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1 Abstract

In this thesis discrepancies between two different formalisms of higher-spin interactions in the context of massless four dimensional space-time will be explored. The first two chapters lay the foundation for one formulation each, while in the third their respective cubic interactions will be derived and analysed. There is difference in the amount and type of available couplings that will be noted. This discrepancy will then be explored specifically on the 3-3-2 vertex, before two attempts at a solution will be executed. Both the simple counter-term as well as the (A)dS limit explored in chapter 4 will not yield positive results. The final chapter is concerned with mixed symmetry fields. For those a higher-derivative action could be found in the $\{2,1\}$ case, where a generalisation could be the solution to the problem that is being treated. At the end an ansatz for a potential spin-3 equivalent will be given from which this generalisation may be executed in a follow-up work.

In dieser Arbeit wird die Diskrepanz zweier verschiedener Formalismen für Interaktionen von higher-spin Feldern ohne Masse in vierdimensionaler Raum-Zeit erörtert. Die ersten beiden Kapitel legen die Grundlagen dar, bevor im dritten die Interaktionen selbst erarbeitet und analysiert werden. Dabei wird ein Unterschied in der Anzahl der zulässigen Kopplungen in den beiden Formalismen festgestellt. Anhand des 3-3-2 Vertex werden Lösungsversuche unternommen. Die Möglichkeit eines Counter-terms wird erwogen ebenso wie ein möglicher Effekt im Limes von (A)dS zu flachen Geometrien in Kapitel vier. Das letzte Kapitel beschäftigt sich mit mixed-symmetry Feldern. Für diese konnte im Rahmen des $\{2,1\}$ Falles eine Wirkung mit einer höheren Zahl von Ableitungen gefunden werden. Dies könnte, wenn verallgemeinert, eine Lösung des gestellten Problems sein. Am Schluss wird ein Ansatz für ein analoge Konstruktion für spin-3-Felder vorgeschlagen.

Inhaltsverzeichnis

2 General Introduction to Higher-Spin Theories

The intrinsic spin s of a particle is an important concept in modern physics, as it assigns the field its corresponding representation of the Lorentz group. This group contains all possible transformations that do not alter physical information of a given system. Therefore it has strong links to causality and to the behaviour of all particles currently known in the standard model. Curiously enough only a small handful of values for s are currently known to play vital roles in nature: Zero for the various scalar fields (like the Higgs), $\frac{1}{2}$ for fermions (leptons and quarks) and one for the force exchanging gauge bosons (photons, gluons, the W and Z boson). General relativity, even without a consistent quantum description, can be assosiated with a bosonic spin. As linearised GR is a spin-2 gauge theory, we can thus view GR as a non-linear spin-2 gauge theory. This accounts for all particles currently known. This implies that spins higher than two are, to our current knowledge, not present in nature. This gives rise to various questions. If we were unable to detect them so far, do they even exist and if they do, what kind of physics would they govern? Lacking any experimental data we can only postulate their supposed properties, based on the knowledge we have of particles with $s \leq 2$. To keep things contained, this thesis will mainly concern itself with bosons, particles where s takes the value of a natural number. This introduction chapter orients itself on three different publications. A compact introductory paper by Kessel [17] that mainly elaborates on the same points mentioned here. For those that wish a more detailed overview of the historic developements and a overview of higher spin theory in general the book [4] by Bengtsson is recommended. The publication [20] is best suited for those that wish to dive into the mathematical aspects of these topics.

At first let us take a look at a well known spin-1 theory, electromagnetism. The object of importance in this theory is the vector potential A^μ. The potential is governed by a set of four equations:

$$\Box A_\mu - \partial_\mu \partial \cdot A = 0 \tag{2.1}$$

that are invariant under a gauge transformation ζ

$$\delta A_\mu = \partial_\mu \zeta. \tag{2.2}$$

With this at hand, we can then define a Lorentz invariant object, the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.3}$$

which encodes the equations of motion as

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.4}$$

For our purposes it is useful to fix the gauge with the Lorenz condition

$$\partial_\mu A^\mu = 0. \tag{2.5}$$

This reduces the equations of motion to a simple wave equation

$$\Box A_\mu = 0. \tag{2.6}$$

In a similar vein to this spin-1 field, we can take a look at a typical spin-2 field $h_{\mu\nu}$, describing a perturbation in free gravity

$$R_{\mu\nu} = \Box h_{\mu\nu} - \partial_{(\mu}\partial \cdot h_{\nu)} + \partial_\mu \partial_\nu h' = 0, \tag{2.7}$$

where $h' = h^\mu{}_\mu$. The gauge transformation admits a gauge parameter ζ_μ and is given by

$$\delta h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu. \tag{2.8}$$

We should also mention the contracted Bianchi identity

$$\nabla_\mu R^\mu{}_\nu = \frac{1}{2}\nabla_\nu R. \tag{2.9}$$

For completeness it should be noted that in this thesis and most higher-spin literature the following (anti-)symmetrisation convention is used

$$A_{(\mu}B_{\nu)} = A_\mu B_\nu + A_\nu B_\mu. \tag{2.10}$$

The field strength is given by the Riemann tensor

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\beta\nu} + \Gamma^\rho_{\nu\beta}\Gamma^\mu_{\alpha\rho} - \partial_\beta \Gamma^\mu_{\alpha\nu} - \Gamma^\rho_{\nu\alpha}\Gamma^\mu_{\beta\rho}, \tag{2.11}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols associated with the metric. As a gauge condition we may find the Donder condition especially useful,

$$\partial \cdot h_\mu - \frac{1}{2}\partial_\mu h' = 0, \tag{2.12}$$

$$\Box \zeta_\mu = 0, \tag{2.13}$$

reducing the equations of motion to a simple wave equation

$$\Box h_{\mu\nu} = 0. \tag{2.14}$$

Electromagnetism and linearised General Relativity written in this form look strikingly similar. This pattern also holds for more general lower spin theories (like Yang-Mills theories). Assuming higher-spin fields behave in the same fashion as their lower spin counterparts, we can postulate their behaviour by generalising the equations we have seen above. The first derivations of such a set of equations for generalised integer spin fields was published by C. Fronsdal [13]. Therefore the generalised set-up we will work with is known as Fronsdal theory. The central object of this theory is the Fronsdal tensor

$$F_{\mu_1 \dots \mu_s} = \Box \varphi_{\mu_1 \dots \mu_s} - \partial_{(\mu_1} \partial \cdot \varphi_{\mu_2 \dots \mu_s)} + \partial_{(\mu_1} \partial_{\mu_2} \varphi_{\mu_3 \dots \mu_{s-2})\lambda}{}^{\lambda}, \tag{2.15}$$

with associated equations of motion

$$F_{\mu_1 \dots \mu_s} = 0. \tag{2.16}$$

The fields involved are called Fronsdal fields and admit a gauge transformation

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \zeta_{\mu_2 \dots \mu_s)}, \tag{2.17}$$

that can be related to the behaviour of (2.15) under gauge transformations

$$\delta F_{\mu_1 \dots \mu_s} = 3 \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \zeta_{\mu_4 \dots \mu_s)\nu}{}^{\nu}. \tag{2.18}$$

This implies that, if we want to have a Fronsdal tensor that is invariant under gauge transformations of the participating fields, we have to require

$$\zeta_{\mu_1 \dots \mu_{s-3}\nu}{}^{\nu} = 0. \tag{2.19}$$

The number of degrees of freedom for a massless particle with spin s in a given dimension d is given by the difference of two binomial coefficients arising from combinatorics [1]

$$\binom{d-3+s}{s} - \binom{d-5+s}{s-2}. \tag{2.20}$$

For four dimension, this will always be two. These two states are refereed to as helicities and are usually labelled spin up $+s$ and spin down $-s$.

With this formulation in hand it is possible for us to write down an action for our higher-spin fields

$$S = \frac{1}{2} \int \mathrm{d}^d x \; \varphi^{\mu_1 \dots \mu_s} \mathcal{F}_{\mu_1 \dots \mu_s}, \tag{2.21}$$

where

$$\mathcal{F}_{\mu_1 \dots \mu_s} = F_{\mu_1 \dots \mu_s} - \eta_{(\mu_1 \mu_2} F_{\mu_3 \dots \mu_s)\nu}{}^{\nu}. \tag{2.22}$$

[1]We will explore this formula in detail in Appendix B 8.2

We can check that this action results in the correct equations of motion

$$\begin{aligned}\delta S &= \frac{1}{2}\int \mathrm{d}^d x\ [\delta(\varphi^{\mu_1\ldots\mu_s})\mathcal{F}_{\mu_1\ldots\mu_s} + \varphi^{\mu_1\ldots\mu_s}\mathcal{F}_{\mu_1\ldots\mu_s}(\delta\varphi)]\\ &= \int \mathrm{d}^d x\ \delta(\varphi^{\mu_1\ldots\mu_s})\mathcal{F}_{\mu_1\ldots\mu_s},\end{aligned}$$

where the condition for a vanishing variation can be met by simply requiring that $\mathcal{F}$ vanishes, this is in turn is met when the Fronsdal tensor itself vanishes. If we now recall (2.22) in terms of fields and act upon it with a derivative, we obtain

$$\partial \cdot \mathcal{F}_{\mu_1\ldots\mu_{s-1}} - \frac{1}{2}\partial_{(\mu_1}\mathcal{F}_{\mu_2\ldots\mu_{s-1})\lambda}{}^{\lambda} = -\frac{3}{2}\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi_{\mu_4\ldots\mu_{s-1})\lambda\rho}{}^{\lambda\rho}, \tag{2.23}$$

which can be seen as a generalisation of the contracted Bianchi identity (2.9). As $\mathcal{F}$ has to vanish when the equations of motions are enforced we conclude

$$\phi_{\mu_1\ldots\mu_{s-4}\lambda\rho}{}^{\lambda\rho} = 0, \tag{2.24}$$

a double traceless constraint on the fields. In the two previous examples we could find a specific choice for our gauge that reduced the equations of motion to a wave equation. This can also be achieved for (2.16) with the generalised Donder condition

$$G_{\mu_2\ldots\mu_s} = \partial \cdot \varphi_{\mu_2\ldots\mu_s} - \frac{1}{2}\partial_{(\mu_2}\varphi'_{\mu_3\ldots\mu_s)} = 0. \tag{2.25}$$

This fixing reduces the equations of motion to

$$\Box\varphi_{\mu_1\ldots\mu_s} = 0. \tag{2.26}$$

For our purposes it is a lot more practical to perform the calculations on-shell. In this setting we have access to on-shell gauge fixing. The transverse-traceless gauge (sometimes TT gauge in the general literature) will be the most convenient. One can think of it as requiring that both parts of the generalised Donder condition vanish on their own

$$\partial \cdot \varphi = 0, \tag{2.27}$$
$$\varphi^{\lambda}{}_{\lambda\mu_3\ldots\mu_s} = 0. \tag{2.28}$$

A system fulfilling theses two equations in addition to the wave equation is commonly referred to as a Fierz system. However we have to pay a price for this. Our gauge parameter can now no longer be arbitrarily chosen, but must instead comply with a similar set of equations

$$\Box\zeta = 0 \qquad \partial\cdot\zeta = 0 \qquad \zeta' = 0. \tag{2.29}$$

If we want to treat all fields of any arbitrary spin on an equal footing it is useful to introduce generating functions

$$\varphi(x,u) = \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\mu_1....\mu_s}(x) u^{\mu_1}....u^{\mu_s}, \tag{2.30}$$

where the u^μ act as dummy place holders, that when acted upon by a partial derivative, free up an index. Furthermore before any physical quantity, like the action integral, is performed, these must be set to zero. An alternative name for these functions that is common in the associated literature is "tower of higher-spin fields". These admit a gauge transformation

$$\delta\varphi(x,u) = u \cdot \partial_x \zeta(x,u). \tag{2.31}$$

In the TT gauge our generating function obeys the following equations

$$\partial_x \cdot \partial_x \varphi = 0, \tag{2.32}$$

$$\partial_x \cdot \partial_u \varphi = 0, \tag{2.33}$$

$$\partial_u \cdot \partial_u \varphi = 0, \tag{2.34}$$

where the second equation encodes the vanishing of divergences, while the last encodes tracelessness.

3 The Light-Front

Dynamics and their calculations are not necessarily restricted to a covariant formalism. It is completely viable to select a coordinate frame and fix a gauge before proceeding with computations. We may lose the convenience that comes with a covariant notation, but in turn calculations become more restrained. The purpose of this section is to review this process and to show that this approach can exist entirely separate from covariant notions. A particular set of coordinates that is of special interest for this thesis are the light-front coordinates [2]

$$x^{\pm} := \frac{x^0 \pm x^3}{\sqrt{2}}. \tag{3.1}$$

For the special case of four dimensions we can simplify further by introducing the complex coordinates

$$z = \frac{x^1 + ix^2}{\sqrt{2}} \qquad \bar{z} = \frac{x^1 - ix^2}{\sqrt{2}}. \tag{3.2}$$

For this thesis the following signature of the metric is used

$$g^{\mu\nu} = (-, +, +, +). \tag{3.3}$$

Later on we will take a look into cubic interactions, where a noticable discrepancy between the light-front and the covariant couplings enters the picture. Adressing this will be the focal point of this thesis. This formulation was first introduced by Bengstsson in [5] to derive cubic interactions between higher-spin fields. The development of higher spins in the light-front formalism was largely influenced by Metsaev, to the point where some refer to fields in this setting as Metsaev fields. He provided a good overview over his methodology in [18]. The basis of this section is covered in [21].

3.1 ISO(3,1) in the Light-Front

For a rigorous definition of this method, which is required if interactions are to be considered, we have to start from the Poincaré algebra. It is composed of translations P^μ and Lorentz transformations $J^{\mu\nu}$

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [J^{\mu\nu}, P^\rho] &= P^\mu \eta^{\nu\rho} - P^\nu \eta^{\mu\rho}, \\ [J^{\mu\nu}, J^{\rho\sigma}] &= J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}\eta^{\mu\rho} - J^{\mu\rho}\eta^{\nu\sigma} + J^{\nu\rho}\eta^{\mu\sigma}. \end{aligned} \tag{3.4}$$

[2] The light-cone formalism also has interesting impacts on gauge theories in general. For example Yang-Mills theory is reduced to a scalar theory in the adjoint global symmetry group. Gravity gets reduced to a theory of two scalars with no diffeomorphisms or other kinds of symmetries.

For a free theory there are only quadratic contributions, but we will receive higher order corrections when interactions enter the picture. We have to distinguish between two kinds of generators, kinematical and dynamical. This distinction is of great importance for quantisation, as it is needed to impose the canonical commutation relations. Even though this thesis will only concern itself with classical field theory, it is still important to consider quantisation for something that at the end of the day should result in a quantum theory. The kinematic generators do not receive corrections when quantised or under interactions and stay quadratic in the fields. These encode and stabilise the Cauchy surface of time evolution. The dynamical generators are, as the name suggests, not invariant and will receive higher order corrections. The stability subgroup of the Poincaré group ISO(3,1) for the canonical equal time choice $t = t_0$ consists of spatial translations and rotations. Boosts and time translations on the other hand do not keep the surface intact. In the light-front the canonical choice is $x^+ = 0$, this means we treat x^+ as our time direction and P^- as the Hamiltonian. Since we fixed x^+ only $(d-1)$ generators need to be deformed, where d is the number of dimensions of the space-time. This is also where the lack of a compact covariance notation starts to show, as, without the notion of tensors, we have to track and analyse all components individually. The generators are

$$\text{kinematical :} \qquad P^+, P^a, J^{a+}, J^{+-}, J^{ab} \tag{3.5}$$

$$\text{dynamaical :} \qquad P^-, J^{a-}. \tag{3.6}$$

In this notation we imply F^a to mean $(F^z, F^{\bar{z}})$. As time evolution for any operator is governed by the Hamilton equation $\dot{G} = -i[G, H]$, we can solve the relations of the Poincaré algebra for any specific time and expect them to hold for all time. The most convenient in most cases will be $x^+ = 0$, as it rids us of any terms with explicit time dependence in our operators.[3] The relations (3.4) are important as they place restriction on possible interacting terms, so solving these will give us a system of equations for possible (self-)interactions. As we are already in a fixed coordinate system, we have exhausted all gauge freedom, so there is no need to consider gauge violations when constructing our interacting terms later on. It should also be noted that P^- is identical to the Hamiltonian. Any operator G can be split into two parts $G = G_2 + G_{int}$, the first being the free part, which is quadratic in the fields, and a second part that contains our higher order corrections. Using the commutators of the algebra (3.4) we may divide all commutators into four different categories:

[K,K]=K These commutators are not interesting for our purposes, as they will stay unchanged for all time and not receive any correction whatsoever.

[K,D]=K We have two types of relations here, those that result in zero and those that do not. 0-type relations place restrictions on our dynamical generators, the others imply that the interacting term commute with the corresponding kinematic operator.

[3] Should we wish to re-install the time dependence, we can simply reconstruct it using the equations of motion.

[K,D]=D These place further constrictions on our interacting corrections. One can think of them as enforcing good behaviour under symmetries in the light-front.

[D,D]=0 The last set of commutators is the most important one as its contents make up the equations we need to solve to derive our interaction terms

$$[J^{a-}, J^{b-}] = 0 \qquad [J^{a-}, P^-] = 0. \tag{3.7}$$

The other three sets are listed for completeness in Appendix A (8.1). We explicitly split

$$H = P^- = H_2 + H_{int}, \qquad J^{a-} = J_2^{a-} + x^a H_{int} + J_{int}^{a-}, \tag{3.8}$$

the interacting from the free parts of our dynamical generators, where the explicit H_{int} dependence for the correction J_{int}^{a-} was chosen to receive more manageable constraints. A total of seven constraints can be extracted:

$$[H_{int}, T] = 0 \qquad\qquad T = P^+, P^a, J^{a+}, J^{ab} \tag{3.9}$$

$$[J^{a-}, T] = 0 \qquad\qquad T = P^+, P^a, J^{a+}. \tag{3.10}$$

These constraints can be explicitly solved, given an ansatz for H and J^{a-}. After that is done we only need to solve the second equation of (3.7) to arrive at our desired solution for an interacting Hamiltonian.

3.2 Free Fields

If we want to advance our understanding of this formalism we have to pick a specific theory and start working out the commutators. As we mentioned previously the neat part about working in four dimensions is that we only have to care about two degrees of freedom (2.20), one if we count scalar fields separately. Due to the coordinate system we chose[4] these two degrees of freedom can be described as two fields $\varphi^{\pm s}(x)$ that are complex conjugate to each other. In four dimensions the Poisson relation for two fields with helicities μ and ν, momenta p and q are determined by the Dirac bracket in the following way

$$[\varphi^\mu(p, x^+), \varphi^\nu(q, x^+)] = \delta^{\mu,-\nu} \frac{\delta^3(p+q)}{2p^+}. \tag{3.11}$$

To reiterate we will work in the setting $x^+ = 0$ and will omit it from now on in arguments. With this in mind we have the following set of kinematic generators [5]

[4] Why this happens exactly can be gleaned in a much more convenient way in the next subsection.

[5] It should be noted that the generator p^+ is sometimes referred to as β or even γ in higher-spin literature specifically.

for a field with helicity ν

$$P^+ = p^+, \qquad P = p, \qquad \bar{P} = \bar{p}, \tag{3.12}$$

$$J^{z+} = -p^+ \partial_{\bar{p}}, \qquad J^{\bar{z}+} = -p^+ \partial_p, \qquad J^{+-} = -\partial_{p^+} p^+, \tag{3.13}$$

$$J^{z\bar{z}} = p\partial_p - \bar{p}\partial_{\bar{p}} - \nu. \tag{3.14}$$

The dynamic generators on the free level for such a field are then given by the following

$$J_2^{z-} = \partial_{\bar{p}} \frac{p\bar{p}}{p^+} + p\partial_{p^+} + \nu \frac{p}{p^+}, \tag{3.15}$$

$$J_2^{\bar{z}-} = \partial_p \frac{p\bar{p}}{p^+} + \bar{p}\partial_{p^+} - \nu \frac{\bar{p}}{p^+}, \tag{3.16}$$

$$H_2 = \frac{p\bar{p}}{p^+}. \tag{3.17}$$

The form for H_2 can be derived from the fact that we work with massless fields, J's are derived via the usual definition for the angular momentum $J^{ab} = x^a \partial_b - x^b \partial_a$. To proceed we have to take a small detour into charges. We can use the fact that charges should commute with all possible generators, once the equations of motion are enforced, to receive constraints on our higher order corrections. A charge Q_ζ associated with a generator O_ζ of the Poincaré algebra together with a Killing vector ζ, along which the charge is conserved, can be constructed as follows

$$Q_\zeta = \int p^+ \mathrm{d}^3 p \quad \varphi^{-\mu}_{-p} O_\zeta(p, \partial_p) \varphi^\mu_p. \tag{3.18}$$

The algebra can then be realised by utilising commutators/Poisson brackets

$$\delta_\zeta \varphi^\mu(p, x^+) = [\varphi^\mu(p, x^+), Q_\zeta]. \tag{3.19}$$

The general formula for an arbitrary functional $F[\varphi]$ is then an integral

$$[F(\varphi), Q_\zeta] = \int \mathrm{d}\rho \quad O_\zeta(\rho) \varphi^\mu_\rho \frac{\partial}{\partial \varphi^\mu_\rho} F[\varphi]. \tag{3.20}$$

With this we have all the tools in hand to compute our potential light-front vertices without resorting to a covariant formulation.

