{t} \int_{E_0}^{p} \omega_{p_\infty,0} \in \mathrm{i}\mathbb{R} \quad \textit{for } p \in \Sigma. \tag{4.47}$$

Moreover, we have set

$$\Theta(p,x,t) = \frac{\theta(\underline{z}(p,x,t))}{\theta(\underline{z}(p,0,0))} \frac{\theta(\underline{z}(p^*,0,0))}{\theta(\underline{z}(p^*,x,t))}. \tag{4.48}$$

Proof. 1. For the proof of the jump condition we need the scattering relations (4.20) and (4.22). We have

$$J(p)^{-1} = \begin{pmatrix} 1 & \overline{R(p)\Theta(p,x,t)}\mathrm{e}^{-t\phi(p)} \\ -R(p)\Theta(p,x,t)\mathrm{e}^{t\phi(p)} & 1 - |R(p)|^2 \end{pmatrix} \tag{4.49}$$

and we will show $m_+(p)J(p)^{-1} = m_-(p)$. Note

$$\frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} = \Theta(p,x,t)\mathrm{e}^{t\phi(p)}.$$

To see that the jump condition holds for the first component of $m_-(p,x,t)$ we calculate

$$\begin{aligned}
T(p)&\frac{\psi_-(p,x,t)}{\psi_{q,-}(p,x,t)} - R(p)\Theta(p,x,t)\mathrm{e}^{t\phi(p)}\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)} \\
&= T(p)\frac{\psi_-(p,x,t)}{\psi_{q,-}(p,x,t)} - R(p)\frac{\psi_{q,+}(p,x,t)}{\psi_{q,-}(p,x,t)}\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)} \\
&= \frac{1}{\psi_{q,-}(p,x,t)}\left(T(p)\psi_-(p,x,t) - R(p)\psi_+(p,x,t)\right) \\
&= \frac{\overline{\psi_+(p,x,t)}}{\psi_{q,-}(p,x,t)} = \frac{\psi_+(p^*,x,t)}{\psi_{q,+}(p^*,x,t)}, \quad \text{for } p \in \Sigma.
\end{aligned}$$

For the second component of $m_-(p)$ we compute

$$
\begin{aligned}
&T(p)\frac{\psi_-(p,x,t)}{\psi_{q,-}(p,x,t)}\overline{R(p)\Theta(p,x,t)}\mathrm{e}^{-t\phi(p)} + (1-|R(p)|^2)\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)}\\
&= \frac{\psi_-(p,x,t)}{\psi_{q,-}(p,x,t)}\overline{R(p)}\frac{\psi_{q,-}(p,x,t)}{\psi_{q,+}(p,x,t)} + |T(p)|^2\,\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)}\\
&= \Big(\overline{\psi_+(p,x,t)} + R(p)\psi_+(p,x,t)\Big)\,\frac{\overline{R(p)}}{\psi_{q,+}(p,x,t)} + |T(p)|^2\,\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)}\\
&= \frac{\overline{\psi_+(p,x,t)}}{\psi_{q,+}(p,x,t)}\overline{R(p)} + |R(p)|^2\,\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)} + |T(p)|^2\,\frac{\psi_+(p,x,t)}{\psi_{q,+}(p,x,t)}\\
&= \frac{1}{\psi_{q,+}(p,x,t)}\,\Big(\overline{\psi_+(p,x,t)R(p)} + \psi_+(p,x,t)\Big)\\
&= \frac{1}{\psi_{q,+}(p,x,t)}\overline{T(p)\psi_-(p,x,t)} = T(p^*)\frac{\psi_-(p^*,x,t)}{\psi_{q,-}(p^*,x,t)}, \quad \text{for } p\in\Sigma.
\end{aligned}
$$

Here we have extended our definition of T to Σ such that it is equal to $T(z)$ on Σ_+ and equal to $\overline{T(z)}$ on Σ_-. Similarly for $R(z)$. In particular, the condition on Σ_+ is just the complex conjugate of the one on Σ_- since we have $R(p^*) = \overline{R(p)}$ and $m_\pm(p^*) = \overline{m_\pm(p)}$ for $p\in\Sigma$.

2. By Riemann's vanishing theorem (cf. [Teschl00, Theorem A.13]) the divisor of the Baker–Akhiezer function ψ_q satisfies

$$
\big(\psi_q(p,x,t)\big) = \mathcal{D}_{\underline{\hat{\mu}}(x,t)} - \mathcal{D}_{\underline{\hat{\mu}}}. \tag{4.50}
$$

Moreover, the transmission coefficient $T(p)$ has simple poles at the eigenvalues ρ_j, $1\le j\le N$. Thus the divisor conditions (4.42) are indeed fulfilled.

The pole conditions follow from the fact that the transmission coefficient $T(p)$ is meromorphic in $\mathcal{K}_g\setminus\Sigma$ with simple poles at ρ_j and its residues are given by (4.25).

3. The symmetry condition (4.44) obviously holds by the definition of the function $m(p)$.

4. The normalization (4.45) is immediately clear from the asymptotic behavior of $m(p)$ near p_∞, which will be derived later on (cf. Theorem 4.9).

□

Remark 4.7. *Let $\rho_j\notin\{\mu_k(x,t)\}_{k=1}^g$. Note that the pole conditions (4.43) can be equivalently formulated in the following way:*

$$
\begin{aligned}
\operatorname{Res}_{\rho_j} m(p) &= \lim_{p\to\rho_j} m(p)\begin{pmatrix} 0 & 0\\ \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g(\rho_j-\mu_k)} & 0\end{pmatrix},\\
\operatorname{Res}_{\rho_j^*} m(p) &= \lim_{p\to\rho_j^*} m(p)\begin{pmatrix} 0 & \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g(\rho_j-\mu_k)}\\ 0 & 0\end{pmatrix}.
\end{aligned}\tag{4.51}
$$

For notational convenience we will abbreviate

$$J_j = \begin{pmatrix} 0 & 0 \\ \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g(\rho_j-\mu_k)} & 0 \end{pmatrix}$$

$$J_j^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g(\rho_j-\mu_k)} \\ 0 & 0 \end{pmatrix}.$$

Moreover, one pole condition, e.g. the one at ρ_j, suffices, since the one at ρ_j^* follows by the symmetry condition:

$$\begin{aligned} \mathrm{Res}_{\rho_j^*} m(p) &= \lim_{p\to\rho_j^*}(p-\rho_j^*)m(p) = \lim_{p^*\to\rho_j}(p^*-\rho_j)m(p) \\ &= \lim_{p^*\to\rho_j}(p^*-\rho_j)m(p^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \lim_{p\to\rho_j} m(p)J_j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \lim_{p\to\rho_j} m(p^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \lim_{p\to\rho_j^*} m(p)J_j^*. \end{aligned}$$

The following asymptotic behavior of the transmission coefficient for p near p_∞ is valid:

Lemma 4.8. *Assume* (4.2). *Then the transmission coefficient $T(z)$ has the following asymptotic behavior*

$$T(z) = 1 + \frac{1}{2\mathrm{i}\sqrt{z}}\int_{-\infty}^{\infty}(V-V_q)(y,t)dy + o\big(\frac{1}{\sqrt{z}}\big) \tag{4.52}$$

as $z\to\infty$.

Proof. Use (4.10) to compute the asymptotic behavior of the Wronskian $W(\psi_-,\psi_+)$. Then inserting this expression into the definition of the transmission coefficient (4.21) yields

$$T(z) = \frac{W(\psi_{q,-}(z),\psi_{q,+}(z))}{W(\psi_-(z),\psi_+(z))} = 1 + \frac{1}{2\mathrm{i}\sqrt{z}}\int_{-\infty}^{\infty}(V-V_q)(y,t)dy + o(z^{-1/2}).$$

□

We have the following asymptotic behavior of $m(p)$ for p near p_∞:

Theorem 4.9. *The function $m(p)$ defined in (4.36) satisfies*

$$m(p) = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2\mathrm{i}\sqrt{z}}\int_x^{\infty}(V-V_q)(y,t)dy\begin{pmatrix} -1 & 1 \end{pmatrix} + o\big(\frac{1}{\sqrt{z}}\big), \quad p=(z,\pm) \tag{4.53}$$

for p near p_∞.

Proof. Take $p = (z, +)$, then using (4.10) one obtains the asymptotics for the second component of $m(p)$

$$\frac{\psi_{q,+}(z,x,t)}{\psi_{+}(z,x,t)} = 1 + \tilde{V}_{+}(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} + o\big(z^{-1/2}\big). \tag{4.54}$$

To get the asymptotic behavior of the first component use Lemma 4.8 and again (4.10) to get

$$\begin{aligned} T(z)\frac{\psi_{-}(z,x,t)}{\psi_{q,-}(z,x,t)} =& \big(1 - \frac{1}{2\mathrm{i}\sqrt{z}}(\tilde{V}_{-}(x,t) + \tilde{V}_{+}(x,t)) + o(z^{-1/2})\big)\cdot \\ & \cdot \big(1 + \tilde{V}_{-}(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} + o(z^{-1/2})\big) = 1 - \tilde{V}_{+}(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} + o(z^{-1/2}). \end{aligned}$$

Hence we have verified (4.53). For $p = (z, -)$ we get the same asymptotic behavior of $m(p)$ because of the symmetry condition (4.44). □

For our further analysis it will be convenient to rewrite the pole conditions as jump conditions following the idea of Deift, Kamvissis, Kriecherbauer, and Zhou [Deift96]. For that purpose we choose ε so small that the discs $|\pi(p) - \rho_j| < \varepsilon$ are inside the upper sheet Π_+ and do not intersect with the other contours around the spectral bands chosen as described in Section 5.4. Then we have the following

Lemma 4.10. *Redefine $m(p)$ in a neighborhood of ρ_j respectively ρ_j^* in the following way:*

$$m(p) = \begin{cases} m(p)\begin{pmatrix} 1 & 0 \\ \frac{\gamma_j(p,x,t)}{\pi(p)-\rho_j} & 1 \end{pmatrix}, & {|\pi(p)-\rho_j|<\varepsilon \atop p\in\Pi_+}, \\ m(p)\begin{pmatrix} 1 & \frac{\gamma_j(p,x,t)}{\pi(p)-\rho_j} \\ 0 & 1 \end{pmatrix}, & {|\pi(p)-\rho_j|<\varepsilon \atop p\in\Pi_-}, \\ m(p), & \text{else}, \end{cases} \tag{4.55}$$

where $\gamma_j(p,x,t)$ is a function which is analytic in $0 < |\pi(p) - \rho_j| < \varepsilon$ and satisfies

$$\lim_{p\to\rho_j} \gamma_j(p,x,t)\frac{\psi_q(p^*,x,t)}{\psi_q(p,x,t)} = \lim_{p\to\rho_j^*} \gamma_j(p,x,t)\frac{\psi_q(p^*,x,t)}{\psi_q(p,x,t)} = \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^{g}(\rho_j-\mu_k)}. \tag{4.56}$$

For example, we can choose

$$\gamma_j(p,x,t) = \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^{g}(\rho_j-\mu_k)}\frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} \tag{4.57}$$

or

$$\gamma_j(p,x,t) = \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho_j)}{\prod_{k=1}^{g}(\pi(p)-\mu_k)}\frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)}. \tag{4.58}$$

Then $m(p)$ is meromorphic away from Σ and satisfies (4.40), (4.44), (4.45), the divisor conditions change according to

$$(m_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*}, \qquad (m_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)}, \tag{4.59}$$

and the pole conditions are replaced by the jump conditions

$$\begin{aligned} m_+(p) &= m_-(p) \begin{pmatrix} 1 & 0 \\ \frac{\gamma_j(p,x,t)}{\pi(p)-\rho_j} & 1 \end{pmatrix}, \quad p \in \Sigma_\varepsilon(\rho_j), \\ m_+(p) &= m_-(p) \begin{pmatrix} 1 & \frac{\gamma_j(p,x,t)}{\pi(p)-\rho_j} \\ 0 & 1 \end{pmatrix}, \quad p \in \Sigma_\varepsilon(\rho_j^*), \end{aligned} \tag{4.60}$$

where

$$\Sigma_\varepsilon(p) = \{q \in \Pi_\pm \,:\, |\pi(q) - z| = \varepsilon\}, \qquad p = (z, \pm), \tag{4.61}$$

is a small circle oriented counterclockwise around p on the same sheet as p.

Proof. Everything except for the pole conditions follows as in the proof of Theorem 4.6. That the pole conditions (4.43) are indeed replaced by the jump conditions (4.60) as $m(p)$ is redefined as in (4.55) can be shown by a straightforward calculation. □

Solitary waves are special solutions of the KdV equation in the sense that they correspond to reflectionless potentials, i.e., $R \equiv 0$. This can be best seen by choosing an initial profile which corresponds to a reflectionless potential and then solve this initial value problem for the KdV equation by the inverse scattering method (cf. [Drazin90, Section 4.5]). The solutions obtained in that way are the single-soliton solution if one eigenvalue is present, or, more general, the N-soliton solution corresponding to N eigenvalues. The velocities (and thus the amplitudes) and relative positions of these solitary waves are determined by the eigenvalues and the norming constants, respectively. Since our method makes use of the fact that one deduces the Riemann–Hilbert problem from scattering theory, it is quite obvious that also in the Riemann–Hilbert approach each discrete eigenvalue corresponds to a soliton.

The next thing we will do will thus be to deduce the one-soliton solution of our Riemann–Hilbert problem, i.e., the solution in the case where only one eigenvalue ρ corresponding to one bound state is present and the reflection coefficient $R(p)$ vanishes identically on $\mathcal{K}_g$.

Lemma 4.11 (One-soliton solution)**.** *Suppose there is only one eigenvalue and a vanishing reflection coefficient, that is, $\mathcal{S}_+(H(t)) = \{R(p) \equiv 0,\ p \in \Sigma;\ (\rho, \gamma)\}$. Let*

$$c_{q,\gamma}(\rho, x, t) = 1 + \gamma W_{(x,t)}(\dot{\psi}_q(\rho, x, t), \psi_q(\rho, x, t)) \tag{4.62}$$

and

$$\psi_{q,\gamma}(p, x, t) = \psi_q(p, x, t) + \frac{\gamma}{z - \rho} \frac{\psi_q(\rho, x, t) W_{(x,t)}(\psi_q(\rho, x, t), \psi_q(p, x, t))}{c_{q,\gamma}(\rho, x, t)}. \tag{4.63}$$

Here the dot denotes a derivate with respect to ρ and $W_{(x,t)}(f,g) = (f(x)g'(x) - f'(x)g(x))$ is the usual Wronski determinant, where the prime denotes the derivative with respect to x.

Then the unique solution of the Riemann–Hilbert problem (4.40)–(4.45) is given by

$$m_0(p) = \begin{pmatrix} f(p^*,x,t) & f(p,x,t) \end{pmatrix}, \qquad f(p,x,t) = \frac{\psi_{q,\gamma}(p,x,t)}{\psi_q(p,x,t)}.$$

In particular

$$\int_x^\infty (V - V_q)(y,t)dy = 2\gamma \frac{\psi_q(\rho,x,t)^2}{c_{q,\gamma}(\rho,x,t)}, \tag{4.64}$$

or

$$(V - V_q)(x,t) = -4\gamma \frac{\psi_q(\rho,x,t)\psi_q'(\rho,x,t)}{c_{q,\gamma}(\rho,x,t)} + 2\gamma \frac{\psi_q(\rho,x,t)^2 c_{q,\gamma}'(\rho,x,t)}{c_{q,\gamma}(\rho,x,t)^2}. \tag{4.65}$$

Proof. Since we assume the reflection coefficient to vanish, the jump along Σ disappears. Moreover, since the symmetry condition (4.44) has to be satisfied it follows that the solution of the Riemann–Hilbert problem (4.40)–(4.45) has to be of the form $m_0(p) = \begin{pmatrix} f(p^*,x,t) & f(p,x,t) \end{pmatrix}$. The divisor conditions (4.42) follow from (4.50) and by construction of $\psi_{q,\gamma}$. It is obvious that the normalization condition (4.45) holds. Thus it is only left to check the pole conditions (4.43). For that purpose we compute

$$\begin{aligned}
\lim_{p\to\rho}(z-\rho)f(p^*) &= \frac{\gamma(\rho,x,t)}{c_{q,\gamma}(\rho,,x,t)} W_{(x,t)}(\psi_q(\rho,x,t), \psi_q(\rho^*,x,t)) \\
&= -\frac{\gamma(\rho,x,t)}{c_{q,\gamma}(\rho,,x,t)} \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho)}{\prod_{k=1}^g(\rho-\mu_k)},
\end{aligned}$$

where we defined

$$\gamma(p,x,t) = \gamma \frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} = \gamma\Theta(p,x,t)\mathrm{e}^{t\phi(p)},$$

and we used (4.35). Moreover,

$$\begin{aligned}
\lim_{p\to\rho} f(p) &= 1 + \frac{\gamma}{c_{q,\gamma}(\rho,x,t)} \lim_{p\to\rho} \frac{W_{(x,t)}(\psi_q(\rho,x,t),\psi_q(p,x,t))}{z-\rho} \\
&= 1 + \frac{\gamma}{c_{q,\gamma}(\rho,x,t)} \big[\psi_q(\rho,x,t) \lim_{p\to\rho} \frac{\psi_q'(p,x,t) - \psi_q'(\rho,x,t)}{z-\rho} - \\
&\quad - \psi_q'(\rho,x,t) \lim_{p\to\rho} \frac{\psi_q(p,x,t) - \psi_q(\rho,x,t)}{z-\rho}\big] \\
&= 1 + \frac{\gamma}{c_{q,\gamma}(\rho,x,t)} W_{(x,t)}(\psi_q(\rho,x,t), \dot{\psi}_q(\rho,x,t)) = \frac{1}{c_{q,\gamma}(\rho,x,t)}.
\end{aligned}$$

Hence we see that the pole conditions (4.43) are satisfied.

The formula (4.64) follows after expanding around $p = p_\infty$, that is,

$$\begin{aligned} f(p,x,t) &= 1 + \frac{\gamma}{(z-\rho)c_{q,\gamma}(\rho,x,t)}\psi_q(\rho,x,t)\Big(\psi_q(\rho,x,t)m_q(p,x,t) - \psi_q'(\rho,x,t)\Big) \\ &= 1 \mp \frac{\gamma}{c_{q,\gamma}(\rho,x,t)}\psi_q(\rho,x,t)^2\frac{1}{\mathrm{i}\sqrt{z}} + O(z^{-1}), \quad p=(z,\pm), \end{aligned}$$

where we have used that the Weyl–Titchmarsh m-function has the following asymptotic expansion for p near p_∞ (cf. [Mikikits-Leitner09, Lemma 6.1])

$$m_{q,\pm}(z,x,t) = \frac{\psi_{q,\pm}'(z,x,t)}{\psi_{q,\pm}(z,x,t)} = \pm\mathrm{i}\sqrt{z} + \frac{V_q(x,t)}{2\mathrm{i}\sqrt{z}} + O(z^{-1}), \quad p=(z,\pm). \tag{4.66}$$

Thus comparing with (4.53) proves the equation (4.64).

To see uniqueness, let $\tilde{m}_0(p)$ be a second solution which must be of the form $\tilde{m}_0(p) = \big(\tilde{f}(p^*) \quad \tilde{f}(p)\big)$ by the symmetry condition. Since the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ is nonspecial, the Riemann–Roch theorem implies $\tilde{f}(p) = \alpha f(p) + \beta$ for some $\alpha, \beta \in \mathbb{C}$. But the pole condition implies $\beta = 0$ and the normalization condition implies $\alpha = 1$. □

5. Conjugation and Deformation

5.1. The stationary phase points and the nonlinear dispersion relation

There is a correspondence between the energy λ of the Lax operator H_q and the propagation speed v at which the corresponding parts of the solutions of the KdV equation travel. In this section we want to investigate this correspondence, which can be considered as an analog of the classical dispersion relation. If we set

$$v(\lambda) = \lim_{\varepsilon\to 0} \frac{-12\,\mathrm{Re}\big(\mathrm{i}\int_{E_0}^{(\lambda+\mathrm{i}\varepsilon,+)} \omega_{p_\infty,2}\big)}{\mathrm{Re}\big(\mathrm{i}\int_{E_0}^{(\lambda+\mathrm{i}\varepsilon,+)} \omega_{p_\infty,0}\big)} = \lim_{\varepsilon\to 0} \frac{-12\,\mathrm{Im}\int_{E_0}^{(\lambda+\mathrm{i}\varepsilon,+)} \omega_{p_\infty,2}}{\mathrm{Im}\int_{E_0}^{(\lambda+\mathrm{i}\varepsilon,+)} \omega_{p_\infty,0}}, \tag{5.1}$$

the nonlinear dispersion relation is given by

$$v(\lambda) = \frac{x}{t}. \tag{5.2}$$

Recall that the Abelian differentials are given by (3.17) and (3.20).

For $\rho \in \mathbb{R}\backslash\sigma(H_q)$ we have by using (2.6)

$$v(\rho) = \frac{-12\int_{E_0}^{(\rho,+)} \omega_{p_\infty,2}}{\int_{E_0}^{(\rho,+)} \omega_{p_\infty,0}}, \tag{5.3}$$

i.e.,

$$v(\rho) = \frac{x}{t} \quad \Leftrightarrow \quad \phi(\rho, \frac{x}{t}) = 0, \tag{5.4}$$

where ϕ, defined in (4.47), is the phase of factorization problem (4.40). In other words, $v(\rho)$ is precisely the velocity of a soliton corresponding to the eigenvalue ρ.

For $\lambda \in \sigma(H_q)$ both numerator and denominator vanish on $\sigma(H_q)$. This follows by again using (2.6). Hence from (3.17) and (3.20) and by using de l'Hospital we get

$$v(\lambda) = -\frac{12\prod_{j=0}^{g}(\lambda - \tilde{\lambda}_j)}{\prod_{j=1}^{g}(\lambda - \lambda_j)}, \tag{5.5}$$

that is,

$$v(\lambda) = \frac{x}{t} \quad \Leftrightarrow \quad \phi'(\lambda, \frac{x}{t}) = 0. \tag{5.6}$$

Definition 5.1 (Stationary phase point). *A point $p = (z, \pm)$ is said to be a stationary phase point if*

$$\phi'(z, \frac{x}{t}) = 0, \tag{5.7}$$

where the prime denotes differentiation with respect to z.

Invoking (3.17) and (3.20), one obtains

$$\phi'(z, \frac{x}{t}) = 12\mathrm{i}\frac{\prod_{j=0}^{g}(z - \tilde{\lambda}_j)}{R_{2g+1}^{1/2}(z)} + \mathrm{i}\frac{x}{t}\frac{\prod_{j=1}^{g}(z - \lambda_j)}{R_{2g+1}^{1/2}(z)}, \tag{5.8}$$

and hence the stationary phase points are given by

$$12\prod_{j=0}^{g}(z - \tilde{\lambda}_j) + \frac{x}{t}\prod_{j=1}^{g}(z - \lambda_j) = 0. \tag{5.9}$$

Due to the normalization of our Abelian differentials, the numbers λ_j, $0 \le j \le g$, are real and different with precisely one lying in each spectral gap, say λ_j in the j'th gap. Similarly, $\tilde{\lambda}_j$, $0 \le j \le g$, are real and different and $\tilde{\lambda}_j$, $1 \le j \le g$, sits in the j'th gap. However $\tilde{\lambda}_0$ can be anywhere (see [Teschl00, Sect. 13.5]).

Thus if $\lambda \in \sigma(H_q)$ equation (5.6) tells us that $v(\lambda) = x/t$ if and only if λ is a stationary phase point.

The following lemma clarifies the dependence of the stationary phase points on x/t.

Lemma 5.1. *Denote by $z_j(v)$, $0 \le j \le g$, the stationary phase points, where $v = x/t$. Set $\lambda_0 = -\infty$ and $\lambda_{g+1} = \infty$, then*

$$\lambda_j < z_j(v) < \lambda_{j+1} \tag{5.10}$$

and there is always at least one stationary phase point in each spectral gap. Moreover, $z_j(v)$ is monotone decreasing with

$$\lim_{\eta \to -\infty} z_j(v) = \lambda_{j+1} \quad \text{and} \quad \lim_{\eta \to \infty} z_j(v) = \lambda_j. \tag{5.11}$$

Proof. Since the Abelian differential $\omega_2 + v\omega_0$ is normalized, there is at least one stationary phase point in each gap, and they are all different. Setting

$$\tilde{q}(z) = \prod_{j=0}^{g}(z - \tilde{\lambda}_j), \qquad q(z) = \prod_{j=1}^{g}(z - \lambda_j), \tag{5.12}$$

we calculate

$$\begin{aligned} 12\tilde{q}\big(z_j(v)\big) + \eta q\big(z_j(v)\big) &= 0 \\ 12\frac{\partial \tilde{q}}{\partial z_j}\frac{\partial z_j}{\partial \eta} + q(z_j) + v\frac{\partial q}{\partial z_j}\frac{\partial z_j}{\partial v} &= 0 \end{aligned}$$

and finally obtain

$$\frac{\partial z_j}{\partial v} = \frac{-q(z_j)}{12\frac{\partial \tilde{q}}{\partial z_j} + v\frac{\partial q}{\partial z_j}}. \tag{5.13}$$

Hence $z_j(v)$ is monotone decreasing and stays between λ_j and λ_{j+1}. Note that the denominator in (5.13) cannot vanish since the z_j, $j = 0, \dots, g$, are all different. □

In other words Lemma 5.1 tells us the following: As $v = x/t$ runs from $-\infty$ to ∞ we start with $z_g(v)$ coming from ∞ towards E_{2g}, while the other stationary phase points z_j, $j = 0, \dots, g-1$ stay in their spectral gaps until $z_g(v)$ has passed E_{2g} and therefore left the first spectral band $[E_{2g}, \infty)$. After this has happened, the next stationary phase point $z_{g-1}(v)$ can leave its gap (E_{2g-1}, E_{2g}) while $z_g(v)$ remains there, traverses the next spectral band $[E_{2g-2}, E_{2g-1}]$ and so on. Finally $z_0(v)$ traverses the last spectral band $[E_0, E_1]$ and moves to $-\infty$. So, depending on x/t there is at most one single stationary phase point belonging to the union of the bands $\sigma(H_q)$, say $z_j(x/t)$. On the Riemann surface, there are two such points z_j and its flipping image z_j^* which may (depending on x/t) lie in Σ.

From the picture described above we conclude that there can occur three cases.

1. One stationary phase point, say z_j, belongs to the interior of a band $[E_{2j}, E_{2j+1}]$, $j = 0, \dots, g$ (setting $E_{2g+1} = \infty$) and all other stationary phase points lie in open gaps.
2. No stationary phase point belongs to $\sigma(H_q(t)) = \bigcup_{j=1}^{g}[E_{2j-2}, E_{2j-1}] \cup [E_{2g}, \infty)$, $t \in \mathbb{R}$.
3. $z_j = z_j^* = E_j$ for some j and all other stationary phase points lie in open gaps.

We will only consider case 1 and case 2. In the first case we have

$$\phi''(z_j)/\mathrm{i} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{R_{2g+1}^{1/2}(z_j)} > 0. \tag{5.14}$$

According to these two cases we will define corresponding regions in the (x,t)-plane in the following way:

Definition 5.2 (Oscillatory and soliton region)**.** *In the (x,t)-plane we can consider the first two cases described above as two different regions:*

Case 1 defines the oscillatory region, that is,

$$D = \bigcup_{j=0}^{g} D_j = \bigcup_{j=0}^{g} \{(x,t) | z_j(x/t) \in (E_{2j}, E_{2j+1})\}.$$

Case 2 defines the soliton region, that is,

$$\{(x,t) | \zeta(x/t) \in \mathbb{R} \setminus \sigma(H_q(t))\}.$$

We now can establish that $v(\lambda)$ is monotone.

Lemma 5.2. *The function $v(\rho)$ defined in* (5.1) *is continuous and strictly monotone decreasing. Moreover, it is a bijection from $\mathbb{R}$ to $\mathbb{R}$.*

Proof. That $v(\lambda)$ defined in (5.1) is continuous is obvious except at the band edges $\lambda = E_j$. However, in this case (5.1) becomes (5.5) by using de l'Hospital. The function $v(\lambda)$ defined in (5.5) is obviously continuous at the band edges E_j since λ_j lies in the j'th gap and thus does not hit the band edges.

Furthermore, for large ρ we have

$$\lim_{|\rho|\to\infty} \frac{v(\rho)}{-12\rho/\log(|\rho|)} = 1, \tag{5.15}$$

which shows $\lim_{\rho\to\pm\infty} v(\rho) = \mp\infty$.

In the regions where there is one stationary phase point $z_j(v) \in \sigma(H_q)$ we know that $z_j(v)$ is the inverse of $v(\lambda)$ and monotonicity follows from Lemma 5.1. In the other regions we have $v(\zeta(z)) = z$. By the implicit function theorem the derivative ζ', where the prime denotes differentiation with respect to z, exists. Using definition (5.1) and differentiating with respect to z yields

$$\zeta' = -R_{2g+1}^{1/2}(\zeta)\frac{\mathrm{Im}\int_{E_0}^{(\zeta,+)}\omega_{p_\infty,0}}{12\prod_{j=0}^{g}(\zeta-\tilde{\lambda}_j) + v\prod_{j=1}^{g}(\zeta-\lambda_j)} = -R_{2g+1}^{1/2}(\zeta)\frac{\mathrm{Im}\int_{E_0}^{(\zeta,+)}\omega_{p_\infty,0}}{\prod_{j=0}^{g}(\zeta-z_j(v))}. \tag{5.16}$$

This shows strict monotonicity since $\mathrm{Im}\int_{E_0}^{(\zeta,+)}\omega_{p_\infty,0} > 0$ for $\zeta \in \mathbb{R}\backslash\sigma(H_q)$ and $z_j(v) \le \zeta(v) \le z_{j-1}(v)$ for $\zeta(v) \in (E_{2j-1}, E_{2j})$ (if we set $z_{-1} = \infty$, $z_{g+1} = -\infty$, $E_{-1} = -\infty$, $E_{2g+2} = \infty$). To see the last claim we can argue as follows: If $\zeta(v)$ were below $z_j(v)$ at some point it would decrease as v decreases whereas $z_j(v)$ increases as v decreases. This contradicts the fact that both must hit at E_{2j-1}. Similarly we see that $\zeta(v)$ stays below $z_j(v)$. □

In summary, we can define a function $\zeta(x/t)$ via

$$v(\zeta) = \frac{x}{t}. \tag{5.17}$$

In particular, different solitons travel at different speeds and don't collide with each other or the parts corresponding to the continuous spectrum.

Moreover, there is some ζ_0 for which $v(\zeta_0) = 0$ and hence there can be *stationary solitons* provided $\zeta_0 \notin \sigma(H_q)$.

