universität
wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„The null distance on a spacetime“

verfasst von / submitted by

Carla Mladek, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Mag. Mag. Dr. Michael Kunzinger

Abstract

Unlike Riemannian manifolds, Lorentzian manifolds do not carry an intrinsic distance function induced by the metric. This thesis explores one way of solving this problem, namely the null distance, introduced by C. Sormani and C. Vega in 2016. One considers spacetimes (i.e. Lorentzian manifolds with a causal structure) admitting a time function τ and defines the null length of a piecewise causal curve as the sum of the τ-differences of its breakpoints. The null distance $\hat{d}_\tau$ of two points in the spacetime is the infimum of the null length of all piecewise causal curves joining the two points.
A priori $\hat{d}_\tau$ is a pseudometric but if the time function additionally satisfies an anti-Lipschitz condition it is definite, i.e. a metric. In this case it also induces the manifold topology. In chapter 5 we discuss the question under which circumstances the null distance encodes the causal structure, i.e. $\hat{d}_\tau(p,q) = \tau(q) - \tau(p) \Leftrightarrow p \leq q$. C. Sormani and C. Vega showed in 2016 that this is true in GRW spacetimes when the time function is of the form $\tau(t,x) = \phi(t)$ with $\phi' > 0$. A. Burtscher and L. García-Heveling proved in 2022 that this holds if τ is a temporal Cauchy time function and the spacetime is globally hyperbolic. It also holds locally for temporal functions. Chapter 6 contains the result that a spacetime is globally hyperbolic if and only if there is a time function such that the manifold with null distance (with respect to this time function) becomes a complete metric space. In chapter 4 one can find bi-Lipschitz bounds for the null distance.

Abstract

Im Gegensatz zu Riemannschen Mannigfaltigkeiten besitzen Lorentzsche Mannigfaltigkeiten keine intrinsische Distanzfunktion, die durch die Metrik induziert wird. Diese Arbeit befasst sich mit einer Möglichkeit dieses Problem zu lösen, nämlich mit der Nulldistanz, die 2016 von C. Sormani und C. Vega eingeführt wurde. Man betrachtet Raumzeiten (d.h. Lorentzsche Mannigfaltigkeiten mit einer kausalen Struktur), die eine Zeitfunktion τ besitzen. Darauf definiert man die Nulllänge einer stückweisen kausalen Kurve als die Summe der τ-Differenzen ihrer Bruchstellen. Die Nulldistanz $\hat{d}_\tau$ zweier Punkte in der Raumzeit ist das Infimum der Nulllängen aller stückweise kausalen Kurven zwischen den beiden Punkten.
A priori ist $\hat{d}_\tau$ eine Pseudometrik. Wenn die Zeitfunktion zusätzlich eine Anti-Lipschitz-Bedingung erfüllt ist sie definit, also eine Metrik. In diesem Fall induziert sie auch die Mannigfaltigkeitstopologie. In Kapitel 5 diskutieren wir die Frage, unter welchen Umständen die Nulldistanz die kausale Struktur kodiert, d.h. $\hat{d}_\tau(p,q) = \tau(q) - \tau(p) \Leftrightarrow p \leq q$. C. Sormani und C. Vega bewiesen 2016, dass dies für GRW-Raumzeiten gilt, wenn die Zeitfunktion die Form $\tau(t,x) = \phi(t)$ mit $\phi' > 0$ hat. A. Burtscher und L. García-Heveling bewiesen 2022, dass das gilt, wenn τ eine temporale Cauchy-Zeitfunktion ist und die Raumzeit global hyperbolisch ist. Außerdem kodiert $\hat{d}_\tau$ die kausale Struktur lokal, wenn τ eine temporale Funktion ist. Kapitel 6 enthält das Resultat, dass eine Raumzeit genau dann global hyperbolisch ist, wenn es eine Zeitfunktion gibt, sodass die Mannigfaltigkeit mit der Nulldistanz (bezüglich dieser Zeitfunktion) ein vollständiger metrischer Raum ist. In Kapitel 4 finden sich bi-Lipschitz Schranken für die Nulldistanz.

Acknowledgements

First of all, I want to thank my supervisor, Univ.-Prof. Dr. Michael Kunzinger, for his helpful feedback and suggestions.
Special thanks go to Tobias Beran for countless discussions and for helping me with a counterexample.
I also want to thank my friends and colleagues who took the time to proofread this thesis, in particular Lena, Pascal and Sophie.
Last but not least, I want to thank my parents for their never ending support.

Contents

1 Introduction

In this chapter I give prerequisites for defining the null distance. After introducing spacetimes and their basic properties I define time functions and relate their existence to the causal structure of the spacetime. I also introduce the Riemannianization of a spacetime, a concept that will be used in many proofs later on.

1.1 Notation

Definition 1.1.1 (pseudometric). *Let X be a set. A map $d : X \times X \to \mathbb{R}$ is called a* ***pseudometric****, if for all $x, y, z \in X$ the following conditions hold:*

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

A set together with a pseudometric (X, d) is called pseudometric space.

Definition 1.1.2 (metric). *Let X be a set. A map $d : X \times X \to \mathbb{R}$ is called a* ***metric****, if for all $x, y, z \in X$ the following conditions hold:*

1. $d(x, x) = 0$
2. $d(x, y) > 0$*, if* $x \neq y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

A set together with a metric (X, d) is called metric space.

Definition 1.1.3 (path). *Let I be an interval and let (X, d) be a metric space. A continuous map $\gamma : I \to X$ is called a* ***path****.*

Definition 1.1.4 (curve). *Let I be an interval and let M be a smooth manifold. A continuous map $c : I \to M$ is called a* ***curve****.*

Definition 1.1.5 (piecewise smooth curve). *Let M be a smooth manifold and consider a curve $c : [a, b] \to M$. We call c* ***piecewise smooth*** *if there exists a partition $a = s_0 < s_1 \ldots < s_k = b$ of $[a, b]$ such that $c|_{[s_{i-1}, s_i]}$ is smooth for all $i = 1, \ldots, k$. The points $s_0 < s_1 \ldots < s_k$ are called* ***breakpoints*** *of c. (This definition works similarly with intervals of the form (a, b), $[a, b)$ and $(a, b]$.)*

1.2 Spacetimes

In this section I will define spacetimes and introduce some of their properties. For further background on semi-Riemannian geometry note the references [Bär20], [BEE96], [KS21], [MS08] and [O'N83].

Definition 1.2.1 (semi-Riemannian metric). *Let M be a smooth manifold. A* ***semi-Riemannian metric*** *g on M is a smooth $(0,2)$-tensorfield on M such that $g(p)$ is symmetric, nondegenerate and has the same index for each $p \in M$.*

Definition 1.2.2 (semi-Riemannian manifold). *A* ***semi-Riemannian manifold*** *is a pair (M,g) where M is a smooth manifold and g is a semi-Riemannian metric on M.*

Definition 1.2.3 (arclength). *Let (M,g) be a semi-Riemannian manifold and consider $c : [a,b] \to M$ a piecewise smooth curve in M with breakpoints $a = s_0 < s_1 \ldots < s_k = b$. We define the* ***arclength*** *of c with respect to g as*

$$L_g(c) = \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \|c'(t)\|_g dt$$

where $\|c'(t)\|_g$ is defined as $\sqrt{|g_{c(t)}(c'(t), c'(t))|}$.

Definition 1.2.4 (causal character of vectors). *Let (M,g) be a semi-Riemannian manifold and $v \in T_pM \setminus \{0\}$ for some $p \in M$. We call v* ***spacelike/timelike/null*** *if $g_p(v,v) > 0/< 0/= 0$. The vector $0 \in T_pM$ is defined to be spacelike. A vector is called* ***causal*** *if it is either timelike or null.*

Definition 1.2.5 (causal character of curves). *Let (M,g) be a semi-Riemannian manifold and let $c : I \to M$ be a piecewise smooth curve in M. We call c* ***spacelike/timelike/null*** *if $c'(t)$ is spacelike/timelike/null for all $t \in I$, t not a breakpoint of c. We call a curve c such that $c'(t)$ is either timelike or null for all $t \in I$, t not a breakpoint of c.*

Definition 1.2.6 (Riemanninan manifold). *A* ***Riemannian manifold*** *is a semi-Riemannian manifold (M,g) such that g_p has index 0 for all $p \in M$.*

Definition 1.2.7 (Riemannian distance). *Let (M,g) be a Riemannian manifold and let $p,q \in M$. We define the* ***Riemannian distance*** *of p and q as*

$$d_g(p,q) = \inf\{L_g(c) \,|\, c \text{ is a piecewise smooth curve from } p \text{ to } q\}$$

Theorem 1.2.8 (Riemannian distance function). *Let (M,g) be a connected Riemannian manifold. The map $d_g : M \times M \to \mathbb{R} : (p,q) \mapsto d_g(p,q)$ is a metric on M and induces the manifold topology.*

Proof. See [KS21], Theorem 2.3.9. □

Note that a curve on a Riemannian manifold (M,g) is a path in the metric space (M,d_g) where d_g denotes the induced Riemannian distance function on M.

Definition 1.2.9 (Lorentzian manifold). *A **Lorentzian manifold** is a semi-Riemannian manifold (M, g) of dimension $n \geq 2$ such that g_p has index 1 for all $p \in M$.*

Remark 1.2.10 (distance on Lorentzian manifolds). *Note that there is no analogous theorem to Theorem 1.2.8 in Lorentzian geometry. In fact there is no intrinsic distance function on Lorentzian manifolds that is related to the metric. This is a problem that the null distance tries to solve.*

Now we want to introduce a time structure on Lorentzian manifolds.

Remark 1.2.11. *Let (M, g) be a Lorentzian manifold. Then for each $p \in M$ the tangent space T_pM at p is isometric to the Minkowski space i.e $\mathbb{M}^{n+1} := (\mathbb{R}^{n+1}, g_{\mathbb{M}^{n+1}})$ with $g_{\mathbb{M}^{n+1}} = -dx_0^2 + dx_1^2 + \ldots + dx_n^2$. It is easy to see that the set of causal vectors consists of two connected components. Now we want to pick one of the two connected components in T_pM in a way that depends smoothly on $p \in M$.*

Definition 1.2.12 (timelike vectorfield). *Let (M, g) be a Lorentzian manifold. A smooth vectorfield $X : M \to TM$ is called **timelike** if for all $p \in M$ we have $g_p(X(p), X(p)) < 0$.*

Definition 1.2.13 (time oriented manifold). *Let (M, g) be a Lorentzian manifold. A **time orientation** is a timelike vector field on (M, g). A Lorentzian manifold with a time orientation is called a **time oriented manifold**.*

Remark 1.2.14. *Note that in particular timelike vector fields are nowhere vanishing. Therefore not all manifolds admit a time orientation. Consider for example S^n for n even. This manifold does not admit a smooth nowhere vanishing vectorfield so it is also not time orientable.*
*Also a time orientation picks for each $p \in M$ one of the connected components of the set of causal vectors of T_pM (the one that contains $X(p)$). Causal vectors are called **future directed** if they are contained in the chosen connected component and **past directed** otherwise.*

Definition 1.2.15 (spacetime). *A **spacetime** is a connected, time oriented Lorentzian manifold.*

Definition 1.2.16 (future/past directed curves). *Let (M, g) be a spacetime and let $c : [a, b] \to M$ be a piecewise smooth causal (timelike) curve in M. We call c **future directed** if for all $t \in [a, b]$, t not a breakpoint of c, we have that $c'(t)$ is a future directed causal (timelike) vector. We call c **past directed** if for all $t \in [a, b]$, t not a breakpoint of c, we have that $-c'(t)$ is a future directed causal (timelike) vector.*

Definition 1.2.17. *Let (M, g) be a spacetime and let $p, q \in M$.*

$$p \ll q \Leftrightarrow \exists \text{future directed timelike curve from } p \text{ to } q$$
$$p < q \Leftrightarrow \exists \text{future directed causal curve from } p \text{ to } q$$
$$p \leq q \Leftrightarrow p < q \text{ or } p = q$$

Definition 1.2.18 (chronological future/past)**.** *Let (M,g) be a spacetime and let $U \subseteq M$. Define:*

$$I^+(U) = \{q \in M \mid \exists p \in U : p \ll q\}$$
$$I^-(U) = \{p \in M \mid \exists q \in U : p \ll q\}$$

We call $I^+(U)$ the ***chronological future*** *of U and $I^-(U)$ the* ***chronological past*** *of U.*

Definition 1.2.19 (causal future/past)**.** *Let (M,g) be a spacetime and let $U \subseteq M$. Define:*

$$J^+(U) = \{q \in M \mid \exists p \in U : p \leq q\}$$
$$J^-(U) = \{p \in M \mid \exists q \in U : p \leq q\}$$

We call $J^+(U)$ the ***causal future*** *of U and $J^-(U)$ the* ***causal past*** *of U.*

Definition 1.2.20 (timelike diamond)**.** *Let (M,g) be a spacetime and let $p,q \in M$ with $p \ll q$. The* ***timelike diamond*** *spanned by p and q is the set $I^+(p) \cap I^-(q)$.*

Definition 1.2.21 (causal diamond)**.** *Let (M,g) be a spacetime and let $p,q \in M$ with $p \leq q$. The* ***causal diamond*** *spanned by p and q is the set $J^+(p) \cap J^-(q)$.*

Definition 1.2.22 (time separation function)**.** *Let (M,g) be a spacetime. The* ***time separation function*** *(or Lorentzian distance function) $d : M \times M \to [0,\infty]$ is defined in the following way. For $p,q \in M$ we have:*

$$d(p,q) = \begin{cases} \sup\{L_g(c) \mid c : [a,b] \to M \text{ future directed causal curve from } p \text{ to } q\} & p < q \\ 0 & p \not< q \end{cases}$$

Example 1.2.23 (time separation function)**.** *An example where the time separation function admits ∞ is the* ***Lorentz Cylinder*** *i.e. $M := \mathbb{M}^2/\mathbb{Z}e_0$, see Figure 1.1. Here we have $J^+(p) = J^-(p) = M$ for all $p \in M$. In particular for $p,q \in M$ there exist future directed causal curves α from p to q and β from q to p. Of course also the concatenations $\alpha\beta\alpha$, $\alpha\beta\alpha\beta\alpha$, ... are future directed causal curves from p to q whose lengths approach ∞. Therefore $d(p,q) = \infty$ for all $p,q \in M$ i.e. $d \equiv \infty$.*

Note that of course the time separation function is not a metric.

Proposition 1.2.24 ([Bär20], Proposition 2.26)**.** *Let (M,g) be a spacetime and let d be the time separation function. Then we have for all $p,q,o \in M$ with $p \leq q$ and $q \leq o$:*

1. $d(p,q) > 0 \Leftrightarrow p \ll q$
2. $d(p,q) + d(q,o) \leq d(q,o)$
3. *d is lower semi-continuous*

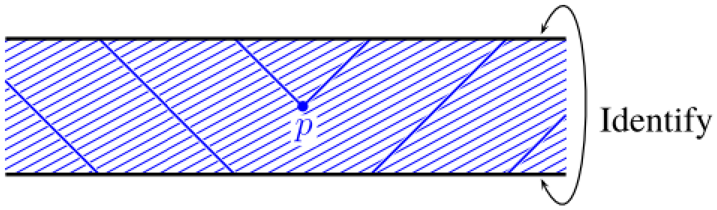

Figure 1.1: Lorentz Cylinder, [Bär20], page 25.

Proof. See [Bär20], Proposition 2.26. □

Definition 1.2.25 (causal)**.** *A spacetime (M, g) is called **causal** if there exists no closed causal curve in M.*

Example 1.2.26 (non-causal spacetime)**.** *Consider the Lorentz Cylinder i.e. $M := \mathbb{M}^2/\mathbb{Z}e_0$ (see Figure 1.1). There we have for any point p that $I^+(p) = I^-(p) = M = J^+(p) = J^-(p)$. In particular there exist closed causal (even timelike) curves.*

Note that a causal spacetime can still contain points p such that for arbitrarily small neighborhoods V of p there exist causal curves starting in V, leaving V and going back to V. For example consider the Lorentz Cylinder, fix a point p in it and remove lines as in Figure 1.2. This spacetime is causal but we can find curves starting and ending arbitrarily close to p and leaving V. This motivates the following definition:

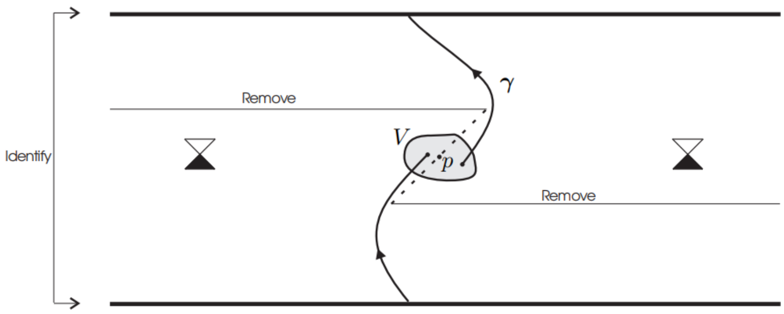

Figure 1.2: Spacetime that is not strongly causal, [MS08], page 28.

Definition 1.2.27 (strongly causal)**.** *Consider a spacetime (M, g). If for all points $p \in M$ and all neighborhoods U of p there exists a smaller neighborhood $V \subseteq U$ of p such that no future directed causal curve with starting point and endpoint in V leaves U, then we call (M, g) **strongly causal**.*

Clearly we have:

Lemma 1.2.28. *Any strongly causal spacetime is causal.*

Definition 1.2.29 ($g < \tilde{g}$)**.** *Let M be a smooth manifold and let $g, \tilde{g}$ be two Lorentzian metrics on M. Let $J^+(p)$ denote the causal future of a point $p \in M$ with respect to g and let $\tilde{I}^+(p)$ denote the timelike future of p with respect to $\tilde{g}$. We write $g < \tilde{g}$ if at every point $p \in M$ we have $J^+(p) \subseteq \tilde{I}^+(p)$.*

Note that in this case also $\overline{J^+(p)} \subseteq \tilde{I}^+(p)$ holds. Also of course this implies $\overline{J^-(p)} \subseteq \tilde{I}^-(p)$

Definition 1.2.30 (stably causal)**.** *A spacetime (M, g) is called **stably causal** if there exists another spacetime metric $\tilde{g}$ on M with $g < \tilde{g}$ such that $(M, \tilde{g})$ is causal.*

Proposition 1.2.31. *Spacetimes that are stably causal are also strongly causal.*

Proof. See [MS08], Proposition 3.57. □

Definition 1.2.32 (inextendible curve)**.** *A future directed causal curve $\gamma : [a, b) \to M$ is called **future inextendible** if $\lim_{t\to b^-} \gamma(t)$ does not exist. A past directed causal curve $\gamma : (a, b] \to M$ is called **past inextendible** if $\lim_{t\to a^+} \gamma(t)$ does not exist. We call a causal curve $\gamma : (a, b) \to M$ **inextendible** if it is future and past inextendible.*

Remark 1.2.33. *Causal curves γ on a spacetime are either future or past directed. If we go through them backwards (denoted by $-\gamma$) they change their time orientation. For a future directed causal curve γ being inextendible means the following: γ is future inextendible and $-\gamma$ is future inextendible. This works analogously for past directed causal curves.*

Definition 1.2.34 (future imprisoned causal curves)**.** *Let (M, g) be a spacetime and let $\gamma : [a, b) \to M$ be a future directed causal curve. Then γ is called **future imprisoned** if there exists some $t \in [a, b)$ and some compact set $K \subseteq M$ such that for all $s \in [t, b)$ we have $\gamma(s) \in K$.*

Definition 1.2.35 (partially future imprisoned causal curves)**.** *Let (M, g) be a spacetime and $\gamma : [a, b) \to M$ be a future directed causal curve. Then γ is called **partially future imprisoned** if there exists some sequence $(t_n)_{n\in\mathbb{N}}$ in $[a, b)$ with $\lim_{n\to\infty} t_n = b$ and a compact set $K \subseteq M$ such that for all $n \in \mathbb{N}$ we have $\gamma(t_n) \in K$.*

Note: Similarly we can define (partially) past imprisoned causal curves.

Proposition 1.2.36 ([BEE96], Proposition 3.13)**.** *Let (M, g) be a strongly causal spacetime and $\gamma : [a, b) \to M$ be a future inextendible future directed causal curve. Then γ is not (partially) future imprisoned in any compact set $K \subseteq M$.*

Proof. Consider such a curve γ and a compact subset $K \subseteq M$ such that for some $t_0 \in [a, b)$ we have $\gamma(t_0) \in K$. We show that there exists an $s \in [a, b)$ such that for all $t > s$ we have $\gamma(t) \notin K$. Assume that this is not the case i.e. for each $s \in [a, b)$ there exists an $t > s$ with $\gamma(t) \in K$. Then there is a sequence $(t_i)_{i\in\mathbb{N}}$ in $[a, b)$ with

$\lim_{n\to\infty} t_n = b$ such that $\gamma(t_n) \in K$ for all $n \in \mathbb{N}$. Because K is compact $(\gamma(t_n))_{n\in\mathbb{N}}$ admits a subsequence (which we will also denote by $(\gamma(t_n))_{n\in\mathbb{N}}$) that converges to a point $p \in K$. But γ is future inextendible, which yields that there must exist a different sequence $(s_n)_{n\in\mathbb{N}}$ in $[a,b)$ with $\lim_{t\to\infty} s_n = b$ such that $(\gamma(s_n))_{n\in\mathbb{N}}$ does not converge to p. This means that there exists a neighborhood U of p such that for each $m \in \mathbb{N}$ there is some $n > m$ with $\gamma(s_n) \notin U$. We choose such a neighborhood U and denote the subsequence of $(\gamma(s_n))_{n\in\mathbb{N}}$ of elements that are not contained in U again by $(\gamma(s_n))_{n\in\mathbb{N}}$. Because the sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ both converge to b we can find subsequences that alternate i.e. $s_n < t_n < s_{n+1} < t_{n+1} < ....$ Now we can consider the curve $\gamma|_{[t_i,t_{i+1}]}$. For i large enough this curve starts in U and ends in U. But because $s_{i+1} \in [t_i, t_{i+1}]$ and $\gamma(s_{n+1}) \notin U$ we know that this curve also leaves U and returns to U in between. Also $(\gamma(t_n))_{n\in\mathbb{N}}$ converges to p, which means that for each neighborhood V of p we have a curve starting and ending in V but leaving U at some point. This is a contradiction to (M,g) being strongly causal. Therefore our assumption must be wrong, which yields that γ leaves K at some point and does not return. $\square$

Definition 1.2.37 (global hyperbolicity)**.** *A spacetime (M,g) is called **globally hyperbolic** if it is causal and all causal diamonds ($J^+(p) \cap J^-(q)$ for $p,q \in M$) are compact.*

Definition 1.2.38 (Cauchy surface)**.** *A **Cauchy surface** of a spacetime (M,g) is a subset $S \subseteq M$ that is intersected exactly once by every inextendible causal curve.*

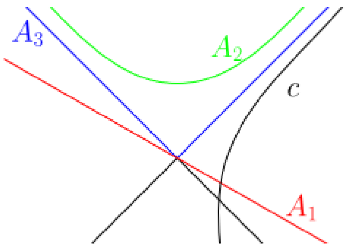

Figure 1.3: Cauchy surfaces, [Bär20], page 65.

Example 1.2.39. *Consider Figure 1.3. We look at the 2-dimensional Minkowski space $\mathbb{M}^2$. A_1 (a spacelike hypersurface) and A_3 (the future null cone of a point) are Cauchy surfaces. A_2 (the 1-dimensional hyperbolic space) is not a Cauchy surface. For example it is not intersected by the inextendible causal curve c.*

1.3 Time functions

In this section I will define (generalized) time functions and introduce some of their properties. They will play an important role when defining the null distance in Chapter 2. For further background see [BS05], [MS08].

