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„Classification of tight contact structures on the solid torus and some handlebodies."

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Abstract

In this paper we present a complete classification of tight contact structures on the solid torus and a classification of tight contact structures on some handlebodies. After we introduce some fundamental ideas of contact geometry such as Giroux's theory of convex surfaces we will introduce the cutting and and gluing argument introduced by Honda that leads to a combinatorial algorithm that classifies tight contact structures on handlebodies. In the following we are going to apply this algorithm to the solid torus and handlebodies.

In diesem Papier präsentieren wir eine komplette Klassifikation von straffen Kontaktstrukturen auf dem Volltorus und eine Klassifikation von straffen Kontaktstrukturen auf verschiedenen Handkörpern. Nach einer Einführung in die fundamentalen Ideen der Kontaktgeometrie, wie Giroux's Theorie von konvexen Flächen, werden wir die zerschneide und klebe Argumente von Honda diskutieren. Diese führen zu einem combinatorischen Algorithmus zur Klassifikation der straffen Kontaktstrukturen auf Handkörpern. Im Folgenden werden wir diesen Algorithmus auf den Volltorus und Handkörper anwenden.

Aknowledgements

I would like to dedicate this thesis to both of my parent. Their tremendous support through my whole life and especially through my study time helped me to become the person I am today and to peruse my interest in mathematics. I would also like to thank my brother, my grandparents and my family. Special thanks also to my advisor Vera Vértesi.

Contents

1 Introduction

3-dimensional contact geometry is one of the major fields of low-dimensional topology. Roughly speaking a contact manifold of dimension $2n+1$ is a manifold M together with a hyperplane field ξ that admits a non-degeneracy condition (see Definition 2.1). In dimension 3 this translates to a non-integrable plane field. Contact structures are special in the field of geometry, in so far, that they do not admit any local invariants (see Theorem 2.9).

Although there are no local invariants in contact manifolds, there exists a dichotomy of contact structures into *tight* and *overtwisted* contact structures. We call an embedded disk into a contact 3-manifold *overtwisted* if its boundary is tangent to the hyperplane field. A contact structure is *overtwisted* if it contains an overtwisted disk and *tight* otherwise. This dichtomy first was discovered by Bennequin [1] and further developed by Eliashberg [2], a main result being, that there is a unique tight contact structure on B^3 for a fixed boundary characteristic foliation. This Theorem of Eliashberg is one of the main building block of the cut and paste techniques introduced by Honda [7].

Another building block is Giroux's [5] theory of convex surfaces. Roughly speaking an embedded surface $\Sigma \subset M$ is *convex* if it admits an neighborhood where ξ is I-invariant. Due to results by Giroux including the flexebility Theorem (see 2.21) we can understand the tubular neighbourhood of Σ by the isotopy class of a multicurve Γ on Σ which we call the *dividing set* (see Theorem 2.16).

Let M be a handlebody of genus g with convex boundary, boundary characteristic foliation $\mathcal{F}$ and dividing set Γ. Honda then introduced in [7] a technique to cut open M along convex compressing disks $D_1, \ldots, D_g \subset M$ rounding the edges, to obtain a manifold diffeomorfic to B^3 with a well defined contact structure. For M to be tight it is a necessary condition, that the contact structure on B^3 is tight. Eliashberg's result regarding a unique tight contact structure on B^3 leads to the idea, that the dividing sets on $D_1, \ldots, D_g$ give rise to a unique contact structure on M that is tight on B^3. Honda showed, that the condition for the contact structure on B^3 to be tight is not sufficient of ξ to be tight.

To find a sufficient criterion for the tightness of ξ Honda [8] defined an combinatorial algorithm, only depending on the dividing set on $\partial M, D_1, \ldots, D_g$. He therefore introduced a directed graph G. The vertices (called *configurations*) of that graph are the possible dividing sets for $D_1, \ldots, D_g$ that do not have homotopically trivial curves. A dividing set on D_k consists of n_k arcs where $n_k = 1/2\#(D_k \cap \Gamma)$. Therefore the number of vertices is finite. We then call a vertex potentially allowable if it gives rise to a tight contact structure on B^3. The edges, called *allowable state transition*, are transitions of the dividing set of one D_k that are induce by a theoretical *bypass attachment* that is trivial on the cut open manifold B^3. The bypass is a tool introduced by Honda [7] (see

2.27). Honda then proves, that the connected components of G that only contain potentially allowable vertices are in a 1-1 correspondence with the tight contact structures on M with boundary characteristic foliation $\mathcal{F}$. The vertices of such a component are called *allowable*. This algorithm is really remarkable as it reduces a complicated geometrical question to a combinatorial algorithm.

After introducing everything in detail we are going to apply Honda's algorithm to the solid Torus $S^1 \times D^2$. A dividing set Γ on $\partial(S^1 \times D^2)$ can be classified by (n, p, q) (introduced by Honda) where $2n$ is the number of components of Γ, p is the number of times each component of Γ twists around $S^1 \times \{0\}$ and q is the number of times each component twists around $\{0\} \times \partial D^2$. We then denote the number on tight contact structures related to (n, p, q) by $N(n, p, q)$. We will argue, that $N(n, p, q) = N(n, -p, -q) = N(n, p, q \pm p)$ and hence we assume $p > q \geq 0$. At first we are going to find some easy allowable state transitions that depend on *boundary parallel arcs* of the dividing set Γ_1 of D_1 which are essentially arcs that cut off a half disc of D_1 (see 3.23). This then gives, that we can connect Γ_1 to some $\tilde{\Gamma}_1$ via allowable state transitions, which contains a specific structure of boundary parallel arcs called *boundary parallel set* (see 3.27). We then prove that the algorithm without changing that boundary parallel set can be viewed as the algorithm for $(n(n, p, q), p(n, p, q), q(n, p, q))$ (called the *reduction* of (n, p, q)) where $n(n, p, q) \cdot p(n, p, q) < np$ (see 3.31).

We then want to understand $(n(n, p, q), p(n, p, q), q(n, p, q))$. For $n \geq 2$ we have $(n(n, p, q), p(n, p, q), q(n, p, q)) = (n-1, p, q)$ (see 3.42). Consider the continuous fraction expansion of p/q.

$$p/q = [r_0, \dots, r_h] = r_0 - \cfrac{1}{r_1 - \frac{1}{r_2 - \dots \frac{1}{r_h}}}, \text{ for } r_i \geq 2.$$

For $n = 1$ let $\tilde{p}/\tilde{q} = [r_0, \dots, r_h - 1]$, we then have $(n(n, p, q), p(n, p, q), q(n, p, q)) = (1, \tilde{p}, \tilde{q})$ (see 3.42).

We are going to introduce *normal forms* using the reduction Theorem and the possibility of achieving boundary parallel sets. Normal forms are specific dividing sets that any allowable configuration can be connected to via allowable state transition. By counting those normal forms we find an upper bound for the number of tight contact structures

$$N(n, p, q) \leq C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1).$$

In the last section we are going to prove that all normal forms are allowable see 3.65 and that two normal forms don't lie in the same connected component of G which essentially is the following theorem:

Theorem 1.1.

$$N(n, p/q = [r_0, \dots, r_h]) = C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1).$$

This equality first was discovered by Zhenkun Li and Jessica J. Zhang [10]. In the last chapter we are going to state some results for more general handlebodies.

2 Preliminaries

In this section we introduce the basic notations and ideas that needed to understand and apply Honda's algorithm [8]. We will further explain and introduce the algorithm.

2.1 Contact structures

This section serves as a brief introduction into contact geometry.

2.1.1 First definitions

Definition 2.1. A *contact manifold* (M, ξ) is a $(2n+1)$ dimensional manifold M equipped with a maximally nonintegrable hyperplane field ξ i.e. locally there is a 1-form α such that $\xi = \ker \alpha$ with $\alpha \wedge d\alpha^n \neq 0$. We then call ξ a *contact structure* and α a *contact 1-form* that locally defines ξ. A *contact diffeomorphism* or *contactomorphism* $\phi : (M_1, \xi_1) \to (M_2, \xi_2)$ between two contact structures (M_1, ξ_1) and (M_2, ξ_2) is a diffeomorphism such that $\phi_* \xi_1 = \xi_2$.

The object of this thesis is to study tight contact structures on handlebodies, hence we will concentrate on the 3-dimensional case i.e. a locally defining contact 1-form α should satisfy $\alpha \wedge d\alpha \neq 0$.

Definition 2.2. A contact 3-manifold (M, ξ) is called *positive* if it is cooriented [1] by a global contact 1-form with $\alpha \wedge d\alpha > 0$.

Remark 2.3. A locally (respectively a globally) defining 1-form α is unique up to multiplication by a nonzero smooth function f and as

$$(f\alpha) \wedge d(f\alpha) = (f\alpha) \wedge (df \wedge \alpha + f d\alpha) = f^2 \alpha \wedge d\alpha,$$

all locally defining 1-forms satisfy the non integrability condition. Furthermore if ξ is positive, then all global contact 1-forms α satisfy $\alpha \wedge d\alpha > 0$.

Furthermore in dimension 3 the non-integrability condition implies local non-integrability in the sense that for $\eta, \nu \in \xi$ being two local vector fields we have $[\eta, \nu](x) \notin \xi(x)$ for all $x \in M$ where η, ν. This is easy to see as $(\alpha \wedge d\alpha)(-, \eta, \nu) = \alpha(-) \cdot d\alpha(\eta, \nu) = \alpha(-) \cdot -\alpha([\eta, \nu])$ which then implies $\alpha([\eta, \nu])(u) \neq 0$.

[1] A coorientation for ξ is by definition a choice of trivialization to TM/ξ. This is equivalent to a choice of a global 1-form α with kernel ξ.

Example 2.4. Let $(\mathbb{R}^3, \xi_0)$ be a contact structure, where ξ_0 is given by the globally defining 1-form $\alpha_0 = dx_3 - x_2 dx_1$. This contact structure is called *standard contact structure* see Figure 2.1. In fact the non integrability condition is satisfied as

$$(dx_3 - x_2dx_1) \wedge d(dx_3 - x_2dx_1) = (dx_3 - x_2dx_1) \wedge -dx_2 \wedge dx_1 = dx_1 \wedge dx_2 \wedge dx_3.$$

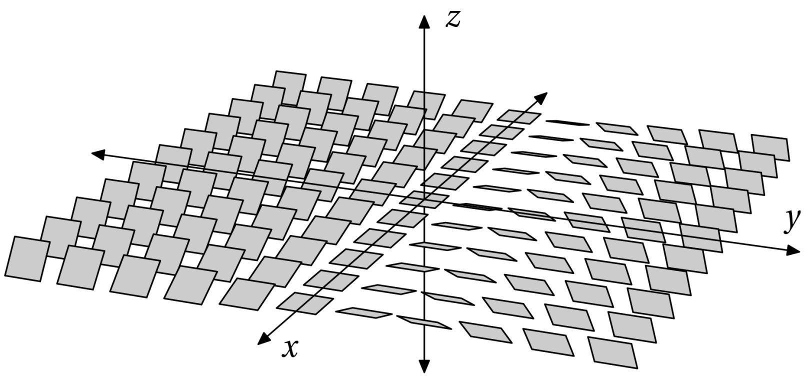

Figure 2.1: The picture shows the standard contact structure ξ_0.(The picture is from [12])

The standard contact structure on $\mathbb{R}^3$ gives rise to another description of a contact structure:

Theorem 2.5. ([6, p.3]) *Every contact* 3*-manifold* (M, ξ) *locally looks like* $(\mathbb{R}^3, \xi_0)$, *i.e. for every* $p \in M$ *there exist an open set* $p \in U \subseteq M$ *and* $0 \in V \subseteq \mathbb{R}^3$ *and a contactomorphism* $\phi : (U, \xi) \to (V, \xi_0)$.

Remark 2.6. The non-integrability property in Definition 2.1 is a local property. Therefore the preceding Theorem gives rise to another definition of a contact 3-manifold i.e. a 3-manifold M together with a hyperplane field ξ, such that (M, ξ) locally looks like $(\mathbb{R}^3, \xi_0)$.

In some literature contact manifolds are even defined as above.

Definition 2.7. Let (M, ξ) be a contact 3-manifold. A *contact vector field* $X \in \mathfrak{X}(M)$ is a vector field such that $\mathrm{Fl}_t^X : (M, \xi) \to (M, \xi)$ is a contact diffeomorphism for all t.

Definition 2.8. Let (M, ξ) be a contact 3-manifold.

i) A properly embedded disk $D \subset M$ is called an *overtwisted disk* iff $\xi = T_pD$ for all $p \in \partial D$.

ii) (M, ξ) is called *overtwisted* iff it contains an overtwisted disk.

iii) (M, ξ) is called *tight* iff it is not overtwisted.

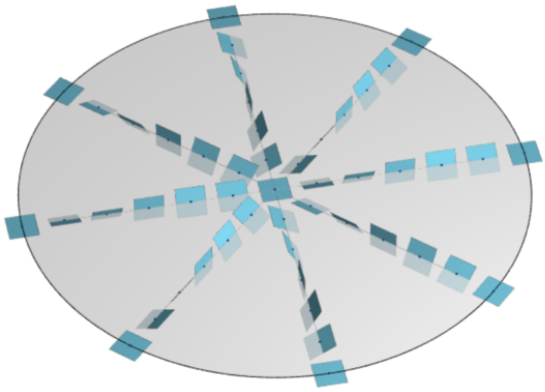

Figure 2.2: The picture shows an overtwisted disk. (The picture is from [11])

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Figure 2.2 shows an overtwisted disk. This gives rise to a dichotomy of contact structures into tight and overtwisted ones. There are more general definitions of tight and overtwisted contact structures that are equivalent to the one above and give a better understanding of this dichotomy. For this thesis it is enough to have a rough understanding of what a tight contact structure is and thus we are not going to go into that further. If the reader is interested into that, more information about overtwisted contact structures and the dichotomy can be found at [4, section 4.5].

Theorem 2.9 (Bennequin)**.** *The standard contact structure* $(\mathbb{R}^3, \xi_0)$ *is tight.*

This is an important result as every contact structure locally looks like the standard one (as mentioned in Theorem 2.5). Hence being tight respectively overtwisted is a global invariant of a contact structure.

2.1.2 Characteristic foliations

An important tool to study contact 3-manifolds is to study properly embedded surfaces. Let Σ be such a properly embedded surface, then a contact structure induces a singular line field on Σ called the *characteristic foliation*. The characteristic foliation gives rise to a classification of the contact structure in a neighborhood of Σ.

Definition 2.10. Let (M, ξ) be a contact 3-manifold and let Σ be a properly embedded surface. The *characteristic foliation* induced on Σ from ξ is the singular foliation $\Sigma_\xi(p) = \Sigma_p \cap \xi_p$. The singular points then are the points p where $T_p\Sigma = \xi_p$.

Integration of the characteristic foliation gives rise to *orbits* (also called *leaves*) of non-singular points.

Remark 2.11. Let (M, ξ) be positive and Σ an oriented properly embedded surface. Let p be a non-singular point of Σ_ξ and $n \in T_p\Sigma$ an oriented normal vector for ξ. Now choose $v \in T_pL$ such that (v, n) is an oriented basis of $T_p\Sigma$. This imposes an orientation on the leaf of p and thus on the characteristic foliation Σ_ξ.

Definition 2.12. We call an isolated singularity *positive* resp. *negative* if the orientation of Σ coincides resp. does not coincide with the orientation of ξ at that point.

Let us give a short description of isolated singularities and closed orbits. There are two types of singularities that are hyperbolic in the dynamical sense i.e. *nodes* and *saddles*. Let us choose coordinates (x, y) on Σ such that the origin is an isolated singularity. Let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ be a local vector field that represents Σ_ξ. If the determinant of the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

is positive resp. negative at that origin, we call the singularity *node* resp. *saddle*.
Let γ be a closed orbit and let $S^1 \times [\varepsilon, -\varepsilon]$ be an annular neighborhood of γ (where $\gamma = S^1 \times \{0\}$). The *return* map $\Phi : [\varepsilon, -\varepsilon] \to [\varepsilon, -\varepsilon]$ defined in a small neighborhood of $(0, 0)$ then maps s to t where $(0, t)$ is the subsequent point where the orbit of $(0, s)$ intersects $\{0\} \times [\varepsilon, -\varepsilon]$. If $(0, t)$ does not exist Φ is not defined on s. We say γ is *hyperbolic* if $d\phi(0)$ is not the identity.

Theorem 2.13. *Let Σ be a C^∞-generic closed oriented surface in a contact 3-manifold (M, ξ), then its characteristic foliation is of Morse-Smale type, i.e.:*

i) The singularities and closed orbits are hyperbolic in the dynamical sense,

ii) there are no saddle-saddle connections [2]*,*

iii) every $p \in \Sigma$ limits to some isolated singularity or closed orbit in forward and in backward time.

2.1.3 Twisting number

Definition 2.14. Let (M, ξ) be a cooriented contact 3 manifold. We then call an integral 1-dimensional manifolds L of ξ Legendrian.

The coorientation of ξ gives a framing on L i.e. a choice of homotopy class of trivializations of the normal bundle to L [3]. For a framing Fr we define the *twisting number* $t(L, Fr)$ to be the number of counterclockwise 2π twists of ξ along L relative to Fr. If L is a connected component of the boundary of a compact surface Σ, $T\Sigma$ induces a natural framing Fr_Σ. If Σ is a Seifert surface i.e. $\partial\Sigma = L$ we call $tb(L) := t(L, Fr_\Sigma)$ the *Thurston-Bennequin invariant.*

[2] This means, that there are no obits connecting two saddle singularities.
[3] The *normal* bundle of L is defined as $TM|_L/TL$.

2.1.4 Convex surfaces

In this section we will introduce the notion of a convex surface (introduced by Giroux [5]). Such surfaces give rise to a description of the contact manifold in their small neighborhood and are comparatively well understood do to flexibility and genericity results we will discuss here.

Definition 2.15. An oriented, properly embedded surface Σ in a contact 3-manifold (M, ξ) is called *convex* if there exists a contact vector field $X \in \mathfrak{X}(M)$ that is transverse to Σ.

We will assume that X coincides with the normal orientation of Σ. Furthermore we assume that all convex surfaces are either closed, or compact with Legendrian boundary. As we assume M to be compact we have that X is complete and hence that there is a smooth map

$$\begin{array}{ccc} \Sigma \times \mathbb{R} & \to & M \\ (x,t) & \mapsto & \mathrm{Fl}_t^X(x) \end{array}$$

As the flow is transverse to Σ, this gives rise to a region $\Sigma \times (-\varepsilon, \varepsilon)$ that is a diffeomorphism onto its image. Thus on $\Sigma \times (-\varepsilon, \varepsilon)$ we can write the contact 1-form α as

$$\alpha = f dt + \beta,$$

with $f \colon \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\beta(x,t)$ a 1-form on Σ for fixed t. As the flow of a contact vector field by definition is a contactomorphism for fixed t we have that f and β are independent of t. The condition that our contact structure is positive can be written as

$$\begin{aligned} (f dt + \beta) \wedge (df \wedge dt + d\beta) &= f dt \wedge d\beta + \beta \wedge d\beta + \beta \wedge df \wedge dt && (2.1) \\ &= f dt \wedge d\beta + \beta \wedge df \wedge dt > 0. && (2.2) \end{aligned}$$

The second equality holds as $\beta(x,t)$ is a 1-form on Σ that is independent of t and hence $i_{dt} d\beta = 0$.

Definition 2.16. Let (M, ξ) be a contact 3-manifold and Σ a convex surface with contact vector field $X \in \mathfrak{X}(M)$. We then call $\Gamma_\Sigma = \{X \in \xi\}$ the *dividing set.*

If we use the representation of a neighborhood of Σ above we can also write $\Gamma_\Sigma = \{f = 0\}$. This is not hard to see as by choice $\frac{\partial}{\partial t}$ coincides with the contact vector field X. The dividing set is an important tool to study convex surfaces. Let us now look at some properties of the dividing set introduced by Giroux [5] (the formulation can be found in [6]):

1. Γ_Σ is a multicurve i.e. a properly embedded 1-manifold.

   Proof. If $f = 0$ equation (2.2) becomes $\beta \wedge df \wedge dt > 0$ and hence $df \neq 0$ and thus $f = 0$ is an embedded 1-manifold. □

2. $\Gamma_\Sigma \pitchfork \Sigma_\xi$.

Proof. As above for $f = 0$ we have $\beta \wedge df \wedge dt > 0$. As f is independent of t we have that df is independent of t. Now let $v \in \Sigma_\xi$ then $df(v) \neq 0$ because else we would have $\beta(p,0) = df(p,0)$ and thus $\beta \wedge df \wedge dt = 0$ at $(p,0)$. Hence on Γ_Σ we have $\{df = 0\}$ is transverse to $\{\beta = 0\}\}$. □

3. Let $R_+ = \{f > 0\} = \{\alpha(v) > 0\} \subset \Sigma$ and $R_- = \{f < 0\} = \{\alpha(v) > 0\} \subset \Sigma$. We then have, that Σ_ξ goes from R_+ to R_-

   Proof. From 2. we have that for $p \in \Gamma_\Sigma$ that $\beta(p,0) \wedge df(p,0) > 0$. Now let $n \in T_p\Sigma$ with $\beta(n) > 0$, $df(n) = 0$ and $v \in T_p\Sigma$ with $df(v) < 0$. We then have (v,n) is an oriented basis for Σ as

   $$(\beta \wedge df)(v,n) = \beta(v) * df(n) - \beta(n) * df(v) = -\beta(n) * df(v) > 0.$$

   □

4. The isotopy class of Γ_Σ is independent of the contact vector field v.

5. Γ_Σ is nonempty.

   Proof. We have that $\alpha = f dt + \beta$. Lets assume Γ_Σ is empty. This is equivalent to $f \neq 0$ and thus we can choose another 1-form $\alpha' = \frac{\alpha}{f} = dt + \beta'$. The non-integrability condition then reads as $dt \wedge d\beta' > 0$ and hence we have $d\beta' > 0$. As β' resp. $d\beta'$ are

   $$\int_\Sigma d\beta' > 0.$$

   Furthermore we have

   $$\int_\Sigma d\beta' = \int_{\partial\Sigma} \beta' = 0,$$

   as $\partial\Sigma$ is either empty or Legendrian. Thus Γ_Σ can not be empty. □

We now introduce the *Legendrian realization principle*, which tells us when a multicurve on a convex surface Σ can be made into a Legendrian one after an isotopy as in the Flexibility theorem. This is an important ingredient for the cutting procedure that is needed to apply Honda's algorithm.

Definition 2.17. A union C of closed curves $\gamma_1, \ldots, \gamma_r$ on a convex surface Σ is called *nonisolating* if the following conditions hold

1. $C \pitchfork \Gamma_\Sigma$,

2. γ_i and γ_j, $i \neq j$ are pairwise disjoint,

3. every component of $\Sigma \backslash (\Gamma_\Sigma \cup C)$ has a boundary component that intersects Γ_Σ.

Theorem 2.18 (Legendrian realization principle [7]). *Let C be a nonisolating collection of closed curves on a convex surface Σ which is closed or compact with Legendrian boundary. Then there exists an allowable isotopy ϕ_t, $t \in [0,1]$ so that:*

1. $\phi_0 = id$,
2. $\phi_t(\Sigma)$ *are convex for all* t,
3. ϕ_t *leaves* $\partial\Sigma$ *and* Γ_Σ *fixed,*
4. $\phi_1(C)$ *is Legendrian.*

Definition 2.19. Let (M,ξ) be a contact 3-manifold. Let $\Sigma \subset M$ be a surface. We say Σ has *standard collared boundary* if every boundary component L of Σ has a *standard collar* i.e. an annular neighborhood $A = L \times [0,1] = \mathbb{R}/\mathbb{Z} \times [0,1] \subset \Sigma$ with coordinated (x,y) and $L = \mathbb{R}/\mathbb{Z} \times \{0\}$. Furthermore there is a neighborhood $A \times [-1,1]$ with coordinates (x,y,t) and contact 1-form $\alpha = \sin(2\pi nx)dy + \cos(2\pi nx)dt$ with $\xi = \{\alpha = 0\}$.

Proposition 2.20 ([7]). *Let (M,ξ) be a contact* 3 *manifold. Let $\Sigma \subset M$ be a compact, oriented, properly embedded surface with Legendrian boundary, such that every boundary-component L satisfies $t(L, Fr_\Sigma) \leq 0$. Then there exists a C^0-small perturbation near $\partial\Sigma$ (fixing $\partial\Sigma$) such that Σ has standard collared boundary, followed by a C^∞-small perturbation (fixing an annular neighborhood of $\partial\Sigma$), which makes Σ convex.*

Theorem 2.21. *A C^∞-generic closed surface Σ is convex.*

Definition 2.22. Let Σ be a surface, Γ_Σ its dividing set and $\mathcal{F}$ be a singular foliation on Σ. We say $\mathcal{F}$ is *adopted* to Γ_Σ if there is some contact structure ξ' on $\Sigma \times I$ that is I-invariant (i.e. Σ is convex under ξ') with $\Sigma_{\xi'} = \mathcal{F}$ and Γ_Σ is also a dividing for ξ'.

Let us now introduce the flexibility theorem. It essentially says, that the contact topological information in a neighborhood of of a convex surface can be determined by the dividing set rather than the characteristic foliation.

Theorem 2.23 (Giroux's Flexibility Theorem [5]). *Assume Σ is convex with characteristic foliation Σ_ξ, contact vector field v and dividing set Γ_Σ. Let $\mathcal{F}$ be a singular foliation on Σ that is adopted to Γ_Σ. Then there is an isotopy ϕ_s, $s \in [0,1]$, of Σ in (M,ξ) such that:*

1) $\phi_0 = id$ *and* $\phi_s|_{\Gamma_\Sigma \cup \partial\Sigma} = id$ *for all* s,

2) $\phi_s(\Sigma) \pitchfork v$ *for all* s,

3) $\phi_1(\Sigma)$ *has characteristic foliation* $\mathcal{F}$.

An isotopy that satisfies 2) *is called allowable isotopy.*

Let us now introduce Giroux's criterion. It is a criterion on the dividing set of a convex surface, which determines whether a small neighborhood of the surface is tight or overtwisted.

Theorem 2.24 (Giroux's criterion). *If $\Sigma \neq S^2$ is a convex surface in a contact manifold (M, ξ), then Σ has a tight neighborhood if and only if Γ_Σ has no homotopically trivial curves. If $\Sigma = S^2$, Σ has a tight neighborhood if and only if $\#\Gamma_\Sigma = 1$.*

Theorem 2.25 (Eliashberg [2]). *Assume that there exists a contact structure ξ on a neighborhood of ∂B^3 which makes ∂B^3 convex with $\#\Gamma_{\partial B^3} = 1$. Then there exists a unique extension of ξ to a tight contact structure on B^3 up to isotopy which fixes the boundary.*

2.1.5 Dividing set on disks

The main convex surface we are going to be dealing with are convex disks with Legendrian boundary. Thus we are going to look into them more detailed. So let D be a convex disk with Legendrian boundary in a contact 3-manifold (M, ξ) and let X be a contact vector field transverse to D. By applying Proposition 2.20 we can perturb D, while fixing ∂D, into a convex disk with standard collared boundary, inside an arbitrary small neighborhood of D. Therefore we usually assume that D has such a standard collar. Hence we have a neighborhood $\mathbb{R}/\mathbb{Z} \times [0, 1] \times [-1, 1]$ of ∂D with coordinates (x, y, t), $\partial D \cong \mathbb{R}/\mathbb{Z} \times \{0\} \times \{0\}$ and defining contact 1-form $\sin(2\pi n x)dy + \cos(2\pi n x)dt$. We then choose a contact vector field that coincides with dt on that neighborhood (this is a partition of unity argument together with the fact that the space of contact vector fields is a vector space). By Giroux criterion (Theorem 2.22) Γ_D can not have any closed curves and thus it is a collection of arcs. In this picture we have a natural identification of $\mathbb{Z}_{2n}$ with $\Gamma_D \cap \partial D$ i.e. $k = 0, \dots 2n-1$ is identified with $(k/2n - 1/4n, 0, 0)$. The dividing set Γ_D then can be thought of as a subset of $\mathbb{Z}_{2n}^2$. Furthermore R_+ resp. R_- is the union of connected components of $D \backslash \Gamma_D$ that intersect a region $(a, a + 1/2n) \times \{0\} \times \{0\}$ where a is positive resp. negative. Let us now consider $A = \{(a_0, b_0), \dots, (a_{n-1}, b_{n-1})\} \subset \mathbb{Z}_{2n}^2$. Every isotopy class of an arc (a_k, b_k) divides ∂D into two disjoint sets and hence also divides $\mathbb{Z}_{2n}$ into the sets $a_k + 1, \dots, b_k - 1$ and $b_k + 1, \dots, a_k - 1$. If every other arc in A has both endpoints in one of those sets we can realize A as a multicurve on D and hence as a possible dividing set. This observation yields the following lemma.