3.3 The Light-Front from Covariant Gauge Fixing for Higher-Spins

As we have seen before, it would be wrong to see the light-front formulation as derivative of a grander covariant theory, however a particular bit of information can be gleaned much easier about this system, when starting from the covariant formalism. In the light-front the

Fierz system (2.28)(2.26) reduces to

$$(-\partial^+\partial^- + \partial\bar{\partial})\varphi(x,u) = 0, \tag{3.21}$$

$$(-\partial^+\partial_u^- - \partial^-\partial_u^+ + \partial\bar{\partial}_u + \bar{\partial}\partial_u)\varphi(x,u) = 0, \tag{3.22}$$

$$(-\partial_u^+\partial_u^- + \partial_u\bar{\partial}_u)\varphi(x,u) = 0, \tag{3.23}$$

where both the space-time coordinates as well as our dummy variables have been rewritten in light-cone coordinates. But merely choosing a coordinate system is not enough to exhaust all the possible freedom in the system. There is still a remaining gauge symmetry

$$\delta_\zeta\varphi(x,u) = (-u^+\partial^- - u^-\partial^+ + u\bar{\partial} + \bar{u}\partial)\zeta, \tag{3.24}$$

which can be fixed by requiring

$$\partial_u^+\varphi(x,u) = 0. \tag{3.25}$$

Applying this new condition to (3.23) alters the system in the following way

$$(-\partial^+\partial^- + \partial\bar{\partial})\varphi(x,u) = 0, \tag{3.26}$$

$$(-\partial^+\partial_u^- + \partial\bar{\partial}_u + \bar{\partial}\partial_u)\varphi(x,u) = 0, \tag{3.27}$$

$$\partial_u\bar{\partial}_u\varphi(x,u) = 0. \tag{3.28}$$

While the equation of motion has not changed at all, the other two have some profound implications. The second equation implies that ∂_u^- can be rewritten in such a way that any explicit dependence on it may be removed

$$\partial_u^-\varphi(x,u) \to \frac{1}{\partial^+}(\partial\bar{\partial}_u + \bar{\partial}\partial_u)\varphi(x,u). \tag{3.29}$$

All fields are now only depend on z and $\bar{z}$. Furthermore the last equation of (3.28) implies that our fields are also holomorphic in each respective entry, allowing the generating function to be written in the following way

$$\varphi(x,u) = \sum_{s\geq 0}\left(\varphi_{-s}(x)u^s + \varphi_{+s}(x)\bar{u}^s\right). \tag{3.30}$$

Here two scalar fields $\varphi_s(x) = \varphi_{z(s)}(x)$ and $\varphi_{-s}(x) = \varphi_{\bar{z}(s)}(x)$ encode the two physical helicity states.

4 The Cubic Vertex

In this section we are going to examine cubic interaction vertices for higher-spin fields in each formalism respectivly before comparing them. Before we can begin we have to understand how the construction of such consistent self interactions is facilitated in the first place. The procedure we will make use of is the Noether procedure [14].

4.1 The Noether Procedure

The Noether procedure is named after Emmy Noether as it draws heavily from her research into the connections between symmetry and conserved quantities. The goal of this process is to derive contributions for the actions as well as the gauge symmetry consisting of higher orders in the involved fields

$$S = S^{(2)} + S^{(3)} + ... \qquad \delta_\epsilon \varphi = \delta_\epsilon^{(0)} \varphi + \delta_\epsilon^{(1)} \varphi + ... \quad , \tag{4.1}$$

where a contribution of order n, as in the expression contains n fields, is denoted by a superscript (n). This scheme is not limited to a certain number and can, given resources are available, be done up to any value of n. The gauge invariance of the action is then an infinite set of coupled equations

$$\delta_\epsilon S = 0 \Rightarrow \begin{cases} \delta_\epsilon^{(0)} S^{(2)} = 0 \\ \delta_\epsilon^{(0)} S^{(3)} + \delta_\epsilon^{(1)} S^{(2)} = 0 \\ \delta_\epsilon^{(0)} S^{(4)} + \delta_\epsilon^{(1)} S^{(3)} + \delta_\epsilon^{(2)} S^{(2)} = 0 \end{cases} \tag{4.2}$$

and so on.
We can solve this from top down, since the free action and gauge transformation are known to us. To solve the second equation, we can look at it under that the condition that the free equations of motion are enforced. As the variation of $S^{(2)}$, induced by any of our gauge transformations, should vanish on-shell in general, we deduce that on-shell the variation of $S^{(3)}$ has to vanish as well. The result can then be plugged into the original equation and we obtain $\delta_\epsilon^{(1)}$. The process can be visualised as

$$S^{(2)}, \delta_\epsilon^{(0)} \xrightarrow{\text{Eq 2, on-shell}} S^{(3)} \xrightarrow{\text{Eq 2}} \delta_\epsilon^{(1)} \xrightarrow{\text{Eq 3, on-shell}} S^{(4)} \xrightarrow{\text{Eq 3}} \delta_\epsilon^{(2)} \dots \, . \tag{4.3}$$

We should also keep in mind that the full non-linear gauge transformation should form an algebra

$$\delta_{\epsilon_1} \delta_{\epsilon_2} \varphi - \delta_{\epsilon_2} \delta_{\epsilon_1} \varphi = \delta_{[\epsilon_1, \epsilon_2]} \varphi + \text{trivial symmetry of the action} \tag{4.4}$$

where the commutator $[\epsilon_1, \epsilon_2]$ is, in principle, field dependent. For most usages only the lowest order is of interest

$$\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\varphi - \delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\varphi = \delta^{(0)}_{[\epsilon_1,\epsilon_2]^{(0)}}\varphi \tag{4.5}$$

as it is field independent and can be computed from $\delta^{(1)}_{\epsilon}$ alone. For our purposes we will remain focused on the derivation of on-shell $S^{(3)}$.

4.2 In the Covariant Formulation

For our cubic interaction vertex we can make the following generic ansatz [10]

$$S^{(3)} = \int \mathrm{d}^4x \quad V(\partial_{x_I}, \partial_{u_I})\varphi^1(x_1, u_1)\varphi^2(x_2, u_2)\varphi^3(x_3, u_3)|_{x_I=x, u_I=0}, \tag{4.6}$$

where $V(\partial_{x_I}, \partial_{u_I})$ is a gauge compatible, covariant differential operator. In order to rewrite the vertex operator in such a way we have assigned each participating field a unique set of space-time coordinates. Once the operator is applied and all derivatives are assigned to their corresponding field, on can set $x_I = x$, $I = 1, 2, 3$ and receive the cubic interaction term.

To calculate our differential operator we can make use of the gauge compatibility condition. If $V(\partial_{x_I}, \partial_{u_I})$ has this property, it should commute with the gauge transformation

$$[V(\partial_{x_I}, \partial_{u_I}), u_I \cdot \partial_{x_I}] \approx 0, \qquad I = 1, 2, 3 \tag{4.7}$$

where $\approx$ denotes an equality up to the free equations of motion. If we remind ourselves of the various restrictions that are placed on the system (2.34), we realise that for parity even vertices one has only the six following variables

$$Y_I = \partial_{u_I} \cdot \partial_{x_{I+1}} \qquad Z_I = \partial_{u_{I+1}} \cdot \partial_{u_{I-1}}. \tag{4.8}$$

The other possible combinations are not independent as they can be traced back to these operators by using partial integration. As an example we can examine $\partial_{u_2} \cdot \partial_{x_1}$

$$\begin{aligned}
\partial_{u_2} \cdot \partial_{x_1}\varphi_1\varphi_2\varphi_3 &= \partial_\mu\varphi_1\varphi_2^\mu\varphi_3 \\
\xrightarrow{\text{Part. Int.}} -\varphi_1\partial_\mu(\varphi_2^\mu\varphi_3) &= -\varphi_1\varphi_2^\mu\partial_\mu\varphi_3 \\
&= -\partial_{u_2} \cdot \partial_{x_3}\varphi_1\varphi_2\varphi_3 \\
&= -Y_3\varphi_1\varphi_2\varphi_3.
\end{aligned}$$

To start the deduction process we should take a look a the commutators of the individual variables with the gauge transformations of the system.
At first we look at the Y_I's:

$$\begin{aligned}
[Y_J, u_I \cdot \partial_{x_I}] &= [\partial_{u_J} \cdot \partial_{x_{J+1}}, u_I \cdot \partial_{x_I}] \\
&= [\partial^\mu_{u_J}, u^\nu_I]\partial_{x^\mu_{J+1}}\partial_{x^\nu_I} \\
&= \partial_{x_{J+1}} \cdot \partial_{x_I}\delta^{JI} \\
\xrightarrow{\text{Part. Int.}} &= -\partial^2_{x_I}\delta^{JI} \approx 0.
\end{aligned}$$

For Z_I's the commutator gets more interesting:

$$\begin{aligned}[Z_J, u_I \cdot \partial_{x_I}] &= [\partial_{u_{J+1}} \cdot \partial_{u_{J-1}}, u_I^\mu]\partial_{x_I^\mu} \\ &\xrightarrow{\text{J=I+1}} \approx \partial_{u_{I+2}} \cdot \partial_{x_I} = -Y_{I-1}, \\ &\xrightarrow{\text{J=I-1}} \approx \partial_{u_{I-2}} \cdot \partial_{x_I} = Y_{I+1}, \\ &\xrightarrow{\text{I=J}} \approx 0.\end{aligned}$$

We can summarize these results as

$$[Y_J, u_I \cdot \partial_{x_I}] = 0, \tag{4.9}$$

$$[Z_I, u_I \cdot \partial_{x_I}] = 0, \tag{4.10}$$

$$[Z_I, u_{I\pm1} \cdot \partial_{x_{I\pm1}}] = \mp Y_{I\mp1}. \tag{4.11}$$

This implies two things:

- Our vertex can depend in any way on the Y_I's as these variables just commute through and won't affect the gauge symmetry. We will however chose a polynomial dependence on these variables to ensure that our vertex is polynomial in the u derivatives, as they serve a function akin to creation operators for our generating functions.
- Any contribution of Z_I must be accompanied by some factor of Y_J to compensate its behaviour in the commutator.

We can achieved this by only allowing Z_I dependence to occur in the form of the variable G, where

$$G = Y_1Z_1 + Y_2Z_2 + Y_3Z_3. \tag{4.12}$$

Now we have all the information we need to write down our covariant cubic vertex operator

$$V = \sum_{n=0}^{s_{min}} \lambda_n^{s1,s2,s3} G^n Y_1^{s1-n} Y_1^{s2-n} Y_3^{s3-n}, \tag{4.13}$$

here $\lambda_n^{s1,s2,s3}$ is a coupling constant, which can be determined by looking at vertices of higher order, and s_{min} is the lowest value out of the three spins.

In four dimensions we have a special relation between G and the Y_I's, as there is not enough algebraic freedom with four space-time variables to view Y and Z independent of each other [6]

$$Y_1Y_2Y_3G \approx 0. \tag{4.14}$$

[6] Such dimension dependent relations are usually referred to as Schouten identities in higher-spin literature.

Using this identity our vertex collapses to mere two contributions, a maximal ($s_1 + s_2 + s_3$ derivatives) and a minimal derivative one ($s_1 + s_3 - s_3$ derivatives)

$$V_{cov} = g_{min} G^{s_3} Y_1^{s_1 - s_3} Y_2^{s_2 - s_3} + g_{non} Y_1^{s_1} Y_2^{s_2} Y_3^{s_3}, \tag{4.15}$$

where w.o.l.g. we shall assume $s_3 \leq s_2 \leq s_1$.

4.3 In the Light-Front Formulation

In order to derive the self-interactions in the light-front formalism we remind ourself of the commutators of the Poincaré algebra we discussed in the previous chapter[7] consisting of the constraints (3.10) and the dynamic equations (3.7). We start by making the following ansatz for the dynamic generators

$$H = H_2 + \sum_n \int \mathrm{d}^{3n} p \; \delta\Big(\sum_i p_i\Big) h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \varphi^{\lambda_1}_{p_1} \dots \varphi^{\lambda_n}_{p_n}, \tag{4.16}$$

$$J^{z-} = J^{z-}_2 + \sum_n \int \mathrm{d}^{3n} p \; \delta\Big(\sum_i p_i\Big) \Big[j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} - \frac{1}{n} h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \Big(\sum_j \frac{\partial}{\partial \bar{p}_j}\Big)\Big] \varphi^{\lambda_1}_{p_1} \dots \varphi^{\lambda_n}_{p_n}, \tag{4.17}$$

$$J^{\bar{z}-} = J^{\bar{z}-}_2 + \sum_n \int \mathrm{d}^{3n} p \; \delta\Big(\sum_i p_i\Big) \Big[\bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} - \frac{1}{n} h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \Big(\sum_j \frac{\partial}{\partial p_j}\Big)\Big] \varphi^{\lambda_1}_{p_1} \dots \varphi^{\lambda_n}_{p_n}, \tag{4.18}$$

where our goal is to determine the unknowns $h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}$, $j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}$ and $\bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}$. We have to insert a delta function here to enforce the conservation of the momenta p^+ and p^a as their respective operators should commute with the interacting Hamiltonian we try to build. This makes it so that we have satisfied the constraints related to the kinematic momenta. This leaves us with the constraints related to the angular momentum. Inserting our ansatz into the constraints

[7] This section follows the same sources as section 3 with the addition of [3], since all of the mentioned publications are focused on interactions in the light-front.

(3.10) results in the following

$$J^{a+}: \qquad \Big(\sum_k p_k^+ \frac{\partial}{\partial p_k^a}\Big) h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.19}$$

$$J^{a+}: \qquad \Big(\sum_k p_k^+ \frac{\partial}{\partial p_k^a}\Big) j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.20}$$

$$J^{a+}: \qquad \Big(\sum_k p_k^+ \frac{\partial}{\partial p_k^a}\Big) \bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.21}$$

$$J^{-+}: \qquad \sum_k p_k^+ \frac{\partial}{\partial p_k^+} h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.22}$$

$$J^{-+}: \qquad \sum_k p_k^+ \frac{\partial}{\partial p_k^+} j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.23}$$

$$J^{-+}: \qquad \sum_k p_k^+ \frac{\partial}{\partial p_k^+} \bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.24}$$

$$J^{z\bar{z}}: \qquad \Big[\sum_k (p\partial_p - \bar{p}\partial_{\bar{p}}) + \sum_k \lambda_k\Big] h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.25}$$

$$J^{z\bar{z}}: \qquad \Big[\sum_k (p\partial_p - \bar{p}\partial_{\bar{p}}) + \sum_k \lambda_k - 1\Big] j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0, \tag{4.26}$$

$$J^{z\bar{z}}: \qquad \Big[\sum_k (p\partial_p - \bar{p}\partial_{\bar{p}}) + \sum_k \lambda_k + 1\Big] \bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n} \sim 0. \tag{4.27}$$

Here ~ 0 implies an equality up to an overall delta-function $\delta^{d-1}(\sum_k p_k)$. The commutators with J^{a+} imply that the dynamic corrections depend on only a specific combinations of momenta

$$J^{a+}: \qquad \Big(\sum_k p_k^+ \frac{\partial}{\partial p_k^a}\Big) \to P^a_{km} = p^a_k p^+_m - p^a_m p^+_k. \tag{4.28}$$

The indices m and k stand for respective particles partaking in the interaction. With this we can rewrite our corrections as

$$h_{\lambda_1,\dots,\lambda_n}(p_1,\dots,p_n) = h_{\lambda_1,\dots,\lambda_n}(P_{km}, \bar{P}_{km}, p_k^+), \tag{4.29}$$

$$j_{\lambda_1,\dots,\lambda_n}(p_1,\dots,p_n) = j_{\lambda_1,\dots,\lambda_n}(P_{km}, \bar{P}_{km}, p_k^+), \tag{4.30}$$

$$\bar{j}_{\lambda_1,\dots,\lambda_n}(p_1,\dots,p_n) = \bar{j}_{\lambda_1,\dots,\lambda_n}(P_{km}, \bar{P}_{km}, p_k^+). \tag{4.31}$$

This also has an effect on our remaining constraints

$$J^{-+}: \qquad \Big[P\frac{\partial}{\partial P}+\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k p_k^+\frac{\partial}{\partial p_k^+}\Big]h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0, \tag{4.32}$$

$$J^{-+}: \qquad \Big[P\frac{\partial}{\partial P}+\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k p_k^+\frac{\partial}{\partial p_k^+}\Big]j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0, \tag{4.33}$$

$$J^{-+}: \qquad \Big[P\frac{\partial}{\partial P}+\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k p_k^+\frac{\partial}{\partial p_k^+}\Big]\bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0, \tag{4.34}$$

$$J^{z\bar{z}}: \qquad \Big[P\frac{\partial}{\partial P}-\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k \lambda_k\Big]h^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0, \tag{4.35}$$

$$J^{z\bar{z}}: \qquad \Big[P\frac{\partial}{\partial P}-\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k \lambda_k-1\Big]j^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0, \tag{4.36}$$

$$J^{z\bar{z}}: \qquad \Big[P\frac{\partial}{\partial P}-\bar{P}\frac{\partial}{\partial \bar{P}}+\sum_k \lambda_k+1\Big]\bar{j}^{p_1,\dots,p_n}_{\lambda_1,\dots,\lambda_n}\sim 0. \tag{4.37}$$

Everything that was stated in this chapter so far is true for all possible vertices in four dimensions. To solve the problem for cubic vertices we proceed with the following observation; for three participating particles the combinations $P_{1,2}$, $P_{2,3}$ and $P_{3,1}$ are all equal due to the conservation of momentum enforced by the delta distribution. Therefore we have only two independent momenta, that we can represent in a cyclic-invariant way

$$P^a=\frac{1}{3}[(p_1^+-p_2^+)p_3^a+(p_2^+-p_3^+)p_1^a+(p_3^+-p_1^+)p_2^a]. \tag{4.38}$$

Originating from this is a new way to write the conservation of momenta

$$\sum_j \frac{\partial}{\partial p_j}P=0. \tag{4.39}$$

Also we can write H_2 at three-level as

$$\sum_i H_2(p_i)=\frac{P\bar{P}}{p_1^+p_2^+p_3^+}=\frac{P\cdot P}{2p_1^+p_2^+p_3^+}. \tag{4.40}$$

With this in hand we can proceed to the dynamic constraints. We start with $[H,J^{a-}]=0$

$$[H,J^{a-}]_{(3)}=[H_3,J_2^{a-}]+[H_2,J_3^{a-}]=0, \tag{4.41}$$

where the subscript implies the third order. Using (4.39) we may glean that

$$\sum_i H_2(p_i)j_3=\sum_i (J^{z-})^T h_3, \tag{4.42}$$

$$\sum_i H_2(p_i)\bar{j}_3=\sum_i (J^{\bar{z}-})^T \bar{h}_3, \tag{4.43}$$

with the transposed generator

$$(J^{z-})^T = -\frac{p\bar{p}}{p^+}\frac{\partial}{\partial \bar{p}} - p\frac{\partial}{\partial p^+} + \lambda\frac{p}{p^+}. \tag{4.44}$$

In order to solve the constraints (4.37) our h_3 needs to posses a certain degree of homogeneity in the momenta

$$h_3 \sim \frac{\bar{P}^{\lambda_1+\lambda_2+\lambda_3}}{(p_1^+)^{\lambda_1}(p_2^+)^{\lambda_2}(p_3^+)^{\lambda_3}}. \tag{4.45}$$

Due to the relation $P\bar{P} \sim \Box$ we can use field redefinitions to obtain forms for our generators that are holomorphic [21]. This exhausts the freedom that our generators have up to two coupling constants

$$h_{\lambda_1,\lambda_2,\lambda_3} = C^{\lambda_1,\lambda_2,\lambda_3}\frac{\bar{P}^{\lambda_1+\lambda_2+\lambda_3}}{(p_1^+)^{\lambda_1}(p_2^+)^{\lambda_2}(p_3^+)^{\lambda_3}} + \bar{C}^{-\lambda_1,-\lambda_2,-\lambda_3}\frac{P^{-\lambda_1-\lambda_2-\lambda_3}}{(p_1^+)^{-\lambda_1}(p_2^+)^{-\lambda_2}(p_3^+)^{-\lambda_3}}, \tag{4.46}$$

$$j_{\lambda_1,\lambda_2,\lambda_3} = +\frac{2}{3}C^{+\lambda_1,+\lambda_2,+\lambda_3}\frac{\bar{P}^{\lambda_1+\lambda_2+\lambda_3-1}}{(p_1^+)^{+\lambda_1}(p_2^+)^{+\lambda_2}(p_3^+)^{+\lambda_3}}\Lambda^{\lambda_1,\lambda_2,\lambda_3}, \tag{4.47}$$

$$\bar{j}_{\lambda_1,\lambda_2,\lambda_3} = -\frac{2}{3}\bar{C}^{-\lambda_1,-\lambda_2,-\lambda_3}\frac{P^{-\lambda_1-\lambda_2-\lambda_3-1}}{(p_1^+)^{-\lambda_1}(p_2^+)^{-\lambda_2}(p_3^+)^{-\lambda_3}}\Lambda^{\lambda_1,\lambda_2,\lambda_3}, \tag{4.48}$$

where

$$\Lambda^{\lambda_1,\lambda_2,\lambda_3} = p_1^+(\lambda_2-\lambda_3) + p_2^+(\lambda_3-\lambda_1) + p_3^+(\lambda_1-\lambda_2). \tag{4.49}$$

$\bar{C}^{-\lambda_1,-\lambda_2,-\lambda_3}$ and $C^{\lambda_1,\lambda_2,\lambda_3}$ are coupling constants that can be fixed by looking at higher order corrections. In order to obtain the full vertex we have to sum over the possible values for the various λ, which results in our final light-front vertex [18]

$$\begin{aligned} V_{l-f} = \sum_{|s_1|,|s_2|,|s_3|}\Big[&\frac{(il)^{s1+s2+s3}}{\Gamma(s_1+s_2+s_3)}(\partial_{x_1}^+)^{-s_1}(\partial_{x_2}^+)^{-s_2}(\partial_{x_3}^+)^{-s_3}\bar{P}^{s_1+s_2+s_3}\varphi_{+s_1}\varphi_{+s_2}\varphi_{+s_3} \\ &+ \frac{(-il)^{-s1-s2-s3}}{\Gamma(-s_1-s_2-s_3)}(\partial_{x_1}^+)^{s_1}(\partial_{x_2}^+)^{s_2}(\partial_{x_3}^+)^{s_3}P^{-s_1-s_2-s_3}\varphi_{-s_1}\varphi_{-s_2}\varphi_{-s_3}\Big]. \end{aligned} \tag{4.50}$$

Here the values for the coupling constants that can be obtained by looking at higher order corrections have been inserted. l is a constant of length dimension.