Definition 1.3.1 ((generalized) time function)**.** *Let (M, g) be a spacetime. A **generalized time function** is a map $\tau : M \to \mathbb{R}$ that is strictly increasing along all nontrivial future-directed causal curves, i.e. $\forall p, q \in M$:*

$$p < q \Rightarrow \tau(p) < \tau(q)$$

*If τ is, furthermore, continuous, then it is called a **time function**.*

Definition 1.3.2 (temporal function)**.** *A **temporal function** is a smooth map $\tau : M \to \mathbb{R}$ with past-pointing timelike gradient $\nabla\tau$.*

Lemma 1.3.3. *Every temporal function is also a time function.*

Proof. Let $\tau : M \to \mathbb{R}$ be a temporal function and $c : [a, b] \to M$ a nontrivial future directed causal curve. Then we have $(\tau \circ c)'(t) = d\tau|_{c(t)}c'(t) = \langle \nabla\tau(t), c'(t) \rangle > 0$. The inequality follows from $\nabla\tau$ being past directed timelike and c' being future directed causal (see [O'N83], Lemma 5.29 and [O'N83] Chapter 5, Example 3a). Therefore τ is strictly monotonically increasing along nontrivial future directed causal curves. It follows that τ is a time function. □

Theorem 1.3.4 ([BS05], Theorem 1.2)**.** *Any spacetime which admits a time function also admits a temporal function.*

Proof. See [BS05], Theorem 1.2. □

Theorem 1.3.5 ([SV16], Theorem 2.3)**.** *On a spacetime (M, g) the following are equivalent:*

1. *M admits a time function.*
2. *M admits a temporal function.*
3. *M is stably causal.*

Proof. The fact that 1 and 2 are equivalent follows from [BS05], Theorem 1.2. By [CGM16], Theorem 4.12, we get that 1 and 3 are equivalent. □

Definition 1.3.6 (Cauchy time function)**.** *A **Cauchy time function** is a time function $\tau : M \to \mathbb{R}$ whose level sets ($\tau^{-1}(s)$, $s \in \mathbb{R}$) are Cauchy surfaces.*

Theorem 1.3.7 ([BH22], Theorem 1.3)**.** *A spacetime is globally hyperbolic if and only if it admits a Cauchy time function.*

Proof. In [BS05], Theorem 1.1 implies that every globally hyperbolic spacetime admits a Cauchy temporal function. Conversely, if a spacetime (M, g) admits a Cauchy time function it admits a Cauchy surface. By [O'N83], Corollary 14.39, it follows that (M, g) is globally hyperbolic. □

Corollary 1.3.8 (global hyperbolicity $\Rightarrow$ stable causality)**.** *Any spacetime that is globally hyperbolic is stably causal.*

Proof. By Theorem 1.3.7 any globally hyperbolic spacetime admits a Cauchy time function. By Theorem 1.3.5 it follows that the spacetime is stably causal. □

1.4 Riemannianizing a spacetime

In this section I will introduce a useful method to construct a Riemannian metric out of a given Lorentzian metric and vice versa. This will be needed in countless proofs later. For further background see [Veg21].

Example 1.4.1 (Euclidean space and Minkowski space)**.** *Consider the smooth manifold* $\mathbb{R}^{n+1} = \{(t = x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ *(with the standard atlas). Equipping it with the Euclidean metric* $g_{\mathbb{E}^{n+1}}$ *(a Riemannian metric) makes it the Euclidean space* $\mathbb{E}^{n+1}$*. Using the Minkowski metric* $g_{\mathbb{M}^{n+1}}$ *(a Lorentzian metric) yields the Minkowski space* $\mathbb{M}^{n+1}$*. These metrics are given by:*

$$\begin{aligned} g_{\mathbb{E}^{n+1}} &= dt^2 + dx_1^2 + \ldots + dx_n^2 \\ g_{\mathbb{M}^{n+1}} &= -dt^2 + dx_1^2 + \ldots + dx_n^2 \end{aligned}$$

Comparing $g_{\mathbb{E}^{n+1}}$ *and* $g_{\mathbb{M}^{n+1}}$*, we notice that they differ only by a single minus sign. Subtracting* $2dt^2$ *from* $g_{\mathbb{E}^{n+1}}$ *yields* $g_{\mathbb{M}^{n+1}}$*, adding* $2dt^2$ *to* $g_{\mathbb{M}^{n+1}}$ *yields* $g_{\mathbb{E}^{n+1}}$*. This motivates the following theorem.*

Proposition 1.4.2 ([Veg21], Proposition 3.1)**.** *Let* M *be a smooth manifold and let* X, Y *be smooth vector fields on* M*.*

1. *Let* g *be a Riemannian metric on* M*. If* Z *is a smooth nowhere vanishing vector field on* M*, then we can define the* $(0,2)$*-tensor* g_Z^L *by*

$$g_Z^L(X,Y) := -\frac{2}{\|Z\|_g^2} g(X,Z)g(Y,Z) + g(X,Y).$$

   This is a Lorentzian metric and we call it the ***Lorentzification of g*** *with respect to* Z*.*

2. *Let* g *be a Lorentzian metric on* M*. If* T *is a smooth timelike vector field on* M*, then we can define the* $(0,2)$*-tensor* g_T^R *by*

$$g_T^R(X,Y) := \frac{2}{\|T\|_g^2} g(X,T)g(Y,T) + g(X,Y).$$

   This is a Riemannian metric and we call it the ***Riemannianization of g*** *with respect to* T*.*

Proof. First of all, we show that we can get from the Euclidean space to the Minkowski space and the other way round by using these two operations. We consider $\mathbb{R}^{n+1}$ with the standard atlas induced by the global chart $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : x \mapsto x$. Now make $\mathbb{R}^{n+1}$ the Minkowski space by equipping it with the Minkowski metric $g_{\mathbb{M}^{n+1}}$ and consider the smooth timelike vector field $T = \partial_t = \partial_0$. We have $\|\partial_t\|_g^2 = 1$ and get:

$$g_{\partial_t}^R(X,Y) = 2g_{\mathbb{M}^{n+1}}(X,\partial_t)g_{\mathbb{M}^{n+1}}(Y,\partial_t) + g_{\mathbb{M}^{n+1}}(X,Y)$$

Pick a point $p \in \mathbb{R}^{n+1}$. Choose a basis in $T_p\mathbb{R}^{n+1}$ and write $X|_p = ((X|_p)_0, \dots, (X|_p)_n)$ in these coordinates. Using the definition of $\partial_0|_p$ and $g_{\mathbb{M}^{n+1}}$ yields:

$$\begin{aligned}
g^R_{\partial_t}|_p(X|_p, Y|_p) &= 2g_{\mathbb{M}^{n+1}}|_p(X|_p, \partial_t|_p)g_{\mathbb{M}^{n+1}}|_p(Y|_p, \partial_t|_p) + g_{\mathbb{M}^{n+1}}|_p(X|_p, Y|_p) \\
&= 2(X|_p)_0(Y|_p)_0 - (X|_p)_0(Y|_p)_0 + \sum_{i=1}^n (X|_p)_i(Y|_p)_i \\
&= \sum_{i=0}^n (X|_p)_i(Y|_p)_i \\
&= g_{\mathbb{E}^{n+1}}(X|_p, Y|_p)
\end{aligned}$$

This means that the Riemannianization of the Minkowski metric $g_{\mathbb{M}^{n+1}}$ with respect to ∂_t is the Euclidean metric $g_{\mathbb{E}^{n+1}}$. Analogously we get that the Lorentzification of the Euclidean metric $g_{\mathbb{E}^{n+1}}$ with respect to ∂_t is the Minkowski metric $g_{\mathbb{M}^{n+1}}$.
This will now be used to prove the theorem. Consider a smooth $(n+1)$-dimensional manifold M with Lorentzian metric g. Let T be a smooth timelike vector field on M. Then for all $p \in M$ we have that $(T_pM, g|_p)$ is isometric to the Minkowski space $\mathbb{M}^{n+1} = (\mathbb{R}^{n+1}, g_{\mathbb{M}^{n+1}})$. Pick one $p \in M$. Because T is a timelike vector field, $T|_p \in T_pM$ is a timelike vector. Extend $v_0 = \frac{T(p)}{\|T\|_g}$ to an ONB $(v_0, \dots, v_n)$ of T_pM. Pick two arbitrary vector fields X, Y on M and write $X|_p$ and $Y|_p$ with respect to our ONB $(v_0, \dots, v_n)$, as $X|_p = ((X|_p)_0, \dots, (X|_p)_n)$, $Y|_p = ((Y|_p)_0, \dots, (Y|_p)_n)$. Then we have:

$$\begin{aligned}
g^R_T|_p(X|_p, Y|_p) &= \frac{2}{\|T\|_g^2} g|_p(X|_p, T|_p)g|_p(Y|_p, T|_p) + g|_p(X|_p, Y|_p) \\
&= 2g|_p(X|_p, \frac{T|_p}{\|T\|_g})g|_p(Y|_p, \frac{T|_p}{\|T\|_g}) + g|_p(X|_p, Y|_p) \\
&= 2g|_p(X|_p, v_0)g|_p(Y|_p, v_0) + g|_p(X|_p, Y|_p) \\
&= 2(X_p)_0(Y_p)_0 - ((X_p))_0(Y_p)_0 + \sum_{i=1}^n (X_p)_i(Y_p)_i \\
&= \sum_{i=0}^n (X_p)_i(Y_p)_i
\end{aligned}$$

So $(T_pM, g^R_T|_p)$ is isometric to the Euclidean space $\mathbb{E}^{n+1}$. Since we chose p arbitrarily this is true for all $p \in M$. Therefore (M, g^R_T) is a Riemannian manifold. Similarly one can show that (M, g^L_Z) is a Lorentzian manifold. □

Remark 1.4.3. *Since any spacetime admits a smooth timelike vector field, any spacetime can be riemannianized.*

The following theorem explains how time functions can be used to riemannianize a Lorentzian metric.

Corollary 1.4.4 ([Veg21], Corollary 3.4)**.** *Let (M, g) be a spacetime and $\tau : M \to \mathbb{R}$ be a smooth time function with timelike gradient $\nabla\tau$. Then*

$$g_\tau^R := \frac{2}{\|\nabla\tau\|_g^2} d\tau^2 + g$$

is a Riemannian metric tensor. In particular if $\nabla\tau$ is unit timelike we get $g_\tau^R := 2d\tau^2 + g$.

Proof. This follows directly from Proposition 1.4.2. Use $T = \nabla\tau$ and remember that $g(\nabla\tau, X) = d\tau(X)$ for all vector fields X by the definition of the gradient. □

Remark 1.4.5 (Notation)**.** *In the situation of Corollary 1.4.4 we will denote the Riemannian distance function (with respect to g_τ^R) by d_τ^R and the Riemannian arc length (with respect to g_τ^R) by L_τ^R.*

2 The null distance

2.1 Definition

In this section we define the null distance, introduced by C. Sormani and C. Vega in 2016. We consider a spacetime (M, g) admitting a generalized time function τ. For a piecewise causal curve we define the null length as the sum of the τ-differences at the breakpoints. The null distance of two points is the infimum over the null length of all piecewise causal curves connecting them. The definitions can be found in [SV16], Section 3.

Definition 2.1.1 (piecewise causal curve)**.** *Let (M, g) be a spacetime. We call a curve $\beta : [a, b] \to M$* ***piecewise causal*** *if there exists a partition $a = s_0 < s_1 \ldots < s_k = b$ of the domain $[a, b]$ such that $\beta|_{[s_{i-1}, s_i]}$ is future directed or past directed causal. This means β is allowed to travel backwards and forwards in time, see Figure 2.1. We denote $\beta_i := \beta|_{[s_{i-1}, s_i]}$ and write $\beta = \beta_1\beta_2 \ldots \beta_k$.*

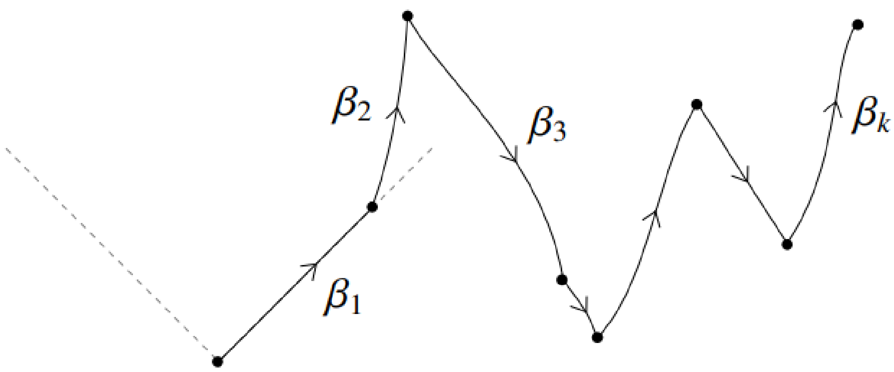

Figure 2.1: Piecewise causal curve, [SV16], page 9.

Definition 2.1.2 (null length)**.** *Let (M, g) be a spacetime and let $\tau : M \to \mathbb{R}$ be a generalized time function on (M, g). Consider a piecewise causal curve $\beta : [a, b] \to M$ with breakpoints $a = s_0 < s_1 < \ldots < s_k = b$. The* ***null length*** *of β is defined to be*

$$\hat{L}_\tau(\beta) = \sum_{i=1}^{k} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|.$$

Definition 2.1.3 (null distance)**.** *Let (M, g) be a spacetime and let $\tau : M \to \mathbb{R}$ be a generalized time function. The* ***null distance*** *of $p, q \in M$ is defined to be*

$$\hat{d}_\tau(p, q) := \inf(\{\hat{L}_\tau(\beta) \,|\beta \textit{ piecewise causal curve from } p \textit{ to } q\}).$$

Remark 2.1.4. *In [SV16], Sormani and Vega used the term "distance function" when writing about (pseudo)metrics. In some places it remained open whether a distance function is defined to be definite or not. To avoid any confusion, I will refrain from using the term in this context and instead make use of the well defined terms "metric" and "pseudometric". The term "distance function" will only be used when writing about the Riemannian distance function or the null distance function.*

2.2 Basic Properties

In this section some basic properties of the null distance will be introduced. In particular, we show that the null distance is a well defined pseudometric and conformally invariant. There are also counterexamples that illustrate that the null distance is neither necessarily definite nor encodes the causal structure of the spacetime. See [SV16], Sections 3.3 and 3.5.

The following theorem ensures that the null distance is defined for any two points of a spacetime (with a generalized time function).

Lemma 2.2.1 ([SV16], Lemma 3.5)**.** *Let (M, g) be a spacetime. For any two points $p, q \in M$ there exists a piecewise causal curve joining them.*

Proof. Pick two points $p, q \in M$. Recall that M is a connected manifold (because it is a spacetime) which yields that M is pathconnected as well. So we can pick a path α from p to q. The image of α is compact i.e. every open cover of the image of α has a finite subcover. Recall that the timelike past of a point $(I^-(x))$ is an open set and the set of all timelike pasts of points cover M. So there exists a finite number m of points $\{x_2, \dots, x_{2m}\}$ in M such that the image of α is covered by the timelike pasts of the points, $I^-(x_2), \dots, I^-(x_{2m})$. Without loss of generality we assume $p \in I^-(x_2)$, $q \in I^-(x_{2m})$ and $I^-(x_{2i}) \cap I^-(x_{2i+2}) \neq \emptyset$. Now because $p \in I^-(x_2)$ there exists a future directed causal curve from p to x_2, and we call this curve β_1. Also because $I^-(x_{2i}) \cap I^-(x_{2i+2}) \neq \emptyset$ for all $i \in \{2, \dots, m-1\}$ we can choose a point in this intersection and a future directed curve β_{2i+1} from this point to x_{2i+2} as well as a past directed curve β_{2i} from x_{2i} to this point. Then $\beta = \beta_1 \dots \beta_{2m}$ is a piecewise causal curve from p to q. $\square$

Lemma 2.2.2 ([SV16], Lemma 3.8)**.** *Let (M, g) be a spacetime. For any generalized time function τ, the induced null distance $\hat{d}_\tau$ is a pseudometric on M, satisfying the following conditions for all $p, q, z \in M$:*
1) $\hat{d}_\tau : M \times M \to [0, \infty)$
2) $\hat{d}_\tau(p, q) = \hat{d}_\tau(q, p)$
3) $\hat{d}_\tau(p, q) \leq \hat{d}_\tau(p, z) + \hat{d}_\tau(z, q)$
4) $\hat{d}_\tau(p, p) = 0$

Proof. $\hat{d}_\tau$ being finite follows from Lemma 2.2.1. The rest of the theorem follows directly from the definition of the null distance (Definition 2.1.3). $\square$

Example 2.2.3 (The null distance is not a priori definite, [SV16] Proposition 3.4)**.** *Consider the Minkowski plane $\mathbb{M}^2$ and the time function $\tau = t^3$. Look at the points $p = (t, 0)$ and $q = (t, l)$ and the curve η_k that connects p and q by bouncing between the t and $t + l/2k$ slices, see Figure 2.2.*

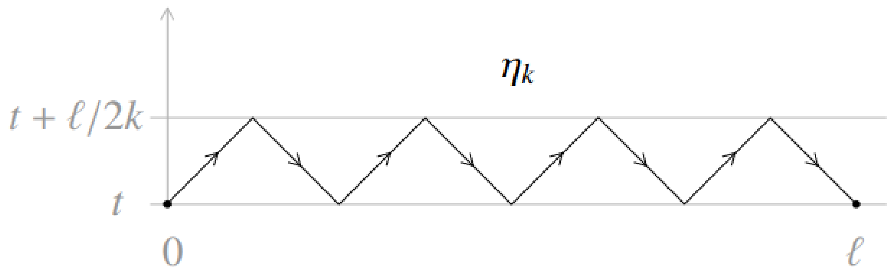

Figure 2.2: η_k [SV16], page 12.

The null length of η_k can be calculated by

$$\hat{L}_\tau(\eta_k) = ((t + l/2k)^3) - t^3)2k = \frac{(t + l/2k)^3 - t^3}{\frac{l}{2k}} l$$

So for the null distance of p and q we get the following:

$$\hat{d}_\tau(p, q) \leq \lim_{k \to \infty} \hat{L}_\tau(\eta_k) = 3t^2 l$$

Thus points of the form of p and q with $t = 0$ have null distance 0 despite not being the same point. Therefore $\hat{d}_\tau$ is not definite for $M = \mathbb{M}^2$ the Minkowski plane and $\tau = t^3$.

In Chapter 3 we deal with the question under which circumstances we have a definite null distance. In particular we will see a condition on τ that ensures that $\hat{d}_\tau$ is definite.

Definition 2.2.4 (conformally equivalent)**.** *Consider a spacetime (M, g). A metric h on M is **conformally equivalent** to g if there exists a smooth map $f : M \to (0, \infty)$ such that $g = fh$. In this case, considering (M, h) instead of (M, g) is called a **conformal change**. Properties of a spacetime that are invariant under conformal changes are called **conformally invariant**.*

Remark 2.2.5. *The causal structure is conformally invariant. A vector $v \in M$ is timelike/spacelike/null with respect to g if and only if it is timelike/spacelike/null with respect to $fg = h$ for some smooth $f : M \to (0, \infty)$.*

Proposition 2.2.6 ([SV16], Proposition 3.9)**.** *Let (M, g) be a spacetime and τ be a generalized time function on M. Then the null distance $\hat{d}_\tau$ is conformally invariant.*

Proof. This follows directly from the fact that the causal structure is conformally invariant. □

Remark 2.2.7 ([SV16], Lemma 3.6 (3), Lemma 3.10)**.** *Let (M, g) be a spacetime and let $\tau : M \to \mathbb{R}$ be a generalized time function. From the definition of the null distance the following is true for all β connecting $p, q \in M$:*

$$\hat{L}_\tau(\beta) \geq \max_{x \in im(\beta)} \tau(x) - \min_{x \in im(\beta)} \tau(x) \geq |\tau(q) - \tau(p)|$$

So it follows that for the null distance we have:

$$\hat{d}_\tau(p, q) \geq |\tau(q) - \tau(p)|$$

From this we immediately get:

$$\hat{d}_\tau(p, q) = 0 \Rightarrow \tau(q) = \tau(p)$$

Lemma 2.2.8 (causality property, [SV16], Lemma 3.11)**.** *Let (M, g) be a spacetime and let $\tau : M \to \mathbb{R}$ be a generalized time function. Then the null distance $\hat{d}_\tau$ satisfies the following condition:*

$$p \leq q \Rightarrow \hat{d}_\tau(p, q) = \tau(q) - \tau(p)$$

Proof. Pick $p, q \in M$ with $p \leq q$. Then from τ being a generalized time function it follows that $|\tau(q) - \tau(p)| = \tau(q) - \tau(p)$. Applying Remark 2.2.7 yields $\hat{d}_\tau(p, q) \geq \tau(q) - \tau(p)$. For the other direction let β be any future directed causal curve from p to q. Then we get $\hat{L}_\tau(\beta) = \tau(q) - \tau(p)$. It follows that $\hat{d}_\tau(p, q) \leq \tau(q) - \tau(p)$. So $\hat{d}_\tau(p, q) = \tau(q) - \tau(p)$ is true. □

Remark 2.2.9. *The converse is not true in general! Take a look at Example 2.2.3 again. Choose $p = (t_p, p_S), q = (t_q, q_S) \in \mathbb{M}^2$ with $t_p < 0 < t_q$. Denote $p_0 := (0, p_S)$ and $q_0 := (0, q_S)$. So we have $p < p_0$ and $q_0 < q$. From Example 2.2.3 we also have $\hat{d}_\tau(p_0, q_0) = 0$ with respect to the time function $\tau = t^3$. Now we get*

$$\tau(q) - \tau(p) \leq \hat{d}_\tau(p, q) \leq \hat{d}_\tau(p, p_0) + \hat{d}_\tau(p_0, q_0) + \hat{d}_\tau(q_0, q) = \hat{d}_\tau(p, p_0) + \hat{d}_\tau(q_0, q)$$

by Remark 2.2.7 for the first inequality and by the triangle inequality for the second inequality. Lemma 2.2.8 yields:

$$\hat{d}_\tau(p, p_0) + \hat{d}_\tau(q_0, q) = \tau(p_0) - \tau(p) + \tau(q) - \tau(q_0) = \tau(q) - \tau(p)$$

So in total we get $\hat{d}_\tau(p, q) = \tau(q) - \tau(p)$ despite not necessarily $p \leq q$. This means that the converse of Lemma 2.2.8 is not true in general.

Definition 2.2.10 (encoding causality)**.** *The null distance $\hat{d}_\tau$ on a spacetime (M, g) with respect to a generalized time function τ is said to* ***encode causality*** *if the following holds:*

$$p \leq q \Leftrightarrow \hat{d}_\tau(p, q) = \tau(q) - \tau(p)$$

Chapter 5 addresses the question under which circumstances this property holds.
Encoding causality is stronger than definiteness:

Lemma 2.2.11 ([SV16], Lemma 3.12)**.** *Let (M,g) be a spacetime with generalized time function τ such that $\hat{d}_\tau$ encodes the causality of (M,g) i.e.*

$$p \leq q \Leftrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$$

Then $\hat{d}_\tau$ is definite.

Proof. Let (M,g) be a spacetime with generalized time function τ such that $\hat{d}_\tau$ encodes the causality of (M,g). Choose $p,q \in M$ with $\hat{d}_\tau(p,q) = 0$. From Remark 2.2.7 it follows that $\tau(p) = \tau(q)$ i.e. $\tau(q) - \tau(p) = 0$. So the right-hand side of the hypothesis is satisfied ($\hat{d}_\tau(p,q) = 0 = \tau(q) - \tau(p)$) and we get $p \leq q$. But there can be no nontrivial future directed causal curve from p to q because otherwise $\tau(q) > \tau(p)$ would be true, a contradiction to $\tau(p) = \tau(q)$. So we get $p = q$. Therefore $\hat{d}_\tau$ is definite. □

Lemma 2.2.12 ([SV16], Lemma 3.16)**.** *Let (M,g) be a spacetime, τ a generalized time function on M and $\lambda > 0$ a constant. Then we have:*

$$\lambda \hat{d}_\tau = \hat{d}_{\lambda\tau}$$

Proof. This follows directly from the definition of the null distance (Definition 2.1.3). □

Lemma 2.2.13 ([SV16], Lemma 3.17)**.** *Let τ_1, τ_2 be generalized time functions on a spacetime (M,g). Then we have*

$$\hat{d}_{\tau_1} = \hat{d}_{\tau_2} \Leftrightarrow \tau_1 = \tau_2 + C$$

for some constant C.

