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„Views on subject matter – preservice mathematics teachers' noticing of and beliefs about subject matter and its role for teaching"

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to good enough teachers striving to become better

"It's hard to be a role model, because what you really need to do is
show students how imperfect people can be and still succeed."

Karen Uhlenbeck

1 Introduction

1.1 What future teachers know, think and see

When Felix Klein (1908/1932) coined the term “double discontinuity” at the beginning of the last century, he could not have known that the challenge of making university mathematics relevant to future teachers would remain a salient topic even in 2023, and that efforts to understand the problems involved would lead to research spanning the wide variety of theoretical lenses that exist in the modern world of mathematics education.

In the introduction to his lecture series on “Elementary Mathematics from an Advanced Standpoint”, Klein describes two separate challenging transitions for future mathematics teachers. The first discontinuity lies in the vastly different ways of doing mathematics at university compared to the school mathematics preservice teachers were thus far used to. The second discontinuity describes the difficulties in applying the kind of mathematics studied at university even for such students who overcome the challenges of the first discontinuity. Even for prospective teachers who end up being successful in their university studies of mathematics, this often does not translate to feeling confident in their (mathematical) capabilities as teachers of mathematics.

Now, in order to investigate the questions surrounding this double discontinuity, one needs to engage, first, with the **subject matter** itself – what makes school mathematics different from university mathematics, and how is it possible that the majority of university mathematicians and teacher educators see deeply meaningful connections between these two, while many preservice teachers struggle to understand why they should have to study the latter in order to teach the former?

Next, one might distinguish different areas of university mathematics and pause to ponder their respective connections to the mathematical ideas dealt with in school – the question of **professional knowledge** thus naturally arises: What should good mathematics teachers know, after all? Asking the even more general question “What makes a mathematics teacher good?” leads to the idea of **expertise**, and determining the answer to that question ultimately hinges on an analysis of what it is teachers do – a kind of “job analysis” as to what the demands of their profession are.

Since what prospective teachers take away from their professional preparation amounts to more than just factual knowledge or procedural skills in mathematics and its teaching, their **beliefs, affects and attitudes** also need to be addressed. How do prospective teachers’ beliefs about the mathematical aspects of a teachers’ job influence their own learning at university? How do their attitudes toward mathematics in turn influence their teaching of the subject to their pupils?

Another question that arises: Do teachers realise the mathematical demands of their profession yet still believe that advanced mathematics as it is presented to them at university doesn't help them meet those requirements? Or is it that they do not see the intellectually challenging mathematical parts of the work of teaching? Here, the field of **professional vision** or **noticing** naturally comes into play.

And lastly, there has been no talk so far of possible solutions!

How to solve this century-old dilemma and have students realize the benefits of becoming subject matter experts, at least to a certain degree, for their future professional activities? Since teacher education happens at universities, this is a question of **tertiary mathematics education** and a question of **(preservice) professional development** for teachers. How much mathematics should we as teacher educators teach, what mathematics, and how should we best teach it? How much teaching practice should be provided in a teacher accreditation programme, what kind of teaching practice should be provided, and at which time?

My dissertation cannot hope to answer all the questions I just posed – entire subfields of research in mathematics education have been dedicated to finding answers to every single one of them. My aim was to investigate how preservice teachers' often problematised views of subject matter in the work of teaching can be changed for the better, where I mean "views" both in the sense of preservice teachers' literal perception (What do they perceive and pay attention to when it comes to subject matter?) and in the metaphorical sense of their opinions on the topic (What do they believe about subject matter?). Since how we see the world can shape our understandings and beliefs about it, and vice versa, it seemed fitting to look at preservice teachers' noticing and beliefs in parallel.

Throughout chapter 2, I aim to provide an overview of the history and main results of each of the fields of study whose relevance to the double discontinuity I have highlighted above: research on teacher expertise, their knowledge, beliefs, noticing, and teacher education. These areas of research are all intimately related and thus many of the cited works make meaningful statements about more than one of these questions, or specifically analyse the connections that exist between them.

The theory chapter then closes with an argument for conceptualising the three constructs of subject matter knowledge, subject matter noticing and beliefs about subject matter as an interrelated triangle, which suggests that achieving changes in any of the three corners is impeded unless applying a holistic strategy that also encompasses the other two. This leads to the rationale of using a holistic instructional framework, the Four Component Instructional Design (van Merriënboer & Kirschner, 2017) to design an intervention targeting preservice teachers' noticing of and beliefs about subject matter.

Chapter 3 lists the specific research questions that the study was designed to shed light on, as well as the specific hypotheses I formulated at the start of the project.

In chapter 4, the research methodology that was employed to answer those questions and test the hypotheses is explained and justified. The first part of the chapter deals with the "big picture" research design, elaborating on the details of the quasi-experimental method that was used. Next, an in-depth look into the design of the learning environment based on the 4C/ID framework is provided, illustrating the situated elements of the intervention whose effects are later examined. Next, the development of the instruments of measurement – the pre/posttest booklet consisting of a noticing, beliefs and subject matter questionnaire, respectively – is reported. The last part of the methodology chapter then describes the methods of data preparation and data analysis that were applied to the data gathered during the two semesters of data collection.

Results of both the quantitative and qualitative data analyses are reported in chapter 5.

The discussion in chapter 6 then interprets the results with regard the hypotheses noted in chapter 3, relates them back to the broader research questions that I had set out to explore and points out their connection to past findings in the literature. Furthermore, limitations of the study and newly arisen questions are discussed, which immediately leads into recommendations for further investigations. A section on the implications these results have for mathematics teacher education closes this chapter, and the thesis.

1.2 "I'm never going to need this" – the question of subject matter

> "The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching."
>
> (Klein, 1932, p. 1)

According to anecdotal evidence retold time and again by multitudes of university educators (who infer this from their students' actions or are told by their students directly, be it in exclamations of frustration, in private conversation, or on official feedback forms), the university courses taken in mathematics very often form a decidedly *unpleasant* memory in the minds of prospective teachers. Preservice teachers tend to believe the subject matter related parts of the curriculum cover too much material that seems out of touch with their future realities as teachers, are too demanding and do not adequately prepare them for their future profession.

There is an abundance of studies reporting these experiences and beliefs, laying out the different ways in which many preservice teachers fail to establish a positive relationship with the subject matter which they will be teaching later on and do not develop a reliable mathematical identity. Meanwhile, many teacher educators lament their university students' attitudes and find these defensive psychological structures, which often go hand in hand with a lack of knowledge of mathematical content (Bungartz &

Wynands, 1998), to be highly problematic. In general, there are recurring discussions about how much content knowledge mathematics teachers need in order to teach effectively as well as concern about whether current or future mathematics teachers possess that kind of subject matter knowledge and are thus equipped to provide quality teaching.

Hefendehl-Hebeker (2013) presents the heart of these parties' conflicting views and concerns comprehensively. She also points out that one of the reasons preservice teachers may be frustrated with the amount and the level of content knowledge required by teacher training programmes could be that prospective teachers do not have a clear picture of the mathematical requirements of their future profession. Preservice teachers enter training programmes with a preconceived notion of what it is a mathematics teacher does which is based upon their own experiences as students. It can be problematic if this initial view on the work of teaching – which naturally lacks insight into the often highly mathematical work a teacher engages in "behind the scenes" – does not change even as preservice teachers handle observation tasks or reflect on their own practice lessons over the course of their training.

Thus, it appears that prospective teachers' inability to attend to subject matter related aspects of teaching may negatively influence those preservice teachers' beliefs about the relevance of subject matter courses in their studies. At the same time, not considering questions of subject matter to be essential for successful teaching, preservice teachers may intentionally focus on other aspects of classroom situations and choose to pay less attention to the mathematical content of a lesson.

On the other hand, expertise studies have made clear that weak content knowledge in a specific area inhibits professional ways of perceiving situations which contribute substantially to expert performance. In essence, low content knowledge may keep prospective teachers from observing the big role content knowledge plays in mathematics teaching, and as a byproduct confirm their negative views about the justification behind their training programmes' content knowledge requirements.

At the same time, initial gaps in content knowledge could be mended over time by teachers with a sufficiently optimistic disposition and some skill in paying attention to mathematical aspects of their work. However, negative attitudes and beliefs can stand in the way of this, as Winter (2003) puts it:

"The teachers' initially deficient knowledge of mathematics will only have particularly negative consequences if the teachers – born out of a persisting resistance against mathematics – do not show a willingness to openly engage with the [mathematical] objects themselves and with questions and impulses provided by the children, to discuss problems of subject-specific teaching with their colleagues, and if they do not have opportunities to further their subject-specific education." (ibid, 97, own translation)

Thus, attempts to solve the problem of prospective teachers' suboptimal mathematical skills and knowledge which disregard efforts to actively counter these beliefs in favour of requiring simply "more" mathematics in the curriculum, may be detrimental in the long term. A smaller body of knowledge acquired in a way that contributed to a sense of mathematical self-efficacy might well allow prospective teachers to later identify gaps in their knowledge during their professional activities, discuss them with colleagues or seek out professional development in order to learn the missing material.

Based upon these interdependencies, I conceptualised the triad of mathematical subject matter knowledge, beliefs about (the importance of) mathematical subject matter, and noticing of mathematical subject matter in teaching situations as illustrated in Figure 1 below.

One challenge in changing these three components is that each of them seems to be, among other things, a prerequisite or at least essential facilitator for achieving positive change in the others. Thus, at a first glance, it appears that all components may need to be jointly addressed in order to achieve a desirable outcome.

If one were to only tackle students' knowledge, for example, by expanding the curriculum to include further content courses, preservice teachers might end up perceiving an even wider gap between the academic mathematics required of them and the work they expect to be doing in the future as mathematics teachers. If this affects their beliefs about the importance of those content courses negatively and they become further alienated from mathematics as a result, the net subject matter knowledge they gain even after receiving those credits may be minimal.

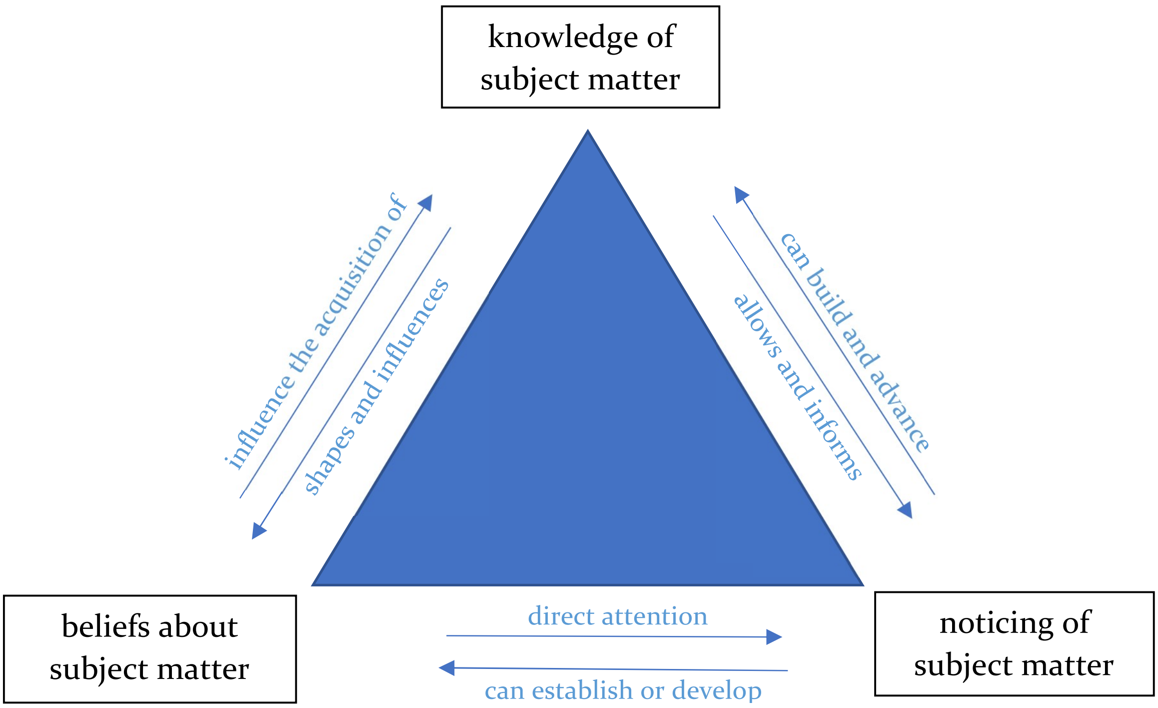


Figure 1: Subject matter knowledge - beliefs - noticing triangle

If, on the other hand, one were to try to change preservice teachers' beliefs about the importance of content knowledge without providing them with opportunities to improve that knowledge, it could be hypothesised that defensive mechanisms of trying to protect one's image as a "good mathematics teacher" may emerge (cf. "blame and justification", Mason, 2002, p. 10). Also, many teachers' beliefs about the importance of one subject area over another in their teacher training programme are centered around the question of applicability to teaching situations – without showing preservice teachers in which ways content knowledge can be beneficial or how a lack of content knowledge can negatively affect a classroom interaction, these beliefs may be difficult to change.

The intervention I designed and tested set out to combat these issues by trying to see whether student teachers could learn to activate the knowledge they had acquired in their subject matter courses while performing typical teacher tasks – thus giving preservice teachers an opportunity to revisit mathematical content in a situated environment, apply the cognitive resources they had already acquired and hopefully take away both a sense of theoretical applicability of mathematical subject matter (this can be *useful*!) and a sense of self-efficacy when applying their mathematical knowledge (*I* can make use of this!).

The following chapter on the relevant literature will present central theoretical conceptualisations of the ideas that need to be interwoven in this kind of research – teacher expertise, including teacher knowledge, teacher beliefs, teacher noticing and the use of situated learning in teacher education – as well as provide an overview of the current state of research in each of these areas and past projects aimed at alleviating the problems described.

At the end of this upcoming chapter, I will specify for each construct the conceptualisation I chose to use in the theoretical framework of my research project and provide definitions of the terms I will be referring to when reporting my own findings.

1.3 A note on terminology

Throughout the text, I will be referring to university students attending a teacher accreditation programme/teacher training as **prospective teachers**, **preservice teachers (PSTs)** or **student teachers**.

In general, the word **student** or **pupil** will be used when talking about school students. The word **student** when referring to a university student will only be used when the meaning is abundantly clear from the context.

Furthermore, the terms "**content knowledge**" and "**subject matter knowledge**" will be used interchangeably when referring to (preservice) teachers' knowledge of mathematics.

2 Situating the study within the literature

2.1 What makes a “good” teacher?

2.1.1 Expertise and expert teachers

What distinguishes experts in a particular domain from laypeople, novices of the field or even intermediate practitioners? This question has been a fertile ground for research in the cognitive sciences since the 1960s.

One of the first examples to be analysed were grandmasters of chess. Studies (de Groot, 1965; Chase & Simon, 1973) showed that, among other characteristics, experts possessed particular modes of perceiving situations that allowed them to act in ways that were superior to the actions of less advanced practitioners. These perceptual skills (Glaser & Chi, 1998) of experts can be directly linked to the modern concept of *noticing* which is central to the study at hand. However, before zooming in on this idea of a special type of professional vision, I will provide an overview of core features of expertise that have been reliably deduced.

Research on expertise is usually conducted in one of two main ways (Bromme, 2001). One are investigations into what cognitive, affective and motivational personal prerequisites are necessary for expert performance, while a second popular mode of research in this area involves contrasting experts’ performance with the performance of novices at the same task.

Glaser and Chi’s (1998) summary of key characteristics of expert performance include points that focus on experts’ cognitive structures, such as their extensive domain-specific know-how. This is taken to be the reasoning behind the fact that experts’ excellence at tasks in their own domain does not seem to automatically transfer to other domains, which implies that expertise cannot be explained as some sort of generalised “problem-solving ability” possessed by experts across fields.

However, many of the other characteristics these authors list assume a reference to non-expert performance, such as

- experts’ abilities to perceive *larger* patterns in their domains and represent problems in that domain at a *deeper* level than novices
- experts’ tendency to spend a *relatively large* amount of time analysing a problem before carrying out a solution attempt, while at the same time being *faster* overall in providing a solution
- experts' display of *more extensive* self-monitoring skills while solving a problem, as well as
- experts' *superior* short- and long-term memory.

A notion strongly linked to expertise in the literature is the concept of “profession”, often used as a qualifier of certain types of highly effective performance (e.g. “professional action” or “professional vision”) or as a description of members of a group of experts within a domain ("professionals"). While one may excel at a specialized task and thus be considered to show expert level performance, for this to be referred to as professional action usually implies the situatedness of the task within a larger, more or less consolidated field whose practitioners share certain understandings and characteristics (Shulman, 1998). Shulman notes six attributes that all professions have in common:

- the obligation of service to others, as in a "calling";
- *understanding* of a scholarly or theoretical kind;
- a domain of skilled performance or *practice*;
- the exercise of *judgment* under conditions of unavoidable uncertainty;
- the need for *learning from experience* as theory and practice interact; and
- a professional *community* to monitor quality and aggregate knowledge. (ibid, p. 516, emphasis in original)

While many historical objects of study in the research on expertise, such as chess players or typists, arguably do not fulfil these criteria, the teaching *profession* as it is typically understood seems to be encompassed by the notion.

There have been some questions (Bromme, 2001) as to whether the original, cognitive idea of expertise can be applied to a process as interpersonal as teaching. After all, determining which teacher is "expert" and which one is not for the purpose of expert-novice comparisons is nowhere near trivial in teaching, where neither competitions nor clear, agreed-upon measures for teaching quality exist.

However, Berliner (2001) noted rather early on that the extensive research program by the National (US) Board of Professional Teacher Standards had demonstrated that "the use of data from the study of experts in other fields is [...] warranted" and that "experts in teaching do, indeed, share characteristics of experts in more prestigious fields such as chess, medical diagnosis, and physics problem solving" (ibid, p. 471).

Considering the research landscape of the past decades, the expert paradigm in studying (mathematics) teaching has certainly been firmly established and has proven to be a valuable framing for research as well as the development of teacher accreditation programs (see e.g. Yang & Leung, 2011; Cheng, 2017; Hoth et al., 2019).

At the time of the publication of Anderson and Taner's (2023) "metasummary" of expertise in teaching, more than 550 empirical papers had been published on the topic since the 1980s. In their systematic review, the authors report the following main findings regarding the cognition of expert teachers: Expert teachers are characterised by a well-developed knowledge base, a high level of awareness of characteristics of a teaching situation which impact student learning, as well as a set of typical beliefs. (Anderson and Taner also specifically point out the importance of the knowledge base as underpinning both expert teachers' "awareness" as well as "beliefs".)

Depending on which aspects of a teacher's work are considered and which focus is adopted for the analysis, different types of expertise in mathematics teaching are distinguished in the literature. Russ et al. (2011) set out to capture the variety of conceptions by describing four “images” of expertise:

- Mathematics Teacher as Diagnostician
- Mathematics Teacher as Conductor
- Mathematics Teacher as Architect
- Mathematics Teacher as River Guide

The *Diagnostician* image involves a teacher expertly making sense of the mathematical ideas in students’ speech, artifacts and behaviour and is generally accompanied by cognitivist, explicitly subject-specific ideas of teaching. The *Conductor* image envisions the expert teacher successfully directing and shaping mathematical discourse among the students, which involves both cognitivist perspectives as well as a more situated point of view (since interactions with the community of learners are central). The *Architect* image emphasizes the expertise needed to select, adapt and implement tasks from an abundance of curriculum materials and is, as such, closely related to the concept of design research (cf. “Mathematics education as a design science”; Wittmann, 1995). Lastly, the *River Guide* image focuses on spontaneous exchanges – student questions, unexpected solutions or errors, interruptions – within a planned lesson, during which an experts’ flexibility and problem-solving abilities are needed. These on-the-spot reactions require cognitive skills such as subject matter knowledge in a highly situated way, since particularities of a rapidly changing classroom environment are analysed and responded to (cf. the dimension of “Contingency”; Rowland et al., 2005).

2.1.2 Professional action and teacher competencies

In the kind of “professional” context as distinguished by Shulman (1998) above, expertise is usually understood to contain affective or motivational psychological components. In Weinert’s conception of “action competence” [Handlungskompetenz], for example, “intellectual abilities, content-specific knowledge, cognitive skills, domain-specific strategies, routines and subroutines, motivational tendencies, volitional control systems, personal value orientations, and social behaviors [form] a complex system” (Weinert, 2001, 51).

Many modern research programmes investigating teaching expertise base their notions of “professional action” in a *combination* of cognitive and affective aspects as in Weinert’s conceptualisation above – e.g. MT21 (Blömeke et al., 2008), TEDS-M (Blömeke et al., 2010) and TALIS (OECD, 2020). In the case of the COACTIV study (Kunter et al., 2011), for example, the model of teachers’ professional competencies includes cognitive components referred to as “competencies in the stricter sense of the word” as well as additional aspects, such as

- professional values, convictions, subjective theories, normative preferences and goals
- motivational orientations and
- professional self-regulation skills (Kunter et al., 2011, p. 33)

Schoenfeld (2019) also argues that cognitive components such as teacher knowledge should be understood as part of a more general framework of teacher proficiency that includes teachers' perceptions, inclinations and orientations.

The *Five Dimensions of a Powerful Mathematics Classroom* approach (ibid) can be considered to approach the question of teacher expertise from the angle of what a mathematics teacher should be able to provide to their students (see Figure 2).

The Five Dimensions of Powerful Mathematics Classrooms				
The Mathematics	**Cognitive Demand**	**Equitable Access to Mathematics**	**Agency, Ownership, and Identity**	**Formative Assessment**
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful mathematical thinkers. Discussions are focused and coherent, providing opportunities to learn mathematical ideas, techniques, and perspectives, make connections, and develop productive mathematical habits of mind.	*The extent to which students have opportunities to grapple with and make sense of important mathematical ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle."*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematical content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about mathematical ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

Figure 2: The Five Dimensions of Powerful Mathematics Classrooms (Schoenfeld, 2019, p. 3)

For example, in order to support rich mathematics (dimension 1) in their classroom, teachers "should know the mathematics they are expected to teach, in a deep way; they should understand how students come to grips with that mathematics, and be prepared to respond productively to what students, in the moment, reveal of their understanding" (ibid, p. 3). This paints a picture of teacher knowledge about mathematics itself and about its teaching, as well as of situational abilities like interpreting students' utterances and spontaneously responding.

Or, in order to encourage "productive struggle" in accordance with dimension 2, a teacher would need to adjust cognitive demand, which "calls for socio-emotional, organizational, and mathematical understandings" (Schoenfeld, 2019, p. 8). Teacher knowledge of mathematics is also involved in establishing sociomathematical norms that act as a framework for a safe and productive environment for classroom discussions (dimension 3).

As becomes apparent in the analytic approach taken by Schoenfeld, content knowledge is repeatedly mentioned as a central part of a teacher's expertise. However, rather than list it as its own category alongside other notions such as professional attitudes or knowledge of children's development, which are also required from a teacher, Schoenfeld flips the focus around and showcases which of a successful teacher's activities require sound content knowledge, appropriate belief systems, and other components.

This way of trying to determine key requirements for an expert teacher's performance can be related to the way "job-analysis" research (e.g. Bass & Ball, 2003) on teaching expertise and the required professional knowledge is conducted – first focusing on core tasks of a mathematics teacher and then analysing those tasks to determine which kind of background would be needed to expertly perform them.

Another term that is often used in the context of talking about professionals' "expertise" is *professional competency*. Like the notion of expertise, professional competency usually encompasses both explicit and implicit knowledge of practitioners, as well as their ability to act in a certain ("professional") manner, while the distinction between experts and non-experts is less central to these concepts. (Of course, it is still clearly implied, since professional competencies are used normatively, with those displaying more of them being judged as a more proficient practitioner compared to those displaying fewer competencies.)

In the large-scale 2008 study TEDS-M (Blömeke et al., 2010), professional competencies of future mathematics teachers (see Figure 3) were characterized along two dimensions reminiscent of the broad cognitive – affective categorization.

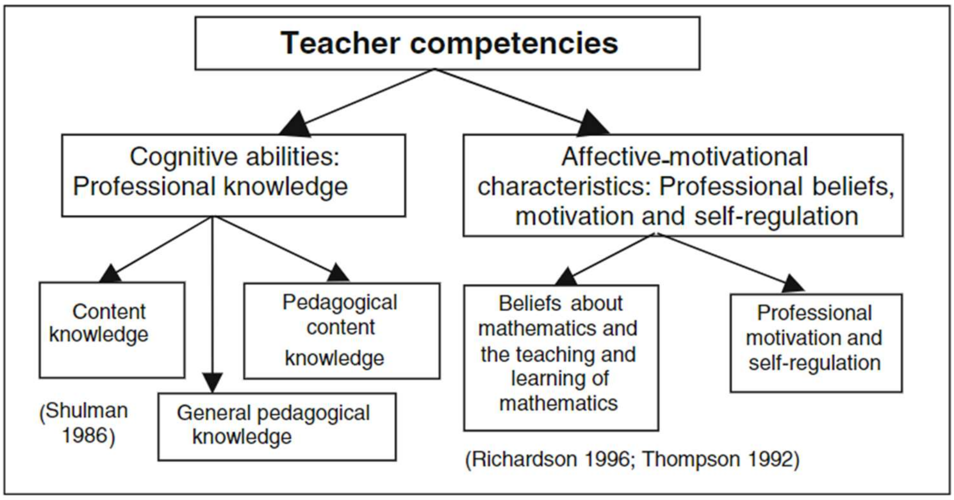


Figure 3: Conceptualisation of teacher competencies (Döhrmann et al., 2012, p. 327)

The cognitive aspects of professional competency were evaluated in TEDS-M by looking at participants' professional knowledge, while teacher beliefs represented the affective dimension. Not using an external metric to distinguish experts from less proficient participants, the study was able to deduce three competence levels from the internal structure of the achievement results (Kaiser et al., 2017).

For both primary and lower secondary (future) mathematics teachers, the following results were obtained:

Those at the highest competence level of content knowledge possessed strong structural content knowledge and were able to use this content knowledge in flexible ways, while those at lower levels had less sophisticated structural insights in the subject matter and struggled more to use their knowledge for argumentation and problem solving. The participants with the highest competence level of pedagogical content knowledge anticipated student difficulties and readily interpreted students' artifacts, while those at lower levels had more problems evaluating the correctness of student answers or strategies chosen by the teacher.

In the follow-up study TEDS-FU, the expert-novice paradigm and the skill of professional noticing was explicitly included in order to gain a more situated perspective. Here, professional competency was characterized by "a high degree of knowledge integration with multiple links and a categorical perception of teaching according to situations" (Kaiser et al., 2017, p. 171) and the competence facets *Perceiving*, *Interpreting* and *Decision making* (PID) were distinguished as contributing to the intermediary "situation-specific skills" that are conceptualised as being located somewhere between internal attributes of a person like cognition or affect and their externally observable performance (see Figure 4).

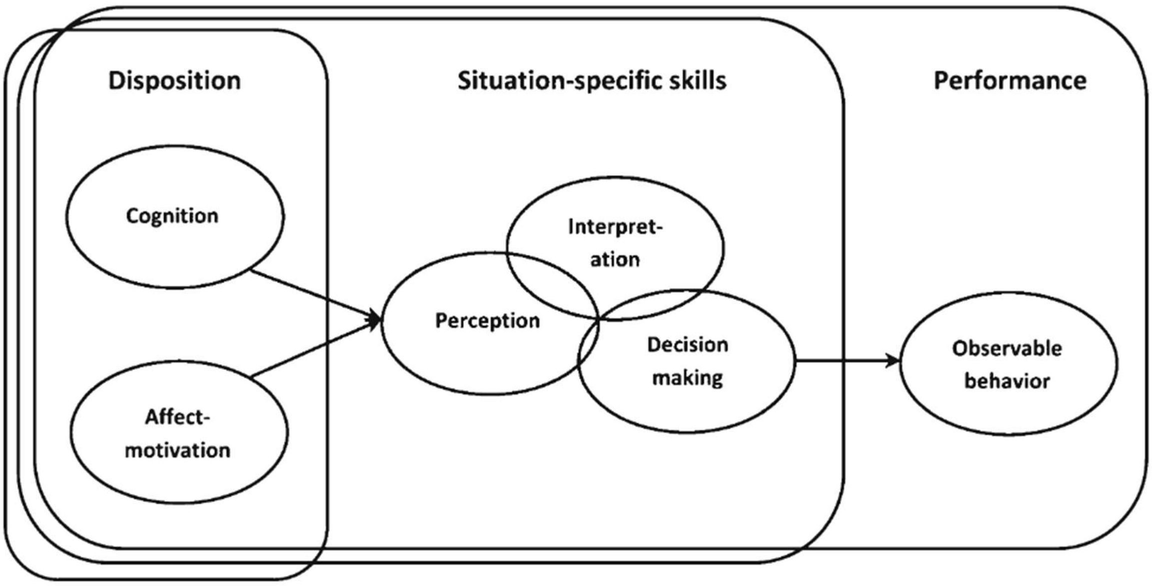


Figure 4: Teacher competence as a continuum (Blömeke et al., 2015, p. 7)

In these studies on teacher expertise and teacher competence, some recurring themes become apparent:

A teacher's expertise seems to rely heavily on specific *knowledge structures*, but is also made up of *affective, emotional dispositions* regarding their work. Aside from these underlying characteristics, there are *situated aspects* of teacher competence that can only be described and observed in the actual work of teaching ("situation-specific skills", Kaiser et al., 2017; "awareness", Anderson & Taner, 2023). Furthermore, these components are interrelated; professional knowledge influences experts' perception of situations, and the motivational facets are strongly related to the way content knowledge is accumulated and drawn upon as a tool for problem solving.

In order to be able to, ultimately, explore what the role of subject matter knowledge is for good teaching and how it can therefore be best represented in teacher training, it stands to reason that the concepts of teachers' (subject matter) knowledge, teachers' beliefs (about subject matter) and their situated skills (applied to subject matter) should be discussed.

From this list, I turn to the notion of *professional knowledge* first.

2.2 Professional knowledge of teachers

Talking about teachers' professional *knowledge* – while it is hardly contentuous that it is a central component of teacher expertise – is not a straightforward issue:

Firstly, there exist different conceptualisations as to what should count as "knowledge" in a strict sense versus more broadly (cf. big K Knowledge vs. small k knowledge; Schoenfeld, 2019). Secondly, different types of knowledge can be talked about, like "implicit" and "explicit" knowledge. Thirdly, it is not only the number of facts someone knows that matters – the specific structure of the knowledge possessed by an individual can make a big difference, as Kaiser et al. (2017) note. Lastly, thinking about knowledge as an abstract "standalone" quality of a teacher which, in a separate step, influences their actions in the classroom, leads to different questions, perspectives and research methods than a notion of situational teacher knowledge that cannot be fully disentangled from the classroom context itself.

The study of teacher knowledge is thus embedded in the general philosophical and psychological literature on the question of what constitutes knowledge. For this reason, the beginning of this chapter shall provide a brief overview of relevant concepts, some of which aren't necessarily specific to mathematics or mathematics education.

2.2.1 What is knowledge?

Anderson (1976) described two different types of knowledge: *declarative knowledge* and *procedural knowledge*. In the context of mathematics, an item of declarative knowledge would be the the $\varepsilon - \delta$-definition of continuity, while being able to prove the continuity of a specific function using this definition would be considered procedural knowledge. Two examples from general pedagogy might be knowing the Piagetian (Piaget, 1960) stages of development (declarative knowledge) versus being able to plan a developmentally appropriate lesson for eight-year-olds.

This idea goes back to Ryle's (1946) distinction of *knowing-that* vs. *knowing-how*. However, Ryle already pointed out that these two cannot be completely separated from each other, as he posited that "true" knowledge of a fact must necessarily go hand in hand with some ability to make use of this fact – being able to identify questions which it can help answer, recognizing situations in which it is applicable and, at least theoretically, being able to put it to use.

Hiebert and Lefevre (1986), in their analysis of different types of knowledge within mathematics, note a lack of investigations into the *interrelations* between the declarative knowledge and procedural knowledge, and especially between the knowledge of concepts and procedures as they are relevant to the teaching of mathematics. They categorize mathematical knowledge into *conceptual knowledge* and *procedural knowledge*.

The former is knowledge that is rich in relationships and cannot be, by definition, an isolated piece of information (using the example of $\varepsilon - \delta$-continuity above, a layperson who does not understand the terms used in the definition but is able to write it down can hardly be said to "know" the requirements for continuity). Procedural knowledge according to Hiebert and Lefevre consists of two distinct parts, one being the formal symbolic system of mathematics, and the other being the algorithms, rules or constraints for completing mathematical tasks.

Gray and Tall (1994) note that the same mathematical symbols often represent conceptual entities *as well as* processes (think of $\lim_{n\to\infty} f_n$ representing both the mental process of "taking f_n to the limit as n approaches ∞" *and* the numerical value of that limit, if it exists, just as, for a convergent series, $\sum_{i=0}^{\infty} x_i$ represents both an infinite summation *and* the resulting sum itself – a number). They conclude that being able to use the benefits of this ambiguous notation sets apart successful mathematicians (in the broadest sense) from people struggling to think mathematically, and define the category of *procept* as encompassing both conceptual and procedural knowledge of an idea.

While encapsulating – or *reifying* (cf. Sfard, 1989) – a process as a mental object is a specific step of cognitive development for Beth and Piaget (2013), Gray and Tall go further in saying that "the mathematician simplifies matters by replacing the complexity of process-concept duality by the convenience of process-product ambiguity" (Gray & Tall, 1994, p. 120). That is, it is one thing to be able to see the concept abstracted from the process, and another, far subtler and more useful thing to be able to appreciate their equivalence and to effortlessly move between the two.

Another way of looking at different types of knowledge involves analyzing how explicitly we can talk about what we can and cannot do, how well we can pinpoint what it is that we know, and in what ways we can showcase that knowledge.

Polanyi (1966) notes on this topic that "we can know more than we can tell", and calls the kind of knowledge that one can act by but not necessarily articulate, or even be aware of, *tacit knowledge*. Neuweg (2015) describes a similar feature of some forms of knowledge with the term *implicit knowledge*. This may be, for example, the knowledge employed when solving a routine task, "intuitively" choosing the most efficient method among many possible ones (e.g. determining the solution(s) of a system of equations). Depending on the person and the context, the practitioner often does not actively think about, for example, past experiences upon which their current action is based, and may not even be able to articulate the specific underlying knowledge when questioned.

Conscious knowledge of one's own cognitive processes and the monitoring of one's mental activities is often called *metacognition* or *metacognitive knowledge* (Flavell, 1979; Veenman et al., 2006). Kuhn (2000) describes a distinction within metacognitive knowledge akin to the dichotomy declarative/procedural knowledge: One can separate knowing that one knows something (*metacognitive knowing* in Kuhn's terminology) and knowing how to apply a procedure, select a strategy or find a problem's solution (*metastrategic knowing*). This kind of metacognition begins developing in early childhood years, but can be expanded upon well into adulthood: In mathematics, for example, proving something using the definitions, theorems and proof techniques one knows involves strategic knowledge about which theorems could be of use in this situation, or which solution methods it make sense to try. Research has shown that undergraduate students of mathematics possess far less such strategic knowledge than doctoral students (Weber, 2011).

Lastly, some kinds of knowledge seem to "arise" or at least be observable only in specific kinds of situations, i.e. "in action". De Jong and Ferguson-Hessler (1996) talk about *situational knowledge* as "knowledge about situations as they typically appear in a particular domain [which enables] the solver to sift relevant features out of the problem statement (selective perception) and, if necessary, to supplement information in the statement" (ibid, p. 106).

Here, one might notice an overlap between the concept of knowledge and the concept of noticing as they are being used by different authors – selective perception, in this case, being framed as part of a special kind of knowledge, while other authors (e.g. Dreher & Kuntze, 2015; Schoenfeld, 2011) conceptualize this specific way of perceiving as part of a "noticing skill" that is influenced by an individual's knowledge but distinct from it.

While categorising knowledge into such schemes on the basis of different cognitive structures is thus very fruitful, it is also illuminating to observe which contents of subject matter the construct teacher knowledge spans. After all, a mathematics teacher should know the mathematical content to be taught (and maybe even more!), but also have knowledge of their students, pedagogical strategies, the current mathematics curriculum, and so on. The following chapter summarizes the literature on this kind of "knowledge landscape" of mathematics teachers.

2.2.2 Mathematics teacher knowledge

Possibly the most influential paper on the subject of teacher knowledge is Shulman's (1986) work, introducing as a pair of terms "content knowledge" and "pedagogical content knowledge". In it, Shulman observes that what teachers of, for example, the school subject mathematics know is not merely a subset of the knowledge that mathematicians possess, nor is their pedagogical knowledge limited to "general" (in the sense of domain-unspecific) psychological considerations of their students' emotions or classroom group dynamics. Rather, there exists a distinct category of knowledge that comprises the application of pedagogical considerations to the specific subject matter.

> "Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice.
>
> Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons."
>
> (Shulman, 1986, p. 9)

According to Shulman, pedagogical content knowledge is also distinctive insofar as it is the "category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (Shulman, 1987, p. 8). Thus, pedagogical content knowledge represents both the additional specialised knowledge a teacher of a subject has over a "mere" subject matter expert, as well as the surplus knowledge a subject teacher possesses compared to other pedagogical professionals such as coaches or social workers (or teachers of other school subjects).

Aside from these crucial notions of *content knowledge*, general *pedagogical knowledge* and *pedagogical content knowledge*, Shulman also lists a variety of other categories of knowledge that a teacher should have, including

- curriculum knowledge
- knowledge of learners and their characteristics
- knowledge of educational contexts, as well as
- knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.

The category of *curriculum knowledge*, for Shulman, encompassed far more than being aware of the contents, demands and restrictions of the mathematics curriculum for the topics to be taught. Curriculum knowledge in this sense includes knowing of and knowing how to sensibly use tools for teaching the specific content of the curriculum (textbooks, visual aids, experiments, teacher demonstrations, …), being aware of the kinds of topics discussed in the pupils' other school subjects at a specific grade level (providing the basis for interdisciplinary examples or projects, a form of "horizontal integration"), as well as having an appreciation for recurring themes within the mathematics curriculum that lend themselves well to "vertical integration" – the repeated analysis of ideas at different levels of mathematical sophistication and abstraction (cf. Bruner, 1960).

Similar categorisations of teacher knowledge can also be found in Bromme (1992), who describes his "topology" of teacher knowledge as consisting of

- content knowledge about mathematics as an academic discipline
- knowledge of the mathematics taught in schools
- philosophy of school mathematics
- pedagogical knowledge and
- content-specific pedagogical knowledge.

Bromme thus actually characterises the teachers' knowledge of mathematics in three different ways – teachers should know some formal mathematics of the kind that is studied at universities as an academic discipline, plus the kinds of mathematics typically taught at school that are ultimately determined by the mathematics curriculum, *and* have some meta-level insight about mathematics as a special field of inquiry and social sphere, for example, knowledge regarding the subject's epistemology and accepted types of discourse within the mathematical community.

In the same year as Bromme, Fennema and Franke (1992) distinguished four sections of research relating to mathematics teacher knowledge:

- *Knowledge of Mathematics* (different forms of content knowledge),
- *Knowledge of Mathematical Representations* (knowledge about how mathematics should be transformed to be suitable for instruction),
- *Knowledge of Students* (e.g. knowledge of children's development and cognitions),
- *General Knowledge of Teaching and Decision-Making* (incl. classroom management techniques and lesson planning strategies based on previous teaching experiences)

These categories differ from Bromme's topology, in that they separate two different kinds of "generic" (not content-specific) pedagogical knowledge: More theoretical background knowledge on students' mental processes that informs how teachers approach their work, and the day-to-day expertise in handling a group of learners in spontaneous interactions or orchestrating specific learning opportunities. On the other hand, the notion of listing subject matter knowledge of mathematics separately from knowledge about how to best make that knowledge accessible to learners again reflects the CK/PCK distinction in this summary of the research landscape of the early 1990s.

Subdivisions of pedagogical content knowledge itself can be found in the COACTIV study programme, where they are called "facets of knowledge" of the "domain of knowledge" PCK (see Figure 5).

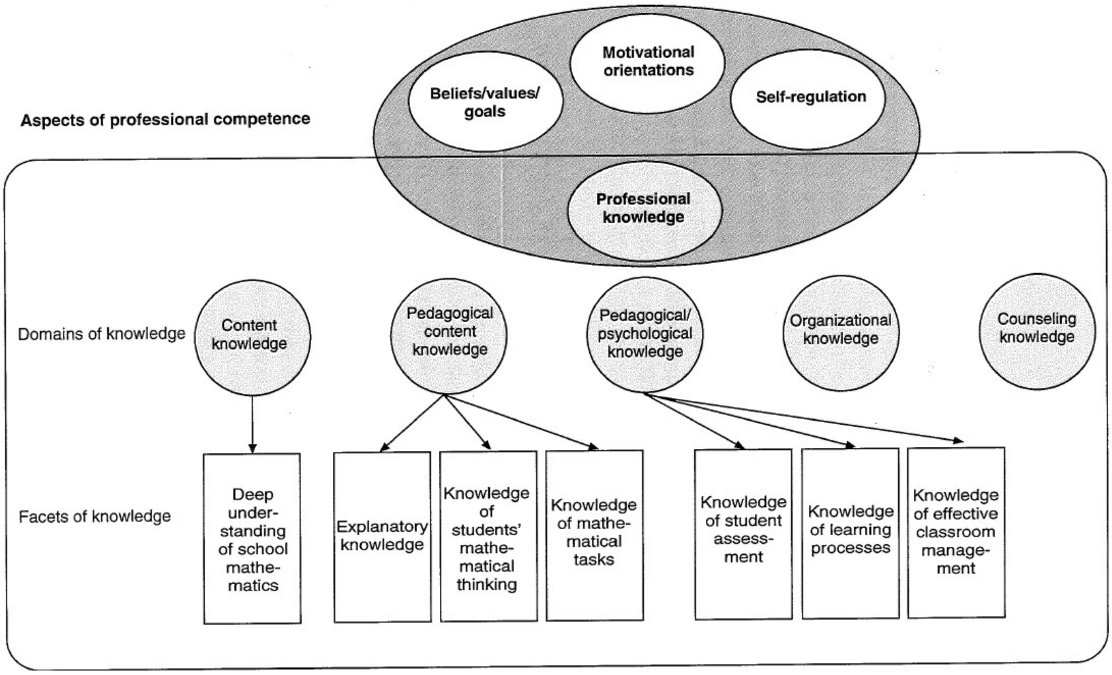


Figure 5: COACTIV conceptualisation of professional competence (Baumert & Kunter, 2013, p. 29)

Pedagogical content knowledge is described to contain the facets of "explanatory knowledge", "knowledge of students' mathematical thinking" and "knowledge of mathematical tasks". Curriculum knowledge as such is not expressed as a separate facet of knowledge in the COACTIV conceptualisation of teacher competence.

Ball, Thames and Phelps (2008) have continued this work of trying to make more explicit what the terms "content knowledge" and "pedagogical content knowledge" of mathematics teachers might encompass. In their MKT "egg" model (see Figure 6 on the next page), the mathematical knowledge a teacher possesses falls under subject matter knowledge, or pedagogical content knowledge, with three subtypes each.

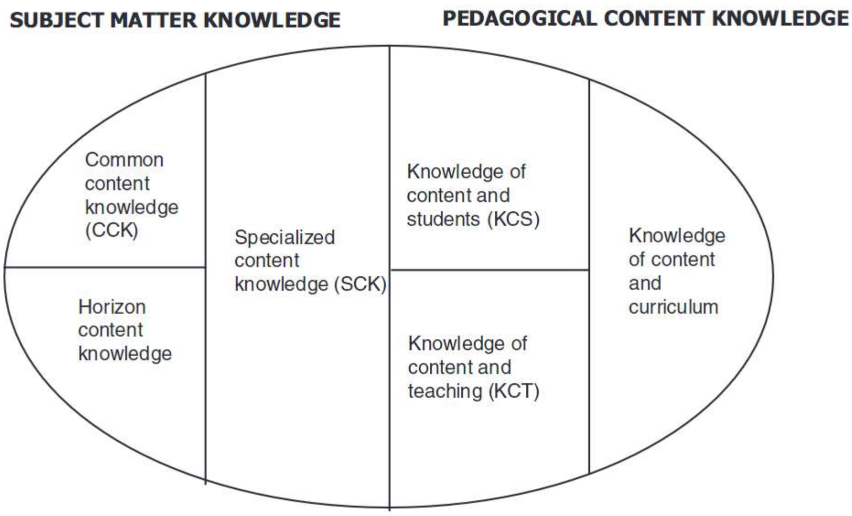


Figure 6: Subtypes of subject matter knowledge and pedagogical content knowledge (Ball et al., 2008, p. 403)

Based on knowledge requirements for primary school teachers, the authors specified three subcategories of the kind of mathematics school teachers of the subject should know: For one, *common content knowledge* (CCK), consisting of "everyday mathematics" (e.g. methods for solving a system of equations) known to many people who don't specifically work in mathematics education. In contrast, *specialized content knowledge* (SCK) consists of mathematical ideas or activities that few people outside of mathematics education will think about and/or regularly use (for example, being aware of different representations or examples of a type of mathematical object and their respective benefits and limitations when teaching). The *horizon knowledge* subcategory describes an awareness for connections both across the school mathematics curriculum as well as between concepts studied at school and more advanced mathematical ideas.

Pedagogical content knowledge is also further broken up by Ball and colleagues into

- *knowledge of content and students* (anticipating what mathematical ideas students will find confusing, motivating or challenging, interpreting students' mathematical thinking based on their speech or artifacts produced, etc.),
- *knowledge of content and teaching* (sequencing mathematical activities in a specific way, choosing examples of mathematical ideas and wording mathematical explanations to achieve specific teaching goals, etc.), and
- *knowledge of content and curriculum* (defined explicitly analogously to Shulman's category of "curriculum knowledge").

Comparing the roles of CK and PCK (with their respective subdomains) between models such as the COACTIV conceptualisation of professional competency and the MKT framework is not trivial, as notions of prerequisite knowledge for primary school teacher may not serve to describe the type of knowledge required to teach secondary school mathematics.

Dreher et al. (2018) note that in addition to the analysis provided by Speer et al. (2015) on the problematic distinction between CCK and SCK at a secondary level, models of mathematics teacher knowledge for secondary schools would (instead?) greatly benefit from separately considering "school mathematical knowledge" versus "academic mathematical knowledge". Dreher and colleagues describe "academic mathematics" as the kind of mathematics usually published and taught at universities, typically presented in a rigorous, deductive and axiomatic manner. "School mathematics" on the other hand, is usually more concerned with providing students with conceptual understandings based on well-chosen examples and locally adequate reasonings, rather than rigorous and general definitions and proofs.

Another reason for discrepancies between German-speaking conceptualisations of teacher knowledge like COACTIV and Anglo-American ones like the MKT framework may be the different research traditions and resulting notions of content knowledge versus pedagogical content knowledge. While the German term "fachdidaktisches Wissen" (subject-specific pedagogical knowledge) is commonly translated with "pedagogical content knowledge", it is worth noting that the two notions stem from different academic contexts. While Shulman (1987, p. 8) conceptualised PCK as an "*amalgam*" of content and pedagogy in the classroom context, the German tradition of "subject matter didactics" (Stoffdidaktik) has often focused on pedagogical content knowledge as a way of elementarising mathematical concepts in an intellectually honest manner (Scheiner & Buchholtz, 2022). Scheiner et al. (2023) add to this discussion by further analysing the differences between "specialized content knowledge" as it is found in the MKT conceptualisation of mathematics teacher knowledge (Ball et al., 2008) versus "subject matter didactic knowledge" in the German-speaking community of mathematics education. Due to these inherently subject matter focused conceptualisations of "PCK" in this German tradition, it has been difficult to distinguish and operationalise the two knowledge types in German research programmes – content knowledge and pedagogical content knowledge – and they are also often empirically found to be strongly correlated (Buchholtz et al., 2014).

Distinguishing CK from PCK, however, is not central to the research endeavour of this doctoral thesis, since the focus specifically lies on all highly content-related tasks of teaching and the knowledge and perception of mathematics required to be able to fulfil these expertly.

Thus, in the intervention, PSTs' attention was specifically drawn to applications of either "pure" content knowledge or the type of pedagogical content knowledge that would not be possible without a strong focus on mathematical content. Using either type of mathematics teacher knowledge would in theory allow PSTs to see the relevance of a thorough subject matter preparation as part of their teacher education.

2.2.3 Subject matter knowledge

Kilpatrick et al. (2015) shift the focus of analysis by trying to investigate which kinds of "mathematical understanding" a teacher of mathematics in secondary school should possess (motivating the grade-level limitation by noting that both the range of mathematical content, the emphasis on formality and abstraction as well as the cognitive development of the students in this age group are very different from the often-studied context of elementary schools). The researchers specifically point out a difference between pedagogical content knowledge according to Shulman (1986) and their conceptualisation of "mathematical understanding for secondary teaching" (MUST), which according to them encompasses more than content knowledge enriched by pedagogical understanding.

The MUST framework seeks to provide three different "lenses" through which the mathematical understanding of a secondary teacher can be viewed:

- **Mathematical Proficiency** – spanning conceptual knowledge, procedural fluency, strategic problem-solving competence, adaptive reasoning abilities, a productive disposition toward mathematics as well as historical and cultural knowledge about mathematics
- **Mathematical Activity** – including mathematical noticing, mathematical reasoning, mathematical creation, and the integration of different strands of mathematical activities in processes such as modelling a real-world problem
- **Mathematical Context of Teaching** – involving analyzing mathematical ideas for teaching purposes, understanding and assessing students' mathematical thinking, knowing the curriculum, and reflecting on one's own practice

The concept of MUST thus functions as less of a categorization scheme and more of a framework to explore different facets of content knowledge that a teacher should possess. This reminds of the notion of *Profound Understanding of Fundamental Mathematics* (PUFM), developed by Ma (2010).

Ma describes a teacher's PUFM as "well-developed, interconnected knowledge packages" (2010, p. 102) combined with general attitudes toward mathematics such as justifying claims with arguments, keeping consistency across various contexts and striving to know "why as well as how". This profound understanding of mathematical topics involves connecting a topic with other, more (or less) conceptually advanced ideas in mathematics, as well as connecting it to ideas in different areas of mathematics, both of which require a thorough map of the "terrain" of mathematics. Teaching based on such understanding should, according to Ma, have four mathematics-related properties:

- **Connectedness** – being aware of parallels between different concepts across subdomains and grade levels, thus allowing students to obtain a unified body of knowledge instead of isolated facts
- **Multiple Perspectives** – knowing different aspects of an idea and solution methods, as well as being able to evaluate advantages and disadvantages of different approaches, thus allowing students themselves to gain flexible understanding of mathematics

- **Basic Ideas** – displaying appropriate mathematical attitudes and relating topics back to the main ideas of the subject (cf. "fundamental ideas"; Bruner, 1960), thus guiding students to conduct their own authentic mathematical activities
- **Longitudinal Coherence** – having an understanding of the mathematics curriculum as a whole, exploiting previously studied concepts and preparing the ground for ideas to be developed later

A teacher being aware of and making use of these four ideas in his or her teaching would be demonstrating "profound understanding" of their subject.

Focusing again on secondary teachers, Heinze et al. (2016) expand upon the concept of mathematics-related teacher knowledge by introducing a third category – school-related content knowledge (SRCK) – to the traditional split into content knowledge and pedagogical content knowledge. This special type of content knowledge involves insight into the sequencing of different concepts within the curriculum in terms of mathematical structure, knowledge about how to adequately accomplish an "intellectually honest" (Bruner, 1960) didactic reduction for various mathematical ideas, as well as knowledge about the connections between various topics in the secondary mathematics curriculum and academic mathematics.

SRCK thus differs from CK in that it is content knowledge that is specifically relevant to (secondary) school teaching and, in general, not knowledge the absence of which would be detrimental to professionals using mathematics in fields outside education. At the same time, it is distinct from PCK as it does not itself involve pedagogical considerations – rather, it provides a basis for making informed pedagogical decisions, e.g. by knowing that certain explanations or proofs hinge on mathematical ideas that have not (yet) been covered at a certain grade level and as such will not efficiently lend themselves for teaching purposes. (Being able to judge the correctness or adequacy of a mathematical proof would constitute CK. Knowing three different proofs of a theorem with varying prerequisites for three different grade levels is a part of SRCK. Choosing the most illuminating or motivating proof among the mathematically "permissible" ones at the students' level of mathematical sophistication, or picking the most insightful or memorable way of representing the ideas involved in that proof to students would fall under PCK.)

SRCK is also described to involve three "facets": curricular knowledge about the structure of school mathematics, and knowledge about interrelations between school mathematics and academic mathematics in a *top-down* as well as a *bottom-up direction* (Dreher et al., 2018). This is related to Wasserman et al.'s (2017) B*uilding Up from* and S*tepping Down to Practice* processes, which describe the way in which a teacher might relate situations, remarks and artifacts that arise in a teaching context to his or her academic content knowledge, form a mathematically informed opinion, and then have to find a way to convey this meaning to students in a way that is appropriate given their mathematical background.

Following this analysis of the theoretical nature of mathematics teachers' content knowledge, the next subchapter is concerned with a review of empirical findings regarding the level and the effects of mathematics teachers' content knowledge.

2.2.4 State of research: subject matter knowledge

> "It is obvious that knowledge of mathematics is basic to being able to help someone else learn it. Teachers must understand concepts and procedures themselves in order to select and construct fruitful tasks and activities for their pupils, as well as to flexibly interpret and appraise pupils' ideas." (Ball, 1988, p. 43)

Fennema and Franke (1992) contend that one of the most common explanations (within the community of mathematics education) for students not learning mathematics is their teachers' own inadequate mathematical knowledge. Or, as Post and colleagues (1991, p. 195) put it, "many teachers simply do not know enough mathematics".

Both statements imply two hypotheses that can be empirically tested: That teachers' subject matter knowledge tends to be poor, and that this lack of subject matter knowledge has an adverse effect on their students' learning gains. For a long time, assessments of teacher content knowledge and its effects on student performance tended to fall short of providing adequate support for a direct relationship one way or another, for example, because they used proxy measures (such as the number of mathematics courses taken) instead of directly measuring content knowledge or employed correlational, rather than experimental, methods (Fennema & Franke, 1992).

Monk (1994), for example, reported that while there is a positive relationship between the number of mathematics courses a mathematics teacher has taken and their students' performance, the effect became smaller and smaller after only a few mathematics courses (i.e. having taken three instead of one mathematics course benefitted one's students, but having taken seven instead of five made a much smaller difference in terms of additional student achievement). Also, it was found that courses in *mathematics education* actually contributed more to student performance gains than courses in mathematics.

More than two decades later, Ferrini-Mundy et al. (2016) still criticise that even large-scale studies trying to establish causation in this regard usually fail to directly assess teachers' subject matter knowledge. Exceptions are Hill et al. (2005) using the Mathematical Knowledge for Teaching (MKT) construct to measure teachers' content knowledge, or the COACTIV study (Baumert & Kunter, 2011).

Hill and colleagues studied first and third graders (with a total of 699 teachers) and found a significant relationship between teachers' mathematical knowledge for teaching and their students' performance gains, with an effect size similar to the effect student background variables had on this dependent variable, thus outperforming the effect sizes of other independent variables such as teacher background and time spent on mathematics instruction each day.

In the COACTIV study, 181 teachers and their 10th grade students were studied. Here, the effect of pedagogical content knowledge on student achievement gains was found to be significant with a substantial effect size, with mathematical content knowledge showing a much lower predictive effect (constituting the prerequisite basis for sound pedagogical content knowledge in the theoretical conceptualisation).

Both the COACTIV study as well as Hill et al. (2008) additionally looked into the relationship between subject matter knowledge and indicators of teaching quality (as opposed to indicators of teaching effectiveness like student performance).

Hill et al. found in their qualitative analysis of 10 teachers that high-knowledge teachers more consistently avoided errors, used more rigorous mathematics in instruction, gave more examples of rich mathematics and showed skill in picking appropriate examples, responding to students and providing their students with opportunities to learn. While lower-knowledge teachers also displayed some of these elements, they were not as consistent and thus these elements were just as often found to be lacking in their lessons.

The COACTIV project in turn analysed teaching quality along four dimensions: cognitive potential of tasks, curricular level of tasks (accordance with the curriculum for the 10th grade), constructive support (including adaptive simplification, constructive handling of mistakes, and appropriate pacing of interactions), and classroom management. Pedagogical content knowledge was shown (Baumert & Kunter, 2011) to have an effect on the first three dimensions of quality teaching, while good classroom management appeared to be possible even without a teacher's adequate pedagogical content knowledge. Considering content knowledge separately, a similar picture as the one relating to student achievement gains is painted by same study: Content knowledge by itself does not show a significant effect on the cognitive potential of the tasks given to students or the individual constructive support provided to them. Only the curricular level of tasks could be shown to be correlated with higher content knowledge.

Zooming out from these two examples, reviews of the literature conclude that overall, empirical studies of directly measurable relationships between the extent of teachers' content knowledge and student achievement have shown mixed results (Fennema & Franke, 1992; Monk, 1994; Ferrini-Mundy et al., 2016). Further taking into account the likelihood of many null results remaining unpublished, the empirical picture remains foggy as to the extent that teacher subject matter knowledge predicts student performance and teaching quality.

Still, the positive findings in the two recent large-scale studies together with the theoretical considerations as to the importance of intimately knowing the subject matter as a teacher suggest that a teacher's mathematics-specific knowledge plays a substantial role for quality teaching and desirable learning outcomes for students.

If this is indeed the case, it seems especially problematic that many mathematics teacher educators perceive (prospective) mathematics teachers' content knowledge to be rather poor and even well-designed interventions and sophisticated efforts to restructure study programs to better serve preservice teachers in this way show only small effects (Buchholtz & Kaiser, 2013; O'Meara et al., 2017; Pournara, 2015).

One of the reasons additional and improved opportunities to learn mathematics in teacher education do not lead to the attainment gains hoped for may be the beliefs prospective teachers hold about mathematics itself and the relevance of the mathematics-related courses in their training programmes. The concept of beliefs will be introduced in the following chapter, with a focus on teacher beliefs specifically relating to mathematics content knowledge.

2.3 Teacher beliefs

2.3.1 What are beliefs?

Teacher beliefs have been the subject of extensive research for multiple decades now. As far back as the early 90s, a multitude of publications on the topic existed (Thompson, 1992). Many papers were using differing definitions and conceptualisations of the phenomenon as well as the mechanisms by which beliefs were formed or by which these beliefs in turn influenced other aspects of a person's characteristics.

One widely cited definition of beliefs is the one given by Philipp (2007):

> [beliefs are] psychologically held understandings, premises, or propositions about the world that are thought to be true (ibid, p. 259)

Philipp goes on to state a few core characteristics of beliefs in the same paragraph, in particular, making distinctions between beliefs and attitudes (manners of acting, feeling, or thinking that show one's disposition or opinion) as well as between beliefs and knowledge: "Beliefs are *more cognitive*, are *felt less intensely*, and are *harder to change* than attitudes. Beliefs might be thought of as *lenses that affect one's view of some aspect of the world* or as *dispositions toward action*. Beliefs, unlike knowledge, may be held with *varying degrees of conviction* and are *not consensual*" (ibid, emphasis added). Importantly, while beliefs are conceptualised as relatively stable, research shows that with proper interventions, beliefs can be changed (e.g. Panero et al., 2023).

"Consensuality" here refers to the fact that most people holding a certain belief are aware that others may believe differently, and that those differing beliefs cannot be disproven in a way analogous to the socially accepted mechanisms for evaluating and arguing the validity of statements of knowledge.

The COACTIV study's working definition expands upon this notion by defining beliefs [Überzeugungen] to be "lasting existential propositions about phenomena or objects in the world that are subjectively considered to be true, contain explicit as well as implicit parts and influence the way one encounters the world" (Voss et al., 2011, p. 235). This is consistent with Philipp's definition, while additionally pointing out the implicit nature of many beliefs.

Integrating different definitions, Op't Eynde et al. (2002) define students' [mathematics-related] beliefs as "the implicitly or explicitly held subjective conceptions students hold to be true [that influence their mathematical learning and problem solving]" (ibid, p. 16).

Despite some discrepancies and open ontological questions, relatively robust findings about beliefs can be extracted from a survey of the literature (Pajares, 1992):

- beliefs tend to form early on in an individual's life
- long-held beliefs become progressively more resistant to change
- pre-existing beliefs influence how people take in and process new information
- beliefs influence how people approach tasks and solve problems

With regard to teachers, this means that preservice teachers enter university with strong beliefs about school and teaching already ingrained by more than a decade of education. Additionally, because beliefs strongly affect people's actions, preservice teachers' beliefs should be taken into consideration when designing teacher training programmes.

One of the biggest challenges in describing beliefs as a construct has been delineating a clear border between the interrelated concepts of beliefs and knowledge (Ashton, 2014). Conceptualisations that were not clear on the nature of the relationship between knowledge and beliefs were one of the main reasons for Pajares' (1992) judgment that beliefs were "a messy construct". Österholm (2010), for example, notes that even the often-made distinction between knowledge and beliefs in terms of their affective component (cf. Philipps, 2007) is disputable. Certain items of knowledge may very well be affectively loaded, such as when the "discovery" of the piece of information was accompanied by feelings of surprise, joy or dismay. At the same time, it could be argued that beliefs themselves aren't affective and instead are an influence *on* our emotional reactions, among other factors.