Lemma 5.3 (Stationary solitons)**.** *There exists a unique ζ_0 such that $v(\zeta_0) = 0$. Moreover, if $\zeta_0 \in \sigma(H_q)$ or $\tilde{\lambda}_0 \in \sigma(H_q)$, then $\zeta_0 = \tilde{\lambda}_0$. In particular, $\zeta_0 \in \sigma(H_q)$ if and only if $\tilde{\lambda}_0 \in \sigma(H_q)$.*

Proof. Existence and uniqueness of ζ_0 follows since v is a bijection. It is left to show that $\zeta_0 = \tilde\lambda_0$ if $\zeta \in \sigma(H_q)$ or $\tilde\lambda_0 \in \sigma(H_q)$. Assume $\zeta_0 \in \sigma(H_q)$. Then using $v(\zeta_0) = 0$ and (5.5) we get

$$\prod_{j=0}^{g}(\zeta_0 - \tilde\lambda_j) = (\zeta_0 - \tilde\lambda_0)\prod_{j=1}^{g}(\zeta_0 - \tilde\lambda_j) = 0.$$

Since $\tilde\lambda_j \in (E_{2j-1}, E_{2j})$, $j = 1, \dots, g$ it follows $\zeta_0 = \tilde\lambda_0$. Now suppose $\tilde\lambda_0 \in \sigma(H_q)$ and again use (5.5) to get

$$v(\tilde\lambda_0)\prod_{j=1}^{g}(\tilde\lambda_0 - \lambda_j) = 0.$$

Since $\lambda_j \in (E_{2j-1}, E_{2j})$, $j = 1, \dots, g$ we obtain $v(\tilde\lambda_0) = 0$ and thus $\zeta_0 = \tilde\lambda_0$. □

5.2. The partial transmission coefficient

Apart from the transmission coefficient $T(p)$ corresponding to the jump on Σ we will also need the so-called partial transmission coefficient $T(p,x,t)$ which corresponds to the jump along $C(x/t) = \Sigma \cap \pi^{-1}\big((-\infty, \zeta(x/t))\big)$. The usefulness of $T(p,x,t)$ will become clear later on in Section 5.3. Before defining the partial transmission coefficient it is necessary to introduce the Blaschke factor.

Definition 5.3 (Blaschke factor)**.** *The factor*

$$B(p,\rho) = \exp\Big(g(p,\rho)\Big) = \exp\Big(\int_{E_0}^{p}\omega_{\rho\,\rho^*}\Big) = \exp\Big(\int_{E(\rho)}^{\rho}\omega_{p\,p^*}\Big), \quad \pi(\rho) \in \mathbb{R}, \tag{5.18}$$

where $E(\rho)$ is E_0 if $\rho < E_0$ and either E_{2j-1} or E_{2j} if $\rho \in (E_{2j-1}, E_{2j})$, $1 \le j \le g$ is called Blaschke factor. Here $\omega_{p\,p^*}$ denotes an Abelian differential of the third kind with simple poles at p and p^*. Note that the Blaschke factor $B(p,\rho)$ is a multivalued function with a simple zero at ρ and simple pole at ρ^* satisfying $|B(p,\rho)| = 1$, $p \in \partial\Pi_+$. It is real-valued for $\pi(p) \in (-\infty, E_0)$ and satisfies

$$B(E_0,\rho) = 1 \quad \text{and} \quad B(p^*,\rho) = B(p,\rho^*) = B(p,\rho)^{-1} \tag{5.19}$$

(see e.g., [Teschl07]).

Later on it will be useful to work with explicit formulas of such differentials appearing in (5.18). Hence we give the following

Remark 5.4. *Abelian differentials of the third kind $\omega_{q_1\,q_2}$, with simple poles at q_1 and q_2, corresponding residues $+1$ and -1, vanishing a-periods, and holomorphic on $\mathcal{K}_g \setminus \{q_1, q_2\}$, are explicitly given by ([Gesztesy03, Appendix B])*

$$\omega_{p_1\,p_2} = \Big(\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_1)}{2\big(\pi - \pi(p_1)\big)} - \frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_2)}{2\big(\pi - \pi(p_2)\big)} + P_{p_1\,p_2}(z)\Big)\frac{d\pi}{R_{2g+1}^{1/2}}, \tag{5.20}$$

$$\omega_{p_1\,p_\infty} = \Big(-\frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(p_1)}{2\big(\pi - \pi(p_1)\big)} + P_{p_1\,p_\infty}(z)\Big)\frac{d\pi}{R_{2g+1}^{1/2}}, \tag{5.21}$$

where $p_1, p_2 \in \mathcal{K}_g \setminus \{p_\infty\}$ *and* $P_{p_1 p_2}(z)$ *and* $P_{p_1 p_\infty}(z)$ *are polynomials of degree* $g-1$ *which have to be determined from the normalization* $\int_{a_\ell} \omega_{p_1 p_2} = 0$ *and* $\int_{a_\ell} \omega_{p_1 p_\infty} = 0$, *respectively. In particular,*

$$\omega_{pp^*} = \Big(\frac{R_{2g+1}^{1/2}(p)}{\pi - \pi(p)} + P_{pp^*}(\pi) \Big) \frac{d\pi}{R_{2g+1}^{1/2}}. \tag{5.22}$$

The following lemma defines a divisor of new poles ν_j, $j = 1, \dots, g$. As one will see in the proof of Lemma 5.6 these poles are introduced in such a way that the single-valuedness of the partial transmission coefficient $T(p,x,t)$ is guaranteed.

Lemma 5.5. *[Kamvissis07b] Define a divisor* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *of degree* g *via*

$$\underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}) = \underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) + \underline{\delta}(x/t), \tag{5.23}$$

where

$$\delta_\ell(x/t) = 2 \sum_{\rho_k < \zeta(x/t)} \underline{A}_{E_0}(\hat{\rho}_k) + \frac{1}{2\pi \mathrm{i}} \int_{C(x/t)} \log(1 - |R|^2) \zeta_\ell, \quad \ell = 1, \dots, g, \tag{5.24}$$

where $C(x/t) = \Sigma \cap \pi^{-1}((-\infty, \zeta(x/t))$ *and* $\zeta(x/t)$ *is defined in* (5.17). *Then* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *is nonspecial and* $\pi(\hat{\nu}_j(x,t)) = \nu_j(x,t) \in \mathbb{R}$ *with precisely one in each spectral gap.*

Proof. One checks that δ_ℓ is real. Hence it follows from [Teschl00, Lem. 9.1] that the ν_j are real and that there is one in each gap. In particular, the divisor $\mathcal{D}_{\underline{\hat{\nu}}}$ is nonspecial by [Teschl00, Lem. A.20]. □

Definition 5.4 (Partial transmission coefficient)**.** *We define the* partial transmission coefficient *as*

$$T(p,x,t) = \frac{\theta\big(\underline{z}(p_\infty,x,t) + \underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)} \frac{\theta\big(\underline{z}(p,x,t)\big)}{\theta\big(\underline{z}(p,x,t) + \underline{\delta}(x/t)\big)} \cdot \\ \cdot \Big(\prod_{\rho_k < \zeta(x/t)} \exp\Big(- \int_{E_0}^p \omega_{\rho_k \, \rho_k^*} \Big) \Big) \exp\Big(\frac{1}{2\pi \mathrm{i}} \int_{C(x/t)} \log(1 - |R|^2) \omega_{p \, p_\infty} \Big), \tag{5.25}$$

where $\underline{\delta}(x,t)$ *is defined in (5.24) and* $\omega_{p_1 p_2}$ *is the Abelian differential of the third kind with poles at* p_1 *and* p_2 *(cf. Remark 5.4).*

The function $T(p,x,t)$ *is meromorphic in* $\mathcal{K}_g \backslash C(x/t)$ *with first order poles at* $\rho_k < \zeta(x/t)$, $\hat{\nu}_j(x,t)$ *and first order zeros at* $\hat{\mu}_j(x,t)$.

Lemma 5.6. $T(p,x,t)$ *satisfies the following scalar meromorphic Riemann–Hilbert problem:*

$$\begin{aligned} & T_+(p,x,t) = T_-(p,x,t)(1 - |R(p)|^2), \quad p \in C(x/t), \\ & (T(p,x,t)) = \sum_{\rho_k < \zeta(x/t)} \mathcal{D}_{\rho_k^*} - \sum_{\rho_k < \zeta(x/t)} \mathcal{D}_{\rho_k} + \mathcal{D}_{\underline{\hat{\mu}}(x,t)} - \mathcal{D}_{\underline{\hat{\nu}}(x,t)}, \\ & T(p_\infty, x, t) = 1. \end{aligned} \tag{5.26}$$

Moreover,

1.
$$T(p^*,x,t)T(p,x,t) = \prod_{j=1}^{g} \frac{z-\mu_j}{z-\nu_j}, \qquad z=\pi(p),$$

2. $\overline{T(p,x,t)} = T(\overline{p},x,t)$ *and in particular* $T(p,x,t)$ *is real-valued for* $p \in \Sigma$.

Proof. For the purpose of making the proof easier to follow we will split it into four steps.

Step 1: Solving the Riemann–Hilbert problem

On the Riemann sphere a scalar Riemann-Hilbert problem is solved by the Plemelj–Sokhotsky formula. This means that the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\varphi(\lambda)}{z-\lambda} d\lambda, \tag{5.27}$$

where L denotes a simple smooth oriented arc and $\varphi(z)$ is a function, which is Hölder continuous on L, solves the Riemann–Hilbert problem

$$\Phi_+(z) - \Phi_-(z) = \varphi(z), \quad z \in L. \tag{5.28}$$

Here $\Phi_+(z)$ and $\Phi_-(z)$ denote the limiting values of $\Phi(z)$ as z approaches L from the left and the right, respectively. For more details on scalar Riemann–Hilbert problems we refer to [Ablowitz03, Chapter 7] and [Muskhelishvili53].

On the Riemann surface $\mathcal{K}_g$ we need to replace the Cauchy kernel $\frac{d\lambda}{\lambda - z}$ by the Abelian differential of the third kind $\omega_{p\, p_\infty}$. Now note that solving the Riemann–Hilbert problem

$$T_+(p) = T_-(p)(1-|R(p)|^2), \quad p \in C(x/t), \tag{5.29}$$

for T is equivalent to finding the solution $\log(T)$ of

$$\log T_+(p) - \log T_-(p) = \log\big(1-|R(p)|^2\big), \quad p \in C(x/t). \tag{5.30}$$

Therefore by the analogue of the Plemelj–Sokhotsky formula for the Riemann surface $\mathcal{K}_g$ we get

$$T(p) = c \cdot \exp\Big(\frac{1}{2\pi \mathrm{i}} \int_{C(x/t)} \log(1-|R|^2)\omega_{p\, p_\infty}\Big), \tag{5.31}$$

where the normalization constant c has to be chosen such that the divisor and normalization condition are fulfilled (cf. Step 3).

Step 2: Single-valuedness

Recall that the Riemann theta function satisfies the quasi-periodic property (3.13), i.e.,

$$\theta(\underline{z} + \underline{m} + \underline{\tau}\, \underline{n}) = \exp\Big\{2\pi \mathrm{i}\Big(-\langle \underline{n}, \underline{z}\rangle - \frac{\langle \underline{n}, \underline{\tau}\, \underline{n}\rangle}{2}\Big)\Big\}\theta(\underline{z}), \quad \underline{n}, \underline{m} \in \mathbb{Z}^g, \tag{5.32}$$

where $\underline{\tau}$ is the matrix of b-periods defined in (2.37) and $\langle .,. \rangle$ denotes the scalar product in $\mathbb{R}^g$ (cf., e.g. [Farkas92] or [Teschl00, App. A]). By definition both the theta functions (as functions on $\mathcal{K}_g$) and the exponential term are only defined on the fundamental polygon $\hat{\mathcal{K}}_g$ of $\mathcal{K}_g$ and do not extend to single-valued functions on $\mathcal{K}_g$ in general. Hence we have to verify that (5.25) gives rise to a single-valued function on $\mathcal{K}_g$.

Let us start by looking at the values from the left/right on the cycle b_ℓ. Since our path of integration in $\underline{z}(p,x,t)$ is forced to stay in $\hat{\mathcal{K}}_g$, the difference between the limits from the right and left is the value of the integral along a_ℓ. So by (3.13) we have

$$\frac{\theta\big(\underline{z}_r(p,x,t)\big)}{\theta\big(\underline{z}_r(p,x,t)+\underline{\delta}(x/t)\big)} \longrightarrow \frac{\theta\big(\underline{z}_l(p,x,t)+\int_{a_\ell}\underline{\zeta}\big)}{\theta\big(\underline{z}_l(p,x,t)+\underline{\delta}(x/t)+\int_{a_\ell}\underline{\zeta}\big)} = \\ = \frac{\theta\big(\underline{z}_l(p,x,t)\big)}{\theta\big(\underline{z}_l(p,x,t)+\underline{\delta}(x/t)\big)},$$

where $\underline{z}_r(p,x,t)$ and $\underline{z}_l(p,x,t)$ denote the value of $\underline{z}(p,x,t)$ if p approaches b_ℓ from the right and left, respectively. Similarly, since $\omega_{p\,p_\infty}$ is normalized along a_ℓ cycles, the limits from the left/right of $\omega_{p\,p_\infty}$ coincide. So the limits of the exponential terms from different sides of b_ℓ match as well.

Next, let us compare the values from the left/right on the cycle a_ℓ. Since our path of integration in $\underline{z}(p,x,t)$ is forced to stay in $\hat{\mathcal{K}}_g$, the difference between the limits from the right and left is the value of the integral along b_ℓ. Thus by (3.13) we have

$$\frac{\theta\big(\underline{z}_r(p,x,t)\big)}{\theta\big(\underline{z}_r(p,x,t)+\underline{\delta}(x/t)\big)} \longrightarrow \frac{\theta\big(\underline{z}_l(p,x,t)+\int_{b_\ell}\underline{\zeta}\big)}{\theta\big(\underline{z}_l(p,x,t)+\underline{\delta}(x/t)+\int_{b_\ell}\underline{\zeta}\big)} = \\ = \frac{\theta\big(\underline{z}_l(p,x,t)\big)}{\theta\big(\underline{z}_l(p,x,t)+\underline{\delta}(x/t)\big)} \cdot \exp(-2\pi\mathrm{i}\delta_\ell).$$

On the other hand, since $\omega_{p\,p_\infty}$ is normalized along a_ℓ cycles, we deduce

$$\exp\Big(\frac{1}{2\pi\mathrm{i}}\int_{C(x/t)} \log(1-|R|^2)\omega_{p_r\,p_\infty}\Big) \longrightarrow \tag{5.33}$$

$$\longrightarrow \exp\Big(\frac{1}{2\pi\mathrm{i}}\int_{C(x/t)} \log(1-|R|^2)\big(\omega_{p_l\,p_\infty}+2\pi\mathrm{i}\zeta_\ell\big)\Big) \tag{5.34}$$

and using (A.21) in [Teschl00] we have

$$\begin{aligned}
\prod_{\rho_k<\zeta(x/t)} \exp\Big(-\int_{E_0}^{p_r} \omega_{\rho_k\,\rho_k^*}\Big) &= \exp\Big(-\sum_{\rho_k<\zeta(x/t)} \int_{E_0}^{p_r} \omega_{\rho_k\,\rho_k^*}\Big) = \\
&= \exp\Big(-\sum_{\rho_k<\zeta(x/t)} \int_{E(\rho_k)}^{\rho_k} \omega_{p_r\,p_r^*}\Big) \longrightarrow \\
&\longrightarrow \exp\Big(-\sum_{\rho_k<\zeta(x/t)} \big(\int_{E(\rho_k)}^{\rho_k} \omega_{p_l\,p_l^*} + 4\pi\mathrm{i}\int_{E(\rho_k)}^{\rho_k} \zeta_\ell\big)\Big) = \qquad (5.35)\\
&= \exp\Big(-\sum_{\rho_k<\zeta(x/t)} \big(\int_{E(\rho_k)}^{\rho_k} \omega_{p_l\,p_l^*} + 4\pi\mathrm{i}\underline{A}_{E_0}(\hat\rho_k)\big)\Big). \qquad (5.36)
\end{aligned}$$

Thus by our definition of $\underline{\delta}$ in (5.24) the jumps of the ratio of theta functions and the exponential terms compensate each other, which shows that (5.25) is single-valued.

Step 3: Poles and zeros

T is a (locally) holomorphic solution of our Riemann–Hilbert problem, which is 1 at p_∞ by our choice of the second pole of the Cauchy kernel $\omega_{p\,p_\infty}$. The ratio of theta functions is meromorphic with simple zeros at $\hat\mu_j$ and simple poles at $\hat\nu_j$ because of Riemann's vanishing theorem and the choice of the divisor $\mathcal{D}_{\underline{\hat\nu}(x,t)}$ defined by (5.23). From the product of the Blaschke factors we get that T has simple poles at ρ_k and simple zeros at ρ_k^* for which $\rho_k < \zeta(x/t)$ is valid. Moreover, the normalization is chosen again such that the ratio of theta functions is one at p_∞.

Step 4: Uniqueness

To prove uniqueness let $\tilde T$ be a second solution and consider $\tilde T/T$. Then $\tilde T/T$ has no jump and the Schwarz reflection principle implies that it extends to a meromorphic function on $\mathcal{K}_g$. Since the poles of T cancel the poles of $\tilde T$, its divisor satisfies $(\tilde T/T) \ge -\mathcal{D}_{\underline{\hat\mu}(x,t)}$. Since $\mathcal{D}_{\underline{\hat\mu}(x,t)}$ is nonspecial, $\tilde T/T$ has to be a constant by the Riemann–Roch theorem. Setting $p = p_\infty$, we see that this constant is one, that is, $\tilde T = T$ as claimed.

Finally, $\overline{T(p,x,t)} = T(\overline{p},x,t)$ follows from uniqueness since both functions solve (5.26).

□

Remark 5.7. *Alternatively to (5.25) there exists also an other representation of the partial transmission coefficient. First observe that*

$$\int_{C(x/t)} f\omega_{p\,p_\infty} = \int_{C(x/t)} f\big(\omega_{p\,p^*} + \omega_{p^*\,p_\infty}\big),$$

and hence by using $\int_{C(x/t)} f\omega_{p^* p_\infty} = -\int_{C(x/t)} f\omega_{p p_\infty}$ *we have*

$$\int_{C(x/t)} f\omega_{p p_\infty} = \frac{1}{2}\int_{C(x/t)} f\omega_{p p^*}, \tag{5.37}$$

where f *can be any symmetric function* $f(q) = f(q^*)$*. Thus, by using symmetry,* $|R(p^*)| = |R(p)|$ *for* $p \in C(x/t)$*, the partial transmission coefficient defined in (5.25) can also be expressed in the following form*

$$\begin{aligned} T(p,x,t) =& \frac{\theta\big(\underline{z}(p_\infty,x,t)+\underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)} \frac{\theta\big(\underline{z}(p,x,t)\big)}{\theta\big(\underline{z}(p,x,t)+\underline{\delta}(x/t)\big)} \cdot \\ &\cdot \Big(\prod_{\rho_k<\zeta(x/t)} \exp\Big(-\int_{E(\rho_k)}^{\rho_k}\omega_{p p^*}\Big)\Big) \exp\Big(\frac{1}{4\pi\mathrm{i}}\int_{C(x/t)} \ln(1-|R|^2)\omega_{p p^*}\Big). \end{aligned} \tag{5.38}$$

Later on we will also need the expansion of $T(p,x,t)$ around p_∞. For that purpose note the following

Theorem 5.8. *The asymptotic expansion of the partial transmission coefficient for* p *near* p_∞ *is given by*

$$T(p,x,t) = 1 \pm \frac{T_1(x,t)}{\sqrt{z}} + O(\frac{1}{z}), \qquad p = (z,\pm), \tag{5.39}$$

where

$$\begin{aligned} T_1(x,t) = &- \sum_{\rho_k<\zeta(x/t)} 2\int_{E(\rho_k)}^{\rho_k}\omega_{p_\infty,0} + \frac{1}{2\pi\mathrm{i}}\int_{C(x/t)} \log(1-|R|^2)\omega_{p_\infty,0} \\ &- \mathrm{i}\partial_x \ln\left(\frac{\theta\big(\underline{z}(p_\infty,x,t)+\underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)}\right), \end{aligned}$$

where $\omega_{p_\infty,0}$ *is the Abelian differential of the second kind defined in (3.17).*

Proof. From $\frac{d^k}{dz^k}\omega_{p\,E_0} = k!\omega_{p_\infty,k-1}$ we get that

$$\omega_{p\,E_0} = \omega_{p_\infty\,E_0} + \sum_{k=1}^{\infty}\zeta^k\omega_{p_\infty,k-1}, \qquad \zeta = z^{-1/2},$$

$$\omega_{p^*\,E_0} = \omega_{p_\infty\,E_0} + \sum_{k=1}^{\infty}\zeta^k\omega_{p_\infty,k-1}, \qquad \zeta = -z^{-1/2}.$$

Using this it follows

$$\omega_{p p^*} = \omega_{p\,E_0} - \omega_{p^*\,E_0} = 2\sum_{k=1}^{\infty}\omega_{p_\infty,2k-2}\zeta^{2k-1}, \quad \zeta = z^{-1/2}. \tag{5.40}$$

Thus, referring to the expression (5.38) of the partial transmission coefficient one can check by expanding the ratio of theta functions, like it is done in the proof of Theorem 3.6, and using (5.40) that the expansion (5.39) is indeed valid. □

Remark 5.9. *Once the last stationary phase point has left the spectrum, that is, once $C(x/t) = \Sigma$, we have $T(p, x, t) = T(z)^{\pm 1}$, $p = (z, \pm)$ (compare [Teschl07]).*

5.3. Conjugation

This section deals with the conjugation of the Riemann–Hilbert problem. We will see that conjugation provides a useful method for passing from a Riemann–Hilbert problem involving solitons to one without.

Since to each discrete eigenvalue there corresponds a soliton, it follows that solitons are represented in Riemann–Hilbert problems by pole conditions like (4.60). For this reason we will study how poles can be dealt with in this section. For this purpose we will follow closely the presentation of [Krüger09a, Section 4].

In order to remove the poles there are two cases to distinguish. If $\rho_j > \zeta(x/t)$, the jump at ρ_j is exponentially close to the identity and there is nothing to do.

Otherwise, if $\rho_j < \zeta(x/t)$, we need to use conjugation to turn the jumps at these poles into exponentially decaying ones, again following [Deift96]. It turns out that we will have to handle the poles at ρ_j and ρ_j^* in one step in order to preserve symmetry and in order to not add additional poles elsewhere.

Moreover, the conjugation of the Riemann–Hilbert problem also serves another purpose, namely that the jump matrix can be separated into two matrices, one containing an off-diagonal term with $\exp(-t\phi)$ and the other with $\exp(t\phi)$. Without conjugation this is not possible for the jump on $C(x/t) = \Sigma \cap \pi^{-1}\big((-\infty, \zeta(x/t))\big)$, since in this case there also appears a diagonal matrix if one wants to separate the jump matrix. As we will see, the partial transmission coefficient $T(p, x, t)$, which was introduced exactly for that purpose in Section 5.2, will play a major role in the conjugation step.

For easy reference we note the following result.

Lemma 5.10 (Conjugation)**.** *Assume that $\widetilde{\Sigma} \subseteq \Sigma$. Let D be a matrix of the form*

$$D(p) = \begin{pmatrix} d(p^*) & 0 \\ 0 & d(p) \end{pmatrix}, \tag{5.41}$$

where $d : \mathbb{M}\backslash\widetilde{\Sigma} \to \mathbb{C}$ is a sectionally analytic function. Set

$$\tilde{m}(p) = m(p)D(p), \tag{5.42}$$

then the jump matrix transforms according to

$$\tilde{v}(p) = D_-(p)^{-1} v(p) D_+(p). \tag{5.43}$$

$\tilde{m}(p)$ satisfies the symmetry condition (4.44) *if and only if $m(p)$ does. Furthermore, $\tilde{m}(p)$ satisfies the normalization condition* (4.45) *if $m(p)$ satisfies* (4.45) *and $d(p_\infty) = 1$.*

Proof. For $p \in \Sigma$ one computes

$$\tilde{m}_+(p) = m_+(p)D_+(p) = m_-(p)v(p)D_+(p) = \tilde{m}_-(p)D_-(p)^{-1}v(p)D_+(p) = \tilde{m}_+(p)\tilde{v}(p),$$

and hence $\tilde{v}(p) = D_-(p)^{-1}v(p)D_+(p)$. The symmetry condition for $\tilde{m}$ follows from

$$\tilde{m}(p^*) = m(p^*)D(p^*) = m(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d(p) & 0 \\ 0 & d(p^*) \end{pmatrix} = \tilde{m}(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The normalization can be investigated by a straightforward calculation. □

Lemma 5.11. *Introduce*

$$\tilde{B}(p,\rho) = C_\rho(x,t)\frac{\theta(\underline{z}(p,x,t))}{\theta(\underline{z}(p,x,t) + 2\underline{A}_{E_0}(\rho))}B(p,\rho). \tag{5.44}$$

Then $\tilde{B}(.,\rho)$ is a well defined meromorphic function, with divisor

$$(\tilde{B}(.,\rho)) = -\mathcal{D}_{\underline{\hat{\nu}}} + \mathcal{D}_{\underline{\hat{\mu}}} - \mathcal{D}_{\rho^*} + \mathcal{D}_\rho, \tag{5.45}$$

where ν is defined via

$$\underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\underline{A}_{E_0}(\rho). \tag{5.46}$$

Furthermore,

$$\tilde{B}(p_\infty,\rho) = 1, \tag{5.47}$$

if

$$C_\rho(x,t) = \frac{\theta(\underline{z}(p_\infty,x,t) + 2\underline{A}_{E_0}(\rho))}{\theta(\underline{z}(p_\infty,x,t))}. \tag{5.48}$$

Proof. We start by checking single-valuedness. This is done like in the proof of Lemma 5.6. The a-periods follow from normalization. For the b periods, we compute for $E \in (E_{2l-1}, E_{2l})$ using (3.13)

$$\begin{aligned}\frac{\theta\big(\underline{z}_r(p,x,t)\big)}{\theta\big(\underline{z}_r(p,x,t) + 2\underline{A}_{E_0}(\rho)\big)} &\longrightarrow \frac{\theta\big(\underline{z}_l(p,x,t) + \int_{b_\ell}\underline{\zeta}\big)}{\theta\big(\underline{z}_l(p,x,t) + 2\underline{A}_{E_0}(\rho) + \int_{b_\ell}\underline{\zeta}\big)} = \\ &= \frac{\theta\big(\underline{z}_l(p,x,t)\big)}{\theta\big(\underline{z}_l(p,x,t) + 2\underline{A}_{E_0}(\rho)\big)} \cdot \exp\big(-2\pi\mathrm{i}(2A_{E_0,l}(\rho))\big),\end{aligned}$$

and by using (A.21) in [Teschl00] we have

$$\begin{aligned}B(p_r,\rho) = \exp\Big(\int_{E(\rho)}^{\rho}\omega_{p_r\,p_r^*}\Big) &\longrightarrow \exp\Big(\int_{E(\rho_k)}^{\rho_k}\omega_{p_l\,p_l^*} + 2\pi\mathrm{i}\big(2\int_{E(\rho_k)}^{\rho_k}\zeta_\ell\big)\Big) = \\ &= \exp\Big(\int_{E(\rho_k)}^{\rho_k}\omega_{p_l\,p_l^*}\Big)\exp\big(2\pi\mathrm{i}(2A_{E_0,l}(\rho))\big)\end{aligned}$$

and the claim follows.

The normalization condition (5.47) follows by a computation using (5.19). □

Now let us start with an easy case, namely to show how to conjugate the jump corresponding to one eigenvalue ρ.

Lemma 5.12. *Assume that the Riemann–Hilbert problem for m has jump conditions near ρ and ρ^* given by*

$$\begin{aligned} m_+(p) &= m_-(p) \begin{pmatrix} 1 & 0 \\ \frac{\gamma(p)}{\pi(p)-\rho} & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho), \\ m_+(p) &= m_-(p) \begin{pmatrix} 1 & -\frac{\gamma(p^*)}{\pi(p)-\rho} \\ 0 & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho^*), \end{aligned} \tag{5.49}$$

and satisfies a divisor condition

$$(m_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}^*}, \qquad (m_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}}. \tag{5.50}$$

Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near ρ and ρ^* given by

$$\begin{aligned} \tilde{m}_+(p) &= \tilde{m}_-(p) \begin{pmatrix} 1 & \frac{\tilde{B}(p,\rho^*)(\pi(p)-\rho)}{\gamma(p)\tilde{B}(p^*,\rho^*)} \\ 0 & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho), \\ \tilde{m}_+(p) &= \tilde{m}_-(p) \begin{pmatrix} 1 & 0 \\ -\frac{\tilde{B}(p^*,\rho^*)(\pi(p)-\rho)}{\gamma(p^*)\tilde{B}(p,\rho^*)} & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho^*), \end{aligned} \tag{5.51}$$

divisor condition

$$(\tilde{m}_1) \geq -\mathcal{D}_{\underline{\hat{\nu}}^*}, \qquad (\tilde{m}_2) \geq -\mathcal{D}_{\underline{\hat{\nu}}}, \tag{5.52}$$

where $\mathcal{D}_{\underline{\hat{\nu}}}$ is defined via

$$\underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{E_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\underline{A}_{E_0}(\rho), \tag{5.53}$$

and all remaining data conjugated (as in Lemma 5.10) by

$$D(p) = \begin{pmatrix} \tilde{B}(p^*,\rho^*) & 0 \\ 0 & \tilde{B}(p,\rho^*) \end{pmatrix}. \tag{5.54}$$

Proof. Denote by U the interior of $\Sigma_\varepsilon(\rho)$. To turn γ into γ^{-1}, introduce D by

$$D(p) = \begin{cases} \begin{pmatrix} 1 & \frac{\pi(p)-\rho}{\gamma(p)} \\ -\frac{\gamma(p)}{\pi(p)-\rho} & 0 \end{pmatrix} \begin{pmatrix} \tilde{B}(p^*,\rho^*) & 0 \\ 0 & \tilde{B}(p,\rho^*) \end{pmatrix}, & p \in U, \\ \begin{pmatrix} 0 & -\frac{\gamma(p^*)}{\pi(p)-\rho} \\ \frac{\pi(p)-\rho}{\gamma(p^*)} & 1 \end{pmatrix} \begin{pmatrix} \tilde{B}(p^*,\rho^*) & 0 \\ 0 & \tilde{B}(p,\rho^*) \end{pmatrix}, & p^* \in U, \\ \begin{pmatrix} \tilde{B}(p^*,\rho^*) & 0 \\ 0 & \tilde{B}(p,\rho^*) \end{pmatrix}, & \text{else}, \end{cases}$$

and note that $D(p)$ is meromorphic away from the two circles. Now set $\tilde{m}(p) = m(p)D(p)$. The claim about the divisors follows from noting where the poles of $\tilde{B}(p,\rho)$ are. □

Observe that our original jump matrix (4.41) has the following important factorization

$$J(p) = b_-(p)^{-1} b_+(p), \tag{5.55}$$

where

$$b_-(p) = \begin{pmatrix} 1 & R(p^*)\Theta(p^*)\mathrm{e}^{-t\phi(p)} \\ 0 & 1 \end{pmatrix}, \qquad b_+(p) = \begin{pmatrix} 1 & 0 \\ R(p)\Theta(p)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}.$$

Later in Section 5.4 it will become clear that this is the right factorization for $p \in \Sigma\backslash C(x/t)$, i.e., $\pi(p) > \zeta(x/t)$. Similarly, we have

$$J(p) = B_-(p)^{-1} \begin{pmatrix} 1 - |R(p)|^2 & 0 \\ 0 & \frac{1}{1-|R(p)|^2} \end{pmatrix} B_+(p), \tag{5.56}$$

where

$$B_-(p) = \begin{pmatrix} 1 & 0 \\ -\frac{R(p)\Theta(p)\mathrm{e}^{t\phi(p)}}{1-|R(p)|^2} & 1 \end{pmatrix}, \qquad B_+(p) = \begin{pmatrix} 1 & -\frac{R(p^*)\Theta(p^*)\mathrm{e}^{-t\phi(p)}}{1-|R(p)|^2} \\ 0 & 1 \end{pmatrix}.$$

This constitutes the right factorization for $p \in C(x/t)$, i.e., $\pi(p) < \zeta(x/t)$. Here we have used $\overline{R(p)} = R(p^*)$, for $p \in \Sigma$. To get rid of the diagonal part in the factorization corresponding to $\pi(p) < \zeta(x/t)$ and to conjugate the jumps near the eigenvalues we need the partial transmission coefficient defined in (5.25). Note that Lemma 5.12 can be applied iteratively to conjugate the eigenvalues $\rho_j < \zeta(x/t)$: start with the poles $\mu = \mu^0$ and apply the lemma setting $\rho = \rho_1$. This results in new poles $\mu^1 = \nu$. Then repeat this with $\mu = \mu^1$, $\rho = \rho_2$, and so on.