Proof. Going from the right-hand side to the left-hand side is immediate from the definition of the null distance (Definition 2.1.3). Conversely suppose $\hat{d}_{\tau_1} = \hat{d}_{\tau_2}$. Pick $p_1, p_2 \in M$ and $\beta : [a,b] \to M$ a piecewise causal curve from p_1 to p_2 with breakpoints at $a = s_0 < s_1 < \ldots < s_k = b$. Denote $x_i := \beta(s_i)$ and $\beta_i := \beta|_{[s_{i-1},s_i]}$. If β_i is a future directed causal curve we get by Lemma 2.2.8:

$$\tau_1(x_i) - \tau_1(x_{i-1}) = \hat{d}_{\tau_1}(x_i, x_{i-1}) = \hat{d}_{\tau_2}(x_i, x_{i-1}) = \tau_2(x_i) - \tau_2(x_{i-1})$$

If β_i is a past directed causal curve we similarly get:

$$\tau_1(x_{i-1}) - \tau_1(x_i) = \hat{d}_{\tau_1}(x_i, x_{i-1}) = \hat{d}_{\tau_2}(x_i, x_{i-1}) = \tau_2(x_{i-1}) - \tau_2(x_i)$$

This yields

$$(\tau_2 - \tau_1)(x_i) = (\tau_2 - \tau_1)(x_{i-1}) \tag{2.1}$$

for all breakpoints, in particular $(\tau_2 - \tau_1)(p_1) = (\tau_2 - \tau_1)(p_2)$. Because we chose $p_1, p_2 \in M$ arbitrarily we get $\tau_1 = \tau_2 + C$ for some constant C. □

Lemma 2.2.14 ([SV16], Lemma 3.18)**.** *Let τ_1, τ_2 be generalized time functions on a spacetime (M,g) and let $\lambda > 0$. Then we have*

$$\hat{d}_{\tau_1} = \lambda \hat{d}_{\tau_2} \Leftrightarrow \tau_1 = \lambda \tau_2 + C$$

Proof. This follows directly from Lemma 2.2.12 and Lemma 2.2.13. □

2.3 Minimal curves

In this section we look at minimal curves with respect to the null distance, i.e. curves $\beta : [a,b] \to M$ with $\hat{d}_\tau(\beta(a),\beta(b)) = \hat{L}_\tau(\beta)$. See [SV16], Section 3.6.

Definition 2.3.1 (minimal curve)**.** *Let (M,g) be a spacetime, $p,q \in M$ and $\beta : [a,b] \to M$ a piecewise causal curve from p to q. We call β a **minimal curve** if $\hat{L}_\tau(\beta) = \hat{d}_\tau(p,q)$.*

Note: For a minimal curve $\beta : [a,b] \to M$ we have for each interval $[u,v] \subseteq [a,b]$ also $\hat{L}_\tau(\beta|_{[u,v]}) = \hat{d}_\tau(\beta(u),\beta(v))$. Otherwise we could replace $\beta|_{[u,v]}$ in β with a shorter curve and get a contradiction to β being minimal.

Corollary 2.3.2 ([SV16], Corollary 3.19)**.** *Let (M,g) be a spacetime and $p,q \in M$ with $p \leq q$. Then every causal curve β from p to q is a minimal curve with $\hat{L}_\tau(\beta) = \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$. In particular, causal curves are null distance realizing.*

Proof. This follows directly from the definition of minimal curves (Definition 2.3.1) and the definition of the null distance (Definition 2.1.3). □

Definition 2.3.3 (achronal)**.** *Let (M,g) be a spacetime. A subset $S \subseteq M$ is called **achronal** if no two points in S can be joined by a timelike curve.*

Lemma 2.3.4 ([SV16], Lemma 3.20)**.** *Let (M,g) be a spacetime and let $\beta : [a,b] \to M$ be a minimal piecewise causal curve. Then either β is causal, or β is an achronal piecewise null geodesic, which changes time orientation at each breakpoint.*

Note: Of course breakpoints are not uniquely determined, we can always add unnecessary ones. The lemma states that there exists a choice of breakpoints such that β changes its time orientation at them and β restricted to the smaller intervals is a null geodesic. β being an achronal curve means that the image of β is an achronal set.

Proof. Let $\beta : [a,b] \to M$ be a minimal piecewise causal curve and let $a = s_0 < \ldots < s_k = b$ be the breakpoints of β. Case 1: β is future or past causal. Then we are done. Case 2: β is not future or past causal. Then there exists a breakpoint s_i where β changes direction in time. Without loss of generality we assume that β changes from future to past causal at s_i and get $\beta_i = \beta|_{[s_{i-1},s_i]}$ future directed causal and $\beta_{i+1} = \beta|_{[s_i,s_{i+1}]}$ past directed causal. Now suppose there exists some $u_0 \in [s_{i-1},s_i]$ such that $\beta(u_0) \ll \beta(s_i)$. Since the relation $\ll$ is open (see [O'N83] Lemma 14.3) there exists $v_0 \in (s_i,s_{i+1}]$ such that $\beta(u_0) \ll \beta(v_0)$. For a future directed timelike curve α_0 from $\beta(u_0)$ to $\beta(v_0)$ we get:

$$\begin{aligned}
\hat{d}_\tau(\beta(u_0),\beta(v_0)) \leq \hat{L}_\tau(\alpha_0) &= \tau(\beta(v_0)) - \tau(\beta(u_0)) \\
&= \tau(\beta(s_i)) - \tau(\beta(u_0)) + \tau(\beta(v_0)) - \tau(\beta(s_i)) \\
&< \tau(\beta(s_i)) - \tau(\beta(u_0)) + \tau(\beta(s_i)) - \tau(\beta(v_0)) \\
&= \hat{L}_\tau(\beta|_{[u_0,v_0]})
\end{aligned}$$

This contradicts the fact that β is minimal (recall the remark after the definition of a minimal curve). Therefore for all $u_0 \in [s_{i-1},s_i]$ we have $\beta(u_0) \leq \beta(s_i)$ and $\beta(u_0) \not\ll$

$\beta(s_i)$. So in particular we get $\beta(s_{i-1}) \not\ll \beta(s_i)$. If β_i were not a null geodesic then timelike curves from $\beta(s_{i-1})$ to $\beta(s_i)$ (arbitrarily near to β_i) would exist (see [O'N83] Proposition 10.46), a contradiction to $\beta(s_{i-1}) \not\ll \beta(s_i)$. It follows that β_i is a future null geodesic. Similarly we get that β_{i+1} is a past null geodesic. We now extend this to β. Suppose $a = s_0 < \ldots < s_{i-2} < s_{i-1} < s_i$ and look at $\beta_{i-1} = \beta|_{[s_{i-2},s_{i-1}]}$. If β_{i-1} is future causal we have $\beta(s_{i-2}) \leq \beta(s_i)$ and $\beta(s_{i-2}) \not\ll \beta(s_i)$ by the same arguments as above. So $\beta_{i-1}\beta_i = \beta|_{[s_{i-2},s_i]}$ is a future null geodesic. If β_{i-1} is past causal it is a past null geodesic analogously to above (we assumed without loss of generality that β changes from future to past directed at s_i, now use the same arguments with β changing from past to future directed at s_{i-1}). So we have that β is an piecewise null geodesic, which changes its time orientation at each breakpoint. Clearly β is achronal (otherwise we could make β shorter by joining two timelike related points). □

Remark 2.3.5. *Note that the previous Lemma is the reason why in [SV16] Sormani and Vega called $\hat{d}_\tau$ the **null** distance.*

Corollary 2.3.6 ([SV16], Corollary 3.21)**.** *Let (M, g) be a spacetime. Pick two points $p, q \in M$ with p and q not causally related, and let β be a piecewise causal curve from p to q. If β has a timelike subsegment, then there is a shorter piecewise causal curve α from p to q, i.e. $\hat{L}_\tau(\alpha) < \hat{L}_\tau(\beta)$.*

Proof. This follows directly from the proof of Lemma 2.3.4. □

2.4 Topological Properties

In this section we will introduce some topological properties of the null distance. The main result of this section is that the null distance induces the manifold topology if and only if $\hat{d}_\tau$ is continuous and definite. For further background see [SV16], Section 3.4.

Lemma 2.4.1 ([SV16], Lemma 3.13)**.** *Let τ be a generalized time function on a spacetime (M, g). Then $\hat{d}_\tau$ is bounded on causal diamonds i.e.*

$$p \leq x, y \leq q \Rightarrow \hat{d}_\tau(x, y) \leq 2(\tau(q) - \tau(p))$$

Proof. Let β be a piecewise causal curve from x to y consisting of a future directed causal curve from x to q and a past directed causal curve from q to y. Then by definition of the null length (Definition 2.1.3) and generalized time functions (Definition 1.3.1) we get:

$$\hat{L}_\tau(\beta) = |\tau(q) - \tau(x)| + |\tau(y) - \tau(q)| \leq |\tau(q) - \tau(p)| + |\tau(p) - \tau(q)| = 2(\tau(q) - \tau(p))$$

□

This result will be used to prove the following theorem.

Proposition 2.4.2 ([SV16], Proposition 3.14)**.** *Let (M, g) be a spacetime and let τ be a generalized time function on M. The null distance $\hat{d}_\tau$ is continuous if and only if τ is continuous (i.e. τ is a time function).*

Proof. Let τ be continuous. Pick $x, y \in M$ and choose α_x a future directed timelike curve through $\alpha_x(0) = x$ and α_y a future directed timelike curve through $\alpha_y(0) = y$. Let $\delta > 0$ be so small that α_x, α_y are defined on $[-\delta, \delta]$. Then for $x' \in I^+(\alpha_x(-\delta)) \cap I^-(\alpha_x(\delta))$ and $y' \in I^+(\alpha_y(-\delta)) \cap I^-(\alpha_y(\delta))$ we get by using first the triangle inequality and then Lemma 2.4.1:

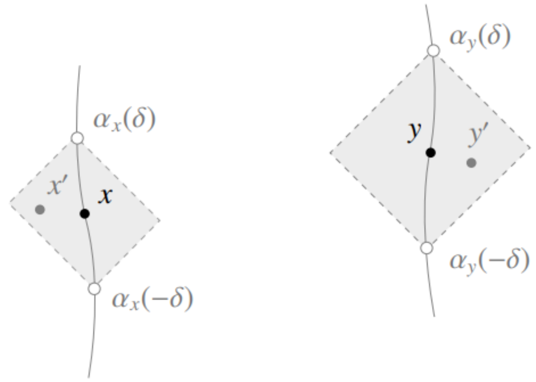

Figure 2.3: [SV16], page 16.

$$\begin{aligned}|\hat{d}_\tau(x,y) - \hat{d}_\tau(x',y')| &\leq \hat{d}_\tau(x,x') + \hat{d}_\tau(y,y') \\ &\leq 2(\tau(\alpha_x(\delta)) - \tau(\alpha_x(-\delta))) + 2(\tau(\alpha_y(\delta)) - \tau(\alpha_y(-\delta)))\end{aligned}$$

Because τ is continuous the right-hand side goes to 0 for $\delta \to 0$. This means that $\hat{d}_\tau$ is continuous. Now let $\hat{d}_\tau$ be continuous. Pick $q_0 \in M$ and $(q_0 \neq) p_0 \in I^-(q_0)$. Then for all $x \in I^+(p_0)$ we get from Theorem 2.2.8: $\tau(x) = \hat{d}_\tau(p_0, x) + \tau(p_0)$. This implies that τ is continuous in p_0. Because we chose p_0 arbitrarily it follows that τ is continuous everywhere. □

Remark 2.4.3. *The null distance $\hat{d}_\tau$ is a pseudometric on (M, g) and therefore induces a topology on (M, g). A base of this topology is given by the so called **open null balls** i.e. the sets $\hat{B}^\tau_r(x) = \{y \in M \mid \hat{d}_\tau(x,y) < r\}$. We call this topology the **null topology**.*

Lemma 2.4.4. *Let M be a manifold with manifold topology $\mathcal{T}$ and let d be a metric on M. Then d induces a topology $\widetilde{\mathcal{T}}$ on M that is coarser than the manifold topology if and only if d is continuous (with respect to $\mathcal{T}$).*

Proof. Let d be continuous (with respect to $\mathcal{T}$). Then also $d(x, \cdot)$ is continuous for all $x \in M$. This implies that $d(x,\cdot)^{-1}(-\infty, \epsilon) = \{y \in M \mid d(x,y) < \epsilon\}$ is open in $\mathcal{T}$ for all $\epsilon > 0$ and all $x \in M$. But these are exactly the d-balls, which form a basis of $\widetilde{\mathcal{T}}$. So we get $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$ i.e. $\widetilde{\mathcal{T}}$ is coarser then $\mathcal{T}$. Conversely assume that $\widetilde{\mathcal{T}}$ is coarser than $\mathcal{T}$. We know that $d : (M, \widetilde{\mathcal{T}}) \times (M, \widetilde{\mathcal{T}}) \to \mathbb{R}$ is continuous when considering the product topology: For any $\epsilon > 0$ and $x, y \in M$, choose $\delta = \frac{\epsilon}{2}$. Consider $x' \in B^d_\delta(x)$ and $y' \in B^d_\delta(y)$. From the triangle inequality we get $|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y') \leq 2\delta = \epsilon$. $\mathcal{T}$ is finer than $\widetilde{\mathcal{T}}$ and this also holds for the product topologies. It follows that $d : (M,\mathcal{T}) \times (M,\mathcal{T}) \to \mathbb{R}$ is continuous. □

Proposition 2.4.5 ([SV16], Proposition 3.15). *Let (M,g) be a spacetime and let τ be a generalized time function on M. The topology on M induced by $\hat{d}_\tau$ coincides with the manifold topology if and only if τ is continuous and $\hat{d}_\tau$ is definite.*

Proof. Suppose that the null topology coincides with the manifold topology. This means that the null topology is coarser and finer than the manifold topology. It being coarser is equivalent to $\hat{d}_\tau$ being continuous. $\hat{d}_\tau$ being continuous is equivalent to τ being continuous (by Theorem 2.4.2). Also the null topology is Hausdorff (because manifold topologies are Hausdorff) and therefore it follows that $\hat{d}_\tau$ is definite.
Suppose that τ (hence $\hat{d}_\tau$) is continuous and $\hat{d}_\tau$ is definite. $\hat{d}_\tau$ being continuous is equivalent to the null topology being coarser than the manifold topology. So in particular we get that the open null balls are open in the manifold topology. Now we need to show that the null topology is also finer than the manifold topology i.e. every open set in the manifold topology is also open in the null topology. Let U be a subset of M that is open in the manifold topology and choose an arbitrary $x_0 \in U$. Let h be a Riemannian metric on M and d_h be the corresponding Riemannian distance function. Now choose some $\epsilon > 0$ such that the closure of $B = \{y \in M \mid d_h(x_0,y) < \epsilon\}$, is a compact set contained in U (we can do so because because manifolds are locally compact and d_h induces the manifold topology). Because $\hat{d}_\tau$ is continuous and definite and ∂B is compact we get:

$$\epsilon_0 = \min_{z \in \partial B} \hat{d}_\tau(x_0, z) > 0$$

Now fix $y_0 \notin B$. Let $\beta : [a,b] \to M$ be a piecewise causal curve from $x_0 = \beta(a)$ to $y_0 = \beta(b)$ and let z_0 be the first point on ∂B that β meets. Let β_0 denote the initial part of β from x_0 to z_0. Then we have $\beta_0 \subseteq \bar{B}$ and get:

$$\hat{L}_\tau(\beta) \geq \hat{L}_\tau(\beta_0) \geq \epsilon_0$$

Taking the infimum over all such β from x_0 to y_0 yields $\hat{d}_\tau(x_0,y_0) \geq \epsilon_0$. So for all $y_0 \notin B$ we have $\hat{d}_\tau(x_0,y_0) \geq \epsilon_0$. This means $\hat{B}_{\epsilon_0}(x_0) \subseteq B \subseteq U$. Because we chose x_0 arbitrarily this means, U is also open in the null topology, since for every $x_0 \in U$ we can find an $\epsilon_0 > 0$ such that $\hat{B}_{\epsilon_0}(x_0) \subseteq U$. □

3 Definiteness

This chapter addresses the question under what conditions the null distance is definite. Note that the null distance depends on the underlying spacetime as well as the choice of the generalized time function. In the first subsection we will prove that on every stably causal spacetime we can find a time function such that the null distance (with respect to this time function) is definite. In the second section we will encounter a condition on generalized time functions that ensures the definiteness of the null distance (with respect to this generalized time function). For further background see [SV16] Section 4.

3.1 Existence of a definite null distance function

Lemma 3.1.1 ([SV16], Lemma 4.1)**.** *Let (M, g) be a spacetime and let $\tau : M \to \mathbb{R}$ be a smooth map with past pointing timelike unit gradient $\|\nabla\tau\|_g = 1$. Then $\hat{d}_\tau$ is definite.*

Proof. Consider the Riemannianization g_τ^R of g with respect to τ (see Proposition 1.4.2 and Corollary 1.4.4). Recall that we denoted the Riemannian distance function (with respect to g_τ^R) by d_τ^R and the Riemannian arc length (with respect to g_τ^R) by L_τ^R. Now fix two arbitrary points p, q in M and a piecewise causal curve $\beta : [a, b] \to M$ with breakpoints $a = s_0 < s_1 \ldots < s_k = b$. from p to q. Then we get, using the definition of the arc length and the fact that $g(\beta', \beta') \leq 0$ (because β is causal):

$$
\begin{aligned}
L_\tau^R(\beta) &= \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \|\beta'(t)\|_{g_\tau^R} dt \\
&= \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \sqrt{|2(d\tau(\beta'(t)))^2 + g(\beta'(t), \beta'(t))|} dt \\
&\leq \sqrt{2} \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} |d\tau(\beta'(t))| dt \\
&= \sqrt{2} \sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} |(\tau \circ \beta)'(t)| dt \\
&= \sqrt{2} \hat{L}_\tau(\beta)
\end{aligned}
$$

Taking the infimum over all piecewise causal curves from p to q we get $d_\tau^R(p, q) \leq \inf L_\tau^R(\beta) \leq \sqrt{2} \inf \hat{L}_\tau(\beta) = \sqrt{2} \hat{d}_\tau(p, q)$. In particular it follows that $\hat{d}_\tau$ is definite (because $\hat{d}_\tau^R$ is definite). $\square$

Proposition 3.1.2 ([SV16], Proposition 4.2)**.** *Let (M, g) be a stably causal spacetime (which is equivalent to (M, g) admitting a time function, see Theorem 1.3.5). Then we can find a time function τ on (M, g) such that $\hat{d}_\tau$ is definite.*

Proof. (M, g) admitting a time function is equivalent to (M, g) admitting a temporal function (see Theorem 1.3.5), i.e. some $\tau : M \to \mathbb{R}$ smooth with $\nabla\tau$ past pointing timelike. By scaling τ we get $\|\nabla\tau\|_g = 1$. Now we can use Lemma 3.1.1 and get the result. $\square$

3.2 Anti-Lipschitz condition

In this section we introduce the so-called anti-Lipschitz condition. The main result of this section is that the null distance $\hat{d}_\tau$ is definite if and only if τ satisfies the anti-Lipschitz condition locally.

Definition 3.2.1 (locally anti-Lipschitz)**.** *Let (M, g) be a spacetime and let $U \subseteq M$. A map $f : M \to \mathbb{R}$ is **anti-Lipschitz on U** if there exists a continuous metric d_U on U such that for all $x, y \in U$:*

$$x \leq y \Rightarrow f(y) - f(x) \geq d_U(x, y)$$

*We say that f is **locally anti-Lipschitz** if for each $x \in M$ there exists a neighborhood U of x in M such that f is anti-Lipschitz on U.*

Remark 3.2.2. *In [SV16], the continuity of d_U in the anti-Lipschitz condition is missing but implicitly assumed throughout the paper. It is used for example in the proof of the next theorem.*

Lemma 3.2.3 ([SV16], Lemma 4.3)**.** *Let (M, g) be a spacetime and let τ be a generalized time function on M. If there exists some neighborhood U in M such that f is anti-Lipschitz on U (with respect to a continuous metric d_U on U) then $\hat{d}_\tau$ distinguishes points in U. This means: $\forall p \in U : \forall q \in M \setminus \{p\} : \hat{d}_\tau(p, q) > 0$. In particular, $\hat{d}_\tau$ is definite on U.*

Proof. Let $p \in U$ and $q \in M \setminus \{p\}$. Choose a precompact (i.e. the closure is compact) open neighborhood B of p such that $\bar{B} \subseteq U$ and $q \notin B$. Let β be a piecewise causal curve from p to q and $z_0 \in \partial B$ the first point on the boundary of B which β meets. Let $\beta_0 \subseteq \bar{B} \subseteq U$ denote the initial part of β from p to z_0 and let $p = x_0, \ldots, x_k = z_0$ denote the breakpoints. Because $\beta_0 \subseteq U$ we can use the triangle inequality for d_U (the continuous metric on U) and the definition of locally anti-Lipschitz (Definition 3.2.1):

$$\hat{L}_\tau(\beta) \geq \hat{L}_\tau(\beta_0) = \sum_{i=1}^{k} |\tau(x_i) - \tau(x_{i-1})| \geq \sum_{i=1}^{k} d_U(x_i, x_{i-1}) \geq d_U(p, z_0) \geq d_U(p, \partial B)$$

Taking the infimum over all such β yields: $\hat{d}_\tau(p, q) \geq d_U(p, \partial B) > 0$. For the last inequality we used the definiteness of d_U ($d_U(p, x) \neq 0$ for all $x \in \partial B$ because $p \notin \partial B$)

and the continuity of d_U ($d_U(p, \partial B) = \inf\{d_U(x,p) \,|\, x \in \partial B\} = \min\{d_U(x,p) \,|\, x \in \partial B\} \neq 0$).