Lemma 2.26. *Let D be a convex disk in a contact 3-manifold that has Legendrian boundary. If $tb(\partial D) = -n$ then the number of isotopy classes of possible dividing sets for D is the n-th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. We first assume w.l.o.g. that D has a collared Legendrian boundary. We will prove this by induction. By the above observation we know, that the number of dividing sets on D is the number of ways to pair the $2np$ endpoints on ∂D that belong to the dividing set without intersection (i.e. to realize those connections as a multicurve). If $n = 1$ there is only one way to pair the 2 points. So let the assumption hold for $n = l$. Assume there is an arc $(0, k)$, then a point in $\{k + 1, \dots, 2n - 1\}$ can not be connected to a point $\{1, \dots, k-1\}$. Thus the points $\{1, \dots, k-1\}$ resp. $\{k+1, \dots, 2n-1\}$ must be pairwise connected (k must be odd because if not we would pair an odd number of points which is not possible). The number of ways to pair $\{1, \dots, k-1\}$ by induction is $C_{(k-1)/2}$ and the number of ways to pair the points $\{k+1, \dots, 2n-1\}$ is $C_{(2n-k-1)/2}$. As all those choices give rise to different dividing sets, and all choices are different we get the formula

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

The use of C_0 in the formula is not a problem as there is only one choice to pair 0 points. This sum then corresponds to the $n-th$ Catalan number (see [9]). $\square$

2.1.6 Edge rounding

The following result is a result in [7] that uses the work of [3]. Let Σ_1 and Σ_2 be two convex surfaces with Legendrian boundary, that have some common connect boundary component $L \subset \partial\Sigma_1 \cap \partial\Sigma_2$. Then similar as in Proposition 2.18 there exists a parametrization of a tubular neighborhood $\{x^2+y^2 \leq \varepsilon\} \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ of $L = \{0\} \times \{0\} \times \mathbb{R}/\mathbb{Z}$ with coordinates (x, y, z) and contact one form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, for some $n \in \mathbb{Z}^+$ such that there is a a C^0-small perturbation of Σ_1, Σ_2 close to L such that

$$\Sigma_1 \cap \{x^2+y^2 \leq \varepsilon\} = \{x = 0, 0 \leq y \leq \varepsilon\},$$

and

$$\Sigma_2 \cap \{x^2+y^2 \leq \varepsilon\} = \{y = 0, 0 \leq x \leq \varepsilon\}.$$

We take d/dx to determine Γ_{Σ_1} and $-d/dy$ to determine Γ_{Σ_2}. We call such a parametrization a *normal parametrization* for L, Σ_1, Σ_2 and ξ. Now rounding the edge at L as $\{(x, y)|(x-\varepsilon)^2+(y-\varepsilon)^2 = \varepsilon^2\}$ gives rise to a new surface Σ. The dividing set Γ_Σ arises by connecting the dividing curve $z = k/(2n)$ on Σ_1 to the dividing curve $z = k/(2n)-1/(4n)$ of Σ_2, see figure 2.3. The important part for us is, how the dividing set of the surface Σ can be compute from the dividing set of Σ_1 and Σ_2.

2.1.7 Bypasses

Let us now introduce the notion of a bypass (introduced byh Honda [7]).

Definition 2.27. Let Σ be a convex surface. A *bypass* is a half-disk D with $\partial D = \alpha \cup \beta$ Legendrian arcs for which the following hold:

1. $\alpha = \Sigma \cap D$ (we call α the *attaching arc*),
2. $\Gamma_\Sigma \cap \alpha = \{p_1, p_2, p_3\}$ where p_1, p_2, p_3 are distinct points,
3. $\alpha \cap \beta = \{p_1, p_3\}$,

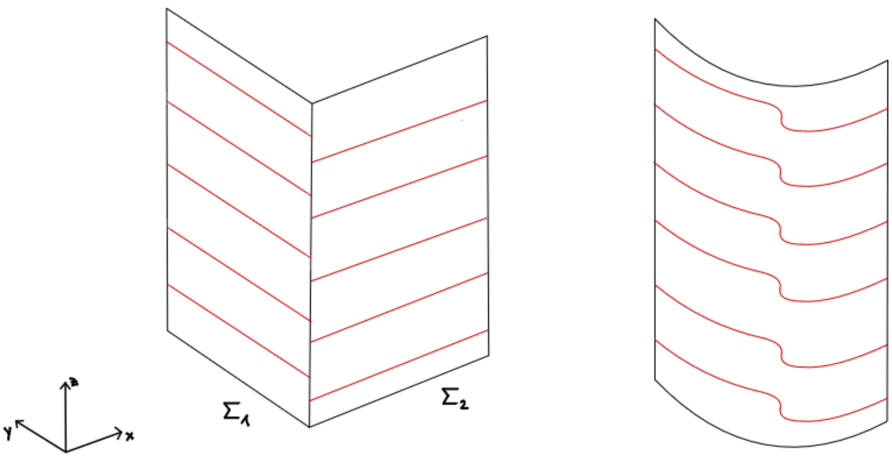


Figure 2.3: Left is the surface with edged and right is the surface after edge rounding. The blue lines are the dividing curves.

4. for one orientation of D, p_1 and p_3 are both positive elliptic singular points of the characteristic foliation on D, p_2 is negative elliptic, and all the singular points along β are positive.

A bypass gives rise to a perturbation of Σ over the bypass half-disk D which changes the dividing curve of Σ:

Theorem 2.28 ([7])**.** *Assume D is a bypass for a convex surface Σ. Then there exists a neighborhood of $\Sigma \cup D \subset M$ diffeomorphic to $\Sigma \times [0,1]$, such that Σ_0 and Σ_1 are convex, $\Sigma \times [0,\varepsilon]$ is I-invariant, $\Sigma = \Sigma_\varepsilon$ and Γ_{Σ_1} is obtained from Γ_Σ by performing the Bypass attachment operation shown in Figure 2.5. Let R_+, R_- resp. R'_+, R'_- be the positive and negative region of Σ_0 resp. Σ_1. Then $\chi(R_+) - \chi(R_-) = \chi(R'_+) - \chi(R'_-)$.*

If the orientation of Σ coincides with the normal orientation induced by D we call the attachment an *attachment from the front* and the dividing set changes as in figure 2.5, and if not we call the attachment an *attachment from the back* and the dividing set changes as in figure 2.6. It often makes sense to look how a bypass attachment changes the dividing set without knowing that the bypass disk exists. We therefore consider an arc δ and assume it is the attaching arc of some bypass disk. The arcs in figure 2.7 are defined as the *trivial* and *disallowed* bypasses (attached from the front). In fact a trivial bypass does not change the isotopy class of the dividing curve while the disallowed one adds two trivial components to the dividing set. The following Lemma gives a criterion on the existence of a trivial bypass.

Lemma 2.29 ([7])**.** *Let Σ be an oriented, convex surface in a contact manifold (M,ξ), and let $\Sigma \times [0,1]$ be a $[0,1]$ invariant neighborhood of $\Sigma = \Sigma \times \{0\}$, oriented such that if t*

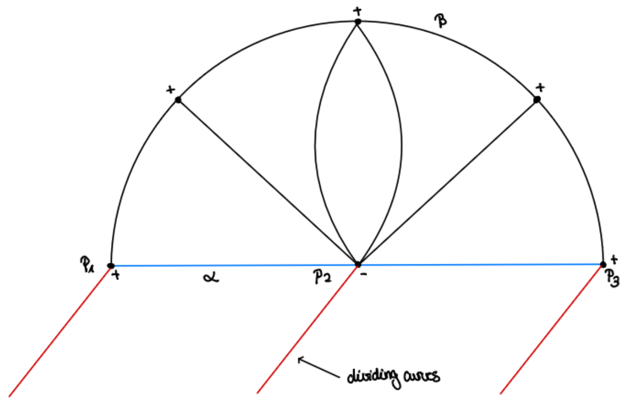


Figure 2.4: A bypass disk.

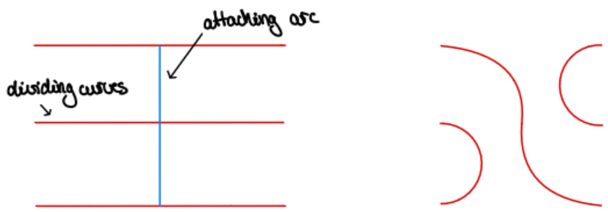


Figure 2.5: A bypass attachment. The left shows the dividing set before and the right the dividing set after the attachment. The orientation of Σ coincides with the normal orientation of D.

is the coordinate for $[0,1]$ *then* $\partial/\partial t$ *is the oriented normal for* $\Sigma \times \{t\}$*. Let* δ *be a trivial attaching arc of a possible bypass then a bypass disk exists in* $\Sigma \times [0,1]$ *with attaching arc* δ*.*

2.1.8 Compression

Definition 2.30. Let $\Sigma \subset M$ be an embedded surface without boundary, we call an embedded disk D in M a *compressing disk*, iff $D \cap \Sigma = \partial D$ and D is transverse to Σ at ∂D.

Let D be a compressing disk in M and let $D \times [-1,1]$ be a tubular neighborhood of D with $D \times [-1,1] \cap \Sigma = \partial D \times [-1,1]$. Then $\Sigma \backslash D \times (-1,1)$ is a surface (with corners). This surface is called the *compressed surface* and the surgery is called *compression.*

Let (M,ξ) be a contact 3-manifold where ∂M is convex i.e. there is an $I=[0,\infty)$-invariant neighborhood $\partial M \times I$ of $\partial M = \partial M \times \{0\}$ for ξ. When we apply C^0-small perturbations, the flexibility Theorem or the Legendrian realization principal close to

Figure 2.6: A bypass attachment from the back.

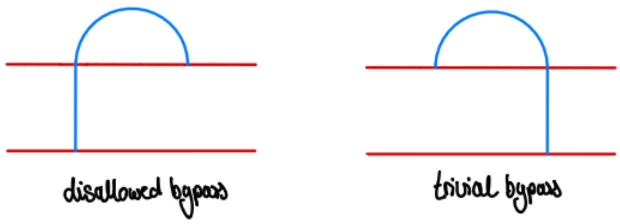


Figure 2.7: Trivial and disallowed bypasses.

the boundary of M we use that $(M, \xi) \cong (M \backslash (\partial M \times [0,k)), \xi|_{M \backslash (\partial M \times [0,k))})$ and apply it to $M \backslash (\partial M \times [0,k))$ for $k >> 0$. We are not going to mention that in the following and just assume that the procedures work on the boundary as well.

Let us now transfer the idea of compression to the contact world. In a tight contact 3-manifold (M, ξ) with convex boundary, a compressing disk D for ∂M can be made convex. First of all we apply the Legendrian realization principal to the boundary of D which yields an isotopy on M that gives us a new disk $\tilde{D}$ with Legendrian boundary. Now $-n = tb(\partial \tilde{D}) < 0$ (As M is tight). Now we can use Proposition 2.18 to perturb $\tilde{D}$ into a convex disk. Then as in the edge rounding chapter we can C^0-perturb ∂M and D such that there is a tubular neighborhood $\{x^2 + y^2 \leq \varepsilon\} \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ of $\partial D = \{0\} \times \{0\} \times \mathbb{R}/\mathbb{Z}$ with coordinates (x, y, z) and contact one form $\alpha = \sin(2\pi n z)dx + \cos(2\pi n z)dy$, such that $D \cap \{x^2 + y^2 \leq \varepsilon\} = \{x = 0, 0 \leq y \leq \varepsilon\}$, and $\partial M \cap \{x^2 + y^2 \leq \varepsilon\} = \{y = 0, 0 \leq x \leq \varepsilon\}$. Compressing $\tilde{D}$ then gives a surface with corners. Applying the edgerounding tools we discussed earlier we can make the surface with corners into a smooth convex surface and thus get a new tight contact 3 manifold with convex boundary. From now on we only write $M \backslash D$ and mean the manifold that arises from the surgery of compression and edgerounding discussed above. This surgery also works for multiple disjoint disks.

2.2 Honda's algorithm

The purpose of this section is to introduce Honda's [8] algorithm (the following results and definitions are all introduced in Honda [8]). Let M be compact and oriented 3-manifold. Let Γ be some multicurve on ∂M and let $\mathcal{F}$ be a singular foliation that is adopted to Γ.

Definition 2.31 ([8])**.** We define $\mathcal{I}(M, \mathcal{F})$ to be the isotopy classes of tight contact structures on M whose characteristic foliation is $\mathcal{F}$. Furthermore let $\mathcal{I}^*(M, \mathcal{F}) = \mathcal{I}(M, \mathcal{F}) \cup \{*\}$, where $*$ is a single point corresponding to the overtwisted contact structures.

Remark 2.32. Let $\mathcal{F}$ and $\mathcal{F}'$ be two characteristic foliations that are adopted to Γ. The Flexibility theorem gives rise to a natural bijection (proven in [7])

$$\mathcal{I}(M, \mathcal{F}) \xrightarrow{\sim} \mathcal{I}(M, \mathcal{F}').$$

Thus we sometimes write $\mathcal{I}(M, \Gamma)$ instead of $\mathcal{I}(M, \mathcal{F})$.

Let us now discuss the handlebody case i.e. let M be a handlebody of genus g. Let $D_1, \ldots, D_g$ be compression disks such that $M \backslash (D_1 \cup \cdots \cup D_g) \cong B^3$. Let ξ be a contact structure defined in a neighborhood of $\Sigma = \partial M$ and $D_1, \ldots, D_g$ such that ∂M and $D_1, \ldots, D_g$ all have a tight neighborhood and all are convex. As compression is a surgery that takes place in a small neighborhood of the compressed and the compressing surfaces $B^3 = M \backslash (D_1 \cup \cdots \cup D_g)$ is a well defined manifold and ξ is a contact structure defined in a neighborhood of ∂B^3 such that ∂B^3 is convex. Then by Giroux's criterion ∂B^3 has a tight neighborhood if and only if $\#\Gamma_{\partial B^3} = 1$. If that is the case, by Theorem 2.25 there is a unique extension of ξ to a tight contact structure on B^3 up to isotopy fixing the boundary. Hence if $\#\Gamma_{B^3} = 1$ there is a unique extension of ξ on M which is uniquely defined by the characteristic foliations on Σ and $D_1, \ldots, D_g$ up to isotopy fixing the characteristic foliation on Σ and $D_1, \ldots, D_g$. Thus the characteristic foliations on Σ and $D_1, \ldots, D_g$ give rise to a unique element of $\mathcal{I}^*(M, \Sigma_\xi)$. By the flexibility Theorem and Remark 2.32 it is the dividing sets of ∂M and $D_1, \ldots, D_g$ that determines the element in $\mathcal{I}^*(M, \Gamma_\Sigma)$.

We now just want to consider $\Gamma, \Gamma_1, \ldots, \Gamma_g$. To do that we need the extra assumption that all ∂D_k admit a normal parametrization. This yields that $\partial D_k \cap \Gamma$ and $\partial D_k \cap \Gamma_k$ are alternating. On the other hand if $\partial D_k \cap \Gamma$ and $\partial D_k \cap \Gamma_k$ are alternating we can choose a contact structure η in a neighborhood on $\partial M \cup D_1 \cup \cdots \cup D_g$ such that ∂M is adopted to ∂M_η and Γ_k are adopted to $(D_k)_\eta$ and such that ∂D_k admit a normal parametrization (we are not going to go through that construction but it can easily be constructed with the tools we know by now). This leads to the following definition.

Definition 2.33 ([8])**.** Let M be an oriented handlebody of genus g, Γ be a multicurve on $\Sigma = \partial M$ that does not have homotopically trivial curves, such that there is a characteristic foliation $\mathcal{F}$ adopted to Γ. Let $D_1, \ldots, D_g$ be compressing disks for Σ with $-n_k = tb(\partial D_k) = 1/2 \cdot \#(\partial D_k \pitchfork \Gamma) = 1/2 \cdot \#(\partial D_k \cap \Gamma)$ such that $M \backslash (D_1 \cup \cdots \cup D_g) \cong B^3$.

We define $\mathcal{C}$ to consist of all possible dividing sets $C = (\Gamma_1, \dots, \Gamma_g)$ of $(D_1 \cup \dots \cup D_g)$ such that for every $k = 1, \dots, g$ we have Γ_k does not have close curves and $\partial D_k \cap \Gamma$ and $\partial D_k \cap \Gamma_k$ are alternating. We call $\mathcal{C}$ the *configuration space*, $C \in \mathcal{C}$ a *configuration disk* and $\partial D_k \cap \Gamma_k$ the *endpoints* of Γ_k.

Remark 2.34. By our observations if we choose a characteristic foliation $\mathcal{F}$ that is adopted to Γ, then an element in $C \in \mathcal{C}$ yields a unique element in $\mathcal{I}^*(M, \mathcal{F})$ i.e. there is a well defined surjective map

$$\psi\colon \quad \mathcal{C} \quad \to \quad \mathcal{I}^*(M, \mathcal{F}).$$

To better deal with a configuration space we want to use the picture we introduced before, where a dividing set on D_k can be represented by a subset of $\mathbb{Z}^2_{2n_k}$. We further want to define R_+ and R_- on each D_k. As in a configuration space we neither have chosen an orientation for the D_k, nor a characteristic foliation adopted to Γ_k we are going to make a choice.

Definition 2.35. An *enumeration of a configuration* $\mathcal{C}$ is a choice of orientation for each D_k and a choice of counterclockwise identification of $\partial D_k \backslash \Gamma$ with $\mathbb{Z}_{2n_k}$.

Remark 2.36. Let $\mathcal{C}$ be enumerated. Then choose a normal parametrization of $\partial D, \partial M, D$ (defined in Subsection 2.1.6) such that the identification of $\Gamma_\Sigma \cap D_k$ with $\mathbb{Z}_{2n_k}$ (defined in Subsection 2.1.5) corresponds to the enumeration on $\mathcal{C}$. We have seen that R_+ and R_- are well defined for D_k. Furthermore they do not depend on the choice of normal parametrization. A component of $D_k \backslash \Gamma_k$ is in R_+ resp. R_- if there is a component of $\partial D_k \backslash \Gamma_k$ connecting a to $a + 1$ where a is even resp. odd.

Definition 2.37 ([8])**.** Let $C \in \mathcal{C}$ be a configuration such that $\Gamma_{\partial B^3}$ has only one connected component i.e. a closed curve. Then we call C *potentially allowable.*

A potentially allowable configuration gives rise to a contact structure on M for which the cut open manifold is tight. This is a necessary condition that the contact structure on M is tight. Honda's algorithm shows that this is not sufficient.

Let C be a potentially allowable configuration. Then every D_k gives rise to two disks $D^1_k, D^2_k \subset \partial B^3$. Therefore an abstract bypass attachment on D_k with attaching arc α gives rise to a bypass attachment on ∂B^3 from the back with attaching arc $\alpha^1 \subset D^1_i$ or $\alpha^2 \subset D^2_i$. If α^1 resp. α^2 is that attaching arc, then Lemma 2.29 gives that there exists a bypass disk in B^3 with attaching arc α^1 resp. α^2 if the corresponding bypass attachment is trivial i.e. if the bypass attachment yields a connected dividing set on ∂B^3. The change of the dividing set induced by that bypass attachment yields a new configuration which yields the same element in $\mathcal{I}^*(M, \Gamma)$. This suggests the following definition.

Definition 2.38 ([8])**.** Let $C \in \mathcal{C}$ be potentially allowable and let $C' \in \mathcal{C}$. We call $C \mapsto C'$ an *allowable state transition*, iff :

1. C is potentially allowable,

2. C' can be obtained by an abstract bypass attachment on $D_i \in C$ that is neither trivial nor disallowed,

3. the abstract bypass attachment, applied to the cut-open manifold for C (i.e. to $\Gamma_{\partial B^3}$) from the interior of B^3 is a trivial bypass attachment.

Remark 2.39. Let $C = (\Gamma_1, \ldots, \Gamma_g), C' = (\Gamma'_1, \ldots, \Gamma'_g) \in \mathcal{C}$ be allowable such that $C \mapsto C'$, then $C \mapsto C'$ implies $C' \mapsto C$. This holds as for every abstract bypass attachment applied to some Γ_k with attaching arc α that gives Γ'_k there is an abstract bypass attachment on Γ'_k with attaching arc α' that yields Γ_k and checking whether one or the other is an allowable state transition yields the same curve on ∂B^3.

This leads to the idea of defining a graph G that has points $\mathcal{C}$ and edges $I \subset \mathcal{C} \times \mathcal{C}$ that are precisely the allowable state transitions.

Definition 2.40 ([8]). $C \in \mathcal{C}$ is called *allowable*, iff every $C' \in \mathcal{C}$ that is connected to C in G is potentially allowable. Denote the set of allowable configuration by $\mathcal{C}_0 \subset \mathcal{C}$ and set of connected components in $\mathcal{C}_0$ by $\pi_0(\mathcal{C}_0)$. For an allowable C we denote by $[C]$ the connected component of C in G (on $\mathcal{C}_0$, G is reflexive).

Let $\mathcal{F}$ be characteristic foliation adopted to Γ, let $C \in \mathcal{C}$ and let ξ be induced by C. Then Honda proved, that if there is an overtwisted disk in M that does not appear in B^3, then there is a sequence of perturbations of $D_1, \ldots, D_g$, such that D does not intersect $D_1, \ldots, D_g$ anymore, and hence that the overtwisted disk appears in B^3. He further proved that the change of the dividing sets on the D_k is induced by a sequence of allowable state transitions. Honda further used a Theorem by Giroux stating,that an isotopy of ξ relative to the boundary induces a change of diving set on the D_k that as well is induced by a sequence of allowable state transitions. This then leads to the following theorem.

Theorem 2.41 ([8]). *ψ restricts to a surcjective map $\mathcal{C}_0 \to \mathcal{I}(M, \Gamma)$ which again factors trough the bijection*

$$\pi_0(\mathcal{C}_0) \quad \overset{\sim}{\to} \quad \mathcal{I}(M, \Gamma).$$

This yields an algorithm to find the number of tight contact structures of a handlebody M with fixed characteristic foliation $\mathcal{F}$, that is adopted to some multicurve Γ:

1. Find compressing disks $D_1, \ldots, D_n$ such that $M/(D_1 \cup \cdots \cup D_n) = B^3$,

2. find all the allowable state transitions for all potentially allowable configuration $\mathcal{C}$,

3. find all the connected components of G that only contain potentially allowable configurations,

4. the number of those connected components then is the number of tight contact structures.

2.2.1 Applying Honda's algorithm

Let M be an oriented handelbody of genus g and Γ be a multicurve on ∂M such that there is a characteristic foliation adopted to it. Further let $D_1, \ldots, D_g$ be compressing disks for ∂M with $M \backslash (D_1 \cup \cdots \cup D_g) \cong B^3$, $2n_k = \partial D_k \pitchfork \gamma = \partial D_k \cap \Gamma$ for all $k = 1, \ldots, g$ and $\mathcal{C}$ be the corresponding configuration space. Let $C = (\Gamma_1, \ldots, \Gamma_k) \in \mathcal{C}$ be potentially allowable. Then every D_k and Γ_k has two copies in the cut open manifold $M \backslash (D_1 \cup \cdots \cup D_g) \cong B^3$ (before edgerounding). The copies of D_k have opposite orientation in ∂B^3 (we choose the submanifold orientation for ∂B^3). As $\mathbb{R}^2 \cong S^2 - \{p\}$ for some point $p \in S^2 \backslash \Gamma_{S^2}$ we can sketch the dividing set of the cut open manifold in a 2-dimensional plane (induced by an orientation preserving map). In that plane we indicate the number in $\mathbb{Z}_{2n_k}$ corresponding to the endpoints of Γ_k. Furthermore we sketch the two copies of each D_k next to each other to indicate which disks correspond to each other. As edgerounding only changes a small neighborhood of ∂D_k edges as in Figure 2.8. After rounding we still indicate the copies of D_k, the Γ_k and the endpoints of Γ_k. An abstract bypass on C with attaching arc α on Γ_k then corresponds to a bypass from the back with attaching arc α^1 where α^1 is one of the copies of α in the representation and vise versa. Attaching α^1 from the back yields a new dividing set and if it is connected

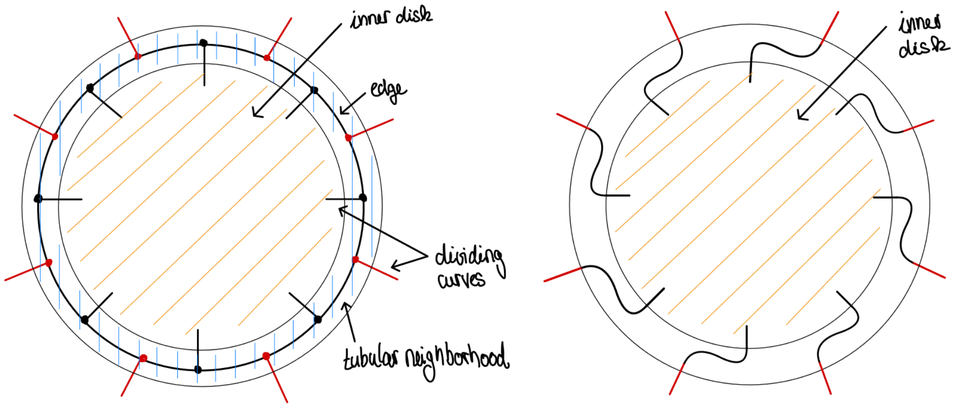


Figure 2.8: The left picture shows a configuration disk in B^3 before edge rounding, the right picture shows the same disk after edge rounding.

then the corresponding state transition is allowable. Let α^2 be the other copy of α in the representation. Then attaching α^1 from the back and α^2 from the front yields a new configuration which corresponds to the state transition induced by α. Therefore we work with representations of configuration to apply the algorithm. Furthermore as we only change the dividing sets that correspond to Γ_k and we only check for connectedness of a curve we sometimes in the representation replace the curves corresponding to Γ by curves that are connecting the same endpoints of the copies of D_k as Γ does. As we only care about the connectedness of the curve we will later also allow intersections in these representations.

Example 2.42. Let us consider the torus $M = \mathbb{R}/\mathbb{Z} \times D$. We choose a compressing disk $D_1 = \{0\} \times D$. Let Γ be a multicurve on ∂M that has 4 components that each intersect ∂D one time. The representation of an arbitrary configuration before edge-

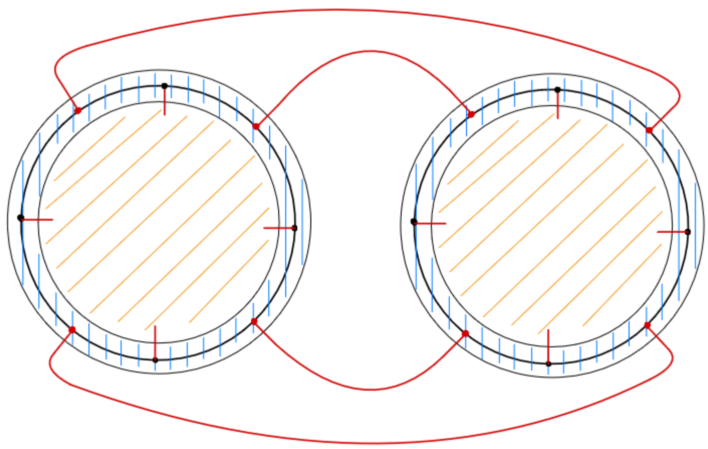


Figure 2.9: Before edge-rounding.

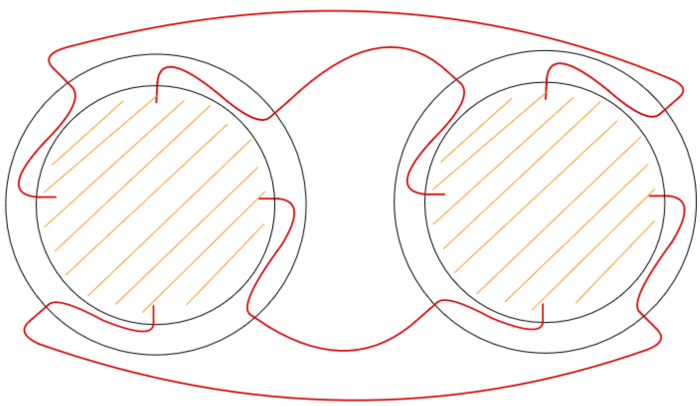


Figure 2.10: After edge-rounding.

rounding is shown in figure 2.9 and after edge-rounding is shown in figure 2.10. There are only two configurations. The representations of those can be seen in figure 2.11. As neither one has a nontrivial abstract bypass there can not be allowable state transitions. Furthermore both configurations are potentially allowable and as they form a connected component in the graph G both are allowable i.e. $|\mathcal{C}_0| = 2$. We will see more complicated examples in the following chapters.

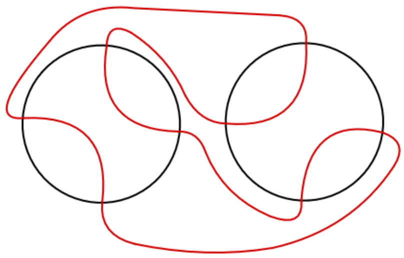


Figure 2.11: The tow configurations.

Let us try to understand how complicated the algorithm can get, by understanding how many points G has.

Lemma 2.43. *The number of configurations in $\mathcal{C}$ is $C_{n_1} \cdots C_{n_g}$.*

Proof. This follows from Lemma 2.24. $\square$

2.2.2 Generating possible dividing sets

We now introduce a technique, to generate a new possible dividing set on a handlebody from a possible dividing set Γ: Let M be a handlebody with possible dividing set Γ on ∂M that has a tight neighborhood i.e. that has no homotopically trivial connected components and has a characteristic foliation $\mathcal{F}$ adopted to it. Let $L = \mathbb{R}/\mathbb{Z}$ be a curve on ∂M, that intersects Γ transversally $2n$ times. The take a small tubular neighborhood $T = [-\varepsilon, \varepsilon] \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ of $L = \{0\} \times \{0\} \times \mathbb{R}/\mathbb{Z}$ with coordinates (x, y, z) with $\partial M \cap T = [-\varepsilon, \varepsilon] \times \{0\} \times \mathbb{R}/\mathbb{Z}$. Let $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$. Then choose a characteristic foliation $\mathcal{F}$ on ∂M that on $\partial M \cap T$ is induced by α. Then cutting T cutting T open along $\{0\} \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ gives rise to two manifolds $T_1 = [-\varepsilon, -\tilde{\varepsilon}] \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ and $T_2 = [\tilde{\varepsilon}, \varepsilon] \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ for some $\tilde{\varepsilon} < \varepsilon$. Gluing $\{-\tilde{\varepsilon}\} \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ to $\{\tilde{\varepsilon}\} \times [0, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ with the identification map $(-\tilde{\varepsilon}, y, z) \mapsto (\tilde{\varepsilon}, y, z - k/n)$ gives a new manifold where α is a well defined. This yields a surgery on ∂M in a neighborhood of L that gives a new surface Σ' and a with new multicurve Γ' where Σ' is diffeomorfic to ∂M. Thus under such an isomorphism Γ' induces a multicurve on ∂M that has a characteristic foliation that is adopted to it and hence if it has no homotopically trivial curves we can apply Honda's alogorithm to it.