All in all we will now make the following conjugation step: abbreviate

$$\gamma_k(p,x,t) = \frac{2\mathrm{i} R_{2g+1}^{1/2}(\rho_k)}{\prod_{l=1}^g (\rho_k - \mu_l)} \frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} \gamma_k$$

and introduce

$$D(p) = \begin{cases} \begin{pmatrix} 1 & \frac{\pi(p)-\rho_k}{\gamma_k(p,x,t)} \\ -\frac{\gamma_k(p,x,t)}{\pi(p)-\rho_k} & 0 \end{pmatrix} D_0(p), & {\scriptstyle |\pi(p)-\rho_k|<\varepsilon \atop p\in\Pi_+},\ \rho_k < \zeta(x/t), \\ \begin{pmatrix} 0 & -\frac{\gamma_k(p^*,x,t)}{\pi(p)-\rho_k} \\ \frac{\pi(p)-\rho_k}{\gamma_k(p^*,x,t)} & 1 \end{pmatrix} D_0(p), & {\scriptstyle |\pi(p)-\rho_k|<\varepsilon \atop p\in\Pi_-},\ \rho_k < \zeta(x/t), \\ D_0(p), & \text{else}, \end{cases} \tag{5.57}$$

where

$$D_0(p) = \begin{pmatrix} T(p^*,x,t) & 0 \\ 0 & T(p,x,t) \end{pmatrix}.$$

Note that $D(p)$ is meromorphic in $\mathcal{K}_g\backslash C(x/t)$ and that we have

$$D(p^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.58}$$

Now we conjugate our problem using $D(p)$ and observe that, since $T(p,x,t)$ has the same behavior as $T(p)$ for p a band edge, the new vector $m^2(p) = m(p)D(p)$ is again continuous near the band edges.

Theorem 5.13 (Conjugation)**.** *The function $m^2(p) = m(p)D(p)$, where $D(p)$ is defined in (5.57), is meromorphic away from $C(x/t)$ and satisfies:*

1. *The jump condition*
$$m^2_+(p) = m^2_-(p)J^2(p), \quad p \in \Sigma, \tag{5.59}$$
where the jump matrix is given by
$$J^2(p) = D_{0-}(p)^{-1}J(p)D_{0+}(p), \tag{5.60}$$
2. *the divisor conditions*
$$(m^2_1) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \quad (m^2_2) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}, \tag{5.61}$$
All jumps corresponding to poles, except for possibly one if $\rho_k = \zeta(x/t)$, are exponentially decreasing. In that case we will keep the pole condition which is now of the form:
$$\begin{aligned} &\Big(m^2_1(p) + \frac{\gamma_k(p,x,t)}{\pi(p)-\rho_k}\frac{T(p^*,x,t)}{T(p,x,t)}m^2_2(p)\Big) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \ \textit{near } \rho_k,\\ &\Big(\frac{\gamma_k(p^*,x,t)}{\pi(p)-\rho_k}\frac{T(p,x,t)}{T(p^*,x,t)}m^2_1(p) + m^2_2(p)\Big) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}, \ \textit{near } \rho_k^*. \end{aligned} \tag{5.62}$$
3. *the symmetry condition*
$$m^2(p^*) = m^2(p)\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},$$
4. *and the normalization*
$$m^2(p_\infty) = \begin{pmatrix} 1 & 1\end{pmatrix}.$$

Proof. Invoking Lemma 5.10 and (4.40) we see that the jump matrix $J^2(p)$ is indeed given by (5.60). Using the divisor conditions (5.26) and (4.59) we obtain
$$(m^2_1(p)) = (m_1(p)T(p^*)) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \quad (m^2_2(p)) = (m_2(p)T(p)) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}. \tag{5.63}$$
That means the shifting of the poles from $\hat{\mu}_j(x,t)$ to $\hat{\nu}_j(x,t)$ stems from conjugating m with the matrix D_0.

Moreover, using Lemma 5.12 one easily sees that the jump corresponding to $\rho_k < \zeta(x/t)$ (if any) is given by
$$\begin{aligned} \tilde{v}(p) &= \begin{pmatrix} 1 & \frac{T(p,x,t)(\pi(p)-\rho_k)}{\gamma_k(p,x,t)T(p^*,x,t)}\\ 0 & 1\end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho_k),\\ \tilde{v}(p) &= \begin{pmatrix} 1 & 0\\ -\frac{T(p^*,x,t)(\pi(p)-\rho_k)}{\gamma_k(p^*,x,t)T(p,x,t)} & 1\end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho_k^*), \end{aligned} \tag{5.64}$$

and by Lemma 5.10 the jump corresponding to $\rho_k > \zeta(x/t)$ (if any) reads

$$\begin{aligned} \tilde{v}(p) &= \begin{pmatrix} 1 & 0 \\ \frac{\gamma_k(p,x,t)T(p^*,x,t)}{T(p,x,t)(\pi(p)-\rho_k)} & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho_k), \\ \tilde{v}(p) &= \begin{pmatrix} 1 & -\frac{\gamma_k(p^*,x,t)T(p,x,t)}{T(p^*,x,t)(\pi(p)-\rho_k)} \\ 0 & 1 \end{pmatrix}, \qquad p \in \Sigma_\varepsilon(\rho_k^*). \end{aligned} \tag{5.65}$$

That is, all jumps corresponding to the poles $\rho_k \neq \zeta(x/t)$ are exponentially decreasing. That the pole conditions are of the form (5.62) in the case $\rho_k = \zeta(x/t)$ can be checked directly: just use the pole conditions of the original Riemann–Hilbert problem (4.43) and the divisor condition (5.26) for $T(p,x,t)$. Furthermore, by (4.44) and (5.58) one checks that the symmetry condition for m^2 is fulfilled. From $T(p_\infty,x,t) = 1$ we finally deduce

$$m^2(p_\infty) = m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \tag{5.66}$$

which finishes the proof. □

Using (5.55) for $p \in \Sigma \setminus C(x/t) = \Sigma \cap \pi^{-1}\big((\zeta(x/t),\infty)\big)$ the jump matrix J^2 can be factorized as

$$\begin{aligned} J^2 &= \begin{pmatrix} 1-|R|^2 & -\frac{T(p,x,t)}{T(p^*,x,t)}\overline{R\Theta}\mathrm{e}^{-t\,\phi} \\ \frac{T(p^*,x,t)}{T(p,x,t)}R\Theta\mathrm{e}^{t\,\phi} & 1 \end{pmatrix} = D_0^{-1} J D_0 = D_0^{-1} b_-^{-1} b_+ D_0 = \\ &= D_0^{-1} b_-^{-1} D_0 D_0^{-1} b_+ D_0 = (\tilde{b}_-)^{-1}\tilde{b}_+, \end{aligned}$$

where $\tilde{b}_\pm = D_0^{-1} b_\pm D_0$, that is,

$$\tilde{b}_- = \begin{pmatrix} 1 & \frac{T(p,x,t)}{T(p^*,x,t)}R(p^*)\Theta(p^*)\mathrm{e}^{-t\phi(p)} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+ = \begin{pmatrix} 1 & 0 \\ \frac{T(p^*,x,t)}{T(p,x,t)}R(p)\Theta(p)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}. \tag{5.67}$$

For $p \in C(x/t) = \Sigma \cap \pi^{-1}\big((-\infty,\zeta(x/t))\big)$ we can factorize J^2 in the following way

$$\begin{aligned} J^2 &= \begin{pmatrix} 1 & -\frac{T_+(p,x,t)}{T_-(p^*,x,t)}R(p^*)\Theta(p^*)\mathrm{e}^{-t\,\phi(p)} \\ \frac{T_+(p^*,x,t)}{T_-(p,x,t)}R(p)\Theta(p)\mathrm{e}^{t\,\phi(p)} & 1-|R(p)|^2 \end{pmatrix} = D_{0-}^{-1} J D_{0+} = \\ &= D_{0-}^{-1} B_-^{-1} \begin{pmatrix} 1-|R(p)|^2 & 0 \\ 0 & (1-|R(p)|^2)^{-1} \end{pmatrix} B_+ D_{0+} = \\ &= D_{0-}^{-1} B_-^{-1} \begin{pmatrix} \frac{T_-(p^*,x,t)}{T_+(p^*,x,t)} & 0 \\ 0 & \frac{T_-(p,x,t)}{T_+(p,x,t)} \end{pmatrix} B_+ D_{0+} = \\ &= D_{0-}^{-1} B_-^{-1} D_{0-} D_{0+}^{-1} B_+ D_{0+} = (\tilde{B}_-)^{-1}\tilde{B}_+, \end{aligned}$$

where $\tilde{B}_\pm = D_\pm^{-1} B_\pm D_\pm$, that is,

$$\tilde{B}_- = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(p^*,x,t)}{T_-(p,x,t)}\frac{R(p)\Theta(p)}{1-|R(p)|^2}\mathrm{e}^{t\,\phi(p)} & 1 \end{pmatrix}, \quad \tilde{B}_+ = \begin{pmatrix} 1 & -\frac{T_+(p,x,t)}{T_+(p^*,x,t)}\frac{R(p^*)\Theta(p^*)}{1-|R(p)|^2}\mathrm{e}^{-t\,\phi(p)} \\ 0 & 1 \end{pmatrix}. \tag{5.68}$$

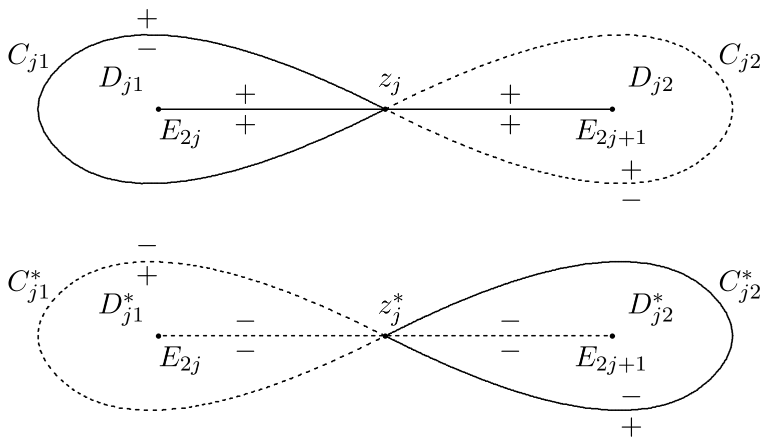

Figure 5.1.: The lens contour near a band containing a stationary phase point z_j and its flipping image containing z_j^*. Views from the top and bottom sheet. Dotted curves lie in the bottom sheet. [Kamvissis07b]

Here we have used (5.56).

Note that by $\overline{T(p,x,t)} = T(\overline{p},x,t)$ we have

$$\frac{T_-(p^*,x,t)}{T_+(p,x,t)} = \frac{T_-(p^*,x,t)}{T_-(p,x,t)} \frac{1}{1-|R(p)|^2} = \frac{\overline{T_+(p,x,t)}}{T_+(p,x,t)}, \quad p \in C(x/t), \tag{5.69}$$

respectively

$$\frac{T_+(p,x,t)}{T_-(p^*,x,t)} = \frac{T_+(p,x,t)}{T_+(p^*,x,t)} \frac{1}{1-|R(p)|^2} = \frac{\overline{T_-(p^*,x,t)}}{T_-(p^*,x,t)}, \quad p \in C(x/t). \tag{5.70}$$

5.4. Deformation

In our next step we make a contour deformation and move the corresponding parts of the jumps into regions where the off-diagonal terms are exponentially decreasing. That means that we will move those parts of the jump matrices containing $\exp(-t\phi)$ (resp. $\exp(t\phi)$) into regions where $\mathrm{Re}(\phi(p,x/t)) > 0$ (resp. $\mathrm{Re}(\phi(p,x/t)) < 0$).

In the oscillatory region, that is in the case where one stationary phase point, say z_j, lies in the interior of its spectral band (E_{2j}, E_{2j+1}), we introduce the "lens" contour near that band as shown in Figure 5.1, which is taken from [Kamvissis07b].

The oriented paths $C_j = C_{j1} \cup C_{j2}$, $C_j^* = C_{j1}^* \cup C_{j2}^*$ are meant to be close to the band $[E_{2j}, E_{2j+1}]$.

Concerning the other bands $[E_{2k}, E_{2k+1}]$, $k \neq j$, $k = 0, \dots, g$ (setting $E_{2g+1} = \infty$), one simply constructs "lens" contour near each of the other bands $[E_{2k}, E_{2k+1}]$ and $[E_{2k}^*, E_{2k+1}^*]$ as shown in Figure 5.2, which is taken from [Kamvissis07b].

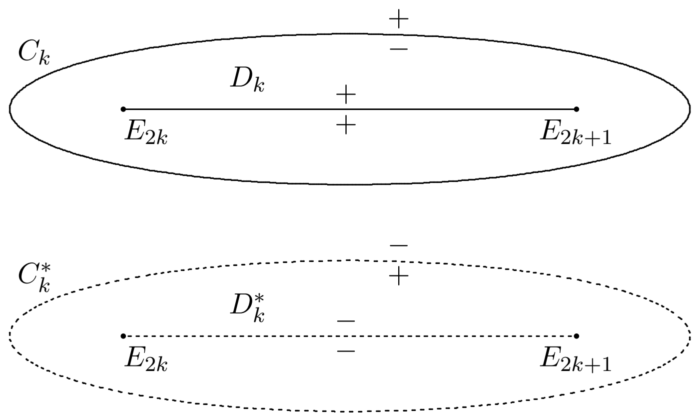

Figure 5.2.: The lens contour near a band not including any stationary phase point. Views from the top and bottom sheet. [Kamvissis07b]

The oriented paths C_k, C_k^* are meant to be close to the band $[E_{2k}, E_{2k+1}]$. In particular, these loops must not intersect with any of the loops around the eigenvalues ρ_j.

In the soliton region, that is in the case where no stationary phase point lies in $\sigma(H_q(t))$, we choose "lens" contours near all bands $[E_{2k}, E_{2k+1}]$, $k = 0, \dots, g$ (setting $E_{2g+1} = \infty$) as shown in Figure 5.2, which is taken from [Kamvissis07b].

We will next investigate the sign of $\mathrm{Re}(\phi)$.

Lemma 5.14. *In the soliton resp. oscillatory region the sign of* $\mathrm{Re}(\phi)$ *is given by:*

1. *Soliton region:*
   In the case of no stationary phase point lying in a spectral band, we have for all bands $[E_{2k}, E_{2k+1}]$

$$\mathrm{Re}(\phi(p)) < (>) 0, \quad p \in D_k, \quad \pi(p) > (<) \zeta(x/t). \tag{5.71}$$

2. *Oscillatory region:*
   In the case of one stationary phase point $\zeta(x/t) = z_j(x/t)$ *lying in the interior of its corresponding spectral band* (E_{2j}, E_{2j+1})*, we have*

$$\mathrm{Re}(\phi(p)) > 0, \quad p \in D_{j1}, \qquad \mathrm{Re}(\phi(p)) < 0, \quad p \in D_{j2},$$

   and concerning the remaining spectral bands (5.71) still holds.

Proof. Let us assume that we have already shown

$$\mathrm{Im}\big(\phi'(p)\big) < 0, \quad p \in \sigma(H_q(t)), \quad \pi(p) < \zeta(x/t), \tag{5.72}$$

$$\mathrm{Im}\big(\phi'(p)\big) > 0, \quad p \in \sigma(H_q(t)), \quad \pi(p) > \zeta(x/t). \tag{5.73}$$

If we denote $z = x + \mathrm{i}y$, then by Cauchy-Riemann we have

$$\mathrm{Im}\big(\phi'(z)\big) = \frac{1}{2}\Big(\mathrm{Im}(\frac{\partial \phi}{\partial x}) - \mathrm{Re}(\frac{\partial \phi}{\partial y})\Big) = -\frac{\partial \mathrm{Re}\phi}{\partial y}, \tag{5.74}$$

which proves the claim as long as C_{j1}, C_{j2} respectively D_k are close enough to the band $[E_{2j}, E_{2j+1}]$ respectively $[E_{2k}, E_{2k+1}]$. A similar picture appears in the lower sheet.

So it is left to show (5.72) and (5.73). We will do so by induction on $g \in \mathbb{N}_0$. We start with $g = 0$: For $z \in [E_0, \infty)$ we have

$$\mathrm{Im}\big(\phi'(z)\big) = \mathrm{Im}\Big(12\mathrm{i}\frac{(z - \tilde{\lambda}_0)}{|R_{2g+1}^{1/2}(z)|}\Big) = 12\frac{z - \tilde{\lambda}_0}{|R_{2g+1}^{1/2}(z)|}. \tag{5.75}$$

Hence for $z_0 \in (E_0, \infty)$ we indeed get

$$\begin{aligned} \mathrm{Im}\big(\phi'(z)\big) &< 0, \quad z \in (E_0, z_j) \\ \mathrm{Im}\big(\phi'(z)\big) &> 0, \quad z \in (z_j, E_1), \end{aligned}$$

and for $z_0 \in (-\infty, E_0)$ we have

$$\mathrm{Im}\big(\phi'(z)\big) > 0, \quad z \in (E_0, E_1).$$

Now let us assume (5.72) and (5.73) hold for $g-1 \in \mathbb{N}_0$, then we will make the inductive step by showing that it also holds for g. In the case of genus g we have one additional spectral band, say $[E_0, E_1]$, the others stay the same as in the case of genus $g - 1$ but are now enumerated appropriately. Using (2.6) one gets

$$\mathrm{Im}\big(\phi'(z)\big) = \mathrm{Im}\Big(12\mathrm{i}\frac{\prod_{j=0}^{g}(z - \tilde{\lambda}_j)}{R_{2g+1}^{1/2}(z)} + \mathrm{i}\frac{x}{t}\frac{\prod_{j=1}^{g}(z - \lambda_j)}{R_{2g+1}^{1/2}(z)}\Big) = (-1)^{g+k}\varphi(z),$$

where

$$\varphi(z) := 12\frac{\prod_{j=0}^{g}(\mathrm{Re}(z) - \tilde{\lambda}_j)}{|R_{2g+1}^{1/2}(z)|} + \frac{x}{t}\frac{\prod_{j=1}^{g}(\mathrm{Re}(z) - \lambda_j)}{|R_{2g+1}^{1/2}(z)|}, \tag{5.76}$$

for $z \in [E_{2k}, E_{2k+1}]$, $k = 0, \dots, g$ (setting $E_{2g+1} = \infty$). In principal there are three cases which might occure:

1. $z_0 \in (E_0, E_1)$ and all other stationary phase points lie in open gaps:

   a) g is odd. Then from Figure 5.3 one deduces

   $$\mathrm{Im}\big(\phi'(z)\big) = \begin{cases} -\varphi(z) < 0 & \text{for } z \in (E_0, z_0), \\ -\varphi(z) > 0 & \text{for } z \in (z_0, E_1). \end{cases} \tag{5.77}$$

   b) g is even. From Figure 5.4 one similarly obtains

   $$\mathrm{Im}\big(\phi'(z)\big) = \begin{cases} \varphi(z) < 0 & \text{for } z \in (E_0, z_0), \\ \varphi(z) > 0 & \text{for } z \in (z_0, E_1). \end{cases} \tag{5.78}$$

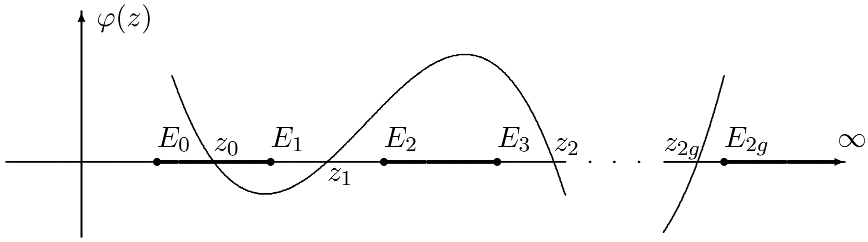

Figure 5.3.: A sketch of the function $\varphi(z)$ defined in (5.76) for g odd and $z_0 \in (E_0, E_1)$. The thick lines denote the spectral bands.

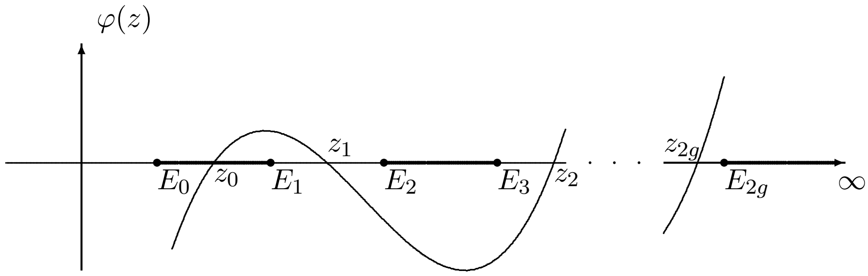

Figure 5.4.: A sketch of the function $\varphi(z)$ defined in (5.76) for g even and $z_0 \in (E_0, E_1)$. The thick lines denote the spectral bands.

2. $z_0 \in (E_1, E_2)$ and maybe some other stationary phase point z_j lies in its corresponding spectral band (E_{2j}, E_{2j+1}), $j = 1, \dots, g$. Note that depending on z_1 having or not having already passed its spectral band (E_2, E_3), z_1 can lie either in one of the gaps (E_1, E_2), (E_3, E_4) or in its spectral band (E_2, E_3).

   a) g is odd. Then from Figure 5.5 one deduces

   $$\mathrm{Im}\big(\phi'(z)\big) = -\varphi(z) < 0 \quad \text{for } z \in (E_0, E_1). \tag{5.79}$$

   b) g is even. From Figure 5.6 one obtains

   $$\mathrm{Im}\big(\phi'(z)\big) = \varphi(z) < 0 \quad \text{for } z \in (E_0, E_1). \tag{5.80}$$

3. $z_0 \in (-\infty, E_0)$ and all other stationary phase points lie in a spectral gap.

   a) g is odd. Then from Figure 5.7 one deduces

   $$\mathrm{Im}\big(\phi'(z)\big) = -\varphi(z) > 0 \quad \text{for } z \in (E_0, E_1). \tag{5.81}$$

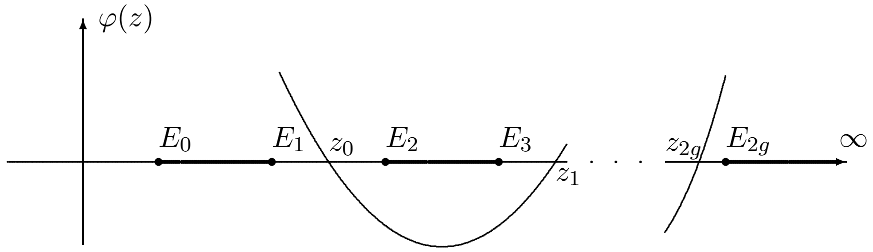

Figure 5.5.: A sketch of the function $\varphi(z)$ defined in (5.76) for g odd and $z_0 \in (E_1, E_2)$. The thick lines denote the spectral bands.

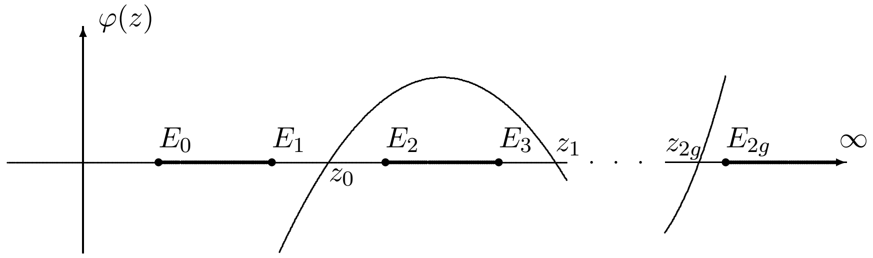

Figure 5.6.: A sketch of the function $\varphi(z)$ defined in (5.76) for g even and $z_0 \in (E_1, E_2)$. The thick lines denote the spectral bands.

b) g is even. From Figure 5.8 one obtains

$$\mathrm{Im}\big(\phi'(z)\big) = \varphi(z) > 0 \quad \text{for } z \in (E_0, E_1). \tag{5.82}$$

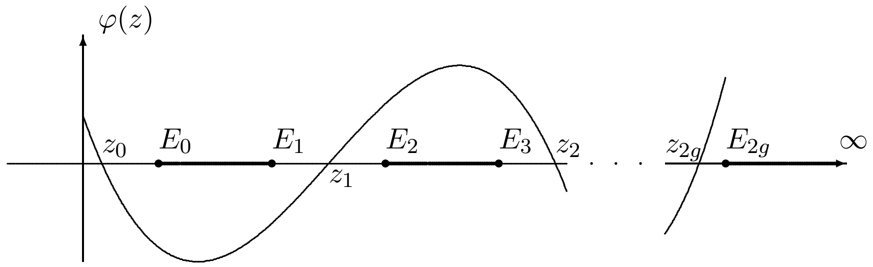

Figure 5.7.: A sketch of the function $\varphi(z)$ defined in (5.76) for g odd and $z_0 \in (-\infty, E_0)$. The thick lines denote the spectral bands.

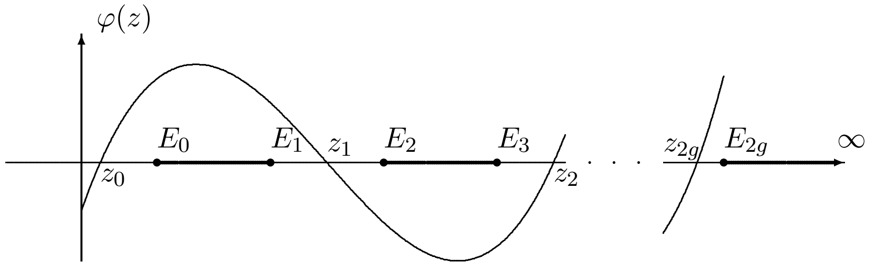

Figure 5.8.: A sketch of the function $\varphi(z)$ defined in (5.76) for g even and $z_0 \in (-\infty, E_0)$. The thick lines denote the spectral bands.

4. $z_0 \in (-\infty, E_0)$ and all other stationary phase points lie in a spectral gap.

   a) g is odd. Then from Figure 5.7 one deduces

$$\mathrm{Im}\big(\phi'(z)\big) = -\varphi(z) > 0 \quad \text{for } z \in (E_0, E_1). \tag{5.83}$$

   b) g is even. From Figure 5.8 one obtains

$$\mathrm{Im}\big(\phi'(z)\big) = \varphi(z) > 0 \quad \text{for } z \in (E_0, E_1). \tag{5.84}$$

In all of the three cases the sign for $\mathrm{Im}\big(\phi'(z)\big)$ for $z \in (E_{2k}, E_{2k+1})$, $k = 1, \dots, g$ can be obtained from the induction hypothesis, which makes the proof complete. □

Having investigated the sign of $\mathrm{Re}(\phi)$ we are now able to redefine the Riemann–Hilbert problem for m^2 in such a way that the jumps of the new Riemann–Hilbert problem will lie in the regions where they are exponentially close to the identity for large times.

Theorem 5.15 (Deformation)**.** *Define m^3 by*

$$\begin{aligned}
m^3 &= m^2 \tilde{B}_+^{-1}, \quad p \in D_k,\ k < j,\\
m^3 &= m^2 \tilde{B}_-^{-1}, \quad p \in D_k^*,\ k < j,\\
m^3 &= m^2 \tilde{B}_+^{-1}, \quad p \in D_{j1},\\
m^3 &= m^2 \tilde{B}_-^{-1}, \quad p \in D_{j1}^*,\\
m^3 &= m^2 \tilde{b}_+^{-1}, \quad p \in D_{j2},\\
m^3 &= m^2 \tilde{b}_-^{-1}, \quad p \in D_{j2}^*,\\
m^3 &= m^2 \tilde{b}_+^{-1}, \quad p \in D_k,\ k > j,\\
m^3 &= m^2 \tilde{b}_-^{-1}, \quad p \in D_k^*,\ k > j,\\
m^3 &= m^2, \quad \textit{else},
\end{aligned} \tag{5.85}$$

where the matrices $\tilde{b}_\pm$ *and* $\tilde{B}_\pm$ *are defined in* (5.67) *and* (5.68), *respectively. Here we assume that the deformed contour is sufficiently close to the original one. Then the function* $m^3(p)$ *satisfies:*

1. *The jump condition*

$$m^3_+(p) = m^3_-(p) J^3(p), \text{ for } p \in \Sigma^3, \tag{5.86}$$

where the jump matrix J^3 *is given by*

$$\begin{aligned} J^3 &= \tilde{B}_+, \quad p \in C_k, \quad k < j, \\ J^3 &= \tilde{B}_-^{-1}, \quad p \in C_k^*, \quad k < j, \\ J^3 &= \tilde{B}_+, \quad p \in C_{j1}, \\ J^3 &= \tilde{B}_-^{-1}, \quad p \in C_{j1}^*, \\ J^3 &= \tilde{b}_+^{-1}, \quad p \in C_{j2}, \\ J^3 &= \tilde{b}_-, \quad p \in C_{j2}^*, \\ J^3 &= \tilde{b}_+, \quad p \in C_k, \quad k > j, \\ J^3 &= \tilde{b}_-^{-1}, \quad p \in C_k^*, \quad k > j, \end{aligned} \tag{5.87}$$

2. *the divisor conditions*

$$(m^3_1) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \quad (m^3_2) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}, \tag{5.88}$$

3. *the symmetry condition*

$$m^3(p^*) = m^3(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.89}$$

4. *and the normalization*

$$m^3(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \tag{5.90}$$

Proof. To see that the jump matrix J^3 is indeed given by (5.87) note

$$\begin{aligned} m^3_+ &= m^2_+ = \big(m^2_+ \tilde{B}_+^{-1}\big)\tilde{B}_+ = m^3_- \tilde{B}_+, \quad \text{for } p \in C_k, \quad k < j, \\ m^3_+ &= m^2_+ = \big(m^2_+ \tilde{b}_+^{-1}\big)\tilde{b}_+ = m^3_- \tilde{b}_+, \quad \text{for } p \in C_k, \quad k > j, \\ m^3_+ &= m^2_+ = \big(m^2_+ \tilde{B}_+^{-1}\big)\tilde{B}_+ = m^3_- \tilde{B}_+, \quad \text{for } p \in C_{j1}, \\ m^3_+ &= m^2_+ \tilde{b}_+^{-1} = m^2_- \tilde{b}_+^{-1} = m^3_- \tilde{b}_+^{-1}, \quad \text{for } p \in C_{j2}. \end{aligned}$$

Analogously one shows

$$\begin{aligned} m^3_+ &= m^2_+ \tilde{B}_-^{-1} = m^2_- \tilde{B}_-^{-1} = m^3_- \tilde{B}_-^{-1}, \quad \text{for } p \in C_k^*, \quad k < j, \\ m^3_+ &= m^2_+ \tilde{b}_-^{-1} = m^2_- \tilde{b}_-^{-1} = m^3_- \tilde{b}_-^{-1}, \quad \text{for } p \in C_k^*, \quad k > j. \\ m^3_+ &= m^2_+ \tilde{B}_-^{-1} = m^2_- \tilde{B}_-^{-1} = m^3_- \tilde{B}_-^{-1}, \quad \text{for } p \in C_{j1}^*, \\ m^3_+ &= m^2_+ = \big(m^2_+ \tilde{b}_-^{-1}\big)\tilde{b}_- = m^3_- \tilde{b}_-, \quad \text{for } p \in C_{j2}^*. \end{aligned}$$

Finally, the jump along Σ disappears:

$$\begin{aligned} m^3_+ &= m^2_+\tilde{B}_+^{-1} = \big(m^2_-\tilde{B}_-^{-1}\tilde{B}_+\big)\tilde{B}_+^{-1} = m^2_-\tilde{B}_-^{-1} = m^3_-, \quad \text{for } p\in C(x/t),\\ m^3_+ &= m^2_+\tilde{b}_+^{-1} = \big(m^2_-\tilde{b}_-^{-1}\tilde{b}_+\big)\tilde{b}_+^{-1} = m^2_-\tilde{b}_-^{-1} = m^3_-, \quad \text{for } p\in \Sigma\setminus C(x/t). \end{aligned}$$

Invoking (5.61) one can easily verify that m^3 indeed satisfies

$$(m^3_1) \ge -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \quad (m^3_2) \ge -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}. \tag{5.91}$$

To see that m^3 fulfills the symmetry condition we have to distinguish cases. Suppose $p\in D_{j1}\cup\bigcup_{k<j} D_k$, then by using the symmetry of m^2 one gets

$$\begin{aligned} m^3(p^*) &= m^2(p^*)\tilde{B}_-^{-1}(p^*) = \\ &= m^2(p)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\tilde{B}_+^{-1}(p)\begin{pmatrix}0&1\\1&0\end{pmatrix} = \\ &= m^2(p)\tilde{B}_+^{-1}(p)\begin{pmatrix}0&1\\1&0\end{pmatrix} = m^3(p)\begin{pmatrix}0&1\\1&0\end{pmatrix}. \end{aligned}$$

Next let $p\in D_{j2}\cup\bigcup_{k>j} D_k$. Then we obtain

$$\begin{aligned} m^3(p^*) &= m^2(p^*)\tilde{b}_-^{-1}(p^*) = \\ &= m^2(p)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\tilde{b}_+^{-1}(p)\begin{pmatrix}0&1\\1&0\end{pmatrix} = \\ &= m^2(p)\tilde{b}_+^{-1}(p)\begin{pmatrix}0&1\\1&0\end{pmatrix} = m^3(p)\begin{pmatrix}0&1\\1&0\end{pmatrix}. \end{aligned}$$

Analoguously one can prove the symmetry condition for m^3 if $p\in D^*_{j1}\cup\bigcup_{k<j} D^*_k$ or $p\in D^*_{j2}\cup\bigcup_{k>j} D^*_k$. For all other p we have $m^3 = m^2$ and thus nothing is to show.