$\square$

Proposition 3.2.4 ([SV16], Proposition 4.5)**.** *Let (M,g) be a spacetime and let τ be a generalized time function on M. Then $\hat{d}_\tau$ is definite if and only if τ is locally anti-Lipschitz.*

Proof. From the left-hand side to the right-hand side: By Remark 2.2.7 $\hat{d}_\tau$ is a pseudometric. By assumption it, furthermore, is definite i.e. a metric. Therefore we can use $\hat{d}_\tau = d_U$ everywhere. The converse follows from Lemma 3.2.3 and the definition of locally anti-Lipschitz. $\square$

When combining Proposition 3.2.4, Proposition 2.4.2 and Proposition 2.2.6 we get the following important result:

Theorem 3.2.5 ([SV16], Theorem 4.6)**.** *Let τ be a time function on a spacetime (M,g). If τ is locally anti-Lipschitz, then the null distance $\hat{d}_\tau$ is a conformally invariant metric on (M,g), which induces the manifold topology.*

Lemma 3.2.6 ([GKOS01], Lemma 3.2.4)**.** *Let M be a manifold and let h_1, h_2 be two Riemannian metrics on M. Let d_1, d_2 denote the induced Riemannian distance functions on M. Then for each compact subset $K \subseteq M$ there exist two constants $\epsilon_0 > 0$ and $C > 0$ such that for all $p \in M$ and for all $0 < \epsilon < \epsilon_0$ the following holds:*

$$B^{d_2}_\epsilon(p) \subseteq B^{d_1}_{C\epsilon}(p)$$

Proof. First of all, we prove that for all compact subsets $K \subseteq M$ there exists a constant $C > 0$ such that for all $p \in K$ and all $v \in T_pM$ the following inequality holds:

$$h_1(p)(v,v) \leq C h_2(p)(v,v) \tag{3.1}$$

Without loss of generality we can assume that K is contained in the domain of a chart (ϕ_α, U_α). After identifying T_pM and $\mathbb{R}^n$, we define the map:

$$f : \phi_\alpha(U_\alpha) \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, (x,v) \mapsto \frac{h_1(\phi_\alpha^{-1}(x))(v,v)}{h_2(\phi_\alpha^{-1}(x))(v,v)}$$

Note that this map is well defined because $h_2(p)$ is positive definite for each $p \in M$. Let $\|v\|_{g_{\mathbb{E}^n}}$ denote the Euclidean norm of v and $B^{g_{\mathbb{E}^n}}_1(0)$ the ball of radius 1 around 0 with respect to the Euclidean metric. Because $h_i(p)$ is bilinear for all $p \in M$ and $i \in \{1,2\}$, we have $h_i(p)(v,v) = \|v\|^2_{g_{\mathbb{E}^n}} h_i(p)(\frac{v}{\|v\|_{g_{\mathbb{E}^n}}}, \frac{v}{\|v\|_{g_{\mathbb{E}^n}}})$ and therefore $f(x,v) = f(x, \frac{v}{\|v\|_{g_{\mathbb{E}^n}}})$. This and the fact that $\phi_\alpha(K)$ and $\partial B^{g_{\mathbb{E}^n}}_1(0)$ are compact and f is continuous yield the following:

$$\sup_{\substack{x \in \phi_\alpha(K) \\ v \in \mathbb{R}^n \setminus \{0\}}} f(x,v) = \sup_{\substack{x \in \phi_\alpha(K) \\ v \in \partial B^{g_{\mathbb{E}^n}}_1(0)}} f(x,v) < \infty$$

This means that there exists a constant $C > 0$ such that for all $p \in K$ and all $v \in T_pM$, $h_1(p)(v,v) \leq Ch_2(p)(v,v)$ holds. We know that every $p \in M$ possesses a basis of neighborhoods that consists of (geodesically) convex sets (see for example [KS21], Theorem 2.2.7). Also M is locally compact. Therefore, for any $p \in M$, we can choose a (geodesically) convex neighborhood U_p of p with respect to h_2, that is relatively compact. Denote by C_p the square root of the constant in equation (3.1), using $\overline{U_p}$ as compact set. Pick $q_1, q_2 \in U_p$ and denote by α the unique radial geodesic in U_p joining q_1 and q_2 (see [KS21], Proposition 2.1.15). We can assume that we chose U_p small enough such that α is minimal (see [KS21], Proposition 2.3.6) and get

$$d_1(q_1, q_2) \leq L_{h_1}(\alpha) \leq C_p L_{h_2}(\alpha) = C_p d_2(q_1, q_2).$$

Now choose $K \subseteq M$ arbitrarily. For each $p \in K$ pick $C_p > 0$ and U_p as defined above and $\epsilon_p > 0$ such that $B^{d_2}_{\epsilon_p}(p) \subseteq U_p$. Because, clearly, $\bigcup\limits_{p \in K} B^{d_2}_{\frac{\epsilon_p}{2}}(p)$ is an open cover of K, it follows that there exists a finite subcover $K \subseteq \bigcup\limits_{i=1,\dots,m} B^{d_2}_{\frac{\epsilon_{p_i}}{2}}(p_i) =: U$. Define $C := \max\{C_{p_i} \,|\, i = 1, \dots, m\}$ and $\epsilon_0 := \min\{d_2(K, \partial U), \epsilon_{\frac{p_1}{2}}, \dots, \epsilon_{\frac{p_m}{2}}\}$. Finally we choose $p \in K$, $\epsilon < \epsilon_0$ and $q \in B^{d_2}_{\epsilon}(p)$ and show that $q \in B^{d_1}_{C\epsilon}(p)$. By construction there exists an $i \in \{1, \dots, m\}$ such that $p \in B^{d_2}_{\frac{\epsilon_{pi}}{2}}$ i.e. $d_2(p,q) \leq \epsilon_{\frac{\epsilon_{p_i}}{2}}$. Also we know that $d(p,q) < \epsilon \leq \frac{\epsilon_{p_i}}{2}$. Together, these results yield that $p, q \in B^{d_2}_{\epsilon_{p_i}}(p_i) \subseteq U_{p_i}$. Therefore $d_1(p,q) \leq C_{p_i} d_2(p,q)$ holds and $q \in B^{d_1}_{C\epsilon}(p)$ follows. □

Lemma 3.2.7 ([GKOS01], Lemma 3.2.5)**.** *Let h_1 and h_2 be two Riemannian metrics on a manifold M and denote the induced Riemannian distance functions by d_1 and d_2. For any two compact subsets $K, K' \subseteq M$ there exists a constant $C > 0$ such that for all $p \in K$ and $q \in K'$ we have:*

$$d_2(p,q) \leq C d_1(p,q)$$

Proof. Assume that there exist two compact subsets $K, K' \subseteq M$ such that the statement is not true. Then we can find two sequences, $(p_n)_{n\in\mathbb{N}}$ in K and $(q_n)_{n\in\mathbb{N}}$ in K', such that $d_2(p_n, q_n) > n d_1(p_n, q_n)$ holds. Because K and K' are compact we can choose convergent subsequences (which we will also denote by $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$). Note that a sequence is convergent with respect to d_1 if and only if it is convergent with respect to d_2, because d_1 and d_2 both induce the manifold topology. Let $(p_n)_{n\in\mathbb{N}}$ converge to $p \in M$ and $(q_n)_{n\in\mathbb{N}}$ converge to $q \in M$. By taking the limit for $n \to \infty$ on both sides of our assumption $(d_2(p_n, q_n) > n d_1(p_n, q_n))$ we get that $p = q$. Choose a relatively compact neighborhood V of $p \in M$. By Lemma 3.2.6, there exist $\epsilon_0 > 0$ and $\alpha > 0$ such that for all $x \in V$ and for all $0 < \epsilon < \epsilon_0$ we have $B^1_\epsilon(x) \subseteq B^2_{\alpha\epsilon}(x)$, where $B^i_\epsilon(x) = \{y \in M \,|\, d_i(x,y) < \epsilon\}$. In particular this is true for $x = p$. For $n \in \mathbb{N}$ large enough $p_n \in B^1_\epsilon(p)$ and $q_n \in B^1_\epsilon(p)$ and therefore $d_2(p_n, q_n) \leq \alpha d_1(p_n, q_n)$. This contradicts our assumption if we choose an $\alpha < n$. □

Lemma 3.2.8 ([SV16], Lemma 4.7)**.** *Let (M, g) be a spacetime and let h be any Riemannian metric on M. For a function $f : M \to \mathbb{R}$ the following conditions are equivalent:*

1. *For all $p \in M$ there exists a neighborhood U of p and a positive constant $C > 0$ such that for all $x, y \in U$:*

$$x \leq y \Rightarrow f(y) - f(x) \geq C d_h(x, y)$$

2. *For all $p \in M$ there exists a neighborhood U of p and a Riemannian metric h_U on U such that for all $x, y \in U$:*

$$x \leq y \Rightarrow f(y) - f(x) \geq d_{h_U}(x, y)$$

Proof. This follows immediately from Lemma 3.2.7. □

Lemma 3.2.9 ([SV16], Lemma 4.7)**.** *Let (M, g) be a spacetime and let $f : M \to \mathbb{R}$ be a map. If one (hence both) of the equivalent conditions in Lemma 3.2.8 are satisfied then f is locally anti-Lipschitz, i.e. for all $p \in M$ there exists a neighborhood U of p and a continuous metric d_U on U such that for all $x, y \in U$:*

$$x \leq y \Rightarrow f(y) - f(x) \geq d_U(x, y)$$

Proof. We get from Lemma 3.2.8 *2.* to the locally anti-Lipschitz condition by choosing $d_U = d_{h_U}$. □

Remark 3.2.10. *In [SV16], Lemma 4.7, Sormani and Vega state that also the converse of Lemma 3.2.9 is true i.e. if $f : M \to \mathbb{R}$ is locally anti-Lipschitz, then one (hence both) of the equivalent conditions of Lemma 3.2.8 is satisfied. They justify this claim by stating that on a manifold all metrics are locally Lipschitz equivalent. However, this is not true. Consider for example the following situation:*
Let $M = \mathbb{R}^2$ be the manifold and for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ let

$$\begin{aligned} d_1(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ d_2(x, y) &= \sqrt{|x_1 - y_1| + |x_2 - y_2|} \end{aligned}$$

which are two continuous metrics on $\mathbb{R}$. These metrics are not locally Lipschitz equivalent. Otherwise for each point $p \in M$ there would exist a neighborhood U of p and a constant $C > 0$ such that for all $x, y \in U$ we have:

$$d_2(x, y) \leq C d_1(x, y)$$

But then

$$\sqrt{|x_1 - y_1| + |x_2 - y_2|} \leq C(|x_1 - y_1| + |x_2 - y_2|)$$

so,

$$\frac{1}{\sqrt{|x_1 - y_1| + |x_2 - y_2|}} \leq C$$

This is a contradiction, because for $x \to y$ the left-hand side approaches ∞ and can therefore not be bounded by a constant $C > 0$.

In fact the inversion of Lemma 3.2.9 does not hold in general. I thank Tobias Beran for constructing a counterexample and include it here.

Example 3.2.11 (The converse of Lemma 3.2.9 does not hold)**.** *Consider the two-dimensional Minkowski space $\mathbb{M}^2 = (\mathbb{R}^2, g_{\mathbb{M}^2})$ and let $d_{\mathbb{E}^2}$ be the Riemannian distance function induced by the Euclidean metric $g_{\mathbb{E}^2}$ on $\mathbb{R}^2$. We define the maps $f_1, f_2, f : \mathbb{R}^2 \to \mathbb{R}$ the following way:*

$$
\begin{aligned}
f_1(t,x) &= \left(t - \frac{x}{2}\right)^3 \\
f_2(t,x) &= \left(t + \frac{x}{2}\right)^3 \\
f(t,x) &= \frac{f_1(t,x) + f_2(t,x)}{2}
\end{aligned}
$$

f_1 is a time function. Pick two points $p, q \in \mathbb{M}^2$ with $(s,x) = p \leq q = (t,y)$. We therefore know that $t \geq s$ and $t - s \geq |y - x|$ holds. The following observation yields the claim:

$$f_1(q) \geq f_1(p) \Leftrightarrow (t - \frac{y}{2})^3 \geq (s - \frac{x}{2})^3 \Leftrightarrow t - \frac{y}{2} \geq s - \frac{x}{2} \Leftrightarrow t - s \geq \frac{y}{2} - \frac{x}{2}$$

Similarly we get that f_2 is a time function. It follows that f is a time function because it is the sum of two time functions scaled by a positive number. Note that f and $d_{\mathbb{E}^2}$ do not satisfy the first condition of Lemma 3.2.8. For $p = (0,0)$ and any neighborhood U of p we can choose some sequence $(q_n = (t_n, 0))_{n\in\mathbb{N}}$ in U with $t_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = 0$. It follows that for all $n \in \mathbb{N}$ we have $p \leq q_n$. But there exists no $C > 0$ such that for all $n \in \mathbb{N}$ we have $t_n^3 = f(q_n) - f(p) \geq C d_{\mathbb{E}^2}(p, q_n) = C t_n$ because $\lim_{n\to\infty} t_n = 0$.
Now we want to show that f is locally anti-Lipschitz i.e. for each $p \in \mathbb{R}^2$ there exists a neighborhood U of p and a continuous metric d_U on U such that for all $x, y \in U$:

$$x \leq y \Rightarrow f(y) - f(x) \geq d_U(x,y)$$

We know that when considering $\hat{d}_f$, the null distance on $\mathbb{M}^2$ with respect to f (a time function), $x \leq y \Rightarrow f(y) - f(x) = \hat{d}_f(x,y)$ always holds. If we can show that $\hat{d}_f$ is definite (and therefore a metric) we have a counterexample for the converse of Lemma 3.2.9. To achieve this, we first show that f_1 is locally anti-Lipschitz away from the line $t = \frac{x}{2}$. Each point above the line is of the form $\tilde{p} = (\tilde{s}, \tilde{x})$ with $\tilde{s} > \frac{\tilde{x}}{2} + \tilde{\epsilon}$ for some $\tilde{\epsilon} > 0$. For such a point we can choose a neighborhood U such that for all points $\tilde{q} = (\tilde{t}, \tilde{y}) \in U$ we have $\tilde{t} > \frac{\tilde{y}}{2} + \epsilon$ for some $\epsilon < \tilde{\epsilon}$. Choose such a point $\tilde{p} \in \mathbb{R}^2$ and such a neighborhood U of $\tilde{p}$. Consider $p = (s,x), q = (s+t, x+y) \in U$ with $p \leq q$. This means $s > \frac{x}{2} + \epsilon$, $t > 0$ and $t \geq |y|$. We have:

$$f_1(q) - f_1(p) = \left(s + t - \frac{x+y}{2}\right)^3 - \left(s - \frac{x}{2}\right)^3$$

Now we use the fact that for $a, b \in \mathbb{R}$ we have $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. We choose $a = s+t-\frac{x+y}{2}$ and $b = s - \frac{x}{2}$ and notice that $(a^2+ab+b^2) > \epsilon^2$ because $a, b > \epsilon$. Therefore the above equation can be estimated in the following way:

$$\left(s+t-\frac{x+y}{2}\right)^3 - \left(s-\frac{x}{2}\right)^3 \geq \left(t-\frac{y}{2}\right)\epsilon^2$$

Using $t > |y|$ yields

$$(t-\frac{y}{2})\epsilon^2 \geq \left(\frac{t}{3}+\frac{2}{3}|y|-\frac{y}{2}\right)\epsilon^2 \geq \left(\frac{t}{6}+\frac{|y|}{6}\right)\epsilon^2 = \frac{\epsilon^2}{6}\tilde{d}(p,q)$$

when considering the continuous metric $\tilde{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto |x_1 - y_1| + |x_2 - y_2|$. So on U we get that f_1 is anti-Lipschitz with respect to the continuous metric $\frac{\epsilon^2}{6}\tilde{d}|_U$. For points below the line we can proceed analogously. It follows that f_1 is locally anti-Lipschitz away from the line $t = \frac{x}{2}$. Similarly one can show that f_2 is locally anti-Lipschitz away from the line $t = -\frac{x}{2}$. It follows that f is locally anti-Lipschitz away from 0 (the intersection of the two lines). This means that $\tilde{f} := f|_{\mathbb{R}^2\setminus\{(0,0)\}}$ is locally anti-Lipschitz and therefore $\hat{d}_{\tilde{f}}$ is a metric on $\mathbb{R}^2 \setminus \{(0,0)\}$. Note that because f is continuous we have that $\hat{d}_f|_{\mathbb{R}^2\setminus\{(0,0)\}} = \hat{d}_{\tilde{f}}$. If we have a piecewise causal curve α in $\mathbb{M}^2$ connecting two points $p, q \in \mathbb{R}^2 \setminus \{(0,0)\}$, we can approximate it by a sequence of piecewise causal curves in $\mathbb{R}^2 \setminus \{(0,0)\}$ converging to α. Because of the continuity of f also the null length of the curves converges to the null length of α. Furthermore $\hat{d}_f((0,0),(0,0)) = 0$ by the definition of the null distance. For $p \neq (0,0)$ we can always choose an open ball $B_\epsilon(p)$ around p (with respect to $d_{\mathbb{E}^2}$, the Euclidean distance function on $\mathbb{R}^2$) such that $(0,0) \notin B_\epsilon(p)$. Then we have $\hat{d}_f(p,(0,0)) \geq \hat{d}_f(p, \partial B_\epsilon(p)) = \hat{d}_{\tilde{f}}(p, \partial B_\epsilon(p)) > 0$. The first inequality holds because each piecewise causal curve from p to $(0,0)$ needs to leave $B_\epsilon(p)$ at some point, the last inequality holds because $\hat{d}_{\tilde{f}}$ is definite and continuous. It follows that $\hat{d}_f$ is definite and therefore f is locally anti-Lipschitz.

4 Bi-Lipschitz bounds

This chapter contains a result obtained by A. Burtscher and L. García-Heveling in [BH22], Section 2. Namely that the null distance $\hat{d}_\tau$ with respect to an admissible time function τ is Lipschitz equivalent to any Riemannian distance function on all compact subsets of the spacetime (see [BH22], Theorem 2.3). The proof makes use of a splitting theorem obtained by O. Müller and M. Sánchez, see [MS11], Lemma 3.5.

4.1 Admissible time functions

In this section admissible time functions will be defined and some important examples will be stated.

Definition 4.1.1 (admissible time function)**.** *Let $\tau : M \to \mathbb{R}$ be a time function on a spacetime (M, g), let h be a Riemannian metric on M and let d_h be the induced Riemannian distance function. If for all $p \in M$ there exists a neighborhood U of p in M and a constant $C \geq 1$ such that for all $x, y \in U$*

$$x \leq y \Rightarrow \frac{1}{C} d_h(x, y) \leq \tau(y) - \tau(x) \leq C d_h(x, y)$$

*holds, then τ is called an **admissible time function**.*

Remark 4.1.2. *Note that this definition is independent of the choice of the Riemannian metric h. Let the condition be satisfied for the Riemannian distance function d_h on the neighborhood U of p. Because M is locally compact, we can choose a compact neighborhood K of p with $K \subseteq U$. By Lemma 3.2.7 d_h is bi-Lipschitz equivalent to any other Riemannian distance function d_g on K, i.e. there exist constants $D, E > 0$ such that $D d_h(p, q) \leq d_g(p, q) \leq E d_h(p, q)$ for all $p, q \in K$. This means that d_g also satisfies the condition on K, a compact neighborhood of p. Moreover, note that the first inequality implies that the locally anti-Lipschitz condition is satisfied, see Lemma 3.2.8 and Lemma 3.2.9. The second inequality is a Lipschitz condition.*

First of all, note that the following fact is also true in Riemannian geometry:

Lemma 4.1.3 (smooth $\Rightarrow$ locally Lipschitz)**.** *Let (M, h) be a Riemannian manifold and d_h be the induced Riemannian distance function. Let $f : M \to \mathbb{R}$ be a smooth map. It follows that f is locally Lipschitz continuous i.e. $\forall x \in M : \exists U$ neighborhood of x: $\exists$ $C > 0$: $\forall p, q \in U$:*

$$|f(q) - f(p)| \leq C d_h(p, q)$$

Proof. Pick such a smooth $f : M \to \mathbb{R}$. By definition of smooth maps on manifolds we get that for each point $p \in M$ there exists a chart (ϕ, U) around p such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. Note, that the map $f \circ \phi^{-1}$ goes from an open subset of $\mathbb{R}^n$ to $\mathbb{R}$. Therefore the smoothness of $f \circ \phi^{-1}$ implies local Lipschitz continuity i.e. $\forall p \in \phi(U) : \exists W$ open neighborhood of p in $\phi(U)$: $\exists C > 0$: $\forall x, y \in W$:

$$|(f \circ \phi^{-1})(y) - (f \circ \phi^{-1})(x)| \leq C\|y - x\|_{\mathbb{E}^n} \tag{4.1}$$

Equivalently, when denoting $\tilde{x} = \phi^{-1}(x)$ and $\tilde{y} = \phi^{-1}(x)$ we can rewrite 4.1 the following way:

$$|f(\tilde{y}) - f(\tilde{x})| \leq C\|\phi(\tilde{y}) - \phi(\tilde{x})\|_{\mathbb{E}^n} \tag{4.2}$$

If we manage to show that locally $\|\phi(\tilde{y}) - \phi(\tilde{x})\|_{\mathbb{E}^n} \leq \tilde{C} d_h(\tilde{x}, \tilde{y})$ holds for some constant $\tilde{C} > 0$, we are done. Let p be a point in M and choose a neighborhood U of p small enough such that U is contained in a chart domain, U is contained in a compact set (M is locally compact), U is geodesically convex and for any two points $x, y \in U$ the radial geodesic $\rho_{xy} : [0, 1] \to U$ in U joining x and y has minimal length. Now we pick two points $x, y \in U$ and get

$$d_h(x, y) = L_{d_h}(\rho_{xy}).$$

Let $(\phi, \tilde{U})$ be a chart of M in p such that U is contained in the chart domain $\tilde{U}$. Then we consider the pullback $(\phi^{-1})^*(h|_{\tilde{U}})$ of $h|_{\tilde{U}}$ under ϕ^{-1} and calculate $L_{(\phi^{-1})^*(h|_{\tilde{U}})}(\phi \circ \rho_{xy})$.

$$\begin{aligned}
L_{(\phi^{-1})^*(h|_{\tilde{U}})}(\phi \circ \rho_{xy}) &= \int_0^1 \sqrt{((\phi^{-1})^*(h|_{\tilde{U}}))((\phi \circ \rho_{xy})(t))(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)), T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)))}dt \\
&= \int_0^1 \sqrt{h|_{\tilde{U}}(\phi^{-1}(\phi(\rho_{xy}(t))))(T_{(\phi\circ\rho_{xy})(t)}\phi^{-1}(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t))), T_{(\phi\circ\rho_{xy})(t)}\phi^{-1}(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t))))}dt \\
&= \int_0^1 \sqrt{h|_{\tilde{U}}(\rho_{xy}(t))(\rho'_{xy}(t), \rho'_{xy}(t))}dt \\
&= L_{d_h}(\rho_{xy})
\end{aligned}$$

Note that $(\phi^{-1})^*(h|_{\tilde{U}})$ is a Riemannian metric on $\phi(\tilde{U})$. We know that $(\phi^{-1})^*(h|_{\tilde{U}})$ is a $(0, 2)$-tensor. Its symmetry and positive definiteness follow directly from $h|_{\tilde{U}}$ having these properties. For the nondegeneracy we also use that ϕ^{-1} is a diffeomorphism and therefore $T_p\phi^{-1}$ is a linear isomorphism.
We know from the proof of Lemma 3.2.6 that for all compact subsets K of a manifold M and for all h_1, h_2 Riemannian metrics on M there exists a constant $C > 0$ such that for all $p \in K$ and all $v \in T_pM$ the following inequality holds:

$$h_1(p)(v, v) \leq Ch_2(p)(v, v)$$

We chose U to be contained in a compact set, therefore w.l.o.g. we can assume that also $\tilde{U}$ is contained in a compact set. So when choosing $M = \phi(\tilde{U})$, $h_1 = g_{\mathbb{E}^n}|_{\phi(\tilde{U})}$ and

$h_2 = (\phi^{-1})^*(h|_{\tilde{U}})$ we can find a constant $C > 0$ such that for all $t \in [0,1]$ we have

$$g_{\mathbb{E}^n}|_{\phi(\tilde{U})}(\phi(\rho_{xy}(t)))(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)), T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t))) \leq C(\phi^{-1})^*(h|_{\tilde{U}})(\phi(\rho_{xy}(t)))(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)), T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)))$$

It follows, that

$$\begin{aligned} L_{g_{\mathbb{E}^n}}(\phi \circ \rho_{xy}) &= \int_0^1 \sqrt{g_{\mathbb{E}^n}|_{\phi(\tilde{U})}(\phi(\rho_{xy}(t)))(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)), T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)))}dt \\ &\leq \int_0^1 \sqrt{C(\phi^{-1})^*(h|_{\tilde{U}})(\phi(\rho_{xy}(t)))(T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)), T_{\rho_{xy}(t)}\phi(\rho'_{xy}(t)))}dt \\ &= \sqrt{C}L_{(\phi^{-1})^*(h|_{\tilde{U}})}(\phi \circ \rho_{xy}) \end{aligned}$$

Note, that $\phi \circ \rho_{xy}$ is a piecewise smooth curve connecting $\phi(x)$ and $\phi(y)$. Therefore $\|\phi(x) - \phi(y)\|_{g_{\mathbb{E}^n}} \leq L_{g_{\mathbb{E}^n}}(\phi \circ \rho_{xy})$. Combining this with the results from above we get

$$\|\phi(x) - \phi(y)\|_{g_{\mathbb{E}^n}} \leq L_{g_{\mathbb{E}^n}}(\phi \circ \rho_{xy}) \leq \sqrt{C}L_{(\phi^{-1})^*(h|_{\tilde{U}})}(\phi \circ \rho_{xy}) = \sqrt{C}L_{d_h}(\rho_{xy}) = \sqrt{C}d_h(x,y)$$

which proves the claim. □

Lemma 4.1.4 ([BH22], Lemma 2.2)**.** *Temporal functions are admissible time functions.*

Proof. The locally Lipschitz condition follows from the smoothness of temporal functions. The locally anti-Lipschitz condition follows from [SV16], Theorem 4.16. □

Now we introduce a specific class of admissible time functions. They are called regular cosmological time functions and were defined by Andersson, Galloway and Howard in [AGH98].