Let L be a curve on ∂M that is transverse to Γ with $n = \#(\partial M \cap \Gamma)$. Then choose a parametrization of a neighborhood $[-\varepsilon, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ with $L = \{0\} \times \mathbb{R}/\mathbb{Z}$ parametrized by (x, y) such that $\Gamma \cap [-\varepsilon, \varepsilon] \times \mathbb{R}/\mathbb{Z} = [-\varepsilon, \varepsilon] \times \{1/2n, 2/2n \ldots, 2n/2n\}$. We then say we twist the dividing set by $k \in \mathbb{Z}$ around S^1 and mean to replace Γ on $[-\varepsilon, \varepsilon] \times \mathbb{R}/\mathbb{Z}$ by a dividing curve connecting $\{l/2n\} \times \{-\varepsilon\}$ to $\{(l + 2k)/2n\} \times \{\varepsilon\}$. By the above observation the new multicurve has a characteristic foliation adopted to it. The isotopy class of the new curve does not depend on the orientation of L.

Definition 2.44. We say we *twist* Γ *by* k *around* L and mean the procedure above.

If L is bound by a disk the sketch in figure 2.12 (a) indicates such a twist, the sketch in figure 2.12 (b) shows a twist by 2.

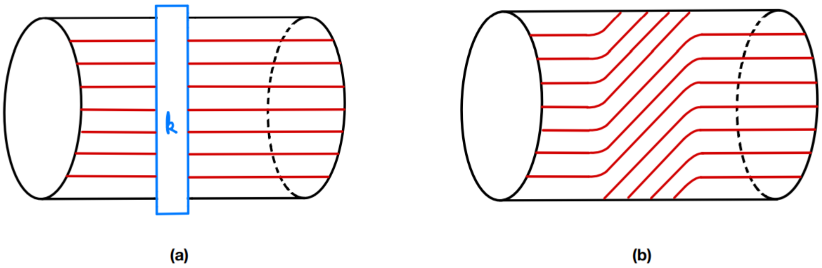


Figure 2.12: Twisting by k.

2.3 Continuous fractions

Continuous fractions play an important role in the classification of tight contact structures on handlebodies. Let $p, q \in \mathbb{N}$ such that $gcd(p, q) = 1$ and $(p, q) \neq (1, 1)$. We then write

$$p/q = [r_0, \ldots, r_h] = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \ldots \frac{1}{r_h}}}, \text{ for } r_i \geq 2$$

.

Remark 2.45. $r_0, \ldots, r_h$ are unique. This can be proven by induction: As $0 < \frac{1}{[r_1, \ldots, r_h]} < 1$, we have that $r_0 = \lceil \frac{p}{q} \rceil$ then

$$\frac{p}{q} = r_0 - \frac{qr_0 - p}{q} = r_0 - \frac{1}{\frac{q}{qr_0 - p}}.$$

Then by induction $\frac{q}{qr_0 - p} = [r_1, \ldots, r_h]$. This argument gives a way to compute $r_0, \ldots, r_h$. Sometimes we will write $(n, [r_0, \ldots, r_h])$ instead of (n, p, q). Furthermore there is an inductive way to compute p, q from $r_0, \ldots, r_h$, i.e. let $p_h = r_h$ and $q_h = 1$ and define $p_{i-1} = r_{i-1} \cdot p_i - q_i$, $q_{i-1} = p_i$. Then $(p, q) = (p_0, q_0)$. We sometimes write $[r_0, \ldots, r_h - 1]$ which is not well defined if $r_h - 1 = 1$ but we can still use the definition to obtain well defined $p/q = [r_0, \ldots, r_h - 1]$.

We will use the notation

$$a \mod b$$

and mean the smallest number $r \geq 0$ such that $r \equiv c \pmod b$.

Lemma 2.46. *Let $p/q = [r_0, \ldots, r_h]$ and $\tilde{p}/\tilde{q} = [r_1, \ldots, r_h]$. Then let $s, \tilde{s} \in \mathbb{N}$ such that*

$$s \cdot 2q \bmod 2p = \tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p}.$$

Let $0 < r$ *be the smallest number such that* $(s+r) \cdot 2q \bmod 2p < 2\tilde{p}$*, then*

$$(s+r) \cdot 2q \bmod 2p = (\tilde{s}+1) \cdot 2\tilde{q} \bmod 2\tilde{p}.$$

Proof. Fist of all we note that $q = \tilde{p}$ and $p = r_0\tilde{p} - \tilde{q}$ and thus

$$2p - r_0 \cdot 2q = 2(r_0 \cdot \tilde{p} - \tilde{q}) - r_0 \cdot 2\tilde{p} = -2\tilde{q}. \tag{2.3}$$

Further note that

$$s \cdot 2q \bmod 2p < 2\tilde{p} = 2q \tag{2.4}$$

and

$$\tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} = s \cdot 2q \bmod 2p = s \cdot 2\tilde{p} \bmod (2(r_0\tilde{p} - \tilde{q})). \tag{2.5}$$

Let $0 < r$ be the smallest number such that

$$2p \leq s \cdot 2q \bmod 2p + r \cdot 2q \tag{2.6}$$

then as $2q < 2p$

$$(s \cdot 2q \bmod 2p + r \cdot 2q) \bmod 2p = s \cdot 2q \bmod 2p + r \cdot 2q - 2p. \tag{2.7}$$

If we exchange p and q in (2.6) we get

$$2(r_0 \cdot \tilde{p} - \tilde{q}) \leq s \cdot 2\tilde{p} \bmod 2(r_0 \cdot \tilde{p} - \tilde{q}) + r \cdot 2\tilde{p}. \tag{2.8}$$

We distinguish 2 cases.
Case 1:

$$s \cdot 2\tilde{p} \bmod 2(r_0 \cdot \tilde{p} - \tilde{q}) < 2(\tilde{p} - \tilde{q}).$$

This implies

$$s \cdot 2q \bmod 2p + 2\tilde{q} < 2\tilde{p}.. \tag{2.9}$$

Then (2.8) implies $r = r_0$. We then have

$$\begin{aligned}
(s+r) \cdot 2q \bmod 2p &= (s \cdot 2q \bmod 2p + r \cdot 2q) \bmod 2p \\
&\overset{(2.7)}{=} s \cdot 2q \bmod 2p + r \cdot 2q - 2p \\
&\overset{(2.3)}{=} s \cdot 2q \bmod 2p + 2\tilde{q} \\
&\overset{(2.5)}{=} \tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} + 2\tilde{q} \\
&\overset{(2.9)}{=} (\tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} + 2\tilde{q}) \bmod 2\tilde{p} \\
&= (\tilde{s}+1) \cdot 2\tilde{q} \bmod 2\tilde{n}\tilde{p}.
\end{aligned}$$

Case 2:

$$s \cdot 2\tilde{p} \bmod 2(r_0 \cdot \tilde{p} - \tilde{q}) \geq 2(\tilde{p} - \tilde{q}).$$

This implies

$$s \cdot 2q \bmod 2p - 2(\tilde{p} - \tilde{q}) \tag{2.10}$$

Then (2.8) and (2.4) implie $r = r_0 - 1$. We then have

$$\begin{aligned}
(s + r) \cdot 2q \bmod 2p &= (s \cdot 2q \bmod 2p + r \cdot 2q) \bmod 2p \\
&\overset{(2.7)}{=} s \cdot 2q \bmod 2p + r \cdot 2q - 2p \\
&\overset{(2.3)}{=} s \cdot 2q \bmod 2p - 2(\tilde{p} - \tilde{q}) \\
&\overset{(2.5)}{=} \tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} - 2(\tilde{p} - \tilde{q}) \\
&= (\tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} - 2(\tilde{p} - \tilde{q})) \bmod 2\tilde{p} \\
&= (\tilde{s} + 1) \cdot 2\tilde{q} \bmod 2\tilde{n}\tilde{p}.
\end{aligned}$$

As r by (2.6) is the smallest candidate such that $(s + r) \cdot 2q \bmod 2p < 2\tilde{p}$ the Lemma is true. □

Lemma 2.47. *Let $p/q = [r_0, \ldots, r_h]$. Let further*

$$h_{\max} := \begin{cases} 0 & \text{if } r_0, \ldots, r_h = 2 \\ \max\{0 \leq i \leq h | r_i > 2\} & \text{else.} \end{cases}$$

Then

$$[r_0, \ldots, r_h - 1] = [r_0, \ldots, h_{\max} - 1].$$

Proof. Let $r_h > 2$ then $h_{\max} = h$ and thus the statement is true. If $h = 0$ then $h_{\max} = 0$ and thus the statement is true. If $h \neq 0$ and $r_h = 2$ we have

$$\begin{aligned}
[r_0, \ldots, r_h - 1] &= r_0 - \cfrac{1}{r_1 - \ldots \cfrac{1}{r_{h-1} - \frac{1}{r_h - 1}}} \\
&= r_0 - \cfrac{1}{r_1 - \ldots \cfrac{1}{r_{h-1} - \frac{1}{1}}} \\
&= r_0 - \cfrac{1}{r_1 - \ldots \cfrac{1}{r_{h-1} - 1}} \\
&= [r_0, \ldots, r_{h-1} - 1].
\end{aligned}$$

Induction then gives the statement of the lemma. □

Lemma 2.48. *Let $p/q = [r_0, \ldots, r_h]$, $h \geq 1$, let $\tilde{p}/\tilde{q} = [r_0, \ldots, r_h - 1]$ and let $\bar{p}/\bar{q} = [r_0, \ldots, r_{h-1}]$. Then $p - \tilde{p} = \bar{p}$ and $q - \tilde{q} = \bar{q}$.*

Proof. We use induction on h. If $h=1$ then $p=r_0r_1-1$ and $q=r_1$. Furthermore $\tilde{p}=(r_1-1)r_0-1$ and $\tilde{q}=r_1-1$. Hence $p-\tilde{p}=r_0r_1-1-((r_1-1)r_0-1)=r_0$ and $q-\tilde{q}=r_0-(r_0-1)=1$. Therefore $\bar{p}=r_0$ and $\bar{q}=1$. Now assume the statement is true for $h\leq l$ and let $h=l+1$. Let $p_1/q_1=[r_1,\dots,r_h]$, $\bar{p}_1/\bar{q}_1=[r_1,\dots,r_{h-1}]$. Then $\tilde{p}_1/\tilde{q}_1\overset{3.44}{=}[r_1,\dots,r_h-1]$. We have the following identities:

$$
\begin{aligned}
p &= r_0p_1-q_1\\
q &= p_1\\
\bar{p}_1 &= r_0\bar{p}-\bar{q}\\
\bar{q}_1 &= \bar{p}\\
\tilde{p} &= r_0\tilde{p}_1-\tilde{q}_1\\
\tilde{q} &= \tilde{p}_1.
\end{aligned}
$$

Then induction gives $p_1-\tilde{p}_1=\bar{p}_1$ and $q_1-\tilde{q}_1=\bar{q}_1$. Taking all together we ge

$$
\begin{aligned}
p-\tilde{p} &= r_0p_1-q_1-(r_0\tilde{p}_1-\tilde{q}_1)\\
&= r_0(p_1-\tilde{p}_1)-(q_1-\tilde{q}_1)\\
&= r_0\bar{p}_1-\bar{q}\\
&= barp,
\end{aligned}
$$

and

$$
\begin{aligned}
q-\tilde{q} &= p_1-\tilde{p}\\
&= \bar{p}_1\\
&= \bar{q}.
\end{aligned}
$$

□

Corollary 2.49. *Let $p/q=[r_0,\dots,r_h]$ and $\tilde{p}/\tilde{q}=[r_0,\dots,r_h-1]$, then $p>\tilde{p}$ and $q\geq\tilde{q}$.*

Proof. The case $h=0$ is trivial as $p=r_0, q=1$ and $\tilde{p}=r_0-1, \tilde{q}=1$. The case $h\geq 1$ is a consequence of 2.47. □

3 Solid Torus

The classification of the number of tight contact structures on the solid torus $S^1 \times D^2$ for some fixed dividing set on its boundary $S^1 \times \partial D^2$ is a key ingredient for the results on more general handlebodies.

Let us first consider the class of possible dividing sets. Let $\partial(S^1 \times D^2) = S^1 \times \partial D^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = T^2$ with coordinates (x, y). A multicurve Γ on T^2 with $2n$ components and without homotopically trivial curves components can be isotoped such that it consist of $2n$ curves with slope q/p where $\gcd(p, q) = 1$. W.l.o.g. we use the convention $p \geq 1$ if the slope is $\neq 0$ and $p = 1, q = 0$ if the slope is 0 (Such a characterization can be found in [7]). On a solid torus we can perform Dehn twists to get a diffeomorphism of T^2 that gives rise to a contactomorphism by pushing the contact structure forward. This changes the dividing set from (n, p, q) to $(n, p, kp + q)$ for a consecutive number of Dehn twists that correspond to k. Therefore both dividing sets (n, p, q) and $(n, p, kp+q)$ correspond to the same number of tight contact structures.

Definition 3.1. For a dividing set Γ that corresponds to (n, p, q) we write $N(n, p, q)$ and mean the number of tight contact structures on T^2 that are adopted to Γ up to isotopy fixing the boundary. The compressing disk $\{0\} \times D$ gives a configuration space which we denote by $\mathcal{C}(n, p, q)$. The corresponding graph is denoted by $G(n, p, q)$.

Remark 3.2. We have $N(n, p, q) = N(n, p, kp + q)$. Thus w.l.o.g. we assume, that $p > q \geq 0$.

The dividing set corresponding to (n, p, q) can also be represented by a dividing set Γ on the torus with $2np$ components parallel to $\mathbb{R}/\mathbb{Z} \times \{0\}$ i.e. a dividing set corresponding to $(np, 1, 0)$ that is twisted nq times around $\{0\} \times \mathbb{R}/\mathbb{Z}$. With that in mind a configuration viewed on B^3 corresponding to (n, p, q) obtained by taking $\{0\} \times D^2$ as the compressing disk can be sketched as in figure 3.1 (b) by sketching the copies of $\{0\} \times D^2$ on ∂B^3 as lines with identified endpoints. We then can get rid of the nq twisting which results in figure 3.1 (c). This will be the representation of a configuration that we work with.

Remark 3.3. As mentioned the exact behavior of the dividing set between the two copies of the compressing disk is not important to the algorithm, but only the information which points on the boundary of the two copies of the compressing disk are connected. Thus in figure 3.1 there are curves that intersect. A configuration then is a way of connecting the boundary points on the left resp. the right disk (remember the boundary of those disks is viewed as a straight line with identified endpoints) such that the pictures are mirrored.

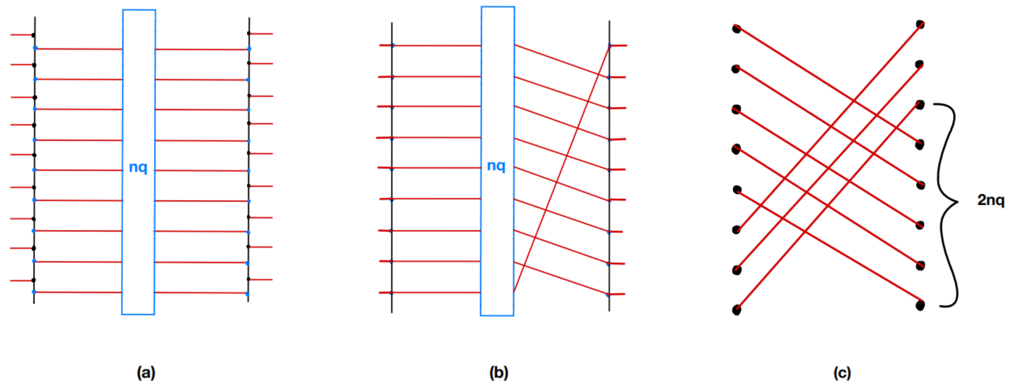


Figure 3.1: Picture (a) shows the cut open torus before rounding and picture (b) after rounding. Picture (c) shows the cut open torus after rounding and resolving the nq twist. The dividing set corresponds to (n, p, q)

Definition 3.4. We call figure 3.1 (c) a *representation* of a configuration (if we fill in the dividing set). If we don't fill in the dividing set figure 3.1 (c) is called the *frame*, np is called the *size* and $2nq - 1$ is called the *slope*.

Remark 3.5. For an enumerated configuration space the enumeration carries on to the representation of a configuration. For the right choice of identifying the boundary of the inner disks with a straight line, the frame looks as in figure 3.2 (a).

For the frame of an enumerated configuration space it is sometimes useful to change the enumeration.

Definition 3.6. We call a change of enumeration induced by a map $\mathbb{Z}_{np} \to \mathbb{Z}_{np}$, $z \mapsto z + k$ a *rotation* of the enumeration and the change of enumeration induced by the map $\mathbb{Z}_{np} \to \mathbb{Z}_{np}$, $z \mapsto -z$ a *mirroring* of the enumeration.

Remark 3.7. We sometimes work with a frame that we do not identify with a cut open torus and apply Honda's algorithm. In that case the size and the slope of the frame characterize the frame completely.

3.1 First results

Let us first go through some general results i.e. we consider $\mathcal{C}(n, 1, 0)$ and $\mathcal{C}(1, p, 1)$. The ideas presented here can be generalized for an arbitrary choice of (n, p, q). The basic idea is, that for a configuration C in one of those configuration spaces the existence of a boundary parallel arc gives rise to an identification of C with a configuration in a more simple configuration space a configuration space that has less elements.

Lemma 3.8. *Every configuration in $\mathcal{C}(n, 1, 0)$ is potentially allowable.*

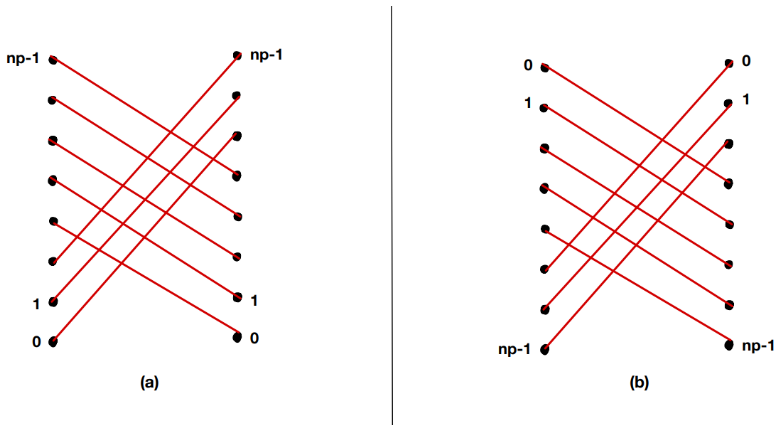


Figure 3.2: Picture (a) shows an enumerated frame. Picture (b) shows a frame of an enumerated configuration space.

Proof. Every configuration has a boundary parallel arc. The representation has slope -1 and thus we can perform the operation in figure 3.3. This operation gives a rep-

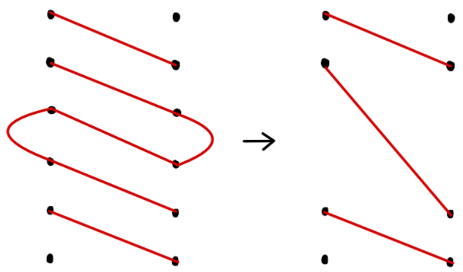


Figure 3.3: Operation for a boundary parallel arc on a representation with slope -1.

resentation with slope -1 and $2(n-1)$ points i.e. corresponds to the representation of the configuration obtained by deleting the boundary parallel arc corresponding to the dividing set $(n-1,1,0)$. The new configuration is potentially allowable if and only if the first configuration is potentially allowable (the operation does not change the connectedness of the representation). Now induction gives the hypothesis as for $(1,1,0)$ every configuration is potentially allowable (the representation has two points and slope -1). $\square$

Before we prove that for $G(n,1,0)$ there are no allowable state transitions, we need a result regarding boundary parallel arcs.

Lemma 3.9. *Let D be a configuration disk with dividing set Γ and $tb(\partial D) = -n$. If γ is the attaching arc of an abstract bypass then γ intersects all boundary parallel arcs of*

Γ *if and only if* $n = 3$.

Proof. Assume $n > 3$, then there must be a component ν of Γ that does not γ as γ only intersects 3 ars. If ν is boundary parallel we are done. If not ν divides D into two parts, one of which contains γ. The other part must contain a nonzero number of components of Γ. Now choose a component of Γ contained in that part and use induction. If $n < 3$ there are only trivial and disallowed abstract bypasses (we do not consider those in the algorithm). □

Lemma 3.10. *$G(n, 1, 0)$ has no allowable state transitions.*

Proof. Let γ be the attaching arc of an abstract bypass in the representation of a configuration. Lets assume, that there is a boundary parallel arc that does not intersect γ. We then can perform the operation in figure 3.3 and the condition that the attachment is potentially allowable is the same in both pictures as the bypass does not change the boundary parallel arc. Using induction we get a picture in which there is no boundary parallel arc that does not intersect γ. By Proposition 3.5 the size of the frame is 3. For $n = 3$ there are only 4 different configurations that admit an abstract bypass that is neither trivial nor disallowed. All of those are rotations of one another and hence by an allowable rotation the abstract bypass attachment looks as in figure 3.4 (a) or (b). We

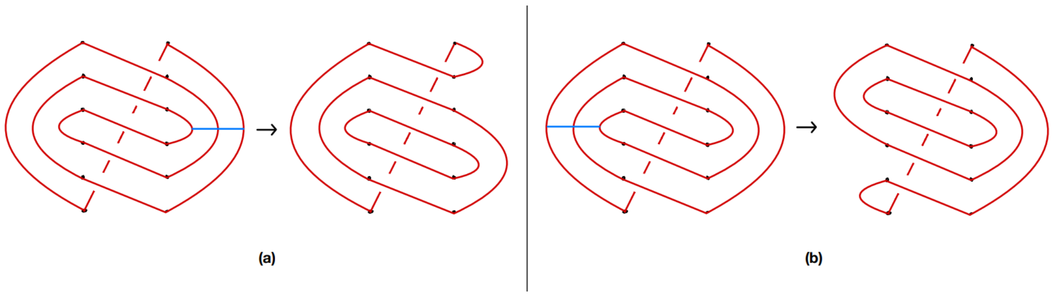


Figure 3.4: Possible bypasses after removing boundary parallel arcs.

see that applying such a bypass results in 3 different curves and thus the state transition is not allowable. □

Proposition 3.11. $N(n, 1, 0) = C_n$.

Proof. As all configurations are potentially allowable and there are now allowable state transitions we have that $\mathcal{C} = \mathcal{C}_0$ and that $\pi_0(\mathcal{C}_0) = \#\mathcal{C}_0 = \#\mathcal{C}$. Then Lemma 2.38 gives $\#\mathcal{C} = C_n$. □

Remark 3.12. In other words, if we apply the algorithm to a frame with slope -1 and size n we get C_n allowable components of the graph G.

Lemma 3.13. *In $\mathcal{C}(1, p, 1)$, all configurations are potentially allowable.*

Proof. The frame of a representation has slope 1. If we mirror the a representation of a configuration horizontally we get a frame with slope -1. We have already seen, that such a representation consists of a connected curve. As mirroring a representation does not change the circumstance whether the curve is connected or not we obtain, that all configurations are potentially allowable. □

Lemma 3.14. *In $G(1, p, 1)$ all state transitions are allowable.*

Proof. Let γ be the attaching arc of an abstract bypass on the representation of a configuration. In the manner of Lemma 3.6 we can get rid of boundary parallel arcs that do not intersect γ inductively and by Lemma 3.5 we end up with a representation of size 3 and slope 1. As all configurations of size 3 that admit a nontrivial non degenerate abstract bypass are rotations of one another the bypass attachment (after some rotation of the inner disk) looks as in figure 3.5 (a) or (b). The resulting dividing set is connected

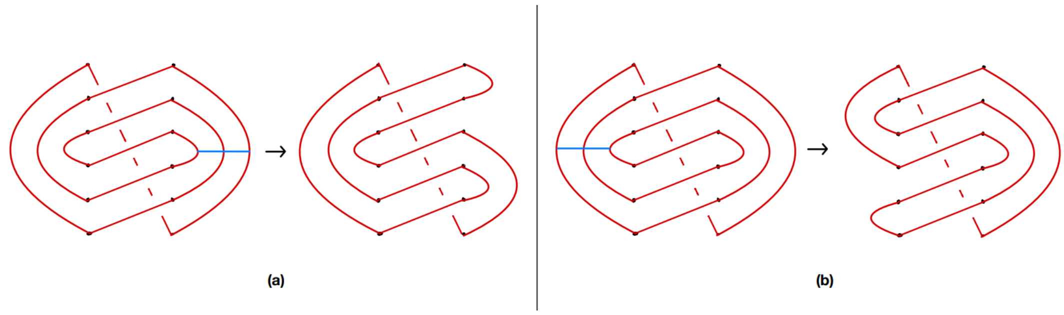


Figure 3.5: Possible bypasses after removing boundary parallel arcs.

and thus every state transition is allowable. □

We now want to prove, that for an enumerated configuration space $\chi(R_+)$ is an invariant of of the connected component of an allowable configuration.

Lemma 3.15. *Let D be a convex disk in a contact 3-manifold with $tb(\partial D) = -n$ and dividing set Γ. Then $\chi(R_+) + \chi(R_-) = n + 1$.*

Proof. As every connected component of R_+ and R_- is homeomorph to a disk we have that $\chi(R_+) + \chi(R_-)$ is the number of connected components of $R_+ \cup R_-$ i.e. the number of connected components of $D \backslash \Gamma$. Using induction on the number of components of a multicurve Γ' on a disk D' gives, that the number of connected components of $D' \backslash \Gamma'$ is the number of connected components of Γ' plus 1. This yields the lemma. □

Lemma 3.16. *Let $\mathcal{C}(n, p, q)$ be enumerated. Then $\chi(R_+)$ is an invariant for each allowable component on $\mathcal{C}(n, p, q)$.*

Proof. By Theorem 2.28 $\chi(R_+) - \chi(R_-)$ is invariant under bypass attachments and as by Lemma 3.15 $\chi(R_+) + \chi(R_-) = np + 1$, we get, that $\xi(R_+)$ is an invariant. □

Example 3.17. Let $\mathcal{C}(n,p,q)$ be enumerated. Then the configuration with dividing set

$$((0,1),\dots,(2(k-1),2(k-1)+1),(2k,-1),(2k+1,(2(k+1)),\dots,(2(np-2)+1,2(np-1)))$$

has $\chi(R_+)=k$. With the previous proof this gives, that $\chi(R_+)$ can be $1,\dots,np$.

Remark 3.18. The previous lemmas give us that $N(1,p,1)\geq p$ as the $\chi(R_+)$ is an invariant for a connected component of $G(1,p,1)$ and as further all components are allowable..

Definition 3.19. Let $\mathcal{C}(n,p,q)$ be enumerated and $C\in\mathcal{C}(n,p,q)$. We then call $\chi(R_+(C))$ the *partition number* of C.

Lemma 3.20. *Let $\mathcal{C}(1,p,1)$ be enumerated. Then for a pair of configurations with the same partition number ($\chi(R_+)$) there is a series of abstract bypass attachments connecting them.*

Proof. Let $C\in\mathcal{C}(1,p,1)$ with $k=\xi(R_+(C))$. Then assume that C contains the boundary parallel arcs $(0,1),\dots,(2(l-1),2(l-1)+1)$. Choose l to be maximal. If $l<k$, then there must be a not boundary parallel arc $(a,b)\neq(2l,-1)$ as else C is as in Example 3.17 and thus $\chi(R_+(C))=l<k$ which by assumption is wrong. Let us assume there is a boundary parallel arc $(2l+1,2(l+1))$ then choose an attaching arc α that intersects $(2l+1,2(l+1)),(a,b)$ and another arc not in $(0,1),\dots,(2(l-1),2(l-1)+1)$. A bypass with attaching arc α from the front or the back (does not matter) gives an allowable state transition $C\to C'$ where C' contains $(0,1),\dots,(2(l-1),2(l-1)+1)$ and does not contain $(2l+1,2(l+1))$. Then a bypass attachment from the back on C' with attaching arc intersecting the arcs that have and endpoint $2l,2l+1$ or $2l+2$ gives a state transition $C'\to\tilde{C}$ where $\tilde{C}$ contains $(0,1),\dots,(2l,2l+1)$. By induction we thus can connect C via allowable state transitions to the configuration in Example 3.17 that corresponds to k. □

Proposition 3.21. $N(1,p,1)=p$.

Proof. This is a result of the previous observation that all state transitions are allowable and the fact that two configurations are connected via a sequence of abstract bypass moves if and only if they have the same partition number (this is just a combi.nation of Lemma 3.18 and Lemma 3.15). □

3.2 Finding allowable state transitions

First we are going to introduce some allowable state transitions that exist for an allowable configuration. Then we are going to show allowable configuration can be connected to a configuration containing a boundary parallel set via a sequence of allowable state transitions.

Definition 3.22. Let $C \in \mathcal{C}(n,p,q)$ be a configuration. We then call a number of arcs $\alpha_1, \dots, \alpha_k$ in C a *sub-configuration* of C if the endpoints of $\alpha_1, \dots \alpha_k$ as a set are consecutive i.e. of the form $s, \dots, e$ for some $s, e \in \mathbb{Z}_{2np}$. We call s the *start* and e the *end* of the sub-configuration. We call k the *size* of the sub-configuration.

Let us first consider how an abstract bypass from the back in the representation of a potentially allowable configuration looks like. Let Γ be the dividing curve and α the attaching arc. Remember that for a potentially allowable configuration the representation contains exactly one closed curve. Then a bypass attachment looks either as figure 3.6 (a) or (b). The key principal is then: A boundary point and the middle point are connected via an arc of Γ that does not intersect α in another point. If that arc is as in a trivial bypass attachment from the back, we get, that α attached from the back is trivial. This is not surprising as it is just another way of saying what a trivial bypass is, but it gives us a good method to check it in a representation.