Finally it is straightforward to check that the normalization of m^2 at p_∞ yields

$$m^3(p_\infty) = \begin{pmatrix}1 & 1\end{pmatrix}.$$

□

Here we have assumed that the reflection coefficient $R(p)$ appearing in the jump matrices admits an analytic extension in the corresponding regions. Of course this is not true in general, but we can always evade this obstacle by approximating $R(p)$ by analytic functions. We refer to the discussion in Chapter 7 for the details.

5.5. The long-time asymptotics inside the soliton region

First we will define the so-called limiting KdV solution. Note that the definition is motivated by (4.53) and (5.39).

Definition 5.5 (Limiting KdV solution). *To define a limiting KdV solution we have to distinguish between the oscillatory and the soliton region.*

(i) In the oscillatory region we define the limiting KdV solution $V_l(x,t)$ by

$$\int_x^\infty (V_l - V_q)(y,t)dy = 2\mathrm{i}T_1(x,t), \tag{5.92}$$

where $T_1(x,t)$ is defined as in Theorem 5.8, that is,

$$\begin{aligned}\int_x^\infty (V_l - V_q)(y,t)dy = &- \sum_{\rho_k<\zeta(x/t)} 4\mathrm{i}\int_{E(\rho_k)}^{\rho_k} \omega_{p_\infty,0} + \frac{1}{\pi}\int_{C(x/t)} \log(1-|R|^2)\omega_{p_\infty,0} \\ &+ 2\partial_x \ln\left(\frac{\theta\big(\underline{z}(p_\infty,x,t)+\underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)}\right),\end{aligned} \tag{5.93}$$

$$\delta_\ell(x/t) = 2\sum_{\rho_k<\zeta(x/t)} \underline{A}_{E_0}(\hat{\rho}_k) + \frac{1}{2\pi\mathrm{i}}\int_{C(x/t)} \log(1-|R|^2)\zeta_\ell, \tag{5.94}$$

where $R = R_+(\lambda,t)$ is the associated reflection coefficient, ζ_ℓ is a canonical basis of holomorphic differentials, and $C(x/t) = \Sigma\cap\pi^{-1}\big((-\infty,\zeta(x/t))\big)$ is oriented such that the upper sheet is to the left.

(ii) In the soliton region we define the limiting KdV solution $V_{l,v}(x,t)$ by

$$\int_x^\infty (V_{l,v} - V_q)(y,t)dy = 2\mathrm{i}T_1(x,t), \tag{5.95}$$

where $T_1(x,t)$ is defined as in Theorem 5.8, that is,

$$\begin{aligned}\int_x^\infty (V_{l,v} - V_q)(y,t)dy = &- \sum_{\rho_j<\zeta(v)} 4\mathrm{i}\int_{E(\rho_j)}^{\rho_j} \omega_{p_\infty,0} + \frac{1}{\pi}\int_{C(v)} \log(1-|R|^2)\omega_{p_\infty,0} \\ &+ 2\partial_x \ln\left(\frac{\theta\big(\underline{z}(p_\infty,x,t)+\underline{\delta}(v)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)}\right),\end{aligned} \tag{5.96}$$

or equivalently

$$(V_{l,v} - V_q)(x,t) = -2\partial_x^2 \ln\left(\frac{\theta\big(\underline{z}(p_\infty,x,t)+\underline{\delta}(v)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)}\right), \tag{5.97}$$

with

$$\delta_\ell(v) = 2\sum_{\rho_j<\zeta(v)} \underline{A}_{E_0}(\hat{\rho}_j) + \frac{1}{2\pi\mathrm{i}}\int_{C(v)} \log(1-|R|^2)\zeta_\ell,$$

where $R = R_+(\lambda,t)$ is the associated reflection coefficient, ζ_ℓ is a canonical basis of holomorphic differentials, and $c_k = v(\rho_k)$ denotes the velocity of the k'th soliton defined by (5.3). Moreover, $C(v) = \Sigma\cap\pi^{-1}\big((-\infty,\zeta(v))\big)$ is oriented such that the upper sheet is to the left. If $v = x/t$ we set $V_l(x,t) = V_{l,x/t}(x,t)$.

The next two theorems concern the long-time asymptotics of the perturbed KdV solution $V(x,t)$ in the soliton region. They tell us that in the soliton region for large times t the perturbed solution $V(x,t)$ splits into a number of stable solitons traveling on the limiting KdV solution $V_{l,v}(x,t)$.

Theorem 5.16. *Assume*

$$\int_{-\infty}^{+\infty} (1+|x|^{1+n})(|V(x,t)-V_q(x,t)|)dx < \infty, \tag{5.98}$$

for some integer $n \geq 1$ and abbreviate by $c_k = v(\rho_k)$ the velocity of the k'th soliton given by (5.3). Then the asymptotics in the soliton region, $\{(x,t)|\, \zeta(x/t) \in \mathbb{R}\backslash\sigma(H_q)\}$, are the following:

Let $\varepsilon > 0$ sufficiently small such that the intervals $[c_k-\varepsilon, c_k+\varepsilon]$, $1 \leq k \leq N$, are disjoint and lie inside $v(\mathbb{R}\backslash\sigma(H_q))$.

If $|\frac{x}{t} - c_k| < \varepsilon$ for some k, the solution is asymptotically given by a one-soliton solution on top of the limiting solution:

$$\int_x^{+\infty} (V - V_{l,c_k})(y,t)dy = 2\frac{\tilde{\gamma}_k \psi_{l,c_k}(\rho_k,x,t)^2}{c_{l,k}(x,t)} + O(t^{-l}), \tag{5.99}$$

for any $l \geq 1$, where

$$c_{l,k}(x,t) = 1 + \tilde{\gamma}_k W_{(x,t)}(\dot{\psi}_{l,c_k}(\rho_k,x,t), \psi_{l,c_k}(\rho_k,x,t)) \tag{5.100}$$

and

$$\begin{aligned}\tilde{\gamma}_k = \gamma_k &\left(\frac{\theta(\underline{z}(\rho_k,0,0)+\underline{\delta}(c_k))}{\theta(\underline{z}(\rho_k,0,0))}\right)^2 \left(\prod_{\rho_j<\zeta(c_k)} \exp\left(2\int_{E_0}^{\rho_k} \omega_{\rho_j\,\rho_j^*}\right)\right)\cdot \\ &\cdot \exp\left(\frac{-1}{\pi\mathrm{i}}\int_{C(c_k)} \log(1-|R|^2)\omega_{\rho_k\,p_\infty}\right).\end{aligned} \tag{5.101}$$

Here $\psi_{l,v}(p,x,t)$ is the Baker–Akhiezer function corresponding to the limiting solution $V_{l,v}(x,t)$ defined above and the dot denotes differentiation with respect to ρ_k.

If $|\frac{x}{t} - c_k| \geq \varepsilon$, for all k, the solution is asymptotically close to the limiting solution:

$$\int_x^{\infty} (V - V_l)(y,t)dy = O(t^{-l}), \tag{5.102}$$

for any $l \geq 1$.

Proof. We apply Theorem A.7 in the following way:

If $|\zeta(x/t) - \rho_k| > \varepsilon$ for all k we can choose $\gamma_0^t = 0$ and w_0^t by removing all jumps corresponding to poles from w^t. The error between the solutions of w^t and w_0^t is exponentially small in the sense of Theorem A.7, that is, $\|w^t - w_0^t\|_\infty \leq O(t^{-l})$ for any $l \geq 1$.

We have the one soliton solution (cf. Lemma 4.11) $\hat{m}_0(p) = \big(\hat{f}(p^*,x,t) \quad \hat{f}(p,x,t)\big)$, where $\hat{f}(p) = 1$ for p large enough. Using Theorem 5.8 we compute

$$\begin{aligned} m(p) =& \hat{m}_0(p) \begin{pmatrix} T(p^*,x,t)^{-1} & 0 \\ 0 & T(p,x,t)^{-1} \end{pmatrix} \\ =& \Big(1 + \tfrac{T_1(x,t)}{\sqrt{z}} + O(z^{-1}) \quad 1 - \tfrac{T_1(x,t)}{\sqrt{z}} + O(z^{-1})\Big). \end{aligned}$$

Comparing this expression with (4.53) yields

$$\int_x^\infty (V - V_q)(y,t)dy = 2\mathrm{i}T_1(x,t) + O(t^{-l}),$$

and thus by our definition of the limiting solution we finally have

$$\int_x^\infty (V - V_l)(y,t)dy = \int_x^\infty \Big((V - V_q)(y,t) - (V_l - V_q)(y,t)\Big)dy = O(t^{-l}),$$

for any $l \geq 1$. This proves the second part of the theorem.

If $|\zeta(x/t) - \rho_k| < \varepsilon$ for some k, we choose $\gamma_0^t = \tilde{\gamma}_k$ and $w_0^t \equiv 0$. Again we conclude that the error between the solutions of w^t and w_0^t is exponentially small, that is, $\|w^t - w_0^t\|_\infty \leq O(t^{-l})$, for any $l \geq 1$. By Lemma 4.11 we have the one soliton solution $\hat{m}_0(p) = \big(\hat{f}(p^*,x,t) \quad \hat{f}(p,x,t)\big)$, with

$$\hat{f}(p,x,t) = 1 + \frac{\tilde{\gamma}_k}{z - \rho_k} \frac{\psi_{l,c_k}(\rho_k,x,t)W_{(x,t)}(\psi_{l,c_k}(\rho_k,x,t), \psi_{l,c_k}(p,x,t))}{\psi_{l,c_k}(p,x,t)c_{l,k}(x,t)},$$

for p large enough, where $\tilde{\gamma}_k$ is defined as in (5.101). We will again use

$$m(p) = \hat{m}_0(p) \begin{pmatrix} T(p^*,x,t)^{-1} & 0 \\ 0 & T(p,x,t)^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\hat{f}(p^*,x,t)}{T(p^*,x,t)} & \frac{\hat{f}(p,x,t)}{T(p,x,t)} \end{pmatrix},$$

and now expand $\hat{f}(p)$ as in the proof of Lemma 4.11. Finally a comparison with (4.53) yields

$$\int_x^\infty (V - V_q)(y,t)dy = 2\mathrm{i}T_1(x,t) + 2\frac{\tilde{\gamma}_k \psi_{l,c_k}(\rho_k,x,t)^2}{c_{l,k}(x,t)} + O(t^{-l}),$$

and hence by our definition of the limiting solution (5.96)

$$\int_x^\infty (V - V_{l,c_k})(y,t)dy = 2\frac{\tilde{\gamma}_k \psi_{l,c_k}(\rho_k,x,t)^2}{c_{l,k}(x,t)} + O(t^{-l}),$$

for any $l \geq 1$, which makes the proof complete. □

To be able to prove the long-time asymptotic behavior for $V - V_{l,c_k}$ we will need the following lemma.

Lemma 5.17. *We have*

$$T(z)\frac{\psi_-(z,x,t)}{\psi_{q,-}(z,x,t)}\frac{\psi_+(z,x,t)}{\psi_{q,+}(z,x,t)} = 1 + \frac{1}{2}(V - V_q)(x,t)\frac{1}{z} + o(z^{-1}). \tag{5.103}$$

Proof. We will use the following representation of the Jost solutions

$$\psi_\pm(z,x,t) = \psi_{q,\pm}(z,x,t)\exp\Big(\mp\int_x^{\pm\infty}\big(m_\pm(z,y,t) - m_{q,\pm}(z,y,t)\big)dy\Big), \qquad (5.104)$$

where

$$m_\pm(z,x,t) = \pm\frac{\psi'_\pm(z,x,t)}{\psi_\pm(z,x,t)}, \quad m_{q,\pm}(z,x,t) = \pm\frac{\psi'_{q,\pm}(z,x,t)}{\psi_{q,\pm}(z,x,t)}$$

are the Weyl–Titchmarsh functions. Here the prime denotes differentiation with respect to x. By [Mikikits-Leitner09, Lemma 6.1] the Weyl m-functions have the following asymptotic expansion for large z

$$m_\pm(z,x,t) \asymp \mathrm{i}\sqrt{z} \pm \sum_{n=1}^{\infty}\frac{\chi_n(x,t)}{(\pm 2\mathrm{i}\sqrt{z})^n},$$

with coefficients defined recursively via

$$\chi_1(x,t) = V(x,t), \quad \chi_{n+1}(x,t) = -\frac{\partial}{\partial x}\chi_n(x,t) - \sum_{m=1}^{n-1}\chi_{n-m}(x,t)\chi_m(x,t).$$

Similarly, we have

$$m_{q,\pm}(z,x,t) \asymp \mathrm{i}\sqrt{z} \pm \sum_{n=1}^{\infty}\frac{\chi_{q,n}(x,t)}{(\pm 2\mathrm{i}\sqrt{z})^n},$$

with coefficients determined by the recursive formulas

$$\chi_{q,1}(x,t) = V_q(x,t), \quad \chi_{q,n+1}(x,t) = -\frac{\partial}{\partial x}\chi_{q,n}(x,t) - \sum_{m=1}^{n-1}\chi_{q,n-m}(x,t)\chi_{q,m}(x,t).$$

Thus, we obtain the following formulas

$$m_\pm(z,x,t) \asymp \mathrm{i}\sqrt{z} + V(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} \pm V'(x,t)\frac{1}{4z} + O(z^{-3/2}),$$
$$m_{q,\pm}(z,x,t) \asymp \mathrm{i}\sqrt{z} + V_q(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} \pm V_q'(x,t)\frac{1}{4z} + O(z^{-3/2}),$$

and therefore

$$(m_\pm - m_{q,\pm})(z,x,t) \asymp (V - V_q)(x,t)\frac{1}{2\mathrm{i}\sqrt{z}} \pm (V - V_q)'(x,t)\frac{1}{4z} + O(z^{-3/2}).$$

Plugging this expansion into (5.104) we get

$$\frac{\psi_-(z,x,t)}{\psi_{q,-}(z,x,t)}\frac{\psi_+(z,x,t)}{\psi_{q,+}(z,x,t)} \asymp \exp\Big(-\int_{-\infty}^{+\infty}(V - V_q)(y,t)dy\frac{1}{2\mathrm{i}\sqrt{z}} + \\ + \big(\int_{-\infty}^{x}(V - V_q)'(y,t)dy - \int_x^{+\infty}(V - V_q)'(y,t)dy\big)\frac{1}{4z} + O(z^{-3/2})\Big). \qquad (5.105)$$

Using [Mikikits-Leitner09, Theorem 6.2] we have the following asymptotic expansion for $\log T(z)$ around $z=\infty$:

$$\log T(z) \asymp \mathrm{i}\sqrt{z}\sum_{k=1}^{\infty}\frac{\tau_k}{z^k},$$

where the quantities τ_k are given by

$$\tau_k = \int_{-\infty}^{\infty}\frac{\chi_{2k-1}(x,t)-\chi_{q,2k-1}(x,t)}{(-1)^k 2^{2k-1}}dx.$$

From this we deduce

$$T(z) = \exp\Big(\int_{-\infty}^{+\infty}(V-V_q)(y,t)dy\frac{1}{2\mathrm{i}\sqrt{z}} + O(z^{-3/2})\Big). \tag{5.106}$$

Finally, combining (5.105) and (5.106) we get

$$\begin{aligned} T(z)\frac{\psi_-(z,x,t)}{\psi_{q,-}(z,x,t)}\frac{\psi_+(z,x,t)}{\psi_{q,+}(z,x,t)} =& \exp\Big((V-V_q)(x,t)\frac{1}{2z}+o(z^{-1})\Big)\\ =& 1+(V-V_q)(x,t)\frac{1}{2z}+o(z^{-1}), \end{aligned}$$

which proves the claim. □

Theorem 5.18. *Assume (5.98) and abbreviate by $c_k = v(\rho_k)$ the velocity of the k'th soliton given by (5.3). Then the asymptotics in the soliton region, $\{(x,t)|\,\zeta(x/t)\in\mathbb{R}\backslash\sigma(H_q)\}$, are the following:*

Let $\varepsilon>0$ sufficiently small such that the intervals $[c_k-\varepsilon, c_k+\varepsilon]$, $1\le k\le N$, are disjoint and lie inside $v(\mathbb{R}\backslash\sigma(H_q))$.

If $|\frac{x}{t}-c_k|<\varepsilon$ for some k, the solution is asymptotically given by a one-soliton solution on top of the limiting solution:

$$(V-V_{l,c_k})(x,t) = 2\tilde{\gamma}_k^2\frac{\psi_{l,c_k}(\rho_k,x,t)^4}{c_{l,k}(x,t)^2} - 4\tilde{\gamma}_k\frac{\psi_{l,c_k}(\rho_k,x,t)\psi'_{l,c_k}(\rho_k,x,t)}{c_{l,k}(x,t)} \tag{5.107}$$

for any $l\ge 1$, where

$$c_{l,k}(x,t) = 1+\tilde{\gamma}_k W_{(x,t)}(\dot{\psi}_{l,c_k}(\rho_k,x,t),\psi_{l,c_k}(\rho_k,x,t))$$

and

$$\begin{aligned} \tilde{\gamma}_k = \gamma_k\left(\frac{\theta(\underline{z}(\rho_k,0,0)+\underline{\delta}(c_k))}{\theta(\underline{z}(\rho_k,0,0))}\right)^2\left(\prod_{\rho_j<\zeta(c_k)}\exp\left(2\int_{E_0}^{\rho_k}\omega_{\rho_j\,\rho_j^*}\right)\right)\cdot\\ \cdot\exp\left(\frac{-1}{\pi\mathrm{i}}\int_{C(c_k)}\log(1-|R|^2)\omega_{\rho_k\,p_\infty}\right). \end{aligned} \tag{5.108}$$

Here $\psi_{l,c_k}(p,x,t)$ is the Baker–Akhiezer function corresponding to the limiting solution $V_{l,v}(x,t)$ defined above and the dot denotes differentiation with respect to ρ_k.

If $|\frac{x}{t} - c_k| \geq \varepsilon$, for all k, the solution is asymptotically close to the limiting solution:

$$(V - V_l)(x,t) = O(t^{-l}), \tag{5.109}$$

for any $l \geq 1$.

Proof. If $|\zeta(x/t) - \rho_k| > \varepsilon$ for all k we can choose $\gamma_0^t = 0$ and w_0^t by removing all jumps corresponding to poles from w^t. The error between the solutions of w^t and w_0^t is exponentially small in the sense of Theorem A.7, that is, $\|w^t - w_0^t\|_\infty \leq O(t^{-l})$ for any $l \geq 1$. We have the one soliton solution (cf. Lemma 4.11) $\hat{m}_0(p) = \begin{pmatrix} \hat{f}(p^*,x,t) & \hat{f}(p,x,t) \end{pmatrix}$, where $\hat{f}(p) = 1$ for p large enough. As in the proof of the previous theorem we have

$$m(p) = \begin{pmatrix} \frac{\hat{f}(p^*,x,t)}{T(p^*,x,t)} & \frac{\hat{f}(p,x,t)}{T(p,x,t)} \end{pmatrix},$$

and thus

$$m_1 \cdot m_2 = \hat{f}(p^*,x,t)\hat{f}(p,x,t)T(p^*,x,t)^{-1}T(p,x,t)^{-1} = 1 + \sum_j (\mu_j - \nu_j)\frac{1}{z} + O(z^{-1}),$$

where we made use of (cf. Lemma 5.6)

$$T(p^*,x,t)T(p,x,t) = \prod_{j=0}^{g} \frac{z - \mu_j}{z - \nu_j}. \tag{5.110}$$

On the other hand from Lemma 5.17 we have

$$m_1 \cdot m_2 = 1 + (V - V_q)(x,t)\frac{1}{2z} + o(z^{-1}). \tag{5.111}$$

Comparing these two expressions for $m_1 \cdot m_2$ yields

$$(V - V_q)(x,t) = 2\sum_j (\mu_j - \nu_j) + O(t^{-l}),$$

or

$$(V - V_l)(x,t) = O(t^{-l}),$$

where in the last step we have used the representations

$$\begin{aligned} V_q(x,t) &= \sum_{j=0}^{2g} E_j - 2\sum_{j=0}^{g} \mu_j(x,t), \\ V_l(x,t) &= \sum_{j=0}^{2g} E_j - 2\sum_{j=0}^{g} \nu_j(x,t). \end{aligned} \tag{5.112}$$

If $|\zeta(x/t) - \rho_k| < \varepsilon$ for some k, we choose $\gamma_0^t = \tilde{\gamma}_k$ and $w_0^t \equiv 0$. Again we conclude that the error between the solutions of w^t and w_0^t is exponentially small, that is, $\|w^t - w_0^t\|_\infty \le O(t^{-l})$, for any $l \ge 1$. By Lemma 4.11 we have the one soliton solution $\hat{m}_0(p) = \big(\hat{f}(p^*, x, t) \quad \hat{f}(p, x, t)\big)$, with

$$\hat{f}(p, x, t) = 1 + \frac{\tilde{\gamma}_k}{z - \rho_k} \frac{\psi_{l,c_k}(\rho_k, x, t) W_{(x,t)}(\psi_{l,c_k} q(\rho_k, x, t), \psi_{l,c_k}(p, x, t))}{\psi_{l,c_k}(p, x, t) c_{l,k}(x, t)},$$

for p large enough, where $\tilde{\gamma}_k$ is defined as in (5.108). The function $\hat{f}(p, x, t)$ can be expanded near $p = p_\infty$ as it is done in the proof of Lemma 4.11

$$\hat{f}(p, x, t) = 1 \mp \tilde{\gamma}_k \frac{\psi_{l,c_k}(\rho_k, x, t)^2}{c_{l,k}(x, t)} \frac{1}{\mathrm{i}\sqrt{z}} - \tilde{\gamma}_k \frac{\psi_{l,c_k}(\rho_k, x, t)\psi'_{l,c_k}(\rho_k, x, t)}{c_{l,k}(x, t)} \frac{1}{z} + O(z^{-3/2}).$$

Then we can compute

$$\begin{aligned} m_1 \cdot m_2 =& \hat{f}(p^*, x, t)\hat{f}(p, x, t) T(p^*, x, t)^{-1} T(p, x, t)^{-1} \\ =& 1 + \Big[\tilde{\gamma}_k^2 \frac{\psi_{l,c_k}(\rho_k, x, t)^4}{c_{l,k}(x, t)^2} - 2\tilde{\gamma}_k \frac{\psi_{l,c_k}(\rho_k, x, t)\psi'_{l,c_k}(\rho_k, x, t)}{c_{l,k}(x, t)} \\ &+ \sum_j (\mu_j - \nu_j)\Big] \frac{1}{z} + O(z^{-3/2}), \end{aligned}$$

where we again made use of (5.110). Comparing this expression for $m_1 \cdot m_2$ with (5.111) yields

$$(V - V_q)(x, t) = 2\tilde{\gamma}_k^2 \frac{\psi_{l,c_k}(\rho_k, x, t)^4}{c_{l,k}(x, t)^2} - 4\tilde{\gamma}_k \frac{\psi_{l,c_k}(\rho_k, x, t)\psi'_{l,c_k}(\rho_k, x, t)}{c_{l,k}(x, t)} + 2\sum_j (\mu_j - \nu_j),$$

or

$$(V - V_{l,k})(x, t) = 2\tilde{\gamma}_k^2 \frac{\psi_{l,c_k}(\rho_k, x, t)^4}{c_{l,k}(x, t)^2} - 4\tilde{\gamma}_k \frac{\psi_{l,c_k}(\rho_k, x, t)\psi'_{l,c_k}(\rho_k, x, t)}{c_{l,k}(x, t)}.$$

Note that we again used the representations (5.112). □

Note that the asymptotics for $V - V_{l,c_k}$ given by expression (5.107) can be checked to be equal the derivative of (5.99) with respect to x by using

$$\begin{aligned} c_{l,k}(x, t) &= 1 + \tilde{\gamma}_k W_{(x,t)}(\dot{\psi}_{l,c_k}(\rho_k, x, t), \psi_{l,c_k}(\rho_k, x, t)) \\ &= 1 + \tilde{\gamma}_k \int_x^{+\infty} |\psi_{l,c_k}(\rho_k, y, t)|^2 dy. \end{aligned}$$

6. The Riemann-Hilbert problem in the oscillatory region

In Chapter 5 we have seen that in the case $z_j \in \bigcup_{k=1}^{g}[E_{2k-2}, E_{2k}] \cup [E_{2g}, \infty)$ we can reduce everything to a Riemann–Hilbert problem for $m^3(p)$ such that the jumps are "$\mathbb{I}$ + exponentially decaying" except in small neighborhoods of the stationary phase points z_j and z_j^*.

Let us start by showing how to decouple the Riemann–Hilbert problem for $m^3(p)$ into simpler parts such that everything is reduced to finding the solution of these simpler parts.

6.1. Decoupling

Let us denote by $\Sigma^C(z_j)$ and $\Sigma^C(z_j^*)$ the parts of Σ^3 inside a small neighborhood of z_j and z_j^*, respectively. The next theorem will enable us to get the solution of our original problem by solving the two Riemann–Hilbert problems on the small crosses $\Sigma^C(z_j)$ and $\Sigma^C(z_j^*)$ separately.

Theorem 6.1 (Decoupling)**.** *Consider the Riemann–Hilbert problem*

$$\begin{aligned} m_+(p) &= m_-(p)J(p), \quad p \in \Sigma, \\ m(p_\infty) &= \begin{pmatrix} 1 & 1 \end{pmatrix} \end{aligned} \tag{6.1}$$

and let $0 < \beta < \alpha \le 2\beta$, $\rho(t) \to \infty$, and some points $z_k \in \Sigma$, $k = 1, \dots, n$, be given.

Suppose each of the points z_k, $k = 1, \dots, n$, has a neighborhood $D_k \subset \mathcal{K}_g$, which should be sufficiently small such that both the L^2 and L^∞ norms of J are $O(t^{-\alpha})$ away from these neighborhoods. Moreover, suppose that the solution of the problem with jump $J(p)$ restricted to a neighborhood $\tilde{D}_k \subset D_k$ has a solution given by

$$M_k(p) = \mathbb{I} + \frac{1}{\rho(t)^\beta} \frac{M_k}{z - z_k} + O\big(\rho(t)^{-\alpha}\big), \quad p \in \mathcal{K}_g \setminus \tilde{D}_k. \tag{6.2}$$

Then the solution $m(p)$ is given by

$$m(p) = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{\rho(t)^\beta} \begin{pmatrix} 1 & 1 \end{pmatrix} \sum_{k=1}^{n} M_k \underline{\Omega}_p^{\hat{\underline{\nu}}}(z_k) + O\big(\rho(t)^{-\alpha}\big), \tag{6.3}$$

where the error term depends on the distance of p to Σ.