Definition 4.1.5 (cosmological time fuction)**.** *Let (M, g) be a spacetime. The **cosmological time function** on (M, g) is defined to be $\tau : M \to (0, \infty] : q \mapsto \tau(q)$ with*

$$\tau(q) = \sup_{p<q} d(p,q)$$

where d denotes the time separation function. Denote by $\mathcal{C}^-(q)$ the set of all past directed causal curves c in M that start at q. Then we can use the equivalent definition

$$\tau(q) = \sup\{L(c) \mid c \in \mathcal{C}^-(q)\}.$$

Definition 4.1.6 (regular cosmological time function)**.** *Let (M, g) be a spacetime with cosmological time function τ. Then τ is called **regular** if it satisfies the following two conditions:*

1. *(M, g) has finite existence times i.e $\tau(q) < \infty$ for all $q \in M$*
2. *$\tau \to 0$ along every past inextendible causal curve*

Lemma 4.1.7 ([BH22], Lemma 2.2)**.** *Regular cosmological time functions are admissible time functions.*

Proof. The locally Lipschitz condition follows from [AGH98] Theorem 1.2. The locally anti-Lipschitz condition follows from [SV16], Theorem 5.4. □

4.2 Statement and implications

This section contains the result we want to prove and two corollaries.

Theorem 4.2.1 ([BH22], Theorem 2.3)**.** *Consider a spacetime (M, g) with $\tau : M \to \mathbb{R}$ an admissible time function and h a Riemannian metric on M. Then for all $K \subseteq M$ compact there exists a constant $C \geq 1$ such that for all $p, q \in K$ the following is true:*

$$\frac{1}{C} d_h(p, q) \leq \hat{d}_\tau(p, q) \leq C d_h(p, q)$$

Corollary 4.2.2 ([BH22], Corollary 2.4)**.** *Consider a smooth manifold M and two different spacetime metrics g_1 and g_2 with admissible time functions τ_1 and τ_2. Then for all $K \subseteq M$ compact there exists a $C \geq 1$ such that for all $p, q \in K$ the following is true:*

$$\frac{1}{C} \hat{d}_{\tau_1}(p, q) \leq \hat{d}_{\tau_2}(p, q) \leq C \hat{d}_{\tau_1}(p, q)$$

Proof. Using Theorem 4.2.1 on (M, g_1) and τ_1 and on (M, g_2) and τ_2 (with the same h both times) yields: For all $K \subseteq M$ compact there exist $C_1, C_2 \geq 1$ such that for all $p, q \in K$ the following is true:

$$\frac{1}{C_1} d_h(p, q) \leq \hat{d}_{\tau_1}(p, q) \leq C_1 d_h(p, q)$$
$$\frac{1}{C_2} d_h(p, q) \leq \hat{d}_{\tau_2}(p, q) \leq C_2 d_h(p, q).$$

It is easy to see that we can use $C = C_1 C_2$ and get the result. □

Corollary 4.2.3 ([BH22], Corollary 2.5)**.** *Consider a spacetime (M, g) with two admissible time functions τ_1 and τ_2. Then for all $K \subseteq M$ compact there exists a $C \geq 1$ such that for all $p, q \in K$ the following is true:*

$$\frac{1}{C} \hat{d}_{\tau_1}(p, q) \leq \hat{d}_{\tau_2}(p, q) \leq C \hat{d}_{\tau_1}(p, q)$$

Proof. This is a special case of Corollary 4.2.3 with $g_1 = g_2 = g$. □

4.3 Comparison of d_τ^R and $\hat{d}_\tau$

In this section we show that the null distance $\hat{d}_\tau$ with respect to a temporal function τ is locally Lipschitz equivalent to the riemannianized distance function d_τ^R. This result will be used in the following section to prove Theorem 4.2.1. For further background see [BH22], Section 2.

Theorem 4.3.1 ([MS11], Lemma 3.5)**.** *Let (M, g) be a spacetime that admits a temporal function τ. Then the metric admits an orthogonal decomposition of the form*

$$g = -\alpha d\tau^2 + \bar{g}$$

where $\alpha = \frac{1}{\|\nabla \tau\|_g^2}$ and $\bar{g}$ is a symmetric 2-tensor, which vanishes on $\nabla \tau$ and is positive definite on the complement.

Proof. See [MS11], Lemma 3.5. $\square$

Remark 4.3.2. *Consider a spacetime (M, g) with a temporal function τ. From Theorem 4.3.1 we get that $g = -\alpha d\tau^2 + \bar{g}$. Now we use Corollary 1.4.4 to riemannianize g:*

$$\begin{aligned} g^R_\tau &= 2\frac{d\tau^2}{\|\nabla\tau\|^2_g} + g \\ &= 2\frac{d\tau^2}{\|\nabla\tau\|^2_g} - \alpha d\tau^2 + \bar{g} \\ &= 2\frac{d\tau^2}{\|\nabla\tau\|^2_g} - \frac{d\tau^2}{\|\nabla\tau\|^2_g} + \bar{g} \\ &= \frac{d\tau^2}{\|\nabla\tau\|^2_g} + \bar{g} \\ &= \alpha d\tau^2 + \bar{g} \end{aligned}$$

Because $\hat{d}_\tau$ is conformally invariant, we can assume without loss of generality that $g = -d\tau^2 + \tilde{g}$ and $g^R_\tau = d\tau^2 + \tilde{g}$ with $\tilde{g} = \alpha^{-1}\bar{g}$. Recall that we denote the Riemannian distance function (with respect to g^R_τ) by d^R_τ and the Riemannian arc length (with respect to g^R_τ) by L^R_τ.

Lemma 4.3.3 ([BH22], Lemma 2.8)**.** *Consider a spacetime (M, g) with temporal function τ and a piecewise causal curve $\beta : [a, b] \to M$. Then we have*

$$\hat{L}_\tau(\beta) \leq L^R_\tau(\beta) \leq \sqrt{2}\hat{L}_\tau(\beta).$$

Therefore for all $p, q \in M$ we have

$$d^R_\tau(p, q) \leq \sqrt{2}\hat{d}_\tau(p, q).$$

Proof. Consider a piecewise causal curve $\beta : [a, b] \to M$ and let $a = s_0 < \ldots < s_n = b$ denote the breakpoints. Because we know that τ is smooth and the tangent vectors β' exist almost everywhere we get

$$\begin{aligned} \hat{L}_\tau(\beta) &= \sum_{i=1}^{n} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))| \\ &= \sum_{i=1}^{n} |\int_{s_{i-1}}^{s_i} (\tau \circ \beta)'(s) ds| \\ &= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} |(\tau \circ \beta)'(s)| ds \\ &= \int_a^b |(\tau \circ \beta)'(s)| ds \end{aligned}$$

Note for the third equality: Because β is a piecewise causal curve, β is either future directed causal or past directed causal in each $[s_{i-1}, s_i]$. Therefore $(\tau \circ \beta)$ is either monotonically increasing or monotonically decreasing in each $[s_{i-1}, s_i]$. If follows that $(\tau \circ \beta)'$ is either positive or negative in each $[s_{i-1}, s_i]$, which is the reason why we can pull the absolute value inside of the integral. Now we also write $L^R_\tau(\beta)$ in terms of an integral:

$$\begin{aligned} L^R_\tau(\beta) &= \int_a^b \|\beta'(s)\|_{g^R_\tau} ds \\ &= \int_a^b \sqrt{|g^R_\tau(\beta'(s), \beta'(s))|} ds \\ &= \int_a^b \sqrt{|d\tau^2(\beta'(s), \beta'(s)) + \tilde{g}(\beta'(s), \beta'(s))|} ds \\ &= \int_a^b \sqrt{|d\tau(\beta'(s))|^2 + \|\beta'(s)\|^2_{\tilde{g}}} ds \end{aligned}$$

Note for the fourth equality: $\tilde{g}$ vanishes on $\nabla\tau$ and is positive definite on the complement, so in particular we have $\tilde{g}(\beta'(t), \beta'(t)) \geq 0$. Also, of course, $d\tau^2(\beta'(t), \beta'(t)) \geq 0$. We immediately get that $\hat{L}_\tau(\beta) \leq L^R_\tau(\beta)$. For the second inequality note that β is piecewise causal i.e. $g(\beta'(t), \beta'(t)) \leq 0$. But $g(\beta'(t), \beta'(t)) = -d\tau^2(\beta'(t), \beta'(t)) + \tilde{g}(\beta'(t), \beta'(t))$ and therefore $\tilde{g}(\beta'(t), \beta'(t)) \leq d\tau^2(\beta'(t), \beta'(t))$. Also note that $(\tau \circ \beta)'(s) = d\tau(\beta'(t))$. Using these results we get:

$$\begin{aligned} L^R_\tau(\beta) &= \int_a^b \sqrt{|d\tau(\beta'(s))|^2 + \|\beta'(s)\|^2_{\tilde{g}}} ds \\ &\leq \int_a^b \sqrt{2|d\tau(\beta'(s))|^2} ds \\ &= \sqrt{2} \int_a^b |(\tau \circ \beta)'(s)| ds \\ &= \sqrt{2} \hat{L}_\tau(\beta) \end{aligned}$$

□

Lemma 4.3.4 ([BH22], Lemma 2.9)**.** *Consider a spacetime (M, g) with a temporal function τ. For all $p \in M$ there exists a neighborhood $U \subseteq M$ such that for all $x, y \in U$ we have:*

$$\hat{d}_\tau(x, y) \leq 4 d^R_\tau(x, y)$$

Proof. See [BH22], Lemma 2.9. □

4.4 Proof

Now we generalize these results for admissible time functions and Riemannian metrics.

Lemma 4.4.1 ([BH22], Lemma 2.10)**.** *Consider a spacetime with admissible time function τ. Let h be a Riemannian metric on (M, g). Then for each $x \in M$ there exists a neighborhood U of x in M and a constant $C \geq 1$ such that for all $p, q \in U$ the following holds:*

$$\frac{1}{C} d_h(p,q) \leq \hat{d}_\tau(p,q) \leq C d_h(p,q)$$

Proof. Note that it is enough to show the result for one Riemannian metric, because Riemannian distance functions are Lipschitz equivalent on all compact sets (recall Lemma 3.2.7) and we can choose U to be precompact. We pick a point $x \in M$ and show separately that the two inequalities hold locally around x (for different Riemannian metrics h). The lemma follows by using the intersection of the two resulting neighborhoods of x as U.

Step 1: First inequality. This follows from the definition of admissible time functions: Pick a Riemannian metric h, a neighborhood U of x and a constant $C \geq 1$ as in Definition 4.1.1. Because $\hat{d}_\tau$ induces the manifold topology we can pick $r > 0$ such that $\hat{B}^\tau_{3r}(x) \subseteq U$. We show that for all $p, q \in V := \hat{B}^\tau_r(x)$ we have $\frac{1}{C} d_h(p,q) \leq \hat{d}_\tau(p,q)$. Pick $p, q \in V$. From the definition of the null distance it follows that we can find a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of piecewise causal curves from p to q such that the null length of the curves approaches the null distance of p and q i.e. $\hat{L}_\tau(\alpha_n) \leq \hat{d}_\tau(p,q) + \frac{1}{n}$. For $n > \frac{1}{r}$ we get that α_n is entirely contained in U. So pick such an $n > \frac{1}{r}$ and denote by $s_0 < s_1 < \ldots < s_k$ the breakpoints of α_n. By using the triangle inequality, the definition of admissible time functions and the definition of α_n we get:

$$\begin{aligned}
\frac{1}{C} d_h(p,q) &\leq \frac{1}{C} \sum_{i=1}^{k} d_h(\alpha_n(s_{i-1}), \alpha_n(s_i)) \\
&\leq \sum_{i=1}^{k} \hat{d}_\tau(\alpha_n(s_{i-1}), \alpha_n(s_i)) \\
&= \hat{L}_\tau(\alpha_n) \leq \hat{d}_\tau(p,q) + \frac{1}{n}
\end{aligned}$$

The claim follows because this is true for all $n \in \mathbb{N}$.

Step 2: Second inequality. Recall that a spacetime admits a time function if and only if it admits a temporal function (see Theorem 1.3.5). So we can choose a temporal function $\tilde{\tau}$ on (M, g) and consider the Riemannianization $g^R_{\tilde{\tau}}$ of g with respect to $\tilde{\tau}$. Now pick a neighborhood U of x and a constant $C \geq 1$ as in Definition 4.1.1 for $h = g^R_{\tilde{\tau}}$. Let $V \subseteq U$ be a geodesically convex neighborhood of x with respect to the Riemannian metric $g^R_{\tilde{\tau}}$. Without loss of generality we assume that V is also contained in the neighborhood from Lemma 4.3.4. Now we show that for all $p, q \in V$ we have $\hat{d}_\tau(p,q) \leq 5C d^R_{\tilde{\tau}}(p,q)$. (Recall that this is sufficient because $d^R_{\tilde{\tau}}$ is a Riemannian distance function and therefore Lipschitz equivalent to every other Riemannian distance function on compact sets.) Pick $p \neq q \in V$ (the case $p = q$ is trivial) and denote by σ_{pq} the minimizing geodesic in V (with respect to $d^R_{\tilde{\tau}}$) from p to q. Then we can find a piecewise causal curve α from p to q such that $L^R_{\tilde{\tau}}(\alpha) < 5L^R_{\tilde{\tau}}(\alpha)$ (see for example [BH22], proof of Lemma 2.9, page 9-11).

Let $s_0 < s_1 < \ldots < s_k$ be the breakpoints of α. We get from τ being an admissible time function:

$$\begin{aligned}
\hat{d}_\tau(p,q) \leq \hat{L}_\tau(\alpha) &= \sum_{i=1}^{k} \hat{d}_\tau(\alpha(s_i), \alpha(s_{i-1})) \\
&\leq C \sum_{i=1}^{k} d^R_{\tilde{\tau}}(\alpha(s_i), \alpha(s_{i-1})) \\
&\leq C L^R_{\tilde{\tau}}(\alpha) \\
&< 5C L^R_{\tilde{\tau}}(\sigma_{pq}) \\
&= 5C d^R_{\tilde{\tau}}(p,q)
\end{aligned}$$

This proves the claim. □

Now we use this result to prove the statement, i.e. Theorem 4.4.1 generalized for all compact sets.

Theorem 4.4.2 ([BH22], Theorem 2.3)**.** *Consider a spacetime (M,g) with $\tau : M \to \mathbb{R}$ an admissible time function and h a Riemannian metric. Then for any $K \subseteq M$ compact there exists a constant $C \geq 1$ such that for all $p,q \in K$ the following is true:*

$$\frac{1}{C} d_h(p,q) \leq \hat{d}_\tau(p,q) \leq C d_h(p,q)$$

Proof. Consider a compact subset $K \subseteq M$. Assume that the second inequality is not true for K and any $C > 0$. Then there exist sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ in K such that the following is true:

$$\hat{d}_\tau(p_n, q_n) > n d_h(p_n, q_n)$$

Because $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are sequences in a compact set we can consider convergent subsequences (which we will also denote by $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$) converging to p and q in M, respectively. Note that a sequence converges with respect to $\hat{d}_\tau$ if and only if it converges with respect to d_h, because they both induce the manifold topology. From our assumption $\hat{d}_\tau(p_n, q_n) > n d_h(p_n, q_n)$ and the fact that $\hat{d}_\tau(p,q) < \infty$ it follows that $p = q$. For any neighborhood U of p we get that $p_n, q_n \in U$ for n big enough. This means that by Theorem 4.4.1 for some $\widetilde{C} \geq 1$ the following is true:

$$\widetilde{C} d_h(p_n, q_n) \geq \hat{d}_\tau(p_n, q_n) > n d_h(p_n, q_n)$$

This is a contradiction for n large. It follows that $\hat{d}_\tau(p,q) \leq C d_h(p,q)$ for some constant $C \geq 1$. Analogously the other inequality can be proved. □

Remark 4.4.3. *Note that the assumptions of Theorem 4.2.1 are sharp. For counterexamples see [BH22] Example 2.12 and 2.13.*

5 Encoding Causality

This chapter addresses the question under which circumstances the null distance encodes the causal structure of a spacetime. The first section contains an encoding causality result for generalized Robertson Walker (GRW) spacetimes obtained by C. Sormani and C. Vega in 2016. In the second section we show that the null distance with respect to a Cauchy temporal function encodes causality in globally hyperbolic spacetimes, proved by A. Burtscher and L. García-Heveling in 2022. Slight adaptations of the proof yield that the null distance with respect to a temporal function encodes the causal structure locally in any spacetime. This result can be found in the final section of this chapter.

5.1 In GRW spacetimes with complete Riemannian fibers

This section contains an encoding causality result for generalized Robertson Walker (GRW) spacetimes obtained by C. Sormani and C. Vega in 2016. It states that in GRW spacetimes with complete Riemannian fibers the null distance with respect to a time function of the form $\tau(t,x) = \phi(t)$ with ϕ smooth and $\phi' > 0$ encodes causality. We introduce two lemmas and use them to get the encoding causality result for $\tau = t$. After that we extended it to time functions of the desired form. For further background see [SV16], Section 3.7.

Definition 5.1.1 (Warped product)**.** *Let (M, g) and (H, h) be semi-Riemannian manifolds and let $f : M \to (0, \infty)$ be smooth. The **warped product** $M \times_f H$ is the smooth product manifold $M \times H$ equipped with the semi-Riemannian metric tensor*

$$\pi_0^*(g) + (f \circ \pi_0)^2 \pi_1^*(h)$$

where $\pi_0 : M \times H \to M$ denotes the projection to M and $\pi_1 : M \times H \to H$ denotes the projection to H.

Definition 5.1.2 (Lorentzian warped product)**.** *Let (M, g) be a semi-Riemannian manifold with index 1 and dimension $n \geq 1$ and let (H, h) be a Riemannian manifold. Let $f : M \to (0, \infty)$ be smooth. The **Lorentzian warped product** $M \times_f H$ is the smooth product manifold $M \times H$ equipped with the Lorentzian metric tensor*

$$\pi_0^*(g) + (f \circ \pi_0)^2 \pi_1^*(h)$$

where $\pi_0 : M \times H \to M$ denotes the projection to M and $\pi_1 : M \times H \to H$ denotes the projection to H.

Definition 5.1.3 (Generalized Robertson Walker (GRW) spacetimes)**.** *Consider an open interval $I \subseteq \mathbb{R}$ equipped with the metric $-dt^2$ and a Riemannian manifold (S, h). Let $f : I \to (0, \infty)$ be a smooth map. A* ***Generalized Robertson Walker spacetime*** *is the Lorentzian warped product $I \times_f S$ i.e. the smooth product manifold $I \times S$ together with the Lorentzian metric tensor*

$$\pi_0^*(-dt^2) + (f \circ \pi_0)^2 \pi_1^*(h)$$

where $\pi_0 : I \times S \to I$ denotes the projection to I and $\pi_1 : I \times S \to S$ denotes the projection to S.

Notation: For the metric $\pi_0^*(-dt^2) + (f \circ \pi_0)^2 \pi_1^*(h)$ we just write $-dt^2 + f^2(t)h$. For a point $p \in I \times_f S$ we write $p = (t_p, \hat{p})$ with $t_p \in I$ and $\hat{p} \in S$.

Lemma 5.1.4 ([SV16], Lemma 3.22)**.** *Let $I \times_f S$ be a GRW spacetime with metric tensor $-dt^2 + f^2(t)h$. Consider the time function $t : I \times_f S \to I, (t_p, \hat{p}) \mapsto t_p$, which projects to the first component. Then the null distance $\hat{d}_t$ (induced by this time function) is definite.*

Proof. Denote $I \times_f S$ by M and $-dt^2 + f^2(t)h$ by g. We want to show that for $p, q \in M$ with $p \neq q$ we have $\hat{d}_t(p, q) \neq 0$. Because of Remark 2.2.7 we have that $\hat{d}_t(p, q) \geq |t(q) - t(p)| = |t_q - t_p|$. This means for $p \neq q$ with $t_q \neq t_p$ we have that $\hat{d}_t(p, q) > 0$. So it is enough to consider $p, q \in M$ with $t_p = t_q =: t_0$. Because $t_0 \in I$ and I is an open interval we can choose $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subseteq I$. Now consider a piecewise causal curve c from p to q. We want to show that the null length of α with respect to t is bounded away from 0 and distinguish two cases. Case 1: $t \circ \alpha \not\subseteq [t_0 - \delta, t_0 + \delta]$. Then we have $\hat{L}_t(\alpha) > 2\delta$. Case 2: $t \circ \alpha \subseteq [t_0 - \delta, t_0 + \delta]$. Denote $\alpha = \alpha_1 \alpha_2 \ldots \alpha_k$ with α_i being a future or past directed causal curve. So for each $i = 1, \ldots, k$ either α_i or $-\alpha_i$ (i.e. α_i backwards) is a future directed causal curve. This means we can reparameterize α_i as a curve $\alpha_i : [t_i, t_i + \delta_i] \to M : t \mapsto (t, \sigma_i(t))$ with $[t_i, t_i + \delta_i] \subseteq [t_0 - \delta, t_0 + \delta]$ and such that $\sigma_i : [t_i, t_i + \delta_i] \to S$ is a piecewise smooth curve in S. This means $\hat{L}_t(\alpha_i) = \delta_i$. By the definition of GRW spacetimes we have (denoting by $(\alpha_i(t))_0$ the component of $\alpha_i(t)$ in I and by $(\alpha_i(t))_1$ the component of $\alpha_i(t)$ in S):

$$\begin{aligned} g_{\alpha_i(t)}(\alpha_i'(t), \alpha_i'(t)) &= -(\alpha_i'(t)_0)^2 + f((\alpha_i(t))_0)^2 h_{(\alpha_i(t))_1}((\alpha_i'(t))_1, (\alpha_i'(t))_1) \\ &= -1 + f^2(t) h_{\sigma_i(t)}(\sigma_i'(t), \sigma_i'(t)) \end{aligned}$$

Using that α_i is causal (i.e. $g_{\alpha_i(t)}(\alpha_i'(t), \alpha_i'(t)) \leq 0$ for all $t \in [t_i, t_i + \delta_i]$) we get that

$$h_{\sigma_i(t)}(\sigma_i'(t), \sigma_i'(t)) \leq \frac{1}{f^2(t)}$$

for all $t \in [t_i, t_i + \delta_i]$. Also f admits a minimum $m := \min\{f(t) \,|\, t \in [t_0 - \delta, t_0 + \delta]\}$ on this compact interval. This yields

$$L_h(\sigma_i) = \int_{t_i}^{t_i + \delta_i} \sqrt{h_{\sigma_i(t)}(\sigma_i'(t), \sigma_i'(t))} dt \leq \int_{t_i}^{t_i + \delta_i} \frac{1}{f(t)} dt \leq \frac{\delta_i}{m}$$

Recall that $\hat{L}_t(\alpha_i) = \delta_i$, so we have $\hat{L}_t(\alpha_i) \geq mL_h(\sigma_i)$. Therefore

$$\hat{L}_t(\alpha) = \sum_{i=1}^{k} \hat{L}_t(\alpha_i) \geq m \sum_{i=1}^{k} L_h(\sigma_i)$$

Note that after exchanging σ_i with $-\sigma_i$ where necessary we have that $\sigma = \sigma_1 \dots \sigma_k$ is a piecewise smooth curve in S joining $\hat{p}$ and $\hat{q}$. We also know that $\hat{p} \neq \hat{q}$ holds because $p \neq q$ and $t_p = t_q$. This yields $L_h(\sigma) \geq d_h(\hat{p}, \hat{q}) > 0$. Using that the arclength of curves is invariant under monotonous reparameterizations (see [KS21], Lemma 2.3.2) we get:

$$m \sum_{i=1}^{k} L_h(\sigma_i) = mL_h(\sigma) \geq md_h(\hat{p}, \hat{q})$$

Together with the result above we have that $\hat{L}_t(\alpha) \geq md_h(\hat{p}, \hat{q})$. Because we chose α from p to q arbitrarily with $t \circ \alpha \subseteq [t_0 - \delta, t_0 + \delta]$ this yields that the null length of all such α is bounded away from 0 by the constant $md_h(\hat{p}, \hat{q})$. Together with case 1 we get that the null distance of p and q with respect to the time function t is greater then 0, which yields the claim. $\square$