4.4 Comparison

To figure out how the vetrex (4.50) differs from its covariant counterpart we may examine triplets of spins (s_1, s_2, s_3) and analyse the total amount of couplings the corresponding vertices have access to [23].

- $s_1 = s_2 = s_3 = s$:
  There are exactly two couplings with $3s$ and s derivatives each.

- $s_1 = s_2 = s$ and $s_3 \neq s$
  There are three couplings with $2s + s_3$, $|2s - s_3|$ and s_3 derivatives.

- $s_1 \neq s_2 \neq s_3$
  Four different couplings exists with $s_1 + s_2 + s_3$, $|s_1 + s_2 - s_3|$, $|s_1 + s_3 - s_2|$ and $|s_3 + s_2 - s_1|$ derivatives each.

So given a triplet of spins we are able to obtain up to four distinctive couplings, which is in contrast to the structure of the covariant vertex (4.13), where only two couplings for any triplet are possible. These two couplings can then be divided into four non-trivial combinations of helicities. The first coupling is obtained when all three fields are in the same helicity state. The second one is obtained if the state of the field with the lowest spin differs from the two higher one. Combinations where one of the higher valued spins is present in a different state are notably absent, which are precisely those additional ones we have obtained in the light-front. If we tried to express these helicity combinations in the covariant vertex we will run into a significant hindrance. As we will observe it is not avoidable to write down a division through Y_I's, if we want to cast the light front vertex into a covariant formulation. This is unfortunate as the objects Y_I^{-1} lack any tensorial interpretation, stemming from their dependence on the dummy variables u^μ, as these should be set to zero before any calculation for a physical quantity can commence. This anomaly was first discussed by Bengtsson in [6] before being expanded upon by later authors. To understand why this rather peculiar problem is not present in the light-front we shall rewrite the covariant into light-front contributions. At first let us recall the basic building block for the covariant form

$$k^{(n)}_{s_1,s_2,s_3} = G^n Y_1^{s_1-n} Y_2^{s_2-n} Y_3^{s_3-n}, \tag{4.51}$$

where the restriction

$$n \leq s_{min}, \tag{4.52}$$

is in place to keep the expression polynomial in ∂_u and $\bar{\partial}_u$. Now we use the definition of the light-front coordinates to express the variables of the differential vertex operator

$$Y_I \to -(\partial^+_{x_I})^{-1}(\bar{P}\partial_{u_I} + P\bar{\partial}_{u_I}), \tag{4.53}$$

$$Z_I \to \bar{\partial}_{u_{I+1}}\partial_{u_{I-1}} + \bar{\partial}_{u_{I-1}}\partial_{u_{I+1}}, \tag{4.54}$$

$$G \to \bar{\partial}_{u_1}\bar{\partial}_{u_2}\partial_{u_3}\Big(\frac{\partial^+_{x_3}}{\partial^+_{x_1}\partial^+_{x_2}}P\Big) + \partial_{u_1}\partial_{u_2}\bar{\partial}_{u_3}\Big(\frac{\partial^+_{x_3}}{\partial^+_{x_1}\partial^+_{x_2}}\bar{P}\Big) + cyclic. \tag{4.55}$$

To proceed we examine selected parts of $k^{(n)}_{s_1,s_2,s_3}$ for two different orders n and helicity combinations

$$(k^{(0)}_{s_1,s_2,s_3})_{---} \to (\partial^+_{x_1})^{-s_1}(\partial^+_{x_2})^{-s_2}(\partial^+_{x_3})^{-s_3}P^{s_1+s_2+s_3}\bar{\partial}^{s_1}_{u_1}\bar{\partial}^{s_2}_{u_2}\bar{\partial}^{s_3}_{u_3}, \tag{4.56}$$

$$\begin{aligned}(k^{(n)}_{s_1,s_2,s_3})_{--+} \to\ & (-1)^{s_1+s_2+s_3-n}(\partial^+_{x_1})^{-s_1}(\partial^+_{x_2})^{-s_2}(\partial^+_{x_3})^{-s_3}P^{s_1+s_2-n}\bar{P}^{s_3-n}\\ & \bar{\partial}^{s_1}_{u_1}\bar{\partial}^{s_2}_{u_2}\partial^{s_3}_{u_3},\end{aligned} \tag{4.57}$$

here only the first halve of each term is displayed, because the other half can be obtained by complex conjugation. The first term is present in both formulations and can be transcribed without any changes. For the other term however the situation is different. Due to the (anti-)holomorphy condition on φ in the light-front, terms that would be not polynomial in ∂_u, drop out by default without the need to exclude specific values for n. This implies that the more rigid light-front constraints allow for a higher limit for n. Our only concern for the additional contributions is now only the issue of locality. Therefore we need to look at locality violating contributions

$$P^m\bar{P}^n \qquad \text{can be removed via EoM,} \tag{4.58}$$

$$\frac{P^m}{\bar{P}^n} \qquad \text{can be removed via EoM,} \tag{4.59}$$

$$\frac{1}{P^m\bar{P}^n} \qquad \text{can not be removed via EoM.} \tag{4.60}$$

While the first two terms are proportional to the equations of motion, the last one is not and needs to be activly avoided for a robust description. For this we can enforce

$$n \le \min(s_1+s_2, s_3) = j \leftrightarrow s_1+s_2+s_3-2n = 0, \tag{4.61}$$

giving us a new limit for our summation index n, which faithfully reproduces the full light-front vertex. This knowledge enables us to rewrite the light-front vertices into a "covariant" form

$$V_{Formal} = M(Y_1, Y_2, Y_3, G)\varphi_1\varphi_2\varphi_3, \tag{4.62}$$

$$M = \sum_{s_i=0}^{\infty}\Big[\sum_{n=\{0,j\}}\frac{(il)^{s_1+s_2+s_3-2n}}{\sqrt{s_1!\,s_2!\,s_3!}\Gamma(s_1+s_2+s_3-2n)}Y_1^{s_1}Y_2^{s_2}Y_3^{s_3}\Big(\frac{G}{Y_1Y_2Y_3}\Big)^n\Big], \tag{4.63}$$

which is strictly formal as Y_I^{-1} lacks any tensorial interpretation.

4.5 Outlining the Problem

To keep things precise we will focus on the triplet $s_1 = s_2 = 3,\ s_3 = 2$, with the assumption that any potential result may be generalised to all such vertices. The vertex in question takes the following form in the covariant (4.15)

$$V_{cov} = g_{min} G^2 Y_1 Y_2 + g_{non} Y_1^3 Y_2^3 Y_3^2, \tag{4.64}$$

and the light-front (4.50)

$$\begin{aligned} V_{l-f} =& \frac{(-il)^8}{8} \left((\partial^+_{x_1})^{-3} (\partial^+_{x_2})^{-3} (\partial^+_{x_3})^{-2} \bar{P}^8 \varphi_3\varphi_3\varphi_2 + (\partial^+_{x_1})^{-3} (\partial^+_{x_2})^{-3} (\partial^+_{x_3})^{-2} P^8 \varphi_{-3}\varphi_{-3}\varphi_{-2} \right) \\ &+ \frac{(-il)^4}{4} \left((\partial^+_{x_1})^{3} (\partial^+_{x_2})^{3} (\partial^+_{x_3})^{-2} \bar{P}^4 \varphi_3\varphi_3\varphi_{-2} + (\partial^+_{x_1})^{-3} (\partial^+_{x_2})^{-3} (\partial^+_{x_3})^{-2} P^4 \varphi_{-3}\varphi_{-3}\varphi_{2} \right) \\ &+ \frac{(-il)^2}{2} \left((\partial^+_{x_1})^{-3} (\partial^+_{x_2})^{3} (\partial^+_{x_3})^{-2} \bar{P}^2 \varphi_{-3}\varphi_3\varphi_{2} + (\partial^+_{x_1})^{-3} (\partial^+_{x_2})^{3} (\partial^+_{x_3})^{-2} P^2 \varphi_{-3}\varphi_{3}\varphi_{-2} \right) \\ &+ \frac{(-il)^2}{2} \left((\partial^+_{x_1})^{3} (\partial^+_{x_2})^{-3} (\partial^+_{x_3})^{-2} \bar{P}^2 \varphi_{3}\varphi_{-3}\varphi_{2} + (\partial^+_{x_1})^{3} (\partial^+_{x_2})^{-3} (\partial^+_{x_3})^{-2} P^2 \varphi_{3}\varphi_{-3}\varphi_{-2} \right), \end{aligned} \tag{4.65}$$

formulation respectively. The first line corresponds to the vertex with the maximal amount of derivatives, while the second corresponds to the minimal one. The last two terms, proportional to $(-il)^2$, correspond to the additional light-front coupling. In the "formal covariant" formulation this coupling has the form

$$\begin{aligned} Y_1^3 Y_2^3 Y_3^2 \Big(\frac{G}{Y_1Y_2Y_3}\Big)^3 &= \frac{G^3}{Y_3} = G^2 \left(Z_3 + \frac{Z_1Y_1 + Z_2Y_2}{Y_3} \right) \\ &= \left((Z_3Y_3)^2 + (Z_2Y_2)^2 + (Z_1Y_1)^2 + 2(Z_3Y_3Z_2Y_2 + Y_2Z_2Y_1Z_1 + Y_3Z_3Z_1Y_1) \right) \\ &\quad \left(Z_3 + \frac{Z_1Y_1 + Z_2Y_2}{Y_3} \right) \\ &= Z_3^3Y_3^2 + 3Z_3(Z_2Y_2)^2 + 3Z_3(Z_1Y_1)^2 + 3Z_3^2Y_3Z_2Y_2 + 3Z_3^2Y_3Z_1Y_1 \\ &\quad + 5Z_1Z_2Z_3Y_1Y_2 + \frac{(Z_1Y_1 + Z_2Y_2)^3}{Y_3}. \end{aligned} \tag{4.66}$$

The second to last term may be discarded as it is proportional to the equations of motion as well as the holomorphy constraint as found in (3.28)

$$Z_1Z_2Z_3Y_1Y_2 = (\partial^+_{x_1})^{-1} (\partial^+_{x_2})^{-1} \left(\bar{\partial}^3_{u_1} \partial^3_{u_2} + \partial^3_{u_1} \bar{\partial}^3_{u_2} \right) \partial_{u_3} \bar{\partial}_{u_3} P\bar{P}. \tag{4.67}$$

It is advantageous to reassure us that this problem is tied to the covariant formulation. Utilising (4.55) we may rewrite the expression bit by bit, starting with the term with the explicit division

$$(Z_1Y_1 + Z_2Y_2) = -(\partial^+_{x_1})^{-1}(\bar{P}\partial_{u_1}\bar{\partial}_{u_2}\partial_{u_3} + \bar{P}\partial_{u_1}\partial_{u_2}\bar{\partial}_{u_3} + P\bar{\partial}_{u_1}\bar{\partial}_{u_2}\partial_{u_3} + P\bar{\partial}_{u_1}\partial_{u_2}\bar{\partial}_{u_3})$$
$$- (\partial^+_{x_2})^{-1}(\bar{P}\bar{\partial}_{u_1}\partial_{u_2}\partial_{u_3} + \bar{P}\partial_{u_1}\partial_{u_2}\bar{\partial}_{u_3} + P\bar{\partial}_{u_1}\bar{\partial}_{u_2}\partial_{u_3} + P\partial_{u_1}\bar{\partial}_{u_2}\bar{\partial}_{u_3}). \quad (4.68)$$

To proceed we will assume that the two fields φ_1 and φ_2 are completely identical. This allows us to swap $x_1 \leftrightarrow x_2$ in different parts to compact our result

$$(Z_1Y_1 + Z_2Y_2) = -2(\partial^+_{x_1})^{-1}(\bar{P}\partial_{u_1}\bar{\partial}_{u_2}\partial_{u_3} + \bar{P}\partial_{u_1}\partial_{u_2}\bar{\partial}_{u_3} + P\bar{\partial}_{u_1}\bar{\partial}_{u_2}\partial_{u_3} + P\bar{\partial}_{u_1}\partial_{u_2}\bar{\partial}_{u_3}). \quad (4.69)$$

There are two more parts to the numerator

$$(Z_1^2Y_1^2 + Z_2^2Y_2^2) = 2(\partial^+_{x_1})^{-2}(\bar{P}^2\partial^2_{u_1}\bar{\partial}_{u_2}{}^2\partial^2_{u_3} + \bar{P}^2\partial^2_{u_1}\partial^2_{u_2}\bar{\partial}_{u_3}{}^2 + P^2\bar{\partial}_{u_1}{}^2\bar{\partial}_{u_2}{}^2\partial^2_{u_3}$$
$$+ P^2\bar{\partial}_{u_1}{}^2\partial^2_{u_2}\bar{\partial}_{u_3}{}^2), \quad (4.70)$$
$$2Z_1Y_1Z_2Y_2 = 2(\partial^+_{x_1}\partial^+_{x_2})^{-1}(\bar{P}^2\partial^2_{u_1}\partial^2_{u_2}\bar{\partial}_{u_3}{}^2 + P^2\bar{\partial}_{u_1}{}^2\bar{\partial}_{u_2}{}^2\partial^2_{u_3}). \quad (4.71)$$

We may ignore mixed terms as they are proportional to the last constraint of (3.28). Adding all of these contributions together will give us the total numerator

$$(Z_1Y_1 + Z_2Y_2)(Z_1^2Y_1^2 + Z_2^2Y_2^2 + 2Z_1Y_1Z_2Y_2) =$$
$$=2(\partial^+_{x_1})^{-3}(\bar{P}^3\partial^3_{u_1}\bar{\partial}_{u_2}{}^3\partial^3_{u_3} + \bar{P}^3\partial^3_{u_1}\partial^3_{u_2}\bar{\partial}_{u_3}{}^3 + P^3\bar{\partial}_{u_1}{}^3\bar{\partial}_{u_2}{}^3\partial^3_{u_3} + P^3\bar{\partial}_{u_1}{}^3\partial^3_{u_2}\bar{\partial}_{u_3}{}^3)$$
$$+ 6(\partial^+_{x_1})^{-2}(\partial^+_{x_2})^{-1}(\bar{P}^3\partial^3_{u_1}\partial^3_{u_2}\bar{\partial}_{u_3}{}^3 + P^3\bar{\partial}_{u_1}{}^3\bar{\partial}_{u_2}{}^3\partial^3_{u_3}). \quad (4.72)$$

Now we take the denominator Y_{s_3} into account

$$\frac{(Z_1Y_1 + Z_2Y_2)^3}{Y_3} = (\partial^+_{x_3})(P\bar{\partial}_{u_3} + \bar{P}\partial_{u_3})^{-1}\times$$
$$(2(\partial^+_{x_1})^{-3}(\bar{P}^3\partial^3_{u_1}\bar{\partial}_{u_2}{}^3\partial^3_{u_3} + \bar{P}^3\partial^3_{u_1}\partial^3_{u_2}\bar{\partial}_{u_3}{}^3 + P^3\bar{\partial}_{u_1}{}^3\bar{\partial}_{u_2}{}^3\partial^3_{u_3} + P^3\bar{\partial}_{u_1}{}^3\partial^3_{u_2}\bar{\partial}_{u_3}{}^3)$$
$$+ 6(\partial^+_{x_1})^{-2}(\partial^+_{x_2})^{-1}(\bar{P}^3\partial^3_{u_1}\partial^3_{u_2}\bar{\partial}_{u_3}{}^3 + P^3\bar{\partial}_{u_1}{}^3\bar{\partial}_{u_2}{}^3\partial^3_{u_3}))$$
$$= 2(\partial^+_{x_1})^{-3}\partial^+_{x_3}(\bar{P}^2\partial^3_{u_1}\bar{\partial}_{u_2}{}^3\partial^2_{u_3} + P^2\bar{\partial}_{u_1}{}^3\partial^3_{u_2}\bar{\partial}_{u_3}{}^2)$$
$$+ (\partial^+_{x_1})^{-2}\partial^+_{x_3}(2(\partial^+_{x_1})^{-1} + 3(\partial^+_{x_2})^{-1})\frac{\bar{P}^3\partial^3_{u_1}\partial^3_{u_2}\bar{\partial}_{u_3}{}^3 + P^3\bar{\partial}_{u_1}{}^3\bar{\partial}_{u_2}{}^3\partial^3_{u_3}}{P\bar{\partial}_{u_3} + \bar{P}\partial_{u_3}}. \quad (4.73)$$

The second term may seem problematic here, but we can show that it does not contribute to the coupling whatsoever. For this we need to simply evoke the principle of locality in the light-front (4.60). This allows us to equate any term proportional to $P/\bar{P}$ to the equations of motion, hence removing the term. The final result is then

$$\frac{(Z_1Y_1 + Z_2Y_2)^3}{Y_3} = 2(\partial^+_{x_1})^{-3}\partial^+_{x_3}(\bar{P}^2\partial^3_{u_1}\bar{\partial}_{u_2}{}^3\partial^2_{u_3} + P^2\bar{\partial}_{u_1}{}^3\partial^3_{u_2}\bar{\partial}_{u_3}{}^2), \quad (4.74)$$

a valid expression in the light-front. The remaining terms are straightforward

$$
\begin{aligned}
Z_3^3 Y_3^2 &= 2(\partial^+_{x_3})^{-2}(\bar{P}^2 \partial^3_{u_1} \bar{\partial}_{u_2}{}^3 \partial^2_{u_3} + P^2 \partial^3_{u_1} \bar{\partial}_{u_2}{}^3 \bar{\partial}_{u_3}{}^2), && (4.75)\\
3Z_3(Z_2Y_2)^2 &= 3(\partial^+_{x_2})^{-2}(\bar{P}^2 \bar{\partial}_{u_1}{}^3 \partial^3_{u_2} \partial^2_{u_3} + P^2 \partial^3_{u_1} \bar{\partial}_{u_2} \bar{\partial}_{u_3}{}^2), && (4.76)\\
3Z_3(Z_1Y_1)^2 &= 3(\partial^+_{x_1})^{-2}(\bar{P}^2 \partial^3_{u_1} \bar{\partial}_{u_2}{}^3 \partial^2_{u_3} + P^2 \bar{\partial}_{u_1}{}^3 \partial_{u_2} \bar{\partial}_{u_3}{}^2), && (4.77)\\
3Z_3^2 Y_3 Z_2 Y_2 &= 3(\partial^+_{x_2})^{-1}(\partial^+_{x_3})^{-1}(\bar{P}^2 \bar{\partial}_{u_1}{}^3 \partial^3_{u_2} \partial^2_{u_3} + P^2 \partial^3_{u_1} \bar{\partial}_{u_2} \bar{\partial}_{u_3}{}^2), && (4.78)\\
3Z_3^2 Y_3 Z_1 Y_1 &= 3(\partial^+_{x_1})^{-1}(\partial^+_{x_3})^{-1}(\bar{P}^2 \partial^3_{u_1} \bar{\partial}_{u_2}{}^3 \partial^2_{u_3} + P^2 \bar{\partial}_{u_1}{}^3 \partial_{u_2} \bar{\partial}_{u_3}{}^2), && (4.79)
\end{aligned}
$$

and up to a few $(\partial^+_{x_i})$ the terms match with (4.65). To resolve this we remind us that each term in the regular light-front coupling posses three derivatives and five integrations, we can restore those by inserting pairs $(\partial^+_{x_i})^{-1}\partial^+_{x_i}$ of ones and integrate by parts.