Proof. To prove the theorem we will make use of the theory developed in Appendix A with the Cauchy kernel $\underline{\Omega}_p^{\hat{\underline{\nu}}}$ defined in (A.8). We make the assumption that $m(p)$ exists. Introduce $\tilde{m}(p)$ by

$$\tilde{m} = \begin{cases} m(p)M_k(p)^{-1}, & p \in \overline{D}_k, \\ m(p), & \text{else}, \end{cases} \tag{6.4}$$

where $\overline{D}_k$ denotes the closure of D_k. The Riemann–Hilbert problem for $\tilde{m}(p)$ has jumps given by

$$\tilde{J}(p) = \begin{cases} M_k(p)^{-1}, & p \in \partial D_k, \\ M_k(p)J(p)M_k(p)^{-1}, & p \in \Sigma \cap (D_k^\circ \setminus \overline{\tilde{D}}_k), \\ \mathbb{I}, & p \in \Sigma \cap \tilde{D}_k^\circ, \\ J(p), & \text{else}, \end{cases} \tag{6.5}$$

where ∂D_k denotes the boundary of D_k, D_k° and $\tilde{D}_k^\circ$ the interior of D_k and $\tilde{D}_k$, respectively, and $\overline{\tilde{D}}_k$ the closure of $\tilde{D}_k$. By assumption the jumps are $O\big(\rho(t)^{-\beta}\big)$ on the boundary ∂D_k and even $O\big(\rho(t)^{-\alpha}\big)$ on the rest (both in L^2 and L^∞ norms). In particular as in Lemma B.4 we get

$$\|\tilde{\mu} - \begin{pmatrix} 1 & 1 \end{pmatrix}\|_2 = O\big(\rho(t)^{-\beta}\big).$$

Hence we deduce

$$\begin{aligned} m(p) &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{\tilde{\Sigma}} \tilde{\mu}\tilde{w}\underline{\Omega}_p^{\hat{\underline{\nu}}} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \sum_{k=1}^{n} \int_{\partial D_k} \tilde{\mu}\big(M_k(p)^{-1} - \mathbb{I}\big)\underline{\Omega}_p^{\hat{\underline{\nu}}} + O\big(\rho(t)^{-\alpha}\big) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2\pi\mathrm{i}}\rho(t)^{-\beta} \begin{pmatrix} 1 & 1 \end{pmatrix} \sum_{k=1}^{n} M_k \int_{\partial D_k} \frac{1}{\pi - z_k}\underline{\Omega}_p^{\hat{\underline{\nu}}} + O\big(\rho(t)^{-\alpha}\big) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \rho(t)^{-\beta} \begin{pmatrix} 1 & 1 \end{pmatrix} \sum_{k=1}^{n} M_k\underline{\Omega}_p^{\hat{\underline{\nu}}}(z_k) + O\big(\rho(t)^{-\alpha}\big), \end{aligned}$$

which finishes the proof. □

6.2. The long-time asymptotics inside the oscillatory region

In Chapter 5 we have reduced everything to the solution of the Riemann–Hilbert problem

$$\begin{aligned} & m_+^3(p) = m_-^3(p)J^3(p), \\ & (m_1^3) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)^*}, \quad (m_2^3) \geq -\mathcal{D}_{\underline{\hat{\nu}}(x,t)}, \\ & m^3(p^*) = m^3(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & m^3(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{aligned}$$

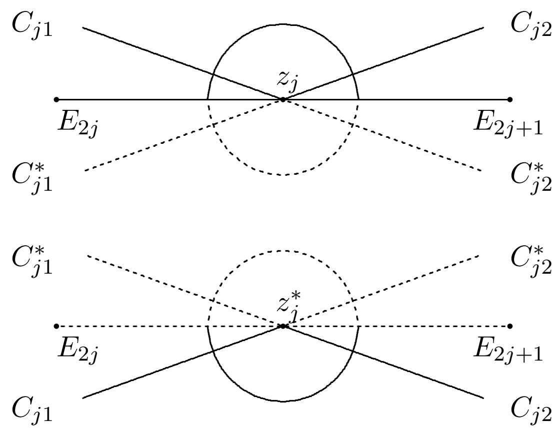

Figure 6.1.: The small cross containing the stationary phase point z_j and its flipping image containing z_j^*. Views from the top and bottom sheet. Dotted curves lie in the bottom sheet. [Kamvissis08]

where the jump matrix J^3 is given by (5.87). We have performed a deformation in such a way that the jumps J^3 on the oriented paths C_k, C_k^* for $k \neq j$ are of the form "$\mathbb{I}$ + exponentially small" asymptotically as $t \to \infty$. The same is true for the oriented paths C_{j1}, C_{j2}, C_{j1}^*, C_{j2}^* at least away from the stationary phase points z_j, z_j^*. On these paths, and in particular near the stationary phase points (cf. Figure 6.1, which is taken from [Kamvissis08]), the jumps read

$$
\begin{aligned}
J^3(p) = \tilde{B}_+(p) &= \begin{pmatrix} 1 & -\frac{T(p,x,t)}{T(p^*,x,t)}\frac{R(p^*)\Theta(p^*)}{1-R(p^*)R(p)}\mathrm{e}^{-t\,\phi(p)} \\ 0 & 1 \end{pmatrix}, \quad p \in C_{j1},\\
J^3(p) = \tilde{B}_-^{-1}(p) &= \begin{pmatrix} 1 & 0 \\ \frac{T(p^*,x,t)}{T(p,x,t)}\frac{R(p)\Theta(p)}{1-R(p^*)R(p)}\mathrm{e}^{t\,\phi(p)} & 1 \end{pmatrix}, \quad p \in C_{j1}^*,\\
J^3(p) = \tilde{b}_+(p) &= \begin{pmatrix} 1 & 0 \\ \frac{T(p^*,x,t)}{T(p,x,t)}R(p)\Theta(p)\mathrm{e}^{t\,\phi(p)} & 1 \end{pmatrix}, \quad p \in C_{j2},\\
J^3(p) = \tilde{b}_-^{-1}(p) &= \begin{pmatrix} 1 & -\frac{T(p,x,t)}{T(p^*,x,t)}R(p^*)\Theta(p^*)\mathrm{e}^{-t\,\phi(p)} \\ 0 & 1 \end{pmatrix}, \quad p \in C_{j2}^*.
\end{aligned}
\tag{6.6}
$$

To reduce our Riemann–Hilbert problem to the one corresponding to the two crosses we proceed as follows: We take a small disc D around $z_j(x/t)$ and project it to the complex plane using the canonical projection π defined in (2.29). Now consider the (holomorphic) Riemann–Hilbert problem in the complex plane with the very jump obtained by projection and normalize it to be $\mathbb{I}$ near ∞.

Before one can apply Appendix B to our problem one has to make the following change of coordinates

$$\xi = \sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z - z_j), \qquad z = z_j + \xi\sqrt{\frac{\mathrm{i}}{\phi''(z_j)}}, \tag{6.7}$$

such that the phase reads

$$\phi(z) = \phi(z_j) + \frac{\mathrm{i}}{2}\xi^2 + O(\xi^3). \tag{6.8}$$

Note that by (2.6) we have

$$\frac{\phi''(z_j)}{\mathrm{i}} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{R_{2g+1}^{1/2}(z_j)} > 0. \tag{6.9}$$

The corresponding Riemann–Hilbert problem will be solved in Appendix B. To apply Theorem B.1 we need the behavior of the jump matrix J^3, that is, the behavior of $T(p,x,t)$ near the stationary phase points z_j and z_j^*.

The following Lemma gives more information on the singularities of $T(p,x,t)$ near the stationary phase points z_j, $j = 0,\dots,g$ and the band edges E_j, $j = 0,\dots,2g+1$ (setting $E_{2g+1} = \infty$).

Lemma 6.2. *For p near a stationary phase point z_j or z_j^* (not equal to a band edge) we have*

$$T(p,x,t) = (z - z_j)^{\pm \mathrm{i}\nu} e^{\pm}(z), \quad p = (z, \pm), \tag{6.10}$$

where $e^{\pm}(z)$ has continuous limits near z_j and

$$\nu = -\frac{1}{2\pi}\log(1 - |R(z_j)|^2) > 0. \tag{6.11}$$

Here $(z - z_j)^{\pm \mathrm{i}\nu} = \exp(\pm \mathrm{i}\nu \log(z - z_j))$, where the branch cut of the logarithm is along the negative real axis.

For p near a band edge $E_k \in C(x/t)$ we have

$$T(p,x,t) = T^{\pm 1}(z)\tilde{e}^{\pm}(z), \quad p = (z, \pm), \tag{6.12}$$

where $\tilde{e}^{\pm}(z)$ is holomorphic near E_k if none of the ν_j is equal to E_k and $\tilde{e}_{\pm}(z)$ has a first order pole at $E_k = \nu_j$ else.

Proof. By factorizing the jump according to $1 - |R(p)|^2 = (1 - |R(z_j)|^2)\frac{1-|R(p)|^2}{1-|R(z_j)|^2}$ we can rewrite (5.25) in the following way

$$\begin{aligned} T(p,x,t) = &\exp\Big(\mathrm{i}\nu \int_{C(x/t)} \omega_{p\,p_\infty}\Big) \frac{\theta\big(\underline{z}(p_\infty,x,t) + \underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty,x,t)\big)} \frac{\theta\big(\underline{z}(p,x,t)\big)}{\theta\big(\underline{z}(p,x,t) + \underline{\delta}(x/t)\big)} \cdot \\ &\cdot \exp\Big(-\sum_{\rho_k < \zeta(x/t)} \int_{E(\rho_k)}^{\rho_k} \omega_{p\,p^*} + \frac{1}{2\pi\mathrm{i}}\int_{C(x/t)} \log\Big(\frac{1 - |R|^2}{1 - |R(z_j)|^2}\Big)\omega_{p\,p_\infty}\Big), \end{aligned} \tag{6.13}$$

where ν is defined in (6.11). Now consider the Abelian differential $\omega_{p\,p^*}$ for $p \in \mathcal{K}_g\backslash\{p_\infty\}$ which is explicitly given by the formula (5.22). We have

$$\frac{1}{2}\int_{C(x/t)} \omega_{p\,p^*} = \pm\log(z - z_j) \pm \alpha(z_j) + O(z - z_j), \quad p = (z, \pm). \tag{6.14}$$

Then from (5.37) we deduce

$$\int_{C(x/t)} \omega_{p\,p_\infty} = \pm\log(z - z_j) \pm \alpha(z_j) + O(z - z_j), \quad p = (z, \pm), \tag{6.15}$$

from which the first claim follows.

For the second claim note that the function

$$t(p) = \begin{cases} T(z), & p = (z, +) \in \Pi_+, \\ T(z)^{-1}, & p = (z, -) \in \Pi_-, \end{cases}$$

satisfies the following (holomorphic) Riemann–Hilbert problem

$$\begin{aligned} & t_+(p) = t_-(p)(1 - |R(p)|^2), \quad p \in \Sigma, \\ & t(p_\infty) = 1. \end{aligned}$$

Thus $T(p, x, t)/t(p)$ has no jump along $C(x/t)$ and is therefore holomorphic near $C(x/t)$ away from band edges $E_k = \nu_j$ (where there is a simple pole) by the Schwarz reflection principle. □

Moreover,

Lemma 6.3. *We have*

$$e^{\pm}(z) = \overline{e^{\mp}(z)}, \qquad p = (z, \pm) \in \Sigma \setminus C(x/t) \tag{6.16}$$

and

$$\begin{aligned} e^+(z_j) = & \exp\big(\mathrm{i}\nu\alpha(z_j)\big)\frac{\theta\big(\underline{z}(p_\infty, x, t) + \underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(p_\infty, x, t)\big)}\frac{\theta\big(\underline{z}(z_j, x, t)\big)}{\theta\big(\underline{z}(z_j, x, t) + \underline{\delta}(x/t)\big)} \cdot \\ & \cdot \exp\Big(- \sum_{\rho_k < \zeta(x/t)} \int_{E(\rho_k)}^{\rho_k} \omega_{z_j\,z_j^*} + \frac{1}{4\pi\mathrm{i}}\int_{C(x/t)} \log\Big(\frac{1 - |R|^2}{1 - |R(z_j)|^2}\Big)\omega_{z_j\,z_j^*}\Big), \end{aligned} \tag{6.17}$$

where

$$\alpha(z_j) = \lim_{p\to z_j}\Big(\frac{1}{2}\int_{C(x/t)} \omega_{p\,p^*} - \log\big(\pi(p) - z_j\big)\Big). \tag{6.18}$$

Here $\alpha(z_j) \in \mathbb{R}$ and $\omega_{p\,p^*}$ is real on $C(x/t)$.

Proof. The first claim follows from the fact that

$$T(p^*, x, t) = T(\overline{p}, x, t) = \overline{T(p, x, t)} \quad \text{for } p \in \Sigma \setminus C(x/t).$$

The second claim clearly follows from (6.13) and (5.37). □

By Lemma 6.2 one deduces that near the stationary phase points the jumps are given by

$$\begin{aligned}
\hat{B}_{+} &= \begin{pmatrix} 1 & -\left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{2\mathrm{i}\nu} \frac{\overline{r}}{1-|r|^2}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in L_{j1}, \\
\hat{B}_{-}^{-1} &= \begin{pmatrix} 1 & 0 \\ \left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{-2\mathrm{i}\nu} \frac{r}{1-|r|^2}\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in L_{j1}^{*}, \\
\hat{b}_{+} &= \begin{pmatrix} 1 & 0 \\ \left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{-2\mathrm{i}\nu} r\mathrm{e}^{t\,\phi} & 1 \end{pmatrix}, \quad p \in L_{j2}, \\
\hat{b}_{-}^{-1} &= \begin{pmatrix} 1 & -\left(\sqrt{\frac{\phi''(z_j)}{\mathrm{i}}}(z-z_j)\right)^{2\mathrm{i}\nu} \overline{r}\mathrm{e}^{-t\,\phi} \\ 0 & 1 \end{pmatrix}, \quad p \in L_{j2}^{*},
\end{aligned} \tag{6.19}$$

where (cf. (6.10))

$$r = R(z_j)\Theta(z_j,x,t)\frac{\overline{e^{+}(z_j)}}{e^{+}(z_j)}\left(\frac{\phi''(z_j)}{\mathrm{i}}\right)^{\mathrm{i}\nu} \tag{6.20}$$

and L_{j1}, L_{j2}, L_{j1}^{*}, L_{j2}^{*} are defined as indicated in Figure 6.1. The error terms will satisfy appropriate Hölder estimates, that is

$$\|\hat{B}_{+}(p) - \hat{B}_{+}(p)\| \le C|z-z_j|^{\alpha}, \qquad p=(z,+) \in C_{j1}, \tag{6.21}$$

for any $\alpha < 1$ and similarly for the other matrices. Thus the assumptions of Theorem B.1 are satisfied and we can conclude that the solution on $\pi\big(\Sigma^{C}(z_j)\big)$ is of the form

$$\begin{aligned}
M(z) &= \mathbb{I} + \frac{1}{\xi}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\beta(t) \\ \overline{\beta(t)} & 0 \end{pmatrix} + O(t^{-\alpha}) = \\
&= \mathbb{I} + \frac{M_0}{z-z_j}\frac{1}{t^{1/2}} + O(t^{-\alpha}),
\end{aligned} \tag{6.22}$$

where

$$M_0 = \mathrm{i}\sqrt{\frac{\mathrm{i}}{\phi''(z_j)}}\begin{pmatrix} 0 & -\beta(t) \\ \overline{\beta(t)} & 0 \end{pmatrix}, \tag{6.23}$$

$$\beta(t) = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(r)+\arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-t\phi(z_j)}t^{-\mathrm{i}\nu}, \tag{6.24}$$

and $1/2 < \alpha < 1$. Now we lift this solution in the complex plane back to the small disc D on the Riemann-surface $\mathcal{K}_g$ by setting

$$M(p) = \begin{cases} M(z), & p \in D, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & p \in D^{*}. \end{cases} \tag{6.25}$$

Thus we conclude that the solution on $\Sigma^C(z_j)$ is given by

$$M^C(p) = \mathbb{I} + \frac{1}{t^{1/2}} \frac{M_0}{z - z_j} + O(t^{-\alpha}), \quad p = (z, +), \tag{6.26}$$

and the one on $\Sigma^C(z_j^*)$ reads

$$\tilde{M}^C(p) = \mathbb{I} + \frac{1}{t^{1/2}} \frac{\overline{M_0}}{z - z_j} + O(t^{-\alpha}), \quad p = (z, -). \tag{6.27}$$

Then as in the proof of Theorem 6.1 we define

$$m^4(p) = \begin{cases} m^3(p) M^C(p)^{-1}, & p \in D, \\ m^3(p) \tilde{M}^C(p)^{-1}, & p \in D^*, \\ m^3(p), & \text{else} \end{cases} \tag{6.28}$$

and finally get

$$\begin{aligned} m^4(p) &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 \end{pmatrix} M_0}{t^{1/2}} \underline{\Omega}_p^{\hat{\underline{\nu}}}(z_j) - \frac{\begin{pmatrix} 1 & 1 \end{pmatrix} \overline{M_0}}{t^{1/2}} \underline{\Omega}_p^{\hat{\underline{\nu}}}(z_j^*) + O(t^{-\alpha}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \sqrt{\frac{\mathrm{i}}{\phi''(z_j) t}} \times \\ &\times \Big(\mathrm{i}\overline{\beta} \Omega_p^{\hat{\underline{\nu}}^*, p_\infty}(z_j) - \mathrm{i}\beta \Omega_p^{\hat{\underline{\nu}}^*, p_\infty}(z_j^*) \quad -\mathrm{i}\beta \Omega_p^{\hat{\underline{\nu}}, p_\infty}(z_j) + \mathrm{i}\overline{\beta} \Omega_p^{\hat{\underline{\nu}}, p_\infty}(z_j^*) \Big) \\ &+ O(t^{-\alpha}), \end{aligned} \tag{6.29}$$

where the Cauchy kernel $\Omega_p^{\hat{\underline{\nu}}, p_\infty}(z_j)$ is defined as in Appendix A, namely by

$$\Omega_p^{\hat{\underline{\nu}}, p_\infty}(z_j) = \omega_{p\, p_\infty}(z_j) + \sum_{k=1}^{g} \sum_{\ell=1}^{g} c_{k\ell}(\hat{\underline{\nu}}) \int_{p_\infty}^{p} \omega_{\hat{\nu}_\ell, 0} \zeta_k(z_j). \tag{6.30}$$

Since we need the asymptotic expansions around p_∞ we note

Lemma 6.4. *We have*

$$\Omega_p^{\hat{\underline{\nu}}, p_\infty}(z_j) = \Lambda_1^{\hat{\underline{\nu}}}(z_j) \zeta + \Lambda_2^{\hat{\underline{\nu}}}(z_j) \zeta^2 + O\big(\zeta^3\big) \tag{6.31}$$

for $\zeta = z^{-1/2}$ being the local chart near p_∞ and

$$\Lambda_1^{\hat{\underline{\nu}}}(z_j) = \omega_{p_\infty, 0}(z_j) - \sum_{k=1}^{g} \sum_{\ell=1}^{g} c_{k\ell}(\hat{\underline{\nu}}) \alpha_{g-1}(\hat{\nu}_\ell) \zeta_k(z_j), \tag{6.32}$$

$$\Lambda_2^{\hat{\underline{\nu}}}(z_j) = \omega_{p_\infty, 1}(z_j) - \frac{1}{2} \sum_{k=1}^{g} \sum_{\ell=1}^{g} c_{k\ell}(\hat{\underline{\nu}}) \zeta_k(z_j), \tag{6.33}$$

where $\omega_{q,k}$, $k = 0, 1, \dots$, is an Abelian differential of the second kind with a single pole of order $k + 2$ at q.

Remark 6.5. *The Abelian differential appearing in the previous lemma is explicitly given by*

$$\omega_{\hat{\nu},0} = \frac{R_{2g+1}^{1/2} + R_{2g+1}^{1/2}(\hat{\nu}) + \frac{R'_{2g+1}(\hat{\nu})}{2R_{2g+1}^{1/2}(\hat{\nu})}(\pi-\nu) + P_{\hat{\nu},0}\cdot(\pi-\nu)^2}{2(\pi-\nu)^2 R_{2g+1}^{1/2}} d\pi, \tag{6.34}$$

with $P_{\hat{\nu},0}$ a polynomial of degree $g-1$ which has to be determined from the normalization. We will use the notation

$$P_{\hat{\nu},0}(z) = \sum_{j=0}^{g-1} \alpha_j(\hat{\nu}) z^j. \tag{6.35}$$

Concerning the Abelian differential $\omega_{p_\infty,0}$ we refer to Lemma 3.3. The differential $\omega_{p_\infty,1}$ is given by

$$\omega_{p_\infty,1} = \Big(-\frac{R_{2g+1}^{1/2}}{2} + P_{p_\infty,1}\Big)\frac{d\pi}{R_{2g+1}^{1/2}}, \tag{6.36}$$

where $P_{p_\infty,1}$ is a polynomial of degree $g-1$ which has to be determined by the vanishing a_j-periods as usual.

Proof. Note that using the local coordinate $\zeta = z^{-1/2}$ near $p_\infty = (\infty,\infty)$ we have (cf. [Teschl07, Theorem 4.1])

$$\omega_{pp_\infty} = \sum_{k=1}^{g} \zeta^k \omega_{p_\infty,k-1} = \omega_{p_\infty,0}\zeta + \omega_{p_\infty,1}\zeta^2 + O(\zeta^3).$$

Moreover, we have

$$\int_{p_\infty}^{p} \omega_{\hat{\nu}_\ell,0} = \omega_{\hat{\nu}_\ell,0}(\zeta)\big|_{\zeta=0}\zeta + \Big(\frac{d}{d\zeta}\omega_{\hat{\nu}_\ell,0}\Big)\Big|_{\zeta=0}\frac{\zeta^2}{2} + O(\zeta^3).$$

Thus we get the following expansion of $\Omega_p^{\underline{\hat{\nu}},p_\infty}(z_j)$ for p near p_∞:

$$\begin{aligned}\Omega_p^{\underline{\hat{\nu}},p_\infty}(z_j) =& \omega_{p_\infty,0}\zeta + \omega_{p_\infty,1}\zeta^2 \\ &+ \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat{\nu}})\Big[\omega_{\hat{\nu}_\ell,0}(\zeta)\big|_{\zeta=0}\zeta + \Big(\frac{d}{d\zeta}\omega_{\hat{\nu}_\ell,0}\Big)\Big|_{\zeta=0}\frac{\zeta^2}{2}\Big]\zeta_k(z_j) + O(\zeta^3) \\ =& \Big[\omega_{p_\infty,0} + \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat{\nu}})\omega_{\hat{\nu}_\ell,0}(\zeta)\big|_{\zeta=0}\zeta_k(z_j)\Big]\zeta \\ &+ \Big[\omega_{p_\infty,1} + \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat{\nu}})\frac{1}{2}\Big(\frac{d}{d\zeta}\omega_{\hat{\nu}_\ell,0}\Big)\Big|_{\zeta=0}\zeta_k(z_j)\Big]\zeta^2 + O(\zeta^3)\end{aligned}$$

and therefore

$$\Lambda_1^{\underline{\hat\nu}}(z_j) = \omega_{p_\infty,0}(z_j) + \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat\nu})\omega_{\hat\nu_\ell,0}(\zeta)\big|_{\zeta=0}\zeta_k(z_j), \tag{6.37}$$

$$\Lambda_2^{\underline{\hat\nu}}(z_j) = \omega_{p_\infty,1}(z_j) + \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat\nu})\frac{1}{2}\Big(\frac{d}{d\zeta}\omega_{\hat\nu_\ell,0}\Big)\Big|_{\zeta=0}\zeta_k(z_j). \tag{6.38}$$

For the purpose of deriving explicit expressions for $\omega_{\hat\nu_\ell,0}(\zeta)\big|_{\zeta=0}$ and $\big(\frac{d}{d\zeta}\omega_{\hat\nu_\ell,0}\big)\big|_{\zeta=0}$ we have to expand the Abelian differential $\omega_{\hat\nu_\ell,0}$ near $\zeta=0$:

$$\begin{aligned}
\omega_{\hat\nu,0}(\zeta) =& \frac{R_{2g+1}^{1/2}(\zeta^{-2}) + R_{2g+1}^{1/2}(\hat\nu) + \frac{R'_{2g+1}(\hat\nu)}{2R_{2g+1}^{1/2}(\hat\nu)}(\zeta^{-2}-\nu) + P_{\hat\nu,0}\cdot(\zeta^{-2}-\nu)^2}{2(\zeta^{-2}-\nu)^2 R_{2g+1}^{1/2}(\zeta^{-2})}(-2\zeta^{-3})d\zeta = \\
=&\Big(\frac{1}{2\zeta^{-4}(1-\zeta^2\nu)^2} + \frac{R_{2g+1}^{1/2}(\hat\nu)}{2\zeta^{-2g-5}(1-\zeta^2\nu)^2\big(1-\frac{1}{2}(\sum_j E_j)\zeta^2+O(\zeta^4)\big)} + \\
&+\frac{R'_{2g+1}(\hat\nu)}{4R_{2g+1}^{1/2}(\hat\nu)\zeta^{-2g-3}(1-\zeta^2\nu)\big(1-\frac{1}{2}(\sum_j E_j)\zeta^2+O(\zeta^4)\big)} + \\
&+\frac{\alpha_{g-1}(\hat\nu)\zeta^{-2g+2}+O(\zeta^{-2g+4})}{2\zeta^{-2g-1}\big(1-\frac{1}{2}(\sum_j E_j)\zeta^2+O(\zeta^4)\big)}\Big)(-2\zeta^{-3})d\zeta = \\
=&\Big(-\zeta(1+2\zeta^2\nu+O(\zeta^4)) - \zeta^{2g+2}R_{2g+1}^{1/2}(\hat\nu)\big(1+(\frac{1}{2}(\sum_j E_j)+2\nu)\zeta^2+O(\zeta^4)\big) \\
&-\frac{R'_{2g+1}(\hat\nu)}{2R_{2g+1}^{1/2}(\hat\nu)}\zeta^{2g}\big(1+(\frac{1}{2}(\sum_j E_j)+\nu)\zeta^2+O(\zeta^4)\big) \\
&-\zeta^{2g-2}\big(\alpha_{g-1}(\hat\nu)\zeta^{-2g+2}+O(\zeta^{-2g+4})\big)\big(1+\frac{1}{2}(\sum_j E_j)\zeta^2+O(\zeta^4)\big)\Big)d\zeta
\end{aligned}$$

and thus for $g \ge 1$ we get

$$\omega_{\hat\nu,0}(\zeta) = -\alpha_{g-1}(\hat\nu) - \zeta + O(\zeta^2). \tag{6.39}$$

This expansion is only valid for the case $g \ge 1$, in the case $g=0$ we do not need the expansion of the Abelian differential $\omega_{\hat\nu,0}$, since the sums in (6.37) and (6.38) vanish anyway. Finally, we deduce

$$\omega_{\hat\nu_\ell,0}(\zeta)\big|_{\zeta=0} = -\alpha_{g-1}(\hat\nu_\ell), \qquad \big(\frac{d}{d\zeta}\omega_{\hat\nu_\ell,0}\big)\big|_{\zeta=0} = -1$$

and we indeed derive the claimed formulas for $\Lambda_1^{\underline{\hat\nu}}(z_j)$ and $\Lambda_2^{\underline{\hat\nu}}(z_j)$. □

Note that the following relations are valid

$$\begin{aligned}
&\omega_{p_\infty,0}(z_j^*) = -\omega_{p_\infty,0}(z_j), \\
&\omega_{p_\infty,1}(z_j^*) + \omega_{p_\infty,1}(z_j) = -1,
\end{aligned} \tag{6.40}$$

and

$$c_{k\ell}(\underline{\hat{\nu}}^*) = -c_{k\ell}(\underline{\hat{\nu}}), \qquad \zeta_k(z_j^*) = -\zeta_k(z_j). \tag{6.41}$$

Moreover, the coefficients $\alpha_j(\hat{\nu})$, $j = 0, \dots, g-1$ of the polynomial $P_{\hat{\nu},0}$ fulfill the relation

$$\alpha_j(\hat{\nu}^*) = -\alpha_j(\hat{\nu}), \qquad j = 0, \dots, g-1. \tag{6.42}$$

This can be checked easily in the following way: the coefficients $\alpha_j(\hat{\nu})$, $j = 0, \dots, g-1$ are determined by the normalization of the Abelian differential $\omega_{\hat{\nu},0}$, i.e.

$$\int_{a_j} \omega_{\hat{\nu},0} = 0, \qquad j = 0, \dots, g-1.$$

That means

$$0 = \int_{a_j} \omega_{\hat{\nu},0} = \underbrace{2\int_{E_{2j}}^{E_{2j+1}} \frac{R_{2g+1}^{1/2}(\hat{\nu}) + \frac{R'_{2g+1}(\nu)}{2R_{2g+1}^{1/2}(\hat{\nu})}(z-\nu)}{2(z-\nu)^2 R_{2g+1}^{1/2}} dz}_{=:c_j(\hat{\nu})} + \int_{a_j} \frac{P_{\hat{\nu},0}}{2R_{2g+1}^{1/2}} d\pi,$$

and thus

$$\int_{a_j} \frac{P_{\hat{\nu},0}}{2R_{2g+1}^{1/2}} d\pi = -c_j(\hat{\nu}),$$

$$\frac{P_{\hat{\nu},0}}{2R_{2g+1}^{1/2}} d\pi = -\sum_{k=1}^{g} c_k(\hat{\nu})\zeta_k.$$

Since we have $c_j(\hat{\nu}^*) = -c_j(\hat{\nu})$ it follows $P_{\hat{\nu}^*,0} = -P_{\hat{\nu},0}$, that is, the coefficients of the polynomial indeed have to fulfill (6.42).

Finally, the next two theorems, i.e., Theorem 6.6 and Theorem 6.7, establish the long-time asymptotic behavior of the perturbed solution $V(x,t)$ in the oscillatory region. More precisely, in this region of the (x,t)-plane the perturbed KdV solution $V(x,t)$ approaches a modulated solution with a decay rate of $O(t^{-1/2})$.

Theorem 6.6. *Assume*

$$\int_{-\infty}^{+\infty} (1+|x|)^7 (|V(x,t) - V_q(x,t)|) dx < \infty. \tag{6.43}$$

and let D_j be the sector $D_j = \{(x,t) : z_j(x/t) \in [E_{2j}+\varepsilon, E_{2j+1}-\varepsilon]\}$ for some $\varepsilon > 0$. Then the asymptotic is given by

$$\int_x^{+\infty} (V - V_l)(y,t) dy = 4\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}} \mathrm{Re}\big(\beta(x,t)\big) \Lambda_1^{\underline{\hat{\nu}}}(z_j) + O(t^{-\alpha}), \tag{6.44}$$

for any $1/2 < \alpha < 1$ uniformly in D_j as $t \to \infty$. Here

$$\frac{\phi''(z_j)}{\mathrm{i}} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{R_{2g+1}^{1/2}(z_j)} > 0 \tag{6.45}$$

(where $\phi(p,x/t)$ is the phase function defined in (4.47) and $R_{2g+1}^{1/2}(z)$ the square root of the underlying Riemann surface $\mathcal{K}_g$),

$$\Lambda_1^{\underline{\hat\nu}}(z_j) = \omega_{p_\infty,0}(z_j) - \sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat\nu})\alpha_{g-1}(\hat\nu_\ell)\zeta_k(z_j), \tag{6.46}$$

with $\omega_{p_\infty,0}$ an Abelian differential of the second kind with a second order pole at p_∞ (cf. Lemma 3.3), $c_{k\ell}\big(\underline{\hat\nu}(x,t)\big)$ some constants defined in (A.6),

$$\begin{aligned}\beta(x,t) =& \sqrt{\nu}\mathrm{e}^{\mathrm{i}\left(\pi/4-\arg(R(z_j))+\arg(\Gamma(\mathrm{i}\nu))+2\nu\alpha(z_j)\right)}\Big(\frac{\phi''(z_j)}{\mathrm{i}}\Big)^{-\mathrm{i}\nu}\mathrm{e}^{-t\phi(z_j)}t^{-\mathrm{i}\nu}\cdot\\ &\cdot\frac{\theta\big(\underline{z}(z_j,0,0)\big)}{\theta\big(\underline{z}(z_j,x,t)+\underline{\delta}(x/t)\big)}\frac{\theta\big(\underline{z}(z_j^*,x,t)+\underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(z_j^*,0,0)\big)}\times\\ &\cdot\exp\Big(-\sum_{\rho_k<\zeta(x/t)}\int_{E(\rho_k)}^{\rho_k}\omega_{z_j\, z_j^*}+\frac{1}{2\pi\mathrm{i}}\int_{C(x/t)}\log\Big(\frac{1-|R|^2}{1-|R(z_j)|^2}\Big)\omega_{z_j\, z_j^*}\Big),\end{aligned} \tag{6.47}$$

where $\Gamma(z)$ is the gamma function,

$$\nu = -\frac{1}{2\pi}\log\big(1-|R(z_j)|^2\big) > 0, \tag{6.48}$$

and $\alpha(z_j)$ is a constant defined in (6.18).