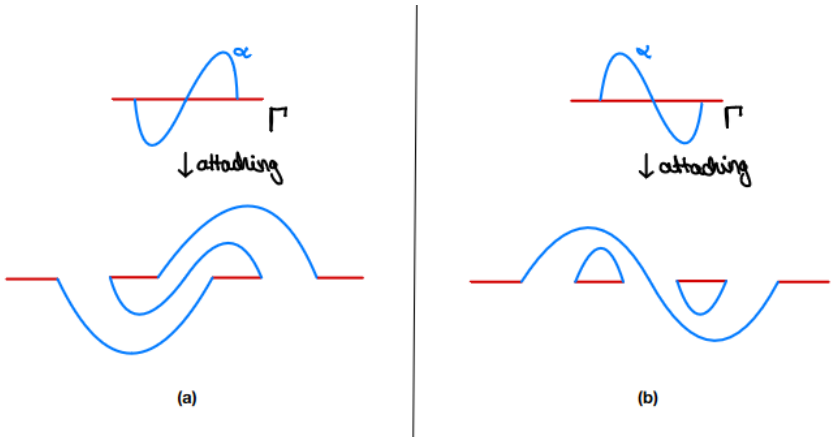


Figure 3.6: Possible inner bypasses on the representation of a potentially allowable configuration. (a) is trivial and (b) is disallowed.

Let us prove the following Lemma that gives us a set of allowable state transitions for every boundary parallel arc contained in a configuration.

Lemma 3.23. *Let $\mathcal{C}(n,p,q)$ be enumerated and let $C \in \mathcal{C}(n,p,q)$ be a allowable. Let further (k_1, k_2) be an arc of C, then:*

i) an abstract bypass from the front with attaching arc that intersects the $k_1 + 2nq - 1$ arc at one endpoint and the $k_2 + 2nq - 1$ arc at the middle is an allowable state transition.

ii) an abstract bypass from the back with attaching arc that intersects the $k_2 - (2nq - 1)$ arc at one endpoint and the $k_1 - (2nq - 1)$ arc at the middle point is an allowable state transition.

Proof. Figure 3.7 shows a part of the representation of C which has slope $2nq - 1$. The principle introduced above then gives that the bypass in figure 3.7 attached from the back are trivial. Bypass (a) corresponds to a bypass attached from the back of D_1 and bypass (b) corresponds to a bypass attached from the front of D_1. □

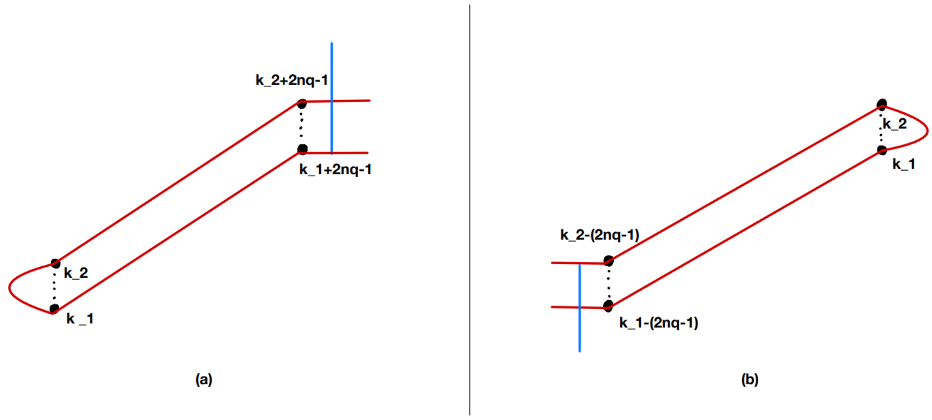


Figure 3.7: Trivial bypasses following from a boundary parallel arc.

Remark 3.24. We will mostly use the Lemma on boundary parallel arcs i.e. let $(l, l+1)$ be a boundary parallel arc in an allowable configuration C. Then a bypass from the front that intersects the $l + 2nq - 1$ arc at one endpoint, the $l + 2nq$ arc at the middle point and th $l + 1 + 2nq$ arc at the other endpoint is an allowable state transition creating an arc $(l + 2nq, l + 1 + 2nq)$. We have, that if C contains $(l + 2nq, l + 1 + 2nq)$ the bypass attachment is trivial and if C contains $(l - 1 + 2nq, l + 2nq)$ then $(l, l + 1)$ on the left disk of the representation and $(l - 1 + 2nq, l + 2nq)$ on the right disk of the representation give a circle, which by assumption can not be as C is allowable. This yields, that the bypass attachment can only change arcs that have exactly one endpoint in $l-1+2nq, l+2nq, l+1+2nq$. As similar statement holds for the bypass attachment creating $(l-2nq, l+1-2nq)$ i.e. and arc of C gets changed if it has exactly one endpoint in $l - 2nq, l + 1 - 2nq, l + 2 - 2nq$.

Lemma 3.25. *Let $\mathcal{C}(n, p, q)$ be enumerated. Let further $C \in \mathcal{C}(n, p, q)$ be a allowable. Assume there is a sub-configuration $((a_1, b_1), \ldots, (a_k, b_k))$ of size $k \in \mathbb{N}$ with $k \leq nq - 1$ and $k + nq \leq np$. Let s be the start and e be the end of the sub-configuration, then:*

(i) there is a sequence of allowable state transitions that connects C to a configuration $C' \in \mathcal{C}(n, p, q)$ has sub-configurations $((a_1, b_1), \ldots, (a_k, b_k))$ and $((a_1 + 2nq, b_1 + 2nq), \ldots, (a_k + 2nq, b_k + 2nq))$. Furthermore every arc in C that does not have an endpoint that is in $\{s - 1 + 2nq, \ldots, e + nq\}$ is also an arc in C',

(ii) there is a sequence of allowable state transitions that connects C to a configuration $C' \in \mathcal{C}(n, p, q)$ that has sub-configurations $((a_1, b_1), \ldots, (a_k, b_k))$ and $((a_1 - 2nq, b_1 - 2nq), \ldots, (a_k - 2nq, b_k - 2nq))$. Furthermore every arc in C that does not have an endpoint that is in $\{e + 1 - 2nq, \ldots, s - 2nq\}$ is also an arc in C'.

Proof. Let us prove the first statement via induction. For $k = 1$ we can apply Lemma 3.28. Let us assume the statement holds for $k \leq l$ and let $k = l+1$. Then we distinguish 2 cases:

Case 1: There is an arc (s, e), w.l.o.g. let $(a_1, b_1) = (s, e)$. We then use induction on the sub-configuration $((a_2, b_2), \dots, (a_k, b_k))$ that has size l, start $s+1$ and end $e-1$, which yields a configuration $\tilde{C}$ and the move only changes arcs that have at least one endpoint in $\{s+1-1+2nq, e-1+2nq\}$. As $k \leq nq-1$ and $k+nq \leq np$ we have that neither s nor e are in this set and thus (s, e) is an arc in $\tilde{C}$. We then have that $\tilde{C}$ has sub-configurations $((a_2, b_2, \dots, (a_k, b_k))$ and $((a_2+2nq, b_2+2nq), \dots, (a_k+2nq, b_k+2nq))$.

Case 2: We can divide the sub-configuration into two sub-configurations, one, that has all endpoints in $s, \dots, e_1$ and another, that has all endpoints in $e_1+1, \dots, e$. If we use induction on the sub-configuration that contains the arc with endpoint e first we obtain an additional sub-configuration that has all endpoints in $e_1+1+2nq, \dots e+2nq$. Applying induction on the sub-configuration that contains the arc with endpoint s then gives a sub-configuration that only changes arcs that have endpoint in $s-1+2nq, \dots e_1+2nq$ and hence does not change the sub-configuration generated by the first use of induction.

The second statement is just the first statement by choosing another enumeration obtained by giving D_1 the opposite orientation. □

3.3 The (almost) boundary parallel set

In the previous section we have seen the idea of simplifying the representation of a configuration by isotoping the frame of the representation. In the cases $(n, p, q) = (1, p, 1), (1, p, 0)$ such a simplification dependeds on one boundary parallel arc. In this section we will generalize that idea. We will therefore introduce the notion of a boundary parallel set and an almost boundary parallel which are essentially structures of boundary parallel arcs that a configuration can contain. A configuration containing such a set can then be simplified by isotoping the representation such that we eliminate the boundary parallel set in it. This essentially gives rise to a simplified view of the algorithm restricted to configurations that contain a specific (almost) boundary parallel set. In the cases $(n, p, q) = (1, p, 1), (1, p, 0)$ a boundary parallel arc is exactly a boundary parallel set.

Let $\mathcal{C}(n, p, q)$ be enumerated and let $C \in \mathcal{C}(n, p, q)$ be a configuration. Let $(l, l+1)$ and $(l+2nq, l+2nq+1)$ be boundary parallel arcs in C. Then in a representation of C there exists an isotopy as in Figure 3.8. This isotopes the copy of $(l, l+1)$ on the lef disk of the representation and the copy of $(l+2nq, l+2nq+1)$ on the right disk of the representation. If there is a pair of arcs $(l, l+1)$ and $(l+2nq-2, l+2nq+1-2)$ in C there exists such an isotopy as in Figure 3.8. The fundamental idea is to take a boundary parallel arc $(l, l+1)$ that is contained in C and apply Lemma 3.23 to create a new boundary parallel $(l+2nq, l+1+2nq)$. If we do that inductively without changing $(l, l+1)$ nor the arcs created before we obtain a set of boundary parallel arcs which we call the boundary parallel set. The hope is then to apply the isotopies introduced above to simplyfy the representation. Let us first try to understand this inductive application of Lemma 3.23 better to give a more precise definition of the (almost) boundary parallel

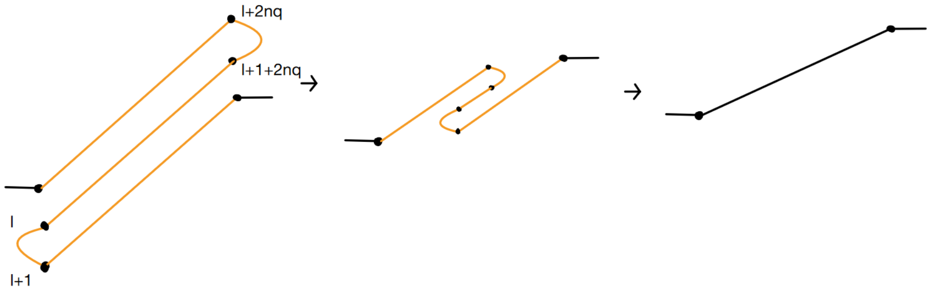


Figure 3.8: The figure shows an isotopy of $(l, l+1)$ on the lef disk of the representation and the copy of $(l+2nq, l+2nq+1)$ on the right disk of the representation.

set.

Lemma 3.26. *Let $\mathcal{C}(n,p,q)$ be enumerated and let $C \in \mathcal{C}(n,p,q)$ be an allowable configuration, such that C has a boundary parallel arc $(l, l+1)$. Let $k \in \mathbb{N}$ be minimal such that*

i) $(k+1) \cdot 2nq \equiv 0 \pmod{2np}$ *, or*

ii) $(k+1) \cdot 2nq \equiv 2 \pmod{2np}$.

Then there is a sequence of allowable state transitions that connects C to a configuration C' that has boundary parallel arcs

$$(l, l+1), (l+1 \cdot 2nq, l+1 \cdot 2nq+1), \ldots, (l+k \cdot 2nq, l+k \cdot 2nq+1).$$

Proof. Let $i \in \mathbb{N}$ be the maximal number we can inductively apply Lemma 3.23 to create new boundary parallel arcs without changing $(l, l+1)$ or the boundary parallel arc created before. This leads to a configuration C' that has boundary parallel arcs $(l, l+1), \ldots, (l+i \cdot 2nq, l+i \cdot 2nq+1)$. Now apply Lemma 3.23 to the boundary parallel arc $(l+i \cdot 2nq, l+i \cdot 2nq+1)$ and get a configuration $\tilde{C}$ that has a boundary parallel arc $(l+i \cdot 2nq+2nq, l+i \cdot 2nq+2nq+1) = (l+(i+1) \cdot 2nq, l+(i+1) \cdot 2nq+1)$. Now every arc in C' that does not have an endpoint in $l+(i+1) \cdot 2nq-1, \ldots, l+(i+1) \cdot 2nq+1$ is also an arc in $\tilde{C}$. Assume there is some arc $(l+\tilde{i} \cdot 2nq, l+\tilde{i} \cdot 2nq+1)$ with $\tilde{i} \leq i$ that has an endpoint in $l+(i+1) \cdot 2nq-1, \ldots, l+(i+1) \cdot 2nq+1$ i.e. that arc is getting changed by the allowable state transition. Then $(l, l+1)$ has an endpoint in $l+(i-\tilde{i}+1) \cdot 2nq-1, \ldots, l+(i-\tilde{i}+1) \cdot 2nq+1$ and thus

$$\begin{aligned} l+(i-\tilde{i}+1) \cdot 2nq-1 &\equiv l+1 \pmod{2np} \\ \text{or } l+(i-\tilde{i}+1) \cdot 2nq+1 &\equiv l+1 \pmod{2np}. \end{aligned}$$

This is equivalent to $(i-\tilde{i}+1) \cdot 2nq \equiv 2 \pmod{2np}$ or $(i-\tilde{i}+1) \cdot 2nq \equiv 0 \pmod{2np}$. This shows, that $i \leq k$. Furthermore if $i < k$ by the above observation applying Lemma 3.26

to $(l+i\cdot 2nq, l+i\cdot 2nq+1)$ does not change an arc of $(l, l+1), \ldots, (l+i\cdot 2nq, l+i\cdot 2nq+1)$ which yields $i = k$. $\square$

Let us now give a definition of the (almost) boundary parallel set.

Definition 3.27. Let $\mathcal{C}(n,p,q)$ be enumerated. Let further $k \in \mathbb{N}$ be the minimal number such that

i) $(k+1)\cdot 2nq \equiv 0 \pmod{2np}$, or

ii) $(k+1)\cdot 2nq \equiv 2 \pmod{2np}$.

We call the set of arcs

$$B = \{(l, l+1), (l+1\cdot 2nq, l+1\cdot 2nq+1), \ldots, (l+k\cdot 2nq, l+k\cdot 2nq+1)\} \subset \mathbb{Z}^2_{2np}$$

the *boundary parallel set* generated by $(l, l+1)$. Sometimes we also view B as the set of vertices. We call $k+1$ the *size* of B. Assume there is a $0 \leq \tilde{k} < k$ with $(\tilde{k}+1)\cdot 2nq \equiv 4 \pmod{2nq}$ we call

$$A_* = \{(l, l+1), (l+1\cdot 2nq, l+1\cdot 2nq+1), \ldots, (l+\tilde{k}\cdot 2nq, l+\tilde{k}\cdot 2nq+1)\} \subset \mathbb{Z}^2_{2np}$$

the *boundary parallel* 4*-set* and if $1 \leq \bar{k}$ we call

$$A = \{(l, l+1), (l+1\cdot 2nq, l+1\cdot 2nq+1), \ldots, (l-2+\tilde{k}\cdot 2nq, l-2+\tilde{k}\cdot 2nq+1)\} \subset \mathbb{Z}^2_{2np}$$

the *almost boundary parallel set* generated by $(l, l+1)$. Sometimes we also view A, A_* as the set of vertices. We call $\tilde{k}+1$ the *size* of A, A_*.

Remark 3.28. The almost boundary parallel set can not exist in every configuration space. It is a special case that can occur for specific choices of (n,p,q). The possible existence can then be formulated as follows. Let $B = \{(l, l+1), \ldots, (l+k\cdot 2nq, l+k\cdot 2nq+1)\}$ be the boundary parallel set generated by $(l, l+1)$. Then the almost boundary parallel set generated by $(l, l+1)$ can exist if there is $1 \leq \bar{k} \leq k$ such that $(\bar{k}+1)\cdot 2nq \equiv 4 \pmod{2np}$, i.e. we want $(l+4, l+5) \in B$ and $2nq \neq 2$.

Definition 3.29. Let $\mathcal{C}(n,p,q)$ be enumerated and let and let K be a set of boundary parallel arcs. We denote the set of configurations in $\mathcal{C}(n,p,q)$ that contain K by $\mathcal{C}^K(n,p,q)$.

3.3.1 Reduction

Before we prove the reduction Theorem for the (almost) boundary parallel set let us first consider an example of a reduction.

Example 3.30. Let $\mathcal{C}(1,5,2)$ be enumerated such that $C \in \mathcal{C}(1,5,2)$ contains the boundary parallel set B generated by $(0,1)$. Then C looks like figure 3.9 (a). The blue arcs of the frame are the arcs that have endpoints that both intersect a boundary parallel arc of the boundary parallel set. The blue arcs of C are the boundary parallel set. Then

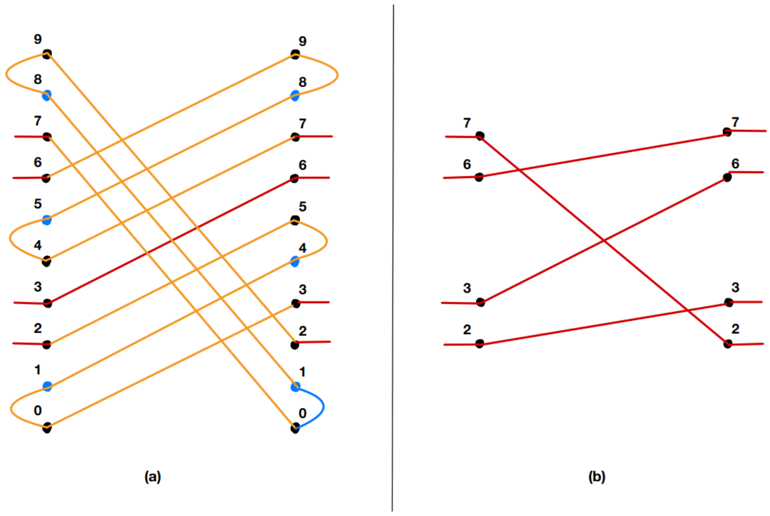


Figure 3.9: Figure (a) shows the boundary parallel arcs in C_1. Figure (b) shows figure (a) after getting rid of the boundary parallel arcs.

we can get rid of the boundary parallel arcs by using the isotopy move shown in figure 3.10. We can see the result in figure 3.9 (b) which is a frame of slope 1 and size 2. As the isotopy does not change the other arcs, figure 3.9 (b) corresponds to a configuration $C^B \in \mathcal{C}(1,2,1)$. A bypass in C^B then corresponds to a bypass in C that does not intersect the boundary parallel set and vise versa.

We would like to see that a move as in the Example exist for every configuration containing an (almost) boundary parallel set.

Lemma 3.31. *Let $\mathcal{C}(n,p,q)$ be enumerated with $np \neq 1$, $q \geq 1$ and B resp. A be the boundary parallel set resp. the almost boundary parallel set (if such a set exists) generated by $(l,l+1)$ of size $k+1$ resp. $\bar{k}+1$. Let $K \in \{A,B\}$ $i(l)$ be 1 when l is even and 0 else. Let*

$$\begin{array}{rccc} \alpha^K\colon & \mathbb{Z}_{2np}\backslash K & \rightarrow & \mathbb{Z}_{2np-2(k+1)} \\ & i & \mapsto & \#(\{l,\dots,i\}-K)-i(l). \end{array}$$

Then there are $\tilde{n},\tilde{p},\tilde{q}$ and a map

$$\begin{array}{rccc} \psi^K\colon & \mathcal{C}^K(n,p,q) & \rightarrow & \mathcal{C}(\tilde{n},\tilde{p},\tilde{q}) \\ & C & \mapsto & C^K. \end{array}$$

where C^K has arcs $(\alpha(a_1),\alpha(a_2))$ for every arc (a_1,a_2) in C that is not in K. Furthermore there is an allowable state transition $C \to \tilde{C}$ for $C,\tilde{C} \in \mathcal{C}^A(n,p,q)$ if and only if there is an allowable state transition $C^K \to \tilde{C}^K$.
If $K=B$, then $\tilde{n}\tilde{p} = np-(k+1)$ and $\tilde{n}\tilde{q}=1$ if $nq=1$ resp. $2\tilde{n}\tilde{q} = \#(\{l,\dots,l+2nq-1\}-B)$ if $nq\neq 1$.
If $K=A$, then let A_* be the boundary parallel 4-set generated by $(0,1)$, then $\tilde{n}\cdot\tilde{q} = np-(\bar{k}+1)$ and $2\tilde{n}\tilde{p} = \#(\{l,\dots,l+2nq-1\}-A_*)$.

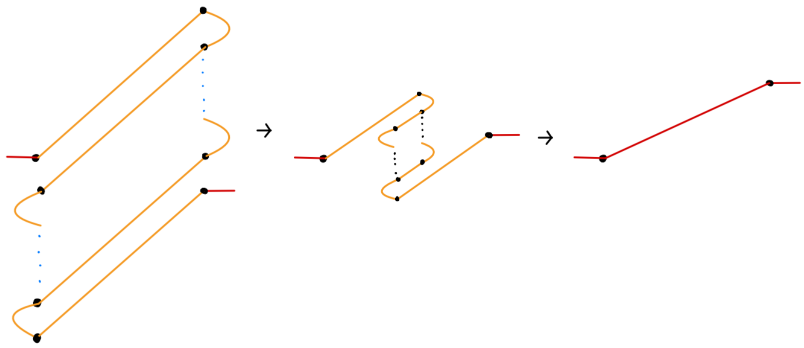


Figure 3.10: Simplifying move.

Proof. Let $C \in \mathcal{C}^K(n,p,q)$. We distinguish 3 cases:

Case 1: $K = B = ((0,1),\ldots,(k \cdot 2nq, k \cdot 2nq+1))$ and $(k+1)\cdot 2nq \bmod 2np = 0$. As k is the smallest number such that the equation holds we have $(k+1)\cdot 2nq$ must be $\mathrm{scm}(2np,2nq)$ and as $\gcd(p,q)=1$ we have that $\mathrm{scm}(2np,2nq) = 2npq$. Hence $k+1=p$. Assume $n=1$ and thus $np=p$. As $k+1=p$ all arcs in C are in B. As $np \neq 1$ we have, that there is an $1 \leq \tilde{k} \leq k$ with $(\tilde{k}\cdot 2nq, \tilde{k}\cdot 2nq+1 = (2,3)$ which is not possible as by assumption $\tilde{k}\cdot 2nq \bmod 2np \neq 2$. Therefore $n \geq 2$. As $(k+1)\cdot 2nq \bmod 2np = 0$ we have that $(l \cdot 2nq, l\cdot 2nq+1)$ for each $l \in \mathbb{Z}$ is a boundary parallel arc in B. Therefore every maximal subset of consecutive arcs in B viewed on the left disk in the representation looks as in figure 3.10 and thus we can apply the simplifying move. If we do that for every maximal subset we get a new picture (without the boundary parallel set). By enumerating the endpoints of arcs that are not in B via α as in the Lemma this picture corresponds to a representation of C^B with a new frame of size $2np-(k+1)$.

Case 2: $K = B = ((0,1),\ldots,(k\cdot 2nq, k\cdot 2nq+1))$ is a boundary parallel set and $(k+1)\cdot 2nq \bmod 2np = 2$. If $n \geq 2$ we have that $(k+1)\cdot 2nq \bmod 2np$ is divisible by 4 as $2nq$ and $2np$ are divisible by 4. As 2 is not divisible by 4 we get $n=1$. Then every maximal subset of consecutive arcs in $((0,1),\ldots,((k-1)\cdot 2nq,(k-1)\cdot 2nq+1))$ viewed on the left disk in the representation looks as in figure 3.10 and thus we can apply the simplifying move. The only copies of arcs of the boundary parallel set that are left are a copy of $(k\cdot 2nq, k\cdot 2nq+1)$ on the left disk and a copy of $(0,1)$ on the right disk. They then look as in figure 3.11 and thus we can apply the shown simplifying move. By enumerating the endpoints of arcs that are not in B via α^B as in the Lemma this picture corresponds to a representation of C^B with a new frame of size $2np-(k+1)$.

Case 3: $K = A = ((0,1),\ldots,(-2+\bar{k}\cdot 2nq, -2+\bar{k}\cdot 2nq+1))$ is an almost boundary parallel set. Then the arc $(-2+\bar{k}\cdot 2nq, -2+\bar{k}\cdot 2nq+1)$ viewed on the left disk in the representation looks as in figure 3.11 and and viewed on the right disk in the representation looks as in figure 3.11 as well. Thus we can apply the simplifying move. All remaining arcs on the left disk in the representation look as in figure 3.10 and thus we can apply the simplifying move. By enumerating the endpoints of arcs that are not

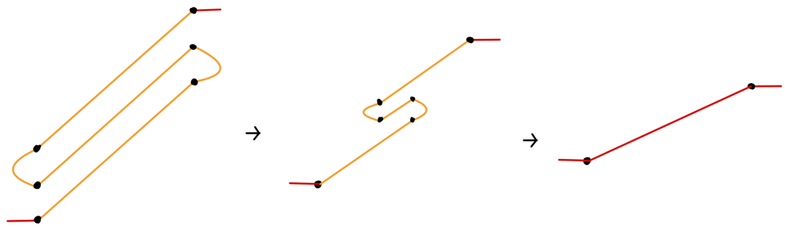


Figure 3.11: Simplifying move.

in A via α^A as in the Lemma this picture corresponds to a representation of C^A with a new frame of size $2np - 2(\bar{k}+1)$ and hence $n(n,p,q)p(n,p,q) = np - (\bar{k}+1)$.

Let us prove the equations for $\tilde{n}\tilde{q}$. Let us consider the following identity.

$$(2nq, 2nq+1) \in K \tag{3.1}$$

If (3.1) does not hold the frame arc with endpoints 2 on the left disk of the representation and $2nq+1$ on the right disk of the representation does not intersect K and thus it does not get changed by the simplifying moves. Thus in the new representation it corresponds to a frame arc connecting $\alpha(2) = 0$ on the left disk and $\alpha(2nq+1) = \#(\{0,\dots,2nq+1\} - K) - 1$ on the right disk. If (3.1) does not hold we either have $K = B$ and $nq = 1$ which by the above yields $\tilde{n}\tilde{q} = 1$, or we have $K = A$ and $\bar{k} = 1$. This then yields, that the boundary parallel 4-set generated by $(0,1)$ is $A_* = A - \{(2nq-2, 2nq-1)\} + \{(2nq, 2nq+1)\}$ which yields $2\tilde{n}\tilde{q} = \#(\{0,\dots,2nq+1\} - A) - 1 = \#(\{0,\dots,2nq-1\} - A_*) - 1$.

Let us consider the case where (3.1) is satisfied. Then the frame arc with endpoints 2 on the left disk of the representation and $2nq+1$ on the right disk of the representation gets changed by 3.10 and hence gets isotoped to a frame arc in the new representation that connects 0 on the left disk to $\#(\{0,\dots,2nq+1\} - K) - 1$ on the right disk. As by assumption (3.1) is satisfied $\#(\{0,\dots,2nq+1\} - K) - 1 = \#(\{0,\dots,2nq-1\} - K) - 1$. For $K = B$ this yields the equation $2\tilde{n}\tilde{q} = \#(\{0,\dots,2nq-1\} - B) - 1$. Let $K = A$. Then

$$A = A_* - \{(\bar{k}\cdot 2nq, \bar{k}\cdot 2nq+1)\} + \{(-2+\bar{k}\cdot 2nq, -2+\bar{k}\cdot 2nq+1)\}.$$

As $(0,1) \neq (\bar{k}\cdot 2nq, \bar{k}\cdot 2nq+1) \neq (2nq, 2nq+1)$ we either have

$$\begin{gathered}(\bar{k}\cdot 2nq, \bar{k}\cdot 2nq+1), (-2+\bar{k}\cdot 2nq, -2+\bar{k}\cdot 2nq+1) \in \{0,\dots,2nq-1\} \text{ or,}\\ (\bar{k}\cdot 2nq, \bar{k}\cdot 2nq+1), (-2+\bar{k}\cdot 2nq, -2+\bar{k}\cdot 2nq+1) \notin \{0,\dots,2nq-1\}.\end{gathered}$$

Therefore $2\tilde{n}\tilde{q} = \#(\{0,\dots,2nq-1\} - A) - 1 = \#(\{0,\dots,2nq-1\} - A_*) - 1$.

As the simplifying moves in all cases don't change the connectedness of the curve in the representation of C, we get that C is potentially allowable if and only if C^K is potentially allowable. As the simplifying moves in all cases do not change the arcs that are not in K we get that there is a 1-1 correspondence of bypasses in C that don't intersect K and bypasses in C. Furthermore the question if they are allowable is an if and only if correspondence. Let K be generated by $(l, l+1)$. Then we can rotate the enumeration of C such that K is generated by $(0,1)$. Then we apply the prove above and get C^K. If l is odd, hence if $l \bmod 2 = 1$ we rotate the enumeration of C^K such that 0 corresponds to 1. Hence $\alpha^B(i) = \alpha^{(0,1)}(i-l) - i(l)$, which is just the map introduced in the lemma. □

Remark 3.32. Let $\mathcal{C}(n,p,q)$ be enumerated and let $B = ((l, l+1), \ldots, (l+k \cdot 2nq, l+k \cdot 2nq+1))$ be the boundary parallel set generated by $(l, l+1)$. The proof of Lemma 3.36 shows that there is a fundamental difference the boundary parallel set for $n \geq 2$ and for $n = 1$. For $n \geq 2$ applying Lemma 3.32 to a boundary parallel arc of B does not change B. Hence every boundary parallel arc in B can generate B. Therefore to get a well defined map α^B it is important to say which arc generated B. For $n = 1$ applying Lemma 3.29 to $(l+k \cdot 2nq, l+k \cdot 2nq+1)$ would create a boundary parallel set generated by $(l+2nq, l+2nq+1)$. Induction then gives, that an allowable configuration is connected to a configuration that contains the boundary parallel set generated by $(l+(k+1) \cdot 2nq, l+(k+1) \cdot 2nq+1) = (l+2, l+3)$. Using induction again gives that C is connected to a configuration that contains the boundary parallel set generated by $(l+2i, 2+2i+1)$ for every $i \in \mathbb{N}$.