4.6 Possible Counter-Terms

Confronted with the main problem of this thesis there are multiple options before us. As we have seen in the previous section most terms contributing to (4.66) are covariant. What we could therefore try is searching for a term that could replace the not covariant part. Such an object is commonly referred to as a counter-term. We could do this by working out the behaviour under gauge transformations for the covariant contributions; therefore obtaining a differential equation that we could solve. To check this behaviour we can simply apply the gauge transformation (2.31) in the same manner we did to derive the covariant vertex in (4.7). Thus we need to compute four different commutators for three different values of I in the transformation each. For this the commutators (4.11) are vital and will be applied in all of the following results. Lets start with the first term

$$
\begin{aligned}
[Z_3^3 Y_3^2, u^I \partial_{x_I}] &= [Z_3^3, u^I \partial_{x_I}] Y_3^2 = 4[Z_3, u^I \partial_{x_I}] Z_3^2 Y_3^2 \\
\implies \quad [Z_3^3 Y_3^2, u^1 \partial_{x_1}] &= -4 Z_3^2 Y_3^2 Y_2, && (4.80)\\
[Z_3^3 Y_3^2, u^2 \partial_{x_2}] &= 4 Z_3^2 Y_3^2 Y_1, && (4.81)\\
[Z_3^3 Y_3^2, u^3 \partial_{x_3}] &= 0, && (4.82)
\end{aligned}
$$

where the calculation is straight forward. We proceed in the same fashion for all the other commutators

$$
\begin{aligned}
3[Z_3(Z_2Y_2)^2, u^I \partial_{x_I}] &= 3[Z_3 Z_2^2, u^I \partial_{x_I}] Y_2^2 = 3(3Z_3 Z_2 [Z_2, u^I \partial_{x_I}] + Z_2^2 [Z_3, u^I \partial_{x_I}]) Y_2^2 \\
\implies \quad 3[Z_3(Z_2Y_2)^2, u^1 \partial_{x_1}] &= 9 Z_3 Z_2 Y_3 Y_2^2 - 3 Z_2^2 Y_2^3, && (4.83)\\
3[Z_3(Z_2Y_2)^2, u^2 \partial_{x_2}] &= 3 Y_1 (Z_2 Y_2)^2, && (4.84)\\
3[Z_3(Z_2Y_2)^2, u^3 \partial_{x_3}] &= -9 Z_3 Z_1 Y_1 Y_2^2, && (4.85)
\end{aligned}
$$

$$
\begin{aligned}
3[Z_3(Z_1Y_1)^2, u^I\partial_{x_I}] =& 3[Z_3Z_1^2, u^I\partial_{x_I}]Y_1^2 = 3(3Z_3Z_1[Z_1, u^I\partial_{x_I}] + Z_1^2[Z_3, u^I\partial_{x_I}])Y_1^2 \\
\implies 3[Z_3(Z_1Y_1)^2, u^1\partial_{x_1}] &= -3Y_2(Z_1Y_1)^2, & (4.86) \\
3[Z_3(Z_1Y_1)^2, u^2\partial_{x_2}] &= -9Z_3Z_1Y_3Y_1^2 + 3Y_1(Z_1Y_1)^2, & (4.87) \\
3[Z_3(Z_1Y_1)^2, u^3\partial_{x_3}] &= 9Z_3Z_1Y_1Y_2^2, & (4.88)
\end{aligned}
$$

$$
\begin{aligned}
3[Z_3^2Y_3Z_2Y_2, u^I\partial_{x_I}] =& 3[Z_3^2Z_2, u^I\partial_{x_I}]Y_3Y_2 = 3(3Z_3Z_2[Z_3, u^I\partial_{x_I}] + Z_3^2[Z_2, u^I\partial_{x_I}])Y_3Y_2 \\
\implies 3[Z_3^2Y_3Z_2Y_2, u^1\partial_{x_1}] &= -9Z_3Z_2Y_3Y_2^2 + 3Z_3^2Y_3^2Y_2, & (4.89) \\
3[Z_3^2Y_3Z_2Y_2, u^2\partial_{x_2}] &= 9Z_3Z_2Y_3Y_2Y_1, & (4.90) \\
3[Z_3^2Y_3Z_2Y_2, u^3\partial_{x_3}] &= -3Z_3^2Y_3Y_2Y_1, & (4.91)
\end{aligned}
$$

$$
\begin{aligned}
3[Z_3^2Y_3Z_1Y_1, u^I\partial_{x_I}] =& 3[Z_3^2Z_1, u^I\partial_{x_I}]Y_3Y_1 = 3(3Z_3Z_1[Z_3, u^I\partial_{x_I}] + Z_3^2[Z_1, u^I\partial_{x_I}])Y_3Y_1 \\
\implies 3[Z_3^2Y_3Z_1Y_1, u^1\partial_{x_1}] &= -9Z_3Z_1Y_3Y_2Y_1, & (4.92) \\
3[Z_3^2Y_3Z_1Y_1, u^2\partial_{x_2}] &= 9Z_3Z_1Y_3Y_1^2 - Z_3^2Y_3^2Y_1, & (4.93) \\
3[Z_3^2Y_3Z_1Y_1, u^3\partial_{x_3}] &= 3Z_3^2Y_3Y_2Y_1. & (4.94)
\end{aligned}
$$

We also need to take the commutator of the parts proportional to the equation of motion into account, as the closure of the gauge should hold off-shell as well

$$
\begin{aligned}
5[Z_1Z_2Z_3Y_1Y_2, u^I\partial_{x_I}] &= 5[Z_3Z_2Z_1, u^I\partial_{x_I}]Y_2Y_1 \\
&= 5(Z_1Z_2[Z_3, u^I\partial_{x_I}] + Z_1Z_3[Z_2, u^I\partial_{x_I}] + Z_2Z_3[Z_1, u^I\partial_{x_I}])Y_1Y_2 \\
\implies 5[Z_1Z_2Z_3Y_1Y_2, u^1\partial_{x_1}] &= -Z_1Z_2Y_1Y_2^2 + Z_1Z_3Y_1Y_2Y_3, & (4.95) \\
5[Z_1Z_2Z_3Y_1Y_2, u^2\partial_{x_2}] &= Z_1Z_2Y_1^2Y_2 - Z_2Z_3Y_1Y_2Y_3, & (4.96) \\
5[Z_1Z_2Z_3Y_1Y_2, u^3\partial_{x_3}] &= -Z_1Z_3Y_1^2Y_2 + Z_2Z_3Y_1Y_2^2. & (4.97)
\end{aligned}
$$

Adding all these contributions together gives us the final form of our commutators

$$
\begin{aligned}
(Y_3)^{-1}[G^3 - (Z_1Y_1 + Z_2Y_2)^3, u^1\partial_{x_1}] &= -3Z_1^2Y_1^2Y_2 - 6Z_1Z_2Y_1Y_2^2 - 3Z_2^2Y_2^3 \\
&= -3Y_2(Y_1Z_1 + Y_2Z_2)^2, & (4.98) \\
(Y_3)^{-1}[G^3 - (Z_1Y_1 + Z_2Y_2)^3, u^2\partial_{x_2}] &= 3Z_1^2Y_1^3 + 6Z_1Z_2Y_1^2Y_2 + 3Z_2^2Y_1Y_2^2 \\
&= 3Y_1(Y_1Z_1 + Y_2Z_2)^2, & (4.99) \\
(Y_3)^{-1}[G^3 - (Z_1Y_1 + Z_2Y_2)^3, u^3\partial_{x_3}] &= 0. & (4.100)
\end{aligned}
$$

To check this result we could compute the contribution from the non covariant term and check if the two would cancel each other.

Now we are able to propose conditions that a potential counter-term C has to satisfy

$$[C, u^1\partial_{x_1}] = 3Y_2(Y_1Z_1 + Y_2Z_2)^2, \tag{4.101}$$
$$[C, u^2\partial_{x_2}] = -3Y_1(Y_1Z_1 + Y_2Z_2)^2, \tag{4.102}$$
$$[C, u^3\partial_{x_3}] = 0. \tag{4.103}$$

These can then be transcribed into differential equation that we will attempt to solve. For this we view C as $C(Y_I, Z_I)$ and translate the commutator into a differential

$$\frac{\partial C}{\partial(\partial_{u_i})}\partial_{x_I} \sim [C, u^i\partial_{x_I}].$$

The derivative of C may then be expressed as

$$\frac{\partial C}{\partial(\partial_{u_I})}\partial_{x_I} = \frac{\partial C}{\partial Y_j}\frac{\partial Y_j}{\partial(\partial_{u_i})}\partial_{x_i} + \frac{\partial C}{\partial Z_j}\frac{\partial Z_j}{\partial(\partial_{u_i})}\partial_{x_i} = -\frac{\partial C}{\partial Z_j}\epsilon_{ij}{}^{k}Y_k, \tag{4.104}$$

which gives us

$$-\frac{\partial C}{\partial Z_2}Y_3 + \frac{\partial C}{\partial Z_3}Y_2 = 3Y_2(Y_1Z_1 + Y_2Z_2)^2, \tag{4.105}$$
$$\frac{\partial C}{\partial Z_1}Y_3 - \frac{\partial C}{\partial Z_3}Y_1 = -3Y_1(Y_1Z_1 + Y_2Z_2)^2, \tag{4.106}$$
$$\frac{\partial C}{\partial Z_1}Y_2 = \frac{\partial C}{\partial Z_2}Y_1. \tag{4.107}$$

To solve this set of equations we can employee the methods of characteristics. The solution is comprised of a special and a homogeneous solution. As a special solution we may pick the one we already find in the original term

$$C_s = \frac{(Y_1Z_1 + Y_2Z_2)^3}{Y_3}. \tag{4.108}$$

Now we need to compute the homogeneous solution C_0 and observe if it is capable of removing it. The last of the three equations reveals to us that

$$C_0 = f(Y_1Z_1 + Y_2Z_2, Y_3), \tag{4.109}$$

where f is an arbitrary function. We can plug this result into the homogeneous version of the second equation

$$\frac{\partial C_0}{\partial Z_1}Y_3 - \frac{\partial C_0}{\partial Z_3}Y_1 = 0, \tag{4.110}$$

which possess the solution

$$C_0 = f(Y_1Z_1 + Y_2Z_2 + Y_3Z_3) = f(G). \tag{4.111}$$

We may use the homogeneous version of the final equation to check our results. Therefore the solution for the potential counter-term is

$$C = C_s + C_0 = \frac{(Y_1 Z_1 + Y_2 Z_2)^3}{Y_3} + f(G). \tag{4.112}$$

As no $f(G)$ is capable of removing the non-covariant contribution C_s we conclude that this method is not what we where looking for.

5 (A)dS vertices in the flat limit

Now that the grasp on the problem is more firm we may judge more exotic methods necessary. In this section I explain my aspirations to use the connection between the covariant (A)dS vertices and their flat counterparts to gain insight into the additional light-front couplings. The motivation for this stems from a paper by Sleight and Taronna [23]. In this publication the authors explore the structure constants of higher-spin algebras for light-front and (A)dS vertices in four space time dimensions. As the two results match and, according to them, heavily relay on the exotic couplings, they suggest there must be a well-defined link between these two. In curved space the lower-derivative structures are natural as they are a result of the commutator arising from the covariant derivatives. They muse over various implications and possible ways on how to proceed. Among those is a proposal about the nature of the flat limit connecting these vertices. The authors suggest that the flat limit may not be as straight forward as we think and that a remainder may manifest as the lower-derivative couplings we see in the light-front. There are some other proposals as well, but this one is explicitly calculable. Metsaev proposed in [19] the extension of the light-front formalism into the (A)dS space. The flat limit of this extension maps each coupling with a unique flat-space counterpart and therefore will most likely not contain additional information. The (A)dS vertex corresponding to our $s_1 = s_2 = 3, s_3 = 2$ example we picked earlier can be found in the Ph.D. thesis of Taronna [24]. This work will also serve as the main reference for this section, as it concerns itself with the derivation of higher-spin vertices in (A)dS . As we are only interested in the flat limit of the resulting Lagrangian and will not make use of the mathematical tools, we will not elaborate much on the foundations of higher-spins in curved space-time and instead point those interested to said thesis. The result for the minimal interacting vertex is given by

$$S^{(3)}_{(A)dS} = -\frac{2}{3} g^{332,2} \int \mathrm{d}^{d+1}X \; \delta(\sqrt{X^2} - L) \; G^2 \; \partial_{U_1} \cdot \partial_{X_2} \; \partial_{U_2} \cdot \partial_{X_1} \\ \times \phi^{(3)}(X_1, U_1)\phi^{(3)}(X_2, U_2)\phi^{(2)}(X_3, U_3)|_{X_i = X, \; U_i = 0}, \tag{5.1}$$

where all of the involved objects are written in the ambient-space, frame-like formalism and L is the (A)dS radius. In the ambient-space formalism the curved space one may want to study is embedded in a flat Minkowski space of one dimension higher, to reduce the complexity of the involved quantities especially the derivatives. While frame-like and metric approaches differ, one can go from one formalism to the other with a given algorithm [12]. The nature of this vertex we try to study however is not affected by this translation. Therefore we can carry on and perform calculations as we are used to. The translation from ambient into

(A)dS intrinsic quantities, as outlined by the Author, results in

$$
\begin{aligned}
S^{(3)}_{(A)dS} = -\frac{8}{3}g^{332,2}\int \mathrm{d}^d x\sqrt{-\det(g^{\mu\nu})}\\
\times\Big[&(\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot D_2 - \partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot D_1 + \frac{1}{2}\partial_{u_1}\cdot\partial_{u_2}\ \partial_{u_3}\cdot(D_1-D_2))^2\partial_{u_1}\cdot D_2\ \partial_{u_2}\cdot D_1\\
&+\frac{4}{L^2}\partial_{u_1}\cdot\partial_{u_2}[(\partial_{u_2}\cdot\partial_{u_3})^2(\partial_{u_1}\cdot D_2)^2 + (\partial_{u_1}\cdot\partial_{u_3})^2(\partial_{u_2}\cdot D_1)^2\\
&\qquad\qquad\qquad - 3\partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot D_2\ \partial_{u_2}\cdot D_1]\\
&+\frac{3}{L^2}(\partial_{u_1}\cdot\partial_{u_2})^2[\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot D_2 - \partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot D_1\\
&\qquad\qquad\qquad + \frac{1}{6}\partial_{u_1}\cdot\partial_{u_2}\ \partial_{u_3}\cdot(D_1-D_2)]\partial_{u_3}\cdot(D_1-D_2)\\
&-\frac{5}{4L^2}(\partial_{u_1}\cdot\partial_{u_2})^2[\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot D_2\ \partial_{u_3}\cdot D_1 + \partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot D_1\ \partial_{u_3}\cdot D_2]\\
&-\frac{7d+29}{2L^4}(\partial_{u_1}\cdot\partial_{u_2})^2\partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot\partial_{u_3}\Big]\varphi^{(3)}(x_1,u_1)\varphi^{(3)}(x_2,u_2)\phi^{(2)}(x_3,u_3)|_{x_i=x,\ u_i=0},
\end{aligned}
\tag{5.2}
$$

where D_i denotes the corresponding covariant derivative. Before we execute the actual calculation we should reassure us that the parts not proportional to $[D_i, D_j] \backsim L^{-2}$ produce the covariant minimal interacting vertex. For this we simply replace $D_i \to \partial_{x_i}$ to reach

$$
\left(\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot\partial_{x_2} - \partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\partial_{x_1} + \frac{1}{2}\partial_{u_1}\cdot\partial_{u_2}\ \partial_{u_3}\cdot(\partial_{x_1}-\partial_{x_2})\right)^2\partial_{u_1}\cdot\partial_{x_2}\ \partial_{u_2}\cdot\partial_{x_1}, \tag{5.3}
$$

which we can rewrite into the covariant building blocks Y_I and Z_I through their definition (4.8) and

$$
\begin{aligned}
\partial_{u_{i+1}}\cdot\partial_{x_i}\ \varphi(x_1,u_1)\psi(x_2,u_2)\chi(x_3,u_3) &= \partial_\mu\varphi(x_1,u_1)\psi(x_2,u_2)^\mu\chi(x_3,u_3)\\
&= -\varphi(x_1,u_1)\psi(x_2,u_2)^\mu\partial_\mu\chi(x_3,u_3)\\
&= -\partial_{u_i}\cdot\partial_{x_{i+1}}\ \varphi(x_1,u_1)\psi(x_2,u_2)\chi(x_3,u_3)\\
&= -Y_i\ \varphi(x_1,u_1)\psi(x_2,u_2)\chi(x_3,u_3),
\end{aligned}
\tag{5.4}
$$

where we made use of partial integration and the gauge conditions (2.28). With this in mind the term (5.3) takes on the form

$$
-\left(Z_1Y_1 - Z_2Y_2 + \frac{1}{2}Z_3(Y_1+Y_2)\right)^2 Y_1Y_2. \tag{5.5}
$$

With a partial integration over x_i as well as over u_i we may show that

$$
\frac{1}{2}(Y_1+Y_2) = Y_3, \tag{5.6}
$$

leading us to

$$- (Z_1Y_1 - Z_2Y_2 + Z_3Y_3)^2 Y_1Y_2 = -G^2Y_1Y_2, \tag{5.7}$$

and therefore we recover the corresponding flat vertex in full. The numeric factors may differ, but this could be a result of the different values for the interaction constants. We can perform the same procedure with the terms proportional to L^{-2} and observe what happens. To provide a better overview we will treat each term consecutively

$$\begin{aligned}&4\partial_{u_1}\cdot\partial_{u_2}[(\partial_{u_2}\cdot\partial_{u_3})^2(\partial_{u_1}\cdot\partial_2)^2+(\partial_{u_1}\cdot\partial_{u_3})^2(\partial_{u_2}\cdot\partial_{x_1})^2-3\partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot\partial_{x_2}\ \partial_{u_2}\cdot\partial_{x_1}]\\&=4Z_3[Z_1^2Y_1^2+Z_2^2Y_2^2+3Z_1Z_2Y_1Y_2],\end{aligned} \tag{5.8}$$

$$\begin{aligned}&3(\partial_{u_1}\cdot\partial_{u_2})^2[\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot\partial_{x_2}-\partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot\partial_{x_1}+\frac{1}{6}\partial_{u_1}\cdot\partial_{u_2}\ \partial_{u_3}\cdot(\partial_{x_1}-\partial_{x_2})]\partial_{u_3}\cdot(\partial_{x_1}-\partial_{x_2})\\&=-6Z_3^2\left[Z_1Y_1-Z_2Y_2-\frac{1}{3}Z_3Y_3\right]Y_3,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&-\frac{5}{4}(\partial_{u_1}\cdot\partial_{u_2})^2[\partial_{u_2}\cdot\partial_{u_3}\ \partial_{u_1}\cdot\partial_{x_2}\ \partial_{u_3}\cdot\partial_{x_1}+\partial_{u_1}\cdot\partial_{u_3}\ \partial_{u_2}\cdot\partial_{x_1}\ \partial_{u_3}\cdot\partial_{x_2}]\\&=-\frac{5}{4}Z_3^2[Z_1Y_1Y_3+Z_2Y_2Y_3].\end{aligned} \tag{5.10}$$

Adding all of these contribution up results in

$$2Z_3^3Y_3^2+4Z_3(Z_1Y_1)^2+4Z_3(Z_2Y_2)^2-\frac{29}{4}Z_3^2Y_3Z_1Y_1+\frac{19}{4}Z_3^2Y_3Z_2Y_2, \tag{5.11}$$

a not so promising result when compared against (4.66). While most of the on-shell terms are obtained, they do posses a diverse variety of numerical prefactors, that can not be explained away by a difference in the interaction parameters. To make matters even worse, the term absent from this procedure is exactly the one proportional to Y_3^{-1}, the one we wanted to explain most of all. All of this leads us to the following conclusion: The similarities we observed between these lower derivative terms stems more likely than not from the restricted nature of higher-spins in four space-time dimensions. The theory is highly restricted in terms of potential well-behaved operators, when constructing meaningful expressions. Our explorations into the properties of such objects in sections 2 and 4.2 provide sufficient foundation for this claim. The connection between these two is undoubtedly given as provided by the authors that proposed this eventuality in the first place, to find it however, we must dig deeper into the subject matter to eventually unveil it.

6 A Potential Solution using Mixed Symmetry Fields

As our attempts so far where not successful in providing us with evidence sufficient enough to continue into a particular direction we have to rethink our approach. This raises the question if the problem may be connected to the Fronsdal fields themselves and the generalisation procedure we decided upon 2 was just not general enough? If that is the case, mixed-symmetry fields may provide us with what we are looking for.

Representations of the Lorentz group are usually either completely symmetric or completely anti-symmetric. While we have worked in the symmetric setting thus far we are not bound to it as the two formulations contain the same amount of physical information. These are the most common, but by no means all possible representations. One can find descriptions that utilise families of indices instead of single ones. These objects are called mixed-symmetry fields

$$\varphi_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots} \tag{6.1}$$

and are the focus of this section. First explored in the form of the Hook field, which we will also do a bit further down the line, by Curtright [11] they are usually explored in the context of string theory. For these fields the spin property is no longer labelled by a single half-integer, but instead by a Young diagram with $s \leq \frac{d-2}{2}$ rows. We are going to elaborate on that topic a bit later. Usually these objects are not used in standard four dimensional theories, as simply treating them like their fully (anti-)symmetric counterparts will result in a theory with no physical degrees of freedom. However there are potential solutions to this conundrum, whose results we could also utilise for our lower-derivative cubic light-front interactions. In a paper by Mkrtchyan and Joung [15] they discussed a higher-derivative action for the generalised spin-2 case that allows for physical degrees of freedom to propagate between massless fields. Utilising this could enable new ways to write down covariant interactions between higher-spin particles, which in turn could potentially be compatible with the light-front formulation. We will not be looking into these higher-derivative actions for generalised fields, but we will try to lay as much groundwork as possible.