Proof. The asymptotics can be read off by using

$$m(p) = m^4(p)\begin{pmatrix}\frac{1}{T(p^*,x,t)} & 0\\ 0 & \frac{1}{T(p,x,t)}\end{pmatrix} \tag{6.49}$$

for p near p_∞ and comparing with (4.53). From that one deduces

$$\begin{aligned}m_2(p) =& m_2^4(p)T(p)^{-1} =\\ =& 1+\Big(\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\big(\mathrm{i}\beta\Lambda_1^{\underline{\hat\nu}}(z_j)-\mathrm{i}\overline{\beta}\Lambda_1^{\underline{\hat\nu}}(z_j^*)\big)-T_1(x,t)+O(t^{-\alpha})\Big)\frac{1}{\sqrt{z}}+O(z^{-1}),\end{aligned}$$

where we have used (6.29), (6.31) and (5.39). Comparing this asymptotic expansion with (4.53) yields

$$\int_x^{+\infty}(V-V_q)(y)dy = 2\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\big(\beta\Lambda_1^{\underline{\hat\nu}}(z_j)-\overline{\beta}\Lambda_1^{\underline{\hat\nu}}(z_j^*)\big)+2\mathrm{i}T_1(x,t)+O(t^{-\alpha}).$$

Invoking (6.40), (6.41) and (6.42) one gets

$$\begin{aligned}\Lambda_1^{\underline{\hat\nu}}(z_j^*) &= -\Lambda_1^{\underline{\hat\nu}}(z_j),\\ \Lambda_1^{\underline{\hat\nu}^*}(z_j) &= \Lambda_1^{\underline{\hat\nu}}(z_j),\end{aligned} \tag{6.50}$$

and therefore

$$\int_x^\infty (V - V_q)(y,t)dy = 4\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\mathrm{Re}\big(\beta(x,t)\big)\Lambda_1^{\underline{\hat{\nu}}}(x,t) + 2\mathrm{i}T_1(x,t) + O(t^{-\alpha}). \quad (6.51)$$

Finally, using the definition of the limiting solution (5.93) proves the claim. Note that one obtains the same result if one compares the expressions for the component m_1. □

Theorem 6.7. *Assume (6.43) and let D_j be the sector $D_j = \{(x,t) : z_j(x/t) \in [E_{2j} + \varepsilon, E_{2j+1} - \varepsilon]\}$ for some $\varepsilon > 0$. Then the asymptotic is given by*

$$(V - V_l)(x,t) = 4\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[\mathrm{Im}\big(\beta(x,t)\big) - \mathrm{i}\mathrm{Re}\big(\beta(x,t)\big)\sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\underline{\hat{\nu}})\zeta_k(z_j)\Big] + O(t^{-\alpha}) \quad (6.52)$$

for any $1/2 < \alpha < 1$ uniformly in D_j as $t \to \infty$. Here

$$\frac{\phi''(z_j)}{\mathrm{i}} = \frac{\prod_{k=0,k\neq j}^{g}(z_j - z_k)}{R_{2g+1}^{1/2}(z_j)} > 0. \quad (6.53)$$

(where $\phi(p, x/t)$ is the phase function defined in (4.47) and $R_{2g+1}^{1/2}(z)$ the square root of the underlying Riemann surface $\mathcal{K}_g$), $c_{k\ell}\big(\underline{\hat{\nu}}(x,t)\big)$ some constants defined in (A.6),

$$\begin{aligned}\beta(x,t) =&\sqrt{\nu}\mathrm{e}^{\mathrm{i}\big(\pi/4-\arg(R(z_j))+\arg(\Gamma(\mathrm{i}\nu))+2\nu\alpha(z_j)\big)}\Big(\frac{\phi''(z_j)}{\mathrm{i}}\Big)^{-\mathrm{i}\nu}\mathrm{e}^{-t\phi(z_j)}t^{-\mathrm{i}\nu}\cdot\\ &\cdot\frac{\theta\big(\underline{z}(z_j,0,0)\big)}{\theta\big(\underline{z}(z_j,x,t)+\underline{\delta}(x/t)\big)}\frac{\theta\big(\underline{z}(z_j^*,x,t)+\underline{\delta}(x/t)\big)}{\theta\big(\underline{z}(z_j^*,0,0)\big)}\cdot\\ &\cdot\exp\Big(-\sum_{\rho_k<\zeta(x/t)}\int_{E(\rho_k)}^{\rho_k}\omega_{z_j\,z_j^*} + \frac{1}{2\pi\mathrm{i}}\int_{C(x/t)}\log\Big(\frac{1-|R|^2}{1-|R(z_j)|^2}\Big)\omega_{z_j\,z_j^*}\Big),\end{aligned} \quad (6.54)$$

where $\Gamma(z)$ is the gamma function,

$$\nu = -\frac{1}{2\pi}\log\big(1 - |R(z_j)|^2\big) > 0, \quad (6.55)$$

and $\alpha(z_j)$ is a constant defined in (6.18).

Proof. From Lemma 5.17 we have

$$m_1 \cdot m_2 = 1 + (V - V_q)(x,t)\frac{1}{2z} + o(z^{-1}). \quad (6.56)$$

On the other hand the following expansion is also valid

$$\begin{aligned}m_1\cdot m_2 =&m_1^4\cdot m_2^4\cdot T(p^*,x,t)^{-1}\cdot T(p,x,t)^{-1} =\\ =&\Big(1-\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[\big(\mathrm{i}\overline{\beta}\Lambda_2^{\underline{\hat{\nu}}^*}(z_j)-\mathrm{i}\beta\Lambda_2^{\underline{\hat{\nu}}^*}(z_j^*)\big)+\big(\mathrm{i}\overline{\beta}\Lambda_2^{\underline{\hat{\nu}}}(z_j^*)-\mathrm{i}\beta\Lambda_2^{\underline{\hat{\nu}}}(z_j)\big)\Big]\frac{1}{z}\Big)\cdot\\ &\cdot\Big(1+\big(\sum_j(\mu_j-\nu_j)\big)\frac{1}{z}+\cdots\Big)+O(t^{-\alpha}),\end{aligned} \quad (6.57)$$

where we made use of (cf. Lemma 5.6)

$$T(p^*,x,t)T(p,x,t) = \prod_{j=0}^{g} \frac{z-\mu_j}{z-\nu_j}.$$

Comparing the two expressions (6.56) and (6.57) we conclude

$$\begin{aligned}(V-V_q)(x,t) = &-2\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[\big(\mathrm{i}\overline{\beta}\Lambda_2^{\hat{\underline{\nu}}^*}(z_j) - \mathrm{i}\beta\Lambda_2^{\hat{\underline{\nu}}^*}(z_j^*)\big) + \big(\mathrm{i}\overline{\beta}\Lambda_2^{\hat{\underline{\nu}}}(z_j^*) - \mathrm{i}\beta\Lambda_2^{\hat{\underline{\nu}}}(z_j)\big)\Big]\\ &+2\sum_j(\mu_j-\nu_j) + O(t^{-\alpha})\end{aligned}$$

or

$$(V-V_l)(x,t) = -2\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[\big(\mathrm{i}\overline{\beta}\Lambda_2^{\hat{\underline{\nu}}^*}(z_j) - \mathrm{i}\beta\Lambda_2^{\hat{\underline{\nu}}^*}(z_j^*)\big) + \big(\mathrm{i}\overline{\beta}\Lambda_2^{\hat{\underline{\nu}}}(z_j^*) - \mathrm{i}\beta\Lambda_2^{\hat{\underline{\nu}}}(z_j)\big)\Big] + O(t^{-\alpha}),$$

where we have used the representations

$$\begin{aligned}V_q(x,t) &= \sum_{j=0}^{2g} E_j - 2\sum_{j=0}^{g}\mu_j(x,t),\\ V_l(x,t) &= \sum_{j=0}^{2g} E_j - 2\sum_{j=0}^{g}\nu_j(x,t).\end{aligned}$$

Hence inserting the expression for $\Lambda_2^{\hat{\underline{\nu}}}(z_j)$ and using (6.40) and (6.41) we can compute

$$\begin{aligned}(V-V_l)(x,t) &= -2\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[-(\mathrm{i}\overline{\beta}-\mathrm{i}\beta) + (\mathrm{i}\overline{\beta}+\mathrm{i}\beta)\sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\hat{\underline{\nu}})\zeta_k(z_j)\Big] + O(t^{-\alpha})\\ &= 4\sqrt{\frac{\mathrm{i}}{\phi''(z_j)t}}\Big[\mathrm{Im}(\beta) - \mathrm{i}\mathrm{Re}(\beta)\sum_{k=1}^{g}\sum_{\ell=1}^{g} c_{k\ell}(\hat{\underline{\nu}})\zeta_k(z_j)\Big] + O(t^{-\alpha}),\end{aligned}$$

which completes the proof. □

7. Analytic Approximation

This chapter is based on [Grunert09, Section 6] and [Krüger09b, Section 6] but will be quite more general in the sense that it concerns Riemann–Hilbert factorization deformations on a Riemann surface rather than in the complex plane.

Here we want to present the necessary changes in the case where the reflection coefficient does not have an analytic extension in those regions of the Riemann surface where the parts of the jump matrices are shifted to. The idea is to use an analytic approximation and to split the reflection coefficient in an analytic part plus a small rest. The analytic part will be moved to regions of the Riemann surface while the rest remains on $\Sigma = \pi^{-1}\big(\sigma(H_q)\big)$. This needs to be done in such a way that the rest is of $O(t^{-1})$ and the growth of the analytic part can be controlled by the decay of the phase.

We will use a splitting based on the following Fourier transform of the reflection coefficient

$$R(p) = \int_{-\infty}^{+\infty} F(x)\psi_q(p,x,0)dx, \tag{7.1}$$

where $\psi_q(p,x,t)$ denotes the time-dependent Baker–Akhiezer function and $F(x) = F_{+,R}(x,0,0)$ with

$$F_{+,R}(x,y,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Sigma} R(p)\psi_q(p,x,t)\psi_q(p,y,t)\frac{\mathrm{i}\prod_{j=1}^{g}(\pi(p)-\mu_j)}{2R_{2g+1}^{1/2}(p)}d\pi(p) \tag{7.2}$$

the reflection coefficient part of the Marchenko kernel (cf. [Egorova09c, eq. (3.3)]). If we make use of (4.3) the above expression for $R(p)$ is equal to

$$R(p) = \int_{-\infty}^{+\infty} F(x)\theta_q(p,x,0)\exp\big(\mathrm{i}xk(p)\big)dx. \tag{7.3}$$

Our decay assumption (4.2) implies $F(x) \in L^1(\mathbb{R})$ and the estimate (cf. [BoutetdeMonvel08, Lemma 4.2])

$$|F(x)| \le c(x)\int_{x/2}^{+\infty} |(V-V_q)(r,0)|dr, \tag{7.4}$$

where $c(x)$ is a continuous function that decreases monotonically as $x \to \infty$, implies $x^l F(x) \in L^1(0,\infty)$.

Lemma 7.1. *Suppose $F \in L^1(\mathbb{R})$, $x^l F(x) \in L^1(0,\infty)$ and let $\beta > 0$ be given. Then we can split the reflection coefficient according to*

$$R(p) = R_{a,t}(p) + R_{r,t}(p),$$

such that $R_{a,t}(p)$ is analytic for in the region $0 < \mathrm{Im}(k(p)) < \varepsilon$ and

$$|R_{a,t}(p)\mathrm{e}^{-\beta t}| = O(t^{-l}), \quad 0 < \mathrm{Im}(k(p)) < \varepsilon, \tag{7.5}$$
$$|R_{r,t}(p)| = O(t^{-l}), \quad p \in \Sigma. \tag{7.6}$$

Proof. We choose

$$R_{a,t}(p) = \int_{-K(t)}^{\infty} F(x)\theta_q(p,x,0)\exp\big(\mathrm{i}xk(p)\big)dx$$

with $K(t) = \frac{\beta_0}{\varepsilon}t$ for some positive $\beta_0 < \beta$. Then, for $0 < \mathrm{Im}(k(p)) < \varepsilon$,

$$\begin{aligned}\left|R_{a,t}(k)\mathrm{e}^{-\beta t}\right| &\le C\mathrm{e}^{-\beta t}\int_{-K(t)}^{\infty} |F(x)|\mathrm{e}^{-\mathrm{Im}(k(p))x}dx\\ &\le C\mathrm{e}^{-\beta t}\mathrm{e}^{K(t)\varepsilon}\|F\|_1 = \|F\|_1\mathrm{e}^{-(\beta-\beta_0)t},\end{aligned}$$

which proves the first claim. Similarly, for $p \in \Sigma$,

$$|R_{r,t}(k)| \le C\int_{K(t)}^{\infty}\frac{x^l|F(x)|}{x^l}dx \le C\frac{\|x^lF(x)\|_{L^1(0,\infty)}}{K(t)^l} \le \frac{\tilde{C}}{t^l}$$

□

We choose

$$\beta = \begin{cases} \min_{p\in C_k} -\mathrm{Re}(k(p)) > 0, & \pi(p) > \zeta(x,t),\\ \min_{p\in C_k} \mathrm{Re}(k(p)) > 0, & \pi(p) < \zeta(x,t).\end{cases} \tag{7.7}$$

Before we can split the jump matrices $\tilde{b}_\pm$ and $\tilde{B}_\pm$ defined in (5.67) and (5.68), respectively, by splitting the reflection coefficient according to Lemma 7.1 we make the following observation.

The matrices $\tilde{B}_\pm$ have at first sight more complicated off diagonal entries than the matrices $\tilde{b}_\pm$, but a closer look shows that they have indeed the same form. To remedy this we will rewrite $\tilde{B}_\pm$ in terms of the left rather than the right scattering data. For this purpose let us use the notation $R_r(p) \equiv R_+(p)$ for the right and $R_l(p) \equiv R_-(p)$ for the left reflection coefficient. Moreover, let $T_r(p,x,t) \equiv T(p,x,t)$ be the right and $T_l(p,x,t) \equiv T(p)/T_r(p,x,t)$ be the left partial transmission coefficient.

With this notation we have

$$J^2(p) = \begin{cases}\tilde{b}_-(p)^{-1}\tilde{b}_+(p), & \pi(p) > \zeta(x/t),\\ \tilde{B}_-(p)^{-1}\tilde{B}_+(p), & \pi(p) < \zeta(x/t),\end{cases} \tag{7.8}$$

where

$$\tilde{b}_- = \begin{pmatrix} 1 & \frac{T_r(p,x,t)}{T_r(p^*,x,t)}R_r(p^*)\Theta(p^*)\mathrm{e}^{-t\phi(p)}\\ 0 & 1\end{pmatrix}, \quad \tilde{b}_+ = \begin{pmatrix} 1 & 0\\ \frac{T_r(p^*,x,t)}{T_r(p,x,t)}R_r(p)\Theta(p)\mathrm{e}^{-t\phi(p)} & 1\end{pmatrix},$$

and

$$\tilde{B}_- = \begin{pmatrix} 1 & 0 \\ -\frac{T_{r,-}(p^*,x,t)}{T_{r,-}(p,x,t)} \frac{R_r(p)\Theta(p)}{|T(p)|^2} \mathrm{e}^{t\,\phi(p)} & 1 \end{pmatrix},$$

$$\tilde{B}_+ = \begin{pmatrix} 1 & -\frac{T_{r,+}(p,x,t)}{T_{r,+}(p^*,x,t)} \frac{R_r(p^*)\Theta(p^*)}{|T(p)|^2} \mathrm{e}^{-t\,\phi(p)} \\ 0 & 1 \end{pmatrix}.$$

Using (4.22) we can write

$$\tilde{B}_- = \begin{pmatrix} 1 & 0 \\ \frac{T_l(p^*,x,t)}{T_l(p,x,t)} R_l(p)\Theta(p)\mathrm{e}^{-t\phi(p)} & 1 \end{pmatrix}, \quad \tilde{B}_+ = \begin{pmatrix} 1 & \frac{T_l(p,x,t)}{T_l(p^*,x,t)} R_l(p^*)\Theta(p^*)\mathrm{e}^{-t\phi(p)} \\ 0 & 1 \end{pmatrix}.$$

If we split now $R_t(p) = R_{a,t}(p) + R_{r,t}(p)$ according to Lemma 7.1 we obtain

$$\begin{aligned} \tilde{b}_\pm(p) &= \tilde{b}_{a,t,\pm}(p)\tilde{b}_{r,t,\pm}(p) = \tilde{b}_{r,t,\pm}(p)\tilde{b}_{a,t,\pm}(p), \\ \tilde{B}_\pm(p) &= \tilde{B}_{a,t,\pm}(p)\tilde{B}_{r,t,\pm}(p) = \tilde{B}_{r,t,\pm}(p)\tilde{B}_{a,t,\pm}(p). \end{aligned}$$

Here $\tilde{b}_{a,t,\pm}(p)$, $\tilde{b}_{r,t,\pm}(p)$ (resp. $\tilde{B}_{a,t,\pm}(p)$, $\tilde{B}_{r,t,\pm}(p)$) denote the matrices obtained from $\tilde{b}_\pm(p)$ (resp. $\tilde{B}_\pm(p)$) by replacing $R_r(p)$ (resp. $R_l(p)$) with $R_{a,t}(p)$, $R_{r,t}(p)$, respectively. Now we can move the analytic parts into regions of the Riemann surface as in Section 5.4 while leaving the rest on Σ. Hence, rather than (5.87), the jump now reads

$$J^3(p) = \begin{cases} \tilde{b}_{a,t,+}(p), & p \in C_k, \quad \pi(p) > \zeta(x/t), \\ \tilde{b}_{a,t,-}(p)^{-1}, & p \in C_k^*, \quad \pi(p) > \zeta(x/t), \\ \tilde{b}_{r,t,-}(p)^{-1}\tilde{b}_{r,t,+}(p), & p \in \pi^{-1}((\zeta(x/t), +\infty)), \\ \tilde{B}_{a,t,+}(p), & p \in C_k, \quad \pi(p) < \zeta(x/t), \\ \tilde{B}_{a,t,-}(p)^{-1}, & p \in C_k^*, \quad \pi(p) < \zeta(x/t), \\ \tilde{B}_{r,t,-}(p)^{-1}\tilde{B}_{r,t,+}(p), & p \in \pi^{-1}(-\infty, \zeta(x/t))). \end{cases} \tag{7.9}$$

By construction we have $\hat{v}(k) = \mathbb{I} + O(t^{-l})$ on the whole contour and the rest follows as in Chapter 5.

In the oscillatory region we need to take the small vicinities of the stationary phase points into account. Since the phase is cubic near these points, we cannot use it to dominate the exponential growth of the analytic part away from the unit circle. Hence we will take the phase as a new variable and use the Fourier transform with respect to this new variable. Since this change of coordinates is singular near the stationary phase points, there is a price we have to pay, namely, requiring additional smoothness for $R(p)$. We begin with

Lemma 7.2. *Suppose $R(p) \in C^5(\Sigma)$. Then we can split $R(p)$ according to*

$$R(p) = R_0(p) + (\pi(p) - \pi(z_j))H(p), \qquad p \in \Sigma \cap D_{j,1}, \tag{7.10}$$

where $R_0(p)$ is a real rational function on $\mathcal{K}_g$ such that $H(p)$ vanishes at z_j, z_j^* of order three and has a Fourier transform

$$H(p) = \int_{\mathbb{R}} \hat{H}(x)\mathrm{e}^{x\phi(p)}dx, \tag{7.11}$$

with $x\hat{H}(x)$ integrable. Here ϕ denotes the phase defined in (4.47).

Proof. We begin by choosing a rational function $R_0(p) = a(z) + b(z)R_{2g+1}^{1/2}(p)$ such that $a(z)$, $b(z)$ are real-valued polynomials which are chosen such that $a(z)$ matches the values of $\mathrm{Re}(R(p))$ and its first four derivatives at z_j and $\mathrm{i}^{-1}b(z)R_{2g+1}^{1/2}(p)$ matches the values of $\mathrm{Im}(R(p))$ and its first four derivatives at z_j. Since $R(p)$ is C^5 we infer that $H(p) \in C^4(\Sigma)$ and it vanishes together with its first three derivatives at z_j, z_j^*.

Note that $\phi(p)/\mathrm{i}$, where ϕ is defined in (4.47) has a maximum at z_j^* and a minimum at z_j. Thus the phase $\phi(p)/\mathrm{i}$ restricted to $\Sigma \cap D_{j,1}$ gives a one to one coordinate transform $\Sigma \cap D_{j,1} \to [\phi(z_j)/\mathrm{i}, \phi(z_j^*)/\mathrm{i}]$ and we can hence express $H(p)$ in this new coordinate (setting it equal to zero outside this interval). The coordinate transform locally looks like a cube root near z_j and z_j^*, however, due to our assumption that H vanishes there, H is still C^2 in this new coordinate and the Fourier transform with respect to this new coordinates exists and has the required properties. □

Moreover, as in Lemma 7.1 we obtain:

Lemma 7.3. *Let $H(p)$ be as in the previous lemma. Then we can split $H(p)$ according to $H(p) = H_{a,t}(p) + H_{r,t}(p)$ such that $H_{a,t}(p)$ is analytic in the region* $\mathrm{Re}(\phi(p)) < 0$ *and*

$$|H_{a,t}(p)\mathrm{e}^{\phi(p)t/2}| = O(1),\ \mathrm{Re}(\phi(p)) < 0, \mathrm{Im}(k(p)) \le 0, \quad |H_{r,t}(p)| = O(t^{-1}),\ p \in \Sigma. \tag{7.12}$$

Proof. We choose $H_{a,t}(p) = \int_{-K(t)}^{\infty} \hat{H}(x)\mathrm{e}^{x\phi(p)}dx$ with $K(t) = t/2$. Then we can conclude as in Lemma 7.1:

$$|H_{a,t}(p)\mathrm{e}^{\phi(p)t/2}| \le \|\hat{H}(x)\|_1|\mathrm{e}^{-K(t)\phi(p)+\phi(p)t/2}| \le \|\hat{H}(x)\|_1 \le const$$

and

$$|H_{r,t}(p)| \le \int_{-\infty}^{-K(t)} |\hat{H}(x)|dx \le const\sqrt{\int_{-\infty}^{-K(t)} \frac{1}{x^4}dx} \le const\frac{1}{K(t)^{3/2}} \le const\frac{1}{t}.$$

□

By construction $R_{a,t}(p) = R_0(p) + (\pi(p) - \pi(z_j))H_{a,t}(p)$ will satisfy the required Lipschitz estimate in a vicinity of the stationary phase points (uniformly in t) and all jumps will be $\mathbb{I} + O(t^{-1})$. The remaining parts of Σ can be handled analogously and hence we can proceed as in Section 6.2.

8. Summary

In the thesis at hand we studied the Korteweg–de Vries (KdV) equation

$$V_t(x,t) = 6V(x,t)V_x(x,t) - V_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

This equation can be considered to be one of the prototype equations in soliton theory. The KdV equation is a completely integrable nonlinear wave equation. Special solutions of this equation are solitary waves (solitons), i.e., waves traveling at a constant speed and not changing their shape for all times.

Completely integrable nonlinear partial differential equations of the soliton-type have so-called algebro-geometric solutions. Solutions belonging to this huge class can be derived by means of algebraic geometry (especially using the theory of compact Riemann surfaces) and can be explicitly written down in terms of Riemann theta functions. Due to the quasi-periodic property of such functions (see equation (3.13)) these solutions are often called quasi-periodic. In particular, this class of solutions contains all soliton-like solutions.

The goal of this thesis was to investigate the long-time asymptotics of KdV solutions V which are short-range perturbations of such quasi-periodic KdV solutions V_q.

The long-time behavior of solutions in the classical (free) case is well-understood. In this case the background V_q is constant and the "perturbed" solution V is a rapidly decaying classical KdV solution, which, e.g., can be derived by the inverse scattering method. In the long-time limit the classical solution V shows the following behavior:

(i) in the so-called soliton region of the (x,t)-plane the classical KdV solution V splits up into a number of solitons travelling on the constant background, which are generated by the discrete spectrum of the corresponding Lax operator.

(ii) In the oscillatory region of the (x,t)-plane there can be observed an oscillating part emerging from the continuous spectrum of the Lax operator, which decays exponentially with time and in the limit approaches the constant background.

As already mentioned above, in this thesis we studied the quasi-periodic background case. Let g be the genus of the hyperelliptic curve associated with the unperturbed background solution V_q. Then the continuous spectrum of the Lax operator $H_q = d^2/dx^2 + V_q$ consists of $g+1$ spectral bands. The eigenvalues of the Lax operator lie in the spectral gaps. We were able to derive the following long-time asymptotic behavior of KdV solutions V that can be considered short-range perturbations of the quasi-periodic background solution V_q:

(i) in the soliton region of the (x,t)-plane, which consists of $g+1$ areas (corresponding to the spectral gaps), the perturbed solution V splits up into a number of solitons, each of them corresponding to one eigenvalue. For large times these solitons do not travel on the quasi-periodic background V_q but on a modulated KdV solution (cf. Definition 5.5). The exact result is given by Theorem 5.18.

(ii) In the oscillatory region of the (x,t)-plane, which consists of $g+1$ areas (corresponding to the spectral bands), the perturbed solution V does not approach the quasi-periodic solution V_q but again the so-called limiting KdV solution with a decay rate of $O(t^{-1/2})$. For the explicit asymptotic term we refer to Theorem 6.7.

This result turned out to be quite surprising given that – taking the classical result concerning the free case into account – one may have expected that in the quasi-periodic case the perturbed solution V asymptotically approaches the background solution V_q. Nevertheless, the quasi-periodic case studied in this thesis is the more general one which contains the classical case (choose $g = 0$).

The method which was used to derive this long-time asymptotics relies on the fact that the inverse spectral problem can be equivalently formulated as a Riemann–Hilbert factorization problem (RHP) defined on the underlying hyperelliptic curve. Then one can use the nonlinear steepest descent method for oscillatory RHP, which can be considered an analogue to the stationary phase method to approximate oscillating integrals. In that way the RHP with a jump on the spectral bands can be reduced to a local RHP which has a jump on a small cross centered at a stationary phase point lying in a spectral band. The advantage is that the localized RHP can be explicitly solved, which enables one to derive the explicit asymptotic terms.

A. Singular integral equations

In this appendix, which is based on [Kamvissis07b, Chapter 5] and [Krüger09c, Appendix A] we will show how to transform a meromorphic Riemann–Hilbert problem with simple poles at ρ, ρ^* of the form

$$\begin{aligned}
&m_+(p) = m_-(p)J(p), \qquad p \in \Sigma,\\
&(m_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*} - \mathcal{D}_{\underline{\rho}}, \qquad (m_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)} - \mathcal{D}_{\underline{\rho}^*},\\
&\Big(m_1(p) + \frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho)}{\prod_{k=1}^g(\rho-\mu_k)} \frac{\gamma}{\pi(p)-\rho} \frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} m_2(p)\Big) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*}, \text{ near } \rho,\\
&\Big(\frac{2\mathrm{i}R_{2g+1}^{1/2}(\rho)}{\prod_{k=1}^g(\rho-\mu_k)} \frac{\gamma}{\pi(p)-\rho} \frac{\psi_q(p,x,t)}{\psi_q(p^*,x,t)} m_1(p) + m_2(p)\Big) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x,t)}, \text{ near } \rho^*,\\
&m(p^*) = m(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\
&m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix},
\end{aligned} \tag{A.1}$$

into a singular integral equation. Since we require the symmetry condition we need to adapt the Cauchy kernel such that the symmetry is indeed preserved.

Moreover, we keep the single soliton as an inhomogeneous term, which will play the role of the leading asymptotics in our applications.

For notational simplicity we will abbreviate $L^p(\Sigma) = L^p(\Sigma, \mathbb{C}^{2\times 2})$.

Hypothesis H. A.1. *Let Σ consist of a finite number of smooth oriented curves in $\mathcal{K}_g$ which intersect at most finitely many times with all intersections being transversal. Assume that the contour is invariant under $p \mapsto p^*$. It is oriented such that under the mapping $p \mapsto p^*$ sequences converging from the positive side to Σ are mapped to sequences converging to the negative side. Moreover, suppose the jump matrix J is continuous and can be factorized according to $J = b_-^{-1} b_+ = (\mathbb{I} - w_-)^{-1}(\mathbb{I} + w_+)$, where $w_\pm = \pm(b_\pm - \mathbb{I})$ are continuous and satisfy*

$$w_\pm(p^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_\mp(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \Sigma. \tag{A.2}$$

In order to respect the symmetry condition we will restrict our attention to the set $L^2_s(\Sigma)$ of square integrable functions $f : \Sigma \to \mathbb{C}^2$ such that

$$f(p) = f(p^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

Clearly this will only be possible if we require our jump data to be symmetric as well (i.e., Hypothesis A.1 holds).

In [Beals84] it is shown that the solution of a Riemann–Hilbert problem in the complex plane can be reduced to the solution of a singular integral equation. Since in our case the underlying space is not the complex plane but the Riemann surface $\mathcal{K}_g$ defined in Chapter 2 we have to generalize the Cauchy kernel appropriately.

We follow the construction from [Rodin88, Sec. 4]. Allowing poles at the nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ the corresponding Cauchy kernel is given by

$$\Omega_p^{\underline{\hat{\mu}},\rho} = \omega_{p\,\rho} + \sum_{j=1}^{g} I_j^{\underline{\hat{\mu}},\rho}(p)\zeta_j, \tag{A.4}$$

where

$$I_j^{\underline{\hat{\mu}},\rho}(p) = \sum_{\ell=1}^{g} c_{j\ell}(\underline{\hat{\mu}}) \int_\rho^p \omega_{\hat{\mu}_\ell,0}. \tag{A.5}$$

Here $\omega_{\hat{\mu}_\ell,0}$ is the (normalized) Abelian differential of the second kind with a second order pole at $\hat{\mu}_\ell$. Thus $I_j^{\underline{\hat{\mu}},\rho}(p)$ has first order poles at the points $\underline{\hat{\mu}}$. By $\omega_{p\,\rho}$ we denote the Abelian differential of the third kind with simple poles at p and ρ (cf. Remark 5.4).

The constants $c_{j\ell}(\underline{\hat{\mu}})$ are chosen such that $\Omega_p^{\underline{\hat{\mu}},\rho}$ is single valued. That is,

$$\int_{b_k} dI_j^{\underline{\hat{\mu}},\rho} = \sum_{\ell=1}^{g} c_{j\ell} \int_{b_k} \omega_{\hat{\mu}_\ell,0} = \sum_{\ell=1}^{g} c_{j\ell}\eta_k(\hat{\mu}_\ell) = \delta_{jk}, \tag{A.6}$$

where $\zeta_k = \eta_k(z)dz$ is the chart expression in a local chart near $\hat{\mu}_\ell$ (here the b_k periods are evaluated using the usual bilinear relations, see [Farkas92, Sect. III.3] or [Teschl00, Sect. A.2]). That the matrix $\eta_k(\hat{\mu}_\ell)$ is indeed invertible can be seen as follows: If $\sum_{k=1}^{g} \eta_k(\hat{\mu}_\ell)c_k = 0$ for $1 \le \ell \le g$, then the divisor of $\zeta = \sum_{k=1}^{g} c_k\zeta_k$ satisfies $(\zeta) \ge \mathcal{D}_{\underline{\hat{\mu}}}$. But since we assumed the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ to be nonspecial, $i(\mathcal{D}_{\underline{\hat{\mu}}}) = 0$, we have $\zeta = 0$ implying $c_k = 0$.

We will always assume that $\mathcal{D}_{\underline{\hat{\mu}}}$ does not hit our contour Σ.