Even though the specific nature of beliefs has not been definitively settled, research into the beliefs (systems) of teachers is a widely established subfield in mathematics education nowadays. Its relevance is particularly salient with regard to the effect different teacher beliefs may or may not have on teachers' own professional development, their resulting instructional decisions and thus, ultimately, on their students' learning. The following chapter will therefore elaborate more specifically on the theoretical landscape of mathematics teachers' beliefs.

2.3.2 Mathematics teachers' beliefs

Mathematics teachers' beliefs can be conceptualised as a *system of beliefs* (Rokeach, 1968; Green, 1971) within which there exist beliefs of differing degrees of centrality to a person, as well as beliefs about various domains (McLeod, 1992), such as beliefs about mathematics as a subject, or about the teaching of mathematics, beliefs about the self as a problem solver, or about one's identity as a mathematics teacher, and beliefs about education and the role of teachers in a broader sense.

Ernest (1989) furthermore describes three broad beliefs about the nature of mathematics itself:

- mathematics as an open field of human inquiry (problem-solving view)
- mathematics as a unified, static body of interconnected truths which are discovered (Platonist view)
- mathematics as a useful collection of unrelated facts, rules and skills (instrumentalist view)

According to Hiebert and Berk (2020), one goal of teacher preparation is to change some preservice teachers' "beliefs what mathematics is and what it means to know mathematics" (ibid, p. 327). Convincing people to relearn something they believe they already know is a difficult endeavour, especially if this relearning process requires more effort than the original learning process they remember. However, the importance of correcting PST's impressions about mathematics teaching and learning which they gathered being students themselves is of great importance, as these ideas about teaching shape their later perception of situations in their own teaching practice (Calderhead & Robinson, 1991; Nesmith, 2008).

In the early 90s, it was assumed that teachers' beliefs shape their teaching practice, but there were varying empirical results across different beliefs domains (e.g. teachers' conceptions of the nature of mathematics having a more clearly observable effect on their teaching than their beliefs about mathematics instruction) (Thompson, 1992). In the COACTIV project (Kunter et al., 2011), it could be shown that teacher beliefs had significant effects on the cognitive activation potential of the observed mathematics lessons, which in turn was significantly correlated with student outcomes. The fact that teachers' beliefs and knowledge affect their teaching styles (Zhang, 2022; in a study with 92 secondary teachers) and, via these teaching practices, their students' achievement (Wang et al., 2022; in a study with 1463 teachers and their 12,437 students in grade 4), could also be shown in recent studies of Chinese teachers.

In general, research tends to conceptualise a dialectic relationship between *beliefs* and practice. The discrepancies between beliefs professed by teachers versus their actions have been explained as a conflict between an individual's beliefs and the actual classroom context (Hannula et al., 2016). In accordance with this view, then, beliefs seem closely connected to classroom-based teacher *knowledge* as situated cognition (Rhoads & Weber, 2016; Davis & Simmt, 2006) as well as to *noticing* as a skill for making sense of teaching situations (Wessels, 2018). For this reason, in this study, the three constructs are considered in combination.

In the particular context of teacher training and PSTs' subject matter education, PSTs' beliefs about the relation of school mathematics to university mathematics and the relevance of learning academic mathematics at university for school are of particular interest (cf. Isaev & Eichler, 2017). Based on the results above, it can be assumed that PSTs' beliefs about the applicability of knowledge of advanced mathematics will have an effect on their perception of teaching situations, which will in turn influence their teaching practices.

The next subchapter will provide a survey of the literature regarding teachers' and PSTs' beliefs about the subject matter education which they undergo during teacher training.

2.3.3 State of research: beliefs about subject matter knowledge

Toward the end of the last century, multiple studies in Germany showed a general lack of understanding by preservice teachers as to why it was necessary for them to study higher mathematics at university.

Tietze (1990) found that secondary school teachers' self-image has moved away from considering themselves academic mathematicians. Many of the respondents thought that the mathematical content in the teacher training programmes was too extensive and didn't match the goal of training future educators. At the same time, the teachers felt ill-prepared to teach applications of mathematics and talked about their difficulties when selecting and presenting appropriate examples and exercises.

Danckwerts (1983) had also noted that secondary preservice teachers found it difficult to transfer and make use of their general and abstract knowledge of analysis within the context of school teaching. This shift in beliefs having occurred over time could be observed in the data, as most of these comments were made by younger teachers – only 6 % of teachers who had completed their studies before 1963 reported similar views.

Bungartz and Wynands (1998) replicated these findings about young or prospective teachers not feeling adequately prepared to teach mathematics in their study of 176 secondary "trainee teachers" [Referendar*innen; meaning teachers roughly in their first two years of work after finishing their university education] in Nordrhein-Westfalen, Germany. Respondents who considered the mathematical content courses of the teacher training programmes to be too demanding tended to leave university with less mathematical proficiency as well as a weaker mathematical self-image and lower confidence in their own skills.

Pieper-Seier (2002) contrasted beliefs of 177 prospective secondary mathematics teachers and 275 university students studying in a non-teaching mathematics major (Diplomstudium). Her results show that the preservice teachers do not manage to develop a resilient, affect-based relationship toward their subject, mathematics. In general, they do not show any particular interest toward mathematics as an academic subject and research area. Pieper-Seier concludes that in a subject as work-intensive as mathematics, this general dissatisfaction with their studies coupled with less-than-helpful views of their subject make it harder for preservice teachers to succeed academically.

Some of the beliefs that teachers hold about their professional education seem to be contradictory at a first glance. In the study mentioned above, Tietze (1990) reports that many teachers consider the mathematical, subject-matter focused part of the accreditation programmes as too extensive, while at the same time complaining about leaving the programme not feeling competent in their own mathematical skills.

Wu (2011) finds this unsurprising, questioning the "trickle down hypothesis" inherent in many teacher accreditation programmes, according to which subject matter knowledge can be discussed in a way that is removed from the context of teaching, and future teachers will be able to transfer this knowledge learnt in the university context to their professions as schoolteacher automatically.

Zazkis and Leikin (2010) also report that "students, even those identified in school as high-achieving students, experience unexpected difficulties when beginning undergraduate mathematics courses, and many teachers perceive their undergraduate studies of mathematics as having little relevance to their teaching practice."

Bungartz und Wynands (1998) obtained similar results when asking first year teachers about their recently ended studies. Additionally, they found that solid knowledge of mathematics and confidence in one's own mathematical abilities tended to be paired with less dissatisfaction with the subject matter related parts of the curriculum. In other words, those most dissatisfied with their subject matter education were the ones who were and felt least prepared mathematically, and vice versa.

Prospective teachers differ from mathematics majors in their handling of successes and failures during their training. According to Curdes et al. (2003; with 739 partipants across both groups), student teachers often blame failures on a lack of (innate) abilities of theirs rather than a lack of effort on their own part. Similarly, when succeeding at a task, they tend to attribute this to their own capabilities and not necessarily hard work or productive studying. Overall, preservice teachers lose a lot of confidence in their own mathematical abilities over the course of the first semesters. However, this feeling of disillusionment when entering university and becoming acquainated with academic mathematics is not particular to students who go on to become teachers, see for example Bressoud (2015) or Di Martino and Gregorio (2023).

Törner (1999) looked into this emotional dimension of studying (university) mathematics in a qualitative study. Analysing 18 student essays that six secondary preservice teachers wrote in a longitudinal design (the PSTs being asked to write an essay at three different times points each) revealed a range of negative emotions connected to mathematics, the student teachers' descriptions ranging from "not too pleasant" to "brutal". According to the conference paper, those university students who successfully completed their studies had somehow survived and come to terms with these subjective hardships, but hadn't processed their experiences in a way that would have allowed them to retrospectively gain an appreciation and take away a positive impression of their studies in mathematics.

Due to these experiences, and due to the fact that they felt teacher training was too theoretical and removed from the practice of teaching, as many as 50 % of prospective teachers reported having thought about quitting or changing majors in a 2006 study of 741 secondary teachers and mathematics majors (Mischau & Blunck, 2006). Blömeke

(2009) also reports in a longitudinal study (n = 760) that prospective teachers of mathematics experience more subjective hardship in their studies compared to non-teacher mathematics majors (while receiving worse grades) and think about quitting more often.

At the same time, as I mentioned above, teacher educators and members of the research community raise concerns (Lambert & Wittmann, 2015) about the diminishing number of content courses in mathematics teacher training due to the addition of "generic" (not subject-related) teaching classes and increased hours of practical teaching experience.

This is in stark contrast to the fact that PSTs seem to want to relate the contents of their studies to their future profession as mathematics teachers "from day one" and thus – unless there is some intervention or explication provided – will likely consider purely theoretical courses in mathematics, mathematics education and pedagogy as less meaningful than the practical parts of the programme (Cramer et al., 2009).

There are two important points that one can conclude from these results: On the one hand, there is a clear dissatisfaction with the amount and the demands of university mathematics, both of which apparently overwhelm many preservice teachers. On the other hand, preservice teachers also feel a strong dissatisfaction with their own mathematical competence (as shown in Liebendörfer, 2018). Hence, it does not seem to be the case that educators are arguing for a stronger focus on mathematical subject matter while student teachers generally feel mathematically well-prepared for their future jobs. Rather, completing the mathematically demanding university classes does not appear to instill a sense of competence in the prospective teachers. This has far-reaching consequences:

Even those university students who report equal preference for both of their subjects (in Austria and Germany, teachers usually choose two subjects to study in teacher training and later teach at schools) seldom choose a mathematical topic on which to write their final theses (Bungartz & Wynands, 1998). They may be less ambitious and aim for more humble goals due to an expectation and fear of failure (Pieper-Seier, 2002; Curdes et al., 2003). Compared to other students of mathematics, their relationship to their own field of study remains strongly ambivalent (Mischau & Blunck, 2006) – a highly problematic characteristic of mathematics teachers, part of whose profession should arguably involve appearing as an "ambassador" for mathematics and instilling within the students a sense of what mathematics can do and what thinking and acting mathematically can look like.

Based on the notions that PSTs may not be inclined to learn advanced mathematics at university because they do not see the necessity for it in in their future profession and that even mathematically competent PSTs struggle to apply their content knowledge within teaching contexts, an investigation into teachers' perception of classroom situations and professional tasks of teachers outside the classroom seemed appropriate. After all, pointing out situations in which expertly assessing the subject matter requirements of a teacher task and learning how to proceed based on this assessment could change PSTs' beliefs regarding the relevance of subject matter via an increased awareness of the need for and the use of sound subject matter knowledge in their future profession.

The following chapter will therefore delve into the topic of teachers' professional vision, also known as teacher noticing. After an introduction into the theoretical construct and a review of relevant empirical findings about mathematics teachers' noticing, a case will be made for a specific analytic lens that focuses on PSTs' perception of the mathematical content present in typical teaching situations.

2.4 Teacher noticing

2.4.1 What is noticing?

The domain-specific way of perceiving professional situations termed "professional vision" or "noticing" has been identified as a key skill that sets apart experts from novices of a field (Goodwin, 1994). As such, noticing is now recognised as an established component of expertise (National Research Council, 2000).

In the complex classroom environment, employing selective attention is of great importance, as teaching situations can be characterised by their multidimensionality and simultaneity (at any point during the lesson, there are a *large number* of things happening *at once*), as well as their immediacy (events during a lesson follow a fast pace and many teaching actions therefore need to be taken quickly) (Sabers et al., 1991).

Expert teachers outperform novices and advanced beginners when it comes to monitoring and interpreting meaningful events, as well as offering hypothetical reasons for and solutions to problems that are identified (ibid; Carter et al., 1988). At the same time, in line with expertise studies in other areas such as chess (de Groot, 1966), experts could not be shown to have superior memory when it came to less relevant details of instructional scenes.

Another way in which experts' perception differs from that of novices is that experts tend to categorise problems according to different criteria. The latter consider situations to be related if they share surface characteristics, while experts focus on deeper, structural similarities (Schoenfeld & Herrmann, 1982).

While one can describe the ways in which successful teachers notice what their less effective colleagues do not, there is another lens through which one might consider noticing, namely by approaching noticing to be less of a skill that one acquires (or doesn't), and more like a practice which a person striving to become a better teacher may adopt – "a systematic method for conducting research into one's own practice" (Mason, 2020, p. 231). This view of noticing has been sitting at the core of Mason's (2002) work from the start:

"[The Discipline of Noticing] focuses attention on enhancing awareness by sharpening and enriching those moments when you get a taste of freedom as you participate in a creative moment. It focuses attention on changing what can be changed, if it is deemed appropriate to change, not fussing about what cannot be changed for social, cultural or institutional reasons." (ibid, 7)

My theoretical framework chapter will show that it is other, more skill-based noticing contructs that my own idea of "noticing of mathematical subject matter" is based upon, and the reader will find that these constructs are much more similar to the initially described "noticing as part of expertise" conceptualisation rather than Mason's view of noticing as a form of and a tool for professional development. However, before proceeding to describe those other constructs, I would like to include a short digression on John Mason's ideas regarding noticing, since they have been so formative and important to the development of this research area that a dissertation centered around noticing would not feel complete without it.

According to Mason, noticing can be a catalyst for professional improvement if it opens up new, improved ways of acting in the future. In analogy to the components of professional vision described in the expertise studies, this disciplined noticing requires a special perceptual skill (I must be able to perceive relevant details of the situation in the moment) as well as background knowledge (alternative actions must be familiar in order for them to occur to me). Noticing can then function to productively channel the energy produced by challenging situations or problems in the classroom (which can otherwise easily dissolve into blame and justification), thus "setting up the possibility of acting differently in the future" (Mason, 2002, 10).

Mason distinguishes intentional noticing from *ordinary-noticing*. The latter to him describes a more general capability of perceiving, such that a person may remember seeing an incident if another person references it later. However, without an external prompt, the person merely perceiving the incident would not "think twice" about specific details of the incident. In opposition to this, and in line with Schön's concept of a "reflective practicioner" (1987, 2017) who is consciously aware of their practice, Mason delineates disciplined noticing as involving both *marking* and *recording* professional incidents.

Marking means identifying something about the incident that one might remark upon out of one's own volition, either in the moment or later on, to oneself or to others. Disciplined noticing for Mason culminates in *recording*. This can involve making a mental note to oneself or recording the incident on some physical medium.

Mason considered this form of deliberate reflection on one's own practice to be applicable to teaching and caring professions in general. In the next subchapter, I will describe a few central noticing constructs that were devised specifically for research in mathematics education.

2.4.2 Mathematics teacher noticing

One of the first noticing conceptualisations specifically related to mathematics education was developed directly linked with a professional development activity. In Sherin and van Es' study (2002), teachers attended so-called "videoclubs" – spaces where they could watch and discuss videotaped lessons. The researchers thus set out to capture the *development* of this skill, i.e. the "learning of noticing".

Noticing was described by Sherin and van Es as consisting of two main components: Employing selective attention by *attending to* what's important or noteworthy about a certain situation, and *interpreting* the input of that selective perception in an informed fashion. The latter involves both relating the specific situation that was observed to one's own knowledge of broader teaching and learning principles, as well as using one's knowledge of the context of that specific situation to reason about what it was that one observed.

Almost two decades later (van Es & Sherin, 2021), this original noticing construct was revisited by the authors and expanded into a new framework for noticing.

The new framework added the idea of *actively disregarding* non-central aspects of the observed situation in the "attending" component. Over the course of the years, the researchers had found that teachers' attention shifted *away* from certain objects of noticing instead of developing by simply adding to the list of things they noticed. Rather than adopting additional foci, the teachers *reprioritized* what they paid attention to.

Similarly, the "interpreting" part of noticing was elaborated upon. Central to successful interpreting of an observed phenomenon is not just being able to make sense of that phenomenon on the basis of one's knowledge and professional experience, but also, as a prerequisite, having adopted a *stance of inquiry* that allows the teacher to regard an event as something worth being considered and explained.

Lastly, the two component framework was expanded to involve a new, third component: *shaping*. Noticing according to this notion also entails purposefully creating situations in which additional information about the objects of one's noticing can be gained. Thus, a teacher may notice a student relying heavily on a particular solution method and *shape* their further conversation so as to find out more about how this student approaches problems and weighs alternative ways to proceed. Other authors (Scheiner, 2020) have recently picked up this idea in an attempt to more accurately represent the fact that teachers do not passively take in sensory information about their classroom but, instead, often actively seek out such information.

The constructs of "professional noticing of children's mathematical thinking" used by Jacobs et al. (2010) as well as the PID-model (Perception, Interpretation, Decision-making) of situated teacher competence (Blömeke et al., 2015; Kaiser et al., 2017) also employ threefold structures.

The PID model calls the first dimension of employing selective attention in order to take in specific details of a multidimensional classroom situation "perceiving (particular events)" while Jacobs et al. refer to this as "attending to (children's mathematical thinking)". Both describe the second step of the noticing process as "interpreting", that is, combining what was perceived in the first dimension with one's own background knowledge and experience, thus contextualising the sensory information. With a somewhat different focus than the "shaping" component in van Es and Sherin's expanded framework, the PID model and Jacob et al.'s construct culminate in "decision-making"/"deciding how to respond" based on one's perception and interpretation of events.

In all of these conceptualisations of teacher noticing, cognitive factors such as professional knowledge as well as affective qualities of the person noticing play an important role. Theoretical knowledge can influence perception as it affects what one can recognise as a meaningful unit and what one deems important and thus noteworthy. Lacking knowledge about alternative solution methods, a teacher may pass over a student having made a substantive mathematical observation simply because it was not the response they were expecting. Likewise, a teacher who prefers their students to solve problems in the way that was introduced during class may actively disregard alternative strategies in order to streamline their teaching according to what they consider most effective for the lesson to proceed.

Similarly, our knowledge can shape the second dimension of noticing – our interpretation of events – by providing a collection of past experiences to which the phenomena observed can be compared, as well as possible explanations which can be explored. A teacher who has wide-ranging experience in teaching and is well-educated in a specific area of mathematics or pedagogy may recognise certain student behaviours as being indicative of mathematical giftedness, or a disability, or emotional disregulation, or a typical content-based misconception.

Our beliefs can also interfere with the interpretation process as they influence what we consider to be the most plausible explanation. A teacher who believes that children are naturally curious about patterns and the reasons for why certain patterns occur in nature may interpret half of their students seeming disengaged with a lesson differently than will a teacher who believes that children usually need external incentives in order to want to investigate mathematical phenomena.

Lastly, deciding on a response hinges on the variety of options available to us as well as on our values, beliefs and motivations for what we want to achieve by responding to the situation at hand (the teacher's personal goals, of course, exist next to institutionally prescribed curricular goals). The more aware one is of different ways of explaining a mathematical idea, the more informed the decision to choose one of those approaches can be. Choosing one way to do things over another often depends on our value judgments and personal priorities: A teacher who is convinced that visual communication is most effective when explaining concepts to students can only choose to provide a visual representation for a concept if one comes to mind. At the same time, another teacher may be aware of multiple ways of visually representing the mathematical situation, but disregard them because of a personal belief that students need to focus on symbolically representing the problem.

Depending on the conceptualisation, some noticing frameworks consider noticing to be a separate *professional skill* that draws upon one's professional knowledge (Jacobs et al., 2010, Dreher & Kuntze, 2015), a form of *tacit knowledge* itself (Yang & Ricks, 2012) or a kind of *mediator* between a teacher's internal characteristics, such as professional knowledge, and their observable actions (Kaiser et al., 2017).

Numerous empirical results *about* noticing, how to acquire it and its effects and dependencies have been found over the last decades. In the next subchapter, I will present an overview of the state of empirical research into the effects and prerequisites of teacher noticing.

2.4.3 State of research: noticing

A large subset of research into teacher noticing has been conducted in the field of mathematics education research, compared to the teaching and learning of other subjects, such as foreign language education (Weger, 2019). Another characteristic of the research landscape on noticing is that, in the majority of studies, the authors interested in "teacher noticing" often only investigate the noticing of preservice teachers still studying at university (usually for practical reasons). As such, there is still a lack of studies whose subjects are in-service teachers (Santagata et al., 2021).

From studies comparing preservice teachers with in-service or even expert teachers, we know that preservice teachers tend to exhibit professional noticing that is located somewhere above a layperson's level of noticing but is clearly distinct from expert noticing (Schäfer & Seidel, 2015). One example of where preservice teachers' noticing is still lacking is the professional task of perceiving and interpreting students' work. Prospective teachers typically struggle with this, and their own mathematical knowledge plays an important role in whether or not they are successful (Stahnke et al., 2016).

Teachers with high amounts of professional knowledge tend to notice possible reasons for a student's error more accurately, while teachers with low professional knowledge focus more on aspects that are not directly connected to the student's learning (Hoth et al., 2018). At the same time, even experienced teachers can sometimes struggle to interpret their students' utterances and coordinate multiple mathematical ideas in the classroom. However, expert teachers are aware of the *importance* of making decisions backed by solid professional knowledge, even if they do not always succeed (Hino & Funahashi, 2022).

Kersting et al. (2012) found that while there was no direct link between mathematical knowledge for teaching and instructional quality, mathematical knowledge for teaching and noticing skills were significantly correlated, and there was another significant relationship between teacher's noticing skills and their instructional quality. Furthermore, teachers' ability to perceive classroom situations is correlated with teaching experience, most importantly, the *relative* amount teachers report to have spent on teaching when considering their overall workload (König et al., 2015). It thus seems that subject matter knowledge paired with intensive involvement in the teaching process is what leads to improvement in noticing, rather than simply accumulating years of experience on the job.

Aside from knowledge and experience, a teacher's beliefs and goals are also an influence on situation-specific skills like their noticing, which in turn correlate with aspects of instructional practice (Stahnke et al., 2016). However, while beliefs are a predictor for noticing, noticing does not act as a mediator between teachers' beliefs and their classroom actions – paying attention to and then noticing something does not automatically lead to taking an appropriate action in the classroom (Griful-Freixenet, 2020).

When it comes to the methodology of measuring teacher noticing in classroom situations, many studies make use of a video vignette format. Capturing specific scenes and presenting teachers with selected videoclips is an efficient and focused way of investigating their noticing which allows researchers to gain information about what teachers notice in the moment *while* watching a video (Kersting et al., 2021) or what they report having noticed after having watched a vignette once or twice (Seidel & Stürmer, 2014). Some studies further reduce complexity and thus increase focus by employing text vignettes that *describe* teaching situations rather than show them in a video format. Text vignettes allow for more systematic variation in the classroom events that are being displayed, while, at the same time, reducing the amount of information to be found in a vignette (Dreher & Kuntze, 2015). This can be beneficial if efficiently directing the teachers' gaze (for example, due to time constraints) is desirable, while of course diminishing the degree of validity of this representation and making the professional noticing that can be displayed less similar to the type of noticing that would take part during the equivalent "real" teaching situation. Aside from the vignette format, eye-tracking is another popular choice of collecting data about teachers' noticing (e.g. Maatta et al., 2021), being especially suited for investigating questions about teachers' in-the-moment noticing.

Video has also been used to successfully *improve* (preservice) teachers' noticing skills in professional development programs (Stahnke et al., 2016). In general, it is believed that noticing is a learnable skill (Jong et al., 2021). Developing preservice teachers' noticing skills, even in early stages of a teacher education programme, has been shown to be possible, but short interventions may have limited effects as noticing typically takes a long time to develop (Stockero et al., 2017). Interventions meant to train noticing using video analyses can also lead to changes in some affective dimensions (Jong et al., 2021) as learning to notice aspects of a situation that one had not previously paid attention to can shift previously held views, e.g. a teacher learning to notice signs of their students' boredom or frustration may change their beliefs regarding the importance of adaptive exercises.

Effects of courses intended to train noticing skills have been shown to be lasting (Barnhart & van Es, 2015) testing three months after completion of the course); at the same time, different dimensions of noticing develop at different speeds and improvement in one dimension of noticing, such as "pure" perception of events does not automatically translate to improvement in the other dimensions like the interpretation of said events. It has been found that it is easiest to train participants' abilities to attend to specific features of classroom situations and relatively harder to achieve improvement on the decision-making/responding level (Yang, König & Kaiser, 2021). On a positive note, there is evidence that whatever improvement is achieved in university-based noticing interventions can indeed be applied to teaching in the classroom (Stockero, 2021).

Some findings about the growth of noticing can be replicated in other cultural contexts, such as the fact that teaching experience is an important factor when developing one's noticing proficiency (Han & Paine, 2010; Lu et al., 2020). The specific educational context has to be taken into consideration when explaining the observed phenomena, however, as the reason for the same findings may be different across cultures. For example, when studying the growth of noticing among Chinese teachers, one has to take into account the strong influence of school-based professional development that is ingrained in Chinese culture. In this cultural context, it is unsurprising that Chinese teachers keep improving during their years as a teacher, as deliberate professional development throughout their career is an institutionalised part of their professional life.

In general, cross-cultural noticing research shows similarities across cultures of mathematics education as well as differences. Dreher et al. (2021) found that both Taiwanese and German experts consider it important to attend to students' thinking. However, Taiwanese teachers tended to focus on correctness and try to identify misunderstandings on the students' side, while German teachers focused more on the students' way of thinking as a resource and tried to value their students' strategies and foster the mathematical ability which was being demonstrated.

Damrau et al. (2022) also found similarities as well as differences in their case study of teacher noticing in Australia, China, and Germany. All three teachers described in the case study frequently commented on the *effectiveness* of certain instructional elements, as well as on time management. However, compared to the Chinese and the German teacher, the Australian teacher reflected much less on her students' abilities and whether she had under- or overestimated them when preparing the lesson. For all three teachers, the aspects of their teaching that they focused on during the interviews heavily depended on the particular lessons taught, indicating that a teacher's focus when noticing during instruction changes strongly depending on which topic is being taught and how the individual lesson unfolds.

Aside from studies investigating, purely descriptively, what it is that teachers pay attention to and how they go about directing their attention, there has also been a variety of research focused on whether and how teachers notice particular aspects of teaching. To list a few examples, studies have been conducted on the noticing of the development of specific skills such as children's early numeracy (Schack et al., 2013), noticing of students' understanding of the specific mathematical ideas such as the derivative concept (Sánchez-Matamoros, 2015), noticing of certain teaching elements such as the use of representations (Friesen & Kuntze, 2021) or noticing of the affordances of typical problems (Choy & Dindyal, 2021).

Particularly relevant for the topic of this dissertation are studies that investigated the perception of mathematical content in teaching. In general, I have found a dearth of research investigating noticing of mathematical content specifically, with most studies on noticing focusing on explicitly PCK aspects of teaching when dealing with subject-specific noticing, or altogether non-subject specific issues such as diversity. This was another reason why I thought it would be particularly worthwhile to explore "subject matter noticing" as such: The question of "Where do preservice teachers locate the content when looking at a teaching situation?" seemed relevant, clearly distinct from other kinds of noticing, and yet understudied.

Stephens (2008), for example, found that in the subject area of algebra, usually only the overt manipulation of symbols was categorized by participants as "algebra", while more implicit aspects such as relational thinking weren't recognised as such. Noticing the mathematical content inherent in teacher tasks such as introducing a topic, choosing activities and representations, and working with students' answers is, unsurprisingly, again particularly difficult for *preservice* teachers (Vondrova & Zalska, 2012).

Another tendency relating to the perception of mathematical content in teaching is for some teachers to *claim* that advanced (i.e. university level) mathematical knowledge is important for teaching but struggle to give specific content-based examples of how this knowledge would be relevant to one's everyday teaching. It seems that it is far easier for teachers to relate the importance of knowing advanced mathematics to general pedagogical principles such as conveying a passion for mathematics or the ability to provide special tasks for gifted children (Zazkis & Leikin, 2010).

Before describing the project's chosen theoretical framework, the following chapter on the topic of teacher education will end the review of the literature.

2.5 Teacher education

2.5.1 The challenge of fragmentation

When designing and implementing courses, modules or even entire training programmes for future mathematics teachers, a specific set of challenges is often encountered – some aspects of which are specific to *mathematics* teacher training, while others represent more general obstacles to learning that any effective teacher training environment has to find ways to overcome.

The overarching theme of the difficulties to be described in the following is that of a *fragmentation* (Kreutz et al., 2020; Merrienboer & Kirschner, 2017; Ball, 2000, Grevholm & Bergsten, 2004) in some sense.

If left unchecked, structural fragmentation can lead to a lack of coherence on the supply side: For example, two educational institutions that do not share a joint vision and are involved in the training of the same cohort may convey contradicting information to preservice teachers. As a result, the preservice teachers on the receiving end might be forced to either disregard some parts of their training, or compartmentalise the knowledge they acquired. (This compartmentalisation can also occur for other reasons, such as different pieces of information, while not necessarily being at odds with each other, never being experienced as related to one another.) The ways in which mathematics teacher education often appears fragmented can be thought of in the following threefold manner, ranging from most to least specific to mathematics teacher training:

1) fragmentation within the (presentation of the) mathematical content itself – one aspect of the *double discontinuity*
2) fragmentation between different content areas such as subject matter, pedagogical subject matter, teaching methods and pedagogy
3) fragmentation between the institutions of theory-focused universities and practice-providing schools (as well as, in countries like Austria, institutions like the Pedagogical Colleges/Colleges for Teacher Education [Pädagogische Hochschulen] which operate on a sort of "intermediate" level of theory and practice)

In the following pages, I will briefly elaborate on each of these types of fragmentation.

School mathematics vs. university mathematics

Firstly, there is a particularity to mathematics insofar as the ways in which mathematics is commonly taught at school and the ways in which it is usually presented in lectures at university are famously (Klein, 1908) at odds with each other. This is the case because for most students, entering university marks the first time they encounter the presentation format typical of research mathematics, the schema of *definition – theorem – proof*. At school, even if proofs regularly appear in mathematics lessons, the sequencing tends to follow a much more illustrative example-based, genetic and applied approach, rather than the abstract, deductive structure upon which mathematics as an academic discipline is based (Rach & Heinze, 2013). Additionally, many of the mathematical concepts the students might recognise from their time at school suddenly appear only indirectly in many lectures, for example, as special cases of much more abstract concepts that now encompass only the common structural features of many mathematical objects familiar to the students. (Think of the set of real numbers as an example of a field and of a vector space.)

The unfamiliar situation is further complicated by a different style of problem posing that is typical of academic mathematics but not regularly practiced in many classrooms: While in school, many students were used to being presented with an algorithm or other step-by-step means of solving an exercise and then practicing the application of this solution method by performing analogous tasks, at university, it is typically expected of students to come up with some solution methods on their own, more or less creatively utilising the tools provided in the accompanying lecture. Felix Klein referred to this difficult transition from school mathematics to university mathematics as the *first discontinuity*.

The transition from school mathematics to the “culture of university mathematics” is a challenge in and of itself, and for preservice teachers, the associated feelings of frustration are often aggravated by the fact that many see no connection to the mathematics teaching they experienced as students in school, and thus the effort required to adopt these new ways of thinking may seem unnecessary, according to their vision of their future profession (Hefendehl-Hebeker, 2013).

For the students who do adopt this culture of university mathematics successfully enough to finish their studies (mathematics has been shown to have some of the highest dropout rates, cf. Dieter et al., 2008; Heublein et al., 2010), the transition back to school teaching represents the *second discontinuity*: It is often unclear for prospective teachers just how they could utilise their knowledge of academic mathematics in order to become effective teachers.

In theory, courses on pedagogical content knowledge could (partially) bridge this gap, however, sometimes not even the university courses on a certain mathematical area and the corresponding “mathematics teaching/mathematics education” explicitly reference each other because the staff teaching either course is not held responsible for this form of integration. Due to this fact, many preservice teachers simply compartmentalise their body of knowledge and thus the abstract concepts and structural connections that in fact underlie all of school mathematics “remain only a more or less pleasant memory” (Klein, 1908, p. 1), with little influence upon their practice of teaching.

Multiple domains of teacher knowledge

The second type of fragmentation is rooted in the fact that mathematical subject matter, teaching methods, pedagogical and psychological components of teaching, and practical skills such as classroom, time and resource management are naturally taught by a wide variety of different educators, each specialised in their respective field. This makes sense from an expertise point of view, however, it can lead to the impression that these content areas are drawn upon during teaching practice in a way that is separable from one another, and that there is a natural tension between them in the sense of a zero-sum game. (“How can I even begin to focus on presenting mathematically rich tasks when I cannot even get my students to sit still and focus for two minutes?”)

It is especially telling that the components of “mathematical subject matter” and “mathematics teaching” [Fachdidaktik] are often regarded by preservice teachers as two distinct entities, with the latter being seen as important and desirable, and the other sometimes considered to be represented excessively within the curriculum – despite research mathematicians and mathematics educators agreeing that knowledge of mathematics teaching cannot exist separately from the subject matter itself (Ball, 2000).

Training at different institutions

Thirdly, since teacher training programmes usually include theoretical as well as practical preparation, and most universities do not have the resources to provide in-house teaching opportunities for their students, a cooperation between universities and practice schools is necessary in almost all cases. This can become a problem in two ways:

If the theoretical conceptions of "good teaching" conveyed at university do not align with the notions about teaching that are conveyed by the practitioners acting as role models and mentors to student teachers (Dewey, 1904; Shulman, 1998), preservice teachers have to make up their mind over which ideas to take on for themselves. Since they experience their practice mentors as being more closely connected to school teaching, many of them might conclude that university educators are misguided about the realities of the classroom and thus disregard their input (Oonk et al., 2019).

However, even if there is no discrepancy between the two images of best practice between institutions, the mere fact that teaching practice is outsourced to another institution (from the point of view of the university), or that the theoretical preparation is assumed to have taken place fully at another institution (from the point of view of the practice school) can lead to universities teaching theory without any effort to relate it to practice. Conversely, it can lead to mentor teachers rarely mentioning mathematical subject matter (Strong & Baron, 2004; Crasborn et al., 2011; Futter, 2016; Sebök, 2019), possibly out of an understanding that the PSTs' theoretical education is not their responsibility and has already been carried out in full.

The first of these may result in the acquisition of *inert knowledge* (Bransford et al., 1988, Perkins & Simmons, 1988) on the side of the preservice teachers at university, while the second forfeits the opportunity to "activate" this knowledge and show its relation to practice. Thus (like in the case of a stark second discontinuity), even mathematically well-prepared student teachers may not know *how to apply* their theoretical knowledge to teaching situations, and as a result, during their teaching practice, feel further validated in their suspicions that the academic knowledge they acquired is not useful to "real teaching".

Combating the problem of fragmentation

Focusing on each of these ways that fragmentation occurs, different approaches for solutions suggest themselves:

In order to combat the double discontinuity, it is necessary to relate the mathematical objects and procedures of school mathematics to the more abstract concepts of university mathematics – essentially the program that Felix Klein himself began when he endeavoured to teach "elementary mathematics from an advanced standpoint" (Klein, 1932).

In order to relate different types of teacher knowledge to each other, their connections need to be emphasised: Which kinds of classroom management problems could actually be rooted in a lack of mathematically challenging problem sets (causing boredom), or in developmentally inappropriate tasks (causing unproductive frustration)? How can a teacher utilise their mathematical content knowledge to generate problem sets for their students or decide between two methods of presentation (Cuoco, 2001)?

In order to relate theory to practice, one can go both ways – make theory more practice-oriented, for example, by enriching university courses with references to teaching situations and including different representations of practice (Grossman et al., 2008), or make practice include references to theory, for instance, by encouraging mentors to explicitly verbalise their internal mathematical considerations when modelling how to prepare for a lesson.

The intervention described in this doctoral thesis endeavoured to make university teaching more practice-oriented (in the spirit of the third fragmentation) by employing a *situated* approach to discussing questions of subject matter. This approach included mainly text vignettes as representation of practice.

At the same time, relating the subject matter back to the teaching situation depicted typically happens via the consideration of corresponding aspects of pedagogical content knowledge – thus the second fragmentation between different kinds of knowledge bases a teacher must draw upon was also highly relevant. In doing so, parallels are automatically drawn between the subject areas of school mathematics and concepts of advanced mathematics, e.g. when analysing the introduction of a certain topic in a school textbook and contrasting it with the presentation the students had now internalised over the course of their university studies (one of the tasks of the intervention) – thus, even the first fragmentation was addressed at times.

The following subchapter will present an overview of the literature on situated learning, and the notion of typical tasks of (mathematics) teachers.

2.5.2 Situated learning environments

Today, we know that most learning is highly *situated*, meaning that our understanding of concepts and procedures is highly dependent on the context in which we learnt them (Sfard, 2013). One might be able to apply knowledge "in real life" in order to solve problems which closely resemble the exercises that were presented in the classroom, but struggle to use the same pieces of information in a flexible way to approach new and unfamiliar looking situations. This is often caused by a *transfer paradox* (van Merriënboer et al., 1997), by which the methods which most effectively train specific, isolated skills are not the methods most conducive to successfully integrating those skills with the rest of one's knowledge base and applying them flexibly and judiciously.

Although there are more and more programmes which seek to provide opportunities for prospective teachers to learn how to integrate the knowledge gained in each of their courses, "the challenge of integrating subject matter knowledge and pedagogy in the contexts of their work" (Ball, 2000, p. 242) systematically is still often left to the preservice teachers themselves.

However, this kind of integration is nowhere near trivial, and as such, might simply never occur for many practicing teachers, if they are left to their own devices once they leave university. Not being able to apply one's content knowledge, in particular, can have dire consequences for one's mathematics teaching: "Teachers may not be able to size up their textbooks and adapt them effectively; they may omit topics central to students' futures or make modifications that distort key ideas" (ibid, 243). Or, as Stahnke et al. (2016) report, teachers may end up having difficulties interpreting tasks and identifying their educational potential. In short, a lack of integration and transfer of one's theoretical knowledge impedes a teacher in performing their day to day tasks.

Llinares (2014) contends that one should therefore focus learning in professional teacher education on naming and analysing these typical teacher tasks: "The identification of tasks which constitute the mathematics teacher's professional practice is relevant because it allows to relate the teacher's knowledge and *the use of knowledge in context*" (ibid, p. 78). Ball (2000) had already put this on the agenda, naming as contenders tasks such as

- "figuring out what students know;
- choosing and managing presentations of ideas;
- appraising, selecting and modifying textbooks; and
- deciding among alternative courses of action" (p. 244).

Rowland et al.'s (2005) "Knowledge Quartet" is essentially the distillation of the results of a survey of different knowledge-backed tasks a mathematics teacher has to perform. They are categorised into four different types of tasks, named *Foundation, Transformation, Connection* and *Contingency*. Table 1 shows some teacher tasks behind each of these categories:

Foundation	**Transformation**	**Connection**	**Contingency**
Identifying errors	Choosing representations	Making decisions about sequencing	Responding to children's ideas
Using a textbook	Choosing examples	Making connections between concepts	Using opportunities

Table 1: Examples from Rowland et al.'s (2005) Knowledge Quartet

More recent research has investigated further responsibilities in a teacher's professional life, such as

- adapting historical texts for use in the classroom (Chorlay, 2022)
- orchestrating classroom discussions (Xu & Mesiti, 2022)
- adapting textbooks during lesson planning (Shinno & Mizoguchi, 2021; cf. "teachers as curriculum redesigners", Cai & Hwang, 2021, p. 1405)

and often centres the mathematics-specificity of these specific tasks, stressing that they cannot be expertly conducted without having and using one's own mathematical knowledge as a teacher (Llinares, 2019). This is in line with one of the reasons Fritzlar (2006) gives for why efforts to reform mathematics teaching in a more problem-oriented way seem to be met with a fair amount of resistance despite the efforts having a well-argued point – such teaching is cognitively highly complex and mathematically demanding of teachers.

Moreover, the kind of mathematical knowledge that is demanded of teachers who create inquiry-based and problem-oriented learning environments cannot be easily described as a set of mathematical statements that can be exhaustively studied at university such that one can be sure to leave the institution as a mathematically competent teacher.

Cuoco (2001) laments the lack of in-service professional development focusing on furthering teachers' subject matter knowledge and adds: "Combined with the notion that mathematics is an established body of facts, this places many teachers in the frightening situation where their mathematical expertise is defined, not by what they can figure out, but by the facts they learned as undergraduates. With such a mindset, questions from students can be very intimidating" (p. 169).

This is not to say that the mathematical spontaneity required of teachers *can* be enacted without a certain basis of factual and procedural knowledge that should be covered in a teacher training programme. Rather, situated approaches can highlight (possibly hidden) ways in which theoretical knowledge of and about mathematics necessarily permeates teaching in the classroom. For example, the reflective tasks designed by teacher educators that usually constitute a core component of preservice teachers' practice-based modules expose certain theoretical lenses (Artmann, 2019) about mathematics teaching and learning which preservice teachers themselves may or may not be aware of. Artmann describes teacher educators' impressions that rather than presenting theory in opposition to a naïve, "theory-less" approach, it can be more beneficial to analyse the benefit of research-backed theoretical approaches in contrast to whatever "everyday theories" or common beliefs preservice teachers brought along from their own times as students.

However, more explicitly content-based situated interventions like the one described in O'Meara et al. (2017) seem equally relevant in order to point out the many connections between the *specific mathematical ideas* of school mathematics and university mathematics that do exist.

The next subchapter will present a selection of endeavours of the last few decades that sought to improve teacher education by decreasing fragmentation. The interventions presented share the conviction that more "situated" approaches to learning subject matter mathematics and its applications to teaching lead to more usable knowledge and to more positive beliefs regarding the relevance of subject matter to the day to day life of a mathematics teacher.

There are multiple design strategies and modes of implementation that can be employed when following a situated approach (cf. Wasserman et al. 2023), many of which have some basis in the idea of analysing teaching practice based on the typical tasks of a mathematics teacher and then pointing out to preservice teachers (or letting preservice teachers explore) the subject matter required in order to competently execute those tasks.

2.5.3 Situated efforts to combat fragmentation

With the nation-wide "Qualitätsoffensive LehrerInnenbildung", a large amount of funding was made available to study teacher education in Germany between 2014 and 2023. In this subchapter, I will describe a few of them and highlight specifically their respective intervention methods, the dependent variables and the methods of data collection as well as the results that were found. A selection of earlier projects and projects in other countries will also be presented with an analogous focus.

Hanke and Schäfer (2018) describe the development of a design-based research programme over the course of three years with the goal of improving the mathematical subject matter courses preservice teachers have to take by introducing a close collaboration between research mathematicians, researchers in mathematics education and lecturers. This was meant to enable PSTs to draw connections between theory and practice on a content level, but also to influence preservice teachers' perception of the necessity of subject matter knowledge for competent teaching.

A mathematical content course in complex analysis was restructured in order to combine subject matter knowledge of complex analysis, pedagogical content knowledge about task design as well as "specific school-related knowledge" – all of which was to be used in order to create lesson plans for topics in complex analysis for upper secondary school. Further, an "experiment day" for school students was designed and implemented by the preservice teachers. At the end of that day, the PSTs were asked to reflect on their experiences. Additionally, they wrote a report on their project including a reflective piece in writing.

The materials produced by the students, and the reflections they made during the interviews as well as in their written reports were then analysed. The researchers came to the conclusion that, although the PSTs' reports were diverse, the design principle of integrating practical experiences was appropriate for initiating a changing perception in the PSTs' eyes regarding the applicability of subject matter knowledge.

Schadl, Ufer and Rachel (2019) similarly employed design-based research with the goal of increasing awareness of the relevance of mathematical content courses for the preservice teachers' future profession. Additionally, it was hoped that preservice teachers would become more proficient in the application of mathematical and pedagogical knowledge to teaching tasks. One of the interventions which were introduced was a seminar designed to allow the perservice teachers to reflect on the ideas in the school mathematics curriculum from the advanced standpoint of university mathematics.

Using a pre- and posttest design, Schadl et al. investigated the effects and found that after one semester, the participants' subject matter knowledge had significantly improved, and the relevance they assigned to subject matter knowledge for their future profession had been raised as well. However, the absolute level of subject matter knowledge remained fairly low despite the improvement, and not all future implementations of the seminar could replicate the significant findings regarding the perceived relevance. Considering these facts, the authors raise the question whether the preservice teachers bring along sufficient subject matter knowledge which could then be applied to teaching situations over the course of such a seminar.

Eichler and Isaev (2023) investigated the effects of integrating school-related tasks into the problem sets of tutorials accompanying mathematics lectures for preservice teachers, employing the “questionnaire about the double discontinuity” (Isaev & Eichler, 2017) to assess beliefs changes regarding the relevance of university mathematics to the teaching profession. They randomly assigned students to treatment and control groups and found a significant effect of these situated additional tasks. Without the intervention, initally positive beliefs about the relevance of the subject matter to be studied significantly decreased over the course of the semester. With the intervention, the beliefs remained stable. Their questionnaire was used in this dissertation in order to assess beliefs about mathematical subject matter.

As a last example from the “Qualitätsoffensive LehrerInnenbildung”, Bauer et al. (2020) describe the restructing of the teacher training programme in Marburg as a whole to include multiple modules in which subject matter knowledge and practical, school-related experiences are connected. The modules start with an immersion into school life via a two-week practical phase right at the beginning of the programme, then go on to connect the learning of university mathematics with considerations on a meta level (Which steps in a proof fulfil which function? How do specifically chosen concept definitions correspond to our concept images? (cf. Vinner, 1983) What might be “uses” of a mathematical proof aside from verification?). After a more extensive practical experience, the last module of the training programme finishes with a reflective session, looking back at how successfully one was able to transfer the subject matter perspective acquired at university into one’s teaching practice.

The researchers investigated the effects of the curriculum change by analysing multiple reflective essays the students had to submit over the course of their studies, and conclude that longitudinally, the preservice teachers’ attitudes and beliefs regarding proofs do change. They add that the types of activities fostered in the newly designed modules are described by the preservice teachers’ as a completely new way of doing mathematics which they did not experience in this way in either the subject matter lectures or the accompanying tutorials.

Of course, as the double discontinuity is far from being a new problem, there were earlier endeavours with the aim of making subject matter feel more relevant to prospective teachers. The idea of integrating specifically teaching-related tasks into the subject matter modules from the beginning had been tried before (Bauer, 2013; Prediger, 2013), just like extending this integration of pure subject matter and applications of mathematical knowledge in school teaching to entire modules of a teacher training programme was not unheard of (Bauer & Partheil, 2009; Beutelspacher et al.; 2012).

(There has also been a variety of research conducted trying to support PSTs during the difficult transition from school mathematics to university mathematics (the first discontinuity) for example, by providing annotated or cloze solutions (Ableitinger & Hermann, 2013). While the first discontinuity may be considered on its own, as would be reasonable when analysing the transition of university students aiming to become mathematicians instead of teachers, the second discontinuity cannot sensibly be talked about without taking into account the first discontinuity. After all, unlike Felix Klein suggested, it is rarely the case that preservice teachers who find university mathematics to be completely unrelated to what they learnt in school go on to embrace university mathematics for its own sake and simply do not know how to implement their comprehensive mathematical notions in the classroom. As the chapter on beliefs outlined in detail, the first discontinuity often leads to such disillusionment that advanced mathematical knowledge is not successfully acquired at all.)

Similar projects with slightly different framings can be found internationally. Wasserman et al. (2017; Weber et al., 2020) refer to instructional processes they call "Building Up From Practice" and "Stepping Down to Practice" which address the connections between teaching school mathematics and university mathematics.

"Building Up from Practice" involves taking a realistic classroom situation and identifying the relevant mathematical ideas in it, which may be most effectively analysed utilising knowledge from academic mathematics. "Stepping Down to Practice" then puts the focus on transforming the insights a teacher might have gained (using their advanced mathematical knowledge) into teaching actions that directly benefit school students on their respective level of mathematical sophistication.

Instead of providing a separate course for PSTs, the advanced mathematical content course (real analysis) itself was restructured in this situated fashion. As Wasserman et al. (2019) report, this approach showed promising results in influencing how much attention was subsequently paid to the *mathematical scope* in lessons (as observed by the researchers that were held by participating preservice teachers). Additionally, although the intervention had been undertaken in a real analysis course, the attention to scope seemed to transfer to various mathematical topics like functions in algebra or figures in geometry.

O'Meara et al. (2017) describe courses specifically designed to confront preservice teachers with the challenges that can arise during teaching when teachers lack certain kinds of subject matter knowledge. Preservice teachers were initially given tasks in which they had to present or explain a mathematical concept or to give explanations in order to experience the challenges firsthand. After this experience, experts in mathematics and mathematics teaching coached peer group work in which similar problems were discussed collaboratively and typical misconceptions and difficulties were reflected upon.

With their focus on relating university subject matter education to the kind of school mathematics that teachers will be teaching, these various interventions already share common threads. However, as the presented variety of projects also shows, the specific ways in which researchers and educators approach bridging the gap differ in their emphases on certain aspects of the discontinuities between academic and school mathematics.

Wasserman et al. (2023) use a survey of the recent literature on similar projects to derive a theoretical framework which can help categorise and identify projects' differences and similarities. They propose a division of related efforts into four categories, depending on how mathematical and didactical concepts, beliefs and practices, as well as (the teaching of) academic mathematics and school mathematics are related to each other (see Figure 7 on the next page).

Projects “sharpening mathematical connections” aim to provide students with explicit connections between concepts of university mathematics and ideas in school mathematics on a content level. Interventions “highlighting disciplinary practices” point out ways in which PSTs’ knowledge of mathematical practices as experienced at university can inform their didactical practices as teachers trying to provide experiences of authentic mathematical activity to their secondary students. Approaches “modeling didactical practices” foreground specific experiences which PSTs can experience as learners at university that could then be transferred into their own ways of teaching at school.

Lastly, design strategies that focus on “incorporating applications to school teaching” seek to illustrate explicitly the relationships between mathematical subject matter and the work of teaching. It is this approach that was chosen for the intervention described in this project, as the focus on “typical teacher tasks” and their subject matter demands was considered most likely to change PSTs’ views on the relevance of subject matter to their everyday work as teachers.

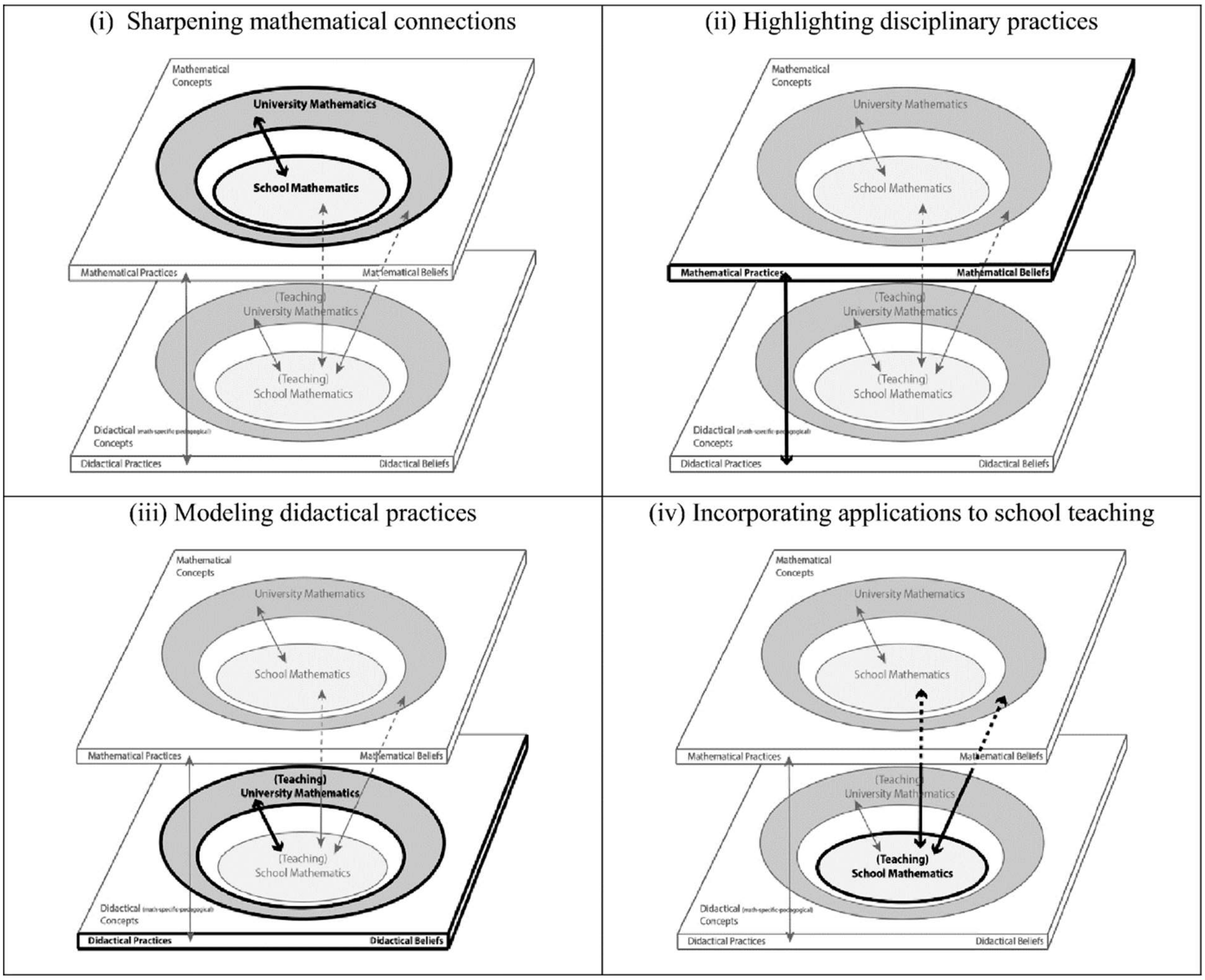


Figure 7 Theoretical framework for relevance interventions Wasserman et al. (2023, p. 730)

As a guiding design framework, the Four Component Instructional Design (van Merriënboer & Kirschner, 2017) was chosen as particularly fitting, as its core idea is that learning of any material should take place through the engagement with learning tasks that are as similar as possible to the real life situations in which educators hope their students' learnings will ultimately be applied.

This framework is introduced in the following chapter.

2.5.4 Four Component Instructional Design

The Four Component Instructional Design (4C/ID), first described by van Merriënboer et al. (1992) in the 1990s, is a comprehensive and versatile framework for designing situated learning environments. It has been further developed (van Merriënboer & Kirschner, 2017) and implemented in various fields that require *complex learning*, including mathematics teacher education (Leuders, 2019).

4C/ID is intended as a tool for providing effective instruction for so-called *complex learning*, which is defined as "integrating knowledge, skills, and attitudes; coordinating qualitatively different constituent skills; and often transferring what is learned in school or training settings to daily life and work settings" (van Merriënboer & Kirschner, 2017, p. 2). Remembering the earlier chapters on the professional competencies of teachers – including their knowledge, beliefs and situated skills such as noticing – this sounds like the type of learning training programmes for (mathematics) teachers might aim for.

The authors of the 4C/ID framework maintain that for such learning to take place, it is necessary to design *holistic* learning environments, which are described in opposition to the results of *atomistic* design strategies that share the problems of compartmentalisation, fragmentation and the so-called transfer paradox.

In essence, many training programmes separately teach domains that are interwoven in practice (*compartmentalisation*), for example offering courses on subject matter, on child development and on the use of technology – but rarely provide opportunities to study situations in which the knowledge from these classes needs to be drawn upon in a synthesised manner. Within the domains, complex skills are often not addressed directly but broken down into parts and subparts which are then practiced in separation, without the step of then (re-)integrating the constituent techniques and practicing the entire complex skill (*fragmentation*). All of this culminates in the *transfer paradox*, termed a "paradox" for the following reason: While focusing on mathematical reasoning in one course and on giving emotionally sensitive feedback in another may result in more effective learning in each of these domains compared to the learning that would have occurred in a class integrating the two components, it is often the case that the two component skills then cannot be successfully integrated to reap the benefits of having acquired a high amount of proficiency in each.

The 4C/ID framework assumes that effective learning environments which combat these problems of compartmentalisation, fragmentation and transfer paradox can be analysed in four basic categories:

- learning tasks,
- supportive information,
- procedural information, and
- part-task practice.

A *learning task* is any activity that must be performed or studied by the learners of a training programme, and should always be based on a real-life task in the target situation (e.g. one of the typical "jobs" of mathematics teachers mentioned earlier).

Supportive information is then any information that helps the learners with non-routine aspects of the learning tasks, such as problem solving or decision-making (e.g. information about typical levels of abstract reasoning in children of certain ages, information about different modes of presentation). *Procedural information* in 4C/ID terminology helps with performing routine aspects of the task at hand that can be described algorithmically (e.g. information on how to use GeoGebra to display a function graph with a dynamically moving tangent line).

Part-task practice is useful for constituent skills where a high amount of automaticity leads to strong improvement in the performance of the complex skill, such that additional, separate training of these constituent skills seems justified (e.g. training teachers to solve linear equations or calculate simple derivatives "in their head" in order to be able to spontaneously spot routine mistakes without having to interrupt the flow of a lesson). Part-task practice is singled out in the framework as not being a necessary component in all complex learning environments and should only be employed if the degree of automaticity adds a significant amount of quality to the overall complex skills that need to be performed.

The 4C/ID framework continues to be extensively used and its effects investigated; to name a few examples of its implementation, in secondary physics (Melo, 2018) as well as social science (Maddens et al., 2020) in continuing education for health professionals (Susilo et al., 2013), in computer science and programming (Güney, 2019), in medical education (Vandewaetere et al., 2015) and vocational training for vehicle painters (Mulders, 2022). A recent meta analysis (Costa et al., 2022) suggests that the use of 4C/ID in educational programmes tends to have a high impact on performance, especially in higher grade levels, and concludes that the 4C/ID model should be employed more often in tertiary learning environments.

The implementation of 4C/ID in multiple sections of the teacher training programme has been successfully tried in Freiburg (Kreutz et al., 2020). In particular, the framework proved a useful tool in the redesign of an analysis module in mathematics teacher education (Leuders & Wessel, 2020). Before this, 4C/ID had not been widely used in teacher education (van Merriënboer, 2020).

Leuders and Wessel designed a course based on the principles of 4C/ID and had held it three times at the time of publication. The researchers concluded that 4C/ID can be applied profitably in the context of teaching real analysis to preservice teachers, but that there is a challenge that arises due to heterogenous subject matter knowledge of the participants. In this implementation, it was decided to structure the course in terms of real-life task classes on the one hand (first "analysing and interpreting student thinking", then "designing introductions to new concepts", lastly "planning phases of adaptive practice") and at the same time, in terms of the canonical sequencing of concepts in real analysis (first only discussing functions, sequences and limits, then adding tasks involving derivative, and so on). Although it requires a high amount of resources and attention on the side of the educators, this type of instruction was reported to be highly motivational by the students.

The evaluation of the course showed that preservice teachers particularly valued theoretical input when it was accompanied by a class discussion with the lecturer. Further, key terms occurring again and again due to deliberate sequencing was noted to be helpful, leading to effects of automatisation which the preservice teachers welcomed. In terms of criticism, some participants expressed a wish for more support by the lecturers especially during the beginning stages of a task class, while some directly stated that sample solutions would have been useful in the beginning in order to get a sense of what an adequate end product could look like.

In my dissertation project, I considered these reported experiences from Freiburg as somewhat of a "feasibility study" and sought to quantify the learning effects of a similar intervention. It seems that 4C/ID is a useful framework for the design of tertiary learning environments in mathematics teacher education in terms of student approval and the subjective experiences of both lecturers and preservice teachers. Can the suggested effects on preservice teachers' professional skills be shown experimentally? In particular, I was interested in the effect of a 4C/ID intervention on preservice teachers' noticing of subject matter related parts of their future profession and, relatedly, in their beliefs regarding subject matter.

The following chapter concludes the theoretical part of this dissertation by making a case for the research design I chose to answer these questions, and describing the theoretical framework I worked with, including my conceptualisation of "subject matter noticing" as a specific type of a mathematics educators' professional vision.

2.6 Purpose of the study

2.6.1 Research gap

Investigating ways in which the "double discontinuity" in teacher education can be ameliorated seems a worthwhile endeavour – and one that has been taken up by a large number of researchers already. Within the group of teacher educators aiming to equip prospective teachers with the mathematical content knowledge they will require and to foster productive attitudes towards mathematics, situated approaches have been widely argued for and implemented.

Researching the noticing of (preservice) mathematics teachers and finding ways to improve this professional skill has also been a popular focus in the past two decades and thus the question to be answered in this section is: What can another project in situated learning or noticing add to our body of knowledge about preservice teachers and their education?

In my dissertation, I posit that the *intersection of the two lenses*, the inherent connections between professional noticing and situated learning research have been underanalysed and underutilised, especially when it comes to preservice teachers' *noticing of* and *beliefs about subject matter*. I aim to make a first attempt at filling this gap by empirically showing the potential of a situated intervention in university teacher training when it comes to developing PSTs' subject matter noticing and positively influencing their beliefs about subject matter.

Considering that many mathematics education researchers describe one of the main reasons for preservice teachers' skepticism towards an extensive mathematical preparation in teacher training as a tendency to "underestimate the mathematical requirements of their future profession" (Hefendehl-Hebeker, 2013, p. 4), it seems only fitting that one goal of teacher education should be to *develop a sensitivity for mathematical aspects of (teaching) situations*. This sensitivity is what I will conceptualise as "subject matter noticing".

To provide some illustration by contrast, preservice teachers often appear keenly aware of the challenges of classroom management awaiting them, and, *anticipating* difficulties in this regard, usually embrace courses in the curriculum which focus on how to deal with students' behavioural issues. Targeting a change in preservice teachers' perception of their professional lives is therefore based on the following hypothesis: People who *expect* to be mathematically challenged in their day-to-day jobs as teachers will naturally want to become more mathematically proficient over the course of their university studies, and may show more persistence and frustration tolerance in the face of struggle as they are *intrinsically motivated to learn* instead of *extrinsically motivated to study* in order to fulfil a curriculum requirement.