6.1 General mixed symmetry fields

This subsection is based on a publication by Campoleoni [9] and will mostly follow the conventions used therein. To set-up these new types of representations we can mostly follow the route we took in the first chapter 2. We simply take a look at the generic higher spin case and iterate upon it. In his work he did not use irreducible representations, as the additional restriction can be quite cumbersome to deal with. If we wanted to work with irreducible representations we would need to enforce restrictions of the type

$$
\begin{aligned}
\varphi_{(\mu_1\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2},\ldots)} &= 0,\\
\varphi_{(\mu_1\ldots\mu_{s_1},\nu_1)\nu_2\ldots\nu_{s_2},\ldots} &= 0,\\
\varphi_{(\mu_1\ldots\mu_{s_1}|,\nu_1\ldots\nu_{s_2},|\lambda_1)\ldots\lambda_{s_3},\ldots} &= 0,\\
\varphi_{\mu_1\ldots\mu_{s_1},(\nu_1\ldots\nu_{s_2},\lambda_1)\ldots\lambda_{s_3},\ldots} &= 0,\\
&\vdots
\end{aligned}
\tag{6.2}
$$

where all possible combinations of this type need to be considered. Therefore we will work with general representations in this section. However for our final result we shall take these into account.

That stated we can proceed by generalise our gauge transformation through the introduction of a separate and independent gauge parameter for each of the index families. The result is as follows

$$
\delta\varphi_{\mu_1\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2},\ldots} = \partial_{(\mu_1}\Lambda^{(1)}_{\mu_2\ldots\mu_{s_1}),\nu_1\ldots\nu_{s_2},\ldots} + \partial_{(\nu_1|}\Lambda^{(2)}_{\mu_1\ldots\mu_{s_1},|\nu_2\ldots\nu_{s_2}),\ldots}, \tag{6.3}
$$

where we want to emphasise that the indices in between two bars are exempt from the (anti-)symmetrization. As one might guess from this example bookkeeping all these possible indexes will be quit exhaustive with the notation that we are currently using, therefore we will be hard pressed to introduce additional notation to keep our equations compact. We will go about this by more or less leaving out individual space-time indices and shifting our focus towards the index families. We then apply these indices to denote which family a given operator should act upon. With a raised index we want to denote for example a fully symmetrized gradient

$$
\partial^i\varphi := \partial_{(\mu_1^i|}\varphi_{\ldots,|\mu_2^i\ldots\mu_{s_i+1}^i),\ldots}, \tag{6.4}
$$

while we want to denote a divergence by

$$
\partial_i\varphi := \partial^\lambda\varphi_{\ldots,\lambda\mu_1^i\ldots\mu_{s_i-1}^i,\ldots}. \tag{6.5}
$$

In short the position of the family index in the operator conveys if an index is supposed to be added, indicated by a raised index, or if an index is supposed to be removed via contraction, symbolised by the lower index. All raised index operation include a full symmetrization over the corresponding family, this is needed to treat all members of the addressed family on equal footing. This condensed notation allows us to denote the full gauge transformation of the field extremely compact

$$
\delta\varphi = \partial^i\Lambda_i. \tag{6.6}
$$

This form presents us also with the presence of gauge-for-gauge transformations for mixed-symmetry fields. We can then acknowledge that

$$\delta\Lambda_i = \partial^j \Lambda_{[ij]}, \tag{6.7}$$

is an operation that leaves φ invariant. As was discussed in the previous chapters it is important to also consider traces of these higher-symmetry fields therefore we introduce a trace operator

$$T_{ij}\varphi := \varphi_{\dots,\ \ \mu^i_1\dots\mu^i_{s_i-1},\dots,\lambda\mu^j_2\dots\mu^j_{s_j-1},\dots}{}^{\lambda}. \tag{6.8}$$

With the tools in hand we can write down the generalisation of the Fronsdal tensor, the Fronsdal-Labastida kinetic tensor

$$\mathcal{F}[\varphi] := \Box\varphi - \partial^i\partial_i\varphi + \frac{1}{2}\partial^i\partial^j T_{ij}\varphi, \tag{6.9}$$

which was verified to be invariant under the gauge transformations (6.6) if the constraints

$$T_{(ij}\Lambda_{k)} = 0, \tag{6.10}$$

are satisfied. The associated field equations is then

$$\mathcal{F} = 0. \tag{6.11}$$

A full and rigorous proof of this statement can be found in [2]. To check that this generalisation behaves properly we can take a look at what degrees of freedom they propagate. The generating functions u^μ come in handy for this

$$\varphi := \frac{1}{s_1!\dots s_N!} u^{1\mu_1}\dots u^{1\mu_{s_1}} u^{2\nu_1}\dots u^{2\nu_{s_2}}\dots\ \varphi_{\mu_1\dots\mu_{s_1},\nu_1\dots\nu_{s_2},\dots}\ . \tag{6.12}$$

Analysis gets a bit simpler in momentum space where (6.9) is given by

$$p^2\varphi - (p\cdot u^i)\left(p\cdot\frac{\partial}{\partial u^i}\right)\varphi + \frac{1}{2}(p\cdot u^i)(p\cdot u^j)T_{ij}\varphi = 0. \tag{6.13}$$

We can then introduce a decomposition for the field

$$\varphi = (p\cdot u^i)\xi_i + \hat{\varphi}, \tag{6.14}$$

to split off the $(p\cdot u^i)$ dependent part from φ proper

$$\frac{\partial}{\partial(p\cdot u^i)}\hat{\varphi} = 0. \tag{6.15}$$

When we insert this decomposition into the equations of motion (6.13) we realise something important

$$p^2\hat{\varphi} - (p\cdot u^i)\left(p\cdot\frac{\partial}{\partial u^i}\right)\hat{\varphi} + \frac{1}{2}(p\cdot u^i)(p\cdot u^j)T_{ij}\hat{\varphi} + \frac{1}{6}(p\cdot u^i)(p\cdot u^j)(p\cdot u^k)T_{(ij}\ \xi_{k)}, \tag{6.16}$$

as each of the terms carries a different potency of $(p\cdot u^i)$ they must all vanish independently giving rise to four different equations

$$p^2\hat{\varphi} = 0, \tag{6.17}$$

$$\left(p\cdot\frac{\partial}{\partial u^i}\right)\hat{\varphi} = 0, \tag{6.18}$$

$$T_{ij}\hat{\varphi} = 0, \tag{6.19}$$

$$T_{(ij}\xi_{k)} = 0. \tag{6.20}$$

This has two major implications. The first one being that ξ_k is pure gauge as it is subject to the constraint (6.10) that must be satisfied for all gauge parameters. The other three give us the second insight. The equation (6.17) leads us to the mass-shell condition $p^2 = 0$, thus we can switch into the light-front coordinate system (3.1) to obtain

$$\frac{\partial}{\partial u^{i-}}\hat{\varphi} = 0, \tag{6.21}$$

$$\frac{\partial}{\partial u^{i+}}\hat{\varphi} = 0, \tag{6.22}$$

from (6.15) and (6.18) respectively. $\hat{\varphi}$ contains therefore no components along the two light-front directions, while (6.19) enforces tracelessness for the transverse components. These properties point toward that the DoF propagated by (6.11) are representations of the little group $O(d-2)$. This is not all we can do before moving on to a proper Lagrange formulation. We recall that for standard Fronsdal fields a set of double traceless conditions (2.24) must hold in order to obtain an invariant Lagrangian. One can identify a similar set of constraints for the case with mixed symmetry, via an equivalent of the Bianchi identities (2.23)

$$\partial_i\mathcal{F} - \frac{1}{2}\partial^j T_{ij}\mathcal{F} + \frac{1}{12}\partial^j\partial^k\partial^l T_{(ij}T_{kl)}\varphi = 0. \tag{6.23}$$

If we want the last term to vanish independently of the first two we have to employ the following conditions

$$\partial^l T_{(ij}T_{kl)}\varphi = 0, \tag{6.24}$$

which serve as our generalisation of the double traceless condition. Here we observe a stark difference between Lagrangians that contain only one family of indices and the mixed-symmetry ones. Here and in (6.10) only certain linear combinations of the various traces have to vanish, not all possible double traces that the Fronsdal theory requires. This trait can then be expended analogously to the field-strength tensor involved in the Lagrangian

$$T_{(ij}T_{kl)}\mathcal{F} = 0. \tag{6.25}$$

Therefore this new mixed-symmetry Lagrangian can contain additional terms proportional to those non vanishing traces of $\mathcal{F}$. In the beginning we can classify the various possible traces of $\mathcal{F}$ as

$$T_{i_1j_1}\dots T_{i_pj_p}\mathcal{F}, \tag{6.26}$$

according to the corresponding irreducible representation. We can now consider that all tensors T_{ij} must behave identically and therefore we will always have a result containing an even number of family indices. To treat all of them on equal footing we have to consider that these indices are symmetrised in some fashion. At first we investigate the fully symmetric case. Combinations of four different indices should suffice as they are contained in all subsequent cases. Various forms are possible starting from $T_{(ij}T_{lk)}$ to more exotic ones like $T_{m(i}T_{j|n}T_{p|k}T_{l)q}$ or $T_{(ij}T_{k|m}T_{|l)n}$. All these possible combinations can be set to zero utilising the constraints (6.25). The second one may seem like it could have some surviving remnant, but it can be shown to be composed of the other combinations

$$T_{m(i}T_{j|n}T_{p|k}T_{l)q} = T_{(ij}T_{kl}T_{mn}T_{pq)} - T_{(ij}T_{lk)}T_{(mn}T_{pq)} - T_{(ij}T_{k|(m}T_{|l)n}T_{pq)}. \tag{6.27}$$

Therefore we have to exclude total symmetry. There is however another method, two-column projection. We can denote these as

$$\mathcal{F}^{[p]}_{i_1j_1,\dots,i_pj_p} := Y_{2^p}T_{i_1j_1}\dots T_{i_pj_p}\mathcal{F}, \tag{6.28}$$

where Y_{2^p} represents the Young projector onto the $\{2,\dots,2\}$ irreducible representation of the corresponding permutation group. These objects are symmetric under the exchange of pairs of family indices and therefore sufficient for our purpose. The name two-column stems from its connection to the Young diagram, a useful tool to write down representations of various symmetry groups.[8] If we only have one index pair, the result is simply the trace operator

$$\mathcal{F}^{[1]}_{ij} = T_{ij}. \tag{6.29}$$

For the case of two different pairs we have to work a bit. If we take a regular tensor with two pairs of indices $S_{i_1j_1,i_2j_2}$ it obeys the restriction $S_{(i_1j_1,i_2)j_2}$ stemming from its behaviour

[8]The application of Young diagrams is vast and varied, so this thesis can only utilise some glimpses of information these objects may provide. While the properties important for this thesis are outlined in 8.3, those who desire a deeper introduction may consult [7]

under total symmetrisation (6.3). This tensor is then represented by a Young-tableaux

$$\begin{array}{|c|c|}\hline i_1 & j_1 \\ \hline i_2 & j_2 \\ \hline \end{array}, \tag{6.30}$$

where we assign one box to each of the indices. With the two-column projection we now desire to project it onto a pair of tensors T_{ij} with a pair of indices each. For this we anti-symmetrise over the columns and symmetrise over the rows. This allows us to combine the indices in four different ways, while the projector must be proportional to said combination. For our tensor S this results in

$$\begin{aligned} Y_{2^2}\ S_{i_1j_1,i_2j_2} &= c\ (T_{i_1j_1}T_{i_2j_2} - T_{i_2j_1}T_{i_1j_2} + T_{i_2j_2}T_{i_1j_1} - T_{i_1i_2}T_{j_1j_2}) \\ &= c\ (2T_{i_1j_1}T_{i_2j_2} - T_{i_1(j_2}T_{i_2)j_1}), \end{aligned} \tag{6.31}$$

where the constant c has yet to be determined. As we are talking about a projection we can simply use the defining property $P^2 = P$ of projections to determine c

$$\begin{aligned} (Y_{2^2})^2 S_{i_1j_1,i_2j_2} =& c^2(2(2T_{i_1j_1}T_{i_2j_2} - T_{i_1(j_2}T_{i_2)j_1}) - (2T_{i_1j_2}T_{i_2j_1} - T_{i_1(j_1}T_{i_2)j_2}) \\ & - (2T_{i_1i_2}T_{j_1j_2} - T_{i_1(j_1}T_{j_2)j_2})) \\ =& c^2(6T_{i_1j_1}T_{i_2j_2} - 3T_{i_1(j_2}T_{i_2)j_1}) \\ \overset{!}{=}& c(2T_{i_1j_1}T_{i_2j_2} - T_{i_1(j_2}T_{i_2)j_1}), \end{aligned}$$

where we deduce that $c = 1/3$. Therefore the two column projection for our traces is equal to

$$\mathcal{F}^{[2]}_{i_1j_1,i_2j_2} = \frac{1}{3}(2\ T_{i_1j_1}T_{i_2j_2} - T_{i_1(i_2}T_{j_2)j_1})\mathcal{F}. \tag{6.32}$$

All other projections are constructable via this algorithm, but these two suffice for the purposes of this thesis. At last we have to think of operators that can be used to contract the projections (6.28) in a natural fashion to create a Lorentz invariant Lagrange containing these traces. Such a natural partner would be the associated Hermitian transpose

$$\langle\varphi, T\psi\rangle = 2\,\langle\eta\varphi, \psi\rangle\,, \tag{6.33}$$

where the factor 2 is purely conventional. The following operators will enable us to do just that

$$\eta^{ii}\varphi = \eta_{(\mu_1^i\mu_2^i|}\varphi_{\dots,|\mu_3^i\dots\mu_{s_i+2}^i),\dots}, \tag{6.34}$$

$$\eta^{ij}\varphi = \frac{1}{2}\sum_{n=1}^{s_i+1}\eta_{\mu_n^i(\mu_1^j|}\varphi_{\dots,\dots\mu_{n-1}^i\mu_{n+1}^i\dots,\dots,|\mu_2^j\dots\mu_{s_j+1}^j),\dots}, \qquad i \neq j, \tag{6.35}$$

where the factor 1/2 stems from the convention mentioned in the scalar product above. Following the conventional literature we see that the ansatz

$$\mathcal{L} = \frac{1}{2}\left\langle \varphi, \sum_{p=0}^{N} k_p \eta_{i_1 j_1} \dots \eta_{i_p j_p} \mathcal{F}^{[p]}_{i_1 j_1, \dots, i_p j_p} \right\rangle, \tag{6.36}$$

for the Lagrangian with the scalar product

$$\langle \varphi, \psi \rangle := \frac{1}{s_1! \dots s_n!} \varphi_{\mu_1^1 \dots \mu_{s_1}^1, \dots, \mu_1^n \dots \mu_{s_n}^n} \psi^{\mu_1^1 \dots \mu_{s_1}^1, \dots, \mu_1^n \dots \mu_{s_n}^n}, \tag{6.37}$$

will bear fruit. N represents the number of index families, it is not possible to introduce more parts into the sum as that would require 2^{N+1} projectors which is always zero due to the system constraints. To fix the factor k_p we can look at the gauge variation of (6.36)

$$\begin{aligned}\delta\mathcal{L} = -\sum_{p=0}^{N} \frac{1}{2^{p+1}} \Big\langle\ T_{i_1 j_i} \dots T_{i_p j_p} \Lambda_k\ ,\ k_p\ \partial_k \mathcal{F}^{[p]}_{i_1 j_1, \dots, i_p j_p} \\ +\ (p+1) k_{p+1}\ \partial^l \mathcal{F}^{[p+1]}_{i_1 j_1, \dots, i_p j_p, kl}\ \Big\rangle.\end{aligned} \tag{6.38}$$

To proceed we employ the same trick as we did for the for the trace components. The Labastida constraint (6.10) allows only certain combinations of traces of the gauge parameter to be different from zero. Therefore only a small selection of projections contribute

$$\begin{aligned}\delta\mathcal{L} = -\sum_{p=0}^{N} \frac{1}{2^{p+1}} \Big\langle\ Y_{\{2^p,1\}}\ T_{i_1 j_i} \dots T_{i_p j_p} \Lambda_k\ ,\ k_p\ Y_{\{2^p,1\}}\ \partial_k \mathcal{F}^{[p]}_{i_1 j_1, \dots, i_p j_p} \\ +\ (p+1) k_{p+1}\ \partial^l \mathcal{F}^{[p+1]}_{i_1 j_1, \dots, i_p j_p, kl}\ \Big\rangle.\end{aligned} \tag{6.39}$$

The additional divergences contained in the variation cause the shift from $Y_{\{2^p\}}$ to $Y_{\{2^p,1\}}$ projectors, the gradient term does not require it, as its structure is already $\{2^p, 1\}$ projected by default. For $p = 0$ the right hand side takes on the form

$$k_0 \partial_k \mathcal{F} + k_1 \partial^l T_{kl} \mathcal{F}. \tag{6.40}$$

Fixing now $k_0 = 1$ and $k_1 = -\frac{1}{2}$ this expression is identical to the Bianchi identities (6.23) with the double traces set to zero. The terms vanish therefore without the introduction of further structures. The factors after can be fixed in a similar manner, by comparing them to suitable traces of these identities

$$\partial_k T_{i_1 j_1} \dots T_{i_p j_p} \mathcal{F} - \frac{1}{2} \sum_{n=1}^{p} \partial_{(i_n} T_{j_n)k} \prod_{r \neq n}^{p} T_{i_r j_r} \mathcal{F} - \frac{1}{2} \partial^l T_{i_1 j_1} \dots T_{i_p j_p} T_{kl} \mathcal{F} = 0, \tag{6.41}$$

while taking $\{2^p, 1\}$ projections into account. The result is then

$$k_{p+1} = -\frac{k_p}{(p+1)(p+2)}, \tag{6.42}$$

where $k_0 = 1$ leads us to

$$k_p = \frac{(-1)^p}{p!\,(p+1)!}, \tag{6.43}$$

as the form of these unknown factors. Therefore we arrive at the final form for our Bose mixed-symmetry Lagrangian

$$\mathcal{L} = \frac{1}{2}\left\langle \varphi \ , \ \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!}\ \eta_{i_1 j_1} \dots \eta_{i_p j_p} \mathcal{F}^{[p]}_{i_1 j_1, \dots, i_p j_p} \right\rangle . \tag{6.44}$$

6.1.1 The Symmetric vs The Anti-Symmetric Convention

A question we have yet to answer is the one of what spin is for these new fields. Before, when we only had one index family, the answer to this question was simply given by the amount of available space-time indices. With Young diagrams we can generalise this approach relativity easily. For a regular spin s boson $\varphi_{\mu_1,\mu_2,\dots,\mu_s}$we have the tableaux

$$\begin{array}{|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 \\ \hline \end{array} \dots \begin{array}{|c|}\hline \mu_s \\ \hline \end{array}, \tag{6.45}$$

where the indices where again assigned to one box each. Continuing this train of thought we may identify the number of columns for such a diagram with the spin number. This answer gives us only a partial answer, as we do not know how these indices would be distributed across the various families. The distribution depends heavily on the convention one chooses at the beginning. If we decide that our field is symmetric in its indices, the columns are the basis of the Young diagram and represent a family each. For example

$$\varphi_{\mu_1\mu_2\mu_3,\ \nu_1\nu_2,\ \rho} \quad \backsim \quad \begin{array}{|c|c|c|}\hline \mu_1 & \mu_2 & \mu_3 \\ \hline \nu_1 & \nu_2 & \multicolumn{1}{c}{} \\ \cline{1-2} \rho & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \ , \tag{6.46}$$

would be a field with three families and of spin three due to its three total columns. As the number of columns in each row has to be lower or equal to the number present in the previous one[9], it is safe to identify the first. If we chose the anti-symmetric convention, the

[9]As all things in physics this is also up to a convention. For this thesis the English notation will be used and the statement holds. However in Francophone countries a different convention, the French notation, was developed, wherein the diagram is read from bottom to top and therefore the number of columns for each row would always be higher or equal to the one before.

rows would be manifest as index families and our field would look slightly different

$$\varphi_{\mu_1\nu_1\rho,\ \mu_2\nu_2,\ \mu_3} \quad \backsim \quad \begin{array}{|c|c|c|} \hline \mu_1 & \mu_2 & \mu_3 \\ \hline \nu_1 & \nu_2 & \multicolumn{1}{c}{} \\ \cline{1-2} \rho & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\ , \tag{6.47}$$

as the ßpin caringïndices are now distributed evenly across the various families. Both contain the same amount of physical information so it is more of a situational choice, as depending on the objective one or the other may be more convenient. To switch between these different notations one need to select suitable (anti-)symmetrisations over the indices of the tensor in question. For simplicity we will consider two tensors $\varphi^{(A)}_{\mu_1\mu_2,\nu}$ and $\varphi^{(S)}_{\mu_1,\mu_2\nu}$ that fulfil the same roll in the anti- and symmetric convention respectively. We can then express these objects through each other to switch between conventions

$$\varphi^{(S)}_{\mu_1,\mu_2\nu} = c_S(\varphi^{(A)}_{\mu_1\mu_2,\nu} + \varphi^{(A)}_{\mu_1\nu,\mu_2}), \tag{6.48}$$

$$\varphi^{(A)}_{\mu_1\mu_2,\nu} = c_A(\varphi^{(S)}_{\mu_1,\mu_2\nu} - \varphi^{(S)}_{\mu_2,\mu_1\nu}), \tag{6.49}$$

where the switch of the index μ_2 is warranted by the difference in the classification of index families in each respective convention. The numbers c_A and c_S are normalisation factors to be determined given the tensors involved. The process gets more involved with raising number of space-time indices and families, but the general idea is the same.