A.1. The Cauchy operator and its properties

Define the following generalized Cauchy operator

$$(Cf)(p) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma f\, \underline{\Omega}_p^{\underline{\hat{\mu}},\rho}, \quad p \notin \Sigma, \tag{A.7}$$

for a vector valued function $f : \Sigma \to \mathbb{C}^2$. The generalized Cauchy kernel $\underline{\Omega}_p^{\underline{\hat{\mu}},\rho}$ is given by

$$\underline{\Omega}_p^{\underline{\hat{\mu}},\rho} = \begin{pmatrix} \Omega_p^{\underline{\hat{\mu}}^*,\rho^*} & 0 \\ 0 & \Omega_p^{\underline{\hat{\mu}},\rho} \end{pmatrix}, \tag{A.8}$$

where $\Omega_p^{\underline{\hat{\mu}},\rho}$ is the Cauchy kernel given by (A.4). In the case where no poles are present we set

$$\underline{\Omega_p^{\hat{\nu}}} = \begin{pmatrix} \Omega_p^{\underline{\hat{\nu}}^*,p_\infty} & 0 \\ 0 & \Omega_p^{\underline{\hat{\nu}},p_\infty} \end{pmatrix}. \tag{A.9}$$

Invoking Remark 5.4, Remark 6.5 and the property $c_{j\ell}(\underline{\hat{\mu}}^*) = -c_{j\ell}(\underline{\hat{\mu}})$ one gets the symmetry property

$$\underline{\Omega_{p^*}^{\hat{\mu},\rho}}(q^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underline{\Omega_p^{\hat{\mu},\rho}}(q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.10}$$

Furthermore, we define the operators

$$(C_\pm f)(q) = \lim_{p\to q\in\Sigma} (Cf)(p), \tag{A.11}$$

where the subscript denotes the non-tangential boundary limit from Π_+ or Π_-, respectively.

Theorem A.2. *Assume Hypothesis A.1. The operators $C_\pm : L^2_s(\Sigma) \to L^2_s(\Sigma)$ defined in (A.11) are bounded. The bound can be chosen independent of the divisor as long as it stays some finite distance away from Σ. Moreover, the operators $C_\pm$ satisfy the Plemelj formulae*

$$\begin{aligned} (C_+ f)(q) - (C_- f)(q) &= f(q), \\ (C_+ f)(q) + (C_- f)(q) &= \frac{1}{\pi \mathrm{i}} \fint_\Sigma f\, \underline{\Omega_q^{\hat{\mu},\rho}}, \end{aligned} \tag{A.12}$$

where $\fint$ denotes the principal value integral, and

$$(Cf)(\rho^*) = \begin{pmatrix} 0 & * \end{pmatrix}, \qquad (Cf)(\rho) = \begin{pmatrix} * & 0 \end{pmatrix}. \tag{A.13}$$

Furthermore, C restricts to $L^2_s(\Sigma)$, that is

$$(Cf)(p^*) = (Cf)(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \mathcal{K}_g \backslash \Sigma \tag{A.14}$$

for $f \in L^2_s(\Sigma)$. If $w_\pm$ satisfy (A.2), we also have

$$C_\pm(f w_\mp)(p^*) = C_\mp(f w_\pm)(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \Sigma. \tag{A.15}$$

Proof. In a chart $z = z(p)$ near $q_0 \in \Sigma$, the differential $\Omega_q^{\underline{\hat{\mu}},\rho} = (\frac{1}{z - z(q)} + O(1))dz$ and hence the first part follows as in the Cauchy case on the complex plane (cf. [Muskhelishvili53]) using a partition of unity. By the definition of the Cauchy operator one can easily check that the property (A.13) is fulfilled. Moreover, invoking (A.2) and (A.10) one can verify by a straightforward calculation that (A.14) and (A.15) also hold. ☐

A.2. The connection between singular integral equations and Riemann–Hilbert problems

Let the operator $C_w : L^2_s(\Sigma) \to L^2_s(\Sigma)$ be defined by

$$C_w f = C_+(f w_-) + C_-(f w_+) \tag{A.16}$$

for a vector valued $f \in L^2_s(\Sigma)$, where

$$w_+ = b_+ - \mathbb{I} \quad \text{and} \quad w_- = \mathbb{I} - b_-. \tag{A.17}$$

From Lemma 4.11 we have that the solution corresponding to the jump $J \equiv \mathbb{I}$ is given by

$$m_0(p) = \begin{pmatrix} f(p^*, x, t) & f(p, x, t) \end{pmatrix}, \qquad f(p, x, t) = \frac{\psi_{q,\gamma}(p, x, t)}{\psi_q(p, x, t)},$$
$$\psi_{q,\gamma}(p, x, t) = \psi_q(p, x, t) + \frac{\gamma}{z - \rho} \frac{\psi_q(\rho, x, t) W_{(x,t)}(\psi_q(\rho, x, t), \psi_q(p, x, t))}{c_{q,\gamma}(\rho, x, t)}.$$

The divisor of f satisfies $(f) \ge -\mathcal{D}_{\underline{\hat{\mu}}} - \mathcal{D}_{\underline{\rho}^*}$. Since we assumed $\mathcal{D}_{\underline{\hat{\mu}}}$ to stay away from Σ, we have $m_0 \in L^2(\Sigma)$. The next theorem shows how one can transform a Riemann–Hilbert problem like (A.1) into a singular integral equation and vice versa.

Theorem A.3. *Assume Hypothesis A.1.*
Suppose m solves the Riemann–Hilbert problem (A.1). Then

$$m(p) = (1 - c_0) m_0(p) + \frac{1}{2\pi \mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s)) \underline{\Omega}_p^{\underline{\hat{\mu}}, \rho}, \tag{A.18}$$

where

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1} \quad \textit{and} \quad c_0 = \left(\frac{1}{2\pi \mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s)) \underline{\Omega}_{p_\infty}^{\underline{\hat{\mu}}, \rho} \right)_1.$$

Here $(m)_j$ denotes the j'th component of the vector m. Furthermore, μ solves

$$(\mathbb{I} - C_w)\mu(p) = (1 - c_0) m_0(p). \tag{A.19}$$

Conversely, suppose $\tilde{\mu}$ solves

$$(\mathbb{I} - C_w)\tilde{\mu}(p) = m_0(p), \tag{A.20}$$

and

$$\tilde{c}_0 = \left(\frac{1}{2\pi \mathrm{i}} \int_\Sigma \tilde{\mu}(s)(w_+(s) + w_-(s)) \underline{\Omega}_{p_\infty}^{\underline{\hat{\mu}}, \rho} \right)_1 \neq 1,$$

then m defined via (A.18), with $(1 - c_0) = (1 - \tilde{c}_0)^{-1}$ and $\mu = (1 - \tilde{c}_0)^{-1} \tilde{\mu}$, solves the Riemann–Hilbert problem (A.1) and $\mu = m_\pm b_\pm^{-1}$.

Proof. First note that by (A.15) $(\mathbb{I} - C_w)$ satisfies the symmetry condition and hence so do $(\mathbb{I} - C_w)^{-1} m_0$ and m.

So if m solves (A.1) and we set $\mu = m_+ b_+^{-1} = m_- b_-^{-1}$, then m satisfies the following additive jump

$$\begin{aligned} m_+ - m_- &= m_-(b_-^{-1} b_+ - \mathbb{I}) = m_- b_-^{-1}(b_+ - b_-) = \mu\big((b_+ - \mathbb{I}) + (\mathbb{I} - b_-)\big) \\ &= \mu(w_+ + w_-). \end{aligned}$$

Now define $\tilde{m}$ by the right hand side of (A.18), then one can show that $\tilde{m}$ satisfies the same additive jump as m:

$$\begin{aligned} m_+ - m_- &= (1 - c_0)(m_{0,+} - m_{0,-}) + (C_+ - C_-)(\mu w_+) + (C_+ - C_-)(\mu w_-) \\ &= \mu(w_+ + w_-). \end{aligned}$$

Here we made use of (A.12) and the fact that m_0 is the solution to the Riemann–Hilbert problem with jump $J = \mathbb{I}$ and thus $m_{0,+} = m_{0,-}$.

The uniqueness result Theorem A.6 yields $m - \tilde{m} = \alpha m_0$ for some $\alpha \in \mathbb{C}$. By the normalization of m and $\tilde{m}$ at p_∞ we finally get that $\alpha = 0$, that is $m = \tilde{m}$.

It is left to show that $\mu = m_\pm b_\pm^{-1}$ solves the singular integral equation (A.19). For that purpose note that if m is given by (A.18), making use of the Plemelj formulae (A.12) yields

$$\begin{aligned} m_\pm &= (1 - c_0)m_0 + C_\pm(\mu w_-) + C_\pm(\mu w_+) \\ &= (1 - c_0)m_0 + C_\pm(\mu w_-) + C_\mp(\mu w_+) + C_\pm(\mu w_+) - C_\mp(\mu w_+) \\ &= (1 - c_0)m_0 + C_w(\mu) \pm \mu w_\pm \\ &= (1 - c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu(\mathbb{I} \pm w_\pm) \\ &= (1 - c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm. \end{aligned}$$

So by the definition of μ we have $m_\pm = \mu b_\pm$ and hence

$$\mu b_\pm = (1 - c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm,$$

which shows that $(1 - c_0)m_0 - (\mathbb{I} - C_w)\mu = 0$, that means μ indeed solves (A.19).

Conversely, suppose $\tilde{\mu}$ solves the singular integral equation (A.20). Let m be defined via (A.18) with $(1 - c_0) = (1 - \tilde{c}_0)^{-1}$ and $\mu = (1 - \tilde{c}_0)^{-1}\tilde{\mu}$, then using the same calculation as above implies

$$\begin{aligned} m_\pm &= (1 - \tilde{c}_0)^{-1} m_0 - (1 - C_w)(1 - \tilde{c}_0)^{-1}\tilde{\mu} + (1 - \tilde{c}_0)^{-1}\tilde{\mu} b_\pm \\ &= (1 - \tilde{c}_0)^{-1} m_0 - (1 - \tilde{c}_0)^{-1} m_0 + (1 - \tilde{c}_0)^{-1}\tilde{\mu} b_\pm = \mu b_\pm, \end{aligned}$$

which shows that m defined via (A.18) solves the Riemann–Hilbert problem (A.1). □

In the oscillatory region, that is in the case $\gamma = 0$, we have $m_0(p) = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Choosing $\rho = p_\infty$ such that $c_0 = \tilde{c}_0 = 0$ yields the following Corollary:

Corollary A.4. *Assume Hypothesis A.1.*

Suppose m solves the Riemann–Hilbert problem (A.1) with $\gamma = 0$. Then

$$m(p) = \mathbb{I} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \mu(s)(w_+(s) + w_-(s)) \underline{\Omega}_p^{\hat{\nu}}, \tag{A.21}$$

where the Cauchy kernel $\underline{\Omega}_p^{\hat{\nu}}$ is defined as in (A.9) and

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1}.$$

Furthermore, μ solves

$$(\mathbb{I} - C_w)\mu = \mathbb{I}. \tag{A.22}$$

Conversely, suppose $\tilde{\mu}$ solves (A.22) then m defined by the integral formula (A.21) solves the Riemann–Hilbert problem (A.1).

Hence we have a formula for the solution of our Riemann–Hilbert problem $m(p)$ in terms of $(\mathbb{I} - C_w)^{-1} m_0(p)$ and thus we have to consider the question of bounded invertibility of the operator $\mathbb{I} - C_w$. This will follow from Fredholm theory (cf. e.g. [Zhou89]) as the next section shows.

A.3. An existence and uniqueness result for symmetric Riemann–Hilbert problems on a Riemann surface

The aim of this section is to prove a result that guarantees existence and uniqueness for symmetric Riemann–Hilbert factorization problems, which are set on an hyperelliptic curve with real branch cuts. We will use the same idea as presented in [Kamvissis07b, Appendix A], whereas our result will be slightly more general since it includes the case where solitons are present.

Lemma A.5. *Assume Hypothesis A.1.*

The operator $\mathbb{I} - C_w$ is Fredholm of index zero,

$$\operatorname{ind}(\mathbb{I} - C_w) = 0. \tag{A.23}$$

Proof. Since one can easily check

$$(\mathbb{I} - C_w)(\mathbb{I} - C_{-w}) = (\mathbb{I} - C_{-w})(\mathbb{I} - C_w) = \mathbb{I} - T_w, \tag{A.24}$$

where

$$T_w = T_{++} + T_{+-} + T_{-+} + T_{--}, \qquad T_{\sigma_1 \sigma_2}(f) = C_{\sigma_1}[C_{\sigma_2}(f w_{-\sigma_2}) w_{-\sigma_1}],$$

it suffices to check that the operators $T_{\sigma_1\sigma_2}$ are compact ([Prössdorf78, Thm. 1.4.3]).

Indeed, suppose $f_n \in L^2(\Sigma)$ converges weakly to zero. Without loss of generality we can assume f_n to be continuous. We will show that $\|T_w f_n\|_{L^2} \to 0$.

Let us temporally assume that w is analytic in a neighborhood of Σ. Then according to the definition of $C_\pm$, we can slightly deform the contour Σ to some contour $\Sigma_\pm$ close to Σ, on the left, and have, by Cauchy's theorem,

$$T_{++}f_n(p) = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma_+} (C(f_n w_-)(s) w_-(s)) \underline{\Omega_p^{\hat{\mu},\rho}}.$$

Note that $(C(f_n w_-)w_-)(p) \to 0$ as $n \to \infty$ and

$$|(C(f_n w_-)w_-)(p)| < const \, \|f_n\|_{L^2} \|w_-\|_{L^\infty} < const.$$

Thus, using the dominated convergence theorem yields $\|T_{++}f_n\|_{L^2} \to 0$ as desired.

Consider the operator $\mathbb{I} - \varepsilon C_w = \mathbb{I} - C_{\varepsilon w}$ for $0 \le \varepsilon \le 1$. As is shown in [Prössdorf78, Thm. 1.3.8] $\mathrm{ind}(\mathbb{I} - \varepsilon C_w)$ is continuous with respect to ε, but since it is an integer, it has to be constant, that is

$$\mathrm{ind}(\mathbb{I} - C_w) = \mathrm{ind}(\mathbb{I}) = 0.$$

Moreover, in the case where w is not analytic one can take an analytic approximation by using the partition of unity. □

By the Fredholm alternative we know that to show the bounded invertibility of the operator $\mathbb{I} - C_w$ it suffices to check that $\ker(\mathbb{I} - C_w) = 0$. The latter is equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

We will consider the Riemann–Hilbert problem (A.1) where the pole conditions are now rewritten as mentioned in Remark 4.7, i.e.,

$$\begin{aligned}
& m_+(p) = m_-(p) J(p), \qquad p \in \Sigma, \\
& (m_1) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)^*} - \mathcal{D}_{\underline{\rho}}, \qquad (m_2) \ge -\mathcal{D}_{\underline{\hat{\mu}}(x,t)} - \mathcal{D}_{\underline{\rho}^*}, \\
& \mathrm{Res}_{\rho_j} m(p) = \lim_{p \to \rho_j} m(p) \begin{pmatrix} 0 & 0 \\ \frac{2\mathrm{i} R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g (\rho_j - \mu_k)} & 0 \end{pmatrix} = \lim_{p \to \rho_j} m(p) J_j, \\
& \mathrm{Res}_{\rho_j^*} m(p) = \lim_{p \to \rho_j^*} m(p) \begin{pmatrix} 0 & \frac{2\mathrm{i} R_{2g+1}^{1/2}(\rho_j)\gamma_j}{\prod_{k=1}^g (\rho_j - \mu_k)} \\ 0 & 0 \end{pmatrix} = \lim_{p \to \rho_j^*} m(p) J_j^*, \\
& m(p^*) = m(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
& m(p_\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}.
\end{aligned} \tag{A.25}$$

Let us, for notational convenience, abbreviate

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and note that $J_j^* = \sigma J_j \sigma$.

Theorem A.6 (Existence and uniqueness)**.** *Assume Hypothesis A.1. Consider the Riemann–Hilbert problem (A.25), where Σ is an oriented contour, consisting of a union of finitely many smooth arcs, which is symmetric under sheet exchange plus conjugation ($\Sigma = \overline{\Sigma}^*$) such that*

(i) $J(p^*) = J^*(\overline{p})$, *for* $p \in \Sigma \setminus \pi^{-1}(\sigma(H_q))$,

(ii) $\mathrm{Re}(J(p)) = \frac{1}{2}(J(p) + J^*(p))$ *is positive definite for* $p \in \pi^{-1}(\sigma(H_q))$,

(iii) $J_j\sigma + \sigma J_j^*$ *is purely imaginary, where the imaginary part is nonnegative,*

(iv) J *is analytic in a neighborhood of* Σ.

The divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *is such that* $\mu_j \in [E_{2j-1}, E_{2j}]$, $j = 1, \dots, g$.

Then the Riemann–Hilbert problem (A.25) on $\mathcal{K}_g$ *has a unique solution.*

Note that the $+$-side of the contour is mapped to the $-$-side under sheet exchange. In particular, the theorem holds if $J = \mathbb{I}$, that is there is no jump, on $\pi^{-1}(\sigma(H_q))$.

Proof. As mentioned above we have to show that $\ker(\mathbb{I} - C_w) = 0$. For that purpose suppose that $(\mathbb{I} - C_w)\mu = 0$ holds for some $\mu \in L^2_s(\Sigma)$. Next, set

$$\tilde{m}(p) = (C_w)(\mu)(p) \quad \text{for } p \in \mathcal{K}_g\backslash\Sigma.$$

One can then show that $\tilde{m}(p)$ solves the Riemann–Hilbert problem (A.1) except for the normalization condition which is now given by $\tilde{m}(p_\infty) = \begin{pmatrix}0 & 0\end{pmatrix}$. That is, $\tilde{m}(p)$ solves the corresponding vanishing Riemann–Hilbert problem.

Next we want to apply Cauchy's integral theorem to $\tilde{m}(p)\tilde{m}^*(\overline{p}^*)$. To handle the poles of $\tilde{m}$ we will multiply it by a meromorphic differential $d\Omega$ which has zeros at $\underline{\mu}$ and $\underline{\mu}^*$ and a simple pole at p_∞ such that finally the differential $\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p)$ is holomorphic away from the contour.

More precisely, let

$$d\Omega = \frac{\prod_{j=1}^{g}(\pi - \mu_j)}{R_{2g+1}^{1/2}} d\pi \tag{A.26}$$

and note that $\mathrm{i}\big(\prod_j (z - \mu_j)\big) R_{2g+1}^{-1/2}(z)$ is a Herglotz function. That is, it has positive imaginary part in the upper half-plane (and it is purely imaginary on $\sigma(H_q)$). Hence $\tilde{m}(p)\overline{\tilde{m}^T(p)}d\Omega(p)$ will be positive on $\pi^{-1}(\sigma(H_q))$.

Next, consider the integral

$$\int_D \tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p), \tag{A.27}$$

where D is a $\overline{*}$-invariant contour consisting of one small loop in every connected component of $\mathcal{K}_g\backslash\Sigma$. We will deform D to a $\overline{*}$-invariant contour consisting of two parts, one, say D_+, wrapping around the part of Σ lying on Π_+ and the $+$ side of $\pi^{-1}(\sigma(H_q))$ and the other being $D_- = \overline{D_+}^*$.

For each component Σ_ν of $\Sigma\backslash\pi^{-1}(\sigma(H_q))$ there are two contributions to the integral on the deformed contour:

$$\int_{\Sigma_\nu} \tilde{m}_+(p)\tilde{m}_-^*(\overline{p}^*)d\Omega(p) = \int_{\Sigma_\nu} \tilde{m}_-(p)J(p)\tilde{m}_-^*(\overline{p}^*)d\Omega(p),$$
$$\int_{-\Sigma_\nu} \tilde{m}_-(p)\tilde{m}_+^*(\overline{p}^*)d\Omega(p) = \int_{-\Sigma_\nu} \tilde{m}_-(p)J^*(\overline{p}^*)\tilde{m}_-^*(\overline{p}^*)d\Omega(p).$$

Using condition (i) one sees that the two integrals cancel each other.

Taking this into account one gets

$$\begin{aligned}\int_D \tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) &= \int_{\pi^{-1}(\sigma(H_q))} \big(\tilde{m}_+(p)\tilde{m}_-^*(\overline{p}^*) + \tilde{m}_-(p)\tilde{m}_+^*(\overline{p}^*)\big)d\Omega(p)\\ &= \int_{\pi^{-1}(\sigma(H_q))} \tilde{m}_-(p)(J(p)+J^*(\overline{p}^*))\tilde{m}_-^*(\overline{p}^*)d\Omega(p) \ge 0, \quad \text{(A.28)}\end{aligned}$$

where we have used condition (ii) to see that this integral is positive. On the other hand by Cauchy's residue theorem we have

$$\int_D \tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) = \sum_{j=1}^N \big(\mathrm{Res}_{\rho_j}\tilde{m}(p)\tilde{m}^*(\overline{p}^*) + \mathrm{Res}_{\rho_j^*}\tilde{m}(p)\tilde{m}^*(\overline{p}^*)\big)d\Omega(p). \quad \text{(A.29)}$$

Thus, our next aim will be to calculate the residues of $\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p)$. The function $\tilde{m}(p)d\Omega(p)$ has the following representations near the poles ρ_j resp. ρ_j^*:

$$\tilde{m}(p) = \begin{cases} \frac{A_j}{z-\rho_j} + B_j + O(z-\rho_j) & \text{near } \rho_j,\\ \frac{C_j}{z-\rho_j} + D_j + O(z-\rho_j^*) & \text{near } \rho_j^*,\end{cases} \quad \text{(A.30)}$$

where A_j, B_j, C_j, and D_j are vectors. Since $\tilde{m}(p)$ has to fulfill the symmetry condition

$$\tilde{m}(p^*) = \tilde{m}(p)\sigma,$$

the following restrictions have to be satisfied:

$$C_j = A_j\sigma \quad \text{and} \quad D_j = B_j\sigma. \quad \text{(A.31)}$$

Moreover, the pole condition

$$\mathrm{Res}_{\rho_j}\tilde{m}(p) = \lim_{p\to\rho_j} \tilde{m}(p)J_j,$$

yields the conditions

$$A_j = B_jJ_j \quad \text{and} \quad A_jJ_j = 0, \quad \text{(A.32)}$$

which are indeed both fulfilled since in our case $J_j^2 = 0$. Thus, we are ready to compute the residue of $\tilde{m}(p)\tilde{m}^*(\overline{p}^*)$ at ρ_j:

$$\begin{aligned}\mathrm{Res}_{\rho_j}\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) &= \mathrm{Res}_{\rho_j}\Big(\big(\frac{B_jJ_j}{z-\rho_j} + B_j\big)\big(\frac{(B_jJ_j\sigma)^*}{z-\rho_j} + (B_j\sigma)^*\big)\Big)d\Omega(p)\\ &= \mathrm{Res}_{\rho_j}\Big(\big(\frac{B_jJ_j}{z-\rho_j} + B_j\big)\big(\frac{\sigma J_j^*B_j^*}{z-\rho_j} + \sigma B_j^*\big)\Big)d\Omega(p)\\ &= B_jJ_j\sigma B_j^* + B_j\sigma J_j^*B_j^* = B_j(J_j\sigma + \sigma J_j^*)B_j^*,\end{aligned}$$

where $J_j^* = \sigma J_j \sigma$. The residue at ρ_j^* can be computed similarly, but it also follows easily by the symmetry condition

$$\begin{aligned}
\mathrm{Res}_{\rho_j^*} \tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) &= \lim_{p\to\rho_j^*}(p-\rho_j^*)\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p)\\
&= \lim_{p^*\to\rho_j}(p^*-\rho_j)\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p)\\
&= \lim_{p^*\to\rho_j}(p^*-\rho_j)\tilde{m}(p^*)\sigma\sigma^*\tilde{m}^*(\overline{p})d\Omega(p)\\
&= \lim_{p\to\rho_j}(p-\rho_j)\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p)\\
&= \mathrm{Res}_{\rho_j}\tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) = B_j(J_j\sigma + \sigma J_j^*)B_j^*.
\end{aligned}$$

We are now ready to compute the integral (A.29):

$$\int_D \tilde{m}(p)\tilde{m}^*(\overline{p}^*)d\Omega(p) = 4\pi\mathrm{i}\sum_{j=1}^{g} B_j(J_j\sigma + \sigma J_j^*)B_j^* \le 0, \tag{A.33}$$

where we made use of condition (iii) to see that the integral is non-positive. Finally, comparing (A.28) and (A.33) it follows that $\tilde{m} = 0$ and hence $\mu = 0$, which shows that indeed $\ker(\mathbb{I} - C_w) = 0$.

To prove uniqueness, suppose there were two solutions. Their difference would satisfy the same jump and would vanish at p_∞. Using the above argument for $\tilde{m}$ we see that the difference would have to vanish everywhere.

□

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps w_0 and w with $\|w - w_0\|_\infty$ and $\|w - w_0\|_2$ small, where

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)}. \tag{A.34}$$

and

$$\|w\|_2 = \|w_+\|_{L^2(\Sigma)} + \|w_-\|_{L^2(\Sigma)} \tag{A.35}$$

For such a situation we have the following result:

Theorem A.7. *Assume that for some data w_0^t the operator*

$$\mathbb{I} - C_{w_0^t} : L^2_s(\Sigma) \to L^2_s(\Sigma) \tag{A.36}$$

has a bounded inverse, where the bound is independent of t, and let $\rho = \zeta_0$, $\gamma^t = \gamma_0^t$.

Furthermore, assume w^t satisfies

$$\|w^t - w_0^t\|_\infty \le \alpha(t) \text{ and } \left\|w^t - w_0^t\right\|_2 \le \alpha(t) \tag{A.37}$$

for some function $\alpha(t) \to 0$ as $t \to \infty$. Then $(\mathbb{I} - C_{w^t})^{-1} : L^2_s(\Sigma) \to L^2_s(\Sigma)$ also exists for sufficiently large t and the associated solutions of the Riemann–Hilbert problems (A.1) only differ by $O(\alpha(t))$.

Proof. By the boundedness of the Cauchy transform we conclude

$$\begin{aligned}\|(C_{w^t} - C_{w_0^t})(f)\|_2 &= \|C_+(f(w_-^t - w_{0,-}^t)) + C_-(f(w_+^t - w_{0,+}^t))\|_2 \\ &\le c\,\|f\|_{L^2(\Sigma)} \left\|w^t - w_0^t\right\|_\infty\end{aligned}$$

and thus by (A.37) we obtain

$$\|C_{w^t} - C_{w_0^t}\|_{L^2(\Sigma)\to L^2(\Sigma)} = O(\alpha(t)).$$

Hence by the second resolvent identity, we infer that $(\mathbb{I} - C_{w^t})^{-1}$ exists for large t and

$$\|(\mathbb{I} - C_{w_0^t})^{-1} - (\mathbb{I} - C_{w^t})^{-1}\|_{L^2(\Sigma)\to L^2(\Sigma)} = O(\alpha(t)).$$

Next, observe that since μ solves (A.19) we get

$$\mu^t - \mu_0^t = (\mathbb{I} - C_{w^t})^{-1}(1 - c_0)m_0(k) - (\mathbb{I} - C_{w_0^t})^{-1}(1 - c_0)m_0(k) \in L_s^2(\Sigma)$$

and can therefore compute

$$\begin{aligned}\|\mu^t - \mu_0^t\|_{L^2(\Sigma)} &= \|[(\mathbb{I} - C_{w^t})^{-1} - (\mathbb{I} - C_{w_0^t})^{-1}](1 - c_0)m_0(k)\|_{L^2(\Sigma)} \\ &\le \|(\mathbb{I} - C_{w_0^t})^{-1} - (\mathbb{I} - C_{w^t})^{-1}\|_{L^2(\Sigma)\to L^2(\Sigma)}\|(1 - c_0)m_0(k)\|_{L^2(\Sigma)} \\ &= O(\alpha(t)).\end{aligned}$$

This now implies that for $p \in \mathcal{K}_g \backslash \Sigma$

$$\begin{aligned}\left|m^t(k) - m_0^t(k)\right| =& \left|\frac{1}{2\pi\mathrm{i}}\int_\Sigma \mu^t(s)(w_+^t(s) + w_-^t(s)) - \mu_0^t(s)(w_{0,+}^t(s) + w_{0,-}^t(s))\underline{\Omega}_p^{\hat{\underline{\mu}},\rho}\right| \\ \le& \frac{1}{2\pi}\left|\int_\Sigma \mu^t(s)(w_+^t + w_-^t(s) - (w_{0,+}^t(s) + w_{0,-}^t(s)))\underline{\Omega}_p^{\hat{\underline{\mu}},\rho}\right| \\ &+ \frac{1}{2\pi}\left|\int_\Sigma (\mu^t(s) - \mu_0^t(s))(w_{0,+}^t(s) + w_{0,-}^t(s))\underline{\Omega}_p^{\hat{\underline{\mu}},\rho}\right| \\ \le& c\|\mu_\infty^t\|_{L^\infty(\Sigma)}\|w^t - w_0^t\|_2 + c\|\mu_2^t\|_{L^2(\Sigma)}\|w^t - w_0^t\|_\infty \\ &+ c\|\mu^t - \mu_0^t\|_{L^2(\Sigma)}\|w_0^t\|_\infty = O(\alpha(t)).\end{aligned}$$

Here μ_∞^t denotes the L^∞ part of μ^t and similarly μ_2^t denotes the L^2 part of μ^t.

Now we want to investigate the case $p \in \Sigma$. Since by definition $w_\pm = \mathbb{I} \pm w_\pm$ we have

$$w^t - w_0^t = (b_+^t - b_{0,+}^t) + (b_-^t - b_{0,-}^t)$$

and hence

$$\begin{aligned}\|b_\pm^t - b_{0,\pm}^t\|_2 &\le \|w^t - w_0^t\|_2 = O(\alpha(t)), \\ \|b_\pm^t - b_{0,\pm}^t\|_\infty &\le \|w^t - w_0^t\|_\infty = O(\alpha(t)).\end{aligned}$$

Thus we can conclude

$$\begin{aligned}\|m^t_\pm(k) - m^t_{0,\pm}(k)\|_{L^2(\Sigma)} =&\|\mu^t b^t_\pm - \mu^t_0 b^t_{0,\pm}\|_{L^2(\Sigma)}\\ \leq&\|\mu^t b^t_\pm + \mu^t_0 b^t_\pm - \mu^t_0 b^t_\pm - \mu^t_0 b^t_{0,\pm}\|_{L^2(\Sigma)}\\ \leq&\|\mu^t - \mu^t_0\|_{L^2(\Sigma)}\|b^t_\pm\|_\infty + \|\mu^t_{0,\infty}\|_{L^\infty(\Sigma)}\|b^t_\pm - b^t_{0,\pm}\|_2\\ &+ \|\mu^t_{0,\infty}\|_{L^2(\Sigma)}\|b^t_\pm - b^t_{0,\pm}\|_\infty = O(\alpha(t)),\end{aligned}$$

where similarly as above $\mu^t_{0,\infty}$ denotes the L^∞ part of μ^t_0 and $\mu^t_{0,2}$ denotes the L^2 part of μ^t_0. □

B. The Riemann-Hilbert problem on a small cross

This chapter is taken from Krüger and Teschl [Krüger09b, Appendix A]. It is devoted to solving the Riemann–Hilbert problem on a small cross.