In the status quo, it seems that preservice teachers can look at professional tasks and conclude that little more than secondary mathematics knowledge was being used by their own mathematics teachers (or whoever else they are basing their impression on), while expert teachers and teacher educators look at the same tasks and consider many of them to be highly linked to more advanced mathematical considerations. In fact, even practicing teachers in general (Zazkis & Leikin, 2010) struggle to provide content-specific instances of the relevance of the type of mathematical knowledge they learnt at university to their practice. Framed like this, it appears there is a difference in the professional vision of preservice and expert teachers when it comes to *noticing the mathematical subject matter required to perform well* in professional situations.

Thus, one might evaluate situated interventions in terms of the effect they have on preservice teachers' professional noticing of subject matter, because changes in this noticing could be expected to influence the teachers' inclination towards and success in learning and using advanced mathematical knowledge.

“I need to pass this course, but I won’t actually need to use the contents in my job… so I might as well just do what it takes to pass the exam”

vs.

“I need to pass this course, *and* the contents are going to be crucial for my job, so I need to pass the exam, and *also* meaningfully engage with the material”

While there are situated interventions being evaluated along various axes (teacher beliefs often being one of them), and multiple interventions showing that teacher noticing can be improved by professional education, I did not find any situated studies interventions aimed at reducing the effects of the double discontinuity that evaluated intervention effects on mathematics teachers’ professional noticing.

Moreover, while different foci for noticing have been researched (e.g. “noticing of children’s mathematical thinking”, “noticing of changes in representation”, etc.), I did not find conceptualisations of noticing as a professional skill that specifically centered *the subject matter* inherent in the completion of various teacher tasks. (In a dissertation project investigating the double discontinuity in the subject area of linear algebra, Blum-Barkmin (2022) conceptualises the “advanced standpoint” allowing teachers to pay attention to deeper subject matter aspects of school mathematics teaching as a professional competence, which is perhaps the most closely related conception albeit under a different framing.)

There *are* frameworks for analysing teaching, such as the Knowledge Quartet (Rowland et al., 2005), or the Three Points framework (Yang & Ricks, 2012), which embody the idea of actively looking for the key *mathematical* aspects of a teaching situation, but provide no conceptualisation of a “consistent awareness of mathematical aspects of one’s professional actions” that would constitute a “key component of teaching expertise” (van Es & Sherin, 2021, p. 17). In my theoretical framework, therefore, I outline how I conceptualise “subject matter noticing” as part of a teacher’s greater professional vision (see Figure 8).

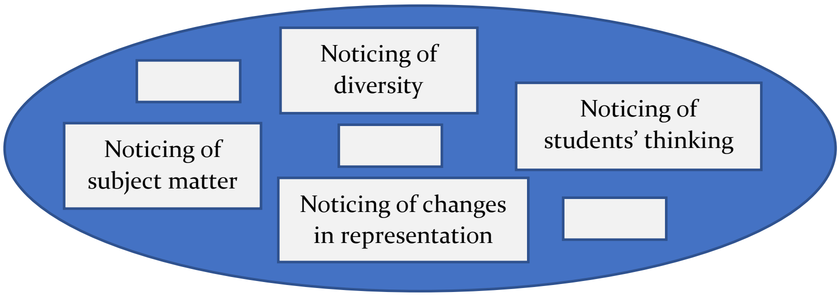


Figure 8: Objects of a mathematics teacher’s professional noticing

The relationship between preservice teachers’ subject matter knowledge and their noticing skills has also been described as an important research focus related to content and noticing (Dindyal et al., 2021), and the connections between the two seem obvious from a theoretical point of view: Subject matter knowledge itself is surely a prerequisite to the noticing of subject matter and most likely also has a large influence on beliefs about the importance of subject matter knowledge (it is hard to see and value something which you don't know).

I therefore also measured participants' subject matter knowledge in order to be able to look into the theoretical relationships between the three variables of subject matter noticing, beliefs about subject matter and subject matter knowledge on an empirical level. However, significant changes in subject matter knowledge as an effect of the intervention were not anticipated. The intervention, after all, sought to connect the subject matter knowledge preservice teachers already possessed with opportunities to apply it profitably when making teaching decisions, and to offer an opportunity to the preservice teachers to "develop an intimate dialogue between their mathematical knowledge and their didactic sensitivity" (Moreno-Armella, 2021, p. 621).

In order to sufficiently narrow down the scope of the intervention to a one-semester course, I decided to focus on teacher tasks with subject matter pertaining to real analysis. Felix Klein has stated that "the discontinuity between school and university, of which I have often spoken, is greatest precisely in the field of infinitesimal calculus" (Klein, 2016, p. 257), and thus restricting the subject matter to topics of real analysis seemed a particularly fitting choice.

4C/ID offered itself as a framework guiding the situated design of the intervention as its implementation in real analysis courses for preservice teachers had been successfully tried in Freiburg. It thus appeared to be a promising framework for designing "opportunities for learning that are aimed at helping teachers use subject matter knowledge to figure out what their students know, to pose questions, to evaluate and modify their textbooks wisely, to design instructional tasks, to manage classroom discussions, and to explain the curriculum to parents" (Ball, 2000, p. 246) – i.e. to relate mathematical subject matter to PSTs' professional tasks of the future.

In a sense, my dissertation project seeks to add to these positive first results reported by Leuders and Wessel (2020) and analyse causal effects of a similar intervention on PSTs' *beliefs about subject matter* and *noticing of subject matter* by employing a quasi-experimental research design.

Before proceeding to formulate specifically which research questions I therefore sought to answer with my study, the following two subchapters will

a) clarify which precise theoretical conceptualisation of the constructs subject matter knowledge, beliefs about subject matter and noticing of subject matter I use, and
b) describe in more detail which kind of situated intervention I was planning on implementing and evaluating.

2.6.2 Theoretical framework

2.6.2.1 Subject matter knowledge

I define **subject matter knowledge** as the mathematical knowledge that a prospective or practicing teacher possesses, encompassing both conceptual as well as procedural aspects. Importantly, subject matter knowledge means purely mathematical knowledge that has not yet been related to the actions of teaching in which it may become relevant. Using Shulman's (1986) terms, I specifically mean content knowledge (CK) as opposed to pedagogical content knowledge (PCK).

This mathematical knowledge contains concepts and techniques which are part of the school curriculum and may have already been acquired during a teacher's time as a school student. However, it also goes beyond that and further encompasses some "Advanced Mathematical Knowledge" (AMK) which Zazkis and Leikin (2010, p. 264) describe as any "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college". It is worth noting that the AMK preservice teachers acquire at university will usually contain both (advanced) school-related content knowledge (SRCK) and purely academic content knowledge, as Heinze et al. (2016) differentiate.

Within the category of school-related content knowledge, one might further distinguish between elaboration on concepts which preservice teachers know from their own school experiences, and completely new ideas, which are first introduced at university but then related back to familiar school mathematical concepts.

It is assumed that while preservice teachers are aware of many of the mathematical objects of school-related subject matter before entering university, their understandings of these objects – such as a function, a sequence, or even an equation – are often radically challenged, altered and enhanced as a result of their continuing mathematical education. This of course applies not only to mathematical objects, but also to mathematical relationships, procedures and strategies.

Other mathematical ideas and procedures, on the other hand, are truly mentioned for the first time during the preservice teachers' university studies. Examples would be the notions of a vector space, continuous differentiability or Gaussian elimination. Some of these concepts involve new ways of thinking about familiar relationships (e.g. school students have been performing operations on the rational numbers according to the field axioms for years without having learnt the definition of a field), while some of the concepts truly go beyond school mathematics (e.g. distinguishing pointwise versus uniform convergence of a sequence) – this would constitute the border to purely "academic content knowledge" in the sense of Heinze et al. (2016).

When measuring beliefs about subject matter knowledge, the two subsets of university-level mathematics – one school related, one purely academic – were not distinguished and the general term "university mathematics" was used in the items to describe any and all mathematical subject matter that the preservice teachers had encountered at university. (It needs to be noted, however, that at the University of Vienna, the subject matter courses taken by preservice teachers are separate and specifically designed to fit the needs of prospective teachers and thus contain fairly little purely academic mathematics compared to the corresponding mathematics courses taken by mathematics students who are not attending the teacher training programme.)

The situated intervention focused primarily on showcasing opportunities in which one of these kinds of *advanced* mathematical knowledge could be used beneficially by a mathematics teacher. As such, the "subject matter knowledge" required to solve the tasks presented in the intervention was generally of **advanced, but school-related** nature. (When the mathematical concepts or relationships themselves were of secondary level, the teacher tasks that needed to be performed required a form of meta-knowledge or reflection which could not typically be expected even of a high-performing secondary student.)

2.6.2.2 Beliefs about subject matter

The set of problematic beliefs about subject matter which motivated this study stem, to a large extent (though not exclusively), from the double discontinuity. Therefore, when measuring the participants' beliefs, two types of negative beliefs that roughly correspond to the first and to the second discontinuity were measured: beliefs regarding the *coherence between school mathematics and advanced mathematics on a content-level*, and beliefs about *the anticipated relevance of advanced mathematics for the teaching of school mathematics*.

The questionnaire developed by Isaev and Eichler (2017) about "beliefs concerning the double discontinuity" appeared to be a great fit to assess changes in the participants' views regarding this set of beliefs. With permission, their questionnaire was adopted for measuring PSTs' beliefs. My conceptualisation of beliefs in general as well as of beliefs regarding the double discontinuity therefore goes in line with the theoretical assumptions on which their development of the measurement tool was based.

Beliefs about subject matter are considered to be a component of "mathematics-related affect" (Hannula, 2012, p. 137) and defined, according to Philipp (2007), as "psychologically held understandings, premises, or propositions" (p. 259). In the context of the double discontinuity, this means preservice teachers' propositions about the connections between school mathematics and advanced ("university") mathematics on the one hand, and propositions about the relevance of such advanced mathematics to the preservice teachers' later job of teaching school mathematics, on the other hand.

While beliefs are conceptualised as relatively stable dispositions of a person (Pajares, 1992), it has been shown that it is possible to influence or change them (Liljedahl et al., 2012; Panero et al., 2023), specifically as a result of appropriate methods in university teacher training – Eichler and Isaev (2023) themselves have used their questionnaire for this purpose. Influencing preservice teachers' beliefs is especially auspicious because of evidence that changes in beliefs go hand in hand with changes in displayed behaviour (Calderhead, 1996). More positive dispositions regarding the usefulness of subject matter knowledge could therefore arguably lead to more "mathematically mindful" teaching actions in the sense of paying increased attention to mathematical aspects of those teaching actions – demonstrating the situated skill of "subject matter noticing", which I will now describe.

2.6.2.3 Noticing of subject matter

I conceptualise "noticing" in the sense of a teacher's professional vision (Goodwin, 1994) as part of a teacher's expertise, which in particular means that noticing is used in a normative sense – this type of vision sets apart novices from experts in the field. Noticing is the specific skill of paying attention to relevant details of a situation while disregarding less crucial aspects, and it draws upon a teacher's professional knowledge as well as their beliefs.

Subject matter noticing is conceptualised as *theme-specific noticing* in the sense of Dreher and Kuntze (2015), with the theme being *the mathematical subject matter* inherent in a teaching situation. This includes, on the one hand, mathematical subject matter that is directly perceivable. For example, the headline "Real functions and their graphs" on a blackboard should make it clear to most people choosing to pay attention to the board's contents that, apparently, (things called) real functions and graphs are part of that teaching situation's mathematical setting. On the other hand, there is mathematical subject matter that requires further background knowledge and an intermediate "interpretative" step in order to be identified. A mathematics teacher (but not a layperson!) might realise that in order to connect the notion of a real function to its two-dimensional graph, the relation which the function provides between elements of its domain and elements of its codomain should be thought of as a set of ordered pairs of numbers $(x, f(x))$, which then correspond to points of the graph on the xy-plane.

Subject matter noticing thus encompasses a sensitivity to subject matter considerations which directly or indirectly affect a teaching situation (in a broad sense, including lesson preparation and post-lesson work such as correcting homework or grading tests). Subject matter noticing allows teachers to pay special attention to uniquely mathematical features of a situation and to use their subject matter knowledge to make sense of said situation and react productively. Successful subject matter noticing may help a teacher understand and adequately answer the student question "Why is there only a vertical line test, but not a horizontal line test for graphs?", or to design tasks that target specific (mis)understandings, such as asking students to explain why a function whose graph only contains points (x, y) with $|y| \leq 10$ must be bounded and give an example of an unbounded function whose graph only contains points (x, y) with $|x| \leq 10$. (While knowing which types of faulty understandings are "common" among learners of mathematics may fall under the category of pedagogical content knowledge, there is purely mathematical subject matter knowledge which is required to understand *why* these misunderstandings are problematic and which counterexamples are *mathematically suitable* to respond to them that can be clearly delineated.)

As these elaborations show, this broad view of subject matter noticing as encompassing more than mere "seeing" lends itself naturally to three-part conceptualisations (see

Figure 9) like the ones that can be found in the constructs of Jacobs et al.'s (2010) or Kaiser et al.'s (2017) work:

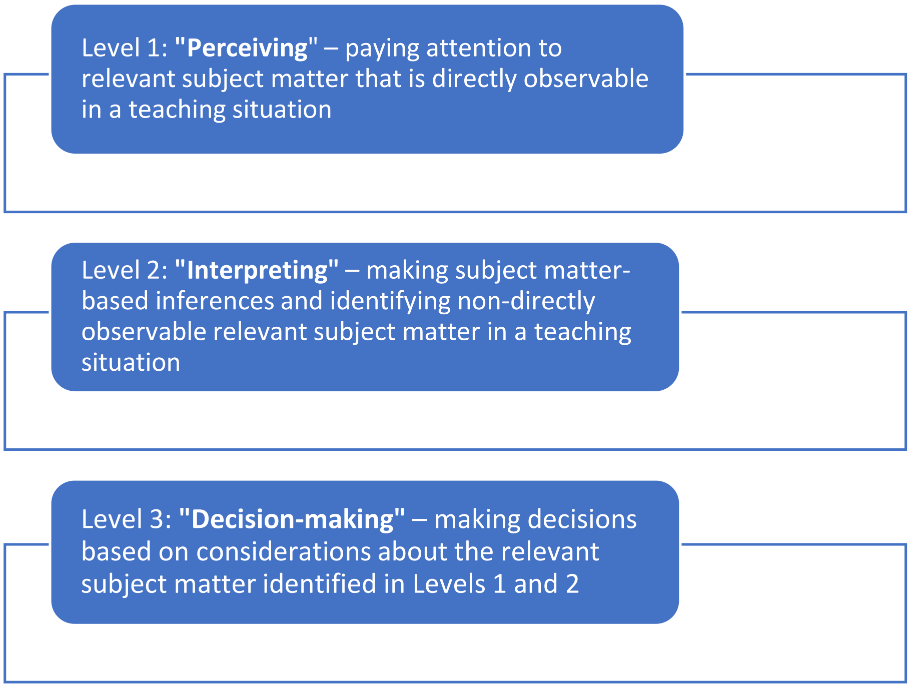


Figure 9: Three-part conceptualisation of subject matter noticing

To illustrate this, let us look at a text vignette describing the following teaching situation:

> You are a teacher looking through a practice booklet, searching for problems you might adapt as tasks for your students. You come across the following item:
>
> *Problem 72: Consider the following approximation:* $(1+x)^2 \approx 1+2x$.
>
> *a) Calculate the approximation error* $r(x)$.
>
> *b) Show that* $\frac{r(x)}{x}$ *approaches* 0 *as* $x \to 0$.

On the first level, a teacher may note that students solving this problem need to be familiar with the concept of an approximation and what it means for an approximation to have a certain degree of error – that can be expressed as a function – associated with it. The teacher might notice that, in this specific instance, it is a quadratic term that is being approximated by a linear term. Also, the concept of a limit is of relevance, as evidenced by the wording of the second task.

Of course, a person with a limited mathematical background may not pay attention to the distinction between a quadratic and a linear term, or may not be able to read the limit notation in the second question. However, since we are considering mathematics teacher noticing in particular, all of these characteristics are more or less directly observable and therefore categorised as belonging to the first level of "Perceiving".

On the second level, a teacher may note that the approximation is arrived at by expanding the squared expression and disregarding the resulting quadratic term involving the variable x – that term x^2 of course representing the "error" $r(x)$ which is referenced in part a) of the problem. Additionally, the teacher may realise that this is a good approximation *for small instances of* x, in which the squaring of a number lesser than 1 results in an even smaller number – the disregarded summand x^2 grows smaller at a faster rate than the variable x itself. The teacher could make the connection that this is precisely what one is supposed to prove in part b) of the problem: That the ratio of $r(x)$ to x tends to 0 as x tends to 0, i.e. that $r(x)$ "grows small more quickly" than x itself does. All of these "secondary perceptions" already involve mathematical considerations and interpretations which go beyond reading what is printed on the page.

Finally, at the "decision-making" third level, the teacher might notice and bear in mind the connection between the linear approximation given in this problem and the topic of derivatives. As one option, they may decide to include a similar task in order to let students explore the fact that the local tangent t is in fact, the *best* linear approximation of the quadratic function f at a specific point x of its domain, and that this fact can be characterized mathematically by demanding that the ratio between $r(x) = f(x) - t(x)$ and x tends to 0 as one approaches that point x.

The last part of this subchapter outlines the relationships I assume between the constructs of subject matter knowledge, subject matter beliefs and subject matter noticing.

2.6.2.4 The triangle of subject matter: knowledge – beliefs - noticing

> "Situation-specific skills function as the processes through which knowledge and beliefs become relevant in practice. At the same time, changes in competence would not be possible if teachers did not deliberately attend to and interpret practice and make decisions that create new knowledge and new beliefs. Therefore, practice offers opportunities both to refine perception, interpretation, and decision making and to increase one's knowledge and change one's beliefs. In other words, **when teachers deliberately design lessons to access and examine student mathematical thinking, their subsequent reflections on their teaching is enhanced by specific evidence of student thinking they were able to collect during teaching**." (Santagata & Yeh, 2016, p. 163)

My theoretical framework rests on the assumption that what holds true for students' mathematical thinking according to Santagata and Yeh, is also the case for subject matter, i.e. the "bi-directionality between knowledge, beliefs, skills and practice" (ibid) exists in an analogous way for the categories of

- subject matter knowledge
- beliefs about subject matter, and
- noticing of subject matter (as an instance of a situation-specific skill).

This assertion is backed by a variety of literature: Domain knowledge heavily influences noticing (Glaser & Chi, 1998; Seidel & Stürmer, 2014; Mason, 2002) as well as a person's beliefs (Liebendörfer, 2018; Stylianou et al., 2015; Stein, 1996). Beliefs, on the other hand, have an effect on what people notice (Meschede et al, 2017; Roose et al., 2019; Goldsmith et al., 2014) and can also influence the acquisition of new knowledge (Mangels et al., 2006; Dahl et al., 2005; Bandura, 1997). Lastly, by the mechanism described in the quotation above, what a person notices can contribute to new knowledge being formed (Llinares, 2014), and shape a person's beliefs (Guskey, 2020).

I shall make the causal connections I thus assume to be plausible explicit for the three variables relating to mathematical subject matter:

Subject matter knowledge → beliefs about subject matter
Subject matter knowledge → noticing of subject matter

Preservice teachers' (lack of) proficiency in the mathematics studied at university can be assumed to have a significant impact on their beliefs regarding the usefulness of such advanced mathematics. After all, it is difficult to see the benefit of something one does not know, remember or understand. At the same time, those preservice teachers with a greater understanding of advanced mathematics are also much more likely to spot instances of it in teaching situations – possibly leading to better noticing skills. Again, it is hard to recognise and be able to make a note of something which one is not familiar with.

Noticing of subject matter → beliefs about subject matter
Noticing of subject matter → subject matter knowledge

Being good at recognising basic as well as advanced mathematical aspects of teaching situations will presumably lead to the impression that teaching happens in a mathematically rich and challenging environment and the belief that therefore sound subject matter knowledge, including mathematics that goes beyond school-level mathematics, is needed. Noticing of subject matter in teaching situations can also lead to discoveries which may enhance one's own mathematical understanding, providing, for example, more extensive personal example spaces or new insight into the mathematical reasons behind of common misconceptions or errors (e.g. recognising an error as an overgeneralization of a correct rule).

Subject matter beliefs → noticing of subject matter
Subject matter beliefs → subject matter knowledge

Lastly, beliefs are a filter through which we perceive the world, and as such, a person who is convinced of the importance of subject matter (including advanced mathematics) for teaching can be assumed to showcase a greater sensitivity for it and thus have a higher likelihood of spotting instances of it in practice. At the same time, preservice teachers' willingness and dedication for learning advanced mathematics will presumably be influenced by whether or not they believe it to be relevant to their future careers.

These hypothesised relationships between subject matter knowledge, beliefs about subject matter knowledge and the noticing of subject matter can be visualised in the form of a triangle (see Figure 10):

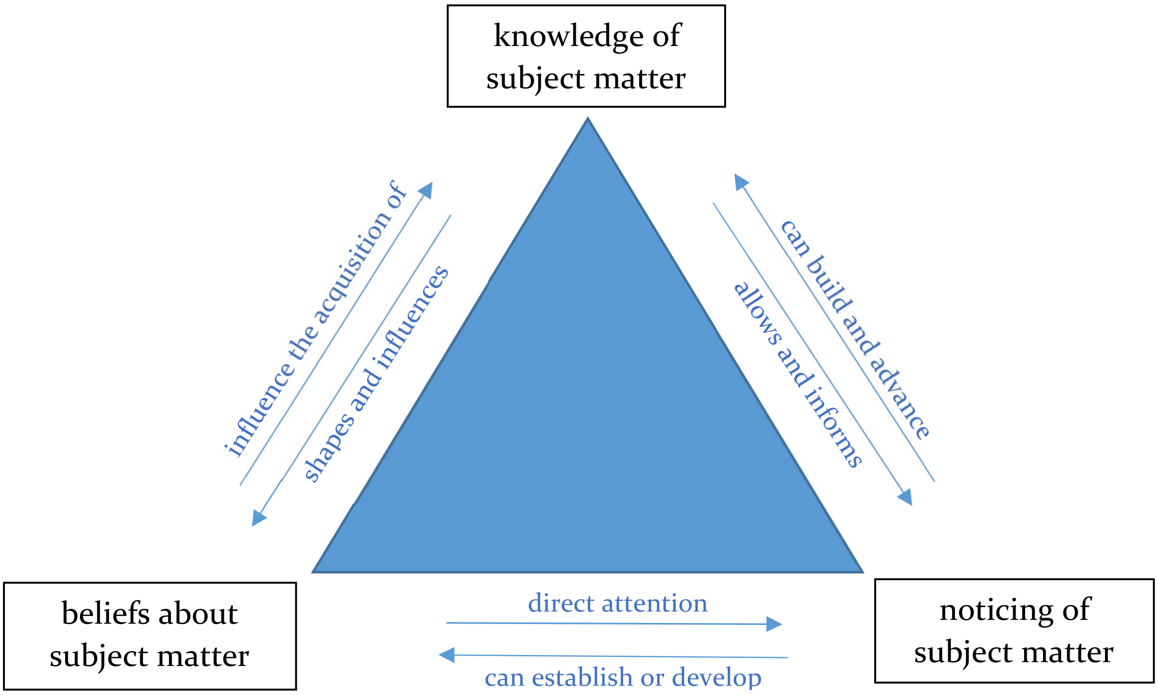


Figure 10: Possible causal interactions between knowledge of, beliefs about and noticing of subject matter

Assuming that the reciprocal nature of these dependencies holds, one must conclude that changing one variable is likely to prove difficult as long as the other two corners remain fixed. A holistic design strategy such as the one developed by the authors of the Four Component Instructional Design framework is promising, then, because of its potential to address all three components jointly:

- revisiting and potentially revising one's own subject matter knowledge directly in relation to specific teacher tasks in which it becomes relevant,
- actively practicing teacher tasks with a sensitivity for the subject matter requirements and affordances (cf. Choy & Dindyal, 2021) of typical teaching situations, and
- gathering positive experiences that may influence beliefs regarding the usefulness of having a comprehensive understanding of subject matter.

These considerations resulted in certain desired characteristics when designing the intervention using the 4C/ID framework, which are described in the following section.

2.6.3 Desiderata for the situated intervention

The intervention was designed to fulfil the following criteria as much as possible:

- **provide a supportive environment for practicing consciously using subject matter knowledge (SMK) for teacher tasks**

My main goal was to invite participants to revisit, reflect on, and if necessary, adjust some notions in their SMK, and to guide them towards using these subject matter considerations to their benefit when performing teacher tasks. Instead of expecting them to be able to use the knowledge they had acquired in an analysis lecture in a completely new situation, the intervention was supposed to allow participants to grow their ability to consciously consider subject matter in teaching situations by providing appropriate amounts of scaffolding including feedback and expert guidance. Doing so, I hoped that over time the preservice teachers would grow more used to and more comfortable in spontaneously and flexibly applying their SMK when performing teacher tasks.

At the same time, I considered it important to enable participants to subjectively experience themselves as becoming better able to apply their own SMK. A key part of this meant continually trying to meet participants at their level of SMK (which could then potentially be a starting point on which new understandings could be built by looking more closely at the mathematical problem at hand) and use self-differentiating prompts whenever possible. One the one hand, these experiences could result in preservice teachers over time becoming more inclined to notice and think about SMK when reflecting on a teaching situation, because they had made personal positive experiences when "acting mathematically" as teachers.

On the other hand, these activities could have an effect on the PSTs' beliefs about subject matter: In the case of students whose views on the importance of subject matter for teaching is, in part, based on a sort of coping mechanism because they do not feel adequately prepared (cf. Bungartz & Wynands, 1998) mathematically, such moments of success might positively influence their views about the relevance they personally assign to SMK.

- **display SMK as necessary and useful when responding to typical teaching situations**

When designing the tasks in the intervention, a key consideration was designing each situation in such a way that the practical benefit of purposefully paying attention to the SMK pertaining to that teacher task would become obvious. This meant showing that preservice teachers could activate their SMK to their own benefit when confronted with a problem, but also highlighting problems which may have resulted from a lack of SMK considerations beforehand. Being aware of the potential of SMK aspects of any teaching situation to enhance or hinder the teaching process could teach preservice teachers to look for those aspects and to establish this as part of their mental routine when considering a teaching action – simply, to notice the affordances of mathematical subject matter in any teaching situation. Hence, the intervention could improve PSTs' noticing of subject matter by making them more inclined to look for it as they experience it as a net benefit to their professional lives.

At the same time, on a beliefs level, this experience could lead skeptical preservice teachers to change their mind about the relevance of SMK, as they are confronted with situations in which it is truly beneficial to the everyday tasks of a teacher.

- **showcase an abundance of SMK in typical teaching situations**

By choosing a variety of teaching situations in my vignettes which featured rich mathematics that was inherent to the task the teacher had to perform, the participants should be able to see for themselves the highly SMK-dependent nature of a teacher's profession instead of merely being told about its relevance. Being aware that there often is more to "the mathematics of a teaching situation" than may be apparent from the perspective of a student might teach preservice teachers to actively look for and notice SMK "beyond the surface", again leading to improved subject matter noticing.

Regarding the effect on PSTs' beliefs, seeing SMK appear again and again in authentic teacher tasks could have a positive influence on preservice teachers' views on the importance of subject matter knowledge for their future profession.

- **point out interconnections of SMK with other kinds of teacher knowledge**

Considering SMK in connection with other types of teacher knowledge and practical skills which many preservice teachers expect to be more relevant to their profession compared to SMK, I hoped to show that, more often than not, one of these components could not be meaningfully employed without relating it to the others.

By using vignettes in which, for instance, decisions about teaching methods needed to be justified by analysing the subject matter that was to be delivered, or in which pedagogical issues could be anticipated or dealt with by considering the SMK pertinent to the situation, participants of the intervention should be incentivised to think of SMK as part of an interrelated professional knowledge pool instead of a separate section of knowledge that could be disregarded "in favour" of other kinds of skills.

This could, again, incentivise preservice teachers to be more strongly attuned to SMK aspects even in teaching situations they primarily associate with other types of knowledge (e.g. a problem of classroom management) – leading to better subject matter noticing –, but also change their beliefs about the meaningfulness of SMK as they realise that it cannot be disentangled from parts of teaching they already consider vital.

The 4C/ID framework's focus on learning tasks that are based on authentic real-life tasks and the use of scaffolding by using various kinds of supportive information seemed highly suitable for achieving these aims of the intervention design.

Whether this type of intervention would really produce the expected results with regard to subject matter noticing and beliefs about subject matter constituted my main research questions, which will now be formulated in the following section.

3 Research questions and hypotheses

Effects of the intervention

The study's main interest was to gather information about the effectiveness of a situated intervention in terms of beliefs changes regarding subject matter and changes in subject matter noticing. As such, the two primary research questions can be formulated in the following way:

- Q_1: How does the situated intervention affect preservice teachers' noticing of subject matter?
- Q_2: How does the situated intervention affect preservice teachers' beliefs about subject matter?

My hypotheses were that the situated intervention would positively influence both PSTs' subject matter noticing as well as their beliefs about subject matter:

H1: The situated intervention will lead to significant improvements in subject matter noticing.

H2: The situated intervention will lead to significantly more positive beliefs regarding subject matter.

In the case that both H1 and H2 would evaluate positively, I was also interested in finding out whether for the participants of the intervention group, any changes in subject matter noticing would go hand in hand with changes in beliefs about subject matter.

- Q_3: How are changes in the PSTs' noticing of subject matter related to changes in the PSTs' beliefs about subject matter?

My hypothesis was that these changes would co-occur:

H3: Changes in participants' noticing of subject matter are significantly correlated with changes in participants' beliefs about subject matter.

Triangle-structure of the three variables noticing, beliefs and knowledge

As a secondary, theoretical interest, I was also interested in exploring the empirical structure of the three variables in my data, with the following research question in mind:

- Q_4: What correlations appear between the PSTs' subject matter knowledge, subject matter noticing and beliefs about subject matter?

Based on the triangle conceptualisation I have laid out in my theoretical framework, I expected all three variables to be significantly correlated with each other at both time points:

H4: The three variables subject matter noticing, beliefs about subject matter and subject matter knowledge are all significantly correlated at both T1 and T2.

Aside from these overall correlations, I was also curious whether there would be any differences in the data structure between the experimental groups, and whether these structures would present differently between T1 and T2. However, I approached this part of the analysis purely exploratively and did not form or seek to reject any particular (null) hypotheses.

Participants' subjective experiences of the seminar

Lastly, I had gathered qualitative data as further source of evidence of participants' self-reported beliefs regarding aspects of the double discontinuity and how they experienced the intervention in relation to this. The PSTs' statements in these reflective essays would hopefully provide some context to the quantitative data about beliefs collected in the questionnaire:

- Q5: What do PSTs think about the relevance of advanced mathematics for their future profession after having participated in the intervention?
- Q6: How did PSTs experience the situated intervention in relation to this topic?

Both of these questions were considered without the aim of rejecting any particular (null) hypotheses.

The research methodology that was decided upon in order to answer these questions will be described in the following chapter.

4 Methodology

4.1 Research design

In order to be able to make causal inferences about the effects of my intervention, a quasi-experimental design was chosen, with another seminar group of preservice teachers acting as a control group.

4.1.1 Overall timeline

The infographic in Figure 11 below provides a chronological overview of the phases of my dissertation project, which ran from January 2020 until the end of 2022. A brief description of each phase follows below.

Figure 11: Project timeline

Step 1: Conceptual work

Once the problem situation of preservice teachers' negative dispositions regarding university mathematics and the epistemic interest in the affordances of situated learning in this issue had been identified, a review of the existing literature was conducted. Research on beliefs quickly emerged as a useful framing to talk about preservice teachers' negative dispositions and Isaev and Eichler's (2017) questionnaire about the double discontinuity appeared to be a good fit for the purposes of my inquiry. While "noticing" as a concept seemed to offer a fruitful lens through which to analyse preservice teachers' cognitive and affective developments in a situated manner, the noticing of mathematical subject matter as such had not yet been studied within this framework. Therefore, I developed the idea of subject matter noticing in analogy to other theme-specific noticing constructs (cf. Dreher & Kuntze, 2015).

Step 2: Designing and piloting the questionnaire

After choosing this conceptual basis, I set out to operationalise "subject matter noticing" and develop a measurement tool that could capture preservice teachers' proficiency in noticing mathematical subject matter. This process involved the formulation of a set of 16 text vignettes which were then sent to six mathematicians and mathematics educators who I expected to show expert level noticing when it came to mathematical subject matter. On the basis of these experts' ratings, the four items to be used in the questionnaire were selected and the first part of the coding manual was developed. Then the entire questionnaire – including the beliefs section and the subject matter knowledge section – was piloted with a group of preservice teachers. Based on the results of the pilot study, the second part of the coding manual for the noticing section of the questionnaire was developed and small changes were made to the questionnaire.

Step 3: Designing the intervention

The design of the intervention was undertaken according to the principles of the Four Component Instructional Design (4C/ID) framework (van Merriënboer & Kirschner, 2017). As a first step, three real-life tasks on which the learning tasks would be based were selected. Then the mathematical topics that would be covered in the seminar were decided upon. Next, learning tasks were formulated and sequenced according to increasing levels of independent work that would be required of the preservice teachers. Different types of supportive and procedural information were designed which would be provided to the learners throughout the intervention – handouts and subject matter quizzes acting as "(re)freshers" for the mathematical knowledge required to solve the learning tasks, checklists of leading questions the students might consider when analysing a teaching situation, as well as worked examples to showcase expert approaches to analysing and solving the problems. Different types of expert and peer feedback were planned at appropriate points of the intervention and the mode of assessment was decided upon.

Step 4: Conducting the quasi-experiment

The intervention was conducted twice in order to maximise the number of participants – once in the winter term of 2021/22, and once in the following summer term of 2022. In the first round, the intervention was led jointly by the author and a mathematics researcher with a special interest in university teacher training in mathematics. In the second round, the author held the intervention by herself.

The course chosen for the intervention was called "seminar subject-specific teaching" [Seminar Fachdidaktik] in which the curriculum allowed for a wide variety of topics. The control group in each round was a "Seminar for teaching practice" [Praxisseminar], which is a course accompanying students undertaking extensive practical experience over the course of the semester. Both groups wrote the pre- and posttest in the first and last sessions of their courses, with the intervention group additionally being asked to submit reflective essays afterwards.

Step 5: Data analysis

After the intervention had been conducted, the data which had been gathered was processed and analysed in order to answer the research questions. For the qualitative data of the noticing section of the pretest and posttest questionnaires, the previously developed coding manuals allowed deductive coding and quantification of the participants' open answers. The quantified data was then analysed using SPSS, employing both parametric and non-parametric tests. For the qualitative data of the reflective essays, an inductive coding strategy was chosen. The combination of data thus gathered was used to evaluate the hypotheses and answer the research questions.

4.1.2 Practical implementation and institutional boundary conditions

For curricular reasons, it was not possible to implement the intervention in a course to which participants could be randomly assigned. Also, in order to make the seminar more attractive to students and gain as many participants as possible, some information was given beforehand (see supplement 10.7) as to the contents of the intervention.

At the university of Vienna, the teacher accreditation program is split into a Bachelor of Education and a Master of Education curriculum. Prospective teachers have to choose two school subjects at the beginning of their Bachelor's degree. The regular (minimum) length of study allocated for these degree programs is 8 and 4 semesters, respectively.

Preservice teachers have some practical components in the Bachelor's curriculum (accumulating to a total of 7 ECTS points, out of which 3 are specific to mathematics teaching), but the largest part of their practical experience follows in the Master's degree, where the practical phase [Praxisphase] in both chosen subjects takes up a total of 30 ECTS points.

While it is not devised as such, the lived reality of the majority of Master's students (and a growing number of Bachelor's students due to the lack of mathematics teachers in

Austria) is that they are already active as teachers and complete their Master's degree parallel to their employment at a school.

The Master's curriculum contains an elective "seminar subject-specific teaching" the specific contents of which can take on a different focus each semester depending on the lecturer. This was the main argument for implementing the situated intervention in this manner. Additionally, since the intervention required some solid pre-existing knowledge of analysis which would then be applied to teaching situations, it did not make sense to position it much earlier in the curriculum. It should be noted that this elective can also be chosen by Bachelor's students, which is why a few participants of the intervention group were (advanced) Bachelor's students and not Master's students.

The teacher training programme at the University of Vienna itself does not have an affiliated practice school, as such, the intervention had to be designed in a way that was as close to authenticity as possible while not being able to provide PSTs with real teaching experiences with pupils.

The control group was chosen to be the accompanying "seminar for teaching practice" which has to be taken by all Master's students who were completing their teaching practice during that semester. Preservice teachers who partook in the teaching practice were automatically assigned to one of the seminar groups and thus no self-selection effects were likely in the case of the control group.

Each course lasted one semester, spanning around 13 weeks of classes. The pretest was administered in the first week of classes, and the posttest in the last session. In the case of the intervention group, the participants were asked to write their reflective essays after having taken the posttest. Figure 12 illustrates the set-up of the experiment.

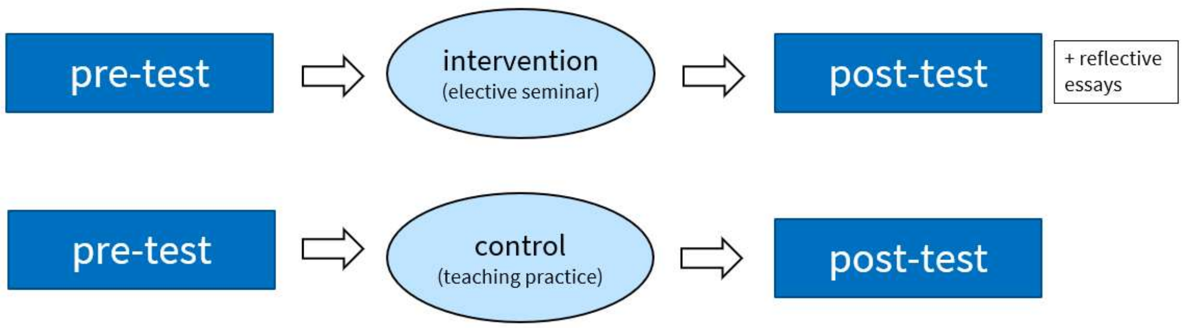


Figure 12: Illustration of the quasi-experimental study design

This procedure was implemented once during the winter term of 2021/22 and repeated identically in the summer term of 2022.

4.1.3 Analysis in Austrian school curricula and in secondary teacher training

The groundstone for the analytic concepts of upper secondary school is laid with the notion of functional relationships which is introduced in years 7 and 8 of lower secondary school, with the focus being a qualitative understanding and the ability to read and interpret different ways of presenting functional relationships between variables (as graphs, tables, or using formulae).

According to the upper secondary school curriculum (BMBWF, 2018), starting with year 10, the concept of function should be formalised and the following analytical concepts are to be taught (topics set in cursive are only required for schools types with more than 3 mathematics lessons per week):

Year 10:

- Abstract notion of a function as a map between two sets
- Different kinds of real functions and their characteristics (monotonicity, symmetry, periodicity, extrema)
- Composition of functions, inverse functions
- Sequences as real functions with the domain being the natural numbers
- Characteristics of sequences (monotonicity, boundedness, convergence)
- Convergent and divergent series

Year 11:

- Difference quotient and differential quotient
- Derivative of a function, higher order derivatives
- Describing monotonicity and curvature, extreme points and inflection points of a function using the derivative
- Continuity of a function
- *Differentiability of a function, link between a function's differentiability and its continuity*

Year 12:

- Definite integral and its interpretations in different contexts
- Indefinitive integral/antiderivative
- Fundamental theorem of calculus
- Basic difference equations and differential equations

In order to be able to teach these topics proficiently, the secondary teacher training programme in mathematics at the University of Vienna includes the following required (partially) analysis-related courses:

- Introduction to mathematics – lecture plus tutorial (1st semester of BEd)
- Analysis in one variable – lecture plus tutorial (4th semester of BEd)
- Teaching analysis – lecture plus tutorial (5th semester of BEd)
- Multivariable linear algebra and analysis – lecture plus tutorial (1st semester of MEd)

Most of the courses on mathematical subject matter in general (not just in analysis) are a part of the Bachelor's curriculum. The listed course on multivariable linear algebra and analysis is one of two subject matter courses in the Master of Education curriculum.

The second one is an elective subject matter course to be chosen from a pool of lectures on “higher mathematics”, which include analytic topics such as differential equations or complex analysis.

4.2 Designing the intervention

The intervention was designed, wherever possible, following the steps and guidelines of the Four Component Instructional Design (4C/ID) framework by Merriënboer & Kirschner (2017).

The four design components of 4C/ID are *Learning Tasks, Supportive Information, Procedural Information*, and *Part-Task Practice*, the last of which is an optional component for recurrent task aspects in which a high level of automaticity is necessary (such as knowing how to perform different types of stitches as a surgeon). Since conscious reflection on (possibly unexpected) mathematical aspects of teaching stood at the centre of the intervention to be designed, any highly automatised aspects of mathematics teaching (e.g. routinely scanning students’ solutions or statements for errors in arithmetic or basic algebra) were set aside. As such, the design process focused on purposefully choosing and interleaving the first three components of the framework.

In the current version of the 4C/ID, the design procedure is further streamlined into *Ten Steps to Complex Learning*, some of which are described as being optional, depending on the type of training programme to be designed and the degree to which the nature of the tasks which are to be trained have already been analysed elsewhere. In the case of mathematics teacher training, which has been researched for many decades, certain prerequisite steps of the analysis (like identifying prerequisite mental models, such as the necessary mathematical subject matter itself) could be skipped because a range of existing resources on these domains can be consulted.

The intervention focused on typical, yet *non-routine* tasks of teachers, hence the steps aimed at analysing and planning instruction for recurrent tasks were partially skipped.

Only the steps in bold will be described in detail below:

1. **Design learning tasks**
2. **Design performance assessments**
3. **Sequence learning tasks**
4. **Design supportive information**
5. Analyze cognitive strategies
6. Analyze mental models
7. **Design procedural information**
8. Analyze cognitive rules
9. Analyze prerequisite information
10. Design Part-Task Practice

4.2.1 Designing learning tasks

> "Thus for me a successful professional development session is one in which participants can actually imagine themselves acting differently in some situation in the future which they recognise." (Mason, 2009, p. 5)

The core of the 4C/ID framework are *learning tasks* designed to reflect as authentically as possible some real-life tasks of, in our case, mathematics teachers. Real-life tasks are set apart from many problems given to students in traditional training programmes by being, in general,

- *ill-structured*, meaning that some crucial elements of the situation may not be provided, and that there may be multiple (or even no) solutions, requiring the problemsolver to make educated judgments instead of following a procedure,
- *multidisciplinary*, in that they require the integration of multiple areas of knowledge in order to be solved, and
- very often *team tasks*, as professionals in any area seldom work alone and are often encouraged or even required to collaborate with colleagues both from within their domain as well as with professionals of other fields.

Choice of real-life tasks

The real-life tasks of teachers can be conceptualised as a profile of requirements that educators are expected to meet (cf. Bromme (1997) for teachers in general, Prediger (2019) focusing on language-responsive mathematics teaching, and Prediger et al. (2022) using the concept to analyse teacher educators' tasks).

Different non-exhaustive catalogues of teacher tasks have been proposed, for example under the term *job analysis*, with examples like evaluating a textbook's approach to a topic, selecting and designing a task, choosing and using representations, and analyzing and responding to student errors (Bass & Ball, 2003).

In order to be able to recreate the typical situations a mathematics teacher may find themselves in as *authentically* as possible, any tasks which would require the involvement of real pupils and typically occur *while* teaching were dismissed. Alternatives such as preservice teachers play-acting as school students or pretending to respond in-the-moment to a recording were deemed too far removed from reality to represent an adequate approximation of practice (cf. Grossman, 2009) for the purpose of the seminar the perceived authenticity of which was central to the design.

Building on these notions, three real-life tasks of mathematics teachers were chosen

- choosing, adapting and utilising exercises
- providing explanations and giving illustrative examples
- evaluating student solutions and giving feedback

These tasks can all fall under the category of "lesson preparation"/"follow-up work"[1] that is often undertaken outside the classroom, in the teacher's office or home – conditions which could be more easily mimicked within the university building. (I say "can", because, of course, teachers also give *spontaneous* feedback and explanations, or make up exercises or examples on the spot – however, it is just as typical for a teacher to perform each of these tasks outside the classroom.)

Thus, the intervention was designed to get student teachers to practice each of these three real-life requirements by engaging with learning tasks – resulting in a 12-week program made up of three thematic sections of four weeks each (see Figure 13).

Figure 13: Intervention design in three thematic sections

While I will analyse some translated items as examples of the typical wording of the text vignettes below, Table 2 on the next page shows a selection of the types of scenarios depicted in the text vignettes used in the intervention (36 learning tasks were created in total):

[1] Lindmeier (2011) delineates this task set including "pre-instructional reflections" and "post-instructional analyses" (p. 107) quite similarly to my intended meaning when defining the "reflective competencies component" of her structure model on teachers' knowledge and competencies.

Table 2: Examples of scenarios depicted in the intervention learning tasks

Choosing, adapting and utilizing exercises	**Providing explanations and giving illustrative examples**	**Evaluating student solutions and giving feedback**
designing an appropriate task around the mathematical story about the chessboard and rice grains	analysing, criticising and selecting Youtube "explainer" videos to recommend to students wanting to revise the topic of optimisation	evaluating a student's solution to a homework problem, providing written feedback and three follow-up problems
modifying a procedural homework task about derivatives so as to be challenging despite students' ability to use technology for solving it	writing down feedback about a specific teacher explanation for a colleague who had asked for mutual lesson observation	looking through formulations about continuity in students' learning diaries and writing responses to the students
sighting tasks about integrals from the standardised Austrian school-leaving exams and devising appropriate problems for students to practice the skills required to solve them	creating discussion provoking Kahoot quiz questions about the concept of limit with interesting distractors and anticipating directions which the student discussions may take	sighting students' examples for "sequences", drawing conclusions about these students' concept images and finding interesting examples to further their understanding
choosing which problem to include when planning a lesson revisiting the concept of function with the class	offering guidance to students who asked about and are confused by one of Zeno's paradoxa	writing feedback to students' answers to a single choice question which required a justification for the answer

Choice of subject matter

(Ma-)Thematically, the seminar was focused on topics in analysis. To present a broad overview of ideas from analysis, four main concepts within the field of analysis – all of which are relevant to Austrian school curricula – were identified:

- function
- limit
- derivative
- integral

(These overlap with the *fundamental calculus concepts* identified by Thompson and Harel (2021) in a review of the literature – limit, derivative, and integral – in conjunction with their own addition of *ideas foundational to calculus learning* – variable, function, rate of change, and accumulation – in the same paper.)

Each section, centering on one real-life task of teachers, was meant to contain a range of learning tasks exploring each of these mathematical concepts. For example, when dealing with the real-life teacher task of examining student solutions and giving appropriate feedback, I ensured that across the entirety of that section, at least one learning task each was presented featuring a student solution concerning the central mathematical topic of function, limit, derivative or integral, respectively. Additionally, while an even split could not be guaranteed within one section, the goal was to present learning tasks covering each of these topics in roughly equal amounts throughout the semester.

Wording of the text vignettes and instructions for PSTs

I considered introducing various theoretical lenses of analysis for the situations the PSTs would be confronted with in the text vignettes, such as the Knowledge Quartet (Rowland et al., 2005), or the Three Points framework (Yang & Ricks, 2012), but ultimately decided against it.

Firstly, only some of the learning tasks would have easily lent themselves to this approach and consistency and recurring structures (e.g. typical leading questions to ask oneself as a mathematics teacher) were considered paramount when trying to let student teachers learn to imitate the experts' strategies.

Secondly, the key was to show a way of carefully attending to professional actions that the students could "accept" as a *realistic* mode of doing the everyday work of a teacher. Any formal framework from the academic setting of research on teaching and learning may have impeded their impression of authenticity.

Lastly, and importantly, I wanted to provide the student teachers with opportunities to make use of their *existing* knowledge and thus show them its applicability and problem solving potential. Introducing additional frameworks or technical terms would have added another layer of theory to the various types of prerequisite knowledge the student teachers already needed to integrate in order to obtain their solutions. Aside from further raising the level of difficulty, doing so may have sent a counterproductive message: "There's a need for something *in addition* to what your classes so far have taught you – in order to be able to act professionally, learning to flexibly apply what you already know is *not* going to be enough."

When it came to actually writing the text vignettes and phrasing the learning prompts for each task, I focused on the desiderata I outlined in chapter 2.6.3 and tried to optimise for those aspects.

Firstly, I tried to find variations on real life tasks which could be most authentically represented in the seminar environment. The following text vignette depicts a situation in which a mathematics teacher might realistically think about appropriate and inappropriate explanations their students may offer *before* the lesson takes place.

Your students (10^{th} grade) are finishing the topic "functions and their properties" (not yet having studied derivatives) and are supposed to fill out "reverse profiles", i.e. find one function each having a certain combination of properties, for instance:

A discontinuous function $f: \mathbb{R} \to \mathbb{R}$ with exactly one local maximum and one local minimum.

You include some "impossible profiles" (and communicate this to your students) – i.e. combinations of properties which cannot be held by one function simultaneously. In those cases, your students' task is to give an explanation as to why no example of such a function can be given.

The "impossible profiles" which you compiled are the following:

- a continuous function $f: [0,1] \to \mathbb{R}, f(0) = 1, f(1) = -1$ without a zero point
- a strictly monotonically increasing function $f: \mathbb{R} \to \mathbb{R}$ with a (global or local) maximum
- a continuous function $f: [0,1] \to \mathbb{R}$ without a global minimum or maximum
- a continuous function $f: [0,1] \to [0,1]$ which does not touch the line $y = x$

You are preparing for your lesson the next day by double checking that all of these are, indeed, impossible to fulfil. Then you consider which reasonings you expect your students to give, which difficulties and misunderstandings you may anticipate and which aids you may provide to support your students in working through the task.[2]

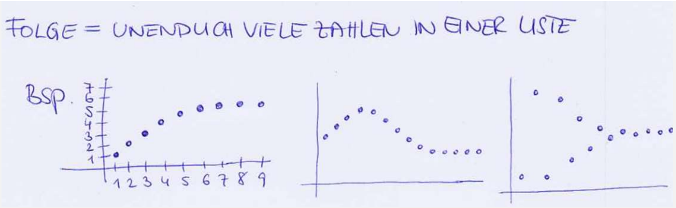
In addition to providing a realistic scenario in which a teacher may think through various ways of explaining a phenomenon during lesson planning, this kind of vignette could also serve as a certain "best practice" example as to which kinds of considerations might be useful in preparation for similar lessons.

Secondly, wherever appropriate, the sequence of learning prompts participants had to work through was designed so as to take an organic beginning based on the teaching situation depicted in the vignette, follow up with some meta-level considerations based on the advanced mathematical knowledge available to preservice teachers, only to then be broken back down to school mathematics level by returning to the teaching situation and reasoning about the teaching action to be taken. This is reminiscent of Wasserman et al.'s (2017) idea of "building up from" and "stepping down to practice" when thinking about using advanced mathematical knowledge in a situated manner.

[2] As this was a "modeling" task in which the lecturer(s) walked the preservice teachers through their own thought processes when confronted with this situation, no specific questions or prompts are listed here.

The vignette below illustrates this progression of subquestions posed to the participants.

As part of a quiz assessing your students' (10th grade) current level of knowledge, you ask them to describe in their own words what a sequence is and to give three examples of a sequence. Looking through their submissions, you notice that the majority of student answers fall into one of the following four categories:



[sequence = infinitely many numbers in a list; Ex.]

eine Folge ist eine Abfolge von Zahlen (Reihenfolge wichtig)
z.B. (1,2,3,4,5,6,...) natürliche Zahlen
(2,4,6,8,10,12,...) gerade Zahlen
(1,1,2,3,5,8,13...) Fibonacci-Zahlen

[a sequence is a succession of numbers (order important); e.g. natural numbers; even numbers; Fibonacci numbers]

Folgen sind Listen von nacheinander aufgezählten Zahlen, die einem bestimmten Muster folgen

$a_k = k^2$ (1,4,9,16,25,...)
$a_k = k+3$ (4,5,6,7,8,...)
$a_k = 1-k$ (0,−1,−2,−3,−4,...)

[sequences are lists of successively enumerated numbers following a certain pattern]

eine Folge ist wie eine Funktion, bei der x nur ganze Zahlen als Werte annimmt, z.B.

•) $f(0)=1$, $f(1)=3$, $f(2)=5$, usw. $f(x)=2x+1$
•) $g(x)=\frac{17x-2}{3}$ ($g(0)=-\frac{2}{3}$, $g(1)=5$, ...)
•) $h(x)=x^2+x$ ($h(0)=0$, $h(1)=2$, $h(2)=5$, ...)

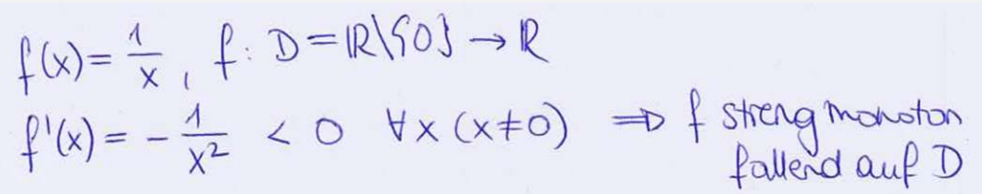
[a sequence is like a function which only takes whole numbers for x, e.g.]

1) Which elements of a formal definition of sequences are represented in each of the students' choice of phrasing? Which are missing or are misrepresented?
2) Which basic ideas [Grundvorstellungen] for sequences seem to be the focus of each student? Are there aspects of the concept which are not mentioned in any of the submissions?
3) Which of the examples given by the students are indicative of a comprehensive view of sequences, point to a restricted view of the concept?
4) What would you focus on when planning a lesson to do a recap on the topic of sequences, after having sighted these submissions?

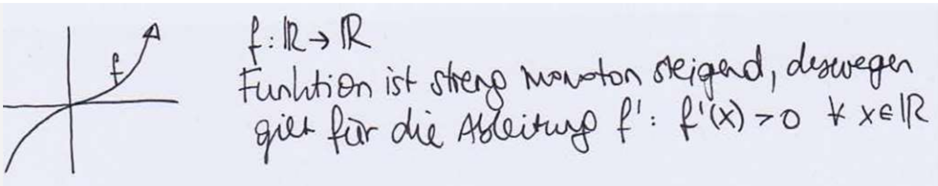
In the first prompt above, the seminar participants are invited to revisit the formal definition of a sequence and contrast it with the "definitions" as given by the students. Then, the next prompts lay out ways in which such an analysis can provide insight about the students' ideas and misconceptions. Finally, participants are supposed to return to the teaching situation and use the understandings gained in order to make an informed decision about how to proceed in the following lesson.

A third principle I employed was to use student solutions as well as student reasonings as a way of addressing potential misunderstandings or limited notions about concepts the preservice teachers may hold themselves. As noted in Hiebert and Berk (2020), there can be value in "studying children's thinking about mathematics because children's thinking can uncover mathematical content that PTs need to re-learn" (p. 330). Tasks featuring this were partly incorporated in order to confront some of the participants with the demands for precision and comprehensive understanding needed in order to correctly assess and provide feedback for student solutions. The text vignette below featured five faulty or incomplete student solutions.

Your students (11th grade) just spent a week independently practicing (as part of an open learning environment in which you provided them with an abundance of materials relating to the topic of *Discussing functions*/*'curve sketching' using the derivative* and were present to assist as a coach whenever needed). At the end of the week, the students are supposed to hand in their work as an (ungraded) portfolio, so that you can evaluate their learning gains. Looking through the portfolios, you notice, among other things, the following sections:



[f is strictly monotonically decreasing on D]



[function is strictly monotonically increasing, therefore the following holds for the derivative f']

text vignette continues on the next page

$$f: \mathbb{R} \to \mathbb{R},\ f(x) = x^4 + 2$$
$$f'(x) = 4x^3 \qquad 0 = 4x^3 \Leftrightarrow x = 0$$
$$f''(x) = 12x^2 \qquad f''(0) = 0 \Rightarrow \text{Sattelstelle bei } x = 0$$

[saddle point at $x = 0$]

$$f(x) = x^3$$
$$f'(x) = 3x^2 = 0 \Leftrightarrow x = 0 \quad \text{waagrechte Tangente bei } x=0$$
$$\Rightarrow \text{Extremstelle}$$

[horizontal tangent line at $x = 0$ $\Rightarrow$ extreme point]

$$f(x) = \begin{cases} 1 & x \le 1 \\ 2 & \text{sonst } (x>1) \end{cases} \qquad f'(x) = \begin{cases} 0 & x \le 1 \\ 0 & x > 1 \end{cases} = 0 \quad \forall x \in \mathbb{R}$$
$$\Rightarrow f \text{ hat keine Extremstellen und ist eine konstante Funktion}$$

[f has no extreme points and is a constant function]

Please assess the following:

1. Which statements made by the students are correct? Which reasonings are adequate?
2. Upon which theorems/rules do the students base their argumentation lines? Correct those parts in which students are misinterpreting the statement of a theorem or applying it incorrectly!
3. What would you focus on after this one week of "open learning", having seen these excerpts? Please make a lesson plan for a lesson in which you address and resolve the issues that appeared in the work above! (Please do so in detail: Which contents do you talk about and in which order? Which examples or counterexamples would you choose? Do you use technology to illustrate some of your points and if so, how? etc.)

(This learning task was, in fact, specifically mentioned in a reflective essay as having made an impression about the mathematical competency required to teach well: "By working through the tasks, my opinion has changed insofar as I now believe (I didn't before!) that it can't hurt to know more than your students, and in particular, that sometimes, in order to answer questions, it really is necessary to know details that may not be featured in the school textbook. One of the reasons for this were, for example, the student solutions for the function discussions, which at first all appeared to be correct.")

Finally, I wanted the participants to take away a sense of professionalization in the sense that they felt more compelled but also more prepared to autonomously utilise their professional knowledge in order to make conscientious teaching decisions. Because of this, I tried to phrase most prompts in an open way, suggesting that in any teaching situation, while there are arguments to be made about better and less beneficial responses, there is usually a variety of productive ways in which a teacher might respond.

For example, my final "action prompts", after having gone through the more reflective tasks, often read like:

"Write down a few sentences of feedback for this student and add three further exercises for them to try and implement your feedback."

"Create your own fact sheet about 'sequences' on roughly 1 page of A4 paper, including definitions, examples, and some basic exercises that are meant to further illustrate the concept (convergence of sequences does *not* have to be included on this fact sheet)."

"Write some short feedback among peers that you think could help your colleague understand the issue. How would *you* respond to the situation if you were in your colleague's shoes? Write an explanation on the students' level."

The participants in the seminar were thus invited to be creative and make decisions as they saw fit, provided that they could justify those on a professional level, which meant including any relevant subject matter considerations.

4.2.2 Sequencing the learning tasks

In line with the principle of *scaffolding* (Wood et al., 1976; Puntambekar, 2022), within each of the three sections designed to span four weeks, four levels of independence were distinguished:

1. lesson: plenary, worked-example, incomplete solutions
2. lesson: well-structured group tasks, to be finished in class
3. lesson: well-structured group tasks, started in class and to be completed at home
4. lesson: relatively open individual tasks

The first lesson of each section introduced the class of learning tasks based on a new real-life task of teachers. In it, the expert(s) showcased their approach to solving a typical situation from the working life of a mathematics teacher. Doing so, the solution as well as the meta-cognitive processes guiding the process of finding it and evaluating different options were verbalised, taking the student teachers "along for the ride" of professional action in this particular circumstance (modeling an *expert completion* of the task).

The idea was to utilise the *worked example effect* (Tricot & Sweller, 2014) by which novices in a domain make more substantial progress at the early stages of acquiring new skills when presented with model solutions alongside problems to learn from (as participants had requested in the project described by Leuders and Wessel (2020)). By not only showing the resulting product (lesson plan, task adaptation, feedback given) but also making the internal thought processes accessible to participants, I sought to provide PSTs with additional insight into “the professional reasoning underlying the practitioners’ actions” (Grossman et al., 2009, p. 2066).

In order to keep these sessions interactive, participants’ prior knowledge was activated by asking them first to try to imagine for five minutes how *they* would currently approach such a situation. Then, during the presentation of the experts’ path to their solution, participants were encouraged to fill in gaps at opportune moments – effectively, utilizing the concept of *incomplete solutions* in an oral version which at the same time provided room for an adaptive form of teaching and relating to the seminar group.

For instance, when discussing two different textbook approaches to the introduction of the integral, the educator might comment that as a next step, she would do a quick mental checklist of the mathematical aspects and *Grundvorstellungen* (“basic mental models”, vom Hofe & Blum, 2016) of the concept of integral to see which of them were centered in each of the textbook introductions and which ones would only be addressed later (or not at all). Participants could now be asked to recall the subject matter aspects and Grundvorstellungen for the integral which they had covered in their course on the teaching of analysis in an earlier semester. The expert would elaborate on each concept or provide missing notions as needed. As a further step, the participants might be asked to share their own thoughts and judgments about the textbook extracts before being provided with the experts’ analysis.