6.2 The Hook field

The Hook field is the simplest mixed-symmetry field and the generalisation of the (anti-)symmetric spin-2 field. It serves as our starting point for our discussion of the higher-derivative action and thus we will review its properties explicitly. A word of caution before we start with the investigation. So far we have worked in the symmetric convention, but for this and the following section we will work in the anti-symmetric convention, to keep with the notation of the primary material. The final results however will be translated to the symmetric approach.

This object has been studied in detail for the first time by Curtright in this paper [11]. Our discussion will center around the information provided in [15] as we want to work towards the Weyl-Lagrangian. In index notation the field takes the form

$$\varphi_{\mu_1\mu_2,\nu}, \tag{6.50}$$

with the symmetries

$$\varphi_{\mu_1\mu_2,\nu} = -\varphi_{\mu_2\mu_1,\nu}, \qquad \varphi_{\mu_1\mu_2,\nu} + \varphi_{\mu_2\nu,\mu_1} + \varphi_{\nu\mu_1,\mu_2} = 0, \tag{6.51}$$

where the second one restricts us to irreducible representations. It should be noted that we have established no (anti-)symmetry between the different families and we will leave this general and not restrict ourselves.
To better analyse some properties of these new representations we will make use of Young tableaux. Young diagrams are helpful tools for representation theory, as they fix how a field transforms under index permutation and how it Lorentz transforms. We will mainly use it to work out the physical degrees of freedom of a given representation. For the Hook field we have the following tableau:

$$\begin{array}{|c|c|}\hline \mu_1 & \nu \\ \hline \mu_2 & \\ \cline{1-1}\end{array}\ . \tag{6.52}$$

The action for this object can be cast into a form resembling general relativity

$$S_E[\varphi] = \langle \varphi | G \varphi \rangle \, . \tag{6.53}$$

The corresponding scalar product is defined as

$$\langle A|B\rangle = \frac{1}{m!\, n!} \int \mathrm{d}^d x \sqrt{|g|} A^{\mu_1 \dots \mu_m, \nu_1 \dots \nu_n}(x) B_{\mu_1 \dots \mu_m, \nu_1 \dots \nu_n}(x). \tag{6.54}$$

G acts as a generalisation of the Einstein tensor from GR and can be defined analogously through a Ricci like tensor

$$G[\varphi]_{\mu_1\mu_2,\nu} = F[\varphi]_{\mu_1\mu_2,\nu} - \eta_{\nu[\mu_1} F[\varphi]_{\mu_2]\lambda,}{}^{\lambda} \tag{6.55}$$

with

$$F[\varphi]_{\mu_1\mu_2,\nu} = \partial^2 \varphi_{\mu_1\mu_2,\nu} + 2\partial_{[\mu_1}\partial^{\lambda}\varphi_{\mu_2]\lambda,\nu} - \partial_\nu \partial^\lambda \varphi_{\mu_1\mu_2,\lambda} - 2\partial_\nu \partial_{[\mu_1}\varphi_{\mu_2]\lambda,}{}^{\lambda}. \tag{6.56}$$

We can simplify this by making use of a generalised Kronecker delta

$$\delta^{\rho\lambda\gamma\delta}_{\mu\nu\alpha\beta} = 4!\, \delta^{\rho}_{[\mu}\delta^{\lambda}_{\nu}\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta]} \tag{6.57}$$

leading to

$$G[\varphi]_{\mu_1\mu_2,}{}^{\nu} = -\frac{1}{2}\delta^{\nu\lambda\gamma\delta}_{\mu_1\mu_2\alpha\beta}\partial^\alpha \partial_\lambda \varphi_{\gamma\delta,}{}^{\beta}. \tag{6.58}$$

As in the previous chapter each index family carries its own gauge parameter

$$\epsilon_{\mu\nu} = \epsilon_{\nu\mu} \qquad\qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \tag{6.59}$$

resulting in two different gauge symmetries for the system

$$\delta_\epsilon \varphi_{\mu_1\mu_2,\nu} = 2\partial_{[\mu_1}\epsilon_{\mu_2]\nu} \qquad \delta_\theta \varphi_{\mu_1\mu_2,\nu} = \partial_\nu \theta_{\mu_1\mu_2} - \partial_{[\mu_1}\epsilon_{\mu_2]\nu}. \tag{6.60}$$

These two symmetries however are not irreducible and admit a gauge-for-gauge symmetry

$$\epsilon_{\mu\nu}(\zeta) = \partial_{(\mu}\zeta_{\nu)} \qquad \theta_{\mu\nu}(\zeta) = \partial_{[\mu}\zeta_{\nu]}. \tag{6.61}$$

At the end we want to count the degrees of freedom of this representation. This can be achieved by utilising covariant counting, a process the is explained in detail in appendix B 8.2. For our Hook field all involved quantities are of order 1 therefore

$$S_E \quad : \quad \ydiagram{2,1}_{GL(4)} \ominus 2\left(\ydiagram{2}_{GL(4)} \oplus \ydiagram{1,1}_{GL(4)}\right) \oplus 2\,\ydiagram{1}_{GL(4)} \oplus \ydiagram{1}_{GL(4)}, \tag{6.62}$$

where the additional one box diagram stems from the trace free condition 6.10 on the gauge parameters. The number of dimensions of a representation associated with GL(d) is equivalent to the number of equations and this follows immediately from its Young diagrams[10]

$$\mathrm{DoF}[S_E] = \frac{5\cdot 4\cdot 3}{3} - 2\left(\frac{4\cdot 3}{2} + \frac{5\cdot 4}{2}\right) + 3\cdot 4 = 0 \tag{6.63}$$

resulting in no accessible physical degrees of freedom as we expected. For the later parts of this thesis it should be noted that this is not the only counting available to us. This method requires us to take the tracelessness of the participants into account separately. For more complex fields we have to include the vanishing of double-traces (6.25) as well, making this approach more unwieldy. We can instead use Hamilton analysis[11], where we us the representations of SO(3) instead of GL(4). For example the dimension of a spin-2 field in SO(3) is given by the number of components a traceless, symmetric 3×3 matrix possesses, 5. For the Hook field this analysis is equivalent to

$$\ydiagram{2,1}_{SO(3)} \ominus \left(\ydiagram{2}_{SO(3)} \oplus \ydiagram{1,1}_{SO(3)}\right) \oplus \ydiagram{1}_{SO(3)}, \tag{6.64}$$

resulting in zero

$$5 - (5+3) + 3 = 0. \tag{6.65}$$

[10] We discuss this in Appendix C

[11] It may not be as known in this form, as it is usually discussed in the context of primary and secondary constraints of the system

6.2.1 The Symmetric Hook

As discussed in (6.1.1) it is possible to switch between the symmetric and anti-symmetric convention rather easily and without loss of information. This validates the results for the higher derivative action in the symmetric convention and assures us its existence. The only thing we need to do is to transcribe the Lagrangian to the symmetric convention, which we can achieve by transcribing G. Taking a look at (6.49) reveals us the symmetric counterpart

$$G^{(S)}_{\mu_1,\mu_2\nu} = \frac{1}{2}(G^{(A)}_{\mu_1\mu_2,\nu} + G^{(A)}_{\mu_1\nu,\mu_2}) \tag{6.66}$$

$$= \frac{1}{2}(F^{(A)}_{\mu_1\mu_2,\nu} - \eta_{\nu[\mu_1}F^{(A)}_{\mu_2]\lambda,}{}^{\lambda} + F^{(A)}_{\mu_1\nu,\mu_2} - \eta_{\mu_2[\mu_1}F^{(A)}_{\nu]\lambda,}{}^{\lambda}), \tag{6.67}$$

where the factor 1/2 stems from a consistency check. To proceed we simply have to express F^A through $F^{(S)}$:

$$F^{(A)}_{\mu_1\mu_2,\nu} = c(F^{(S)}_{\mu_1,\mu_2\nu} - F^{(S)}_{\mu_2,\mu_1\nu}) = \frac{c}{2}(F^{(A)}_{\mu_1\mu_2,\nu} + F^{(A)}_{\mu_1\nu,\mu_2} - F^{(A)}_{\mu_2\mu_1,\nu} - F^{(A)}_{\mu_2\nu,\mu_1})$$

$$= \frac{c}{2}3F^{(A)}_{\mu_1\mu_2,\nu}, \tag{6.68}$$

where the execution of the last step relied on the annihilation of $F^{(A)}$ under total anti-symmetry. Therefore we can conclude that $c = 2/3$. With this we may execute the transformation

$$\begin{aligned} G^{(S)}_{\mu_1,\mu_2\nu} = \frac{1}{2}\frac{2}{3}(&F^{(S)}_{\mu_1,\mu_2\nu} - F^{(S)}_{\mu_2,\mu_2\nu} + F^{(S)}_{\mu_1,\nu\mu_2} - F^{(S)}_{\nu,\mu_1\mu_2} \\ &- \eta_{\nu[\mu_1}F^{(S)}_{\mu_2],\lambda}{}^{\lambda} + \eta_{\nu[\mu_1|}F^{(S)\lambda}{}_{,|\mu_2]\lambda} - \eta_{\mu_2[\mu_1}F^{(S)}_{\nu],\lambda}{}^{\lambda} + \eta_{\mu_2[\mu_1|}F^{(S)\lambda}{}_{,|\nu]\lambda}) \\ = \frac{1}{3}(&3F^{(S)}_{\mu_1,\mu_2\nu} - 3\eta_{\nu\mu_2}F^{(S)}_{\mu_1,\lambda}{}^{\lambda} - 3\eta_{\mu_1(\nu|}F^{(S)\lambda}{}_{,|\mu_2)\lambda}), \end{aligned} \tag{6.69}$$

and arrive at the result

$$G^{(S)}_{\mu_1,\mu_2\nu} = F^{(S)}_{\mu_1,\mu_2\nu} - \eta_{\nu\mu_2}F^{(S)}_{\mu_1,\lambda}{}^{\lambda} - \eta_{\mu_1(\nu|}F^{(S)\lambda}{}_{,|\mu_2)\lambda}. \tag{6.70}$$

6.3 The Higher-Derivative Action of the Hook field

If we desire an alternative description of our theory, it helps to change some basic conditions and observe any changes that may occur. For our higher-derivative action this happens to be the move from a flat to a curved background. For starts we will make use of the following convention

$$[\nabla_\mu, \nabla_\nu]V^\rho_\lambda = R_{\mu\nu,\lambda}{}^{\sigma}V^\rho_\sigma - R_{\mu\nu,\sigma}{}^{\rho}V^\sigma_\lambda, \tag{6.71}$$

$$R_{\mu\nu,\lambda}{}^{\sigma} = \frac{2\Lambda}{(d-1)(d-2)}(g_{\mu\lambda}\delta^\rho_\nu - g_{\nu\lambda}\delta^\rho_\mu), \tag{6.72}$$

where Λ is the cosmological constant and $g_{\mu\nu}$ is our (A)dS metric. We can begin setting our theory in a curved background by replacing partial derivatives ∂_μ with covariant ones ∇_μ in the action (6.53) and the gauge symmetries

$$\delta^{\Lambda}_{\epsilon}\varphi_{\mu_1\mu_2,\nu} = 2\nabla_{[\mu_1}\epsilon_{\mu_2]\nu} \qquad\qquad \delta^{\Lambda}_{\theta}\varphi_{\mu_1\mu_2,\nu} = \nabla_{\nu}\theta_{\mu_1\mu_2} - \nabla_{[\mu_1}\epsilon_{\mu_2]\nu}. \tag{6.73}$$

There is however a catch to this naive approach; these new covariant derivatives are not commuting with each other. If they do not commute, a ambiguous mass-like term is introduced into the Lagrangian that is dependent on the commutator. This term can be fixed by enforcing the invariance of the action under the available gauge transformations. During this procedure we may notice that the two symmetries require different conditions to fix the mass-like term. A Lagrangian that sustains both flat-space symmetries is therefore not possible. If we were to pick the gauge transformation $\delta^{\Lambda}_{\epsilon}$ we get

$$G^{\Lambda}_{S}[\varphi]_{\mu_1\mu_2}{}^{\nu} = \frac{1}{2}\delta^{\nu\lambda\gamma\delta}_{\mu_1\mu_2\alpha\beta}\nabla^{\alpha}\nabla_{\lambda}\varphi_{\gamma\delta,}{}^{\beta}. \tag{6.74}$$

With the subscript S we denote that this Einstein tensor is gauge invariant with respect to the transformation induced by the symmetric gauge parameter. There is also a gauge-for-gauge transformation present

$$\epsilon_{\mu\nu} = \Big(\nabla_{(\mu}\nabla_{\nu)} + \frac{2\Lambda}{(d-1)(d-2)}g_{\mu\nu}\Big)\zeta. \tag{6.75}$$

If choose to keep the other gauge transformation $\delta^{\Lambda}_{\theta}$ instead, we obtain a mass-like term for the Einstein tensor

$$G^{\Lambda}_{A} = G^{\Lambda}_{S} - m^2_{\Lambda}\mathcal{I}, \tag{6.76}$$

where we associate A with the object invariant under antisymmetric gauge transformation governed by the parameter $\theta_{\mu\nu}$. The mass-squared parameter is dependent on the cosmological constant

$$m^2_{\Lambda} = -\frac{4(d-3)}{(d-2)(d-1)}\Lambda. \tag{6.77}$$

The additional mass-term operator is given by

$$\mathcal{I}[\varphi]_{\mu_1\mu_2,}{}^{\nu} = \frac{1}{2}\delta^{\nu\alpha\beta}_{\mu_1\mu_2\gamma}\phi_{\alpha\beta,}{}^{\gamma}. \tag{6.78}$$

An interesting observation is that, as the mass-like term is dependent on the cosmological constant, that in AdS only G^{Λ}_{A} is unitary while G^{Λ}_{S} is only unitary in dS. The respective actions are

$$S^{\Lambda}_{E,S}[\varphi] = \left\langle\varphi\middle|G^{\Lambda}_{S}\varphi\right\rangle, \qquad S^{\Lambda}_{E,A}[\varphi] = \left\langle\varphi\middle|G^{\Lambda}_{A}\varphi\right\rangle. \tag{6.79}$$

Let us count the DoF of these two systems. For this we again subtract the number of gauge-identities times their order from the number of equations of motion before adding the number of gauge-for-gauge transformations times their order back in

$$S^{\Lambda}_{E,S} \quad : \quad \ydiagram{2,1}_{GL(d)} \ominus \left(2\, \ydiagram{2}_{GL(d)} \oplus \ydiagram{1,1}_{GL(d)} \right) \oplus 3 \bullet \oplus \ydiagram{1}_{GL(d)}, \tag{6.80}$$

$$S^{\Lambda}_{E,A} \quad : \quad \ydiagram{2,1}_{GL(d)} \ominus \left(\ydiagram{2}_{GL(d)} \oplus 2\, \ydiagram{1,1}_{GL(d)} \right) \oplus \ydiagram{1}_{GL(d)}. \tag{6.81}$$

We can decompose this further to the DoF present in the light-front, where we have $SO(d-2)$ as our group

$$S^{\Lambda}_{E,S} \quad : \quad \ydiagram{2,1}_{SO(d-2)} \oplus \ydiagram{1,1}_{SO(d-2)}, \tag{6.82}$$

$$S^{\Lambda}_{E,A} \quad : \quad \ydiagram{2,1}_{SO(d-2)} \oplus \ydiagram{2}_{SO(d-2)}. \tag{6.83}$$

Compared to the situation we have in flat space, the hook actions describe additional degrees of freedom, corresponding to the light-front propagation of a two-form field $\ydiagram{1,1}$ and a mass-less spin-two field $\ydiagram{2}$ respectively. In our four dimensional case however the hook modes vanish and we are left with ordinary mass-less spin-two or two-form fields.

6.3.1 The Weyl Action

The aim of the game is now to find an action that could sustain both gauge symmetries. As we have seen before this is not possible with an action containing a regular amount of derivatives. There is however a four-derivative action that manages to be invariant under both symmetries in a relatively undemanding form

$$S^{\Lambda}_{W}[\varphi] = -\left\langle \varphi \middle| G^{\Lambda}_{S} \mathcal{I}^{-1} G^{\Lambda}_{A} \varphi \right\rangle, \tag{6.84}$$

where

$$\mathcal{I}^{-1}[\varphi]_{\mu_1\mu_2}{}^{\nu} = \varphi_{\mu_1\mu_2}{}^{\nu} - \frac{2}{d-2} \delta^{\nu}_{[\mu_2} \varphi_{\mu_1]\alpha,}{}^{\alpha}. \tag{6.85}$$

We use the subscript W to indicate, that this action also admits a Weyl symmetry

$$\delta^{\Lambda}_{\alpha} \varphi_{\mu_1\mu_2,\nu} = g_{\nu[\mu_1} \alpha_{\mu_2]}. \tag{6.86}$$

To see this we have to utilise that any linear combination of gauge symmetries is itself a gauge symmetry. If we set

$$\epsilon_{\mu\nu}(\alpha) = \nabla_{(\mu}\alpha_{\nu)}, \qquad \theta_{\mu\nu}(\alpha) = -\frac{1}{3}\nabla_{[\mu}\alpha_{\nu]}, \tag{6.87}$$

the linear combination will result in the Weyl transformation

$$(\delta_{\epsilon(\alpha)} + \delta_{\theta(\alpha)})\varphi_{\mu_1\mu_2,\nu} = \frac{\Lambda}{(d-1)(d-2)} g_{\nu[\mu_1}\alpha_{\mu_2]}. \tag{6.88}$$

This is striking similar to the gauge-for-gauge transformation (6.61) and one can think of it as the (A)dS remnants of that symmetry.
In the flat limit we obtain

$$S_W^{\text{flat}}[\varphi] = -\left\langle \varphi \middle| G\mathcal{I}^{-1}G\varphi \right\rangle, \tag{6.89}$$

with the flat space Weyl transformation

$$\delta_\alpha \varphi_{\mu_1\mu_2,\nu} = \eta_{\nu[\mu_1}\alpha_{\mu_2]}. \tag{6.90}$$

This result is analogous to the Weyl action for gravity, which is uniquely fixed by Weyl transformations and diffeomorphisms. We will call S_W^{flat} the Weyl action from here on out.

6.3.2 DoF of the Weyl Action

As the Weyl action contains more than two derivatives, our usual methods fall a bit short. One can however rewrite it as a regular two derivative action at the price of having to introduce an auxiliary field $f_{\mu_1\mu_2,\nu}$

$$S_W^\Lambda[\phi] \backsimeq m_\Lambda^2(\left\langle \varphi \middle| G_S^\Lambda \phi \right\rangle + 2\left\langle \varphi \middle| G_S^\Lambda f \right\rangle + m_\Lambda^2 \left\langle f \middle| \mathcal{I} f \right\rangle). \tag{6.91}$$

The field redefinition

$$\varphi_{\mu_1\mu_2,\nu} = h_{\mu_1\mu_2,\nu} - f_{\mu_1\mu_2,\nu}, \tag{6.92}$$

allows us to diagonalise the action

$$S_W^\Lambda[\varphi] \backsimeq m_\Lambda^2(S_{E,S}^\Lambda[h] - S_{E,A}^\Lambda[f]). \tag{6.93}$$

Both of these rewriting procedures are singular in the flat limit, while the flat limit for regular Weyl action (6.84) is well defined and both contain the same number of degrees of freedom. If we analyse the DoF of S_W^Λ in terms of helicity modes the result is, at least schematically

$$m_\Lambda^2\left(\ydiagram{2,1}_{\,h}+\Lambda\,\ydiagram{1,1}_{\,h}\right)-m_\Lambda^2\left(\ydiagram{2,1}_{\,f}+\Lambda\,\ydiagram{2}_{\,f}\right). \tag{6.94}$$

We see that each of the hook fields is accompanied by a two mode. The h-field is unitary in AdS and non-unitary in dS while for the f field the opposite holds. Taking the signs of the factor m_Λ^2 into account both supplementary fields are non-unitary in both cases. This leads us to the conclusion that the theory itself is non-unitary for dimensions $d \geq 5$ where hook modes propagate. In four dimensions where hook modes do not propagate the theory is described by a spin-two field and a pseudo scalar $\ydiagram{1,1}_{\,h} \approx \bullet_h$ where none of them are unitary. This however can be fixed by introducing an additional sign into the original action. The result is therefore a description of a unitary system composed of a scalar and a spin-two field in four dimensions via the Weyl action. As the DoF for any value of Λ, and therefore including flat space, are the same, we have thus constructed a flat space action that contains non trivial interactions between two massless particles in flat space.