Lemma 5.1.5 ([SV16], Lemma 3.23)**.** *Let $I \times_f S$ be a GRW spacetime with metric tensor $-dt^2 + f^2(t)h$ and let (S, h) be a complete connected Riemannian manifold. Then we have:*

$$p \leq q \Leftrightarrow d_h(\hat{p}, \hat{q}) \leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt$$

Proof. Denote $I \times_f S$ by M and $-dt^2 + f^2(t)h$ by g. Consider $p, q \in M$ with $p \leq q$. By definition there exists a future directed causal curve from p to q. We pick such a curve and call it α. Similarly to the proof of the previous lemma we can reparameterize α such that it is a (future directed causal) curve of the following form: $\alpha : [t_p, t_q] \to M, t \mapsto (t, \sigma(t))$ where $\sigma : [t_p, t_q] \to S$ is a piecewise smooth curve in S from $\hat{p}$ to $\hat{q}$. Then α being a causal curve and the definition of g yield for all $t \in [t_p, t_q]$

$$0 \geq g_{\alpha(t)}(\alpha'(t), \alpha'(t)) = -1 + f(t)^2 h(\sigma'(t), \sigma'(t))$$

and therefore

$$\frac{1}{f(t)^2} \geq h(\sigma'(t), \sigma'(t)).$$

Because σ is a piecewise smooth curve from $\hat{p}$ to $\hat{q}$ we get

$$d_h(\hat{p}, \hat{q}) \leq L_h(\sigma) = \int_{t_p}^{t_q} \sqrt{h(\sigma'(t), \sigma'(t))} dt \leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt$$

which yields the claim. Now assume conversely that we have $p, q \in M$ such that

$$d_h(\hat{p}, \hat{q}) \leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt$$

holds. We denote $d := d_h(\hat{p}, \hat{q})$. Now we want to construct a future directed causal curve from p to q to show that $p \leq q$. Because (S, h) is a complete connected Riemannian manifold we can find a minimizing geodesic from $\hat{p}$ to $\hat{q}$ (see [KS21], Corollary 2.4.3). We pick such a geodesic, denote it by σ and assume without loss of generality that σ is parameterized by h-arclength. Then we have $\sigma : [0, d] \to S$ and $L_h(\sigma) = d$. Now we define the monotonically increasing map:

$$\psi : [t_p, t_q] \to [0, \infty), t \mapsto \int_{t_p}^{t} \frac{1}{f(s)} ds$$

By our assumption and because ψ is continuous there exists a $t_0 \in [t_p, t_q]$ such that $\psi(t_0) = d$. We now use σ and ψ to define:

$$\alpha : [t_p, t_q] \to M, t \mapsto \begin{cases} (t, \sigma(\psi(t))) & t \in [t_p, t_0] \\ (t, \hat{q}) & t \in [t_0, t_q] \end{cases}$$

This is a causal curve from p to q. Note that $\alpha(t_p) = (t_p, \sigma(\psi(t_p))) = (t_p, \sigma(0)) = (t_p, \hat{p})$ and $\alpha(t_q) = (t_q, \hat{q})$.
Also for $t \in [t_0, t_q]$ we have $g_{\alpha(t)}(\alpha'(t), \alpha'(t)) = -1 + f(t)^2 h(0, 0) = -1 \leq 0$. For $t \in [t_p, t_0]$ we have $g_{\alpha(t)}(\alpha'(t), \alpha'(t)) = -1 + f(t)^2 h_{\sigma(\psi(t))}(\sigma'(\psi(t))\psi'(t), \sigma'(\psi(t))\psi'(t))$. Using that σ was assumed to be parameterized by h-arclength we have:

$$h_{\sigma(\psi(t))}(\sigma'(\psi(t))\psi'(t), \sigma'(\psi(t))\psi'(t)) = \psi'(t)^2 h_{\sigma(\psi(t))}(\sigma'(\psi(t)), \sigma'(\psi(t))) = \psi'(t)^2$$

Therefore we can rewrite the condition for α being causal on $[t_p, t_0]$ the following way (for $t \in [t_p, t_0]$):

$$\begin{aligned} g_{\alpha(t)}(\alpha'(t), \alpha'(t)) \leq 0 &\Leftrightarrow -1 + f(t)^2 h_{\sigma(\psi(t))}(\sigma'(\psi(t))\psi'(t), \sigma'(\psi(t))\psi'(t)) \leq 0 \\ &\Leftrightarrow f(t)^2 \psi'(t)^2 \leq 1 \\ &\Leftrightarrow f(t)\psi'(t) \leq 1 \end{aligned}$$

The fact that $\psi'(t) = \frac{1}{f(t)}$ yields the claim. So α is a future directed causal curve from p to q and therefore $p \leq q$. □

We now use these results to prove that the null distance with respect to the time function $\tau = t$ encodes causality in GRW spacetimes.

Lemma 5.1.6 ([SV16], Lemma 3.24)**.** *Let $I \times_f S$ be a GRW spacetime with metric tensor $-dt^2 + f^2(t)h$ and let (S, h) be a complete connected Riemannian manifold. Then we have:*

$$p \leq q \Leftrightarrow \hat{d}_t(p, q) = t(q) - t(p)$$

Proof. Note that $p \leq q \Rightarrow \hat{d}_t(p, q) = t(q) - t(p)$ is always true so it is enough to show the converse direction. Denote $I \times_f S$ by M and $-dt^2 + f^2(t)h$ by g. Let $p, q \in M$ with $\hat{d}_t(p, q) = t(q) - t(p) = t_q - t_p$. We proceed similarly to the proof of Lemma 5.1.4.

Because I is an open interval we can pick $\delta > 0$ such that $[t_p - \delta, t_q + \delta] \subseteq I$. We also pick $\epsilon \in (0, \delta)$. From the definition of the null distance there exists a piecewise causal curve α from p to q with null length $\hat{L}_t(\alpha) \leq t_q - t_p + \epsilon$. Then we have $t \circ \alpha \subseteq [t_p - \delta, t_q + \delta]$. Denote $\alpha = \alpha_1 \alpha_2 \dots \alpha_k$ with α_i being a future or past directed causal curve. So for each $i = 1 \dots, k$ either α_i or $-\alpha_i$ (i.e. α_i backwards) is a future directed causal curve. This means we can reparameterize α_i as a curve $\alpha_i : [t_i, t_i + \delta_i] \to M : t \mapsto (t, \sigma_i(t))$ such that $\sigma_i : [t_i, t_i + \delta_i] \to S$ is a piecewise smooth curve in S. We know that $\hat{L}_t(\alpha_i) = \delta_i$ holds. After exchanging σ_i with $-\sigma_i$ where necessary we have that $\sigma = \sigma_1 \dots \sigma_k$ is a piecewise smooth curve in S joining $\hat{p}$ and $\hat{q}$. Therefore we have (using the definition of g in the last step):

$$d_h(\hat{p}, \hat{q}) \leq L_h(\sigma) = \sum_{i=1}^{k} L_h(\sigma_i) = \sum_{i=1}^{k} \int_{t_i}^{t_i+\delta_i} \sqrt{h_{\sigma_i(t)}(\sigma_i'(t), \sigma_i'(t))} dt \leq \sum_{i=1}^{k} \int_{t_i}^{t_i+\delta_i} \frac{1}{f(t)} dt$$

Now we want to rewrite the sum of integrals on the right-hand side. If we extract $\int_{t_p}^{t_q} \frac{1}{f(t)} dt$ we are left with a sum of integrals over $\frac{1}{f(t)}$. The sum of the length of the intervals we integrate over is exactly $\hat{L}_t(\alpha) - (t_q - t_p) \leq \epsilon$ because it is just the extra ups and downs in time that α performs. Now using $m := \inf\{f(t) \mid t \in [t_p - \delta, t_q + \delta]\}$ to bound $\frac{1}{f(t)}$ by $\frac{1}{m}$ (from above) we have:

$$\begin{aligned} \sum_{i=1}^{k} \int_{t_i}^{t_i+\delta_i} \frac{1}{f(t)} dt &\leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt + (\hat{L}_t(\alpha) - (t_q - t_p)) \frac{1}{m} \\ &\leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt + \frac{\epsilon}{m} \end{aligned}$$

Because we chose ϵ arbitrarily, $d_h(\hat{p}, \hat{q}) \leq \int_{t_p}^{t_q} \frac{1}{f(t)} dt$ holds. Using Lemma 5.1.5 we get that $p \leq q$ which yields the claim. $\square$

The next theorem is the main result of this section. It is a generalization of the previous lemma.

Theorem 5.1.7 ([SV16], Theorem 3.25). *Let $I \times_f S$ be a GRW spacetime with metric tensor $-dt^2 + f^2(t)h$ and (S, h) a complete connected Riemannian manifold. Let $\tau(t, x) = \phi(t)$ be a smooth time function on $I \times_f S$ with $\phi' > 0$. Then we have:*

$$p \leq q \Leftrightarrow \hat{d}_\tau(p, q) = \tau(q) - \tau(p)$$

Proof. Denote $I \times_f S$ by M and $-dt^2 + f^2(t)h$ by g. Recall that the null distance is conformally invariant. Our goal is to find a conformally equivalent metric $\tilde{g}$ to g such that the null distance with respect to this metric encodes causality. First of all note that $\tau = \phi \circ t$ and therefore:

$$d\tau = \phi'(t) dt$$

(This follows immediately from the definition of the exterior derivative and the chain rule: For all $p \in M$ we have $d\tau|_p = T_p\tau = T_p(\phi \circ t) = T_{t(p)}\phi \circ T_p t = \phi'(t(p))dt|_p$.) Now we define $F := \phi' \circ \phi^{-1}$ and get $F(\tau) = \phi'(\phi^{-1}(\tau)) = \phi'(t) > 0$. This yields:

$$dt = \frac{1}{F(\tau)} d\tau$$

We define $\rho(\tau) := F(\tau) f(\phi^{-1}(\tau)) (= F(\tau) f(t))$ and rewrite g as

$$g = -dt^2 + f^2(t)h = \frac{1}{F^2(\tau)}(-d\tau^2 + \rho^2(\tau)h) = \frac{1}{F^2(\tau)}\tilde{g}$$

with $\tilde{g} = -d\tau + \rho^2(\tau)h$. We have that $\tilde{g}$ and g are conformally equivalent metrics on M because $\frac{1}{F^2(\tau)} > 0$. So $\hat{d}_\tau$ with respect to g encodes causality if and only if $\hat{d}_\tau$ with respect to $\tilde{g}$ encodes causality. Because ϕ is smooth and $\phi' > 0$ we can identify $(I \times S, -d\tau^2 + \rho^2(\tau)h)$ with $(\phi(I) \times S, -dt^2 + \rho^2(t)h)$ where $t : \phi(I) \times S \to \phi(I), (t_p, \hat{p}) \to t_p$. Also $\hat{d}_\tau$ with respect to $-d\tau^2 + \rho^2(\tau)h$ can be identified with $\hat{d}_t$ on $\phi(I) \times S$ with respect to $-dt^2 + \rho^2(t)h$ on $I \times S$. From Lemma 5.1.6 we get that the null distance $\hat{d}_t$ on $(\phi(I) \times S, -dt^2 + \rho^2(t)h)$ encodes causality, which yields the claim. $\square$

5.2 In globally hyperbolic spacetimes with Cauchy temporal functions

This section contains a result obtained by A. Burtscher and L. García-Heveling in [BH22], Section 3. Namely that in globally hyperbolic spacetimes the null distance $\hat{d}_\tau$ with respect to a Cauchy temporal function τ encodes causality.

Definition 5.2.1 (causal relation)**.** *On a spacetime (M, g) we define the* ***causal relation*** *$J^+ \subseteq M \times M$ in the following way: $J^+ = \{(p, q) \in M \times M \mid p \leq q\}$ where $p \leq q$ denotes the fact, that there exists a future directed causal curve from p to q.*

Clearly we have:

Lemma 5.2.2 (causal relation)**.** *Let (M, g) be a spacetime. The causal relation is a reflexive, antisymmetric and transitive relation on $M \times M$.*

Definition 5.2.3 (null distance relation)**.** *On a spacetime (M, g) with a generalized time function τ we define the* ***null distance relation*** *$\hat{R}^+_\tau \subseteq M \times M$ as follows: $\hat{R}^+_\tau = \{(p, q) \in M \times M \mid \hat{d}_\tau(p, q) = \tau(q) - \tau(p)\}$.*

Lemma 5.2.4 (null distance relation)**.** *Let (M, g) be a spacetime with a locally anti-Lipschitz time function τ. Then the null distance relation is a reflexive, antisymmetric and transitive relation on $M \times M$ with $J^+ \subseteq \hat{R}^+_\tau$.*

Proof. $J^+ \subseteq R^+_{\hat{\tau}}$ follows from Lemma 2.2.8. Reflexivity is clear by the definition of the null distance ($\hat{d}_\tau(p, p) = 0 = \tau(p) - \tau(p)$). For transitivity consider $(p, q), (q, v) \in \hat{R}^+_\tau$

i.e. $\hat{d}_\tau(p,q) = \tau(q) - \tau(p)$ and $\hat{d}_\tau(q,v) = \tau(v) - \tau(q)$. By the triangle inequality we get $\hat{d}_\tau(p,v) \leq \hat{d}_\tau(p,q) + \hat{d}_\tau(q,v) = \tau(q) - \tau(p) + \tau(v) - \tau(q) = \tau(v) - \tau(p)$. But the reverse inequality holds by Remark 2.2.7, which yields equality. For antisymmetry consider $(p,q),(q,p) \in R^+_\tau$. $\tau(q) - \tau(p) = \hat{d}_\tau(p,q) = \hat{d}_\tau(q,p) = \tau(p) - \tau(q)$ yields $\hat{d}_\tau(p,q) = 0$. The locally anti-Lipschitz condition ensures definiteness, see Lemma 3.2.4. This gives $p = q$ and therefore antisymmetry. □

5.2.1 Conformally equivalent complete Riemannian metric

Consider a spacetime (M,g) and the riemannianized metric g^R_T with respect to some smooth timelike vector field T. Then g^R_T does not have to be complete, even if (M,g) is globally hyperbolic. Note the following counterexample:

Example 5.2.5. *Consider the Minkowski space $(\mathbb{R}^{n+1}, g_{\mathbb{M}^{n+1}})$. Let $p = (-1,0,\dots,0)$ and $q = (1,0,\dots,0)$. Denote the timelike diamond $I^+(p) \cap I^-(q)$ by D. Then D is an open subset of $\mathbb{R}^{n+1}$ and $(D,(g_{\mathbb{M}^{n+1}})|_D)$ is a globally hyperbolic spacetime. From the proof of 1.4.2 we get that $((g_{\mathbb{M}^{n+1}})|_D)^R_{\partial_t} = (g_{\mathbb{E}^{n+1}})|_D$. But $(D,(g_{\mathbb{E}^{n+1}})|_D)$ is not complete.*

But for a spacetime (M,g) with riemannianized metric g^R_T we can use the following theorem to get a conformally equivalent complete Riemannian metric

$$g_{\tilde{R}} = \Omega^2 g^R_\tau \tag{5.1}$$

with $\Omega : M \to [1,\infty)$ smooth. The corresponding Riemannian distance and length are denoted by $L_{\tilde{R}}$ and $d_{\tilde{R}}$.

Theorem 5.2.6 ([NO61], Theorem 1)**.** *Let (M,g) be a Riemannian manifold. Then there exists a conformally equivalent metric $\tilde{g}$ on M such that $(M,\tilde{g})$ is complete.*

Proof. Let (M,g) be a Riemannian manifold and let d be the induced Riemannian distance function. For $x \in M$ and $r > 0$ denote by $B^d_r(x)$ the open r-Ball (with respect to d) around x. Now we can define $r(x) := \sup(\{r > 0 \,|\, \overline{B^d_r(x)} \text{ is compact}\})$ for $x \in M$. Assume that for one $x \in M$ we have $r(x) = \infty$. Then for any bounded subset $A \subseteq M$ there exists $r > 0$ such that $A \subseteq \overline{B^d_r(x)}$. If A is also closed, it follows from the compactness of $\overline{B^d_r(x)}$ that A is compact. This means that the Heine-Borel property holds, which by the Hopf-Rinow Theorem (see [KS21], 2.4.2) is equivalent to the completeness of (M,g). So if for one $x \in M$ we have $r(x) = \infty$ we can choose $\tilde{g} = g$ and are done. Now assume that for all $x \in M$ we have $r(x) < \infty$. We then show that in this case for all $x,y \in M$ we have $|r(x) - r(y)| \leq d(x,y)$. Case 1: $y \notin \overline{B^d_{r(x)}(x)}$ and $x \notin \overline{B^d_{r(y)}(y)}$. Then we have:

$$d(x,y) > r(x) + r(y) > |r(x) - r(y)|$$

Case 2: $y \notin \overline{B^d_{r(x)}(x)}$ and $x \in \overline{B^d_{r(y)}(y)}$. Then we have $r(x) < d(x,y) \leq r(y)$. From $x \in \overline{B^d_{r(y)}(y)}$ we get $r(x) \geq d(x, \partial B^d_{r(y)}(y)) = r(y) - d(x,y)$. (Because a closed ball around x that is subset of the compact set $\overline{B^d_{r(y)}(y)}$ is again compact.) It follows that

$$d(x,y) \geq r(y) - r(x) = |r(x) - r(y)|$$

Case 3: $y \in \overline{B^d_{r(x)}(x)}$ and $x \in \overline{B^d_{r(y)}(y)}$. Similar to case 2 we get $d(x,y) \geq r(y) - r(x)$ and $d(x,y) \geq r(x) - r(y)$ which yields $d(x,y) \geq |r(x) - r(y)|$.
If follows that the map $r : M \to \mathbb{R}^+ : x \mapsto r(x)$ is continuous. Note that the map indeed goes to $\mathbb{R}^+$ because M is locally compact and therefore $r(x) > 0$ everywhere. Now we can use a partition of unity to construct a smooth map $\Omega : M \to [1,\infty)$ such that $\Omega(x) > \frac{1}{r(x)}$ for all $x \in M$. For this we choose a precompact open cover $(U_\alpha)_{\alpha\in A}$ of M and a subordinate partition of unity $(\mathcal{X}_\alpha)_{\alpha\in A}$. Note that we can choose such a partition of unity because M is Hausdorff and second countable. Note that $a_\alpha := \frac{1}{\inf(\{r(x)\,|\,x\in\overline{U_\alpha}\})} + 1$ is well defined, because the infimum is attained (r is continuous and $\overline{U_\alpha}$ is compact). Then the map $\Omega := \sum_{\alpha\in A} a_\alpha \mathcal{X}_\alpha : M \to [1,\infty)$ is smooth and satisfies $\Omega(x) > \frac{1}{r(x)}$ for all $x \in M$. We define the conformally equivalent Riemannian metric $\tilde{g} := \Omega^2 g$ with induced Riemannian distance function $\tilde{d}$ and show that $(M,\tilde{g})$ is complete. Note that it is enough to show that $B^{\tilde{d}}_{\frac{1}{3}}(x) \subseteq B^d_{\frac{r(x)}{2}}(x)$ for all $x \in M$. From this condition it follows that $\overline{B^{\tilde{d}}_{\frac{1}{3}}(x)}$ is compact for all $x \in M$. If we consider a Cauchy sequence $(x_k)_{k\in\mathbb{N}}$ in M (with respect to $\tilde{d}$) we know that there exists $\tilde{k} \in \mathbb{N}$ such that for all $k \geq \tilde{k}$ we have $\tilde{d}(x_k, x_{\tilde{k}}) < \frac{1}{3}$. This means that after the index $\tilde{k}$ the sequence stays inside the compact set $\overline{B^{\tilde{d}}_{\frac{1}{3}}(x_{\tilde{k}})}$ and is therefore convergent. Now let us show that $B^{\tilde{d}}_{\frac{1}{3}}(x) \subseteq B^d_{\frac{r(x)}{2}}(x)$ holds for all $x \in M$. We pick $x, y \in M$ with $y \notin B^d_{\frac{r(x)}{2}}(x)$ (i.e. $d(x,y) \geq \frac{r(x)}{2}$) and show that also $y \notin B^{\tilde{d}}_{\frac{1}{3}}(x)$. Consider a piecewise smooth curve $\alpha : [a,b] \to M$ from x to y. We know that, when denoting the arc length with respect to g by L, the following holds:

$$L(\alpha) \geq d(x,y) \geq \frac{r(x)}{2} \tag{5.2}$$

Now consider $\tilde{L}(\alpha)$, the arc length of α with respect to $\tilde{g}$.

$$\begin{aligned}\tilde{L}(\alpha) &= \int_a^b \|\alpha'(t)\|_{\tilde{g}} dt = \int_a^b \sqrt{\tilde{g}|_{\alpha(t)}(\alpha'(t),\alpha'(t))}dt = \int_a^b \sqrt{\Omega^2(\alpha(t)) g|_{\alpha(t)}(\alpha'(t),\alpha'(t))}dt \\ &= \int_a^b \Omega(\alpha(t))\sqrt{g|_{\alpha(t)}(\alpha'(t),\alpha'(t))}dt\end{aligned}$$

Note that $\Omega \circ \alpha : [a,b] \to \mathbb{R}$ is a continuous map on a compact set and therefore there exists $c \in [a,b]$ such that $\Omega(\alpha(c)) \leq \Omega(\alpha(t))$ for all $t \in [a,b]$. We choose such a c and get:

$$\begin{aligned}\int_a^b \Omega(\alpha(t))\sqrt{g|_{\alpha(t)}(\alpha'(t),\alpha'(t))}dt &\geq \Omega(\alpha(c)) \int_a^b \sqrt{g|_{\alpha(t)}(\alpha'(t),\alpha'(t))}dt \\ &= \Omega(\alpha(c))L(\alpha) > \frac{L(\alpha)}{r(\alpha(c))}\end{aligned}$$

Now we use $r(\alpha(c)) - r(x) \leq |r(\alpha(c)) - r(x)| \leq d(x, \alpha(c)) \leq L(\alpha)$ and get $r(\alpha(c)) \leq d(x, \alpha(c)) + r(x) \leq L(\alpha) + r(x)$. This inequality, the result from abvove and (5.2) yield

$$\tilde{L}(\alpha) > \frac{L(\alpha)}{r(\alpha(c))} \geq \frac{L(\alpha)}{L(\alpha) + r(x)} \geq \frac{L(\alpha)}{L(\alpha) + 2L(\alpha)} = \frac{1}{3}$$

Because this works for every piecewise smooth curve α from x to y we get, that $\tilde{d}(x, y) \geq \frac{1}{3}$ i.e $y \notin B^{\tilde{d}}_{\frac{1}{3}}(x)$ which yields the claim. $\square$

Lemma 5.2.7 ([BH22], Lemma 3.4)**.** *Let (M, g) be a spacetime and g^R_T and let $g_{\tilde{R}}$ be the conformally equivalent Riemannian metrics on M, defined above. Consider a compact subset $K \subseteq M$. Then we can find constants $C_K \geq c_K \geq 1$ such that for all piecewise smooth curves γ in K we have*

$$L^R_T(\gamma) \leq L_{\tilde{R}}(\gamma) \leq c_K L^R_T(\gamma)$$

and for all $x, y \in K$ we have

$$d^R_T(x, y) \leq d_{\tilde{R}}(x, y) \leq C_K d^R_T(x, y).$$

Proof. The first inequality for piecewise smooth curves follows immediately from the definition of $g_{\tilde{R}}$. For the second inequality choose $c_K = \max_{x \in K} \Omega(x)$. The first inequality for distances follows immediately from the definition of $g_{\tilde{R}}$. The second inequality for distances follows from Lemma 3.2.7. Note that $C_K \geq c_K$ follows because minimizing curves may leave K. $\square$

5.2.2 Metric derivative

In this section we define the metric derivative and show that it coincides with the norm of the analytic derivative whenever it exists. This will be used later in the proof of the main result.