The second lesson of each task class dealt with teaching situations the participants were asked to discuss in groups of three to five. In order to do so, they could reference the handout listing *leading questions* which accompanied each of those first, “modeling” lessons of a task class, as well as the subject matter “freshers” available at any time both in print and as a PDF for those participants working on their laptops/tablets. The first was supposed to help guide participants’ problem solving if they did not know how to proceed, the second to provide a more efficient subject matter reminder, if needed. Two groups each separately discussed the same text vignette, and then met to exchange ideas on the ways they had each approached the situation. During these work phases, the educators were in the room, providing assistance when requested and taking corrective courses of action when a group showed difficulties or exhibited misunderstandings.

In the third lesson of each task class, participants were again asked to work in groups. However, the learning tasks required them to hand in a written solution which they could finish writing up after class. Supportive information in the form of feedback was strongly utilized at this stage. In a first round, groups received written peer feedback from the other group having worked on the same learning task, which they could then incorporate before handing their work in for grading. This final submission was then assessed by the educators and the group of participants received both a grade as well as detailed expert feedback on their work.

In the final lesson of a section the participants started working on more extensive individual tasks they then had a few weeks to complete. Assessment of the individual tasks was split into two halves: Participants were encouraged to submit a preliminary version which the educator provided expert feedback on. Participants could then incorporate this feedback into the final version of their work. (Due to two short semesters with only 11 seminar lessons each after subtracting the lessons in which pretest and posttest were conducted, the individual tasks of the third task class ended up not being used.)

This design feature sought to incorporate one of the learnings Leuders and Wessel (2020) reported after their implementation – participants had expressed a wish for *more extensive and adequate feedback*. While participants' views on the value of the peer feedback received was mixed, throughout the course as well as in the reflective essays, PSTs expressed a high level of appreciation for the individual feedback received from the experts.

4.2.3 Designing performance assessments

Since there was no test or exam to be taken at the end of the seminar (the type of course the student teachers were taking was "prüfungsimmanent", meaning continuous assessment throughout the semester), the *contents of the assessments* couldn't be separated from the tasks worked on in the seminar itself.

Regarding the *mode of assessment*, some general principles were decided upon:

<u>Principle 1: no grade-relevant assessment for the first two lessons of a section</u>

In order to let the participants observe the experts' working method in a relaxed atmosphere and feel as safe as possible asking any questions that came to their mind, I decided that the lessons toward the beginning of a new section should be free of any assessment. Another benefit I hoped to achieve was for student teachers to more openly admit to gaps in their prerequisite knowledge of analysis and its teaching, which could be used to modify or expand supportive information material.

<u>Principle 2: peer feedback before expert feedback as an intermediate step</u>

As mentioned, in the third lesson of a section, student teachers were instructed to provide specific written peer feedback. The quality of this peer feedback was part of the grade they received for this task. Only after having received the peer feedback (and being granted time to incorporate it) did the participants have to submit their final papers for grading and expert feedback.

<u>Principle 3: expert feedback and option to resubmit</u>

In the fourth and last lesson of a section, both a first version and a final version of the solution were graded. After submitting the first version, student teachers received written expert feedback that they could use to improve upon their work before submitting it as a final version. There was no penalty for skipping this step and re-submitting the first version unchanged. In this case, the grade assigned to the first version was also considered the grade for the final version of the task. Whenever the final version was assigned a different grade from the first version, the arithmetic mean of the two grades was calculated and used to give a final grade. (Even though in theory, I would like to have further discounted the first version as a "trial" version in order to encourage students to communicate their ideas freely and dare to make mistakes, the 50/50 split was decided upon to make it less likely that participants would forfeit the opportunity for expert feedback entirely.)

4.2.4 Designing procedural and supportive information

"Supportive information" in 4C/ID is meant to bridge a gap between learners' prior knowledge and what they need to know in order to successfully perform the non-routine parts of the tasks in question. In contrast, "procedural information" in the context of the 4C/ID framework specifies how to perform recurrent, rule-based aspects of said tasks. Rule-based in this context refers to more-or-less algorithmic procedures that can and should be performed in the same manner everytime they are appropriate, as opposed to using one's knowledge about the defining parameters of a situation (which could also be called "rules") to solve a problem without violating those restrictions.

While some aspects of my learning tasks necessarily involved routine aspects of teaching (finding a derivative, creating a spreadsheet, looking up a definition in a textbook), the flexible and thus per definition non-algorithmic application of domain knowledge was the main focus of the intervention. As such, most information provided to participants falls under the umbrella of supportive information.

Procedural information was mostly provided in the form of

- corrective feedback by experts
- just-in-time subject matter quizzes

Corrective feedback was given by the educators either verbally during class when listening to or participating in group discussions, or in written form when assessing the work handed in after lessons 3 or 4 of a task class. This was direct, mathematical feedback concerning participants' misconceptions about some topic or, often, regarding imprecise language which they had used.

The subject matter quizzes were implemented as multiple choice questions on Moodle, with self-evaluating features which allowed participants to take the quiz as often as needed. All possible answers were annotated with explanations as to why they were correct or incorrect. My intention was to "jog" participants' memory of certain topics before discussing them in class, but also to confront them with the state of their subject matter knowledge regarding those topics without the external pressure or judgment of being graded on these quizzes. This felt important in order to reach the goals of the intervention, one of which was to combat any feelings of alienation or non-belonging with regard to subject matter. The combination of design elements was meant to convey a sentiment akin to the one voiced here:

"We have to design our courses to suit the students we get. We believe our students are, in general, quite capable of coping with, and even enjoying, undergraduate mathematics provided it is presented in a suitable way. [...] Many of our own disenchanted undergraduates serving out their time are potentially good mathematicians." (Appleby et al., 1998, p. 15)

In contrast to this kind of procedural information, supportive information is made up of

a) "information on the organization of the task domain and how to solve problems within that domain,
b) examples illustrating this domain-specific information, and
c) cognitive feedback on the quality of task performance." (Merriënboer & Kirschner, 2017, 142)

Aside from cognitive feedback provided by the educators, the following supportive information was designed to assist participants and scaffold their learning:

- subject matter "freshers" to make referencing the necessary mathematical background knowledge easier during class time
- "leading questions" handouts (given to participants in each modeling lesson)
- written expert solutions (given to participants in each modeling lesson)
- the wording and sequencing of the subtasks themselves

The subject matter freshers were handouts on the big topics of "functions", "sequences and series", "derivative" and "integral", summarizing the most important definitions and statements of theorems which would be relevant to the seminar tasks. Additionally, the respective subject-matter didactical concepts of "aspects" and "basic ideas" of the main objects were also listed and "recapped" in a concise manner. The idea of the freshers was to organise information the participants would most likely need to look up in their textbooks or lecture notes in a practical way so that as little time as possible would be lost while working on learning tasks in the seminar environment. The freshers are available (in the original German) in the supplements.

The handouts containing "leading questions" one may think about when performing a certain teacher task were handed out as a scaffold during the modelling lessons of the seminar. They were meant to be used by participants to get started during the first, short brainstorming sections of those lessons, and then to help visualise the structure of considerations the educators were displaying while modeling their performance of the teacher task in question. After the fact, participants could take the handout home for use in their own (future) teaching, or to assist in the work on the more ill-structured individual learning tasks of the seminar toward the end of a block.

The written expert solutions to the tasks in the modelling lessons resembled the subject matter analysis of teaching situations as showcased in the model solutions to the noticing test items, which are provided in the following chapter. These solutions were provided to participants after the modelling lesson. PSTs could study them in private at their own learning pace and consider them to be examples of an “expert-level” solution to a learning task in the seminar. This made communicating with PSTs about expectations as to the extent and the quality of subject matter analysis, especially with regard to the individual tasks that made up a large part of the total grade, much easier.

In the upcoming chapter, I will turn to the instruments with which the effects of this intervention were measured and describe the development and validation of the pre and posttest questionnaire.

4.3 Instruments of data collection

4.3.1 Noticing questionnaire

4.3.1.1 Choice of format and content

In order to assess the PSTs’ noticing skills, I decided to work with text vignettes which the participants could re-read and respond to with little to no time pressure. This measures subject matter noticing as part of a reflective competence (cf. “reflection-on-action"; Schön, 2017, “reflective competences”; Lindmeier, 2011) that is shown after the fact or in anticipation of a situation (as opposed to “reflection in action”, ibid).

This reflects the choice of real-life tasks in the intervention, which focused on tasks of lesson preparation or post-lesson work. Aside from one representative vignette each from the three task classes featured in the intervention, a fourth vignette was added to the noticing questionnaire which required reflection on a teacher task which could not authentically be practiced within the intervention. Originally, the reasoning behind this was for the data to allow for an analysis of any transfer effects by comparing the effects of the intervention on the PST’s performance at task classes that were specifically practiced with the effects regarding the PSTs’ performance on this task that was not part of the intervention. However, this line of analysis was ultimately abandoned for reasons of scope.

As such, the items were designed to cover the following teacher tasks (abbreviations in brackets):

- choosing, adapting and using mathematical tasks (tasks)
- evaluating students‘ written solutions and giving feedback, and (solutions)
- anticipating or responding to questions and preparing explanations (explanations)
- interpreting student utterances and reacting appropriately (dialogue)

The format of text vignettes in measurement instruments has been shown to be viable in the research domain of noticing (e.g. Friesen et al., 2018). However, because the theme-specific category of *subject matter noticing* had not been researched so far, I needed to develop my own measurement instrument for subject matter noticing (from now on, I will call this the noticing questionnaire) in order to assess any intervention effects on PSTs’ noticing skills.

The intervention had been chosen to focus on topics from analysis, therefore the noticing questionnaire was also going to revolve around mathematical ideas pertaining to analysis and appearing in the Austrian high school mathematics curriculum. (Questions of the transferability of subject matter noticing, in this case to other mathematical domains, naturally arise. However, investigating this was sadly also beyond the scope of the project.)

The final noticing questionnaire thus includes items from the topics

- functions and their properties
- sequences, series and convergence
- derivatives
- integrals

These are the same four concepts that were also covered in the learning tasks of the intervention.

4.3.1.2 Design of the items and subject matter analysis

As a first step, I surveyed the literature for instances of analysis-related tasks, student dialogues, teacher explanations and student solutions. My goal was to make the vignettes as authentic as possible by basing them on reported real-life situations, whenever I could. (Of course, complete authenticity cannot be achieved when using text vignettes which at the very least selectively represent a teaching situation. My goal was to design text vignettes that would be subjectively perceived by PSTs as realistic depictions of plausible teaching situations. While I did not specifically gather data on PSTs' impressions with regard to the questionnaire vignettes, the similarly designed learning tasks of the intervention seemed to be widely accepted as authentic by the participants who, overall, lauded the situated nature of the intervention.)

Some of the examples of the "task" category I copied from available resources and built an appropriate item around them. In other cases, such as when I found interesting student-teacher dialogues quoted in research papers, I shortened, expanded or modified them in order to fit my item. In the end, only one student solution to a real task from an Austrian mathematics textbooks was completely fictitious.

The first collection of items I arrived at included 16 items – four from each content category and four from each task class, as Table 3 on the following page shows.

	Function	Convergence	Derivative	Integral
Tasks	single choice task about equivalence of functions	textbook task exploring connection between approximation, convergence and derivative	single choice item about connection between position, velocity, and acceleration	exploratory task about different methods of area estimation
Solutions	student solution mistaking image of a shape for graph of related function	student solution to task using series to calculate total distance travelled	two student answers about derivative at extremum	two student solutions for area between two curves
Explanation	teacher explains proper function notation	teacher "proves" that $2.\overline{9} \neq 3$	teacher answers question about chain rule	teacher is asked about connection between FTC and integration as summation
Dialogue	students discuss whether real root function is multi-valued	two students discuss necessary and sufficient conditions for convergence	teacher discusses meaning of a partial derivative in context	students discuss how to use integral to calculate area partially below the x-axis

Table 3: Initial 16 items developed for the noticing questionnaire

4.3.1.3 Phrasing of the prompts

In concordance with my three-tiered concept of *subject matter noticing*, I initally considered asking my subjects' to answer three questions regarding each of the vignettes.

When a video vignette is shown, participants may be asked to describe the scene they just witnessed, assessing the perceptual component of their noticing separately from interpretation. This was the case, for example, in Jacobs et al. (2011), who used the prompt: "Please describe in detail what Rex said and did in response to this tadpole problem." (p. 103)

Next, a second question can ask for a contextualisation of the scene, including how the acts and statements witnessed can be interpreted with regard to not directly observable characteristics of the scene, for example, in terms of the students' mathematical thinking. Lastly, a third question might ask the participants to imagine taking action in response to what they've seen and justify their decision.

In contrast to video-based studies, my study used text vignettes, which means a much less complex environment in which much of the "perceiving" portion of noticing – namely, the omission of much of the "irrelevant" information for answering the questions at hand (cf. the expansion of van Es & Sherin's (2021) framework) – had already been taken care of in the item design process. Thus, the surplus value from having participants simply describe what they see in a particular item seemed minimal, especially since for this to make more sense, the text vignette would have to be removed temporarily.

Due to this and the second, practical, reason of time constraints, I decided to merge the first two levels of my noticing conceptualisation into one prompt that required the participants to demonstrate both their perceiving as well as their interpreting skills.

The second prompt would then concern participants' decision-making. Depending on the task class, this meant either incorporating a task into a lesson plan, preparing expository explanations or responding to student questions, giving feedback on written solutions or responding productively to student discussions. In each case, the participants were specifically asked to give a reason behind their decision to respond in a certain way. The final prompts read as follows in Table 4:

	Prompt 1	Prompt 2
Tasks	Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the necessary mathematical prerequisites to obtain a solution, or about the mathematical content areas that could be conveyed or strengthened using this task.)	Imagine that you are a teacher and would like to use this task. Please sketch the structure of a lesson in which you might implement this task and give an explanation for your decision.
Solutions	Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the task in question, the learners' mathematical skills or lack of mathematical skills, if applicable.)	Imagine that you are a teacher and would like to provide written feedback on these solutions via Moodle. Please compose such a response to both students and give an explanation for your decision.
Dialogue	Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the task in question, the learners' mathematical skills or lack of mathematical skills, if applicable.)	Imagine that you are a teacher and would like to react to these statements during a phase of group work. Please write down a response addressing the students and give an explanation for your decision.
Explanations	Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the mathematical background of the student question and about the mathematical content the teacher touches upon in his explanation.)	Imagine that you are the teacher of this class and would like to answer the student's question. Please write down your own response to this student's statements and give an explanation for your decision.

Table 4: Final prompts in the noticing questionnaire

It is crucial to note that the decisionmaking prompts do not ask the participants to "notice subject matter". They ask them to make a teaching decision and justify it, which requires a much more complex set of considerations.

As such, there are obvious pedagogical or methodological aspects to the PSTs' answers which were disregarded in the coding process because of the specific focus on subject matter in this project. Thus, it was not the overall "quality" of the PSTs' decisions and reasons behind their decisions, which was evaluated. Instead, coding involved extracting from those decisions and their given rationale evidence for any *subject matter* noticing.

This procedure was chosen in analogy to Jacobs et al.'s (2011) coding strategy, in which participants responded to a similarly broad prompt, the responses to which are then analysed with regard to the exhibited noticing of children's mathematical thinking:

"[...] we were not evaluating whether the suggested moves were the best moves (given that we do not believe that best moves even exist). Instead, we tracked this participant's extensive consideration of Rex's understandings of the previous problem and her awareness of the importance of his future thinking in solving the tadpole problem."

4.3.1.4 Rating of items by experts

The initial 16 items were next presented to six people who were expected to display expert level *subject matter noticing*: Three research mathematicians and three mathematics education researchers, all of whom were either postdocs or professors at different universities in Austria, Germany or Norway (the group of experts included two women – one each from mathematics research and from mathematics education).

Mathematicians were considered to be experts in the noticing of mathematical subject matter alongside mathematics education researchers. Both groups were expected to notice mathematical subject matter in various situations that included mathematical but also other aspects. The research mathematicians could not be expected to be as familiar with school mathematics as the mathematics education researchers, however, I did not expect this to pose a problem due to the fact that – similarly to the coding principles employed for analysing responses to the noticing questionnaire – I was only interested in extracting any purely mathematical contents. Aspects of subject matter conceptualisations such as school-related content knowledge (SRCK; Dreher et al., 2018) that require teacher-specific meta-knowledge such as knowledge about the sequencing of mathematical topics in the curriculum or the specific ways in which certain mathematical concepts are typically introduced in school were not relevant to my analysis.

Indeed, when evaluating the purely mathematical content of mathematicians' and mathematics education researchers' responses, there was a large overlap and no apparent systematic discrepancies between the two groups (the difference between mathematicians' and mathematics education researchers' evaluation was not quantitatively analysed). These overlaps in the experts' answers also provide a partial validation of the conceptualisation of "subject matter noticing" as separate from, for example, "noticing of children's mathematical thinking", which is inherently (also) based on pedagogical content knowledge. The mathematicians among the experts would have been unlikely to exhibit this kind of noticing.

The items were sent to these experts along with only the first set of prompts (perceiving/interpreting dimension) for reasons of scope. The idea was to validate the noticing questionnaire as a measurement instrument of subject matter noticing by basing the coding manual on empirical data about what a group of experts in mathematical subject matter consistently noticed.

Rather than deciding what was noteworthy about the items I had chosen on the basis of my theoretical analysis (which I had conducted previously in the process of designing and selecting appropriate items with the idea of producing 'mathematically rich' items), I wanted to evaluate preservice teachers' noticing in comparison to the *empirically observed* noticing that experts demonstrated when looking at the same text vignettes.

The goal was to use the expert ratings in order to arrive at a consensus of subject matter considerations which *all experts* had made a note about for each item and use this consensus as the basis for my coding manual. After all, different people – even experts in a field – might note slightly different aspects of a situation depending on the situation observed, their own background and interests, but also the format in which they are questioned (like the participants would be in the final pre/posttests, the experts were asked to provide written answers to my questions in a similar time frame).

Considering the intersection of all expert answers allowed me to deduce a reasonable standard in the form of a list of "minimum requirements" for adequate subject matter noticing. Participants' noticing scores would therefore reflect the degree to which PSTs successfully noticed what any expert in the field would most likely notice about the subject matter aspects of a certain situation.

4.3.1.5 Development of coding guidelines I

The expert ratings were evaluated based on procedures of qualitative analysis such as those described by Mayring (2015) or Flick (2009). The goal of these procedures is to use a structured and documented series of steps in order to analyse the contents of qualitative data – in my case, the subject matter evaluation the experts provided in response to my item prompts – in a systematic and transparent way.

The Mayring model for qualitative content analysis is a well known procedure which as been developed and used since 1983 (Mayring, 1983). Figure 14 on the next page shows the steps outlined in Mayring's content analysis schematically.

The collected **material** comprised the six experts' written answers to my open ended prompts, which were returned digitally in the "answer sheet" Word documents I had delivered to them alongside the 16 text vignettes. Most experts used a bulletpoint style, while some answered, at least partially, in full sentences (**formal characteristics**). Since the material was collected specifically for my research purposes, the **context of origin** was quite clear. As mentioned above, I informed the experts that about 10 minutes of time were allotted per item, so as to create a situation akin to the time frame in which the participants of the study would be analysing the text vignettes.

The **direction of analysis** was the professional vision, specifically the experts' noticing of subject matter. The material was analysed with the intention of deducing those subject matter aspects of the vignettes which the six experts thought to be concurrently noteworthy.

The **theoretical differentiation of the research** question has been outlined in the chapter on the theoretical framework of the study: As the experts only received the prompts relating to the first two levels of subject matter noticing (Perceiving and Interpreting), my interest in analysing their responses was directed accordingly.

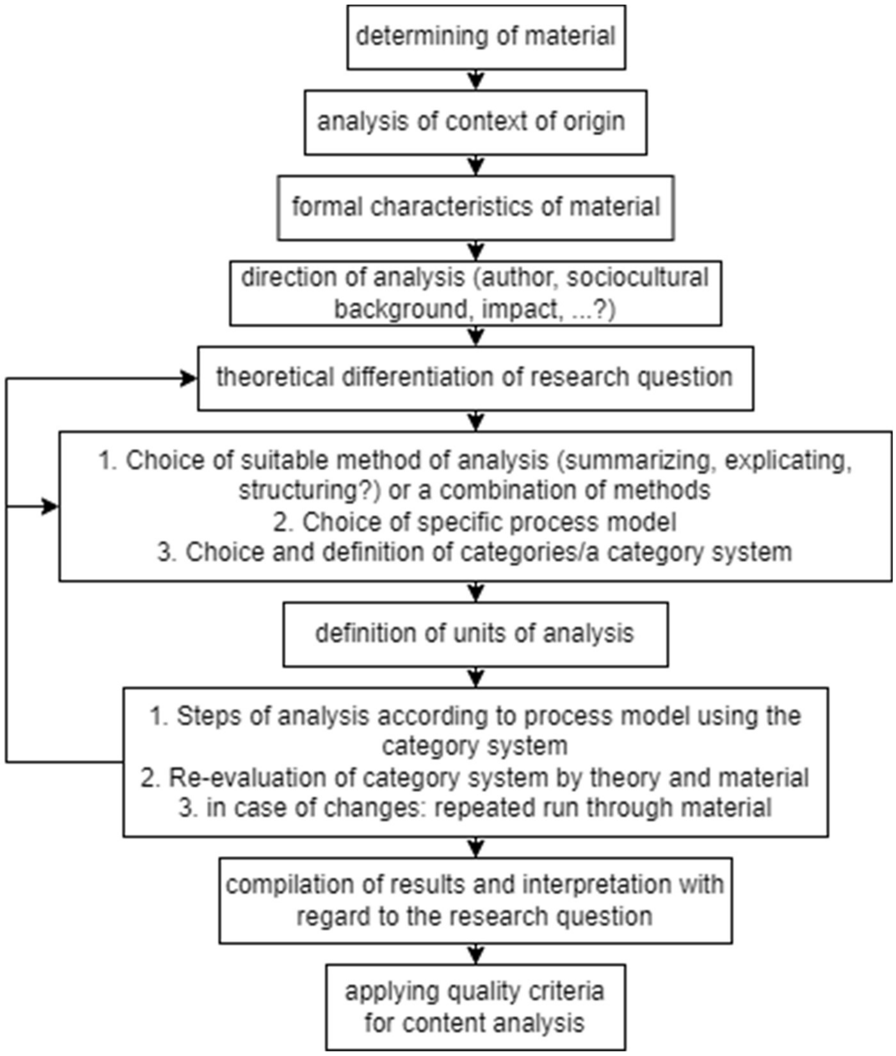


Figure 14: Mayring model of qualitative content analysis, diagram by author

In summary, the questions I wanted the qualitative content analysis to answer were:

- Which contents of subject matter knowledge are noticed and remarked upon (explicitly or implicitly) in each vignette by experts of the field?
- More specifically: Which contents of subject matter knowledge are noticed and remarked upon (explicitly or implicitly) in each vignette by *all of the six experts*?

The **method of analysis** I primarily used was that of *structuring*, occassionally also employing *summarizing*. My first goal was to arrive at a system of categories for each item, made up of all separate contents of mathematical subject matter which the experts identified in their answers. To this end, I proceeded according to the following **process model**:

First, I free-coded the first four expert ratings, looking for delimitable contents of subject matter that I could base a category upon. The **units of analysis** were quite variable, as some experts would touch upon two different mathematical ideas within one short sentence, while others elaborated on one concept for an entire paragraph.

However, since the final assessment was to be conducted on a binary scale (Was this category mentioned in the rating of the item or not?), this did not present a problem. Neither the frequency with which a category was touched upon (one expert mentioned the concept of a function three separate times within his rating) within a single rating, nor the level of elaboration (one expert mentioned "convergence" as a one-word bulletpoint, while elaborating on another concept of similar complexity for five lines) was relevant to the establishment of the category system. As soon as "function" and "convergence" were identified within the rating of an item, the respective category was considered fulfilled.

These first, at times rather specific, categories were then summarised into clearly distinguishable categories ("codes"). Using this preliminary category system, I re-analysed the same four expert ratings, paying close attention to the categories

- being exhaustive, i.e. all statements in the ratings should be able to be attributed to a specific code, and
- well-defined, in the sense that each sensible unit of analysis is clearly attributable to *one* code.

After this second run through proved successful and confirmed the summarization of the initial free-codes into categories, I wanted to test the wider applicability of the category system by using it to analyse the remaining two expert ratings. For this purpose, I wrote a preliminary coding guideline for the first two prompts using code examples from the first four ratings. The coding guideline for the third prompt, concerning decisionmaking, I planned to develop inductively using the material collected in the pilot study.

Each answer to the first set of prompts was to be assigned one out of three "levels", depending on the level of agreement with the expert noticing of subject matter:

- level 2: highly developed subject matter noticing
- level 1: partially developed subject matter noticing
- level 0: no clearly developed subject matter noticing

The levels were based on two criteria: The total number of codes that were applicable (here, I counted the total number of categories of subject matter that were mentioned by at least one of the experts), and the degree of agreement with the expert consensus (categories of subject matter mentioned by all of the experts). Using this method, I attempted to capture and appreciate both the breadth and the focus of the participants' noticing skills.

For each category from the expert consensus (these were usually two or three codes per vignette, see example below), the participant received two points, for each of the other categories, one. Once a participant had scored more than half of all possible points for a vignette *and* had mentioned all consensus categories, level 2 (highly developed subject matter noticing) was assigned. If a participant had scored more than half of all possible points for a vignette but at least one of the consensus categories were missing, level 1 (partially developed subject matter noticing) was assigned. In the case of participants who scored less or equal to half of all possible points on a vignette, level 0 (no clearly developed subject matter noticing) was assigned for this item. This was only slightly changed after the pilot study (adjusting some cut-off points) and thus for an illustration I would like to refer the reader to the finished coding guidelines in the supplements.

This procedure was designed to satisfy the following desiderata:

- Each expert rating would have been assigned level 2.
- Level 1 could be reached with a *certain* sensitivity to subject matter aspects of a vignette, despite not finding the right "focus" of the expert consensus as to which subject matter categories were crucial.
- Level 2 could only be reached by participants displaying a *high* level of sensitivity to subject matter by mentioning all of the categories deemed most important by experts.

Coding of the last two expert ratings according to these guidelines showed that they were widely applicable, with only minor changes needing to be made: The categories of the expert consensus remained the same, but some vignettes' category systems were simplified by leaving out categories only mentioned by exactly one out of six experts. (Thus, the remaining codes for each vignette were contents of mathematical subject matter mentioned in at least two to six of the expert ratings.) As was expected and required for validity, the expert ratings of all vignettes received the highest level of subject matter noticing according to these coding guidelines.

4.3.1.6 Choosing and adapting the final four questionnaire items

Based on the expert ratings, four items were chosen for the final noticing questionnaire (see the translated items in section 4.2.1.9, and the original questionnaire items in German in the supplements).

I chose these items based on a few considerations:

- breadth of subject matter contents as identified by the experts (richness of item)
- at least two distinct subject matter contents belonging to an expert consensus
- both subject matter contents which could be identified by mere "perceiving" and contents that required some degree of "interpreting" in order to be noticed

As a further example of the latter distinction:

Consider the task "Solve the following equation using the binomial formula: $(x + 7)^2 = 81$" versus the task "Simplify the following term: $(9 - x)(x + 9)$". In the first instance, noticing the content category "binomial equations" only requires first-level noticing (perceiving) as it is mentioned explicitly. In the second case, only second-level noticing (interpreting) reveals this category – professional vision allows a person to make inferences about the task's affordances, about possible solution methods, and so on.

The wordings of the prompts were adjusted on the basis of the information gathered from the expert ratings. For example, I noticed that some of the experts had tried to answer the two prompts separately from one another in a way that was not intended by me. I had only set two distinct questions in an attempt to "tease out" as much information as possible.

While I wanted to phrase my prompts in a maximally open way in order to leave the participants' noticing as uninfluenced by my style of questioning as possible, I had been afraid that too general a question might not evoke any substantial response at all. Using the data from the expert rating, however, I concluded that students would be even more likely than the experts (who, comfortable in their role of subject matter "authorities" had even made remarks and suggestions unrelated to my prompts) to follow "instructions" and the structure provided very closely. As such, too specific a question might encourage them to leave out certain aspects of their noticing which I may have been interested in.

For example, I specifically asked about the subject matter prerequisites of a task and the subject matter a task was aiming to convey. Experts additionally provided criticism of the task itself and analysed mathematical affordances of the task. The new prompts were designed to explicitly leave room for participants to include such considerations while still containing impulses for those participants who did not know where to start and what to write.

At this stage, I also re-evaluated and revised the prompts for the third level of subject matter noticing (decisionmaking). Instead of asking in a rather open fashion about how the participants would proceed in the situation shown in a vignette, each prompt was redesigned to include a clear invitation to produce a product – e.g. a lesson sketch – and provide an explanation for their choices.

4.3.1.7 Piloting the questionnaire

Due to the pandemic, almost all lectures had to be performed remotely at the University of Vienna during the semester of the pilot study. For this reason, the questionnaire (I piloted the entire questionnaire, including the beliefs and subject matter knowledge sections, which will be elaborated upon in the respective sections) was adapted to an online format which could be integrated into Moodle (using the quiz function which allowed for anonymised responses and simple ways of exporting the data) and filled in by participants digitally.

Overall, participants scored rather low throughout the sample, with even the one positive outlier scoring only a little more than half of all possible points. This went in line with my expectations that without intervention, subject matter noticing would not be well-developed in preservice teachers, and there was substantial room for improvement regarding this professional skill.

The pilot study also provided information as to the length of time it would typically take participants to answer the noticing prompts, and whether there would be any misunderstandings as to the type of response I was looking for. Results regarding the feasibility of assessing subject matter noticing in this way were promising, and only minor changes to the questionnaire and its implementation were made between pilot study and the main study's pre- and posttests.

Finally, PSTs' responses in the pilot study helped develop the coding guidelines for the second prompt in each text vignette.

4.3.1.8 Development of coding guidelines II

Participants' answers to the two prompts per item were evaluated on the basis of two separate coding manuals for each prompt. Both were initally based on the expert ratings and the subject matter contents derived from them. As a reminder, the expert consensus resulted in special, "consensus" subject matter contents that were deemed especially important to notice. These will from now on be referred to as core contents.

The (core) contents of each item (for the items see the section below) are listed in Table 5. Core contents are shaded grey.

Item A	**Item B**	**Item C**	**Item D**
concept of a function	sequence	connection between functions and their derivatives	integration as summation
domain (of a function)	concept of convergence	local properties of functions	fundamental theorem of calculus
mathematical language and notation	logical reasoning	linear approximation	limit
equality of functions	cluster points	extreme points	concept of approximation
concept of a variable	ε-neighbourhood	slope	integral as area
difference between function and function value	infinity vs finiteness		
	mathematical language		

Table 5: (Core) contents for each item

Both coding manuals are based on these mathematical contents. The coding manual for prompt 1 (P1) screens for the breadth of the participants' subject matter noticing as well as the focus on the most relevant contents of subject matter. Table 6 on the next page summarises the final version after slight adaptations based on the pilot study. The number of points required for a score of 1 as opposed to 0 varies slightly from item to item, as the number of (core) contents derived from the expert ratings is not the same for each item.

Item	max. points achievable	Score: 0	Score: 1	Score: 2
A	8	0-4 points AND at least one core content is missing	5+ points AND at least one core content is missing OR 4 points AND both core contents are featured	5+ points AND both core contents are featured
B	10	0-5 points	6+ points AND at least one core content is missing OR 5 points AND two core contents are featured	6+ points AND all core contents are featured
C	8	0-4 points AND at least one core content is missing	5+ points AND at least one core content is missing OR 4 points AND all core contents are featured	5+ points AND all core contents are featured
D	8	0-4 points	5+ points AND at least one core content is missing OR 4 points AND two core contents are featured	5+ points AND all core contents are featured

Table 6: Overview of coding guidelines for Prompt 1

The coding manual for prompt 2 (P2) was developed after the material from the pilot study had been sighted – while it was still based on the experts' mathematical contents, I wanted to see which kinds of responses the prompt would elicit at all in order to make sure the coding manual would be applicable.

The coding manual for P2 was supposed to evaluate how well participants could spontaneously integrate mathematical contents into their decisionmaking process as (prospective) teachers (the prompt itself, in contrast to P1, did not ask them to specifically include mathematical considerations when explaining or justifying their decision). Even more strongly than in the answers to P1, the coding manual reflected the necessity to consider the *most relevant* mathematical contents of an item when making a teaching decision. As such, the consideration of the *core contents* in the participants' answers was evaluated.

Another difference to the coding manual for P1 is that different levels of sophistication when addressing the same content were distinguished in the coding manual for P2. (This led to scores 0-3 for P2 as opposed to the range 0-2 for P1.)

For each core content, it is evaluated whether the participant

- does not consider this content at all
- considers this content, but not adequately
- adequately considers this content.

When making the distinction between adequate and non-adequate consideration of certain content, the two central criteria were mathematical correctness and the specificity of statements. This was decided after having sighted the material and realised that focusing on mathematical correctness alone, a participant superficially mentioning a concept would often receive more points than a participant who tried to analyse the relevance of a mathematical concept in detail. This seemed problematic since it is entirely possible that the person only mentioning certain contents as bulletpoints may have equally or even more flawed understandings that simply aren't verbalised to be judged when coding the answer.

Imprecisions in the participants' mathematical expression were considered mildly when the context made it plausible that they were not indicative of any conceptual misunderstandings. This was decided upon after seeing that participants across the board had difficulty stating mathematical ideas in a precise manner, but that some imprecisions seemed to point to gross errors in thinking while others "merely" suggested lack of attention to detail or lack of appreciation of the importance of speaking with precision (which is, of course, also a key part of doing mathematics).

Depending on how many of the core contents were considered to which qualitative extent, participants received a score from 0 to 3. The flowchart in Figure 15 on the following page illustrates the decision process. The coding manual in the supplements contains detailed descriptions that helped decide what "adequate" meant for each specific content area, including examples from both the expert ratings and the pilot study. Non-core contents were supposed to be considered only when they were adequately – i.e. specifically and correctly – analysed in the participants' responses. However, this part of the decision tree ended up barely relevant.

To determine the interrater reliability, five out of the 14 questionnaires of the pilot study were rated by a second researcher using the coding guidelines. Out of the resulting 20 ratings, 17 were concordant. The scores where both raters did not agree were discussed and the second researcher's feedback was used to specify and improve upon both coding guidelines.

Aside from concordance, Cohen's κ was calculated in order to consider that a certain amount of agreement is to be expected by pure chance. I calculated both unweighted and weighted Cohen's κ, since in my ordinal rating system, it made sense to distinguish a slight disagreement about scores from a major one (see Table 7).

	Prompt 1 (scores 0-2)	Prompt 2 (scores 0-3)
Percentual agreement	95 %	90 %
Cohen's κ (unweighted)	0.8667	0.8502
Cohen's κ (linearly weighted)	0.8246	0.8851

Table 7: Interrater reliability

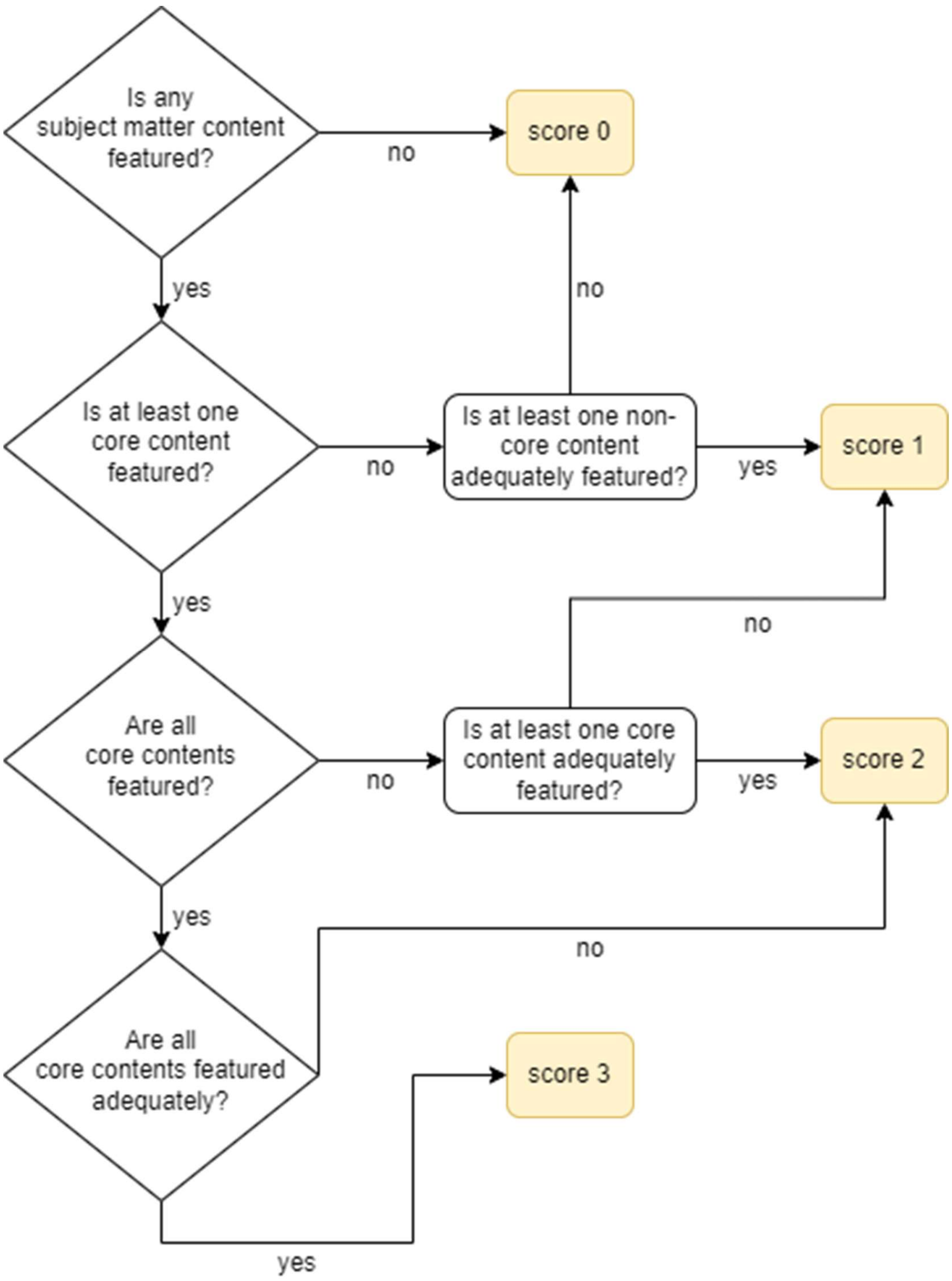


Figure 15: Flowchart illustrating coding guidelines for Prompt 2

Five months later, I myself recoded the same five questionnaires in order to determine intrarater reliability over time (see Table 8).

	Prompt 1 (scores 0-2)	Prompt 2 (scores 0-3)
Percentual agreement	85 %	80 %
Cohen's κ (unweighted)	0.6629	0.7015
Cohen's κ (linearly weighted)	0.7458	0.7714

Table 8: Intrarater reliability

These statistics were calculated using the online tool by Vassar Stats and SPSS, which also yields a p-value. All of the κ-values can be considered very good to excellent (Fleiss et al., 2003). The unweighted Cohen's κ are significant at a level of $p < 0.01$. This means that it is highly unlikely that the observed agreement in the rating of the items is a result of chance.

One thing to note is that due to some unequal marginal distributions observed in the prevalance across scores, with some scores not being given or only given very few times, the explanatory power of Cohen's κ is limited. However, due to the very high values I had calculated, and considering that other reliability measures have limitations of their own, I did not pursue this further.

In the last subsection of this chapter on the noticing questionnaire, a translated version of the final noticing questionnaire, a subject matter analysis of each item, as well as a potential sample solution to the decisionmaking dimension of the second prompt is presented. The original questionnaire can be found in the supplements.

4.3.1.9 The final noticing questionnaire

Item A

1. Let f be a function given by $f(x) = \sin(x) + \cos(x)$ and let g be a function given by $g(u) = \sin(u) + \cos(u)$, for all real numbers x und u. **Which of the statements below is correct?**
   (a) f and g are identical functions.
   (b) If x and u are different, then f and g are different functions.
   (c) There is not enough information provided in order to determine whether f and g are identical.
2. **True or false?** If $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x + 2$, then we can say that the functions f and g are identical.

Prompt 1: Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the necessary mathematical prerequisites to obtain a solution, or about the mathematical content areas that could be conveyed or strengthened using this task.)

Prompt 2: Imagine that you are a teacher and would like to use this task. Please sketch the structure of a lesson in which you might implement this task and give an explanation for your decision.

Item A – Analysis of subject matter

This item contains two single choice tasks based on quiz questions from Cornell University's *Good Questions Project* (Miller et al., 2006). These address the students' understanding of the concept of function as a triple of domain, codomain and some sort of prescription relating elements of the domain to elements of the codomain.

The task defines two functions, conventionally named *f* and *g*, on all of $\mathbb{R}$. Both functions are given by equations which only differ in the naming of the independent variable *x* vs. *y*.

The answer to the first quiz question is presumably **a) *f* and *g* are identical functions**, since the functions' domains as well as function equations are the same. The codomain is not explicitly stated, but since a restriction is not otherwise specified, we could implicitly assume it is $\mathbb{R}$ again, since it seems we are dealing the real sine and the cosine functions. Of course, a case can be made for answer **c) There is not enough information provided in order to determine whether f and g are identical**, arguing that since the codomains could sensibly be defined as any subset of $\mathbb{R}$ containing $[-1,1]$, we cannot be sure whether these two functions are definitionally the same.

The distractor **b) If *x* and *u* are different, then *f* and *g* are different functions** refers either to the misconception that choosing a different letter as a variable has an effect on the essential characteristics of the function that is being defined, or to an oversight of the fact that the sets of values which *x* and *u* represent have been defined in the text as being identical.

The answer to the second quiz question is **false**, as mere function equations a function do not make – in this case this is apparent since choosing the largest possible domain for a function specified by each of these equations results in two non-identical functions. One may be tempted to simply cancel the term in the denominator of the first expression without paying attention to the fact that this is only an equivalence transformation when $x \neq 2$. Accordingly, the maximal domains are $\mathbb{R}\backslash\{x \neq 2\}$ and $\mathbb{R}$, respectively, and thus both functions defined on their maximal domains would be non-identical.

Example of decision-making informed by subject matter analysis

Based on this analysis of mathematical content relevant to the item, one might use these quiz questions to check the learners' understanding of the abstract concept of a function as a relation between two sets, and the thus resulting necessary parts of a function's definition.

This could be implemented in the form of a Kahoot-style quiz, or via the group quiz feature of Microsoft Teams. It could both provide the learners with some feedback with regard to their own competence, as well as allow the teacher to quickly gain some insight into the class's level of understanding. Allowing for discussion and questions in a following plenary discussion allows for certain misconceptions to be addressed before closing the lesson. Question 1 allows a revisit of the flexibility we have when choosing names for variables, while Question 2 shows a specific example that can be discussed and used as further illustration of the important fact that stating a function's domain is a crucial, and non-superfluous part of giving its definition.

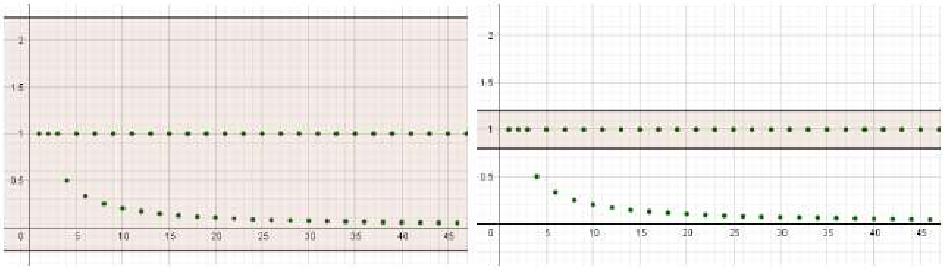
Furthermore, Question 2 presents an opportunity to talk about the degrees of freedom we have when choosing a particular domain for a function we would like to define. One might discuss how a function definition does not entail one specific domain that can be deduced from it, but does lead to a certain "upper bound" by implying a maximal domain (just like knowledge of a function equation and the function's domain implies a certain minimal codomain, namely the range of that function) within which the domain can, in general, be arbitrarily chosen. Thus, the functions in Question 2 *could* be the same, if both domains were defined to only include real numbers unequal to 2.

Alternatively to this proceeding, quiz questions such as these may also serve as an initial motivation for introducing a formal definition of function and arguing for the necessity of each component of that canonical definition. For this, students should bring along at least an intuitive understanding of a function as a mapping, and at least superficial familiary with the types of functions mentioned in the tasks, i.e. rational functions and trigonometric functions.

Item B

During group work, the learners are discussing the difference between the following two conditions in relation to a sequence of real numbers:

- **Condition A.** For all ε-strips the following holds: Infinitely many terms of the sequence are inside the ε-strip, if it is centered around L.
- **Condition B.** For all ε-strips the following holds: Only a finite number of sequence terms are outside the ε-strip, if it is centered around L.



In order to do so, they experiment with ε-strips of various widths and consider the images thus generated (see above). They arrive at the following conclusion:

L1: The second condition eliminates this possibility [points to the second image].
L2: Hmhm [agreeing].
L1: But the first condition would still work ... since infinitely many points are contained. So we need the second condition.
L2: Exactly, that's basically the reason why we said earlier, condition A is wrong.
L1: Yeah.
L2: Because there are counterexamples, like this sequence.

Prompt 1: Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the task in question, the learners' mathematical skills or lack of mathematical skills, if applicable.)

Prompt 2: Imagine that you are a teacher and would like to react to these statements during a phase of group work. Please write down a response addressing the students and give an explanation for your decision.

Item B – Analysis of subject matter

This item was inspired by an article by Roh and Lee (2017) and is built around part of a dialogue described in that paper between two learners engaging with materials on the topic of convergence. Despite the authors quoting university students, such an experimental engagement with the concept of limit and the process of finding a suitable definition is appropriate for illustrating the relevant ideas and talking about mathematics in higher secondary school, as well.

The item shows the learners being presented with two different conditions. **Condition A** is a phrasing typically used to characterise an accumulation point of a sequence, while **Condition B** would be a typical wording when trying to characterise the limit of a sequence. From the students' dialogue, it becomes apparent that they are trying to find a sufficiently specific condition for the limit of a sequence.

The learners experiment, using technology, which of the two conditions are met for certain sequences (and contenders for the sequence limit L) and which are not. To do so, they vary the width of an epsilon strip centered around their chosen L and consider the sequence terms lying inside or outside that epsilon strip, respectively.

It seems that the learners have an intuitive understanding about which types of sequences should be "identified" by the sought-after condition (convergent sequences) and which types of sequences should be eliminated (non-convergent sequences) by requiring an appropriate condition to be met. The images show them varying the epsilon strip for a divergent sequence with two accumulation points 1 and 0, with the learners currently having chosen 1 as L around which to center the epsilon strip.

L1 observes that condition B is not met for this sequence and L, while condition A is applicable. They are not satisfied with this: It seems that they are looking for a condition that would *not* be met for this kind of sequence. L2 agrees and suggests that this experiment confirms a conjecture they had previously arrived at. L2 calls the inadequacy of condition A for their purposes "condition A is wrong", arguing that there are "counterexamples, like this sequence".

The learners thus show no problems with the intuitive concept of a sequence as a mathematical object and the related notions of finite or infinite numbers of sequence terms. Both pursue their goal of finding a condition which characterises convergent sequences by eliminiting divergent ones confidently (and successfully).

They seem to understand that this single instance of a divergent sequence meeting condition A already makes this condition inadequate for what they are trying to achieve. Thus, there's an implicit understanding of the requirement of generality in mathematics.

Their phrasing „condition A is wrong" shows an attunement to the purpose of the task, which is to differentiate between appropriate and inappropriate conditions. However, the phrasing itself is sloppy, especially since wrong/false in mathematics has a very specific meaning which cannot be sensibly transferred to talk about a condition. In general, a condition can either be met or not. In their case, they are talking about the (in)adequacy of the condition, most likely using “condition A is wrong” as shorthand for “condition A is the wrong answer” or “condition A is not the one we need”.

Similarly, the term “counterexample” is used by the students in an understandable, if somewhat unorthodox way. In essence, they mean to say that this example is sufficient to argue the inadequacy of condition A for their purposes.

However, in spoken mathematical language, certain phrases are colloquially “abused” as long as all participants involved understand the subtleties swept under the table. If the last statement is to be interpreted as “This is a counterexample to the statement that condition A can be used to characterise convergent sequences”, the use of the word counterexample actually holds.

<u>Example of decision-making informed by subject matter analysis</u>

Many points above can be commented upon and the students praised for their efficient approach in determining the usefulness of different conditions. At the same time, one might help these seeming high-performers by inviting them to more clearly articulate their thoughts and arguments:

“Brilliant, you have found a great example of a sequence which meets condition A, but not condition B. That sounds really promising. Could you explain to me again why *this means that condition A cannot be the one we are looking for?”*

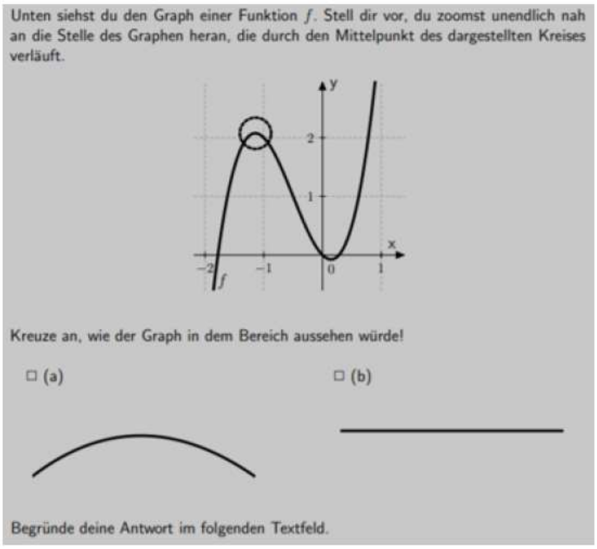
Asking this question unintrusively corrects the talk of “wrong conditions” (by restating it as “not the condition we are looking for”) and asks the learners to make their argument explicit, e.g. by saying outright: “Well, this sequence is clearly not convergent, and at the same time, condition A is met. So condition A is not strict enough, if we only want to include convergent sequences.”

Another interesting question to pursue could be: What does all of this mean for condition B? Is it a sufficient condition, or a necessary one, or both? Therefore, a follow-up question may sound like this:

“Quite right, condition A does not appear to guarantee us convergence. What about condition B? Will all convergent sequences be meeting this condition? What do you think? And vice versa: Is a sequence that meets condition B automatically a convergent one? Maybe you can start by trying this out for the different kinds of sequences we have encountered so far.”

Item C

For the task displayed on the left, two learners choose answer (b) and provide the following explanations:



Die Extremstellen, in diesem Fall ein Hochpunkt besitzt keine Steigung. Im Ableitungsgraph befindet sich dort auch ein Nullpunkt.

Wenn man unendlich nah heran-zoomt, dann wird man eine gerade Linie sehen, weil der gesamte Graph nur aus kleinen Geraden besteht, die aneinander gereiht sind.

[Translation of the text of the task: "Below you can see the graph of a function f. Imagine zooming in infinitely close to the part of the graph in the center of the circle that is displayed. [...] Select what the graph would look like in this area!"

Translation of explanation 1: The extreme points [sic], in this case a maximum, does not have a slope. In the derivative graph, there is also a zero point there."

Translation of explanation 2: "When you zoom in infinitely close, you are going to see a straight line because the entire graph is made up of small lines that are strung together."]

Prompt 1: Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the task in question, the learners' mathematical skills or lack of mathematical skills, if applicable.)

Prompt 2: Imagine that you are a teacher and would like to provide written feedback on these solutions via Moodle. Please compose such a response to both students and give an explanation for your decision.

Item C – Analysis of subject matter

Item C is based on a figure from a dissertation by Klinger (2018), containing two handwritten student answers.

The task that is posed relates to the idea of a "function microscope" which goes back to Arnold Kirsch (1995), who suggested it as a mental tool in order to visualise local qualities of functions. The invitation to zoom in „infinitely close" to a certain point of the graph cannot be perfectly clear due to the diffuse notion of "infinity" that it entails. However, the context suggests that the authors of the task thought of some sort of mental limit process taking place which would reveal the locally linear nature of the differentiable graph. Students with this basic idea of the nature of "nice" (differentiable) functions would therefore choose answer (b).

However, the images portrayed alongside the answer options leave room for potentially unwanted lines of argumentation. For example, one might argue that (a) is correct, since a differentiable curve can be *approximated* locally by a straight line, but that this straight line will, in general, only coincide with the curve at a single point in any sufficiently small neighbourhood. A student may just as well think that no matter how closely one zooms in (and here the problematic notion of "infinitely close" comes into play), no section of what appears to be a polynomial function of degree two or higher will behave in a linear fashion.

Aside from this criticism of the task itself, the student answers can be analysed as follows:

The first learner seems to have understood something fundamental about the behaviour of a derivative function in the neighbourhood of extreme points: When a function has an extremum at x, then the derivative function's graph has a zero at that point.

At the same time, the colloquial phrase about the function "not having a slope" could be indicative of a misunderstanding, since the function has a slope at all points, even where that slope happens to be equal to 0. (The wording of the first sentence is equally imprecise, since it is not the extreme point that has a slope, but the (graph of the) function at this point.)

The second learner responds to the idea of zooming in "infinitely" close without hesitation, but argues based on the misconception that the function graph is essentially made up of straight line segments. The way the answer is worded makes it sound like the infinite zooming process reveals the graph's "true nature", rather than having a nuanced understanding of the connection between the graphs global nonlinearity and its local linearity in the sense of being able to be increasingly well-approximated by its tangent functions the further one zooms in.

Example of decision-making informed by subject matter analysis

In the case of the first answer, one might point out the subtlety of differentiating between "a non-existent slope" and "a slope equal to zero". Like in response to item B, one may choose to unintrusively correct the sloppy talk about the “extreme point’s slope”, as well:

“Great, you correctly point out the connection between the derivative’s extreme points and the function’s zeros! Take care when talking about slopes, though. You say that *the graph does not* have *a slope at the extreme point*, but remember that the slope is equal to the value of the derivative at this point. So at the maximum point, right, it would be equal to 0. So the graph at this point has a slope of 0. Yeah, that’s probably what you meant – but it’s an important distinction, since there *are* functions with points at which the derivative, and thus a meaningful idea of “slope”, truly does not exist – can you think of (or remember) one?”

The misconception suggested by the second learner’s answer is more difficult to correct. Maybe this student lacks the understanding that a sequence does not have to assume its limit (similar to a sequence which can converge to 1 while all its terms are strictly lesser than 1, a function can be locally “linearisable”, even though on every single interval it behaves non-linearly). Additionally, the illustration used in the task itself may have further cemented this viewpoint. One possibility of countering it or at least raising helpful questions might be an analogy:

“Right, you correctly found that, when you zoom in really closely, our eye won’t be able to see any bending of the curve at all anymore! The graph will look straight. But this is only true for each point you zoom into like this, not for the graph as a whole. If the graph truly were just made up of tiny straight lines stuck together, then there would have to be ‘corners’, right? This is not the case for these kinds of functions, they are smooth everywhere. But they can be approximated very well by their tangent lines, when you zoom in really closely, because in very small environments, they are *really similar* to straight lines. Imagine the globe as a large sphere. If you are standing on an open field, and your friend is standing on another field a few hundred meters away, both of you have the *impression* that you are standing on a plane. However, if you were, then those planes would have to intersect somewhere and build an ‘edge”. An astronaut can see that this is not the case because what you are standing on only *appears* to be a plane, because you are looking at it from up close. So you can basically act *as if* it is a plane, because this approximation works really well when you’re up close. But zoomed out, the astronaut can tell that there’s a little bit of curvature all over the sphere.”

Item D

While working on the task below, a student mutters:

> I get how to do the calculation, but it says minus - I thought this was about summation? What even is the integral then?

$$\int_0^2 x^3 dx \quad = \quad \frac{1}{4}x^4\Big|_0^2 \quad = \quad \frac{16}{4} - 0 \quad = \quad 4$$

As an explanation, the teacher sketches a graph and draws an upright rectangle between the graph and the x-axis, and says:

> Okay, once again: So this is our y [points to the rectangle's height], and this is our small change in x [points to the rectangle's width and marks it as Δx]. So *this* times *this* is the area of the rectangle. And if we go another small step further, we get a new rectangle, which we can add to it [draws a second rectangle to the right of the first rectangle].
>
> So essentially, the integral is an approximation of this curve. When you make the rectangles infinitely small, you approach the curve more and more closely, while adding up these small blocks... so because you are calculating the areas of these blocks and adding them up, you eventually end up with the entire area.

Prompt 1: Which various contents of mathematical subject matter are addressed in this item? (Think, for example, about the mathematical background of the student question and about the mathematical content the teacher touches upon in his explanation.)

Prompt 2: Imagine that *you* are the teacher of this class and would like to answer the student's question. Please write down your own response to this student's statements and give an explanation for your decision.

Item D – Analysis of subject matter

This item is loosely based on a paper by Wagner (2018) and problematises the differences and compatibilities between a conceptual understanding of the Riemann/Darboux integral as the limit of *sums* versus the calculation of the integral value by using a *difference* of antiderivative values according to the fundamental theorem of calculus.

The student flawlessly computes a definite integral, implicitly using the fundamental theorem and subtracting the value of the antiderivative at the left boundary of the interval from the value of the antiderivative at the right boundary of the interval. At the same time, she seems to associate „integration" with the idea of summation, which she has trouble reconciling with what she is computing. This is presumably what leads her to ask "What even is the integral then?".

The teacher does not seem to pick up on this discrepancy, which which lies at the heart of the student's confusion, and so he goes back to recapping the concept of the integral as a sum of areas of rectangles. In doing so, he proceeds rather imprecisely:

In the beginning of his explanation, no limiting process is mentioned. Then he *does* suggest such a process by using the word "infinitely", but it remains unclear whether the "final" sum is *equal* to the area we are trying to determine, or whether it remains a (good) approximation until the end. (The teacher's last sentence implies the former interpretation, but his focusing on the integral as an approximation instead facilitates the latter understanding.) In terms of helping the students think of the integral as a mathematical object in its own right, suggesting that the integral is an approximation is problematic as it emphasises the idea of a process, making the development of an object understanding more difficult.

Additionally, in the German original, the teacher repeatedly talks about "Fläche" (area as a two-dimensional surface) instead of "Flächeninhalt" (area as one of the measures associated with a two-dimensional surface), when he means the latter. He further claims that the integral is an "approximation of the curve", again, in contradiction to the fact that the definite integral, being a number, is rather an approximation of the *area* "under the curve" – or more precisely, "a balance of areas" between the curve and the x-axis.

Example of decision-making informed by subject matter analysis

Depending on which previous knowledge one can assume and how extensive an explanation the situation allows, one might answer this student's very perceptive question in the following way:

"Yes, you remember correctly that we used sums of smaller areas in order to define the integral as the total area under curve. You might remember, however, that it was quite a tedious task finding a suitable series of sums of rectangle areas and then computing the limit of that series of sums. This was why we soon went on to talk about that important mathematical theorem – the fundamental theorem of calculus. This statement basically shows us a „trick" by which we can compute integrals much more easily: We just need to find an antiderivative of our function and then subtract the value of the antiderivative function at the “start” of the interval from the value of the antiderivative function at the “end” of our interval. So the integral can be calculated in two ways, one helps us understand what we can picture in our heads when using the word “integral” and working with it, the other one helps us by making the actual calculation very easy, once we know an antiderivative.”

Or even shorter:

„Right, we defined the integral as the limit of an infinite series, which means we defined it as a sum. However, the minus in your computation doesn’t stem from this definition, it stems from a method of actually computing integrals, given by the theorem we called “fundamental theorem of calculus”. The theorem tells us a quick way of working with integrals. You can still think of integrals as these sums, and simply know that the minus in your computation stems from this useful particular formula that mathematicians have found to make life with integrals a bit easier.”

Of course, given the time resources, the idea behind the fundamental theorem and thus the reason for the subtraction of antiderivative values can be explained at an appropriate mathematical level, as well. This would involve recapitulating what the integral function is and how it is related to antiderivatives.

4.3.2 Beliefs questionnaire

Since I was interested in preservice teachers' beliefs regarding the content and the usefulness of their subject matter related courses in mathematics, the questionnaire Isaev and Eichler (2017) had devised seemed like a fitting measurement instrument. It was originally used to assess preservice teachers' beliefs changes resulting from the implementation of special, teaching-specific exercises [Lehramtsaufgaben] in the tutorials of the foundational mathematics courses those preservice teachers have to take in the first semesters of their training.

The questionnaire contains items relating to two subdimensions of the construct of "beliefs concerning the double discontinuity", one addressing the first discontinuity – a disconnect preservice teachers experience between the school mathematics they were familiar with before entering university and the academic mathematics they are confronted with in their teacher training. As such, items of this subdimension specifically talk about the relationship between school mathematics and advanced/"university" mathematics. This subdimension will be called *content relationship* (CR) in the following.

The second subdimension then comprises beliefs about the second discontinuity – preservice teachers' struggle to see the relevance of their academic studies of mathematics and resulting doubts about the usefulness of academic mathematics for their future profession. This subdimension will be called *relevance for profession* (RP) in the following.