Remark 3.33. It is important to realize that α^K is a map that depends on a choice. Furthermore $(\tilde{n}, \tilde{p}, \tilde{q})$ do not depend on the generator but only on $K = B$ resp. $K = A$. This leads to the following Definition.

Definition 3.34. Let $\mathcal{C}(n,p,q)$ be enumerated and let $C \in \mathcal{C}(n,p,q)$. Let B resp. A be the boundary parallel resp. the almost boundary parallel set generated by $(l, l+1)$. If C contains K we call C^K the *reduction* of C by K. Furthermore we define maps

$$n, p, q, n_a, p_a, q_a \colon \quad \mathbb{N}^3 \quad \to \quad \mathbb{N},$$

given by

$$(n(n,p,q), p(n,p,q), q(n,p,q)) = (\tilde{n}, \tilde{p}, \tilde{q})$$

for $K = B$ and

$$(n_a(n,p,q), p_a(n,p,q), q_a(n,p,q)) = (\tilde{n}, \tilde{p}, \tilde{q})$$

where $(\tilde{n}, \tilde{p}, \tilde{q})$ are obtained by Lemma 3.31. We call $(n(n,p,q), p(n,p,q), q(n,p,q))$ and $(n_a(n,p,q), p_a(n,p,q), q_a(n,p,q))$ *reductions* for (n,p,q).

3.3.2 The underlying structure of the (almost) boundary parallel set

The reductions of (n,p,q) can be computed by understanding the boundary parallel set. We are now going to see an important relation between continuous fractions and the boundary parallel (4)-set. We are going to consider the case $n = 1$. Let us first consider the following example that gives the main idea of what we want to prove next.

Example 3.35. Let $\mathcal{C}(1,[3,2])$ be enumerated and let $C \in \mathcal{C}(1,[3,2])$ contain the boundary parallel set B generated by $(0,1)$. Let further $\mathcal{C}(1,[2])$ be enumerated and let $\tilde{C} \in \mathcal{C}(1,[2])$ contain the boundary parallel set $\tilde{B}$ generated by $(0,1)$.Figure 3.12 (a) shows B in a representation of C and figure 3.12 (b) shows $\tilde{B}$ in the representation of $\tilde{C}$. We view (b)

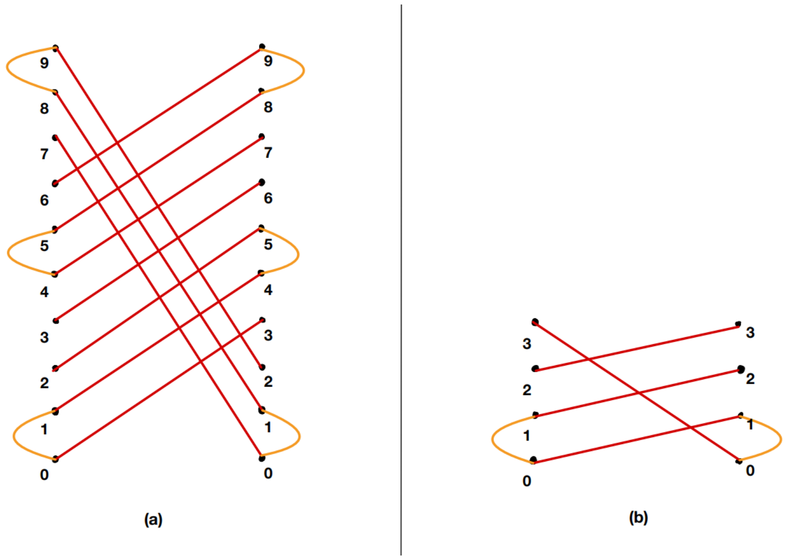


Figure 3.12: Figure (a) shows the boundary parallel set of C_1 and figure (b) shows the boundary parallel set of C^1

as building block of figure (a) i.e. figure (a) arise from stacking figure (b) 3 times on top of itself and then cutting away the last $2 \cdot (3 \cdot 2 - 3)$ points.

This stacking idea is basically the statement we want to prove. Roughly speaking we want to prove, that if we look at the representation of the boundary parallel set obtained from $(n, p/q = [r_0, \dots, r_h])$ and generated by $(0,1)$ then it is obtained from stacking the representation of a boundary parallel set obtained form $(n, [r_1, \dots, r_h])$ and generated by $(0,1)$ r_0 times on top of ech other and cutting away the last $2 \cdot (r_0 \cdot q - p)$ points. Let us look at the following construction.

Remark 3.36. Let $p/q = [r_0, \dots, r_h]$ with $h \geq 1$ and $\tilde{p}/\tilde{q} = [r_1, \dots, r_h]$. Then define a sequence $s_0, s_1, \dots$ as follows. Let $s_0 = 0$. Let $0 < r_i$ be the smallest number such that $((s_i + r_i) \cdot 2q) \bmod 2p < 2\tilde{p}$. Then $s_{i+1} = s_i + r_i$. By Lemma 2.16 we have

$$(s_i \cdot 2q) \bmod 2p = (i \cdot 2\tilde{q}) \bmod 2\tilde{p}.$$

Let $l < 2\tilde{p}$ be even. Let $0 < s$ be minimal such that $s \cdot 2q \bmod 2p = l$ and let $0 < \tilde{s}$ be minimal such that $\tilde{s} \cdot 2\tilde{q} \bmod 2\tilde{p} = l$. Define

$$B_l \coloneqq \{(0,1), \dots, (s \cdot 2q, s \cdot 2q + 1)\} \subset \mathbb{Z}^2_{2p}$$
$$\tilde{B}_l \coloneqq \{(0,1), \dots, (\tilde{s} \cdot 2\tilde{q}, \tilde{s} \cdot 2\tilde{q} + 1)\} \subset \mathbb{Z}^2_{2\tilde{p}}.$$

Then by the choice of s_i we have that there is $0 \leq m \leq s$ with $m \cdot 2q \bmod 2p < 2\tilde{q}$ if and only if $m = r_i$ for some $i < \tilde{s}$ and thus if and only if $m \cdot 2q \bmod 2p = i \cdot 2\tilde{q} \bmod 2\tilde{p}$. Hence we have

$$\begin{aligned}&(2a, 2a+1) \in B_l \text{ with } 2a \bmod 2np < 2\tilde{q}\\ \Leftrightarrow &(2a \bmod 2np, (2a+1) \bmod 2np) \in \tilde{B}_l.\end{aligned}$$

Be aware that $B_2, \tilde{B}_2$ are boundary parallel sets and $B_4, \tilde{B}_4$ are boundary parallel 4-sets. This leads to the following Corollaries describing the precise stacking idea.

Corollary 3.37. *Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ be an enumerated with $h \geq 1$ and let $B = \{(0,1), \ldots, (k \cdot 2q, k \cdot 2q+1)\}$ be the boundary parallel set generated by $(0,1)$. Furthermore let $\mathcal{C}(1, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h])$ be enumerated and let $\tilde{B} = \{(0,1), \ldots, (\tilde{k} \cdot 2\tilde{q}, \tilde{k} \cdot 2\tilde{q}+1)\}$ be the boundary parallel set of generated by $(0,1)$. Then there is an $(a, a+1) \in B$ if and only if*

$$((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in \tilde{B}$$

Proof. Let $a \in \mathbb{Z}_{2np}$ with $a \bmod 2p \geq 2q$, then by the construction of the boundary parallel set we have $(a, a+1) \in B$ if and only if $(a - 2q, a - 2q + 1) \in B$. By the construction of the boundary parallel set we have $(a, a+1) \in B$ with $a \bmod 2p \geq 2q$ if and only if $(a - 2q, a - 2q + 1) \in B$. Note that in remark 3.36 we defined B_l and $\tilde{B}_l$ and we have $B = B_2$ and $\tilde{B} = \tilde{B}_2$. Therefore

$$\begin{aligned}(a, a+1) \in B &\Leftrightarrow ((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in B\\ &\overset{3.41}{\Leftrightarrow} ((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in \tilde{B}.\end{aligned}$$

□

Corollary 3.38. *Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ be an enumerated and let $B = ((0,1), \ldots, (k \cdot 2q, k \cdot 2q+1))$ be the boundary parallel set generated by $(0,1)$. Then*

$$(2, 2+1), \ldots, (r_h - 2, r_h - 1) \notin B.$$

Proof. This is just an inductive argument. As for $h = 0$ we have $\tilde{B} = \{(0,1)\}$ the assumption holds for $h = 1$. Let $\mathcal{C}(1, p/q = [r_1, \ldots, r_h])$ be enumerated and let $\tilde{B}$ be the boundary parallel set generated by $(0,1)$. If $h \geq 1$ Corollary 3.42 gives, that

$$(2, 2+1), \ldots, (r_h - 2, r_h - 1) \notin B \Leftrightarrow (2, 2+1), \ldots, (r_h - 2, r_h - 1) \notin \tilde{B}.$$

□

Corollary 3.39. *Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ be an enumerated with $h \geq 3$ and let $A_* = ((0,1), \ldots, (\bar{k} \cdot 2q, \bar{k} \cdot 2q+1))$ be the boundary parallel 4 set generated by $(0,1)$. Furthermore let $\mathcal{C}(1, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h])$ be enumerated with $n\tilde{p} \geq 3$. Then the boundary parallel 4-set $\tilde{A}_* = ((0,1), \ldots, (\tilde{\bar{k}} \cdot 2\tilde{q}, \tilde{\bar{k}} \cdot 2\tilde{q}+1))$ of $\mathcal{C}(n, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h])$ generated by $(0,1)$ exists. Then there is an $(a, a+1) \in A_*$ if and only if*

$$((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in \tilde{A}_*$$

Proof. Let $a \in \mathbb{Z}_{2np}$ with $a \bmod 2p \geq 2q$, then by the construction of the boundary parallel 4-set we have $(a, a+1) \in A_*$ if and only if $(a-2q, a-2q+1) \in A_*$. Note that in remark 3.36 we defined B_l and $\tilde{B}_l$ and we have $A_* = B_4$ and as $\tilde{q} > 2$ (as $h \geq 3$) $\tilde{A}_* := \tilde{B}_4$ is the boundary parallel 4-set of $\mathcal{C}(1, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h])$ generated by $(0,1)$ (this proves the existence). Therefore

$$\begin{aligned}(a, a+1) \in A_* &\Leftrightarrow ((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in A_*. \\ &\overset{3.41}{\Leftrightarrow} ((a \bmod 2p) \bmod 2q, (a \bmod 2p) \bmod 2q + 1) \in \tilde{A}_*.\end{aligned}$$

□

3.3.3 The reductions of (n, p, q).

The relation between the boundary parallel set and continuous fractions gives us the necessary tools to prove some theorems about the reductions of (n, p, q). We are first going to concentrate on the case $n = 1$ and then prove a total classification of the reductions of (n, p, q).

Lemma 3.40. *Let $p/q = [r_0, \ldots, r_h])$ and $\tilde{p}/\tilde{q} = [r_1, \ldots, r_h]$ with $h \geq 2$. Then $n(1,p,q) = n(1, \tilde{p}, \tilde{q})$, $p(1,p,q) = r_0 \cdot p(1,\tilde{p},\tilde{q}) - q(1,\tilde{p},\tilde{q})$ and $q(1,p,q) = p(1,\tilde{p},\tilde{q})$.*

Proof. Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ be enumerated and let $C \in \mathcal{C}(1,p,q)$ contain the boundary parallel set $B = ((0,1), \ldots, (\bar{k} \cdot 2q, \bar{k} \cdot 2q + 1))$ generated by $(0,1)$. Let further $\mathcal{C}(1, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h])$ be enumerated and let $\tilde{C} \in \mathcal{C}(1, \tilde{p}/\tilde{q} = [r_1, \ldots, r_h]$ contain the boundary parallel set $\tilde{B} = ((0,1), \ldots, (\tilde{k} \cdot 2\tilde{q}, \tilde{k} \cdot 2\tilde{q} + 1))$ generated by $(0,1)$. Then Lemma 3.31 gives the following identities:

$$2 \cdot n(1,p,q)p(1,p,q) = 2p - 2\bar{k} + 1) = \#(\{0, \ldots, 2p-1\} - B), \tag{3.2}$$
$$2 \cdot n(1,p,q)q(1,p,q) = \#(\{0, \ldots, 2q-1\} - B), \tag{3.3}$$
$$2 \cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}) = 2\tilde{p} - 2(\tilde{k}+1) = \#(\{0, \ldots, 2\tilde{p}-1\} - \tilde{B}), \tag{3.4}$$

We have

$$\begin{aligned}2 \cdot n(1,p,q)q(1,p,q) &\overset{(3.3)}{=} \#(\{0, \ldots, 2q-1\} - B) \\ &\overset{3.37}{=} \#(\{0, \ldots, 2q-1\} - \tilde{B}\}) \\ &\overset{3.4}{=} 2 \cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q})\end{aligned}$$

Note that $p = r_0\tilde{p} - \tilde{q}$ and $q = \tilde{p}$. Let

$$I_i := \{i \cdot 2\tilde{p}, \ldots, (i+1) \cdot 2\tilde{p} - 1\} \subset \mathbb{Z}_{2p}$$

and

$$I := \{(r_0 - 1) \cdot 2\tilde{p}, \cdots, 2(r_0\tilde{p} - \tilde{q}) - 1\} \subset \mathbb{Z}_{2p}.$$

Then

$$\#(\{0,\dots,2p-1\}-B) = \#\left(\left(\bigsqcup_{i=0}^{r_0-2} I_i \sqcup I\right) - B\right) \tag{3.5}$$

$$= \sum_{i=0}^{r_0-2} \#(I_i - B) + \#(I-B) \tag{3.6}$$

Then

$$\begin{aligned}\#(I_i - B) &\overset{3.37}{=} \#(I_0 - \tilde{B})\\ &\overset{3.4}{=} 2\cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}).\end{aligned}$$

Furthermore

$$\begin{aligned}\#(I-B) &\overset{3.37}{=} \#(\{0,\dots,2(\tilde{p}-\tilde{q})-1\}-\tilde{B})\\ &\overset{3.37}{=} \#(\{2\tilde{q},\dots,2\tilde{p}-1\}-\tilde{B})\\ &= \#(\{0,\dots,2\tilde{p}-1\}-\tilde{B}) - \#(\{0,\dots,2\tilde{q}-1\}-\tilde{B})\\ &\overset{3.31}{=} 2\cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}) - 2\cdot n(1,\tilde{p},\tilde{q})q(1,\tilde{p},\tilde{q}).\end{aligned}$$

All together we get

$$\begin{aligned}2\cdot n(1,p,q)p(1,p,q) &\overset{(3.2)\&(3.6)}{=} \sum_{i=0}^{r_0-2} \#(I_i - B) + \#(I-B)\\ &= r_0\cdot 2\cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}) - 2\cdot n(1,\tilde{p},\tilde{q})q(1,\tilde{p},\tilde{q}).\end{aligned}$$

Then as

$$\begin{aligned}&2\cdot n(1,p,q)\\ =&\gcd\left(2\cdot n(1,p,q)q(1,p,q), 2\cdot n(1,p,q)p(1,p,q)\right)\\ &\gcd\left(2\cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}), r_0\cdot 2\cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}) - 2\cdot n(1,\tilde{p},\tilde{q})q(1,\tilde{p},\tilde{q})\right)\\ =&2\cdot n(1,\tilde{p},\tilde{q}),\end{aligned}$$

the Lemma follows. □

Let us prove a similar version for the almost boundary parallel set.

Lemma 3.41. *Let $p/q = [r_0,\dots,r_h]$ and $\tilde{p}/\tilde{q} = [r_1,\dots,r_h]$ with $h \geq 3$. Then $n_a(1,p,q) = n_a(1,\tilde{p},\tilde{q})$, $p_a(1,p,q) = r_0\cdot p_a(1,\tilde{p},\tilde{q}) - q_a(1,\tilde{p},\tilde{q})$ and $q_a(1,p,q) = p_a(1,\tilde{p},\tilde{q})$.*

Proof. Let $\mathcal{C}(1,p/q = [r_0,\dots,r_h])$ be enumerated, let $C \in \mathcal{C}(1,p,q)$ contain the almost boundary parallel set generated by $(0,1)$ and let $A_* = ((0,1),\dots,(\bar{k}\cdot 2q, \bar{k}\cdot 2q+1))$ be the boundary parallel 4-set generated by $(0,1)$. Let further $\mathcal{C}(1,\tilde{p}/\tilde{q} = [r_1,\dots,r_h])$ be enumerated, let $\tilde{C} \in \mathcal{C}(1,\tilde{p}/\tilde{q} = [r_1,\dots,r_h]$ contain the almost boundary parallel set generated by $(0,1)$ and let $\tilde{A}_* = ((0,1), \ldots, (\tilde{k} \cdot 2\tilde{q}, \tilde{k} \cdot 2\tilde{q}+1))$ be the boundary parallel 4-set generated by $(0,1)$. Then Lemma 3.31 gives the following identities:

$$2 \cdot n_a(1,p,q)p_a(1,p,q) = 2p - 2(\bar{k}+1) = \#(\{0, \ldots, 2p-1\} - A_*), \tag{3.7}$$
$$2 \cdot n_a(1,p,q)q_a(1,p,q) = \#(\{0, \ldots, 2q-1\} - A_*), \tag{3.8}$$
$$2 \cdot n_a(1,\tilde{p},\tilde{q})p_a(1,\tilde{p},\tilde{q}) = 2\tilde{p} - 2(\tilde{k}+1) = \#(\{0, \ldots, 2\tilde{p}-1\} - \tilde{A}_*), \tag{3.9}$$

We have

$$\begin{aligned} 2 \cdot n_a(1,p,q)q_a(1,p,q) &\overset{(3.8)}{=} \#(\{0, \ldots, 2q-1\} - A_*) \\ &\overset{3.39}{=} \#(\{0, \ldots, 2\tilde{p}-1\} - \tilde{A}_*\})) \\ &\overset{(3.9)}{=} 2 \cdot n(1,\tilde{p},\tilde{q})p(1,\tilde{p},\tilde{q}) \end{aligned}$$

Note that $p = r_0\tilde{p} - \tilde{q}$ and $q = \tilde{p}$. Let

$$I_i \coloneqq \{i \cdot 2\tilde{p}, \ldots, (i+1) \cdot 2\tilde{p} - 1\} \subset \mathbb{Z}_{2p}$$

and

$$I \coloneqq \{(r_0 - 1) \cdot 2\tilde{p}, \cdots, 2(r_0\tilde{p} - \tilde{q}) - 1\} \subset \mathbb{Z}_{2p}.$$

Then

$$\#(\{0, \ldots, 2p-1\} - A_*) = \#\left(\left(\bigsqcup_{i=0}^{r_0-2} I_i \sqcup I\right) - A_*\right) \tag{3.10}$$
$$= \sum_{i=0}^{r_0-2} \#(I_i - A_*) + \#(I - A_*) \tag{3.11}$$

Then

$$\begin{aligned} \#(I_i - A_*) &\overset{3.39}{=} \#(I_0 - \tilde{A}_*) \\ &\overset{(3.9)}{=} 2 \cdot n_a(1,\tilde{p},\tilde{q})p_a(1,\tilde{p},\tilde{q}). \end{aligned}$$

Furthermore

$$\begin{aligned} \#(I - A_*) &\overset{3.39}{=} \#(\{0, \ldots, 2(\tilde{p} - \tilde{q}) - 1\} - \tilde{A}_*) \\ &\overset{3.39}{=} \#(\{2\tilde{q}, \ldots, 2\tilde{p} - 1\} - \tilde{A}_*) \\ &= \#(\{0, \ldots, 2\tilde{p} - 1\} - \tilde{A}_*) - \#(\{0, \ldots, 2\tilde{q} - 1\} - \tilde{A}_*) \\ &\overset{3.31}{=} 2 \cdot n_a(1,\tilde{p},\tilde{q})p_a(1,\tilde{p},\tilde{q}) - 2 \cdot n_a(1,\tilde{p},\tilde{q})q_a(1,\tilde{p},\tilde{q}). \end{aligned}$$

All together we get

$$\begin{aligned} 2 \cdot n_a(1,p,q)p_a(1,p,q) &\overset{(3.7)\&(3.11)}{=} \sum_{i=0}^{r_0-2} \#(I_i - A_*) + \#(I - A_*) \\ &= r_0 \cdot 2 \cdot n_a(1,\tilde{p},\tilde{q})p_a(1,\tilde{p},\tilde{q}) - 2 \cdot n_a(1,\tilde{p},\tilde{q})qa(1,\tilde{p},\tilde{q}). \end{aligned}$$

Then as

$$
\begin{aligned}
&2 \cdot n_a(1,p,q)\\
=&\gcd\left(2 \cdot n_a(1,p,q)q_a(1,p,q), 2 \cdot n_a(1,p,q)p_a(1,p,q)\right)\\
=&\gcd\left(2 \cdot n_a(1,\tilde{p},\tilde{q})p_a(1,\tilde{p},\tilde{q}), r_0 \cdot 2 \cdot n_a(1,\tilde{p},\tilde{q})pa(1,\tilde{p},\tilde{q}) - 2 \cdot n_a(1,\tilde{p},\tilde{q})q_a(1,\tilde{p},\tilde{q})\right)\\
=&2 \cdot n_a(1,\tilde{p},\tilde{q}),
\end{aligned}
$$

the Lemma follows. □

This leads to the following classification of a reduction for a boundary parallel set.

Lemma 3.42. *Let* $(n, p/q = [r_0, \ldots, r_h])$.

(i) If $n \geq 2$*, then*

$$n(n,p,q) = n-1,\ p(n,p,q) = p \text{ and } q(n,p,q) = q,$$

(ii) if $n = 1$*, then*

$$n(n,p,q) = 1,\ p(n,p,q)/q(n,p,q) = [r_0, \ldots, r_h - 1].$$

Proof. Let $B = \{(0,1), \ldots, (k \cdot 2nq, k \cdot 2nq + 1)\}$ be the boundary parallel set generated by $(0,1)$. We distinguish 4 cases.

Case 1: $n \geq 2$. In the prove of Lemma 3.31 we have seen, that $k+1 = p$. As further $i \cdot 2nq \bmod 2np$ is divisible by $2n$ we get, that $B = \{(0,1), (2n, 2n+1), \ldots, (k \cdot 2n, k \cdot 2n + 1)\}$. Thus

$$\#(\{0, \ldots, 2nq-1\} - B) = 2(n-1)q.$$

Then Lemma 3.31 gives $n(n,p,q)p(n,p,q) = np - (k+1) = np - p = (n-1)p$ and $2n(n,p,q)q(n,p,q) = 2(n-1)q$. Therefore

$$
\begin{aligned}
n(n,p,q) &= n-1\\
p(n,p,q) &= p\\
q(n,p,q) &= q.
\end{aligned}
$$

Case 2: $n = 1$ and $h = 0$. Then $q = 1$ and $k = 0$. Thus Lemma 3.31 gives $n(1,p,q)q(1,p,q) = 1$ and $n(1,p,q)p(1,p,q) = p - (k+1) = p - 1 =$ and thus

$$
\begin{aligned}
n(1,p,1) &= 1\\
p(1,p,1) &= p-1\\
q(1,p,1) &= 1.
\end{aligned}
$$

Hence $p/q = [r_0 - 1]$.

Case 3: $n = 1$ and $h = 1$. Then $p = r_0 r_1 - 1$ and $q = r_0$. Therefore $r_1 2q \equiv$

2 (mod $2p$) and hence $B = \{(0,1), \ldots, (r_1 - 1) \cdot 2r_0, (r_1 - 1) \cdot 2r_0 + 1\}$. Therefore $2n(1,p,q)q(1,p,q) = \#(\{0, \ldots, r_0{-}1\} - \{(0,1), \ldots, (r_1{-}1) \cdot 2r_0, (r_1{-}1) \cdot 2r_0{+}1\}) = 2r_0{-}2$ and $n(1,p,q)p(1,p,q) = p - (k+1) = r_0r_1 - 1 - r_1 = (r_0 - 1)r_1 - 1$. This gives

$$\begin{aligned} n(1,p,1) &= 1 \\ p(1,p,1) &= (r_0 - 1)r_1 - 1 \\ q(1,p,1) &= r_0 - 1. \end{aligned}$$

Hence $p/q = [r_1, r_0 - 1]$.

Case 4: $n = 1$ and $h \geq 2$. Let $\tilde{p}/\tilde{q} = [r_1, \ldots, r_h]$. Then induction gives $n(1, \tilde{p}, \tilde{q}) = 1$ and $p(1, \tilde{p}, \tilde{q})/q(1, \tilde{p}, \tilde{q}) = [r_1, \ldots, r_h - 1]$. Then

$$\begin{aligned} n(1,p,q) &\overset{3.39}{=} n(1, \tilde{p}, \tilde{q}) = 1, \\ p(1,p,q) &\overset{3.39}{=} r_0 \cdot p(1, \tilde{p}, \tilde{q}) - q(1, \tilde{p}, \tilde{q}) \\ q(1,p,q) &\overset{3.39}{=} p(1, \tilde{p}, \tilde{q}). \end{aligned}$$

Thus $p/q = [r_0, \ldots, r_h - 1]$. □

We now prove the classification of the reduction of (n, p, q) for an almost boundary parallel set.

Lemma 3.43. *Let $(n, p/q = [r_0, \ldots, r_h])$ then a boundary parallel 4-set exists if and only if $n = 2$ and $q \neq 1$ or $r_h = 2$ and $h \geq 2$. Then*

(i)

$$n_a(2,p,q) = 1,\ p_a(2,p,q)/q_a(2,p,q) = [r_0, \ldots, r_h, 2],$$

(ii) if $r_h = 2$ and $h \geq 2$ then,

$$n_a(1,p,q) = 2,\ p_a(1,p,q)/q_a(1,p,q) = [r_0, \ldots, r_{h-1} - 1].$$

Proof. Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated. We have that the boundary parallel 4-set generated by $(0,1)$ (if it exists) is proper subset of the boundary parallel set $B = \{(0,1), \ldots, k \cdot 2nq, k \cdot 2nq + 1\}$ generated by $(0,1)$. This implies, that $(4,5) \in B$. Furthermore we want $0 \leq \tilde{k} \leq k$ with $\tilde{k} \cdot 2nq \equiv 4 \pmod{2np}$ such that $\tilde{k} \neq 1$, which is equivalent to $2nq \neq 4$.

If $n \geq 2$ we have seen in the proof of 3.42 that

$$B = \{(0,1), \ldots, ((p-1) \cdot 2n, (p-1) \cdot 2n + 1)\},$$

hence $(4,5) \in B$ if and only if $n = 2$. Then the boundary parallel 4-set exists if and only if $2np \neq 4$ i.e. $p \neq 1$. This proves the case for $n \geq 2$.

If $n = 1$ Corollary 3.43 gives, that if $r_h \geq 3$ then $(4,5)$ is not in the boundary parallel set and hence there is no boundary parallel 4-set. So let $r_h = 2$. If $h = 0$ the boundary parallel set has cardinality 1 and thus there can not be an almost boundary parallel set.

If $h = 1$ we have $q = 2$. Hence $2nq = 4$ and thus there is no almost boundary parallel set. If $h \geq 2$ then $q \geq 3$ and thus $2nq > 4$ which implies that there is an almost boundary parallel set. This proves the case for $n = 1$.

Let $A_* = (0,1), \dots, (\bar{k} \cdot 2nq, \bar{k} \cdot 2nq + 1)$ be the boundary parallel 4-set generated by $(0,1)$. We now distinguish 3 cases.

Case 1: $n = 2$. We have for $\tilde{k}, m \in \mathbb{N}$

$$(\tilde{k}+1) \cdot 4q = 2m \pmod{4p} \Leftrightarrow (\tilde{k}+1) \cdot 2q = m \pmod{2p} \tag{3.12}$$

This yields $(\bar{k}+1) \cdot 4q = 4 \pmod{4p} \Leftrightarrow (\bar{k}+1) \cdot 2q = 2 \pmod{2p}$ and hence $\bar{k}+1$ is the size of the boundary parallel set for $(1,p,q)$ and hence

$$2p - (\bar{k}+1) = p + p(1,p,q).$$

Furthermore

$$\begin{aligned}
2 \cdot n_a(2,p,q)q_a(2,p,q) =& \#\big(\{0,\dots,4q-1\} - \{(0,1),\dots,(\bar{k}\cdot 4q, \bar{k}\cdot 4q+1)\}\big) \\
=& \#\big(\{2,3,2+4,3+4,\dots,2+4(q-1),3+4(q-1)\} \\
& \quad \sqcup \{0,1,4,1+4,\dots,4(q-1),1+4(q-1)\} \backslash \\
& \quad \{(0,1),\dots,(\bar{k}\cdot 4q, \bar{k}\cdot 4q+1)\}\big) \\
=& 2q + \#\big(\{0,1,4,1+4,\dots,4(q-1),1+4(q-1)\} \backslash \\
& \quad \{(0,1),\dots,(\bar{k}\cdot 4q, \bar{k}\cdot 4q+1)\}\big) \\
\overset{(3.12)}{=}& 2p + 2p(1,p,q).
\end{aligned}$$

By Lemma 3.42 we have

$$\begin{aligned}
p(1,p,q)/q(1,p,q) &= [r_0,\dots,r_h - 1] \\
&\overset{2.47}{=} [r_0,\dots,r_h, 2-1].
\end{aligned}$$

By Lemma 2.48 we get $(p + p(1,p,q))/(q + q(1,p,q)) = [r_0,\dots,r_h,2]$ which proves the case for $n = 2$.