Definition 5.2.8 (absolute continuity on $\mathbb{R}^n$)**.** *Let I be an interval and $f : I \to \mathbb{R}^n$ a map. We call f* ***absolutely continuous*** *on I if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all finite sets of disjoint open intervals $(a_i, b_i)_{i=1,\dots,n}$ with $[a_i, b_i] \subseteq I$ we have*

$$\sum_{i=1}^{n} |a_i - b_i| < \delta \Rightarrow \sum_{i=1}^{n} \|f(a_i) - f(b_i)\|_{\mathbb{E}^n} < \epsilon$$

Note that this can also be defined with an arbitrary open subset of $\mathbb{R}$ instead of I.

Definition 5.2.9 (local absolute continuity on $\mathbb{R}^n$)**.** *Let I be an interval and $f : I \to \mathbb{R}^n$ a map. We call f* ***locally absolutely continuous*** *on I if for all $[a, b] \subseteq I$ the map $f|_{[a,b]}$ is absolutely continuous.*

Note that for a closed interval I the terms absolutely continuous and locally absolutely continuous coincide.

Definition 5.2.10 (absolute continuity on manifolds)**.** *Let I be a closed interval and M be a connected manifold. A curve $c : I \to M$ is called **absolutely continuous** on I if for each chart (ϕ, U) of M the composition (a map from a subset of $\mathbb{R}$ to $\mathbb{R}^n$)*

$$\phi \circ c : (I \cap c^{-1}(U)) \to \phi(U)$$

is locally absolutely continuous.

Lemma 5.2.11 (Lipschitz continuity ⇒ absolute continuity)**.** *Let (M, g) be a connected Riemannian manifold and $c : I \to M$ a curve. If c is (globally) Lipschitz continuous (with respect to d_g) then c is absolutely continuous.*

Proof. We want to show that for all charts (ϕ, U) of the manifold and all $[a, b] \subseteq (I \cap c^{-1}(U))$ the map $(\phi \circ c)|_{[a,b]}$ is absolutely continuous. This follows trivially if for all charts (ϕ, U) the map $(\phi \circ c)|_{[a,b]}$ is Lipschitz continuous. Because c is already Lipschitz continuous by assumption, it is enough to show that there exists $L > 0$ such that for all $p, q \in c([a, b])$ we have:

$$Ld_g(p, q) \geq \|\phi(q) - \phi(p)\|_{\mathbb{E}^n}$$

For this we pick a chart (ϕ, U) of M and consider $\tilde{g} := (\phi)^*(g_{\mathbb{E}^{n-1}})$ the pullback of $g_{\mathbb{E}^{n-1}}$ under ϕ. Then $\tilde{g}$ is a Riemannian metric on U and

$$d_{\tilde{g}}(p, q) = d_{g_{\mathbb{E}^n}}(\phi(p), \phi(q)) = \|\phi(q) - \phi(p)\|_{\mathbb{E}^n}$$

holds for all $p, q \in U$. We know that $c([a, b])$ is a compact subset of U so it follows from Lemma 3.2.7 that there exists an $L > 0$ such that for all $p, q \in c([a, b])$ we have:

$$d_{\tilde{g}}(p, q) \leq Ld_g(p, q)$$

Combining these results we obtain that there exists $L > 0$ such that for all $p, q \in c([a, b])$ we have

$$Ld_g(p, q) \geq \|\phi(q) - \phi(p)\|_{\mathbb{E}^n}$$

which yields the claim. □

Definition 5.2.12 (metric derivative)**.** *Let (X, d) be a metric space and let $\gamma : I \to X$ be a path. The **metric derivative** of γ at $t \in I$ is defined to be the following limit (whenever it exists):*

$$|\dot{\gamma}|(t) = \lim_{\delta \to 0} \frac{d(\gamma(t), \gamma(t + \delta))}{|\delta|}$$

*If for one $t \in I$ this limit exists we call γ **metrically differentiable** in t.*

Proposition 5.2.13 ([Bur15] Corollary 3.8)**.** *Let (M, g) be a connected Riemannian manifold and $c : I \to M$ a curve. If c is an absolutely continuous map then we have*

$$|\dot{c}|(t) = \|c'(t)\|_{g_{c(t)}}$$

whenever the limit is defined.

Proof. For an arbitrary $t \in I$ pick a normal neighborhood U of $c(t)$ in M such that the radial geodesics in U through $c(t)$ are globally minimizing and $\exp_{c(t)} : \tilde{U} \to U$ is a diffeomorphism for some neighborhood $\tilde{U}$ of $0_{c(t)}$ in $T_{c(t)}M$. Pick $\delta > 0$ small enough such that $c([t-\delta, t+\delta]) \subseteq U$. Using that the length of the radial geodesic $\sigma_{c(t)c(t+\delta)}$ from $c(t)$ to $c(t+\delta)$ is just $\| \exp^{-1}_{c(t)}(c(t+\delta))\|_{g_{c(t)}}$ we have:

$$d_g(c(t), c(t+\delta)) = L_g(\sigma_{c(t)c(t+\delta)}) = \| \exp^{-1}_{c(t)}(c(t+\delta))\|_{g_{c(t)}}$$

We multiply both sides with $\frac{1}{\delta}$ and take the limit:

$$\lim_{\delta \to 0^+} \frac{d(c(t), c(t+\delta))}{\delta} = \lim_{\delta \to 0^+} \left\| \frac{\exp^{-1}_{c(t)}(c(t+\delta))}{\delta} \right\|_{g_{c(t)}} = \left\| \frac{d}{d\delta}\bigg|_0 \exp^{-1}_{c(t)}(c(t+\delta)) \right\|_{g_{c(t)}}$$

The fact that $T_0 \exp_{c(t)} = id_{T_{c(t)}M}$ yields the result:

$$\left\| \frac{d}{d\delta}\bigg|_0 \exp^{-1}_{c(t)}(c(t+\delta)) \right\|_{g_{c(t)}} = \|T_0 \exp^{-1}_{c(t)}(c'(t))\|_{g_{c(t)}} = \|c'(t)\|_{g_{c(t)}}$$

□

5.2.3 Further tools

This section contains results that are used in the proof of Theorem 5.2.19.

Definition 5.2.14 (uniformly equicontinuous). *Consider two metric spaces (X, d) and $(Y, \tilde{d})$ and $(f_i : X \to Y)_{i \in I}$ a family of maps between them. This family is called **uniformly equicontinuous** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$ and all $i \in I$ we have*

$$d(x, y) < \delta \Rightarrow \tilde{d}(f_i(x), f_i(y)) < \epsilon$$

Theorem 5.2.15 (Arzelà-Ascoli). *Let X be a topological space that is Hausdorff and second countable. Let (M, g) be Riemannian manifold that is complete and denote the induced Riemannian distance function by d_g. Consider a sequence $(f_n : X \to M)_{n \in \mathbb{N}}$ of uniformly equicontinuous maps such that for each $x \in X$ the set $\bigcup_{n \in \mathbb{N}} f_n(x)$ is bounded (with respect to d_g). Then there exists a subsequence of $(f_n : X \to M)_{n \in \mathbb{N}}$ that converges uniformly to a continuous map $f : X \to M$ on each compact subset of X.*

Proof. See [Mun00], Theorem 47.1. □

Lemma 5.2.16 ([BEE96], Lemma 3.65). *Let (M, g) be a complete Riemannian manifold. If a piecewise smooth curve $c : [0, 1) \to M$ has finite g-length then there exists a point $p \in M$ with $\lim_{t \nearrow 1} c(t) = p$.*

Proof. Denote by d the induced Riemannian distance function of (M,g) and by L the (finite) g-length of c. Now consider the set $U := \overline{B^d_L(c(0))}$. Then the image of c is contained in U because for all $t \in [0,1)$ we have $d(c(0),c(t)) \leq L_g(c|_{[0,t]}) \leq L$. Additionally it follows from the Hopf-Rinow Theorem (see [KS21], Theorem 2.4.2) that U is compact (because it is closed and bounded). We fix a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,1)$ with $\lim_{n\to\infty} t_n = 1$ and consider $(c(t_n))_{n\in\mathbb{N}}$, a sequence in U. From the compactness of U it follows that $(c(t_n))_{n\in\mathbb{N}}$ has a convergent subsequence with a limit $p \in U$. Assume that $\lim_{n\to\infty} c(t_n) \neq p$. Then (by the definition of convergence) there exists $\epsilon > 0$ such that c leaves $B^d_\epsilon(p)$ infinitely many times. But this and the convergence of the subsequence is a contradiction to the length of c being finite. Therefore we must have $\lim_{n\to\infty} c(t_n) = p$. The claim follows because this works for all such sequences $(t_n)_{n\in\mathbb{N}}$. ☐

Lemma 5.2.17. *Let (M,g) be a spacetime with time function τ and $p,q \in M$. Let $c : [a,b] \to M$ be a piecewise causal curve from p to q with $\hat{L}_\tau(c) \leq \tau(q) - \tau(p) + \delta$ for some $\delta > 0$. Then for all $[s,t] \subseteq [a,b]$ we have $\hat{L}_\tau(c|_{[s,t]}) \leq \tau(c(t)) - \tau(c(s)) + \delta$.*

Proof. Assume that there exists $[s,t] \subseteq [a,b]$ such that $\hat{L}_\tau(c|_{[s,t]}) > \tau(c(t)) - \tau(c(s)) + \delta$ holds. But then we have

$$\begin{aligned}\hat{L}_\tau(c) &= \hat{L}_\tau(c|_{[a,s]}) + \hat{L}_\tau(c|_{[s,t]}) + \hat{L}_\tau(c|_{[t,b]})\\ &> |\tau(c(s)) - \tau(p)| + |\tau(c(t)) - \tau(c(s)) + \delta| + |\tau(q) - \tau(c(t))|\\ &\geq \tau(q) - \tau(p) + \delta\end{aligned}$$

which is a contradiction. Therefore the claim must be true. ☐

Lemma 5.2.18 (future inextendible future directed causal curves exist at any point)**.** *Let (M,g) be a Semi-Riemannian manifold and $p \in M$ an arbitrary point. Then there exists a future inextendible future directed causal curve starting at p.*

Proof. For a given point $p \in M$ pick a vector v in the future causal cone of T_pM. Consider the unique maximal geodesic $c_v : I \to M$, with $c_v(0) = p$ and $c_v'(0) = v$. We know that c_v is future directed causal because it is so at 0 and geodesics have causal character. Now restrict the maximal interval $I = (a,b) \ni 0$ to $[0,b)$ and denote the new future directed causal curve starting at p with $\tilde{c_v}$. This curve is, furthermore, future inextendible. Otherwise it would be extendible as a geodesic to $[0,b]$ (see [KS21], Corollary 2.2.8) which is a contradiction to (a,b) being the maximal domain of c_v. ☐

Note that this works completely analogously for past directed or timelike curves.

5.2.4 Statements and proofs

Finally we can prove the main result:

Theorem 5.2.19 ([BH22], Theorem 1.9)**.** *Let (M,g) be a globally hyperbolic spacetime with a Cauchy temporal function $\tau : M \to \mathbb{R}$. Then $\hat{d}_\tau$ encodes causality i.e. $J^+ = \hat{R}^+_\tau$.*

The idea of the proof is the following: For $(p,q) \in R_\tau^+$ we want to construct a future directed causal curve α from p to q. For this we first consider a sequence of piecewise causal curves $(\alpha_n)_{n\in\mathbb{N}}$ from p to q whose null length approaches $\tau(q)-\tau(p)$. We extend them to future directed future inextendible causal curves and use the Arzelà-Ascoli Theorem (Theorem 5.2.15) to get a continuous limit curve α. After this we show that the curve is future directed causal and finally that q lies on α.

Proof of Theorem 5.2.19. The fact that $J^+ \subseteq \hat{R}_\tau$ follows from the definition of the null distance, so we only have to show the converse inclusion. Therefore pick $(p,q) \in \hat{R}_\tau$ i.e. $\hat{d}_\tau(p,q) = \tau(q)-\tau(p)$. If $p = q$ then of course $(p,q) \in J^+$ and we are done. So assume $p \neq q$. Because $\hat{d}_\tau$ is definite (because temporal functions are locally anti-Lipschitz) we have $\tau(q)-\tau(p) > 0$.

Step 1: Construct α

We know that $\hat{d}_\tau(p,q)$ is defined as the infimum of the null length of piecewise causal curves joining p and q. Therefore we can find a sequence $(\alpha_n : [0,L_n] \to M)_{n\in\mathbb{N}}$ of piecewise causal curves in M joining p and q such that:

$$\hat{L}_\tau(\alpha_n) \leq \tau(q) - \tau(p) + \frac{1}{n} \tag{5.3}$$

By Lemma 5.2.17 we know that for all $[s,t] \subseteq [0,L_n]$ we have:

$$|\tau(\alpha_n(t)) - \tau(\alpha_n(s))| \leq \hat{L}_\tau(\alpha_n|_{[s,t]}) \leq \tau(\alpha_n(t)) - \tau(\alpha_n(s)) + \frac{1}{n} \tag{5.4}$$

Without loss of generality we can assume that for all $n \in \mathbb{N}$ α_n is parameterized by $g_{\tilde{R}}$-arclength (for the definition of $g_{\tilde{R}}$ see (5.1)). Also by Lemma 5.2.18 we can choose a future inextendible future directed causal curve $\tilde{\alpha}$ starting at q. Because $(M, g_{\tilde{R}})$ is a complete Riemannian manifold Lemma 5.2.16 implies that $\tilde{\alpha}$ has infinite $g_{\tilde{R}}$-length. We attach $\tilde{\alpha}$ at each α_n and call the resulting curves again $\alpha_n : [0,\infty) \to M$. We can assume that $\tilde{\alpha}$ is also parameterized by $g_{\tilde{R}}$-arclength. (Note that we need the fact that $\tilde{\alpha}$ has infinite $g_{\tilde{R}}$-length to have α_n parameterized by arclength on $[0,\infty)$.) We then have for all $n \in \mathbb{N}$ and for all $s,t \in [0,\infty)$ with $s \leq t$:

$$d_{\tilde{R}}(\alpha_n(s), \alpha_n(t)) \leq L_{\tilde{R}}(\alpha_n|_{[s,t]}) = |s-t| \tag{5.5}$$

Also note that the local estimate (5.4) holds for all $[s,t] \subseteq [0,\infty)$ when considering the new $\alpha_n : [0,\infty) \to M$ because the attached curve $\tilde{\alpha}$ is future directed causal. Our next step is to use the Arzelà-Ascoli Theorem (Theorem 5.2.15) on the sequence $(\alpha_n)_{n\in\mathbb{N}}$. From (5.5) it follows that $(\alpha_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous (with respect to $d_{\tilde{R}}$) and that for all $t \in [0,\infty)$ and all $n \in \mathbb{N}$ we have $\alpha_n(t) \in \overline{B_t^{d_{\tilde{R}}}(p)}$. This implies that the conditions of the theorem are satisfied so there exists a subsequence of $(\alpha_n)_{n\in\mathbb{N}}$ (which we will again denote by $(\alpha_n)_{n\in\mathbb{N}}$) that converges uniformly (with respect to $d_{\tilde{R}}$) on each $[a,b] \subseteq [0,\infty)$ to a continuous map $\alpha : [0,\infty) \to M$. Note that the subsequence $(\alpha_n)_{n\in\mathbb{N}}$ still satisfies (5.3). Also (5.5) implies that α is (globally) Lipschitz continuous (with respect to $d_{\tilde{R}}$) and for all $s,t \in [0,\infty)$ we have:

$$d_{\tilde{R}}(\alpha(s), \alpha(t)) \leq |s-t| \tag{5.6}$$

Step 2: Show α is future directed causal

Now we want to show that α is future directed causal. From the Lipschitz continuity it follows by Rademacher's Theorem that $\alpha'(t)$ exists for almost all $t \in [0, \infty)$. From Lemma 5.2.11 we know that α is absolutely continuous. Using Proposition 5.2.13 we get that the metric derivative of α (with respect to $d_{\tilde{R}}$) and the $d_{\tilde{R}}$-norm of the analytic derivative coincide whenever they exist. This and (5.6) give:

$$0 \leq \|\alpha'(t)\|_{g_{\tilde{R}}} = \lim_{h \to 0} \frac{d_{\tilde{R}}(\alpha(t), \alpha(t+h))}{|h|} \leq \lim_{h \to 0} \frac{|h|}{|h|} = 1 \tag{5.7}$$

Now we pick $s \in [0, \infty)$ such that $\alpha'(s)$ exists and consider for some $\epsilon > 0$ the closed ball $\overline{B_\epsilon^{d_{\tilde{R}}}(\alpha(s))}$. Note that $\overline{B_\epsilon^{d_{\tilde{R}}}(\alpha(s))}$ is compact by the Hopf-Rinow Theorem (see [KS21], Theorem 2.4.2) because it is closed and bounded. By (5.6) we have that $\alpha([s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]) \subseteq \overline{B_{\frac{\epsilon}{2}}^{d_{\tilde{R}}}(\alpha(s))}$. Because $(\alpha_n)_{n \in \mathbb{N}}$ converges uniformly to α on all $[a, b] \subseteq [0, \infty)$ we have that for n large enough $\alpha_n([s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]) \subseteq \overline{B_\epsilon^{d_{\tilde{R}}}(\alpha(s))}$. Using (5.4) we get:

$$\lim_{n \to \infty} \hat{L}_\tau(\alpha_n|_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]}) = \tau(\alpha\Big(s + \frac{\epsilon}{2}\Big)) - \tau(\alpha\Big(s - \frac{\epsilon}{2}\Big)) \tag{5.8}$$

Also we get combining Lemma 4.3.3 and Lemma 5.2.7 (and using that α_n is parameterized by $g_{\tilde{R}}$-arclength for all $n \in \mathbb{N}$) when choosing $c_\epsilon = \max\{\Omega(x) | x \in \overline{B_\epsilon^{d_{\tilde{R}}}(\alpha(s))}\} \geq 1$:

$$\epsilon = L_{\tilde{R}}(\alpha_n|_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]}) \leq c_\epsilon L_\tau^R(\alpha_n|_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]}) \leq \sqrt{2} c_\epsilon \hat{L}_\tau(\alpha_n|_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]})$$

This and (5.8) yield:

$$\frac{1}{\sqrt{2} c_\epsilon} \leq \frac{1}{\epsilon} \lim_{n \to \infty} \tilde{L}_\tau(\alpha_n|_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]}) = \frac{\tau(\alpha\Big(s + \frac{\epsilon}{2}\Big)) - \tau(\alpha\Big(s - \frac{\epsilon}{2}\Big))}{\epsilon} \tag{5.9}$$

Now we consider $\epsilon \to 0$. The right-hand side of (5.9) converges to $(\tau \circ \alpha)'(s)$ while $c_\epsilon \to \Omega(\alpha(s))$ because Ω is continuous. Therefore we have:

$$0 < \frac{1}{\sqrt{2} \Omega(\alpha(s))} \leq (\tau \circ \alpha)'(s)$$

Also by the definition of g_τ^R ($g = -2d\tau^2 + g_\tau^R$) and this result we have:

$$\begin{aligned} g_{\alpha(s)}(\alpha'(s), \alpha'(s)) &= -2d\tau^2(\alpha'(s), \alpha'(s)) + g_\tau^R(\alpha'(s), \alpha'(s)) \\ &= -2((\tau \circ \alpha)'(s))^2 + \|\alpha'(s)\|_{g_\tau^R}^2 \\ &\leq -\Omega(\alpha(s))^{-2} + \Omega(\alpha(s))^{-2} \|\alpha'(s)\|_{\tilde{R}}^2 \leq 0 \end{aligned}$$

where the last inequality follows from (5.7). Because we can do the above calculations with almost every $s \in [0, \infty)$ α' is future directed causal almost everywhere. If follows that α is a future directed causal curve.

Step 3: Show that q lies on α
If we can show that $q = \alpha(t)$ for some $s \in [0, \infty)$ we are done because then $\alpha|_{[0,s]}$ is a future directed causal curve from p to q i.e. $(p, q) \in J^+$. For this we distinguish two cases:
Case 1: $\exists s \in [0, \infty) : \tau(\alpha(s)) = \tau(q)$. We show that in this case we have $q = \alpha(s)$. Because $\tilde{d}_\tau$ and $d_{\tilde{R}}$ both induce the manifold topology we have that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $p \in M$ we have $d_{\tilde{R}}(\alpha(s), p) < \delta \Rightarrow \hat{d}_\tau(\alpha(s), p) < \epsilon$. Pick some $\epsilon > 0$. From the Arzelà-Ascoli Theorem (Theorem 5.2.15) we got (in Step 1) that $\alpha_n(s)$ converges to $\alpha(s)$ with respect to $d_{\tilde{R}}$. This and the argument above yield that for n large enough we have

$$\hat{d}_\tau(\alpha_n(s), \alpha(s)) < \epsilon \tag{5.10}$$

Because τ is continuous we also have for n sufficiently large $|\tau(q)-\tau(\alpha_n(s))| = |\tau(\alpha(s))-\tau(\alpha_n(s))| < \epsilon$. Choose $n > \frac{1}{\epsilon}$ large enough to satisfy this inequality and (5.10). Using the local estimate (5.4) we get assuming $s \leq L_n$

$$\hat{d}_\tau(\alpha_n(s), q) \leq \hat{L}_\tau(\alpha_n|_{[s,L_n]}) \leq \tau(q) - \tau(\alpha_n(s)) + \frac{1}{n} < 2\epsilon \tag{5.11}$$

(Note that the case $s > L_n$ works analogously because the local estimate is true for all $[s, t] \subseteq [0, \infty)$ because $\tilde{\alpha}$ is future directed causal, see Step 1.) Combining (5.10) and (5.11) we get that $\hat{d}_\tau(q, \alpha(s)) < 3\epsilon$. Because we chose ϵ arbitrarily (and $\hat{d}_\tau$ is definite) it follows that $q = \alpha(s)$.
Case 2: $\forall s \in [0, \infty) : \tau(\alpha(s)) \neq \tau(q)$. Because $\tau \circ \alpha$ is strictly increasing and continuous and $\tau(q) > \tau(p)$ this is equivalent to $\tau(\alpha(s)) < \tau(q)$ for all $s \in [0, \infty)$. We show that this yields a contradiction. We know that τ is a Cauchy time function, which means that each level set $\tau^{-1}(s)$ is intersected exactly once by every inextendible causal curve. It follows that α is not future inextendible: $\tau(p) < \tau(q)$ and α intersects the level set $\tau^{-1}(\tau(p))$ but not $\tau^{-1}(\tau(q))$. This means $\lim_{t\to\infty} \alpha(t) =: x \in M$ exists and $\tau(x) \leq \tau(q)$ because $\tau \circ \alpha$ is continuous. If $x = q$ we are done (extend α to a future directed causal curve from p to x). So we assume $x \neq q$. Then we can choose a precompact open neighborhood W of $\alpha([0, \infty)) \cup \{x\}$ such that $q \notin \overline{W}$. Our goal is now to generate a contradiction by showing that α must leave W at some point. We do this using the curves $(\alpha_n)_{n\in\mathbb{N}}$. Because $\alpha_n(0) = p \in W$ and $\alpha_n(L_n) = q \notin \overline{W}$ there must exist a point in $[0, \infty)$ where α leaves W for the first time. We define:

$$b_n = \sup(\{s \,|\, \forall t \in [0, s] : \alpha_n(t) \in W\}) \in (0, L_n)$$

Clearly $\alpha_n(b_n) \in \partial W$. Because $\overline{W}$ is compact we can use Lemma 5.2.7 and Lemma 4.3.3 and get (using also that α_n is parameterized by $g_{\tilde{R}}$-arclength) for some $C > 0$:

$$b_n = L_{\tilde{R}}(\alpha_n|_{[0,b_n]}) \leq \sqrt{2}C\hat{L}_\tau(\alpha_n|_{[0,b_n]}) \leq \sqrt{2}C(\hat{L}_\tau(\alpha_n|_{[0,L_n]})) \leq \sqrt{2}C(\tau(q) - \tau(p) + \frac{1}{n}) \tag{5.12}$$

This means that $(b_n)_{n\in\mathbb{N}}$ is bounded from above by $a := \sqrt{2}C(\tau(q) - \tau(p) + 1)$ (and from below by 0). Consequently $b := \limsup_{n\to\infty} b_n \in [0, a)$ exists and in particular there exists a subsequence of $(b_n)_{n\in\mathbb{N}}$ that converges to b (which we will also denote by $(b_n)_{n\in\mathbb{N}}$ and use from now on). Also we know that α_n converges to α on all compact intervals, in particular on $[0, a]$. Pick $\epsilon > 0$. Then for n large enough we have:

$$d_{\tilde{R}}(\alpha(b_n), \alpha_n(b_n)) < \epsilon \tag{5.13}$$

Note that $(\alpha_n(b_n))_{n\in\mathbb{N}}$ is a sequence in ∂W, a compact set (because it is a closed subset of the compact set $\overline{W}$). Therefore there exists a convergent subsequence (which we will again denote by $(\alpha_n(b_n))_{n\in\mathbb{N}}$) that converges (with respect to d_g) to some $y \in \partial W$. For n large enough we have therefore:

$$d_{\tilde{R}}(\alpha_n(b_n), y) < \epsilon \tag{5.14}$$

Combining these results we have for n large enough:

$$d_{\tilde{R}}(\alpha(b_n), y) \leq d_{\tilde{R}}(\alpha(b_n), \alpha_n(b_n)) + d_{\tilde{R}}(\alpha_n(b_n), y) < 2\epsilon$$

Because we chose ϵ arbitrarily it follows that $d_{\tilde{R}}(\alpha(b), y) = 0$, which yields $\alpha(b) \in \partial W$, a contradiction to α being contained in the open set W. □

Remark 5.2.20. *The assumption of global hyperbolicity in Theorem 5.2.19 cannot be omitted. From Theorem 1.3.7 we know that spacetimes are globally hyperbolic if and only if they admit a Cauchy time function. The following is an example of a (not globally hyperbolic) spacetime that admits a (not Cauchy) temporal function such that the null distance does not encode causality.*

Example 5.2.21 ([AB22], Example 3.23)**.** *Consider the two-dimensional Minkowski space $\mathbb{M}^2$ without the point $(1, 1)$. Let $p = (0, 0)$, $q = (2, 2)$ and $\tau = t$. Clearly $p \not\leq q$ but we have $\hat{d}_\tau(p, q) = 2 = 2 - 0 = \tau(q) - \tau(p)$. This can be seen by taking an approximating sequence of piecewise causal curves from p to q, see Figure 5.1.*

This shows that Theorem 5.2.19 is wrong without the assumption of global hyperbolicity.