The questionnaire contained 18 Likert scale items, made up of eight items of subdimension CR and ten items of subdimension RP. The final, open-format question in the original asked participants to share any further thoughts on the topic. As my questionnaire required long open-format answers in the noticing section, and since I was going to ask the participants of the intervention group to write down their thoughts in reflective essays later on, I omitted this question.

4.3.3 SMK questionnaire

Because I conceptualise subject matter noticing as highly dependent on a teacher's subject matter knowledge (SMK), I wanted to include a section in my questionnaire which would allow me to gain some insight into the SMK participants brought along in the relevant area of analysis. For reasons of scope, I could not include a "comprehensive" test of participants' knowledge of analysis, and I was also hesitant to include open questions that would resemble an "exam" during the pre- and posttests.

For this reason, I decided to use items that were somehow related to the analysis knowledge needed for successful subject matter noticing in the first part of the questionnaire (thus narrowing the content-wise scope of the SMK section). Furthermore, I chose to include only single choice items.

The final eight single choice items of the SMK questionnaire are partly comprised of questions (adapted) from Bauer (2019), Cornell University's *Good Questions Project*

(Miller et al., 2006) and Hughes-Hallet and Pilzer (2003). Their contents range from mathematical logic, functions, sequences, derivatives and integration (the full questionnaire – in German – with correct solutions can be found in the supplements).

I present one translated example of an SMK item below (the correct solution is checkmarked). This item was chosen as related to the prerequisite knowledge for Item A of the noticing questionnaire, since similarly to Item A, knowledge about the relevance of the domain of a function is required to solve it.

Somebody tells you: "$f(x) = \frac{1}{x}$ is monotonically decreasing."
☐ This is true.
☐ This is false, "monotonically increasing" is correct.
☐ This is false, because it does not make sense to talk of monotonicity of functions.
☒ There is not enough information provided to make a statement about monotonicity.

The pilot study showed a wide range of SMK scores, with no floor or ceiling effects. Considering the results on an item level, none of the eight items was correctly answered by all participants. However, no participants answered question 7 correctly. After some discussions with other researchers, the original question was replaced with a more clearly worded alternative on the same topic (integration).

4.3.4 Demographic and other data collected

After the noticing, SMK and beliefs sections of the questionnaire, participants were asked about the following demographic and personal data:

- age
- gender
- second subject[3]
- other degree programmes they were attending
- number of semesters in teacher training
- analysis-related courses already passed
- teaching experience in a regular school environment (practical experiences that were part of the teaching degree excluded)
- teaching experience in a different setting
- grade on the final school-leaving exam

Surprisingly, the participants in the first cohort were hesitant to provide their age and I only received this data from about 1/3 of all participants (across both the intervention

[3] In Austria, the secondary teacher training programme requires preservice teachers to choose two subjects, so all of my participants were in training to become a teacher in mathematics and one other subject.

and the control groups). For this reason, and because (almost all) the participants were of similar age in any case, I deleted the question about age from the questionnaire for the second cohort.

4.3.5 Reflective essays

In order to be able to better contextualise the quantitative results of the beliefs questionnaire I used, I decided to incorporate one last assignment into my intervention seminar. After some consideration of scope, I decided to ask participants of the intervention to respond to some prompts in form of a reflective essay.

Collecting qualitative data this way meant that I forfeited the comparison with qualitative data from my control group, as well as the comparison with qualitative statements the same participants might have made before the intervention. However, since the qualitative data was meant to play a supplemental role in explaining the quantitative results of the quasi-experiment, I accepted these restrictions.

The participants were incentivised to submit a reflective essay by making the submission worth 10 % of their final grade for the course. At the same time, I did not want the social desirability bias to be further exacerbated (as it was part of their grade, this data could not be collected anonymously), which is why I communicated to the participants that I would be assessing their essays on a binary scale, with every participant who submitted an essay at all receiving a "1" (the top mark in Austria) for the assignment, and every participant who failed to do so receiving a "5" (the lowest mark in Austria) on the assignment.

In the end, almost all participants handed in a reflective essay and I collected a total of 44 reflective essays across both cohorts of the intervention group.

The prompts for the reflective essays were the following:

Please address the following questions in your reflective essay (about 1-2 pages) – you can answer the questions separately, or discuss them in a continuous text. Your thoughts on these topics are much more important than the form, and there are, as should be evident from the phrasing of the questions, no wrong answers or opinions!

1.
   a) What did you expect from the mathematics-specific* part of your teacher training when you started your studies?
   b) Which of these expectations were met, which turned out to be wrong?
2.
   a) How would you assess the relevance of the mathematics-specific part of your teacher training for your profession as a teacher?
   b) How did you experience the seminar with respect to this – did your assessment change as a result of participating in this course?
   c) If so, is there a specific scene/a specific task/a specific aspect which changed your thinking in this regard?

*Note on terminology: By *mathematics-specific* part of your teacher training, I mean all courses that were substantially about mathematics – these are going to be the academic mathematics and school mathematics courses, but also, depending on their content, other lectures, seminars, etc. in which you engaged with mathematical content (as opposed to, for instance, courses that focused more on teaching methods or the pedagogy of teaching mathematics).

In my dissertation, I only evaluated the second half of these prompts, as they were directly related to the research question and the results of the quantitative beliefs results.

The methods of data analysis, including the analysis of this qualitative data, are described in the next subchapter.

4.4 Methods of data processing and data analysis

4.4.1 Quantifying and processing the questionnaire data

The open answers to the prompts of the noticing questionnaire were coded by me according to the coding manuals which can be found in the supplements. Before coding the noticing data, all questionnaires were pseudonymised to reduce systematic biases during the coding process that may have resulted. (This means questionnaires were not identifiable to the coder(s) as belonging to either intervention group or T1/T2. However, all data from cohort 1 was coded separately from cohort 2, as soon as it was available.) Each participant's overall noticing score, as well as partial scores for prompt 1 and prompt 2 were recorded.

In the case of the beliefs section of the questionnaire, no pseudonymisation was necessary as answers were given in the form of Likert scores and no subjective rating was undertaken. The answers selected by participants were directly translated into numerical values. The questionnaire concerning beliefs about the double discontinuity contained both negatively and positively phrased items. Responses were numericized in such a way that the expression of positive beliefs by choosing a Likert scale option always resulted in a higher numerical value. Thus, participants' beliefs scores were calculated as the sum of points across all Likert scale items, with higher scores representing more positive beliefs about subject matter. Each participant's overall beliefs score, as well as partial scores in the beliefs components "content relationship" and "relevance for profession" were recorded.

In the case of the SMK data, the participants' scores were calculated and recorded as the number of correctly answered Single Choice questions.

4.4.2 Statistical analyses of the questionnaire data

Research questions Q1 and Q2 concerned effects of the intervention on PSTs' subject matter noticing and their beliefs about subject matter.

To answer these research questions, 2x2 mixed analyses of variance (ANOVA) were conducted for each of the variables of noticing and beliefs, as well as their respective subcomponents (P1 and P2 scores in the case of noticing; CR and RP scores in the case of beliefs).

Before conducting the mixed ANOVAs, the statistical reprequisites for this parametric test were investigated. As the questionnaire data collected did not fulfil some of these requirements, additional non-parametric tests (Mann-Whitney U test, Wilcoxon signed rank test) were conducted to corroborate the significant results that were yielded by the mixed ANOVAs.

A power analysis was additionally conducted to determine the minimum effect sizes a (quasi-)experimental study with this power could detect and put into perspective the null results. I used G*Power 3.1.9.71 to attain the following: The study design at hand with N = 75 can detect statistically significant effects upwards of Cohen's $f(U) = 0.33$. To find effects of size $f(U) = 0.25$ (a "medium" sized effect according to Cohen (1988), and according to Kraft (2020), arguably larger when considered in situated educational contexts), for example, already 130 participants would have been required.

Research question Q3 concerned the relationship between changes in PSTs' subject matter noticing and beliefs about subject matter over the course of the intervention. If changes in both variables had occurred (as a result of investigating Q1 and Q2), this would have been pursued further using correlational or regression analysis.

Research question Q4 concerned the theoretical relationship between the three constructs of the subject matter knowledge – subject matter noticing – subject matter beliefs triangle. Correlational analyses were conducted at T1 and T2, both for the whole sample as well as the subsamples of each experimental group (IG and CG) in order to determine significant correlations and thus try to support the theoretical conceptualisation. As multiple correlations were calculated for each subsample, the Bonferroni correction was applied in each case and the alpha level (the threshold for statistical significance) was adjusted accordingly.

4.4.3 Missing or disregarded data

The entire data set of four participants in total was excluded from all analyses (3 CG, 1 IG) because participants either only wrote the pretest or only wrote the posttest.

Some data sets miss answers to specific questions that were, on a case by case basis, more or less vital to the analyses I conducted. Below I will summarise how I handled each case.

Noticing questionnaire

In case of the noticing questionnaire, entirely empty pages were given a score of 0, and the data nevertheless included in all analyses.

Participants were allotted more than enough time to respond to the noticing prompts, and unanswered prompts were therefore attributed to participants' complete lack of ability or lack of inclination to answer the prompts, both of which correspond best to zero points in noticing (which was conceptualised to represent PSTs' competency in attending to subject matter features of the item, including volitional and skillbased components).

One person in cohort 1 of the control group did not hand in one page of the noticing questionnaire at T2 (or that page was lost) – in this case I substituted the noticing score they had received for that item at T1.

Unlike in the (few) cases described above in which parts of the response forms were left blank, I considered it unlikely that this participant had kept that page on purpose (being unwilling or unable to answer the prompts). Since I expected their noticing not to have decreased substantially over the course of one semester, I used the existing noticing score for that item as a realistic substitute. It seemed a better substitute than a score of 0, corresponding to complete loss of subject matter noticing on this item.

SMK questionnaire

A few participants also left single items unanswered in case of the Single Choice subject matter knowledge section. Again, unanswered questions were considered to correspond to participants not knowing the answer to the question posed (they were possibly applying strategies used in MC tests where false answers resulted in negative points, in case they were unsure), which is why I chose to award the same number of points as would have been given in case of a wrong answer (0).

Beliefs questionnaire

When single Likert scale items (answer ranging from 5 – completely agree to 0 – do not at all agree) were left unchecked, those missing values were substituted by the average value of 2.5. The reason behind this is that, not knowing the participants' true beliefs regarding the item, I wanted to substitute points corresponding to a fully "moderate" view of the statement in question, instead of the item contributing to their overall beliefs score with 0 points, corresponding to a maximally positive or negative view, depending on the item. (Additionally, in case that the statement had been left blank on purpose, I considered it especially unlikely that the participant held an extreme view on the matter.) This kind of correction was undertaken for only 7 items in total, and never for more than one item per participant.

Two participants checked two boxes on a Likert scale item – in both cases I substituted the arithmetic mean of the two boxes they had checked as I assumed they had been unsure which one to pick, which may mean that their true assessment lies somewhere between two values.

Data collected but disregarded

The data collected when asking about the number of semesters the participants had been studying at university so far turned out to be unreliable as participants were unsure whether to answer with only the number of semesters in their current study (the Master's program) or the cumulative number (including their Bachelor's degree). This data was therefore also not included in any analyses.

The participants' age could not be used for any analyses as many participants refused to provide this information. In the first cohort, 17 out of 39 participants declined to answer this question, which is why I did not include the question at all in the second implementation with the other cohort.

Changes made due to participants (potentially) misunderstanding the questionnaire

In the online implementation of the posttest (cohort 1, who was forced to write the test remotely due to lockdown measures because of the pandemic), one of the analysis lectures was mistakenly listed twice in the multiple choice item asking about analysis-related courses the participants had already passed. This may have resulted in one participant giving a contradictory answer, potentially as a result (i.e. they claimed to have passed course X in the pretest, but did not check this option in the posttest). This posttest data was corrected to the pretest value.

Also, students claiming to study both BEd and MEd at the same time were treated as having given the answer MEd, since you cannot start the Master of Education at the University of Vienna without having finished your Bachelor's degree.

4.4.4 Qualitative analysis of the essays

The qualitative essay data was analysed using the method of *Thematic Analysis* (Braun & Clarke, 2012). I proceeded according to the six steps suggested by the authors:

1. Becoming familiar with the data
2. Generating initial codes
3. Searching for themes
4. Reviewing themes
5. Defining themes
6. Writing up my results

It has to be noted that the process of conducting a thematic analysis does not necessarily have to proceed in such a linear fashion, especially when dealing with complex data sets. Over the course of my analysis, I moved back and forth between steps four, five and six multiple times before turning to record my results.

According to the procedure outlined above, I started by first reading the reflective essays of the first cohort multiple times in order to become well-acquainted with the data. The first reading I undertook without taking notes, while the second reading was done while taking a first set of notes and comments (not yet looking to identify each statement with a theme, but rather highlight parts that to me seemed specifically salient, surprising, or even unclear). Participants had submitted 1-3 page long essays that were either split up into answers to each separate prompt I had provided or were one coherent piece of writing touching upon each of the prompts in a less structured manner. Some participants spontaneously included personal feedback to the lecturer(s) in their essays and/or specific suggestions for their study program. Sections like these were disregarded in the following analysis, which focused on answering research questions regarding participants' beliefs about the relevance of subject matter related courses and their perception of the intervention in this regard.

While thematic analysis can proceed deductively according to theory-based themes, I chose to explore the data inductively. Hence, the second step consisted in the generation of first codes, as I went through the data again, systematically writing down keywords or -phrases as the author of each essay seemed to move on to a new idea.

This step was conducted for the entirety of the essay data submitted by the first cohort. Next, these initial codes were summarized into coherent themes that appeared to meaningfully describe a certain (usually recurring), delimitable idea. This first half of the data was then recoded using the themes I had arrived at and slight changes were made to the themes, with the goal of making the list as exhaustive as possible while keeping each theme unambiguous.

I then set aside the list of themes until the data from the second cohort had been collected.

Once this had occurred, I read and reread the new essays and then attempted to code them using the themes identified from the first 22 essays. After necessary adjustments had been made and new themes had been added to the pool of themes, I reconsidered the essays from the first cohort, paying special attention to the newly added themes.

After this step, the final list of themes was compiled, each theme was given a short description, and representative quotations from the essays were selected to be included in the report on the results of the qualitative analysis.

To provide a very rough quantified overall impression of participants' beliefs as reported in their reflective essays, I categorised each essay according to whether overall, that participant reported opinions that suggested

- clearly positive
- mixed, but overall positive
- mixed, but overall negative, or
- clearly negative

beliefs towards the relevance of their subject matter education to their future jobs as mathematics teachers. To add a measure of interrater reliability to this quantifying step, a fellow researcher recoded six of the 44 essays (13.6 %). The four ratings belonging to the latter three categories were all concordant, while the second rater categorised two ratings I had coded as "clearly positive" as "mixed, but overall positive".

This discrepancy was resolved in discussion, deciding that participants voicing strong appreciation for a solid and comprehensive subject matter education who nevertheless criticised, for instance, some of the teaching methods or institutional structures at university would still be listed as having "clearly positive" views on the relevance of subject matter courses for their future profession.

However, it has to be noted that this distinction between beliefs about the relevance of an "ideal" subject matter education and beliefs about the relevance of the subject matter education PSTs received may point to the perceived inconsistencies between the subject matter beliefs as assessed by the Likert scale items and the opinions PSTs voiced in their essays, which will be elaborated upon in the discussion chapter.

5 Results

5.1 Descriptive statistics and quality of instruments

In this chapter, I will provide an overview of the properties of the data I included when conducting my statistical analyses. This includes descriptive statistics for the whole sample, as well as for each of the two cohorts and the two experimental groups, separately. Then the results of tests concerning statistical prerequisites for the parametric tests employed in the later sections are reported and lastly, indicators of the quality of the self-developed parts of the questionnaire are discussed.

5.1.1 General sample characteristics

As can be seen in Table 9, the final sample which my calculations are based upon consisted of 75 participants. In both cohorts, the control group (CG) was somewhat smaller, with 16 and 13 participants compared to the intervention groups' (IG) 22 and 24 participants, respectively. The typical surplus of female preservice teachers is noticable, but not too extreme, in this sample featuring 45 female participants and 30 male participants.

Table 9: General sample characteristics

	Cohort 1 CG	**Cohort 2 CG**	**Cohort 1 IG**	**Cohort 2 IG**
n (m/f)	16 (7/9)	13 (8/5)	22 (4/18)	24 (11/13)
MEd/BEd	16/0	13/0	13/9	21/3
mean # of analysis-related courses*	6,38	6,31	5,36	6,13
teaching experience prior to intervention**	8/5/1	11/1/1	4/3/15	9/2/13
mean school-leaving grade in mathematics	1,73	1,67	1,8	1,81

Note. *mean out of max. 8 **extensive/ moderate/low to none

Both courses, the seminar "subject matter specific teaching" as well as the seminar "teaching practice" are located in the Master of Education curriculum, however, the former can also be taken by Bachelor's students looking for an elective. As such, all the students in my control group were Master's students, while there were 12 Bachelor's students in total in my intervention group (across both cohorts).

A t-test for independent groups revealed no significant difference between intervention group participants and control group participants when it came to the final grades in high school mathematics [Maturanote]. However, unsurprisingly (being all Master's students), control group participants ($M=6.35$, $SD=0.798$) reported having passed a significantly higher number of courses in analysis than the intervention group participants ($M=5.79$, $SD=0.999$) did, $t(76)=2.653$, $p< .05$, with an fairly large effect size of Cohen's $d = 0.61$.

Teaching experience varied strongly between control groups and intervention groups, as many of the Master's students attending the teaching practice seminar were already active as school teachers and were finishing their Master's degree parallel to this occupation. Thus, only two participants of the control group had little to no experience teaching in schools, while the majority of the intervention group (28 out of 46) reported the same. This can be explained, on the one hand, by the number of Bachelor's students attending the seminar "subject matter specific teaching", as well as by the fact that for many students, the teaching practice seminar is (one of) the last courses they take at university, often having commenced their careers at some point during their Master's degree.

5.1.2 Level and distribution of scores at T1 vs. T2

The general level of untrained subject matter noticing was rather low (remember that the maximum was 20), and so was the level of subject matter knowledge, with all groups exhibiting a mean of less than half of all possible points (the maximum being 8). A "perfectly positive" beliefs score being 90 points, averages in the 40s also left sufficient room for improvement.

As the tables 10 and 11 illustrate, there was a rather wide range of SMK scores and noticing scores across groups, and multiple participants scoring 0 points on one of these measures. Beliefs scores showed neither floor nor ceiling effects but also exhibited a broad range. SMK scores showed no outliers, and there were only very few outliers in the case of noticing and beliefs scores. Furthermore, the median values for all three variables were sufficiently close to the arithmetic mean that in the following analyses I choose to primarily report mean values.

Table 10: Descriptive statistics for the variables noticing, beliefs and SMK at T1

	M	SD	min	max	n
Noticing (IG)	4.47	3.09	0	15	47
Noticing (CG)	4.81	3.11	0	14	31
Beliefs (IG)	45.79	15.02	5.5	88	47
Beliefs (CG)	41.36	13.42	16	76	31
SMK (IG)	3.53	1.41	1	7	47
SMK (CG)	3.55	1.82	0	7	31

Table 11: Descriptive statistics for the variables noticing, beliefs and SMK at T2

	M	SD	min	max	n
Noticing (IG)	6.30	3.44	0	13	46
Noticing (CG)	4.20	2.85	0	10	30
Beliefs (IG)	47.51	15.26	18	86	46
Beliefs (CG)	43.10	14.90	12	76	30
SMK (IG)	3.91	1.64	1	7	46
SMK (CG)	3.53	1.50	1	7	30

5.1.3 Checking for normality

Before using a mixed ANOVA, I checked whether the data was normally distributed within the between-subjects factor *group* – i.e. whether the control group data and the intervention group data were normally distributed in the case of each of my three independent variables.

At T1, this was the case for the beliefs scores (both groups, according to both Kolmogorov-Smirnov (CG: p = .200, IG: p=.200) and Shapiro-Wilk (CG: p = .496, IG: p = .857) and some of the SMK scores (only control group, only according to Shapiro-Wilk, p = .171). The SMK scores of the intervention group were not normally distributed according to either of the tests, and neither were the noticing scores of either group, for details see Table 12.

Table 12: Tests of Normality at T1

	group	Statistic (Kolmogorov -Smirnov)**	df	p-value	Statistic (Shapiro-Wilk)	df	p-value
beliefs at T1	CG	.119	29	.200*	.968	29	.496
	IG	.085	46	.200*	.986	46	.857
SMK at T1	CG	.188	29	.010	.949	29	.171
	IG	.160	46	.005	.945	46	.031
noticing at T1	CG	.172	29	.029	.921	29	.032
	IG	.200	46	.000	.898	46	.001

Note. *lower bound of the true significance, **Lilliefors Significance Correction

Histograms of the SMK and noticing scores revealed that both distributions were rather heavily skewed toward the left. In the case of SMK as measured in the intervention group, there was also an issue with a second peak occuring around the lowest scores.

When examining the data at T2, only the SMK scores of the intervention group remain non-normally distributed (see Table 13).

Table 13: Tests of Normality at T2

	group	Statistic (Kolmogorov -Smirnov)**	df	p-value	Statistic (Shapiro-Wilk)	df	p-value
beliefs at T1	CG	.073	29	.200*	.990	29	.990
	IG	.116	46	.144	.966	46	.198
SMK at T1	CG	.143	29	.134	.954	29	.236
	IG	.152	46	.010	.926	46	.006
noticing at T1	CG	.134	29	.198	.954	29	.232
	IG	.110	46	.200*	.963	46	.144

Note. *lower bound of the true significance, **Lilliefors Significance Correction

Because I wanted to look more closely at the two component dimensions of both the beliefs construct as well as my subject matter noticing construct, I also examined normality assumptions for the relevant subsamples with regard to these two component variables, separately.

For each of the subdimensions of subject matter beliefs (beliefs concerning the double discontinuity), that is, *beliefs regarding content relationship* and *beliefs regarding relevance for profession*, the normality assumption can be maintained in each group, at both T1 and T2. However, this is, in general, not the case for the subdimensions of my noticing contructs (which is not surprising as the total scores were not normally distributed, either). Hence, the considerations and interpretations of the results that are to follow need to be taken with an appropriate grain of salt when considering the results of the parametric tests conducted with the variables representing these subdimensions of *perceiving and interpreting* (P1) versus *decision-making* (P2) of my noticing construct.

Next, I consider the rest of the requirements that need to be fulfilled in order to be able to meaningfully interpret the data arrived at by conducting an ANOVA.

5.1.4 Further prerequisites for ANOVA

Firstly, an ANOVA requires the dependent variable to be measured as ratio or at least **interval data** (data that is ordinal and where comparing differences is meaningful). Strictly speaking, neither of my three variables contain "interval data" in a sense as clear as a person's height or temperature. However, it is common practice to consider data such as clinical test scores or IQ points as metric data for the purposes of statistical tests, which is why I will do so in the following.

Secondly, the data should show **homogeneity of variance** between the levels of the within-subjects factor time. This I checked using Box's test of equality of covariance matrices in SPSS. The significant results ($M = 22.639$, $p = .491$) indicate insufficient evidence for inequality of covariance between the groups.

Thirdly, if there were more than 2 between-subjects factors (more than one intervention and one control group), **sphericity** would need to be checked, however, in my study design, this is not the case

Fourthly, the **error variances** of the dependent variables should be equal across groups. Homoscedasticity can be tested using Levene's test, which yielded non-significant results ($p > .05$) for all subsamples. Therefore, the null hypothesis that error variances are equal should not be rejected.

Lastly, after considering whether the dependent variables' **residuals are normally distributed** (I tested for normal distribution of the variables themselves, a stronger condition), I checked for any **extreme outliers** in the sample that would need to be disregarded.

There were 11 outliers in total, with no more than two outliers in each of the 12 groups (2 within-subjects factors x 2 between-subjects factors x 3 dependent variables). However, none of these constituted extreme outliers (being more than 3 times the interquartile range away from the first/third quartile), and since they were also fairly equally distributed across groups, I chose to leave the sample as is and not exclude any data.

5.1.5 Quality of the self-developed SMK and noticing questionnaires

Due to the rather low number of participants (78 participants across both cohorts were considered for the following analyses), testing for structural validity of my SMK and noticing questionnaires with typical statistical instruments such as factor analyses and interpreting those results was only possible in a very limited manner (MacCallum et al., 1999; White, 2022).

The statistical prerequisites for conducting a exploratory factor analysis (EFA) on my self-developed instruments were tested and showed that based on the Kaiser-Meyer-Olkin Measure of Sampling Adequacy, the sample was insufficient to perform an EFA on the eight-item SMK test (KMO = .47) and on the combined 12-item SMK-noticing questionnaire (KMO = .56).

This meant that I was only able to perform a (to be cautiously interpreted) factor analysis on the four items which assessed the noticing variable. However, this was also the most interesting questionnaire section to consider in this manner. Since the SMK variable represents a formative indicator, one would not expect the items used to measure it to map onto a single component of "knowledge of analysis". Meanwhile, for the noticing variable that could potentially be conceptualised as a reflective indicator, it was certainly interesting to consider whether the prerequisite subject matter knowledge that was different for each item would lead to a fragmented component structure in a factor analysis or reflect a unidimensional nature.

I performed an EFA using principal component analysis and the varimax rotation with Kaiser normalization on the four items of the noticing questionnaire. The Kaiser-Meyer-Olkin measure of KMO=0.657 indicated acceptable (Cohen et al., 2018) to mediocre (Kaiser & Rice, 1974) sampling adequacy and the results of Bartlett's Test for Sphericity were significant, $\chi^2 = (6, 78) = 24.081$ ($p < .001$), which indicates the data's suitability for factor analysis.

Only one component could be extracted, which is why the initial component matrix (including the extracted communalities) is presented below. Table 14 on the following page shows that noticing item D accounts for a much smaller amount of variance than do the other three items, and only items A and B surpass the typically required threshold of 0.5 for communalities. While the principal component analysis did not reveal multiple factors, this is insufficient information to conclude that the noticing test truly assesses a single construct.

Table 14: Component matrix and extracted communalities (N = 78)

	Component	Communalities (Extraction)
noticing_A_t1	.757	.573
noticing_B_t1	.718	.515
noticing_C_t1	.692	.479
noticing_D_t1	.392	.153

Extraction Method: Principal Component Analysis. 1 component extracted.

Calculating Cronbach's alpha values for the eight SMK test items results in α = .376 (which is again unsurprising due to the formative nature of the variable and the wide range of analytic concepts included, cf. Taber, 2018).

Cronbach's alpha for the four noticing test items was calculated to be α = .545. This alpha value for the noticing test is again well below the commonly used reliability threshold of α = 0.7 (Cohen et al., 2018). On the one hand, a low number of test items and heterogeneity of test items, both of which are given in this case, may also lead to underestimation of reliability (Tavakol & Dennick, 2011). At the same time, the low alpha value could of course also reflect the high dependence of subject matter noticing on the participants' SMK. If SMK is thought of as a formative variable and noticing is strongly dependent on SMK, then participants with specific knowledge structures may exhibit stark differences in their noticing from item to item.

For both the noticing test as well as the SMK, items exhibited varying rates of solutions. Table 15 below summarises the solution rates for all 12 items.

Table 15: Solution rates for items of the noticing and the SMK questionnaire (N=78)

	solution rate
noticing_A_t1	.17
noticing_B_t1	.23
noticing_C_t1	.29
noticing_D_t1	.24
smk_q1_t1	.42
smk_q2_t1	.29
smk_q3_t1	.56
smk_q4_t1	.79
smk_q5_t1	.60
smk_q6_t1	.49
smk_q7_t1	.13
smk_q8_t1	.24

As can be seen, solution rates for the noticing items were rather low throughout, with minor fluctuations, with participants' achieving 17 % of all possible points for the most difficult item and 29 % of all possible points for the least difficult item, on average.

In case of the SMK items, the observed empirical difficulties were much more divergent, with question 4 being correctly solved by almost 80 % of all respondents and question 7 only being answered correctly 13 % of the time. This may be based on the fact that the subject matter knowledge questionnaire contained both items about knowledge which were more immediately applicable to school teaching as well as related, more theoretical "background knowledge" items. (To take the two extreme examples mentioned above, item 7 is a question about integrability, while item 4 asks about a function's extreme points based on a graph of that function's derivative.)

Neither the SMK nor the noticing test show ceiling or floor effects, with no participants achieving perfect scores, and only a very small number of participants scoring 0 points on either the SMK (2 out of 78) or the noticing (4 out of 78) tests.

In both cohorts, control group participants' noticing scores and SMK scores remained stable across testing at T1 and T2, giving a positive indication as to the instruments' reliability across time.

Repeated measures t-tests (noticing, SMK) were performed for both the control group overall, as well as each cohort separetely, with no significant differences being detected.

Table 16 below summarises the relevant means and standard deviations

Table 16: Mean values (std. deviation) of scores in control group

	M (SD) cohort 1 (N=16)	M (SD) cohort 2 (N=13)
SMK_t1	3.38 (2.06)	3.69 (1.65)
SMK_t2	3.25 (1.44)	3.92 (1.61)
noticing_t1	4.81 (3.49)	4.92 (2.50)
noticing_t2	4.88 (3.03)	3.15 (2.38)

While the data was not suitable for using factor analysis to check whether the SMK test and the noticing test truly measure different traits of preservice teachers, the development of participants' SMK and noticing scores was contrasted to gain further information.

Again, two repeated measures t-tests (noticing, SMK) were performed with the intervention group data from T1 and T2, respectively. The results indicate that there was no significant difference between the intervention group's SMK at T1 ($M=3.5$, $SD=1.41$) and at T2 ($M=3.91$, $SD=1.64$). At the same time, the intervention group's noticing at T2 ($M=4.54$, $SD=3.08$) was significantly higher than at T1 ($M=6.30$, $SD=3.44$), $t(45)=4.11$, $p<.001$.

While these results cannot make definitive statements as to the discriminatory quality of the noticing test between subject matter noticing and the underlying SMK, they are in line with the assumption that despite subject matter noticing being highly dependent on prerequisite subject matter knowledge, the two tests likely measure different traits. The factor analyses suggesting that the construct measured by the noticing test can be conceptualised as one-dimensional while the SMK items do not appear so also indicates that it is unlikely to be a redundant measure.

In the upcoming chapter, results of the analyses pertaining to each of the seven research questions will be reported. Interpretation and discussion of those results then follow in chapter 6.

5.2 Research question Q1

As a reminder, research question Q1 reads as follows:

- Q_1: How does the situated intervention affect preservice teachers' noticing of subject matter?

5.2.1 Effect on the noticing variable

A 2x2 mixed ANOVA with repeated measures was performed on the variable "noticing".

There were no significant main effects of either the within-subjects factor "time" nor the between-subjects factor "group". This means that there is no indication ($p = .135$) that overall (across both groups), there was a difference in noticing between T1 and T2. Furthermore, there is no indication ($p = .163$) that overall (across both time points), there was a difference in noticing between the intervention group and the control group.

However, there was a significant interaction effect time*group $F(1{,}73) = 14.468$ ($p < .01$) with $\eta_p^2 = .165$, which corresponds to the effect size of Cohen's $f = \sqrt{\frac{\eta_p^2}{1-\eta_p^2}} = .44$. Posthoc pairwise comparisons (LSD) indicated that it was the noticing scores of the experimental group at T2 which were significantly ($F(1{,}73) = 8.293$, $p < .01$) higher than those of the control group at T2. All other comparisons were non-significant.

In order to make the positive results obtained by the mixed ANOVA – the requirements for which were not perfectly met for all relevant subgroups – more robust, two associated non-parametric tests were carried out.

The Mann-Whitney U test yielded a significant result ($p < .01$) when comparing the noticing scores of the intervention group at T2 with the noticing scores of the control group at T2. The results indicate that we should reject the null hypothesis that the probability that a randomly drawn score from one population will exceed a randomly drawn score from the second population is exactly 50 %. Knowing that the intervention group has higher noticing scores at T2 than does the control group, rejecting this hypothesis means that participants who took part in the intervention have significantly higher noticing scores on the posttest than participants who took part in the control condition.

Similarly to the results above, the only significant result ($p < 0.01$) when carrying out the Wilcoxon signed rank test was obtained when looking at the differences in noticing scores obtained by participants of the intervention group at T1 versus at T2. Again, knowing from our descriptives that noticing scores had increased between T1 and T2 for this group, this result indicates that participants in the intervention group showed significantly higher noticing after the intervention than they did before.

5.2.2 Effect on the component variables P1 and P2

Effects on the noticing dimensions assessed by prompt 1 (perceiving & interpreting) and prompt 2 (decisionmaking) were also considered separately.

The conducted 2x2 mixed ANOVAs with repeated measures on the variables "P1" and "P2" yielded the following results:

Neither of the component variables showed any main effects of the factors "time" or "group", analogously to the noticing variable. This means that there is no indication ($p = .453$) that overall (across both groups), there was a difference in noticing dimension P1 between T1 and T2. Furthermore, there is no indication ($p = .544$) that overall (across both time points), there was a difference in noticing dimension P1 between the intervention group and the control group.

There is no indication ($p = .158$) that overall (across both groups), there was a difference in noticing dimension P2 between T1 and T2. There is no indication ($p = .098$) that overall (across both time points), there was a difference in noticing dimension P2 between the intervention group and the control group.

A significant interaction effect could be shown for both P1, $F(1,73)=5.341$ ($p< .05$) and P2, $F(1,73)=12.479$ ($p< .05$). The effect sizes were $\eta_p^2 = 0.068$ and $\eta_p^2 = 0.146$, corresponding to Cohen's $f = .27$ (P1) and $f= .41$ (P2), respectively. Posthoc pairwise comparisons (LSD) for P1 showed that this interaction effect stems from the noticing scores of the experimental group at T2, which were higher than those of the control group at T2, although the difference in the posthoc analysis was non-significant ($p= .083$). Posthoc tests for P2 yielded a significant result ($p< .01$) in the same direction for the same comparison and non-significant results for the two others.

(Interpreting the results for P1, it has to be noted that Box's M test yielded significant results, meaning that homogeneity of variances was not given for all groups of this component variable. This was not a problem for P2.)

5.3 Research question Q2

Research question 2 reads as follows:

- Q_2: How does the situated intervention affect preservice teachers' beliefs about subject matter?

5.3.1 Effect on the beliefs variable

A 2x2 mixed ANOVA with repeated measures was performed on the variable "beliefs".

There were no main effects of either the within-subjects factor "time" nor the between-subjects factor "group". This means that there is no indication ($p = .166$) that overall (across both groups), beliefs about subject matter changed between T1 and T2. Furthermore, there is no indication ($p = .258$) that overall (across both time points), participants of the intervention group and the control group held different beliefs about subject matter.

The interaction *group*time* also did not yield significant results ($p = .928$). The lack of an interaction effect indicates that the null hypothesis that the intervention did not change preservice teachers' beliefs about subject matter should not be rejected.

As non-parametric tests are generally less powerful than their parametric alternatives, in the absence of significant results, none were employed to further validate these findings.

5.3.2 Effect on the beliefs subdimensions CR and RP

Two 2x2 mixed ANOVAs with repeated measures were performed on the variables "content relationship" (CR) and "relevance for profession" (RP).

There were no main effects of either the within-subjects factor "time" nor the between-subjects factor "group" for the variable CR. This means that there is no indication ($p = .296$) that overall (across both groups), beliefs pertaining to the subdimension CR changed between T1 and T2. Furthermore, there is no indication ($p = .360$) that overall (across both time points), participants of the intervention group and the control group held different beliefs pertaining to the subdimension CR.

The interaction group*time also did not yield significant results ($p = .933$) for the variable CR. The lack of an interaction effect indicates that the null hypothesis that the intervention did not change preservice teachers' beliefs pertaining to the subdimension CR should not be rejected.

Similarly, there were no main effects of either the within-subjects factor "time" nor the between-subjects factor "group" for the variable RP. This means that there is no indication ($p = .197$) that overall (across both groups), beliefs pertaining to the subdimension RP changed between T1 and T2. Furthermore, there is no indication ($p = .222$) that overall (across both time points), participants of the intervention group and the control group held different beliefs pertaining to the subdimension RP.

The interaction group*time also did not yield significant results ($p = .898$) for the variable RP. The lack of an interaction effect indicates that the null hypothesis that the intervention did not change preservice teachers' beliefs pertaining to the subdimension RP should not be rejected.

5.4 Research question Q3

Research question 3 reads as follows:

- Q_3: How are changes in the PSTs' noticing of subject matter related to changes in the PSTs' beliefs about subject matter?

Since no significant changes had occurred to participants' beliefs about subject matter, this research question was not further investigated. The results of the mixed ANOVAs indicate that the intervention improved participants' subject matter noticing, while leaving their beliefs about subject matter unaffected.

5.5 Research question Q4

Research question 4 reads as follows:

- Q_4: What correlations appear between the PSTs' subject matter knowledge, subject matter noticing and beliefs about subject matter?

5.5.1 SMK – Noticing – Beliefs triangle at T1 and T2

As multiple correlations were calculated in this analysis, the Bonferroni correction was applied to the standard significance threshold $\alpha = .05$. The reported correlations below are all still significant when compared to the adjusted value of $\bar{\alpha} = \frac{0.05}{6} = .008$.

When calculating two-tailed Pearson correlation coefficients between my three dependent variables noticing, beliefs and SMK for the entire sample (N=75) at T1, all three variables appear to be significantly correlated (see Table 17). As can be seen in Table 18, at T2, all three two-tailed Pearson correlations are again significant.

Due to my data not fulfilling all assumptions for the parametric Pearson's correlation r, I additionally calculated the non-parametric coefficient Kendall's Tau. All three correlations are corroborated by the two non-parametric measure which also becomes significant (at the adjusted $\bar{\alpha}$-level) for all three combinations at each time points.

Table 17: Two-tailed Pearson correlation coefficient r at T1

	SMK	Noticing	Beliefs
SMK	1	.345 (p = .002)	.410 (p < .001)
Noticing		1	.406 (p < .001)
Beliefs			1

Note. Exact p-values (where possible) featured for comparison against $\bar{\alpha}$

Table 18: Two-tailed Pearson correlation coefficient r at T2

	SMK	Noticing	Beliefs
SMK	1	.462 (p < .001)	.363 (p = .001)
Noticing		1	.464 (p < .001)
Beliefs			1

Note. Exact p-values (where possible) featured for comparison against $\bar{\alpha}$

These results indicate that all three measured variables are indeed positively correlated, as the triangle conceptualisation of the relationship between subject matter knowledge, subject matter beliefs and subject matter noticing suggests.

5.5.2 Triangle relationship in the subsamples

Again, as multiple correlations were calculated, the Bonferroni correction was applied to the standard significance threshold $\alpha = .05$. The reported correlations below are all still significant when compared to the adjusted value of $\bar{\bar{\alpha}} = \frac{0.05}{12} = .0041$.

I was interested in whether the correlations persisted in each of the treatment groups separately. (Significant results in this case are less likely due to the Bonferroni correction and since the total number of cases per group and thus power is decreased.)

At T1, for the control group (N=29), significant correlations could be shown between noticing and SMK as well as between noticing and beliefs about subject matter, while in the case of the intervention group, there were significant correlations only between SMK and subject matter beliefs at this $\bar{\bar{\alpha}}$-level.

At T2, there were no significant correlations found in the control group's data. Intervention group data now shows no significant correlations between SMK and subject matter beliefs anymore, whereas now noticing is significantly correlated with the other two variables SMK and subject matter beliefs.

The non-parametric Kendall's Tau corroborates all the significant results at the same $\bar{\bar{\alpha}}$-level, except for the intervention group data at T1, where the correlation between SMK and beliefs yields $\tau = .303$ ($p = 0.06$).

Tables 19 and 20 summarise these results (Pearson's r values listed only where significant correlation was found):

Table 19: Two-tailed Pearson correlation coefficient r control group T1 vs T2

Control T1	SMK	Not.	Beliefs	Control T2	SMK	Not.	Beliefs
SMK	1	.525 (p = .003)			1		
Not.		1	.524 (p = .004)			1	
Beliefs			1				1

Note. Exact p-values (where possible) featured for comparison against $\bar{\bar{\alpha}}$

Table 20: Two-tailed Pearson correlation coefficient r intervention group T1 vs T2

Control T1	SMK	Not.	Beliefs	Control T2	SMK	Not.	Beliefs
SMK	1		.458 (p = .001)		1	.516 (p < 0.001)	
Not.		1				1	.523 (p < 0.001)
Beliefs			1				1

Note. Exact p-values (where possible) featured for comparison against $\bar{\bar{\alpha}}$

In short, any significant correlations between variables seem to disappear from T1 to T2 in the case of the control group. In particular, neither of the two other variables previously correlated with noticing – SMK and beliefs – show a statistically significant relationship with subject matter noticing at the end of the semester in this group's data.

In the intervention group, on the other hand, the situation seems reversed. Whereas previously only SMK and beliefs were significantly correlated, at the end of the semester, that correlation has fallen below the threshold of statistical significance, while the other two relationships – between noticing and SMK and between noticing and subject matter beliefs, respectively – are now detectable in this experimental group's data.

5.6 Research questions Q5 and Q6

Research questions 5 and 6 read as follows:

- Q_5: What do PSTs think about the relevance of advanced mathematics for their future profession after having participated in the intervention?
- Q_6: How did PSTs experience the situated intervention in relation to this topic?

The first two prompts in the reflective essays, which asked participants about their expectations going into the study programme and their evaluation of how their realities as a students have so far lived up to those expectations, will not be analysed in this dissertation. The essay prompts relevant to these research questions are listed again here:

a) How would you assess the relevance of the mathematics-specific part of your teacher training for your profession as a teacher?
b) How did you experience the seminar with respect to this – did your assessment change as a result of participating in this course?
c) If so, is there a specific scene/a specific task/a specific aspect which changed your thinking in this regard?

I will give both a very rough quantified overview of participants' overall impression of the relevance of mathematics-specific courses in their studies as expressed in these essays, as well as provide some insight into the breadth of answers I received regarding their reasons for their impressions.

5.6.1 Overall impression regarding relevance of subject matter

Out of the 44 intervention group participants who submitted reflective essays after the end of the semester, 16 gave a definitive positive answer regarding the relevance of advanced mathematics to their future profession, 14 indicated a generally preferable view (writing about both positive and negative aspects, but concluding in the positive direction), while 5 participants remained generally critical (naming both arguments for and against advanced mathematics being relevant, but concluding in the negative direction) and 9 participants reported a definitive negative answer to the question.

To give a second rough quantitative overview of the tenor of the reflective essays, I classified the answers regarding possible changes in beliefs (prompts 2b and 2c) into four categories as well. As many participants did not directly answer the question, the largest category ended up being "not stated". However, many of the essays coded as such contained detailed descriptions on how the preservice teachers experienced the seminar in relation to their (future) jobs as teachers, which is why in chapter 5.6.8 I specifically analyse the themes mentioned by these PSTs.

In the following, I will present the themes that participants mentioned when arguing for or against the relevance of advanced mathematics for their profession as school teachers, as well as when reporting on the changes or the lack of change in their views regarding this question as a result of attending the seminar. I will use a variety of direct quotations from the reflective essays which I have translated into English.

5.6.2 In favour of the relevance of advanced mathematics

Unspecific concessions

Some participants provided a general, but vague, positive stance regarding the relevance of advanced mathematics for their future profession. These did not lend themselves well for distinguishing thematic reasons behind their conclusion and because of it, it remained unclear to me how genuine those stated opinions were. Some statements sounded like they may have been the result of social desirability bias, wanting to give answers that pleased me as their lecturer and as a researcher. Examples include:

> "Fundamental subject matter knowledge is definitely important for one's teaching."

> "Having certain background knowledge never hurts and the accuracy of one's teaching also depends on subject matter competence."

> "It is surely important to know more than one's students."

The vagueness of these latter kinds of statements did not allow me to determine what benefit the PSTs truly saw in studying advanced mathematics in their training. The other codes that *did* emerge later on I was able to summarise into four main themes:

1. Views on the role of a mathematics teacher
2. Experiences a mathematics teacher should have had
3. Implicit benefits for teaching mathematics
4. Explicit benefits for teaching mathematics

In the following, I will elaborate on each of these, and provide some illustrative examples for each of the codes that were summarized under this theme, when applicable.

Views on the role of a mathematics teacher

The first theme is about the roles of mathematics teachers within the school system, and how learning advanced mathematics at university can contribute to better fulfilling these various responsibilities.

Teacher as a representative of their subject

Three participants noted that as a representative in their school for mathematics as a discipline, a teacher should have acquired extensive background knowledge by practicing mathematics themselves. In this role, the teacher should then be able to authentically answer the question of "why" it makes sense to study mathematics.

One of the participants put it like this:

> "We as teachers are somehow those who can give students some perspective and help them find their own way. [...] I have to know my subject much better [than my students do] in order to be able to adequately represent it."

Teacher as an expert and role model

Four participants noted the positive impression a teacher should make on their students as somebody who is highly competent in their field and thus worth learning from. Some specifically mentioned the word "expert" or "expertise", while others, such as the participant in the following quotation, characterised a high level of subject matter knowledge as being a constitutive part of being a mathematics teacher.

> "I would say that the contents that are being conveyed [at university] are valuable when it comes to developing a high level of mathematical understanding, which in my opinion is a very important skill of a mathematics teacher."

Experiences a mathematics teacher should have had

I denote this as a separate theme because while the first theme concerns a mathematics teacher's relationship with and appearance in the outside world, here, PSTs were interested in a teacher's personal relationship with mathematics. The preservice teachers made fewer references to what difference in perceivable behaviour would result from teachers gaining these experiences, and focused more on the teachers' internal experiences themselves.

Experiencing mathematics as an academic discipline

Three participants believed that having experienced firsthand (something closer to) the way mathematics is practiced as an academic experience provides intrinsic value to school teachers of mathematics.

One student called this "developing a sense for what it means to practice mathematics", while another wrote:

> "The study program is of great importance as it provides a view of mathematics which is not conveyed, in that sense, at school. [...] You can see the point of the theory which singles out mathematics as its own discipline and not just as a tool for solving problems in other subjects such as physics, chemistry, ... This is important so one can see the necessity of one's own work."

Experiencing mathematics itself as joyful and interesting

One participant asserted that "a mathematics teacher should have at least a little fun doing mathematics and should be interested in pure mathematics". This was reported in the context of how a mathematics teacher can convey a certain image of mathematics to their students (see "explicit benefits" below), but since this comment voiced such a strongly normative opinion about the mathematics teacher's *personal internal experience*, I include it under this theme.

Implicit benefits for teaching mathematics

Students who had positive or mixed views on learning advanced mathematics at university named both "explicit" and "implicit" benefits, which I categorised by whether the participants' talked about improving some of the teachers' qualities (like their professional knowledge, or attitudes) or resources, or explicitly mentioned how the work of teaching might be affected by having studied advanced mathematics.

Increasing mathematical competence

Prospective teachers learn a lot over the course of their studies. Ten PSTs specifically mentioned that studying advanced mathematics had closed gaps in their knowledge and made them more competent than they had been when they had entered the study programme. These participants either reported having gained a more profound understanding in general, or of topics which had been neglected in their previous studies (or for which they had no previous affinity). One participant explicitly stated that they were "not the best mathematician" and thus "maybe it was necessary for [them] to study this much in order to walk away with a basic understanding of things". A representative example of the statements PSTs made in this category is the following:

> "If the study program only covered school content knowledge and its pedagogical content knowledge, then [my] understanding of more complex topics certainly wouldn't be as pronounced, or one would have to acquire all of this in a more nuanced way by oneself as a teacher."

Increasing one's subjective feeling of competence

On a related note, five PSTs also commented that mastering advanced mathematics makes teachers feel more qualified and thus feel more assured of their own competence while teaching. This added self-confidence which seems to be important to these participants as it influences their subjective experience of being a mathematics teacher. One of them puts it this way:

> "Even just comparing my level of knowledge before my studies with my current level, it becomes apparent to me how much the subject matter courses have influenced my sense of competence. Even if certain contents were discussed in highly complex detail and I probably won't need the majority of the proofs I have encountered again, these still improved my mathematical thinking and vastly expanded my background knowledge. And it is precisely these factors – together with a strong sense of competence – that are essential in my eyes for feeling confident as a teacher [...]."

Acquiring relative ease in dealing with school mathematics

Even if they will not be directly applying (some of the) advanced mathematical knowledge they studied in their programme, two PSTs expected that, having mastered advanced mathematics, the mathematics that will be required of teachers in school should be relatively easy. Dealing with the mathematical demands of situations that arise during teaching should accordingly become more efficient. One participant put it this way:

> „Proofs or theorems from the textbook can be understood with a quick glance, since one is used to "much more complicated things" and one has acquired an experienced eye for these things."

The other PST said that gaining a useful overview of the separate topics will make it easier to concern oneself with them on a "more basic level".

Learning how to autonomously fill in gaps in one's own understanding

Even if not everything that will be needed in the future is studied, the mathematical basis is laid for learning or re-learning mathematics by oneself as a teacher. This includes reminding oneself of facts that may have been forgotten. One participant ventured that "things that you have already learned once are still there in the background and can be re-acquired much faster". Another stressed the benefit of knowing where to efficiently look up information (like answers to some specific student questions) one might not know by heart:

> "I think it is especially important, when confronted with questions, to immediately know what this could be about and where one could find relevant information."

Acquiring directly useful resources

Although the lack of specific takeaways in the sense of ready-made lessons plans or other resources that can be used in the classroom setting was bemoaned by some of the participants with more negative views regarding the utility of advanced mathematics, one preservice teacher also mentioned this in a positive light: "Via conversations with the lecturers of the mathematics courses and discussions with my colleagues I also got to know useful resources."

Explicit benefits for teaching mathematics

The following points are the codes in which participants were able to more specifically connect benefits to classroom teaching actions with having learned advanced mathematics.

Being able to provide better explanations and exposition

The more a teacher knows about their subject, the more flexibly and reliably they can answer unexpected student questions or respond to other teaching situations. Participants stressed being able to answer questions about "why" something was the case mathematically. Some made references to this being especially important when the mathematics class is "good" (meaning high-achieving students) and questions why something is true or necessary.

Knowing "the mathematics behind" what is taught at school is considered to be important by many participants, and they expect this extra background knowledge to be useful for various aspects of explaining. One participant specifically stressed the relevance for upper secondary levels.

Advanced mathematics or deeper mathematical understanding, according to participants, was needed "especially where there is a need for different explanations, or where there are misconceptions", but also more generally "in order to be able to convey the contents correctly, confidently, and precisely". Multiple participants highlighted the importance of advanced mathematical understanding so that one is "able to adress more specific questions/concerns as a teacher" that might arise.

Two participants also referred to their own experiences in teaching where their more advanced mathematical knowledge helped them in addressing student questions.

Being able to convey joy and enthusiasm for the subject

Three participants were of the opinion that a teacher who is more well-versed in their subject will be able to better convey a passion for their subject and show students what makes doing mathematics a worthwhile activity. One preservice teacher put it like this:

"This way, it is certainly easier to show more of a passion for the subject, or to ignite that passion in the students and convey how much joy there is in exploring, understanding and finding out, … i.e. in gaining a sense of accomplishment."

Being able to make strategic decisions using meta-knowledge
One preservice teacher referenced being able to choose between different paths forward as a teacher. They contended that a teacher needs to know more about the subject than what they are telling the student because they need to decide which aspects are important, which options there are to choose from and which potential for misunderstandings there is when acquiring a certain piece of knowledge or skill. They used a metaphor from their secondary subject, music, for this:

"When I teach a child how to play piano, I also wouldn't need to be able to play much more than that child. But in order to provide the child with a good, well-founded education on the instrument, it is important to know for myself, what is important on this instrument. Which kinds of techniques there are, which mistakes you yourself have made. Instruction you can pass on quickly, but this true „know-how" has a totally different significance and can result in a much more solid foundation."

Being able to use one's own learning experience to better understand student difficulties
Another participant reported that one can use their own experience learning certain topics to anticipate difficulties or misunderstandings and gain insight into learners' needs when studying mathematics: "Especially my own learning experience appears to me to be a good basis from which to better respond to my future students' needs. I can comprehend which difficulties are given when learning certain topics, and how one grapples with them as a learner."

5.6.3 Against the relevance of advanced mathematics

Assertion of lack of relevance without further justification
Some participants, as with the positive judgments, gave no specific reasons for their opinion and just stated outright that they did not believe advanced mathematical content to be relevant for their future profession. Some used a sort of circular reasoning:

"I think that these are less important for the real work of teaching, since a lot of the concepts taught at university are not relevant for the classroom."

"However, I believe that the advanced mathematics part of my studies are not quite as relevant for my teaching."

Among the answers that participants gave that featured specific reasons for why they do not consider advanced mathematics to be (very) relevant to their future profession, I was able to distinguish five main themes:

1. "Too much" advanced mathematics in curriculum in absolute terms
2. "Too much" advanced mathematics in curriculum relative to more relevant content
3. Missing link between theory and practice lessens relevance
4. Inadequate for demands of future profession
5. Argument based on one's own teaching experience

Again, these summarise one to three codes each and will now be described (and illustrated with exemplary quotations from the preservice teachers' reflective essays).

"Too much" advanced mathematics in curriculum in absolute terms

A number of students think that in some sense, the advanced mathematics they have to study over the course of the program either goes into too much detail or covers too wide a spectrum outside what will be directly usable in school teaching. This is either reflected by a general view that preservice teachers are being "overprepared" for the realistic demands of their future jobs, or by the naming of specific topics which the participants deemed irrelevant.

Courses "overprepare" for many future situations

Two students concluded quite generally that – while they see the connections between advanced mathematics and the elementary mathematics that is taught at school – actually applying their knowledge of advanced mathematics in the course of their work is unrealistic. As one of them puts it:

> „[U]niversity mathematics has nothing to do with the mathematics I teach. As a student/teacher I do of course see the connections and how the complex generalisations can be applied to elementary mathematics, but I often get the feeling that it is all quite out of touch with reality."

The other participant frames this in terms of the imagined mathematical level of the group of students they would be teaching (cf. the examples of positive evaluations referencing questions by "particularly gifted/interested/high-achieving" students):

> "The majority of courses seems to be preparing me for problems and questions of a class that has a utopian level, which teachers aren't confronted with."

Two other participants single out a school type in which advanced mathematical knowledge is deemed especially irrelevant: Lower secondary schools [Mittelschule] that are not part of a "Gymnasium" (academic school in Austria's double system that makes a distinction between more academically oriented "Gymnasien" and schools that focus more on preparing the 10-14 year olds for going on to start an apprenticeship or continue at a vocational school). One of them says that when teaching at a „Mittelschule", especially in a class with lower-achievers, "the mathematics degree is too extensive, since here we mainly focus on the basics".

Specific topics not relevant

One participant listed specific topics (groups and rings) which they deemed less relevant for their profession, and said that learning these was only important for the next semester in their degree.

"Too much" advanced mathematics in curriculum relative to more relevant content

Some participants thought that, while it might make sense to have an extensive subject matter education in mathematics, the limited number of courses in a degree resulted in other contents they deemed *more* important to their future profession being cut short in favour of mathematical content.

Too extensive at the cost of practical teaching skills & teaching practice

Three participants bemoaned that courses teaching school mathematics, subject matter specific or generic teaching methods courses, and/or practical experiences would be more important to spend time on, even at the expense of some of the subject matter education. Again, one participant points out that this is a problem especially for those teaching at lower secondary schools. An exemplary statement:

> "I still think that the tension between subject matter versus teaching/practical applications in our degree is resolved too strongly in favour of subject matter."

Too extensive at the cost of moving away from actual topics of school mathematics

Two participants reported feeling like they had moved so far away from the mathematical topics they needed to teach in their profession that, starting to work as a teacher, they would need to reacquire skills in the actual "school mathematics" which they had partially forgotten. One of them said that they had been teaching for three months so far and they "had to, especially in the beginning, but even now, go back and review many things, because you just move so far away from the school curriculum".

Missing link between theory and practice lessens relevance

By far the most common reason provided for feeling like advanced mathematics is not very relevant is that students struggle to relate their understandings of advanced mathematical concepts to the context of teaching topics of school mathematics. Making these connections is something that does not happen automatically for many preservice teachers:

> "I personally had/have trouble connecting the subject matter related parts [of the curriculum] with the school mathematics part."

> "Sometimes I have also been/I also am of the opinion [that advanced mathematics is irrelevant] but only when professors sadly do not manage to create a context for school […]."

This applies in part even to subject matter specific courses that should be tailored to school mathematics and its teaching: "There was very little I could take away from many school mathematics lectures/tutorials or subject matter specific teaching seminars, or even see how these contents would be relevant for school."

One participant notes that learning how to apply the knowledge they learnt at university would have been desirable. This is difficult to do in parallel to learning it for yourself for the first time:

> "Even though the tutorials in parallel to the subject matter lectures are great, you don't really learn how to teach your pupils something, since you are still in your role as a learner yourself, even if you have to explain things [in the context of the tutorial]."

Another preservice teacher uses the common criticism that the six year degree (4 years Bachelor of Education and another 2 years Master of Education, which most students nowadays complete "on the side", while teaching part or full time) contains rather little practical experience: "Thus, you also cannot try out the different mathematical contents at school or see what it is like to teach mathematics."

Inadequate for demands of future profession

A number of participants actually reported that they thought the subject matter which was provided to them as part of their degree did not seem to adequately prepare them for their role as a teacher.

Too little subject matter for certain school types

One teacher mentioned that due to the broad spectrum of school types in Austria that each have their own curricula, they actually felt like they had had studied too little mathematics, for example, in order to teach at one of the technical secondary schools [Höhere Technische Lehranstalten] where there tends to be a strong focus on science, engineering and mathematics.

Lack of teaching for understanding

Three participants suggested that they had trouble seeing the relevance of advanced subject matter courses because they personally had not been able to learn much from them, and/or because the lecturers "unfortunately did not manage to explain the contents in a way that was understandable".

Lack of retention

Five participants mentioned forgetting large parts of what they studied in the university subject matter courses – although not only negatively, as the following shows:

> "I have definitely forgotten quite a bit – especially from the first semesters – but I have acquired a very good basic structure of knowledge and understanding through these courses."

The other four preservice teachers were very critical regarding this fact. Some reported having learned many definitions and proofs of theorems by heart which they forgot soon after having passed their exams:

> "It is not necessary for daily life at school to know about 500 different mathematical theorems and be able to prove them, and this factual knowledge cannot be retained by anyone much longer than a few months after the respective exam."

Others turned the causal relationship around ("[…] since prospective teachers will not be able to do much with it in school and thus will forget the content sooner") or said that due to the fact that prospective teachers will not be using the knowledge they acquire in some lectures for years (before they start teaching at a school), they are likely to forget most details before being able to use it.

Specific expectations not fulfilled

In the case of three of the participants, it seemed that they (implicitly) had had very specific expectations as to what the university training would provide, which were not met. Two of them seemed to believe that the contents of the university courses would be accessible or at least immediately transferable to the contents they would present to their pupils at school:

> "[Some of the] explanations and ideas [which are presented in the school mathematics lectures] are, especially for weaker students, too abstract."

Or, as the other preservice teacher puts it:

"The way I experienced the mathematics degree, you are moving in such abstract dimensions that aren't comprehensible for students and will therefore contribute little to my teaching."

The third participant seemed to have the expectation of leaving their university education with a set of ready-to-be-used teaching materials:

"The only thing that I may have gained is a better understanding for mathematical tasks in general. When I start teaching, I still have to plan every single lesson from scratch and have, apart from one seminar and the teaching practice, no further work materials that I can use in school."

Argument based on one's own teaching experience

School experience has shown it to be irrelevant

Lastly, four participants argued that based on their own experience having started to teach, they can conclude that the subject matter related courses in their studies ended up not being relevant to their professional activities. Two participants noticed not being able to apply their knowledge of advanced mathematics in their teaching and thus concluded that the relevance for it must be rather small:

"During my almost two years in school I have to admit that I was barely able to apply my knowledge and my understandings from university. [...] The more teaching experience I gain, the more I question my training as a teacher at university."

In contrast, two participants concluded only slightly differently and were disappointed that they felt ill-prepared for their future profession:

"I am missing the whole pedagogical content knowledge so much, and I am happy to have a team teacher who keeps helping me. Grading, what is important about a topic, difficult/easy tasks for students, connections to everyday life, the way to teach mathematics and make it more understandable, and much more, are all things I was lacking and I am still working on learning them."

5.6.4 Positive changes in beliefs due to intervention

Thirteen participants explicitly stated that taking part in the seminar had changed their views on the relevance of advanced mathematics for their profession for the better. They reported that the intervention had shown them in what way advanced mathematics can help make more informed decisions as teacher, respond more competently to students and avoid making mistakes of their own:

"The seminar played an important role making these connections [between advanced mathematics and school mathematics] more clear [...] it helped me learn how to *look at school mathematics tasks through the "lens of advanced mathematics"*, so to say, *and make well-informed decisions as a mathematics teacher based on that*."

„I did not expect many of the students' answers/questions/remarks that we discussed in the seminar. Usually I could tell whether they were true or false. But to be completely sure, it is really important to grapple with the definitions and theorems of analysis [...] this subject matter specific teaching seminar showed me that the *advanced mathematics lectures really make sense when you need to quickly evaluate students' questions and answers.*"

While the latter student reported in the rest of their essay that they had had a generally positive view of the relevance of learning advanced mathematics and the seminar had “reminded” them of this and further cemented their view, some students explicitly stated that the intervention had made them change their minds:

“Especially having had to analyse tasks from the standpoint of advanced mathematics and then explain it in a way that is suitable for learners. That made me realise that what I had learned wasn’t as “useless” as I had always thought. […] I noticed that mathematical knowledge is relevant and helps with my understanding.”