Case 2: $n = 1$ and $h = 2$. As $r_2 = 2$ we have $p = r_0(r_2 r_1 - 1) - r_2 = r_0(2r_1 - 1) - 2$ and $q = 2r_1 - 1$. Therefore $r_0 \cdot 2q \equiv 4 \pmod{2p}$ and hence

$$A_* = \{(0,1),\dots,(r_0-1)\cdot 2q, (r_0-1)\cdot 2q + 1\}.$$

Therefore $\bar{k} + 1 = r_0$ and thus

$$\begin{aligned}
n_a(1,p,q)p_a(1,p,q) &= p - r_0 \\
&= r_0(2r_1 - 1) - 2 - r_0 \\
&= r_0(2r_1 - 2) - 2 \\
&= 2(r_0(r_1 - 1) - 1).
\end{aligned}$$

Furthermore

$$\begin{aligned}2 \cdot n_a(1,p,q)q_a(1,p,q) &= \#(\{0,\ldots,2q-1\} - \{(0,1),\ldots,((r_0-1)\cdot 2q,(r_0-1)\cdot 2q+1)\})\\ &= 2(q-1)\\ &= 2r_1-2\end{aligned}$$

Then $\gcd(2(r_0(r_1-1)-1), 2r_1-2) = 2$ and thus $n_a(1,p,q) = 2$, $p_a(1,p,q) = r_0(r_1-1)-1$ and $q_a(1,p,q) = r_1 - 1$ which implies $p_a(1,p,q)/q_a(1,p,q) = [r_0, r_1 - 1]$.

Case 3: $n = 1$ and $h \geq 3$. Let $\bar{p}/\bar{q} = [r_1,\ldots,r_h]$. Then by Lemma 3.47 we have $n_a(1,p,q) = n_a(1,\tilde{p},\tilde{q})$, $p_a(1,p,q) = r_0 p_a(1,\tilde{p},\tilde{q}) - q_a(1,\tilde{p},\tilde{q})$ and $q_a(1,p,q) = q_a(1,\tilde{p},\tilde{q})$. Induction on h then gives the assumption. □

3.3.4 Normal forms

As we now completely understand the reduction induced by a boundary parallel set we next want to make use of it. We will introduce the notion of a normal form which roughly speaking is a configuration that contains a boundary parallel set such that the reduction contains a boundary parallel set and so on. We will further introduce a subset of normal forms such that every allowable configuration is connected to such a normal form. By counting those normal forms we will find an upper bound for the number of tight contact structures on the torus.

Definition 3.44. Let $\mathcal{C}(n, p/q = [r_0,\ldots,r_h])$ be enumerated. Then $C \in \mathcal{C}(n, p/q = [r_0,\ldots,r_h])$ has *normal form* if either $n = p = q = 1$ or C contains a boundary parallel set B generated by $(l, l+1)$ and C^B has *normal form* (this is an inductive definition).

Let us now prove, that every allowable configuration can be connected to a configuration that has normal form.

Lemma 3.45. *Let $\mathcal{C}(n, p/q = [r_0,\ldots,r_h])$ be enumerated and $C \in \mathcal{C}(n, p/q = [r_0,\ldots,r_h])$ be allowable. Then there is a sequence of allowable state transitions, that connect C to a configuration that has normal form.*

Proof. Let $(l, l+1)$ be a boundary parallel arc in C. Then by Lemma 3.31 there is a sequence of allowable state transitions that connects C to a configuration $\tilde{C}$ that contains the boundary parallel set generated by $(l, l+1)$. Then $\tilde{C}^B \in \mathcal{C}(n(n,p,q), p(n,p,q), q(n,p,q))$. Induction then gives the statement of the lemma. □

We will first consider the case $n = 1$. This has the advantage, that we can connect an allowable configuration containing some $(l, l+1)$ to a configuration that contains the boundary parallel set generated by $(l+2i, l+1+2i)$ for every $i \in \mathbb{N}$ see Remark 3.32. This leads to the following definition of specific normal forms.

Definition 3.46. Let $\mathcal{C}(1, p/q = [r_0,\ldots,r_h])$ be enumerated. Then $C \in \mathcal{C}(1, p/q = [r_0,\ldots,r_h])$ has *normal form* $(l_0,\ldots,l_h)$ with $l_i \leq r_i - 2$ for $0 \leq i \leq h-1$ and $l_h \leq r_h - 1$ iff one of the following holds

(i) $n = q = p = 1$,

(ii) if $l_h \geq 1$ then C contains the boundary parallel set B generated by $(0,1)$ and C^B has normal form $(l_0, \dots l_h - 1)$ if $r_h > 2$ resp. $(l_0, \dots, l_{h_{\max}})$ if $r_h = 2$.

(iii) if $l_h = 2$ then C contains the boundary parallel set B generated by $(1,2)$ and C^B has normal form $(l_0, \dots, l_{h_{\max}})$.

Let us prove that every allowable configuration for $n = 1$ is connected to such a normal form.

Lemma 3.47. *Let $\mathcal{C}(1, p/q = [r_0, \dots, r_h])$ be enumerated and $C \in \mathcal{C}(1, p/q = [r_0, \dots, r_h])$ be allowable. Then there is a sequence of allowable state transitionsthat connect C to a configuration $\tilde{C} \in \mathcal{C}(1, p/q = [r_0, \dots, r_h])$ that has normal form $(\tilde{l}_0, \dots, \tilde{l}_h)$.*

Proof. We use induction on $n(r_0 + \dots + r_h)$. Let $(l, l+1)$ be a boundary parallel arc in C. We argued in remark 3.32 that there is sequence of allowable state transitionsthat connects C to a configuration $\tilde{C}$ that contains the boundary parallel set generated by $(0,1)$ if $l \bmod 2 = 0$ resp. the boundary parallel set generated by $(1,2)$ if $l \bmod 2 = 1$. The induction argument gives $\tilde{C}^B$ is connected to a configuration $\bar{C}_1$ that has some normal form $(l_0, \dots, l_{h_{\max}})$ which by Lemma 3.31 implies that $\tilde{C}$ is connected to a configuration $\bar{C}$ without changing B such that $\bar{C}^B = \bar{C}_1$. Let us distinguish the two cases.

Case 1: B is generated by $(0,1)$. If $h_{\max} = h$ i.e. iff $r_h > 2$ then $\bar{C}$ has normal form $(l_0, \dots, l_{h_{\max}} + 1)$. If $h_{\max} < h$ i.e. iff $r_h = 2$ then $\bar{C}$ has normal form $(l_0, \dots, l_{h_{\max}}, \dots, l_h)$ with $l_h = 1$ and $l_i = 0$ for $h_{\max} < i < h$.

Case 2: B is generated by $(1,2)$. If $h_{\max} < h$ i.e. iff $r_h = 2$ then $\bar{C}$ has normal form $(l_0, \dots, l_{h_{\max}}, \dots, l_h)$ with $l_i = 0$ for $h_{\max} < i \leq h$. If $h_{\max} = h$ i.e. iff $r_h > 2$ and $l_{h_{\max}} \geq 1$ then $\bar{C}^B$ has boundary parallel arc $(0,1)$. By 3.32 there is a sequence of allowable state transitionsthat connects $\bar{C}^B$ to a configuration $\hat{C}_1$ that has boundary parallel arc $(2,3)$. By Lemma 3.31 this corresponds to a sequence of allowable state transitions, that connect $\bar{C}$ to some $\hat{C}$ without changing B such that $\hat{C}^B = \hat{C}_1$. By Corollary 3.37 we have $(4,5) \notin B$ and thus $\alpha^B(4) = 2$ and $\alpha^B(5) = 3$ which yields that $(4,5)$ is a boundary parallel arc in $\hat{C}$. If we chose $(l, l+1) = (4,5)$ in the beginning of the prove we can apply case 1. $\square$

We now want to see that two configurations that have normal form $(l_0, \dots, l_h)$ have different partition number $\chi(R_+)$ and thus two normal forms, that are allowable can not belong to the same connected component in $G(1,p,q)$. We first prove the following interim result.

Corollary 3.48. *Let $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ be enumerated and let $C \in \mathcal{C}(n, p/q = [r_0, \dots, r_h])$ contain a boundary parallel set resp. an almost boundary parallel set B generated by $(l, l+1)$ that has size $k+1$. Then*

(i) if $l \bmod 2 = 0$ then $\chi(R_+(C)) = k + 1 + \chi(R_+(C^B))$,

(ii) if $l \bmod 2 = 1$ then $\chi(R_+(C)) = \chi(R_+(C^B))$.

Proof. By Lemma 3.36 α^B sends an even point to an even point and an odd point to an odd point. Therefore every positive resp. negative region in C^B corresponds to a positive resp. negative region in C. The only regions in C that do not correspond to a region in C^B are the cut off half disks of the boundary parallel arcs in B. If $l \bmod 2 = 0$ then $(l, l+1)$ cuts of a positive half disk and hence every boundary parallel arc in B cuts of a positive half disk which implies, that $\chi(R_+(C)) = k + 1 + \chi(R_+(C^B))$. If $l \bmod 2 = 1$ then $(l, l+1)$ cuts of a negative half disk and hence $\chi(R_+(C)) = \chi(R_+(C^B))$. □

Lemma 3.49. *Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ be enumerated. Let $p_i/q_i = [r_0, \ldots, r_{i-1}]$ for $1 \le i \le h$. Let C be the configuration corresponding to $(l_0, \ldots, l_h)$ then*

$$\chi(R_+(C)) = l_0 + 1 + \sum_{i=1}^{h} l_i p_i.$$

Proof. We use induction on $r_0 + \cdots + r_h$. If $r_0 = 2$ and $h = 0$ the normal form (0) by construction has partition number 1 and the normal form (1) has partition number 2 (there are only two configurations). Let $r_0 + \cdots + r_h \ge 3$. We distinguish 2 cases.

Case 1: $l_h \ge 1$. Then there is a boundary parallel set $B = \{(0,1), \ldots, (k\cdot 2q, k\cdot 2q+1)\}$ in C that is generated by $(0,1)$. By Corollary 3.48 we thus have that $\chi(R_+(C)) = k + 1 + \chi(R_+(C^B))$.

If $h = 0$ we have $k+1 = 1$. As C^B has normal form $(l_h - 1)$ by induction $\chi(R_+(C^B)) = l_h$ and thus $\chi(R_+(C)) = l_h + 1$.

If $h \ge 1$ by Lemma 2.48 and Lemma 3.42 we have $k + 1 = p_h$. If $r_h = 2$ we have $\chi(R_+(C^B)) = l_0 + 1 + \sum_{i=1}^{h_{\max}} l_i p_i$. Furthermore as $r_i = 2$ for $h_{\max} < i \le h - 1$ we have $0 \le l_i \le r_i - 2 \Rightarrow l_i = 0$ for $h_{\max} < i \le h - 1$. This implies

$$\chi(R_+(C)) = l_0 + 1 + \sum_{i=1}^{h-1} l_i p_i + p_h = l_0 + 1 + \sum_{i=1}^{h} l_i p_i.$$

If $r_h > 2$ we have that C^B has normal form $(l_0, \ldots, l_h - 1)$ and thus

$$\chi(R_+(C^B)) = l_0 + 1 + \sum_{i=1}^{h-1} l_i p_i + (l_h - 1)p_h + p_h = l_0 + 1 + \sum_{i=1}^{h} l_i p_i.$$

Case 2: $l_h = 0$. Then there is a boundary parallel set $B = \{(1,2), \ldots, (k \cdot 2q + 1, k \cdot 2q + 2)\}$ in C that is generated by $(1,2)$. By Corollary 3.48 we thus have $\chi(R_+(C)) = \chi(R_+(C^B))$. Then C^B has normal form $(l_0, \ldots, l_{h_{\max}})$. We have $r_i = 2$ for $h_{\max} < i \le h - 1$ and thus $l_i \le r_i - 2 \Rightarrow l_i = 0$ for $h_{\max} < i \le h - 1$. Furthermore by assumption $l_h = 0$ and thus $l_i = 0$ for $h_{\max} < i \le h$ and thus

$$\chi(R_+(C)) = l_0 + 1 + \sum_{i=1}^{h_{\max}} = l_0 + 1 + \sum_{i=1}^{h}.$$

□

Corollary 3.50. *Let $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$. Then two normal forms $(l_0, \ldots, l_h)$ and $(\tilde{l}_0, \ldots, \tilde{l}_h)$ have the same partition number $(\chi(R_+))$ if and only if $l_i = \tilde{l}_i$ for $0 \leq i \leq h$.*

Proof. We use induction on h. For $h = 0$ this is obviously true. So let $h \geq 1$. Since $r_i = 2$ for $(h-1)_{\max} < i < h-1$ and $l_i \leq r_i - 2$ for $(h-1)_{\max} < i < h-1$ we have $l_i = 0$ for $(h-1)_{\max} < i < h-1$. Therefore

$$l_0 + 1 + \sum_{i=1}^{h-1} l_i p_i = l_0 + 1 + \sum_{i=1}^{(h-1)_{\max}} l_i p_i.$$

Now $(l_0, \ldots, l_{(h-1)_{\max}})$ is a normal form in $\mathcal{C}(1, \tilde{p}/\tilde{q} = [r_0, \ldots, r_{(h-1)_{\max}}])$. Then induction gives that

$$l_0 + 1 + \sum_{i=1}^{h-1} l_i p_i = \tilde{l}_0 + 1 + \sum_{i=1}^{h-1} \tilde{l}_i p_i \tag{3.13}$$

$$\Leftrightarrow l_i = \tilde{l}_i \text{ for } 0 \leq i \leq h-1. \tag{3.14}$$

Furthermore Example 3.17 implies that the maximal partition number $(\chi(R_+))$ of a normal form $(l_0, \ldots, l_{h_{\max}})$ can be maximal $\tilde{p}$. By Lemma 2.47 we have $\tilde{p}/\tilde{q} = [r_0, \ldots, r_{h-1} - 1]$. By definition $p_h/q_h = [r_0, \ldots, r_{h-1}]$. Then Corollary 2.49 gives $\tilde{p} < p_h$. This then gives the following

$$l_h p_h < l_0 + 1 + \sum_{i=1}^{h} l_i p_i < (l_h + 1) p_h,$$

$$\tilde{l}_h p_h < \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i < (\tilde{l}_h + 1) p_h.$$

Then

$$l_0 + 1 + \sum_{i=1}^{h} l_i p_i = \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i$$

$$\Rightarrow l_h = \tilde{l}_h.$$

which then implies

$$l_0 + 1 + \sum_{i=1}^{h-1} l_i p_i = \tilde{l}_0 + 1 + \sum_{i=1}^{h-1} \tilde{l}_i p_i.$$

By 3.14 this is true if $l_i = \tilde{l}_i$ for $0 \leq i \leq h-1$. Together this gives the lemma. □

This now leads to the following Theorem giving an upper bound for the tight contact structures adopted to a multicurve related to $(1, p, q)$.

Theorem 3.51. *Let* $(1, p/q = [r_0, \dots, r_h])$ *and*

$$r = (r_0 - 1) \cdots (r_{h-1} - 1) r_h,$$

then every allowable component in $G(1, p/q = [r_0, \dots, r_h])$ *is generated by a normal form i.e.*

$$N(1, p/q = [r_0, \dots, r_h]) \leq r.$$

Furthermore two normal forms can not generate the same allowable component in $G(1, p/q = [r_0, \dots, r_h])$.

Proof. By Lemma 3.47 an allowable component in $G(1, p/q = [r_0, \dots, r_h])$ contains a normal form $(l_0, \dots, l_k)$. as $0 \leq l_i \leq r_i - 2$ for $0 \leq i \leq h-1$ and $l_h \leq r_h - 1$ there are $r = (r_0 - 1) \cdots (r_{h-1} - 1) r_h$ different normal forms and thus the $N(1, p/q = [r_0, \dots, r_h]) \leq r$. □

Let us now consider the case $n \geq 2$. We want to see that the partition number $(\chi(R_+))$ of any $n-1$-th reduction of a normal form is unique. This then leads to a better understanding of normal forms for $n \geq 2$.

Corollary 3.52. *Let* $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ *be enumerated and let* $C \in \mathcal{C}(n, p/q = [r_0, \dots, r_h])$ *have normal form. Let further* $\tilde{C}, \bar{C} \in \mathcal{C}(n, p/q = [r_0, \dots, r_h])$ *be* $n-1$*-th reductions of* C*. Then* $\tilde{C}$ *and* $\bar{C}$ *have the same partition number* $(\chi(R_+))$*. We characterize it by* $l_0, \dots, l_h$ *as*

$$l_0 + 1 + \sum_{i=1}^{h} l_i p_i.$$

Then there is a unique number $n(l)$ *such that*

$$\chi(R_+(C)) = l_0 + 1 + \sum_{i=1}^{h} l_i p_i + n(l) p.$$

Proof. Let $\chi(R_+(\tilde{C})) = \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i$ and let $\chi(R_+(\bar{C})) = \bar{l}_0 + 1 + \sum_{i=1}^{h} \bar{l}_i p_i$. By Lemma 3.42 the size of a boundary parallel set for $n \geq 2$ is p and by Corollary 3.48 every reduction contributes either 0 or p to the partition number $(\chi(R_+))$. Hence we have that there are $n(\tilde{l})$ and $n(\bar{l})$ such that

$$\chi(R_+(C)) = \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i + n(\tilde{l}) p = \bar{l}_0 + 1 + \sum_{i=1}^{h} \bar{l}_i p_i + n(\bar{l}) p.$$

We have $1 \leq \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i \leq p$ and $1 \leq \bar{l}_0 + 1 + \sum_{i=1}^{h} \bar{l}_i p_i \leq p$. Then assume w.l.o.g. let $n(\bar{l}) > n(\tilde{l})$. This implies

$$\begin{aligned} \tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i + n(\tilde{l}) p - \left(\bar{l}_0 + 1 + \sum_{i=1}^{h} \bar{l}_i p_i + n(\bar{l}) p \right) &\geq 1 - p + (n(\bar{l}) - n(\tilde{l})) p \\ &> 0. \end{aligned}$$

This implies $n(\tilde{l}) = n(\bar{l})$ and thus

$$\tilde{l}_0 + 1 + \sum_{i=1}^{h} \tilde{l}_i p_i = \bar{l}_0 + 1 + \sum_{i=1}^{h} \bar{l}_i p_i,$$

which by Corollary 3.50 implies $\bar{l}_i = \tilde{l}_i$ for $0 \leq i \leq h$. □

This yields that every allowable C has partition number $(\chi(R_+))$ characterized by $(n(l), l_0, \ldots, l_h)$. We have seen that a boundary parallel set for $n \geq 2$ generated by $(l, l+1)$ looks like $\{(l, l+1), (l+2n, l+1+2n), \ldots, (l+k \cdot 2n, l+1+k \cdot 2n)\}$ i.e. it contains the shifted arcs of $(l, l+1)$ by a multiple of $2n$. Because normal forms are constructed by boundary parallel sets the hope is, that a normal form contains shifted sets rather then arcs. This motivates the following definition.

Definition 3.53. Let $A = \{(a_0, b_0), \ldots, (a_i, b_i)\} \subset \mathbb{Z}_{2nq}$. Then

$$A + 2n \coloneqq \{(a_0 + 2n, b_0 + 2n), \ldots, (a_i + 2n, b_i + 2n)\} \text{ for } n \in \mathbb{N}.$$

If the size of A is n let w.l.o.g a_0 be the start and b_i be the end of A. Then define

$$\begin{aligned} A^{-1} &\coloneqq \{(a_0, b_0), \ldots, (a_{i-1}, b_{i-1}), (b_i - 2n, a_i)\}, \\ A^1 &\coloneqq \{(b_0, a_0 + 2n), (a_1, b_1) \ldots, (a_i, b_i)\}. \end{aligned}$$

Let us now prove that such shifted sets exist in a normal form.

Lemma 3.54. *Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated with $n \geq 2$. Let $C \in \mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ have normal form. Then there there is $1 \leq i \leq n-1$ and there are two subconfiguration S_1 with endpoints $\{s, \ldots, s+2i-1\}$ and S_2 with endpoints $\{s+2i+1, , \ldots, s+2(n-1)\}$ (note that S_2 can be empty) in C that satisfy*

$$S_1 + 2nj \subset C \text{ for all } j \in \mathbb{N}, S_2 + 2nj \subset C \text{ for all } j \in \mathbb{N}.$$

If we have S_1, S_2 as above then define $i(s) = 1$ if s is even and $i(s) = 0$ if s is odd. Let further $S = \cup_{j \in \mathbb{N}}((S_1 + 2nj) \cup (S_2 + 2nj))$ then the map

$$\begin{array}{rccc} \alpha^S\colon & \mathbb{Z}_{2np} \backslash B & \to & \mathbb{Z}_{2np} \\ & i & \mapsto & \#(\{s, \ldots, i\} - B) - i(s). \end{array}$$

induces C^S and $\psi^S\colon \mathcal{C}^S(n, p/q = [r_0, \ldots, r_h]) \to \mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ with $\mathcal{C}(1, p/q = [r_0, \ldots, r_h])$ as in Lemma 3.31. α^S corresponds to a successive reduction of S.

Proof. **First part:** We will use induction. For $n = 2$ we have that C contains a boundary parallel set B and we have seen already that

$$B = \{(l, l+1), \ldots, (l+k \cdot 2n, l+1+k \cdot 2n)\}$$

with $(l + (k+1) \cdot 2n, l+1+(k+1) \cdot 2n) = (l, l+1)$. If we set $S_1 = \{(l, l+1)\}$ and $S_2 = \emptyset$ then $B = S = \cup_{i \in \mathbb{N}}(S_1 + 2ni)$ and by applying Lemma 3.31 we get the statement for $n = 2$. So let $n \geq 3$. Let $B = \{(l, l+1), \dots, (l + k \cdot 2n, l + 1 + k \cdot 2n)\}$ be a boundary parallel set in C. The distribution of the boundary parallel arcs in B imply $\alpha^B((-) + 2n) = \alpha^B((-)) + 2(n-1)$. Thus we have that $\alpha^B(\{a, \dots, a + 2n - 1\} - B) = \{\alpha^B(a), \dots, \alpha^B(a) + 2(n-1) - 1\}$. Now by induction C^B contains two sub-configurations $\tilde{S}_1$ and $\tilde{S}_2$ and $\tilde{S} = \cup_{i \in \mathbb{N}}((\tilde{S}_1 + 2(n-1)i) \cup (\tilde{S}_2 + 2(n-1)i))$ as in the lemma. Then $\tilde{S}_1, \tilde{S}_2$ have endpoints in $\{\tilde{s}, \dots, \tilde{s} + 2(n-1) - 1\}$. Let $\alpha^B(s) = \tilde{s}$. Then $\tilde{S}_1$ resp. $\tilde{S}_2$ correspond to some $\bar{S}_1$ resp. $\bar{S}_2$ in C that have endpoints in $\{s, \dots, s + 2n - 1\}$. As we have have $\tilde{S}_1 + 2(n-1)i$ reps. $\tilde{S}_2 + 2(n-1)i$ correspond to $\bar{S}_1 + 2ni$ resp. $\bar{S}_2 + 2ni$ for all $i \in \mathbb{N}$ we have that $\bar{S}_1 + 2ni$ resp. $\bar{S}_2 + 2ni$ is in C for all $i \in \mathbb{N}$. Assume that B has not arc, that has both endpoints in $\{s, \dots, s + 2n - 1\}$. This implies that B must contain the arc $(s-1, s)$ (because of the shifted structure of B) which is not possible as s is the start of $\bar{S}_1$. Hence B contains an arc α that has both endpoints in $\{s, \dots, s + 2n - 1\}$.

- Assume $\tilde{S}_2 = \emptyset$. This implies $\bar{S}_2 = \emptyset$.

  If there is no point between α and $\bar{S}_1$ then $\bar{S}_1 \cup \alpha$ is a sub-configuration and we get that $S_1 = \bar{S}_1 \cup \alpha$ and $S_2 = \emptyset$ satisfy the conditions of the lemma.

  Assume there is one point between $\bar{S}_1$ and α then $\bar{S}_1$ is a sub-configuration and $S_1 = \bar{S}_1$ and $S_2 = \{\alpha\}$ satisfy the conditions of the lemma.

  Assume there are 2 points between $\bar{S}_1$ and α. Then $(\bar{S}_1 + 2n) \cup \{\alpha\}$ is a sub-configuration and $S_1 = (\bar{S}_1 + 2n) \cup \{\alpha\}$ and $S_2 = \emptyset$ satisfy the conditions of the lemma.

- Assume $\tilde{S}_2 \neq \emptyset$.

  If there are 3 points between $\bar{S}_1$ and $\bar{S}_2$ then α must lie between $\bar{S}_1$ and $\bar{S}_2$ (this is as there is only one point between $\tilde{S}_1$ and $\tilde{S}_2$). Then either $\bar{S}_1 \cup \{\alpha\}$ resp. $\bar{S}_2 \cup \{\alpha\}$ is a sub-configuration which yields that $S_1 = \bar{S}_1 \cup \{\alpha\}$ and $S_2 = \bar{S}_2$ resp. $S_1 = \bar{S}_1$ and $S_2 = \bar{S}_2 \cup \{\alpha\}$ satisfy the conditions of the Lemma (there is only one point between S_1 and S_2 as α is between $\bar{S}_1$ and $\bar{S}_2$).

  Assume there is one point between $\bar{S}_1$ and $\bar{S}_2$. If there is one point between α and $\bar{S}_1$ resp. α and $\bar{S}_2$ then $\alpha = (s, s+1)$ resp. $\alpha = (s + 2n - 2, s + 2n - 1)$ and hence $S_1 = (\bar{S}_2 - 2n) \cup \{\alpha\}$ and $S_2 = \bar{S}_1$ resp. $S_1 = (\bar{S}_1 + 2n) \cup \{\alpha\}$ and $S_2 = \bar{S}_2$ satisfy the conditions of the lemma. If there is no point between α and $\bar{S}_1$ resp. α and $\bar{S}_2$ then $S_1 = \bar{S}_1 \cup \{\alpha\}$ and $S_2 = \bar{S}_2$ resp. $S_1 = \bar{S}_1$ and $S_2 = \bar{S}_2 \cup \{\alpha\}$ satisfy the conditions of the lemma.

Second part: The case $n = 2$ holds because S is the boundary parallel set generated by $(s, s+1)$ and Lemma 3.31 then gives the statement. For $n \geq 3$ we use induction. Let α be a boundary parallel arc in $S_1 \cup S_2$ that does not have an endpoint that is the start of S_1 (must exist because if the size of S_1 is 1 the size of S_2 is ≥ 1 and we can choose α in S_2, if the size of S_1 is ≥ 2 then S_1 must contain a boundary parallel arc that does not have an endpoint being the start of S_1). Then let $B = \{(a, a+1), \dots, (a + 2nk, a + 2nk + 1)\}$. Then define $\tilde{S}_1 = \alpha^B(S_1 - B)$ and $\tilde{S}_2 = \alpha^B(S_2 - B)$. Then $\tilde{S}_1$ and $\tilde{S}_2$ are as in the lemma. Let $\tilde{S} = \cup_{i \in \mathbb{N}}((\tilde{S}_1 + 2ni) \cup (\tilde{S}_2 + 2ni))$. Then by induction α^S satisfies the conditions of the lemma. Now regard

$$\alpha^{\tilde{S}}(\alpha^B((-))): \quad \mathbb{Z}_{2np} - S \quad \rightarrow \quad \mathbb{Z}_{2p}.$$

This is well defined as $\alpha^B(\mathbb{Z}_{2np} - S) = \mathbb{Z}_{2(n-1)p} - \tilde{S}$. Furthermore

$$\begin{aligned}
\alpha^{\tilde{S}}(\alpha^B(x)) &= \alpha^{\tilde{S}}(\#(\{a,\dots,x\} - B)) - i(a)) \\
&= \#(\{\#(\{a,\dots,s\} - B) - i(a),\dots,\#(\{a,\dots,x\} - B) - i(a)\} - \tilde{S}) - i(s) \\
&= \#(\{\#(\{a,\dots,s\} - B),\dots,\#(\{a,\dots,x\} - B)\} - \tilde{S}) - i(s) \\
&= \#(\{\#(\{s,\dots,x\} - B)\} - \tilde{S}) - i(s) \\
&= \#(\{s,\dots,x\} - S) - i(s) \\
&= \alpha^S(x).
\end{aligned}$$

As $\alpha^B((-))$ resp. $\alpha^{\tilde{S}}$ give a one to one correspondence between $\mathcal{C}^S(n,p,q)$ and $\mathcal{C}^{\tilde{S}}(n-1,p,1)$ resp. $\mathcal{C}^{\tilde{S}}(n-1,p,q)$ and $\mathcal{C}(1,p,q)$ we get that α^S gives a one to one correspondence between $\mathcal{C}^S(n,p,q)$ and $\mathcal{C}(1,p,q)$ and thus the statement is true. $\square$

This leads to another definition of a normal form i.e. a configuration C that contains S_1, S_2, S as in the Lemma such that C^S is in normal form. Assume C is allowable. Then $\chi(R_+(C))$ is characterized by $(n(l), l_0, \dots, l_h)$ which as we have seen is an invariant of the connected component of C. Then $\chi(R_+(C^S))$ has partition number characterized by $(l_0,\dots,l_h)$ and thus can be connected to the normal form $(l_0,\dots,l_h)$. Hence that question if two allowable configurations are connected can be reduced to the question if the partition number $(\xi(R_+))$ is the same and wether we can connect both to a structure containing some S_1, S_2, S as in the lemma. We are actually able to simplify that condition.

Let C be allowable and let C contain S_1, S_2, S as in the lemma. Then C^S contains a boundary parallel arc, which by the structure of S yields a subconfiguration of size n in S. We are now going to reduce the criterion above to the existence of a subconfiguration of size n. Let us first prove the following interim result concerning subconfigurations of size n.