5.3 Local result for temporal functions

Locally the null distance with respect to a temporal function encodes causality on all stably causal spacetimes:

Theorem 5.3.1 ([BH22], Theorem 3.3)**.** *Let (M, g) be a spacetime with a temporal function $\tau : M \to \mathbb{R}$. Then every point $x \in M$ has a neighborhood U in M such that:*

$$J^+ \cap (U \times U) = \hat{R}^+_\tau \cap (U \times U)$$

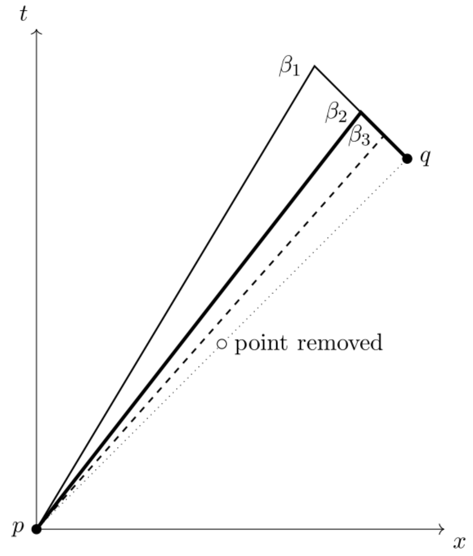

Figure 5.1: [AB22], page 27.

Note that in the proof of Theorem 5.2.19 the global hyperbolicity of the spacetime was only used in Step 3 (when using that the temporal fuction τ is a Cauchy time function). Everything else works also for non globally hyperbolic spacetimes with (not Cauchy) temporal functions.

Proof of Theorem 5.3.1. Remember that $J^+ \subseteq \hat{R}^+_\tau$ is always true so we only need to show that $J^+ \supseteq \hat{R}^+_\tau$ holds locally. Pick a point $x \in M$. Because M is locally compact and $\hat{d}_\tau$ induces the manifold topology (due to the fact that temporal functions are locally anti-Lipschitz and Theorem 3.2.5) we can choose an $r > 0$ such that the open null ball $\hat{B}^\tau_{3r}(x)$ is precompact. Denote $U := \hat{B}^\tau_r(x)$ and pick $p, q \in U$ with $(p, q) \in \hat{R}^+_\tau$. So we have $\hat{d}_\tau(p, q) = \tau(q) - \tau(p)$ and furthermore by the triangle inequality $\hat{d}_\tau(p, q) \leq \hat{d}_\tau(p, x) + \hat{d}_\tau(x, q) < 2r$.

Step 1: Constructing α

Because of the definition of the null distance we can pick a sequence of piecewise causal curves $(\alpha_n : [0, L_n] \to M)_{n \in \mathbb{N}}$ from p to q whose null lengths approach $\hat{d}_\tau(p, q)$ i.e. $\lim_{n\to\infty} \hat{L}_\tau(\alpha_n) = \tau(q) - \tau(p)$. We choose $0 < \epsilon < \hat{d}_\tau(p, q)$ such that $\hat{d}_\tau(p, q) + \epsilon < 2r$ and assume without loss of generality that for all $n \in \mathbb{N}$ we have $\hat{L}_\tau(\alpha_n) < \hat{d}_\tau(p, q) + \epsilon$. Because this means that $\hat{L}_\tau(\alpha_n) < 2r$ and α_n goes from p to q with $p, q \in U := \hat{B}^\tau_r(x)$ we get that $\alpha_n([0, L_n]) \subseteq \hat{B}^\tau_{3r}(x)$ for all $n \in \mathbb{N}$. Also τ is a temporal function so we can consider the riemannianization g^R_τ of g with respect to τ. Without loss of generality we assume that all α_n are parameterized by g^R_τ-arclength, which yields $L_n = L^R_\tau(\alpha_n)$. Using Lemma 4.3.3 we get:

$$\hat{d}_\tau(p, q) \leq \hat{L}_\tau(\alpha_n) \leq L^R_\tau(\alpha_n) = L_n \leq \sqrt{2}\hat{L}_\tau(\alpha_n) \leq \sqrt{2}(\hat{d}_\tau(p, q) + \epsilon) \tag{5.15}$$

Now we want all the curves $(\alpha_n)_{n\in\mathbb{N}}$ to be defined on the same interval $[0, L]$ with $L := \sqrt{2}(\hat{d}_\tau(p, q) + \epsilon)$. For this we attach a future directed causal curve $\widetilde{\alpha_n}$ to the curve α_n at the point q. We assume that $\widetilde{\alpha_n}$ is parameterized by g^R_τ-arclength with $L^R_\tau(\widetilde{\alpha_n}) := \widetilde{L_n} := L - L_n$. Denote the new sequence again by $(\alpha_n : [0, L] \to M)_{n\in\mathbb{N}}$. Note that for ϵ small enough we have the following:

$$\widetilde{L_n} = L - L_n < \sqrt{2}(\hat{d}_\tau(p, q) + \epsilon) - \hat{d}_\tau(p, q) < 2r(\sqrt{2} - 1) + \sqrt{2}\epsilon < r \tag{5.16}$$

Because $\widetilde{\alpha_n}$ starts at $q \in U = \hat{B}^\tau_r(x)$ and $\hat{L}_\tau(\widetilde{\alpha_n}) \leq \widetilde{L_n} < r$ (because of Lemma 4.3.3) this yields that $\alpha_n([0, L]) \subseteq \overline{\hat{B}^\tau_{3r}(x)}$. Now consider $(\overline{\hat{B}^\tau_{3r}(x)}, g^R_\tau|_{\overline{\hat{B}^\tau_{3r}(x)}})$. This is a compact Riemannian manifold and therefore complete (see [KS21], Corollary 2.4.6). Also $(\alpha_n)_{n\in\mathbb{N}}$ is a uniformly equicontinuous family of curves because they are parameterized by g^R_τ-arclength and of course $\bigcup_{n\in\mathbb{N}}(\alpha_n(t))$ is bounded for all $t \in [0, L]$. This means we can use the Arzelà-Ascoli Theorem (Theorem 5.2.15) and get that there exists a subsequence (which we will also denote by $(\alpha_n)_{n\in\mathbb{N}}$) that converges uniformly (with respect to d^R_τ) to some continuous $\alpha : [0, L] \to \overline{\hat{B}^\tau_{3r}(x)}$ on the whole domain $[0, L]$. Also (from the arclength parameterization of α_n) it follows that α is Lipschitz continuous (with respect to d^R_τ) with Lipschitz constant 1.

Step 2: α is future directed causal

Exchange $g_{\tilde{R}}$ with g^R_τ in step 2 of the proof of Theorem 5.2.19. We can do this because the completeness of $g_{\tilde{R}}$ was not used in this step.

Step 3: q lies on α

We know that $\alpha_n(L_n) = q$ for all $n \in \mathbb{N}$. Also $(L_n)_{n\in\mathbb{N}}$ is a sequence in the compact set $[0, L]$ and therefore has a convergent subsequence. Pick such a subsequence and denote the limit by L_∞. Now pick some $\delta > 0$. Then we have for n big enough (along the subsequence):

$$\begin{aligned} d^R_\tau(\alpha(L_\infty), q) &\leq d^R_\tau(\alpha(L_\infty), \alpha_n(L_\infty)) + d^R_\tau(\alpha_n(L_\infty), \alpha_n(L_n)) + d^R_\tau(\alpha_n(L_n), q) \\ &\leq \delta + |L_n - L_\infty| + 0 < 2\delta \end{aligned}$$

which yields $q = \alpha(L_\infty)$. So q lies on α and therefore $(p, q) \in J^+$. □

6 Global hyperbolicity and null distance

This chapter contains the result obtained by A. Burtscher and L. García-Heveling that a spacetime (M, g) is globally hyperbolic if and only if there exists a time function τ such that $(M, \hat{d}_\tau)$ is a complete metric space, see Theorem 6.2.1. For further background see [BH22], Section 4.

6.1 Preparations

In fact we will prove that in globally hyperbolic spacetimes we can choose the time function such that $\hat{d}_\tau$ is a complete metric to be h-steep temporal. In this section we introduce this definition and state a proposition obtained by A. Burtscher and B. Allen in [AB22] that is used in the proof.

Definition 6.1.1 (h-steep temporal function)**.** *Let (M, g) be a spacetime and let h be a complete Riemannian metric on M. A smooth map $\tau : M \to \mathbb{R}$ is called an **h-steep temporal function** if for all $p \in M$ and for all causal vectors $v \in T_pM$*

$$d\tau(v) \geq \|v\|_h$$

is true.

Theorem 6.1.2 ([BS18], Theorem 3)**.** *A spacetime (M, g) admits an h-steep temporal function if and only if it is globally hyperbolic.*

Proof. See [BS18], Theorem 3. □

Proposition 6.1.3 ([AB22], Proposition 3.14)**.** *Consider a spacetime (M, g) with a time function τ. If there exists a complete metric d on M that induces the manifold topology and a constant $C > 0$ such that for all $p, q \in M$ the following condition holds*

$$Cd(p, q) \leq \hat{d}_\tau(p, q)$$

then $(M, \hat{d}_\tau)$ is complete.

Remark 6.1.4. *Note that d is continuous with respect to the manifold topology. By assumption d induces the manifold topology, in particular it induces a topology that is coarser then the manifold topology, which is equivalent to d being continuous, see Lemma 2.4.4.*

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to $\hat{d}_\tau$ in M. From the assumption we get that $(p_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence with respect to d. Because d is complete it follows that $(p_n)_{n\in\mathbb{N}}$ converges to a point p in M with respect to d. We show that $(p_n)_{n\in\mathbb{N}}$ converges to the same point p in M with respect to the null distance $\hat{d}_\tau$. Both d and $\hat{d}_\tau$ induce the manifold topology. For d this is true by assumption, for $\hat{d}_\tau$ it holds by Proposition 2.4.5 (τ is continuous because it is a time function and it is definite by the assumption $Cd(p,q) \leq \hat{d}_\tau(p,q)$ with d definite). Choose an arbitrary $\epsilon > 0$. It follows that we can find an $\epsilon' \in (0,\epsilon)$ and an $\epsilon'' > 0$ such that we have $B_{\epsilon''}(p) \subseteq \hat{B}_{\epsilon'}(p) \subseteq B_\epsilon(p)$. Because p_n converges to p with respect to d, we can find $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}, m \geq n$ we have $p_m \in B_{\epsilon''}(p) \subseteq \hat{B}_{\epsilon'}(p) \subseteq \hat{B}_\epsilon(p)$. So for each $\epsilon > 0$ we can find $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}, m \geq n$ we have $p_m \in \hat{B}_\epsilon(p)$. This means that $\lim_{n\to\infty} \hat{d}_\tau(p_n, p) = 0$. □

Theorem 6.1.5 ([AB22], Theorem 1.6)**.** *Let (M,g) be a spacetime and τ a time function on M. If τ is anti-Lipschitz on M with respect to a complete metric that induces the manifold topology, then $(M,\hat{d}_\tau)$ is complete.*

Proof. We show that the assumptions of Proposition 6.1.3 are satisfied, which will give the claim. Choose $p,q \in M$ arbitrarily. If p and q are causally related we have $|\tau(p) - \tau(q)| = \hat{d}_\tau(p,q)$, see Lemma 2.2.8. We know that τ is anti-Lipschitz on M with respect to a complete metric d. Let C be the anti-Lipschitz constant. Then we have $\hat{d}_\tau(p,q) = |\tau(p) - \tau(q)| \geq Cd(p,q)$, so the condition is satisfied. Let p and q not be causally related. For each $\epsilon > 0$ we can find a piecewise causal curve $\beta : [a,b] \to M$ from p to q with $\hat{L}(\beta) \leq \hat{d}_\tau(p,q) + \epsilon$. Note that such curves exist between any $p,q \in M$ by Lemma 2.2.1 and the definition of the null distance. Let $a = s_0 < \ldots < s_n = b$, $p_i = \beta(s_i)$ be the break points of β. By using the definition of the null length, the anti-Lipschitz condition on τ and the triangle inequality we get:

$$\hat{d}_\tau(p,q) \geq \hat{L}_\tau(\beta) - \epsilon = \sum_{k=1}^{n} |\tau(p_k) - \tau(p_{k-1})| - \epsilon \geq C \sum_{k=1}^{n} d(p_k, p_{k-1}) - \epsilon \geq Cd(p,q) - \epsilon$$

Because we chose $\epsilon > 0$ arbitrarily we have $\hat{d}_\tau(p,q) \geq Cd(p,q)$. So we can use Proposition 6.1.3 and get that $(M,\hat{d}_\tau)$ is complete. □

6.2 Statement and Proof

Theorem 6.2.1 ([BH22], Theorem 4.1)**.** *Let (M,g) be a spacetime.*

1. *If τ is a time function such that $(M,\hat{d}_\tau)$ is a complete metric space then τ is a Cauchy time function. In particular (M,g) is globally hyperbolic.*

2. *If (M,g) is globally hyperbolic then there exists a h-steep Cauchy temporal function τ (with respect to a complete Riemannian metric h) such that $(M,\hat{d}_\tau)$ is a complete metric space.*

Proof. Let τ be a time function such that $(M, \hat{d}_\tau)$ is a complete metric space and assume that τ is not a Cauchy time function. This means there exists an $s \in \mathbb{R}$ such that the level set $\tau^{-1}(s)$ is not a Cauchy surface. So there exists an inextendible causal curve γ in (M, g) that has not exactly one point of intersection with $\tau^{-1}(s)$. Without loss of generality we assume that $\gamma : \mathbb{R} \to M$ is a future directed nontrivial future inextendible curve. We know that γ does not intersect $\tau^{-1}(s)$ in exactly one point. But it cannot intersect τ^{-1} in more than one point because $\tau \circ \gamma$ is strictly monotonously increasing (because τ is a time function and γ a future directed causal curve). Therefore γ cannot intersect $\tau^{-1}(s)$ at all and it follows that $\lim_{t\to\infty} \tau(\gamma(t)) < \infty$ or $\lim_{t\to-\infty} \tau(\gamma(t)) > -\infty$. Without loss of generality we can assume that $\lim_{t\to\infty} \tau(\gamma(t)) < \infty$. Now consider the sequence $(p_n)_{n\in\mathbb{N}}$ in M with $p_n = \gamma(n)$. The points p_n are causally related among each other (because the are in the image of γ, a causal curve) so we have by Lemma 2.2.8: $\hat{d}_\tau(p_n, p_m) = |\tau(p_n) - \tau(p_m)|$. Because $\tau \circ \gamma : \mathbb{R} \to \mathbb{R}$ is strictly increasing and bounded from above we get that $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(M, \hat{d}_\tau)$. By assumption $(M, \hat{d}_\tau)$ is complete, therefore $(p_n)_{n\in\mathbb{N}}$ converges to some point $p \in M$, $p \in \bar{\gamma}$. Consider a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $a_n \to \infty$ for $n \to \infty$. We want to show that the sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n := \gamma(a_n)$ also converges to $p \in M$. By the arguments from above $(q_n)_{n\in\mathbb{N}}$ converges to some $q \in M$, $q \in \bar{\gamma}$. Consider the sequence $(z_n)_{n\in\mathbb{N}} := (p_1, q_1, p_2, q_2, \ldots)$. Because all the z_n are causally related we have $\hat{d}_\tau(z_n, z_m) = |\tau(z_n) - \tau(z_m)|$. $\tau \circ \gamma : \mathbb{R} \to \mathbb{R}$ is strictly increasing and bounded from above, therefore we get that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(M, \hat{d}_\tau)$. By assumption $(M, \hat{d}_\tau)$ is complete, therefore $(z_n)_{n\in\mathbb{N}}$ converges to some point. If follows that this point has to be $p = q$. This contradicts γ being inextendible, therefore τ must be a Cauchy time function. By Theorem 1.3.7 this is equivalent to (M, g) being globally hyperbolic.

Now assume that (M, g) is globally hyperbolic. By Theorem 6.1.2 this is equivalent to the existence of an h-steep temporal function on (M, g) (h being a complete Riemannian metric on M). Now we want to use Theorem 6.1.5. Consider d_h (the induced Riemannian distance function of h), a complete metric on M that induces the manifold topology. We show that τ is anti-Lipschitz on M with respect to d_h i.e. for all $x, y \in M$ we have

$$x \leq y \Rightarrow \tau(y) - \tau(x) \geq d_h(x, y).$$

So let us pick $x, y \in M$ with $x \leq y$ and $\gamma : [0, 1] \to M$ a future directed causal curve from $\gamma(0) = x$ to $\gamma(1) = y$. By using the definition of h-steepness and the Riemannian arc length we get:

$$\begin{aligned}
\tau(y) - \tau(x) &= \int_0^1 d\tau(\gamma'(s))ds \\
&\geq \int_0^1 \sqrt{h(\gamma'(s), \gamma'(s))}ds \\
&\geq L_h(\gamma) \\
&\geq d_h(x, y)
\end{aligned}$$

So we have that τ is anti-Lipschitz on M with respect to d_h and therefore by Theorem 6.1.5 $(M, \hat{d}_\tau)$ is complete. $\square$

Remark 6.2.2. *In particular, Theorem 6.2.1 implies: (M, g) is globally hyperbolic if and only if there exists a time function τ such that $(M, \hat{d}_\tau)$ is a complete metric space.*

Remark 6.2.3. *In the second part of the proof we did not use that τ is Cauchy. This leads to a new proof of Corollary 6.2.4, originally due to Bernard and Suhr in [BS18].*

Corollary 6.2.4 ([BH22], Corollary 4.4)**.** *Let (M, g) be a spacetime. If τ is an h-steep temporal function on M with respect to a complete Riemannian metric h, then τ is Cauchy.*

Proof. Let τ be an h-steep temporal function on M with respect to a complete Riemannian metric h. By Theorem 6.2.1 (2) we get that $(M, \hat{d}_\tau)$ is complete. (We can use this because in the proof we did not use that τ is Cauchy.) Then by Theorem 6.2.1 (1) it follows that τ is Cauchy. $\square$

Remark 6.2.5. *Note that Theorem 6.2.1 does not work without the h-steepness condition. For a counterexample see [BH22], Example 4.5.*

Bibliography

[AB22] Allen, B., Burtscher, A. Properties of the null distance and spacetime convergence. *Int. Math. Res. Not.*, No. 10, 7729–7808, 2022.

[AGH98] Andersson, L., Galloway, G. J., Howard, R. The cosmological time function, *Classical Quantum Gravity 15*, No. 2, 309–322, 1998.

[Bär20] Bär, C. *Lorentzian Geometry, Lecture Notes, Summer Term 2004*, Version January 7 2020, 2020.

[BEE96] Beem, J. K., Ehrlich P. E., Easley K. L. *Global Lorentzian geometry*, volume 202, Marcel Dekker, Inc., 1996.

[BH22] Burtscher, A., García-Heveling, L. Global hyperbolicity through the eyes of the null distance. *arXiv preprint arXiv:2209.15610v2*, 2022.

[BS05] Bernal, A., Sánchez, M. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Commun. Math. Phys. 257*, No. 1, 43–50, 2005.

[BS18] Bernard, P., Suhr, S. Lyapounov functions of closed cone fields: from Conley theory to time functions. *Commun. Math. Phys. 359*, No. 2, 467–498, 2018.

[Bur15] Burtscher, A. Y. Length structures on manifolds with continuous Riemannian metrics. *New York J. Math.* 21:273–296, 2015.

[CGM16] Chruściel, P. T., Grant, J., Minguzzi, E. On differentiability of time functions, *Ann. Henri Poincaré 17*, No. 10, 2801-2824, 2016.

[Ger70] Geroch, R. Domain of dependence. *J. Mathematical Phys. 11*, No. 2, 437–449, 1970.

[GKOS01] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer R. *Geometric theory of generalized functions with Applications to General Relativity*, Kluwer Academic Publishers, 2001.

[Gro07] Gromov, M. *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser Boston Inc., 2007. Based on the 1981 French original.

[HE73] Hawking, S. W., Ellis, G. F. R. *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, No. 1, Cambridge University Press, 1973.

[Kir94] Kirchheim, B. Rectifiable metric spaces: Local structure and regularity of the Hausdorff measure, *Proceedings of the American Mathematical Society: Volume 121, Number 1*, 1994.

[KS21] Kunzinger, M., Steinbauer, R. Riemannian Geometry. *Lecture notes, University of Vienna*, 2021.

[MS08] Minguzzi, E., Sánchez, M. The causal hierarchy of space-times, Alekseevsky, Dmitri V. (ed.) et al., Recent developments in pseudo-Riemannian geometry. Zürich: European Mathematical Society (ISBN 978-3-03719-051-7/pbk). ESI Lectures in Mathematics and Physics, 299-358 (2008).

[MS11] Müller, O., Sánchez, M. Lorentzian manifolds isometrically embeddable in $\mathbb{L}^N$, *Trans. Amer. Math. Soc.*, 363 (10):5367–5379, 2011.

[Mun00] Munkres, J.R., *Topology*, Prentice Hall, Inc., 2000.

[NO61] Nomizu, K., Ozeki, H. The existence of complete Riemannian metrics. *Proc. Amer. Math. Soc.* 12:889–891, 1961.

[O'N83] O'Neill, B. *Semi-Riemannian geometry with applications to relativity*, volume 103. Academic Press Inc., 1983.

[SS22] Sakovich, A., Sormani, C. The null distance encodes causality. *arXiv preprint arXiv:2208.01975v1*, 2022.

[SV16] Sormani, C., Vega, C. Null distance on a spacetime. *Classical Quantum Gravity 33*, No. 8, Article ID 085001, 29 p., 2016.

[Veg21] Vega, C. Spacetime distances: an exploration. *arXiv preprint arXiv:2103.01191v2*, 2021.