One student analysed that the work in the seminar had impacted her way of perceiving the nuances of teaching mathematics – by learning about the challenges and demands of providing high quality mathematics instruction, they realised how much their effectiveness as teachers depends on their own mathematical competence:

“Our way of responding to our (future) students’ work is very dependent on whether we as teachers even manage to perceive what is true or false, and what the reasons behind the students’ answers are. Mathematics does not at all mean that an answer is either correct or incorrect, and this has become much more apparent to me after this seminar. One often thinks that evaluating and grading student work in mathematics is much easier than it is in, for example, languages, but this has turned out to be a misconception for me.”

Multiple students reflected on their subject matter knowledge as a result of the intervention. This went in two directions – two students reported having realised that their proficiency in mathematics was not as high as they had thought:

“Furthermore I have realised once more how sloppy I myself am when it comes to advanced mathematics. I had lots of wrong answers when doing the quizzes. I do understand quickly in those situations, why the answer was wrong, but I can’t do it perfectly myself since I haven’t been engaging with that so much at the moment. I noticed how easy it is to make a wrong statement which negatively impacts the students’ learning gains.”

“By working through the tasks, my opinion has changed insofar as I now believe (I didn’t before!) that it can’t hurt to know more than your students, and in particular, that sometimes, in order to answer questions, it really is necessary to know details that may not be featured in the school textbook. One of the reasons for this were, for example, the student solutions for the function discussions, which at first all appeared to be correct.”

Two other students gave an account of having found a new appreciation for how much mathematical knowledge they actually use when performing teaching tasks. They reported previously not having realised how much they benefitted from the advanced mathematics skills they have acquired in their teacher training:

„What changed was mainly my assessment of my own knowledge. Critically analysing the tasks/children’s answers, I noticed that I know much more about mathematics than I would have given myself credit for. It made me see that the degree has helped me look at tasks/answers in a much more critical and more subject specific way.”

„It gave me a greater appreciation for these general mathematical skills, which I probably would not have had as easily without this seminar. Especially those moments where my advanced mathematical knowledge allowed me to solve problems were moments in which I learned to appreciate my broad mathematical knowledge.”

As is to be expected, between participants who reported having noticed some shift in their perception and participants who did not change their views, there was room for preservice teachers who noted some changes in their beliefs but said that these new impressions did not change their opinion as a whole. One participant put it explicitly:

„The seminar shifted my opinion a little because I can now see that there is good use for mathematical knowledge when it comes to various student problems. [...] However, I am still of the opinion that overall it is too much mathematics which one will never have to use."

5.6.5 Negative changes in beliefs due to intervention

Interestingly, one participant actually reported feeling even more cemented in her opinion that, compared to seminars such as the intervention, the [mathematics-specific] lectures had not contributed to her professionalisation. They were frustrated to find that despite having attended all these advanced mathematics courses, they were not capable of solving the intervention tasks using their subject matter knowledge:

"Like I have mentioned, I have always found the seminars and courses to be the good part of my degree and this has not changed. If anything, I have become even a little angrier at the curriculum, because I had to, more often than I'd like to admit, research the background knowledge (e.g. the theorems that were relevant) in the freshers, scripts or on the internet. Despite knowing that I had heard all of it before, I just couldn't remember many of the things."

In the discussion section, I will try to relate this account specifically to the quantitative results of the beliefs test, as it may illuminate why the questionnaire concerning the double discontinuity may not have assessed the right construct for my purposes.

5.6.6 Participants explicitly reporting no changes in beliefs

Among the participants stating that their beliefs had not been substantially changed as a result of the intervention, there were both participants who said that their views on the relevance of advanced mathematics had been positive even at the start and had now been further cemented, as well as participants who clearly stated that, while the intervention was a good course, it had not changed their beliefs.

Most of these participants fell into the first category, which I will illustrate with a few exemplary sections from the essays:

"My assessment in this regard has not changed due to the seminar, it was only confirmed. I knew beforehand already that it is enormously important to grapple with difficult contents over a longer period of time."

"My beliefs about this topic did not really change over the course of the seminar. However, that is mostly due to the case that I had previously attended many courses held by [the co-lecturer in the first implementation of the intervention[4]]. This topic is often addressed there, and I have also privately discussed and reflected on the structure of the degree many times with my colleagues."

"Thus, I believe the mathematics-specific part of my degree to be relevant and important. [...] Since the seminar confirmed my beliefs, I do not have anything to add here."

[4] The student writing this account participated in the second cohort which was taught by the author alone.

Out of the three participants who specifically stated that their beliefs hadn't changed, two pointed out that the seminar was "very useful" and "contributed to their training as a future teacher", but that it had not changed their evaluation of the question of relevance. (The third one merely stated that their beliefs had not changed, without any further qualifiers.)

5.6.7 Aspects that affected beliefs about relevance

As a reminder, question 2c had asked the participants:

If so[5], is there a specific scene/a specific task/a specific aspect which changed your thinking in this regard?

The preservice teachers who responded to this prompt explicitly gave a variety of answers. Some of them focused on specific teacher tasks or task classes that had been particularly successful in showing them the relevance of advanced mathematics for their profession. Others highlighted one or multiple tasks all revolving around a certain mathematical concept whose application to teaching was illuminated to them. A third group of participants remarked upon specific ways in which work in the seminar was structured that effected change in their views.

Specific tasks and task classes

Incidentally, all three task classes I had chosen to structure the intervention around received at least one explicit mention among the answers to this question.

Task: Designing a worksheet

Two participants specifically mentioned the task in which the preservice teachers were asked to prepare a worksheet for their students that contained a variety of problems around a specific topic. Additionally, I had asked them to provide a solution sheet that was phrased in such a way that it would be suitable for students studying by themselves.

> "The task which most shaped my thinking was the creation of the worksheet as an individual task. […] This is, I think, a job that requires and promotes so many of a teacher's competencies. You have to think about how to advance the topic within a problem and use the knowledge the students already possess, how to structure a problem by yourself in a way the topic is best conveyed (e.g. self-differentiating problems [Blütenaufgaben]) and how to phrase everything in a way that it is comprehensible, fluent, and above all, mathematically correct."

> "When designing the worksheet (individual task), I realised that I had never created a worksheet from scratch (during my teacher training) before. Before that I had arranged lesson sequences or taught classes at school using the textbook as assistance, but had not "invented", written or created many of my own questions."

[5] Referring to question 2b: How did you experience the seminar with respect to this – did your assessment change as a result of participating in this course?

Task: Student utterances or student solutions

One participant highlighted the importance of tasks from the task classes that dealt with students' reactions to certain situations or student work that had to be evaluated.

"Especially those tasks in which we discussed real teaching situations and responded to the students' misconceptions. On the fly, I'd recall the task about upper and lower sums in integral calculus."

Task class: Giving explanations

One participant singled out the task class "giving explanations" and how it had shown them how mathematical knowledge and didactical knowledge are interconnected in the work of teaching.

„I was reminded again by this seminar, especially during the „giving explanations" section, how important it is to have a solid grasp of your own subject. It's generally all too easy to make mistakes when phrasing things, but it happens even more often, when you are not clear about the topic in general. [...] And even small imprecisions in my own understanding will surely have an effect on my students' understandings about a topic."

Specific mathematical topics

A number of participants drew attention to certain mathematical topics that kept re-occurring over the course of the seminar. They reported that they had gained further insight into these areas on a subject matter level and had experienced useful applications of that understanding for their teaching.

"The tasks about the **limit of a sequence**, or the example of "Achilles and the tortoise", in particular, have led to considerations regarding the mathematical correctness, but also a comprehensible way of communicating with students. The question whether 0.999... is equal to 1 also showed me the importance of mathematical background knowledge for teachers."

"One of the most valuable classes for me was the lesson in which we talked about **continuity and points of discontinuity**. The sentence which most stuck to me, and which I am surely going to use when tutoring or later teaching in a classroom, is "in mathematics, there is only high jump, no long jump" [this was said referring to jump discontinuities versus functions with disconnected domains]."

"The group work we did about **sequences** (1. group work drug levels) showed me that I do have one or the other mathematical weaknesses which I will be needing during teaching. I learned that subject matter knowledge is important and helps with my understanding."

Specific ways of working within the structure of the situated seminar

Another group of participants highlighted not specific tasks or task classes but more general ways in which the situated intervention was designed. This referred to elements of the 4C/ID framework like the supportive information I had provided in the form of a list of leading questions that can be used to analyse tasks, explanations or student work, or the different opportunities for expert and peer feedback that were incorporated.

"The seminar played an important role making these connections [between advanced mathematics and school mathematics] more clear [...] it helped me learn how to look at school mathematics tasks through the "lens of advanced mathematics", so to say, and make well-informed decisions as a mathematics teacher based on that. Especially **analysing tasks or teaching situations with the help of given leading questions** has helped me arrive at this insight."

„The thing I most took away from this course was to not only evaluate answers as true or false, but that one should question why the student answered in a certain way. I specifically remember the first in-class group task. Our group focused on one topic (the domain). Comparing notes with the other group, we noticed that they had focused on something completely different. **Both groups had lost sight of the big picture, and only after consulting with each other were we able to create a coherent problem for students.** This moment showed me that one does not only have to consider details, but one's entire understanding and knowledge of mathematics can be useful."

5.6.8 Participants not addressing the question of change of beliefs

A large part of participants' essays did not explicitly address the question of "change in beliefs". However, they did describe their experience of the intervention, which I would like to present in an overview below. This way of reading the essays can also serve as an implementation check as to whether the intervention was able to provide the kinds of experiences it was designed to.

Again, I have tried to identify themes within these essays in order to gain an impression as to what participants remembered and found noteworthy about having attended the seminar. The main themes these participants discussed when talking about their experience were:

- Connections between theory and practice
- The experience of practicing typical teacher tasks
- Perceived level of "realistic" situatedness of the seminar
- Perceived benefits they took away from the seminar
- Overcoming personal struggle
- Opportunities for in-depth engagement with mathematics
- Changed perception of teaching as a profession
- Benefits of (peer) discussions and group work

As I have previously done, I will highlight a few exemplary statements from each of these themes to give a better impression of the types of reports featured in the participants' essays.

Connections between theory and practice
Multiple students specifically pointed out that the seminar had addressed the discontinuity that I had set out to address, namely to bridge the gap between the advanced mathematics as they had been studying at university and the tasks they would be required to perform as teachers later on.

"It is a good approach to close the seemingly unbridgeable gap between school mathematics and advanced mathematics."

"I liked this course because it appeared that somebody had thought about how to make this gap between university and profession a little 'easier'".

"The seminar definitely focused on conveying theory in classroom teaching, both verbally (explanations and feedback) as well as in a written form (tasks). To me, that is already one of the aspects that move school and university a bit closer together."

The experience of practicing typical teacher tasks

Multiple participants appreciated that they had been provided opportunities to practice (and be modeled) the correct execution of teacher tasks. As a result, they reported feeling better prepared and more confident.

> "Furthermore, it is important to practice such crucial tasks of a teacher and to be shown the correct way of performing them during your university studies, because no knowledge within us is useful it cannot be correctly and comprehensibly transferred."

> "In my view, this seminar helped my preparation to become a teacher very much, since we worked with many examples using students' answers and solutions to identify and discuss misconceptions and problems."

> "This was the first time during my university studies in which I considered students' solutions to problems, which gives me the confidence to fix students' difficulties in order to further [their] understanding."

> „It has been very helpful to discuss students' mistakes or look at school textbooks and compare these with advanced mathematics."

Perceived level of "realistic" situatedness of the seminar

Most participants who commented on this topic did so in a positive way, pointing out that the seminar tasks were close to situations which would arise at school, and that the usefulness of advanced mathematics in realistic teaching tasks was demonstrated.

> "The tasks were quite similar to those in upper secondary school."

> "This seminar was very helpful to me, especially since it was very practice-oriented."

> "In conclusion, I would say that contents of advanced mathematics were only used when this made sense in a school setting."

However, one participant had a different impression, ending her report with the following comment:

> "Nevertheless I would like to remark at this point that the (mathematical) level of the tasks is, at least in my experience, not realistic."

Perceived benefits they took away from the seminar

Participants listed a great variety of benefits they attributed to having attended the seminar. They claimed to feel better prepared for entering work as a teacher, to have learned about typical student difficulties before having had the opportunity to teach a class, and to have an easier time identifying students' problems with a topic.

> "Additionally, I have learned how to deal with a variety of views and erroneous connections made by students and how to address these in my lessons."

> "I find it easier now to apply what I had learned. I mean that I can, for instance, find appropriate materials and analyse them. I was missing a seminar like this in my Bachelor's degree and I now try to analyse all exercises better, considering how they help students or whether other materials would be more appropriate for this group."

Multiple participants also said that the seminar had helped them identify gaps in their own understanding.

"In my opinion, it was a training exercise for teaching and one was able to identify mistakes, both on the students' side and also our, the teachers', side. One then has time to think about it beforehand (during the seminar), in order to close those gaps before entering the world of work, which I like a lot."

"I liked that we discussed topics in the seminar which, to be honest, showed me that I do have gaps in my subject matter knowledge. You were forced to intensively address these, even if you are not currently teaching those topics. Just because you passed all the exams in your degree does not mean that you are an expert in that area. Oftentimes, you think to yourself 'of course I can do it', and then it turns out, you can't."

Overcoming personal struggle

Two participants noted that at first, they struggled with the requirements of the seminar. However, both reported that they overcame this rather quickly and were able to productively take part in the seminar after that.

"I have to admit that I was completely overwhelmed with this 'new challenge' in the beginning, which I had not encountered in this way during my university studies. But with the help of the peer feedback and the feedback provided by the lecturer I was able to quickly learn the ropes."

"At the start of the seminar, when we filled out the questionnaire, I was a little disconcerted for a moment because it was rather hard for me to answer the questions. This caused my assessment to sway for a moment. But this went away quickly and I really learned a lot over the course of the seminar."

Opportunities for in-depth engagement with mathematics

Multiple participants not only commented on learning to apply their advanced mathematical knowledge, but also more deeply engaging with the mathematical content per se.

"In general, that one can exchange views with colleagues about student difficulties and get into the topics more deeply than was possible in the lectures."

"I think that the exchange with others (be it lecturers or "just" other students) is very valuable and provides one with the opportunity to gain a somewhat more profound understanding or a new perspective."

"I was afforded the opportunity to further grapple with the concepts of analysis."

Changed perception of teaching as a profession

Finally, one participant noted that they had gained a new appreciation for the (mathematically) demanding tasks of a teacher.

"So the seminar made it really clear how difficult it is to cover the contents of the curriculum, make your lessons engaging and rich, and to achieve that every single student understands the topic without misconceptions."

6 Interpretation and discussion

In this chapter, I will interpret the results of my analyses as I seek to answer the research questions I have posed and evaluate the hypotheses I have formulated, while considering the limitations of the study design and the data I have collected. Doing so, I will relate my findings to the existing literature on these questions. Additionally, I report on new questions that arose during analysis.

Based on the limitations of my study and these novel considerations, I then summarise what future research into the area of subject matter noticing and related beliefs of preservice teachers might fruitfully look into.

Lastly, I conclude with a section on possible implications of my results for teacher education.

6.1 Evaluation of hypotheses and answering of research questions

6.1.1 Research question Q1

- **Q_1: How does the situated intervention affect preservice teachers' noticing of subject matter?**

H1: The situated intervention will lead to significant improvements in subject matter noticing.

According to the results of the mixed ANOVA, the null hypothesis that the situated intervention did not affect PSTs' noticing should be rejected ($p < .01$). A significant improvement of large effect size (Cohen's $f = .44$) was shown.

The results of the mixed ANOVAs on the component variables P1 and P2 indicate that the intervention led to increased noticing along both these dimensions, and growth in one of the two components did not compensate for stagnation or decline in the other.

As such, there is a clear indication that the situated intervention helped PSTs develop and improve their overall noticing of subject matter as tested by the questionnaire I had developed.

These results go in line with studies which had previously shown noticing to be a trainable skill (Jong et al., 2021; Barnhart & van Es, 2015), as well as studies which reported positive effects of situated interventions based on the 4C/ID framework (Costa et al., 2022). Within mathematics teacher education, the results of the experimental study can add to the research landscape by providing further evidence of causal effects on PSTs' learning in line with what Leuders and Wessel (2020) had reported based on their experiences in utilising 4C/ID in mathematics teacher education.

In general, improving PSTs' noticing skills of subject matter is particularly compelling as successful noticing has been linked to better classroom teaching (Stockero, 2021) and it is widely agreed that an awareness of the role of subject matter during teaching is a crucial element of competent teaching (Ma, 2010; Akinwunmi et al., 2014).

Various vignette formats (Friesen et al., 2018) had been previously used to assess the level of (preservice) teachers' noticing skills, but interventions set out to *train* teacher noticing in mathematics had previously typically been implemented using video recordings of classroom situations (e.g. van Es et al., 2017; Star & Strickland, 2008; Star et al., 2011). As such, the positive results of the study indicate that utilising text vignettes in teacher education seems to be an appropriate method for improving teacher noticing.

Similarly, while there has been literature discussing the less prominent notion of teacher noticing during lesson preparation (Choy, 2014), effects of an intervention specifically focused on the training of teacher tasks that take place outside the classroom had not yet been investigated. Most noticing interventions had so far centered teachers' "in the moment" noticing in the classroom context. The positive results indicate that practicing teacher noticing in these types of tasks is a promising direction for educational design.

As a theoretical addition to the topic, *subject matter noticing* as such had not yet been distinguished in the literature, and thus, neither instruments for testing it nor methods for improving it had been developed or tested. The effects that could be shown provide some evidence that it is possible to train and measure this newly distinguished kind of theme-specific (cf. Dreher & Kuntze, 2015) noticing. As PSTs have been shown to struggle to recognise subject matter in typical teaching situations (Vondrova & Zalska, 2012; Stephens, 2008), the gains in subject matter noticing indicate that similar interventions may offer a way of combating this problem.

Despite these reasons for optimism as to the effectiveness of the intervention based on the calculated effects, the specific circumstances of the project require a critical consideration of alternative explanations for the quantitative results.

Firstly, there is a point to be addressed regarding the question of the self-evidence of the effects found. The intervention set out to develop preservice teachers' proficiency in actively using their subject matter knowledge to perform typical teacher tasks. The instrument assessing subject matter noticing when analysing and responding to typical teacher situations was designed in parallel. Both used text vignettes featuring typical tasks of mathematics teachers.

On the one hand, the prompts were dissimilar – at no point in the intervention were participants directly asked to "mine" text vignettes for relevant subject matter contents (as was the case for the P1 prompts of the noticing questionnaire). Similarly, in constrast to the learning tasks of the intervention, the P2 prompts in the noticing questionnaire asked participants to make a teaching decision without referencing subject matter (as the learning tasks had explicitly asked PSTs to do). Still, one may consider the positive results unsurprising seeing that the preservice teachers had just practiced this kind of engagement with very similar vignettes for an entire semester.

A crucial question is whether PSTs exhibited better noticing in the posttest environment because after this semester of practicing on similar vignettes, they "knew what I was looking for".

Even if my intentions as a researcher were abundantly clear to participants by the time they re-read the questionnaire prompts at the posttest (and even if we assume the unlikely case that they had been completely unaware of my interests at the pretest – in spite of the topic of the seminar and the subject matter focus of the first prompts in the questionnaire), and even if their inclination to notice subject matter spontaneously had not changed over the course of the intervention, the positive results indicate that PSTs were *able* to provide whatever they thought I as a researcher was interested in.

This means that in the most modest interpretation, the existence of sufficient "potential subject matter knowledge to be activated" could be shown. This is not self-evident considering the widespread dismay over PSTs' lack of subject matter knowledge, and the fact that one common challenge to intervention design in this direction is the heterogeneity of PSTs' subject matter knowledge (cf. Leuders & Wessel, 2020).

The subject matter knowledge PSTs arrived with at the beginning of my intervention seems to have allowed them to showcase a much more proficient and content-based view of teaching situations after the intervention than they had shown initially. (Of course, in absolute terms, there was still much room for improvement in PSTs' noticing, which suggests a limit of such interventions. Further professionalisation may need to rely upon changes in the subject matter education of preservice teachers leading up to this point of their education.) There would have been no point in trying to make preservice teachers pay more attention to something they cannot learn to recognise due to a lack of prerequisite knowledge – hence, at the very least, the intervention effects suggest laying such concerns about the feasibility of subject matter noticing interventions to rest.

At the same time, the worst-case scenario outlined above does not appear to be the most likely interpretation of events.

For one, for both participants of the control group as well as participants of the intervention, the university context of the Faculty of Mathematics, as well as the priming effect of prompt 1 explicitly asking about subject matter in combination make it very unlikely that *only* the participants of my seminar became aware of my interest in their thoughts about subject matter, and even then only did so *after* the intervention had been conducted.

It therefore does not seem too likely that the inclination to consider subject matter aspects of the vignettes due to, e.g. a general social desirability bias, was *much* stronger at T2 than at T1. I say "much", because at the same time, it is possible that despite understanding the researcher's intentions from the beginning, PSTs' personal preference for conforming to my expectations rose over the course of the semester in which I taught them.

Empirically, there is further indication that social desirability bias in general seemed to have played a much less pronounced role than may be expected. The null results in the case of beliefs change show that on a measure where conforming to the researcher's expectations would have been much easier – checking Likert scales in order to purposefully signal something contains no skill component, as opposed to showcasing an increase in subject matter noticing which can only be “faked” as far as one’s actual noticing abilities allow – and on which the educator’s/researcher’s likely expectations were much more obvious, no effects could be detected. PSTs did not seem too eager to provide desired answers, according to these results. However, it is possible that the psychological effect of “wanting to perform well” on a measure of competence lead to different motivations of PSTs when giving answers on the noticing questionnaire as opposed to reporting their opinions.

With these caveats, the intervention seems to have meaningfully improved participants’ abilities and readiness to consider subject matter when engaging with typical tasks of teaching.

This is mirrored by the self-reports participants provided in the qualitative data of the reflective essays. It is worth mentioning here that none of the essay prompts had *asked* the participants to talk about their perception of any learning gains or skill development – the questions referring to the seminar had only asked about changes in beliefs regarding the relevance of mathematics-related courses and specific instances of situations in the seminar which had affected those beliefs.

However, spontaneously, many participants talked about having gained meaningful experience in carrying out teacher tasks, and about changes in their approach to these tasks. As a clear example, I would like to highlight one quotation from the ones reported in the results section:

“**I find it easier now to apply what I had learned**. I mean that I can, for instance, find appropriate materials and analyse them. I was missing a seminar like this in my Bachelor’s degree and **I now try to analyse all exercises besser**, considering how they help students or whether other materials would be more appropriate for this group.” [emphasis added]

While this participant does not specifically mention subject matter knowledge (only referring to “what I had learned”), the qualitative data in combination with the quantitative results of the noticing questionnaire suggest that altogether, a definitive change in participants’ way of perceiving and approaching subject matter related parts of their future profession has been achieved.

It is also rather illuminating to consider the characteristics of the participants in the control group. Instead of a non-situated seminar, these preservice teachers had arguably spent one semester performing much *more* situated work, having instead completed their practice semester that involved the planning *and* actual carrying out of lessons in schools and the accompanying seminar. However, instead of having gained expertise in applying their subject matter knowledge to practical tasks, the group’s noticing scores had remained the same.

Of course, there are still a number of caveats to consider before extrapolating these promising first results to the larger population of preservice mathematics teachers in general (such as the lack of randomisation in the study design), especially since questions of scalability are also involved in the case of this intervention that was very resource-intensive for the educators involved. The chapter on limitations of the study will discuss these concerns and then follow up with recommendations for future research as well as possible implications for teacher training.

6.1.2 Research question Q2

- **Q2: How does the situated intervention affect preservice teachers' beliefs about subject matter?**

H2: The situated intervention will lead to significantly more positive beliefs regarding subject matter.

Interpreting the results of the mixed ANOVA, no significant changes resulting from the intervention could be detected. The corresponding null hypothesis can therefore not be rejected and it is entirely possible that this kind of situated intervention will not lead to any improvements in PSTs' beliefs regarding subject matter.

Mixed ANOVAs on the beliefs subdimensions of "content relationship" and "relevance for profession" did not yield significant results, either. This means there is no indication that the intervention led to changes in beliefs along either of these dimensions as measured by the questionnaire concerning the double discontinuity.

Of course, the power of the study can be taken into consideration here. It is possible that with a higher number of participants in both intervention and control groups, a small or medium-sized significant effect could have been detected.

While disappointing, these null results corroborate the state of the literature reporting on the robustness and stability of beliefs over time (Philipp, 2007). This is true in particular for beliefs about the "importance of learning contents" (Rach, 2023). Beliefs *can* be changed (Panero et al., 2023), but this tends to become more and more difficult the longer a person has held a particular belief or belief system (Pajares, 1992).

The seemingly robust beliefs participants hold about the (lack of a) relationship between advanced mathematics, school mathematics and the work of a mathematics teacher are somewhat of a letdown. Similar efforts to change beliefs via integrating practical experiences had at times proven successful (Hanke & Schäfer, 2018), but effects could not always be replicated in further implementations of interventions (Schadl et al., 2019). Adding the results of this study to the research landscape, reliably improving PSTs' perceptions of the relevance of subject matter seems to remain as open and complex a problem as it was before.

Part of the reason for preservice teachers not recognizing the need for sound academic preparation in subject matter is hypothesized to be a lack of awareness regarding the mathematical demands of their future profession (Becher & Biehler, 2015). As such, one idea behind this intervention was to try to counter possible negative beliefs indirectly by improving preservice teachers' skills in recognising subject matter aspects of typical teacher tasks. On the face of it, this route to beliefs change does not appear to have worked.

One of the challenges I had anticipated was that, since the intervention needed to be positioned at a sufficiently late point in the curriculum so as to be able to make use of previously acquired subject matter knowledge, preservice teachers' beliefs about subject matter may have become too entrenched at that point to be changed by a single seminar. While these results do not prove that at this point in their studies, PSTs' beliefs about subject matter cannot be changed (it is entirely possible that beliefs changes could have been achieved with other means), the study's results can unfortunately not provide any evidence of the contrary.

There are multiple reasons, aside from the stability of beliefs in general, and of long-standing beliefs in particular, why this might have been the case. It is possible that more explicitly reflecting on the benefits of subject matter considerations in the context of the seminar and leaving room for participants to actively talk about and exchange opinions on the subject may have resulted in measurable differences in beliefs. As it was conducted, the intervention was designed to directly and openly discuss practical questions of how to best approach teaching situations and which resources to use to one's benefit, but excluding the reflective essay prompts which participants received only after the posttest had been conducted, no other seminar tasks had asked participants to take on a reflective stance with regard to their beliefs.

It is also possible that most participants did not consider themselves to have much knowledge of the kind of advanced mathematics (they thought) the questionnaire was asking about.[6] In that case, the unchanged beliefs as to the usefulness of knowledge which they do not (think they) possess would be unsurprising.

Lastly, there's the hypothesis that the teaching experience already gained in other settings has made a more convincing impression on these students than my – designed to be experienced as situated, but nevertheless, ultimately artificial – seminar environment could have done. (One reason to doubt this, however, is that overall, intervention group participants reported much less teaching experience than the control group and both groups showed similar baselines for beliefs as well as similar changes, or lack of change, in beliefs between T1 and T2.)

Despite the null results and various reasons for their plausibility, the qualitative data gained from the reflective essays raises some questions and suggests that there may have been some changes in beliefs about subject matter which the questionnaire concerning the double discontinuity does not measure. (Of course another research project is needed in order to say anything further in this regard.) In retrospect, even my essay

[6] This is reminiscent of an anecdote reported to me by a colleague. After the lecturer has finished presenting a proof, a student asks: "Professor, does this really constitute a proof?" The lecturer is puzzled: "Yes, why do you think it shouldn't?" – "Because I understand it." Depending on the level of alienation from mathematics as an academic subject, preservice teachers may consider any mathematics they understand and are able to meaningfully apply to teaching "school mathematics" and any other mathematics "university mathematics"!

prompts may have led participants to answer a specific question which is interesting to us as teacher educators but less central to the way in which these future teachers will approach their professional tasks – namely, how PSTs think about the usefulness of the lived reality of their specific subject matter education, as opposed to the relevance of the idea of "subject matter" to their future profession in general.

While it is definitely not the case that participants (in their essays) reported positive changes in beliefs across the board, the appreciation for and high level of engagement with the heavily subject matter focused learning tasks of the seminar provide reason to think that the majority of participants was able to experience a critical and conscious consideration of subject matter aspects as meaningful and useful.

One participant put it like this: "If I was given the opportunity to write a new curriculum, I would add to each advanced mathematics lecture and school mathematics lecture exactly this kind of seminar in mathematics education, in which the contents can be immediately practiced and related to the context of school, in order to 'bridge' the gap between school mathematics and university mathematics."

The wish this PST is communicating is *not* to reduce the amount or the difficulty of mathematics to be studied, but to receive help in applying it productively and seeing connections (which the participant seems to assume do exist!). Or, thinking back to the quotation by one PST talking about mathematics lectures being very useful to their future profession, *except when they don't understand them*, it seems possible that a situated intervention may be able to change participants' views on the potential of solid and extensive subject matter knowledge itself – and still give null results in the questionnaire about "university mathematics" if participants do not feel like they left those courses with that kind of proficiency.

The questionnaire concerning the double discontinuity may have been interpreted in the sense of "this is how I feel regarding the specific courses on subject matter which I have taken as a PST" and less in the sense of "this is how I feel regarding the idea of extensively studying subject matter as a PST". This conjecture seems corroborated by participants appreciating the highly subject-matter focused and time-intensive intervention, while at the same time lamenting the extent of subject matter courses they had so far received. In extreme cases, memorable experiences with regard to the necessity of *subject matter itself* may even lead to more negative beliefs about the relevance of their *subject matter courses*, if PSTs do not experience those courses as helpful in becoming proficient in mathematics. As one PST reported:

"If anything, I have become even a little angrier at the curriculum, because I had to, more often than I'd like to admit, research the background knowledge (e.g. the theorems that were relevant) in the freshers, scripts or on the internet."

Of course, these are hypotheses which need to be further explored in a follow-up studies. With the study design at hand, one can merely conclude that there is no sign that the intervention successfully improved PSTs' subject matter beliefs as measured by the questionnaire concerning the double discontinuity.

6.1.3 Research question Q3

- **Q_3: How are changes in the PSTs' noticing of subject matter related to changes in the PSTs' beliefs about subject matter?**

H3: Changes in PSTs' noticing of subject matter are significantly correlated with changes in PSTs' beliefs about subject matter.

As significant interaction effects could be shown in the mixed ANOVA results for the noticing variable, but no such effects were found in the case of the beliefs variable, it does not seem that improvements in subject matter noticing automatically go hand in hand with more positive beliefs about subject matter. As such, the null hypothesis that changes in PSTs' noticing of subject matter occur independently of changes in PSTs' beliefs about subject matter cannot be rejected.

Considering participants' quantitative questionnaire results in conjunction with the mixed qualititive data, it seems that even very well-received interventions which effect significant changes in being able to apply one's subject matter knowledge, do not seem to necessarily change preservice teachers' views on the role and the utility of mathematics-related courses in their training.

According to the triangle conceptualisation of the constructs subject matter noticing, subject matter knowledge and subject matter beliefs, a theoretical case could be made that improving the noticing of subject matter in teaching situations may foster an appreciation in preservice teachers for the importance of subject matter, as noticing has been shown to be able to influence teacher beliefs in general (Guskey, 2020). In the domain of mathematics education, specifically, one of the reasons for the famous double discontinuity (Klein, 1908) has been attributed to a lack of awareness of the subject matter demands of a mathematics teacher's profession (Hefendehl-Hebeker, 2013). This theory-backed assumption cannot be corroborated by the data collected in this study.

These results show that the connections between subject matter noticing and beliefs about subject matter may not be as straightforward as the "triangle" might suggest – at least not when it comes to *changes* to the corners of that triangle. Measures that show clear effects on a situated variable (subject matter noticing) may leave affective variables (beliefs) unchanged.

It remains to be investigated whether the assumed causal connections need to be revised, and/or whether the constructs themselves involved in this line of argumentation about the interconnectedness of prerequisite knowledge (SMK) for a situated skill (subject matter noticing) and beliefs about the relevance of said prerequisite knowledge need to be further specified and the measurement instruments adjusted accordingly.

6.1.4 Research question Q4

- **Q_4: What correlations appear between the PSTs' subject matter knowledge, subject matter noticing and beliefs about subject matter?**

H4: The three variables subject matter noticing, beliefs about subject matter and subject matter knowledge are all significantly correlated at both T1 and T2.

While the study was unsuccessful at showing that *changes* in the noticing and beliefs variables co-occur, correlational analyses were able to provide some evidence for the hypothesised relationships between the three variables subject matter noticing, beliefs about subject matter and subject matter knowledge at each of the measurement time points.

Situation at T1

Considering the entire sample (N = 75), all three variables *are* indeed positively correlated. This provides evidence for the triangular relationship I had assumed between the three constructs based on various results in the literature.

Of course, the correlational analysis cannot provide evidence as to the directionality of any of these relationships (the theoretical conceptualisation assumes them to be bidirectional). For instance, this means that it is not clear whether PSTs with more positive beliefs regarding subject matter knowledge acquire mathematical knowledge more easily, or whether having gained more profound subject matter knowledge leads to PSTs exhibiting more positive beliefs about SMK (or both), whether there is a third variable affecting both, or whether this is a spurious correlation.

The intervention group showed a somewhat different structure at T1 compared to the total sample and the control group: While PSTs of the intervention group exhibited a significant correlation between SMK scores and beliefs scores, no significant correlation was found between subject matter noticing and either of the two variables. (In the intervention group, at T1, there appear to have been participants with either rather positive beliefs who achieved only low noticing scores despite this, or participants with negative beliefs who nevertheless showed high noticing skills, or both.)

The results indicate some possible self-selection. Since participants knew what the seminar was going to be about, I suspected that possibly, participants with already positive dispositions regarding SMK in general had chosen to attend who were interested in discovering ways of applying their knowledge, but could not do so (yet). SMK and beliefs scores being correlated, it would also make sense that these participants may bring along a high level of SMK which they then struggled to relate to performing teacher tasks.

Situation at T2

In the total sample, all three correlations are still significant at T2. Considering the correlations of either experimental group separately, however, it becomes apparent that this seems to be the result of opposing changes in the intervention group and the control group, respectively.

On the one hand, the control group at T2 did not show any significant correlation anymore between subject matter noticing and SMK or between subject matter noticing and beliefs. Only the correlation between SMK and beliefs remained significant (cf. the situation found in the intervention group at T1).

This is, however, made up for by the intervention group's variable structure: In contrast to T1, now all three variables are significantly correlated, in particular, there is now a significant correlation between participants' SMK and noticing scores. One interpretation could be that now those participants who brought along high SMK are able to put that SMK to use when performing teacher tasks that allow them to exhibit subject matter noticing, whereas those participants with less prerequisite SMK were limited in the amount of subject matter noticing which they could develop as a result of the intervention.

In summary, while the intervention does not appear to have significantly changed participants' beliefs, it may have helped participants activate the potential of already held beliefs regarding subject matter in a way that manifested in more professional teaching actions. This is just an emergent hypothesis, however, that would need to be further explored.

Considering the correlations between the three corners of the SMK – beliefs – noticing triangle, noticing no longer seemed to be meaningfully related to the subject matter knowledge or beliefs about subject matter which the participants of the control group held. Again, one might hypothesise, that participants had stopped showcasing the maximum level of subject matter noticing that might have been possible based on their prerequisite cognitive and affective qualities. Relating these results back to what we know about the lack of consideration of subject matter concerns in mentoring conversations during practical phases of teacher training (Strong & Baron, 2004; Crasborn et al., 2011; Futter, 2016), the impression that exposing preservice teachers to the practicalities of classroom teaching too soon (Dewey, 1904; Shulman, 1998) (or, regardless of the time frame, without proper guidance) could have detrimental effects is emphasized by these results.

6.1.5 Research question Q5

- **Q_5: What do PSTs think about the relevance of advanced mathematics for their future profession after having participated in the intervention?**

Thematic analysis of the reflective essays revealed that while the majority of intervention group participants voiced positive or mostly positive impressions regarding the relevance of university level mathematics for their future profession, there remained almost a quarter of PSTs still holding explicitly negative opinions on this matter. As with the quantitative results of the beliefs questionnaire, this again points to the *stability* of certain beliefs (Pajares, 1992) despite concerted efforts to expose PSTs to evidence of the contrary. The most recently published study (Eichler & Isaev, 2023) undertaken with the questionnaire concerning the double discontinuity, for example, shows that sometimes, even when significant results are attained, the margin of success is rather small – PSTs beliefs were not improved, but the characteristic worsening of beliefs over the course of the semester could be avoided by having participants attend the intervention.

Overall, the analysis of the texts revealed that many PSTs were able to name a variety of reasons for their opinions about advanced subject matter courses, whether those were positive or negative. PSTs who held positive views on the matter named a breadth of reasons that ranged from principled stances about the person of the mathematics teacher and which personal prerequisites they should bring to the table all the way to specific teaching actions which PSTs felt required an advanced understanding of mathematical subject matter. The latter was especially meaningful as PSTs have been shown to experience difficulties relating the importance of knowing mathematics to specific tasks of a teacher rather than general pedagogical principles (Zazkis & Leikin, 2010).

As such, the assertion that prospective teachers in general underestimate the demands of their future profession (Hefendehl-Hebeker, 2013; Becher & Biehler, 2015) is not backed by the content of the reflective essays. Most PSTs displayed a reasonably clear understanding (especially considering that many of them had not yet taught and were asked about *supposed* benefits to their *future* profession) of the benefits of acquiring subject matter knowledge that goes beyond the mathematics taught directly in secondary school.

Maybe even more interesting is that the PSTs with less favourable views also provided a multitude of reasons for their opinions. Of course, as with the participants who merely asserted the importance of subject matter without providing any details on why they might think so, some participants were not able to argue exactly why they thought subject matter related courses were not very relevant to their profession. The rest of the PSTs with unfavourable opinions, however, often voiced nuanced views.

For example, while some participants asserted that "too much" subject matter was covered in general (especially considering what they believed the typical students needed), multiple participants problematised the trade-off between the amount of courses on advanced mathematical subject matter and coursework dedicated to school mathematics, teaching methods or practical experiences. Some of the PSTs who were already teaching specifically lamented feeling not adately prepared for their work life, having to "catch up" on learning about how to teach mathematics on the job.

Similarly to what Bungartz and Wynands (1998) reported, oftentimes participants' statements about the perceived lack of relevance of mathematics-heavy courses were really about the fact that PSTs did *not* feel more mathematically proficient after passing those courses. They reported having trouble understanding the material presented to them and/or retaining whatever knowledge they had taken away from the experience.

The most common reason however, which goes in line with the assumptions of the 4C/ID framework (van Merriënboer & Kirschner, 2017) with regard to traditional learning environments, is that PSTs did not feel equipped to transfer whatever they had learnt in those courses to their professional actions, and as such did not feel like the courses had provided them with a lot of surplus value.

At the same time, some participants who already work as teachers part-time voiced opinions that are reminiscent of the argument (Dewey, 1904; Shulman, 1998) that exposing PSTs to the very immediate, practical challenges of school teaching too soon can have negative consequences as to their willingness and ability to master the underlying subject matter. These PSTs reported not having been able to apply their mathematical knowledge from university to their teaching (as per their own words) and thus conclude that advanced mathematical knowledge must not be very relevant to teaching "in the real world".

Overall, one main learning from these essays seems to be that most PSTs wish to feel confident in attending to their teacher tasks competently and effectively. They value subject matter courses of even advanced content when they feel that those help them be a "better" teacher (by being able to understand students better, by being able to represent their subject better, by being able to have greater insight into mathematics as a whole), and they are skeptical of subject matter courses in contexts where they feel like those courses are not helping them (because they struggle to understand the content and thus gain increased mathematical understanding, because they feel subject matter courses take up space in the curriculum at the cost of other relevant classes, because they don't see ways of using their advanced understanding in the classroom).

This instrumental focus seems to be a long way from empirical results (e.g. Pieper-Seier, 2002) and anecdotal evidence that PSTs become alienated from mathematics per se and are, after a while, not interested in gaining deeper subject matter understandings. Or at least, most of the alienation, if present, according to these essays appears to be a function of university subject matter education not living up to PSTs' expectations in enabling them to learn for themselves and help their future students learn, and not necessarily of a negative relationship with mathematics and the study of even advanced mathematics itself.

6.1.6 Research question Q6

- **Q_6: How did PSTs experience the situated intervention in relation to this topic?**

Many participants did not directly answer the question regarding the change in beliefs.

From the answers that did specifically address this topic, it can be gathered that nine of the 44 PSTs reported a positive change in beliefs for a multitude of different reasons: Some PSTs now saw the point of their past subject matter focused courses because the intervention had made them realise the mathematical demands of the teaching profession, which they had previously not acknowledged. Others noted that the intervention had finally shown them ways of activating and applying their mathematical knowledge during teaching tasks which they already experienced to be challenging.

There were both PSTs who reported appreciating the importance of those courses because the intervention had shown them how much more mathematically proficient they had become over the course of their studies, as well as already PSTs who had realised in the seminar how many mathematical mistakes they had been previously making in their teaching.

However, the latter kind of experience could also lead to a pessimistic takeaway, as was the case for the teacher who reported feeling even more disappointed in her subject matter focused courses, seeing how much she struggled with the mathematical requirements of the teacher tasks that were practiced.

While many PSTs did not state whether or not they saw the relevance of university level subject matter courses differently post-intervention, there were PSTs who explicitly gave reasons as to why their beliefs did *not* change. These participants usually reasoned that they had already gone into the seminar having highly positive beliefs that were only reinforced.

Considering the answers of PSTs as to which parts of the intervention contributed to any change in beliefs, participants noted some activities (such as having to think through the entire design process of a worksheet or reflecting on real student solutions and practicing how to interpret them), mathematical ideas that stuck with them (the limit of the periodic number 0.999… or jump discontinuities versus graphs on disconnected domains) and specific ways of working in the seminar (working with leading questions to solve a complex task, or exchanging ideas with another group who had thought about the same task).

One main takeaway seems to be that depending on the specific circumstances of a PST, the same intervention can lead to a multitude of outcomes regarding those PST's beliefs. Importantly, while the majority of participants who talked about beliefs changes at all did so in a positive direction, the reasons for and the mechanisms behind those beliefs changes seem varied. Some were straightforward (e.g. being confronted with mathematically challenging teacher tasks having an effect on one's beliefs as to how much subject matter knowledge is needed as a teacher), others read more like a detour via one of the other two corners of the noticing triangle (e.g. learning to notice opportunities for applying mathematical knowledge leading to more positive views on the usefulness of that knowledge, or realising the limits of one's own personal mathematical knowledge and thus gaining an appreciation for the need for advanced subject matter education).

All of this suggests that despite a tendency for highly positive reception of the seminar, a clear mechanism as to how beliefs changes can be achieved for PSTs with negative beliefs about the relevance of advanced subject matter education cannot be deduced.

Furthermore, it would have been especially important for a more comprehensive picture to find out more about the participants' who did not specifically state anything about changes in their beliefs. If those participants' beliefs remained the same (as the quantitative data suggests), it would be interesting to gain some insight into what prerequisites on their part were not given that allowed other participants to change their minds at least partially, or, conversely (and, from a design pespective, more importantly), what features of the intervention were missing for these participants to be reached.

6.2 Limitations of the study

Because of the lack of randomisation, self-selection effects and resulting systematic differences that were not tested for in the participants of the intervention group versus the participants of the control group may have influenced the observed outcome. While no baseline differences between samples were observed in the main variables noticing and beliefs or the control variable SMK, other differences may have been present.

The control group was made up of Master's students who can be assumed to be quite representative for their cohort, since in general, every Master's student has to complete the seminar for teaching practice. Within that course, the specific group they are assigned to is further chosen by administrators based on the mentor teacher's timetables, so student choice does not factor into this decision.

On the other hand, the seminar in which the intervention was implemented is established in the curriculum as an elective for Master's students. This means that a self-selection effect could likely have occurred here, especially because the contents of the seminar and its aims of providing practical applications of university mathematics to teaching practice were communicated in advance. Furthermore, even though the vast majority of students in the intervention group were Master's students (like in the control group), the elective could also be taken as a Bachelor's student, which was the case for 12 participants.

The implications of these factors are unclear because apart from the self-selection effect contemplated above (which would probably make significant results more likely), many of the factors could influence results in both ways. For example, while Bachelor's students had taken fewer classes in analysis altogether (there is one joint course on multivariable linear algebra and analysis in first semester of the Master programme), the course in which they learnt the prerequisite knowledge for the activities in the seminar (one-dimensional analysis) was a more distant memory for the Master's students. Second, knowing about the aims of the elective seminar may have attracted both students with an above-average willingness to learn, but it is also plausible that the description sounded especially attractive for disillusioned and skeptical students with a lack of SMK who felt the lack of theory-practice integration quite strongly.

As mentioned in the discussion of results, social desirability bias, especially in the intervention group which was (partially) taught by the researcher herself, cannot be disregarded as an influencing factor. This is especially true in the case of the qualitative data, which was collected non-anonymously. Additionally, no reflective essays were collected from participants of the control group, nor is there qualitative data about participants' views *before* attending the seminar. As such, the contents of the reflective essays as to whether the seminar has achieved any changes in beliefs or competencies should be interpreted with these restrictions in mind. I had mainly collected the qualitative data in order to gain some contextual insight on the results of the main measuring instrument of my study, the quantitative questionnaire. Further, only the quantifying overview step of the qualitative analysis was able to be recoded by another researcher. The categorisation scheme as arrived at via the thematic analysis has hence not been tested for interrater reliability.

Another factor that has to be noted is that the dissertation project took place during the COVID pandemic. While all necessary precautions were taken to be able to hold the seminar in presence according to the policies of the University of Vienna, the last third of the intervention for the first cohort had to be conducted online due to lockdown measures. Furthermore, while all intervention groups filled out the pre- and posttest questionnaires in presence, for the same reason, the first cohort of the control group submitted their posttest questionnaire via Moodle. The posttest was implemented in such a way as to guarantee test modalities as close to regular circumstances as possible (including anonymity), however, it is possible that this change in setting led to some of the decreases in control group participants' willingness or ability to showcase their subject matter noticing.

As there was no appropriate and previously validated test instrument available to measure subject matter noticing, one had to be designed and piloted before the main study could begin. While the basis in multiple expert ratings was selected to increase validity and the resulting extensive coding guidelines resulted in highly satisfactory interrater reliability, the noticing questionnaire was only piloted on a limited sample of 14 Bachelor's students. An additional measure which may have been taken to validate the questionnaire would have been to assess participants' noticing through some other measure (e.g. interviews) and see if the results of the noticing questionnaire match the judgment arrived at by other measures of noticing. However, for reasons of scope, the pilot study was mainly focused on feasibility.

Many typical quantitative measures assessing e.g. reliability and dimensionality of my measurement tools were not conducted due to the low number of participants in the study. Furthermore, especially in case of the SMK questionnaire, only a very rudimentary measure of relevant knowledge of analysis was taken. As such, all results need to be considered critically and interpreted carefully in light of possible measurement errors and questions of validity.

As the noticing questionnaire specifically asked about typical teacher tasks in the domain of analysis and mainly involved teacher tasks which had been practiced in the intervention, no statements about the transferability of subject matter noticing to other domains (such as algebra or stochastics) or other teacher tasks (such as orchestrating classroom discussions) can be inferred. While initially the fourth item on the noticing questionnaire had been included in order to test transferability to a teacher task not specifically practiced in the intervention, I did not pursue this after it became apparent that empirical item difficulties between the four noticing items were vastly different even at the time of the pretest. (If I had had a larger sample, relative gains per item could have been investigated, but even then, considering the fluctuation between empirical item difficulties that may well have resulted from different SMK prerequisites of individual participants made such an detailed analysis appear a little dubious to me. If I had had multiple items assessing the noticing of each teacher task, this would have been a different question. But again, due to time constraints for the pre/posttests and the constraints on the scope of the project overall, a much more extensive noticing questionnaire had not really been within the realm of possibility in this particular implementation of the research project.)

While I have hypothesized about some of the reasons why there seems to be a discrepancy between the null results of the beliefs questionnaire and the predominantly positive impressions participants took from the seminar (even those who reportedly did not change their beliefs regarding the mathematics-related part of their training), the study at hand cannot shed light on whether those hypotheses are correct. If the qualitative data had allowed interaction with participants (e.g. if I had conducted interviews post-intervention instead of collecting the essays), some of the questions (as to the possible need for further distinctions in beliefs about subject matter) that only arose post-hoc may have been answered.

Lastly, even though noticing results were promising, participants' noticing skills were only evaluated at one time point directly after finishing the intervention. Additionally, the pretests and posttest were conducted in the same situated but ultimately artificial environment as the intervention itself. Whether the gains in subject matter noticing really do transfer to real teaching action the classroom context and whether they are stable over time cannot be inferred from this data.

All these considerations build the basis for the recommendations I deduce for further research into preservice teachers' subject matter noticing and their beliefs regarding the utility of studying advanced mathematics during their teacher training.

6.3 Directions for further research

For more robust results, it would be favourable to see this kind of research conducted in a randomised setting. Additionally, being able to observe the stability of results across multiple cohorts rather than just two may further strengthen any interpretation. Similarly, the scalability of any promising intervention depends on being able to systematically implement it regardless of the persons conducting the seminar. It would be interesting to see whether similar results could be achieved by another lecturer using the learning tasks designed.

It would be very interesting to see whether any achieved “inclination” to attend to subject matter considerations would transfer to other tasks of teaching and other subject matter areas, or whether – or rather, to what extent – preservice teachers really need to be given the tools to apply their domain knowledge in specific instances. For example, it is possible that some participants in the intervention are now more likely in general to critically assess subject matter aspects of certain textbook passages or worksheets, regardless of topic. However, even though they may have become aware that being careful with one’s metaphors and the way one phrases certain explanations is crucial, it’s unclear whether they would know how to use their subject matter knowledge of linear algebra or geometry to spot potential sources of misconceptions without having practiced noticing in that area of mathematics. (Wasserman et al. (2019) had shown promising results regarding this transfer for the skill of paying “attention to scope”.)

Furthermore, an intervention which makes use of actual teaching situations which the participants experience as teachers and then reflect on may shed light on the benefits of an added degree of situatedness. For example, in the control group’s seminar accompanying their teaching practice, one could attempt to create an even more situated version of this intervention. Additionally, projects like this may also allow for evaluations of the outcome based on measures of real teaching activity.

Qualitative analyses of the ways in which preservice teachers change their ways of noticing subject matter and of using subject matter to perform teacher tasks would also be instructive, as they might reveal in which ways lack of prerequisite knowledge ends up being the limiting factor and in which ways prerequisite knowledge is available but simply not used. If the latter is primarily the case, one might investigate whether this knowledge is not used due to a belief that it would not be useful, due to a lack of understanding of how one might proceed in doing so, or other reasons.

6.4 Implications for teacher training

The positive results obtained for subject matter noticing combined with the positive reception of the intervention even by those participants who reported unchanged negative beliefs regarding mathematics-related parts of their studies suggest that participants are not necessarily as alienated (cf. Pieper-Seier, 2002) from the subject of mathematics itself as one might have feared – or at least the desire to meaningfully engage with mathematics is still present within them.

When engaging with subject matter in a way that attempts to frame it as connected to the work of teaching (after all, the learning tasks in this study had been text vignettes, so it seems a rather low degree of actual situatedness is necessary as long as the situations depicted in the artificial seminar environment are plausibly authentic), preservice teachers were able to experience grappling with subject matter questions as worthwhile and useful to their teacher education. This means PSTs *can* be motivated and enabled to increase the extent to which they include subject matter considerations into their performance of teacher tasks. The improvements displayed on the posttest suggest that many preservice teachers' subject matter noticing is nowhere near as developed as would be possible given their prerequisites in both subject matter knowledge and beliefs. It seems there is much potential to be activated by purposefully adding such situated elements to teacher education programmes.

In turn, this kind of experience of being able to purposefully "act mathematically" as a teacher may provide PSTs with greater feelings of confidence, both with regard to their subject matter knowledge as well as with regard to their competence as (prospective) teachers more generally. While there is no data to compare it to, the reflective essays of the intervention group participants written after the posttest hint at the fact that many preservice teachers *are* capable of making a distinction between the importance of subject matter itself (mathematics) for the work of teaching and the importance of the subject matter education (mathematics courses) they remember having received as part of their teacher training programme. It seems that the right kind of relevance intervention may allow them to reflect on this difference and allow potential feelings of alienation from their subject itself to be replaced by a in all likelihood more appropriate and certainly less destructive sense of disappointment over curricular structures or the implementation of university courses of varying degrees of effectiveness.

Changing preservice teachers' beliefs about the value of mathematics courses which do not deliberately point out connections to the work of teaching and/or did not succeed in enabling preservice teachers to become familiar and actually master the material presented does not seem to be viable with this form of intervention. This is troubling as negative beliefs about those mathematics courses may inhibit the benefit those teachers may maximally gain from situated interventions.

Despite the intervention's relative success, in absolute terms, noticing scores remained rather low. This means that if the first years of a teacher training programme result in preservice teachers becoming disillusioned with mathematics, it may be hard to counter these effects with situated interventions aiming to bridge the gap between advanced mathematics and the work of teaching later on. Some relevance interventions and potentially more engaging ways of presenting advanced mathematics to preservice teachers may be necessary at an earlier point in order to counteract these effects and provide a more promising basis to work with later on.

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10 Supplements

10.1 Summary

Many preservice mathematics teachers (PSTs) have been known to struggle with the academic mathematics that is part of their university teacher training. This is often related back to educators' impressions that PSTs underestimate the mathematical demands of their future profession and experience their subject matter preparation as too extensive and unhelpful in preparing them for the challenges of teaching which they do anticipate. As a result, many PSTs develop negative beliefs about their teacher training and sometimes experience alienation from mathematics as a subject itself. In this dissertation, the ways in which preservice teachers perceive the subject matter inherent in typical teaching actions is conceptualised as a form of theme-specific teacher noticing, subject matter noticing. Subject matter noticing is assumed to influence and be influenced by a (preservice) teacher's subject matter knowledge (SMK) and their beliefs about subject matter. An situated intervention based on the Four Component Instructional Design framework was designed to allow secondary PSTs to practice consciously applying their SMK to effectively handle typical teacher tasks. Mixed ANOVA results of the quasi-experiment suggest that the intervention was highly effective in training PSTs' subject matter noticing but had no discernible impact on their subject matter beliefs. Correlational analyses indicate that the assumed relationships between PSTs' SMK, subject matter noticing and subject matter beliefs exist as conceptualised. Additionally gathered qualitative data opens up new questions as to how PSTs' problematised beliefs about subject matter can be most usefully conceptualised and measured.

10.2 Zusammenfassung

Es ist bekannt, dass angehende Lehrkräfte häufig Schwierigkeiten haben mit der Hochschulmathematik, der sie in der Lehramtsausbildung begegnen. Dies wird von Ausbildenden häufig damit in Verbindung gebracht, dass Lehramtsstudierende die mathematischen Anforderungen ihres künftigen Berufs unterschätzen und ihre fachmathematische Ausbildungskomponente deshalb als zu umfangreich und als unhilfreich als Vorbereitung auf die von ihnen antizipierten Herausforderungen des Unterrichtens betrachten. Dadurch entwickeln viele Lehramtsstudierende negative Einstellungen zur Lehramtsausbildung und manchmal auch zum Fach Mathematik selbst. In dieser Dissertation wird die Weise, in der Lehrpersonen die inhärenten fachmathematischen Bestandteile typischer Lehrtätigkeiten wahrnehmen als eine Art des themenspezifischen Noticings konzeptualisiert, als fachmathematisches Noticing. Es wird angenommen, dass fachmathematisches Noticing vom Fachwissen (FW) und von den Einstellungen zum Fachwissen beeinflusst wird und diese beeinflusst. Eine situierte Intervention auf Basis des Four Component Instructional Design Frameworks wurde entworfen, um Lehramtsstudierende das bewusste Anwenden ihres FW bei der effektiven Ausführung typischer Lehrtätigkeiten trainieren zu lassen. Ergebnisse einer Mixed ANOVA zur Evaluation der quasi-experimentellen Studie lassen darauf schließen, dass die Intervention das fachmathematische Noticing der Studierenden erfolgreich verbessert hat, jedoch keine feststellbare Auswirkung auf die Einstellungen der Studierenden zur Fachmathematik hatte. Korrelationsanalysen bestätigen die in der theoretischen Konzeptualisierung angenommenen Beziehungen zwischen FW, fachmathematischem Noticing und Einstellungen der Studierenden zur Fachmathematik. Zusätzlich erhobene qualitative Daten eröffnen Fragen zur sinnvollen Konzeptualisierung und adäquaten Messung der problematisierten Einstellungen von Lehramtsstudierenden zur Fachmathematik.

10.3 Pretest and posttest questionnaire

Sehr geehrte Teilnehmer*innen,

vielen Dank für Ihre Bereitschaft, diesen Fragebogen zu bearbeiten! Damit leisten Sie einen wichtigen Beitrag zu meinem Dissertationsprojekt.

Der Fragebogen besteht aus vier Teilen:

- In Teil 1 sind wir an Ihrer Wahrnehmung als (zukünftige) Mathematiklehrende interessiert.
- In Teil 2 möchten wir Teile Ihres fachlichen Hintergrundwissens erheben.
- In Teil 3 werden gewisse Einstellungen und Meinungen abgefragt.
- In Teil 4 werden abschließend einige Daten zu Ihrer Person erhoben.

Wichtige Anmerkungen:

- Die vier Teile sollten jeweils <u>nacheinander</u> bearbeitet werden und werden Ihnen deshalb in dieser Reihenfolge ausgeteilt. Bitte geben Sie die vier Fragebogenteile am Ende gesammelt ab.
- Für die Beantwortung von Teil 1 – dem Hauptteil der Befragung – sind ungefähr <u>60 Minuten</u> vorgesehen. Die Bearbeitung der restlichen Teile nimmt erfahrungsgemäß insgesamt rund <u>15 Minuten</u> in Anspruch.
- In Teil 4 werden <u>keine</u> Daten erhoben, die eine persönliche Identifikation ermöglichen. Um ggf. Ergebnisse mehrerer Erhebungen zusammenführen zu können, wird in diesem Teil für Sie ein Pseudonym aus Buchstaben und Zahlen generiert.

Bei Unklarheiten können Sie sich sowohl vor als auch während der Bearbeitungszeit gerne an mich wenden.

Herzlichen Dank!

Teil 1: Professionelle Wahrnehmung

Auf den nächsten Seiten finden Sie vier Items, zu denen Ihnen jeweils zwei Fragen gestellt werden. Bitte beantworten Sie diese in den dafür vorgesehenen Bereichen des beiliegenden Antwortbogens. Pro Item haben Sie ungefähr 15 Minuten Zeit – lesen Sie sich daher jedes Item in Ruhe durch und versuchen Sie, Ihre Antworten möglichst detailliert zu verfassen.

Um Ihnen Umblättern zu ersparen, ist der Antwortbogen getrennt abgeheftet. Bitte achten Sie auf die korrekte Zuordnung Ihrer Antworten, um zu jedem Item die richtigen Fragestellungen zu bearbeiten.

Item A

1. Sei f eine Funktion, die durch $f(x) = \sin(x) + \cos(x)$ definiert ist, und sei g eine Funktion, die durch $g(u) = \sin(u) + \cos(u)$ definiert ist, für alle reellen Zahlen x und u. **Welche der unten angeführten Aussagen ist richtig?**
   (a) f und g sind die gleiche Funktion.
   (b) Falls x und u verschieden sind, dann sind auch f und g unterschiedliche Funktionen.
   (c) Es ist nicht genügend Information vorhanden, um festzustellen, ob f und g gleich sind.
2. **Wahr oder falsch?** Wenn $f(x) = \frac{x^2-4}{x-2}$ und $g(x) = x + 2$, dann kann man sagen, dass die Funktionen f und g gleich sind.

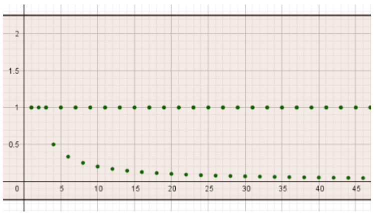
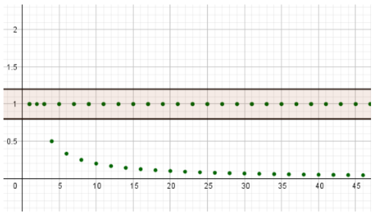
Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie sowohl an die mathematischen Voraussetzungen, die man zur Lösung der Aufgabe benötigt, als auch an die mathematischen Inhalte, die mit dieser Aufgabe vermittelt oder gefestigt werden können.)

Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten diese Aufgabe verwenden. Bitte skizzieren Sie den Aufbau einer Unterrichtssequenz, im Zuge derer Sie diese Aufgabe einsetzen könnten, und begründen Sie Ihre Entscheidung.