Lemma 3.55. *Let $\mathcal{C}(n,p,q)$ be enumerated and let SC be a sub-configuration of size j with endpoints in $\{s,\dots,s+2j-1\}$. Then there is a unique set of sub-configurations SC_i for $0 \le i < N$ with start s_i and end e_i such that $(s_i,e_i) \in S_i$ and $e_i + 1 = s_{i+1}$ for $0 \le i, i+1 < N$. We then have*

$$\sqcup_{0 \le i < N} SC_i = SC.$$

Proof. We have s_0 must be the start of S. Then define the arc in S that belongs to s_0 as (s_0,e_0). Then (s_0,e_0) cuts of a sub-configuration S_0 that has start s_0 and end e_0. Further $(s_0,e_0) \in S_0$. Use induction on $S - S_0$. As we S_0 is unique induction gives that the S_i are unique. $\square$

Lemma 3.56. *Let $\mathcal{C}(n,p,q)$ be enumerated and let C be allowable. If C contains a subconfiguration $SC = \sqcup_{0 \le i < N} SC_i$ (as in Lemma 3.55) of size n. Let $S_1 = SC_0 - \{(a_0,b_0)\}$ and $S_2 = SC - SC_0$ and $S = \cup_{j\in\mathbb{N}}((S_1 + 2nj) \cup (S_2 + 2nj))$. Then C is connected to configuration $\tilde{C}$ by allowable state transitions where $\tilde{C}$ contains SC, S.*

Proof. Now apply Lemma 3.25 inductively $p-1$ times on S_2 (if S_2 is not empty). By the Lemma thus only arcs that have endpoints in

$$\sqcup_{1\leq i\leq p-1}(\{a_1-1,\dots,b_{\bar{N}-1}\}+i\cdot 2nq)=\sqcup_{1\leq i\leq p-1}(\{a_1-1,\dots,b_{\bar{N}-1}\}+i\cdot 2n)$$

get changed and thus S_1 does not get changed. Then apply Lemma 3.25 inductively to S_1 which only changes arcs in

$$\sqcup_{1\leq i\leq p-1}(\{a_0,\dots,b_0-1\}+i\cdot 2nq)=\sqcup_{1\leq i\leq p-1}(\{a_0,\dots,b_0-1\}+i\cdot 2n)$$

and hence does not change $\sqcup_i(\{a_0,\dots,b_0\}+2nq)$ nor (a_0,b_0). Hence we get a configuration $\tilde{C}$ that contains $\bar{SC}$ and S_1,S_2,S as in Lemma 3.63. $\square$

We are now going to introduce an equivalence class of subconfigurations of size n. This is motivated by the previous lemma.

Definition 3.57. Let $\mathcal{C}(n,p/q=[r_0,\dots,r_h])$ be enumerated. Let SC be a sub-configuration and let $SC=\sqcup_{0\leq i<N}SC_i$ as in Lemma 3.56. Then

1. We denote the equivalence relation generated by

$$SC^{-1}\sim SC\sim SC^1$$

   for sub-configurations of size n as $\sim_1$.

2. We denote the equivalence relation generated by

$$(SC_0+2n)\sqcup\sqcup_{1\leq i<N}SC_i\sim SC\sim\sqcup_{0\leq i<N-1}\sqcup(SC_{N-1}-2n)$$

   for sub-configurations of size n as $\sim_2$.

Let us now introduce specific normal forms.

Definition 3.58. Let $\mathcal{C}(n,p/q=[r_0,\dots,r_h])$ be enumerated. Let $C\in\mathcal{C}(n,p/q=[r_0,\dots,r_h])$.

i) Let SC be a sub-configuration of size n with start 0. Let $SC=\sqcup_{0\leq i<N}SC_i$ be as in Lemma 3.55. Let $S_1=SC_0-\{(a_0,b_0)\}$ and $S_2=SC-SC_0$. Let $S=\cup_{j\in\mathbb{N}}((S_1+2nj)\cup(S_2+2nj))$. Then if C contain S and if C^S has normal form $(l_0,\dots,l_h)$ with $0<l_h<r_h-1$ we say C has *normal form* $(SC,l_0,\dots,l_h)$.

ii) Let SC be a sub-configuration of size n and let $SC=\sqcup_{0\leq i<N}SC_i$ be as in Lemma 3.55 where 0 is in SC_0. Let $S_1=SC_0-\{(a_0,b_0)\}$ and $S_2=SC-SC_0$. Let $S=\cup_{j\in\mathbb{N}}((S_1+2nj)\cup(S_2+2nj))$. Then if C contain S and if C^S has normal form $(l_0,\dots,l_h)$ with $l_h=0$ if s_0 is even and $l_h=r_h-1$ if s_0 is odd we say C has *normal form* $(SC,l_0,\dots,l_h)$.

Let us now prove, that every allowable configuration is connected to such a normal form.

Lemma 3.59. *Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated and let $C \in \mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be allowable. Let further C be related to $(l_0, \ldots, l_h)$. Then there is a sequence of allowable state transitions that connects C to a configuration that has normal form $(SC, l_0, \ldots, l_h)$. Furthermore if C contains a sub-configuration $\bar{SC}$ of size n then*

1. *if $0 < l_h < r_h - 1$ then C is connected to a normal form $(\hat{SC}, l_0, \ldots, l_h)$ where $\hat{SC} \sim_1 \bar{SC}$,*
2. *if $0 \neq l_h \neq r_h - 1$ then C is connected to a normal form $(\hat{SC}, l_0, \ldots, l_h)$ where $\hat{SC} \sim_2 \bar{SC}$.*

Proof. W.l.o.g. assume C contain a sub-configuration $\bar{SC}$ of size n (if not bring C into normal form, then it contains such a subconfiguration).

Case 1: $0 < l_h < r_h - 1$. Use Lemma 3.56 to connect C to a configuration $\tilde{C}$ containing $\bar{SC}, \bar{S}_1 = \bar{SC}_0, \bar{S}_2 = \bar{SC} - \bar{SC}_0, \bar{S}$ as in the Lemma. Then for every choice of boundary parallel arc we have that $\tilde{C}^{\bar{S}}$ can be connected to a configuration that contains that arc (this is because of the choice of l_h). Use that to connect $\tilde{C}$ to a configuration $\bar{C}$ that contains $(\bar{b}_0, \bar{a}_0 + 2n)$ without changing $\bar{S}$. This yields that $\bar{C}$ contains $\bar{SC}^1$. The argumentation that C is connected to a configuration that contains $\bar{SC}^{-1}$ works by choosing $\tilde{S}_1 = \sqcup_{0 \leq i < \bar{N}-1} \bar{SC}_i$ and $\tilde{S}_2 = \bar{SC}_{\bar{N}-1} - \{(\bar{a}_{\bar{N}-1}, \bar{b}_{\bar{N}-1})\}$, creating $\tilde{S}$ as in Lemma 3.56 and then creating an arc $(\bar{b}_{\bar{N}-1} - 2n, \bar{a}_{\bar{N}-1})$.

Now let $SC \sim_1 \bar{SC}$ with start 0. Then $SC = \sqcup_{0 \leq i < N} SC_i$. Let $S_1 = SC_0 - \{(a_0, b_0)\}$ and $S_2 = \sqcup_{1 \leq i < N} SC_i$. Then use Lemma 3.56 and the above techniques inductively to connect C to a configuration $\hat{C}$ that contains S_1, S_2 and S as in Lemma 3.54. Then connect $\hat{C}$ to the normal form $(SC, l_0, \ldots, l_h)$ by connecting $\hat{C}^S$ to the normal form $(l_0, \ldots, l_h)$.

Case 2: $0 \neq l_h \neq r_h - 1$. Use Lemma 3.56 to connect C to a configuration $\tilde{C}$ containing $\bar{SC}, \bar{S}_1 = \bar{SC}_0, \bar{S}_2 = \bar{SC} - \bar{SC}_0, \bar{S}$ as in the Lemma. Then (a_0, b_0) corresponds to a boundary parallel arc in $\tilde{C}^{\bar{S}}$ where a_0 is even resp. odd if $\alpha^{\bar{S}}(a_0)$ is even resp. odd and thus if $l_h = r_0 - 1$ resp. $l_h = 0$ (the last relationship is because of the choice of l_h $\tilde{C}^{\bar{S}}$ can not contain a boundary parallel arc $(l, l+1)$ where l is even resp. odd if $l_h = 0$ resp. $l_h = r_h - 1$). Furthermore $\tilde{C}^{\bar{S}}$ can be connected to a configuration containing an arbitrary choice of a boundary parallel arc $(l, l+1)$ where l is even resp. odd if $l_h = r_h - 1$ resp. $l_h = 0$. Hence connect $\tilde{C}$ to a configuration $\bar{C}$ containing $(a_0 + 2n, b_0 + 2n)$ without changing $\bar{S}$. As $\bar{S}$ contains $\{\bar{SC}_0 - (\bar{a}_0, \bar{b}_0)\} + 2n$ we have that $\bar{C}$ contains the sub-configuration $(\bar{SC}_0 + 2n) \sqcup \sqcup_{1 \leq i < \bar{N}-1} \bar{SC}_i$. Furthermore $\tilde{C}$ contains the subconfiguration $\sqcup_{0 \leq i < \bar{N}-2} \bar{SC}_i \sqcup (\bar{SC}_{\bar{N}-1} - 2n)$.

Now let $SC \sim_1 \bar{SC}$ and $SC = \sqcup_{0 \leq i < N} SC_i$ with 0 in SC_0. Let $S_1 = SC_0 - \{(a_0, b_0)\}$ and $S_2 = \sqcup_{1 \leq i < N} SC_i$. Then use Lemma 3.56 and the above techniques inductively to connect C to a configuration $\hat{C}$ that contains S_1, S_2 and S as in Lemma 3.54. Then bring $\hat{C}$ into normal form $(SC, l_0, \ldots, l_h)$ by bringing $\hat{C}^S$ into normal form. □

Lemma 3.60. *Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated. Then the number of normal forms $(SC, l_0, \ldots, l_h)$ is*

$$C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1).$$

Proof. Let us first count the number of normal forms $(SC, l_0, \dots, l_h)$ with $0 < l_h < r_h - 1$. Hence we are looking for the number of sub-configurations of size n that start at 0. This obviously is the same as C_n. Furthermore the number of normal forms $(l_0, \dots, l_h)$ with $0 < l_h < r_h - 1$ is $(r_h - 2) \cdot (r_{h-1} - 1) \cdots (r_0 - 1)$. Hence the number is $C_n \cdot (r_h - 2) \cdot (r_{h-1} - 1) \cdots (r_0 - 1)$.

Let us now count the number of normal forms $(SC, l_0, \dots, l_h)$ where $l_h = 0$ or $l_h = r_h - 1$. Let $SC = \sqcup_{0 \leq i < N} SC_i$ as in Lemma 3.55. If i is the size of SC_0 then there are $2i \cdot C_{i-1} \cdot C_{n-i}$ different possibilities for 0 to be in SC_0 ($C_{i-1} \cdot C_{n-i}$ is the number of configurations SC with start 0 and the size of $SC_0 = i$ and $2i$ is the maximal number such that for $0 \leq k < 2i$ we have that 0 is in $SC_0 - k$). Therefore the number of sub-configurations with 0 in SC_0 is $\sum_{i=0}^{n-1} i \cdot C_{i-1} \cdot C_{n-i}$. As $C_{i-1} \cdot C_{n-i} = C_{(n-i+1)-1} \cdot C_{n-(n-i+1)}$ we have

$$\begin{aligned}
&2i \cdot C_{i-1} \cdot C_{n-i} + 2(n-i+1) \cdot C_{(n-i+1)-1} \cdot C_{n-(n-i+1)} \\
=&(n+1) \cdot C_{i-1} \cdot C_{n-i} + (n+1) \cdot C_{(n-i+1)-1} \cdot C_{n-(n-i+1)}
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{i=0}^{n-1} i \cdot C_{i-1} \cdot C_{n-i} &= \sum_{i=0}^{n-1} (n+1) \cdot C_{i-1} \cdot C_{n-i} \\
&= (n+1) \cdot \sum_{i=0}^{n-1} C_{i-1} \cdot C_{n-i} \\
&\overset{2.24}{=} (n+1) \cdot C_n.
\end{aligned}$$

Furthermore the number of normal forms $(l_0, \dots, l_h)$ with $l_h = 0$ is equal to the number of normal forms $(l_0, \dots, l_h)$ with $l_h = r_h - 1$ is equal to $(r_0 - 1) \cdots (r_{h-1} - 1)$ and thus the number of normal forms $(SC, l_0, \dots, l_h)$ with $l_h = 0$ or $l_h = r_h - 1$ is

$$C_n(n+1)(r_{h-1} - 1) \cdots (r_0 - 1).$$

Therefor the number of normal forms $(SC, l_0, \dots, l_h)$ is

$$\begin{aligned}
&C_n(n+1)(r_{h-1} - 1) \cdots (r_0 - 1) + C_n(r_h - 2)(r_{h-1} - 1) \cdots (r_0 - 1) \\
=&C_n(n + r_h - 1)(r_{h-1} - 1) \cdots (r_0 - 1).
\end{aligned}$$

□

Theorem 3.61.

$$N(n, p, q) \leq C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1).$$

3.4 Equality

We now want to prove that the upper bound we found is actually equal to the number of tight contact structures. The main new idea in this section is, to take a boundary parallel arc α and consider Honda's algorithm on $\mathcal{C}^\alpha(n,p,q)$ (remember that this means to only consider configurations containing α). The following Lemma give a relation between that idea and the reduction $(n(n,p,q), p(n,p,q), q(n,p,q))$.

Lemma 3.62. *Let $\mathcal{C}(n, p/q = [r_0,\ldots,r_h])$ be enumerated. Let α be a boundary parallel arc. Then the number of allowable components in $G^\alpha(n, p/q = [r_0,\ldots,r_h])$ (we mean the algorithm defined for fixed α on $\mathcal{C}^\alpha(n, p/q = [r_0,\ldots,r_h])$) is $N(n(n,p,q), p(n,p,q), q(n,p,q))$.*

Proof. Let Γ' be a multicurve on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} = S^1\times\partial D^2 = \partial(S^1\times D^2)$. In the representation (see 3.13 (a)) we assume, that the copies of α and the frame are the copy of Γ' (see 3.13 (b)) and that the picture is obtained from the cutting process applied for Γ' (this defines Γ' completely). Let Γ be te multivurve of slope q/p with $2n$ components on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ (see 3.13 (c) the blue arc is the attaching arc of the bypass), then Γ' is obtained from Γ by attaching a bypass from the back along an arc in $\partial(\{0\}\times\mathbb{R}/\mathbb{Z})$. Therefore we have to find out, to which class (n',p',q') Γ' belongs.

If $n\geq 2$ then the bypass attachment intersects 3 components of Γ and thus decreases

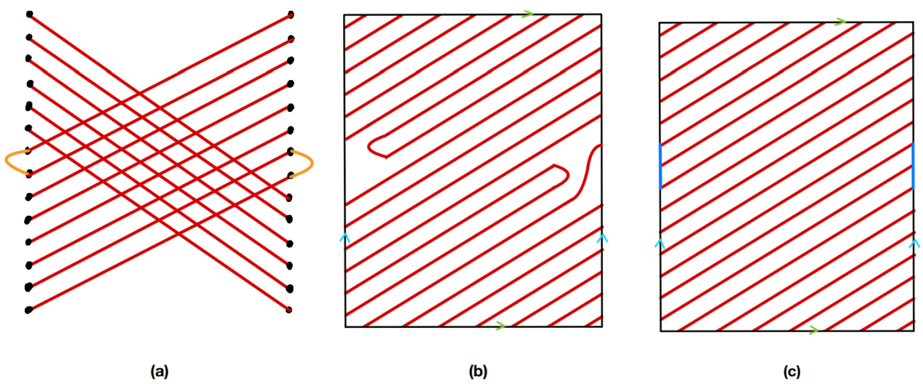
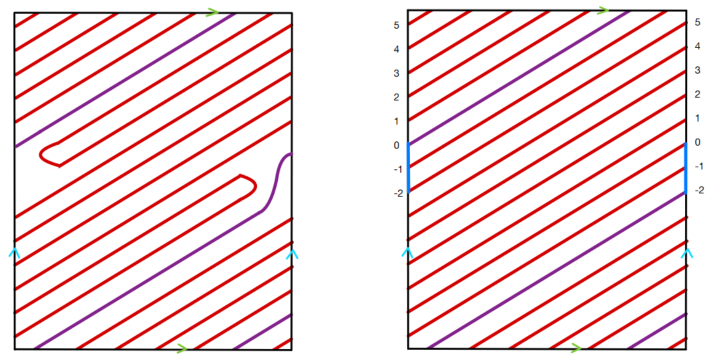


Figure 3.13: The left picture shows the representation of a configuration containing α, the middle picture shows Γ' on T^2 and the right picture shows Γ on T^2.

n by 1, i.e. $n' = n-1$. As there is at least one component that does not get changed by the bypass attachment the slope of that component has to be q/p and thus $p' = p$ and $q' = q$. Therefore $(n,p,q) = (n(n,p,q), p(n,p,q), q(n,p,q))$.

The $n=1$ case is a bit more complicated. We want to count how many times the purple line in Figure 3.14 (a) twists around $\mathbb{R}/\mathbb{Z}\times\{0\}$ which is p' and how many times the purple line twist around $\{0\}\times\mathbb{R}/\mathbb{Z}$ which is q'. We want to relate this to the boundary parallel set. We enumerate the intersections of Γ with $\{0\}\times\mathbb{R}/\mathbb{Z}$ with $\mathbb{Z}_{2np}$ see Figure 3.14 (b). Let us consider the purple arc in 3.14 (b) that starts at 0 on the



(a) (b)

Figure 3.14: The left picture shows a configuration disk in B^3 before edge rounding, the right picture shows the same disk after edge rounding.

left and ends at -2 on the right. The number of times this arc intersects $\{0\} \times \mathbb{R}/\mathbb{Z}$ is p' and the number of times this arc intersects $\mathbb{R}/\mathbb{Z} \times \{0\}$ is q'. This can be translated to algebra. Remark that every arc in the picture connects a on the left to $a + 2nq$ on the right. Let $\hat{k}$ be the smallest number such that $\hat{k} \cdot 2nq \equiv -2 \pmod{2np}$. Then the set $\{0, 2nq, \ldots, \hat{k} \cdot 2nq\} \subset \mathbb{Z}_{2np}$ corresponds to the intersections of that arc with $\{0\} \times \mathbb{R}/\mathbb{Z}$ i.e. $\hat{k} + 1 = p'$ and we further have $\#(\{0, 2nq, \ldots, \hat{k} \cdot 2nq\} \cap \{0, 2, \ldots, 2nq - 2\}) = q'$ (this is as every intersection with $\mathbb{R}/\mathbb{Z} \times \{0\}$ results in such a unique intersection with $\{0\} \times \mathbb{R}/\mathbb{Z}$). This looks very much like the boundary parallel set. Let $k + 1$ be the smallest number such that $(k + 1) \cdot 2nq \equiv 0 \pmod{2np}$ i.e. $k = 2np - 1$ and let $(\bar{k} + 1)$ be the smallest number such that $(\bar{k} + 1) \cdot 2nq \equiv 2 \pmod{2np}$ (this is the same as the boundary parallel set). We then have

$$\{0, 2nq, \ldots, k \cdot 2nq\} = \{0, \ldots, \bar{k} \cdot 2nq, (\bar{k} + 1) \cdot 2nq, \ldots, (\hat{k} + \bar{k} + 1) \cdot 2nq\}.$$

This yields $p' = \hat{k} + 1 = np - \bar{k}$ and hence $p' = p(1, p, q)$. As $(\bar{k} + 1) \cdot 2nq \equiv 2 \pmod{2np}$ we have, that

$$\begin{aligned}&(\{2, 2 + 2nq, \ldots, 2 + \hat{k} \cdot 2nq\} \cap \{2, 4, \ldots, 2nq - 2\}) \\ =&\{0, 2 \ldots, 2nq - 2\} - \{0, 2nq, \ldots, k \cdot 2nq\}.\end{aligned}$$

This is almost what we have in Lemma 3.31 (we have $\{0, \ldots, 2nq\} - B$ has double the points of $\{0, 2 \ldots, 2nq - 2\} - \{0, 2nq, \ldots, k \cdot 2nq\}$). If $\bar{k} = 0$ i.e. if $q = 1$ we thus have $q' = \#(\{0, 2nq, \ldots, \hat{k} \cdot 2nq\} \cap \{0, 2, \ldots, 2nq - 2\}) = 1$. If $\bar{k} \neq 0$ we have

$$\begin{aligned}q(1, p, q) &= \#(\{2, 2 + 2nq, \ldots, 2 + \hat{k} \cdot 2nq\} \cap \{2, 4, \ldots, 2nq - 2\}) \\ &= \#(\{0, 2nq, \ldots, \hat{k} \cdot 2nq\} \cap \{0, 2, \ldots, 2nq - 2\}) \\ &= q'.\end{aligned}$$

This yields $q' = q(n, p, q)$.

All together we get that for arbitrary (n, p, q) we have $(n', p', q') = (n(n, p, q), p(n, p, q), q(n, p, q))$ an thus that the algorithm for fixed alpha can be related to an algorithm that comes form a dividing set characterized by $(n(n, p, q), p(n, p, q), q(n, p, q))$. $\square$

Let us now give an correspondence between $\mathcal{C}^\alpha(n, p, q)$ and $\mathcal{C}(n(n, p, q), p(n, p, q), q(n, p, q))$.

Lemma 3.63. *Let α be a boundary parallel arc. Let $\mathcal{C}^\alpha(n, p/q = [r_0, \ldots, r_h])$ be enumerated and let $C \in \mathcal{C}^\alpha(n, p/q = [r_0, \ldots, r_h])$ be allowable and contain a boundary parallel arc β. Then C is connected to a configuration $\bar{C}$ that has normal form without changing without changing α or β which contains a boundary parallel set containing β. Let further $n \geq 3$ then $\bar{C}$ contains the boundary parallel set generated by α and the boundary parallel set generated by β.*

Proof. Let $\alpha = (a, a+1)$ and $\beta = (b, b+1)$. Apply Lemma 3.23 inductively to β to create arcs $\{(b + 2nq, b + 1 + 2nq), \ldots, (b + k_1 \cdot 2nq, b + 1 + k_1 \cdot 2nq)\}$ where k_1 is the maximal number such that we do not change α, β or the already created arcs. Now apply Lemma 3.23 inductively to create arcs $\{(a - 2nq, a + 1 - 2nq), \ldots, (a - k_2 \cdot 2nq, a + 1 - k_2 \cdot 2nq)\}$ where k_2 is the maximal number such that we do not change $\alpha, \beta, \{(b + 2nq, b + 1 + 2nq), \ldots, (b + k_1 \cdot 2nq, b + 1 + k_1 \cdot 2nq)\}$ or the already created arcs. This yields a configuration $\tilde{C}$ that contains α and

$$B = \{(b - k_2 \cdot 2nq, a + 1 - k_2 \cdot 2nq), \ldots, (b, b + 1), \ldots, (a + k_1 \cdot 2nq, a + 1 + k_1 \cdot 2nq)\}.$$

We distinguish 2 cases. Let us simplify the notation a bit. Choose $(c, c + 1) = (b - k_2 \cdot 2nq, a + 1 - k_2 \cdot 2nq)$ and $k = k_1 + k_2$, then we have

$$B = \{(c, c + 1), \ldots, (c + k \cdot 2nq, c + 1 + k \cdot 2nq).\}$$

Case 1: B is the boundary parallel set generated by $(c, c + 1)$. If $\alpha \in B$ then we can connect $\tilde{C}^B$ to a normal form i.e. connect $\tilde{C}$ to a normal form without changing α or β. Furthermore the normal form contains B and hence if $n \geq 3$ it contains the boundary parallel set generated by α and by β.
If $\alpha \notin B$ then $\psi^B(\alpha)$ is a boundary parallel arc in $\tilde{C}^B$ and thus we can create the boundary parallel set generated by $\psi^B(\alpha)$ and then connect it to a normal form which is a way to connecting C to a normal form without changing α or β. We already saw that for $n \geq 3$ the boundary parallel set generated by $\psi^B(\alpha)$ corresponds to a boundary parallel set in $\mathcal{C}^\alpha(n, p/q = [r_0, \ldots, r_h])$ generated by α and thus the normal form contains the boundary parallel set generated by α and the boundary parallel set generated by β.

Case 2: B is not a boundary parallel set. We want to see, that B must be an almost boundary parallel set. If creating $(c - 2nq, c + 1 - 2nq)$ would change an arc of B then it must be $(c + k \cdot 2nq, c + 1 + k \cdot 2nq)$ (this is argued in the proof of 3.26) i.e. we have

$$\{(c - 2nq, c + 1 - 2nq)\} = \{(c + k \cdot 2nq, c + k \cdot 2nq)\} - 2.$$

Therefore creating $(c + (k+1) \cdot 2nq, c+1+(k+1) \cdot 2nq)$ would changed $(c, c+1)$ and hence B is the boundary parallel set generated by $(c, c+1)$. As this can is not true by assumption we have, that creating $(c + (k+1) \cdot 2nq, c+1+(k+1) \cdot 2nq)$ resp. $(c - 2nq, c+1-2nq)$ would change α. By 3.24 we therefore have, that α has exactly one endpoints in

$$\{c - 1 + (k+1) \cdot 2nq, c+1+(k+1) \cdot 2nq\} \cap \{c - 2nq, c+2-2nq\}.$$

This yields $\alpha = (c+2-2nq, c-1+(k+1)\cdot 2nq) = (c-2+(k+1)\cdot 2nq, c-1+(k+1)\cdot 2nq)$ and therefore $A = B \cup \{\alpha\}$ is an alost boundary parallel set. By Lemma 3.43 it can only exist in 2 cases.

Let $n = 1$ and $r_h = 2$ with $h \geq 2$. We have $\tilde{C}^A \in \mathcal{C}(2, \tilde{p}/\tilde{q} = [r_0, \ldots, r_{h-1}-1])$. Assume there exists a boundary parallel arc $(\tilde{l}, \tilde{l}+1)$ in $\tilde{C}^B$. Then create the boundary parallel set generated by $(\tilde{l}, \tilde{l}+1)$ which as $n = 2$ the boundary parallel set is $\cup_i\{(\tilde{l}+4i, \tilde{l}+1+4i)\}$. As A is an almost boundary parallel set generated by $(c, c+1)$ we have that $c+2, \ldots, c+5$ is not in A and therefore there is an arc in $\cup_i\{(\tilde{l}+4i, \tilde{l}+1+4i)\}$ that corresponds to a boundary parallel arc $(l, l+1)$ with endpoints in $c+2, \ldots, c+5$. We have that c is even resp. odd implies $l_h = 1$ resp. $l_h = 0$ which implies l is even resp. odd which implies $\tilde{l}$ is even resp. odd. Furthermore neither $(c-2q, c+1-2q)$ is in A (this is a direct consequence of the construction of the boundary parallel set) nor $(c+4-2q, c+5-2q)$ is in A (if that would be the case then there would be $\hat{k} \leq k$ such that $\dot{k} \cdot 2q \equiv 4 \pmod{2p}$ and hence $(c+4, c+5)$ is in A which is not the case by the construction of the boundary parallel set). Therefore there is an arc $(\tilde{s}, \tilde{s}+1)$ in $\cup_i\{(\tilde{l}+4i, \tilde{l}+1+4i)\}$ that corresponds to $(c+4-2q, c+5-2q)$ or $(c-2q, c-2q)$. We denote $\bar{C}$ and mean the configuration that corresponds to the above state transitions. If $(c-2nq, c+1-2nq) \in \bar{C}$ then we inductively create new arcs $\{(c - 2 \cdot 2q, c+1-2\cdot 2q), \ldots, (c - \hat{k} \cdot 2q, c+1-\hat{k} \cdot 2q)\}$ where $\hat{k}$ is the maximal number we can apply Lemma 3.23 without changing A or the arcs created before. Then

$$\{(c - \hat{k} \cdot 2q, c+1-\hat{k} \cdot 2q), \ldots, (c + k \cdot 2q, c+1+k \cdot 2q)\}$$

is a boundary parallel set and we can apply case 1. If $(c+(k+1)\cdot 2q, c+1+(k+1)\cdot 2q)$ is in $\bar{C}$ then we inductively create new arcs $(c, c+1), \ldots, (c+\hat{k}\cdot 2q, c+1+\hat{k}\cdot 2q)$ where $\hat{k}$ is th maximal number we can apply Lemma 3.23 without changing A or the arcs created before. Then

$$\{(c, c+1), \ldots, (c + \hat{k} \cdot 2q, c+1+\hat{k} \cdot 2q)\}$$

is a boundary parallel set and we can apply case 1.

Let $n = 2$ and $q \neq 1$ (hence $h \geq 1$). Then by Lemma 3.43 $\tilde{C}^B \in \mathcal{C}(1, [r_0, \ldots, r_h, 2])$. Let $(\tilde{l}, \tilde{l}+1)$ be a boundary parallel arc in $\tilde{C}^A$. $\tilde{C}^A$ can be connected to a configuration that has boundary parallel arc $(\tilde{l}+2i, \tilde{l}+1+2i)$ for all i(as $n^{a(n,p,q)} = 1$).By the same argumentation as above $\tilde{C}^A$ can only a boundary parallel arc $(\tilde{l}, \tilde{l}+1)$ if $\tilde{l}$ and c are even resp. odd. As $(c+(k+1)\cdot 2nq, c+1+(k+1)\cdot 2nq)$ is not in A we can connect $\tilde{C}$ to a configuration $\bar{C}$ that has boundary parallel arc $(c+(k+1)\cdot 2nq, c+1+(k+1)\cdot 2nq)$.