6.4 The Generalised Spin-3 Case

Through the results we have gathered over the past chapters, we can now start to write down an action for mixed symmetry spin-3 fields and analyse the respective degrees of freedom. Our regular fully symmetric formulation of spin-3 can be generalised in two distinct ways illustrated by the Young diagrams

$$\varphi_{\mu_1\mu_2\mu_3,\nu} \backsim \ydiagram{3,1}, \tag{6.95}$$

$$\varphi_{\mu_1\mu_2\mu_3,\nu_1\nu_2} \backsim \ydiagram{3,2}. \tag{6.96}$$

We will be discussing both variants in this section to be thorough, even though the second option is looking much more promising. Should it behave like the Hook case under the higher-derivative action, it alone could lead to a consistent exchange of degrees of freedom between spin-3 and spin-2. The problematic spin-(3,3,2) vertex we analysed could then be given, with the inconsistency eventually resolved, by an interaction between a regular Frondsdal spin-3 field with the mixed-symmetry field $\ydiagram{3,2}$. But so far this is pure speculation. We can proceed by utilising what we learned so far by writing down the respective Lagrangian's and Gauge symmetries of these two objects.

6.4.1 The Lagrangian

As we deal with no more than two index families, the Lagrangian (6.44) simplifies significantly

$$\mathcal{L} = \frac{1}{2}\left\langle \varphi, \left(1 - \frac{1}{2}\eta^{ij}T_{ij} - \frac{1}{36}\eta^{i_1j_1}\eta^{i_2j_2}\left(2T_{i_1j_1}T_{i_2j_2} - T_{i_1(i_2}T_{j_2)j_1}\right)\right)\mathcal{F}\right\rangle. \tag{6.97}$$

At first we can write down the expressions for the traces (6.8)

$$T_{\mu\mu}\varphi = \phi^{\lambda}{}_{\lambda\mu_3\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2}}, \tag{6.98}$$

$$T_{\mu\nu}\varphi = \phi^{\lambda}{}_{\mu_2\ldots\mu_{s_1},\lambda\nu_2\ldots\nu_{s_2}}, \tag{6.99}$$

and its counterpart (6.35),

$$\eta^{\mu\mu} = \eta_{(\mu_1\mu_2}\phi_{\mu_3\ldots\mu_{s_1+2}),\nu_1\ldots\nu_{s_2}}, \tag{6.100}$$

$$\eta^{\mu\nu} = \frac{1}{2}\sum_{n=1}^{s_1+1}\eta_{\mu_n(\nu_1|}\varphi_{\ldots,|\nu_2\ldots\nu_{s_2})}. \tag{6.101}$$

Furthermore only the first two Young projections (6.29) and (6.32) are relevant. Higher orders do not contribute, as we would anti-symmetrise over more index families than the field itself posses. While most components of the Lagrangian are straightforward, $\mathcal{F}^{[2]}$ is not. We have to remember that only some combinations of (6.25) traces are not zero and with only two index families to choose from the scope of this projector gets reduced significantly. We have to distinguish between three different cases

- If all act on the same family: $\mathcal{F}^{[2]}_{\mu\mu,\mu\mu} = 0$,
- If only one family index is different $\mathcal{F}^{[2]}_{\mu\mu,\mu\nu} = 2T_{\mu\mu}T_{\mu\nu} - T_{\mu\nu}T_{\mu\mu} - T_{\nu\mu}T_{\mu\mu} = 0$,
- If the indices are evenly distributed we have the only non zero contribution.

Taking the contractions with the η's into account $\mathcal{F}^{[2]}$ reduces to

$$\eta^{i_1j_1}\eta^{i_2j_2}\mathcal{F}^{[2]}_{i_1j_1,i_2j_2} = 2\left((\eta^{\mu\nu}\eta^{\mu\nu} - \eta^{\mu\mu}\eta^{\nu\nu})T_{\mu\nu}T_{\mu\nu} + (\eta^{\mu\mu}\eta^{\nu\nu} - \eta^{\mu\nu}\eta^{\mu\nu})T_{\mu\mu}T_{\nu\nu}\right)\mathcal{F}. \tag{6.102}$$

As an example we can derive the Lagrangian for the hook field as we have a solution (6.70) to compare against. We can compute the first quantity outright

$$\eta^{\mu\mu}T_{\mu\mu}\mathcal{F} = \eta^{\mu\mu}\mathcal{F}^{\lambda}{}_{\lambda,\nu} = \eta_{\mu_1\mu_2}\mathcal{F}^{\lambda}{}_{\lambda,\nu}. \tag{6.103}$$

For the trace over the second family we have to work a bit more. We could argue that the contribution vanishes outright, as the trace demands more indices than the respective index family contains. To be through however we will work the term out properly by utilising the commutators between T^{ii} and η_{ii}

$$[T_{ii}, \eta^{ii}]h = (d + s_i)h, \tag{6.104}$$

where s_i is the number of family members of the family i. We will perform this calculation once and then omit the argument from here on out. Knowing this we can perform the computation

$$\begin{aligned}\eta^{\nu\nu}T_{\nu\nu}\mathcal{F} &= (T_{\nu\nu}\eta^{\nu\nu} - (d+2))\mathcal{F} \\ &= T_{\nu\nu}\eta_{(\nu_1\nu_2|}\mathcal{F}_{\mu_1\mu_2,|\nu_3)} - (d+2)\mathcal{F} \\ &= 2\eta_{\nu}{}^{\lambda}\mathcal{F}_{\mu_1\mu_2,\lambda} + \eta_{\lambda}{}^{\lambda}\mathcal{F}_{\mu_1\mu_2,\nu} - (d+2)\mathcal{F} = 0,\end{aligned} \tag{6.105}$$

and observe that when we try to contract over more space-time indices than accessible the term vanishes. By this logic the hook field will also receive no contribution from $\mathcal{F}^{[2]}$. This leaves one final term

$$\eta^{\mu\nu}T_{\mu\nu}\mathcal{F} = \eta^{\mu\nu}\mathcal{F}_{\mu_1\lambda}{}^{\lambda} = \frac{1}{2}\sum_{n=1}^{2}\eta_{\nu(\mu_1}\mathcal{F}_{\mu_2)} = \frac{1}{2}\eta_{\nu(\mu_1}\mathcal{F}_{\mu_2)\lambda,}{}^{\lambda}, \tag{6.106}$$

where we have to keep in mind that this term appears twice in the sum. When computing this quantity, we need to keep two important things in mind, that unveil some problems with this type of highly compactified notation

1. As we use the convention (2.10), we omit all combinatoric factors that may occur when calculating (anti-)symmetrisation brackets. For these types of sums that occur in the η's it is therefore implied to write down each different combination once and not count how many times such a factor occurs.

2. When a trace operator T_{ij} acts on a field before η_{kl} does, one has to pay attention to the following: Acting with T_{ij} on a tensor causes it to lose two of its indices through contraction, rendering these inaccessible for the symmetrisation to follow.

Put together the result is

$$\mathcal{L}_{Hook} = \frac{1}{2}\left\langle \varphi, \mathcal{F} - \frac{1}{2}\left(\eta_{\mu_1\mu_2}\mathcal{F}^{\lambda}{}_{\lambda,\nu} + \eta_{\nu(\mu_1}\mathcal{F}_{\mu_2)\lambda,}{}^{\lambda}\right)\right\rangle, \tag{6.107}$$

where the additional factors of $1/2$ not present in (6.70) stem from the convention used for η^{ij}. The calculations for the spin-3 cases are straightforward from here on out. To complete the demonstration we may also compute the gauge and gauge-for-gauge transformations. According to (6.6) the gauge transformation should be

$$\delta h_{\mu_1\mu_2,\nu} = \partial_{(\mu_1}\theta_{\mu_2),\nu} + \partial_{\nu}\epsilon_{\mu_1\mu_2}, \tag{6.108}$$

where ϵ is symmetric and θ is anti-symmetric in its indices. The gauge parameters therefore coincide with the ones given in (6.60). The same holds true for the gauge-for-gauge transformations

(6.7)

$$\delta\theta_{\mu,\nu} = \partial_\mu \theta'_\nu, \tag{6.109}$$

$$\delta\epsilon_{\mu_1\mu_2} = \partial_{(\mu_1}\epsilon'_{\mu_2)}, \tag{6.110}$$

as they coincide with (6.61).

6.4.2 Irreducible Representations

As we alluded to at the beginning of 6.1 we worked so far with fields that are not necessary irreducible. As we want to do just that we need to enforce the constraints (6.3) onto our fields. We can write this set of constraints in a compact way using family indices

$$S^i{}_j\varphi := \varphi_{\dots,(\mu_1^i\dots\mu_{s_i}^i|,\dots,|\mu_1^j)\dots\mu_{s_j}^j,\dots}, \tag{6.111}$$

where the constraints are given by

$$S^i{}_j\varphi = 0 \qquad (i < j). \tag{6.112}$$

The process we will follow is detailed in [8], but we will give a well-rounded overview. Let us start by stating that the operator $S^i{}_j$ commutes with the Fronsdal-Labastida tensor (6.9). Therefore the governing tensor, and by extend our field equations, are not subject to change when enforcing the constraints. However this does not hold true for the gauge transformations. By enforcing that the constraints (6.112) be preserved under the gauge transformation

$$S^i{}_j\delta\varphi = S^i{}_j(\partial^k\Lambda_k) = 0, \tag{6.113}$$

we can observe how the transformations themselves are altered. We have to distinguish between three different cases. If $k \neq i$, $k \neq j$ none of indices are identical and we can just commute $S^i{}_j$ with ∂_k without a second thought. The second case is $k = i$. To see if we may commute the two operators we can simply calculate the term before and after the commutation and compare the results. The calculation is relatively straight forward, as one has to be mindful of potential double symmetrisation, that may lead us to not count our components correctly

$$S^i{}_j(\partial^k\Lambda_k) = \partial_{(\mu_k^1|}(\Lambda_k)_{\dots,|\mu_2^k\dots\mu_{s_k}^k|,\dots,|\mu_{s_{k+1}}^k)\mu_2^j\dots\mu_{s_j}^j,\dots}, \tag{6.114}$$

$$\begin{aligned} \partial^k(S^i{}_j\Lambda_k) &= \partial_{(\mu_k^1|}(S^i{}_j\Lambda_k)_{\dots,|\mu_2^k\dots\mu_{s_{k+1}}^k),\dots,\mu_2^j\dots\mu_{s_j}^j,\dots} \\ &= \partial_{(\mu_k^1|}(\Lambda_k)_{\dots,|\mu_2^k\dots\mu_{s_k}^k|,\dots,|\mu_{s_{k+1}}^k)\mu_2^j\dots\mu_{s_j}^j,\dots}. \end{aligned} \tag{6.115}$$

As the results are identical, we may assume that in this case the operators commute as well. For $k=j$ we this is not the case

$$
\begin{aligned}
S^i{}_j(\partial^k\Lambda_k) &= (\partial^k\Lambda_k)_{\dots,(\mu^i_1\dots\mu^i_{s_i}|,\dots,|\mu^i_{s_{i+1}})\mu^k_2\dots\mu^k_{s_k},\dots} \\
&= \partial_{(\mu^i_1|}(\Lambda_k)_{\dots,|\mu^i_2\dots\mu^i_{s_i}),\dots,\mu^k_2\dots\mu^k_{s_k},\dots} + \partial_{(\mu^k_2|}(\Lambda_k)_{\dots,(\mu^i_1\dots\mu^i_{s_i}|,\dots,|\mu^i_{s_{i+1}})|\mu^k_3\dots\mu^k_{s_k}),\dots} \\
&= \partial^i\Lambda_k + \partial_{(\mu^k_2|}(\Lambda_k)_{\dots,(\mu^i_1\dots\mu^i_{s_i}|,\dots,|\mu^i_{s_{i+1}})|\mu^k_3\dots\mu^k_{s_k}),\dots},
\end{aligned}
\tag{6.116}
$$

where the "inner" symmetrisation contains all indices labelled μ^i, while the "outer" one contains all indices labelled μ^k. Now there is one last term remaining

$$
\partial^k(S^i{}_j\Lambda_k) = \partial_{(\mu^k_2|}(\Lambda_k)_{\dots,(\mu^i_1\dots\mu^i_{s_i}|,\dots,|\mu^i_{s_{i+1}})|\mu^k_3\dots\mu^k_{s_k}),\dots}, \tag{6.117}
$$

that implies

$$
S^i{}_j(\partial^k\Lambda_k) - \partial^k(S^i{}_j\Lambda_k) = \partial^i\Lambda_k \qquad j=k. \tag{6.118}
$$

To obtain the full expression we simply sum over all the individual values of k resulting in

$$
\partial^k\left(S^i{}_j\Lambda_k + \delta^i_k\Lambda_j\right) = 0 \qquad (i<j). \tag{6.119}
$$

This enables us to define a gauge-for gauge transformation

$$
S^i{}_j\Lambda_k + \delta^i_k\Lambda_j = \partial^l\Lambda^i{}_{[kl]j} \qquad (i<j), \tag{6.120}
$$

that represent a set of constraints that we may solve from the bottom up. For our purposes we may treat $\Lambda^i{}_{[kl]j}$ as zero. Now as per definition j is bigger than k which in turn has to be bigger than i, the last parameter in the chain can be cast into the form

$$
S^i{}_j\Lambda_k = 0, \tag{6.121}
$$

making it irreducible by construction. As we only work with fields that contain two index families there is only one additional step we have to undertake so that in total we end up with the requirements

$$
S^1{}_2\Lambda_1 + \Lambda_2 = 0, \tag{6.122}
$$

$$
S^1{}_2\Lambda_2 = 0. \tag{6.123}
$$

Solving these requirements with respect to the unconstrained ones will result in the irreducible gauge parameters.

6.4.3 Results

Let us start with the $\{3,1\}$ case, which shares many similarities with the hook field,

$$\mathcal{L}_{\{3,1\}} = \frac{1}{2}\left\langle \varphi, \mathcal{F} - \frac{1}{2}\left(\eta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3)\lambda}{}^{\lambda}{}_{,\nu} + \eta_{\nu(\mu_1}\mathcal{F}_{\mu_2\mu_3)\lambda,}{}^{\lambda}\right)\right\rangle, \tag{6.124}$$

as the projection $\mathcal{F}^{[2]}$ does not contribute here for similar reasons. The gauge transformations can then be shown to also have a comparable form

$$\delta\varphi_{\mu_1\mu_2\mu_3,\nu} = \partial_{(\mu_1}\Lambda_{\mu_2\mu_3),\nu} + \partial_\nu\Theta_{\mu_1\mu_2\mu_3}. \tag{6.125}$$

The gauge-for-gauge transformations are then

$$\delta\Lambda_{\mu_1\mu_2,\nu} = \partial_\nu\Lambda'_{\mu_1\mu_2}, \tag{6.126}$$
$$\delta\Theta_{\mu_1\mu_2\mu_3} = \partial_{(\mu_1}\Theta'_{\mu_2\mu_3)}, \tag{6.127}$$

where $\Lambda'_{\mu_1\mu_2} = -\Theta'_{\mu_1\mu_2}$ to fulfil the condition of anti-symmetry in the index families stated in (6.7). We can take a quick look at the degrees of freedom as a sort of final consistency check. For $\{3,1\}$ we have

$$\ydiagram{3,1}_{SO(3)} \ominus \left(\ydiagram{2,1}_{SO(3)} \oplus \ydiagram{3}_{SO(3)}\right) \oplus \ydiagram{2}_{SO(3)}, \tag{6.128}$$

resulting in

$$\mathrm{DoF}[S_{\{3,1\}}] = 7 - (7+5) + 5 = 0, \tag{6.129}$$

as expected. The irreducible gauge transformations can be formulated by applying the constraints (6.123) to (6.125). From the second equation we learn that $\Theta_{\mu_1\mu_2\mu_3}$ is irreducible by default and retains its full gauge symmetry. $\Lambda_{\mu_1\mu_2,\nu}$ on the other hand has to be constrained. By default our gauge parameters are not obliged to obey any symmetry apart from the (anti-)symmetric behaviour in the individual families. Λ could posses, for example, different additional symmetries. For example a $\{2,1\}$ field ϕ could posses the additional symmetry $\phi_{\mu_1\mu_2,\nu} = \phi_{\mu_1(\mu_2,\nu)}$. Which would result in it possessing the same symmetries as a $\{3\}$ field $\phi_{\mu_1\mu_2\mu_3}$. Such additional (anti-)symmetries are valid, as so far we have only imposed restriction on the exchange of entire families with other indices, but never have we stated the behaviour of mixed-symmetry fields under the exchange of individual indices from different families. We will introduce a superscript that will help us distinguish between an entire field from its "components" with additional symmetries. For example

$$\begin{aligned}\Lambda^{(3)}_{\mu_1\mu_2,\nu} &= \Lambda_{\mu_1(\mu_2,\nu)},\\ \Lambda^{(3,1)}_{\mu_1\mu_2,\nu_1\nu_2} &= \Lambda_{(\mu_1\mu_2,\nu_1)\nu_2},\\ &\vdots,\end{aligned}$$

and so on. In our case the gauge parameter $\Lambda_{\mu_1\mu_2,\nu}$ could have two components (3) or $(2,1)$. With the first equation of (6.123) the (3) component of Λ may be rewritten as a contribution of the corresponding component of Θ

$$S^1{}_2\Lambda + \Theta = 0 \qquad \Longrightarrow \qquad \Lambda^{(3)}_{\mu_1\mu_2,\nu} = -\frac{1}{3}\Theta^{(3)}_{\mu_1\mu_2\nu}. \tag{6.130}$$

The additional factor stems from combinatorics as

$$\Lambda_{\mu_1\mu_2,\nu} = \Lambda^{(3)}_{\mu_1\mu_2,\nu} + \Lambda^{(2,1)}_{\mu_1\mu_2,\nu}, \tag{6.131}$$

which implies

$$\Lambda_{(\mu_1\mu_2,\nu)} = \Lambda_{\mu_1\mu_2,\nu} + \Lambda_{\mu_1\nu,\mu_2} + \Lambda_{\mu_2\nu,\mu_1} = 3\Lambda^{(3)}_{\mu_1\mu_2,\nu}, \tag{6.132}$$

as $\Lambda^{(3)}$ is symmetric in the indices. This counting is due to the symmetrisation convention (2.10) we picked at the start. Our total irreducible gauge transformation is therefore

$$\delta\varphi = \partial_\nu\Theta_{\mu_1\mu_2\mu_3} + \left(\partial_{(\mu_1}\Lambda^{(2,1)}_{\mu_2\mu_3),\nu} - \frac{1}{3}\partial_{(\mu_1}\Theta^{(3)}_{\mu_2\mu_3)\nu}\right). \tag{6.133}$$

This leaves us with the $\{3,2\}$ case. Unlike the previous cases this one contains contributions of $\mathcal{F}^{(2)}$. Their computation is however straight forward, as only the term (6.102) may contribute. The result is then

$$\begin{aligned}\eta^{i_1j_1}\eta^{i_2j_2}\mathcal{F}^{[2]}_{i_1j_1,i_2j_2} =& \eta_{\mu_1(\nu_1}\eta_{\nu_2)\mu_2}\mathcal{F}_{\mu_3\lambda\rho,}{}^{\lambda\rho} + \eta_{\mu_2(\nu_1}\eta_{\nu_2)\mu_3}\mathcal{F}_{\mu_1\lambda\rho,}{}^{\lambda\rho} + \eta_{\mu_3(\nu_1}\eta_{\nu_2)\mu_1}\mathcal{F}_{\mu_2\lambda\rho,}{}^{\lambda\rho}\\ &- \eta_{\nu_1\nu_2}\eta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3)\lambda\rho,}{}^{\lambda\rho}\\ &+ \eta_{\mu_1(\nu_1}\eta_{\nu_2)\mu_2}\mathcal{F}_{\mu_3\lambda}{}^{\lambda}{}_{,\rho}{}^{\rho} + \eta_{\mu_2(\nu_1}\eta_{\nu_2)\mu_3}\mathcal{F}_{\mu_1\lambda}{}^{\lambda}{}_{,\rho}{}^{\rho} + \eta_{\mu_3(\nu_1}\eta_{\nu_2)\mu_1}\mathcal{F}_{\mu_2\lambda}{}^{\lambda}{}_{,\rho}{}^{\rho}\\ &- \eta_{\nu_1\nu_2}\eta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3)\lambda}{}^{\lambda}{}_{,\rho}{}^{\rho}\\ =&: \mathcal{H},\end{aligned} \tag{6.134}$$

where we introduce the definition to keep the Lagrangian easily readable. The object in question is then of the form

$$\mathcal{L}_{\{3,2\}} = \frac{1}{2}\Big\langle \varphi, \mathcal{F} - \frac{1}{2}\Big(\eta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3)\lambda}{}^{\lambda}{}_{,\nu_1\nu_2} + \eta_{\nu_1\nu_2}\mathcal{F}_{\mu_1\mu_2\mu_3,\lambda}{}^{\lambda}$$
$$+ \eta_{\nu_1(\mu_1}\mathcal{F}_{\mu_2\mu_3)\lambda,\nu_2}{}^{\lambda} + \eta_{\nu_2(\mu_1}\mathcal{F}_{\mu_2\mu_3)\lambda,\nu_1}{}^{\lambda}\Big) + \frac{1}{36}\mathcal{H}\Big\rangle, \tag{6.135}$$

with its gauge

$$\delta\phi_{\mu_1\mu_2\mu_3,\nu_1\nu_2} = \partial_{(\mu_1}\Lambda_{\mu_2\mu_3),\nu_1\nu_2} + \partial_{(\nu_1|}\Theta_{\mu_1\mu_2\mu_3,|\nu_2)}, \tag{6.136}$$

and gauge-for-gauge transformations

$$\delta\Lambda_{\mu_1\mu_2,\nu_1\nu_2} = \partial_{(\nu_1|}\Lambda'_{\mu_1\mu_2,|\nu_2)}, \tag{6.137}$$
$$\delta\Theta_{\mu_1\mu_2\mu_3,\nu} = \partial_{(\mu_1}\Theta'_{\mu_2\mu_3),\nu}, \tag{6.138}$$

respectively, where again $\Lambda'_{\mu_1\mu_2,\nu} = -\Theta'_{\mu_1\mu_2,\nu}$. Checking the degrees of freedom for this representation does not require the methods we used so far, as any diagram with more than four boxes in its first two rows corresponds to the zero representation in $GL(4)$. Such tableaux in four space-time dimensions contain enough anti-symmterisations so that dimension dependent identities set all components equal to zero. The irreducible gauge transformations here follows the same principle as for the previous case, enabling us to replace the $(3,1)$ mode of Λ with the corresponding mode for Θ

$$\delta\varphi = \partial_{(\nu_1|}\Theta_{\mu_1\mu_2\mu_3,|\nu_2)} + \left(\partial_{(\mu_1}\Lambda^{(2,2)}_{\mu_2\mu_3),\nu_1\nu_2} - \frac{1}{2}\partial_{(\mu_1}\Theta^{(3,1)}_{\mu_2\mu_3)(\nu_1,\nu_2)}\right). \tag{6.139}$$

If we want to write down a potential higher derivative action in the vain of the Weyl action for the Hook field (6.53) there are still a lot of things to be done. At first the defining equations have to be translated into the setting of (A)dS space where the breaking of the gauge-for-gauge transformations may be analysed. From there a new action may be proposed, which is invariant under all of the separate gauge-transformations. The final result would the be the flat limit of this new action. This new action alone may be sufficient to derive meaningful interactions between massless higher-spin fields.