Item B

Im Rahmen einer Gruppenarbeit diskutieren die Lernenden den Unterschied der beiden folgenden Bedingungen in Bezug auf eine Zahlenfolge:

- **Bedingung A.** Für jeden ε-Streifen gilt: Es befinden sich unendlich viele Folgenglieder innerhalb des ε-Streifens, wenn dieser auf L zentriert ist.
- **Bedingung B.** Für jeden ε-Streifen gilt: Nur endlich viele Folgenglieder befinden sich außerhalb des ε-Streifens, wenn dieser auf L zentriert ist.

Dazu experimentieren sie mit unterschiedlich breiten ε-Streifen und betrachten ihre so generierten Grafiken (siehe oben). Sie kommen zu folgendem Schluss:

L1: Die zweite Bedingung schließt das hier jetzt aus [zeigt auf das zweite Bild].
L2: Hmhm [bejahend].
L1: Aber die erste Bedingung würde trotzdem gehen... weil unendlich viele Punkte enthalten sind. Also brauchen wir die zweite.
L2: Genau, das ist ja eigentlich der Grund, weshalb wir vorher gesagt haben, die Bedingung A ist falsch.
L1: Genau.
L2: Weil es Gegenbeispiele gibt, wie diese Folge eben.

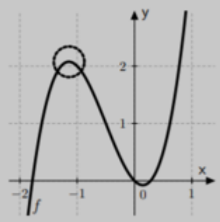
Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie an die gestellte Aufgabe, die mathematischen Kenntnisse der Lernenden und gegebenenfalls fehlendes mathematisches Wissen der Lernenden, das in diesem Kontext relevant wäre.)

Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten zu diesen Aussagen während einer Gruppenarbeitsphase Stellung nehmen. Bitte verfassen Sie eine Reaktion, die sich an die Lernenden richtet, und begründen Sie Ihre Entscheidung.

Item C

Bei der links abgebildeten Aufgabe kreuzen zwei Lernende die Antwort (b) an und geben dafür folgende Begründungen:

Unten siehst du den Graph einer Funktion f. Stell dir vor, du zoomst unendlich nah an die Stelle des Graphen heran, die durch den Mittelpunkt des dargestellten Kreises verläuft.



Kreuze an, wie der Graph in dem Bereich aussehen würde!

☐ (a)

☐ (b)

Begründe deine Antwort im folgenden Textfeld.

Die Extremstellen, in diesem Fall ein Hochpunkt besitzt keine Steigung. Im Ableitungsgraph befindet sich dort auch ein Nullpunkt.

Wenn man unendlich nach heran -zoomt, dann wird man eine gerade Linie sehen, weil der gesamte Graph nur aus kleinen Geraden besteht, die aneinander gereiht sind.

Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie an die gestellte Aufgabe, die mathematischen Kenntnisse der Lernenden und gegebenenfalls fehlendes Fachwissen der Lernenden, das in diesem Kontext relevant wäre.)

Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten via Moodle ein schriftliches Feedback auf diese Bearbeitungen geben. Bitte verfassen Sie eine Rückmeldung an die beiden Lernenden und begründen Sie Ihre Entscheidung.

Item D

Bei der Bearbeitung der untenstehenden Aufgabe murmelt eine Schülerin:

> Ich verstehe eh, wie ich rechnen muss, aber da steht ja minus... ich dachte, man summiert? Was ist das Integral jetzt eigentlich?

$$\int_0^2 x^3 dx \quad = \quad \frac{1}{4}x^4 \Big|_0^2 \quad = \quad \frac{16}{4} - 0 \quad = \quad 4$$

Als Erklärung zeichnet der Lehrer die Skizze eines Graphen, fügt zwischen Graph und x-Achse ein aufrecht stehendes Rechteck ein und erklärt:

> Okay, noch einmal: Also das ist unser y [zeigt auf die Höhe des Rechtecks], und das ist unsere kleine Veränderung in x [zeigt auf die Breite des Rechtecks und beschriftet es mit Δx]. Also ist *das* mal *das* der Flächeninhalt des Rechtecks. Und wenn wir noch einen kleinen Schritt weitergehen, gibt uns das ein neues Rechteck, das wir dazu addieren [zeichnet zweites Rechteck rechts vom ersten Rechteck ein].
>
> Also im Grunde ist das Integral eine Annäherung an diese Kurve. Wenn du die Rechtecke unendlich klein machst, dann kommst du der Kurve näher und näher, während du diese kleinen Blöcke addierst... also weil du den Flächeninhalt dieser kleinen Blöcke berechnest und zusammenrechnest, hast du irgendwann den ganzen Flächeninhalt.

Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie sowohl an die mathematischen Hintergründe der Frage der Schülerin als auch an die mathematischen Inhalte, auf die der Lehrer in seiner Erklärung eingeht.)

Fragestellung 2: Stellen Sie sich vor, *Sie* sind die Lehrperson in dieser Klasse und möchten der Schülerin antworten. Bitte verfassen Sie Ihre eigene Antwort zu den Aussagen der Schülerin und begründen Sie Ihre Entscheidung.

Teil 1: Antwortbogen

Item A – Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie sowohl an die mathematischen Voraussetzungen, die man zur Lösung der Aufgabe benötigt, als auch an die mathematischen Inhalte, die mit dieser Aufgabe vermittelt oder gefestigt werden können.)

Item A – Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten diese Aufgabe verwenden. Bitte skizzieren Sie den Aufbau einer Unterrichtssequenz, im Zuge derer Sie diese Aufgabe einsetzen könnten, und begründen Sie Ihre Entscheidung.

Skizze/Aufbau der Unterrichtssequenz:

Begründung:

Item B – Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie an die gestellte Aufgabe, die mathematischen Kenntnisse der Lernenden und gegebenenfalls fehlendes mathematisches Wissen der Lernenden, das in diesem Kontext relevant wäre.)

Item B – Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten zu diesen Aussagen während einer Gruppenarbeitsphase Stellung nehmen. Bitte verfassen Sie eine Reaktion, die sich an die Lernenden richtet, und begründen Sie Ihre Entscheidung.

An die Lernenden gerichtete Reaktion: Begründung:

Item C – Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie an die gestellte Aufgabe, die mathematischen Kenntnisse der Lernenden und gegebenenfalls fehlendes Fachwissen der Lernenden, das in diesem Kontext relevant wäre.)

Item C – Fragestellung 2: Stellen Sie sich vor, Sie sind eine Lehrperson und möchten via Moodle ein schriftliches Feedback auf diese Bearbeitungen geben. Bitte verfassen Sie eine Rückmeldung an die beiden Lernenden und begründen Sie Ihre Entscheidung.

Feedback an die Lernenden: Begründung:

Item D – Fragestellung 1: Welche verschiedenen Inhalte mathematischen Fachwissens werden in diesem Item thematisiert? (Denken Sie sowohl an die mathematischen Hintergründe der Frage der Schülerin als auch an die mathematischen Inhalte, auf die der Lehrer in seiner Erklärung eingeht.)

Item D – Fragestellung 2: Stellen Sie sich vor, *Sie* sind die Lehrperson in dieser Klasse und möchten der Schülerin antworten. Bitte verfassen Sie Ihre eigene Antwort zu den Aussagen der Schülerin und begründen Sie Ihre Entscheidung.

Antwort an die Schülerin: Begründung:

Teil 2: Fachliches Hintergrundwissen

Auf den nächsten Seiten finden Sie acht Single-Choice-Fragen aus dem Bereich der Analysis. Bitte markieren Sie jeweils die Antwort, die Sie für richtig halten, mit einem Kreuz: ☒

Falls Sie eine Korrektur vornehmen wollen, füllen Sie das Kästchen bei der falschen Antwort komplett aus: ■ und setzen Sie ein Kreuz bei der neuen Antwort: ☒

Frage 1

Jemand sagt: "$f(x) = \frac{1}{x}$ ist monoton fallend."

☐ Dies ist wahr.

☐ Dies ist falsch, denn "monoton steigend" ist richtig.

☐ Dies ist falsch, denn bei Funktionen kann man nicht von Monotonie sprechen.

☐ Es ist nicht genügend Information gegeben, um eine Aussage über Monotonie zu machen.

Frage 2

Es sei eine Funktion $g: D \to \mathbb{R}$ an der Stelle $t = a$ nicht definiert. Dann

☐ existiert der Limes $\lim_{t \to a} g(t)$ nicht.

☐ könnte der Limes $\lim_{t \to a} g(t)$ gleich 0 sein.

☐ muss der Limes $\lim_{t \to a} g(t)$ gegen ∞ streben.

☐ ist keine der oben genannten Aussagen richtig.

Frage 3

Gegeben sei die Folge $\left(0, \frac{1}{1000}, 0, \frac{1}{1000}, 0, \frac{1}{1000}, \ldots\right)$. Welche der folgenden Aussagen ist korrekt?

☐ Alle Folgenglieder liegen in der ε-Umgebung $U_\varepsilon(0)$, wenn man $\varepsilon = \frac{1}{100}$ wählt, daher ist die Folge konvergent.

☐ In jeder ε-Umgebung $U_\varepsilon(0)$ liegen fast alle Folgenglieder, daher ist die Folge konvergent.

☐ Es gibt eine ε-Umgebung von 0, in der nur endlich viele Folgenglieder liegen, daher ist die Folge nicht konvergent.

☐ Es gibt eine ε-Umgebung von 0, in der nicht fast alle Folgenglieder liegen, daher ist die Folge nicht konvergent.

Frage 4

Betrachten Sie die Aussagen (A) und (B):

(A) $\forall \varepsilon > 0 \; \exists n \in \mathbb{N}: \frac{1}{\mathrm{n}} < \varepsilon.$

(B) $\exists n \in \mathbb{N} \; \forall \varepsilon > 0: \frac{1}{n} < \varepsilon.$

Was trifft zu?

☐ Die zwei Aussagen sind äquivalent, und sie sind beide wahr.

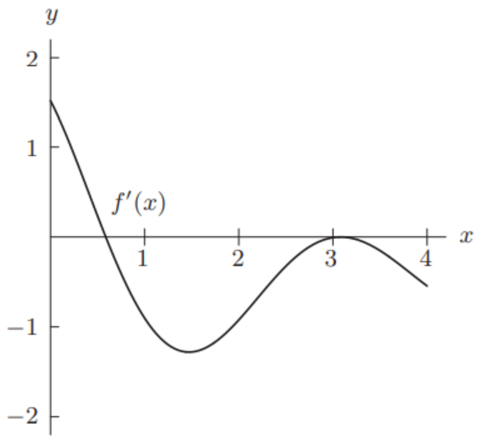
☐ Die zwei Aussagen sind äquivalent, und sie sind beide falsch.

☐ Die Aussagen sind nicht äquivalent, und nur (A) ist wahr.

☐ Die Aussagen sind nicht äquivalent, und nur (B) ist wahr.

Frage 5

Die untenstehende Abbildung stellt den Graphen von f' dar. Welche der folgenden Aussagen über die Funktion f trifft dann zu?



☐ Die Ableitung von f ist an zwei Stellen gleich 0, und beides sind lokale Maximumstellen von f.

☐ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Maximumstelle von f, während die andere eine lokale Minimumstelle von f ist.

☐ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Maximumstelle von f, während die andere weder eine lokale Maximum- noch eine lokale Minimumstelle von f ist.

☐ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Minimumstelle von f, während die andere weder eine lokale Maximum- noch eine lokale Minimumstelle von f ist.

☐ Die Ableitung von f ist nur an einer Stelle gleich 0, und es handelt sich um eine lokale Minimumstelle von f.

Frage 6

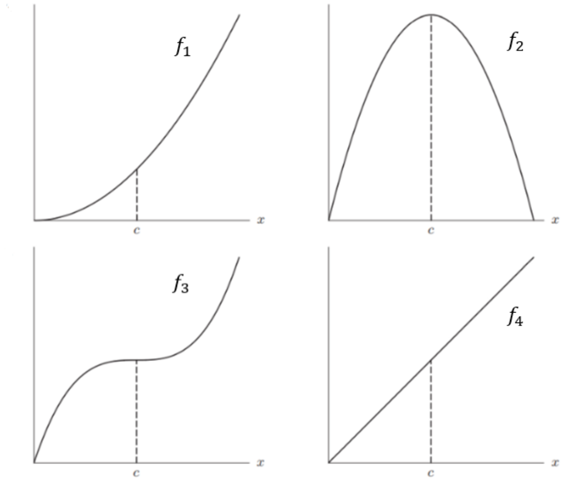
Stellen Sie sich vor, man möchte die unten abgebildeten Polynomfunktionen f_1, f_2, f_3 und f_4 unter Verwendung der jeweils ersten Ableitung $f_n{}'$ $(n \in \{1,2,3,4\})$ in der Nähe der Stelle $x = c$ lokal approximieren. Bei welcher der vier Funktionen würde eine solche Approximation den geringsten Fehler aufweisen?

☐ bei f_1

☐ bei f_2

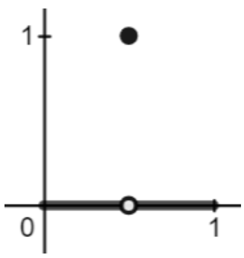
☐ bei f_3

☐ bei f_4



Frage 7

Die Funktion $f\colon [0,1] \to \mathbb{R}$ mit dem Graphen



☐ ist nicht integrierbar, weil sie nicht stetig ist.

☐ ist nicht integrierbar, weil es Untersummen gibt, die gleich 0 sind, aber alle Obersummen gleich 1 sind.

☐ ist integrierbar, weil alle Untersummen gleich 0 sind und es Obersummen gibt, die beliebig nahe an 0 sind.

☐ ist integrierbar, weil es Obersummen gibt, die gleich 0 sind, und alle Untersummen auch gleich 0 sind.

Frage 8

Lesen Sie die folgenden vier Aussagen und wählen Sie dann die richtige Antwortmöglichkeit aus:

Sei f stetig auf dem Intervall $[a, b]$. Dann gilt:

a) $\int_a^b f(x)\,dx$ ist eine Fläche, die vom Graphen von f, der x-Achse und den Geraden $x = a$ sowie $x = b$ eingegrenzt wird.
b) $\int_a^b f(x)\,dx$ ist eine Zahl.
c) $\int_a^b f(x)\,dx$ ist eine Stammfunktion von f.
d) $\int_a^b f(x)\,dx$ existiert möglicherweise nicht.

Antwortmöglichkeiten:

☐ Nur b) ist richtig.

☐ Nur a) und b) sind richtig.

☐ Nur a) und c) sind richtig.

☐ Nur d) ist richtig.

Teil 3: Einstellungen und Meinungen

In diesem Teil sind wir an Ihrer persönlichen Einschätzung interessiert, hier gibt es kein “richtig” oder “falsch” (falls Sie dennoch eine Antwort korrigieren möchten, gehen Sie bitte so vor, wie in der Präambel zu Teil 2 beschrieben wurde).

Einige Fragen oder Aussagen werden Ihnen sehr ähnlich vorkommen, sie sind jedoch nicht identisch. Die vermeintlichen Doppelungen sind absichtlich so gestaltet und erhöhen die Aussagekraft der erhobenen Daten. Lesen Sie deshalb jede Aussage zunächst genau durch und antworten Sie dann möglichst spontan und ehrlich.

	trifft gar nicht zu	trifft nicht zu	trifft eher nicht zu	trifft eher zu	trifft zu	trifft voll zu
Die universitäre Mathematik hat inhaltlich kaum etwas mit der Schulmathematik zu tun.	☐	☐	☐	☐	☐	☐
Es gibt in der universitären Mathematik zu wenig inhaltliche Bezüge zur Schule.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik ist sehr nützlich für den Lehrberuf.	☐	☐	☐	☐	☐	☐
Ohne die universitäre Mathematik könnte ich das Schulfach Mathematik kaum unterrichten.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik werde ich nach meinem Studium kaum wieder benötigen.	☐	☐	☐	☐	☐	☐
Schulmathematik und universitäre Mathematik sind inhaltlich zwei verschiedene Welten.	☐	☐	☐	☐	☐	☐
Es gibt viele inhaltliche Verbindungen zwischen Schulmathematik und der universitären Mathematik.	☐	☐	☐	☐	☐	☐
Schulmathematik und universitäre Mathematik sind inhaltlich aufeinander bezogen.	☐	☐	☐	☐	☐	☐

	trifft gar nicht zu	trifft nicht zu	trifft eher nicht zu	trifft eher zu	trifft zu	trifft voll zu
Das Lernen von Mathematik an der Universität ist nicht so wichtig für den Lehrberuf.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik weist viele inhaltliche Parallelen zur Schulmathematik auf.	☐	☐	☐	☐	☐	☐
Auf die Inhalte der universitären Mathematik werde ich nach meinem Studium noch häufiger zurückgreifen.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik ist nicht relevant für den Lehrberuf.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik bringt mir nichts für meine spätere Tätigkeit als Lehrkraft in der Schule.	☐	☐	☐	☐	☐	☐
Das Lernen von Mathematik an der Universität ist im Hinblick auf Schule reine Zeitverschwendung.	☐	☐	☐	☐	☐	☐
Zwischen der Schulmathematik und der universitären Mathematik gibt es eine unüberwindbare Kluft.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik bleibt nach meinem Studium eine Erinnerung, die auf meinen Unterricht keinen Einfluss hat.	☐	☐	☐	☐	☐	☐
Die universitäre Mathematik hängt inhaltlich stark mit der Schulmathematik zusammen.	☐	☐	☐	☐	☐	☐

Die Relevanz der universitären Mathematik für den Lehrberuf ist ...					
sehr niedrig	niedrig	eher niedrig	eher hoch	hoch	sehr hoch
☐	☐	☐	☐	☐	☐

Teil 4: Persönliche Daten

Zum Schluss stellen wir Ihnen noch einige Fragen über Ihr Studium und Ihre Person, um Effekte von z.B. Geschlecht, Studienfortschritt und bisheriger Unterrichtserfahrung auf Ihre Antworten in den anderen Fragebogenteilen untersuchen zu können.

Welche der angeführten Lehrveranstaltungen haben Sie bereits positiv abgeschlossen (bzw. wurden Ihnen ggf. bereits angerechnet)?
□ VO Einführung in die Mathematik (StEOP)
□ VO Analysis in einer Variable für das Lehramt
□ UE zu Analysis in einer Variable für das Lehramt
□ VO Schulmathematik Analysis
□ UE zu Schulmathematik Analysis
□ VO Lineare Algebra und Analysis in mehreren Variablen für das Lehramt (gemeint ist die LV aus dem MEd-Curriculum – NICHT: "Geometrie und Lineare Algebra für das Lehramt" aus dem BEd-Curriculum)
□ UE zu Lineare Algebra und Analysis in mehreren Variablen für das Lehramt (LV aus dem MEd, siehe oben)

Haben Sie <u>im vergangenen Semester</u> Mathematik im Regelschulunterricht (außerhalb von etwaigen Praktika im Rahmen Ihrer universitären Lehramtsausbildung) unterrichtet?
□ Ja, mehr als drei Monate lang.
□ Ja, bis zu drei Monate lang.
□ Nein.

<u>Falls ja:</u> **Haben Sie im vergangenen Semester in Ihrem Mathematikunterricht Themengebiete behandelt, die sich der Analysis zuordnen lassen?**
□ Ja.
□ Nein.
□ Ich bin mir nicht sicher, ob dieses Thema zur Analysis zählt: ____

Haben Sie <u>im vergangenen Semester</u> Unterrichtserfahrung in Mathematik in anderen Settings, wie z.B. Lerncamps oder private Nachhilfe, gesammelt?
□ Ja, ich habe einmal/ab und zu in einem solchen Setting Mathematik unterrichtet.
□ Ja, ich habe regelmäßig oder häufig in solchen Settings unterrichtet.
□ Nein.

<u>Falls ja:</u> **Haben Sie im vergangenen Semester in einem dieser außerschulischen Lernsettings Themengebiete der Analysis besprochen?**
□ Ja.
□ Nein.
□ Ich bin mir nicht sicher, ob dieses Thema zur Analysis zählt: ____

Welche Note haben Sie bei der schriftlichen Mathematikmatura erhalten?
□ 1
□ 2
□ 3
□ 4
□ Ich habe nicht schriftlich in Mathematik maturiert. Meine Note bei der mündlichen Maturaprüfung in Mathematik war: ___

Welche Note haben Sie im Abschlusszeugnis vor der Matura im Fach Mathematik erhalten?
□ 1
□ 2
□ 3
□ 4
□ 5

10.4 Coding manuals for the noticing section

Codierleitfaden für den Fragebogen zum fachmathematischen Noticing – Fragestellung 1

Für jedes der vier Items ist auf den folgenden Seiten eine Liste von fachmathematischen Inhalten angeführt, die auf der Auswertung der Expert*innen-Ratings basieren. Jeder der fachmathematischen Inhalte wurde von mindestens zwei der sechs befragten Expert*innen genannt.

Die ersten, grau hinterlegten Inhalte der Liste sind die sog. Kerninhalte des jeweiligen Items – diese entsprechen dem Expert*innen-Konsens (d.h. sie wurden in allen Expert*innen-Antworten thematisiert). Je nach Item gibt es zwei oder drei solcher Kerninhalte. Ihre Nennung fließt verstärkt in die Scorevergabe hinein.

Es wird empfohlen, den Codierbogen im Anhang zu verwenden und in einem ersten Durchgang für alle Teilnehmer*innen jene Inhalte zu markieren, die bei der Beantwortung der Fragestellung 1 thematisiert werden – der Codierleitfaden bietet für jeden Code Anhaltspunkte, was dabei als „Thematisierung" zählen soll. Insbesondere die Anker- bzw. Abgrenzungsbeispiele aus den Expert*innenratings und der Pilotierung sollen dabei helfen, bei Grenzfällen eine Entscheidung zu treffen.

Anschließend können auf Basis des Codierbogens alle Scores berechnet werden. Grau hinterlegte Kerninhalte bringen dabei je 2 Punkte, alle anderen Inhalte je einen 1 Punkt. Je nachdem, ob mehr als die Hälfte der so möglichen Punkte erreicht wird, wird der Score 0 bzw. 1 vergeben. Wird mehr als die Hälfte der möglichen Punkte erreicht *und* wurden alle Kerninhalte genannt, so wird der Score auf 2 erhöht.

Eine Übersicht bietet folgende Tabelle:

Item	max. Punkte	Score: 0	Score: 1	Score: 2
A	8	0-4 Punkte, mind. ein Kerninhalt fehlt	5+ Punkte, mindestens ein Kerninhalt fehlt ODER 4 Punkte, 2 Kerninhalte genannt	5+ Punkte, alle Kerninhalte genannt
B	10	0-5 Punkte	6+ Punkte, mindestens ein Kerninhalte fehlt ODER 5 Punkte, 2 Kerninhalte genannt	6+ Punkte, alle Kerninhalte genannt
C	8	0-4 Punkte mind. ein Kerninhalt fehlt	5+ Punkte, mindestens ein Kerninhalt fehlt ODER 4 Punkte, alle Kerninhalte genannt	5+ Punkte, alle Kerninhalte genannt
D	8	0-4 Punkte	5+ Punkte, mindestens ein Kerninhalt fehlt ODER 4 Punkte, 2 Kerninhalte genannt	5+ Punkte, alle Kerninhalte genannt

Bemerkungen bzgl. Sprachqualität der Antworten:

Die Formulierungen „das Charakterisierende an einer Funktion" und „erkennen, was die Abhängigkeit von einer Variable bedeutet, und wie diese gewählt werden kann" sind suboptimal, was Klarheit und Eloquenz betrifft. Dennoch können sie den Inhalten „Funktionsbegriff" bzw. „mathematische Notation und Sprache" zugeordnet werden, da sie keine Fehlvorstellungen vermitteln und auch kein Hinweis für Fehlvorstellungen seitens der Proband*innen sind.

Hingegen ist die Ungenauigkeit „der Unterschied zwischen unendlichen und endlichen Folgengliedern sollte erarbeitet worden sein" missverständlich (es sind wahrscheinlich (un)endlich *viele* Folgenglieder gemeint, aber es wird etwas anderes gesagt) und kann nicht als Inhalt „(Un-)Endlichkeitsbegriff" zählen. Ähnlich verleitet „Umformen einer Funktion" als Ausdrucksweise dazu, eine Funktion mit einer als Term gegebenen Zuordnungsvorschrift gleichzusetzen.

Weitere Beispiele:

- „die SuS sollen sich die Rechtecke als Annäherung an die Funktion vorstellen" (der/die Proband*in übernimmt fehlerhafte Formulierung aus dem Item ohne Kommentar)
- „Hoch- und Tiefpunkte und deren Steigung" (unpräzise und suggeriert, dass Hoch- und Tiefpunkte selbst eine Steigung besitzen)

Diese Antworten(teile) können deshalb nicht als Nennungen der Inhalte „Approximation" und „Steigung" gewertet werden.

Bemerkungen bzgl. mathematisch inkorrekten Aussagen

Aussagen, die offensichtlich machen, dass die Proband*innen den Sachverhalt im Item selbst nicht verstanden haben (z.B. die den Lernenden gestellte Aufgabe selbst falsch gelöst hätten), können selbstverständlich kein Teil von Noticing sein.

Eine Antwort zu Item A enthält beispielsweise den Satz „durch kürzen können sie dann erst sehen, dass die f(x) und g(x) die gleiche Funktion beschreiben". Dieser sagt erstens aus, dass der/die Proband*in der Auffassung ist, dass die richtige Antwort auf die zweite Frage in Item A „Ja" ist. Zweitens ist die Formulierung, dass f(x) und g(x) eine Funktion „beschreiben" unpräzise und irreführend und weist auf weitere Fehlvorstellungen der Verfasserin/des Verfassers hin.

Ebenso diagnostiziert ein/e Proband*in: „[Die Lernenden] können nicht erkennen, dass sich die beiden Bedingungen nicht ausschließen". Während hier die Rolle von Bedingungen thematisiert wird, kommt der Satz für den Inhalt „Logisches Schlussfolgern und Argumentieren" nicht in Frage, da die Argumentationsstruktur der Lernenden in Item B missverstanden wird. Die Lernenden untersuchen, welche Bedingung die stärkere und für ihre Zwecke ausreichende ist.

Solche Sätze bzw. Satzteile sind zu streichen. Es ist möglich, dass Proband*innen darin angesprochene Inhalte in den restlichen Teilen ihrer Antwort erneut und diesmal auf eine akzeptable Weise thematisieren, in diesem Fall wird der jeweilige Inhalt auf Basis jener unproblematischen Äußerungen als „genannt" gewertet.

Beispiel:

Eine Antwort beginnt mit „Den SuS muss auf jeden Fall die Grenzwertdefinition bekannt sein. Sie müssen wissen, dass in der E-Umgebung alle bis auf endlich viele Folgenglieder liegen." Drei Zeilen weiter steht dann der Satz: „Jede Folge hat eine eigene E-Umgebung, ich kann nicht jede für alle Folgen verwenden.

Für konvergente Folgen gilt, dass für alle Glieder der Folge existiert ab einem gewissen Index eine Epsilon-Umgebung, in der alle bis auf endlich viele Folgenglieder liegen."

Letzteren beiden Sätzen kann selbstverständlich kein Inhalt entnommen werden, und sie weisen Fehlvorstellungen hin, was die ε-N-Definition der Konvergenz betrifft. Trotzdem zählt der erste Satz als Erkennung und Nennung des relevanten Inhalts „Konvergenzbegriff". Eine Codierung könnte deshalb so aussehen:

Konvergenzbegriff

Den SuS muss auf jeden Fall die Grenzwertdefinition bekannt sein. Sie müssen wissen, dass in der E-Umgebung alle bis auf endlich viele Folgenglieder liegen. […] ~~Jede Folge hat eine eigene E-Umgebung, ich kann nicht jede für alle Folgen verwenden. Für konvergente Folgen gilt, dass für alle Glieder der Folge existiert ab einem gewissen Index eine Epsilon-Umgebung, in der alle bis auf endlich viele Folgenglieder liegen.~~

Einer der Gründe für diese Vorgehensweise ist, dass – wie auch bei den Expert*innenratings – die Antworten der Proband*innen unterschieldlich knapp gehalten sind. Da der Listenpunkt

- Grenzwertbegriff

in dieser Codierung als Inhalt zählt, obwohl sich dahinter eine komplett falsche Vorstellung dessen, was ein Grenzwert ist, verbergen könnte, soll die Ausführlichkeit von Antworten so nicht „bestraft" werden.

Ein zweiter Grund ist die Tatsache, dass Noticing unterschiedliche Qualitätsstufen hat, die in dieser Codierung nicht erfasst werden. So ist es durchaus interessant (und kann in einer qualitativen Analyse der Antworten ggf. ergänzt werden), zwischen einer reinen Nennung und unterschiedlich differenzierten Analysestufen zu unterscheiden. Ebenso ist es jedoch interessant – und das möchte diese Codierung der Fragestellung 1 erfassen – ob es überhaupt zu einer Erkennung und Nennung eines Inhalts kommt. Der/die Proband*in hat offensichtliche Verständnislücken hinsichtlich der ε-N-Definition des Grenzwerts, aber hat – im Gegensatz zu vielen anderen Proband*innen – identifiziert, dass es sich bei einer der Bedingungen um eine Charakterisierung des Grenzwerts handelt (und diese Tatsache erwähnenswert gefunden).

Item A

Abkürzung	Name	Ankerbeispiele aus den Expert*innenratings	(Abgrenzungs-)Beispiele aus der Pilotierung
FUN	Funktionsbegriff	• Definition von Funktionen • Kenntnis des Funktionsbegriffs • Begriff und Definition der reellwertigen Funktion in einer Variablen **Bemerkung:** Grundsätzlich sollte der Begriff einer Funktion irgendwo als Punkt für sich angeführt sein – es reicht nicht, dass das Wort „Funktion“ zur Formulierung von Kommentaren zum Operieren mit Funktionen, zu Eigenschaften spezieller Funktionen etc. verwendet wird.	„Grundlegendes Wissen über: Funktionen […]“ „Was ist eine Funktion?“ „Wissen über Funktionen als Abbildungen“ „Kriterien, die eine Funktion erfüllen muss“ „das Charakterisierende [sic] an einer Funktion“ „Grundwissen zu Funktionen“ „Verständnis für Funktionen“ „[...] zu wissen, was eine Funktion ist“ **NICHT:** „unterschiedliche Darstellung von Funktionen“ „das Arbeiten mit Funktionen“ „Wissen über Polynomfunktionen, rationale Funktionen, Winkelfunktionen“ „Kenntnisse von Sinus und Cosinus Funktion“ (keine Erwähnung des Funktionsbegriffs an sich und kein Bezug auf charakterisierende Eigenschaften einer Funktion)
DEF	Definitionsbereich	• Begriffswissen Definitionsbereich einer Funktion; Lücken im Definitionsbereich • Definitionsbereich als integraler Bestandteil des Funktionsbegriffs • Eine Funktion besteht nicht nur aus einer Zuordnungsvorschrift, sondern auch aus einer Definitions- und Zielmenge	„Was sind Definitions- und Zielmenge? Wofür sind sie wichtig?“ „(Grundvorstellungen zur) Definitionsmenge“ „der ganze Definitionsbereich [muss durchdacht werden]“ „Definitionsbereich sollte man kennen“ „die Definitionsbereiche und Wertebereiche“

NOT	Mathematische Notation und Sprache	• Schreibweise für die Definition einer Funktion • Irrelevanz der Variablenbezeichnung, Bedeutung des Allquantors • Funktionsterm als Möglichkeit zur Darstellung einer Zuordnungsvorschrift • Variablen [und] deren Bezeichnung	„richtig Lesen – mathematische Sprache verstehen“ „Fachvokabular“ „Variable und die Funktion können beliebig benannt werden“ „dass man 2 Funktionen auch als gleich bezeichnen kann, auch wenn sie unterschiedlich heißen“ „vorherrschenden Annahmen (Funktionen werden immer mit f bezeichnet, Variablen immer mit x) entgegen zu wirken“ „dass f(x) durchaus das Gleiche wie g(u) sein kann, wenn es dementsprechend definiert ist“ „wie [eine Variable] gewählt werden kann“ „unterschiedliche Darstellung von gleichen Funktionen, Verwenden von verschiedenen Variablen“ **NICHT:** „Wissen, dass 1 und die selbe Funktion auf verschiedene Arten darstellbar sind“ „[Funktionen und] deren Aussehen, Darstellung von Funktionen“ (zu ungenau – es ist insbesondere unklar, ob mit „Darstellung“ eine Notation gemeint ist, oder z.B. Arten, Funktionen zu visualisieren – mit einem Graphen, anhand einer Wertetabelle oder eines Pfeildiagramms, etc.)
EQU	Gleichheit von Funktionen	• Prüfkriterien für die Gleichheit von Funktionen • Definition der Gleichheit von Funktionen • Kriterien zur Feststellung bzw. Konventionen zur Festlegung der Gleichheit zweier Funktionen	„nur weil Funktionen das gleiche Bildungsgesetz für den Funktionswert haben, heißt das nicht, dass die Funktionen auch gleich sind“ „wann man zwei Funktionen als ‚gleich‘ bezeichnen kann“ „kontrollieren, ob zwei Funktionen ident sind“ „was bedeutet es, wenn zwei Funktionen gleich sind?“ „das die Funktionen zwar sehr ähnlich sind, jedoch immer alle Eventualitäten und der ganze Definitionsbereich durchdacht werden müssen“
VAR	Variablenbegriff- und Aspekte	• Umgang mit dem Aspektreichtum des Variablenbegriffs • Rolle und Aspekte der Funktionsvariablen • Rolle von x als Unbestimmte [vs.] die Rolle von x als konkret eingesetzten Wert	„Was ist eine (un)abhängige Variable?“ „Verständnis über Variablen“ „Verständnis des Variablenbegriffs“ „Wissen über Variablen“ **NICHT:** „Umformen von Gleichungen mit Variablen“ (Fokus nicht auf dem Begriff Variable, sondern auf zulässigen Operationen) „was die Abhängigkeit von einer Variable bedeutet“

			„was bewirken x und u?“ (zu vage/unklar) „Verwenden von verschiedenen Variablen“ (Fokus auf Notation, nicht auf Begriff oder Funktion der Variablen)
FWF	Unterscheidung: Funktionswert vs. Funktion	• Unterscheidung zwischen Funktionswert und Funktion • f(x) von f zu unterscheiden	„Unterscheidung zwischen Zuordnungsvorschrift, Funktionswert und Funktion“

Item B

Abkürzung	Name	Ankerbeispiele aus den Expert*innenratings	(Abgrenzungs-)Beispiele aus der Pilotierung
KON	Konvergenzbegriff, Definition/ Charakterisierungen	• Verständnis von Konvergenz und Divergenz von Folgen • die Konvergenzbedingung für Folgen • Bedingungen im Zusammenhang mit der Konvergenz einer Folge, Charakterisierungen von Folgenkonvergenz	„Unterschied […] Grenzwert ←→ Häufungspunkt“ „was es für eine Folge bedeutet, einen Grenzwert zu haben“ „Erfassen des Grenzwertbegriffs“ „Definition des Grenzwertes und der Konvergenz bzw. Divergenz“
FOL	(Zahlen)-Folgen als Begriff, ihre Interpretation und ihre Darstellung	• Interpretation einer Zahlenfolge als diskrete Funktion und Darstellung durch einen Funktionsgraphen • Alternierende Folgen, ggf. Teilfolgen • Verstehen des Begriffs „Zahlenfolge“ und die (ersten) Glieder einer solchen graphisch darstellen können **Bemerkung:** Grundsätzlich sollte der Begriff/die Definition „(Zahlen-)Folge“ irgendwo als Punkt für sich angeführt sein – es reicht nicht, dass das Wort zur Formulierung von Kommentaren zur Konvergenz, zu *Folgen*gliedern, zu Eigenschaften spezieller Folgen etc. verwendet wird.	„Definition von Folgen“ „Folgen“ „Folgenbegriff“ „Entwicklung von Zahlenfolgen“ „Kenntnisse von Folgen und auch von Zahlenfolgen“ „Grundverständnis für Folgen“ **NICHT:** „es hängt auch damit zusammen, welche Folgen von den Lernenden zur Veranschaulichung genutzt werden“ (Folgen werden im Rest der Antwort nicht erwähnt oder als Inhalt angeführt, sondern es kommt lediglich das Wort „Folge“ in diesem vagen Abschlusssatz vor)
LOG	Logisches Schlussfolgern und Argumentieren	• Unterschiede erkennen zwischen notwendigen und hinreichenden Bedingungen • Verneinung von mathematischen Aussagen, insbesondere solchen, die Quantorenfolgen enthalten • Implikation (aus einer Bedingung folgt eine andere, stärkere/schwächere Bedingungen)	„widerlegen, begründen, argumentieren“ „die Reflexion von mathematischen Aussagen“

		• Verstehen, dass ein Gegenbeispiel ausreicht, um eine Aussage zu widerlegen	
HÄU	Häufungspunkte	• Häufungspunkte • Abgrenzung zur Bedingung für Häufungspunkte	„Häufungswert einer Folge“ „Grenzwert ←→ Häufungspunkt“
EPS	ε-Umgebung, -Streifen	• Wissen, was eine epsilon-Umgebung („Schlauch“) ist • ε-Streifen variabler Breite zur Veranschaulichung der Konvergenzdefinition und als Testmittel auf Konvergenz • Verstehen des Begriffs „Epsilonstreifen“ **Bemerkung:** Ähnlich wie bei den Inhalten „Funktionen“ und „Folgen“ soll die ε-Umgebung in irgendeiner Form als eigener Inhalt thematisiert werden und nicht nur als Teil von explizierenden Sätzen zur Folgenkonvergenz etc. vorkommen, wie z.B. hier: „[die Lernenden] müssen wissen, dass in der E-Umgebung alle bis auf endlich viele Folgenglieder liegen“.	„was der Epsilon-Streifen bedeuten soll“ „Erfassen des Grenzwertbegriffs mit Hilfe der Epsilon-Umgebung“ „Epsilon-Umgebungen, Epsilonkriterium“ „die E-Umgebung um den Grenzwert“ „was man unter einem ε-Streifen versteht“ **NICHT:** „Definition von ε“ (ungenau/unvollständig)
SPR	Mathematische (Fach-) Sprache und Sprachfertigkeit	• Mathematische Fachbegriffe und Formulierungen: „Epsilonstreifen“, „für jeden“, usw. • Es mangelt an „mathematischer Sprachfertigkeit“ – es wird nicht zwischen den Begriffen „Behauptung“ und „Bedingung“ unterschieden und die schwächere Bedingung wird als „falsch“ bezeichnet	in der Pilotierung wurden bei diesem Item keine (fach-)sprachlichen Aspekte thematisiert, am nähesten kommt noch folgendes Abgrenzungsbeispiel: **NICHT:** „‘Also brauchen wir die zweite“ … brauchen, für was? ‘[…] die Bedingung A falsch ist‘ – Inwiefern falsch?“ (es wird weniger eine mangelhafte Sprachfertigkeit der Lernenden diagnostiziert, als die eigenen Schwierigkeiten, den Inhalt des Gesprächs der beiden Lernenden zu durchdringen, eingestanden)
INF	(Un)endlichkeitsbegriff	• Wissen über Unterschiede zwischen endlichen und unendlichen Mengen • Verstehen der Bedeutung von „endlich viele“ **Bemerkung:**	„Was bedeutet unendlich bzw. endlich viele?“ „Unterschied unendlich ←→ alle bis auf endlich viele“ „was es bedeutet, unendlich viele Folgenglieder […] zu haben“ „was es bedeutet, dass eine Folge gegen Unendlich geht“ „Begriff der Unendlichkeit“

		Wie oben bei anderen Inhalten bereits erläutert, zählen nur explizite Nennungen des Unendlichkeitsbegriffs als eigener Punkt, und keine bloßen Verwendungen der Wörter „endlich/unendlich“ z.B. bei der Charakterisierung von Häufungspunkt und Grenzwert.	„was es heißt, wenn unendlich viele Folgenglieder im Epsilon Bereich sind und was es heißt, wenn alle bis auf endlich viele in diesem Bereich sind“ „Vorstellung vom Unendlichen und Endlichen“

Item C

Abkürzung	Name	Ankerbeispiele aus den Expert*innenratings	(Abgrenzungs-)Beispiele aus der Pilotierung
FAB	Funktionen und ihre Ableitung	• Wissen über den Zusammenhang von Funktion und ihrer Ableitung • Zusammenhang zwischen der Ableitung einer Funktion f an einer Stelle x_0 und der besten linearen Approximation der Funktion an dieser Stelle • Verhalten der 1. Ableitung an Extremstellen • An der Maximumsstelle verschwindet die Ableitung der Funktion	„Bedeutung der Ableitung für die Steigung […] einer Funktion“ „Kenntnis […] was mit Extremstellen passiert, wenn Funktionen abgeleitet werden“ „Argumentation mittels erster Ableitung für die Steigung“ **NICHT:** „Was sagt die Ableitung aus? […] Wo taucht sie in diesem Beispiel auf? Steht hier etwas von Ableitung?“ (unklar, ob der/die Proband*in diese Fragen selbst beantworten kann oder ob Unverständnis darüber ausdrückt wird, was die Ableitung mit der Frage zu tun hat) „Sie müssen über die Ableitungsregeln Bescheid wissen“ „Wie wird [die Ableitung] gebildet?“ (Fokus auf operatives Arbeiten, nicht auf konzeptuelles Verständnis des ZH zwischen Funktion und Ableitung) „(Kenntnisse über) Ableitungsfunktionen“ „Differenzial“ (zu vage)
LOK	Lokales Verhalten von Funktionen	• Verständnis für das lokale Betrachten von Funktionen • die Vorstellung, dass („normale“) Funktionen im Kleinen linear sind • ob beim Hochpunkt auf mikroskopischer Ebene Situation (a) oder (b) vorliegt • die Formulierung „unendlich an die Stelle des Graphen heran[zoomen]“ • „Hineinzoomen“ bedeutet genaugenommen, die Funktionsgleichung in einer kleineren Umgebung um diesen Punkt zu plotten • Idee davon, was „unendlich nah“ heißen kann **Generelle Bemerkungen:** Das ist eine der umfang- und facettenreichsten Inhaltskategorien. Grundsätzlich akzeptieren manche der Expert*innen das didaktische Konzept der „Funktionenlupe“ (nach Arnold Kirsch), während andere diese Formulierung	„Was meint „unendlich nah“?“ „Was könnte ‚unendlich nah heranzoomen‘ bedeuten?“ „SuS wissen, was unendlich nahe ist“ „Kenntnisse über den Grenzwert“ „Funktion als Grenzwert von zusammengefügten Geraden, Bezug auf die E-Umgebung im Maximum“ **NICHT:** „wie sich die Funktionswerte verändern, wenn die Differenz der Variable unendlich klein wird“ (unklar, Bezug auf Stetigkeit?)

		kritisieren. Allen Ratings gemein ist jedoch eine *Sensibilität* für den vorkommenden nicht-trivialen Begriff des „unendlichen“ „Zoomens“. Daher sollen all diese Variationen als Nennung dieses Inhalts zählen: 1. Die explizite Sprache von „lokalen Eigenschaften von Funktionen“ o. Ä. – ohne Problematisieren des Zoom-Begriffs. 2. Der Verweis auf einen impliziten Grenzwertbegriff beim Konzept des „unendlichen Zoomens“ (vgl. „infinitesimale Linearität von differenzierbaren Funktionen“, Westermann et al. 1999). 3. Allgemeines Infragestellen der Formulierung „unendlicher Zoom“ und/oder Kritik an der Darstellung, die suggerieren kann, dass die Funktion in einer (sehr kleinen) Umgebung des Maximums tatsächlich linear verläuft.	„andererseits ist die Argumentation mit, wenn man lange genug und nahe genug reinzoomt, findet man eine Gerade vor, auch nicht falsch“ (es ist nicht klar, ob und in welcher Form die Argumentation akzeptiert wird („andererseits auch nicht falsch“), bzw. was der/die Proband*in selbst mit der Formulierung „findet man eine Gerade vor“ meint) „außerdem wird ihnen der Stetigkeitsbegriff näher gebracht, nämlich dass egal wie lange man an den Funktionsgraphen zoomt, es keine Lücken geben wird“ (Proband*in verwurstelt die Konzepte der Stetigkeit/Vollständigkeit und verwechselt dies mit dem hier vorliegenden Sachverhalt der lokalen Linearisierbarkeit von differenzierbaren Funktionen)
LIN	lineare Approximation	• Wissen über lokale Approximationen von Funktionen, der Ableitung als bestapproximierende Gerade • Tangente als Bestapproximation (implizit) • Lokale Linearisierung von Funktionen • Differenzierbarkeit als lokale lineare Approximation • beste lineare Approximation der Funktion an dieser Stelle	„Tangente als Approximation“
GRA	Funktionsgraph	• Fehlvorstellungen in Bezug auf den Funktionsgraphen • Die Zeichnung oder Darstellung des Graphen auf einem Bildschirm enthält immer weniger Information als die Funktionsgleichung **Bemerkung:** Da auch die Expert*innen nicht einheitlich einen bestimmten Aspekt zum Thema Funktionsgraph thematisiert haben, soll hier jede explizite Erwähnung des (Lesens/Interpretierens von bzw. Verständnisses für) Funktionsgraphen als solcher Inhalt „Funktionsgraph“ zählen.	„den Funktionsgraph interpretieren zu können“
EXT	Extremstellen	• Wissen über Charakteristika von Extremstellen • An einem Hochpunkt verläuft die Tangente an den Graphen horizontal.	„Bedeutung von Hoch- und Tiefpunkt“ „Eigenschaften einer Extremstelle“ „Definition eines Extremwertes“ **NICHT:** „Bezug auf E-Umgebung im [sic] Maximum“ (keine explizite Nennung der Extremstelle bzw. die Eigenschaften von Extremstellen als Inhalt für sich, sondern das Wort wird nur im Zuge der Erläuterung eines anderen Inhalts genannt)

STE	Steigung	• Der mathematische Begriff der Steigung erfasst dagegen jede Änderungstendenz, also auch den Fall 0 • Die Steigung einer Funktion wird „an einzelnen Punkten“ berechnet, man kann also von der Steigung an einer bestimmten Stelle sprechen und die Steigung kann sich von Punkt zu Punkt ändern • Interpretation der 1. Ableitung als Steigung **Bemerkung:** Da auch die Expert*innen nicht einheitlich einen bestimmten Aspekt zum Thema Steigung thematisiert haben, soll hier jede explizite Erwähnung von Steigung als Inhalt „Steigung“ zählen.	„Steigung einer Funktion“ „wissen, dass, wenn die Steigung gleich 0 ist, die Tangente waagrecht liegt/parallel zur x-Achse“ „Wissen zur Steigung einer Funktion“ „Verhalten und Steigung bei einem Extremwert“ „sehen, dass bei dem Höhepunkt die Steigung = 0 ist“ „Argumentation mittels erster Ableitung für die Steigung“ „wie die Steigung am Teil dieser Funktion [sic] aussehen müsste“

Item D

Abkürzung	Name	Ankerbeispiele	(Abgrenzungs-)Beispiele aus der Pilotierung
ISU	Integrale und Summen	• Verständnis des Integrals als verallgemeinerte Summe von Produkten • Grundverständnis des (Riemann-)Integrals als Grenzwert von Rechtecksummen (Erklärung ohne Verbindung zum Hauptsatz, daher kein Beitrag für die Frage der Schülerin) • Summen aus Produkten „kleine Argumentwertdifferenz mal Ordinatenhöhe“ (Integral als verallgemeinerte Summation)	„Bedeutung von Obersumme/Untersumme“ „Vorstellung der Untersumme“ „Summieren der kleinen Flächeninhalte/Summieren der Rechtecke“ „Flächeninhalt als Grenzwert von Produktsummen“ „Ober- und Untersumme“ „die Lehrperson versucht die Rechenregeln anhand der Darstellungsart des Flächeninhalts der Unter- bzw. Obersummen zu erklären“ „der Begriff des Riemann-Integrals als Summe von unendlich vielen Teilflächen“ „Ober- und Untersummen von Funktionen, Grenzwert der Ober- und Untersummen“ „Annäherung des Flächeninhalts durch Ober- und Untersummen“
HAU	Hauptsatz der Differential- und Integralrechnung	• Die Erklärung thematisiert aber nicht das Problem der Schülerin, dass die Berechnung bestimmter Integrale über die Differenz von Stammfunktionswerten erfolgt (Hauptsatz) • Anwendung des Hauptsatzes als Berechnungsinstrument für Integrale • Praktische Berechnung des Integrals mit Hilfe des Stammfunktion durch Subtraktion	„Kenntnis über den Hauptsatz der Integralrechnung“ „Berechnen eines konkreten Integrals über den Hauptsatz“ „bestimmtes Integral (Ober-Untergrenze) – Verbindung zum HSDI“ „die Schülerin stellt hier eine Frage zum HSDI […]“
GRE	Grenzwerte, Begriff des „unendlich Kleinen“	• Grenzwertprozess • Integral als Grenzwert • Grenzübergang zu „unendlich kleinen Rechtecken“ • Delta-x-Begriff	„Bedeutung/Vorstellung von unendlich (unendlich klein)“ „Grenzwert der Ober- und Untersummen“ „Grenzwert von Produktsummen“
APP	Approximation	• Approximationen von Flächeninhalten unterhalb von Funktionsgraphen • Annäherung an die durch die Kurve begrenzte Fläche	„Integral näherungsweise die Summe von Produkten“ „der Begriff der Annäherung“ „Annäherung an das Integral“ „Annäherung von [sic] immer kleineren Rechtecken, Fehler wird immer kleiner“ „Annäherung des Flächeninhalts“

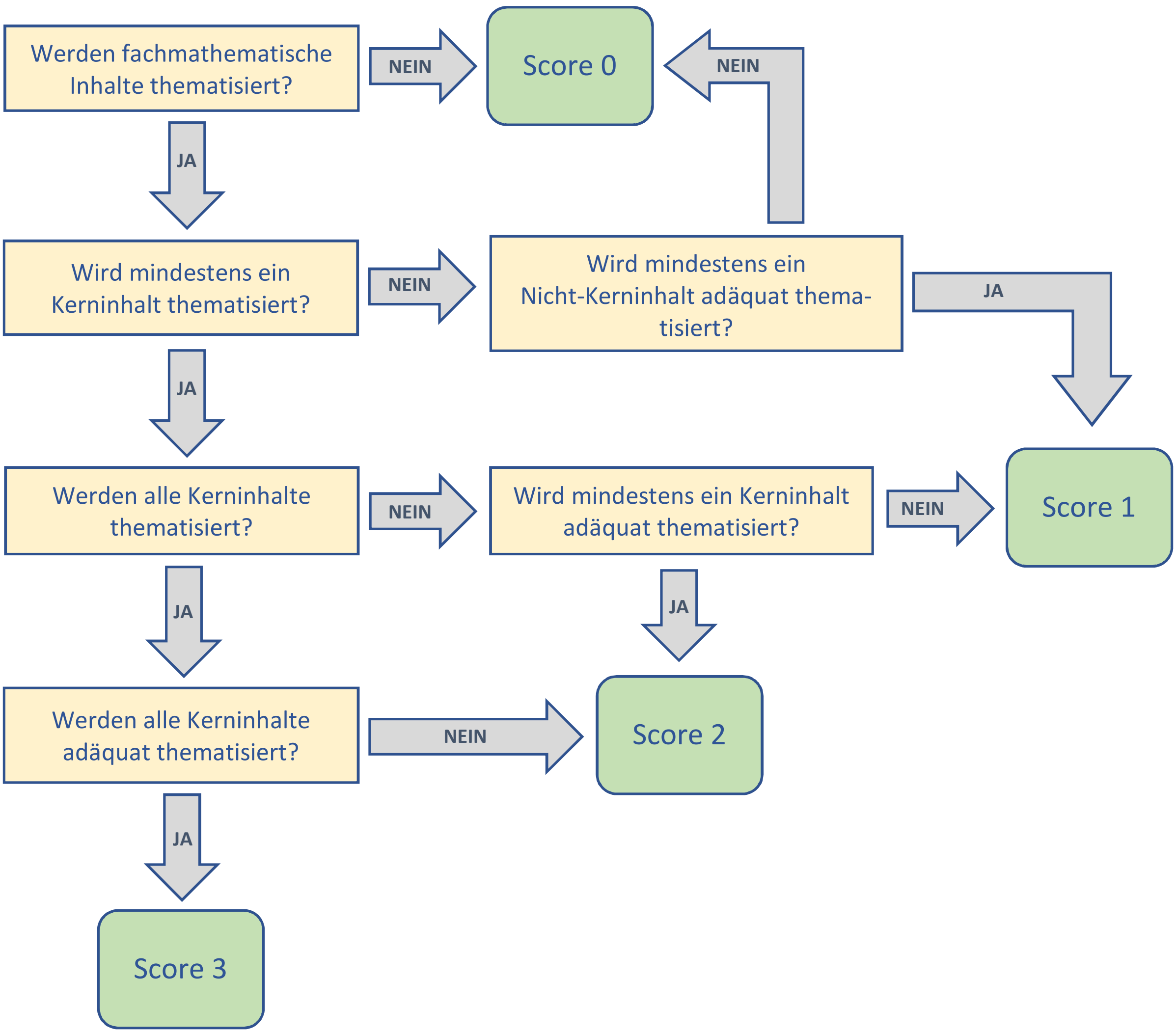
IFI	Integral als (orientierter) Flächeninhalt	• Integral als orientierter Flächeninhalt • graphische Veranschaulichung [des Integrals] durch Flächeninhaltsvorstellung • die Definition des Integrals als Fläche unter einer Funktionskurve **Bemerkung:** In irgendeiner Form sollte hier eine Verbindung zwischen Integral und Flächeninhalt postuliert werden, d.h. der bloße Verweis auf „Flächeninhalte“ reicht nicht aus.	„wie man sich ein Integral bildlich vorstellen kann (als Flächeninhalt)“ „Flächeninhalt als Grenzwert von Produktsummen“ „Annäherung des Flächeninhalts durch Ober- und Untersummen“ **NICHT:** „Flächenberechnung eines Rechtecks“ „Flächenberechnung“ (zu allgemein) „Die SuS sollen [wissen,] dass alles zwischen Funktion und x-Achse der Flächeninhalt sein soll, den sie berechnen“ (kein Bezug zum Integralbegriff)

Codierleitfaden für den Fragebogen zum fachmathematischen Noticing – Fragestellung 2

Das dreistufige Scoring (Scores 0-2), das bei der Fragestellung 1 zur Anwendung kommt, wird bei der Auswertung der Fragestellung 2 verfeinert, indem zwischen dem niedrigsten und dem höchsten Score *zwei Zwischenstufen* unterschieden werden.

- Score 0 wird vergeben, wenn es **keine der Kerninhalte** (siehe Codierleitfaden zu Fragestellung 1) thematisiert werden.
- Score 1 wird vergeben, wenn **mindestens einer, aber nicht alle** Kerninhalte thematisiert werden, die Thematisierung jedoch **unspezifisch** bleibt und/oder **Fehler** aufweist, ODER wenn **mindestens ein Nicht-Kerninhalt** (aus dem Codierleitfaden zu Fragestellung 1) **adäquat** – d.h. fehlerfrei und spezifisch – **thematisiert** wird.
- Score 2 wird vergeben, wenn entweder **alle fachmathematischen Kerninhalte – wenn auch unspezifisch und/oder fehlerhaft** thematisiert werden – ODER wenn **mindestens ein Kerninhalt** – aber nicht alle – **adäquat thematisiert** wird.
- Score 3 wird vergeben, wenn **alle fachmathematischen Kerninhalte adäquat** thematisiert werden.

Das folgende Flowchart stellt den sich daraus ergebenden Prozess der Scorevergabe graphisch dar.



Abgrenzungsbeispiele

Es werden bei den Kerninhalten grob „explizite“ und „implizite“ Inhalte unterschieden, wobei „explizite“ Inhalte solche fachmathematischen Themen sind, die im Text des Items direkt erwähnt werden (und somit durch reines *Attending-to* erkannt werden können). So ist beispielsweise im Text von Item A die Sprache von „Funktionen“; ebenso ist im Text von Item B der Begriff „Folgen“ bereits enthalten.

Manche der Kerninhalte können nur nach zusätzlichem *Interpreting* erkannt werden, da sie im Text des Items nicht direkt genannt werden, so z.B. „Definitionsbereich“ (Item A) oder „Hauptsatz“ (Item D).

Auf die Codierung wirkt sich diese Unterscheidung so aus, dass im Allgemeinen ein niedrigerer Maßstab für die „Thematisierung“ eines impliziten Kerninhalts angelegt wird. Das heißt, die reine – auch knappe – Nennung eines impliziten Kerninhalts führt dazu, dass dieser als „thematisiert“ eingestuft wird. Bei expliziten Kerninhalten muss hingegen klar hervorgehen, dass der Inhalt als Punkt für sich angesprochen wird und das Wort nicht etwa bloß im Rahmen eines Satzes über ein angrenzendes Thema fällt.

(Für die Abgrenzung zwischen „thematisiert“ und „adäquat thematisiert“ hat das **keine** Relevanz, es geht hier ausschließlich um einen nachvollziehbaren Maßstab, bei welchen Inhalten eine beiläufige Erwähnung des Schlagwortes als Thematisierung zählt und bei welchen nicht.)

Item A

Kerninhalt: FUN (Funktionsbegriff)

Nicht thematisiert	thematisiert	Adäquat thematisiert
<u>keinerlei Erwähnung</u>	<u>unspezifisch</u> „die Wiederholung, **was Funktionen sind**" „Wiederholung: **Was ist eine Funktion?**"	„nachdem Funktionen besprochen wurden, **die zu erfüllenden Kriterien** bekannt sind, […] die Ergebnisse zeigen mir, ob die Schülerinnen und Schüler den Inhalt, den Begriff einer Funktion verstanden haben" „Man benötigt eben das Wissen über die Eigenschaften dieser unterschiedlichen Funktionen, aber man darf nie **die Grundeigenschaften von Funktionen** vergessen." „Ich würde zuerst mit dem Funktionsbegriff einsteigen und wiederholen, **was eine Funktion eigentlich zur Funktion macht**." (Funktionsbegriff, explizite Rede von Charakterika, die Funktionen aufweisen müssen)
<u>Zu vage, unklar ob Funktionsbegriff gemeint</u> „Bei der Einführung von Funktionen" „Wenn Funktionen bereits besprochen wurden"		
<u>Falscher Fokus</u> „Im Zusammenhang mit rationalen Funktionen" „Einführung in die Darstellung von Funktionen" „Berechnung von Funktionswerten, Zeichnen von Funktionsgraphen"		

Kerninhalt: DEF (Definitionsbereich)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>Keinerlei Erwähnung</u>	<u>Unpassender Fokus</u> „[…] bei der es um die Bestimmung des **Definitionsbereiches** […]“	„die wichtigen Grundeigenschaften (wie eben **der Definitions- und Zielbereich, der angegeben werden muss, damit eine Funktion Sinn macht**)“ (sprachlich unelegant, aber thematisiert explizit, dass der Definitionsbereich ein notwendiger Bestandteil einer Funktion(-Definition) ist)
	<u>Unspezifisch/Fokus unklar</u> „Die **Definitionsmenge** wird dabei jeweils verändert und die Eigenschaften der Funktion beschrieben“ „Um die wichtige Funktion von Definitions- und Wertemenge nochmals ins Gedächtnis […] zu rufen“	

Item B

Kerninhalte: KON (Konvergenz) / FOL (Folgen)

Da der Folgenbegriff hier eher als Hintergrundwissen auftritt, im Rahmen der Diskussion, die im Item abgebildet wird, vorausgesetzt wird und den Lernenden scheinbar keine Probleme bereitet, sind die beiden Kerninhalte zu einem Inhalt „(Folgen)Konvergenz“ zusammengefasst. (Erwartungsgemäß hat auch niemand ausführlich den Folgenbegriff thematisiert, der/die den Konvergenzbegriff ausgelassen hätte.)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>keinerlei Erwähnung von Folgengrenzwerten/-konvergenz</u>	<u>Unspezifisch</u> „[…] klar machen, was es bedeutet, **wenn eine Folge konvergiert**, was eine Epsilon-Umgebung ist“ (kein Bezug auf das Schüler*innengespräch) Ich würde […] ihnen den **Grenzwertbegriff** sowie die Bedeutung der Epsilonumgebung nochmals erklären.	„‘[…] Müssen beide Bedingungen erfüllt sein, **damit eine Folge konvergiert**?‘ Ziel ist es, die **Epsilon-Bedingung für Konvergenz** genau zu erarbeiten.“ (Kerninhalt spezifisch auf die Situation bezogen genannt und thematisch passende Frage zum Weiterüberlegen gestellt)
	<u>Fehlerhaft</u> „Ich vermute, dass mit L in den Bedingungen der **Grenzwert** gemeint ist. Da es sich um die Häufungswerte 0 und 1 handelt, stimmt die Bedingung B nicht.“ (fehlerhafte Thematisierung, da der Zusammenhang zwischen den Bedingungen, der Variable „L“ und dem Grenzwert scheinbar missverstanden wird) „es ist egal, wie schmal euer Epsilon-Streifen ist – solange er um den **Grenzwert** verläuft“ (fehlerhafte Thematisierung, da der Zweck des Experimentierens des Schüler*innen offenbar nicht voll erfasst wird)	

Kerninhalt: LOG (Logisches Schlussfolgern)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>keinerlei Erwähnung von Aspekten des logischen Schlussfolgerns</u>	<u>Fehlerhaft</u> „klar machen, dass die erste Bedingung die zweite ja nicht ausschließt“ (spezifische Thematisierung eines logischen Aspekts, allerdings „fehlerhaft“ in dem Sinn, dass der Inhalt der Schüler*innendiskussion offenbar nicht erfasst wurde, da den Schüler*innen die Sache, die „klar gemacht“ werden soll, auf Basis des Gesprächsausschnitts bereits klar zu sein scheint)	„Man braucht also die zweite Bedingung. Diese wird also durch die Folge ja nicht erfüllt. Die erste Bedingung ist also nicht ausreichend, das habt ihr richtig erkannt.“ (Logische Struktur der Schüler*innen-Diskussion richtig erfasst und spezifisch thematisiert)
<u>Zu vage</u> „wahrscheinlich würde ich auch hinterfragen, wie sie zu den Schlussfolgerungen gekommen sind und vielleicht ein bisschen mitdiskutieren“ (selbst für einen impliziten Kerninhalt zu vage, da unklar ist, ob ‚Schlussfolgerungen‘ im mathematisch-logischen Sinn oder umgangssprachlich – also z.B. als Synonym für „Konklusio“ – verwendet wird)		„Welche Gegenbeispiele könnt ihr noch liefern, um eure getätigte Aussage zu bestätigen? Was stört euch an Bedingung A? Warum habt ihr A verworfen?“
		„Der Ansatz mit Gegenbeispielen zu arbeiten ist sehr nützlich in diesem Fall. Betrachtet gut, was für jeden Epsilon-Streifen tatsächlich zutrifft und nicht nur auf den abgebildeten.“

Item C

Kerninhalt: FAB (Funktionen und ihre Ableitung)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
keinerlei Erwähnung des Zusammenhangs zwischen Funktion und Ableitung	Irreführend/ungenau „Ebenfalls richtig ist, dass bei der Ableitungsfunktion an dieser Stelle ein Nullpunkt ist. Die Ableitungsfunktion beschreibt jedoch nicht das Aussehen der eigentlichen Funktion, sie beschreibt bloß die Änderung des Steigungsverhaltens.“ (Zusammenhang zwischen Extremstelle und Nullstelle der Ableitungsfunktion thematisiert; fehlerhaft: allgemeiner Zusammenhang zwischen Ableitung globalem Funktionsverlauf wird quasi aktiv bestritten)	„Wird eine Funktion abgeleitet, dann sehen wir uns die Tangentensteigungen an, und du hast recht: Bei Extremstellen ist die Steigung = 0. Bei der 1. Ableitung entsteht also tatsächlich ein Nullpunkt […]“ (stellt ohne gröbere Fehler – mit ein paar sprachlichen Ungenauigkeiten, z.B. Nullstelle/Nullpunkt – den Zusammenhang zwischen den Nullstellen der Ableitung und den Extremstellen der Funktion sowie der Tangentensteigung her)
Sprache nur von Steigung, ohne Bezug zur Ableitung herzustellen „In Extremstellen ist die Steigung = 0.“ „Außerdem würde ich statt „die Extremstelle besitzt keine Steigung“ sagen, „die Steigung der Tangente an der Extremstelle ist 0“. „Die Tangente an der Extremstelle besitzt die Steigung Null.“	(Sprachlich) fehlerhaft „dass wenn ihr in einem Hochpunkt eine Tangente anlegt, keine Steigung vorfindet. Auch, dass sich im Ableitungsgraph hier eine Nullstelle befindet ist richtig.“ „Es wurde richtig erkannt, dass es sich bei -1 um eine Extremstelle handelt. An dieser Stelle steigt der Funktionsgraph nicht, daher wird gelten, dass die erste Ableitung von -1 gleich 0 ist.“	
Zusammenhang bleibt unklar „Loben, dass der erste S Wissen über Extremwerte hat und Ableitungen von Fkt.“ (Zusammenhang zwischen Eigenschaften der Funktion und ihrer Ableitung bleibt unklar)	Unspezifisch „ansonsten hast du […] auch schon gut erkannt, wie die Extremstellen sich verändern, wenn man Funktionen ableitet“ (zusätzlich sprachlich problematisch) Es stimmt zwar, dass die Ableitung an der Extremstelle null ergibt, […].	