Introduce the cross $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_4$ (cf. Figure B.1) by

$$\begin{aligned} \Sigma_1 &= \{u\mathrm{e}^{-\mathrm{i}\pi/4},\, u \in [0,\infty)\} \quad & \Sigma_2 &= \{u\mathrm{e}^{\mathrm{i}\pi/4},\, u \in [0,\infty)\} \\ \Sigma_3 &= \{u\mathrm{e}^{3\mathrm{i}\pi/4},\, u \in [0,\infty)\} \quad & \Sigma_4 &= \{u\mathrm{e}^{-3\mathrm{i}\pi/4},\, u \in [0,\infty)\}. \end{aligned}$$

The contour Σ is oriented such that the real part of z increases in the positive direction. Denote by $\mathbb{D} = \{z \in \mathbb{C},\, |z| < 1\}$ the open unit disc. Throughout this appendix $z^{\mathrm{i}\nu}$ will denote the function $\mathrm{e}^{\mathrm{i}\nu \log(z)}$, where the branch cut of the logarithm is chosen along the negative real axis $(-\infty, 0)$.

Now consider the Riemann–Hilbert problem given by

$$\begin{aligned} m_+(z) &= m_-(z) v_j(z), \quad & z &\in \Sigma_j, \quad j = 1, \dots, 4, \\ m(z) &\to \mathbb{I}, & z &\to \infty, \end{aligned} \tag{B.1}$$

where the jump matrices are given as follows: (v_j for $z \in \Sigma_j$)

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & -R_1(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, \quad & v_2 &= \begin{pmatrix} 1 & 0 \\ R_2(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}, \\ v_3 &= \begin{pmatrix} 1 & -R_3(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, \quad & v_4 &= \begin{pmatrix} 1 & 0 \\ R_4(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}. \end{aligned}$$

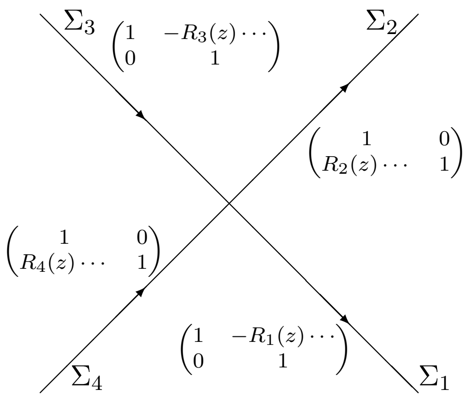

Figure B.1.: Contours of a cross. [Krüger09b]

We can now state the next theorem, which gives us the solution of the Riemann–Hilbert problem (B.1). In the proof we follow the computations of Deift and Zhou [Deift93, Section 3 and 4].

We will allow some variation, in all parameters as indicated.

Theorem B.1. *There is some* $\rho_0 > 0$ *such that* $v_j(z) = \mathbb{I}$ *for* $|z| > \rho_0$*. Moreover, suppose that within* $|z| \le \rho_0$ *the following estimates hold:*

1. *The phase satisfies* $\Phi(0) \in \mathrm{i}\mathbb{R}$, $\Phi'(0) = 0$, $\Phi''(0) = \mathrm{i}$ *and*

$$\pm \operatorname{Re}\big(\Phi(z) - \Phi(0)\big) \;\ge\; \frac{1}{4}|z|^2, \qquad \begin{cases} + & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\ - & \text{else}, \end{cases} \tag{B.2}$$

$$|\Phi(z) - \Phi(0) - \frac{\mathrm{i}z^2}{2}| \;\le\; C|z|^3. \tag{B.3}$$

2. *There is some* $r \in \mathbb{D}$ *and constants* $(\alpha, L) \in (0,1] \times (0,\infty)$ *such that* R_j, $j = 1, \dots, 4$, *satisfy Hölder conditions of the form*

$$|R_1(z) - \overline{r}| \le L|z|^\alpha, \qquad |R_2(z) - r| \le L|z|^\alpha,$$
$$|R_3(z) - \frac{\overline{r}}{1-|r|^2}| \le L|z|^\alpha, \quad |R_4(z) - \frac{r}{1-|r|^2}| \le L|z|^\alpha.$$

Then the solution of the Riemann–Hilbert problem (B.1) satisfies

$$m(z) = \mathbb{I} + \frac{1}{z}\frac{\mathrm{i}}{t^{1/2}} \begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O(t^{-\frac{1+\alpha}{2}}), \tag{B.4}$$

for $|z| > \rho_0$*, where*

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-t\Phi(0)}t^{-\mathrm{i}\nu}, \qquad \nu = -\frac{1}{2\pi}\log(1-|r|^2). \tag{B.5}$$

Furthermore, if $R_j(z)$ *and* $\Phi(z)$ *depend on some parameter, the error term is uniform with respect to this parameter as long as* r *remains within a compact subset of* $\mathbb{D}$ *and the constants in the above estimates can be chosen independent of the parameters.*

Remark B.2. *Note that the solution of the Riemann–Hilbert problem (B.1) is unique. This follows from the usual Liouville argument [Deift00, Lemma 7.18] since* $\det(v_j) = 1$.

Before we are able to prove this theorem, which will be done in the end of Section B.2 (see page 114), we have to work out some necessary preliminaries.

B.1. Approximation

A close look at Theorem B.1 shows, that the actual value of ρ_0 is of no importance. In fact, if we choose $0 < \rho_1 < \rho_0$, then the solution $\tilde{m}$ of the problem with jump $\tilde{v}$, where $\tilde{v}$ is equal to v for $|z| < \rho_1$ and $\mathbb{I}$ otherwise, differs from m only by an exponentially small error.

This already indicates that we should be able to replace $R_j(z)$ by their respective values at $z = 0$. To see this we start by rewriting our Riemann–Hilbert problem as a singular integral equation. We will use the theory similar to the one developed in Appendix A with the only difference being that the problem is now set in the complex plane. We will apply the theory to the case of 2×2 matrix valued functions with $m_0(z) = \mathbb{I}$ and will use the usual Cauchy kernel (since we won't require symmetry in this appendix)

$$\Omega_\infty(s, z) = \mathbb{I}\frac{ds}{s - z}.$$

Moreover, since our contour is unbounded, we will again assume $w \in L^\infty(\Sigma) \cap L^2(\Sigma)$. All results from Appendix A still hold in this case with some straightforward modifications as the only difference is that μ is now a matrix and no longer a vector. Indeed, as in Appendix A (cf. Corollary A.4), in the special case $b_+(z) = v_j(z)$ and $b_-(z) = \mathbb{I}$ for $z \in \Sigma_j$, $j = 1, \dots, 4$, we obtain

$$m(z) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s) w(s) \frac{ds}{s - z}, \tag{B.6}$$

where $\mu - \mathbb{I}$ is the solution of the singular integral equation

$$(\mathbb{I} - C_w)(\mu - \mathbb{I}) = C_w \mathbb{I}, \tag{B.7}$$

that is,

$$\mu = \mathbb{I} + (\mathbb{I} - C_w)^{-1} C_w \mathbb{I}, \qquad C_w f = C_-(wf). \tag{B.8}$$

Here C denotes the usual Cauchy operator given by

$$(Cf)(z) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{f(s)}{s - z} ds, \qquad z \in \mathbb{C}\backslash\Sigma,$$

and we have set $w(z) = w_+(z)$ (since $w_-(z) = 0$).

As our first step we will get rid of some constants and rescale the entire problem by setting

$$\hat{m}(z) = D(t)^{-1} m(z t^{-1/2}) D(t), \tag{B.9}$$

where

$$D(t) = \begin{pmatrix} d(t)^{-1} & 0 \\ 0 & d(t) \end{pmatrix}, \quad d(t) = \mathrm{e}^{t\Phi(0)/2} t^{\mathrm{i}\nu/2}. \tag{B.10}$$

Then one checks that $\hat{m}(z)$ solves the Riemann–Hilbert problem

$$\begin{aligned} \hat{m}_+(z) &= \hat{m}_-(z) \hat{v}_j(z), & z &\in \Sigma_j, \quad j = 1, 2, 3, 4, \\ \hat{m}(z) &\to \mathbb{I}, & z &\to \infty, \quad z \notin \Sigma, \end{aligned} \tag{B.11}$$

where the jump matrices are now given by

$$\hat{v}_j(z) = D(t)^{-1} v_j(zt^{-1/2}) D(t), \quad j = 1, \dots, 4,$$

or explicitly by

$$\begin{aligned}
\hat{v}_1(z) &= \begin{pmatrix} 1 & -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} \\ 0 & 1 \end{pmatrix}, \\
\hat{v}_2(z) &= \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2})-\Phi(0))} & 1 \end{pmatrix}, \\
\hat{v}_3(z) &= \begin{pmatrix} 1 & -R_3(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} \\ 0 & 1 \end{pmatrix}, \\
\hat{v}_4(z) &= \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2})-\Phi(0))} & 1 \end{pmatrix}.
\end{aligned}$$

Our next aim is to show that the solution $\hat{m}(z)$ of the rescaled problem is close to the solution $\hat{m}^c(z)$ of the Riemann–Hilbert problem

$$\begin{aligned}
\hat{m}^c_+(z) &= \hat{m}^c_-(z) \hat{v}^c_j(z), & z \in \Sigma_j, \quad j = 1, 2, 3, 4, \\
\hat{m}^c(z) &\to \mathbb{I}, & z \to \infty, \quad z \notin \Sigma,
\end{aligned} \tag{B.12}$$

associated with the following jump matrices

$$\hat{v}^c_1(z) = \begin{pmatrix} 1 & -\overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, \qquad \hat{v}^c_2(z) = \begin{pmatrix} 1 & 0 \\ r z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix},$$

$$\hat{v}^c_3(z) = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, \quad \hat{v}^c_4(z) = \begin{pmatrix} 1 & 0 \\ \frac{r}{1-|r|^2} z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix}.$$

The difference between the jump matrices $\hat{v}_j$ and $\hat{v}^c_j$, $j = 1, \dots, 4$, can be estimated as follows.

Lemma B.3. *The matrices $\hat{w}^c$ and $\hat{w}$ are close in the sense that*

$$\hat{w}_j(z) = \hat{w}^c_j(z) + O(t^{-\alpha/2} \mathrm{e}^{-|z|^2/8}), \quad z \in \Sigma_j, \quad j = 1, \dots 4. \tag{B.13}$$

Furthermore, the error term is uniform with respect to parameters as stated in Theorem B.1.

Proof. We only give the proof $z \in \Sigma_1$, the other cases being similar. There is only one nonzero matrix entry in $\hat{w}_1(z) - \hat{w}^c_1(z)$ given by

$$W = \begin{cases} -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} + \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2}, & |z| \le \rho_0 t^{1/2}, \\ \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} & |z| > \rho_0 t^{1/2}. \end{cases}$$

A straightforward estimate for $|z| \le \rho_0 t^{1/2}$ shows

$$\begin{aligned}
|W| &= \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2})\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - \overline{r}|\mathrm{e}^{-|z|^2/2} \\
&= \mathrm{e}^{\nu\pi/4} \left| R_1(zt^{-1/2})\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - \overline{r}\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} + \overline{r}\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - \overline{r} \right| \mathrm{e}^{-|z|^2/2} \\
&\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{\mathrm{Re}(-t\hat\Phi(zt^{-1/2}))-|z|^2/2} \\
&\quad + \mathrm{e}^{\nu\pi/4} |\mathrm{e}^{-t\hat\Phi(zt^{-1/2})} - 1| \mathrm{e}^{-|z|^2/2} \\
&\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} t |\hat\Phi(zt^{-1/2})| \mathrm{e}^{-|z|^2/4},
\end{aligned}$$

where $\hat\Phi(z) = \Phi(z) - \Phi(0) - \frac{\mathrm{i}}{2}z^2 = \frac{\Phi'''(0)}{6}z^3 + \dots$. Here we have used $\frac{\mathrm{i}}{2}z^2 = \frac{1}{2}|z|^2$ for $z \in \Sigma_1$, $\mathrm{Re}(-t\hat\Phi(zt^{-1/2})) \le |z|^2/4$ by (B.2), and $|r| < 1$. Furthermore, by (B.3) and (B.4),

$$|W| \le \mathrm{e}^{\nu\pi/4} L t^{-\alpha/2} |z|^{\alpha} \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} C t^{-1/2} |z|^3 \mathrm{e}^{-|z|^2/4}, \qquad \text{(B.14)}$$

for $|z| \le \rho_0 t^{1/2}$. For $|z| > \rho_0 t^{1/2}$ we have

$$|W| \le \mathrm{e}^{\nu\pi/4} \mathrm{e}^{-|z|^2/2} \le \mathrm{e}^{\nu\pi/4} \mathrm{e}^{-\rho_0^2 t/4} \mathrm{e}^{-|z|^2/4}$$

which finishes the proof. □

The next lemma allows us to replace $\hat m(z)$ by $\hat m^c(z)$.

Lemma B.4. *Consider the Riemann–Hilbert problem*

$$\begin{aligned}
m_+(z) &= m_-(z) v(z), && z \in \Sigma, \\
m(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma.
\end{aligned}$$

Assume that $w \in L^2(\Sigma) \cap L^\infty(\Sigma)$. *Then*

$$\|\mu - \mathbb{I}\|_2 \le \frac{c\|w\|_2}{1 - c\|w\|_\infty} \qquad \text{(B.15)}$$

provided $c\|w\|_\infty < 1$, *where* c *is the norm of the Cauchy operator on* $L^2(\Sigma)$.

Proof. We know that $\tilde\mu = \mu - \mathbb{I} \in L^2(\Sigma)$ and satisfies $(\mathbb{I} - C_w)\tilde\mu = C_w \mathbb{I}$. Thus we can estimate $\tilde\mu$ by using Neumann series as follows:

$$\begin{aligned}
\|\tilde\mu\|_2 &= \left\|(\mathbb{I} - C_w)^{-1} C_w\right\|_2 \\
&= \left\|(\mathbb{I} + C_w + C_w^2 + \dots) C_w\right\|_2 \\
&\le \|C_w\|_2 + \left\|C_w^2\right\|_2 + \left\|C_w^3\right\|_2 + \dots \\
&\le c\,\|w\|_2 + c^2\,\|w\|_2\,\|w\|_\infty + c^3\,\|w\|_2\,\|w\|_\infty^2 + \dots \\
&\le c\,\|w\|_2\,(1 + c\,\|w\|_\infty + c^2\,\|w\|_\infty^2 + \dots) \\
&= c\,\|w\|_2 \frac{1}{1 - c\,\|w\|_\infty}.
\end{aligned}$$

Here we have used that

$$\|C_w(f)\|_2 \le c\,\|f\|_2\,\|w\|_\infty. \qquad \text{(B.16)}$$

□

Lemma B.5. *The solution $\hat{m}(z)$ of the Riemann–Hilbert problem (B.11) has a convergent asymptotic expansion*

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}(t) + O(\frac{1}{z^2}) \tag{B.17}$$

for $|z| > \rho_0 t^{1/2}$ with the error term uniformly in t. Moreover,

$$\hat{M}(t) = \hat{M}^c + O(t^{-\alpha/2}). \tag{B.18}$$

Proof. Consider $\hat{m}^d(z) = \hat{m}(z)\hat{m}^c(z)^{-1}$, whose jump matrix is given by

$$\begin{aligned}
\hat{v}^d(z) &= \hat{m}^c_-(z)\hat{v}(z)\hat{v}^c(z)^{-1}\hat{m}^c_-(z)^{-1} \\
&= \hat{m}^c_-(z)(\mathbb{I} + \hat{w}(z))(\mathbb{I} + \hat{w}^c(z))^{-1}\hat{m}^c_-(z)^{-1} \\
&= \hat{m}^c_-(z)(\mathbb{I} + \hat{w}(z))(\mathbb{I} - \hat{w}^c(z))\hat{m}^c_-(z)^{-1} \\
&= \hat{m}^c_-(z)(\mathbb{I} + \hat{w}(z) - \hat{w}^c(z) - \hat{w}(z)\hat{w}^c(z))\hat{m}^c_-(z)^{-1} \\
&= \mathbb{I} + \hat{m}^c_-(z)(\hat{w}(z) - \hat{w}^c(z))\hat{m}^c_-(z)^{-1}.
\end{aligned}$$

By Lemma B.3 we have that $\hat{w} - \hat{w}^c$ is decaying of order $t^{-\alpha/2}$ in the norms of L^1 and L^∞ and hence also in the norm of L^2. Thus the same is true for $\hat{w}^d = \hat{v}^d - \mathbb{I} = \hat{m}^c_-(z)(\hat{w}(z) - \hat{w}^c(z))\hat{m}^c_-(z)^{-1}$. Hence by the previous lemma

$$\|\hat{\mu}^d - \mathbb{I}\|_2 = O(t^{-\alpha/2}).$$

Furthermore, by $\hat{\mu}^d = \hat{m}^d_- = \hat{m}_-(\hat{m}^c_-)^{-1} = \hat{\mu}(\hat{\mu}^c)^{-1}$ we infer

$$\|\hat{\mu} - \hat{\mu}^c\|_2 = \left\|\hat{\mu}^d\hat{\mu}^c - \hat{\mu}^c\right\|_2 = O(t^{-\alpha/2})$$

since $\hat{\mu}^c$ is bounded. Now

$$\begin{aligned}
\hat{m}(z) &= \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \hat{\mu}(s)\hat{w}(s)\frac{1}{s-z}ds \\
&= \mathbb{I} - \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma \hat{\mu}(s)\hat{w}(s)\sum_{l=0}^\infty \left(\frac{s}{z}\right)^l ds \\
&= \mathbb{I} - \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma \hat{\mu}(s)\hat{w}(s)ds + \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma s\hat{\mu}(s)\hat{w}(s)\frac{ds}{s-z}
\end{aligned}$$

shows (recall that $\hat{w}$ has support inside $|z| \le \rho_0 t^{1/2}$)

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}(t) + O(\frac{\|\hat{\mu}(s)\|_2\|s\hat{w}(s)\|_2}{z^2}),$$

where

$$\hat{M}(t) = -\frac{1}{2\pi\mathrm{i}}\int_\Sigma \hat{\mu}(s)\hat{w}(s)ds.$$

Now the rest follows from

$$\hat{M}(t) = \hat{M}^c - \frac{1}{2\pi\mathrm{i}}\int_\Sigma(\hat{\mu}(s)\hat{w}(s) - \hat{\mu}^c(s)\hat{w}^c(s))ds$$

using $\|\hat{\mu}\hat{w} - \hat{\mu}^c\hat{w}^c\|_1 \le \|\hat{w} - \hat{w}^c\|_1 + \|\hat{\mu} - \mathbb{I}\|_2\|\hat{w} - \hat{w}^c\|_2 + \|\hat{\mu} - \hat{\mu}^c\|_2\|\hat{w}^c\|_2$. □

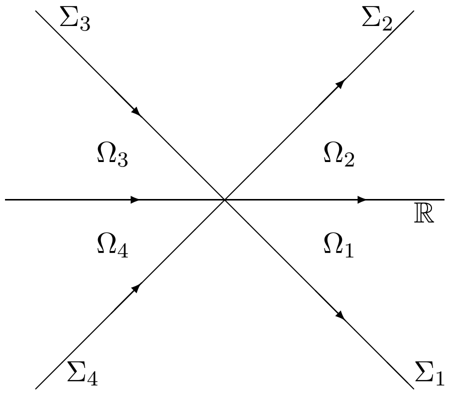

Figure B.2.: Deforming back the cross. [Krüger09b]

B.2. Solving the Riemann–Hilbert problem on a small cross with constant jumps

In Section B.1 we have shown that the solution of the rescaled Riemann–Hilbert problem (B.11) is close to the solution of the Riemann–Hilbert problem (B.12) with jumps being constant with respect to t. Thus, it remains to show:

Theorem B.6. *The solution of the Riemann–Hilbert problem (B.12) is of the form*

$$\hat{m}^c(z) = \mathbb{I} + \frac{1}{z}\hat{M}^c + O(\frac{1}{z^2}), \tag{B.19}$$

where

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix}, \quad \beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}. \tag{B.20}$$

The error term is uniform with respect to r in compact subsets of $\mathbb{D}$. Moreover, the solution is bounded (again uniformly with respect to r).

The aim to prove this Theorem will be the task of the remainder of this section. In order to solve (B.12) we begin with a deformation which moves the jump to $\mathbb{R}$ as follows. Denote the region enclosed by $\mathbb{R}$ and Σ_j as Ω_j, $j = 1, \dots, 4$, as indicated in Figure B.2 and define a new Riemann–Hilbert problem by introducing

$$\tilde{m}^c(z) = \hat{m}^c(z) \begin{cases} D_0(z) D_j, & z \in \Omega_j,\ j = 1, \dots, 4, \\ D_0(z), & \text{else}, \end{cases} \tag{B.21}$$

where

$$D_0(z) = \begin{pmatrix} z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix},$$

and

$$D_1 = \begin{pmatrix} 1 & \overline{r} \\ 0 & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad D_3 = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} \\ 0 & 1 \end{pmatrix} \quad D_4 = \begin{pmatrix} 1 & 0 \\ -\frac{r}{1-|r|^2} & 1 \end{pmatrix}.$$

Lemma B.7. *The function $\tilde{m}^c(z)$ defined in (B.21) satisfies the Riemann–Hilbert problem*

$$\begin{aligned} \tilde{m}^c_+(z) &= \tilde{m}^c_-(z) \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}, \quad z \in \mathbb{R} \\ \tilde{m}^c(z) &= (\mathbb{I} + \frac{1}{z}\hat{M}^c + \dots)D_0(z), \quad z \to \infty, \ \frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}. \end{aligned} \tag{B.22}$$

Proof. First, one checks that $\tilde{m}^c_+(z) = \tilde{m}^c_-(z)D_0(z)^{-1}\hat{v}^c_1(z)D_0(z)D_1 = \tilde{m}^c_-(z)$, $z \in \Sigma_1$ and similarly for $z \in \Sigma_2, \Sigma_3, \Sigma_4$. To compute the jump along $\mathbb{R}$ observe that, by our choice of branch cut for $z^{\mathrm{i}\nu}$, $D_0(z)$ has a jump along the negative real axis given by

$$D_{0,\pm}(z) = \begin{pmatrix} \mathrm{e}^{(\log|z|\pm\mathrm{i}\pi)\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & \mathrm{e}^{-(\log|z|\pm\mathrm{i}\pi)\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix}, \qquad z < 0.$$

Hence the jump along $\mathbb{R}$ is given by

$$D_1^{-1}D_2, \quad z > 0 \quad \text{and} \quad D_4^{-1}D_{0,-}^{-1}(z)D_{0,+}(z)D_3, \quad z < 0,$$

and (B.22) follows after recalling $\mathrm{e}^{-2\pi\nu} = 1 - |r|^2$. □

Now, we can follow [Deift93, (4.17)–(4.51)] to construct an approximate solution.

The idea is the following: taking a look at the Riemann–Hilbert problem (B.22) one easily sees that the derivative $\frac{d}{dz}\tilde{m}^c(z)$ has the same jump as $\tilde{m}^c(z)$ and hence

$$\frac{d}{dz}\tilde{m}^c(z) = n(z)\tilde{m}^c(z),$$

where $n(z)$ is an entire matrix which can be determined from the behavior $z \to \infty$. Since this will just serve as a motivation for our ansatz, we will not worry about justifying any steps.

For z in the sector $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$ (enclosed by Σ_2 and Σ_3) we have $\tilde{m}^c(z) = \hat{m}^c(z)D_0(z)$ and hence

$$\begin{aligned} &\left(\frac{d}{dz}\tilde{m}^c(z) + \frac{\mathrm{i}z}{2}\sigma_3\tilde{m}^c(z)\right)\tilde{m}^c(z)^{-1} \\ &\quad = \left(\mathrm{i}(\frac{\nu}{z} - \frac{z}{2})\hat{m}^c(z)\sigma_3 + \frac{d}{dz}\hat{m}^c(z) + \mathrm{i}\frac{z}{2}\sigma_3\hat{m}^c(z)\right)\hat{m}^c(z)^{-1} \\ &\quad = \frac{\mathrm{i}}{2}[\sigma_3, \hat{M}^c] + O(\frac{1}{z}), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Here we assumed that the solution of the Riemann–Hilbert problem (B.12) is given by (B.19) and inserted it. Since the left hand side has no jump, it is entire and hence by Liouville's theorem a constant given by the right hand side. More precisely,

$$\frac{d}{dz}\tilde{m}^c(z) + \frac{\mathrm{i}z}{2}\sigma_3\tilde{m}^c(z) = \beta\tilde{m}^c(z), \quad \beta = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix} = \frac{\mathrm{i}}{2}[\sigma_3, \hat{M}^c]. \tag{B.23}$$

This differential equation can be solved in terms of parabolic cylinder function which then gives the solution of (B.22).

Lemma B.8. *The Riemann–Hilbert problem (B.22) has a unique solution, and the term $\hat{M}^c$ is given by*

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta_{12} \\ \beta_{21} & 0 \end{pmatrix}, \qquad \beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}. \tag{B.24}$$

Proof. Uniqueness follows by the standard Liouville argument since the determinant of the jump matrix is equal to 1. We find the solution using the ansatz

$$\tilde{m}^c(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix}.$$

From (B.23) we can conclude that the functions $\psi_{jk}(z)$ satisfy

$$\psi_{11}''(z) = -\left(\frac{\mathrm{i}}{2} + \frac{1}{4}z^2 - \beta_{12}\beta_{21}\right)\psi_{11}(z), \qquad \psi_{12}(z) = \frac{1}{\beta_{21}}\left(\frac{d}{dz} - \frac{\mathrm{i}z}{2}\right)\psi_{22}(z),$$
$$\psi_{21}(z) = \frac{1}{\beta_{12}}\left(\frac{d}{dz} + \frac{\mathrm{i}z}{2}\right)\psi_{11}(z), \qquad \psi_{22}''(z) = \left(\frac{\mathrm{i}}{2} - \frac{1}{4}z^2 + \beta_{12}\beta_{21}\right)\psi_{22}(z).$$

That is, $\psi_{11}(\mathrm{e}^{3\pi\mathrm{i}/4}\zeta)$ satisfies the parabolic cylinder equation

$$D''(\zeta) + \left(a + \frac{1}{2} - \frac{1}{4}\zeta^2\right)D(\zeta) = 0$$

with $a = \mathrm{i}\beta_{12}\beta_{21}$ and $\psi_{22}(\mathrm{e}^{\mathrm{i}\pi/4}\zeta)$ satisfies the parabolic cylinder equation with $a = -\mathrm{i}\beta_{12}\beta_{21}$.

Let D_a be the entire parabolic cylinder function defined in [Whittaker27, §16.5] and set

$$\psi_{11}(z) = \begin{cases} \mathrm{e}^{-3\pi\nu/4}D_{\mathrm{i}\nu}(-\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{\pi\nu/4}D_{\mathrm{i}\nu}(\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0, \end{cases}$$
$$\psi_{22}(z) = \begin{cases} \mathrm{e}^{\pi\nu/4}D_{-\mathrm{i}\nu}(-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{-3\pi\nu/4}D_{-\mathrm{i}\nu}(\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0. \end{cases}$$

Using the asymptotic behavior

$$D_a(z) = z^a\mathrm{e}^{-z^2/4}\big(1 - \frac{a(a-1)}{2z^2} + O(z^{-4})\big), \quad z \to \infty, \quad |\arg(z)| \le 3\pi/4,$$

shows that the choice $\beta_{12}\beta_{21} = \nu$ ensures the correct asymptotics

$$\begin{aligned}
\psi_{11}(z) &= z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1 + O(z^{-2})), \\
\psi_{12}(z) &= -\mathrm{i}\beta_{12}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})), \\
\psi_{21}(z) &= \mathrm{i}\beta_{21}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})), \\
\psi_{22}(z) &= z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1 + O(z^{-2})),
\end{aligned}$$

as $z \to \infty$ inside the half plane $\mathrm{Im}(z) \geq 0$. In particular,

$$\tilde{m}^c(z) = \big(\mathbb{I} + \frac{1}{z}\hat{M}^c + O(z^{-2})\big) D_0(z) \quad \text{with} \quad \hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

It remains to check that we have the correct jump. Since by construction both limits $\tilde{m}^c_+(z)$ and $\tilde{m}^c_-(z)$ satisfy the same differential equation (B.23), there is a constant matrix v such that $\tilde{m}^c_+(z) = \tilde{m}^c_-(z)v$. Moreover, since the coefficient matrix of the linear differential equation (B.23) has trace 0, the determinant of $\tilde{m}^c_\pm(z)$ is constant and hence $\det(\tilde{m}^c_\pm(z)) = 1$ by our asymptotics. Moreover, evaluating

$$v = \tilde{m}^c_-(0)^{-1}\tilde{m}^c_+(0) = \begin{pmatrix} \mathrm{e}^{-2\pi\nu} & -\frac{\sqrt{2\pi}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(\mathrm{i}\nu)}\gamma^{-1} \\ \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}\gamma & 1 \end{pmatrix}$$

where $\gamma = \frac{\sqrt{\nu}}{\beta_{12}} = \frac{\beta_{21}}{\sqrt{\nu}}$. Here we have used

$$D_a(0) = \frac{2^{a/2}\sqrt{\pi}}{\Gamma((1-a)/2)}, \qquad D_a'(0) = -\frac{2^{(1+a)/2}\sqrt{\pi}}{\Gamma(-a/2)}$$

plus the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ for the Gamma function. Hence, if we choose

$$\gamma = \frac{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}} r,$$

we have

$$v = \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}$$

since $|\gamma|^2 = 1$. To see this use $|\Gamma(-\mathrm{i}\nu)|^2 = \frac{\Gamma(1-\mathrm{i}\nu)\Gamma(\mathrm{i}\nu)}{-\mathrm{i}\nu} = \frac{\pi}{\nu\sinh(\pi\nu)}$ which follows from Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for the Gamma function.

In particular,

$$\beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}$$

which finishes the proof. □

Now we are ready to prove our main theorem.

Proof of Theorem B.1. Using Theorem B.6 and Lemma B.5 we can compute

$$\begin{aligned} m(z) &= D(t)\hat{m}(zt^{1/2})D(t)^{-1} = \mathbb{I} + \frac{1}{t^{1/2}z}D(t)\hat{M}(t)D(t)^{-1} + O(z^{-2}t^{-1}) \\ &= \mathbb{I} + \frac{1}{t^{1/2}z}D(t)\hat{M}^c D(t)^{-1} + O(t^{-(1+\alpha)/2}) \end{aligned}$$

for $|z| > \rho_0$, where we have used that $D(t)$ is bounded. □

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Curriculum vitae

Personal data

Name:	Alice Mikikits-Leitner
Date of birth:	August 31st, 1984
Place of birth:	Vienna, Austria
Nationality:	Austria
Marital status:	Married

Education

1990–1994:	Volksschule Wittelsbachstraße, Vienna
1994–2002:	Bundesrealgymnasium Kleine Sperlgasse, Vienna
June 2002:	Matura (school leaving examination)
2002- 2007:	undergraduate studies (mathematics) at the Faculty of Mathematics, University of Vienna
June 2007:	Mag. rer. nat. (MSc) in mathematics, University of Vienna
2004 - 2009:	undergraduate studies (physics) at the Faculty of Physics, University of Vienna
October 2009:	Mag. rer. nat. (MSc) in physics, University of Vienna
2007 - 2009:	PhD studies (mathematics) at the University of Vienna

Positions

March 2007 - Feb. 2008:	Teaching Assistant (Tutor), Faculty of Mathematics, University of Vienna
March 2008 - June 2008:	Teaching Assistant (Tutor), University of Natural Resources and Applied Life Sciences (Boku), Vienna
Sept. 2007 - present:	Research Assistant fully supported by the FWF START project Y-330

Scientific talks

29 May 2008:	"Almost everything you ever wanted to know about the Korteweg-de Vries equation", Colloquium for Master and PhD students, University of Vienna, Austria.
25 September 2009:	"Stability of the periodic Korteweg–de Vries equation", ÖMG/DMV-Congress, Graz, Austria.

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