7 Conclusion

All of this together gives us a promising outlook into further investigation of these mixed symmetry fields. As this specific set-up is only possible in four dimensions, it speaks to reason, that it could solve our problem, that is also confined to four dimensional space-time. An exploration of the (A)dS behaviour will most likely pose some difficulty as the involved gauge fields increase in rank and, in the antisymmetric formulation, would force an additional Einstein tensor like structure to enter the higher-derivative action. Higher-derivative actions come with their own bags of problems, but as we are trying to find any solution for our conundrum I shall leave the discussion for another day, should the attempt prove fruitful. Furthermore, as we have seen, it is one of the few straight forward answers this problem could have. Every other proposition that I came across was very nebulous in nature, most of these can be found across the references used in this project, as they require some sort of special extension to the Fronsdal field with so very vague requirements that are hard to work with at the best of times. We can also not exclude that this difference is just a fact of life and a solution may not be possible, which may not be of physical concern as it appearers in the exchange between massless particles. Multiple no-go theorems[12] of quantum mechanics only allow trivial scattering matrices (the identity matrix) to occur in such exchanges in flat space, rendering the vertex physically irrelevant. While I remain optimistic it is something to keep in mind when tackling this problem.

[12] [21] provides a neat introduction

8 Appendix

8.1 A: Commutators for the ISO(3,1) generators in the Light-Front

This appendix provides the reader with the full list of commutators of the generators of the ISO(3,1) algebra discussed in section (3.1).

[K,K]=K

$$[P^+, P^a] = 0 \qquad [P^b, P^a] = 0 \tag{8.1}$$
$$[J^{-+}, P^+] = -P^+ \qquad [J^{-+}, P^a] = 0 \qquad [J^{a+}, P^b] = 0 \tag{8.2}$$
$$[J^{a+}, P^b] = -P^+\delta^{ab} \qquad [J^{ab}, P^+] = 0 \qquad [J^{ab}, P^c] = P^a\delta^{bc} - P^b\delta^{ac} \tag{8.3}$$
$$[J^{-+}, J^{a+}] = -J^{a+} \qquad [J^{-+}, J^{ab}] = 0 \qquad [J^{a+}, J^{b+}] = 0 \tag{8.4}$$
$$[J^{a+}, J^{bc}] = \eta^{ab}J^{c+} - \eta^{ac}J^{b+} \qquad [J^{ab}, J^{cd}] = \text{the usual} \tag{8.5}$$

[K,D]=K

$$[P^+, P^-] = 0 \qquad [P^a, P^-] = 0 \qquad [J^{ab}, P^-] = 0 \tag{8.6}$$
$$[J^{a+}, P^-] = P^a \qquad [J^{a+}, J^{b-}] = J^{+-}\delta^{ac} - J^{ac} \qquad [J^{a-}, P^+] = P^a \tag{8.7}$$

[K,D]=D

$$[J^{-+}, P^-] = P^- \qquad [J^{-+}, J^{a-}] = J^{a-} \qquad [J^{ab}, J^{c-}] = J^{a-}\delta^{bc} - J^{b-}\delta^{ac} \tag{8.8}$$
$$[J^{a-}, P^b] = -P^-\delta^{ab} \tag{8.9}$$

[D,D]=0

$$[J^{a-}, J^{b-}] = 0 \qquad [J^{a-}, P^-] = 0 \tag{8.10}$$

8.2 B: Counting Physical Degrees of Freedom

Counting degrees of freedom is a prevalent concept throughout this thesis. This is seldom explained in a way that allows for easy abstraction to more generalised tensors and therefore going through it in more detail, along the lines of [16], should be worthwhile. As it turns out the physical DoF a system can access are correlated to the "strength" of its governing differential equation. The numerical value for this strength is related to the amount of Cauchy data required to define a general solution to a given system modulo its gauge freedom. To start of we look at a system of PDE's of order m containing the fields $\varphi^i,\ i = 1, \dots f$

$$T_a(\varphi^i, \partial_\mu \varphi^i, \dots, \partial_{\mu_1} \dots \partial_{\mu_m} \varphi^i) \qquad a = 1, \dots, t. \tag{8.11}$$

Now we consider the Taylor expansions of these fields around $x = x_0$

$$\varphi^i(x) = \sum_{p=0}^{\infty} \frac{1}{p!} \varphi^i_{\mu_1 \dots \mu_p} (x - x_0)^{\mu_1} \dots (x - x_0)^{\mu_p}, \tag{8.12}$$

where we use the notation $\partial_\mu \varphi = \varphi_\mu$. Now we replace the derivatives of our fields in (8.11) with the respective Taylor coefficients

$$T_a(\varphi^i, \varphi^i_\mu, \dots, \varphi^i_{\mu_1, \dots, \mu_m}). \tag{8.13}$$

In this state the amount of possible algebraic properties makes a general approach considerable difficulty and while a brute force computation is maybe doable, there is a way to simplify this analysis. We introduce k additional derivatives into the system. This shifts the focus to higher order coefficients in particular and those should be the most sensitive to restrictions, given their nature. After setting $x = x_0$ we obtain the equation

$$\mathcal{J}^{\nu_1 \dots \nu_{m+k}}_{ai\mu_1 \dots \mu_k} \varphi^i_{\nu_1 \dots \nu_{m+k}} + \mathcal{I}_{a\mu_1 \dots \mu_k} = 0, \tag{8.14}$$

that is of algebraic nature where the functions $\mathcal{J}$ and $\mathcal{I}$ depend on coefficients of φ with an order lower than $m + k$. So for each k we obtain a system of linear inhomogeneous equations for $\phi^i_{\nu_1 \dots \nu_{m+k}}$. The determinant factor in this system is the matrix $\mathcal{J} = \mathcal{J}_k$ and it is usually referred to as the symbol matrix in literature. In its explicit form

$$\mathcal{J}^{\nu_1 \dots \nu_{m+k}}_{ai\mu_1 \dots \mu_k} = \mathcal{J}^{(\nu_1 \dots \nu_m}_{ai} \delta^{\nu_{m+1}}_{\mu_1} \dots \delta^{\nu_{m+k})}_{\mu_k}, \qquad \mathcal{J}^{\nu_1 \dots \nu_m}_{ai} = \frac{\partial T_a(\varphi)}{\partial \varphi^i_{\nu_1 \dots \nu_m}}, \tag{8.15}$$

we see that for any given order k an explicit dependence on order 0 remains. From this we may glean that it is possible to identify an integer m in such a way that the space of monomials, governed by the terms $(x - x_0)^\mu$, of order $p > m$ can be fixed by a finite amount of linear inhomogeneous equations dependent on coefficients of order lower than or equal to p. This implies that the system (8.14) can be solved recursively step-by-step, where each step deals only with a finite number of equations. Therefore we can forgo the more involved process otherwise required and analyse this problem using only the methods of linear algebra. Before we continue we place a non-trivial restriction on the PDE system we analysis. From here on out we assume that the system is non singular. Doing this is not necessary, but it causes the rank of the symbol matrix $\mathcal{J}$ to be maximal, reducing the amount of considerations we have to take in the following steps. At the end we will present the full formula where these restrictions are not enforced. The symbol matrix alone is sufficient to fix the homogeneous solution and dominates the size of the solution space. The dimension of this space is linked to the rank of the symbol matrix and the number of unknowns in the system. The later is given by the number of coefficients $\varphi^i_{\nu_1\ldots\nu_{m+k}}$ and is in agreement with the number N_p of linear independent contributions to the Taylor series for a given order $p = m + k$. Using combinatorics we get the number of independent monomials

$$N_p = f\binom{p+d-1}{p} = f\frac{(p+d-1)!}{p!\,(d-1)!}, \tag{8.16}$$

where f is the number of fields governed by the PDE we defined at the start (8.11). To compute the rank of $\mathcal{J}_k$ we have to subtract the amount of its left null-vectors from the total number of equations present in (8.14). We can express this last quantity also through a binomial coefficient

$$t\binom{p-m+d-1}{p-m}, \tag{8.17}$$

where t is the number of PDE's present in the original system (8.11). All of this is assuming that all equations contained in the problem are of the same order. To generalise this expression we can simply sum over all present orders

$$\sum_{n=0}^{\infty} t_n\binom{p-n+d-1}{p-n}, \tag{8.18}$$

where $p = k + m$ and we labelled the total amount of equations of algebraic order zero as t_0, of order one as t_1 and so on. This way of generalising this setting only holds when the system does not contain any hidden integrability conditions, but given a well posed problem this should not be an issue.

Now we need to determine the amount of left null-vectors. As these define identities, we can start the procedure by looking at the gauge identities [13]

$$\hat{L}^a T_a := 0, \tag{8.19}$$

[13] There is a difference between the concept of gauge identites and gauge freedom in generalised settings. We will take about it later in more detail, but as a quick side note one can think of them as the left and right null-vectors of the symbol matrix respectively.

where

$$\hat{L}^a = \sum_{n=0}^{q'} \mathcal{L}^{a\nu_1\ldots\nu_m}\partial_{\nu_1}\ldots\partial_{\nu_m}, \tag{8.20}$$

is a differential operator with coefficients $\mathcal{L}^{a\nu_1\ldots\nu_m}$ dependent on the involved fields and derivatives up to the finite order j. The highest order q' for which not all entries vanish is called the order of the gauge identity. Employing the same differentiation scheme to (8.19) that lead us to (8.14) we arrive at

$$\mathcal{L}^{a\mu_1\ldots\mu_{q'}}\mathcal{J}^{\nu_1\ldots\nu_{m+s+q'}}_{ai\mu_1\ldots\mu_{s+q'}}\varphi^i_{\nu_1\ldots\nu_{m+s+q'}} + \cdots := 0, \tag{8.21}$$

where we added s derivatives to the equation and we ignore the inhomogeneous terms, which are of order $m+s+q'$. Now we may decide to choose s so large that $m+s+q' > j$ holds. This would imply that all coefficients $\varphi^i_{\nu_1\ldots\nu_{m+s+q'}}$ must be identical to zero, otherwise the identity would no longer hold. With this restriction now unearthed we have a limit on the possible left null-vectors for $\mathcal{J}_{s+q'}$. The remaining ones are linearly independent and of the form

$$\mathcal{L}^{a\mu_1\ldots\mu_{s+q'}}_{\lambda_1\ldots\lambda_s} = \mathcal{L}^{a(\mu_1\ldots\mu_{q'}}\delta^{\mu_{q'+1}}_{\lambda_1}\ldots\delta^{\mu_{q'+s})}_{\lambda_s}. \tag{8.22}$$

To estimate their total amount we make use of binomial coefficients once again

$$\binom{s+d-1}{s}, \tag{8.23}$$

where we keep in mind that $s = k - q'$. A system like (8.11) will more likely than not enjoy multiple gauge identities of different total orders $q = q' + m$. If we take l_n to symbolise the amount of identities at total order n, we can write down a generalised expression similar to (8.18)

$$\sum_{n=0}^{\infty} l_n \binom{p-n+d-1}{p-n}, \tag{8.24}$$

given a sufficiently large p. As we now have both the total amount of gauge-identities and equations we can write down the dimension of the solution space of order p

$$N'_p = f\binom{p+d-1}{p} - \sum_{n=0}^{\infty}(t_n - l_n)\binom{p-n+d-1}{p-n}, \tag{8.25}$$

which is equal to the number of independent monomials at order p.

Before we draw any conclusions, there is one last criterion that we have to factor into our consideration; the gauge freedom of the fields. In general a gauge transformation can be written as

$$\delta_\epsilon \varphi^i = \sum_{n=0}^{q} \mathcal{R}^{i\mu_1\ldots\mu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \epsilon, \tag{8.26}$$

where the coefficients $\mathcal{R}^{i\mu_1\ldots\mu_n}$ are functions of the fields and their derivatives. The order of the transformation is given by the number q. From the invariance of the PDE system under this transformation we can deduce that

$$\delta_\epsilon T_a = \hat{U}_a^b T_b, \tag{8.27}$$

where $\hat{U}$ is some differential matrix-valued operator. We continue in the vein of the previous two analyses. At first we Taylor expand the gauge parameter around some value x_0

$$\epsilon(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \epsilon_{\mu_1\ldots\mu_n} (x-x_0)^{\mu_1} \ldots (x-x_0)^{\mu_n}, \tag{8.28}$$

before we apply s derivatives onto equation (8.27) resulting in

$$\mathcal{J}_{ai}^{\mu_1\ldots\mu_m} \mathcal{R}^{i\mu_{m+1}\ldots\mu_{m+q}} \epsilon_{\mu_1\ldots\mu_{m+q+s}} + \cdots = 0, \tag{8.29}$$

where terms of order $m+q+s$ are again represented by the dots. As these identities have to hold for any value of ϵ we can conclude that

$$\mathcal{J}_{ai}^{\mu_1\ldots\mu_{m+s}} \mathcal{R}^{i\nu_{m+1}\ldots\nu_{m+q+s}}_{\mu_1\ldots\mu_{m+s}} = 0, \tag{8.30}$$

where

$$\mathcal{R}^{i\nu_1\ldots\nu_{s+q+m}}_{\mu_1\ldots\mu_{m+s}} = \mathcal{R}^{i(\nu_1\ldots\nu_q} \delta^{\nu q+1}_{\mu_1} \ldots \delta^{\nu_{q+m+s})}_{\mu_{m+s}}. \tag{8.31}$$

Therefore $\mathcal{R}$ contains the set of right null-vectors $\{\mathcal{R}^{i\nu_1\ldots\nu_{s+q+m}}\}$ of the symbol matrix $\mathcal{J}$. We can then compute the span of the right-null vector space

$$\binom{m+s+q+d-1}{m+q+s} = \binom{p+q+d-1}{p+q}. \tag{8.32}$$

Like the other two properties general PDE systems have multiple gauge transformations of different orders so we need to generalise this expression as well. Assuming that all the r_n transformations, of a given order n, are irreducible then in total

$$\sum_{n=0}^{\infty} r_n \binom{p+n+d-1}{q+n}, \tag{8.33}$$

components of the system are pure gauge. With this we can write down the number of physically indistinguishable coefficients at a given order p

$$N''_p = f\binom{p+d-1}{p} - \sum_{n=0}^{\infty}\left[(t_n - l_n)\binom{p-n+d-1}{p-n} + r_n\binom{p+n+d-1}{p+n}\right]. \tag{8.34}$$

We can then estimate the physical degrees of freedom per point $\mathcal{N}$ by

$$\mathcal{N} = \lim_{p\to\infty}\frac{p}{d-1}\frac{N''_p}{N_p}, \tag{8.35}$$

where we multiply the percentage of physically independent monomials of the total monomials for a given order p with the order normed by the dimension minus one. We take the limit here as it is reasonable to assume that the higher orders will dominate and we take the limit to higher orders therefore[14]. We can utilise the asymptotic expansion of the binomial coefficient

$$\binom{n}{k} = \frac{1}{k!}(n^k - (1+2+\dots+(k-1))n^{k-1} + \dots + (-1)\cdot(-2)\dots(-(k-1))), \tag{8.36}$$

to rewrite the fraction

$$\frac{N''_p}{N_p} = (f-t+l-r) + \frac{(d-1)}{p}\sum_{n=0}^{\infty} n(t_n - l_n - r_n) + O\left(\frac{1}{p^2}\right), \tag{8.37}$$

where

$$t = \sum_{n=0}^{\infty} t_n, \qquad l = \sum_{n=0}^{\infty} l_n, \qquad r = \sum_{n=0}^{\infty} r_n. \tag{8.38}$$

The leading term

$$\Delta = f - t + l - r, \tag{8.39}$$

is usually called the compatibility coefficient in the literature. Assuming that it vanishes, a system like this is named totally compatible, we gain the following expression for the degrees of freedom

$$\mathcal{N} = \sum_{n=0}^{\infty} n(t_n - l_n - r_n). \tag{8.40}$$

This condition can be established by the assumption that all right null-vectors arise from gauge symmetries [22]. If we further allow reducible gauge symmetries, transformations etc.

[14] This formula can also be justified by utilising formal integrability, but that is not for this thesis. The interested may consult the book [22] for this formal approach.

the formula has the following form

$$\mathcal{N} = \sum_{n,m=0}^{\infty} n(t_n - (-1)^m(l_n^m - r_n^m)), \tag{8.41}$$

where l_n^m and r_n^m are the numbers for the gauge identities and transformation of a given order n and reducibility order m. For most systems in physics we can simplify this a bit further, as they require no differentiation between gauge identities and gauge transformations and Lagrange equations are typically of second-order. These criteria imply

$$\mathcal{N} = 2\left(t_2 + \sum_{n,m=0}^{\infty} (-1)^{m+1}(n+1)r_n^m\right), \tag{8.42}$$

where a division by the factor 2 will yield the desired formula for the number of physical polarisations. We can express this formula in words for the irreducible case:

$$\text{Nr. EoM - Nr. Gauge-id} \times \text{Their Order} + \text{Nr. GforG-id} \times \text{Their Order} + \ldots$$

8.3 C: Calculating Dimensions of GL(n,D) Irrep. using Young Diagrams

To every irreducible representation of GL(n,D) we can associate a Young Diagram. In such a tableaux symmetric indices are represented by rows and antisymmetric ones are represented by columns. A family of indices is represented by a column in the anti-, or a row in the symmetric representation. A nice overview of their working on various relevant groups in provided by [1] and [7]. Encoding all of this information into these objects enables us to calculate the dimensions of their associated representation. The process is algorithmic and can be understood fairly well by following an example. Let us pick the following diagram

$$\ydiagram{5,3,1}\,, \tag{8.43}$$

comprised of 8 total boxes.

To begin we assign the box in the upper left corner the value n. The other boxes are then filled in correlation to that value. For this we increase the value by 1 along each step to the right and decrease it for each step down. For our example this results in

$$\begin{ytableau} n & n+1 & n+2 & n+3 & n+4 \\ n-1 & n & n+1 \\ n-2 \end{ytableau}\,. \tag{8.44}$$

The result we desire is than the product of all occurring entries[15]

$$p = n^2(n+1)^2(n+2)(n+3)(n+4)(n-1)(n-2). \tag{8.45}$$

The other component we need is another polynomial. For this we enter the hook length of each box into itself. The hook length is determined by the hook length formula, which, simply put, just adds the total amount of boxes to the respective right and below together [16]. In action our tableaux will be filled with the following values

$$\begin{ytableau} 7 & 5 & 4 & 2 & 1 \\ 4 & 2 & 1 \\ 1 \end{ytableau}\,, \tag{8.46}$$

[15] Notice that when trying to anti-symmetricing over n or more lines the result will be 0. Giving us a result that follows from dimensioal identities vary handily.

[16] The Hook length formula can be proven in various ways, probabilistic random walks or utilising the Frobenius theorem just to name a few.

where the relevant value is again a product of all the individual fillings

$$q = 7 \cdot 5 \cdot 4^2 \cdot 2^2 = 2240. \tag{8.47}$$

To get the final answer we simply divide p/q. If we choose $n = 4$ to represent the standard case in four dimensions the representation associated to our test diagram would posses

$$\frac{p}{q} = \frac{4^2(4+1)^2(4+2)(4+3)(4+4)(4-1)(4-2)}{2240} = 360, \tag{8.48}$$

dimensions.

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