Kerninhalt: LOK (lokales Verhalten von Funktionen)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>keine Erwähnung einer lokalen Betrachtung von Funktionen oder des Zoom-Konzepts</u>	<u>Variante A – Konzept der „Funktionenlupe“ wird akzeptiert</u> „Zoomt man unendlich nah hinein, sieht man dann die Steigung an dem Punkt […]“ (unpräzise, keine genaue Erläuterung) „Da du unendlich nahe hineinzoomst, muss die Steigung in dem Intervall also annähernd 0 sein.“ (Vermischung des angedeuteten Grenzwertvollzugs „unendlich nahe“ mit dem abschwächenden „annähernd“ macht die Antwort unklar)	<u>Variante A – Konzept der „Funktionenlupe“ wird akzeptiert</u> „Angenommen, man zoomt also unendlich nahe an den Funktionsgraph an dieser Stelle heran, so wird man, wie richtig erkannt wurde, eine Gerade vorfinden. […] Betrachtet man also den Funktionsgraph um die Stelle -1 herum, so wird die Steigung immer kleiner, je näher man sich der Extremstelle nähert. Wenn man dann unendlich weit an diese Extremstelle heranzoomt, wird die Steigung unendlich klein und der Graph erscheint als Gerade. Es ist mir nicht ganz klar, was die Schülerin oder der Schüler mit der Aussage „der gesamte Graph besteht aus kleinen Geraden“ meint. Daher würde ich ihn/sie bitten, mir die Vorstellung noch einmal genauer zu erklären, damit man abschätzen kann, wo evtl. falsche Vorstellungen oder Annahmen herrschen […].“ (Sprache des „unendlichen Zooms“ wird übernommen und so mithilfe immer kleiner werdender und schließlich „unendlich kleiner“ Steigung argumentiert; „derscheint als Gerade“ dennoch bewusst formuliert; potentielle Fehlvorstellung in der zweiten Antwort identifiziert)

<u>Antworten, die die Vorstellung des„stückweise geraden“ Graphen kritisieren, aber keine Klärung schaffen oder zumindest den Ursprung dieser Vorstellung identifizieren</u> „die Aussagen ‚…weil der gesamte Graph nur aus kleinen Geraden besteht, die aneinander gereiht sind‘ ist nicht ganz korrekt. Eine Funktion besteht aus unendlich vielen Wertepunkten, an die man eine Steigungsfunktion anlegen kann“ „Die Argumentation, dass ein Graph nur aus kleinen aneinandergereihten Graphen besteht, ist nicht ganz richtig. Das sehen wir uns in der nächsten Unterrichtsstunde genauer an.“ „Grundsätzlich besteht eine Funktion aus einer Aneinanderreihung unendlich vieler Punkte und nicht aus kleinen aneinandergereihten Geraden“	<u>Variante B – Konzept des „unendlichen Zooms“ wird abgelehnt</u> „Ihr müsst aufpassen, die Funktion ist auch bei sehr weitem Hineinzoomen an den Punkt ihrer Extremstelle nicht gerade. Sie ist weiterhin gekrümmt, da für die Bedingung eines (globalen) Maximums/Minimums, die Werte links und rechts des Extremuns kleiner/größer sein müssen. Veranschaulicht dies in Geogebra und zoomt immer weiter hinein.“ (einige Probleme: Verwechslung des lokalen/globalen Extremums, wenig hilfreicher Vorschlag des Hineinzoomens in Geogebra – was für die Lernenden erwartungsgemäß genau die gegenteilige Ansicht bestätigen würde) „Rückmeldung für Antwort a) Es stimmt zwar, dass die Ableitung an der Extremstelle null ergibt, jedoch **ist die Extremstelle nur ein Punkt der Funktion**. Somit ist es unwesentlich, wie weit man heranzoomt, die Funktion rund um die Extremstelle weist in diesem Fall dennoch eine Krümmung auf. Rückmeldung für Antwort b) Der Graph besteht nicht aus unendlich vielen ‚kleinen‘ Geraden, der Graph ist vielmehr eine Aneinanderreihung von unendlich vielen Punkten. Steht man beispielsweise an einem Strand eines Sees oder am Meer, dann sieht die Fläche vor einem ebenfalls gerade aus, obwohl man weiß, dass eine Krümmung der Oberfläche vorhanden ist.“ (scheinbarer Widerspruch zwischen den beiden Erklärungen, da Erklärung an b) ja gerade zeigt, weshalb a) meint, es „sähe dort so aus“, außerdem ist die Aussage „eine Extremstelle sei nur ein Punkt der Funktion“ konfus bis falsch)	<u>Variante B – Konzept des „unendlichen Zooms“ wird abgelehnt</u> „Die Extremstelle ist nur eine Stelle, also ein x-Wert. Der Hochpunkt ist nur ein Punkt des Graphen, nur ein Wertepaar. Mit Antwort B müsste das [Steigung 0] aber auf viele Punkte zutreffen.“

Item D

Kerninhalt ISU (Integrale und Summen) / GRE (Grenzwert)

Wie bei Item 2 werden auch hier die beiden Kerninhalte „Integral als Summe“ und „Grenzwert“ in einen gemeinsamen Kerninhalt zusammengezogen, der die Definition des Integrals als Grenzwert von Rechtecksummen meint.

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>keinerlei Erwähnung des Zusammenhangs Integral und Grenzwert</u> <u>oder</u> <u>des Zusammenhangs Integral und Rechtecksummen</u>	<u>Darstellung des Integrals selbst als „Approximation“/“Prozess“</u> „Dabei wird nichts anderes gemacht, als unter der Kurve kleine Rechtecke zu bilden und diese zu summieren. **Wenn diese Rechtecke kleiner werden, dann nähert sich der berechnete Flächeninhalt natürlich immer mehr dem tatsächlichen Flächeninhalt unter der Kurve an. Das macht das Integral.**“ „**Diese Fläche versucht man mithilfe von Rechtecken ungefähr zu berechnen**, da der Flächeninhalt von Rechtecken ja einfach zu berechnen ist […] Wenn ich jetzt jedoch nur ein paar Rechtecke einzeichne […] bleibt mir immer eine gewisse Fläche übrig. Wenn ich jedoch eine größere Anzahl an Rechtecken nehme, [….] werden diese zwar kleiner, aber schmiegen sich viel besser an den Graphen an. **Nehme ich nun unendlich viele** dann komm ich zum richtigen Ergebnis“	<u>(Fast?) adäquate Summenthematisierung, allerdings mit problematischem Ausdruck:</u> „anders gesehen kann man das Integral auch durch kleine Rechtecke berechnen, welche sich unter dem Graphen befinden. Hierbei kann man die Breite der Rechtecke immer kleiner machen, **sodass die Fläche immer mehr dem Integral gleicht**. Der Fehler, der hier zwischen Funktionsgraph, Unter- und Obersumme entsteht, wird hierbei mit immer kleiner werdender Breite der Rechtecke immer kleiner“ „Das Integral beschreibt den Flächeninhalt zwischen dem Graph und der x-Achse. **Dieser Flächeninhalt wird durch die vom L genannten Rechtecke gebildet und deren Aufsummierung**. Vorstellung von Ober- und Untersumme und deren Bedeutung. → Hinführung dazu, dass das Integral annähernd die Summe von Produkten (Summe der Flächeninhalte) ist, wenn Delta x immer kleiner wird und sich dadurch Ober- und Untersumme angleichen“
<u>Sehr vage</u> „[…] nochmal erklären, wie es zur Idee der Approximation der gesamten Fläche durch die einzelnen Flächen kommt“	<u>Generell: unspezifisch/unvollständig/fehlerhaft</u> „Die Schülerin hat es von der Lehrperson **mittels Annäherung von Rechtecken, die addiert werden**, erklärt bekommen. Beim Beispiel wird jedoch subtrahiert und jetzt versteht die Schülerin den Zusammenhang nicht.“ „[…] wir müssen beim Integrieren die Rechtecksflächen zusammenaddieren. […] Wenn wir uns die	

	Flächeninhaltsgröße von diesem Rechteck anschauen wollen, dann schauen wir uns ja Delta-x an und Delta-x ist ja nichts anderes als Differenz von den Eckpunkten. Wir schauen uns ja nur einen Teil der X-Achse an“	

Kerninhalt HAU (Haupsatz der Integral- und Differentialrechnung)

Nicht thematisiert	**thematisiert**	**Adäquat thematisiert**
<u>keinerlei Erwähnung des Hauptsatzes oder seiner Aussagen</u>	<u>Unspezifisch: Thematisierung des Hauptsatzes ohne Herstellung des Zusammenhangs mit der Schüler*innenfrage bzgl. der Subtraktion</u> „→ zuerst rekonstruieren wir eine mögliche Stammfunktion, diese ist über die Integralfunktion gegeben“ → dann berechnen wir ein konkretes Integral von 0 bis 2, das liefert uns eine Zahl → diese ist als orientierter Flächeninhalt definiert. Dann würde ich mit einer Skizze Berandung, orientierter Flächeninhalt und die Subtraktion erklären.“	„es gibt dann den sogenannten Hauptsatz der Integralrechnung, welcher beweist, dass das Integral von f(x)dx berechnet werden kann mit F(obere Grenze)-F(untere Grenze), F ist hier die Stammfunktion von f“
<u>Verweise auf „Integrationsgrenzen“ ohne Erwähnung von Stammfunktionen, Integralfunktion oder Hauptsatz</u> „um die Flächen zu berechnen, die zwischen den beiden Integrationsgrenzen liegt, muss ich die kleine Fläche von der großen Fläche abziehen, um die Flächen zwischen den beiden Grenzen zu bekommen“ „Im konkreten Fall will man nur ein bestimmtes Teilstück unter der Funktion und somit gehört die Summe auch eingegrenzt – sprich es muss auch etwas abgezogen werden.“	<u>Fehlerhaft/unzulänglich</u> „Ich würde Geogebra öffnen und eine einfache integrierbare Funktion wählen. Danach würde ich diese eingeben […]. Dann würde ich einen Teil der Fläche unter der Funktion markieren […] Nun wird mit den SuS das Integral der Funktion berechnet. Diese Funktion geben wir in Geogebra ein und betrachten sie an den Integralgrenzen. Durch Ablesen von F(b) und F(a) erkennen wir, dass wir diese subtrahieren müssen, um die Fläche (bereits von Geogebra ausgerechnet) zu erhalten. Es ist glaube ich zielführender dies anhand von einem Beispiel zu zeigen und damit nochmal speziell den Unterschied zwischen Funktion und Stammfunktion herauszuheben.“ („Unterschied zwischen Funktion und Stammfunktion“ – und klingt nach einer eher „experimentellen“ Begründung des „Minus“ bzw. weniger nach einer Begründung als nach einer Illustration anhand eines Beispiels)	„Hier rechnen wir Minus, da im HSDI (Hauptsatz der Differential- und Integralrechnung) definiert wurde, dass wir Ober- minus Untergrenze der integrierten Funktion (Stammfunktion) rechnen, wenn wir den Flächeninhalt in diesem Bereich wissen wollen. Dies ist eine Methode, um die exakte Fläche zu berechnen.“

10.5 Sample solutions for the SMK questionnaire

Teil 2: Fachliches Hintergrundwissen

Korrekte Antwort: ☒

Inkorrekte Antwort: ☐

Pro korrekter Antwort 1 Punkt, maximal 8 Punkte

Frage 1

Jemand sagt: "$f(x) = \frac{1}{x}$ ist monoton fallend."

☐ Dies ist wahr.

☐ Dies ist falsch, denn "monoton steigend" ist richtig.

☐ Dies ist falsch, denn bei Funktionen kann man nicht von Monotonie sprechen.

☒ Es ist nicht genügend Information gegeben, um eine Aussage über Monotonie zu machen.

Frage 2

Es sei eine Funktion $g: D \to \mathbb{R}$ an der Stelle $t = a$ nicht definiert. Dann

☐ existiert der Limes $\lim_{t\to a} g(t)$ nicht.

☒ könnte der Limes $\lim_{t\to a} g(t)$ gleich 0 sein.

☐ muss der Limes $\lim_{t\to a} g(t)$ gegen ∞ streben.

☐ ist keine der oben genannten Aussagen richtig.

Frage 3

Gegeben sei die Folge $\left(0, \frac{1}{1000}, 0, \frac{1}{1000}, 0, \frac{1}{1000}, \ldots\right)$. Welche der folgenden Aussagen ist korrekt?

☐ Alle Folgenglieder liegen in der ε-Umgebung $U_\varepsilon(0)$, wenn man $\varepsilon = \frac{1}{100}$ wählt, daher ist die Folge konvergent.

☐ In jeder ε-Umgebung $U_\varepsilon(0)$ liegen fast alle Folgenglieder, daher ist die Folge konvergent.

☐ Es gibt eine ε-Umgebung von 0, in der nur endlich viele Folgenglieder liegen, daher ist die Folge nicht konvergent.

☒ Es gibt eine ε-Umgebung von 0, in der nicht fast alle Folgenglieder liegen, daher ist die Folge nicht konvergent.

Frage 4

Betrachten Sie die Aussagen (A) und (B):

(A) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}: \frac{1}{\mathrm{n}} < \varepsilon.$

(B) $\exists n \in \mathbb{N}\ \forall \varepsilon > 0: \frac{1}{n} < \varepsilon.$

Was trifft zu?

☐ Die zwei Aussagen sind äquivalent, und sie sind beide wahr.

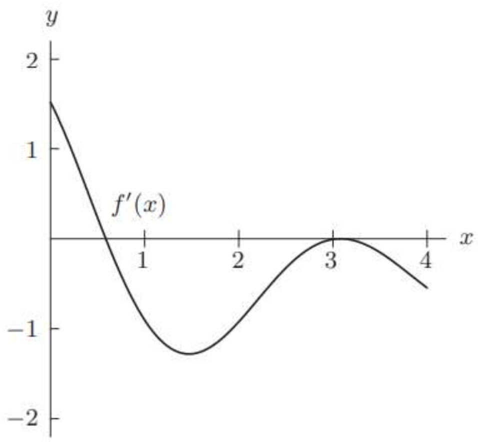
☐ Die zwei Aussagen sind äquivalent, und sie sind beide falsch.

☒ Die Aussagen sind nicht äquivalent, und nur (A) ist wahr.

☐ Die Aussagen sind nicht äquivalent, und nur (B) ist wahr.

Frage 5

Die untenstehende Abbildung stellt den Graphen von f' dar. Welche der folgenden Aussagen über die Funktion f trifft dann zu?



☐ Die Ableitung von f ist an zwei Stellen gleich 0, und beides sind lokale Maximumstellen von f.

☐ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Maximumstelle von f, während die andere eine lokale Minimumstelle von f ist.

☒ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Maximumstelle von f, während die andere weder eine lokale Maximum- noch eine lokale Minimumstelle von f ist.

☐ Die Ableitung von f ist an zwei Stellen gleich 0; eine ist eine lokale Minimumstelle von f, während die andere weder eine lokale Maximum- noch eine lokale Minimumstelle von f ist.

☐ Die Ableitung von f ist nur an einer Stelle gleich 0, und es handelt sich um eine lokale Minimumstelle von f.

Frage 6

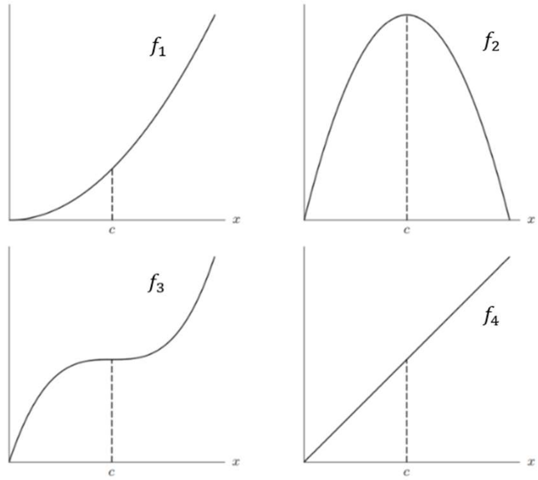
Stellen Sie sich vor, man möchte die unten abgebildeten Polynomfunktionen $f_{1,}$ f_2, f_3 und f_4 unter Verwendung der jeweils ersten Ableitung $f_n{}'$ $(n \in \{1{,}2{,}3{,}4\})$ in der Nähe der Stelle $x = c$ lokal approximieren. Bei welcher der vier Funktionen würde eine solche Approximation den geringsten Fehler aufweisen?

☐ bei f_1

☐ bei f_2

☐ bei f_3

☒ bei f_4



Frage 7

Es sei die Funktion $f: [a, b] \to \mathbb{R}$ Riemann-integrierbar.
Welches Vorgehen liefert den Integralwert $\int_a^b f$?

☐ Man bildet eine Riemann-Summe, deren zugrunde liegende Zerlegung genügend viele Teilpunkte hat.

☒ Man bestimmt das Infimum der Integrale aller Treppenfunktionen, die auf dem Intervall $[a, b]$ größer oder gleich f sind.

☐ Man bildet eine unendliche Reihe aus Rechtecksflächen, indem man das Intervall $[a, b]$ in unendlich viele Teilintervalle zerlegt.

☐ Die 2. und 3. Aussage sind beide richtig.

10.6 Subject matter „freshers“ and leading questions

„Fresher": Funktionen

Def. **Funktion (Zuordnungsdefinition)**

Seien A und B Mengen.
i) Unter einer Funktion oder Abbildung f von A nach B verstehen wir eine Vorschrift, die jedem $a \in A$ genau ein $b \in B$ zuordnet.
ii) Das dem Element a zugeordnete Element b bezeichnen wir mit $f(a)$ und nennen es den Wert der Funktion f an der Stelle a bzw. das Bild von a unter f. Das Element a wird als ein Urbild von b unter f bezeichnet.
iii) Die Menge A wird als Definitionsmenge bzw. Definitionsbereich von f bezeichnet; die Menge B als Zielmenge bzw. -bereich von f.

Def. **Funktion (mengentheoretische Definition)**

Eine Funktion ist ein Tripel $f = (A, B, G)$ bestehend aus einer Menge A, genannt Definitionsbereich, einer Menge B, genannt Zielbereich, und einer Teilmenge G des Produkts $A \times B$ mit den Eigenschaften

i) $\forall a \in A: \exists b \in B: (a, b) \in G$ (d.h. jedes $a \in A$ tritt als erste Komponente eines Paares in G auf)

ii) $\forall a \in A: \forall b_1, b_2 \in B: (a, b_1) \in G \wedge (a, b_2) \in G \Rightarrow b_1 = b_2$ (d.h. stimmen die ersten Komponenten eines Paares überein, dann auch die zweiten)

Die Menge G heißt Graph der Funktion f und wird oft auch mit $G(f)$ bezeichnet. Gilt $(a, b) \in G$, so schreiben wir $f(a) = b$ und wir können den Graphen schreiben als

$$G(f) = \{(a, f(a)) | a \in A\}$$

Def. **Bild, Wertebereich**

Sei $f: A \to B$ eine Abbildung und sei $M \subseteq A$ eine Teilmenge. Wir nennen die Menge

$$f(M) := \{b \in B \mid \exists a \in M: f(a) = b\} = \{f(a) \mid a \in M\}$$

das Bild der Menge M unter f. Speziell nennen wir das Bild der Definitionsmenge A, $f(A)$, den Wertebereich von f.

Def. **Urbild**

Sei wieder $f: A \to B$ eine Abbildung.

i) Sei $N \subseteq B$ eine Teilmenge des Zielbereichs. Wir definieren die Menge

$$f^{-1}(N) := \{a \in A \mid f(a) \in N\}$$

und nennen sie Urbild der Menge N unter f.

ii) Für ein Element $b \in B$ definieren wir das Urbild von b durch $f^{-1}(b) := f^{-1}(\{b\})$. Achtung: Das Urbild von b ist also eine Menge!

Def. **beschränkte Funktion**

Eine Funktion $f: \mathbb{R} \supseteq D \to \mathbb{R}$ heißt beschränkt, falls ihr Bild $f(D) \supseteq \mathbb{R}$ beschränkt ist, d.h. falls

$$\exists M > 0: \forall x \in D: |f(x)| \leq M.$$

Def. **monotone Funktion**

Seien $(A, \preccurlyeq)$ und $(B, \trianglelefteq)$ zwei geordnete Mengen. Eine Abbildung $f: A \to B$ heißt monoton wachsend, falls aus $a \preccurlyeq b$ schon $f(a) \trianglelefteq f(b)$ folgt. Sie heißt monoton fallend, falls sich aus $a \preccurlyeq b$ die Relation $f(a) \trianglerighteq f(b)$ ergibt.

Def. **stetige Funktion**

Sei $f: \mathbb{R} \supseteq D \to \mathbb{R}$ eine reelle Funktion und sei $x_0 \in D$ eine Stelle im Definitionsbereich.

i) f heißt stetig in x_0, falls $\forall \varepsilon > 0: \exists \delta > 0: \forall x \in D$ mit $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

ii) f heißt stetig (auf D), falls f stetig in jedem $x_0 \in D$ ist.

Satz: Sei $f: D \to \mathbb{R}$ und sei $a \in D$. Dann gilt:

$$f \text{ ist stetig in } a \Leftrightarrow \text{Für jede Folge } (x_n) \text{ in } D \text{ gilt: } x_n \to a \Rightarrow f(x_n) \to f(a)$$

Satz: Polynomfunktionen und rationale Funktionen sind stetig auf ihrem gesamten Definitionsbereich.

Def. **Berührpunkt, Häufungspunkt**

Sei $A \subseteq \mathbb{R}$.

i) $a \in \mathbb{R}$ heißt Berührpunkt von A, falls $\forall \varepsilon > 0$: $U_\varepsilon(a) \cap A \neq \emptyset$

ii) $a \in \mathbb{R}$ heißt Häufungspunkt von A, falls $\forall \varepsilon > 0$: $U_\varepsilon(a) \cap A$ enthält unendlich viele Punkte

Def. **Grenzwert einer Funktion**

Sei $f: D \to \mathbb{R}$ eine Funktion und sei a ein Berührpunkt von D. Wir schreiben

$$\lim_{x \to a} f(x) = c, \qquad \text{falls für jede Folge } (x_n) \text{ in } D \text{ mit } x_n \to a \text{ gilt, dass } f(x_n) \to c,$$

wobei hier sowohl $c \in \mathbb{R}$ als auch $c = \pm\infty$ (also uneigentliche Konvergenz) erlaubt sind.

Def. **einseitige/uneigentliche Grenzwerte von Funktionen**

Sei $f: D \to \mathbb{R}$ eine Funktion.

i) Sei a ein Berührpunkt von $D \cap (a, \infty)$. Wir schreiben

$$\lim_{x \searrow a} f(x) = c \text{ oder } \lim_{x \to a^+} f(x) = c$$

und sagen, c ist der rechtsseitige Grenzwert von f gegen a, falls für alle Folgen (x_n) in D mit $x_n > a$ und $x_n \to a$ gilt, dass $\lim_{x_n \to a} f(x_n) = c$. Dabei ist sowohl $c \in \mathbb{R}$ als auch $c = \pm\infty$ (also uneigentliche Konvergenz) erlaubt. Analog dazu definieren wir den linksseitigen Grenzwert, den wir mit $\lim_{x \nearrow a} f(x) = c$ oder $\lim_{x \to a^-} f(x) = c$ bezeichnen.

ii) Falls D nach oben unbeschränkt ist und für jede Folge (x_n) in D mit $x_n \to \infty$ gilt, dass $\lim_{x_n \to \infty} f(x_n) = c$, dann schreiben wir $\lim_{x_n \to \infty} f(x) = c$, wobei wieder $c \in \mathbb{R}$ oder $c = \pm\infty$ erlaubt ist.

Analog dazu definieren wir $\lim_{x_n \to -\infty} f(x)$ für nach unten unbeschränkte Definitionsbereiche.

Satz: Sei $f: \mathbb{R} \supseteq D \to \mathbb{R}$ eine Funktion und $a \in D$. Dann gilt:

$$f \text{ ist stetig in } a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$$

Def. **Supremum**

Sei $E \subseteq M$ Teilmenge der totalgeordneten Menge M. Das Supremum von E ist definiert als die kleinste obere Schranke von E, d.h. $\sup E = \alpha$, falls

i) α ist obere Schranke von E (d.h. $\alpha \geq e \ \forall e \in E$)

ii) kein $\alpha' < \alpha$ ist obere Schranke von E (d.h. $\alpha' < \alpha \ \Rightarrow \alpha'$ ist nicht untere Schranke von E)

Def. **Ordnungsvollständigkeit**

Eine totalgeordnete Menge $(M, \leq)$ heißt ordnungsvollständig, falls jede nichtleere, nach oben beschränkte Teilmenge ein Supremum besitzt.

Def. **der reelle Zahlenraum** $\mathbb{R}$

$\mathbb{R}$ ist der (bis auf Isomorphie eindeutige) ordnungsvollständige geordnete Körper, der $\mathbb{Q}$ als geordneten Unterkörper besitzt.

Satz vom Minimum und Maximum: Sei $f: [a, b] \to \mathbb{R}$ stetig. Dann ist f beschränkt und nimmt ihr Maximum und Minimum auf $[a, b]$ an, d.h. es gilt:

$$\exists y \in [a, b]: f(y) = \min_{x \in [a,b]} f(x) = \min f\,[a, b]$$

$$\exists z \in [a, b]: f(z) = \max_{x \in [a,b]} f(x) = \max f\,[a, b]$$

Nullstellensatz: Sei $f: [a, b] \to \mathbb{R}$ stetig mit $f(a) < 0 < f(b)$. Dann $\exists x_0 \in [a, b]$ mit $f(x_0) = 0$.

(Eine analoge Version gilt natürlich auch für $f(a) > 0 > f(b)$.)

Zwischenwertsatz: Sei $f: [a, b] \to \mathbb{R}$ stetig und $c \in \mathbb{R}$, sodass $f(a) \leq c \leq f(b)$ oder $f(a) \geq c \geq f(b)$, dann

$$\exists x_0 \in [a, b] \text{ mit } f(x_0) = c.$$

Fachdidaktische Aspekte des Funktionsbegriffs

- **Zuordnungsaspekt**

Eine Funktion ist eine Zuordnung zwischen den Elementen zweier Mengen A und B, wobei jedem Element von A genau ein Element von B zugeordnet wird. Dieser Aspekt ist der Kern des Funktionsbegriffs und liegt der „klassischen“ Funktionsdefinition zugrunde.

- **Paarmengenaspekt**

Eine Funktion ist gegeben durch eine Teilmenge G des kartesischen Produkts zweier Mengen A und B mit der Eigenschaft, dass für jedes $a \in A$ genau ein $b \in B$ existiert, sodass $(a, b) \in G$. Dieser Aspekt liegt der mengentheoretischen Definition zugrunde.

Grundvorstellungen zum Funktionsbegriff

- **Zuordnungsvorstellung**

Eine Funktion ordnet jedem Wert einer Größe genau einen Wert einer zweiten Größe zu.

- **Kovariationsvorstellung**

Mit Funktionen wird erfasst, wie sich Änderungen einer Größe auf eine zweite Größe auswirken bzw. wie die zweite Größe durch die erste beeinflusst wird.

- **Objektvorstellung**

Eine Funktion ist ein mathematisches Objekt, das einen Zusammenhang zwischen Größen als Ganzes beschreibt.

„Fresher“: Folgen & Reihen

Def. **Folge**

Sei M eine Menge. Eine Folge in M ist eine Abbildung $a\colon \mathbb{N} \to M$. Gilt $M = \mathbb{R}$, so nennen wir a eine reelle Folge.

Def. **Grenzwert einer Folge**

Sei (a_n) eine reelle Folge und $a \in \mathbb{R}$. Wir sagen, (a_n) konvergiert gegen a, falls

$$\forall \varepsilon > 0 \colon \exists N \in \mathbb{N} \colon |a_n - a| < \varepsilon \quad \forall n \geq N.$$

In diesem Fall heißt a Grenzwert bzw. Limes der Folge (a_n) und wir schreiben $a = \lim_{n\to\infty} a_n$ bzw. $a_n \to a$ für $n \to \infty$.

Def. **Divergenz**

Ist eine Folge (a_n) nicht konvergent (d.h. es gibt kein $a \in \mathbb{R}$ mit $a_n \to a$), dann heißt (a_n) divergent.

Def. **Nullfolge**

Gilt $a_n \to 0$ $(n \to \infty)$, dann heißt (a_n) Nullfolge.

Def. **beschränkte Folge**

Sei (a_n) eine reelle Folge.

i) (a_n) heißt nach oben (bzw. nach unten) beschränkt, falls $\exists K \in \mathbb{R}$, sodass $a_n \leq K$ (bzw. $a_n \geq K$) für alle $n \in \mathbb{N}$.

ii) (a_n) heißt beschränkt, falls (a_n) nach oben und nach unten beschränkt ist.

Satz: Jede konvergente reelle Folge ist beschränkt.

Satz: Jede konvergente reelle Folge hat genau einen Limes.

Def. **Reihe**

Sei (a_n) eine Folge.

i) Für jedes $m \in \mathbb{N}$ definieren wir die m-te Partialsumme (der a_ns) als

$$s_m := \sum_{n=0}^{m} a_n$$

ii) Die Folge $(s_m)_{m\in\mathbb{N}}$ der Partialsummen heißt (unendliche) Reihe mit Gliedern a_n und wird mit

$$\sum_{n=0}^{\infty} a_n$$

bezeichnet.

iii) Konvergiert $(s_m)_{m\in\mathbb{N}}$, so sagen wir auch, die Reihe konvergiert. Wir bezeichnen $\lim_{m\to\infty} s_m$ ebenfalls mit $\sum_{n=0}^{\infty} a_n$ (kurz $\sum a_n$) und nennen das die Summe der Reihe.

Beispiel: **geometrische Reihe**

Sei $x \in \mathbb{R}$ beliebig aber fix. Wir betrachten $\sum_{n=0}^{\infty} x^n$. Die Partialsummen dieser Reihe sind

$$s_m(x) = \begin{cases} m+1 & (x=1) \\ \dfrac{1-x^{m+1}}{1-x} & (x \neq 1) \end{cases}$$

Fall 1: $|x| > 1 \Longrightarrow \sum x^n$ ist divergent, denn die Partialsummen sind unbeschränkt.

Fall 2: $|x| = 1 \Longrightarrow \sum x^n$ ist divergent, denn falls $x = 1$, sind die Partialsummen unbeschränkt, und falls $x = -1$, dann gilt:

$$s_m(x) = \frac{1-(-1)^{m+1}}{1-(-1)} = \begin{cases} 1 & m \text{ gerade} \\ 0 & m \text{ ungerade} \end{cases}$$

Fall 3: $|x| < 1 \Longrightarrow \sum x^n$ ist konvergent und es gilt:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Beispiel: **periodische Dezimalzahlen**

Unendliche Dezimalzahlen sind spezielle Reihen. Hier betrachten wir die periodische Dezimalzahl $x = 0{,}08636363\overline{63}$. Das bedeutet, dass x den folgenden Wert hat:

$$\begin{aligned} x &= \frac{8}{100} + \frac{63}{10000} + \frac{63}{1000000} + \cdots \\ &= \frac{8}{100} + \frac{63}{10^4} + \frac{63}{10^6} + \cdots \\ &= \frac{8}{100} + \sum_{k=0}^{\infty} \frac{63}{10^{4+2k}} \end{aligned}$$

Wir können berechnen:

$$\sum_{k=0}^{\infty} \frac{63}{10^{4+2k}} = \frac{63}{10^4} \sum_{k=0}^{\infty} \frac{1}{10^{2k}} = \frac{63}{10^4} \sum_{k=0}^{\infty} (10^{-2})^k = \frac{63}{10^4}\left(\frac{1}{1-10^{-2}}\right) = \frac{63}{10^4}\frac{100}{99} = \frac{63}{9900}$$

und somit

$$x = \frac{8}{100} + \frac{63}{9900} = \frac{19}{220}$$

Def. **bestimmte Divergenz/uneigentliche Konvergenz**

i) Eine reelle Folge (a_n) heißt uneigentlich konvergent oder bestimmt divergent gegen $+\infty$, falls

$$\forall K \in \mathbb{R}: \exists N \in \mathbb{N}: \; a_n > K \quad \forall n \geq N.$$

In diesem Fall schreiben wir $\lim\limits_{n\to\infty} a_n = +\infty$ bzw. $a_n \to +\infty$.

ii) Eine reelle Folge (a_n) heißt uneigentlich konvergent oder bestimmt divergent gegen $-\infty$, falls $-(a_n) \to \infty$ $(n \to \infty)$ gilt. In diesem Fall schreiben wir $\lim\limits_{n\to\infty} a_n = -\infty$ bzw. $a_n \to -\infty$.

Def. **monotone Folge**

Sei (a_n) eine reelle Folge.

i) (a_n) heißt (streng) monoton wachsend, falls $a_n \leq a_{n+1}$ $(a_n < a_{n+1})$ $\forall n \in \mathbb{N}$.

ii) (a_n) heißt (streng) monoton fallend, falls $a_n \geq a_{n+1}$ $(a_n > a_{n+1})$ $\forall n \in \mathbb{N}$.

iii) Falls es ein $N \in \mathbb{N}$ gibt, sodass die obigen Eigenschaften nur für alle $n \geq N$ gelten, so sagen wir, (a_n) hat die entsprechende Eigenschaft *ab* N.

Satz: Jede nach oben beschränkte und monoton wachsende Folge konvergiert.

(Das gilt auch für Folgen, die erst ab einem gewissen Folgenglied monoton wachsend sind, und ein analoger Satz gilt für nach unten beschränkte und monoton fallende Folgen.)

Def. **b-adische Entwicklung**

Sei $b \in \mathbb{N}, b \geq 2, N \in \mathbb{Z}$ und $a_n \in \{0,1,\ldots,b-1\}$ $(N \leq n \in \mathbb{Z})$. Die Reihe

$$\pm \sum_{n=N}^{\infty} a_n b^{-n}$$

heißt b-adische Entwicklung mit Ziffern a_n $(N \leq n \in \mathbb{Z})$.

Satz: Sei $b \in \mathbb{N}, b \geq 2$, dann gilt:

i) Jede b-adische Entwicklung konvergiert.
ii) Jede reelle Zahl x ist Summe (d.h. Limes) einer b-adischen Entwicklung.

Satz: Jedes $x \in \mathbb{R}$ ist Limes einer Folge in $\mathbb{Q}$.

Fachdidaktische Aspekte des Folgenbegriffs

- **Iterations- bzw. Rekursionsaspekt**

Jedes Folgenglied (außer dem ersten) wird sukzessive aus seinem Vorgänger (bzw. seinen Vorgängern) konstruiert.

- **Aufzählungsaspekt**

Die Folge wird als Auflistung, Aneinanderreihung, Reihenfolge oder Aufzählung von Zahlen oder Objekten betrachtet.

- **Zuordnungsaspekt**

Eine Folge ist eine Funktion, die jeder natürlichen Zahl einen Funktionswert zuordnet.

Grundvorstellungen zum Folgenbegriff

- **Aneinanderreihungsvorstellung**

Eine Folge wird als Aneinanderreihung von Objekten in einer bestimmten Reihenfolge angesehen.

- **Kovariationsvorstellung**

Eine Folge erfasst, wie sich Werte von einem Folgenglied zum nächsten ändern.

- **Zuordnungsvorstellung**

Eine Folge ordnet jeder natürlichen Zahl ein Folgenglied zu.

- **Objektvorstellung**

Eine Folge wird „als Ganzes“ – ein mathematisches Objekt, das bestimmte Eigenschaften (z.B. Monotonie, Konvergenz, Beschränktheit, …) haben kann – betrachtet.

Fachdidaktische Aspekte des Grenzwertbegriffs

- **dynamischer Aspekt**

Dem dynamischen Aspekt des Grenzwertbegriffs liegt die intuitive Vorstellung eines potenziell unendlichen Vorgangs zugrunde, wie z.B. der zeitliche Verlauf eines Medikamentenspiegels oder die sukzessive Konstruktion eines geometrischen Objekts. Der Grenzwert ist – falls er existiert – der Wert, um den sich die Folgenglieder schließlich „einpendeln“.

- **statischer Aspekt**

Ausgangspunkt der statischen Sichtweise ist ein fester Wert, den man betrachtet. Ist dieser Wert der Grenzwert einer Folge, so kann man beliebig kleine Umgebungen dieses festen Wertes betrachten und für jede davon ein Folgenglied identifizieren, ab dem sich alle weiteren Folgenglieder in dieser Umgebung des festen Wertes befinden.

Grundvorstellungen zum Grenzwertbegriff

- **Annäherungsvorstellung**

Das Zustreben oder Annähern der Folgenglieder an einen festen Wert oder ein Objekt liefert die Annäherungsvorstellung als intuitive Vorstellung vom Grenzwert.

- **Umgebungsvorstellung**

Für jede noch so kleine Umgebung um den Grenzwert gilt: Es liegen ab einem bestimmten Folgenglied alle weiteren Glieder in dieser Umgebung.

- **Objektvorstellung**

Grenzwerte werden als mathematische Objekte – etwa (feste) Werte, Matrizen oder geometrische Objekte – angesehen, die durch Folgen – etwa eine Zahlenfolge, eine Folge von Matrizen oder geometrischen Objekten – konstruiert oder definiert werden können.

„Fresher“: Ableitung

Def. **Differenzenquotient**

Sei $I \subseteq \mathbb{R}$ ein Intervall, sei $f: I \to \mathbb{R}$ eine Funktion und $\xi \in I$ fix. Für $x \in I, x \neq \xi$, heißt der Ausdruck

$$\frac{f(x) - f(\xi)}{x - \xi}$$

Differenzenquotient von f bei ξ.

Def. **Differenzierbarkeit und Ableitung**

Sei $I \subseteq \mathbb{R}$ ein Intervall und $f: I \to \mathbb{R}$ eine reelle Funktion.

i) Sei $\xi \in I$. Die Funktion f heißt differenzierbar an der Stelle ξ bzw. differenzierbar in ξ, falls

$$\lim_{\substack{x \to \xi \\ x \neq \xi}} \frac{f(x)-f(\xi)}{x-\xi} \text{ oder, was dasselbe ist, } \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(\xi+h)-f(\xi)}{h}$$

existiert und endlich ist. In diesem Fall nennen wir diesen Grenzwert die Ableitung von f in ξ und schreiben $f'(\xi)$.

ii) Ist f differenzierbar in allen Punkten $\xi \in I$, dann heißt f differenzierbar auf I oder einfach *differenzierbar*.

Satz: Sei $I \subseteq \mathbb{R}$ ein Intervall und sei $f: I \to \mathbb{R}$ eine Funktion. Falls f differenzierbar in $\xi \in I$ ist, dann ist f in ξ auch stetig.

Satz: Sei $f: I \to \mathbb{R}$ eine reelle Funktion auf einem Intervall I und sei $\xi \in I$. Dann gilt:

$$f \text{ ist differenzierbar in } \xi \Leftrightarrow$$
$$\exists a \in \mathbb{R}, \exists \text{ Funktion } r: I \to \mathbb{R}, \text{ sodass } f(\xi + h) - f(\xi) = a \cdot h + r(h) \text{ und } \lim_{0 \neq h \to 0} \frac{r(h)}{h} = 0$$

In diesem Fall gilt $f'(\xi) = a$.

Satz (Kettenregel): Seien $I, J \subseteq \mathbb{R}$ Intervalle und seien $f: I \to \mathbb{R}, g: J \to \mathbb{R}$ reelle Funktionen mit $f(I) \subseteq J$. Ist f differenzierbar in $\xi \in I$ und ist g differenzierbar in $\eta := f(\xi) \in J$, dann ist die Verknüpfung $g \circ f: I \to \mathbb{R}$ differenzierbar in ξ und es gilt:

$$(g \circ f)'(\xi) = g'\big(f(\xi)\big) \cdot f'(\xi)$$

Def. **lokale Extremstellen**

Sei I ein Intervall und sei $f: I \to \mathbb{R}$ eine reelle Funktion.

i) Ein Punkt $\xi \in I$ heißt lokale Maximumstelle von f, falls es eine Umgebung von ξ gibt, auf der f nur Funktionwerte kleiner-gleich $f(\xi)$ annimmt, d.h. falls $\exists \varepsilon > 0: \forall x \in U\varepsilon(\xi) \cap I: f(\xi) \geq f(x)$.

ii) Der Punkt ξ heißt strikte lokale Maximumstelle von f, falls in der obigen Zeile die Gleichheit (von "$\geq$") nur im Punkt $x = \xi$ gilt.

iii) Analog sind (strikte) lokale Minimumstellen von f definiert.

Satz: Sei $f: I \to \mathbb{R}$ differenzierbar und sei ξ ein innerer Punkt von I. Falls ξ lokales Extremum von f ist, dann gilt $f'(\xi) = 0$.

Mittelwertsatz: Sei $f: [a,b] \to \mathbb{R}$ stetig und differenzierbar auf (a,b). Dann gibt es einen Punkt $\xi \in (a,b)$ mit

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

oder (was dasselbe ist)

$$f(b) - f(a) = f'(\xi)(b - a).$$

Satz von Rolle: Sei $f: [a,b] \to \mathbb{R}$ stetig und differenzierbar auf (a,b). Falls $f(a) = f(b)$, dann gibt es ein $\xi \in (a,b)$ mit

$$f'(\xi) = 0.$$

Satz: Sei $f: [a,b] \to \mathbb{R}$ stetig und differenzierbar auf (a,b). Dann gilt:

i) $f'(x) \geq 0 \quad \forall x \in (a,b) \quad \Leftrightarrow \quad f$ ist monoton wachsend auf $[a,b]$

ii) $f'(x) > 0 \quad \forall x \in (a,b) \quad \Rightarrow \quad f$ ist streng monoton wachsend auf $[a,b]$

iii) Beide Punkte gelten analog für $f'(x) \leq 0$ bzw. $f'(x) < 0$ und (streng) monoton fallend.

Satz: Sei $f: (a,b) \to \mathbb{R}$ differenzierbar, sei $\xi \in (a,b)$ und sei f in ξ zweimal differenzierbar. Dann gilt

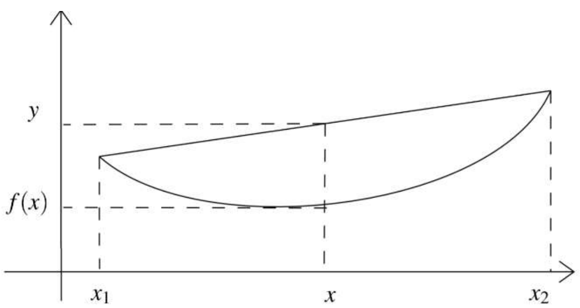
$f'(\xi) = 0$ und $f''(\xi) > 0$ $(f''(\xi) < 0)$ $\quad \Rightarrow \quad f$ hat in ξ ein striktes lokales Minimum (Maximum)

Def. **konvexe/konkave Funktionen**

Sei $f: I \to \mathbb{R}$ eine Funktion auf dem Intervall I.

i) Wir nennen f konvex (linksgekrümmt), falls für alle $x_1, x_2 \in I$ und für alle $\lambda \in [0,1]$ gilt, dass

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$



ii) Wir nennen f konkav (rechtsgekrümmt), falls $-f$ konvex ist.

Satz: Sei $f: (a,b) \to \mathbb{R}$ zweimal differenzierbar. Dann gilt:

$$f \text{ konvex} \Leftrightarrow f''(x) \geq 0 \text{ für alle } x \in (a,b).$$

Fachdidaktische Aspekte des Ableitungsbegriffs

- **Aspekt der Ableitung als Grenzwert von Differenzenquotienten**

Die Ableitung wird als Limes von Differenzenquotienten (im geometrischen Kontext des Funktionsgraphen: Sekantensteigungen; im typischen physikalischen Kontext: mittlere Geschwindigkeiten) aufgefasst und als momentane Änderungsrate (geom.: Tangentensteigung; physik.: Momentangeschwindigkeit) interpretiert.

- **Aspekt der Ableitung als lineare Bestapproximation**

Die Ableitung wird als jene Gerade bzw. lineare Funktion aufgefasst, die das lokale Verhalten der Funktion an einer Stelle am besten approximiert, d.h. Näherungen der Funktionswerte nahe der Stelle x_0 lassen sich aus dem Funktionswert $f(x_0)$ besonders gut berechnen, indem man den Ableitungswert $f'(x_0)$ als „Verstärkungsfaktor" der Änderung in der unabhängigen Variablen – also $(x - x_0)$ – betrachtet.

Grundvorstellungen zum Ableitungsbegriff

- **Grundvorstellung von der Ableitung als lokale Änderungsrate**

Die Ableitung beschreibt die momentane Änderungsrate der durch eine Funktion dargestellten Größe an einer bestimmten Stelle.

- **Grundvorstellung von der Ableitung als Tangentensteigung**

Die Ableitung beschreibt auch geometrisch – als Steigung der lokalen Schmiegetangente – das momentane Änderungsverhalten der Funktionswerte, die durch den Funktionsgraphen als Kurve dargestellt werden.

- **Grundvorstellung von der lokalen Linearität einer Funktion**

Eine differenzierbare Funktion ist lokal linear, d.h. ihre Änderungsrate verhält sich auf einem kleinen Intervall annähernd konstant und das Verhalten der Funktion kann dort „gut" durch eine lineare Funktion beschrieben werden.

- **Grundvorstellung von der Ableitung als Verstärkungsfaktor kleiner Änderungen**

Die Ableitung gibt an, mit welchem Faktor sich eine kleine Änderung der unabhängigen Variablen auf die abhängige Größe auswirkt.

„Fresher": Integral

Def. **Treppenfunktion**

Eine Funktion $\varphi: [a, b] \to \mathbb{R}$ heißt Treppenfunktion, falls es

i) eine endliche Zerlegung $\mathfrak{Z} = \{t_0, t_1, \dots, t_n\}$ des Intervalls $[a, b]$ gibt, d.h. $t_i \in [a, b]$ mit

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$

ii) und Konstanten $c_1, c_2, \dots, c_n \in \mathbb{R}$ existieren, sodass

$$\varphi(t) = c_j \text{ für } t \in (t_{j-1}, t_j) \quad (j = 1, \dots n).$$

Wir bezeichnen die Menge der Treppenfunktionen auf $[a, b]$ mit

$$\mathfrak{T}[a, b] \coloneqq \{\varphi: [a, b] \to \mathbb{R} \mid \varphi \text{ ist Treppenfunktion}\}.$$

Def. **Integral für Treppenfunktionen**

Sei $\varphi \in \mathfrak{T}[a, b]$ mit Zerlegung $\mathfrak{Z} = \{t_0, t_1, \dots, t_n\}$ und Werten $\varphi(t) = c_j$ $\left[t \in (t_{j-1}, t_j), 1 \leq j \leq n\right]$.
Wir definieren das Integral von φ auf $[a, b]$ als

$$\int_a^b \varphi(t)dt \coloneqq \sum_{j=1}^{n} c_j(t_j - t_{j-1})$$

Def. **Riemann-Integral**

Sei $f: [a, b] \to \mathbb{R}$ beschränkt.

i) Wir definieren das Ober- und Unterintegral von f als

$$\int_a^{b*} f(t)dt \coloneqq \inf\left\{\int_a^b \varphi(t)dt \,\middle|\, \varphi \in \mathfrak{T}[a, b], f \leq \varphi\right\}$$

bzw.

$$\int_{a*}^{b} f(t)dt \coloneqq \sup\left\{\int_a^b \psi(t)dt \,\middle|\, \psi \in \mathfrak{T}[a, b], \psi \leq f\right\}$$

ii) Wir nennen f Riemann-integrierbar, falls

$$\int_a^{b*} f(t)dt = \int_{a*}^{b} f(t)dt$$

gilt. In diesem Fall definieren wir das Riemann-Integral von f von a nach b als:

$$\int_a^b f(t)dt \coloneqq \int_a^{b*} f(t)dt$$

Satz: Sei $f: [a, b] \to \mathbb{R}$ beschränkt. Dann gilt:

f ist Riemann-integrierbar

$$\Leftrightarrow$$

$$\forall \varepsilon > 0: \exists \varphi, \psi \in \mathfrak{T}[a, b] \text{ mit } \psi \leq f \leq \varphi \text{ und } 0 \leq \int_a^b \varphi(t)dt - \int_a^b \psi(t)dt \leq \varepsilon$$

Satz: Jedes stetige $f: [a, b] \to \mathbb{R}$ und jedes monotone $f: [a, b] \to \mathbb{R}$ ist Riemann-integrierbar.

Satz: Sei $f: [a, b] \to \mathbb{R}$ stetig. Dann gibt es zu jedem $\varepsilon > 0$ Treppenfunktionen $\varphi, \psi \in \mathfrak{T}[a, b]$ mit den Eigenschaften

i) $\psi \leq f \leq \varphi$ und

ii) $|\varphi(x) - \psi(x)| \leq \varepsilon$ für alle $x \in [a, b]$.

Satz: Seien $f, g: [a, b] \to \mathbb{R}$ Riemann-integrierbar und sei $\lambda \in \mathbb{R}$, dann gilt:

i) $f + g$ ist Riemann-integrierbar und $\int_a^b (f + g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$.

ii) λf ist Riemann-integrierbar und $\int_a^b (\lambda f)(t)dt = \lambda \int_a^b f(t)dt$.

iii) $f \leq g \Rightarrow \int_a^b f(t)dt \leq \int_a^b g(t)dt$

Def. **Stammfunktion**

Sei $I \subseteq \mathbb{R}$ ein Intervall und sei $f: I \to \mathbb{R}$ eine Funktion. Eine Funktion $F: I \to \mathbb{R}$ heißt Stammfunktion von f auf I, falls

$F'(x) = f(x)$ für alle $x \in I$ gilt.

Satz: Sei $F: I \to \mathbb{R}$ eine Stammfunktion von $f: I \to \mathbb{R}$. Für $G: I \to \mathbb{R}$ gilt:

G ist ebenfalls eine Stammfunktion von $f \Leftrightarrow F - G$ ist konstant.

Hauptsatz der Differential- und Integralrechnung:

Sei $f: I \to \mathbb{R}$ stetig und seien a, b beliebige Punkte im Intervall I.

i) Die Funktion $F: I \to \mathbb{R}$ definiert durch

$$F(x) := \int_a^x f(t)dt$$

ist stetig differenzierbar und es gilt $F' = f$ auf I. Insbesondere ist F eine Stammfunktion von f auf I.

ii) Sei F eine (beliebige) Stammfunktion von f, dann gilt:

$$\int_a^b f(t)dt = F(b) - F(a)$$

Fachdidaktische Aspekte des Integralbegriffs

- **Stammfunktionsaspekt**

Der Aspekt des Integrals als Stammfunktion stellt den Zusammenhang zwischen dem Integrieren und dem Differenzieren her. Er ist damit untrennbar mit dem Hauptsatz der Differential- und Integralrechnung verbunden.

- **Produktsummenaspekt**

Ausdrucke des Typs $\sum f(x)\Delta x$ – also (unendliche) Summen von Produkten – spielen formal die tragende Rolle bei der analytischen Präzisierung des Integralbegriffs.

Grundvorstellungen zum Integralbegriff

- **Flächeninhaltsvorstellung**

Die Flächeninhaltsvorstellung macht den Integralbegriff am naiven Flächeninhaltsbegriff fest. Sie betont also den „klassischen Zugang“ zur Integralrechnung, bei dem es das Ziel ist, den Flächeninhalt unter einer (beliebigen, aber „schönen“) Berandung zu bestimmen.

- **Rekonstruktionsvorstellung**

Unter Rekonstruktion in diesem Zusammenhang versteht man einerseits die Rekonstruktion einer Größe aus gegebenen Änderungsraten und andererseits die Rekonstruktion einer Stammfunktion einer gegebenen Funktion.

- **Mittelwertsvorstellung**

Die Mittelwertsvorstellung bringt zum Ausdruck, dass mithilfe des Integrals einer gegebenen Funktion über ein bestimmtes Intervall ein Mittelwert berechnet werden kann, wenn man durch die Länge des Intervalls dividiert.

Inhaltsbezogene Fragen aus “Leitfragen zur Unterrichtsplanung”

Aufgaben

ZIEL

- Welche fachlichen Inhalte und Fähigkeiten kann ich mit dieser Aufgabe vermitteln?
- Welche Aspekte der vorkommenden Begriffe stehen bei dieser Aufgabe im Vordergrund?
- Welche Grundvorstellungen kann ich mit dieser Aufgabe zum jeweiligen Inhalt vermitteln?

RESSOURCEN

- An welches Vorwissen sollen/müssen meine Lernenden bei der Bearbeitung dieser Aufgabe anknüpfen?
- Welche typischen Fehlvorstellungen zu den Begriffen bzw. anderweitige Hürden bei der Aufgabenlösung kann ich antizipieren?
- Welche Darstellungsarten tragen zur Erreichung des Ziels dieser Aufgabe bei?
- Welche didaktischen Prinzipien werden bei dieser Aufgabe verfolgt?
- Welche Medien und Materialien könnten das Ziel dieser Aufgabe unterstützen?

Erklärungen

ZIEL

- Welche fachlichen Inhalte und Fähigkeiten möchte ich mit dieser Erklärung vermitteln?
- Welche Aspekte der vorkommenden Begriffe stehen im Vordergrund meiner Erklärung?
- Welche Grundvorstellungen kann ich mit dieser Erklärung zum jeweiligen Inhalt vermitteln?

RESSOURCEN

- Von welchem Vorwissen der Lernenden gehe ich bei dieser Erklärung aus?
- Welche typische Fehlvorstellungen zu den Begriffen bzw. anderweitige Verständnishürden kann ich antizipieren?
- ggf. Auf welche konkrete Fehlvorstellung bzw. Unklarheit reagiere ich mit meiner Erklärung?
- Welche Darstellungsarten unterstützen mich bei meiner Erklärung?
- Welche didaktischen Prinzipien verfolge ich bei dieser Erklärung?
- Welche Medien und Materialien können das Ziel meiner Erklärung unterstützen?

Schüler*innenlösungen und Feedback

ZIEL:

- Welche fachlichen Inhalte und Fähigkeiten sollten die Lernenden demonstrieren?
- Welche Grundvorstellungen sollten sie aufgebaut haben?
- ggf: Welche spezifischen Repräsentationen/Lösungsstrategien/Hilfsgeräte/… sollten sie einsetzen?

EVALUATION:

- Welche Fehler können identifiziert werden?
- Welche Fehlerursachen – z.B. Fehlvorstellungen – liegen nahe?
- Was fehlt den Lernenden zur Erreichung des Ziels? (notwendige Begriffsaspekte, Grundvorstellungen, Lösungsstrategien, …)

FEEDBACK:

- Welche Hinweise, Beispiele, Fragen können zur Erreichung des Ziels führen?
- Welche Darstellungen, Medien, … könnten als Hilfestellung angeboten werden?

10.7 Announcement text of seminar (German)

Ziele, Inhalte und Methode der Lehrveranstaltung

UNI-THEORIE MIT SCHULALLTAG VERKNÜPFEN - GEHT DAS? (JA!)
Im Seminar werden typische Lehrer*innentätigkeiten geübt und das Verknüpfen und Anwenden von fachmathematischem und fachdidaktischem Wissen bei Tätigkeiten der Unterrichtsvor- und Nachbereitung (Block 1: Aufgaben auswählen, Block 2: Erklärungen vorbereiten, Block 3: Schüler*innenlösungen evaluieren und Feedback geben) trainiert.
Ziel ist es, zuvor theoretisch erworbenes Wissen in authentischen Situationen und bei der Lösung von Problemstellungen aus dem Berufsleben einer Lehrperson anzuwenden und gemeinsam eine reflektierte Haltung zu eigenen Unterrichtsentscheidungen zu praktizieren.
Die Lehrveranstaltung wird im Rahmen eines Dissertationsprojekts begleitend beforscht, weshalb mit der Anmeldung zur Teilnahme gleichzeitig die Bereitschaft bestätigt wird, an zwei nicht notenrelevanten anonymen schriftlichen Erhebungen innerhalb der LV-Zeit teilzunehmen.

Art der Leistungskontrolle und erlaubte Hilfsmittel

Zwei Eckpfeiler des Lehrveranstaltungskonzepts sind anfangs stark ausgeprägte und mit der Zeit kontinuierlich abnehmende Hilfestellungen, sowie systematische Feedbackschleifen. Deshalb finden die ersten Einheiten eines Blocks immer frei von Leistungsbewertung statt und die Ausarbeitungen späterer Aufgabenstellungen können durch Einarbeitung des Feedbacks verbessert werden
.

Mindestanforderungen und Beurteilungsmaßstab

Grundsätzlich besteht 100 % Anwesenheitspflicht (prüfungsimmanente LV).
Sofern möglich (gesetzliche und universitätsweite Vorgaben je nach Infektionslage) wird die Lehrveranstaltung VOR ORT stattfinden.
Die Note setzt sich folgendermaßen zusammen:
- 30 % Endversion der Gruppenaufgaben nach Einarbeitung des Peerfeedbacks & eigenes Peerfeedback an eine andere Gruppe
- 30 % erste Abgabeversion der Einzelaufgaben
- 30 % Endversion der Einzelaufgaben nach Einarbeitung des Feedbacks der LV-Leitung
- 10 % Abschlussreflexion

Es ist – über die Ausarbeitung der Aufgabenstellungen (+Abschlussreflexion) hinaus – KEINE Seminararbeit zu verfassen.

Prüfungsstoff

- (prüfungsimmanente LV)