Then we inductively create new arcs $(c, c+1), \dots, (c+\hat{k} \cdot 2q, c+1+\hat{k} \cdot 2q)$ where $\hat{k}$ is th maximal number we can apply Lemma 3.23 without changing A or the arcs created before. Then

$$\{(c, c+1), \dots, (c+\hat{k} \cdot 2q, c+1+\hat{k} \cdot 2q)\}$$

is a boundary parallel set and we can apply case 1. $\square$

Lemma 3.64. *Let α be a boundary parallel arc. A configuration $C \in \mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ is allowable if and only if it is connected to a configuration $\tilde{C} \in \mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ without changing α that contains a boundary parallel set B containing α generated by some $(l, l+1)$ where $\tilde{C}^B \in \mathcal{C}(n(n,p,q), p(n,p,q), q(n,p,q))$ is allowable. Furthermore there are two allowable configuration $\tilde{C}, \bar{C} \in \mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ that contain a boundary parallel set B generated by some $(l, l+1)$ containing α. Then $\tilde{C}^B$ and $\bar{C}^B$ are connected in $\mathcal{C}(n(n,p,q), p(n,p,q), q(n,p,q))$.*

Proof. Let $C \in \mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ be allowable and let B be a boundary parallel set generated by some $(l, l+1)$ that contains $\alpha = (a, a+1)$. Then there is a smallest $0 \leq k$ such that $\alpha = (l + k \cdot 2nq, l+1+k \cdot 2nq)$. Furthermore $\{(l, l+1), \dots, (l + k \cdot 2nq, l+1+k \cdot 2nq)\}$ is a subset of B that can not contain $(a-2, a-1)$ (if that was the case creating $(l+k \cdot 2nq, l+1+k \cdot 2nq) = (a, a+1)$ in Lemma 3.26 would yield a problem as it would change $(a-2, a-1)$). We inductively apply Lemma 3.23 to create arcs $(a - 2nq, a+1-2nq), \dots (a - k \cdot 2nq, a+1-k \cdot 2nq)$. This would only change $(a, a+1)$ if $a - i \cdot 2nq = a-2$ for some $1 \leq i \leq k$ which by the above observation is not the case. Therefore we can connect C to a configuration $\tilde{C}$ that contains $\{(l, l+1), \dots, (l+k \cdot 2nq, l+1+k \cdot 2nq)\}$ without changing α. Then applying Lemma 3.26 to $(l, l+1)$ in $\tilde{C}$ connects $\tilde{C}$ to a configuration $\bar{C}$ that contains B without changing $\{(l, l+1), \dots, (l+k \cdot 2nq, l+1+k \cdot 2nq)\}$ and hence without changing α. Now $\bar{C}$ can only be allowable if $\bar{C}^B$ is allowable. As the number of allowable components in $\mathcal{C}(n(n,p,q), p(n,p,q), q(n,p,q))$ is the same as the number of allowable components in $\mathcal{C}^\alpha(n,p,q)$ and by the previous observation every allowable component in $\mathcal{C}^\alpha(n,p,q)$ contains a configuration C that contains B such that C^B is allowable and as further the allowable components in $\mathcal{C}^B(n,p,q)$ are send by ψ^B to the allowable components of $\mathcal{C}(n(n,p,q), p(n,p,q), q(n,p,q))$ we have that the allowable components in $\mathcal{C}^B(n,p,q)$ generate all the allowable components of $\mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ and that they are different in $\mathcal{C}^B(n,p,q)$ if and only if they are different in $\mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$. $\square$

Theorem 3.65. *Let $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ be enumerated. Then all normal forms are allowable.*

Proof. We use induction on $n(r_0 + \dots + r_h)$. If $n(r_0 + \dots + r_h) = 1, 2$ then all configurations are allowable as there can not be any state transitions. If $np = 3$ then $n = 1$ and either $r_0 = r_1 = 2$ which implies that a normal form only consists of boundary parallel arcs and hence does not admit any state transition or $r_0 = 3$ which we already discussed.

Let $C \in \mathcal{C}(n, p/q = [r_0, \dots, r_h])$ contain a boundary parallel arc α and a boundary parallel set B that contains α. By induction C^B is allowable if and only if it is connected to a normal form. Furthermore by Lemma 3.64 $C \in \mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ is allowable if and only if C^B is allowable. Then an allowable state transition in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ is an allowable state transition in $\mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ if and only if it does not change α. Furthermore such a state transition creates an potentially allowable configuration in $\mathcal{C}^\alpha(n, p/q = [r_0, \dots, r_h])$ and thus in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$. So let us regard an allowable state transition in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ that changes α. As $p > 3$ we have that there is a boundary parallel arc β in C that does not get changed by the state transition. Then by Lemma 3.63 C is connected to a configuration that contains a boundary parallel set $\tilde{B}$ containing β and having normal form without changing α nor β. By Lemma 3.64 and the induction hypothesis we therefore have that $C \in \mathcal{C}^\beta(n, p/q = [r_0, \dots, r_h])$ must be allowable. Therefore as the allowable state transition does not change β it connects C to a configuration that is potentially allowable in $\mathcal{C}^\beta(n, p/q = [r_0, \dots, r_h])$ and thus in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$. Therefore if C is allowable in $\mathcal{C}^\zeta(n, p/q = [r_0, \dots, r_h])$ for an arbitrary boundary parallel arc ζ in C, then the connected component of C in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ only contains potentially allowable configurations and hence C is allowable in $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$. Therefore all normal forms are allowable. $\square$

Lemma 3.66. *Let $\mathcal{C}(n, p/q = [r_0, \dots, r_h])$ with $n \geq 2$ be enumerated and let $C \in \mathcal{C}(n, p/q = [r_0, \dots, r_h])$ contain a boundary parallel set B generated by α. Let SC and $\tilde{SC}$ be two sub-configurations of size n. Then*

$$\alpha^B(SC - B) \sim_1 \alpha^B(\tilde{SC} - B) \Rightarrow SC \sim_1 \tilde{SC}, \tag{3.15}$$
$$\alpha^B(SC - B) \sim_2 \alpha^B(\tilde{SC} - B) \Rightarrow SC \sim_2 \tilde{SC}. \tag{3.16}$$

Proof. Let $SC = \sqcup_{0 \leq i < N} SC_i$ and $\tilde{SC} = \sqcup_{0 \leq i < \tilde{N}} \tilde{SC}_i$ as in Lemma 3.55. Note that $\alpha^B((-) + 2n) = \alpha^B((-)) + 2(n-1)$ (we already saw that relation). Let us first prove (3.15). We distinguish 4 cases.

Case 1: Let SC_0 not be a subsets of B hence $a_0 \notin B$. Then $\alpha^B(SC - B)$ has start $\alpha^B(a_0)$. Then

$$\alpha^B(SC - B)^1 = \{(\alpha^B(b_0), \alpha^B(a_0) + 2(n-1))\} \sqcup (\alpha^B(SC - B) - \{(\alpha^B(a_0), \alpha^B(b_0))\}).$$

Now assume

$$\alpha^B(\tilde{SC} - B) = \{(\alpha^B(b_0), \alpha^B(a_0) + 2(n-1))\} \sqcup (\alpha^B(SC - B) - \{(\alpha^B(a_0), \alpha^B(b_0))\}).$$

Then

$$\tilde{SC} - B = \{(b_0, a_0 + 2n)\} \sqcup ((SC - B) - \{(a_0, b_0)\}).$$

If $(a_0 + 1, a_0 + 1) \in B$ then $(a_0 + 1, a_0 + 2) \in SC_0$ and we have

$$\begin{aligned} \tilde{SC} &= \{(b_0, a_0 + 2n)\} \sqcup (SC - \{(a_0, b_0)\}) = SC^1 \\ \text{or, } \tilde{SC} &= \{(b_0, a_0 + 2n)\} \sqcup (\{(a_0 + 1, a_0 + 2)\} + 2n) \\ &\quad \sqcup (SC - \{(a_0, b_0)\} - \{(a_0 + 1, a_0 + 2)\}) = ((SC^1)^1)^1. \end{aligned}$$

If $(a_0+1, a_0+2) \notin B$ then

$$\tilde{SC} = \{(b_0, a_0+2n)\} \sqcup (SC - \{(a_0, b_0)\}) = SC^1.$$

This yields that in this case

$$\alpha^B(SC-B)^1 = \alpha^B(\tilde{SC}-B) \Rightarrow \tilde{SC} = ((SC^1)^1)^1 \text{ or } \tilde{SC} = SC^1 \Rightarrow \tilde{SC} \sim_1 SC.$$

Case 2**:** Let SC_{N-1} not be a subsets of B. The identity

$$\alpha^B(SC-B)^{-1} = \alpha^B(\tilde{SC}-B) \Rightarrow \tilde{SC} = ((SC^{-1})^{-1})^{-1} \text{ or } \tilde{SC} = SC^{-1} \Rightarrow \tilde{SC} \sim_1 SC,$$

works by using case 1 for a mirrored enumeration.

Case 3**:** $SC_0 \subset B$ i.e. $SC_0 = \{\alpha\}+2ni$ for some i. Then $\alpha^B(SC-B) = \sqcup_{1\le i<N}\alpha^B(SC_i)$ is as in Lemma 3.55. Then

$$\begin{aligned}
&=\alpha^B(SC-B)^1\\
&=\{(\alpha^B(b_1), \alpha^B(a_1)+2(n-1))\} \sqcup \alpha^B(SC_1 - \{(a_1,b_1)\}) \sqcup \sqcup_{2\le i<N}\alpha^B(SC_i)\\
&=\alpha^B(\{(b_1, a_1+2n)\}) \sqcup \alpha^B(SC_1 - \{(a_1,b_1)\}) \sqcup \sqcup_{2\le i<N}\alpha^B(SC_i).
\end{aligned}$$

Assume $\alpha^B(\tilde{SC}-B) = \alpha^B(SC-B)^1$, which thus yields

$$\tilde{SC} - B = \{(b_1, a_1+2n)\} \sqcup (SC_1 - \{a_1, b_1\}) \sqcup \sqcup_{2\le i<N} SC_i.$$

As $\{\alpha\}+2n(i+1)$ lies between b_1 and a_1+2n we must have

$$\begin{aligned}
\tilde{SC} &= \{(b_1, a_1+2n)\} \sqcup (SC_1 - \{a_1, b_1\}) \sqcup \sqcup_{2\le i<N} SC_i \sqcup (\{\alpha\}+2n(i+1))\\
&= ((SC^1)^1)^1.
\end{aligned}$$

Hence in our case

$$\alpha^B(SC-B)^1 = \alpha^B(\tilde{SC}-B) \Rightarrow \tilde{SC} = ((SC^1)^1)^1 \Rightarrow \tilde{SC} \sim_1 SC.$$

Case 4**:** Let $SC_{N-1} \subset B$. Then arguing

$$\alpha^B(SC-B)^{-1} = \alpha^B(\tilde{SC}-B) \Rightarrow \tilde{SC} = ((SC^{-1})^{-1})^{-1} \Rightarrow \tilde{SC} \sim_1 SC,$$

works by using case 3 for a mirrored enumeration.

All 4 cases together give (3.15).

Let us now consider (3.16). Let $\alpha^B(SC-B) = \sqcup_{0\le i<\bar{N}}\bar{SC}_i$ be as in Lemma 3.55. Let $\bar{SC} = (\bar{SC}_0 + 2(n-1)) \sqcup \sqcup_{1\le i<\bar{N}}\bar{SC}_i$. Let us assume $\alpha^B(\tilde{SC}-B) = \bar{SC}$. We distinguish 2 cases.

Case 1**:** $SC_0 \subset B$. Then $SC_0 = \{\alpha\}+2ni$ for some i. Then $\bar{SC}_0 = \alpha^B(SC_1)$. As $\alpha^B(\tilde{SC}-B) = \bar{SC}$ we have

$$\tilde{SC} - B = (SC_1 + 2n) \sqcup \sqcup_{2\le i<N} SC_i.$$

Then as $\alpha + 2n(i+1)$ lies between $\sqcup_{2\leq i<N} SC_i$ and $(SC_1 + 2n)$ we must have

$$\tilde{SC} = (\{\alpha\} + 2n(i+1)) \sqcup (SC_1 + 2n) \sqcup \sqcup_{2\leq i<N} SC_i.$$

hence $\tilde{SC} \sim_2 SC$.

Case 2: SC_0 is not a subset of B.

If there is $\{\alpha\} + 2ni \in SC_0$ then $\bar{SC}_0 = \alpha^B(SC_0 - (\{\alpha\} + 2ni))$. Therefore we have

$$\tilde{SC} - B = ((SC_0 - (\alpha + 2ni)) + 2n) \sqcup \sqcup_{1\leq i<N} SC_i.$$

Hence $((SC_0 - (\alpha + 2ni)) + 2n) \subset \tilde{SC}_{\tilde{N}-1}$ which implies $SC_0 + 2n \subset \tilde{SC}_{\tilde{N}-1}$. Therefore we have

$$\tilde{SC} = (SC_0 + 2n) \sqcup \sqcup_{1\leq i<N} SC_i.$$

Thus $\tilde{SC} \sim_2 SC$.

If $B \cap SC_0 = \emptyset$ we have $\bar{SC}_0 = \alpha^B(SC_0)$ which implies $\tilde{SC}_{\tilde{N}-1} = SC_0 + 2n$ and hence

$$\tilde{SC} = (SC_0 + 2n) \sqcup \sqcup_{1\leq i<N} SC_i.$$

Therefore $\tilde{SC} \sim_2 SC$.

We defined $\sim_2$ to be generated some relations. Case 1 and 2 give one relation i.e. the relation

$$\alpha^B(\tilde{SC} - B) = (\bar{SC}_0 + 2(n-1)) \sqcup \sqcup_{1\leq i<\tilde{N}} \bar{SC}_i \Rightarrow \tilde{SC} \sim_2 SC.$$

The other relation i.e.

$$\alpha^B(\tilde{SC} - B) = (\bar{SC}_{\tilde{N}-1} - 2(n-1)) \sqcup \sqcup_{0\leq i<\bar{N}-1} \bar{SC}_i \Rightarrow \tilde{SC} \sim_2 SC,$$

can be proven by mirroring the enumeration and use case 1 and 2. Both cases then give (3.16) as they generate the equivalence relation $\sim_2$. □

Lemma 3.67. *Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated. Let SC be a sub-configuration of size n. Then*

1. *If SC has start 0 then there is no other sub-configuration in $[SC]_1$ that has start 0,*
2. *If $SC = \sqcup_{0\leq i<N} SC_i$ and 0 is in SC_0 then there is no other sub-configuration $\tilde{SC}$ in $[SC]_2$ where $\tilde{SC} = \sqcup_{0\leq i<\tilde{N}} \tilde{SC}_i$ with 0 is in $\tilde{SC}_0$.*

Proof. Let us regard 1. If SC has start 0 then the first sub-configurations we can reach by applying ^{1} and $^{-1}$ that have the start being a multiple of $2n$ are $SC+2n$ and $SC-2n$. Therefore every configuration in $[SC]_1$ that has start a multiple of $2n$ must be of the form $SC + 2nj$ for some j. To get a configuration with start being a multiple of $2np$ we have a configuration $SC + 2npj = SC$ for some j and thus the first statement is true.

Let us regard 2. By the generators of $\sim_2$ we have that the first subconfiguration we can reach by generator moves starting with SC $\bar{SC} = \sqcup_{0\leq i<\bar{N}} \bar{SC}_i$ we can reach by generator moves starting with SC where $\bar{SC}_0$ contains $2n$ are $SC + 2n$ and $SC - 2n$. The same argumentation as above then gives the statement for 2. □

Proof of Theorem 1.1.

$$N(n, p/q = [r_0, \ldots, r_h]) = C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1).$$

Proof. Let $\mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be enumerated. We have to show that two different normal forms $(SC, l_0, \ldots, l_h)$ and $(\tilde{SC}, \tilde{l}_0, \ldots, \tilde{l}_h)$ can not be connected by a sequence of allowable state transitions. Then by Lemma 3.65 those normal forms generate the allowable components of $G(n, p, q)$ and that the total number is $C_n(n + r_h - 1) \cdot (r_{h-1} - 1) \cdots (r_0 - 1)$ by Lemma 3.60.

We have already seen, that $(l_0, \ldots, l_h) = (\tilde{l}_0, \ldots, \tilde{l}_h)$. We then use induction on n. **Case** $n = 2$**.** First note that if $n(l)$ is 1 resp. 0 every configuration connected to $(SC, l_0, \ldots, l_h)$ only contains boundary parallel arcs $(l, l+1)$ where l is even resp. odd. Let B be the boundary parallel set generated by some $(l, l+1)$ contained in $(SC, l_0, \ldots, l_h)$. As $B = \cup_i\{(l + 4i, l + 1 + 4i)\}$ (because $n = 2$) there are only 4 different possibilities for a boundary parallel set in $\mathcal{C}(2, p/q = [r_0, \ldots, r_h])$. Then $(SC, l_0, \ldots, l_h)$ can only be connected via allowable state transitions to a configuration that contains $B + 2$ as the other possibilities would contain a boundary parallel arc that it can not contain because of the previous observation. Now assume $(SC, l_0, \ldots, l_h)$ is connected to a configuration that contains $B + 2$. Then there must exist a configuration that is connected to $(SC, l_0, \ldots, l_h)$ that contains an arc α in B and an arc β in $B + 2$ (this is as every boundary parallel arc a configuration connected to $(SC, l_0, \ldots, l_h)$ can have must lie in $B \cup (B + 2)$). Then by Lemma 3.63 we can connect this configuration to a normal form containing α and β which then implies that if $n(l) = 0$ resp. $n(l) = 1$ we have $l_h \neq r_h - 1$ resp. $l_h \neq 0$.

Now assume $0 < l_h < r_h - 1$. By the definition of $(SC, l_0, \ldots, l_h)$ we have that there is a boundary parallel set which does not contain 0. If B is another boundary parallel set, then $(SC, l_0, \ldots, l_h)$ only contains boundary parallel arcs which is not possible as $0 < l_h < r_h - 1$. Thus B does not contain 0. Now B contains $(2, 3)$ resp. $(1, 2)$ if $n(l)$ is even resp. odd and thus $SC = \{(0, 1), (2, 3)\}$ resp. $SC = \{(0, 3), (1, 2)\}$. As further $B + 2$ is the only other boundary parallel set that can be contained in a configuration connected to $(SC, l_0, \ldots, l_h)$ and $B + 2$ contains 0 and thus can not be contained in as normal form $(0 < l_h < r_h - 1)$ and as further SC depends uniquely on B we have $\tilde{SC} = SC$.

If $l_h = 0$ and $n(l) = 1$ resp. $l_h = r_h - 1$ and $n(l) = 0$ then by the above $(\tilde{SC}, l_0, \ldots, l_h)$ must contain B and as SC uniquely depends on B we have $\tilde{SC} = SC$. If $l_h = 0$ and $n(l) = 0$ resp. $l_h = r_h - 1$ and $n(l) = 1$, then every configuration connected to $(SC, l_0, \ldots, l_h)$ must contain B or $B + 2$. If $(\tilde{SC}, l_0, \ldots, l_h)$ contains $B + 2$ we have 0 is in $B + 2$ resp. B and thus $(\tilde{SC}, l_0, \ldots, l_h)$ resp. $(SC, l_0, \ldots, l_h)$ contains only boundary parallel arcs and hence $SC = \tilde{SC}$. Else $(\tilde{SC}, l_0, \ldots, l_h)$ contains B and as SC uniquely depends on B we have $SC = \tilde{SC}$. This finishes the case for $n = 2$.

Let $n \geq 3$. Let $C \in \mathcal{C}(n, p/q = [r_0, \ldots, r_h])$ be allowable and let there be a boundary parallel arc $\alpha \in C$. Then connect C to two configuration $\tilde{C}, \bar{C}$ without changing α that have normal form and contain the boundary parallel set B generated by α. Then $\tilde{C}$ resp. $\bar{C}$ contain sub-configurations $\tilde{SC}$ resp. $\bar{SC}$ of size n. Then $\tilde{C}^B$ resp. $\bar{C}^B$ are connected to normal forms $(\tilde{SC}_1, l_0, \dots, l_h)$ resp. $(\bar{SC}_1, l_0, \dots, l_h)$. By Lemma 3.64 we have that $(\tilde{SC}_1, l_0, \dots, l_h)$ and $(\bar{SC}_1, l_0, \dots, l_h)$ are connected in $\mathcal{C}(n-1, p/q = [r_0, \dots, r_h])$. Now we have $0 < l_h < r_h - 1$ resp. $l_h = 0$ or $l_h = r_h - 1$ if and only if (by induction) $\tilde{SC}_1 \sim_1 \bar{SC}_1$ resp. $\tilde{SC}_1 \sim_2 \bar{SC}_1$ which by Lemma 3.66 implies $\bar{SC} \sim_1 \tilde{SC}$ resp. $\bar{SC} \sim_2 \tilde{SC}$.

Now assume we have an allowable state transition that changes α. Then there is a boundary parallel arc β in C that does not get changed by the state transition. By Lemma 3.63 we can connect C to a normal form $\hat{C}$ that contains the boundary parallel set B generated by α and the boundary parallel set $\tilde{B}$ generated by β without changing α or β. Then by the above observation $\hat{C}$ contains a sub-configuration $\hat{SC}$ of size n. Furthermore as we did not change α nor β $[\hat{SC}]_1$ or $[\hat{SC}]_2$ (depending on l_h) determines the class of sub-configurations of size n that we can find in normal forms that are connected to C without changing α or without changing β (this is by the above observation). This then yield, that the only subconfigurations we can find in a configuration connected to C are equivalent to $\hat{SC}$ ($\sim_1$ or $\sim_2$). By Lemma 3.67 $[\hat{SC}]_1$ or $[\hat{SC}]_2$ only contain one candidate SC for a normal form and hence C can only be connected to the normal form $(SC, l_0, \dots, l_h)$. This finishes the prove. $\square$

4 Handlebodies

Let M be a handlebody of genus g and let $D_1, \dots, D_g$ be compressing disks with $M \backslash (D_1 \cup \dots \cup D_g) \cong B^3$. Then take a multicurve Γ on M defined the following way: For every D_l there is a odd number of connected components a_l of Γ that each have one transversal intersection with ∂D_l and not intersection with ∂D_j for $j \neq k$. Furthermore there is one component of Γ that intersects each ∂D_l transversally exactly one time. Then by construction Γ can not have homotopically trivial curves (there is an intersection of each component ot Γ with at least on ∂D_l that is odd). For every D_l we now take two closed curves L_l^1 and L_l^2 on ∂M that don't intersect each other nor do they intersect $\partial D_1, \dots, \partial D_g$ such that L_l^i for $i = 1, 2$

1. intersect each of the a_l components of Γ belonging to D_l once
2. intersect the unique component of Γ (intersecting all ∂D_j once)
3. don't intersect any other component of Γ.

We then twist Γ k_l^i times around L_l^i for $(i = 1, 2)$. We denote the resulting multicurve Γ'. An example can be seen in Figure 4.1 Let $2n_l = \gcd(a_l + 1, 2k_l^1 + 2k_l^2)$, $p_l = (a_l + 1)/2n_l$

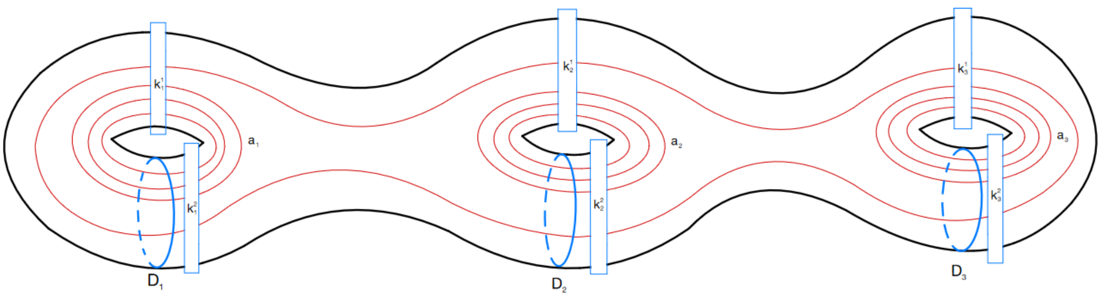


Figure 4.1: The picture shows M, Γ', L_l^1, L_l^2.

and $q_l = (2k_l^1 + 2k_l^2)/2n_l$. This looks like the calculation of (n, p, q) for a given frame. We then have the following theorem:

Theorem 4.1. *Let (M, Γ') be as above, then the number of tight contact structures with fixed boundary characteristic foliation $\mathcal{F}$ adopted to Γ' equals*

$$N(n_1, p_1, q_1) \cdot \dots \cdot N(n_g, p_g, q_g).$$

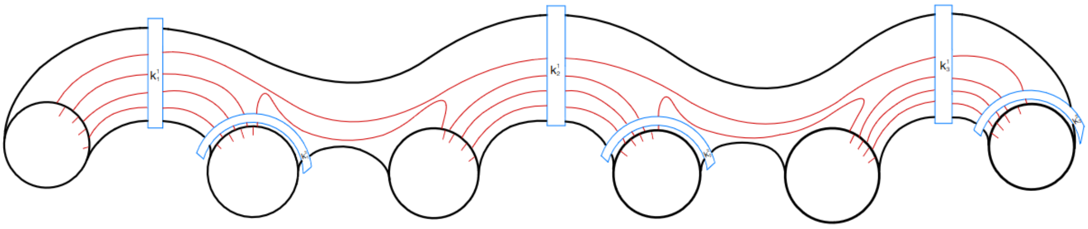


Figure 4.2: The picture shows the cut open handlebody before rounding.

Proof. Let us cut M open along $D_1, \dots, D_g$ (without rounding) see Figure 4.2. We then choose one D_l. We denote the a_l copies of Γ by $A_1, \dots A_{a_l}$ and the unique copy to Γ that also intersects ∂D_l by A. All those closed curves are now arcs with start end end on one of the copies of D_l in B^3. Let us now only regard $A, A_1, \dots, A_{a_l}, D_l, L_l^1, L_l^2$ on B^3 see Figure 4.3. Edge rounding then results in Figure 4.4 (a). The question is now what

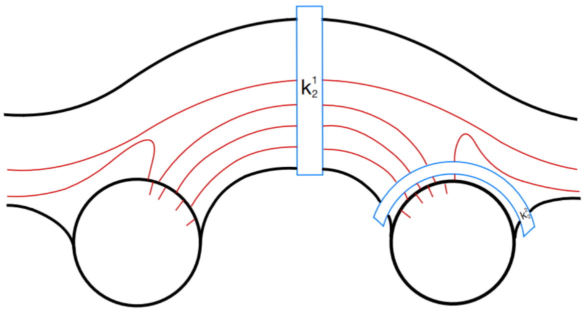


Figure 4.3: The picture shows the cut open handlebody before rounding.

happens with the two arcs leaving the Figure 4.4 (a) on the right (belonging to a copy of A) and the two arcs leaving Figure 4.4 (a) on the left (belonging to a copy of A). As the copies of $A_1, \dots A_{a_l}, L_l^1$ and L_l^2 all intersect the copy of A exactly once and as L_l^1 and L_l^2 divide ∂B^3 into 3 regions we have that there exist 3 parts of each $A_1, \dots A_{a_l}, L_l^1$ (on ∂B^3 after rounding). One part connects the one copy of ∂D_l with the copy of L_l^1, one part connecting the copy of L_l^1 and the copy of L_l^2 and one part connecting the copy of L_l^2 to the other copy of ∂D_l (on ∂B^3). This yields that the two arcs on the right of Figure 4.4 (a) must be connected (after rounding) and that the two arcs on the left of 4.4 (a) must be connected after rounding. This results in Figure 4.4 (b), which immediately results in Figure 4.4 (c). This is the picture we get when we apply the algorithm to the torus where the dividing set on T^2 is characterized by (n_l, p_l, q_l). This yield a configuration is allowable only if Γ_l is allowable for (n_l, p_l, q_l) on the torus. Furthermore there is an allowable state transitions $(\Gamma_1, \dots, \Gamma_l, \dots, \Gamma_g) \mapsto (\Gamma_1, \dots, \Gamma_l', \dots, \Gamma_g)$ if and only if there is an allowable state transition $(\Gamma_l) \mapsto (\Gamma_l')$ for (n_l, p_l, q_l) on the torus. Let $(\Gamma_1, \ldots, \Gamma_l, \ldots, \Gamma_g)$ be a configuration where Γ_l are allowable for (n_l, p_l, q_l) on the torus. Let us assume this is not potentially allowable for M. We have seen, that all arcs of Γ_l are connected to each other (on ∂B^3). Furthermore the copy of A connects all copies of D_l. This directly implies, that $(\Gamma_1, \ldots, \Gamma_l, \ldots, \Gamma_g)$ is potentially allowable. The other observations give that it is in fact allowable and that the number of tight contact structures are $N(n_1, p_1, q_1) \cdot \cdots \cdot N(n_g, p_g, q_g)$. $\square$

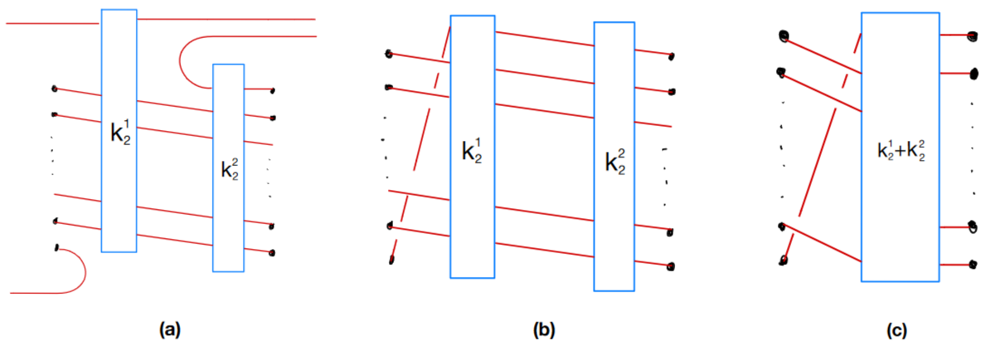


Figure 4.4: The picture shows the dividing set on B^3 after rounding.

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